F. Cajori

ELEMENTS OF ALGEBRA:

EMBRACING ALSO

THE THEORY AND APPLICATION OF LOGARITHMS;

TOGETHER WITH

AN APPENDIX,

CONTAINING

INFINITE SERIES, THE GENERAL THEORY OF EQUATIONS, AND
THE MOST APPROVED METHODS OF RESOLVING
THE HIGHER EQUATIONS.

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PREFACE.

THE object of this treatise is to present to the student a full and systematic course of practical and theoretical *elementary* Algebra. With this object steadily in view, the author has made no effort for the display of mathematical genius, but has assiduously applied himself to the preparation of a text-book in the science. Believing that original discoveries are not best adapted to *beginners*, he has satisfied himself with the humble vocation of collecting, arranging, and illustrating the ample materials already provided. But it is due to himself to say that these materials have all been re-wrought, and not a few of them re-written several times. It has been a constant endeavour to make everything explicit, and also to exhibit it in the simplest possible form. By this means, the author has been enabled to embrace, within a comparatively small compass, a more comprehensive view of the science than can be found in any text-book on the subject now in use.

Among the works which have shared, and still share most largely in the patronage of the public, isolated parts or subjects are treated with great ability and clearness; but, in some instances, these works are remarkably deficient, so far as concerns any methodical arrangement of the subjects introduced, while also other subjects of great importance are omitted altogether. That these books force their way into public patronage is not surprising, when, on the other hand, those treatises which are systematic in the arrangement of topics are, in general, too theoretical and abstract for the convenience or profit of the beginner, or, indeed, of the practical algebraist.

In collecting his materials, the author has consulted the most approved writers upon the subject. It would be difficult, if not impossible, to point out the precise amount of his indebtedness to each; yet he does not hesitate to acknowledge it, nor has any desire of appearing original led him to remodel these materials. Indeed, this has been done only when it was necessary in order to preserve the unity of the work, or to render the subjects more explicit. In arranging and digesting these materials, however, the author has been fettered by no adopted system. Whatever seemed most appropriate to his general object, and in keeping with the general plan of his work, he has freely made use of, at all times having reference to the wants of our schools, and endeavouring to meet them. How far this object has been attained, he now leaves the reader to judge, claiming only for himself that it is a well-meant contribution to elementary education in an important branch of science.

For the article on "Roots of Numbers," as well as for other valuable assistance in the preparation of this work, he is indebted to the Rev. Joseph Cummings, A.B., lecturer on Natural Science in the Amenia Seminary.

In the present work, Algebra has not been regarded merely as an introduction to the higher branches of mathematics, but also as a means of unfolding more clearly the principles and theory of common arithmetic. This is an important consideration. A great portion of the students in our academies and schools do not pursue the mathematical course beyond algebra. Such, aside from the mental discipline acquired in its study, derive their chief advantage from the superior understanding it gives them of common arithmetic; and we speak only the common sentiment of the better-informed school-teachers, when we say that few, if any, are properly qualified to teach *arithmetic* without a knowledge of *algebra*. The author has not, however, limited himself to this object, and believes the work will fully answer all the necessary requisitions of an introduction to the higher branches of mathematics.

The Logic of Algebra is an object that should not be lost sight of in the study; and in order that the student may be exercised in this, every important principle has been explained and demonstrated. But, at the same time, the explanations have been made simple, and the demonstrations put in such a form (especially in the first part of the work), that they can be easily comprehended by those unaccustomed to the rigid demonstrations of analytical algebra. In the higher departments of mathematics, it is undoubtedly desirable that the formality in stating every proposition, and the course of demonstration required by the precise rules of logic, should be adhered to. But in algebra the case is different. The mind of the student must become gradually habituated to the more abstract modes of thinking and precise methods of reasoning; and, as algebra is commenced in so early a part of the course, a certain degree of familiarity, rather than formality, in the statement of propositions and in their proof, becomes not only excusable, but even necessary. The author, however, has studied precision in the statement of propositions, and endeavoured to make his reasoning explicit. In this way has he endeavoured to make the theory obvious and satisfactory.

Believing that a knowledge of the general principles of algebra can be perfected and permanently secured only by frequent and rigid application, the author has endeavoured, throughout the work, to blend *theory* and *practice.* For this purpose, a careful selection of problems and exercises has been made from the most approved authors.

In the ninth section the author has given a clear and concise view of the theory of Logarithms, and a method of calculating common logarithms, or those in general use, so explicit, and yet so simple, that the student well versed in proportion and progression may be able to calculate them with ease and facility.

As the last three sections treat upon subjects that are seldom called into use by the merely practical algebraist, and yet subjects that are indispensable as an introduction to the higher departments of mathematics, they have been thrown into the form of an Appendix.

This work was commenced, and has been carried to its completion, amid the arduous duties incident to the charge of a large and flourishing seminary of learning. Yet labour and care have been bestowed upon every part of it, and that, too, while the author was daily engaged in instructing classes in this interesting and important branch of study; and if, under these circumstances, he has been able to discover the wants of the student, and adapt his work to meet those wants, he will feel amply compensated for his toil.

D. W. Clark.

Amenia Seminary, March, 1843.

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NOTE.

The ill health of the author while the work was in course of publication has prevented him from devoting that personal attention to a final examination of the proofs that he desired. This he would offer as an apology for any errors that may have been overlooked.

ELEMENTS OF ALGEBRA.

SECTION I.

Preliminary Remarks.—Definitions.—Axioms.—Algebraic Method of Notation.

PRELIMINARY REMARKS.

1. The *object* of Mathematical Science is to investigate the relations of quantities and the properties of numbers.

2. *Quantity*, or magnitude, is a general term, embracing everything which admits of increase, diminution, and measurement.* Thus, a given weight or bulk, a sum of money, or a number of yards, are quantities.

3. The *measurement* of quantity is accomplished by means of an assumed unit or standard of measure. This unit must be of the same kind as the quantity. Thus, the measuring unit of money is one dollar; of a line is one inch, foot, or mile, &c.; of area is one square inch, foot, or acre, &c.

4. *Numbers* are symbols adopted to facilitate the investigation of quantities. They represent a unit or an assemblage of units. Thus, 35, 42, and 64 are numbers; but $35, 42 cwt., and 64 acres, are quantities.

5. Whole numbers, as 4, 6, 15, 30, &c., are called *integers.* Broken numbers, as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{5}{16}$, &c., are called *fractions.*

6. Any number which can be divided by 2 without producing a fraction, is called an *even number;* and all numbers which cannot be divided by 2 without producing a fraction, are called *odd numbers.*

7. Numbers are also distinguished into *composite* and *prime*

* See Note A.

numbers. Any number which can be produced by multiplying two or more numbers together, *each* of which is *greater than a unit*, is called a composite number, as 4, 6, 12, 20, &c. Numbers which admit of no exact divisor except themselves and unity, are called prime numbers, as 1, 2, 3, 5, 7, 11, &c.

8. *The foundation of mathematical reasoning is laid in Definitions and Axioms.* The absolute certainty of its conclusions results no less from the exactness of mathematical definitions, and the clearness and simplicity of its first principles, than from the nature of the subjects about which it is employed.

9. A *Definition*, when applied to *language*, is a brief explanation of what is meant by a word or phrase. When applied to a *thing*, it is an analysis of its parts or an enumeration of its principal attributes ; but this analysis or enumeration must be sufficiently extensive and definite to distinguish the thing defined from everything else. Definitions, in mathematics, are used to determine the meaning of the terms, as well as the signs and symbols used.

10. An *Axiom* is a self-evident truth or proposition. They are said to be self-evident, because, as soon as enunciated, they produce in the mind a force of conviction that cannot be increased by any subsequent train of reasoning. This conviction is the result of an instantaneous and intuitive perception of the simple relations involved.

11. By a skilful use of the simple elements of mathematical knowledge, furnished by Definitions and Axioms, we are led on through the most complicated processes of mathematical investigation.

DEFINITIONS.

12. A *problem* is a question proposed which requires a solution ; and the problem is said to be solved when the value of the unknown quantity, involved in the conditions of the question, is discovered.

13. A *theorem* is a general truth, which is to be proved by

a course of mathematical reasoning called a demonstration.

14. A *lemma* is a subsidiary truth previously laid down, in order to render the solution of a problem, or the demonstration of a theorem, more easy.

15. A *proposition* is a common name, applied indifferently to problems, theorems, and lemmas.

16. A *corollary* is an obvious consequence, derived from some proposition already demonstrated, without the aid of any other proposition.

17. A *scholium* is a remark made on one or several preceding propositions, to point out their connexion, their use, their restriction, or their extension.

18. A *hypothesis* is a supposition made either in the enunciation of a proposition or in the course of demonstration.

AXIOMS.

19. The following is a list of mathematical axioms. The list is incomplete, but sufficiently extensive for our present purpose.

1. The whole of a quantity is greater than a part.
2. Quantities equal to the same quantity are equal to each other.
3. If to equal numbers equals be added, the sums will be equal.
4. If from equal numbers equals be subtracted, the remainders will be equal.
5. If equal numbers be multiplied by equals, the products will be equal.
6. If equal numbers be divided by equals, the quotients will be equal.
7. If the same quantity be added to and subtracted from another, the value of the latter will not be altered.
8. If a quantity be multiplied and divided by a number, its value will not be altered.
9. If equal numbers are involved to equal degrees, their powers will be equal.

10. If corresponding roots of equal numbers be taken, they will be equal.
11. If to unequal numbers equals be added, the greater will give the greater sum.
12. If from unequal numbers equals be subtracted, the greater will give the greater remainder.
13. If unequal numbers be multiplied by equals, the greater will give the greater product.
14. If unequal numbers be divided by equals, the greater will give the greater quotient.

ALGEBRAIC NOTATION.

20. *Algebra* is that branch of mathematical science in which the relations of quantities are investigated, and the value of unknown quantities determined, by means of letters and signs.*

21. *Quantities*, in Algebra, are represented by the letters of the alphabet as well as by numbers.

22. The first letters, as a, b, c, &c., are used to represent the known quantities. The last letters, as x, y, &c., are used to represent the unknown quantities.

23. The use of letters to represent quantities is productive of several important advantages.

1. A letter may be made to represent the *unknown* quantity whose value is sought, and then be used in the solution of the problem as though its value were already determined.
2. The long and tedious processes of arithmetic may be greatly abridged by the introduction of letters, since a single letter may be made to represent any quantity, however great it may be.
3. The several quantities which enter into the calculation are preserved distinct from each other, in all their combinations.
4. The requisite operations may be performed with much more readiness, and with less liability of mistake, with letters than with numbers.

* See Note B.

5. The processes of algebra may be used to demonstrate theorems and general rules, inasmuch as a letter may represent every possible value.

24. The *relations* of quantities, or the *operations* to be performed upon them, are represented by signs. This method of notation presents to the eye, at one view, the conditions of the problem, and at the same time facilitates the reduction of it.

25. *Addition* is represented by a horizontal and perpendicular line mutually bisecting each other, as +. Thus, $a+b$ represents that b is to be added to a, and the expression is read "a plus b."

26. *Subtraction* is indicated by a horizontal line prefixed to the quantity to be subtracted, as —. Thus, $a-b$ represents that b is to be subtracted from a, and the expression is read "a minus b."

27. *Multiplication* is indicated by a sign formed something like a Roman X, as ×. Thus, $a \times b$ indicates that a is to be multiplied by b. Sometimes the multiplication is indicated by a dot placed between the quantities to be multiplied, as $a \,.\, b$; or if the quantities are represented by letters, the letters may be written one after another, in alphabetical order, without any sign, as ab. If numerals are to be multiplied, the sign must be expressed. Thus, 4×10, or $4 \,.\, 10$, without the sign, would become 410.

28. *Division* is indicated in three ways. 1. By connecting the divisor to the dividend by a horizontal line with a dot above and another below it, as ÷. Thus, $a \div b$ indicates that a is to be divided by b. 2. By making the dividend the numerator, and the divisor the denominator of a vulgar fraction, as $\frac{a}{b}$. 3. Or by placing the divisor to the right of the dividend, and drawing a perpendicular line between them, and a horizontal line under the divisor, as $a|\underline{b}$.

29. To indicate that the *difference* between two quantities is to be taken without determining which is to be subtracted, a sign like the letter *s* placed horizontally is used, as $\backsim$.

Thus, $a \backsim b$ represents that the difference between a and b is to be taken.

30. *Equality* between two quantities or sets of quantities is indicated by two horizontal lines, as $=$. Thus, $a=b$ represents that a is equal to b, and is read "a equals b."

31. An *equation* is the algebraic expression of two equal quantities connected by the sign of equality. If the algebraic quantities are known, the expression is called an equality.

32. *Inequality* is indicated by two lines forming an angle, like the letter V placed horizontally, the vertex denoting the less of the two quantities, as $>$. Thus, $a>b$ represents that a is greater than b, and is read "a greater than b."

33. An *inequation* is the algebraic expression of two unequal quantities connected by the sign of inequality. If the quantities are known, the expression is called an inequality.

34. *Proportion* is indicated in the same manner as in Common Arithmetic. Thus, $a:b::c:d$ represents that the four quantities a, b, c, and d are proportional, and the expression is read "a is to b as c is to d."

35. A *coefficient* is a numeral figure or a letter prefixed to a quantity to show how many times the quantity is to be taken. Thus, $4a$ shows that a is to be taken four times, as $a+a+a+a=4a$; and ax shows that x is to be taken as many times as there are units in a.

36. When a quantity has no number prefixed to it, 1 is always understood as its coefficient. Thus, a is the same as $1a$.

37. An *Algebraic expression* is a quantity or several quantities written in algebraic language; that is, by the aid of letters and signs.

39. An *algebraic formula* is a general rule or principle stated in algebraic language; that is, by the aid of letters and signs.

39. A *monomial* or *simple* algebraic quantity is one that

may be represented in an algebraic expression, without the aid of the signs *plus* or *minus*. Thus, a, $3ab$, $\frac{1}{2}ab^2$, and $\frac{3}{5}ab^3mx$ are monomials. Monomials are sometimes called terms.

40. *Polynomials*, or *compound* quantities, are expressions containing two or more simple quantities connected by the signs plus or minus. Thus, $a+3ab$ and $a+2b-3c$ are polynomials.

41. A polynomial composed of two terms is called a *binomial;* of three terms, a *trinomial;* of four terms, a *quadrinomial.* If the two terms of a binomial are connected by the sign minus, it is sometimes called a *residual.*

42. To indicate that like operations are to be performed upon all the terms of a polynomial, they must be included in a parenthesis, or have a bar or vinculum drawn over them. Thus, $a-(b+c)$ indicates that the sum of b and c is to be subtracted from a; and $(a+b)\times c$ indicates that the sum of a and b is to be multiplied by c; and $(a+b)\div c$ indicates that the sum of a and b is to be divided by c.

43. If both multiplicand and multiplier, or dividend and divisor, are polynomials, each should be included in a parenthesis, as $(a+b)\times(c+d)$, or $(a+b)\div(c+d)$. And in general, when a sign is prefixed to a parenthesis, it is to be understood as affecting all the terms included in the parenthesis, taken collectively.

44. *Positive* or additive quantities are those to which the sign plus is prefixed. *Negative* or subtractive quantities are those to which the minus sign is prefixed. When no sign is prefixed to the first term of an algebraic expression, the sign plus is always to be understood.

45. A quantity is said to be ambiguous with regard to its sign when it is affected with the double sign $\pm$. Thus, $a\pm b$ represents that b is to be added to or subtracted from a; and the expression is read "a plus or minus b."

46. Equal terms affected by unlike signs, in an algebraic expression, cancel each other, and may be rejected from the expression. Thus, $3a-5b+5b=3a$, since $-5b$ and $+5b$ cancel each other.

47. *Positive and negative quantities* sustain opposite relations with respect to addition; i. e., a negative quantity must be subtracted when a positive quantity would be added, and added when a positive quantity would be subtracted.

48. The numbers which are multiplied together to form a composite number, are called *factors*. Thus, $11abcx$ is a composite number, formed by multiplying the factors 11, a, b, c, and x.

49. A number is said to be *resolved into factors* when two or more numbers are taken, such that, when multiplied together, their product shall equal the given number. Thus, 54 may be resolved into 6×9, or 3×18, or 2×27.

50. The *power* of a number is the product arising from the multiplication of the number by itself, till it has been used as a factor a certain number of times. If the number is taken twice as a factor, the product is called the *second power*; if three times, the product is called the *third power*; if four times, the *fourth power*, &c.

51. The *index*, or *exponent*, is a figure or letter placed to the right and a little above the number, and is used to show the power to which the number is to be involved. The number is to be used as a factor as many times as there are units in the exponent. When no exponent is expressed, 1 is understood.

The first power of a is - - - a, or a^1.
The second power of a is - - $a\times a$, or a^2.
The third power of a is - - $a\times a\times a$, or a^3.
The fourth power of a is - $a\times a\times a\times a$, or a^4.
The mth power of a is $a\times a\times a$ m times, or a^m, &c.

52. If a polynomial is to be involved, its terms should be included in a parenthesis, and the exponent placed without the parenthesis to the right. Thus, $(a+b)^2$ is the algebraic expression of the second power of the sum of a and b.

53. *Involution* is finding the powers of numbers.

54. The *root* of a number is a number which, multiplied into itself till it is taken a certain number of times as a

factor, will produce the given number. The root is called square root, cube root, fourth root, &c., according to the number of times it must be used as a factor to produce the given number.

55. The radical sign, as $\sqrt{}$, or *fractional index*, is used to indicate that the root of a number is to be taken. The denominator of the fractional index denotes the root; and when the radical is used, the figure over the foot of the radical determines the root. Thus,

The square root of a is expressed - $\sqrt{a}$, or $a^{\frac{1}{2}}$.
The cube root of a is expressed - $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$.
The fourth root of a is expressed - $\sqrt[4]{a}$, or $a^{\frac{1}{4}}$
The fifth root of $a+b$ is expressed, $\sqrt[5]{a+b}$, or $(a+b)^{\frac{1}{5}}$, &c.

56. *Evolution* is finding the roots of algebraic numbers.

57. A *power of a root*, or *root of a power*, is a result obtained by involving the root of a number, or by extracting the root of a power. Cases of this kind are indicated as follows:

The second power of the third root of a is $\sqrt[3]{a^2}$, or $(\sqrt[3]{a})^2$, or $a^{\frac{2}{3}}$.

The third power of the fourth root of $a+b$ is $\sqrt[4]{(a+b)^3}$, or $(a+b)^{\frac{3}{4}}$.

58. It should be remarked that the exponent affects only the letter over which it is placed. Thus, in the expression abc^2, the first powers of a and b, and the second power of c, are to be taken. When no coefficient is prefixed to the radical sign, 1 is always understood as the coefficient.

59. *Exponents* should not be confounded with *coefficients*. The exponent *indicates* that the number is to be used as a *factor* a certain number of times. Thus, a^6 represents that a is to be taken six times as a factor, or $a\times a\times a\times a\times a\times a=a^6$. The coefficient indicates that the number is to be used as a *term* a certain number of times. Thus, $6a$ represents that a is to be used six times as a term, or $a+a+a+a+a+a=6a$.

60. The *reciprocal* of a quantity is the quotient arising

from dividing a unit by that quantity. Thus, the reciprocal of a is $\frac{1}{a}$; of $a+b$ is $\frac{1}{a+b}$; and of 4 is $\frac{1}{4}$.

61. The *reciprocal of a power* is the quotient arising from dividing a unit by that power, and is frequently expressed by a negative exponent. Thus,

The reciprocal of a^2 is - - - $\frac{1}{a^2}$, or a^{-2}.
The reciprocal of $4a^3$ is - - $\frac{1}{4a^3}$, or $\frac{1}{4}a^{-3}$.
The reciprocal of $(a+b)^4$ is - $\frac{1}{(a+b)^4}$, or $(a+b)^{-4}$, &c.

62. *Rational quantities* are those whose exact value can be expressed in finite terms. Thus, $4a$, $\frac{4}{3}b$, and $a+3b$, are rational quantities.

63. *Irrational quantities*, or *surds*, are those whose exact value cannot be expressed in finite terms. Thus, since only the approximate value of the square root of 2 can be obtained, $\sqrt{2}$ is called a surd; also $\sqrt{a}$ is a surd.

64. The *measure* or *divisor* of a quantity is that by which it can be divided without leaving a remainder; and when a quantity will divide two or more quantities without leaving a remainder, it is called a *common measure* of those quantities. Thus, $7a$ is a measure of $28a$, since $\frac{28a}{7a}=4$; and $3a$ is a common measure of $12a$ and $21a$, since $\frac{12a}{3a}=4$, and $\frac{21a}{3a}=7$.

65. The *multiple* of a quantity is that which can be divided by the quantity without leaving a remainder. Thus, $28a$ is a multiple of $7a$, since $\frac{28a}{7a}=4$, &c.

66. *Commensurable* quantities are such as have a common measure or divisor. Thus, $12a$ and $21a$ are commensurable, because they have a common divisor, 3 or $3a$.

67. *Incommensurable* quantities are such as have no common measure except unity. Thus, 5 and 7, $3a$ and $10b$, are incommensurable quantities.

68. The *value* of an algebraic expression is the result obtained by substituting for the letters their numerical values, and performing the operations indicated by the signs. Thus, the value of $4a-8b$, on the supposition that $a=12$ and $b=5$, is $4\times 12-8\times 5=48-40=8$.

The value of $\frac{1}{2}a+\frac{3b}{5}$, on the supposition that $a=6$ and $b=10$, is $\frac{1}{2}\times 6+\frac{3\cdot 10}{5}=3+6=9$.

69. The following examples are given for the exercise of the learner. On the supposition that $a=6$, $b=5$, $c=4$, $d=1$, and $m=10$, it is required to find the value of the following algebraic expressions.

1. $a^2+2ab+b^2=6^2+2\,.\,6\,.\,5+5^2=36+60+25=121.$
2. $2a^3-3a^2b+c^3=2\,.\,6^3-3\,.\,6^2\,.\,5+4^3=$
3. $(a+b)\times a^2-5cdm+\frac{a^2}{c}=(6+5)\,.\,6^2-5\,.\,4\,.\,1\,.\,10+\frac{6^2}{4}=$
4. $4b+a\,.\,(2c+c^2)-3m=$
5. $(4c^2+b^3)\,.\,a-(b+3b^2)\,.\,8=$
6. $\frac{a+m-d}{a-d}+\frac{bc-m}{2d}=$
7. $5\sqrt{c^2m^2}-ab\,.\,(3a^2-10m-7d)=$
8. $\frac{3a\cdot m}{9d}\,.\,(3a+b+4m)-6a\,.\,(3b+d)=$
9. $3\,.\,\sqrt{4c^2-3b}+3a\,.\,(2a+b-d)^{\frac{1}{2}}=$
10. $b\,.\,\sqrt{a^2+3d+m}-3bc\sqrt{b+c}=$
11. $\frac{2b+c}{3a-c}-\sqrt{\frac{5b-9\sqrt{c+d}}{2a+c}}=$
12. $(\frac{a\times(d+c)}{m-b}+ab)^{\frac{1}{2}}-\frac{4\times\sqrt{c+b}}{c}=$

70. The value of an algebraic expression is not altered by changing the order of the terms or the order of the factors, if its proper sign be prefixed to each. Thus, $a+b+c-d$ is the same in value as $-d+c+b+a$; and $a\times b\times c\times d$ is the same in value as $d\times c\times b\times a$. For considering the letters of the same value as in art. 69, we shall have $a+b+c-d=6+5+4-1=-1+4+5+6=14$; and $a\times b\times c\times d=6\,.\,5\,.\,4\,.\,1=1\,.\,4\,.\,5\,.\,6=120$.

Note.—It will be found convenient to write the letters in alphabetical order.

71. *Like* quantities or terms are those which consist of the same letters and the same powers, or the same roots of the same letters. Thus, $3a$, $6a$, $5a$ are like quantities; and $7a^2b$, $8a^2b$, and $a\,b$ are also like quantities.

72. *Unlike* quantities or terms are those which consist of different letters, or different powers of the same letters. Thus, $3a$ and $3ab$ are unlike quantities, because they have different letters; and $3a$ and $3a^2$ are unlike, because different powers of the same letter are taken.

73. A polynomial which is composed of like quantities or

terms may be reduced to one term. Thus, $5a+3a=8a$; for, letting $a=6$, and substituting for a its value, the expression will become $5.6+3.6=8.6$, or $30+18=48$. Again, $5a-3a=2a$; for, letting $a=6$, and substituting, as before, $5.6-3.6=2.6$, or $30-18=12$. In the third place, $-5a-3a=-8a$; for substituting, as before, $-5.6-3.6=-8.6$, or $-30-18=-48$. The same method of illustration may be applied, whatever value we assume for a.

Note.—The addition of negative quantities may seem inconsistent at first sight; but it should be recollected that the minus sign merely indicates that the quantities affected by it are to be subtracted; hence the sum or aggregate of those quantities must also be subtracted. Thus, if a merchant loses $30 in one speculation and $18 in another, the sum of his losses is $48, and the algebraic expression of it is —$30—18=—$48.

74. A polynomial which is composed of unlike quantities or terms cannot be reduced to a simpler form. Thus, $5a+3b$ can be reduced to no simpler expression; for, letting $a=6$ and $b=5$, we shall have $5a+3b=5.6+3.5=45$; but $5a+3b$ cannot equal $8a$, since $8.6=48$; neither can it equal $8b$, since $8.5=40$: therefore the polynomial cannot be reduced to a simpler expression.

75. The processes of algebra are employed chiefly in the solution of problems, or in the demonstration of theorems and the investigation of general rules. This is accomplished by means of a series of equations or proportions. But, before entering upon the consideration of these, it will be necessary to make an application of the algebraic method of notation to the fundamental principles of numeration.

SECTION II.

Addition, Subtraction, Multiplication, and Division of Algebraic Quantities.

ADDITION.

76. ADDITION is a method of finding the sum of two or more algebraic quantities.

77. This may be done by connecting the several quantities by their proper signs in one algebraic expression.

78. In order to obtain the simplest expression for the sum of two or more quantities, it is necessary to reduce the like quantities to one term. Accordingly, Addition may be conveniently considered in three cases.

CASE I.

79. In this case the quantities are like, and have like signs.

RULE.

1. *Write the quantities to be added so that the like terms may fall under each other.*

2. *Add the coefficients, and to their sum prefix the common sign, and annex the common letter or letters.*

Note.—For the reason of this and the next rule, see the illustrations in Art. 73.

EXAMPLES.

(1)	(2)	(3)
$3a+2b-5c$	$4ab-2cd$	$7a^2bx+12cb^2$
$5a+6b-c$	$7ab-cd$	$8a^2bx+8cb^2$
$7a+11b-8c$	$15ab-2cd$	$2a^2bx+cb^2$
$a+b-3c$	$ab-12cd$	$3a^2bx+7cb^2$
$16a+20b-17c$	$27ab-17cd$	$20a^2bx+28cb^2$

4. Add $3a^2-2bc$, $4a^2-5bc$, $7a^2-8bc$, and $2a^2-3bc$.

Arranging the terms for addition,

$$\begin{array}{r} 3a^2-2bc \\ 4a^2-5bc \\ 7a^2-8bc \\ 2a^2-3bc \\ \hline \end{array}$$

5. Add $-8ax-2by^2$, $-2ax-6by^2$, $-3ax-by^2$, $-9ax-8by^2$, and $-3ax-by^2$. *Ans.* $-25ax-18by^2$.

6. Add $12ab^2c-8cdx^3+5by^4$, $ab^2c-7cdx^3+3by^4$, and $3ab^2c-cdx^3+2by^4$. *Ans.* $16ab^2c-16cdx^3+10by^4$.

7. Add $3ab+2axy$, $7ab+4axy$, and $12ab+10axy$. *Ans.* $22ab+16axy$.

8. Add $17x^3y-3x^2y^2+4xy^3$, $6x^3y-12x^2y^2+6xy^3$, and $x^3y-3x^2y^2+xy^3$. *Ans.* $24x^3y-18x^2y^2+11xy^3$.

9. Add $8a^2b^3c^4-3b^2x^5$, $12a^2b^3c^4-6b^2x^5$, $13a^2b^3c^4-7b^2x^5$, $2a^2b^3c^4-6b^2x^5$, and $11a^2b^3c^4-13b^2x^5$. *Ans.* $46a^2b^3c^4-35b^2x^5$.

10. Add $60ab-12(a+b)$, $30ab-3(a+b)$, $40ab-7(a+b)$, and $80ab-3(a+b)$. *Ans.* $210ab-25(a+b)$.

CASE II.

80. In this case the quantities are like, but the signs unlike.

RULE.

1. *Write the quantities to be added so that the like terms may fall under each other.*

2. *Take the difference between the sum of the coefficients of the positive, and the sum of the coefficients of the negative terms, and to this difference prefix the sign of the greater sum, and annex the common letter or letters.*

EXAMPLES.

(1)	(2)	(3)
$12ab+6ax$	$7a^2x+13ab^2y^3$	$6cd-12xy$
$-7ab+3ax$	$6a^2x+\ ab^2y^3$	$cd-\ 3xy$
$3ab-7ax$	$-3a^2x-\ 7ab^2y^3$	$-3cd+\ 9xy$
$8ab+2ax$	$10a^2x+\ 7ab^2y^3$	$4cd-\ 6xy$

4. Add $2a-3x^2$, $7a+5x^2$ $-3a+x^2$, and $a+3x^2$. *Ans.* $7a+6x^2$

5. Add $4ab^2c^3-12dx$, $-3ab^2c^3+8dx$, $7ab^2c^3+6dx$, $-3ab^2c^3-7dx$. *Ans.* $5ab^2c^3-5dx$

6. Add $7abz^2-13a^2x^3+9bc^2$, $-8abz^2-a^2x^3-6bc^2$, $3abz^2+4a^2x^3-6bc^2$, and $-abz^2+4a^2x^3-7bc^2$.

Ans. $abz^2-6a^2x^3-10bc^2$.

7. Add $12a^3y^4+13$, $3a^3y^4-7$, $-a^3y^4+8$, $2a^3y^4-9$, and $-3a^3y^4-11$. *Ans.* $13a^3y^4-6$.

8. Add $6x+5ay$, $-3x+2ay$, $x-6ay$, and $2x+ay$.

Ans. $6x+2ay$.

9. Add $-3ab+7a^2b$, $3ab-10a^2b$, $3ab-6a^2b$, $-ab-2a^2b$, and $2ab+7a^2b$. *Ans.* $4ab-4a^2b$.

10. Add $3a(a+b)$, $7a(a+b)$, $-5a(a+b)$, and $3a(a+b.)$

Ans. $8a(a+b)$.

11. Add $7(6x+y-z)^2$, $-8(6x+y-z)^2$, $(6x+y-z)^2$, and $3(6x+y-z)^2$. *Ans.* $3(6x+y-z)^2$.

12. Add $3ab+4a(6y+b)$, $-8ab-9a(6y+b)$, $12ab+13a(6y+b)$, $ab+a(6y+b)$, and $7ab+6a(6y+b)$.

Ans. $15ab+15a(6y+b)$.

CASE III.

81. In this case the quantities are unlike, or some like and others unlike.

RULE.

1. *Reduce the like terms as in the preceding cases.*

2. *To the results thus obtained, connect the terms which cannot be reduced by their proper signs.*

Note.—This rule is founded on the principles illustrated in articles 73 and 74.

EXAMPLES.

1. Add $3ay^2$, $-2xy^2$, $-3y^2x$, $-8x^2y$, and $2xy^2$.

These terms may be arranged thus:
$$\left\{\begin{array}{l} 3ay^2-2xy^2 \\ \quad -3xy^2-8x^2y \\ \quad +2xy^2 \end{array}\right.$$

$$\overline{3ay\ -3xy\ -8x^2y}$$

2. Add $8a^2x^2$, $-3ax$, $7ax$, $-5xy$, $-5ax$, $9xy$, $2a^2x^2$, and xy.

These terms may be arranged thus:
$$\left\{\begin{array}{rrr} 8a^2x^2 & -3ax & \\ & +7ax & -5xy \\ & -5ax & +9xy \\ 2a^2x^2 & & +\ xy \\ \hline 10a^2x^2 & -\ ax & +5xy \end{array}\right.$$

3. Add $10b^2-3a^2x$, $-b^2+2a^2x^2$, $50+2a^2x$, and a^2x^2+120.
Ans. $9b^2+3a^2x^2-a^2x+170$.

4. Add $ab+8$, $cd-3$, 28, and $5cd-4m+2$.
Ans. $ab+6cd+35-4m$.

5. Add $5ab\times c^2d^2$, $7ab\times c^2d^2$, $6abcd$, and $-3ab\times c^2d^2+3abcd$.
Ans. $9ab\times c^2d^2+9abcd$.

6. Add $3ax-21$, $6bc+2$, $ax+15-5bc$, $-8+6ax-bc$, and $11ax+13-3bc$. *Ans.* $21ax-3bc+1$.

7. Add $3m^2-1$, $6am-4m^2+8$, $7-9am+8$, $6m^2-3+am$, and $4m^2-am+12$. *Ans.* $9m^2-3am+31$.

8. Add $13x(a^2-b^2)$, $-6a^2x+12b^2x$, $-10x(a^2-b^2)+13a^2x$, and $3x(a^2-b^2)+3b^2x$. *Ans.* $6x(a^2-b^2)+7a^2x+15b^2x$.

9. Add $4a^2+3b+2c$, $-3a^2+4b+8c$, $9a^2-7b-10c$, and $3a^2-b+c+ax$. *Ans.* $13a^2-b+c+ax$.

10. Add $8ax+2(x+a)+3b$, $9ax+6(x+a)-9b$, and $-7ax-8(x+a)+6b+11x$. *Ans.* $10ax+11x$.

11. Add $5a^2b+12(a-x)^2$, $3a^2b-8+9(a-x)^2$, $12-8a^2b$, and $-13(a-x)^2+3$. *Ans.* $8(a-x)^2+7$.

12. Add $28a^3(x+5y)+21$, $-13a^3(x+5y)+18a$, $-15a^3(x+5y)-8$, and $-13-8a$. *Ans.* $10a$.

13. Add $72ax^4-8ay^3$, $-38ax^4-3ay^4+7ay^3$, $8+12ay^4$, $-6ay^3+12-34ax^4+5ay^3-9ay^4$. *Ans.* $-2ay^3+20$.

14. Add $12a-13ab+16ax$, $8-4m+2y$, $-6a+7ab^2+12y-24$, and $7ab-16ax+4m$. *Ans.* $6a-6ab+14y+7ab^2-16$.

15. Add $17a(x+3ay)+12a^3b^4c^2$, $8-18ay-8a^3b^4c^2-7a(x+3ay)$, $-4+12ay-10a(x+3ay)-4a^3b^4c^2$, and $6ay-4$. *Ans.* 0.

16. Add $3ab^4x^2-8a^3cd$, $-7ab^4x^2+7a^3cd-12$, $32ab^4x^2$, and $12-3a^3cd+ab^4x^2$. *Ans.* $29ab^4x^2-4a^3cd$.

SUBTRACTION.

82. SUBTRACTION is a method of finding the *difference* between two algebraic quantities or sets of quantities. It is the opposite of Addition.

83. We have already seen that a negative quantity is of an opposite nature to a positive quantity (Art. 47), with respect to addition and subtraction: that is, it must be subtracted when a positive quantity would be added, and added when a positive quantity would be subtracted. Hence, for subtraction of algebraic quantities, we have the following general

RULE.

1. *Write the quantity to be subtracted under that from which it is to be taken, placing like terms under each other.*

2. *Change the signs of all the quantities to be subtracted, or conceive them to be changed, and then proceed as in addition.**

EXAMPLES.

	(1)	(2)	(3)
From	$6a-9b$	$11ab-6ac^2-12axy$	$8cd^2-8axy+3bd$
Take	$3a-4b$	$5ab-10ac^2-2axy$	$11cd^2-axy+3bd$
Ans.	$3a-5b$	$6ab+4ac^2-10axy$	$-3cd^2-7axy$

4. From $3ab^2-8a^2cx^3+2b^2c$ take $2ab^2+4a^2cx^3+3b^2c$.

Ans. $ab^2-12a^2cx^3-b^2c$.

* The principles on which this rule is founded may be stated and demonstrated as follows:

1. Subtracting a positive quantity will produce the same result as adding an equal negative quantity.

Represent the sum of two quantities by - - $a+b$

Taking $+b$ away from this expression, there remains a

Adding $-b$ to it, we shall have - - - $a+b-b=a$.

2. Subtracting a negative quantity will produce the same result as adding an equal positive quantity.

Represent the difference of two quantities by - $a-b$

Taking $-b$ away from this expression, there remains a

Adding $+b$ to it, we shall have - - - - $a-b+b=a$.

5. From $12ax-19c^3b+2abx-x^2$ take $9ax-9c^3b-8abx$.
Ans. $3ax-10c^3b+10abx-x^2$

6. From $3a^4b-12cd^2+4y-8ax^2$ take $-4a^4b-10cd^2+4y-5ax^2-2cd^2$. *Ans.* $7a^4b-3ax^2$.

7. From $13a^3d+xy+d$ take $7a^3d-xy+d+hm^2-r^2y^2$.
Ans. $6a^3d+2xy-hm^2+r^2y^2$.

8. From $7a^3bc^2-8+7x$ take $3a^3bc^2-8-dx^2+r$.
Ans. $4a^3bc^2+7x+dx^2-r$.

9. From $13a^4bx^3+17b-8c^2$ take $11a^4bx^3+9b-7c^2$.
Ans. $2a^4bx^3+8b-c^2$.

10. From $16a^2b^2x^3-3+48dx$ take $-4a^2b^2x^3+bx+12cd$.
Ans. $20a^2b^2x^3-3-bx+48dx-12cd$.

11. From $3abc-8xy+25+8b$ take $-11abc+4xy-22-7b$.
Ans. $14abc-12xy+47+15b$.

12. From the sum of $6x^2y-11ax^3$ and $8x^2y+3ax^3$ take $4x^2y-4ax^3$. *Ans.* $10x^2y-4ax^3$.

13. From the sum of $15abc+8cdx-3$ and $24-8abc+2cdx$ take the sum of $12abc-3cdx-8$ and $-4abc+cdx+16$.
Ans. $-abc+12cdx+13$.

14. From the difference between $8ab-12cx$ and $-3ab+4cx$ take the sum of $5ab-7cx$ and $ab+cx$.
Ans. $5ab-10cx$.

15. From the sum of $4ax^2+2x^3+350$, $5ax^2+6x^3+250$, and $9ax^2+12x^3+100$, take the sum of $6ax^2+9x^3+432$, ax^2+5x^3+328, and $5ax^2+x^3+30$. *Ans.* $6ax^2+5x^3-90$.

84. The minus sign, when placed before the marks of parenthesis which include a polynomial, indicates that each term of the polynomial is to be subtracted, or that the result obtained by reducing the terms, if they are like quantities, is to be subtracted. This is done by removing the marks of parenthesis, and changing the signs of the terms included between them. Thus:

1. $3a-(3c-x)=3a-3c+x$.
2. $3abc-(2abc+3x-b)=3abc-2abc-3x+b=abc-3x+b$.
3. $4ax-3c-14-(ax+7c-12)=4ax-3c-14-ax-7c+12=3ax-10c-2$.

4. $7abc-13+8abx-(3abc-14+9abx)=7abc-13+8abx-3abc+14-9abx=4abc+1-abx$.

85. When a number of terms are introduced within the marks of parenthesis, to which the minus sign is prefixed, the signs of the terms should be changed. Thus:

1. $3ax-12a-3b=3ax-(+12a+3b)$.
2. $3abc-6+4ab+3x=3abc-(6-4ab-3x)$.
3. $7xy-12ab-4y-8-b=7xy-(12ab+4y+8+b)$.

86. By the above methods, polynomials may be made to undergo a variety of transformations, which are sometimes of great use in algebraic operations.

87. The word *Addition*, as here used, it will be perceived from the foregoing operations, does not always imply increase or augmentation, nor does the word *Subtraction* always imply diminution. Hence the term *Reduction* has been sometimes employed to express the operations included under addition and subtraction.

MULTIPLICATION.

88. MULTIPLICATION is repeating the multiplicand as many times as there are units in the multiplier. Thus:

1. If a is to be multiplied by b, it must be taken as many times as there are units in b, and the expression would become $a\times b$, or ab.
2. If ab is to be multiplied by cd, it must be taken as many times as there are units in cd, and the expression would become $ab\times cd$, or $abcd$. Hence, *to multiply letters, we write them one after the other, in alphabetical order.*
3. If $4a$ is to be multiplied by $3b$, it must be taken as many times as there are units in $3b$. Thus, $4a\times 3b=4\times a\times 3\times b=4\times 3\times a\times b=12ab$. Hence, *numerical coefficients must be multiplied together, and their product prefixed to the product of the letters.*
4. If $7a^2$ is to be multiplied by $4a^3$, it must be taken as many times as there are units in $4a^3$, and the work may be expressed thus: $7a^2\times 4a^3=7aa\times 4aaa=7\times aa\times 4\times$

$aaa = 7\times4\times aa\times aaa = 28a^5 = 28a^{2+3}$. Hence, *if the same letter is found in both factors, it is multiplied by adding together its exponents, and their sum is the index of the same letter in the product.*

89. With regard to the signs, it should be observed, that if the signs of the two factors are *like*, the sign of the product will be +; but if their signs are *unlike*, the sign of the product will be —. This rule may be illustrated thus:

1. If $+a$ is to be multiplied by $+b$, the multiplication consists in repeating $+a$ as many times as there are units in $+b$; and, consequently, the product is $+ab$.
2. If $-a$ is to be multiplied by $+b$, the multiplication consists in repeating $-a$ as many times as there are units in $+b$; and, consequently, the product is $-ab$.
3. If $+a$ is to be multiplied by $-b$, the minus multiplier indicates that the repetitions of $+a$ are to be subtracted; consequently, the product is $-ab$.
4. If $-a$ is to be multiplied by $-b$, the repetitions of $-a$ will be negative; but the minus multiplier indicates that these repetitions are to be subtracted; consequently, the product is $+ab$.

90. It should also be remarked, that if there are more than two factors, an odd number of negative factors will produce —, and an even number +. Thus, $-a\times-b\times-c=-abc$; for $-a\times-b=+ab$, and $+ab\times-c=-abc$. Again, $-a\times -b\times-c\times-d=+abcd$; for $-a\times-b=+ab$, and $+ab\times-c =-abc$, and $-abc\times-d=+abcd$.

91. The classification of quantities into monomials and polynomials suggests three cases in multiplication, viz.: When the factors are both monomials; when one is a polynomial and the other a monomial; and when both are polynomials.

CASE I.

92. In this case the factors are monomials, and the signs are like or unlike.

RULE.

1. *Multiply together the numerical coefficients of the factors.*

2. *To the product of the coefficients annex the product of the letters, observing that if a letter is contained in both the multiplicand and multiplier, it will be affected with an exponent in the product equal to the sum of its exponents in the factors.*

3. *Prefix to the product the sign required by the principle that like signs produce plus, and unlike signs minus.*

Note.—For an illustration of the principles of this rule, see Arts. 88 and 89.

EXAMPLES.

	(1.)	(2.)	(3.)	(4.)
Multiply	$10a^2bx$	$-7axyz$	$12dm^3n^2x^4$	$-13a^2bc^3d^4x^2$
By	$3ab^2x^4$	$3abcx$	$-8d^2fmn$	$-3abx$
Product	$30a^3b^3x^5$	$-21a^2bcx^2yz$	$-96d^3fm^4n^3x^4$	$+39a^3b^2c^3d^4x^3$

	(5)	(6)	(7)
Multiply	$8a^4b^6c^2d^3x^5y^4$	$-17a^3x^2yz^4$	$-28a^3y^4z$
By	$7a^3b\,c^5d^4x^2y^3$	12	$-12a^3b^4cd^3xyz^2$
Product	$36a^7b^7c^7d^7x^7y^7$	$-204a^3x^2yz^4$	$+336a^6b^6cd^3xy^5z^3$

8. Multiply $11a^2bcd$ by $4a^3b^4cx$. *Ans.* $44a^5b^5c^2dx$.

9. Multiply $18amxy$ by $6anxz$. *Ans.* $108a^2mnx^2yz$.

10. Multiply $96a^2cd^3y$ by $-2ac^4dy^3$. *Ans.* $-192a^3c^5d^4y^4$.

11. Multiply $-32abcd$ by $5abcx$. *Ans.* $-160a^2b^2c^2dx$.

12. Multiply $-12ab$ by $-144ab^3$. *Ans.* $+1728a^2b^4$.

13. Multiply $6a^2b^3$ by $12a^3xy$, and that product by $2ax^3y^2$. *Ans.* $144a^6b^3x^4y^3$.

14. Multiply $8a^2bc$ by $3ab^3c^2$, and that product by $-2a^3bc^3$. *Ans.* $-48a^6b^5c^6$.

15. Multiply $-2a^3x^4y^3$ by $8ax^2y^2$, and the product by $-4a^2xy^3$. *Ans.* $+64a^6x^7y^8$.

16. Multiply $-10bc^3$ by $-3b^5c^2x^3$, and that product by $-4b^3c^2xy$. *Ans.* $-120b^9c^7x^4y$.

17. Multiply the product of $18x^3y$ and $5x^4y^2$ by the product of x^3y^3 and $3xy$. *Ans.* $270x^{11}y^7$.

18. Multiply $13a^3b^4c^7d^5$ by $10a^2b^3x$.

19. Multiply $21a^3b^4c^5d^3x$ by $-12adnx^2$.
20. Multiply $-16a^2x^3$ by -16.
21. Multiply $6ada$ by $3a$, and that product by $12a^2b^3x^4$.
22. Multiply $-7ax$ by $13abx$, and that product by $-12a^3x^2y^4$.
23. Multiply $-6a^3x^4z^5$ by $-12a^3x$, and that product by $-2ax^4z^2$.
24. Multiply the product of $7ax$ and $3axz$ by the product of $8abc^2$ and $2a^3x^3y^2$.
25. Multiply the product of $18az$ and $2a^3x$ by the product of $6ax$ and $-3a^4y^3z^2$.
26. Multiply the product of $-8ax$ and $-12ax$ by the product of $-4ax$ and $3ax$.
27. Multiply $8abcd$ by $12a^2x^3$, and that product by $-8a^3b^2c^3dx$.
28. Multiply the product of $-3abx$ and $4a^2xz$ by the product of $6z$ and $12a^2b^2x^2z^2$.
29. Multiply $12a^3d^4$ by $-8a^3d^4$, and that product by $11a^3d^4$.
30. Multiply the product of $-28a^2c^3d^4$ and $-12a^2c^3d^4x^5$ by the product of $-8a^2c^3d^4x^5y^6$ and $-7a^2c^3d^4x^5y^6z^7$.

CASE II.

93. In this case the multiplicand is a polynomial and the multiplier a monomial.

RULE.

1. *Multiply the letters and coefficients of each term of the multiplicand by the letters and coefficients of the multiplier.*

2. *Prefix to each term of the product the sign required by the principle that like signs produce plus, and unlike signs minus.*

Note.—The simple principle on which this rule rests is, that the sum of all the units in the multiplicand is to be taken as many times as there are units in the multiplier. Thus, if $a+b$ is to be multiplied by c, it is evident that both a and b must be taken as many times as there are units in c; hence, $(a+b)\times c=ac+bc$.

EXAMPLES.

	(1)	(2)
Multiply	$4ab+cd$	$2a^2-3b$
By	$3ac$	$6abd$
Product	$12a^2bc+3ac^2d$	$12a^3bd-18ab^2d$
	(3)	(4)
Multiply	$a+3b-2c$	$2b-7a-3$
By	$-3bx$	$4ab$
Product	$-3abx-9b^2x+6bcx$	$8ab^2-28a^2b-12ab$
	(5)	(6)
Multiply	$3a^2x^3y^4-1+a$	$-12a^2bx^3-4bc^2$
By	$5a^3xy$	$-3a$
Product	$15a^5x^4y^5-5a^3xy+5a^4xy$	$36a^3bx^3+12abc^2$

7. Multiply $11a^3bc^2-13xy$ by $3ax$.
Ans. $33a^4bc^2x-39ax^2y$.

8. Multiply $42c^2-1$ by -4. *Ans.* $-168c^2+4$.

9. Multiply $-30a^2bx^2y+13$ by $-5a^3$.
Ans. $+150a^5bx^2y-65a^3$.

10. Multiply the product of $a+b$ and $3c$ by $8ax$.
Ans. $24a^2cx+24abcx$.

11. Multiply the product of $2a+3b-4c$ and $-2a$ by $8abdx$. *Ans.* $-32a^3bdx-48a^2b^2dx+64a^2bcdx$.

12. Multiply the sum of $3ab+10$ and $ab-8$ by $6ax$.
Ans. $24a^2bx+12ax$.

13. Multiply the sum of $12abx-8ad-3b$ and $8abx-ad+b$ by $3a$. *Ans.* $60a^2bx-27a^2d-6ab$.

14. Multiply the difference between $12a-7bdx$ and $8a+bdx$ by $6abdx$. *Ans.* $24a^2bdx-48ab^2d^2x^2$.

15. Multiply the sum of $16-3y+12bc$ and $y+10-bd$ by $2abdm$. *Ans.* $52abdm-4abdmy+24ab^2cdm-2ab^2d^2m$.

16. Multiply $20a^3b^2x^3y^4-1+17aby+3xz$ by $-9a^4b^2c^3d^2$.
Ans. $-180a^7b^4c^3d^2x^3y^4+9a^4b^2c^3d^2-153a^5b^3c^3d^2y-27a^4b^2c^3d^2xz$.

CASE III.

94. In this case the multiplicand and multiplier are both polynomials.

RULE.

1 *Multiply the letters and coefficients of each term of the multiplicand by the letters and coefficients of each term of the multiplier.*

2. *Prefix to each term of the product the sign required by the principle that like signs produce plus, and unlike signs minus.*

Note 1.—The principle on which this rule is founded is, that in multiplication the sum of the units in the multiplicand is to be taken as many times as is expressed by the sum of the units in the compound multiplier. Thus, if $a+b$ is to be multiplied by $c+d$, it is evident that $a+b$ is to be repeated as many times as there are units in $c+d$; hence, since $c+d$ cannot be reduced to a monomial, we repeat the multiplicand c times, and then d times, and then take the sum of the repetitions. Thus, $a+b$ repeated c times gives $ac+bc$, and $a+b$ repeated d times gives $ad+bd$; and the sum of the repetitions is $ac+bc+ad+bd$.

Note 2.—Like terms in the product should be placed under each other, and the product reduced to its simplest form.

EXAMPLES.

(1)

Multiply	$2a+3b$
By	$a+b$
	$2a^2+3ab$
	$2ab+3b^2$
Product	$2a^2+5ab+3b^2$

(2)

$$\begin{array}{l} 6xy \quad -2z \\ 3ax \quad -5d \\ \hline 18ax^2y-6axz-30dxy+10dz \end{array}$$

(3)

Multiply	$a+b+c$
By	$a-b-c$
	$a^2+ab+ac$
	$-ab-b^2-bc$
	$-ac-bc-c^2$
Product	$a^2-b^2-2bc-c^2$

(4)

Multiply $3ab - b^2y - cd$

By $6ab - 2b^2y + cd$

$$\begin{array}{l} 18a^2b^2 - 6ab^3y - 6abcd \\ \qquad - 6ab^3y \qquad\qquad + 2b^4y + 2cb^2dy \\ \qquad\qquad + 3abcd \qquad - cb^2dy - c^2d^2 \\ \hline \end{array}$$

Product $18a^2b^2 - 12ab^3y - 3abcd + 2b^4y + cb^2dy - c^2d^2$

5. Multiply $b+c+2$ by $b+c+3$.
Ans. $b^2+2bc+5b+c^2+5c+6$.

6. Multiply $2a+3b+c$ by $4a+c+1$.
Ans. $8a^2+12ab+6ac+3bc+c^2+2a+3b+c$.

7. Multiply a^2+b^2 by $a+b$. *Ans.* $a^3+ab^2+a^2b+b^3$.

8. Multiply $3b^2+5bc+7c^2$ by $3b^2+2c^2$.
Ans. $9b^4+15b^3c+27b^2c^2+10bc^3+14c^4$.

9. Multiply $a+b$ by $a-b$. *Ans.* a^2-b^2.

10. Multiply $2a-b$ by $3a^2-1$. *Ans.* $6a^3-3a^2b-2a+b$.

11. Multiply $3ab^2-6$ by $a+4$.
Ans. $3a^2b^2-6a+12ab^2-24$.

12. Multiply $6a+4b$ by $3a-2b$. *Ans.* $18a^2-8b^2$.

13. Multiply $7abc^2+3xy+1$ by $8a^2b^3$.
Ans. $56a^3b^4c^2+24a^2b^3xy+8a^2b^3$.

14. Multiply $4a^2bx^3+3cd$ by $3cdx$, and that product by $4a+b$. *Ans.* $48a^3bcdx^4+36ac^2d^2x+12a^2b^2cdx^4+9bc^2d^2x$.

15. Multiply $a+b+c$ by $8ab$, and that product by $a+b$.
Ans. $8a^3b+16a^2b^2+8a^2bc+8ab^3+8ab^2c$.

16. Multiply $2a+4b$ by $2a-4b$. *Ans.* $4a^2-16b^2$.

17. Multiply $x^3+x^2y+xy^2+y^3$ by $x-y$. *Ans.* x^4-y^4.

18. Multiply x^2+xy+y^2 by x^2-xy+y^2. *Ans.* $x^4+x^2y^2+y^4$.

19. Multiply $xy+1$ by $3a+b$, and that product by $4c$.
Ans. $12acxy+12ac+4bcxy+4bc$.

20. Multiply $3b+2x+h$ by $3a \times b \times 2c \times 2x$.
Ans. $36ab^2cx+24abcx^2+12abchx$.

21. Multiply $3x^3+2x^2y^2+3y^3$ by $2x^3-3x^2y^2+5y^3$.
Ans. $6x^6-5x^5y^2+6x^3y^3-6x^4y^4+15x^3y^3+x^2y^5+15y^6$.

22. Multiply $a^2+b^2+x^2-ab-ax-bx$ by $a+b+x$.
Ans. $a^3-3abx+b^3+x^3$.

23. Multiply $x^4+x^2y^2+y^4$ by x^2-y^2. *Ans.* x^6-y^6.

24. Multiply together $a+b$, a^2+ab+b^2, $a-b$, and a^2-ab+b^2. *Ans.* a^6-b^6.

95. For many purposes, it is sufficient to indicate the multiplication of polynomials, as $(a+b)\times(a+b+c)$; and when the multiplication is performed, the expression is said to be *expanded*.

96. As the multiplier merely expresses the number of times the multiplicand is to be repeated, it is always considered a *number*.

97. The multiplicand may be either a quantity or a number; and, since repeating a quantity cannot change its nature, the product will be of the same nature as the multiplicand.

98. We sometimes speak of multiplying dollars by yards or pounds; but this language, if construed literally, is absurd. To obtain the cost of a given number of articles, we repeat the cost of one article as many times as there are articles purchased.

99. If the multiplier is a unit, the product will be equal to the multiplicand; if it is greater than a unit, the product will be greater than the multiplicand; but if it is less than a unit, the product will be less than the multiplicand. And, in general, with the same multiplicand, the product decreases as the multiplier decreases; and if the multiplier be reduced to 0, the product is 0. Hence, if 0 enters as a factor into any quantity whatever, the value of the expression becomes 0.

DIVISION.

100. Division is finding a quotient which, being multiplied into the divisor, will produce the dividend. It is the converse of multiplication, the product and one of the factors being given to find the other factor.

101. It is evident, from the nature of division, that a factor equal to the divisor must be rejected from the dividend;

or, the coefficient of the dividend must be divided by the coefficient of the divisor, and a factor equal to the literal part of the divisor rejected from the literal part of the dividend. Thus, $8ab \div 4b = \frac{8ab}{4b} = 2a$; for $4b \times 2a = 8ab$.

102. If the same letter is found in both dividend and divisor, and affected with a greater exponent in the dividend than in the divisor, the exponent of the quotient will be that of the dividend diminished by that of the divisor. Thus, $a^7 \div a^5 = aaaaaaa \div aaaaa = \frac{aaaaa \times aa}{aaaaa} = aa = a^2 = a^{7-5}$.

103. If the exponent of the divisor is greater than the exponent of the same letter in the dividend, the exponent of the quotient will be negative. Thus, $a^5 \div a^7 = a^{5-7} = a^{-2}$.

If the divisor contains a letter that is not found in the dividend, the exponent of that letter will be affected with the opposite sign in the quotient. Thus, $a^m \div b^m = a^m b^{-m}$; for $a^m b^{-m} \times b^m = a^m b^{-m+m} = a^m b^0 = a^m$. (Art. 107.)

104. The same rule is observed with regard to the signs in division as in multiplication, i. e., like signs produce plus, and unlike signs minus. Thus,

$+ab \div +b = \ a$; for $+a \times +b = +ab$.

$+ab \div -b = -a$; for $-a \times -b = +ab$.

$-ab \div +b = -a$; for $-a \times +b = -ab$.

$-ab \div -b = +a$; for $+a \times -b = -ab$.

105. The operations to be performed in division may also be conveniently considered in three cases, accordingly as the quantities are both monomials, or as the dividend is a polynomial and the divisor a monomial, or as both are polynomials.

CASE I.

106. In this case, the dividend and divisor are both monomials.

RULE.

1. *Divide the coefficient of the dividend by the coefficient of the divisor.*

2. *Reject the letters common to both dividend and divisor when they have the same exponent; but when the exponents are not*

the same, subtract the exponent of the divisor from that of the dividend, and the remainder will be the exponent of that letter in the quotient.

3. *To the above results, annex the letters of the dividend that are not found in the divisor, and also those of the divisor that are not found in the dividend, observing that the signs of the exponents of the latter are to be changed.*

4. *Prefix to the whole the sign required on the principle that like signs produce plus, and unlike signs minus.*

EXAMPLES.

	(1)	(2)	(3)	(4)
Divide	$8a^2bd$	$-28a^3c^2dx^4y^5$	$35a^2b^3$	$64a^3bc^2d$
By	$2ab$	$7a^3c^2dx^3y^4$	$5a^3b^5$	-16
Quotient	$4ad$	$-4xy$	$7a^{-1}b^{-2}$	$-4a^3bc^2d$

5. Divide $12a^2b^3x$ by $3ab^2$. *Ans.* $4abx$.
6. Divide $48a^3c^2d^4x^3z^2$ by $8a^2d^3x^3$. *Ans.* $6ac^2dz^2$.
7. Divide $42adx^3$ by $-7adx$. *Ans.* $-6x^2$.
8. Divide $72a^7b^3c^2x$ by $8a^3b^2x$. *Ans.* $9a^4bc^2$.
9. Divide $120x^4y^2$ by $-8x^3y$. *Ans.* $-15xy$.
10. Divide $-84bc^2$ by $-12bc$. *Ans.* $+7c$.
11. Divide $256a^7b^6c^2$ by $8a^5b^2c^3$. *Ans.* $32a^2b^4c^{-1}$.
12. Divide $56a^6b^7d^3x^3$ by $8a^3b^5dx^2$. *Ans.* $7a^3b^2d^2x$.
13. Divide $90a^2bc^3dx^4y^5$ by $5a^3bc^4d^2x^5y$. *Ans.* $18a^{-1}c^{-1}d^{-1}x^{-1}y^4$.
14. Divide $98a^4b^4$ by $-49a^3b^2$. *Ans.* $-2ab^2$.
15. Divide $620x^4y^7$ by $4x^3y^5z^2$. *Ans.* $155xy^2z^{-2}$.
16. Divide $-15a^4b^6$ by $5a^2b^4c^3d$. *Ans.* $-3a^2b^2c^{-3}d^{-1}$.
17. Divide $1728a^3b^7d^8$ by $12a^5b^3c^2d^3$. *Ans.* $144a^{-2}b^4c^{-2}d^5$.
18. Divide $684x^5$ by $-12x^{-2}$. *Ans.* $-57x^7$.
19. Divide $3280x^3y^5z^9$ by $40x^2y^3z^6$. *Ans.* $82xy^2z^3$.
20. Divide $62a^3b^2$ by $31a^{-1}b^{-3}d^2$. *Ans.* $2a^4b^5d^{-2}$.
21. Divide the product of $8a^2bd^3$ and $7a^3b^3dx$, by $28a^3bdx$. *Ans.* $2a^2b^3d^3$.
22. Divide the product of $5a^3b^2c^2d^3x^4$ and $-10abc^4d$ by $-2a^3b^3x^2$. *Ans.* $+25ac^6d^4x^2$.

23. Divide the sum of $12a^4b^3c^4$ and $8a^4b^3c^4$ by the sum of $3ab^2c^3+7ab^2c^3$. *Ans.* $2a^3bc$.

24. Divide the difference of $21a^3b^4c^3d^2x^4$ and $15a^3b^4c^3d^2x^4$ by $6a^2bcdx$. *Ans.* $ab^3c^2dx^3$.

107. It sometimes occurs in the operations of division, that a letter becomes affected with the exponent 0. We will, therefore, explain that symbol. Thus, $a^2\div a^2=a^{2-2}=a^0$; or, again, $a^m\div a^m=a^{m-m}=a^0$; but $\frac{a^2}{a^2}=1$, and $\frac{a^m}{a^m}=1$; therefore, since a may represent any quantity whatever, and m any exponent whatever, every quantity affected with the exponent 0 is equal to 1.

This will also explain why $a^m\div b^m=a^mb^{-m}$ (Art. 103); for, multiplying the divisor and quotient together, $b^m\times a^mb^{-m}=a^mb^{-m\ m}=a^mb^0=a^m$.

CASE II.

108. In this case the dividend is a polynomial and the divisor a monomial.

RULE.

1. *Divide each term of the dividend by the divisor, and the resulting quantities, connected by their proper signs, will be the quotient.*

Note.—That each term of the dividend should be divided by the divisor, is evident from the fact that when a polynomial is multiplied by a monomial factor, that factor enters into every term of the polynomial. Thus, $(a+b+c)\times d=ad+bd+cd$; hence, $(ad+bd+cd)\div d=\frac{ad+bd+cd}{d}=\frac{ad}{d}+\frac{bd}{d}+\frac{cd}{d}=a+b+c$.

EXAMPLES.

	(1)	(2)	(3)
Divide	$2ab+6bc$	$12x^2y+39a^4x^2y^3$	$42a^3c^2x+18ab^3c^4x^5$
By	$2b$	$3x^2y$	$-6ac^2x$
Quotient	$a\ +3c$	$4\quad +13a^4y^2$	$-7a^2\ -\ 3b^3c^2x^4$

	(4)	(5)	(6)
Divide	$12a^3b^2c-21a^5b^3c^4$	$15axy+12cd$	$9a^2b^3x-3abc^2$
By	$3a^5bc^2$	3	$3abc^2$
Quotient	$4a^{-2}bc^{-1}-7b^2c^2$	$5axy+4cd$	$3ab^2c^{-2}x-1$

7. Divide $72a^4b^3c^5-8a^3b^2c^7$ by $8a^3b^2c^4$. *Ans.* $9abc-c^3$.

8. Divide $35dm+14dx$ by $7d$. *Ans.* $5m+2x$.

9. Divide $4axy-4a+16ad$ by $4a$. *Ans.* $xy-1+4d$.

10. Divide $3ax^3+6x^2+3ax-15x$ by $3x$.
Ans. $ax^2+2x+a-5$.

11. Divide $3abc+12abx-3a^2b$ by $3ab$. *Ans.* $c+4x-a$.

12. Divide $25a^2bx-15a^2cx^2+5abc$ by $-5ax$.
Ans. $-5ab+3acx-bcx^{-1}$.

13. Divide $20ab^3+15ab^2+10ab+5a$ by $5a$.
Ans. $4b^3+3b^2+2b+1$.

14. Divide the product of $9a^3b^2+7a^2x^3$ and $4a^3c^3d^2$ by $2a^3b$.
Ans. $18a^3bc^3d^2+14a^2b^{-1}c^3d^2x^3$.

15. Divide the product of $12a^3bx^3+3bx^1$ and $8x^2$ by $24bx^3$.
Ans. $4a^3x^2+x^3$.

16. Divide the product of $6axy+7a^3bc^2dx$ and $6x^2+4y^3$ by $2ax$. *Ans.* $18x^2y+21a^2bc^2dx^2+12y^4+14a^2bc^2dy^3$.

CASE III.

109. In this case the dividend and divisor are both polynomials.

RULE.

1. *Arrange the terms of the dividend, and also those of the divisor, with reference to the power of some letter, so that its exponents shall diminish from left to right.*

2. *Divide the first term of the dividend by the first term of the divisor; the result is the first term of the quotient.*

3. *Multiply the whole divisor by this term, and subtract the product from the dividend; the remainder will form a new dividend.*

4. *Divide the first term of this new dividend by the first term of the divisor; the result is the second term of the quotient. Multiply the divisor by this term, and subtract as before.*

5. *Proceed in this manner till the dividend has been exhausted, or till no term of the remainder is divisible by the first term of the divisor.*

Note 1.—In applying the above rule, we find, successively, how often the divisor is contained in parts of the dividend, for the reason that, as the dividend is made up of all its parts.

the divisor is contained in the whole as often as it is contained in all its parts.

Note 2.—If the first term of the dividend is not divisible by the first term of the divisor, after the terms in each have been arranged, the division is impossible.

Note 3.—If the dividend is exactly divisible by the divisor, the dividend will be completely exhausted, leaving no remainder. But if it is not exactly divisible, the division may be continued till the first term of the remainder is not divisible by the first term of the divisor; the remainder should then be placed over the divisor so as to form a fraction.

Note 4.—It will not in all cases be necessary to bring down all the terms of the dividend to form the first remainder.

EXAMPLES.

1. Divide $a^2+2ab+b^2$ by $a+b$.

$$\begin{array}{lll} \text{Dividend,} & a^2+2ab+b^2 & \underline{|a+b,} \ \text{Divisor.} \\ & \underline{a^2+ab\qquad} & a+b, \ \text{Quotient.} \\ & *\quad ab+b^2 & \\ & \quad\ \underline{ab+b^2} & \\ & \quad\ 0 \quad 0 & \end{array}$$

Proof. $(a+b)\times(a+b)=a^2+2ab+b^2$.

2. Divide $12a^5b^2-6a^4b^3+8a^2b^5-4a^3b^4-22a^6b+5a^7$ by $4a^2b^2+5a^4-2a^3b$.

Dividend arranged. *Divisor arranged.*

$$\begin{array}{ll} 5a^7-22a^6b+12a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5 & \underline{|5a^4-2a^3b+4a^2b^2} \\ \underline{5a^7-2a^6b+4a^5b^2\qquad\qquad\qquad\qquad} & \text{Quotient, } a^3-4a^2b+2b^3 \\ *-20a^6b+8a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5 & \\ \underline{-20a^6b+8a^5b^2-16a^4b^3\qquad\qquad\quad} & \\ \quad * \qquad * +10a^4b^3-4a^3b^4+8a^2b^5 & \\ \qquad\qquad\quad \underline{10a^4b^3-4a^3b^4+8a^2b^5} & \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 & \end{array}$$

Proof. $(5a^4-2a^3b+4a^2b^2)\times(a^3-4a^2b+4a^2b^2)=5a^7-22a^6b+12a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$.

3. Divide a^3-1 by $a-1$.

$$
\begin{array}{l|l}
\textit{Dividend.} & \textit{Divisor.} \\
a^3-1 & a-1 \\
\underline{a^3-a^2} & \overline{a^2+a+1}\text{, Quotient.} \\
\quad +a^2-1 & \\
\quad \underline{a^2-a} & \\
\qquad +a-1 & \\
\qquad \underline{a-1} & \\
\qquad\quad 0 &
\end{array}
$$

4. Divide a^6-b^6 by $a-b$.

$$
\begin{array}{l|l}
a^6-b^6 & a-b \\
\underline{a^6-a^5b} & \overline{a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5}. \\
\quad +a^5b-b^6 & \\
\quad \underline{+a^5b-a^4b^2} & \\
\qquad +a^4b^2-b^6 & \\
\qquad \underline{+a^4b^2-a^3b^3} & \\
\qquad\quad +a^3b^3-b^6 & \\
\qquad\quad \underline{a^3b^3-a^2b^4} & \\
\qquad\qquad +a^2b^4-b^6 & \\
\qquad\qquad \underline{+a^2b^4-ab^5} & \\
\qquad\qquad\quad +ab^5-b^6 & \\
\qquad\qquad\quad \underline{ab^5-b^6} & \\
\qquad\qquad\qquad 0 &
\end{array}
$$

5. Divide $a^3+5a^2b+5ab^2+b^3$ by $a+b$. *Ans.* $a^2+4ab+b^2$.

6. Divide $a^3+2a^2b+2ab^2+b^3$ by a^2+ab+b^2. *Ans.* $a+b$.

7. Divide $x^3-9x^2+27x-27$ by $x-3$. *Ans.* x^2-6x+9.

8. Divide x^3+y^3 by $x+y$. *Ans.* x^2-xy+y^2.

9. Divide $6a^3x-9a^2x^2-a^4+4x^4$ by a^2+2x^2-3ax.
Ans. $2x^2+3ax-a^2$.

10. Divide $6ax+2xy-3ab-by+3ac+cy$ by $3a+y$.

11. Divide $2x^3-19x^2+26x-16$ by $x-8$.
Ans. $2x^2-3x+2$.

12. Divide y^5+1 by $y+1$. *Ans.* $y^4-y^3+y^2-y+1$.

13. Divide y^6-1 by $y-1$. *Ans.* $y^5+y^4+y^3+y^2+y+1$.

14. Divide x^2-a^2 by $x-a$. *Ans.* $x+a$.

15. Divide x^4-a^4 by $x-a$. *Ans*: $x^3+ax^2+a^2x+a^3$.

16. Divide $-15a^4+37a^2bc-29a^2dx-20b^2c^2+44bcdx-8d^2x^2$ by $3a^2-5bc+dx$. *Ans*. $-5a^2+4bc-8dx$.

17. Divide $3a^4-8a^2b^2+3a^2c^2+5b^4-3b^2c^2$ by a^2-b^2. *Ans*. $3a^2-5b^2+3c^2$.

18. Divide $20a^5-41a^4b+50a^3b^2-45a^2b^3+25ab^4-6b^5$ by $4a^2-5ab+2b^2$. *Ans*. $5a^3-4a^2b+5ab^2-3b^3$.

19. Divide $9x^6-46x^5+95x^2+150x$ by x^2-4x-5. *Ans*. $9x^4-10x^3+5x^2-30x$.

20. Divide $6x^4-96$ by $3x-6$. *Ans*. $2x^3+4x^2+8x+16$.

21. Divide $432+1152b^2c^2+576b^4c^4$ by $6+12b^2c^2$. *Ans*. $72+48b^2c^2$.

22. Divide $8a^4x^2-8a^3bx^2+8a^3cx^2-11abz+11b^2z-11bcz$ by $a-b+c$. *Ans*. $8a^3x^2-11bz$.

23. Divide $6x^6-5x^5y^2-6x^4y^4+6x^3y^2+15y^3x^3-9x^2y^4+10x^2y^5+15y^6$ by $3x^3+2x^2y^2+3y^2$. *Ans*. $2x^3-3x^2y^2+5y^3$.

24. Divide $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$ by $a^4+4a^3b+6a^2b^2+4ab^3+b^4$. *Ans*. $a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

It is sometimes desirable to resolve a polynomial into its original factors. The principles of division enable us to do this; for, having obtained one of the factors by inspection or trial, the other may be obtained by division. Thus,

1. $4abc+4axy+4abd=4a(bc+xy+bd)$.
2. $72a^2b+4abc=4ab(18a+c)$.
3. $8a^2cx-18acx^2+2ac^5y-30a^6c^9x=2ac(4ax-9x^2+c^4y-15a^5c^8x)$.
4. $x^2+2ax+a^2=(x+a)\times(x+a)$.
5. $x^2-2ax+a^2=(x-a)\times(x-a)$.
6. $42x^3y-28x^2y=7x^2y(6x-4)$.

In the multiplication of compound quantities, when the signs are unlike, some of the terms disappear or are cancelled in the product. These terms will reappear in the division, so that the quotient frequently contains more terms than the dividend. Thus,

$$(a^2-ax+x^2)\times(a+b)=a^3+x^3.$$

But $(a^3+x^3)\div a+x=a^2-ax+x^2$.

110. The division of quantities may sometimes be carried *ad infinitum*. In such cases it will be sufficient to write out a few of the leading terms. Thus,

$$\begin{array}{l|l} 1 & 1-x \\ \cline{2-2} \underline{1-x} & 1+x+x^2+x^3, \text{ \&c.} \\ \quad +x & \\ \quad \underline{+x-x^2} & \\ \qquad +x^2 & \\ \qquad \underline{+x^2-x^3} & \\ \qquad\quad +x^3 & \\ \qquad\quad \underline{+x^3-x^4} & \\ \qquad\qquad\quad x^4 & \end{array}$$

111. In multiplication, the multiplier is always considered a number, but the multiplicand and product may be either numbers or quantities. In division, we have the product and either one of the factors to find the other. Hence, the dividend and divisor may be either numbers or quantities.

112. If the dividend and divisor are both numbers, the quotient will be a number. Thus, $12\div4=3$.

113. If the dividend is a quantity and the divisor a number, the quotient will be a quantity of the same kind as the dividend. Thus, 12 rods$\div4=3$ rods.

114. If the dividend and divisor are both quantities, the quotient will be a number. Thus, 12 rods$\div4$ rods$=4$.

115. From the nature of division, it is evident that the value of the quotient depends upon both the divisor and dividend.

If the dividend be multiplied while the divisor remains the same, the quotient will be multiplied. Thus, $ab\div b=a$; but multiplying the dividend by m, $abm\div b=am$.

Dividing the dividend, while the divisor remains the same, divides the quotient. Thus, $abm\div b=am$; but, dividing the dividend by m, $ab\div b=a$.

If the divisor be multiplied while the dividend remains the same, the quotient will be divided. Thus, $abm \div b = am$; but $abm \div bm = a$.

Dividing the divisor, while the dividend remains the same, multiplies the quotient. Thus, $abm \div bm = a$; but $abm \div b = am$.

116. The student will observe that there is a striking resemblance between the division of compound numbers in algebra, and what is termed "long division" in common arithmetic. But this essential difference should be noted; the several terms are so independent of each other, that after the first term of the quotient has been obtained, and the first remainder brought down for a new dividend, an entirely new arrangement of the terms, with reference to a different letter from that first assumed, may be made in both the divisor and dividend, and the division completed under this new arrangement without affecting the value of the quotient.

SECTION III.

Algebraic Fractions.

REDUCTION OF ALGEBRAIC FRACTIONS.

117. ALGEBRAIC FRACTIONS are perfectly analogous to vulgar fractions in common arithmetic. They express a part or parts of a whole number, or unity.

118. The *denominator* shows the number of parts into which the unit is divided; the *numerator* shows how many of these parts are taken.

119. Every case in division may be expressed in a fractional form, the dividend being used as the numerator, and the divisor as the denominator.

120. The denominator and numerator, taken together, are called *terms* of the fraction.

121. A *proper fraction* is one whose numerator is less than its denominator. Example, $\frac{a-b}{a+b}$.

122. An *improper fraction* is one whose numerator is equal to, or greater than, its denominator. Example, $\frac{a+b}{a-b}$.

123. A *mixed number* is an integer or whole number connected with a fraction by the sign plus or minus. Example, $a+\frac{b}{c}$.

124. A *compound fraction* is the fraction of a fraction, the simple fractions of which it is composed being connected by the word *of*. Example, $\frac{a}{b}$ of $\frac{c}{d}$.

125. The *value* of a fraction is the quotient resulting from the division of the numerator by the denominator. Hence, if the numerator equal the denominator, the value of the fraction is a unit; if the numerator is less than the denominator, the value is less than a unit; and if the numerator is greater than the denominator, the value is greater than a unit.

126. The principles involved in the reduction of Algebraic Fractions are the same as those applied in arithmetic. It will, however, be necessary to trace out those operations in accordance with the method of notation adopted in Algebra.

CASE I.

DISCUSSION OF SIGNS.

127. The sign that is prefixed to the horizontal line drawn between the numerator and denominator, determines whether the value of the fraction is to be added or subtracted.

128. A sign prefixed to one of the terms of the numerator or denominator affects only that term.

129. If the sign *prefixed to the fraction* be changed from + to − or from − to +, the value of the fraction will also be changed from + to −, or the contrary.

Thus, $+\frac{ab}{b}=+ab\div b=a$; but $-\frac{ab}{b}=-ab\div b=-a$.

Again, $+\frac{ab+ac}{a}=(ab+ac)\div a=+b+c$; but $-\frac{ab+ac}{a}=-((ab+ac)\div a)=-(b+c)=-b-c$.

And, $+\frac{ab-ac}{a}=(ab-ac)\div a=b-c$; but $-\frac{ab-ac}{a}=-((ab-ac)\div a)=-(b-c)=-b+c$.

130. If the sign *prefixed to the several terms of the numerator* be changed from + to — or from — to +, the value of the fraction will be changed accordingly.

Thus, $\frac{+ab}{a}=+ab\div a=b$; but $\frac{-ab}{a}=-ab\div a=-b$.

Again, $\frac{ab+ac}{a}=(ab+ac)\div a=b+c$; but $\frac{-ab-ac}{a}=(-ab-ac)\div a=-b-c$.

And, $-\frac{ab-ac}{a}=-((ab-ac)\div a)=-(b-c)=-b+c$; but

$$-\frac{-ab+ac}{a}=-((-ab+ac)\div a)=-(-b+c)=b-c.$$

131. If the sign *prefixed to the several terms of the denominator* be changed from + to — or from — to +, the value of the fraction will be changed accordingly.

Thus, $\frac{ab}{a}=ab\div a=b$; but $\frac{ab}{-a}=ab\div -a=-b$.

Again, $\frac{ab+ac}{a}=b+c$; but $\frac{ab+ac}{-a}=-b-c$.

And, $-\frac{ab-ac}{a}=-(b-c)=-b+c$; but $-\frac{ab-ac}{-a}=-(-b+c)=+b-c$.

132. If *any two* of the above changes are made, the value of the fraction will not be altered.

1. If the signs before the fraction and also before the several terms of the numerator be changed from + to — or from — to +, the value of the fraction will remain the same.

Thus, $\frac{ab}{a}=b$; and $-\frac{-ab}{a}=-(-ab\div a)=-(-b)=+b$.

Again, $\frac{ab+ac}{a}=b+c$; and $-\frac{-ab-ac}{a}=-(-b-c)=+b+c$.

Also, $\frac{ab-ac}{a}=b-c$; and $-\frac{-ab+ac}{a}=-(-b+c)=+b-c$.

2. If the signs before the fraction and before the several terms of the denominator be changed from + to — or from — to +, the value of the fraction will remain the same.

Thus, $\frac{ab}{a}=b$; and $-\frac{ab}{-a}=-(ab\div-a)=-(-b)=+b$.

Again, $\frac{ab+ac}{a}=b+c$; and $-\frac{ab+ac}{-a}=-((ab+ac)\div-a)=-(-b-c)=+b+c$.

Also, $\frac{ab-ac}{a}=b-c$; and $-\frac{ab-ac}{-a}=-((ab-ac)\div-a)=-(-b+c)=+b-c$.

3. If the signs before the several terms of both numerator and denominator be changed from + to — or from — to +, the value of the fraction will remain the same.

Thus, $\frac{ab}{a}=b$; and $\frac{-ab}{-a}=-ab\div-a=+b$.

Again, $\frac{ab+ac}{a}=b+c$; and $\frac{-ab-ac}{-a}=+b+c$.

Also, $\frac{ab-ac}{a}=b-c$; and $\frac{-ab+ac}{-a}=+b-c$.

133. Hence, to make a negative fraction positive without altering its value, *change the sign before the fraction, and also before all the terms of the numerator.*

Thus, $-\frac{a+b}{c+d}=+\frac{-a-b}{c+d}$;

And, $-\frac{a-b+d}{3abd}=+\frac{-a+b-d}{3abd}$;

Also, $-\frac{12}{6}=+\frac{-12}{6}=-2$.

CASE II.

134. To reduce a mixed number to an improper fraction.

RULE.

1. *Make all the fractional parts positive.*

2. *Multiply the quantity to which the fraction is annexed, by the denominator of the fraction, and connect the product, by its proper sign, with the numerator.*

3. *Under this result write the denominator.*

EXAMPLES.

1. Reduce $2a^2x+\frac{3a^2+bx}{4ab}$ to an improper fraction.

Ans. $\frac{8a^3bx+3a^2+bx}{4ab}$.

2. Reduce $3a+\frac{2a^3+x^2}{3a^2-x}$ to an improper fraction.

Ans. $\frac{11a^3-3ax+x^2}{3a^2-x}$.

3. Reduce $2x+y-\frac{3x^2-y^2}{2x-y}$ to an improper fraction.

Ans. $\frac{x^2}{2x-y}$.

4. Reduce $a-b-\frac{a^2-ab}{b}$ to an improper fraction.

Ans. $\frac{2ab-b^2-a^2}{b}$.

5. Reduce $8-b-\frac{a-2}{6}$ to an improper fraction.

Ans. $\frac{50-6b-a}{6}$.

6. Reduce $3a+9-\frac{3a^2-30}{3+a}$ to an improper fraction.

Ans. $\frac{57+18a}{3+a}$.

CASE III.

135. To reduce an improper fraction to a whole or a mixed number.

RULE.

1. *Divide the numerator by the denominator; the quotient will be the integral part.*

2. *If there be a remainder, write the divisor under it, and connect, by its proper sign, the fraction so formed to the integral part.*

EXAMPLES.

1. Reduce $\frac{20a^4-12a^3x}{4a}$ to a whole quantity.

Ans. $5a^3-3a^2x$.

2. Reduce $\frac{a^2+2ab+b^2-1}{a+b}$ to a mixed quantity.

Ans. $a+b-\frac{1}{a+b}$.

3. Reduce $\frac{4abc+8a^2b^3-6ax}{4ab}$ to a mixed quantity.

Ans. $c+2ab^2-\frac{3x}{2b}$.

4. Reduce $\frac{12ax^2+6x^2-3}{6x^2}$ to a mixed quantity.

Ans. $2a+1-\frac{1}{2x^2}$.

5. Reduce $\frac{a^4-x^4}{a^2-ax+x^2}$ to a mixed quantity.

Ans. $a^2+ax-\frac{ax^3+x^4}{a^2-ax+x^2}$.

6. Reduce $\frac{a^3-b^3}{a-b}$ to a whole quantity.

Ans. a^2+ab+b^2.

7. Reduce $\frac{a^5b-3ab^5}{a^3-ab^2}$ to a mixed quantity.

Ans. $a^2+b^3-\frac{2ab^5}{a^3-ab^2}$.

8. Reduce $\frac{2a^2x-22a^3x^2-2}{2a^2x^3}$ to a mixed quantity.

Ans. $\frac{1}{x^2}-\frac{11a}{x}-\frac{1}{a^2x^3}$.

CASE IV.

136. To find the greatest common divisor of two numbers.

In order to obtain a general rule for finding the greatest common divisor, we must observe:

1. If two numbers are respectively divisible by a third, their *sum* or *difference* will also be divisible by the same number. Thus, if a and b are each divisible by c, $a+b$ and $a-b$ will also be divisible by c; for, if c is contained in a eight times and in b twice, in $a+b$ it will be contained ten times, and in $a-b$ six times.
2. If any number is divisible by another, every multiple of that number will also be divisible by the other. Thus, if a is perfectly divisible by b, $2a$, $3a$, $4a$, or ma will also be divisible by b.
3. Hence, if two numbers are divisible by a third, the difference between the larger and any multiple of the smaller of these numbers must also be divisible by that third number. Thus, if c is contained in a eight times and in b twice, it will be contained in $3b$ six times, and in $a-3b$ twice.
4. Also, if the larger of two numbers having a common divisor is divided by the smaller, the remainder will be divisible by the common divisor.

137. From the preceding principles we deduce the following general rule for finding the greatest common measure.

RULE.

1. *Divide one of the given numbers by the other.*

2. *If there be a remainder, divide the first divisor by this remainder.*

3. *Continue to divide in the same manner till there is no remainder; the last divisor will be the greatest common measure.*

Note 1.—If, in the course of the reduction, one factor is found to be common to all the terms of one of the quantities and not of the other, this factor may be cancelled; for, since only one of the numbers is divisible by it, it cannot be a factor of the common divisor.

Note 2.—For a like reason, the dividend may be multiplied by a factor which does not contain a measure of the divisor.

EXAMPLES.

1. Find the greatest common divisor of $a^3-a^2x+3ax^2-3x^3$ and $a^2-5ax+4x^2$.

First Division.

$$\begin{array}{l|l}
a^3-a^2x+3ax^2-3x^3 & a^2-5ax+4x^2 \\
a^3-5a^2x+4ax^2 & a+4x \\
\hline
\quad 4a^2x-ax^2-3x^3 & \\
\quad 4a^2x-20ax^2+16x^3 & \\
\hline
\end{array}$$

Dividing by $19x^2$) $19ax^2-19x^3$

$$a-x$$

Second Division.

$$\begin{array}{l|l}
a^2-5ax+4x^2 & a-x \\
a^2-ax & a-4x \\
\hline
\quad -4ax+4x^2 & \\
\quad -4ax+4x^2 & \\
\hline
\quad\quad 0 &
\end{array}$$

Hence, the greatest common divisor is $a-x$.

2. Find the greatest common divisor of a^3-ab^2 and $a^2+2ab+b^2$.

Dividing a^3-ab^2 by a, we obtain a^2-b^2.

First Division.

$$\begin{array}{l|l}
a^2+2ab+b^2 & a^2-b^2 \\
a^2\qquad\ -b^2 & 1 \\
\hline
\end{array}$$

Dividing by $2b$) $2ab+2b^2$

$$a+b$$

Second Division.

$$\begin{array}{l|l}
a^2-b^2 & a+b \\
a^2+ab & a-b \\
\hline
\quad -ab-b^2 & \\
\quad -ab-b^2 & \\
\hline
\quad\quad 0 &
\end{array}$$

Hence, the greatest common divisor is $a+b$.

3. Find the greatest common divisor of a^4-x^4 and $a^3+a^2x-ax^2-x^3$.

First Division.

$$\begin{array}{l|l} a^4-x^4 & a^3+a^2x-ax^2-x^3 \\ a^4+a^3x-a^2x^2-ax^3 & a-x \\ \hline -a^3x+a^2x^2+ax^3-x^4 & \\ -a^3x-a^2x^2+ax^3+x^4 & \\ \hline \end{array}$$

Dividing by $2x^2$) $2a^2x^2 \quad -2x^4$

$$a^2 \quad -x^2$$

Second Division.

$$\begin{array}{l|l} a^3+a^2x-ax^2-x^3 & a^2-x^2 \\ a^3 \qquad -ax^2 & a+x \\ \hline a^2x \qquad -x^3 & \\ a^2x \qquad -x^3 & \\ \hline 0 & \end{array}$$

Hence, the greatest common divisor is a^2-x^2.

4. Find the greatest common divisor of a^4-b^4 and a^3-b^3.

First Division.

$$\begin{array}{l|l} a^4-b^4 & a^3-b^3 \\ a^4-ab^3 & a \\ \hline \end{array}$$

Dividing by b^3) ab^3-b^4

$$a-b$$

Second Division.

$$\begin{array}{l|l} a^3-b^3 & a-b \\ a^3-a^2b & a^2+ab+b^2 \\ \hline a^2b-b^3 & \\ a^2b-ab^2 & \\ \hline ab^2-b^3 & \\ ab^2-b^3 & \\ \hline 0 & \end{array}$$

Hence, the greatest common divisor is $a-b$.

5. Find the greatest common measure of $3a^4-5a^3b+5a^2b^2-5ab^3+2b^4$ and $6a^3+8a^2b-11ab^2+2b^3$.

First Division.

Multiplying by 2,

$$3a^4-5a^3b+5a^2b^2-5ab^3+2b^4$$

$$\begin{array}{l|l} 6a^4-10a^3b+10a^2b^2-10ab^3+4b^4 & 6a^3+8a^2b-11ab^2+2b^3 \\ 6a^4+8a^3b-11a^2b^2+2ab^3 & a-3b \\ \hline -18a^3b+21a^2b^2-12ab^3+4b^4 & \\ -18a^3b-24a^2b^2+33ab^3-6b^4 & \\ \hline \end{array}$$

Dividing by $5b^2$) $45a^2b^2-45ab^3+10b^4$

$$9a^2-9ab+2b^2$$

Second Division.

Multiplying by 3, $6a^3+8a^2b-11ab^2+2b^3$

$$\begin{array}{l|l} 18a^3+24a^2b-33ab^2+6b^3 & 9a^2-9ab+2b^2 \\ 18a^3-18a^2b+4ab^2 & 2a+14b \\ \hline \end{array}$$

Multiplying by 3, $42a^2b-37ab^2+6b^3$

$$\begin{array}{l} 126a^2b-111ab^2+18b^3 \\ 126a^2b-126ab^2+28b^3 \\ \hline \end{array}$$

Dividing by $5b^2$) $15ab^2-10b^3$

$$3a-2b$$

Third Division.

$$\begin{array}{l|l} 9a^2-9ab+2b^2 & 3a-2b \\ 9a^2-6ab & 3a-b \\ \hline -3ab+2b^2 & \\ -3ab+2b^2 & \\ \hline 0 & \end{array}$$

Hence, the greatest common divisor is $3a-2b$.

6. Find the greatest common divisor of a^4-b^4 and $a^3-a^2b-ab^2+b^3$. *Ans.* a^2-b^2.

7. Find the greatest common divisor of $3a^3-3a^2x+ax^2-x^3$ and $4a^2-5ax+x^2$. *Ans.* $a-x$.

8. Find the greatest common divisor of $4a^3-2a^2-3a+1$ and $3a^2-2a-1$. *Ans.* $a-1$.

9. Find the greatest common divisor of $a^3+9a^2+27a-98$ and $a^2+12a-28$. *Ans.* $a-2$.

10. Find the greatest common divisor of $36a^6b^2-18a^5b^2-27a^4b^2+9a^3b^2$ and $27a^5b^2-18a^4b^2-9a^3b^2$.

Ans. $9a^4b^2-9a^3b^2$.

138. The greatest common divisor of *more than two* numbers may be obtained by finding, in the first place, the greatest common divisor of two of them, and then of that divisor and the third, and so on. The last divisor thus found will be the greatest common divisor of all the quantities.

EXAMPLE.

Find the greatest common divisor of a^3-b^3, $a^3+2a^2b+2ab^2+b^3$, and $a^4+a^2b^2+b^4$.

First Division.

$$\begin{array}{r|l} a^3+2a^2b+2ab^2+\ b^3 & a^3-b^3 \\ \cline{2-2} a^3 \qquad\qquad\quad -\ b^3 & 1 \\ \cline{1-1} 2b).\quad 2a^2b+2ab^2+2b^3 & \\ \cline{1-1} a^2+\ ab+\ b^2 & \end{array}$$

Second Division.

$$\begin{array}{l|l} a^3-\ b^3 & a^2+ab+b^2 \\ \cline{2-2} a^3+a^2b+ab^2 & a-\ b \\ \cline{1-1} \quad -a^2b-ab^2-b^3 & \\ \quad -a^2b-ab^2-b^3 & \\ \cline{1-1} \qquad\qquad 0 & \end{array}$$

Hence, the greatest common divisor of the first two numbers is a^2+ab+b^2.

Third Division.

$$\begin{array}{l|l} a^4+a^2b^2+b^4 & a^2+ab+b^2 \\ \underline{a^4+a^2b^2+a^3b} & a^2-ab+b^2 \\ \quad -a^3b+b^4 & \\ \quad \underline{-a^3b-a^2b^2-ab^3} & \\ \qquad a^2b^2+ab^3+b^4 & \\ \qquad \underline{a^2b^2+ab^3+b^4} & \\ \qquad\qquad 0 & \end{array}$$

Hence, the greatest common divisor of the three numbers is a^2+ab+b^2.

CASE V.

139. To reduce a fraction to its lowest terms.

RULE.

Divide the two terms of the fraction by their greatest common divisor.

Note.—To show that the *value* of the fraction will not be altered by the operation indicated in the preceding rule, we will demonstrate the following theorem:

THEOR. *If both terms of a fraction be divided by the same quantity, its value will not be altered.*

Let abm and am represent the numerator and denominator of an algebraic fraction of any assignable value, m representing any whole or fractional number whatever:

Then the fraction $\frac{abm}{am}=abm\div am=b$.

Dividing both terms of the fraction by the indefinite number m, and reducing $\frac{ab}{a}=ab\div a=b$.

Hence (by Ax. 2), $\frac{abm}{am}=\frac{ab}{a}$; which was to be demonstrated.

EXAMPLES.

1. Reduce $\frac{12ax}{18ab}$ to its lowest terms.

The greatest common divisor is $6a$; hence, $\frac{12ax}{18ab}=\frac{12ax\div 6a}{18ab\div 6a}=\frac{2x}{3b}$, which is the simplest form of the fraction.

2. Reduce $\frac{14a^2x^2y}{21ax^2}$ to its lowest terms. *Ans.* $\frac{2ay}{3}$.

3. Reduce $\frac{21a^2b^3x^4-14a^2b^2x^3+7a^2b^3x^3}{14a^2b^4x^5}$ to its lowest terms.

The greatest common divisor is $7a^2b^2x^3$. *Ans.* $\frac{3bx-2+b}{2b^2x^2}$.

4. Reduce $\frac{18a^2x-45ax^2}{9ab^2x+36a^3x^3}$ to its lowest terms.

The greatest common divisor is $9ax$. *Ans.* $\frac{2a-5x}{b^2+4a^2x^2}$.

5. Reduce $\frac{5a^3bx^3+10abx^5}{a^4x^2+2a^2x^4}$ to its lowest terms.

The greatest common divisor is $a^3x^2+2ax^4$. *Ans.* $\frac{5bx}{a}$.

6. Reduce $\frac{x^4-3ax^3-8a^2x^2+18a^3x-8a^4}{x^3-ax^2-8a^2x+6a^3}$ to its lowest terms.

The greatest common divisor is $x^2+2ax-2a^2$.

Ans. $\frac{x^2-5ax+4a^2}{x-3a}$.

7. Reduce $\frac{2ab^2-a^2b-a^3}{2b^2+3ab+a^2}$ to its lowest terms.

Ans. $\frac{ab-a^2}{b+a}$.

8. Reduce $\frac{6a^3x+9ax^2-12ax-8a^3}{6ax-8a}$ to its lowest terms.

Ans. $\frac{2a^2+3x}{2}$.

9. Reduce $\frac{a^4-b^4}{a^2+b^2}$ to its lowest terms. *Ans.* $\frac{a^2-b^2}{1}$.

10. Reduce $\frac{a^3x+2a^2x^2+2ax^3+x^4}{5a^5+10a^4x+5a^3x^2}$ to its lowest terms.

Ans. $\frac{a^2x+ax^2+x^3}{5a^4+5a^3x}$.

CASE VI.

140. To find the least common multiple of two or more numbers.

The *least common multiple* of two or more numbers is the least number which can be divided by each of them without a remainder. The reason for the following rule will be sufficiently obvious without farther illustration.

RULE.

1. *Resolve the numbers into their prime factors.*

2. *Select all the different factors which occur, observing, when the same factor has different powers, to take the highest power.*

3. *Multiply together the factors thus selected, and their product will be the least common multiple.*

EXAMPLES.

1. Find the least common multiple of $8a^2x^2y$, $12a^3b^3x$, and $16a^4b^2cx$.

Resolving them into their prime factors,

$$8a^2x^2y = 2^3 \times a^2 \times x^2 \times y$$
$$12a^3b^3x = 2^2 \times a^3 \times x \quad \times b^3 \times 3$$
$$16a^4b^2cx = 2^4 \times a^4 \times x \quad \times b^2 \times c$$

The different factors are 2^4, a^4, x^2, y, b^3, 3, and c.

Hence, the least common multiple is $2^4 \times 3 \times a^4 \times b^3 \times c \times x^2 \times y = 48a^4b^3cx^2y$.

2. Find the least common multiple of $12a^2b^3$, $16a^4bc^2$, and $24a$. *Ans.* $48a^4b^3c^2$.

3. Find the least common multiple of $3a^2b$, $5a$, $7a^3c$, $12a^7$, $15a^8$, $18a^2bc$, and $35a^4b^2c^3$. *Ans.* $1260a^8b^2c^3$.

4. Find the least common multiple of $12a^3y + 12a^2by$, $6a^3y^2 + 12a^2by^2 + 6ab^2y^2$, and $4a^2y^2$.

Resolving the numbers into their prime factors,

$$12a^3y + 12a^2by = 12a^2y(a+b) = 2^2 \times 3 \times a^2 \times y \times (a+b)$$
$$6a^3y^2 + 12a^2by^2 + 6ab^2y^2 = 6ay^2(a^2 + 2ab + b^2) = 2 \times 3 \times a \times y^2 \times (a+b)^2$$
$$4a^2y^2 = 2^2 \times a^2 \times y^2$$

The different factors are 2^2, 3, a^2, y^2, and $(a+b)^2$.

Hence, the least common multiple is $2^2 \times 3 \times a^2 \times y^2 \times (a+b)^2 = 12a^2y^2(a+b)^2$.

5. Find the least common multiple of a^2-b^2, $a+b$, and a^2+b^2. *Ans.* a^4-b^4.

6. Find the least common multiple of $a+b$, $a-b$, a^2+ab+b^2, and a^2-ab+b^2. *Ans.* a^6-b^6.

CASE VII.

141. To reduce fractions to equivalent ones having a common denominator.

RULE.

1. *Multiply each numerator into all the denominators, except its own, for the new numerators.*

2. *Multiply all the denominators together for the common denominator.*

Note 1.—It will be perceived that, by the operations indicated in the preceding rule, the *terms* of each fraction are, in effect, multiplied by the product of the other denominators, i. e., the numerator and denominator of each fraction are multiplied by the same number. To show that the *value* of the fractions is not altered by this transformation, it is only necessary to demonstrate the following theorem:

THEOR. *If both terms of a fraction be multiplied by the same number, the value will not be altered.*

Let ab and a represent the numerator and denominator of an algebraic fraction of any assignable quantity:

$$\text{Then the fraction } \frac{ab}{a} = ab \div a = b.$$

Let m represent any whole or fractional number whatever; then multiplying the terms of the fraction by m, we have $\frac{abm}{am} = abm \div am = b$.

Hence, since $\frac{ab}{a} = b$ and $\frac{abm}{am} = b$ (by Ax. 2), $\frac{ab}{a} = \frac{abm}{am}$, which was to be demonstrated.

Note 2.—Mixed numbers should be reduced to improper

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fractions, and all the fractions should be made positive before they are reduced to a common denominator.

Note 3.—Whole numbers or integers can be put under the form of a fraction by writing 1 for a denominator under them, and then be reduced to a common denominator with fractions.

EXAMPLES.

1. Reduce $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{x}{y}$ to equivalent fractions, having a common denominator.

$a\times d\times y=ady$, first numerator.
$c\times b\times y=bcy$, second numerator.
$x\times b\times d=bdx$, third numerator.
$b\times d\times y=bdy$, the common denominator.

Hence, the values of the fractions are $\frac{ady}{bdy}$, $\frac{bcy}{bdy}$, and $\frac{bdx}{bdy}$.

2. Reduce $\frac{5a}{2x}$, $\frac{4b}{2y}$, and $\frac{2d}{3c}$ to equivalent fractions, having a common denominator. *Ans.* $\frac{30acy}{12cxy}$, $\frac{24bcx}{12cxy}$, and $\frac{8dxy}{12cxy}$.

3. Reduce $\frac{4}{ax}$, $\frac{2a}{3b}$, and $\frac{5}{7d}$ to equivalent fractions having a common denominator.

Ans. $\frac{84bd}{21abdx}$, $\frac{14a^2dx}{21abdx}$, and $\frac{15abx}{21abdx}$.

4. Reduce $\frac{2a}{5b}$ and $\frac{a+b}{3c}$ to equivalent fractions having a common denominator. *Ans.* $\frac{6ac}{15bc}$ and $\frac{5ab+5b^2}{15bc}$.

5. Reduce $\frac{a}{b}$, $\frac{3x-2}{c+d}$, and a (or $\frac{a}{1}$) to equivalent fractions having a common denominator.

Ans. $\frac{ac+ad}{bc+bd}$, $\frac{3bx-2b}{bc+bd}$, and $\frac{abc+abd}{bc+bd}$.

6. Reduce $\frac{5a-1}{4x}$, $\frac{3}{5}$, and $\frac{b+2y}{c+d}$ to equivalent fractions having a common denominator.

Ans. $\frac{25ac+25ad-5c-5d}{20cx+20dx}$, $\frac{12cx+12dx}{20cx+20dx}$, and $\frac{20bx+4axy}{20cx+20dx}$.

7. Reduce $\frac{2x+1}{a}$ and $\frac{x+a}{3}$ to equivalent fractions having a common denominator. *Ans.* $\frac{6x+3}{3a}$ and $\frac{ax+a^2}{3a}$.

8. Reduce $\frac{-5a}{7bx}$, $\frac{3c}{d}$, and $\frac{4}{-5}$ to equivalent fractions having a common denominator.

Ans. $\frac{25ad}{-35bdx}$, $\frac{-105bcx}{-35bdx}$, and $\frac{28bdx}{-35bdx}$.

9. Reduce $\frac{a^2+b^2}{2a}$, $\frac{3b}{a-b}$, and $\frac{6a^2}{2a^2+2ab}$ to equivalent fractions having a common denominator.

Ans. $\frac{2a^5-2ab^4}{4a^4-4a^2b^2}$, $\frac{12a^3b+12a^2b^2}{4a^4-4a^2b^2}$, and $\frac{12a^4-12a^3b}{4a^4-4a^2b^2}$.

10. Reduce $\frac{3x^2-2}{4a}$ and $\frac{2x^2-x+4}{a+x}$ to equivalent fractions having a common denominator.

Ans. $\frac{3ax^2+3x^3-2a-2x}{4a^2+4ax}$ and $\frac{8ax^2-4ax+16a}{4a^2+4ax}$.

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142. To reduce fractions to their least common denominator.

RULE.

1. *Find the least common multiple of all the denominators of the given fractions, and it will be the least common denominator.*

2. *Divide the least common denominator by the denominator of each fraction separately, and multiply the quotient by the respective numerators, and the products will be the numerators of the fractions required.*

EXAMPLES.

1. Reduce $\frac{3a^3}{8x^2}$ and $\frac{5ab}{4a^2x^3}$ to their least common denominators.

$$8x^2 = 2^3 \times x^2$$
$$4a^2x^3 = 2^2 \times x^3 \times a^2$$

Hence, the least common multiple is $2^3 \times x^3 \times a^2 = 8a^2x^3$.

Then, $\left.\begin{array}{l}\frac{8a^2x^3}{8x^2} \times 3a^3 = a^2x \times 3a^3 = 3a^5x \\ \frac{8a^2x^3}{4a^2x^3} \times 5ab = 2 \quad \times 5ab = 10ab\end{array}\right\}$ new numerators.

Ans. $\frac{3a^5x}{8a^2x^3}$ and $\frac{10ab}{8a^2x^3}$.

2. Reduce $\frac{3a^2b}{4cx^2}$, $\frac{y}{2x}$, and $\frac{5x^2}{8ac^2}$ to their least common denominators. *Ans.* $\frac{6a^3bc}{8ac^2x^2}$, $\frac{4ac^2xy}{8ac^2x^2}$, and $\frac{5x^4}{8ac^2x^2}$.

3. Reduce $\frac{a}{a^2-x^2}$, $\frac{3b}{4a-4x}$, and $\frac{5x}{a+x}$ to their least common denominator.

Ans. $\frac{4a}{4a^2-4x^2}$, $\frac{3ab+3bx}{4a^2-4x^2}$, and $\frac{20ax-20x^2}{4a^2-4x^2}$.

CASE VIII.

ADDITION OF FRACTIONS.

143. THEOR. *If two fractions have a common denominator, their sum will be equal to the sum of their numerators divided by the common denominator.*

Let $\frac{am}{a}$ and $\frac{an}{a}$ represent two fractions whose common denominator is a;

Then, $\frac{am}{a}+\frac{an}{a}=\frac{am+an}{a}$;

For, $\frac{am}{a}=m$, and $\frac{an}{a}=n$; therefore, $\frac{am}{a}+\frac{an}{a}=m+n$.

But, $\frac{am+an}{a}=(am+an)\div a=m+n$;

Hence (by Ax. 2), $\frac{am}{a}+\frac{an}{a}=\frac{am+an}{a}$, which was to be demonstrated.

144. Hence, to add fractions, we obtain the following general

RULE.

1. *Reduce the fractions to equivalent ones, having a common denominator, and make them all positive.*

2. *Add all the numerators together, and under their sum write the common denominator.*

3. *Reduce the resulting fraction to its lowest terms.*

EXAMPLES.

1. Add together $\frac{3a^2}{2b}$, $\frac{2a}{5}$, and $\frac{3b}{7a}$.

Reducing the fractions to a common denominator,

$$\frac{3a^2}{b}+\frac{2a}{5}+\frac{3b}{7a}=\frac{105a^3}{70ab}+\frac{28a^2b}{70ab}+\frac{30b^2}{70ab}.$$

Adding the numerators of the reduced fractions,

$$\frac{105a^3}{70ab}+\frac{28a^2b}{70ab}+\frac{30b^2}{70ab}=\frac{105a^3+28a^2b+30b^2}{70ab}.\quad \textit{Ans.}$$

2. Add together $\frac{a}{2b}$ and $\frac{3c+x}{a+b}$. *Ans.* $\frac{a^2+ab+6bc+2bx}{2ab+2b^2}$.

3. Add together $\frac{2a+1}{3}$, $\frac{4a+2}{5}$, and $\frac{a}{7}$. *Ans.* $\frac{169a+77}{105}$.

4. Add together $\frac{3+8c}{x+y}$, $\frac{3-8c}{x}$, and $-\frac{4}{b}$.

Ans. $\frac{6bx+3by-8bcy-4x^2-4xy}{bx^2+bxy}$.

5. Add together $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$. *Ans.* $\frac{2a^2+2b^2}{a^2-b^2}$.

6. Add together $\frac{a}{b}$, $\frac{a-3b}{cd}$, and $\frac{a^2-b^2-ab}{bcd}$.

Ans. $\frac{acd-4b^2+a^2}{bcd}$.

7. Add together $\frac{12a^2}{16b^2}$ and $\frac{9a}{8b^2}$. *Ans.* $\frac{6a^2+9ab^2}{8b^2}$.

8. Add together $\frac{a}{a+b}$ and $\frac{b}{a-b}$. *Ans.* $\frac{a^2+b^2}{a^2-b^2}$.

9. Add together $\frac{2a}{a+b}$, $-\frac{3}{4}$, and $-\frac{x}{b}$.

Ans. $\frac{5ab-3b^2-4ax-4bx}{4ab+4b^2}$.

10. Add together $\frac{3}{a}$, $-\frac{a+1}{b-1}$, and $-\frac{a-1}{b+1}$.

Ans. $\frac{3b^2-2ab-2a-3}{ab^2-a}$.

11. Add together $2a+\frac{a+3}{5}$ and $4a+\frac{2a-5}{4}$.

Ans. $6a+\frac{14a-13}{20}$.

12. Add together $a-\frac{8x^2}{b}$ and $b+\frac{2ax}{c}$.

Ans. $a+b+\frac{2abx-8cx^2}{bc}$.

CASE IX.

SUBTRACTION OF FRACTIONS.

145. THEOR. *If two fractions have a common denominator, their difference is equal to the numerators divided by the common denominator.*

Let $\frac{am}{a}$ and $\frac{an}{a}$ represent two fractions whose common denominator is a;

Then, $\frac{am}{a}-\frac{an}{a}=\frac{am-an}{a}$.

For $\frac{am}{a}=m$, and $\frac{an}{a}=n$; therefore, $\frac{am}{a}-\frac{an}{a}=m-n$.

But, $\frac{am-an}{a}=(am-an)\div a=m-n$.

Hence (by Ax. 2), $\frac{am}{a}-\frac{an}{a}=\frac{am-an}{a}$, which was to be demonstrated.

146. Hence, to subtract one fraction from another, we obtain the following general

RULE.

1. *Reduce the fractions to equivalent ones, having a common denominator, and make them all positive.*

2. *Subtract the numerator of the fraction to be subtracted from that of the other fraction, and under their difference write the common denominator.*

EXAMPLES.

1. From $\frac{2a}{3b}$ subtract $\frac{4c}{5d}$.

Reducing the fractions to a common denominator, &c.

$$\frac{2a}{3b}-\frac{4c}{5d}=\frac{10ad}{15bd}-\frac{12bc}{15bd}=\frac{10ad-12bc}{15bd}. \textit{ Ans.}$$

2. From $\frac{a+b}{c}$ subtract $\frac{x}{y}$. *Ans.* $\frac{ay+by-cx}{cy}$.

3. From $\frac{6a^2}{7b}$ subtract $\frac{a^2b+a^3b}{3ax}$.

$$\frac{6a^2}{7b}-\frac{a^2b+a^3b}{3ax}=\frac{18a^3x}{21abx}-\frac{7a^2b^2+7a^3b^2}{21abx}=\frac{18a^3x-7a^2b^2-7a^3b^2}{21abx}. \textit{ Ans.}$$

4. From $\frac{4x}{5}$ subtract $\frac{3x+1}{x+1}$. *Ans.* $\frac{4x^2-11x-5}{5x+5}$.

5. From $\frac{1}{x-y}$ subtract $\frac{1}{x+y}$. *Ans.* $\frac{2y}{x^2-y^2}$.

6. From $\frac{x^2+2}{y}$ subtract $\frac{3}{x^2-2}$. *Ans.* $\frac{x^4-4-3y}{x^2y-2y}$.

7. From $\frac{ax+x^2}{ax-x^2}$ subtract $\frac{ax-x^2}{ax+x^2}$. *Ans.* $\frac{4ax}{a^2-x^2}$.

8. From $5a+\frac{3b}{x}$ subtract $3a+\frac{3x}{c}$. *Ans.* $2a+\frac{3cb-3x^2}{cx}$.

9. From $6m-\frac{4a+1}{2}$ subtract $m+\frac{3}{5}$. *Ans.* $5m-\frac{20a+11}{10}$.

10. From $\frac{a^2b^2}{2a+b^2}$ subtract $\frac{2a^4b^2}{4a^3-ab^4}$.

Ans. $\frac{-a^3b^6-2a^4b^4}{8a^4-2a^2b^4+4a^3b^2-ab^6}=\frac{a^2b^4}{b^4-4a^2}$.

CASE X.

MULTIPLICATION OF FRACTIONS.

147. THEOR. *The product of two fractions is equivalent to the product of the numerators divided by the product of the denominators.*

Let $\frac{a}{b}$ and $\frac{a'}{b'}$ represent any two fractions:

Then will $\frac{a}{b}\times\frac{a'}{b'}=\frac{aa'}{bb'}$:

For, letting v represent the value of the first fraction, and v' the value of the second, we shall have $\frac{a}{b}=v$, and $\frac{a'}{b'}=v'$.

Multiplying the two equalities together (Ax. 5), $\frac{a}{b}\times\frac{a'}{b'}=v.v'$.

Multiplying the equality $\frac{a}{b}=v$ by b (Ax. 5), $a=bv$.

Multiplying the equality $\frac{a'}{b'}=v'$ by b' (Ax. 5), $a'=b'v'$.

Multiplying the last two equalities together (Ax. 5), $aa'=bv\,.\,b'v'=bb'\times vv'$.

Dividing the last equality by bb' (Ax. 6), $\frac{aa'}{bb'}=vv'$.

Hence (by Ax. 2), $\frac{a}{b}\times\frac{a'}{b'}=\frac{aa'}{bb'}$, which was to be demonstrated.

COROL. The product of any number of fractions is equivalent to the product of the numerators divided by the product of the denominators:

Thus, $\frac{a}{b}\times\frac{a'}{b'}\times\frac{a''}{b''}=\frac{aa'a''}{bb'b''}$.

148. From the foregoing theorem we infer the following general rule for the multiplication of fractions.

RULE.

1. *Multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

2. *Reduce the resulting fraction to its lowest terms.*

EXAMPLES.

1. Multiply $\frac{3a^2}{4b}$ by $\frac{5a^3c}{7b^2d}$. *Ans.* $\frac{3a^2}{4b}\times\frac{5a^3c}{7b^2d}=\frac{15a^5c}{28b^3d}$.

2. Multiply $\frac{3a^2+b}{8ax}$ by $\frac{2ac^2}{3dx^3}$.

Ans. $\frac{3a^2+b}{8ax}\times\frac{2ac^2}{3dx^3}=\frac{6a^3c+2abc^2}{24adx^4}=\frac{3a^2c+bc^2}{12dx^4}$.

3. Multiply $\frac{a^2+5b}{a-1}$ by $\frac{3x^2}{a+1}$. *Ans.* $\frac{3a^2x^2+15bx^2}{a^2-1}$.

4. Multiply $\frac{3a^2-5a}{14}$ by $\frac{7x}{2a^3-3a}$. *Ans.* $\frac{3ax-5x}{4a^2-6}$.

5. Multiply $\frac{16a^2x-24x^2}{45b^3x}$ by $\frac{45b^3x}{16a^2x-24x^2}$. *Ans.* 1.

6. Multiply $\frac{a^2-x^2}{cx}$ by $\frac{a^2+x^2}{c+x}$. *Ans.* $\frac{a^4-x^4}{c^2x+cx^2}$.

7. Multiply $\frac{2a}{m}$, $\frac{h-d}{y}$, $\frac{b}{c}$, and $\frac{1}{n-1}$ together.

Ans. $\frac{2abh-2abd}{cmny-cmy}$.

8. Multiply $\frac{5a^2d}{12m}$, $\frac{8m}{5a^3}$, and $\frac{3ah}{2d}$ together. *Ans.* h.

9. Multiply $\frac{a^4-b^4}{a+b}$ by $\frac{a^2}{ab-b^2}$. *Ans.* $\frac{a^4+a^2b^2}{b}$.

10. Multiply $\frac{a^2x-x^3}{a}$ by $\frac{3a}{2ax-2x^2}$. *Ans.* $\frac{3a+3x}{2}$.

Note 1.—Since every integer can be expressed in the form of a fraction by writing 1 under it for a denominator, it is evident that an integer, or whole number, may be multiplied into a fraction by multiplying the numerator of the fraction by the whole number, while the denominator remains the same.

Thus, $a\times\frac{b}{c}=\frac{a}{1}\times\frac{b}{c}=\frac{ab}{c}$.

Note 2.—If the denominator is divisible by a whole number, dividing the denominator multiplies the fraction.

Thus, $\frac{a}{bc}\times c=\frac{a}{bc\div c}=\frac{a}{b}$; for, $\frac{a}{bc}\times c=\frac{a}{bc}\times\frac{c}{1}=\frac{ac}{bc}=\frac{a}{b}$.

CASE XI.

DIVISION OF FRACTIONS.

149. THEOR. *If one fraction be divided by another fraction, the quotient will be equivalent to the product of the fractional dividend multiplied by the fractional divisor inverted.*

Let $\frac{a}{b}$ and $\frac{a'}{b'}$ represent any two fractions:

Then will $\frac{a}{b}\div\frac{a'}{b'}=\frac{ab'}{a'b}$:

For, letting v represent the value of $\frac{a}{b}$, and v' the value of $\frac{a'}{b'}$;

Then, $\frac{a}{b}=v$ and $\frac{a'}{b'}=v'$; and $\frac{a}{b}\div\frac{a'}{b'}=v\div v'$.

Multiplying the equality $\frac{a}{b}=v$ by bb' (Ax. 5), $ab'=bb'v$.

Multiplying the equality $\frac{a'}{b'}=v'$ by bb' (Ax. 5), $a'b=bb'v'$.

Dividing the former by the latter of the last two equalities, $\frac{ab'}{ba'}=\frac{bb'v}{bb'v'}=v\div v'$.

Hence (by Ax. 2), $\frac{a}{b}\div\frac{a'}{b'}=\frac{ab'}{ba'}$, which was to be demonstrated.

150. From the foregoing theorem we infer the following general rule for the division of one fraction by another.

RULE.

1. *Invert the fractional divisor.*

2. *Then proceed as in multiplication, and the product thus found will be the quotient required.*

EXAMPLES.

1. Divide $\frac{3a^2}{4b}$ by $\frac{5c^3}{6d^2}$.

$$\frac{3a^2}{4b}\div\frac{5c^3}{6d^2}=\frac{3a^2}{4b}\times\frac{6d^2}{5c^3}=\frac{18a\,d^2}{20bc^3}=\frac{9a^2d^2}{10bc^3}. \textit{ Ans.}$$

2. Divide $\frac{3a^2+2b}{2b^2+c}$ by $\frac{3a}{5b}$.

$$\frac{3a^2+2b}{2b^2+c}\div\frac{3a}{5b}=\frac{3a^2+2b}{2b+c}\times\frac{5b}{3a}=\frac{15a^2b+10b^2}{6ab+3ac}. \textit{ Ans.}$$

3. Divide $\frac{a}{1-a}$ by $\frac{a}{5}$. *Ans.* $\frac{5}{1-a}$.

4. Divide $\frac{2ab+b^2}{c^3-b^3}$ by $\frac{b}{c-b}$. *Ans.* $\frac{2a+b}{c^2+bc+b^2}$.

5. Divide $\frac{a^4-b^4}{a^2-2ab+b^2}$ by $\frac{a^2+ab}{a-b}$. *Ans.* $\frac{a^2+b^2}{a}=a+\frac{b^2}{a}$.

6. Divide $\frac{3a-3b}{a+d}$ by $\frac{5a-5b}{a+d}$. *Ans.* $\frac{3}{5}$.

7. Divide $\frac{2d^2}{a^3+d^3}$ by $\frac{d}{a+d}$. *Ans.* $\frac{2d}{a^2-ad+d^2}$.

8. Divide $\frac{6a-7}{a+1}$ by $\frac{a-1}{3}$. *Ans.* $\frac{18a-21}{a^2-1}$.

9. Divide $\frac{a+a^2}{3c^2}$ by $\frac{2ac+2a^2c}{7}$. *Ans.* $\frac{7}{6c^3}$.

10. Divide $\frac{2a^6c-2c^7}{a^2c-2ac^2+c^3}$ by $\frac{a^2+ac+c^2}{a-c}$. *Ans.* $2(a^3+c^3)$.

Note 1.—When a fraction is to be divided by a whole number, or a whole number by a fraction, write the whole number in the form of a fraction by making its denominator 1, and then proceed as before.

Thus, $\frac{a}{b}\div c=\frac{a}{b}\div\frac{c}{1}=\frac{a}{b}\times\frac{1}{c}=\frac{a}{bc}$:

And, $a\div\frac{b}{c}=\frac{a}{1}\div\frac{b}{c}=\frac{a}{1}\times\frac{c}{b}=\frac{ac}{b}$.

Note 2.—The reciprocal of a fraction is expressed by the fraction inverted:

Thus, the reciprocal of $\frac{a}{b}$ is $\frac{1}{\frac{a}{b}}$, or $1\div\frac{a}{b}$:

But, $1\div\frac{a}{b}=1\times\frac{b}{a}=\frac{b}{a}$; hence, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

Note 3.—If the numerator or denominator of a fraction has a rational coëfficient, the expression may be reduced to a simpler form, on the principle that multiplying the denominator of a fraction has the same effect upon its value as dividing the numerator; and multiplying the numerator has the same effect as dividing the denominator.

Thus, 1. $\frac{\frac{3}{4}a}{b}=\frac{3}{4}\times\frac{a}{b}=\frac{3a}{4b}$.

2. $\frac{\frac{5}{8}a+\frac{5}{8}}{3a+2b}=\frac{\frac{5}{8}(a+1)}{3a+2b}=\frac{5}{8}\times\frac{a+1}{3a+2b}=\frac{5a+5}{24a+16b}$.

3. $\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$.

SECTION IV.

OF EQUATIONS.

151. An EQUATION is the algebraic expression of two equal quantities connected by the sign of equality.

152. The monomial or polynomial quantity which is written on the left of the sign of equality is called the *first member;* that which is written on the right, the *second member.*

An equation, then, is composed of *two members;* and each member is composed of *one* or *more terms.*

153. The two members of the equation must be composed of quantities of the same kind; that is, dollars must be put equal to dollars, weight equal to weight, &c.

154. Equations are distinguished into different *degrees*, according to the highest *power* of the unknown quantity. If it involve only the first power of the unknown quantity, it is called an equation of the *first degree.* If the highest power of the unknown quantity be the second power, it is called an equation of the *second degree;* if it be the third power, an equation of the *third degree*, &c.

Thus, $x=a$ is an equation of the first degree.

$$\left.\begin{array}{r} x^2=a \\ x^2+x=a \end{array}\right\} \text{ are equations of the second degree.}$$

$$\left.\begin{array}{r} x^3=a \\ x^3+x^2=a \\ x^3+x^2+x=a \end{array}\right\} \text{ are equations of the third degree.}$$

155. The *solution* of a problem is the method of discovering, by analysis, the value of the unknown quantity involved in the conditions of the problem, and consists of two parts.

1. *The translation of the problem from common into algebraic language; or the expression of its conditions in the form of an equation by means of algebraic symbols.*

2. *The reduction of the equation to such a form that the un-*

known quantity may stand by itself, and form one member of the equation, while the known quantities form the other.

156. No general rule can be given for translating the problem from common into algebraic language; only that the algebraic expression shall exhibit the same relations, and indicate the same operations as those implied in the original statement of the problem.

157. *Proportions* may be converted into equations by taking the product of the first and fourth terms for one member, and the product of the second and third for the other. Thus, if $a:b::c:d$, converting the proportion into an equation, we shall have $a\times d=b\times c$; or, if $2:4::8:16$, we shall have $2\times 16=4\times 8$.

158. The *reduction* of an equation involves the following general axiom: *If equal operations be performed upon equal quantities, the results will be equal.* Hence,

1. *If equal quantities be added to both members of an equation, the equality of the members will not be destroyed.*

2. *If equal quantities be subtracted from both members of an equation, the equality will not be destroyed.*

3. *If both members of an equation be multiplied by the same number, the equality will not be destroyed.*

4. *If both members of an equation be divided by the same number, the equality will not be destroyed.*

5. *If both members of an equation be involved to equal powers, the equality will not be destroyed.*

6. *If equal roots of both members of an equation be taken, the equality will not be destroyed.*

159. The *verification* of a problem consists in substituting the value of the unknown quantity for the unknown quantity itself in the given equation, and thereby ascertaining whether it answers the conditions of the problem.

160. Equations are either *numerical* or *literal.* Numerical equations contain numbers only, excepting the unknown quantity. In literal equations, the given quantities are represented by letters.

EQUATIONS OF THE FIRST DEGREE, INVOLVING ONE UNKNOWN QUANTITY.

161. There may be three cases of equations of this nature, viz.: When the known and unknown quantities are connected by *addition* or *subtraction*, by *division*, or by *multiplication.*

CASE I.

162. In this case the unknown quantity is connected to known quantities by addition or subtraction.

Let it be required to reduce the equation $\quad 9x-32=86+8x.$

Adding 32 to both members $\quad 9x-32+32=86+32+8x.$

Subtracting $8x$ from both members $\quad 9x-32+32-8x=86+32+8x-8x.$

Cancelling - - $\quad 9x-8x=86+32.$

Reducing - - $\quad x=118.$

163. These operations will suggest the following general rule when the unknown and known quantities are connected by the signs *plus* or *minus.*

RULE.

1. *Transpose, so that all the unknown quantities may be in the first, and the known in the second member of the equation; observing to affect the terms transposed with the contrary sign.*

2. *Reduce each member to a monomial*

EXAMPLES.

1. Reduce the equation $\quad 6x-5+x=12-x+7x.$

   Transposing - $\quad 6x+x+x-7x=12+5.$

   Reducing - $\quad x=17.$

2. Reduce the equation $14-8x+5=3x+28+6x-2x-16x.$ *Ans.* $x=9.$

3. Reduce the equation $x-12+7x-8x=4-x+20.$ *Ans.* $x=36.$

4. Reduce the equation $-6x-32+10x=84+3x-100.$ *Ans.* $x=16.$

5. Reduce the equation $20-18x+44+x=70-18x-5$.

Ans. $x=1$.

6. Reduce the equation $4x+25+3x=6x+80$.

Ans. $x=55$.

CASE II.

164. In this case the unknown and known quantities are combined by division.

Let it be required to reduce the equation $\frac{3x}{4}-\frac{x}{2}=8$.

Multiplying both members of the equation by 4, the least common multiple of the denominators, the equation becomes - - - - - - $\frac{12x}{4}-\frac{4x}{2}=32$.

Reducing the fractions to whole numbers, $3x-2x=32$.

Reducing the terms - - - - $x=32$.

165. Hence, to free an equation of fractions, we have the following general

RULE.

1. *Multiply both members of the equation by the least common multiple of the denominators.*

2. *Reduce the improper fractions thus produced to whole numbers.*

3. *Transpose and reduce the terms as before.*

Note 1.—Instead of finding the least common multiple of the denominators, the equation may be multiplied by each denominator successively.

Note 2.—When a minus fraction is cleared from its denominator, the sign before each term of the numerator must be changed.

EXAMPLES.

1. Reduce the equation $\frac{4x}{5}-\frac{3x}{4}+2=8$.

The least common multiple of 4 and 5 is 20; multiplying both members by this, the equation becomes

$$\frac{80x}{5}-\frac{60x}{4}+40=160.$$

Reducing the fractions to whole numbers,

$$16x-15x+40=160.$$

Transposing - - - - $16x-15x=160-40.$

Reducing - - - - - $x=120.$

2. Reduce the equation $\frac{2x}{3}-\frac{3x}{4}+\frac{x}{6}=11.$

Multiplying both members by 3, $2x-\frac{9x}{4}+\frac{x}{2}=33.$

Multiplying by 4 - - - $8x-9x+2x=132.$

Reducing - - - - $x=132.$

CASE III.

166. In this case the known and unknown quantities are combined by multiplication.

Let it be required to reduce the equation $\frac{3x}{5}+\frac{x}{4}=17.$

Clearing of fractions - - - $12x+5x=340.$

Reducing the terms - - - $17x=340.$

Dividing the equation by 17 - - $x=20.$

167. Hence, if the unknown quantity, after the equation has been cleared of fractions and the terms reduced, has a coefficient, the reduction may be completed by the following

RULE.

Divide both members of the equation by the coefficient of the unknown quantity.

EXAMPLES.

1. Reduce the equation $13x+31=8x+76.$

Transposing - - - - $13x-8x=76-31.$

Reducing - - - - $5x=45.$

Dividing by coefficient of x - $x=9.$

2. Reduce the equation $\frac{3x}{4}+\frac{5x}{3}-\frac{3x}{5}=5.$

Clearing of fractions $45x+100x-36x=300.$

Reducing - - - $109x=300.$

Dividing - - - $x=\frac{300}{109}=2\frac{82}{109}.$

3. Reduce the equation $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=12$. *Ans.* $x=11\frac{1}{13}$.

4. Reduce the equation $12x+\frac{3x}{2}-1=16$. *Ans.* $1\frac{7}{27}$.

5. Reduce the equation $\frac{5x}{4}-5=\frac{2x+20}{3}$. *Ans.* $x=20$.

6. Reduce the equation $\frac{6x}{5}=\frac{x}{2}+\frac{x}{3}+10$. *Ans.* $x=27\frac{3}{11}$.

7. Reduce the equation $x+\frac{2x}{3}+\frac{x}{2}=26$. *Ans.* $x=12$.

8. Reduce the equation $x+\frac{x}{2}+\frac{3x}{4}=81$. *Ans.* $x=36$.

9. Reduce the equation $x+x+\frac{x}{2}=100-2\frac{1}{2}$.

Ans. $x=39$.

10. Reduce the equation $x+\frac{x}{2}+\frac{3x}{4}+\frac{2x}{7}+\frac{x}{14}=146$.

Ans. $x=56$.

168. Combining the principles discussed in the preceding three cases, we have, for the solution of all equations of the first degree involving only one unknown quantity, the following general

RULE.

1. *Clear the equation of fractions.*

2. *Transpose the terms, so as to bring all the unknown quantities into the first, and the known into the second member of the equation.*

3. *Reduce each member to a monomial.*

4 *Divide the equation by the coefficient of the unknown quantity.*

Note 1.—If the unknown quantity in the result is negative, change the signs of all the terms in the equation.

Thus - - $4x-5x=9-12$.

Reducing - - $-x=-3$.

Changing the signs $x=3$.

Note 2.—To verify the result obtained by the reduction of an equation, substitute the value obtained for the unknown quantity in the first equation, and see if it satisfies the conditions. Thus, substituting 3 for x in the equation above, we have - - $4\times3-5\times3=9-12.$

Multiplying factors $12-15=9-12.$

Reducing - - $-3=-3.$

EXAMPLES.

1. Reduce the equation $x+\frac{x}{2}+\frac{x}{3}=11.$ *Ans.* $x=6.$

2. Reduce the equation $3x+\frac{4x+2}{5}=x+\frac{4}{3}.$ *Ans.* $x=\frac{1}{3}.$

3. Reduce the equation $\frac{11x+1}{3x+5}-2=1.$ *Ans.* $x=7.$

4. Reduce the equation $\frac{3x-8}{11}+\frac{4x}{3}=x+2.$ *Ans.* $x=4\frac{1}{2}.$

5. Reduce the equation $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=94.$ *Ans.* $x=120.$

6. Reduce the equation $\frac{3x}{4}+\frac{x}{10}=x-20-\frac{x}{8}.$

Ans. $x=800.$

7. Reduce the equation $\frac{3x+5}{4}+2b=\frac{3b+10}{6}.$

Ans. $x=\frac{5-18b}{9}.$

8. Reduce the equation $8\frac{3}{4}+\frac{x+1}{5}=4+x-26\frac{1}{4}.$

Ans. $x=39.$

9. Reduce the equation $\frac{2x-5}{18}+\frac{19-x}{3}=\frac{10x-7}{9}-\frac{5}{2}.$

Ans. $x=7.$

10. Reduce the equation $2x-\frac{x+3}{3}+15=\frac{12x+26}{5}$

Ans. $x=12.$

11. Reduce the equation $\frac{x-2}{2}+\frac{x}{3}=20-\frac{x-6}{2}.$

Ans. $x=18.$

12. Reduce the equation $\frac{3a+x}{x}-5=\frac{6}{x}$.

Ans. $x=\frac{3a-6}{4}$.

13. Reduce the equation $\frac{2ax+b}{3}=cx+4a$.

Ans. $x=\frac{12a-b}{2a-3c}$.

14. Reduce the equation $8b+7ax=3x+4c-cx$

Ans. $x=\frac{4c-8b}{7a-3+c}$.

15. Reduce the equation $\frac{x}{12}-\frac{x}{9}+\frac{x}{3}=20-\frac{x}{2}$.

Ans. $x=24\frac{24}{29}$.

16. Reduce the equation $\frac{x}{4}+\frac{x}{5}-\frac{x}{6}+x=2x-43$.

Ans. $x=60$.

17. Reduce the equation $3x+\frac{bx-d}{3}=x+a$.

Ans. $x=\frac{3a+d}{6+b}$.

18. Reduce the equation $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}-\frac{x}{6}=1$.

Ans. $x=\frac{60}{67}$.

19. Reduce the equation $\frac{3x-3}{4}-\frac{3x-4}{3}=5\frac{1}{3}-\frac{27+4x}{9}$.

Ans. $x=9$.

20. Reduce the equation $\frac{15x+15}{3x+6}-4=\frac{6x-12}{x-2}-5$.

Ans. $x=2$.

21. Reduce the equation $\frac{4x+7}{3}+12a=10-\frac{x-5b}{5}$.

Ans. $x=\frac{115+15b-180a}{23}$.

22. Reduce the equation $\frac{31-x}{2}+\frac{15x+8}{13}=-\frac{7x-8}{11}+3x$.

Ans. $x=9$.

23. Reduce the equation $\frac{5x-4}{6}-\frac{3x-7}{10}=7\frac{1}{2}-\frac{8x-1}{3}$.

Ans. $x=2\frac{21}{28}$.

24. Reduce the equation $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$.

Ans. $x=8$.

25. Reduce the equation $\frac{7x+8}{3x-1}-8=\frac{27-36x}{3x-1}+4$.

Ans. $x=1$.

26. Reduce the equation $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$.

Ans. $x=72$.

27. Reduce the equation $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}$.

Ans. $x=7$.

28. Reduce the equation $\frac{18+4x}{5}:3x+6::2:5$.

Ans. $x=3$.

29. Reduce the equation $3x+25a:9x+4b::4:10$.

Ans. $x=\frac{250a-16b}{6}$.

30. Reduce the equation $\frac{9x+25}{7}:7-3x::10:7$.

Ans. $x=\frac{15}{13}$.

31. Reduce the equation $\frac{21-3x}{3}-\frac{4x+6}{9}=6-\frac{5x+1}{4}$.

Ans. $x=3$.

32. Reduce the equation $\frac{6x+8}{11}-\frac{5x+3}{2}=\frac{27-4x}{3}-\frac{3x+9}{2}$.

Ans. $x=6$.

33. Reduce the equation $x+\frac{27-9x}{4}-\frac{5x+2}{6}=5\frac{1}{12}-\frac{2x+5}{3}$
$-\frac{29+4x}{12}$.

Ans. $x=5$.

34. Reduce the equation $\frac{7x-8}{11}+\frac{15x+8}{13}=3x-\frac{31-x}{2}$

Ans. $x=9$.

35. Reduce the equation $\frac{4x+3}{6x-43}:1::2x+19:3x-19$.

Ans. $x=8$.

36. Reduce the equation $5x+\frac{7x+9}{4x+3}=9+\frac{10x^2-18}{2x+3}$.

Ans. $x=3$.

37. Reduce the equation $\frac{20x+36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+3\frac{11}{25}$.

Ans. $x=4$.

38. Reduce the equation $\frac{4x-34}{17}-\frac{258-5x}{3}=\frac{69-x}{2}$.

Ans. $x=51$.

39. Reduce the equation $2x-\frac{4x-2}{13}=\frac{2x+11}{5}-\frac{7-8x}{7}$.

Ans. $x=7$.

40. Reduce the equation $16x+5:\frac{4x+14}{9x+31}::36x+10:1$.

Ans. $x=5$.

PROBLEMS PRODUCING EQUATIONS OF THE FIRST DEGREE, INVOLVING ONLY ONE UNKNOWN QUANTITY.

169. Though no general and definite rule can be given for the translation of a problem into algebraic language, yet the following precepts may be found useful for this purpose.

1. *Let* x *represent the unknown quantity whose value we wish to determine.*

2. *Indicate by the aid of algebraic signs the operations that would be necessary in order to verify the answer were the problem already solved.*

3. *The equation or proportion thus formed may be reduced by the preceding rules.*

PROBLEMS.

1. Two men, A and B, trade in company and gain $680, of which B has 4 times as much as A. What is the share of each?

Let $x=$ number of dollars in A's share;
Then $4x=$ number of dollars in B's share,
And we shall have the equation $x+4x=680$.
Reducing terms - - - $5x=680$.
Dividing by coefficient of x - $x=136$, A's share.
And - - - - - - $4x=544$, B's share.
Verification - - $136+4\times136=680$.

2. What number is that, the sum of whose third part and fourth part is 7?

Let $x=$ the number:

Then $\frac{x}{3}=$ one third,

And $\frac{x}{4}=$ one fourth,

And we shall have the equation $\frac{x}{3}+\frac{x}{4}=7$.

Clearing of fractions, $4x+3x=84$:
Reducing terms - $7x=84$:
Dividing by 7 - - $x=12$. *Ans.*

Verification - $\frac{12}{3}+\frac{12}{4}=4+3=7$.

3. Divide $5000 between A, B, C, and D in such a manner that A shall have $300 more than B, and B $50 more than C, and C $200 more than D. What was the share of each?

Let $x=$ D's share;
Then $x+200=$ C's share;
And $x+250=$ B's share;
And $x+550=$ A's share,

And we shall have the equation $x+x+200+x+250+x+550=5000$:

Transposing - $x+x+x+x=5000-200-250-550$;
Reducing - $4x=4000$;
Dividing by 4 - $x=1000$, D's share:
$x+200=1200$, C's share:
$x+250=1250$, B's share:
$x+550=1550$, A's share.

Verification, $1000+1000+200+1000+250+1000+550=5000$.

Or, reducing, $5000=5000$.

4. It is required to divide the number 84 into two such parts that the greater shall be to the less as 8 to 5.

Let $x=$ the greater part,

And $84-x=$ the less part,

And we have the proportion $x:84-x::8:5$.

Converting the proportion into an equation $5x=672-8x$.

Transposing - - $5x+8x=672$.

Reducing terms - - $13x=672$.

Dividing by 13 - - $x=51\frac{9}{13}$, greater part.

$84-x=32\frac{4}{13}$, less part.

5. It is required to divide \$972 between A and B in such a manner that B may have $\frac{4}{5}$ths as much as A.

Let $x=$ A's share,

And $\frac{4x}{5}=$ B's share,

And we shall have the equation $x+\frac{4x}{5}=972$.

Clearing of fractions - $5x+4x=4860$.

Reducing terms - - $9x=4860$.

Dividing by 9 - - - $x=540$, A's share.

$\frac{4x}{5}=432$, B's share.

6. A man puts out three fifths of his money at 6 per cent. and the remainder at 7 per cent., and at the end of the year receives \$4825 interest. How much money had he?

Let $x=$ the amount:

Then $\frac{3x}{5}=$ the amount at 6 per cent,

And $\frac{2x}{5}=$ the amount at 7 per cent., And, multiplying each amount by its rate, we shall have the equation

$$\frac{3x}{5}\times\frac{6}{100}+\frac{2x}{5}\times\frac{7}{100}=4825.$$

Multiplying factors $\frac{18x}{500}+\frac{14x}{500}=4825.$

Clearing of fractions $18x+14x=2412500:$

Reducing terms - - $32x=2412500:$

Dividing by 32 - - $x=75390\frac{5}{8}$. *Ans.*

7. A can do a piece of work in 8 days; B can do the same work in 12 days; in what time will they do it if both work together?

A will do $\frac{1}{8}$th of the work in one day:

B will do $\frac{1}{12}$th of the work in one day:

Let $x=$ the time it would take them to do the work which is represented by 1:

Then, in x days A will do $\frac{x}{8}$ of the work,

And in x days B will do $\frac{x}{12}$ of the work,

And we shall have the equation $\frac{x}{8}+\frac{x}{12}=1.$

Reducing - - - - $x=4\frac{4}{5}$. *Ans.*

8. A gentleman meeting 5 poor persons, distributed \$4,50 among them, giving to the second twice, to the third three times, to the fourth four times, and to the fifth five times as much as to the first. How much did he give to each? *Ans.* 30, 60, 90, 120, and 150 cents.

9. A man left \$11004 to be divided among his widow, two sons, and three daughters, in such a manner that the widow should have twice as much as both the sons, and each son should have as much as the three daughters. What was the share of each?

Widow's share, \$6288,
Each son's share, \$1572,
Each daughter's share, \$524. } *Ans.*

10. What number is that which, being multiplied by 8, the product increased by 10 times the number, and that sum divided by 12, the quotient shall be 4? *Ans.* $2\frac{2}{3}$.

11. A post is $\frac{1}{5}$ in the earth, $\frac{3}{7}$ in the water, and 13 feet out of the water. What is the length of the post? *Ans.* 35.

12. After paying away $\frac{1}{4}$ and $\frac{1}{7}$ of my money, I had \$85 left in my purse. How many dollars had I at first? *Ans.* 140.

13. Of a battalion of soldiers (the officers being included), $\frac{3}{4}$ are on duty, $\frac{1}{10}$ sick, $\frac{3}{5}$ of the remainder are absent, and there are 48 officers. What is the number of persons in the battalion? *Ans.* 800.

14. In an orchard of fruit-trees, $\frac{1}{2}$ of them bear apples, $\frac{1}{4}$ pears, $\frac{1}{8}$ plums: 7 bear peaches, 3 bear cherries, and 2 quinces. How many trees are there? *Ans.* 96.

15. A farmer being asked how many sheep he had, answered, he had them in 4 pastures: in the first he had $\frac{1}{3}$ of the whole number, in the second $\frac{1}{4}$, in the third $\frac{1}{6}$, and in the fourth he had 18 sheep. How many had he? *Ans.* 72.

16. A and B talking of their ages, A says to B, if $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{7}$ of my age be added to my age, and 2 years more, the sum will be twice my age. What was his age? *Ans.* 84.

17. The rent of an estate is this year 8 per cent. greater than it was last. This year it is \$1890; what was it last year? *Ans.* \$1750.

18. A capitalist receives a yearly income of \$2940; $\frac{4}{5}$ of his money being at 4 per cent. interest, and the remainder at 5 per cent. How much has he at interest? *Ans.* \$70,000.

19. A cistern, containing 60 gallons of water, has three unequal cocks for discharging it. The largest will empty it in 1 hour, the second in 2 hours, and the third in 3 hours. In what time will they empty the cistern if they all run at once? *Ans.* $32\frac{8}{11}$ *minutes.*

20. A farmer wishes to mix 90 bushels of provender, consisting of rye, barley, and oats, so that the mixture may contain $\frac{3}{5}$ as much barley as oats, and $\frac{1}{3}$ as much rye as barley. How much of each must there be in the mixture?

Ans. 50 *bushels of oats*, 30 *of barley, and* 10 *of rye.*

21. A, B, and C trade in company. A puts into their stock \$3 as often as B puts in \$7 and C \$5. They gain \$960. What is each man's share of the gain?

Ans. A's \$192, *B's* \$448, *C's* \$320.

22. A, B, and C trade in company. A puts in \$700, B \$450, and C \$950. They gained \$420. What was the share of each?

Ans. A's \$140, *B's* \$90, and *C's* \$190.

23. At a certain election, the whole number of votes was 673. The candidate chosen had a majority of 11. How many voted for each? *Ans. One* 342, *the other* 331.

24. On canvassing the votes at a certain election, it was found that there was no choice: it was also ascertained that one of the candidates had $\frac{2}{5}$ of the whole number of votes, the other $\frac{3}{8}$ of the whole number, and there were 45 scattering votes. What was the whole number of votes? *Ans.* 200.

25. Three men built 780 rods of fence. The first built 9 rods per day, the second 7, the third 5; the second worked three times as many days as the first, and the third twice as many days as the second. How many days did each work?

26. A gentleman bequeathed \$65,600 to his wife, two sons, and three daughters. The wife was to have \$2000 less than the elder son and \$3000 more than the younger son, and the portion of each of the daughters was \$3500 less than that of the younger son. What was the share of the wife and each son?

\$16,350 elder son's share.
\$14,350 wife's share. *Ans.*
\$11,350 younger son's share.
\$7,850 each daughter's share.

27. A man meeting some beggars, gave each of them 4*d.*, and had 16*d.* left; if he had undertaken to give them 6*d.* apiece, he would have wanted 12*d.* more for that purpose. How many beggars were there, and how many pence had he?

28. A boy being sent to market to buy a certain quantity of meat, found that if he bought beef, which was 4*d.* per pound, he would lay out all the money he was intrusted with; but if he bought mutton, which was $3\frac{1}{2}$*d.* per pound, he would have 2 shillings left. How much meat was he sent for?

29. It is required to divide 85 into two such parts that $\frac{2}{5}$ of the one added to $\frac{4}{7}$ of the other may make 60.

30. When the price of a bushel of barley wanted but 3*d.* to be to the price of a bushel of oats as 8 to 5, four bushels of barley and 90*d.* in money were given for nine bushels of oats. What was the price of each per bushel?

31. A market-woman bought a certain number of eggs at the rate of 2 for a cent, and an equal number at 3 for a cent. She sold the whole lot at the rate of 5 for 2 cents, and lost 4 cents by her trade. How many eggs of each sort had she? *Ans.* 120.

32. The hold of a vessel contained 442 gallons of water, which was emptied out by two buckets, the greater of which, holding twice as much as the other, was emptied twice in 3 minutes, but the less 3 times in 2 minutes, and the whole time of emptying was 12 minutes. How much would each bucket contain?

Ans. 13 *and* 26.

33. The expense of paving a square court at 50 cents per square yard, is the same as that of surrounding it with an iron fence at $1, 75 per foot. How many square feet does it contain? *Ans.* 15,876.

34. Out of a certain sum a man paid his creditors $96; half of the remainder he lent a friend; he then spent

one fifth of what remained, and after all these deductions had one tenth of his money left. How much had he at first? *Ans.* \$128.

35. There are two numbers whose sum is a sixth part of their product, and the greater is to the less as 3 to 2. What are the numbers? *Ans.* 10 *and* 15.

36. A person being asked the hour, answered that it was between 5 and 6, and the hour and minute hands were together. What was the time? *Ans.* 5*h.* 27*m.* $16\frac{4}{11}$*s.*

37. Four places are situated in the order of the four letters, A, B, C, and D. The distance from A to D is 102 miles; the distance from A to B is to the distance from C to D as 2 to 3; and $\frac{1}{4}$ of the distance from A to B, added to $\frac{1}{2}$ of the distance from C to D, is three times the distance from B to C. What is the distance between the places?

36 from A to B
12 from B to C } *Ans.*
54 from C to D

38. A waterman went down a river and returned again in 6 hours. Now with the stream he can row 9 miles an hour, but against it he can make a headway of only 3 miles an hour. How far did he go? *Ans.* $13\frac{1}{2}$ *miles.*

39. A hare is 50 leaps before a hound, and takes 4 leaps to the hound's 3; but 2 of the hound's leaps are equal to three of the hare's. How many leaps must the hound take to catch the hare? *Ans.* 300.

40. There is a certain number consisting of two digits or figures, and their sum is 6. If 18 be added to the number, the sum will consist of the same digits transposed. What is the number? *Ans.* 24.

Note.—As the local value of figures increases in a tenfold ratio from right to left, if $x=$ the left-hand digit, and $6-x=$ the right, then $10x+6-x$ the number.

41. A man and his wife would consume a sack of meal in 15 days. After living together 6 days, the woman

alone consumed the remainder in 30 days. How long would the sack last either of them alone?

The man, $21\frac{3}{7}$ days.
The woman, 50 days. } *Ans.*

42. In the composition of a quantity of gunpowder, the nitre was 10 lbs. more than $\frac{2}{3}$ of the whole; the sulphur $4\frac{1}{2}$ lbs. less than $\frac{1}{6}$ of the whole; the charcoal 2 lbs. less than $\frac{1}{7}$ of the nitre. What was the amount of gunpowder? *Ans.* 69 *lbs.*

43. Two pieces of cloth, of the same price per yard, but of different lengths, were bought, the first for £5, the second for £$6\frac{1}{2}$. If 10 be added to the length of each, their sums will be as 5 to 6. What was the length of each piece? *Ans.* 20 *and* 26.

44. A and B began trade with equal sums of money. The first year A gained £40, and B lost £40. The second year A lost $\frac{1}{3}$ of what he had at the end of the first, and B gained £40 less than twice the sum which A had lost. B then had twice as much money as A How much had each at first? *Ans.* £320.

45. On an approaching war, 594 men are to be raised from three towns, A, B, and C, in proportion to their population. The population of A is to that of B as 3 to 5, and the population of B is to that of C as 8 to 7. How many men must each town furnish?

46. A shepherd, in time of war, was plundered by a party of soldiers, who took $\frac{1}{4}$ of his flock and $\frac{1}{4}$ of a sheep; another party took $\frac{1}{3}$ of what he had left and $\frac{1}{3}$ of a sheep; then a third party took $\frac{1}{2}$ of what remained and $\frac{1}{2}$ of a sheep, after which he had 25 sheep left. How many had he at first? *Ans.* 103.

47. A merchant adds yearly to his capital one third of it, but takes from it, at the end of each year, \$500 for his expenses. At the end of the third year, after deducting the last \$500, he finds his original capital is doubled? What was that capital? *Ans.* \$5550.

48. A labourer was hired for 48 days: for each day he wrought he was to receive 24*s.*, but for each day he was idle he was to forfeit 12*s.* At the end of the time he received 504*s.* How many days did he work?

Ans. 30.

49. A cistern which holds 820 gallons, is filled in 20 minutes by 3 pipes, one of which conveys 10 gallons more, and the other 5 gallons less than the third, per minute. How much flows through each pipe per minute?

Ans. 22, 7, *and* 12 *gallons.*

50. A sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and 8 hours afterward B sets out from the same place, and travels the same road at the rate of 5 miles in 3 hours. How long and how far must B travel before he overtakes A?

Ans. 42 *hours, and* 70 *miles.*

EQUATIONS OF THE FIRST DEGREE INVOLVING MORE THAN ONE UNKNOWN QUANTITY.

170. Most of the problems which we have already considered, involve more than one unknown quantity; but we have been able to solve them by employing but one letter or symbol, as we have found it easy, by means of this letter and the conditions of the problem, to find expressions for the other unknown quantities. In many cases, however, the solution is simplified by representing more than one of the unknown quantities by a letter, and in complicated problems it is frequently necessary to do this.

171. When, by the conditions of the problem, two or more unknown quantities are to be determined, it is necessary that there should be *as many independent equations* as there are *unknown quantities.* When this is not the case, the problem will be *indeterminate.* Equations are *independent* when they express different conditions, and *dependent* when they express the same conditions under different forms.

172. *Elimination* is a method of deriving from the given equations a new equation, from which all the unknown quan-

tities except one shall be excluded. The unknown quantities thus excluded are said to be eliminated; and the resulting equation may be solved by the principles already discussed and applied. Having found the value of one of the unknown quantities, the others may be readily found by substitution.

173. There are *three* principal methods of elimination, viz., 1. *By Comparison;* 2. *By Substitution;* 3. *By Addition or Subtraction.*

OF ELIMINATION WHEN THERE ARE TWO EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

First Method.—By Comparison.

174. This method of elimination rests upon the axiom, that *if each of two things is equal to a third, they are equal to each other.*

Let us take the two equations - $x+y=16,$

$$2x+3y=36.$$

Finding the value of x in the 1st equation, $x=16-y,$

Finding the value of x in the 2d equation, $x=\frac{36-3y}{2}.$

Since each of the two quantities, $16-y$, and $\frac{36-3y}{2}$, is equal to x, they must be equal to each other (Ax.).

Hence we have the equation, - $16-y=\frac{36-3y}{2},$

Reducing - - - - - - $y=4.$

Substituting for y its value in the 1st equation, $x+4=16,$

Reducing - - - - - $x=16-4=12.$

175. Hence, for the elimination of one of two unknown quantities, by *comparison*, we have the following general

RULE.

1. *Find the value of one of the unknown quantities in each of the equations.*

2. *Form a new equation by placing these two values equal to each other.*

Note.—It will generally be found convenient to eliminate that unknown quantity, which is least involved with known quantities.

EXAMPLES.

1. Reduce the equations $4x+y=34$ and $4y+x=16$.

$$4x+y=34,$$
$$x+4y=16,$$

Finding the value of x in the 1st equation, $x=\frac{34-y}{4}$,

Finding the value of x in the 2d equation, $x=16-4y$,

Forming a new equation - $\frac{34-y}{4}=16-4y$,

Reducing - - - - - $y=2$,

And - - - - - - $x=8$.

2. Reduce the equations $2x+3y=16$, and $3x-2y=11$.

Ans. $x=5$, *and* $y=2$.

3. Reduce the equations $\frac{x}{2}+\frac{y}{3}=7$, and $\frac{x}{3}+\frac{y}{2}=8$.

Ans. $x=6$, *and* $y=12$.

4. Reduce the equations $\frac{x}{2}+2y=a$, and $\frac{x}{2}-2y=b$.

Ans. $x=a+b$, *and* $y=\frac{a-b}{4}$.

5. Reduce the equations $\frac{x}{2}+\frac{y}{3}=8$, and $\frac{x}{3}-\frac{y}{2}=1$.

Ans. $x=12$, *and* $y=6$.

6. Reduce the equation $\frac{x}{2}+\frac{y}{3}=9$, and the proportion $x:y::4:3$.

Ans. $x=12$, *and* $y=9$.

7. Reduce the equations $\frac{2x+3y}{6}=8-\frac{x}{3}$, and $\frac{7y-3x}{2}=11+y$.

Ans. $x=6$, *and* $y=8$.

Second Method.—By Substitution.

176. This method of elimination rests upon the principle, that if any equivalent expression be substituted, in an equation, for an unknown quantity, it will satisfy the conditions of that equation.

Let us resume the two equations before used,

$$x+y=16,$$
$$2x+3y=36.$$

Finding a value of x in the 1st equation, $x=16-y$:

Substituting this value for x in the 2d equation,

$$2(16-y)+3y=36;$$

Reducing - - - - - - - $y=4$,

And - - - - - - - - $x=12$.

177. Hence, for the elimination of one or two unknown quantities by *substitution*, we obtain the following general

RULE.

1. *Find the value of one of the unknown quantities in one of the equations.*

2. *In the other equation, substitute this value for the unknown quantity itself, and then reduce as before.*

EXAMPLES.

1. Reduce the equations $x+y=13$, and $x-y=3$.

Ans. $x=8$, *and* $y=5$.

2. Reduce the equations $x-7=3y-21$, and $x+7=2y+14$.

Ans. $x=49$, *and* $y=21$.

3. Reduce the equations $7x=8y$, and $x=y+20$.

Ans. $x=160$, *and* $y=140$.

4. Reduce the equations $x+10=2y$, and $y+10=3x$.

Ans. $x=6$, *and* $y=8$.

5. Reduce the equations $\frac{x}{8}+8y=194$, and $\frac{y}{8}+8x=131$.

Ans. $x=16$, *and* $y=24$.

6. Reduce the equations $4x+\frac{y}{2}=26$, and $\frac{x}{2}+\frac{y}{5}=6$.

Ans. $x=4$, *and* $y=20$.

7. Reduce the proportions $x:y::3:1$, and $\frac{x}{2}:5y-4::3:9$.

Ans. $x=24$, *and* $y=8$.

8. Reduce the equations $\frac{x+8}{4}+6y=21$, and $\frac{y+6}{3}+5x=23$.

Ans. $x=4$, *and* $y=3\frac{2}{3}$.

Third Method.—By Addition or Subtraction.

178. This method of elimination rests upon the principle, that if equals be added to or subtracted from equals, the results will be equal.

As the members of an equation are equal quantities, it will follow that if one equation be added to or subtracted from another, the results will be equal.

Let us resume the two equations before used,

$$x+y=16,$$
$$2x+3y=36.$$

Multiplying the 1st equation by 2 - $2x+2y=32$;

Subtracting the 3d from the 2d equation $y=4$;

Substituting and reducing - - - $x=12$.

Again, let us take the two equations, $3x+5y=28$, $2x-5y=2$.

Adding the two equations - - $5x=30$;

Dividing by 5 - - - - $x=6$;

Substituting and reducing - - $y=2$.

179. Hence, for the elimination of one or two unknown quantities, by *Addition or Subtraction*, we have the following general

RULE.

1. *Multiply or divide the equations in such a manner that the term containing one of the unknown quantities shall be the same in both equations.*

2. *If the signs of these terms are alike, subtract one equation from the other; if unlike, add the two equations together.*

EXAMPLES.

1. Reduce the equations $2x+y=16$, and $3x-3y=6$.

Ans. $x=6$, *and* $y=4$.

2. Reduce the equations $x+y=48$, and $x-y=32$.

Ans. $x=40$, *and* $y=8$.

3. Reduce the equations $5x-3y=9$, and $2x+5y=16$.

Ans. $x=3$, *and* $y=2$.

4. Reduce the equations $30x+40y=270$, and $50x+30y=340$.
Ans. $x=5$, *and* $y=3$.

5. Reduce the equations $3x+7y=79$, and $2y=9+\frac{x}{2}$.
Ans. $x=10$, *and* $y=7$.

6. Reduce the equations $\frac{3x-22}{y}+2=7$, and $\frac{x}{6}+\frac{y}{5}=6$.
Ans. $x=24$, *and* $y=10$.

7. Reduce the equations $\frac{2x-y}{2}+14=18$, and $\frac{2y+x}{3}+16=19$.
Ans. $x=5$, *and* $y=2$.

8. Reduce the equations $\frac{2x+3y}{6}+\frac{x}{3}=8$, and $\frac{7y-3x}{2}-y=11$.
Ans. $x=6$, *and* $y=8$.

9. Reduce the equations $\frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4}$, and $\frac{2y+4}{3}-\frac{2x+y}{8}=\frac{x+13}{4}$.
Ans. $x=7$, *and* $y=10$.

10. Reduce the equations $\frac{x}{6}+\frac{y}{4}=6$, and $\frac{x}{4}+\frac{y}{6}=5\frac{2}{3}$.
Ans. $x=12$, *and* $y=16$.

MISCELLANEOUS EQUATIONS OF THE FIRST DEGREE INVOLVING TWO UNKNOWN QUANTITIES.

1. Reduce the equations $8x+5y=68$, and $12x+7y=100$.
Ans. $x=6$, *and* $y=4$.

2. Reduce the equations $8x+\frac{2y}{3}=20$, and $20x+3y=70$.
Ans. $x=1\frac{1}{4}$, *and* $y=15$.

3. Reduce the equations $\frac{x+y}{3}-2y=2$, and $\frac{2x-4y}{5}+y=\frac{23}{5}$.
Ans. $x=11$, *and* $y=1$.

4. Reduce the equations $\frac{2x-3}{2}+y=7$, and $5x-13y=\frac{67}{2}$.
Ans. $x=8$, *and* $y=\frac{1}{2}$.

5. Reduce the equations $\frac{3x-7y}{3}=\frac{2x+y+1}{5}$, and $8-\frac{x-y}{5}=6$. *Ans.* $x=13$, *and* $y=3$.

6. Reduce the equations $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$.

Ans. $x=\frac{1}{2}$, *and* $y=\frac{1}{3}$.

7. Reduce the equations $\frac{x}{7}+7y=99$, and $\frac{y}{7}+7x=51$.

Ans. $x=7$, *and* $y=14$.

8. Reduce the equations $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27$. *Ans.* $x=60$, *and* $y=40$.

9. Reduce the equations $\frac{x}{2}-12=\frac{y}{4}+13$, and $\frac{x+y}{5}+\frac{x}{3}+16=\frac{2x-y}{4}+27$. *Ans.* $x=60$, *and* $y=20$.

10. Reduce the equations $\frac{7+x}{5}-\frac{2x-y}{4}=3y-5$, and $\frac{5y-7}{2}+\frac{4x-3}{6}=18-5x$. *Ans.* $x=3$, *and* $y=2$.

11. Reduce the equations $4x+\frac{15-x}{4}=2y+5+\frac{7x+11}{16}$, and $3y-\frac{2x+y}{5}=2x+\frac{2y+4}{3}$. *Ans.* $x=3$, *and* $y=4$.

12. Reduce the equations $1-\frac{y-x}{6}+4=y-16\frac{2}{3}$, and $\frac{y}{5}-2=\frac{x}{5}$. *Ans.* $x=10$, *and* $y=20$.

13. Reduce the equations $\frac{2x-3y}{5}=x-2\frac{3}{5}$, and $x-\frac{y-1}{2}=0$.

Ans. $x=1$, *and* $y=3$.

14. Reduce the equations $\frac{y}{4}-\frac{x}{7}+5=6$, and $\frac{y}{5}+4=\frac{x}{14}+6$.

Ans. $x=28$, *and* $y=20$.

15. Reduce the equations $y-3=\frac{x}{2}+5$, and $\frac{x+y}{2}=y-3\frac{1}{2}$.

Ans. $x=2$, *and* $y=9$.

16. Reduce the equations $2y-\frac{x+3}{4}=7+\frac{3x-2y}{5}$, and $4x-\frac{8-y}{3}=24\frac{1}{2}-\frac{2x+1}{2}$. *Ans.* $x=5$, *and* $y=5$.

17. Reduce the equations $x-\frac{3y-2+x}{11}=1+\frac{15x+\frac{4}{3}y}{33}$, and $\frac{3x+2y}{6}-\frac{y-5}{4}=\frac{11x+152}{12}-\frac{3y+1}{2}$.

Ans. $x=8$, *and* $y=9$.

18. Reduce the equations $\frac{80+3x}{15}=18\frac{1}{3}-\frac{4x+3y-8}{7}$, and $10y+\frac{6x-35}{5}=55+10x$. *Ans.* $x=10$, *and* $y=15$.

19. Reduce the equations $x-\frac{3x+5y}{17}+17=5y+\frac{4x+7}{3}$, and $\frac{22-6y}{3}-\frac{5x-7}{11}=\frac{x+1}{6}-\frac{8y+5}{18}$. *Ans.* $x=8$, *and* $y=2$.

20. Reduce the equations $\frac{7x-21}{6}+\frac{3y-x}{3}=4+\frac{3x-19}{2}$, and $\frac{2x+y}{2}-\frac{9x-7}{8}=\frac{3y+9}{4}-\frac{4x+5y}{16}$. *Ans.* $x=9$, *and* $y=4$.

PROBLEMS REQUIRING TWO UNKNOWN QUANTITIES, AND PRODUCING TWO EQUATIONS OF THE FIRST DEGREE.

181.—1. A fruiterer sold to one person 6 lemons and 3 oranges for 42 cents, and to another 3 lemons and 8 oranges for 60 cents. What was the price of each?

Let $x=$ the price of a lemon,

And $y=$ the price of an orange,

Then we shall have the two equations,

$$6x+3y=42,$$
$$3x+8y=60.$$

Transposing in the 1st equation - - $6x=42-3y$;

Dividing by 6 - - - - - - $x=\frac{42-3y}{6}$;

Transposing in the 2d equation - - $3x=60-8y$;

Dividing by 3 - - - - - $x=\frac{60-8y}{3}$;

Forming a new equation from the two values of x,

$$\frac{42-3y}{6}=\frac{60-8y}{3};$$

Reducing - - $y=6$, the price of an orange,

And - - - $x=4$, the price of a lemon.

2. What fraction is that, to the numerator of which, if 1 be added, its value will be $\frac{1}{3}$, but if 1 be added to the denominator, its value will be $\frac{1}{4}$?

Let $x=$ the numerator,

And $y=$ the denominator,

Then $\frac{x}{y}=$ the fraction,

And we shall have the two equations,

$$\frac{x+1}{y}=\frac{1}{3},$$

$$\frac{x}{y+1}=\frac{1}{4}.$$

Clearing the 1st of fractions - $3x+3=y;$

Clearing the 2d of fractions - - $4x=y+1;$

Dividing the 4th equation by 4 - - $x=\frac{y+1}{4};$

Substituting the value of x in the third equation - } $3\times\left(\frac{y+1}{4}\right)+3=y;$

Reducing - - - - - $y=15,$

And - - - - - - $x=4:$

Hence - - - - - - $\frac{x}{y}=\frac{4}{15}$, the fraction required.

3. A boy bought 2 apples and 3 oranges for 13 cents; he afterward bought, at the same rate, 3 apples and 5 oranges for 21 cents. What was the price of each?

Let $x=$ the price of an apple,

And $y=$ the price of an orange,

Then we shall have the two equations,

$$2x+3y=13,$$

$$3x+5y=21.$$

Multiplying the 1st equation by 3, $6x+9y=39$;
Multiplying the 2d equation by 2, $6x+10y=42$;
Subtracting the 3d from the 4th - $y=3$, the price of an orange,
And - - - - - - - $x=2$, the price of an apple.

4. What fraction is that, to the numerator of which if 4 be added, the value is $\frac{1}{2}$, but if 7 be added to its denominator, the value is $\frac{1}{5}$? *Ans.* $\frac{5}{18}$.

5. A and B have certain sums of money: says A to B, "Give me \$15 of your money, and I shall have five times as much as you have left." Says B to A, "Give me \$5 of your money, and I shall have exactly as much as you have left." How many dollars had each?
Ans. A had \$35, *and B* \$25.

6. There are two numbers, such that 3 times the greater added to $\frac{1}{3}$ the less is equal to 36; and if 2 times the greater be subtracted from 6 times the less and the remainder divided by 8, the quotient will be 4. What are the numbers? *Ans.* 9 *and* 11.

7. A person was desirous of relieving a certain number of beggars by giving them 2*s.* 6*d.* each, but found that he had not money enough in his pocket by 3*s.*: he then gave them 2*s.* each, and had 4*s.* to spare. How many shillings had he, and how many beggars did he relieve?
Ans. 32*s. and* 14 *beggars.*

8. A labourer working for a gentleman for 12 days, and having had with him the first 7 days his wife and son, received 74*s.*: he wrought afterward 8 other days, during 5 of which he had with him his wife and son, and he received 50*s.* Required the gain of the labourer per day, and also that of his wife and son.
Ans. Husband 5*s.*, *and the wife and son* 2*s.*

9. A purse holds 19 crowns and 6 guineas. Now 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many will it hold of each? *Ans.* 21 *crowns and* 63 *guineas.*

10. A farmer, with 28 bushels of barley at 2*s.* 4*d.* per bushel, would mix rye at 3*s.* per bushel, and wheat at 4*s.* per bushel, so that the whole mixture may consist of 100 bushels, and be worth 3*s.* 4*d.* per bushel. How many bushels of rye, and how many of wheat, must he mix with the barley? *Ans.* 20 *of rye and* 52 *of wheat.*

11. A and B speculate with different sums: A gains $150, B loses $50, and now A's stock is to B's as 3 to 2. And if A had lost $50, and B gained $100, then A's stock would have been to B's as 5 to 9. What was the stock of each? *Ans. A's.* $300, *and B's* $350.

12. A rectangular bowling-green having been measured, it was observed that, if it were 5 feet broader and 4 feet longer, it would contain 116 feet more; but if it were 4 feet broader and 5 feet longer, it would contain 113 feet more. Required the length and breath.

Ans. Length 12, *and breadth* 9 *feet.*

13. There is a number consisting of two figures, the second of which is greater than the first; and if the number be divided by the sum of its figures, the quotient is 4; but if the figures be inverted, and the number which results be divided by a number greater by 2 than the difference of the figures, the quotient becomes 14. What is the number? *Ans.* 48.

14. A person owes a certain sum to two creditors. At one time he pays them $53, giving to one $\frac{4}{11}$ of the sum due to him, and to the other $3 more than $\frac{1}{6}$ of his debt to him. At a second time he pays them $42, giving to the first $\frac{3}{7}$ of what remains due to him, and to the other $\frac{1}{3}$ of what is due to him. What were the debts?

Ans. $121 *and* $36.

15. Some smugglers discovered a cave which would exactly hold the cargo of their boat, viz., 13 bales of cotton and 33 casks of wine. While they were unloading, a custom-house cutter coming in sight, they sailed away

with 9 casks and 5 bales, leaving the cave $\frac{2}{3}$ full. How many bales or casks would it hold?

Ans. 24 *bales or* 72 *casks.*

16. A and B can perform a piece of work in 16 days. They work together 4 days; then A being called off, B is left to finish it, which he does in 36 days more. In what time would each do it separately?

Ans. A in 24, *and B in* 48 *days.*

17. Two loaded wagons were weighed, and their weights were found to be in the ratio of 4 to 5. Parts of their loads, which were in the proportion of 6 to 7, being taken out, their weights were then found to be in the ratio of 2 to 3; and the sum of their weights was then 10 tons. What were their weights at first?

Ans. 16 *and* 20 *tons.*

18. There is a cistern, into which water is admitted by three cocks, two of which are of exactly the same dimensions. When they are all open, $\frac{5}{12}$ of the cistern is filled in 4 hours; and if one of the equal cocks be stopped, $\frac{7}{9}$ of the cistern is filled in $10\frac{2}{3}$ hours. In how many hours would each cock fill the cistern?

Ans. Each of the equal ones in 32 *hours, and the other in* 24.

19. A has a capital of \$30,000, which he puts out to interest at a certain rate per cent., and he owes \$20,000, on which he pays a certain rate per cent. interest. The interest which he receives exceeds that which he pays by \$800. B has a capital of \$35,000, which he puts out to interest at the same rate per cent. that A pays; he also owes \$24,000, on which he pays interest at the same rate that A receives. The interest which he receives exceeds that which he pays by \$310. What are the two rates of interest? *Ans.* 6 *and* 5 *per cent.*

20. A has a certain capital, which he puts out to interest at a certain rate per cent. B has a capital of \$10,000 more than A, which he puts out to interest at one per cent. more, and receives \$800 more interest than A.

C has a capital of $15,000 more than A, which he puts out at 2 per cent. more, and receives $1500 more interest than A. What is the capital of each, and the three rates of interest? *Ans. A's capital*, $30,000; *B's*, $40,000; *C's*, $45,000; *and the rates of interest* 4, 5, *and* 6 *per cent.*

OF ELIMINATION WHERE THERE ARE THREE OR MORE EQUATIONS INVOLVING AS MANY UNKNOWN QUANTITIES.

182. In the problems hitherto given, each has contained no more than *two* unknown quantities, and two independent equations have been sufficient to express the conditions of the question. Other problems, however, may involve three or more unknown quantities; and if they are determinate, their conditions will give rise to as many independent equations as there are unknown quantities.

183. The principles already discussed, and the rules already given for the elimination of one of two unknown quantities, may also be applied where the number exceeds two.

Thus, if there be three independent equations involving three unknown quantities,

I. *From the three equations involving three unknown quantities, deduce two equations involving only two unknown quantities.*

II. *Then from these two deduce one, involving only one unknown quantity.*

III. *Reduce this equation, or find the numerical or literal value of the unknown quantity involved in it: then substitute this value for the unknown quantity itself, in an equation which involves only that and another unknown quantity whose value may thus be found. The value of the remaining unknown quantity may be found in a similar manner.*

184. If there be four independent equations involving four unknown quantities,

I. *From the four equations deduce three, involving only three unknown quantities.*

II. *Reduce these three equations as before.*

185. If there be n independent equations, involving n un-

known quantities, they may be reduced in a similar manner. For from the n equations involving n unknown quantities, we may deduce $n-1$ equations involving $n-1$ unknown quantities; and from these $n-2$ equations involving $n-2$ unknown quantities, and so on until only one equation remains, involving only one unknown quantity. The value of this being found, the values of all the rest may be determined by substitution, as before.

A calculation may often be very much abridged by the exercise of judgment in stating the question, in selecting the equations from which others are to be deduced, in the manner of performing the reduction, in simplifying fractional expressions, in avoiding radical quantities, &c.

EXAMPLES.

1. Reduce the equations $\left\{\begin{array}{l} x+y+z=29 \\ x+2y+3z=62 \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=8, \\ y=9, \\ z=12. \end{array}\right.$

From the 1st equation - - $x=29-y-z$; (4)

From the 2d equation - - $x=62-2y-3z$; (5)

From the third equation - - $x=20-\frac{2y}{3}-\frac{z}{2}$; (6)

Making the 1st and 2d values of x equal - - } $29-y-z=62-2y-3z$. (7)

Making the 1st and 3d values of x equal - - } $29-y-z=20-\frac{2y}{3}-\frac{z}{2}$. (8)

From the 7th equation - - $y=33-2z$; (9)

From the 8th equation - - $y=27-\frac{3z}{2}$; (10)

Making the two values of y equal - - - } $33-2z=27-\frac{3z}{2}$. (11)

Reducing - - - - - $z=12$; (12)

Substituting for z its value in the 9th equation - } - $y=33-24=9$. (13)

Substituting for y and z their values in the 4th equation } - $x=29-9-12=8$. (14)

2. Reduce the equations $\begin{cases} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-z=63 \end{cases}$ *Ans.* $\begin{cases} x=3, \\ y=7, \\ z=4. \end{cases}$

Multiplying the 1st by 6 - $12x+24y-18z=132.$ (4)
Multiplying the 2d by 3 - $12x-6y+15z=54.$ (5)
Multiplying the 3d by 2 - $12x+14y-2z=126.$ (6)
Subtracting the 5th from the 4th $30y-33z=78.$ (7)
Subtracting the 5th from the 6th $20y-17z=72.$ (8)
Multiplying the 7th by $\frac{2}{3}$ - - $20y-22z=52.$ (9)
Subtracting the 9th from the 8th - $5z=20.$ (10)
Dividing by 5 - - - - - $z=4,$ (11)
And - - - - - - - $y=7,$ (12)
And - - - - - - - $x=3.$ (13)

3. Reduce the equations $\begin{cases} 12x+y+7u=26 \\ 18z+3y+12u=69 \\ 10x+20z+17u=69 \\ 18x+10z+7y=66 \end{cases}$. *Ans.* $\begin{cases} x=\frac{1}{2}, \\ y=6, \\ z=1\frac{1}{2}, \\ u=2. \end{cases}$

From the 1st equation, $y=26-7u-12x.$ (5)
Substituting for y its value in the 2d - - $18z+78-21u-36x+12u=69.$ (6)
Substituting for y its value in the 4th - - $18x+10z+182-49u-84x=66.$ (7)
Transposing and uniting in the 6th - - $18z-9u-36x=-9.$ (8)
Transposing and uniting in the 7th - - $10z-49u-66x=-116.$ (9)
Multiplying the 9th by 2, $20z-98u-132x=-232.$ (10)
Subtracting the 10th from the 3d, $115u+142x=301.$ (11)
Multiplying the 8th by 5, $90z-45u-180x=-45.$ (12)
Multiplying the 9th by 9, $90z-441u-594x=-1044.$ (13)
Subtracting the 13th from the 12th $396u+414x=999.$ (14

From the 14th equation, $x=\frac{999-396u}{414}$. (15)

From the 11th equation, $x=\frac{301-115u}{142}$. (16)

Making these two values of x equal - - $\left\{\frac{999-396u}{414}=\frac{301-115u}{142}\right.$. (17)

Reducing - - - - $u=2$,
And - - - - - $x=\frac{1}{2}$,
And - - - - - $y=6$,
And - - - - - $z=1\frac{1}{2}$.

4. Reduce the equations $\left\{\begin{array}{l}\frac{1}{x}+\frac{1}{y}=\frac{1}{8}, \\ \frac{1}{x}+\frac{1}{z}=\frac{1}{9}, \\ \frac{1}{y}+\frac{1}{z}=\frac{1}{10}\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=14\frac{34}{49}, \\ y=17\frac{23}{41}, \\ z=23\frac{7}{31}.\end{array}\right.$

Adding the three equations, $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{1}{8}+\frac{1}{9}+\frac{1}{10}=\frac{121}{360}$. (4)

Dividing by 2 - - - - $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{121}{720}$. (5)

Subtracting the 1st from the 5th, $\frac{1}{z}=\frac{31}{720}$, or $z=\frac{720}{31}$, (6)

Subtracting the 2d from the 5th, $\frac{1}{y}=\frac{41}{720}$, or $y=\frac{720}{41}$, (7)

Subtracting the 3d from the 5th, $\frac{1}{x}=\frac{49}{720}$, or $x=\frac{720}{49}$. (8)

5. Reduce the equations $\left\{\begin{array}{l}4x+2y-3z=4, \\ 3x-5y+2z=22, \\ x+y+z=12.\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=5, \\ y=1, \\ z=6.\end{array}\right.$

6. Reduce the equations $\left\{\begin{array}{l}\frac{1}{2}x+\frac{1}{3}y+z=46, \\ \frac{1}{4}x-y+\frac{1}{2}z=9, \\ x+\frac{1}{4}y-\frac{1}{8}z=19.\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=20, \\ y=12, \\ z=32\end{array}\right.$

7. Reduce the equations $\left\{\begin{array}{l}x+y+z+w=14, \\ \frac{1}{2}x+\frac{1}{3}y-z+w=3, \\ 3x+2y-\frac{1}{2}z+w=15, \\ x+\frac{2y}{3}+\frac{z}{4}-\frac{3w}{5}=2.\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=2, \\ y=3, \\ z=4, \\ w=5.\end{array}\right.$

8. Reduce the equations $\left\{\begin{array}{l} x+y=52, \\ y+z=82, \\ z+w=68, \\ w+u=30, \\ u+x=32. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=20, \\ y=32, \\ z=50, \\ w=18, \\ u=12. \end{array}\right.$

9. Reduce the equations $xy=28$, $xz=20$, and $yz=35$.

Ans. $x=4$, $y=7$, *and* $z=5$.

10. Reduce the equations $\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=124, \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=94, \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=76. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=48, \\ y=120, \\ z=240. \end{array}\right.$

11. Reduce the equations $xy=100$, $yz=40$, and $x2=160$.

Ans. $x=20$, $y=5$, $z=8$.

12. Reduce the equations $x+100=y+z$, $y+100=2x+2z$, and $z+100=3x+3y$.

Ans. $x=9\frac{1}{11}$, $y=45\frac{5}{11}$, *and* $z=63\frac{7}{11}$.

13. Reduce the equations $\left\{\begin{array}{r} \frac{1}{3}x+3y=23, \\ x+\frac{z}{4}=8, \\ y+3z=31, \\ x+y+\frac{1}{2}z+2w=35. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=6, \\ y=7, \\ z=8, \\ w=9. \end{array}\right.$

14. Reduce the equations $\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62, \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47, \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=24, \\ y=60, \\ z=120. \end{array}\right.$

15. Reduce the equations $\left\{\begin{array}{r} 2x+y-2z=40, \\ 4y-x+3z=35, \\ 3w+u=13, \\ y+w+u=15, \\ 3x-y+3u-w=49. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=20, \\ y=10, \\ z=5, \\ w=4, \\ u=1. \end{array}\right.$

16. Reduce the equations $\left\{\begin{array}{l} x+y+z=53, \\ x+2y+3z=105, \\ x+3y+4z=134. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=24, \\ y=6, \\ z=23. \end{array}\right.$

PROBLEMS PRODUCING THREE OR MORE EQUATIONS, AND REQUIRING AS MANY UNKNOWN QUANTITIES.

1. Three persons divided a sum of money between them in such a manner that the shares of A and B together amounted to $900; the shares of A and C together amounted to $800; and the shares of B and C to $700. What was the share of each?

Let x=A's share,
y=B's share,
z=C's share:

Then - $x+y=900$,
And - $x+z=800$,
And - $y+z=700$.
Reducing, $x=500$, A's share,
$y=400$, B's share,
$z=300$, C's share.

2. A man with his wife and son, talking of their ages, said that his age added to that of his son was 16 years more than that of his wife; the wife said that her age added to that of her son made 8 years more than that of her husband; and that all their ages together amounted to 88 years. What was the age of each?

Ans. Husband 40, *wife* 36, *and son* 12 *years.*

3. Three teachers, A, B, and C, speaking of their respective schools, says A to B, "If you will give me 20 of your scholars, my number will then be to the sum of C's and what you will have left, as 4 to 5." Says B to A and C, "If each of you will give me 10 scholars, my number will be to what you will then have as 5 to 4." Says C to A and B, "If you will give me 10 each, I shall have twice as many as both of you." What was the number of scholars each had?

Ans. A 20, *B* 30, *and C* 40 *scholars.*

4. A cistern is furnished with three pipes, A, B, C. By the pipes A and B it can be filled in 12 minutes, by the pipes B and C in 20 minutes, and by A and C in 15

minutes. In what time will each fill the cistern alone, and in what time will it be filled if all three run together? *Ans. A* 20, *B* 30, *C* 60 *minutes, and the three together in* 10 *minutes.*

5. It is required to divide the number 72 into 4 such parts, that if the first part be increased by 5, the second diminished by 5, the third part multiplied by 5, the fourth part divided by 5, the sum, difference, product, and quotient shall all be equal. *Ans.* 5, 15, 2, *and* 50.

6. Find three numbers, such that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third shall be equal to 62; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third equal to 47; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third equal to 38. *Ans.* 24, 60, *and* 120.

7. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days, how many days will it take each person alone to perform the same work?

Ans. A in $14\frac{34}{49}$, *B in* $17\frac{23}{41}$, *and C in* $23\frac{7}{31}$ *days.*

8. A, B, and C sit down to play, each one with a certain number of shillings: A loses to B and C as many shillings as each of them has. Next B loses to A and C as many as each of them now has: lastly, C loses to A and B as many as each of them now has. At the close of the game, each of them has 16 shillings. How much did each one gain or lose?

Ans. A lost 10*s*., *B gained* 2*s*., *and C* 8*s*.

9. There are two such fractions, that if 3 be added to the numerator of the first, its value is double that of the second; but if 3 be added to the denominator, their values are equal. Now the sum of the two fractions is 9 times as great as their difference; and if the numerator of their product be increased by 10, its value will be equal to that of the first fraction. What are the fractions? *Ans.* $\frac{5}{12}$ *and* $\frac{1}{3}$.

10. Three brothers purchased an estate for $15,000: the

first wanted, in order to complete his part of the payment, $\frac{1}{2}$ of the property of the second; the second would have paid his share with the help of $\frac{1}{3}$ of what the first owned; and the third required, to make the same payment, in addition to what he had, $\frac{1}{4}$ part of what the first possessed. What was the amount of each one's property?

Ans. \$3000, \$4000, *and* \$4250 *respectively.*

11. A merchant has 3 ingots, each composed of gold, silver, and copper, in the following proportions, viz., in the first there are 7 ounces of gold, 8 ounces of silver, and 1 ounce of copper to the pound; in the second, there are 5 ounces of gold, 7 ounces of silver, and 4 ounces of copper; and in the third, 2 ounces of gold, 9 ounces of silver, and 5 ounces of copper to the pound. What parts must be taken from each in order to compose a fourth ingot, in which there shall be $4\frac{15}{16}$ ounces of gold, $7\frac{10}{16}$ ounces of silver, and $3\frac{7}{16}$ ounces of copper to the pound?

Ans. 4 *ounces of gold,* 9 *ounces of silver, and* 3 *ounces of copper.*

12. At an election for two members of Congress, three men offer themselves as candidates: the number of voters for the two successful ones are in the ratio of 9 to 8; and if the first had had 7 more, his majority over the second would have been to the majority of the second over the third as 12 to 7. Now if the first and third had formed a coalition, and had one more voter, they would each have succeeded by a majority of 7. How many voted for each?

Ans. 369, 328, *and* 300 *respectively.*

SECTION V.

Generalization of Algebraic Problems.—Demonstration of General Propositions or Theorems.—Properties of Numbers.—Reduction of Formulas relating to Simple Interest, Compound Interest, and Fellowship.—Discussion of Equations of the First Degree.—Theory of Negative Quantities.—Explanation of Symbols.—Infinity.—Infinitesimal.—Indetermination.—Inequations.

GENERALIZATION OF ALGEBRAIC PROBLEMS.

188. THE solution of many questions does not depend upon the particular numbers given in those questions, but will be the same for any other numbers. By generalizing such questions, we are able to deduce a general method or rule for the solution of all questions whose conditions are similar, or which differ from the proposed only in particular numbers which are given.

The following instances of generalization will serve to introduce the learner into this important branch of Algebra.

First General Problem.

189. The sum of two numbers is a, their difference b; it is required to find the two numbers.

Let $x=$ the greater, and $y=$ the less:

Then, by the conditions - - $x+y=a$,

And - - - - - - $x-y=b$;

Adding the two equations - - $2x=a+b$,

$$x=\frac{a}{2}+\frac{b}{2};$$

Subtracting the 2d from the 1st equation, $2y=a-b$;

$$y=\frac{a}{2}-\frac{b}{2}.$$

Hence, since a and b may represent any numbers whatever, the sum and difference of two quantities being given,

1. To find the greater, *Add the half difference to the half sum.*
2. To find the less, *Subtract the half difference from the half sum.*

EXAMPLES.

1. The sum of two numbers is 24, the difference 6: what are the two numbers?

Let $x=$ the greater, and $y=$ the less:

Then $x=\frac{a}{2}+\frac{b}{2}=\frac{24}{2}+\frac{6}{2}=12+3=15$, the greater,

And $y=\frac{a}{2}-\frac{b}{2}=\frac{24}{2}-\frac{6}{2}=12-3=9$, the less.

2. The sum of two numbers is 56, their difference 12: what are the numbers?
3. It is required to divide \$860 between two men, so that the first may have \$250 more than the second.
4. Two merchants invest in trade \$10,000; the sum invested by the first exceeds that invested by the second by \$1225: what was the sum invested by each?

Second General Problem.

190. The sum of three numbers is a; the excess of the mean above the least, b; and the excess of the greatest above the mean, c. Required the three numbers.

Let $x=$ the least, $y=$ the mean, and $z=$ the greatest:

Then, by the conditions - $x+y+z=a$;

$$y-x=b;$$

$$z-y=c;$$

Reducing these three equations, $x=\frac{a-(2b+c)}{3}$;

$$y=\frac{a+b-c}{3};$$

$$z=\frac{a+b+2c}{3}.$$

Hence, since a, b, and c may represent any values whatever, having given the sum of three numbers, the excess of the mean above the least, and the excess of the greater above the mean:

1. To find the least, *From their sum subtract the sum of twice the mean above the least, and the excess of the greater above the mean, and divide the remainder by* 3.

2. To find the mean, *To their sum add the excess of the mean above the least, and from the result take the excess of the greatest above the mean, and divide the remainder by* 3.

3. To find the greatest, *Add together the sum of the three numbers, the excess of the mean above the least, and twice the excess of the greatest above the mean, and divide the sum by* 3.

EXAMPLES.

1. The sum of three numbers is 440; the excess of the mean above the least is 40; the excess of the greatest above the mean is 60: what are the numbers?

Let x, y, and z represent the numbers.

Then $x=\frac{a-(2b+c)}{3}=\frac{440-(2\times 40+60)}{3}=100$, the least number;

$y=\frac{a+b-c}{3}=\frac{440+40-60}{3}=140$, the mean number;

$z=\frac{a+b+2c}{3}=\frac{440+40+2\times 60}{3}=200$, the greatest number.

2. It is required to divide a prize of $973 among 3 men, so that the second shall have $69 more than the first, and the third $43 more than the second.

3. The sum of three numbers is 15,730; the second exceeds the third by 2320, and the first exceeds the second by 3575: what are the three numbers?

Third General Problem.

191. The sum of 4 numbers is a; the second exceeds the first by b; the third exceeds the second by c; the fourth exceeds the third by d. Required the four numbers.

EXAMPLE.

Find each of the above numbers, on the supposition that $a=3753$, $b=159$, c 275, and $d=389$.

Fourth General Problem.

192. The sum of 2 numbers is a, and if 3 times the first be divided by 2 times the second, the quotient will be b. Required the numbers.

EXAMPLE.

If $a=420$ and $b=8$, what are the numbers?

Fifth General Problem.

193. The sum of two numbers is a, and if the first be divided by 5 and the second by 2, the sum of the quotient will be b. Required the numbers.

EXAMPLE.

If $a=120$ and $b=42$, what are the numbers?

Sixth General Problem.

194. Three men share a certain sum in the following manner, viz.: the sum of A's and B's shares is a; that of A's and C's, b; that of B's and C's, c. What is the sum divided, and the share of each?

EXAMPLES.

1. If $a=\$123$, $b=\$110$, and $c=\$83$, what will be the sum, and the share of each?

Seventh General Problem.

195. A person engaged a workman to labour n days; for each day that he laboured he was to receive a cents, and for each day he was idle he was to pay b cents: at the time of settlement he received c cents. How many days did he labour, and how many was he idle?

Let $x=$ number of days he laboured,

$y=$ number he was idle;

Then, by the conditions,

$$\left.\begin{array}{r} x+y=n \\ ax-by=c \end{array}\right\} \begin{array}{l} \text{or } x=\dfrac{bn+c}{a+b}, \\ \text{and } y=\dfrac{an-c}{a+b}. \end{array}$$

Note.—If the labourer had paid c cents instead of receiving it, the general equations would become,

$$\left.\begin{array}{r} x+y=n \\ by-ax=c \end{array}\right\} \begin{array}{l} \text{or } x=\dfrac{bn-c}{a+b}, \\ \text{and } y=\dfrac{an+c}{a+b}. \end{array}$$

EXAMPLES.

1. If $n=48$, $a=24$, $b=12$, and $c=504$, how many days did he work, and how many was he idle?
2. A labourer was hired for 75 days: for each day that he wrought he was to receive $3, but for each day that he was idle he was to forfeit $7. At the time of settlement he received $125: how many days did he labour, and how many was he idle?
3. A man agreed to carry 20 earthen vessels to a certain place on this condition, viz., that for every one delivered safe he should receive 11 cents, and for every one he broke he should forfeit 13 cents: he received 124 cents. How many did he break?
4. A fisherman, to encourage his son, promises him 9 cents for each throw of the net in which he should take any fish; but the son is to forfeit 5 cents for each unsuccessful throw. After 37 throws the son receives from the father 235 cents. What was the number of successful and unsuccessful throws of the net?

DEMONSTRATION OF GENERAL PROPOSITIONS OR THEOREMS.

196. It was remarked in the introductory section of this work, that algebraic symbols might be applied to the demonstration of general truths or principles. We will now exhibit a few of these applications.

First Theorem.

197. The greater of any two numbers is equal to half their sum added to half their difference, and the less is equal to half their sum *minus* half their difference.

Let a and b represent any two numbers, of which a is the greater and b the less; let their sum be represented by s, and their difference by d:

Then - - - - - - $a+b=s$,

And - - - - - - $a-b=d$;

Adding the equations - - - $2a \quad =s+d$;

Dividing - - - - - $a \quad =\frac{s}{2}+\frac{d}{2}$.

Subtracting the 2d from the 1st equation, $2b=s-d$;

Dividing - - - - - - $b=\frac{s}{2}-\frac{d}{2}$.

Second Theorem.

198. The product of the sum and difference of two numbers is equal to the difference of their squares.

Let a, b, s, and d sustain the same relations as in the preceding theorem:

Then - - - - - $s=a+b$,

And - - - - - $d=a-b$.

Multiplying the two equations, $d.s=(a+b)(a-b)=a^2-b^2$.

Corol. 1.—Dividing the above equation by d, we have

$$s=\frac{a^2-b^2}{d}.$$

Hence, if the difference of the squares of two numbers be divided by the difference of the numbers, the quotient will be their sum.

Corol. 2.—Dividing the same equation by s, we shall have

$$d=\frac{a^2-b^2}{s}:$$

Hence, if the difference of the squares of two numbers be divided by the sum of the numbers, the quotient will be their difference.

Third Theorem.

199. Four times the product of any two numbers is equal to the squares of their sum, diminished by the square of their difference.

Let a, b, s, and d sustain the same relations as in the preceding theorem:

Then - - - - - - $a+b=s$,
$a-b=d$.

Adding the two equations - - $2a=s+d$;
Subtracting the 2d from the 1st - $2b=s-d$;
Multiplying the 3d and 4th - - $4ab=s^2-d^2$.

Fourth Theorem.

200. The sum of the squares of any two numbers is equal to the square of their difference *plus* twice their product.

Let a, b, and d sustain the same relations as before, and let p represent the product of the two numbers:

Then - - - - - - $a-b=d$,
And - - - - - - $ab=p$;
Squaring the members of the 1st equation - - - $a^2-2ab+b^2=d^2$;
Multiplying the 2d equation by 2, $2ab \quad =2p$;
Adding the two equations together - - - - $a^2+b^2=d^2+2p$.

Fifth Theorem.

201. The square of a polynomial expressing the sum of two numbers, is equal to the square of the first term + twice the product of the two terms + the square of the last term.

Let s represent the sum, and $a+b$ the polynomial:

Then - - - - - $s=a+b$;
Squaring the equation - - $s^2=a^2+2ab+b^2$.

Sixth Theorem.

202. The square of a polynomial expressing the differ-

ence between two numbers, is equal to the square of the first term — twice the product of the two terms + the square of the last term.

Let d represent the difference, and $a-b$ the polynomial:

Then - - - - - - $d=a-b$;

Squaring the equation - - $d^2=a^2-2ab+b^2$

Seventh Theorem.

203. The difference of any two equal powers of different numbers, is always divisible by the difference of the numbers.

Let a and b represent any two numbers, a being greater than b:

Then - - - - - $\frac{a^2-b^2}{a-b}=a+b$,

And - - - - - $\frac{a^3-b^3}{a-b}=a^2+ab+b^2$,

And - - - - - $\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3$.

This process may evidently be continued indefinitely; hence we have

$$\frac{a^m-b^m}{a-b}=a^{m-1}+a^{m-2}\times b+a^{m-3}\times b^2+\ \ldots\ a^2b^{m-3}+ab^{m-2}+b^{m-1}.$$

Corol. If $b=1$ in the above formula, the formula will become

$$\frac{a^m-1}{a-1}=a^{m-1}+a^{m-2}+a^{m-3}+\ \ldots\ +a^3+a^2+a+1.$$

Eighth Theorem.

204. The difference of two equal powers of different numbers, is divisible by the sum of the numbers, when the exponent of the power is an even number.

Let a and b sustain the same relations as before:

Then - - - $\frac{a^2-b^2}{a+b}=a-b$,

And - - - $\dfrac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3$,

And - - - $\dfrac{a^6-b^6}{a+b}=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$.

Hence we also conclude (letting m represent any *even* number),

$$\frac{a^m-b^m}{a+b}=a^{m-1}+a^{m-2}\times b+a^{m-3}\times b^2+\ldots+a^2b^{m-3}+ab^{m-2}+b^{m-1}.$$

COROL. If $b=1$, the above formula will become

$$\frac{a^m-1}{a+1}=a^{m-1}-a^{m-2}+a^{m-3}-\ldots\ldots+a^3-a^2+a-1.$$

Ninth Theorem.

205. The sum of any two equal powers of different numbers, is divisible by the sum of the numbers, when the exponent of the power is an odd number.

Let a and b sustain the same relations as before:

Then - - $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2$,

And - - $\dfrac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4$,

And - - $\dfrac{a^7+b^7}{a+b}=a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6$.

Hence we also conclude (letting m represent any *odd* number),

$$\frac{a^m+b^m}{a+b}=a^{m-1}-a^{m-2}\times b+a^{m-3}\times b^2-\ldots\ldots-a^3b^{m-4}+a^2b^{m-3}-ab^{m-2}+b^{m-1}.$$

COROL. If $b=1$, the above formula will become

$$\frac{a^m+1}{a+1}=a^{m-1}-a^{m-2}+a^{m-3}-\ldots\ldots-a^3+a^2-a+1.$$

Tenth Theorem.

206. If a given number be divided into two parts, and those parts multiplied together, the product will be the greatest possible when the parts are equal.

Let $n=$ the given number, $a=$ the greater part, $b=$ the less part, $d=$ the difference between the parts, and $p=$ the product of the two parts:

Then - - - - - - - $ab=p$,

And (Art. 189) - - - - - $a=\frac{n}{2}+\frac{d}{2}$,

And (Art. 189) - - - - - $b=\frac{n}{2}-\frac{d}{2}$.

Multiplying the last two equations together, $ab=\frac{n^2}{4}-\frac{d^2}{4}$.

Hence - - - - - - - $p=\frac{n^2}{4}-\frac{d^2}{4}$.

Now it is evident that p will increase as d diminishes; hence it will be the greatest possible when

$$d=0, \text{ or } p=\frac{n^2}{4}.$$

DEMONSTRATIONS RELATING TO CERTAIN PROPERTIES OF NUMBERS.

207. We will now apply the principles of Algebra heretofore discussed to the demonstration of some singular properties of numbers.

Let it first be premised that the local value of the digits increases in a tenfold ratio from right to left, and that any number is equal to the number of units expressed by the digit in the unit's place, + the number of units expressed by the digit in the ten's place, + the number of units expressed by the digit in the hundred's place, +, &c.

Thus, $3756=6+50+700+3000$, and $12,899=9+90+800+2000+10,000$.

First Proposition.

208. If from any number the sum of its digits be subtracted, the remainder will be divisible by 9.

Let a, b, c, d, &c., represent the digits of any number, a being the digit in the unit's place, b the digit in the ten's place, &c.; also let $N=$ the number, $n=$ the number of the digits, $S=$ the sum of the digits, and $r=10$:

Then $N=a+br \quad +cr^2 \quad +dr^3 \quad +\ldots+xr^{n-1}$,
And $S=a+b \quad +c \quad +d \quad +\ldots+x$.

$N-S= \quad br-b \quad +cr^2-c \quad +dr^3-d \quad +\ldots+xr^{n-1}-x$,
Or $N-S= \quad b(r-1)+c(r^2-1)+d(r^3-1)+\ldots+x(r^{n-1}-1)$, by subtraction.

But (Th. 7, Art. 203) $r-1$, r^2-1, $r^3-1+\ \ldots+r^{n-1}-1$, are divisible by $r-1$, which is equal to 9; hence, N—S is also divisible by 9.

EXAMPLE. 327,856 — (3+2+7+8+5+6)=327,825, and 327,825÷9=36,425.

Second Proposition.

209. If the sum of the digits of any number be divisible by 9, the number itself is divisible by 9.

Let N= any number, and S= the sum of its digits:

Then, since S is supposed to be divisible by 9, let $S=9m$

Since N—S is divisible by 9, let $N-S=9p$:

Then - - $N-S=N-9m=9p$;

Transposing - - $N=9p+9m$;

Resolving into factors, $N=9(p+m)$, which is divisible by 9; consequently, N also is divisible by 9.

EXAMPLE. 51,489÷9=5721, and (5+1+4+8+9)÷9=27÷9=3.

Third Proposition.

210. If the sum of the digits of any number be divisible by 3, the number itself is divisible by 3.

Let N represent any number, and S the sum of its digits, as before; and let $S=3m$, and $N-S=3p$:

Then - - $N-3m=3p$;

Transposing, $N=3p+3m$, which is evidently divisible by 3.

EXAMPLE. 785,142÷3=261,714, and (7+8+5+1+4+2)÷3=27÷3=9.

Fourth Proposition.

211. If from any number the sum of the digits standing in the *odd* places be subtracted, and to it the sum of the

digits standing in the *even* places be added, the result will be divisible by 11.

Let the number be	$a+br$	$+cr^2$	$+dr^3$	&c.:
Add - -	$-a+b$	$-c$	$+d,$	&c.
The result is - .	$br+b$	$+cr^2-c$	$+dr^3+d,$	&c.,
Or - - -	$b(r+1)$	$+c(r^2-1)$	$+d(r^3+1),$	&c.

But (Ths. 8 and 9) $r+1$, r^2-1, r^3+1, &c., are divisible by $r+1$; hence, $b(r+1)+c(r^2-1)+d(r^3+1)$, &c., is divisible by $r+1$, or 11.

EXAMPLE. $(57{,}937-(7+9+5)+(7+3))\div 11=(57{,}937-21+10)\div 11=57{,}926\div 11=5266$.

Fifth Proposition.

212. If the sum of the digits standing in the even places in any number be equal to the sum of the digits standing in the odd places, the number is divisible by 11.

Let $N=$ the number, $S=$ the sum of the even digits, and $s=$ the sum of the odd digits:

Then, by Prop. 4, $N+S-s$ is divisible by 11. But $S-s=0$; therefore N is divisible by 11.

EXAMPLE. $(137{,}456+13-13)\div 11=137{,}456\div 11=12{,}496$.

Sixth Proposition.

213. Every prime number which will exactly divide the product of two factors, one of which is also a prime number, will divide the other.

Let $A\times B$ represent the product of two numbers, which is divisible by P; A being greater than P, and prime with it, or not divisible by it.

Then let us endeavour to find the greatest common divisor of A and P, representing the successive quotients by Q, Q′, &c., and the successive remainders by R, R′, &c.:

$$\begin{array}{l} P)\ A\ (Q \\ \quad QP \\ \quad \overline{\quad R)\ P\ (Q'} \\ \qquad\quad Q'R \\ \qquad\quad \overline{\quad R')\ R\ (Q''} \\ \qquad\qquad\quad Q''R \\ \qquad\qquad\quad \overline{\quad R'', \&c.} \end{array}$$

Making each dividend equal to the product of its divisor and quotient, we have

1. - - - $A=PQ+R$;
2. - - - $P=RQ'+R'$;
3. - - - $R=R'Q''+R''$, &c.

Multiplying the first equation (1.) by B,

$$AB=PQB+RB;$$

Dividing by P - - - - $\frac{AB}{P}=BQ+\frac{BR}{P}$.

By hypothesis, $\frac{AB}{P}$ produces a whole number; and since B and Q are whole numbers, the product BQ is a whole number; hence $\frac{BR}{P}$ is also a whole number.

Multiplying the second equation (2.) by B, and dividing by P, we have

$$B=\frac{BRQ}{P}+\frac{BR'}{P}.$$

We have already shown that $\frac{BR}{P}$ produces a whole number; hence $\frac{BRQ}{P}$ will also produce a whole number. This being the case, $\frac{BR'}{P}$ must also be a whole number. If this operation is continued till the number which multiplies B becomes 1, we shall still have $\frac{B\times 1}{P}$, equal to an entire number; therefore B is divisible by P.

Hence, if a number will exactly divide the product of two numbers, and is prime with one of them, it will divide the other.

REDUCTION OF FORMULAS.

214. The processes of generalization which we have noticed will suggest some methods of demonstrating formulas or general rules.

FORMULAS RELATING TO SIMPLE INTEREST.

215. It is required to deduce general formulas or rules for the computations relating to simple interest.

216. To present the subject in a general point of view, let us consider the *five* things that enter into the calculation, viz., *Principal*, *Interest*, *Rate*, *Time*, and *Amount*.

Let $p=$ principal, $i=$ interest, $r=$ rate per cent., $t=$ time, and $a=$ amount.

Taking the dollar as unity, r will be a fraction, whose denominator is 100. If the given sum be put at interest for one year, then $t=1$; if for a longer period, $t>1$; if for a shorter period, $t<1$. The interest of \$1 will evidently be proportional to the rate and time jointly, or the interest of $\$1=r\times t$. The rate and time being the same, any given principal will be to any other principal as the interest of the former is to the interest of the latter.

Hence - - $\$1 : \$p :: r\times t : i$, or $i=p\times r\times t$.

By making the necessary transformations, we obtain the four following formulas:

1. - - $i=prt.$

2 - - $p=\frac{i}{rt}.$

3. - - $t=\frac{i}{pr}.$

4. - - $r=\frac{i}{pt}.$

These formulas may be enunciated in the form of general rules.

EXAMPLES.

1. What is the interest of $3875,20 for 3 years, at 7 per cent. per annum?
2. What is the interest of $325 for 3 months, at 6 per cent. per annum?
3. The interest of a certain sum is $92,75, the time 3 years and 6 months, and the rate 5 per cent.: what is the sum?
4. The interest received for $4070, at 9 per cent., was $91,575: how long was it at interest?
5. The sum $3671 was put at interest for 6 months, and at the end of the time the interest paid was $183,55: what was the rate per cent.?

217. Since the amount is equal to the principal + the interest, or $a=p+prt$, by making the necessary transformations we shall have

1. - - $a=p+prt.$

2. - - $p=\dfrac{a}{1+rt}.$

3. - - $r=\dfrac{a-p}{pt}.$

4. - - $t=\dfrac{a-p}{pr}.$

These formulas may also be stated in the form of general rules.

EXAMPLES.

1. If $p=\$895$, $r=7$, and $t=4\frac{1}{2}$, what is the value of a?
2. If $a=\$7589{,}50$, $r=8$, and $t=5\frac{1}{8}$, what is the value of p?
3. If $a=\$820{,}20$, $p=\$600$, and $t=\frac{1}{3}$, what is the numerical value of r?
4. If $a=\$525{,}86$, $p=\$35{,}80$, $r=4$, what is the numerical value of t?

FORMULAS RELATING TO COMPOUND INTEREST.

218. In Compound Interest the interest is supposed to remain in the hands of the borrower, and, being added to the

principal at the end of each year, forms a part of the principal for the succeeding year.

219. Let p and r sustain the same relations as before, and let $a=$ the amount for the first year, and, consequently, the principal for the second year, $a'=$ the amount for the second year, $a''=$ the amount for the third year, $a'''=$ the amount for the fourth year, &c. Then, as $1 will be to any given sum as the amount of $1 for one year is to the amount of that given sum for the same time, we shall have

$$1 : p :: 1+r : a, \text{ or } a = p(1+r),$$

And - $$1 : p(1+r) :: 1+r : a', \text{ or } a' = p(1+r)^2,$$

And - $$1 : p(1+r)^2 :: 1+r : a'', \text{ or } a'' = p(1+r)^3,$$

And - $$1 : p(1+r)^3 :: 1+r : a''', \text{ or } a''' = p(1+r)^4,$$

&c.

Let A represent the amount for n years, and we shall have

$$1 : p(1+r)^{n-1} :: 1+r : A, \text{ or } A = p(1+r)^n.$$

Hence, by making the necessary transformations, we obtain the following formulas:*

1. - - $A = p(1+r)^n$.

2. - - $p = \frac{A}{(1+r)^n}$.

3. - - $r = \left(\frac{A}{p}\right)^{\frac{1}{n}} - 1$.

These formulas may be stated in the form of general rules.

EXAMPLES.

1. If $p=\$3250$, $n=8$, and $r=5$, what is the numerical value of A?
2. If $A=\$30,200$, $n=20$, and $r=6$, what is the numerical value of p?
3. If $A=\$1479,15$, $p=\$1000$, and $n=6$, what is the numerical value of r?

* The fourth formula is omitted, since it would involve Logarithms, which are treated of in a subsequent section.

FORMULAS RELATING TO FELLOWSHIP.

220. Two men engage in trade together, and furnish money in proportion to the numbers m and n; they gain a sum represented by g; it is required to deduce formulas for the division of the gain, so that each man shall receive his equitable share.

Let $x=$ the share of the first, and $y=$ the share of the second:

Then - - - - - - $x+y=g$,

And - - - $x : y : : m : n$, or $my=nx$.

Reducing these equations, we obtain

$$x=\frac{mg}{m+n}.$$

$$y=\frac{ng}{m+n}.$$

Hence, to find each man's share of the gain, *Multiply his stock by the whole gain, and divide the product by the whole stock invested.*

EXAMPLE. Two merchants, A and B, gained by trading in company \$20,480. A's stock was \$15,000, B's \$18,000: what was each man's share of the gain?

221. Again: suppose three persons engage in trade, and furnish money in proportion to the numbers m, n, and p; they gain a sum represented by g; it is required to deduce formulas for the division of the gain as before.

Let x, y, and z represent the respective shares of the three persons:

Then we shall have

$$x+y+z=g,$$

$$x : y : : m : n, \text{ or } my=nx,$$

$$x : z : : m : p, \text{ or } mz=px.$$

Reducing the above equations, we obtain

$$x=\frac{mg}{m+n+p}.$$

$$y = \frac{ng}{m+n+p}.$$

$$z = \frac{pg}{m+n+p}.$$

Hence, to find each man's share of the gain, *Multiply his stock by the whole gain, and divide the product by the whole amount of stock invested.*

EXAMPLE. Three merchants, A, B, and C, gained by trading in company \$1100; A's stock was \$1500, B's \$1200, and C's \$850: what was each man's share of the gain?

222. As the above formulas contain four things, viz., *whole stock*, *whole gain*, the particular stock whose share of the gain is to be found, and that share of the gain, it is evident that any one of these may be found if the other three be given.

Letting S= whole stock, S′= stock whose share of the gain is to be found, g= the whole gain, and g'= share of the gain to be found, and substituting these letters in the preceding formulas, they become

1. - - $g' = \frac{S'g}{S}$.

2. - - $g = \frac{Sg'}{S'}$.

3. - - $S' = \frac{Sg'}{g}$.

4. - - $S = \frac{S'g}{g'}$.

EXAMPLES.

1. Two men, A and B, traded in company, with a joint capital of \$8000; they gained \$1250. A's stock was \$3250: what was his share of the gain?

2. Three men, A, B, and C, jointly invest in trade \$2725; they gain \$560, of which A receives as his share \$120, B receives as his share \$160: what was the stock invested by each?

3. Three men, A, B, and C, gain by trading \$6000. A's

stock was \$8000, and he took as his share of the gain \$2800: what was the whole stock invested?

4. Two men, A and B, invest in trade \$3000. A's gain was \$250, and his stock \$2600: what was the whole gain?

223. Let us now consider the cases in which the stock of the partners in trade has been invested for different lengths of time.

224. Two men engage in trade together, and furnish money in proportion to the numbers m and n, for the times t and t'; they gain a sum represented by g: it is required to deduce formulas for the equitable division of the gain.

Let x and y represent the respective shares of the gain: Then we shall have

$$x+y=g,$$
$$nt'x=mty.$$

Reducing these equations,

$$x=\frac{mtg}{mt+nt'}.$$

$$y=\frac{nt'g}{mt+nt'}.$$

These results may be enunciated in the form of a general rule.

225. Again: suppose three persons invest in trade money in proportion to the numbers m, n, and p, for times t, t', and t''; the sum gained is represented by g: it is required to deduce formulas for the equitable division of the gain.

Let x, y, and z represent their respective shares of the gain: then

$$x+y+z=g,$$
$$nt'x=mty,$$
$$pt''x=mtz.$$

Reducing these equations,

$$x=\frac{mtg}{mt+nt'+pt''}.$$

$$y=\frac{nt'g}{mt+nt'+pt''}.$$

$$z=\frac{pt''g}{mt+nt'+pt''}.$$

These results may also be enunciated in the form of a general rule.

EXAMPLE. A, B, and C enter into partnership. A invests \$1200 for 3 years, B \$2000 for 2 years and 9 months, C \$950 for 4 months. They gain \$2400: what is each man's share of the gain?

DISCUSSION OF EQUATIONS OF THE FIRST DEGREE.

226. When a question has been solved in a general manner, that is, by representing the known quantities by letters, it may be proposed to determine what values the unknown quantities will take when particular suppositions are made upon the known quantities. This is called the discussion of that equation.

227. The discussion of the following problem presents nearly all the circumstances that can ever occur in equations of the first degree.

PROBLEM OF THE COURIERS.

A courier sent out from a certain place travels in a right line with a velocity expressed by n. After the first courier had travelled a distance, a second was despatched after him, travelling with a velocity expressed by m. At what distance from the starting point will they be together?

In order to render the conditions of the question more evident, let ED

represent the line upon which the couriers travel, A the starting point, B the point at which the first courier is when the second starts, and C the point at which the second will overtake the first:

Let x=AC, and y=BC:

Then - - - - - $x-y=a$,

And - - - - - $\frac{x}{m}=\frac{y}{n}.$

Reducing these equations, we have

$$x=\frac{am}{m-n}, \text{ and } y=\frac{an}{m-n}.$$

DISCUSSION.

I. Let $m>n$.

228. In this case the values of x and y will be positive, and the solution of the problem will exactly accord with the enunciation; for if the second courier travels faster than the first, they will evidently meet somewhere in the direction AD, and to the right of B.

II. Let $m<n$.

229. In this case the values of x and y will be negative. In order to interpret this result, we observe that, the courier from B travelling faster than the courier from A, the interval between them must increase continually. It is absurd, therefore, to require that they should meet in the direction AD. The negative values of x and y, then, indicate an absurdity in the conditions of the question. To remove this absurdity, we have only to suppose that the two couriers start at the same time from B and A, and travel in the direction BE′, in which case the equations will become

$$y-x=a,$$

And - - - - - - $\frac{x}{m}=\frac{y}{n}.$

Or - - - $x=\frac{am}{n-m}$, and $y=\frac{an}{n-m}$,

which give the values of x and y positive, and indicate that the couriers will come together at C′ instead of C.

III. Let $m=n$.

230. In this case the values of x and y become

$$x=\frac{am}{m-n}=\frac{am}{0},$$

$$y=\frac{an}{m-n}=\frac{an}{0}.$$

In order to interpret this result, let us return to the question. If the couriers travel with equal velocities, it is evident that the interval between them must always continue the same, however far they may travel in either direction. Indeed, on the hypothesis $m=n$, the conditions of the question produce

$$x-y=a,$$

And - - - - - $x-y=0,$

equations which are incompatible with each other. It is therefore absurd to suppose that the couriers will come together on this supposition.

IV. Let $m=n$, and $a=0$.

231. In this case,

$$x=\frac{am}{m-n}=\frac{0\times m}{m-n}=\frac{0}{0}.$$

$$y=\frac{an}{m-n}=\frac{0\times m}{m-n}=\frac{0}{0}.$$

In order to obtain a correct interpretation of this result, it is only necessary to observe that, if the couriers set out each from the same point at the same time, and travel equally fast, there is no particular point at which one can be said to overtake the other, since they will be together, however far, and in whatever direction they may travel. Indeed, on this supposition, the conditions of the problem produce

$$x-y=0,$$
$$x-y=0,$$

two dependant or identical equations. The problem is therefore indeterminate, since we have, in fact, but one equation with two unknown quantities.

THEORY OF NEGATIVE QUANTITIES.

232. It has already been shown,

1. That adding a negative quantity is the same as subtracting an equal positive quantity.
2. That subtracting a negative quantity is the same as adding an equal positive.

3. If a negative quantity be multiplied or divided by a positive, the result will be negative.
4. If a positive quantity be multiplied or divided by a negative, the result will be negative.
5. If a negative quantity be multiplied or divided by a negative, the result will be positive.

233. We will now proceed to show that, if the conditions of the problem are such as to render the unknown quantity essentially negative, it will appear in the result with the minus sign, although it may have been regarded as positive in the statement of the problem.

1. The length of a certain field is a, and its breadth b: how much must be added to its length that its contents may be c?

Let $x=$ the quantity to be added to the length:

Then $a+x=$ whole length.

Since the area of a field is found by multiplying its length by its breadth, we have

$$ab+bx=c.$$

Reducing - - - $x=\frac{c}{b}-a.$

Now, letting $a=8$, $b=5$, and $c=60$, the equation becomes

$$x=\frac{60}{5}-8=12-8=4.$$

This value of x satisfies the conditions of the problem in the precise sense in which it was stated.

Again: letting $a=8$, $b=5$, and $c=30$,

Then - - - - $x=\frac{30}{5}-8=6-8=-2.$

In order to interpret this negative result, let us return to the original equation:

$$ab+bx=c.$$

Substituting - - - $8\times5+5\times-2=30;$

Resolving into factors - - $5(8-2)=30.$

Hence we perceive that, though addition was required by the enunciation, yet it was incompatible with the conditions of the question; and the algebraic result, true to the conditions of the question, detects the error in the enunciation, and shows that x is to be subtracted from instead of being added to the length of the side. Thus,

$$ab-bx=c,$$

Or - - - - - - $x=a-\frac{c}{b}.$

By substitution - - - $x=8-\frac{30}{5}=8-6=2.$

This result answers to the question modified in this manner:

The length of a certain field is a, and its breadth b: how much must be subtracted from its length that its contents may be c?

234. Discuss in like manner the following questions:

2. A father is a years old, and his son b: in how many years will the son be one fourth as old as the father?
3. A man when he was married was a years old, his wife b: how many years before his marriage was he twice as old as she?

EXPLANATION OF SYMBOLS.

INFINITY.

235. A mathematical quantity is said to be infinite when it is supposed to be increased beyond any determinate limits.

The symbols usually adopted by mathematicians to express such quantities are $\frac{A}{0}$ and ∞, A being used to represent any finite quantity.

In order to explain these symbols, let us resume the equations for x and y in the problem of the couriers:

$$x=\frac{am}{m-n}, \text{ and } y=\frac{an}{m-n};$$

Or, if $m=n$ - - $x=\frac{am}{0}$, and $y=\frac{an}{0}$.

236. In order to explain these expressions for the values

of x and y, we will show how these values are affected by assuming different values for m and n.

If $m=3$, and $n=2$,

$$x=\frac{3a}{3-2}=3a, \text{ and } y=\frac{2a}{3-2}=2a.$$

If $m=3$, and $n=2{,}9$,

$$x=\frac{3a}{,1}=30a, \text{ and } y=\frac{2{,}9a}{,1}=29a.$$

If $m=3$, and $n=2{,}99$,

$$x=\frac{3a}{,01}=300a, \text{ and } y=\frac{2{,}99a}{,01}=299a.$$

If $m=3$, and $n=2{,}999$,

$$x=\frac{3a}{,001}=3000a, \text{ and } y=\frac{2{,}999a}{,001}=2999a.$$

If $m=3$, and $n=2{,}9999$,

$$x=\frac{3a}{,0001}=30\,000a, \text{ and } y=\frac{2{,}9999a}{,0001}=29\,999a, \text{ \&c.}$$

Hence we infer, that if the difference between m and n becomes less than any assignable quantity, the values of x and y will be greater than any assignable quantity; and $\frac{A}{0}$ or ∞ is the proper symbol of *infinity*.

237. COROL. Since, by the conditions of the question, $x=y+a$, which will continue to be the case when the values of x and y become infinite, we infer that one mathematical infinity may be greater than another.

INFINITESIMAL.

239. A mathematical quantity is called an infinitesimal, or sometimes nothing, when it is supposed to be decreased below any determinate limits.

The symbols used to express such a quantity are $\frac{A}{\infty}$ or 0.

In order to explain these symbols, let us resume again the equations

$$x=\frac{am}{m-n}, \text{ and } y=\frac{an}{m-n}.$$

239. A course of reasoning similar to that adopted in the preceding case will show that the values of x and y decrease as the difference between m and n increases. Hence, when that difference becomes greater than any assignable quantity, the values of x and y become less than any assignable quantity.

That is, $x=\frac{am}{m-n}=\frac{am}{\infty}$, or the value of x may be expressed by the symbol $\frac{A}{\infty}$ or 0.

And $y=\frac{an}{m-n}=\frac{an}{\infty}$, or the value of y may be expressed by the symbol $\frac{A}{\infty}$ or 0.

240. COROL. Since $x=y+a$, we infer that one infinitesimal may be greater than another.

EXPLANATION OF THE SYMBOL OF INDETERMINATION.

$$\frac{0}{0}.$$

241. A quantity is said to be indeterminate when every possible value will satisfy the conditions of the question.

242. The symbol used to express indetermination is $\frac{0}{0}$.

We have already seen that the equations $x=\frac{am}{m-n}$, and $y=\frac{an}{m-n}$, on the hypothesis $m=n$ and $a=0$, reduce to $x=\frac{0}{0}$, and $y=\frac{0}{0}$; and also that all possible values of x and y will satisfy the conditions of these two equations.

243. We will, however, add another illustration to this case.

Take the expression $\frac{1-x}{1-x}$: if we perform the division, the quotient will be 1; and if we make $x=1$, there will result

$$1. \quad - \quad - \quad - \quad \frac{1-x}{1-x}=1=\frac{0}{0}.$$

2. - - - $\frac{1-x^2}{1-x}=1+x=2=\frac{0}{0}.$

3. - - - $\frac{1-x^3}{1-x}=1+x+x^2=3=\frac{0}{0},$

&c., ad infin.

Hence every possible value will satisfy the conditions $\frac{0}{0}$.

244. It should, however, be observed, that this symbol does not always imply indetermination.

Thus, the expression $x=\frac{a^3-b^3}{a^2-b^2}$, if $a=b$, will become

$$x=\frac{a^3-b^3}{a^2-b^2}=\frac{a^3-a^3}{a^2-a^2}=\frac{0}{0}.$$

But, resolving the terms of the fraction into factors,

$$x=\frac{(a-b)(a^2+ab+b^2)}{(a-b)(a+b)}=\frac{a^2+ab+b^2}{a+b},$$

which, on the supposition $a=b$, becomes

$$x=\frac{a^2+a\times a+a^2}{a+a}=\frac{3a^2}{2a}=\frac{3a}{2}.$$

Hence we conclude that the symbol $\frac{0}{0}$, in Algebra, sometimes indicates the existence of a factor common to the two terms of the fraction, which, in consequence of a particular hypothesis, becomes 0, and reduces the fraction to the form $\frac{0}{0}$.*

INEQUATIONS OR INEQUALITIES.

245. The principles established respecting equations will in most cases also apply to inequations. As there are some exceptions, we will here illustrate the principal transformations which may be made upon inequations, and then apply those transformations to the determination of the *limits* of unknown quantities.

246. Two inequations are said to subsist in the *same sense* when the greater quantity stands at the right in both or at

* See Note C.

the left in both, and in a *contrary sense* when the greater quantity stands at the right in one and at the left in another.

247. 1. We may always add the same quantity to both members of an inequality, or subtract the same quantity from both members, and the inequality will continue in the same sense.

Thus, let - - $2<12$; adding 6 to both sides,
we have - $6+2<12+6$,
Or - - - $8<18$.
Again: let - $-2>-12$:
Then - - $6-2>6-12$, or $4>-6$.

Corol. A term may be transposed from one member of an inequation to the other.

248. 2. If we add the corresponding members of two or more inequations subsisting in the same sense, the inequation which results will exist in the same sense of those added.

Thus - - - $8>5$,
And - - - $10>2$.
Adding - - $10+8>2+5$, or $18>7$.

But if we subtract the corresponding members of one inequation from another subsisting in the same sense, the resulting inequation will not always exist in the same sense.

Thus, from - - - $4<7$
Subtract - - - - $2<3$.
There will remain - $4-2<7-3$, or $2<4$.
But if from - - - $9<10$
We subtract - - - $6<8$,
There will remain - - $9-6>10-8$, or $3>2$.

249. 3. If both members of an inequation be multiplied or divided by any positive whole number, the resulting inequation will exist in the same sense as the inequation multiplied.

Thus - - - - $6<10$;
Multiplying by 3 - - $18<30$.
Or, again - - - - $\frac{1}{3}<\frac{1}{2}$;
Multiplying by 6 - - $2<3$.

COROL. An inequation may be freed from fractions in the same manner as an equation.

250. 4. If we multiply or divide the two members of an inequation by a negative quantity, the resulting inequation will subsist in a contrary sense.

Thus - - - - - $6<10$;
Multiplying by -3 - - - $-18>-30$.
Or, again - - - - $\frac{1}{3}<\frac{1}{2}$;
Multiplying by -6 - - - $-2>-3$.

Hence it also follows, that if we change the sign of each term of an inequation, the inequation which results will exist in a sense contrary to the inequation proposed; for this transformation will be equivalent to multiplying the inequation by -1.

EXAMPLES.

1. Find the limit of the value of x in the inequation

$$7x-\frac{23}{3}>\frac{2x}{3}+5.$$

Clearing of fractions, $21x-23>2x+15$;
Transposing - - $21x-2x>15+23$;
Reducing - - $19x>38$;
Dividing by nineteen, $x>2$.

2. Find the limits of the value of x in the inequations

$$14x+\frac{5}{9}>\frac{4x}{5}+230,$$

$$x-\frac{25}{7}<10+\frac{4x}{5}.$$

Note.—To determine both the limits of x, it is necessary that we have two inequations existing in a contrary sense. These inequations are not combined together like equations, but reduced separately.

3. Find the limits of x in the inequations

$$\frac{x}{5}+\frac{x}{3}>\frac{7}{5}+\frac{2x}{3},$$

$$\frac{x}{7}-\frac{x}{14}<\frac{6}{5}-\frac{x}{10}.$$

4. The double of a number diminished by 5 is greater than 25, and triple the number diminished by 7 is less than double the number increased by 13. Required a number that will satisfy the conditions.

Let $x=$ the number: then, by the question, we have

$$2x-5>25,$$
$$3x-7<2x+13.$$

Resolving these inequalities, we have $x>15$ and $x<20$. Any number, therefore, either entire or fractional, comprised between 15 and 20, will satisfy the conditions.

5. A shepherd being asked the number of his sheep, replied that double their number diminished by 7 is greater than 29, and triple their number diminished by 5 is less than double their number increased by 16. Required a number that will satisfy the conditions.

Resolving the question, we have $x>18$ and $x<21$. Here all the numbers comprised between 18 and 21 will satisfy the inequalities; but since the nature of the question requires that the answer should be an entire number, the number of solutions is limited to two, viz., $x=19$, $x=20$.

6. A market-woman has a number of oranges, such that triple the number increased by 2 exceeds double the number increased by 61, and 5 times the number diminished by 70 is less than four times the number diminished by 9. Required a number that will satisfy the conditions.

7. The sum of two numbers is 32; and if the greater be divided by the less, the quotient will be less than 5, but greater than 2. What are the numbers?

8. A boy being asked how many apples he had in his basket, replied, that the sum of three times the number plus half the number diminished by 5, is greater than 16; and twice the number diminished by one third of the number plus 2, is less than 22. Required the numbers that will satisfy these conditions.

SECTION VI.

Involution and Powers.—Of Monomials.—Of Polynomials.—Binomial Theorem.—Evolution and Roots.—Square Root of Numbers.—Cube Root of Numbers.—General Method of obtaining any Root of Numbers.—Evolution of Monomials.—Of Polynomials.—Calculus of Radicals.

INVOLUTION AND POWERS.

251. *Involution* is the multiplying a number by itself till it has been used as a factor as many times as there are units in the exponent.

252. The product thus produced is called the *power* of that quantity; and the power is designated *first*, *second*, *third*, *fourth*, &c., accordingly as the number has been used once, twice, three times, four times, &c., as a factor.

253. To indicate the involution of a polynomial, or of a monomial composed of several factors, the numbers should be placed within a parenthesis, to the right of which the exponent should be written.

INVOLUTION OF MONOMIALS.

254. In order to obtain a general rule for the involution of monomials, let the following proposition be demonstrated, viz.: The power of the product of two or more factors is equal to the product of their powers.

Let $(ab)^2$ represent the second power of the product of two factors,

And a^2b^2 the product of the second power of the same factors:

Then $(ab)^2=a^2b^2$; for, by the definition of involution (Art. 251), $(ab)^2=ab\times ab=aa\times bb=a^2b^2$.

Again: $(ab)^m=a^mb^m$; for $(ab)^m=ab\times ab\times ab\times \ldots\ldots$ taken m times$=aaa\ldots m$ times$\times bbb\ldots m$ times$=a^mb^m$.

255. Now let it be required to involve $3ab^2$ to the fourth power:

$$(3ab^2)^4 = 3ab^2 \times 3ab^2 \times 3ab^2 \times 3ab^2 = 3 \times 3 \times 3 \times 3 \times aaaa \times b^2b^2b^2b^2$$
$$= 81a^4b^8 = 3^4 \times a^{1\times 4} \times b^{2\times 4}.$$

256. The same reasoning will evidently apply to every case of monomials; hence, for the involution of monomials we have the following general

RULE.

1. *Involve the coefficient to the required power.*

2. *Multiply the exponent of each letter by the exponent which denotes the power to which the monomial is to be involved.*

Note 1.—If the number to be involved is positive, all its powers will be positive (Art. 89). If it be negative, the *even* powers will be positive and the *odd* powers negative (Art. 90).

Note 2.—If the given number be fractional, involve both the numerator and denominator. This results from the principle that the product of fractions is equal to the product of their numerators divided by the product of their denominators (Art. 149). Thus, $\left(\frac{a}{c}\right)^2 = \frac{a}{c} \times \frac{a}{c} = \frac{a^2}{c^2}$.

Note 3.—The above rule is applicable to numbers having negative exponents, since the negative exponent expresses the reciprocal of a power (Art. 61). Thus, $(a^{-3})^2 = a^{-3\times 2} = a^{-6}$; for $a^{-3} = \frac{1}{a^3}$, and $\left(\frac{1}{a^3}\right)^2 = \frac{1}{a^6} = a^{-6}$.

Note 4.—The fourth power of a number is equal to the square of the second power; thus, $(a)^4 = a \times a \times a \times a = aa \times aa = (a^2)^2$. The sixth power is equal to the cube of the second power; thus, $a^6 = a \times a \times a \times a \times a \times a = aa \times aa \times aa = (a^2)^3$, &c.

EXAMPLES.

1. Required the second power of $8a^2b^3$. *Ans.* $64a^4b^6$.
2. Required the third power of $5x^2z$. *Ans.* $125x^6z^3$.
3. Required the third power of $6a^5y^2x$. *Ans.* $216a^{15}y^6x^3$.
4. Required the fourth power of $2a^2b^3c^4$. *Ans.* $16a^8b^{12}c^{16}$.

5. Required the fifth power of $2ab^3x^5$. *Ans.* $32a^5b^{15}x^{25}$.

6. Required the second power of $-6a^3b^8$. *Ans.* $36a^6b^{16}$.

7. Required the third power of $-3abc^2$.

Ans. $-27a^3b^3c^6$.

8. Required the sixth power of $\frac{2}{3}a^4b$. *Ans.* $\frac{64}{729}a^{24}b^6$.

9. Required the seventh power of $-2x^2y$.

Ans. $-128x^{14}y^7$.

10. Required the fourth power of $-4a^2b$.

Ans. $256a^8b^4$.

11. Required the fourth power of $-\frac{2a}{5b^2c^3}$.

Ans. $\frac{16a^4}{625b^8c^{12}}$.

12. Required the second power of $3a^{-2}$. *Ans.* $9a^{-4}$.

13. Required the second power of $\frac{18a^2}{10bc}$. *Ans.* $\frac{324a^4}{100b^2c^2}$.

14. Required the second power of $\frac{225}{82x}$. *Ans.* $\frac{50625}{6724x^2}$.

15. Required the third power of $6a^{-2}b^{-1}$.

Ans. $216a^{-6}b^{-3}$.

16. Required the fourth power of $8ab^{-3}$

Ans. $4096a^4b^{-12}$.

17. Required the fourth power of $10x^3z^m$.

Ans $10000x^{12}z^{4m}$.

18. Required the fifth power of $4a^6xy^2$.

Ans. $1024a^{30}x^5y^{10}$.

19. Required the fifth power of $-3abxy$.

Ans. $-243a^5b^5x^5y^5$.

20. Required the eighth power of $5a^2x$.

Ans. $390625a^{16}x^8$.

21. Required the fourth power of $\frac{8ab}{xyz}$. *Ans.* $\frac{4096a^4b^4}{x^4y^4z^4}$.

22. Required the second power of $-\frac{20x^2}{18yz}$.

Ans. $\frac{400x^4}{324y^2z^2}$.

23. Required the third power of $-\frac{3a^{-2}bc^2d^3}{4a^2c^3d^{-4}}$.

Ans. $-\frac{27a^{-6}b^3c^6d^9}{64a^6c^9d^{-12}}$.

24. Required the fifth power of $4a^2b^3c^4d^5$.

Ans. $1024a^{10}b^{15}c^{20}d^{25}$.

25. Required the seventh power of $-2axcd^2$.

Ans. $-128a^7x^7c^7d^{14}$.

26. Required the nth power of $\frac{4abx}{8cdy}$. *Ans.* $\frac{4^na^nb^nx^n}{8^nc^nd^ny^n}$.

27. Required the eighth power of $-4a^2b^5$.

Ans. $65536a^{16}b^{40}$.

28. Required the fourth power of $10x^3y^2z$.

Ans. $10000x^{12}y^8z^4$.

29. Required the sixth power of $-3a^2b^3c^4d^{-1}$.

Ans. $729a^{12}b^{18}c^{24}d^{-6}$.

30. Required the third power of $12a^{-2}b^{-3}c^{-5}d^{-4}$.

Ans. $1728a^{-6}b^{-9}c^{-15}d^{-12}$.

INVOLUTION OF POLYNOMIALS.

257. *Multiply the polynomial by itself till it has been used as a factor as many times as there are units in the exponent denoting the power to which it is to be raised; the final product will be the power required.*

EXAMPLES.

1. Required the first, second, third, fourth, and fifth powers of the binomial $a+b$.

$(a+b)^1=a+b$ - - - - - - 1st power.

$$\begin{array}{l} a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline \end{array}$$

$(a+b)^2=a^2+2ab+b^2$ - - - - - 2d power.

$$\begin{array}{l} a+b \\ \hline a^3+2a^2b+ab^2 \\ \quad +a^2b+2ab^2+b^3 \\ \hline \end{array}$$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$ - - - - - 3d power.

$$\begin{array}{l} a+b \\ \hline a^4+3a^3b+3a^2b^2+ab^3 \\ \quad a^3b+3a^2b^2+3ab^3+b^4 \\ \hline \end{array}$$

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$ - - 4th power.

$$\begin{array}{l} a+b \\ \hline a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\ \quad a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\ \hline \end{array}$$

$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$ - 5th power.

2. Required the first, second, third, fourth, and fifth powers of the binomial $a-b$.

$(a-b)^1=a-b$ - - - - - - 1st power.

$$\begin{array}{l} a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline \end{array}$$

$(a-b)^2=a^2-2ab+b^2$ - - - - - 2d power.

$$\begin{array}{l} a-b \\ \hline a^3-2a^2b+ab^2 \\ \quad -a^2b+2ab^2-b^3 \\ \hline \end{array}$$

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$ - - - - 3d power.

$$\begin{array}{l} a-b \\ \hline a^4-3a^3b+3a^2b^2-ab^3 \\ \quad -a^3b+3a^2b^2-3ab^3+b^4 \\ \hline \end{array}$$

$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$ - - 4th power.

$$\begin{array}{l} a-b \\ \hline a^5-4a^4b+6a^3b^2-4a^2b^3+ab^4 \\ \quad -a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5 \\ \hline \end{array}$$

$(a-b)^5=a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$ - 5th power.

3. Required the second power of $a+b$.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2. \quad \textit{Ans.} \end{array}$$

Note.—Since a and b may represent any numbers whatever, we infer the following general principle: The square of a binomial is the square of the first term, plus twice the product of the two terms, plus the square of the last term.

4. Required the second power of $a-b$.

$$\begin{array}{l} a-b \\ a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline a^2-2ab+b^2. \quad \textit{Ans.} \end{array}$$

Note.—Since a and b may represent any two numbers, a being greater than b, we infer the following general principle: The square of a residual is the square of the first term, minus twice the product of the two terms, plus the square of the last term.

5. Required the second power of $6a+3b$.
Ans. $36a^2+36ab+9b^2$.

6. Required the second power of $7a-2b$.
Ans. $49a^2-28ab+4b^2$.

7. Required the second power of $2ab+3c$.
Ans. $4a^2b^2+12abc+9c^2$.

8. Required the second power of $5abc-2acd$.
Ans. $25a^2b^2c^2-20a^2bc^2d+4a^2c^2d^2$.

9. Required the third power of $2a+3b$.
Ans. $8a^3+36a^2b+54ab^2+27b^3$.

10. Required the third power of $2a-5b$.
Ans. $8a^3-60a^2b+150ab^2-125b^3$.

11. Required the second power of $a+1$.
Ans. a^2+2a+1.

12. Required the second power of $2a-1$.
Ans. $4a^2-4a+1$.

13. Required the third power of $a+1$.
Ans. a^3+3a^2+3a+1.

14. Required the second power of $\frac{3a-1}{b+c}$.

Ans. $\frac{9a^2-6a+1}{b^2+2bc+c^2}$.

15. Required the third power of $a+b+c$. *Ans.* $a^3+3a^2b+3a^2c+3ab^2+3ac^2+6abc+3b^2c+3bc^2+b^3+c^3$.

16. Required the fourth power of $3a+2bc$.

Ans. $81a^4+216a^3bc+216a^2b^2c^2+96ab^3c^3+16b^4c^4$.

17. Required the fifth power of $6x-2b$. *Ans.* $7776x^5-12960x^4b+8640x^3b^2-2880x^2b^3+480xb^4-32b^5$.

18. Required the second power of $6a+2b-3c$.

Ans. $36a^2+24^4ab-36ac+4b^2-12bc+9c^2$.

19. Required the third power of $2a^2-3x$.

Ans. $8a^6-36a^4x+54a^2x^2-27x^3$.

20. Required the second power of $\frac{2a-3b^2}{8x+y}$.

Ans. $\frac{4a^2-12ab^2+9b^4}{64x^2+16xy+y^2}$.

21. Required the second power of $\frac{3xy^{-4}}{3b-4d}$.

Ans. $\frac{9x^2y^{-8}}{9b^2-24bd+16d^2}$.

22. Required the third power of $\frac{6a^{-1}b^2}{3a-1}$.

Ans. $\frac{216a^{-3}b^6}{27a^3-27a^2+9a-1}$.

23. Required the fourth power of $4a^3b-2c^2$.

Ans. $256a^{12}b^4-512a^9b^3c^2+384a^6b^2c^4-128a^3bc^6+16c^8$.

24. Required the fourth power of $\frac{-8a^2x^3y^{-2}}{a+1}$.

Ans. $\frac{4096a^8x^{12}y^{-8}}{a^4+4a^3+6a^2+4a+1}$.

258. *Remark.*—Any factor may be transferred from the numerator to the denominator, or from the denominator to the numerator of a fraction, by changing the sign of its exponent.

1. $\frac{ax^{-2}}{y}=\frac{a}{y}\times x^{-2}=\frac{a}{y}\times\frac{1}{x^2}$ (Art. 061) $=\frac{a}{x^2y}$.

2. $\frac{3a}{2y^3}=\frac{3a}{2}\times\frac{1}{y^3}=\frac{3a}{2}\times y^{-3}=\frac{3ay^{-3}}{2}$.

3. $\frac{a}{cx^{-2}}=\frac{a}{c}\times\frac{1}{x^{-2}}=\frac{a}{c}\times(1\div x^{-2})=\frac{a}{c}\times\left(1\div\frac{1}{x^2}\right)=\frac{a}{c}\times x^2=\frac{ax^2}{c}$.

4. $\frac{2a^{-3}}{5b^{-2}}=\frac{2}{5}\times a^{-3}\times\frac{1}{b^{-2}}=\frac{2}{5}\times\frac{1}{a^3}\times b^2=\frac{2b^2}{5a^3}$.

5. $\frac{ax^3}{c}=\frac{a}{c}\times x^3=\frac{a}{c}\div\frac{1}{x^3}=\frac{a}{c}\div x^{-3}=\frac{a}{c}\times\frac{1}{x^{-3}}=\frac{a}{cx^{-3}}$.

6. $\frac{ad^2}{xy^{-3}}=ad^2x^{-1}y^3$.

7. $\frac{ax^{-3}}{2by^{-4}}=\frac{ay^4}{2bx^3}$.

8. $\frac{a^{-3}b^{-2}}{3ab}=\frac{1}{3a^4b^3}$.

9. $\frac{3a}{7a^{-3}b^{-1}y^{-4}}=\frac{3a^4by^4}{7}$.

10. $\frac{5abc}{a^{-1}b^{-4}c^{-4}d^{-8}}=5a^2b^5c^5d^8$.

BINOMIAL THEOREM.

259. The method of involving polynomials by repeated multiplications is somewhat tedious, especially when high powers are required. This has led mathematicians to seek for other methods. The most simple method known is the one invented by Sir Isaac Newton, called the *Binomial Theorem.* Its use is very important and extensive in algebraic operations.

260. Let us take the binomial $a+b$, of which a is called the leading quantity, b the following quantity. Involving by the preceding rule, we shall find,

$(a+b)^2=a^2+2ab+b^2$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

$(a+b)^7=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

261. By observing the several results above, the *number of terms* will be found to be greater by 1 than the index denoting the power to which the binomial is to be expanded. Thus,

The square has three terms;
The cube has four terms;
The fourth power has five terms;
The fifth power has six terms;
The sixth power has seven terms;
The seventh power has eight terms;

And if the nth power of $a+b$ were required, the number of terms would be $n+1$. Hence, if the index of a binomial be a positive whole number, the number of terms will be one greater than the number of units contained in the index.

262. By attending to the *exponents* of the letters in the above powers, we shall find that they preserve an invariable order.

In the square, the exponents $\begin{cases} \text{of } a \text{ are } 2, 1, 0; \\ \text{of } b \text{ are } 0, 1, 2. \end{cases}$

In the cube, the exponents $\begin{cases} \text{of } a \text{ are } 3, 2, 1, 0; \\ \text{of } b \text{ are } 0, 1, 2, 3. \end{cases}$

In the fourth power, the exponents $\begin{cases} \text{of } a \text{ are } 4, 3, 2, 1, 0; \\ \text{of } b \text{ are } 0, 1, 2, 3, 4, \end{cases}$ &c.

Two laws are discoverable here:

1. The sums of the exponents of the two letters in each term are equal, and each sum is equal to the index denoting the power to which the binomial was to be raised.
2. The exponent of the leading quantity in the first term is the same as the index denoting the power to which the binomial was to be raised, and decreases regularly by 1; the exponent of the following quantity is 1 in the second term, and increases regularly by 1.

263. If it be required to involve $a+b$ to the power denoted by n, the exponents of a would be

$n, n-1, n-2, n-3, \ldots\ldots -3, 2, 1, 0;$

Of b, 0, 1, 2, 3, ... $n-3$, $n-2$, $n-1$, n.

Or, expressing the letters without the coefficients,

$$(a+b)^n=a^n+a^{n-1}b+a^{n-2}b^2+a^{n-3}b^3+\dots a^3b^{n-3}+a^2b^{n-2}+ab^{n-1}+b^n.$$

264. The same principle may be applied if the exponents be negative or fractional. Thus,

$$(a+b)^{-2}=a^{-2}+a^{-3}b+a^{-4}b^2+a^{-5}b^3+a^{-6}b^4+a^{-7}b^5+, \text{ \&c., ad infin.}$$

Also,

$$(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+a^{-\frac{1}{2}}b+a^{-\frac{3}{2}}b^2+a^{-\frac{5}{2}}b^3+a^{-\frac{7}{2}}b^4+a^{-\frac{9}{2}}b^5+, \text{ \&c., ad infin.}$$

It is evident that the above two series will never terminate, as a *negative* or *fractional* index can never become 0 by the successive subtractions of a unit; hence, when the index of the binomial is negative or fractional, the number of terms in the series will be infinite.

265. The law of the *coefficients* is more complicated, though not less remarkable.

In the preceding series of powers (Art. 259), the coefficients taken separately are,

In the first power - - -	1, 1.
In the second power - - -	1, 2, 1.
In the third power - -	1, 3, 3, 1.
In the fourth power - -	1, 4, 6, 4, 1.
In the fifth power - -	1, 5, 10, 10, 5, 1.
In the sixth power -	1, 6, 15, 20, 15, 6, 1.
In the seventh power -	1, 7, 21, 35, 35, 21, 7, 1.

By examining the above series of coefficients, it will be discovered,

1. That the coefficient of the first term is 1.
2. That the coefficient of the second term is the same as the index denoting the power to which the binomial is to be raised.
3. If the coefficient of any term be multiplied by the index of the leading quantity in the same term, and the

product divided by the index of the following quantity increased by 1, the quotient will be the coefficient of the following term.

266. By recurring to the above series of coefficients, it will be observed that they increase and then decrease in the same ratio, so that the coefficients of terms equally distant from the first and last terms are equal. It is sufficient, then, to find the coefficients of *half the terms;* these, repeated in the inverse order, will give the coefficients for the remaining terms.

267. By inspecting the coefficients farther, we shall discover that in any power of $a+b$, the sum of the coefficients is equal to the number 2 raised to that power. Thus, the sum of the coefficients

In the second power is - - - - $4=2^2$;
In the third power is - - - - $8=2^3$;
In the fourth power is - - - - $16=2^4$;
In the fifth power is - - - - $32=2^5$;
In the sixth power is - - - - $64=2^6$;
In the seventh power is - - - $128=2^7$,

268. If it be required to involve $a+b$ to the power expressed by n, first, taking the letters and exponents without their coefficients, we shall have

$$(a+b)^n=a^n+a^{n-1}b+a^{n-2}b^2+a^{n-3}b^3\ldots a^3b^{n-3}+a^2b^{n-2}+ab^{n-1}+b^n.$$

Let A, B, C, &c., represent the coefficients of the several terms in order, excepting the first and the last, which are always 1.

$A=n$, coefficient of the second term.

$B=\frac{n^2-n}{2}$, coefficient of the third term.

$C=\frac{(n^2-n)(n-2)}{2\times 3}$, coefficient of the fourth term.

The same coefficients may be used in the inverse order for the last terms of the indefinite series. Then we shall have, by restoring the coefficients,

$(a+b)^n=a^n+Aa^{n-1}b+Ba^{n-2}b^2+Ca^{n-3}b^3 \ldots Ca^3b^{n-3}+Ba^2b^{n-2}+Aab^{n-1}+b^n.$

269. We proceed, in the next place, to consider the signs to be prefixed to the several terms produced by the involution of a binomial. When a term is composed of several factors, the sign of the term will evidently depend upon the proper signs of the factors; if an *even* number of them be minus, or if none of them be minus, the quantity will be positive; if an *odd* number of them be minus, the quantity will be negative (Art. 90). Thus, analyzing the fourth power of $a+b$, each term is composed of one numerical and four literal factors, all *plus*, and consequently each term will be positive. Thus,

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

$1\times a\times a\times a\times a=a^4$, the first term;
$4\times a\times a\times a\times b=4a^3b$, the second term;
$6\times a\times a\times b\times b=6a^2b^2$, the third term;
$4\times a\times b\times b\times b=4ab^3$, the fourth term;
$1\times b\times b\times b\times b=b^4$, the fifth term.

The letters and exponents are $a^4+\ a^3b+\ a^2b^2+\ ab^3+b^4$;
The coefficients are - - $1+4\ \ +6\ \ +4\ \ +1.$

Compounding the series, $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$

Again, $(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4.$

$1\times a\times a\times a\times a=+a^4$, the first term;
$4\times a\times a\times a\times -b=-4a^3b$, the second term;
$6\times a\times a\times -b\times -b=+6a^2b^2$, the third term;
$4\times a\times -b\times -b\times -b=-4ab^3$, the fourth term;
$1\times -b+-b\times -b\times -b=+b^4$, the fifth term.

The letters and exponents are $a^4-\ a^3b+\ a^2b^2-\ ab^3+b^4$;
The coefficients are - - $1+4\ \ +6\ \ +4\ \ +1.$

Compounding the series, $(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4.$

270. The signs of the terms are also affected by the sign of the exponent.

Let it be required to expand $(a+b)^{-2}$:

The letters and exponents are $\Big\}$ $a^{-2}+\ a^{-3}b+\ a^{-4}b^2+\ a^{-5}b^3$, &c., ad infin.;
The coefficients are $\Big\}$ $1\ \ -2\ \ \ +3\ \ \ -4$, &c., ad infin.

Multiplying,

$$(a+b)^{-2}=a^{-2}-2a^{-3}b+3a^{-4}b^2-4a^{-5}b^3, \text{ \&c., ad infin.;}$$

Or, $$(a+b)^{-2}=\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}, \text{ \&c., ad infin.}$$

Again: let it be required to expand $(a-b)^{-2}$:

The letters and exponents are $\Big\}$ $a^{-2}-\ a^{-3}b+\ a^{-4}b^2-\ a^{-5}b^3$, &c., ad infin.;
The coefficients are $\Big\}$ $1\ \ -2\ \ \ +3\ \ \ -4$, &c., ad infin.

Multiplying,

$$(a-b)^{-2}=a^{-2}+2a^{-3}b+3a^{-4}b^2+4a^{-5}b^3, \text{ \&c., ad infin.;}$$

Or, $$(a-b)^{-2}=\frac{1}{a^2}+\frac{2b}{a^3}+\frac{3b^2}{a^4}+\frac{4b^3}{a^5}, \text{ \&c., ad infin.}$$

271. The principles of the Binomial Theorem may be stated as follows:

I. *The exponent of the leading quantity in the first term of the power is the same as the index denoting the power to which the binomial is to be raised, and decreases regularly by* 1. *The exponent of the following quantity is* 1 *in the second term, and increases regularly by* 1 *in the succeeding terms.*

II. *The coefficient of the first term is* 1; *that of the second the same as the power to which the binomial is to be raised; and universally, if the coefficient of any term be multiplied by the exponent of the leading quantity in that term, and the product be divided by the exponent of the following quantity* +1, *the result will be the coefficient of the succeeding term.*

Note 1.—The learner will find it convenient to obtain the series of the letters and exponents, and the series of coefficients separately, and then compound them by multiplying their corresponding terms, as in the preceding cases.

Note 2.—The preceding discussions relating to the Binomial Theorem will suggest some methods of verifying the work, and also of abridging it.

EXAMPLES.

1. Required the fourth power of $a+b$.

Expanding letters, &c., $a^4+\ a^3b+\ a^2b^2+\ ab^3+b^4$;
Finding coefficients - $1+4\ \ +6\ \ +4\ \ +1$.

Compounding, $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$. *Ans*

2. Required the fourth power of $a-b$.

The letters and exponents are $a^4-\ a^3b+\ a^2b^2-\ ab^3+b^4$;
The coefficients are - - $1+4\ \ +6\ \ +4\ \ +1$.

Compounding, - $(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

3. Required the fifth power of $a+b$ and of $a-b$.
4. Required the sixth power of $a+x$ and of $a-x$.
5. Required the seventh power of $x+y$ and of $x-y$.
6. Required the eighth power of $a+b$ and of $a-b$.
7. Required the eighth power of $x-y$.
8. Required the fourth power of $1+a$.

Expanding the terms - $1^4+1^3\times a+1^2\times a^2+1^1\times a^3+a^4$;
Finding coefficients - $1+4\ \ +6\ \ +4\ \ +1$.
Compounding, and rejecting the factor 1 - } $1+4a\ \ +6a^2\ \ +4a^3\ \ +a^4$.

9. Required the fourth power of $3a+2b$.

Let $x=3a$, and $y=2b$: then $(3a+2b)^4=(x+y)^4$.

Expanding this last expression, $x^4+\ x^3y+\ x^2y^2+\ xy^3+y^4$:
Finding coefficients - - $1+4\ \ +6\ \ +4\ \ +1$.

Compounding - $(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$.

Restoring the values of x and y,

$$(3a+2b)^4=(3a)^4+4\times(3a)^3\times 2b+6\times(3a)^2\times(2b)^2+4\times 3a\times(2b)^3+(2b)^4.$$

Involving the terms,

$$(3a+2b)^4=81a^4+4\times 27a^3\times 2b+6\times 9a^2\times 4b^2+4\times 3a\times 8b^3+16b^4.$$

Multiplying factors,

$$(3a+2b)^4=81a^4+216a^3b+216a^2b^2+96ab^3+16b^4.$$

10. Required the fifth power of $2cx-4y$.

Let $a=2cx$, and $b=4y$: then $(2cx-4y)^5=(a-b)^5$.

Expanding the terms, $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5$:
Finding coefficients, $1 + 5 + 10 + 10 + 5 + 1.$

Compounding, $(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$
Restoring the values of a and b,

$$(2cx-4y)^5 = 32c^5x^5 - 320c^4x^4y + 1280c^3x^3y^2 - 2560c^2x^2y^3 + 2560cxy^4 - 1024y^5.$$

11. Required the fourth power of a^2+b^3.

Let $x=a^2$, and $y=b^3$: then $(a^2+b^3)^4=(x+y)^4$.

Expanding the terms - $x^4 + x^3y + x^2y^2 + xy^3 + y^4$:
Finding the coefficients - $1 + 4 + 6 + 4 + 1.$

Compounding - $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$
Restoring the values of x and y,

$$(a^2+b^3)^4 = a^8 + 4a^6b^3 + 6a^4b^6 + 4a^2b^9 + b^{12}.$$

12. Expand $(2a^2-5b)^6$.
13. Expand $(3abx+y^2)^5$.
14. Expand $\frac{1}{(a+b)^2} = (a+b)^{-2}$.
15. Expand $(a+x)^{-4}$.
16. Expand $(a-b)^{-5}$.
17. Expand $(6abc-7axy)^5$.
18. Expand $(3x^2-4y)^4$.
19. Expand $(c^3-6ax)^5$.
20. Expand $(3a^3-1)^4$.

272. The powers of any polynomial whatever may be found by the Binomial Theorem. Take, for example, $(a+b+c)^3$. Letting $x=b+c$, we shall have

$$(a+b+c)^3 = (a+x)^3.$$

Expanding - - - $(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$
Restoring the value of x,

$$(a+b+c)^3 = a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3.$$

Expanding and multiplying factors,

$$(a+b+c)^3 = a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3.$$

2. Required the third power of $a-b+c$.
3. Required the third power of $2bc-3x+y$.

EVOLUTION AND ROOTS.

Extraction of the Square Root of Numbers.

273. A power of a number has already been defined to be the result of multiplying the number into itself continually, until the number has been used as a factor as many times as there are units in the exponent denoting the power.

The second power of $6=6\times6=36$.

The third power or cube of $6=6\times6\times6=216$.

The fifth power of $47=47\times47\times47\times47\times47=229,345,007$.

Involution is the method of finding the various powers of numbers.

Evolution is the reverse of this: it explains the method of resolving a number into equal factors, called roots.

274. When a number is resolved into two equal factors, one of the factors is called the Square Root; when resolved into three, the Cube Root; when into four, the Fourth Root, &c.

The first ten numbers are	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
And their squares -	1,	4,	9,	16,	25,	36,	49,	64,	81,	100.

By inspecting this table, it will be perceived that among entire numbers, consisting of one or two figures, there are nine only which are squares of other numbers. The square roots of other numbers, expressed by one or two figures, will be found between two whole numbers differing from each other by unity. Thus, 55, comprised between 49 and 64, has for its square root a number between 7 and 8; 78 has for its square root a number between 8 and 9. The numbers in the second line of the table being the squares of those in the first, the numbers in the first are the square roots of those in the second; therefore the square root of numbers consisting of one or two figures will readily be found by the table.

275. Let it next be required to find the root of a number consisting of more than two figures. It has already been shown that the square of any binomial, as $(a+b)^2=a^2+2ab+b^2$. Every number may be regarded as made up of a cer-

tain number of tens and a certain number of units; thus, 46 is composed of 4 tens and 6 units, and may be expressed thus, $40+6$; the square of which may be obtained in the same manner as the square of $a+b$; thus,

$$\begin{array}{r} 40+6 \\ 40+6 \\ \hline 1600+240 \\ 240+36 \\ \hline 1600+480+36=2116. \end{array}$$

In this result, as in the square of the binomial $a+b$, in which a may represent tens and b units, it will be observed there are three parts, viz.: the square of the tens, $40^2=1600$; twice the product of the tens by the units, $2\times40\times6=480$; and the square of the units, $6^2=36$. These three parts will be found in the second power of every number.

276. We next proceed to reverse this process, and find the square root of 2116.

As the square of 4 tens, or 40, is 1600, and the square of 5 tens, or 50, is 2500, the root can contain only 4 tens.

Subtracting the square of this - - -	2116
Square of 4 tens, or 40 - - - -	1600
	516

This remainder contains twice the product of the tens by the units, plus the square of the units. Now, if we double the tens, which gives 80, and divide 516 by 80, the quotient is the figure of the units, or a figure greater than the units. This quotient figure can evidently never be too small, but it may be too large, as 516, besides containing double the product of the tens by the units, may contain tens arising from the square of the units. The figure representing the units can never be greater than 9. $516\div80=6$. To ascertain whether 6 express the units, we multiply 80 by $6=480$, and subtract it from 516: the remainder is 36; from this subtract the square of the units $6\times6=36$: the remainder is 0; hence 4 tens and six units, or 46, which is the root.

The operation will stand thus:

```
              2116|40+6=46, root.
              1600
              ----
    40×2=80)516
             480
             ---
              36
     6×6=     36
             ---
               0
```

277. The work may be abridged by several modifications. By observing the table of the squares of the numbers 1, 2, 3, 4, &c., it will be perceived that the square of a number consisting of one figure can contain no figure of a higher denomination than tens. If we annex a cipher to the numbers 1, 2, 3, 4, &c., they become

10, 20, 30, 40, 50, 60, 70, 80, 90, 100;

And their squares are

100, 400, 900, 1600, 2500, 3600, 4900, 6400, 8100, 10000.

From which we see that the square of tens will contain no figure of a less denomination than hundreds, nor higher than thousands. When, then, the square root of a number consisting of three or four figures is required, in finding the tens, we may reject the first two figures on the right, as they can in no way influence the result. As the square of hundreds can contain no figure of a less denomination than thousands, when the square root of a number consisting of five or six figures is required, in obtaining the hundreds we may reject four figures at the right hand. When, then, the square root of any number is required, we may divide it into periods of two figures each (if a number consist of an odd number of figures, the last period will contain but one figure), and the number of these periods will be the number of figures in the root. Each of these periods, in connexion with the remainder resulting from the operations on the preceding period, may be used independently of the following periods in obtaining that figure of the root contained in it. In the above example, likewise, in which the square root of 2116 is required, as the product of tens by units evidently can contain no figure less than tens, after subtracting the square of the tens, the next step, the division, may be as well performed after rejecting the cipher from the right of the tens, and the unit figure from the right of the dividend. Moreover, it will be perceived that, instead of finding first twice the product of the tens by the units, and then the square of the units, we may obtain the sum of both numbers by placing the unit figure at the right of the tens in the divisor, and multiplying the result by the unit figure. With these modifications, the work of extracting the square root of 2116 will stand thus:

$$\begin{array}{r|l} 2\dot{1}1\dot{6} & \underline{46} \\ 16 & \\ \hline 86)516 & \\ 516 & \end{array}$$

Find the square of the tens in the first period; subtract, and bring down to the right of the remainder the next period. Divide by twice the tens, rejecting the right-hand figure of the dividend. Place the quotient figure in the root, and at the right of the divisor, and multiply this last number by the quotient figure, and subtract; as there is no remainder, 46 is the root.

Required the square root of 53361.

$$\begin{array}{r|l} \dot{5}3\dot{3}6\dot{1} & \underline{231} \\ 4 & \\ \hline 43)133 & \\ 129 & \\ \hline 461)461 & \\ 461. & \quad \textit{Ans. } 231. \end{array}$$

278. The same process may be extended to any number, however large. From the preceding operations, the following rule for the extraction of the second root will be readily inferred:

RULE.

I. *Separate the number into periods of two figures each, beginning at the right hand: the left-hand period will often contain but one figure.*

II. *Find the greatest square in the first period on the left; write the root in the place of a quotient in division, and subtract the second power from the left-hand period.*

III. *Bring down the next period to the right of the remainder for a dividend, and double the root already found for a divisor. See how many times the divisor is contained in the dividend, exclusive of the right-hand figure, and place the result in the root, and also at the right of the divisor.*

IV. *Multiply the divisor thus augmented by the last figure of the root, and subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

V. *Double the whole root already found for a new divisor, and proceed as before, till all the periods are brought down. The root will be doubled if the right-hand figure of the last divisor be doubled.*

If there is no remainder after all the periods are brought down, the proposed number is a perfect square. If there is a remainder, by the above rule, the root of the greatest square number contained in the proposed number will be obtained.

When the proposed number is not a perfect square, a doubt may arise whether the root found be that of the greatest square contained in the number. This may be determined by the following rule. The square of $a+1$ is a^2+2a+1; whence the square of a quantity greater by unity than a exceeds the square of a by $2a+1$; or, *the difference between the squares of two consecutive numbers is equal to twice the less number augmented by unity.*

Hence the entire part of the root cannot be augmented unless the remainder exceed twice the root found plus unity.

Required the square root of $\dot{1}2\dot{8}7|35$

$$\begin{array}{r} 1287 \\ 9 \\ \hline 65)387 \\ 325 \\ \hline 62 \end{array}$$

Now, as $35\times2+1=71>62$, 35 is the entire part of the root.

EXAMPLES.

1. What is the square root of 451,584? *Ans.* 672.
2. What is the square root of 9,186,961? *Ans.* 3031.
3. What is the square root of 13,032,100? *Ans.* 3610.
4. What is the square root of 4,543,164,409? *Ans.* 67,403.
5. What is the square root of 669,420,148,761? *Ans.* 818,181.

279. From what has been done, it will be perceived that there are many numbers the roots of which are not whole numbers; and although there must be a number which, multiplied into itself, will produce any number whatever, yet these numbers can have no assignable roots, either among whole or fractional numbers. The proof of this depends on the following proposition, which has already been demonstrated (see Art. 213):

Every number, P, which will exactly divide the product, $A\times B$, of two numbers, and which is prime to one of them, will divide the other.

The root of an imperfect power evidently cannot be expressed by a whole number, and, to show that it cannot be expressed by a fraction, let c be an imperfect square; if its root can be expressed by a fractional number, let $\frac{a}{b}$ represent that fractional number: then we shall have

$$\sqrt{c}=\frac{a}{b},$$

Or - - - - $$c=\frac{a^2}{b^2}.$$

If c be not a perfect square, its root will not be an entire number; that is, a will not be divisible by b; but it has been demonstrated that if a is not divisible by b, $a \times a$ or a^2 is not divisible by b or $b \times b = b^2$, whence $\frac{a^2}{b^2}$ cannot be equal to an entire number c.

All numbers, both entire and fractional, have a common measure with unity; on this account they are said to be commensurable; and since the *ratio* of these numbers to unity may always be expressed, they are called *rational* numbers.

The root of a number not a perfect square can have no common measure with unity, as no fraction can be assigned sufficiently small to measure at the same time this root and unity. The roots of such numbers are called *incommensurable* or *irrational numbers*. They are likewise called *surds*.

EXTRACTION OF THE SQUARE ROOT OF FRACTIONS.

280. The square root of a fraction may be found by extracting the square root of the numerator and of the denominator; thus, the square root of $\frac{16}{64}$ is $\frac{4}{8}$. If the numerator or denominator is not a perfect square, the root of the fraction cannot be found exactly, but the root to within less than one of the equal parts of the fraction may readily be found by the following

RULE.

Multiply both terms of the fraction by the denominator which does not change the value of the fraction; then extract the square root of the perfect square nearest the value of the numerator, and place the root of the denominator under it; this fraction will be the approximate root.

Required the square root of $\frac{3}{5}$: multiply both terms by 5, which gives $\frac{15}{25}$, of which $\frac{4}{5}$ is the required root exact to within less than $\frac{1}{5}$. We might multiply both terms of $\frac{15}{25}$ by any perfect square, and thus approximate the root more nearly. Thus, multiplying by 144, it becomes $\frac{2160}{3600}$, the root of which is nearest $\frac{46}{60}$. Thus we have the root of $\frac{3}{5}$ to within less than $\frac{1}{60}$.

The approximate root of a number not a perfect square may be found in a similar manner within a given fraction.

Multiply the proposed number by the square of the denominator of the fraction; then extract the square root of the product to the nearest unit, and divide this root by the denominator of the fraction.

This rule may be demonstrated as follows:

Let a be the number proposed, of which it is required to find the root to within less than $\frac{1}{n}$: $a=\frac{an^2}{n^2}$; let r be the entire part of the root of the numerator an^2; an^2 will be comprised between r^2 and $(r+1)^2$; consequently, the square root of a will be comprised between those of $\frac{r^2}{n^2}$ and $\frac{(r+1)^2}{n^2}$, that is, between $\frac{r}{n}$ and $\frac{(r+1)}{n}$, whence $\frac{r}{n}$ will be the root of a to within less than $\frac{1}{n}$.

Find the square root of 59 to within less than $\frac{1}{12}$.

$$59\times(12)^2=8496.$$

$$\sqrt{8496}=92. \quad \tfrac{92}{12}, \textit{ Ans.}$$

281. The manner of determining the approximate root in decimals is a consequence of the preceding rule.

To obtain the square root of a number within $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c., we multiply, by the preceding rule, the number by $(10)^2$, $(100)^2$, &c., or, what is the same thing, we add to the right of the number two, four, six, &c., ciphers; then extract the square root of the product to the nearest unit, and divide this root by 10, 100, 1000, &c.

The number of ciphers annexed to the whole number should always be double the number of decimal places required to be found in the root. The roots of decimal fractions, whole numbers, and decimals, may be found by the preceding rules. The number of decimals in the proposed number must always be made even by annexing ciphers if necessary. A vulgar fraction may be changed to a decimal fraction before extracting its root, and a mixed number to a whole number and decimal.

EXAMPLES.

1. What is the square root of 31,027 to within ,01 ?
Ans. 5,57.
2. What is the square root of 0,01001 to within ,00001 ?
Ans. 0,10004.
3. What is the square root of $\frac{11}{14}$ to within ,001 ?
Ans. 0,886.
4. What is the square root of $2\frac{13}{15}$ to within 0,0001 ?
Ans. 1,6931.
5. What is the square root of 7 ? *Ans.* 2,645+.
6. What is the square root of $4\frac{1}{9}$? *Ans.* 2,027+.
7. What is the square root of $\frac{11}{17}$? *Ans.* 0,8044+.
8. What is the square root of 0,01001 ?
Ans.
9. What is the square root of 0,0001234 ?
Ans.
10. What is the square root of 227 to within $\frac{1}{10000}$?
Ans. 15,0665.
11. What is the square root of 3,425 to within $\frac{1}{100}$?
Ans. 1,85.
12. What is the square root of $\frac{256}{361}$? *Ans.* $\frac{16}{19}$.
13. What is the square root of $11\frac{11}{16}$ to within ,001 ?
Ans. 3,418.
14. What is the square root of 3 to within ,0000000001 ?
Ans. 1,7320508076.

EXTRACTION OF THE CUBE ROOT OF NUMBERS.

282. The *third power*, or *cube* of a number, is the product arising from multiplying this number into itself till it has been used three times as a factor. The *third* or *cube root* is a number which, being raised to the third power, will produce the proposed number.

The first ten numbers being

	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,
Their cubes are,	1,	8,	27,	64,	125,	216,	343,	512,	729,	1000.

The numbers of the first line are the cube roots of the second.

By inspecting these lines, we perceive there are but nine *perfect cubes* among numbers expressed by one, two, or three figures; the cube root of other numbers consisting of one, two, or three figures, cannot be expressed exactly by means of unity, as may be shown by a process similar to that used in Art. 279.

The cube root of an entire number consisting of not more than three figures, may be obtained by merely inspecting the cubes of the first nine numbers. Thus, the cube root of 125 is 5; the cube root of 30 is 3 plus a fraction, or within one of 3.

To extract the cube root of a number consisting of more than three figures, we present the following

RULE.

1. *Separate the given number into periods of three figures each, beginning at the right hand: the left-hand period will often contain less than three figures.*

2. *Find the greatest cube in the left-hand period, and place its root on the right, in the place of a quotient in division. Subtract the cube of this figure of the root from the first period, and to the remainder bring down the next period, and call this number the dividend.*

3. *Multiply the square of the root just found by* 300 *for a divisor. Find how many times the divisor is contained in the dividend, and place the quotient for a second figure of the root. Multiply the divisor by this second figure, and place the product under the dividend. Multiply the former figure or figures of the root by* 30, *and that product by the square of the last figure, and place the result under the last; under these two products place the cube of the last figure of the root, and call the sum of the last three numbers the subtrahend.*

4. *Subtract the subtrahend from the dividend, and to the remainder bring down the next period for a new dividend; and in finding a divisor and subtrahend, proceed precisely as before, and so continue till all the periods have been brought down.*

We proceed next to the explanation of this rule.

The cube of a binomial, as $a+b=a^3+3a^2b+3ab^2+b^3$. In the number 45, a may represent tens, and b units, or we may find the cube of 45 writing it $40+5$.

$$
\begin{array}{rrrr}
 & & 45=40+ & 5 \\
 & & 40+ & 5 \\
\hline
 & & 200+ & 25 \\
 & 1600+ & 200 & \\
\hline
 & (45)^2=1600+ & 400+ & 25 \\
 & & 40+ & 5 \\
\hline
 & 8000+ & 2000+ & 125 \\
64000+ & 16000+ & 1000 & \\
\hline
\end{array}
$$

$$(45)^3=64000+24000+3000+125=91125$$

283. On inspecting the above examples, it will be perceived that the cube of a number composed of tens and units is equal to the cube of the tens, plus three times the product of the square of the tens by the units, plus three times the tens by the square of the units, plus the cube of the units. Let it now be required to reverse the above process, and find the cube root of 91125.

$$(40)^3=64000,\ (50)^3=125000.$$

Hence the cube root is evidently 40 plus a certain number of units. Subtracting the cube of 40, that is, the cube of the tens, there remains 27125, which contains the remainder of the parts above specified. As it is evident that the third power of tens can have no significant figure below the fourth place, in finding the third root of the tens the three figures on the right may be rejected, as they will not influence the result. As the cube of 100 is 1000000, in obtaining the cube root of hundreds in a number consisting of more than six figures, we may reject the first six figures on the right as not influencing the result; hence any number of which the root is required may be separated into periods of three figures each, each one of the periods may be used separately in connexion with the remainder resulting from the preceding operations, and the number of periods will be the number of figures in the root. The cube of no one of the digits contains more than three figures. In the above example, rejecting the first three figures on the right, the cube root of the tens found in 91 is 4. Subtracting the cube of this (64), and bringing down the next period, the result of the operation is 27125. This must contain, from what has been said, triple the product of the square of the tens by the units, together with two remaining parts already specified. As the square of tens contains no significant figures less than hundreds, we may reject the two right-hand figures from 27125, and dividing the remainder by three times the square of the tens, we should obtain the unit figure. In practice it is found more convenient to use the whole dividend, and to annex two ciphers to the divisor, as, instead of multiplying the square of the tens by three, we multiply by 300. For the same reason, instead of multiplying the product of the square of the units by the tens by 3, we multiply by 30.

Dividing 27125 by the square of the tens (16), multiplied by 300, which $=4800$, the quotient 5 will be the unit figure sought; or it may be too large by 1 or 2, as there may be hundreds arising from the other parts of the root sought: this can only be determined by trial. Having now the tens and the units, and having already subtracted the cube of the tens, we next proceed to subtract the other parts of the cube from the remainder.

The square of the tens, multiplied by 300 and by the units, the last figure of the root - - - -	$=24000$
The tens, multiplied by the square of the units and by 30 - - - - - - -	$=3000$
The cube of the last figure or units - - -	$=\ 125$
Sum - -	27125

As there is no remainder, 45 is the root.

The operation may be exhibited as follows:

$$\begin{array}{rr|l}
 & 91125 & 45 \\
(4)^3 = & 64 & \\
(4)^2 = 16) \times 300 = 4800) & 27125 & \\
4800 \times \ \ 5 = & 24000 & \\
(5)^2 \times 4 \times \ 30 = & 3000 & \\
(5)^3 = 5 \times 5 \times 5 = & 125 & \\
 & \overline{000} &
\end{array}$$

284. Any number, however large, may be considered as composed of units and tens: the process of finding the cube root may therefore be reduced to that of the preceding example.

Required the third root of 9663597.

$$\begin{array}{rrl}
 & 9663597 & |\ 213\text{, root.} \\
(2)^3 = & 8 & \\
(2)^2 \times 300 = \overset{\text{Divisor.}}{1200}) & 1663\text{,} & \text{first dividend.} \\
1200 \times 1 = & 1200 & \\
2 \times 30 \times (1)^2 = & 60 & \\
(1)^3 = & 1 & \\
 & \overline{1261}\text{,} & \text{first subtrahend.} \\
(21)^2 \times 300 = \overset{\text{Divisor.}}{132300}) & 402597\text{,} & \text{second dividend.} \\
132300 \times 3 = & 396900 & \\
21 \times 30 \times (3^2) = & 5670 & \\
(3^3) = & 27 & \\
 & \overline{402597}\text{,} & \text{second subtrahend.} \\
 & \overline{000000} &
\end{array}$$

Should the divisor not be contained in the dividend as prepared above, place a cipher in the root, and bring down the next period to form a new dividend.

The difference between the cubes of two consecutive whole numbers is equal to three times the square of the least number, plus three times this number, plus 1.

Let a and $a+1$ be two consecutive whole numbers.

$$(a+1)^3 = a^3 + 3a^2 + 3a + 1.$$

$$(a+1)^3 - a^3 = 3a^2 + 3a + 1.$$

$$(90)^3 - (89)^3 = 3 \times (89)^2 + 3 \times 89 + 1 = 24031.$$

In extracting the cube root of any number not a perfect cube, if any of the remainders are equal to, or exceed three times the square of the root obtained, plus three times this root, plus 1, the last figure of the root is too small, and must be augmented by at least unity.

285. The third root of a fraction is found by extracting the third root of the numerator and denominator. When the denominator is not a perfect third power, we may obtain the root approximately by multiplying both terms by the square of the denominator; thus, in obtaining the cube root of $\frac{3}{7}$, we multiply both terms by 49; the fraction then becomes $\frac{147}{343}$, the root of which is nearest $\frac{5}{7}$ accurate to within $\frac{1}{7}$. We might multiply both terms of $\frac{147}{343}$ by any perfect cube, and then extract the cube root, and we should approximate still nearer the true root. By a process similar to that explained in the article on square root, we may approximate the third root of a number not a perfect third power, by converting it into a fraction, the denominator of which is a perfect third power. Thus the approximate root of 15 may be found, putting it under the following form:

$$\frac{15\times 12^3}{(12)^3}=\frac{25920}{1728},$$

the third root of which is $\frac{29}{12}$, or $2\frac{5}{12}$ accurate to within less than $\frac{1}{12}$. The root may be obtained with greater accuracy by using some number greater than 12.

In such cases it is most convenient to convert the proposed number into a fraction, the denominator of which shall be the third power of 10, 100, 1000, &c. Let it be required to find the third root of 25 to within ,001; converting 25 into a decimal, the denominator of which is the third power of 1000, viz., 25,000 000000, the third root of which is 2,920 to within ,001, we have then $\sqrt[3]{25}=2,920$ accurate to within less than ,001.

To approximate the third root of an entire number by means of decimals, *we annex to the proposed number three times as many ciphers as there are decimal places required in the root; we then extract the root of the number thus prepared to within a* *unit, and point off for decimals as many places as there are decimal figures required in the root.*

If the proposed number contain decimals, beginning at the place of units, separate the number, both to the right and left, into periods of three figures, annexing ciphers, if necessary, to complete the right-hand period in the decimal part. Then extract the root, and point off for decimals in the root as many places as there are periods in the decimal part of the power.

The third root of a vulgar fraction may be most readily obtained after converting it first into a decimal fraction.

EXAMPLES.

1. What is the cube root of 75686967 ? *Ans.* 423.
2. What is the cube root of 128787625 ? *Ans.* 505.
3. What is the cube root of 205483447701 ? *Ans.* 5901.
4. What is the cube root of 524581674,625 ? *Ans.* 806,5.
5. What is the cube root of 1003,003001 ? *Ans.* 10,01.
6. What is the cube root of 0,756058031 ? *Ans.* 0,911.
7. What is the cube root of 32977340218432 ? *Ans.* 32068.
8. What is the cube root of 473 to within $\frac{1}{20}$? *Ans.* $7\frac{3}{4}$.
9. What is the cube root of 79 to within ,0001 ? *Ans.* 4,2908.
10. What is the cube root of 3,00415 to within ,0001 ? *Ans.* 1,4429.
11. What is the cube root of 0,00101 to within ,01 ? *Ans.* 0,10.
12. What is the cube root of 0,000003442951 ? *Ans.* 0,0151.
13. What is the cube root of $6\frac{1275}{1331}$? *Ans.* $1\frac{10}{11}$.
14. What is the cube root of $\frac{17576}{24389}$? *Ans.* $\frac{26}{29}$.

TO EXTRACT ANY GIVEN ROOT OF A WHOLE NUMBER.

286. Any root exceeding the third, consisting simply of two and three, as factors, may be found by the preceding rules; thus, the fourth root may be found by extracting the square root twice; the sixth root by extracting the third

root, and then the square root of that; the twelfth root by extracting the square root twice, and then the third of the last root. Before proceeding to give a rule for the extraction of any root, we subjoin a table of roots and powers.

Roots,	1	2	3	4	5	6	7	8	9
2d Power,	1	4	9	16	25	36	49	64	81
3d Power,	1	8	27	64	125	216	343	512	729
4th Power,	1	16	81	256	625	1296	2401	4096	6561
5th Power,	1	32	243	1024	3125	7776	16807	32768	59049
6th Power,	1	64	729	4096	15625	46656	117649	262144	531441
7th Power,	1	128	2187	16384	78125	279936	823543	2097152	4782969
8th Power,	1	256	6561	65536	390625	1679616	5764801	16777216	43046721
9th Power,	1	512	19683	262144	1953125	10077696	40353607	134217728	387420489
10th Power,	1	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$$

$$(a+b)^7=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$$

In these examples a and b may represent tens and units in any given number, as 47; and to obtain the root of any given power, we evidently must reverse the process by which the power is obtained from the root.

By carefully attending to the preceding explanations, and the different powers of the binomial $(a+b)$, the reason of the following rule for extracting any given root of a proposed number will readily be discovered:

RULE.

1. *Divide the number into periods of as many figures each as there are units in the index denoting the root.*

2. *Find the first figure of the root by trial, and subtract its power from the left-hand period, and to the remainder bring down the first figure of the next period for a dividend.*

3. *Involve the root to the next inferior power to that which is given, and multiply it by the number denoting the given power, and it will be the divisor.*

4. *Find how many times the divisor is contained in the dividend, and the quotient will be another figure of the root, or* 1 *or* 2 *too large.*

5. *Involve the whole root to the given power, and subtract it from the two left-hand periods of the given number; bring down the first figure of the next period to the remainder for a new dividend, find a new divisor, another figure of the root, and again involving the whole root to the given power, subtract it from the first three left-hand periods. Thus proceed till the whole root is obtained.*

Required the fifth root of 36936242722357.

	36936242722357 | 517
$5^5=$	3125
$5^4\times5=3125$, first divisor.	5686, first dividend.
$(51)^5=$	345025251, subtrahend.
$(51)^4\times5=33826005$, 2d divisor.	243371762, 2d dividend.
$(517)^5=$	36936242722357
	0000000

287. The preceding rule may be put in another form, embracing the same principles, but more consistent with the method by which we have explained the extraction of the second and third roots.

RULE.

1. *Separate the number into periods of as many figures as there are units in the index denoting the root.*

2. *Find by trial the root of the first period: this will be the first figure of the root: place this figure to the left, in a column called* FIRST COLUMN; *then multiply it by itself, and place the product for the first term of a* SECOND COLUMN. *This, multiplied by the same figure, will give the first term of a* THIRD COLUMN. *Thus continue until the number of columns is one less than the units in the index denoting the root.*

Multiply the term in the LAST COLUMN *by the same figure, and subtract the product from the first period, and to the remainder bring down the next period, and it will form the* FIRST DIVIDEND.

Again, add this same figure to the term of the FIRST COLUMN, *multiply the sum by the same figure, and add the product to the term of the* SECOND COLUMN, *which, in turn, must be multiplied by the same figure, and added to the term of the* THIRD COLUMN, *and so on till we reach the* LAST COLUMN, *the term of which will form the* FIRST TRIAL DIVISOR.

Again, beginning with the FIRST COLUMN, *repeat the above process until you reach the column next to the last; and so continue to do until there are as many terms in the* FIRST COLUMN *as there are units in the index denoting the root, observing in each successive operation to terminate in the column of the next inferior order.*

3. *Seek how many times the first trial divisor, when there are annexed to it as many ciphers, less one, as there are units in the index, is contained in the* FIRST DIVIDEND; *the quotient figure will be the second figure of the root.*

Then proceed to form a new series by annexing this figure to the last term in the FIRST COLUMN; *multiply the result by the last figure, and add it to the last term in the* SECOND COLUMN, *advan-*

cing the number to be added two places to the right of the other before adding. Multiply this result by the same figure, and add the product to the last term in the THIRD COLUMN, *having previously advanced it three places to the right of that term; proceed in the same manner to the last term, observing to advance the numbers added to the different columns as many places to the right of the terms as the number expressing the order of the column; that is, advancing the terms of the* FIRST COLUMN *one place, those of the* SECOND COLUMN *two places, &c. Multiply the term thus obtained in the last column by the last figure of the root, and subtract it from the dividend; to the remainder bring down the next period for a new dividend, and proceed to find a divisor and the third figure of the root in the same manner as the second was obtained.*

Proceed in the same manner till all the periods are brought down. If there is still a remainder, the process can be extended by forming periods of ciphers.

Required the third root of 103823.

1*st col.*	2*d col.*	103823 | 47
4	16	64
8	48, first trial divisor.	39823, first div.
127	5689	39823

288. It will be perceived that this method of extracting the cube root is similar to that already explained. Let $a+b=40+7=47 (a+b)^3=a^3+3a^2b+3ab^2+b^3$. We subtract first the cube of the tens, $a^3=64$. We next form the divisor, which is $3a^2=48$; 16, the first term of the second column, is once the square of the tens; 8, the second term of the first column, is double the tens; multiplying this by the tens (4), the product is twice the square of the tens (32); this added to 16, the square of the tens, gives 48, three times the square of the tens. Instead of rejecting two figures on the right of the dividend, we annex two ciphers to the divisor. The second figure of the root is the result of the division. Then there remains to be obtained and subtracted

from the given number, $3a^2b+3ab^2+b^3$: this may be put in another form, thus: $((3a+b)\times b+3a^2)\times b$. On inspecting the above work, it will be perceived that $127=3a+b$; this, multiplied by 7 (b), becomes $889=(3a+b)\times b$; adding $4800=3a^2$, the result is 5689; multiplying, according to the rule, by 7 (b), we have $39823=((3a+b)\times b+3a^2)\times b$. The same explanation will apply, however extended the operations may be.

Required the fifth root of 36936242722357.

1st col.	*2d col.*	*3d col.*	*4th col.*	
5	25	125	625	36936242722357 | 517
10	75	500	3125	3125
15	150	1250	32525251	56862427
20	250	1275251	33826005	32525251
251	25251	1300754	347673946051	2433717622357
252	25503	1326510		2433717622357
253	25756	1344842293		0
254	26010			
2557	2618899			

What is the seventh root of 1231171548132409344?

1st col.	2d col.	3d col.	4th col.	5th col.	6th col.	
3	9	27	81	243	729	1231171548132409344 | 384
6	27	108	405	1458	5103	2187
9	54	270	1215	5103	11568197824	101247154813
12	90	540	2835	808149728	21076554688	92545582592
15	135	945	37231216	1188544608	21753930553102336	870157222124093444
18	189	1110152	47549360	1663938528		870157222124093444
218	20644	1289768	59424240	169343966275584		
226	22452	1484360	72979760			
234	24324	1694440	737528368896			
242	26260	1920520				
250	28260	1932692224				
258	30324					
2664	3043056					

EXAMPLES.

1. Find the fifth root of 418227202051. *Ans.* 211.
2. Find the fourth root of 75450765,3376. *Ans.* 93,2.
3. Find the fifth root of 0,00016850581551.

   Ans. 0,111.
4. Find the fourth root of 2526,88187761. *Ans.* 7,09.
5. Find the sixth root of 2985984. *Ans.* 12.
6. Find the eighth root of 1679616. *Ans.* 6.
7. Find the seventh root of 2. *Ans.* 1,10409, nearly.

EVOLUTION OF MONOMIALS.

289. From a previous demonstration (Art. 156), it is evident that *the root of the product of two or more factors is equal to the product of the roots.* Thus, $\sqrt{a^2b^4c^6}=\sqrt{a^2}\times\sqrt{b^4}\times\sqrt{c^6}$.

Again, by the definition of evolution (Art. 274), $\sqrt{c^6}=c^3$; for $c^3\times c^3=c^{3+3}=c^6$; hence $\sqrt{c^6}=c^{6\div2}$, or $c^{\frac{6}{2}}=c^3$.

And, $\sqrt[3]{8c^6}=2c^2$; for $2c^2\times2c^2\times2c^2=8c^6$; hence $\sqrt[3]{8c^6}=\sqrt[3]{8}\times\sqrt[3]{c^6}=2\times c^{6\div3}$, or $2c^{\frac{6}{3}}=2c^2$.

The same reasoning will evidently apply to every case of monomials. Hence, for the evolution of monomials, we have the following general

RULE.

1. *Extract the required root of the coefficient.*
2. *Divide the exponent of each literal factor by the number denoting the root, and annex the result to the root of the coefficient.*

Note 1.—With regard to the *sign* to be prefixed to the root, it is important to observe,

a. An *odd* root of a number will have the *same sign as the number itself.* Thus, the cube root of a^3, or $\sqrt[3]{a^3}=a$, for $a\times a\times a=a^3$; and the cube root of $-a^3$, or $\sqrt[3]{-a}=-a$, for $-a\times-a\times-a=-a^3$.

b. The *even* root of an affirmative number is *ambiguous.* Thus, the square root of a^2, or $\sqrt{a}=\pm a$; for $a\times a=a^2$, and $-a\times-a=+a$; also, the square root of 16, or $\sqrt{16}=\pm4$, for $4\times4=16$, and $-4\times-4=+16$.

c. The *even* root of a negative number is *impossible.* Thus, the square root of $-a^2$, or $\sqrt{-a^2}$, can be neither $+a$ nor $-a$, for $+a\times+a=+a^2$, and $-a\times-a=+a^2$. Also, the square root of -16, or $\sqrt{-16}$, can be neither $+4$ nor -4.

Note 2.—The root of a fraction is equal to the root of the numerator divided by the root of the denominator. Thus,

$$\sqrt{\frac{a}{c}}=\frac{a^{\frac{1}{2}}}{c^{\frac{1}{2}}};\ \text{for}\ \frac{a^{\frac{1}{2}}}{c^{\frac{1}{2}}}\times\frac{a^{\frac{1}{2}}}{c^{\frac{1}{2}}}=\frac{a^{\frac{1}{2}+\frac{1}{2}}}{c^{\frac{1}{2}+\frac{1}{2}}}=\frac{a}{c}.$$

Note 3.—The above rule for the evolution of monomials is equally applicable where the exponents are negative. Thus, the square root of a^{-4}, or $\sqrt{a^{-4}}=a^{-\frac{4}{2}}=a^{-2}$; for $a^{-4}=\frac{1}{a^4}$; hence, $\sqrt{a^{-4}}=\sqrt{\frac{1}{a^4}}=\frac{1}{a^2}=a^{-2}$.

EXAMPLES.

1. Required the square root of $9a^4b^2$.

$$\sqrt{9a^4b^2}=3a^{\frac{4}{2}}b^{\frac{2}{2}}=3a^2b. \quad \textit{Ans.}$$

2. Required the square root of $64a^6x^4$. *Ans.* $8a^3x^2$.
3. Required the cube root of $27a^6b^3$. *Ans.* $3a^2b$.
4. Required the fourth root of $16a^8x^{12}y^2$. *Ans.* $2a^2x^3y^{\frac{1}{2}}$.
5. Required the square root of $\frac{4a^4b^6}{9x^2y^4}$. *Ans.* $\frac{2a^2b^3}{3xy^2}$.
6. Required the fifth root of $243a^{10}b^5$. *Ans.* $3a^2b$.
7. Required the fourth root of $16a^{-8}b^4$. *Ans.* $2a^{-2}b$.
8. Required the sixth root of $64a^6x^{12}y^3$. *Ans.* $2ax^2y^{\frac{1}{2}}$.
9. Required the third root of $8a^{-3}b^{-6}c^3$. *Ans.* $2a^{-1}b^{-2}c$.
10. Required the square root of $196a^4b^2c^6$. *Ans.* $14a^2b^{-1}c^3$.
11. Required the square root of $784x^2y^4z^6$. *Ans.* $28xy^2z^3$.
12. Required the square root of $\frac{16a^4}{49b^2c^6}$. *Ans.* $\frac{4a^2}{7bc^3}$.
13. Required the cube root of $-27a^6b^2$. *Ans.* $-3a^2b^{\frac{2}{3}}$.
14. Required the nth root of $a^nb^{3n}c^{-2n}$. *Ans.* ab^3c^{-2}.
15. Required the fifth root of $-32a^5b^{10}c^{15}$. *Ans.* $-2ab^2c^3$.
16. Required the fourth root of $81a^2b^4c^6$. *Ans.* $3a^{\frac{1}{2}}bc^{\frac{3}{2}}$.
17. Required the cube root of $-64a^{-3}b^{-6}c^{-12}$.

Ans. $-4a^{-1}b^{-2}c^{-4}$.

18. Required the square root of $441x^8y^2z^6$. *Ans.* $21x^4yz^3$.

19. Required the square root of $576a^4b^{-6}c^{-4}d^{12}$.

Ans. $24a^2b^{-3}c^{-2}d^6$.

20. Required the fifth root of $-243x^{-5}y^{10}z^{-15}$.

Ans. $-3x^{-1}y^2z^{-3}$.

21. Required the square root of $\dfrac{a^{2n}b^2c}{a^m}$. *Ans.* $\dfrac{a^nbc^{\frac{1}{2}}}{a^{\frac{m}{2}}}$.

290. If all the factors of which the monomial is composed are not complete powers of the same name as the root, it is evident that the root of the entire number cannot be obtained. Still the expression may be simplified by removing that factor which is a complete power of the same name as the required root, from under the radical sign. This is done on the principle that *the root of the product is equal to the product of the roots.*

Thus, $\sqrt{8a^2b} = \sqrt{4a^2 \times 2b} = \sqrt{4a^2} \times \sqrt{2b} = 2a\sqrt{2b}$.

And, $\sqrt[3]{24ab^3} = \sqrt[3]{8b^3 \times 3a} = \sqrt[3]{8b^3} \times \sqrt[3]{3a} = 2b\sqrt[3]{3a}$.

And, $\sqrt[n]{6a^nb} = \sqrt[n]{a^n \times 6b} = \sqrt[n]{a^n} \times \sqrt[n]{6b} = a\sqrt[n]{6b}$.

Hence, to reduce radicals to their simplest forms:

1. *Resolve the quantity under the radical sign into two factors, one of which shall be a complete power of the same name as the root.*

2. *Extract the root of this factor, and multiply it by the coefficient of the radical, if it has any, and prefix the result to the radical sign under which the factor that is not a complete power will remain.*

EXAMPLES.

1. Required the simplest form of $\sqrt{32a^4b^2c}$.

Ans. $4a^2b\sqrt{2c}$.

2. Required the simplest form of $\sqrt{98a^2b^6c^4d}$.

Ans. $7ab^3c^2\sqrt{2d}$.

3. Required the simplest form of $\sqrt[3]{24a^3cd^6}$.

Ans. $2ad^2\sqrt[3]{3c}$.

4. Required the simplest form of $\sqrt[3]{54a^6xy^3z^6}$.

Ans. $3a^2yz^2\sqrt[3]{2x}$.

5. Required the simplest form of $\sqrt[4]{32a^4b^8c}$.

Ans. $2ab^2\sqrt[4]{2c}$.

6. Required the simplest form of $\sqrt{\frac{8a^2b}{48x^4y}}$.

$$\sqrt{\frac{8a^2b}{48x^4y}}=\sqrt{\frac{4a^2}{16x^4}\times\frac{2b}{3y}}=\sqrt{\frac{4a^2}{16x^4}}\times\sqrt{\frac{2b}{3y}}=\frac{2a}{4x^2}\times\sqrt{\frac{2b}{3y}}$$

Ans. $\frac{a}{2x^2}\sqrt{\frac{2b}{3y}}$.

7. Required the simplest form of $\sqrt{8a^2b-12a^2x}$.

Ans. $2a\sqrt{2b-3x}$.

$\sqrt{8a^2b-12a^2x}=\sqrt{4a^2\times(2b-3x)}=\sqrt{4a^2}\times\sqrt{2b-3x}=2a\sqrt{2b-3x}$.

8. Required the simplest form of $\sqrt[4]{\frac{32a^4b}{81c^4d}}$.

Ans. $\frac{2a}{3c}\sqrt[4]{\frac{2b}{d}}$.

9. Required the simplest form of $\sqrt[3]{24a^3c-32a^3cx}$.

Ans. $2a\sqrt[3]{3c-4cx}$.

10. Required the simplest form of $\sqrt[3]{a^3+a^3b^2}$.

Ans. $a\sqrt[3]{1+b^2}$.

11. Required the simplest form of $\sqrt{405a^3b^4c^2de}$.

$\sqrt{405a^3b^4c^2de}=\sqrt{81a^2b^4c^2}\times\sqrt{5ade}=9ab^2c\sqrt{5ade}$.

Ans. $9ab^2c\sqrt{5ade}$.

12. Required the simplest form of $\sqrt{605a^7b^4c^5d^3}$.

Ans. $11a^3b^2c^2d\sqrt{5acd}$.

13. Required the simplest form of $\sqrt{1014a^5b^9c^3d}$.

Ans. $13a^2b^4c\sqrt{6abcd}$.

14. Required the simplest form of $\sqrt{-8a^2x}$.

$\sqrt{-8a^2x}=\sqrt{4a^2\times-2x}=\sqrt{4a^2}\times\sqrt{-2x}=2a\sqrt{-2x}$.

Ans. $2a\sqrt{-2x}$.

15. Required the simplest form of $\sqrt{-16}$.

Ans. $4\sqrt{-1}$.

EVOLUTION OF POLYNOMIALS.

291. We might give rules for the extraction of the different roots separately, but it will comport better with our purpose to introduce the student at once to a general rule by which we may evolve any root whatever. The reason for the following rule will be sufficiently obvious if we recur to the formation of powers by the binomial theorem or by actual multiplication; and, indeed, the work verifies itself.

RULE.

1. *Arrange the terms according to the powers of one of the letters, so that the highest power shall stand in the first term, the next highest in the second, &c.*

2. *Find the root of the first term, and place it in the quotient; then subtract its power from the first term, and bring down the second term for a dividend.*

3. *Involve the first term of the root to the next inferior power, and multiply it by the index of the given power for a divisor. Divide the dividend by this divisor, and the quotient will be the second term of the root.*

4. *Involve the terms of the root thus found to the given power, and subtract it from the whole polynomial. Divide the first term of the remainder by the divisor first found; the quotient will be another term of the root.*

5. *Proceed in this manner till the power obtained by the involution of the terms of the root is equal to the given polynomial. This will be the case only when the true root is found.*

EXAMPLES.

1. Required the square root of $4a^2+4ab+b^2$.

$$4a^2+4ab+b^2. \quad |\underline{2a+b.} \quad \textit{Ans.}$$
$$4a^2$$
$$4a)\ * +4ab+$$
$$4a^2+4ab+b^3.$$
$$0$$

2. Required the cube root of $a^6+3a^5-3a^4-11a^3+6a^2+12a-8$.

$$
\begin{array}{l}
a^6+3a^5-3a^4-11a^3+6a^2+12a-8 \quad |\underline{a^2+a-2.} \quad \textit{Ans.} \\
a^6 \\
3a^4)^* \ +3a^5 \\
a^6+3a^5+3a^4+a^3 \\
3a^4)^* \quad * \ -6a^4-12a^3 \\
a^6+3a^5-3a^4-11a^3+6a^2+12a-8. \\
\qquad 0
\end{array}
$$

3. Required the fourth root of $16a^4+96a^3b+216a^2b^2+216ab^3+81b^4$.

$$
\begin{array}{l}
16a^4+96a^3b+216a^2b^2+216ab^3+81b^4. \quad |\underline{2a+3b.} \quad \textit{Ans.} \\
16a^4 \\
32a^3) \ * \ +96a^3b \\
16a^4+96a^3b+216a^2b^2+216ab^3+81b^4. \\
\qquad 0
\end{array}
$$

4. Required the cube root of $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$. *Ans.* $a^2-2ab+b^2$.

5. Required the fifth root of $32a^5-80a^4x+80a^3x^2-40a^2x^3+10ax^4-x^5$. *Ans.* $2a-x$.

6. Required the fifth root of $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$. *Ans.* $a+b$.

7. Required the sixth root of $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$. *Ans.* $a-b$.

292. REMARK 1.—The square of a binomial consists of three parts, viz., the square of the first term, twice the product of the two terms, and the square of the last term. Hence the second power of the simplest polynomial will consist of three terms; and every trinomial in which, when the terms are arranged, the extremes are complete squares, and the middle term is double the product of the square roots of the extremes, is a perfect square, whose root may be found by the following

RULE.

Take the square roots of the two terms that are complete powers, and connect them by the sign prefixed to the other term.

1. $\sqrt{a^2 + 2ab + b^2} = a + b$.
2. $\sqrt{a^2 - 2ab + b^2} = a - b$.
3. $\sqrt{a^2 + 2a + 1} = a + 1$.
4. $\sqrt{a^2 - \frac{4a}{3} + \frac{4}{9}} = a - \frac{2}{3}$.
5. $\sqrt{36y^2 + 36y + 9} = 6y + 3$.
6. $\sqrt{9d^2 - 6dh + h^2} = 3d - h$.
7. $\sqrt{a^2b^2 + 2abcd + c^2d^2} = ab + cd$.
8. $\sqrt{x^2 - \frac{2bx}{m} + \frac{b^2}{m^2}} = x - \frac{b}{m}$.

293. Remark 2.—Since the fourth power of a quantity may be found by squaring the second power, it is evident that the fourth root may be obtained by extracting the square root of the square root.

$$\text{Thus, } \sqrt[4]{a^4} = \sqrt{\sqrt{a^4}} = \sqrt{a^2} = a.$$

$$\text{And, } \sqrt[6]{a^6} = \sqrt[3]{\sqrt{a^6}} = \sqrt[3]{a^3} = a, \text{ \&c.}$$

Hence,

1. *To obtain the fourth root, we may extract the square root of the square root.*

2. *To obtain the sixth root, we may extract the cube root of the square root, &c.*

EXAMPLES.

1. Required the square root and fourth root of $16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4$.

Ans. $4a^2 + 12ab + 9b^2$, and $2a + 3b$.

2. Required the sixth root of $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$. *Ans.* $x - 2$.

3. Required the eighth root of $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$. *Ans.* $a + b$.

4. Required the ninth root of $a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$. *Ans.* $a + b$.

294. Remark 3.—If the polynomial is not a perfect power, it may sometimes be simplified in the same manner as monomials.

1. Required the simplest form of $\sqrt{5a^2+10ab+5b^2}$.

$\sqrt{5a^2+10ab+5b^2}=\sqrt{(a^2+2ab+b^2)\times 5}=\sqrt{a^2+2ab+b^2}\times\sqrt{5}$
$=(a+b)\sqrt{5}$.

2. Required the simplest form of $\sqrt{a^3b+4a^2b^2+4ab^3}$.
Ans. $(a+2b)\sqrt{ab}$.

3. Required the simplest form of $\sqrt[4]{2a^5+8a^4b+12a^3b^2+8a^2b^3+2ab^4}$.
Ans. $(a+b)\sqrt[4]{2a}$.

4. Required the simplest form of $\sqrt[3]{3a^4b-9a^3b^2+9a^2b^3-3ab^4}$.
Ans. $(a-b)\sqrt[3]{3ab}$.

295. REMARK 4.—*Roots* may also be obtained by the Binomial Theorem, since n in the general formula (Art. 268) may be either an integer or a fraction. The series produced by the expansion of a binomial, however, will never terminate, since the successive subtractions of units from the fractional exponent of the leading letter can never reduce that exponent to 0.

EXAMPLES.

1. Expand by the Binomial Theorem $(a+b)^{\frac{1}{2}}$.

The exponents in the successive terms of the result will be as follows:

Of a . . . . .
$\frac{1}{2}$, $\frac{1}{2}-1=-\frac{1}{2}$, $-\frac{1}{2}-1=-\frac{3}{2}$, $-\frac{3}{2}-1=-\frac{5}{2}$, $-\frac{5}{2}-1=-\frac{7}{2}$, &c.

Of b . . . . .
0, 1, 2, 3, 4, &c.

Hence the letters, without their coefficients, will be

$$a^{\frac{1}{2}}+a^{-\frac{1}{2}}b+a^{-\frac{3}{2}}b^2+a^{-\frac{5}{2}}b^3+a^{-\frac{7}{2}}b^4, \text{ \&c., } ad\ infin.$$

Proceeding as in Art. 271, we shall obtain for the coefficients of the successive terms,

1, $\frac{1}{2}$, $\frac{1}{2}\times-\frac{1}{2}\div 2=-\frac{1}{2.4}$, $-\frac{1}{2.4}\times-\frac{3}{2}\div 3=\frac{3}{2.4.6}$, $\frac{3}{2.4.6}\times-\frac{5}{2}\div 4=-\frac{3.5}{2.4.6.8}$, &c.

Or - 1, $\frac{1}{2}$, $-\frac{1}{8}$, $\frac{1}{16}$ $-\frac{1}{128}$, &c., *ad infin.*

Hence, by compounding the series of letters and coefficients, we obtain

$$(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{a^{-\frac{1}{2}}b}{2}-\frac{a^{-\frac{3}{2}}b^2}{8}+\frac{a^{-\frac{5}{2}}b^3}{16}-\frac{a^{-\frac{7}{2}}b^4}{128}, \text{ \&c., } ad\ infin.$$

Transferring a, where it is affected with the negative exponent, to the denominator, as in Art. 258, the expression becomes

$$(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}}-\frac{b^2}{8a^{\frac{3}{2}}}+\frac{b^3}{16a^{\frac{5}{2}}}-\frac{b^4}{128a^{\frac{7}{2}}}, \text{ \&c., } ad\ infin.$$

2. Expand $(a-b)^{\frac{1}{3}}$.

The exponents, obtained as in the last example, are as follows:

Of a . . . . . $\frac{1}{3}$, $\frac{1}{3}-1=-\frac{2}{3}$, $\frac{2}{3}-1=-\frac{5}{3}$, $-\frac{5}{3}-1=-\frac{8}{3}$, &c.
Of b . . . . . 0, 1, 2, 3, &c.

Hence the letters, &c., will be

$$a^{\frac{1}{3}}-a^{-\frac{2}{3}}b+a^{-\frac{5}{3}}b^2-a^{-\frac{8}{3}}b^3, \text{ \&c. } ad\ infin.$$

The coefficients, obtained as before, are

$$1,\ \tfrac{1}{3},\ \tfrac{1}{3}\times-\tfrac{2}{3}\div 2=-\tfrac{2}{3.6},\ -\tfrac{2}{3.6}\times-\tfrac{5}{3}\div 3=\tfrac{2.5}{3.6.9}, \text{ \&c.}$$

Or - - $1+\frac{1}{3}-\frac{2}{3.6}+\frac{2.5}{3.6.9}$, &c., *ad infin.*

Hence - $(a-b)^{\frac{1}{3}}=a^{\frac{1}{3}}-\dfrac{b}{3a^{\frac{2}{3}}}-\dfrac{2b^2}{3.6a^{\frac{5}{3}}}-\dfrac{2.5b^3}{3.6.9a^{\frac{8}{3}}}$, &c., *ad infin.*

EXAMPLES.

1. Expand by the Binomial Theorem $(a+b)^{\frac{1}{2}}$.

Diminishing the exponents $\frac{1}{2}$ successively by 1, the exponents of a, in the successive terms, are $\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{3}{2}$, $-\frac{5}{2}$, &c. (Art. 271.)

The coefficients obtained by the general theorem (Art. 271) are as follows: $1, \frac{1}{2}, \frac{1}{2}\times-\frac{1}{2}\div 2=-\frac{1}{8}, -\frac{1}{8}\times-\frac{3}{2}\div 3=\frac{1}{16}$.

2. Expand $(a+b)^{\frac{2}{3}}$.
3. Expand $(1+x)^{\frac{1}{3}}$.
4. Expand $(a+b)^{\frac{3}{4}}$.
5. Expand $(a-b)^{\frac{1}{3}}$.
6. Expand $(a-x)^{-\frac{1}{2}}$.
7. Expand $b(a^2-b)^{-\frac{2}{3}}$.
8. Expand $(a^3-bx)^{-\frac{1}{2}}$.
9. Expand $\sqrt{2}=(1+1)^{\frac{1}{2}}$.

CALCULUS OF RADICALS.

296. A *radical quantity* is the indicated root of an imperfect power; as, $\sqrt{2}$, $\sqrt{a}$, and $\sqrt[3]{b}$.

Radicals are *similar* when they are composed of the same numbers or letters, placed under the same radical sign or index. Thus, $\sqrt{a}$, $4\sqrt{a}$, and $a\sqrt{a}$ are similar radicals.

Before entering upon equations of the higher degrees, we will consider some of the transformations that may be made upon algebraic expressions involving radicals.

CASE I.

297. *To reduce a rational number to the form of a radical.*

RULE.

1. *Involve the given number to a power of the same name as the root.*

2. *Apply the corresponding radical sign or index to the power thus produced.*

1. Reduce $3a$ to the form of the fourth root.

$3a$ involved to the fourth power equals $81a^4$.

Applying the radical sign, $3a=\sqrt[4]{81a^4}$; or applying the fractional index, $3a=(81a^4)^{\frac{1}{4}}$.

2. Reduce $5a^2b$ to the form of the third root.

Ans. $\sqrt[3]{125a^6b^3}$, or $(125a^6b^3)^{\frac{1}{3}}$.

3. Reduce $2ax^3y^4$ to the form of the eighth root.

Ans. $\sqrt[8]{256a^8x^{24}y^{32}}$, or $(256a^8x^{24}y^{32})^{\frac{1}{8}}$.

4. Reduce $\frac{1}{3}a^2cx^5$ to the form of the fourth root.

Ans. $\sqrt[4]{\frac{1}{81}a^8c^4x^{20}}$, or $(\frac{1}{81}a^8c^4x^{20})^{\frac{1}{4}}$.

5. Reduce $\dfrac{3a^2bx}{4a^4d^3y}$ to the form of the third root.

Ans. $\sqrt[3]{\dfrac{27a^6b^3x^3}{64c^{12}d^9y^3}}$, or $\left(\dfrac{27a^6b^3x^3}{64c^{12}d^9y^3}\right)^{\frac{1}{3}}$.

CASE II.

298. *To introduce a rational coefficient under the radical sign or fractional index.*

We have already seen (Art. 290) that a part of the root may be removed from under radicals of the form $\sqrt[n]{a^nb}$. Thus,

$$\sqrt[n]{a^nb}=\sqrt[n]{a^n\times b}=\sqrt[n]{a^n}\times\sqrt[n]{b}=a\sqrt[n]{b}.$$

Now, by reversing this process,

$$a\sqrt{b}=\sqrt[n]{a^n}\times\sqrt[n]{b}=\sqrt[n]{a^n\times b}=\sqrt[n]{a^nb}.$$

Hence we have the following

RULE.

1. *Raise the rational coefficient to a power of the same name as the root indicated by the radical sign or fractional index.*

2. *Multiply the quantity under the radical sign or index by this power, and place the given radical sign or index over the product.*

EXAMPLES.

1. In the expression $3a\sqrt{2b}$, let the coefficient be introduced under the radical.

$$3a\sqrt{2b}=\sqrt{9a^2}\times\sqrt{2b}=\sqrt{9a^2\times 2b}=\sqrt{18a^2b}. \quad Ans.$$

2. In the expression $4a^2\sqrt[3]{3ax}$, let the coefficient be introduced under the radical. *Ans.* $\sqrt{192a^7x}$.

3. In the expression $2a^2(4xy)^{\frac{1}{3}}$, let the coefficient be introduced under the fractional index.

$$2a^2(4xy)^{\frac{1}{3}}=(8a^6)^{\frac{1}{3}}\times(4xy)^{\frac{1}{3}}=(8a^6\times 4xy)^{\frac{1}{3}}=(32a^6xy)^{\frac{1}{3}}. \quad Ans.$$

4. In the expression $6\sqrt[3]{13}$, let the coefficient be introduced under the radical. *Ans.* $\sqrt[3]{2808}$.

5. In the expression $2ab(2ab^2)^{\frac{1}{3}}$, let the rational cofficient be introduced under the fractional index. *Ans.* $(16a^4b^5)^{\frac{1}{3}}$.

6. In the expression $(a+b)\sqrt{ab}$, let the rational coefficient be introduced under the radical sign.

Ans. $\sqrt{(a+b)^2\times ab}=\sqrt{a^3b+2a^2b^2+ab^3}$.

7. In the expression $\frac{a}{b}\left(\frac{b^2c}{a^2+b^2}\right)^{\frac{1}{2}}$, let the rational coefficient be introduced under the fractional index.

Ans. $\left(\frac{a^2b^2c}{a^2b^2+b^4}\right)^{\frac{1}{2}}$.

CASE III.

299. *To reduce radicals of different indices to equivalent radicals having a common fractional index.*

RULE.

1. *Reduce the indices to a common denominator.*

2. *Involve each quantity to the power expressed by the numerator of the reduced index.*

3. *Take the root denoted by the denominator.*

EXAMPLES.

1. Reduce $a^{\frac{1}{4}}$ and $b^{\frac{1}{6}}$ to a common index.

$$a^{\frac{1}{4}}=a^{\frac{3}{12}}=(a^3)^{\frac{1}{12}}.$$
$$b^{\frac{1}{6}}=b^{\frac{2}{12}}=(b^2)^{\frac{1}{12}}.$$

Ans. $(a^3)^{\frac{1}{12}}$, and $(b^2)^{\frac{1}{12}}$.

2. Reduce $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ to a common index.

Ans. $8^{\frac{1}{6}}$, and $9^{\frac{1}{6}}$.

3. Reduce $\sqrt{\frac{1}{2}}$ and $\sqrt[4]{\frac{1}{4}}$ to a common index.

$$\sqrt{\tfrac{1}{2}}=(\tfrac{1}{2})^{\frac{1}{2}}=(\tfrac{1}{2})^{\frac{4}{8}}=((\tfrac{1}{2})^4)^{\frac{1}{8}}=(\tfrac{1}{16})^{\frac{1}{8}}.$$

$$\sqrt[4]{\tfrac{1}{4}}=(\tfrac{1}{4})^{\frac{1}{4}}=(\tfrac{1}{4})^{\frac{2}{8}}=((\tfrac{1}{4})^2)^{\frac{1}{8}}=(\tfrac{1}{16})^{\frac{1}{8}}.$$

4. Reduce $\left(\frac{a}{2}\right)^{\frac{1}{2}}$ and $\left(\frac{a}{2}\right)^{\frac{1}{3}}$ to a common index.

Ans. $\left(\frac{a^3}{8}\right)^{\frac{1}{6}}$ and $\left(\frac{a^2}{4}\right)^{\frac{1}{6}}$.

5. Reduce $\sqrt[3]{\frac{3}{4}}$, $\sqrt{\frac{1}{3}}$, and $\sqrt[4]{2}$ to a common index.

Ans. $\frac{1}{2}\sqrt[12]{1296}$, $\frac{1}{3}\sqrt[12]{6561}$, and $\sqrt[12]{8}$.

6. Reduce $\sqrt[3]{3\frac{1}{2}}$ and $\sqrt[4]{5\frac{1}{3}}$ to a common index.

Ans. $\sqrt[12]{150\frac{1}{16}}$, and $\sqrt[12]{151\frac{19}{27}}$.

CASE IV.

ADDITION OF RADICALS.

300. If the radicals are not similar, and cannot be made so by reduction, it is evident that the addition can only be expressed.

Thus, $\sqrt{a}+\sqrt{b}$ can be reduced to no simpler form.

301. If the radicals are similar, they may be added by the following

RULE.

Add the coefficients, and to their sum annex the common radical.

Note.—If the radicals are not similar, they may frequently be made so by reduction.

EXAMPLES.

1. Add $4\sqrt{ax}$, $2\sqrt{ax}$, $5\sqrt{ax}$, and $3\frac{1}{2}\sqrt{ax}$.

$$\begin{array}{r} 4\sqrt{ax} \\ 2\sqrt{ax} \\ 5\sqrt{ax} \\ 3\frac{1}{2}\sqrt{ax} \\ \hline 14\frac{1}{2}\sqrt{ax}. \end{array}$$

Ans. $14\frac{1}{2}\sqrt{ax}$.

2. Add $3a\sqrt[3]{\frac{a}{b}}$ and $5c\sqrt[3]{\frac{a}{b}}$. *Ans.* $(3a+5c)\sqrt[3]{\frac{a}{b}}$.

3. Add $\sqrt{8}$ and $\sqrt{32}$. *Ans.* $6\sqrt{2}$.

$$\sqrt{8}=\sqrt{4\times2}=\sqrt{4}\times\sqrt{2}=2\sqrt{2}.$$
$$\sqrt{32}=\sqrt{16\times2}=\sqrt{16}\times\sqrt{2}=4\sqrt{2}.$$
Ans. $6\sqrt{2}$.

4. Add $\sqrt{50}$ and $\sqrt{128}$. *Ans.* $13\sqrt{2}$.

5. Add $(36a^2y)^{\frac{1}{2}}$ and $(25y)^{\frac{1}{2}}$. *Ans.* $(6a+5)\sqrt{y}$.

6. Add $\sqrt[3]{54a^3}$ and $\sqrt[3]{128a^3}$. *Ans.* $7\sqrt[3]{2}$.

7. Add $\sqrt{\frac{3a^2}{5}}$ and $\sqrt{\frac{a^2}{15}}$. *Ans.* $4a\sqrt{\frac{1}{15}}$.

$$\sqrt{\frac{3a^2}{5}}=\sqrt{\frac{9a^2}{15}}=\sqrt{9a^2\times\tfrac{1}{15}}=\sqrt{9a^2}\times\sqrt{\tfrac{1}{15}}=3a\sqrt{\tfrac{1}{15}}.$$
$$\sqrt{\frac{a^2}{15}}=\sqrt{a^2\times\tfrac{1}{15}}=\sqrt{a^2}\times\sqrt{\tfrac{1}{15}}=a\sqrt{\tfrac{1}{15}}.$$
$$4a\sqrt{\tfrac{1}{15}}.$$

8. Add $\sqrt{\frac{50}{147}}$ and $\sqrt{\frac{100}{294}}$. *Ans.* $\frac{10}{21}\sqrt{6}$.

9. Add $\sqrt[3]{\frac{1}{3}}$ and $\sqrt[3]{\frac{3087}{13824}}$. *Ans.* $\frac{15}{24}\sqrt[3]{9}$.

10. Add $\sqrt[3]{b^4y}$ and $\sqrt[3]{by^4}$. *Ans.* $(b+y)\sqrt[3]{by}$.

11. Add $\sqrt[4]{32a^4b}$ and $2a\sqrt[4]{2b}$. *Ans.* $4a\sqrt[4]{2b}$.

12. Add $4(a+x)^{\frac{1}{2}}$ and $(4a^3b^2+4a^2b^2x)^{\frac{1}{2}}$.
Ans. $(4+2ab)\sqrt{a+x}$.

13. Add $\sqrt{\frac{3}{4}}$, $4\sqrt{12}$, and $3\sqrt{\frac{1}{12}}$. *Ans.* $9\sqrt{3}$.

14. Add $\sqrt[3]{192}$ and $\sqrt[3]{24}$. *Ans.* $6\sqrt[3]{3}$.

15. Add $3\sqrt[3]{16a^4b^3}$ and $ab\sqrt[3]{54a}$. *Ans.* $9ab\sqrt[3]{2a}$.

CASE V.

SUBTRACTION OF RADICALS.

302. If the radicals are not similar, and cannot be made so by reduction, the subtraction can only be expressed.

Thus, $\sqrt{a}-\sqrt{b}$ can be reduced to no simpler form.

303. If the radicals are similar, the subtraction may be performed by the following

RULE.

Subtract the coefficient of the subtrahend from the coefficient of the minuend, and to the difference annex the common radical.

Note.—If the radicals are not similar, they may frequently be made so by reduction.

EXAMPLES.

1. From $4ab\sqrt{cd}$ subtract $3ab\sqrt{cd}$. *Ans.* $ab\sqrt{cd}$.

2. From $\sqrt{50a^2}$ subtract $\sqrt{8a^2}$.

$$\sqrt{50a^2}=\sqrt{25a^2\times 2}=\sqrt{25a^2}\times\sqrt{2}=5a\sqrt{2}.$$
$$\sqrt{8a^2}=\sqrt{4a^2\times 2}=\sqrt{4a^2}\times\sqrt{2}=2a\sqrt{2}.$$
$$3a\sqrt{2}.\quad Ans.$$

3. From $\sqrt[3]{192}$ subtract $\sqrt[3]{24}$. *Ans.* $2\sqrt[3]{3}$.

4. From $11\sqrt{96a^2bc^3}$ subtract $3a\sqrt{24bc^3}$. *Ans.* $38ac\sqrt{6bc}$.

5. From $\sqrt{\frac{3}{5}}$ subtract $\sqrt{\frac{5}{27}}$. *Ans.* $\frac{4}{45}\sqrt{15}$.

$$\sqrt{\tfrac{3}{5}}=\sqrt{\tfrac{15}{25}}=\sqrt{\tfrac{1}{25}\times 15}=\sqrt{\tfrac{1}{25}}\times\sqrt{15}=\tfrac{1}{5}\sqrt{15}=\tfrac{9}{45}\sqrt{15}.$$
$$\sqrt{\tfrac{5}{27}}=\sqrt{\tfrac{15}{81}}=\sqrt{\tfrac{1}{81}\times 15}=\sqrt{\tfrac{1}{81}}\times\sqrt{15}=\tfrac{1}{9}\sqrt{15}=\tfrac{5}{45}\sqrt{15}.$$
$$\tfrac{4}{45}\sqrt{15}.$$

6. From $\sqrt{\frac{3}{4}}$ subtract $\sqrt{\frac{3}{9}}$. *Ans.* $\frac{1}{6}\sqrt{3}$.

7. From $3\sqrt[3]{\frac{2}{5}}$ subtract $\frac{1}{4}\sqrt[3]{400}$. *Ans.* $\frac{1}{10}\sqrt[3]{50}$.

8. From $4a\sqrt[3]{1250a^3}$ subtract $\frac{1}{2}\sqrt[3]{640a^6}$. *Ans.* $18a^2\sqrt{10}$.

9. From $\frac{1}{10}\sqrt[4]{567}$ subtract $\frac{1}{8}\sqrt[4]{112}$. *Ans.* $\frac{1}{20}\sqrt[4]{7}$.

10. From $\frac{1}{2}\sqrt[4]{81a^4}$ subtract $\frac{1}{4}\sqrt{9a^2}$. *Ans.* $\frac{3}{4}a$.

CASE VI.

MULTIPLICATION OF RADICALS.

304. If the quantities are not under the same radical sign, and are not roots of the same letters, the multiplication can only be indicated. But since the product of the roots is equal to the root of the product (Art. 156), and since the product of several factors, composed of the same letters or quantities, is obtained by taking one of the factors affected by an exponent equal to the sum of the exponents of the several factors (Art. 88), that is, $a^{\frac{1}{m}}\times a^{\frac{1}{n}}=a^{\frac{n}{mn}}\times a^{\frac{m}{mn}}=a^{\frac{n+m}{mn}}$, we have the following general

RULE.

1. *If the quantities are under the same radical sign or index, multiply them like rational quantities, and place the common radical sign over the product.*

2. *If the quantities are composed of the same letters or numbers, and are affected with different fractional exponents; add these exponents.*

3. *If the radicals have rational coefficients, multiply them, and prefix the product to the product of the radicals.*

EXAMPLES.

1. Multiply $2a\sqrt{3ax}$ by $3b\sqrt{3a^2cx}$.

$$2a\sqrt{3ax}\times 3b\sqrt{3a^2cx}=6ab\sqrt{3ax\times 3a^2cx}=6ab\sqrt{9a^3cx^2}.$$

$$6ab\sqrt{9a^3cx^2}=6ab\sqrt{9a^2x^2\times ac}=6ab\sqrt{9a^2x^2}\times\sqrt{ac}=6ab\times 3ax\sqrt{ac}$$
$$=18a^2bx\sqrt{ac}.\quad \textit{Ans.}$$

2. Multiply $(3a)^{\frac{1}{2}}$ by $(3a)^{\frac{1}{3}}$. *Ans.* $\sqrt[6]{243a^5}$.

$$(3a)^{\frac{1}{2}}=(3a)^{\frac{3}{6}}$$
$$(3a)^{\frac{1}{3}}=(3a)^{\frac{2}{6}}$$
$$(3a)^{\frac{5}{6}}=(243a^5)^{\frac{1}{6}}=\sqrt[6]{243a^5}.$$

3. Multiply $4\sqrt{2}$ by $\sqrt[3]{6}$. *Ans.* $4\sqrt[6]{288}$.

4. Multiply $5\sqrt{5}$ by $3\sqrt{8}$. *Ans.* $30\sqrt{10}$.

5. Multiply $2a\sqrt{a^2+b^2}$ by $-3a\sqrt{a^2+b^2}$.
Ans. $-6a^2(a^2+b^2)$.

6. Multiply $5a\sqrt[3]{a+x}$ by $4b\sqrt[3]{(a+x)^2}$. *Ans.* $20ab(a+x)$.

7. Multiply $\sqrt{\dfrac{2ab}{3c}}$ by $\sqrt{\dfrac{3ad}{2b}}$. *Ans.* $a\sqrt{\dfrac{d}{c}}$.

8. Multiply $(6a^2bc)^{\frac{1}{3}}$ by $(6a^2bc)^{\frac{3}{4}}$. *Ans.* $(6a^2bc)^{\frac{13}{12}}$.

9. Multiply $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, and $a^{\frac{1}{4}}$ together. *Ans.* $a^{\frac{13}{12}}$.

10. Multiply $7\sqrt[3]{18}$ by $5\sqrt[3]{4}$. *Ans.* $70\sqrt[3]{9}$.

11. Multiply $2a^2(a+b)^{\frac{1}{n}}$ by $6ac(a+b)^{\frac{1}{m}}$.
Ans. $12a^3c(a+b)^{\frac{m+n}{mn}}$.

12. Multiply $5a\sqrt[4]{3a^2}$ by $6\sqrt[4]{27a^2b}$. *Ans.* $90a^2\sqrt[4]{b}$.

13. Multiply $4\sqrt[3]{\frac{1}{16}}$ by $3\sqrt{8}$. *Ans.* $12\sqrt[6]{2}$.

14. Multiply $2\sqrt{\frac{12}{7}}$ by $\frac{1}{2}\sqrt[4]{\frac{2}{9}}$. *Ans.* $2\sqrt[4]{\frac{18}{441}}$.

15. Multiply $4a\sqrt{\frac{3a^2}{8b}}$ by $\frac{1}{2}a\sqrt{\frac{3}{2b}}$. *Ans.* $\frac{3a^3}{2b}$.

16. Multiply $a^{\frac{2}{3}}b^{\frac{3}{2}}$ by $a^{\frac{1}{3}}b^{\frac{2}{3}}$. *Ans.* $ab^{\frac{13}{6}}$.

17. Multiply $3a^{-\frac{1}{2}}b^2$ by $3a^{\frac{1}{2}}b^{-2}$. *Ans.* 9.

18. Multiply $7ax^{\frac{1}{4}}$ by $2bx^{\frac{4}{3}}$. *Ans.* $14abx^{\frac{19}{12}}$.

19. Multiply $6a^{\frac{3}{2}}x^{\frac{1}{2}}$ by $11a^{\frac{2}{3}}x$. *Ans.* $^{6}6a^{\frac{13}{6}}x^{\frac{3}{2}}$.

CASE VII.

DIVISION OF RADICALS.

305. If the numbers are not under the same radical sign, nor roots of the same letters, the division can only be indicated.

306. But in those two cases it may be performed by the following general

RULE.

1. *If the quantities are under the same radical sign or index, divide them like rational quantities, and place the common radical sign over the quotient.*

2. *If the quantities are composed of the same letters or numbers, and affected with different fractional exponents, subtract the exponent of the divisor from that of the dividend.*

3. *If the radicals have rational coefficients, divide the coefficient in the dividend by that in the divisor.*

EXAMPLES.

1. Divide $6\sqrt{12a^2b}$ by $3\sqrt{3ab}$. *Ans.* $4\sqrt{a}$.

$$6\sqrt{12a^2b} \div 3\sqrt{3ab} = \frac{6}{3}\sqrt{\frac{12a^2b}{3ab}} = 2\sqrt{4a} = 4\sqrt{a}.$$

2. Divide $14a^{\frac{2}{3}}$ by $7a^{\frac{1}{2}}$. *Ans.* $2a^{\frac{1}{6}}$.

$$14a^{\frac{2}{3}} \div 7a^{\frac{1}{2}} = 14a^{\frac{4}{6}} \div 7a^{\frac{3}{6}} = 2a^{\frac{4}{6}-\frac{3}{6}} = 2a^{\frac{1}{6}};$$

$$\text{Or, } 14\sqrt[3]{a^2} \div 7\sqrt{a} = 14\sqrt[6]{a^4} \div 7\sqrt[6]{a^3} = 2\sqrt[6]{a}.$$

3. Divide $4^{\frac{1}{2}}$ by $4^{\frac{1}{3}}$. *Ans.* $4^{\frac{1}{6}}$.

4. Divide $6\sqrt{54}$ by $3\sqrt{2}$. *Ans.* $6\sqrt{3}$.

5. Divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$. *Ans.* $2\sqrt[3]{4}$.

6. Divide $\sqrt{7}$ by $\sqrt[3]{7}$. *Ans.* $\sqrt[6]{7}$.

7. Divide $8\sqrt{108}$ by $2\sqrt{6}$. *Ans.* $12\sqrt{2}$.

8. Divide $(a^2b^2d^3)^{\frac{1}{6}}$ by $d^{\frac{1}{2}}$. *Ans.* $(ab)^{\frac{1}{3}}$.

9. Divide $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{1}{3}}$. *Ans.* $\sqrt{\frac{3}{2}}$, or $\frac{1}{2}\sqrt{6}$.

10. Divide $\frac{1}{2}\sqrt[3]{\frac{1}{2}}$ by $\frac{1}{3}\sqrt[3]{\frac{1}{3}}$. *Ans.* $\frac{3}{4}\sqrt[3]{12}$.

11. Divide $8a^{\frac{1}{3}}b^{\frac{2}{3}}$ by $2a^{\frac{1}{4}}b^{\frac{1}{2}}$. *Ans.* $4a^{\frac{1}{12}}b^{\frac{1}{2}}$.

12. Divide $21ab^{\frac{13}{6}}$ by $3a^{\frac{1}{3}}b^{\frac{2}{3}}$. *Ans.* $7a^{\frac{2}{3}}b^{\frac{3}{2}}$.

CASE VIII.

INVOLUTION OF RADICALS.

307. Let it be required to involve $6a\sqrt{3b^3}$ to the second power. By the definition of involution (Art. 25), we shall have $(6a\sqrt{3b^3})^2 = 6a\sqrt{3b^3} \times 6a\sqrt{3b^3}$; or, performing the multiplication, $(6a\sqrt{3b^3})^2 = 6a\sqrt{3b^3} \times 6a\sqrt{3b^3} = 36a^2\sqrt{9b^6} = 36a^2 \times 3b^3 = 108a^2b^3$.

Again, let it be required to involve $3a^{\frac{1}{2}}b^{\frac{3}{5}}$ to the third power.

$$(3a^{\frac{1}{2}}b^{\frac{3}{5}})^3 = 3a^{\frac{1}{2}}b^{\frac{3}{5}} \times 3a^{\frac{1}{2}}b^{\frac{3}{5}} \times 3a^{\frac{1}{2}}b^{\frac{3}{5}} = 27a^{\frac{3}{2}}b^{\frac{9}{5}}.$$

308. The above operation will evidently apply to all cases of monomial radicals.

Hence we have the following general

RULE.

1. *Involve the coefficients to the required power.*

2. *If the number is under the radical sign, involve it as if it were rational; over the power place the radical sign, and then reduce the result to its simplest form.*

3. *If the number to be involved is affected by a fractional exponent, multiply the exponent of each letter by the index of the required power.*

EXAMPLES.

1. Required the third power of $3a\sqrt[3]{y}$. *Ans.* $27a^3y$.

2. Required the second power of $4a^{\frac{1}{3}}b\sqrt[3]{6b}$.

Ans. $16a^{\frac{2}{3}}b^2\sqrt{36b^2} = 96a^{\frac{2}{3}}b^2$.

3. Required the third power of $2a^{\frac{2}{3}}b^{\frac{1}{3}}\sqrt{8b^4x}$.

Ans. $1288a^2b^{\frac{6}{3}}\sqrt{2x}$.

4. Required the fourth power of $6\sqrt{\frac{1}{6}}$. *Ans.* 36.

5. Required the third power of $5\sqrt[4]{2a^2b}$.

Ans. $125a\sqrt[4]{8a^2b^3}$.

6. Required the third power of $3\sqrt{\frac{1}{2}}\times 2\sqrt[3]{2}$.

Ans. $216\sqrt{\frac{1}{2}}$.

CASE IX.

EVOLUTION OF RADICALS.

309. From the foregoing operations, the following rule will be sufficiently obvious:

RULE.

1. *Extract the root of the coefficient, if it is a complete power; if not, introduce it under the radical sign or index.*

2. *If the radical sign is used, multiply the figure over the foot of the radical by the index denoting the root to be taken.*

3. *If the fractional index is used, divide the index of each letter by the index of the required root.*

EXAMPLES.

1. Extract the square root of $16a^2b^4\sqrt{4x^6}$.

Ans. $4ab^2\sqrt{2x^3}$.

2. Extract the third root of $27a^6\sqrt[4]{2bd}$. *Ans.* $3a^2\sqrt[12]{2bd}$.

3. Extract the third root of $8a^{\frac{2}{3}}b^{\frac{3}{2}}$. *Ans.* $2a^{\frac{2}{9}}b^{\frac{1}{2}}$.

4. Extract the third root of $64a^{\frac{5}{2}}\sqrt{11b^3}$. *Ans.* $4a^{\frac{5}{6}}\sqrt[6]{11b^3}$.

5. Extract the square root of $24\sqrt[2]{3a}$. *Ans.* $2\sqrt[4]{108a}$.

6. Extract the cube root of $54a^{\frac{1}{2}}b^{\frac{3}{4}}\sqrt[3]{\frac{3}{4}}$. *Ans.* $3a^{\frac{1}{6}}b^{\frac{1}{4}}\sqrt[3]{6}$.

CASE X.

POLYNOMIALS HAVING RADICAL TERMS.

310. We will give, for the exercise of the learner, some examples of polynomials having one or more of their terms radical quantities. These examples may be solved by an application of the rules laid down in the preceding cases.

EXAMPLES.

1. Required the second power of $a+\sqrt{y}$.

Ans. $a^2+2a\sqrt{y}+y$.

2. Required the third power of $a-\sqrt{b}$.

Ans. $a^3-3a^2\sqrt{b}+3ab-\sqrt{b^3}$.

3. Required the second power of $\sqrt{3a}+\sqrt{2x}$.

Ans. $3a+\sqrt{6ax}+2x$.

4. Required the square root of $a+2\sqrt{ab}+b$.

Ans. $\sqrt{a}+\sqrt{b}$.

5. Required the square root of $9a+36\sqrt{3ax}+108x$.

Ans. $3\sqrt{a}+6\sqrt{3x}$.

6. Required the cube root of $a^3+3a^2\sqrt[3]{x}+3a\sqrt[3]{x^2}+x$.

Ans. $a+\sqrt[3]{x}$.

CASE XI.

BINOMIAL AND TRINOMIAL SURDS.

311. Expressions under this form, $\sqrt{a}+\sqrt{b}$, or $a+\sqrt{b}$, are called binomial surds, and may be reduced to rational quantities on the principle that *the product of the sum and difference of two quantities is equal to the difference of their squares.*

Thus the binomial surd $\sqrt{a}+\sqrt{b}$

Multiplied by - - $\sqrt{a}-\sqrt{b}$

$$\begin{array}{l} a+\sqrt{ab} \\ \quad -\sqrt{ab}+b \\ \hline a \qquad\quad +b \end{array}$$

Gives - - - $a+b$, a rational quantity.

312. Trinomial surds may be reduced, first, to binomial surds, then to rational quantities. Thus,

The trinomial surd - $\sqrt{a}+\sqrt{b}-\sqrt{c}$

Multiplied by - - $\sqrt{a}-\sqrt{b}+\sqrt{c}$

$$\begin{array}{lll} a+\sqrt{ab}-\sqrt{ac} & & \\ \quad -\sqrt{ab} & -b+\sqrt{bc} & \\ \qquad\qquad +\sqrt{ac} & +\sqrt{bc}-c & \\ \hline a & -b+2\sqrt{bc}-c & \end{array}$$

Gives - - - $a-b+2\sqrt{bc}-c$.

Let $x=a-b-c$; then we shall have

$$
\begin{array}{ll}
 & x+2\sqrt{bc} \\
\text{Multiplying by} & x-2\sqrt{bc} \\
\hline
 & x^2+2x\sqrt{bc} \\
 & \quad -2x\sqrt{bc}-4bc \\
\hline
 & x^2 \qquad\qquad -4bc \\
\text{Restoring value of } x & (a-b-c)^2-4bc.
\end{array}
$$

3. Find a factor which will make $1+\sqrt{2}$ rational.

$$
\begin{array}{l}
1+\sqrt{2} \\
1-\sqrt{2} \\
\hline
1+\sqrt{2} \\
\quad -\sqrt{2}-2 \\
\hline
1 \qquad -2=-1.
\end{array}
$$

Hence the factor is $1-\sqrt{2}$.

4. Find a factor which will make $\sqrt{10}-\sqrt{2}-\sqrt{3}$ rational.

$$
\begin{array}{ll}
 & \sqrt{10}-\sqrt{2}-\sqrt{3} \\
\text{Multiplying by} & \sqrt{10}+\sqrt{2}+\sqrt{3} \\
\hline
 & 10-\sqrt{20}-\sqrt{30} \\
 & -2+\sqrt{20} \qquad -\sqrt{6} \\
 & -3 \qquad +\sqrt{30}-\sqrt{6} \\
\hline
 & 5 \qquad\qquad -2\sqrt{6} \\
\text{Multiplying by} & 5 \qquad\qquad +2\sqrt{6} \\
\hline
 & 25 \qquad\qquad -10\sqrt{6} \\
 & \qquad\qquad +10\sqrt{6}-24 \\
\hline
 & 25 \qquad\qquad\qquad -24=1.
\end{array}
$$

Hence the factors are $\sqrt{10}+\sqrt{2}+\sqrt{3}$, and $5+2\sqrt{6}$.

5. Find a factor which will make $3-2\sqrt{2}$ rational.

6. Find a factor which will make $\sqrt{6}+3\sqrt{2}-\sqrt{5}$ rational.

313. By the above process, fractions may be cleared from

radical numerators or denominators without altering the value of the fraction, and thus the process of extracting the root be facilitated by confining it either to the numerator or denominator.

1. Let it be required to extract the square root of the fraction $\frac{a}{b}$.

$$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{a}\times\sqrt{a}}{\sqrt{b}\times\sqrt{a}}=\frac{a}{\sqrt{ab}}.$$

2. Extract the square root of the fraction $\frac{a+b}{xy}$.

$$\sqrt{\frac{a+b}{xy}}=\frac{\sqrt{a+b}}{\sqrt{xy}}=\frac{\sqrt{a+b}\times\sqrt{a+b}}{\sqrt{xy}+\sqrt{a+b}}=\frac{a+b}{\sqrt{axy+bxy}}.$$

3. Extract the square root of $\frac{5}{8}$.

$$\sqrt{\frac{5}{8}}=\frac{\sqrt{5}}{\sqrt{8}}=\frac{\sqrt{5}\times\sqrt{5}}{\sqrt{8}\times\sqrt{5}}=\frac{\sqrt{25}}{\sqrt{40}}=\frac{5}{2\sqrt{10}}.$$

4. Reduce the fraction $\frac{\sqrt{2}}{3-\sqrt{2}}$ to an equivalent fraction having a rational denominator.

Ans. $\frac{2+3\sqrt{2}}{7}$.

5. Reduce the fraction $\frac{3}{\sqrt{5}-\sqrt{2}}$ to an equivalent fraction having a rational denominator.

Ans. $\frac{\sqrt{5}+\sqrt{2}}{1}$.

6. Reduce the fraction $\frac{6}{5^{\frac{1}{4}}}$ to an equivalent fraction having a rational denominator.

Ans. $\frac{6\sqrt[4]{125}}{5}$.

7. Reduce the fraction $\frac{8}{\sqrt{3}+\sqrt{2}+1}$ to an equivalent fraction having a rational denominator.

Ans. $4-2\sqrt{6}+2\sqrt{2}$.

8. Reduce the fraction $\frac{\sqrt{10}-\sqrt{12}+\sqrt{5}}{12}$ to an equivalent fraction having a rational numerator.

Ans. $\frac{191}{84\sqrt{10}-20\sqrt{6}+72\sqrt{3}+204\sqrt{5}}$.

CASE XII.

ROOTS OF BINOMIAL SURDS OF THE FORM $a\pm\sqrt{b}$.

314. It is proposed to obtain a formula for extracting the square roots of expressions in the form of $a\pm\sqrt{b}$.

Let - $\sqrt{a+\sqrt{b}}=x+\sqrt{y}$ (1).

Then - $\sqrt{a-\sqrt{b}}=x-\sqrt{y}$ (2).

Squaring both equations,

$a+\sqrt{b}=x^2+2x\sqrt{y}+y$ (3).

$a-\sqrt{b}=x^2-2x\sqrt{y}+y$ (4).

Adding - $2a=2x^2+2y$ (5).

And - - $a=x^2+y$ (6).

Multiplying the first equation by the second,

$\sqrt{a^2-b}=x^2-y.$ (7).

Adding the sixth and seventh equations,

$a+\sqrt{a^2-b}=2x^2$ (8).

Reducing - $x=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}$ (9).

Subtracting the seventh from the sixth equations,

$a-\sqrt{a^2-b}=2y$ (10).

Reducing - $\sqrt{y}=\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$ (11).

Substituting these values of x and $\sqrt{y}$ in the first and second equations,

$$\sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

$$\sqrt{a-\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}-\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

Or, letting $c=\sqrt{a^2-b}$.

1. $$\sqrt{a+\sqrt{b}}=\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}} \quad \text{(A).}$$

2. $$\sqrt{a-\sqrt{b}}=\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}} \quad \text{(B).}$$

EXAMPLES.

1. Extract the square root of $3+2\sqrt{2}$.

Here $a=3$, and $\sqrt{b}=2\sqrt{2}=\sqrt{8}$, or $b=8$, and $c=\sqrt{9-8}=1$.

$$3+2\sqrt{2}=3+\sqrt{8}$$

$$\sqrt{3+\sqrt{8}}=\sqrt{\frac{3+1}{2}}+\sqrt{\frac{3-1}{2}}$$

$$\sqrt{3+2\sqrt{2}}=\sqrt{2}+1. \quad \textit{Ans.}$$

2. Extract the square root of $14-6\sqrt{5}$. *Ans.* $3-\sqrt{5}$.
3. Extract the square root of $11+6\sqrt{2}$. *Ans.* $3+\sqrt{2}$.
4. Extract the square root of $7-2\sqrt{10}$. *Ans.* $\sqrt{5}-\sqrt{2}$.
5. Extract the square root of $94+42\sqrt{5}$. *Ans.* $7+3\sqrt{5}$.
6. Extract the square root of $1+4\sqrt{-3}$.

Ans. $2+\sqrt{-3}$.

7. Extract the square root of $28+10\sqrt{3}$. *Ans.* $5+\sqrt{3}$.
8. Extract the square root of $ab+4c^2-d^2+2\sqrt{4abc^2-abd}$.

Ans. $\sqrt{ab}+\sqrt{4c^2-d^2}$.

9. Find the sum of $\sqrt{16+30\sqrt{-1}}+\sqrt{16-30\sqrt{-1}}$.

Ans. 10.

10. Find the sum of $\sqrt{bc+2b\sqrt{bc-b^2}}+\sqrt{bc-2b\sqrt{bc-b^2}}$.

Ans. $2b$

SECTION VII.

Equations of the Second Degree.

EQUATIONS EXCEEDING THE FIRST DEGREE.

315. The questions heretofore discussed involved only the first power of the unknown quantity, or, if a higher power ever appeared, it was cancelled in the process of the reduction. The enunciation of other questions, however, frequently requires a *power* or *root* of the unknown quantity, and for the solution of such cases we must seek for methods different from any heretofore discussed.

316. Equations of this nature are divided into two classes, viz., *pure* or *incomplete* equations, and *affected* or *complete* equations.

317. A *pure equation* is one which, when reduced to its simplest form, involves only *one power* or *root* of the unknown quantity. Thus,

$x=q$ is a pure equation of the first degree;

$x^2=q^2$ is a pure equation of the second degree, or a pure quadratic equation;

$x^3=q^3$ is a pure equation of the third degree, or a pure cubic equation;

$x^4=q^4$ is a pure equation of the fourth degree, or a pure biquadratic equation, &c.

$\sqrt{x}$, or $x^{\frac{1}{2}}=q^{\frac{1}{2}}$, is a sub-quadratic equation;

$\sqrt[3]{x}$, or $x^{\frac{1}{3}}=q^{\frac{1}{3}}$, is a sub-cubic equation;

$\sqrt[4]{x}$, or $x^{\frac{1}{4}}=q^{\frac{1}{4}}$, is a sub-biquadratic equation, &c.

318. An *affected equation* is one which, when reduced to its simplest form, involves *different powers* or *roots* of the unknown quantity. Thus,

$x^2+px=q$ is an affected quadratic equation;

$x^3+x^2+px=q$ is an affected cubic equation, &c.

PURE EQUATIONS.

319. Pure equations may be readily solved on the principles, 1. *If the same root of both members of an equation be extracted, the results will be equal.* 2. *If both members be involved to the same power, the results will be equal* (Art. 158).

Let us take the equation - - - $x^n = q^n$;
Extracting the nth root - - - - $x = q$.
Again, the equation - - - - $\sqrt[n]{x} = \sqrt[n]{q}$;
Involving to the nth power - - - $x = q$.

Hence, for the reduction of pure equations, we have the following general

RULE.

1. *Reduce the equation to such a form that the power or root of the unknown quantity may stand by itself in the first, and the known quantity by itself in the second member of the equation.*

2. *If the expression containing the unknown quantity is a power, extract the corresponding root of both members.*

3. *If the expression containing the unknown quantity is a root, involve both members to a power of the same name.*

Note 1.—If the *even* root, *i. e.*, the second, fourth, or sixth root of an equation, is to be taken, the resulting second member should be affected by the double sign $\pm$. Thus, the square root of $q^2 = \pm q$; for $+q \times +q = +q^2$, and $-q \times -q = +q^2$: the fourth root of $q^4 = \pm q$; for $q \times q \times q \times q = q^4$, and $-q \times -q \times -q \times -q = +q^4$: and the sixth root of $q^6 = \pm q$; for $q.q.q.q.q.q = q^6$, and $-q \times -q \times -q \times -q \times -q \times -q = q^6$, &c.

Note 2.—Since, if the even root be taken on both sides of the equation, it would be very natural to suppose that the first member, or x, should be affected with the double sign $\pm$, as well as the second member of the equation. Affecting it thus, and arranging the signs in the equation, $\pm x = \pm q$, in every possible manner, we shall have the four equations,

(1). $+x = +q$.	(3). $-x = +q$.
(2). $+x = -q$.	(4). $-x = -q$.

But still we have in reality no more than the first two equations, as the third equation expresses the same relations with the second; for, changing the signs, it becomes

(2). $+x=-q$:

And the fourth expresses the same relations with the first; for, changing the signs, it becomes

(1). $+x=+q$.

EXAMPLES.

1. Find the value of x in the equation $\frac{3x^2}{4}-10=2$.

$$\frac{3x^2}{4}-10=2;$$

Clearing of fractions - - $3x^2-40=8$;

Transposing and reducing - $3x^2=48$;

Dividing - - - - - $x^2=16$;

Evolving - - - - $x=\pm 4$.

Verifying on the supposition that $x=+4$;

$$\frac{3.16}{4}-10=12-10=2.$$

Verifying on the supposition that $x=-4$;

$$\frac{3\times -4^2}{4}-10=\frac{3.16}{4}=12-10=2.$$

2. Find the value of x in the equation $\sqrt{x-16}=8-\sqrt{x}$.

$$\sqrt{x-16}=8-\sqrt{x};$$

Involving both members - $x-16=64-16\sqrt{x}+x$;

Cancelling and transposing $16\sqrt{x}=80$;

Dividing - - - - $\sqrt{x}=5$;

Involving - - - - $x=25$.

Verification - - $\sqrt{25-16}=8-\sqrt{25}$;

Or - - - - - - $\sqrt{9}=8-5=3$.

3. Find the value of x in the equation $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$.

Clearing the equation of fractions - $x^2-ax^2=x$;

Dividing by x - - - - - $x-ax=1$;

Resolving into factors - - - $(1-a).x=1$;

Dividing by the coefficient of x - $x=\dfrac{1}{1-a}$.

4. Find the value of x in the equation $\sqrt{x+a}=\dfrac{a+b}{\sqrt{x-a}}$.

Clearing of fractions - $\sqrt{x^2-a^2}=a+b$;

Involving - - - $x^2-a^2=a^2+2ab+b^2$;

Transposing and reducing - $x^2=2a^2+2ab+b^2$;

Evolving - - - - $x=\pm\sqrt{2a^2-2ab+b^2}$.

5. Find the value of x in the equation $\dfrac{\sqrt{x}+28}{\sqrt{x}+4}=\dfrac{\sqrt{x}+38}{\sqrt{x}+6}$.

Clearing of fractions - $x+34\sqrt{x}+168=x+42\sqrt{x}+152$;

Transposing - - - - $8\sqrt{x}=16$;

Dividing - - - - - $\sqrt{x}=2$;

Involving - - - - - $x=4$.

6. Find the value of x in the equation $\dfrac{\sqrt{ax}-b}{\sqrt{ax}+b}=\dfrac{3\sqrt{ax}-2b}{3\sqrt{ax}+5b}$.

Clearing of fractions,

$$3ax+2b\sqrt{ax}-5b^2=3ax+b\sqrt{ax}-2b^2;$$

Transposing - - - $b\sqrt{ax}=3b^2$;

Dividing by b - - - $\sqrt{ax}=3b$;

Involving - - - - $ax=9b^2$;

Dividing - - - - $x=\dfrac{9b^2}{a}$.

7. Find the value of x in the equation $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\frac{3}{2}\sqrt{\dfrac{x}{x+\sqrt{x}}}$.

Multiplying by $\sqrt{x+\sqrt{x}}$ - $x+\sqrt{x}-\sqrt{x^2-x}=\dfrac{3\sqrt{x}}{2}$;

Transposing and reducing - - $x-\dfrac{\sqrt{x}}{2}=\sqrt{x^2-x}$;

Dividing by $\sqrt{x}$ - - - - $\sqrt{x}-\frac{1}{2}=\sqrt{x-1}$;

Involving - - - - - $x-\sqrt{x}+\frac{1}{4}=x-1$;

Transposing and reducing - - - $\sqrt{x}=\frac{5}{4}$;

Involving - - - - - - $x=\frac{25}{16}$.

8. Find the value of x in the equation $x+\sqrt{a^2+x^2}=\dfrac{2a^2}{\sqrt{a^2+x^2}}$

Clearing of fractions - $x\sqrt{a^2+x^2}+\ a^2+x^2=2a^2$;

Transposing and reducing, $x\sqrt{a^2+x^2}=2a^2-a^2-x^2=a^2-x^2$;

Involving - - $x^2(a^2+x^2)=\ a^4-2a^2x^2+x^4$;

Multiplying factors - - $a^2x^2+x^4=\ a^4-2a^2x^2+x^4$;

Transposing and reducing $3a^2x^2=\ a^4$;

Dividing and evolving - - $x=\dfrac{a}{\sqrt{3}}$, or $a\sqrt{\frac{1}{3}}$.

9. Find the value of x in the equation $3x^2-29=\dfrac{x^2}{4}+510$.

Ans. $x=14$.

10. Find the value of x in the equation $\sqrt{x-32}=\sqrt{x}-\frac{1}{2}\sqrt{32}$.

Ans. $x=50$.

11. Find the value of x in the equation $\sqrt{\dfrac{20x^2-9}{4x}}=\sqrt{x}$.

Ans. $x=\frac{3}{4}$.

12. Find the value of x in the equation $\sqrt{x}+\sqrt{3+x}=\dfrac{6}{\sqrt{3+x}}$.

Ans. $x=1$.

13. Find the value of x in the equation $x+\sqrt{9+x^2}=\dfrac{18}{\sqrt{9+x^2}}$.

Ans. $x=\sqrt{3}$.

14. Find the value of x in the equation

$$x+2=\sqrt{4+x\sqrt{64+x^2}}.$$

Ans. $x=6$.

15. Find the value of x in the equation $\sqrt[3]{2x^3+9x^2+27x}=x+3$.

Ans. $x=3$.

16. Find the value of x in the equation $\sqrt{x-32}=16-\sqrt{x}$.

Ans. $x=81$.

17. Find the value of x in the equation $\dfrac{\sqrt{6x}-2}{\sqrt{6x}+2}=\dfrac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$

Ans. $x=6$.

18. Find the value of x in the equation $\sqrt[3]{x^3-a^3}=\sqrt[3]{3ax^2-3a^2x+8b}$.

Ans. $x=a+2\sqrt[3]{b}$.

19. Find the value of x in the equation $a^2-2ax+x^2=b$.

Ans. $x=a\mp\sqrt{b}$.

20. Find the value of x in the equation $\dfrac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b$.

Multiplying both numerator and denominator of the first member by $a-\sqrt{a^2-x^2}$.

$$\frac{(a-\sqrt{a^2-x^2})^2}{a^2-a^2+x^2}=b.$$

Reducing and clearing of fractions - - - $(a-\sqrt{a^2-x^2})^2=bx^2$;

Evolving - - - - - $a-\sqrt{a^2-x^2}=\pm x\sqrt{b}$;

Transposing - - - - $a\mp x\sqrt{b}=\sqrt{a^2-x^2}$;

Involving - - - - $a^2\mp 2ax\sqrt{b}+bx^2=a^2-x^2$;

Cancelling and transposing - - $bx^2+x^2=\pm 2ax\sqrt{b}$;

Dividing by x - - - - $bx+x=\pm 2a\sqrt{b}$;

Resolving into factors - - $(b+1)x=\pm 2a\sqrt{b}$;

Dividing - - - - - - $x=\dfrac{\pm 2a\sqrt{b}}{b+1}$.

21. Find the value of x in the equation $\dfrac{\sqrt{x}+\sqrt{x-a}}{\sqrt{x}-\sqrt{x-a}}=\dfrac{n^2a}{x-a}$.

Multiplying the numerator and denominator of the first fraction by $\sqrt{x}-\sqrt{x-a}$.

$$\frac{x-(x-a)}{(\sqrt{x}-\sqrt{x-a})^2}=\frac{n^2a}{x-a};$$

Whence - $\dfrac{1}{(\sqrt{x}-\sqrt{x-a})^2}=\dfrac{n^2}{x-a}$.

Evolving - - $\dfrac{1}{\sqrt{x}-\sqrt{x-a}}=\dfrac{\pm n}{\sqrt{x-a}}$;

Clearing of fractions - $\sqrt{x-a}=\pm n(\sqrt{x}-\sqrt{x-a})$;

Multiplying by $\sqrt{x-a}$ - $x-a=\pm n(\sqrt{x^2-ax}-x+a)$,

Or - - - - $x-a=\pm n\sqrt{x^2-ax}-n(x-a)$;

Transposing - $(x-a)+n(x-a)=\pm n\sqrt{x^2-ax}$;

Resolving into factors . . . $(1+n).(x-a)=\pm n\sqrt{x^2-ax}$;

Involving . . . $(1+n)^2.(x-a)^2=n^2(x^2-ax)=n^2x(x-a)$;

Dividing by $x-a$, $(1+n)^2.(x-a)=n^2x$,

Or . . . $(1+n)^2x-(1-n)^2a=n^2x$;

Transposing . . . $(1+n)^2x-n^2x=(1+n)^2a$,

Or . . . $(1+2n+n^2)x-n^2x=(1+n)^2a$;

Adding coefficients of x . . . $(1+2n+n^2-n^2)x=(1+n)^2a$;

Reducing . . . $(1+2n)x=(1+n)^2a$;

Dividing by $1+2n$ . . . $x=\frac{(1+n)^2a}{1+2n}$. *Ans.*

22. Find the value of x in the equation $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b$.

Ans. $x=\frac{2ab}{b^2+1}$.

23. Find the value of x in the equation $\frac{1}{1-\sqrt{1-x^2}}-\frac{1}{1+\sqrt{1-x^2}}=\frac{\sqrt{3}}{x^2}$. *Ans.* $x=\pm\frac{1}{2}$.

24. Find the value of x in the equation $\sqrt{x^3+8}=\sqrt{125-6x^2-12x}$. *Ans.* $x=3$.

25. Find the value of x in the equation $\sqrt{\frac{a+x}{x}}+2\sqrt{\frac{a}{a+x}}=b^2\sqrt{\frac{x}{a+x}}$. *Ans.* $x=\frac{a}{(b\mp1)^2}$.

Examples of Pure Equations containing two or more unknown Quantities.

320. In equations of this kind, unknown quantities may be eliminated by the same principles that were applied in equations of the first degree.

EXAMPLES.

1. Find the values of x and y in the equations $x^2+y=28$, and $\frac{4x^2}{5}-\frac{y}{3}=19$. *Ans.* $x=5$, and $y=3$.

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$$x^2+y=28$$
$$\frac{4x^2}{5}-\frac{y}{3}=19$$

Transposing y, and extracting the square root of the first equation - - - $x=\sqrt{28-y}$

Clearing the second equation of fractions - - $12x^2-5y=285$

Transposing, &c. - - - $x^2=\frac{285+5y}{12}$

Evolving - - - - - $x=\sqrt{\frac{285+5y}{12}}$.

Forming a new equation of the two values of x,

$$\sqrt{28-y}=\sqrt{\frac{285+5y}{12}}$$
$$28-y=\frac{285+5y}{12}$$
$$336-12y=285+5y$$
$$-12y-5y=285-336$$
$$17y=51$$
$$y=3,\text{ and } x=5.$$

2. Find the values of x and y in the equations $3y=x+y$, and $xy=18$. *Ans.* $x=\pm 6$, and $y=\pm 3$.

3. Find the values of x, y, and z in the equations $(x+y+z)^3=8000$, $y^2+2yz-36=64-z^2$, and $2xy+20z=200$. *Ans.* $x=10$, $y=8$, and $z=2$.

4. Find the values of x and y in the equations $5x-5y=4y$, and $x^2+4y^2=181$. *Ans.* $x=9$, and $y=5$.

5. Find the values of x, y, and z in the equations $x^3y=54$, $yz=8$, and $xz=12$. *Ans.* $x=3$, $y=2$, and $z=4$.

6. Find the values of x and y in the equations $x^2+y^2=\frac{13}{x-y}$, and $xy=\frac{6}{x-y}$. *Ans.* $x=3$ or -2, and $y=2$ or -3

$$x^2+y^2=\frac{13}{x-y};$$
$$xy=\frac{6}{x-y};$$

Multiplying the second equation by 2	$2xy=\frac{12}{x-y}$;	(3.)
Subtracting the third from the first	$x^2-2xy+y^2=\frac{1}{x-y}$;	(4.)
Contracting the first member	$(x-y)^2=\frac{1}{x-y}$;	(5.)
Clearing of fractions	$(x-y)^3=1$;	(6.)
Evolving	$x-y=1$;	(7.)
Substituting for $x-y$ its value in the first equation	$x^2+y^2=13$;	(8.)
Substituting for $x-y$ its value in the third equation	$2xy=12$;	(9.)
Adding the eighth and ninth equations	$x^2+2xy+y^2=25$;	
Evolving	$x+y=\pm 5$.	
But	$x-y=1$.	
Hence	$x=3$ or -2, and $y=2$ or -3.	

7. Find the values of x and y in the equations $x^{\frac{2}{3}}+y^{\frac{2}{3}}=13$, and $x^{\frac{1}{3}}+y^{\frac{1}{3}}=5$. *Ans.* $x=27$ or 8, and $y=8$ or 27.

8. Find the values of x and y in the equations $x^8+x^4y^4+y^8=273$, and $x^4+x^2y^2+y^4=21$.
Ans. $x=\pm 2$ or $\pm 2\sqrt{-1}$, and $y=\pm 1$ or $\pm\sqrt{-1}$.

9. Find the values of x and y in the equations $(x^2-y^2).(x-y)=3xy$, and $(x^4-y^4).(x^2-y^2)=45x^2y^2$.
Ans. $x=4$ or 2, and $y=2$ or 4.

10. Find the values of x and y in the equations $x^2y+xy^2=6$, and $x^3y^2+x^2y^3=12$. *Ans.* $x=2$ or 1, and $y=1$ or 2.

11. Find the values of x and y in the equations $\sqrt[4]{x}-\sqrt[4]{y}=3$, and $\sqrt[4]{x}+\sqrt[4]{y}=7$. *Ans.* $x=625$, and $y=16$.

12. Find the values of x and y in the equations $x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}=1009$, and $x^3+x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3=582193$.
Ans. $x=81$ or 16, and $y=16$ or 81.

PROBLEMS PRODUCING PURE EQUATIONS.

321.—1. What two numbers are those whose difference is to the greater as 2 to 9, and the difference of whose squares is 128? *Ans.* 18 and 14.

2. A fisherman being asked how many fish he had caught, replied, "If you add 14 to the number, the square root of the sum, diminished by 8, will equal nothing." How many had he caught? *Ans.* 50.

3. A merchant gains in trade a sum, to which $320 bears the same proportion as five times the sum does to $2500. What is the sum? *Ans.* $400.

4. What number is that, the fourth part of whose square being subtracted from 8, leaves a remainder equal to 4? *Ans.* 4.

5. It is required to divide the number 18 into two such parts, that the squares of these parts may be in the proportion of 25 to 16. *Ans.* 10 and 8.

6. It is required to divide the number 14 into two such parts, that the quotient of the greater part, divided by the less, may be to the quotient of the less, divided by the greater, as 16 to 9. *Ans.* 8 and 6.

7. Two persons, A and B, lay out some money on speculation. A disposes of his bargain for $11, and gains as much per cent. as B lays out; B gains $36, and it appears that A gains four times as much per cent. as B. Required the capital of each. *Ans.* A's $5, and B's $120.

8. A gentleman bought two pieces of silk, which together measured 36 yards. Each of them cost as many shillings by the yard as there were yards in the piece, and their whole prices were as 4 to 1. What were the lengths of the pieces? *Ans.* 24 and 12 yards.

9. A number of boys set out to rob an orchard, each having as many bags as there were boys in all, and each bag capable of containing as many apples as there were boys. They filled their bags, and found the whole number of apples was 1000. What was the number of boys? *Ans.* 10.

10. Several gentlemen made an excursion, each taking the same sum of money. Each had as many servants as there were gentlemen; the number of dollars which each had was double the number of all the servants; and the whole sum of money taken out was $1458. What was the number of gentlemen? *Ans.* 9

11. There is a rectangular field, whose length is to the breadth as 6 to 5. After planting one sixth of the whole, there remained 625 square yards. What are the dimensions of the field?

Ans. The sides are 30 and 25 yards.

12. There are two numbers, which are to each other as 3 to 2, and the difference of their fourth powers is to the sum of their cubes as 26 to 7. What are the numbers?

Ans. 6 and 4.

13. What two numbers are as 5 to 4, and the sum of whose cubes is 5103? *Ans.* 15 and 12.

14. There is a rectangular field containing 360 square rods, and whose length is to its breadth as 8 to 5. What is the length and breadth. *Ans.* Length 24, breadth 15.

15. There are two square fields, the larger of which contains 13941 square rods more than the smaller, and the proportion of their sides is as 15 to 8. What is the length of the sides? *Ans.*

16. Two travellers, A and B, set out to meet each other. They started at the same time, and travelled on the direct road between the two places; and on meeting, it appeared that A had travelled 18 miles more than B, and that A could have gone B's distance in $15\frac{3}{4}$ days, while B would have been 28 days in going A's distance. What was the distance travelled by each? *Ans.* A's 72, B's 54.

17. There are two men whose ages are to each other as 5 to 4, and the sum of the third power of their ages is 137781. What are their ages? *Ans.* 45 and 36 years.

18. Find two numbers, such that the second power of the greater, multiplied by the less, may be equal to 448; and

the second power of the less, multiplied by the greater, may be 392.

19. A man wishes to make a cellar that shall contain 31104 cubic feet, and in such a form that the breadth shall be twice the depth, and the length $1\frac{1}{3}$ the breadth. What must be the length, breadth, and depth?

Ans. Length 48, breadth 36, depth 18.

20. A man wishes to make a cistern that shall contain 500 gallons of wine, in such a form that the length shall be to the breadth as 5 to 4, and the depth to the length as 2 to 5. Now, allowing 231 cubic inches for one wine gallon, what will be the length, breadth, and depth?

AFFECTED EQUATIONS OF THE SECOND DEGREE.

322. Let $2p$ and q be two variable numbers, $2p$ representing the coefficient of the unknown quantity, and q the known quantity; then, however complicated may be the equations which involve the first and second powers of the unknown quantity, they may be reduced to one of the four following forms:

(1.) $x^2+2px=q$. | (3.) $x^2+2px=-q$.

(2.) $x^2-2px=q$. | (4.) $x^2-2px=-q$.

Let us then determine the process by which equations of these forms may be solved.

323. We have already seen that a binomial cannot be a perfect square, and also that the root of a trinomial, which is a perfect square, may be formed by taking the root of the two terms that are complete powers, and connecting them by the sign of the other term (Art. 292). Thus, $\sqrt{x^2+2px+p^2}=x+p$, and $\sqrt{x^2-2px+p^2}=x-p$.

324. We have also seen that the square of a binomial is equal to the square of the first term, plus twice the product of the two terms, plus the square of the last term. Thus,

$$(x+p)^2=x^2+2px+p^2;$$

And the square of the residual, $x-p$, gives

$$(x-p)^2=x^2-2px+p^2.$$

Hence, if p^2 be added to both members of each of the preceding four forms of the affected quadratic equation, the first member of each will be a perfect square. Thus,

(5.) $x^2+2px+p^2=q+p^2$;
(6.) $x^2-2px+p^2=q+p^2$;
(7.) $x^2+2px+p^2=p^2-q$;
(8.) $x^2-2px+p^2=p^2-q$.

325. If we compare p^2 with the coefficient of x, it will be found equal to the square of half of it. Thus, $p^2=\left(\frac{2p}{2}\right)^2$. Hence, when the quadratic equation is reduced to the first, second, third, or fourth power, the first member may be rendered a perfect square by adding the square of half the coefficient of the first power of the unknown quantity to both members of the equation. This is called *completing the square.*

326. Each of the above equations may be reduced by extracting the square root of both members, and making the necessary transformations.

Extracting the square root of the (5), $x+p=\pm\sqrt{q+p^2}$;
Transposing - - - - - $x=-p\pm\sqrt{q+p^2}$.
Extracting the square root of the (6), $x-p=\pm\sqrt{q+p^2}$;
Transposing - - - - - $x=p\pm\sqrt{q+p^2}$.
Extracting the square root of the (7), $x+p=\pm\sqrt{p^2-q}$;
Transposing - - - - - $x=-p\pm\sqrt{p^2-q}$.
Extracting the square root of the (8), $x-p=\pm\sqrt{p^2-q}$;
Transposing - - - - - $x=p\pm\sqrt{p^2-q}$.

327. Hence, for the solution of affected quadratic equations, we have the following general

RULE.

1. *Reduce the equation to one of the above four forms.*

2. *Complete the square by adding to both members of the equation the square of half the coefficient of the first power of the unknown quantity.*

3. *Extract the square root of both members, observing to affect the second member with the double sign* $\pm$, *and complete the reduction by the preceding principles.*

Note.—Equations of this nature also give two values of the unknown quantity. Thus, $x^2+4x=12$.

Completing the square - $x^2+4x+4=12+4=16$.

Extracting the square root - $x+2=\pm\sqrt{16}=\pm 4$.

Transposing - - - - $x=-2\pm 4=2$ or -6.

EXAMPLES.

1. Find the values of x in the equation $3x^2+18x=81$.

$$3x^2+18x=81$$
$$x^2+6x=27$$
$$x^2+6x+9=27+9=36$$
$$x+3=\pm\sqrt{36}=\pm 6$$
$$x=-3\pm 6=3 \text{ or } -9.$$

2. Find the values of x in the equation $3x^2+2x-9=76$.
Ans. $x=5$, or $-5\frac{2}{3}$.

3. Find the values of x in the equation $\frac{x}{2}=\frac{36}{x+2}-4$.
Ans. $x=4$, or -14.

4. Find the values of x in the equation $x^2+48=426+12x-5x^2$. *Ans.* $x=9$, or -7.

5. Find the values of x in the equation $2x^2+12x=-16$.
Ans. $x=-2$, or -4.

6. Find the values of x in the equation $x^2-15x=-54$.
Ans. $x=9$, or 6.

7. Find the values of x in the equation $\frac{x^2}{2}+\frac{x}{4}=\frac{x^2}{5}-\frac{x}{10}+\frac{13}{20}$.
Ans. $x=1$, or $-2\frac{1}{6}$.

8. Find the values of x in the equation $\frac{x^2}{2}+\frac{x}{3}-15=\frac{x^2}{4}+x-14\frac{3}{4}$. *Ans.* $x=3$, or $-\frac{1}{3}$.

9. Find the values of x in the equation $4a^2-2x^2+2ax=18ab-18b^2$. *Ans.* $x=2a-3b$, or $-a+3b$.

10. Find the values of x in the equation $2ax-x^2=-2ab-b^2$. *Ans.* $2a+b$, or $-b$.

A SECOND METHOD OF COMPLETING THE SQUARE.

328. It is frequently impossible to clear the highest power of the unknown quantity from its coefficient without introducing fractional expressions into the equation. But, resuming the four forms of affected quadratic equations, and letting a represent the coefficient of x^2, we shall have

(1.) $ax^2+2px=q$; (3.) $ax^2+2px=-q$;
(2.) $ax^2-2px=q$; (4.) $ax^2-2px=-q$.

Multiplying each of these equations by $4a$, and adding the square of $2p$ to both members, we shall have

$$(5.)\quad 4a^2x^2+8apx+4p^2=4aq+4p^2;$$
$$(6.)\quad 4a^2x^2-8apx+4p^2=4aq+4p^2;$$
$$(7.)\quad 4a^2x^2+8apx+4p^2=-4aq+4p^2;$$
$$(8.)\quad 4a^2x^2-8apx+4p^2=-4aq+4p^2.$$

329. It is evident that the first member of each of these equations is a perfect square; hence, extracting the square root of both members, we shall have

$$(9.)\quad 2ax+2p=\pm\sqrt{4aq+4p^2};$$
$$(10.)\quad 2ax-2p=\pm\sqrt{4aq+4p^2};$$
$$(11.)\quad 2ax+2p=\pm\sqrt{-4aq+4p^2};$$
$$(12.)\quad 2ax-2p=\pm\sqrt{-4aq+4p^2}.$$

Or, reducing,

$$(13.)\quad x=\frac{-2p\pm\sqrt{4aq+4p^2}}{2a};$$
$$(14.)\quad x=\frac{+2p\pm\sqrt{4aq+4p^2}}{2a};$$
$$(15.)\quad x=\frac{-2p\pm\sqrt{-4aq+4p^2}}{2a};$$
$$(16.)\quad x=\frac{+2p\pm\sqrt{-4aq+4p^2}}{2a}.$$

330. Hence the square of an affected quadratic equation may be completed by the following general

RULE.

1. *Multiply both members of the equation by four times the*

coefficient of the second power of the unknown quantity, and add the square of the coefficient of the first power to both members ; the first member will then be a perfect square.

2. *Extract the root, and reduce as before.*

EXAMPLES.

1. Find the values of x in the equation $2x^2+3x=65$.

$$2x^2+3x=65.$$

Completing the square: $16x^2+24x+9=520+9=529$;

Evolving: $4x+3=\pm\sqrt{529}=\pm 23$;

$$4x=-3\pm 23=20, \text{ or } -26;$$

$$x=5, \text{ or } -6\tfrac{1}{2}.$$

2. Find the values of x in the equation $3x^2-9x-4=80$.

Ans. $x=7$, or -4.

3. Find the values of x in the equations $4x-\frac{36-x}{x}=46$.

Ans. $x=12$, or $-\frac{3}{4}$.

4. Find the values of x in the equation $x^2+\frac{2x}{3}-\frac{x}{2}=8+\frac{273}{12}-\frac{5x^2}{6}-\frac{3}{4}$.

Ans. $x=4$, or $-\frac{45}{11}$.

5. Find the values of x in the equation $2x^2+8x+7=\frac{5x}{4}-\frac{x^2}{8}+197$.

Ans. $x=8$, or $-11\frac{3}{17}$.

6. Find the values of x in the equation $\frac{x^2}{2}-\frac{x}{3}+7\frac{3}{8}=8$.

Ans. $x=1\frac{1}{2}$, or $-\frac{5}{6}$.

7. Find the values of x in the equation $\frac{8-x}{2}-\frac{2x-11}{x-3}=\frac{x-2}{6}$.

Ans. $x=6$, or $\frac{1}{2}$.

8. Find the values of x in the equation $5x^2+4x=273$.

Ans. $x=7$, or $-7\frac{4}{5}$.

PARTICULAR CASES OF AFFECTED QUADRATICS.

331. It is evident that every equation of the form

$$x^{2n}+2px^n=q$$

may be solved by the preceding rules; for, let $y=x^n$, and $y^2=x^{2n}$, and the above equation will become

$$y^2+2py=q;$$

Completing the square, $y^2+2py+p^2=q+p^2$;

Evolving and transposing - $y=-p\pm\sqrt{q+p^2}$;

Substituting the value of y - $x^n=-p\pm\sqrt{q+p^2}$;

Evolving - - - - $x=\sqrt[n]{-p\pm\sqrt{q+p^2}}$.

332. Equations also occur in the form

$$x^{\frac{2}{n}}+2px^{\frac{1}{n}}=q.$$

Let $y=x^{\frac{1}{n}}$, then $y^2=x^{\frac{2}{n}}$; and substituting these values,

$$y^2+2py=q;$$

Reducing - - - - $y=-p\pm\sqrt{q+p^2}$,

Or - - - - - - $x^{\frac{1}{n}}=-p\pm\sqrt{q+p^2}$;

Involving to the nth power - - $x=(-p\pm\sqrt{q+p^2})^n$.

These equations may be readily solved without the formality of substitution. Resume the equation

$$x^{2n}+2px^n=q;$$

Completing the square, $x^{2n}+2px^n+p^2=q+p^2$;

Evolving - - - - $x^n+p=\pm\sqrt{q+p^2}$;

Transposing - - - - $x^n=-p\pm\sqrt{q+p^2}$;

Evolving - - - - $x=\sqrt[n]{-p\pm\sqrt{q+p^2}}$.

Resume, also, the equation

$$x^{\frac{2}{n}}+2px^{\frac{1}{n}}=q;$$

Completing the square, $x^{\frac{2}{n}}+2px^{\frac{1}{n}}+p^2=q+p^2$;

Evolving - - - - $x^{\frac{1}{n}}+p=\pm\sqrt{q+p^2}$;

Transposing - - - - $x^{\frac{1}{n}}=-p\pm\sqrt{q+p^2}$;

Involving - - - - $x=(-p\pm\sqrt{q+p^2})^n$.

333. The same principles will apply also to all equations in which there are two terms, simple or compound, and the

exponent of one is double that of the other. Thus, in the equation

$$(x^2+2px+q)^2+(x^2+2px+q)=q',$$

Letting $y=x^2+2px+q$, and $y^2=(x^2+2px+q)^2$, we shall have then - - $y^2+y=q'$,

And - - - - $y=-\frac{1}{2}\pm\sqrt{q'+\frac{1}{4}}$;

Whence - $x^2+2px+q=-\frac{1}{2}\pm\sqrt{q'+\frac{1}{4}}$,

And - - - - $x=-p\pm\sqrt{p^2q-\frac{1}{2}\pm\sqrt{q'+\frac{1}{4}}}$;

Or, in the equation

$$(ax+2b)^2-2p(ax+2b)=q,$$

Letting $y=ax+2b$, and $y^2=(ax+2b)^2$, we shall have

$$y^2-2py=q,$$

And - - - - $y=p\pm\sqrt{q+p^2}$;

Whence - - $ax+2b=p\pm\sqrt{q+p^2}$,

And - - - - $x=\dfrac{p\pm\sqrt{q+p^2}-2b}{a}$.

These equations may also be solved without the formality of substitution.

334. If the indeterminate quantity $q=0$, the affected quadratic will assume the form

$$x^2\pm 2px=0.$$

This equation may be readily solved; for, dividing both members by x, we have

$$x\pm 2p=0;$$

Transposing - - $x=\mp 2p$.

335. Equations involving more than one unknown quantity, as $x^2y^2+2pxy=q$, or $(x^n+y^n)^2+2p(x^n+y^n)=q$, if the exponent of one term is double that of the other, may also be reduced to simpler forms by completing the square and performing the necessary transformations.

336. When there are two unknown quantities similarly involved in the equations, the work may be simplified by the introduction of two additional symbols which shall rep-

resent known functions of the unknown quantities. Thus, in the equations

$$x+y=a,$$
$$x^2+y^2=b;$$

Let $x+y=2s$, and $x-y=2z$; then $x=s+z$, and $y=s-z$.

And - $x^2=(s+z)^2=s^2+2sz+z^2$;

And - $y^2=(s-z)^2=s^2-2sz+z^2$;

Adding - - $x^2+y^2=2s^2+2z^2$;

Consequently, $b=2s^2+2z^2$;

Transposing and reducing - $z^2=\frac{b-2s^2}{2}$, and $z=\pm\sqrt{\frac{b-2s^2}{2}}$;

Hence - - - $x=s+\sqrt{\frac{b-s^2}{2}}$, and $y=s-\sqrt{\frac{b-2s^2}{2}}$;

Or $x=\frac{1}{2}a-\sqrt{\frac{b-4,\text{ or }\frac{1}{4}a^2}{2}}$, and $y=\frac{1}{2}a-\sqrt{\frac{b-4,\text{ or }\frac{1}{4}a^2}{2}}$.

Similar operations will reduce any equations of the same form;

As, $x+y=a$, and $x^3+y^3=b$; or $x+y=a$, and $x^4+y^4=b$; or $x+y=a$, and $x^5+y^5=b$, &c.

337. Again, let us take the equations

$$x+y=a;$$
$$\frac{x}{y}+\frac{y}{x}=b.$$

Clearing the second equation of fractions - $x^2+y^2=bxy$.

Let $x+y=2s$, and $x-y=2z$, as before; then $x=s+z$, and $y=s-z$.

Whence - $x^2=(s+z)^2=s^2+2sz+z^2$;

And - - $y^2=(s-z)^2=s^2-2sz+z^2$.

Adding, $x^2+y^2=2s^2+2z^2=b(s+z).(s-z)=b(s^2-z^2)=bs^2-bz^2$.

Whence - $z^2=\frac{bs^2-2s^2}{2+b}$; or $z=\pm\sqrt{\frac{(b-2)s^2}{b+2}}$;

Or, substituting and reducing,

$$x=s\pm\sqrt{\frac{(b-2)s^2}{b+2}}, \text{ and } y=s\mp\sqrt{\frac{(b-2)s^2}{b+2}}.$$

Restoring the value of s and s^2,

$$x=\tfrac{1}{2}a\pm\sqrt{\frac{(b-2)\frac{1}{4}a^2}{b+2}},\text{ and }y=\tfrac{1}{2}a\mp\sqrt{\frac{(b-2)\frac{1}{4}a^2}{b+2}}.$$

Similar operations will reduce any equations of the same form,

As, $x+y=a$, and $\frac{x^2}{y}+\frac{y^2}{x}=b$, &c.

There are a variety of expedients by which complicate equations may be simplified. The above cases will indicate some of the most general. Others must be left to exercise the skill and ingenuity of the learner.

AFFECTED EQUATIONS INVOLVING ONLY ONE UNKNOWN QUANTITY.

1. Find the values of x in the equation $6x+\frac{35-3x}{x}=44$.

Clearing of fractions - - $6x^2+35-3x=44x$;
Transposing - - - - - $6x^2-47x=-35$;
Completing the square and reducing - $x=7$, or $\frac{5}{6}$.

2. Find the values of x in the equation $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$. *Ans.* $x=4$, or -1.

3. Find the values of x in the equation $\frac{3x}{2}+\frac{2}{3x}=x+\frac{2x-2}{3}$.

Ans. $x=2\pm2\sqrt{2}$.

4. Find the values of x in the equation $3x-\frac{3x-10}{9-2x}=2+\frac{6x^2-40}{2x-1}$. *Ans.* $x=11\frac{1}{2}$, or 4.

5. Find the values of x in the equation $x^6-4x^3=621$.

Ans. $x=3$, or $\sqrt[3]{-23}$.

6. Find the values of x in the equation $\frac{x^6}{2}-\frac{x^3}{4}=-\frac{1}{32}$.

Ans. $x=\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{2}{8}}=\frac{1}{2}\sqrt[3]{2}$.

7. Find the values of x in the equation $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$.

Ans. $x=\frac{1}{8}$, or -8.

8. Find the values of x in the equation $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$.

Ans. $x=243$, or $(-28)^{\frac{5}{3}}$.

9. Find the values of x in the equation $(10+x)^{\frac{1}{2}}-(10+x)^{\frac{1}{4}}$ $=2$.

Ans. $x=6$.

10. Find the values of x in the equation $2(1+x-x^2)-(1+x-x^2)^{\frac{1}{2}}=-\frac{1}{9}$.

Ans. $x=\frac{1}{2}+\frac{1}{6}\sqrt{41}$.

11. Find the values of x in the equation $\sqrt[3]{x^3-a^3}=x-b$.

Ans. $x=\frac{b}{2}\pm\sqrt{\frac{4a^3-b^3}{12b}}$.

12. Find the values of x in the equation $2\sqrt{x-a}+3\sqrt{2x}=\frac{7a+5x}{\sqrt{x-a}}$.

Ans. $x=9a$.

13. Find the values of x in the equation $\frac{4x-5}{x}-\frac{3x-7}{3x+7}=\frac{9x+23}{13x}$.

Ans. $x=2$.

14. Find the values of x in the equation $\frac{3}{6x-x^2}+\frac{6}{x^2+2x}=\frac{11}{5x}$.

Ans. $x=3$.

AFFECTED EQUATIONS INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

1. Given - $x+\sqrt{xy}+y=19$ } to find x and y.
And - - $x^2+xy+y^2=133$ }

Dividing the second by the first equation - - - - $x-\sqrt{xy}+y=7$; (3.)

Adding the first and third equations - - - - $2x+2y=26$, (4.)

Or - - - - - - - - $x+y=13$; (5.)

Substituting, in the first equation, $\sqrt{xy}+13=19$; (6.)

Whence - - - - - - - $\sqrt{xy}=6$, (7.)

And - - - - - - - - $xy=36$; (8.)

Multiplying by 3 - - - - - $3xy=108$; (9.)

Subtracting the ninth from the second equation - - - $x^2-2xy+y^2=25$; (10.)

Evolving - - - - - - $x-y=\pm 5.$ (11.)

Adding the eleventh and fifth equations - - - - $2x=13\pm 5=18$, or 8;

Whence - - - - - $x=9$, or 4.

Subtracting the eleventh from the fifth equation - - - $2y=13\mp 5=8$, or 18;

Whence - - - - - $y=4$, or 9.

2. Given - $x^2+x+y=18-y^2$
And - - - $xy=6$ } to find x and y.

Transposing in the first equation - - $x^2+y^2+x+y=18$; (3.)

Multiplying the second equation by 2 - - - $2xy=12$; (4.)

Adding third and fourth equations $x^2+2xy+y^2+x+y=30$, (5.)

Or - - - - - $(x+y)^2+(x+y)=30$; (6.)

Completing the square - $(x+y)^2+(x+y)+\frac{1}{4}=30+\frac{1}{4}=\frac{121}{4}$; (7.)

Evolving - - - $x+y+\frac{1}{2}=\pm\frac{11}{2}$; (8.)

And - $x+y=\pm\frac{11}{2}-\frac{1}{2}=\frac{10}{2}$ or $-\frac{12}{2}=5$, or -6; (9.)

Whence, from the first equation - - $x^2+y^2=13$, or 24; (10.)

Subtracting the 4th from the 10th equation $x^2-2xy+y^2=1$, or 12; (11.)

Evolving - $x-y=\pm 1$, or $\pm\sqrt{12}=\pm 2\sqrt{3}$; (12.)

Adding 13th and 9th equations $2x=5\pm 1$, or $-6\pm 2\sqrt{3}$;

Whence - - - $x=3$ or 2, or $-3\pm\sqrt{3}$;

Subtract'g 13th from 9th equation $2y=4$ or 6, or $-6\mp 2\sqrt{3}$;

Whence - - - $y=2$ or 3, or $-3\mp\sqrt{3}$.

3. Given - - $4xy=96-x^2y^2$
And - - $x+y=6$ } to find x and y.

Ans. $x=4$ or 2, or $3\pm\sqrt{21}$; and $y=2$ or 4, or $3\mp\sqrt{21}$.

4. Given - - $x+y=8$; And - - $x^3+y^3=152$ } to find x and y.

Ans. $x=5$ or 3, and $y=3$ or 5.

5. Given - - $x+y=7$; And - - $x^4+y^4=641$ } to find x and y.

Ans. $x=5$ or 2, and $y=2$ or 5.

6. Given - - $x+y=6$; And - - $x^5+y^5=1056$ } to find x and y.

Ans. $x=4$ or 2, and $y=2$ or 4.

7. Given - - - $x^{\frac{2}{3}}y^{\frac{3}{2}}=2y^2$; And - - $8x^{\frac{1}{3}}-y^{\frac{1}{2}}=14$ } to find x and y.

Ans. $x=2744$ or 8, and $y=4$ or 9604.

PROBLEMS PRODUCING AFFECTED EQUATIONS.

1. It is required to divide the number 40 into two such parts that the sum of their squares shall be 818.

Ans. 23 and 17.

2. What two numbers are those whose difference is 9, and their sum, multiplied by the greater, produces 266?

Ans. 14 and 5.

3. An officer would arrange 1200 men in a solid body, so that each rank may exceed each file by 59 men. How many must be placed in rank, and how many in file?

Ans. Rank 75, file 16 men.

4. Some bees had alighted upon a tree; at one flight the square root of half of them went away; at another eight ninths of them; two bees then remained. How many alighted on the tree? *Ans.* 72.

5. A mercer bought a piece of silk for £16 4*s.*, and the number of shillings he paid per yard was to the number of yards as 4 to 9. How many yards did he buy, and what was the price per yard? *Ans.* 27 yards, at 12*s.* per yard.

6. There is a field in the form of a rectangular parallelogram, whose length exceeds the breadth by 16 yards, and it contains 960 square yards. Required the length and breadth.

Ans. Length 40, breadth 24 yards.

7. A person being asked his age, answered, "If you add

the square root of it to half of it, and subtract 12 from the sum, there will remain nothing." What was his age ?

Ans. 16.

8. What number is that which, if divided by the product of its digits, the quotient will be 2; but, if 27 be added to the number, the digits will be inverted ? *Ans.* 36.

9. Find two numbers such that their sum, their product, and the difference of their squares may all be equal to one another. *Ans.* $\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$, and $\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$.

10. A and B hired a pasture, into which A put 4 horses, and B as many as cost him 18*s*. a week. Afterward B put in two additional horses, and found that he must pay 20*s*. a week. How many horses had B at first, and at what rate was the pasture hired ?

Ans. B had 6 horses, and the pasture was hired at 50*s*. per week.

11. A labourer dug two trenches, one of which was 6 yards longer than the other, for £17 16*s*., and the digging of each of them cost as many shillings per yard as there were yards in its length. What was the length of each ?

Ans. 10 and 16 yards.

12. A and B set out from two towns which were distant from each other 247 miles, and travelled the direct road till they met. A went 9 miles a day, and the number of days at the end of which they met was greater by 3 than the number of miles which B went in a day. How many miles did each go ? *Ans.* A 117, and B 130 miles.

13. Two merchants each sold the same kind of stuff; the second sold 3 yards more of it than the first, and together they receive 35 crowns. The first said to the second, "I would have received 24 crowns for your stuff;" the other replied, "I would have received $12\frac{1}{2}$ crowns for yours." How many yards did each of them sell ?

Ans. The first sold 15 or 5, the second 18 or 8.

14. A widow possessed $13,000, which she divided into two parts, and placed them at interest in such a manner that

the incomes from them were equal. If she had put out the first portion at the same rate as the second, she would have drawn for this part $360 interest; and if she had placed the second out at the same rate as the first, she would have drawn $490. What were the two rates of interest?

Ans. 7 and 6 per cent.

15. The sum of two numbers is 9, and the sum of their cubes 243. What are the numbers? *Ans.* 3 and 6.

16. The sum of two numbers is 10, and the sum of their fourth powers is 1552. What are the numbers?

Ans. 4 and 6.

17. The sum of two numbers is 7, and the sum of their fifth powers 3157. What are the numbers?

Ans. 5 and 2.

18. There are two square buildings that are paved with stones a foot square each. The side of one building exceeds that of the other by 12 feet, and both their pavements together contain 2120 stones. What are the lengths of them separately? *Ans.* 26 and 38 feet.

19. A regiment of soldiers, consisting of 1066, formed into two squares, one of which has four men more in a side than the other. What number of men are in a side of each of the squares? *Ans.* 21 and 25.

20. The plate of a looking-glass is 18 inches by 12, and is to be framed with a frame of equal width, whose area is to be equal to that of the glass. Required the width of the frame. *Ans.* 3 inches.

21. A square courtyard has a rectangular gravel-walk round it. The side of the court wants two yards of being six times the width of the gravel walk, and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 164. Required the area of the court. *Ans.* 256 yards.

22. There are four towns in the order of the letters A, B, C, and D. The difference between the distances from A to B and from B to C is greater by four miles than the dis-

tance from B to D. Also, the number of miles between B and D is equal to two thirds of the number between A and C; and the number between A and B is to the number between C and D as seven times the number between A and C is to 208. Required the respective distances.

Ans. A B 42, B C 6, C D 26 miles.

DISCUSSION OF THE GENERAL EQUATION OF THE SECOND DEGREE.*

CASE I.

338. It has already been remarked, and we will now proceed to demonstrate, that every affected equation of the second degree necessarily admits of two values for the unknown quantity, and only two.

339. Let us resume the first of the four forms of the affected quadratic (Art. 322).

$$x^2+2px=q; \qquad (1.)$$

Adding p^2 to both members, $x^2+2px+p^2=q+p^2$, (2.)

Or - - - - - - $(x+p)^2=q+p^2$; (3.)

Let - - - - - - - $m^2=q+p^2$; (4.)

Then - - - - - $(x+p)^2=m^2$; (5.)

Transposing - - - $(x+p)^2-m^2=0$; (6.)

Resolving into factors, $(x+p+m).(x+p-m)=0$; (7.)

Dividing by $x+p+m$ - - $x+p-m=0$;

Transposing - - - $x=-p+m$, or $x=-p+\sqrt{q+p^2}$;

Dividing the 7th equation by $x+p-m$ - - } - $x+p+m=0$;

Transposing - - - $x=-p-m$, or $x=-p-\sqrt{q+p^2}$.

Either of these values of x will answer the conditions of the equation.

The same course of demonstration might be applied to the remaining three forms of the quadratic equation.

Hence *every affected equation of the second degree necessarily admits of two values of the unknown quantity, and only two.*

CASE II.

340. We will now resume the results obtained in the four

* This discussion is substantially that of M. Bourdon.

preceding formulas, and enter into such an analysis of them with reference to the relative values of q and p as will determine the particular values of x. These results are (Art. 326),

(1.) $x=-p\pm\sqrt{q+p^2}$; | (3.) $x=-p\pm\sqrt{p^2-q}$;
(2.) $x=+p\pm\sqrt{q+p^2}$; | (4.) $x=+p\pm\sqrt{p^2-q}$.

1. Since the value of x in each equation is expressed by a rational term, with which a radical is connected by the sign $\pm$, in order that this value may be found, the quantity under the radical sign must be positive.

As p^2 is necessarily positive, the value of x may always be found in the first and second equations.

If $q+p^2$ is a perfect square, the exact value of x will be obtained; if it is a surd, its approximate value.

In the third and fourth equations, if

$$q<p^2,$$

the value of x may also be found, either exactly or approximately; but if

$$q>p^2,$$

the value of x will be imaginary, since it will involve the extraction of the square root of a negative quantity.

2. In the first and second equations, since

$$p<\sqrt{q+p^2},$$

the value of x will be positive when the radical is taken positive, and negative when the radical is taken negative.

3. In the third equation, since

$$p>\sqrt{p^2-q},$$

if $q<p^2$, the value of x will be negative; but if $q>p^2$, the value of x will be imaginary.

4. In the fourth equation, since

$$p>\sqrt{p^2-q},$$

if $q<p^2$, the value of x will be positive; but if $q>p^2$, the value of x will be imaginary.

5. If $q=p^2$, the radical expression in the third and fourth equations will be reduced to 0, and the values of x will be,

In the third - - - - $x=-p$,

In the fourth - - - - $x=+p$.

6. If $q=0$, the equation will assume the form $x^2\pm 2px=\pm 0$; and, consequently, $x=\pm 2p$, or ± 0.

7. If $p=0$, the equation will assume the form $x^2=\pm q$; and, consequently, $x=\pm\sqrt{\pm q}$, and the value of x will be imaginary in the third and fourth forms of the quadratic.

8. In the equation $ax^2\pm 2px=\pm q$, if $a=0$, the equation will assume the form $\pm 2px=\pm q$, or be reduced to a simple equation.

CASE III.

341. In order to show why we obtain the imaginary results in the third and fourth equations when $q>p^2$, we will demonstrate that these equations, when $q>p^2$, express conditions that are incompatible with each other.

Resume the equation

$$x^2-2px=-q;$$

Reducing - - - $x=p\pm\sqrt{p^2-q}$.

Designating the first value of x by x', and the second value by x'', we shall have

$$x'=p+\sqrt{p^2-q};$$

$$x''=p-\sqrt{p^2-q};$$

Adding - $x'+x''=p+\sqrt{p^2-q}+p-\sqrt{p^2-q}=2p$.

Hence *the sum of the two values of x is equal to the coefficient of the first power of the unknown quantity, taken with the contrary sign.*

Multiplying the above two equations,

$$x'x''=(p+\sqrt{p^2-q}).(p-\sqrt{p^2-q})=p^2-(p^2-q)=q.$$

Hence *the product of the two values of x is equal to the second member of the equation, taken with the contrary sign.*

Therefore, in the general equation, $x^2-2px=-q$, $2p$ is the sum of two numbers, of which q is the product. Now it has already been demonstrated, that if a quantity be resolved into two factors, their product will be the greatest possible when the factors are equal (Art. 206).

Hence the conditions of the equation limit the value of q; it may vary between the limits 0 and p^2, but can never become greater than $\left(\frac{2p}{2}\right)^2=p^2$.

If, then, we assign to q a value greater than the square of half $2p$, the equation will express conditions which are incompatible with each other, and, consequently, the value of x will be imaginary or impossible. Thus,

Let it be required to divide 16 into two such parts that their product shall be 72.

Let - - $x=$ one of the parts,

Then - $16-x=$ the other;

And, by the conditions of the problem,

$$x.(16-x)=72;$$

Multiplying - - - $16x-x^2=72$;

Changing the signs - - $x^2-16x=-72$;

Completing the square, $x^2-16x+64=64-72=-8$;

Evolving - - - - - $x-8=\pm\sqrt{-8}$;

Transposing - - - - $x=8\pm\sqrt{-8}$.

Thus we obtain an imaginary result, which should be the case, as 16 can be divided into no two factors whose product shall be equal to 72; for, since $2p=16$, p will equal 8, and $p^2=64$, which is the greatest possible product that can be formed of two numbers whose sum is 16.

CASE IV.

344. We will now apply the principles exhibited in this discussion to a few problems, which will give rise to nearly all the circumstances that usually occur in equations of the second degree.

First Problem.

Find upon a line which joins two luminous bodies, A and B, the point where these bodies shine with equal intensity.

Note.—The solution of this problem depends upon the following principle in physics, viz.: *The intensity of light from the same luminous body will be, at different distances, in the inverse ratio of the squares of the distances.*

This being premised, in the indefinite line (1, 2) let A and B represent the respective position of the two lights, and C the point required.

Let a=A B, the distance between the two lights,

And b= the intensity of the light A at the unity distance,

And c= the intensity of the light B at the unity distance.

Let x=A C, the distance from A to the point of equal intensity,

Then $a-x$=B C, the distance from B to the point of equal intensity.

Then, by the above principle in physics, the intensity of A at the distance 1 being b, its intensity at the different distances 2, 3, 4 . . . . x, will be $\frac{b}{4}, \frac{b}{9}, \frac{b}{16} \ldots\ldots \frac{b}{x^2}$, which last term represents its intensity at C. In the same manner, it may be shown that the intensity of B at the distance $a-x$, or at C, is equal to $\frac{c}{(a-x)^2}$. But the conditions of the question require that their intensities be equal at C; hence we have the equation

$$\frac{b}{x^2}=\frac{c}{(a-x)^2};$$

Reducing - $x=\frac{ab}{b-c}\pm\sqrt{\frac{a^2b^2}{(b-c)^2}-\frac{a^2b}{b-c}}$;

Or, simplifying, $x=\frac{a(b\pm\sqrt{bc})}{b-c}$;

But - $b\pm\sqrt{bc}=\sqrt{b}(\sqrt{b}\pm\sqrt{c})$,

And - - $b-c=(\sqrt{b})^2-(\sqrt{c})^2=(\sqrt{b}+\sqrt{c}).(\sqrt{b}-\sqrt{c})$;

Whence - - $x=\frac{a\sqrt{b}(\sqrt{b}\pm\sqrt{c})}{(\sqrt{b}+\sqrt{c}).(\sqrt{b}-\sqrt{c})}$.

Taking $\sqrt{c}$ in the numerator, minus, and dividing the equation by $\sqrt{b}-\sqrt{c}$, we have

$$(1).\quad x=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}};$$

But, taking $\sqrt{c}$ in the numerator, plus, and dividing the equation by $\sqrt{b}+\sqrt{c}$, we have

$$(2). \quad x=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}.$$

Hence we also obtain

$$(1). \quad a-x=\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}};$$

$$(2). \quad a-x=\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}.$$

Discussion.

I. Let $b>c$.

343.—1. The first value of x, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}$, is positive, and less than a; for, $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}$ being a proper fraction, $a.\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}<a$. This value of x, therefore, gives the point C between A and B. It is, moreover, nearer B than A; for, in consequence of $b>c$, we have $\sqrt{b}+\sqrt{b}=2\sqrt{b}>\sqrt{b}+\sqrt{c}$, whence $\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{1}{2}$, and, consequently, $a.\frac{\sqrt{b}}{\sqrt{b}+\sqrt{c}}=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}>\frac{a}{2}$. This, indeed, should be the case, since we have supposed the intensity of A greater than that of B, or $b>c$.

2. The corresponding value of $a-x$, $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$, is also positive, and less than $\frac{a}{2}$; for, since $x>\frac{a}{2}$,

$$a-x=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}<\frac{a}{2}.$$

3. The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, is also positive, but greater than a; for, $\frac{\sqrt{b}}{\sqrt{b}-\sqrt{c}}$ being an improper fraction, $a.$

$\frac{\sqrt{b}}{\sqrt{b}-\sqrt{c}}=\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}>a$. This value of x, therefore, gives the point of equal intensity to the right of B, at C′. This should evidently be the case, since the light from A and B radiates in all directions. This point will, moreover, be nearer the body, the light of which is least intense.

4. The second value of $a-x$, $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, is negative, as it should be, since $x>a$; and the point C′ is in the direction opposite from A.

II. Let $b<c$.

344.—1. The first value of x will be positive, but less than $\frac{a}{2}$; for, $\sqrt{b}+\sqrt{b}=2\sqrt{b}<\sqrt{b}+\sqrt{c}$.

2. The corresponding value of $a-x$ will be positive, but greater than $\frac{a}{2}$; for, since $x<\frac{a}{2}$

$$a-x=\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}>\frac{a}{2}.$$

Thus, on the present hypothesis, the point C will be situated between the two lights, but nearer A than B, which should evidently be the case.

3. The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, is essentially negative, and indicates that the point of equal intensity is situated at C″, in a direction from A opposite to B.

4. The corresponding value of $a-x$ (which, since x is essentially negative, becomes $a+x$), $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, is positive; for, since $\sqrt{c}>\sqrt{b}$, the numerator and denominator will be affected with like signs, and, consequently, the value of the fraction will be plus. This result also indicates that C″ should be to the left of A.

III. Let $b=c$.

345.—1. The firt value of x, $\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{c}}=\frac{a\sqrt{b}}{\sqrt{b}+\sqrt{b}}=\frac{a\sqrt{b}}{2\sqrt{b}}=\frac{a}{2}$.

2. The corresponding value of $a-x$, $\frac{a\sqrt{c}}{\sqrt{b}+\sqrt{c}}$, also equals $\frac{a}{2}$. These two results give the middle point A B for the first required point, and this result conforms to the hypothesis.

3. The second value of x, $\frac{a\sqrt{b}}{\sqrt{b}-\sqrt{c}}$, since $\sqrt{b}=\sqrt{c}$, will be reduced to $\frac{a\sqrt{b}}{0}$, which indicates that no finite value can be assigned to x.

4. The corresponding value of $a-x$, $\frac{-a\sqrt{c}}{\sqrt{b}-\sqrt{c}}$, will also become $\frac{-a\sqrt{c}}{0}$.

These results also agree with the hypothesis; for, as the difference of their intensity decreases, the second values of x and $a-x$ increase, and, when that difference becomes infinitely small, these values must become infinitely large.

IV. Let $b=c$, and $a=0$.

346. The first values of x and $a-x$ become 0, and the second $\frac{0}{0}$. This last character is the symbol of indetermination; for, on returning to the equation of the problem,

$$(b-c)x^2-2abx=-a^2b,$$

this equation becomes

$$0 \,.\, x^2-0 \,.\, x=0,$$

an equation which may be satisfied by any number whatever taken for x. And this agrees with the hypothesis; for, if the bodies have the same intensity, and are placed at the same point, they will shine with equal light upon any point whatever in the line 1—2.

V. Let $a=0$, and b and c unequal.

347. Each of the two values becomes 0 in this case, which proves that on this hypothesis there can be but one point equally illuminated, and that is the point in which the two lights are placed.

Second Problem.

348. Find two such numbers that the difference of their products by the numbers a and b respectively may be equal to a given number s, and the difference of their squares equal to another given number q.

Let x and y represent the numbers sought;

Then - - - $ax-by=s$,

And - - - $x^2-y^2=q$.

Reducing these two equations, we have for the two values of x,

(1). - $x=\frac{as+b\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}$;

(2). - $x=\frac{as-b\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}$.

The two values of y are,

(1). - $y=\frac{bs+a\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}$;

(2). - $y=\frac{bs-a\sqrt{s^2-q(a^2-b^2)}}{a^2-b^2}$.

Discussion.

I. Let $a>b$.

349. In this case a^2-b^2 will be positive; therefore, in order that the values of x and y may be real, we must have

$$q(a^2-b^2)<s^2, \text{ or } q<\frac{s^2}{a^2-b^2}.$$

1. This condition being fulfilled, the first values of x and y will necessarily be positive, and, consequently, will form a direct solution of the problem in the sense in which it is enunciated.

2. The second value of x will be essentially positive; for, $a>b$ gives $as>b\sqrt{s^2-q(a^2-b^2)}$.

The second value of y may be either positive or negative In order that it may be positive, we must have

$bs>a\sqrt{s^2-q(a^2-b^2)}$;

Or, squaring - $b^2s^2>a^2s^2-a^2q(a^2-b^2)$;

Or, transposing - $b^2s^2+a^2q(a^2-b^2)>a^2s^2$;

Or, subtracting b^2s^2 from both sides of the inequation,

$$a^2q(a^2-b^2)>a^2s^2-b^2s^2=s^2(a^2-b^2);$$

Or, dividing - $q>\frac{s^2}{a^2}$.

Thus, if $q>\frac{s^2}{a^2}$, and $q<\frac{s^2}{a^2-b^2}$, the question is susceptible of a real and direct solution, and will give positive values of y.

But, if $q<\frac{s^2}{a^2}$, and $q<\frac{s^2}{a^2-b^2}$, the value of y will be negative; and we shall not obtain a solution of the problem in the sense in which it was enunciated, but of an analogous problem, the equations of which are

$$ax+by=s,$$
$$x^2-y^2=q,$$

and which differ from the proposed equations in this respect only, that s will express the arithmetical *sum* instead of the *difference of their products.*

II. Let $a<b$.

350. In this case a^2-b^2 will be negative, and the values of x and y may be put under the form,

(1). - $x=\frac{-as-b\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}$;

(2). - $x=\frac{-as+b\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}$.

(1). - $y=\frac{-bs-a\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}$;

(2). - $y=\frac{-bs+a\sqrt{s^2+q(b^2-a^2)}}{b^2-a^2}$.

The values of x and y, it is evident, will be real, since the quantity placed under the radical is essentially positive.

Their first values will be negative.

The second value of y may be either positive or negative. In order that it may be positive, we must have $q>\frac{s^2}{a^2}$.

III. Let $a=b$.

351. In this case $a^2-b^2=0$, and the first values of x and y will be

$$x=\frac{2as}{0}, \text{ and } y=\frac{2as}{0}.$$

The second values of x and y will be

$$x=\frac{0}{0}, \text{ and } y=\frac{0}{0}.$$

But, if we solve the given equations on the hypothesis that $a=b$, we shall have

$$x=\frac{a^2q+s^2}{2as}, \text{ and } y=\frac{a^2q-s^2}{2as}.$$

The preceding discussions show the precision with which the algebraic results correspond to all the circumstances of the enunciation of a problem.

SECTION VIII.

Ratio, Proportion, and Progression

RATIO.

352. By *Ratio* is meant the relation which one quantity bears to another with respect to magnitude. The quantities compared must be of the same nature, so as to admit of a common measuring unit. Thus, we compare dollars with dollars, length with length, weight with weight, time with time, &c.

353. The magnitudes of quantities may be compared in two ways.

1. With regard to their *difference*. This is called *Arithmetical ratio*, or ratio by difference.

2. With regard to the number of times one quantity is contained in the other. This is called *Geometrical ratio*, or ratio by quotient.

The Arithmetical ratio of two numbers, as a and b, is expressed, $a-b$, or $a\,..\,b$.

The Geometrical ratio of two numbers, as a and b, is expressed, $a:b$, or $\frac{a}{b}$.

When the ratio is thus expressed, the first term is called the *antecedent*, the last term the *consequent*, and the two terms, taken together, are called a *couplet*.

354. The term arithmetical ratio is only a substitute for the word *difference*, and involves no principle that is not essentially involved in algebraic subtraction.

355. In a geometrical ratio *three* things are involved, viz., the *antecedent*, the *consequent*, and the *ratio;* and any two of these being given, the other may be found.

Let $a=$ antecedent, $c=$ consequent, and $r=$ ratio :

Then, from the geometrical ratio $a:c=r$, we have

$r=\frac{a}{c}$, *i. e.*, ratio $=$ antecedent $\div$ by consequent ;

And, $a=c\,.\,r$, *i. e.*, antecedent $=$ consequent $\times$ by ratio ;

And, $c=\frac{a}{r}$, *i. e.*, consequent $=$ antecedent $\div$ by ratio.

356. When the antecedent is equal to the consequent, the ratio is a unit, or a *ratio of equality.* When the antecedent is greater than the consequent, the ratio is greater than a unit, or a *ratio of greater inequality.* When the antecedent is less than the consequent, the ratio is less than a unit, or a *ratio of less inequality.*

A *compound ratio* is the ratio formed by multiplying the corresponding terms of two or more ratios.

A *duplicate ratio* is the ratio of the squares of the corresponding terms of a ratio ; the *triplicate ratio*, of the cubes of the corresponding terms ; the *sub-duplicate*, of the square roots of the corresponding terms, &c.

357. The ratio, it will be observed, is expressed by a fraction, the antecedent becoming the *numerator* and the consequent the *denominator*. Now it has been demonstrated that, if both terms of the fraction be multiplied or divided by the

same number, the value of the fraction will not be affected. Hence we infer,

1. *If the terms of a ratio be multiplied or divided by the same number, it does not alter the value of the ratio.*

2. *A ratio may be reduced to its lowest terms by dividing its antecedent and consequent by their greatest common measure.*

3. *Ratios may be compared with each other by reducing the fractions which represent their values to equivalent fractions having a common denominator.*

358. The following are some of the more important theorems relating to ratios:

1. A ratio of *greater inequality* is *diminished*, and a ratio of *lesser inequality* is *increased*, by adding the same quantity to both members.

First. Let $a+b:a$, or $\frac{a+b}{a}$, represent a ratio of greater inequality:

Adding x to both terms - $a+b+x:a+x$, or $\frac{a+b+x}{a+x}$;

Then - - - - $a+b:a>a+b+x:a+x$,

Or - - - - - $\frac{a+b}{a}>\frac{a+b+x}{a+x}$;

For, reducing to a common denom. $\frac{a^2+ab+ax+bx}{a(a+x)}>\frac{a^2+ab+ax}{a(a+x)}$.

Second. Let $a-b:a$, or $\frac{a-b}{a}$, represent a ratio of lesser inequality:

Adding x to both terms - $a-b+x:a+x$, or $\frac{a-b+x}{a+x}$;

Then - - - - $a-b:a<a-b+x:a+x$,

Or - - - - - $\frac{a-b}{a}<\frac{a-b+x}{a+x}$;

For, reducing to a common denom. $\frac{a^2+ax-ab-bx}{a(a+x)}<\frac{a^2-ab+ax}{a(a+x)}$.

2. A ratio of *greater inequality* is *increased*, and a ratio of *lesser inequality* is *diminished*, by subtracting the same quantity from both terms.

First. Let $a+b:a$, or $\frac{a+b}{a}$, represent a ratio of greater inequality:

Subtracting x from both terms, $a+b-x:a-x$, or $\frac{x+b-x}{a-x}$;

Then - - - - $a+b:a<a+b-x:a-x$,

Or - - - - - $\frac{a+b}{a}<\frac{a+b-x}{a-x}$;

For, reducing - - $\frac{a^2+ab-ax-bx}{a(a-x)}<\frac{a^2+ab-ax}{a(a-x)}$.

Second. Let $a+b:a$, or $\frac{a-b}{a}$, represent a ratio of lesser inequality:

Subtracting x from both terms, $a-b-x:a-x$, or $\frac{a-b-x}{a-x}$;

Then - - - - $a-b:a>a-b-x:a-x$,

Or - - - - - $\frac{a-b}{a}>\frac{a-b-x}{a-x}$;

For, reducing - $\frac{a^2-ab-ax+bx}{a(a-x)}>\frac{a^2-ab-ax}{a(a-x)}$.

3. A ratio of *greater inequality* compounded with another ratio *increases it;* but a ratio of *lesser inequality* compounded with another ratio *diminishes* it.

First. Let $a+b:a$, or $\frac{a+b}{a}$, represent a ratio of greater inequality,

And - - - - $m:n$, or $\frac{m}{n}$, be any other ratio:

Compounding - - $am+bm:an$, or $\frac{am+bm}{an}$;

Then - - - $m:n<am+bm:an$,

Or - - - - - $\frac{m}{n}<\frac{am+bm}{an}$;

For, reducing - - $\frac{amn}{an^2}<\frac{amn+bmn}{an^2}$.

Second. Let $a-b:a$, or $\frac{a-b}{a}$, represent a ratio of lesser inequality, And - - - - $m : n$, or $\frac{m}{n}$, another ratio:

Compounding - - $am—bm : an$, or $\frac{am—bm}{bm}$;

Then - - - $m : n > am—bm : an$,

Or - - - - - $\frac{m}{n} > \frac{am—bm}{an}$;

For, reducing - - $\frac{amn}{an^2} > \frac{amn—bmn}{an^2}$.

4. *If to the terms of any couplet there be added two other terms having the same ratio, the sums will have the same ratio.*

Let the ratio $a : b$ equal the ratio $c : d$:

Then - - - - $\frac{a+c}{b+d} = \frac{a}{b} = \frac{c}{d}$;

For, since - - - - $a : b = c : d$,

We have - - - - - $\frac{a}{b} = \frac{c}{d}$;

Clearing of fractions - - - $ad = bc$;

Adding cd to both members, $ad + cd = bc + cd$;

Resolving into factors - $d(a+c) = c(b+d)$;

Dividing - - - - $\frac{a+c}{b+d} = \frac{c}{d} = \frac{a}{b}$.

5. *If from the terms of any couplet two other quantities having the same ratio be subtracted, the remainders will have the same ratio.*

Let the ratio $a : b =$ the ratio $c : d$:

Then - - - - - $\frac{a—c}{a—d} = \frac{a}{b} = \frac{c}{d}$;

For, since - - - - - $a : b = c : d$,

We have - - - - - - $\frac{a}{b} = \frac{c}{d}$;

Clearing of fractions - - - - $ad = bc$;

Subtracting cd from both members, $ad—cd = bc—cd$;

Resolving into factors - - $d(a—c) = c(b—d)$;

Dividing - - - - - $\frac{a—c}{b—d} = \frac{c}{d} = \frac{a}{b}$.

EXAMPLES.

1. Reduce the ratios 360 : 315, and 1595 : 667, to their lowest terms.

2. Which is the greater of the two ratios, 11 : 9, and 44 : 35 ?

3. Which is the least of the three ratios, 20 : 17, 22 : 18, and 25 : 23 ?

4. If the consequent be 35, and the antecedent 985, what is the ratio ?

5. If the antecedent be 1512, and the ratio 12, what is the consequent ?

6. If the consequent be 320, and the ratio $\frac{1}{2}$, what is the antecedent ?

7. What is the compound ratio of 12 : 21, 18 : 6, and 24 : 5 ?

8. What kind of a ratio will be produced by compounding $5x+7 : 2x-3$, and $x+2 : \frac{1}{2}x+3$?

9. What kind of a ratio will be produced by compounding $a^2-x^2 : a^2$, $a+x : b$, and $b : a-x$?

10. What kind of a ratio will be produced by compounding $x+y : a$, $x-y : b$, and $b : \frac{x^2-y^2}{a}$?

11. What is the ratio produced by compounding 3 : 7, the duplicate ratio of 3 : 5, and the triplicate ratio of 4 : 3 ?

12. What is the ratio produced by compounding the subduplicate ratio of 49 : 4, and the sub-triplicate ratio of 64 : 125 ?

PROPORTION.

359. Ratio is a comparison of two quantities to ascertain their difference, or how often one is contained in the other.

Proportion is a comparison of two equal ratios.

If the ratios are arithmetical, the proportion is called *arithmetical proportion*, or *proportion by difference.*

If the ratios are geometrical, the proportion is called *geometrical proportion*, or *proportion by quotient.*

360. There are always *two* couplets, or *four* terms, in a proportion. The first and fourth terms are called *extremes ;*

the second and third, *means.* The two antecedents, or the two consequents taken together, are called *homologous* terms. The terms of the same couplet are called, with reference to the proportion, *analogous* terms. Three terms are said to be proportional when the ratio formed by the first and second is equal to the ratio formed by the second and third.

361. As an *arithmetical* proportion, or a proportion by *difference*, is nothing more than a simple form of equation, it is unnecessary to give the subject a separate consideration. It is expressed $a-b=c-d$, or $a\,..\,b=c\,..\,d$.

362. *Geometrical proportion* is expressed by

$$a:b=c:d,$$

Or - $a:b::c:d,$

which expressions are read, "a to b equals c to d," or, "a is to b as c to d."

THEOREMS RELATING TO PROPORTION.

363.—(1.) *If four numbers be proportional, the product of the extremes will be equal to the product of the means.*

Let - - - - $a:b::c:d$;

Then, by equality of ratios, $\frac{a}{b}=\frac{c}{d}$;

Clearing of fractions - - $ad=bc$.

Cor. 1. Any factor may be transferred from one *mean* or *extreme* to the other without destroying the proportion. Thus, if $a:b::cm:dn$, then $an:bm::c:d$.

Cor. 2. If any *three* terms of a proportion be given, the *fourth* can always be ascertained; for if $a:b::c:d$,

Then - - $ad=cb$;

Dividing by d, $a=\frac{cb}{d}$, *i. e.*, the 1st term$=$2d$\times$3d$\div$4th;

Dividing by c, $b=\frac{ad}{c}$, *i. e.*, the 2d term$=$1st$\times$4th$\div$3d;

Dividing by b, $c=\frac{ad}{b}$, *i. e.*, the 3d term$=$1st$\times$4th$\div$2d;

Dividing by a, $d=\frac{cb}{a}$, *i. e.*, the 4th term$=$2d$\times$3d$\div$1st.

364.—(2.) *If the product of any two numbers be equal to the product of two others, these four numbers will constitute a proportion when so arranged that the factors of one product be made the means, and the factors of the other product the extremes.*

Let - - - - - $ad=bc$:

Dividing by db, and reducing, $\frac{a}{b}=\frac{c}{d}$;

Hence - - - - $a:b::c:d$.

365.—(3.) *If three numbers be proportional, the product of the two extremes is equal to the square of the mean.*

Let - - - - $a:b::b:c$:

Then - - - - - $\frac{a}{b}=\frac{b}{c}$,

Or - - - - - $ac=b^2$.

COR. The mean proportional, or geometrical mean, between two numbers is equal to the square root of their product. Thus, if $ac=b^2$, then $b=\sqrt{ac}$.

366.—(4.) *If four numbers be proportional,* 1. *the order of the extremes,* 2. *of the means,* 3. *of the terms of each couplet,* 4. *of the couplets,* 5. *of all the terms, may be inverted without destroying the proportion.*

Let . - - - $a:b::c:d$:

Then - - - - $\frac{a}{b}=\frac{c}{d}$.

Dividing by a, and multiplying by d - - - $\frac{d}{b}=\frac{c}{a}$, $\therefore d:b::c:a$; (1.)

Dividing by c, and multiplying by b - - - $\frac{a}{c}=\frac{b}{d}$, $\therefore a:c::b:d$; (2.)

Inverting the fractions - $\frac{b}{a}=\frac{d}{c}$, $\therefore b:a::d:c$; (3.)

Inverting the order of the members - - - $\frac{c}{d}=\frac{a}{b}$, $\therefore c:a::a:b$; (4.)

Inverting the fractions and changing the order of the terms - - - - $\frac{d}{c}=\frac{b}{a}$, $\therefore d:c::b:a$. (5.)

367.—(5.) *If four numbers be proportional, the Analogous* *or the Homologous terms may be multiplied or divided by the same number without destroying the proportion.*

Let - - - - $a:b::c:d:$

Then - - - - - $\frac{a}{b}=\frac{c}{d}.$

Multiplying the terms of the first fraction by m - - $\Big\}\ \frac{am}{bm}=\frac{c}{d}$, $\therefore am : bm :: c : d$;

Multiplying the terms of the second fraction by m - $\Big\}\ \frac{a}{b}=\frac{cm}{dm}$, $\therefore a : b :: cm : dm$;

Multiplying the equation by m, $\frac{am}{b}=\frac{cm}{d}$, $\therefore am : b :: cm : d$;

Dividing the equation by m, $\frac{a}{bm}=\frac{c}{dm}$, $\therefore a : bm :: c : dm$;

Dividing the terms of the first fraction by m - - - $\Big\}\ \frac{\frac{a}{m}}{\frac{b}{m}}=\frac{c}{d}$, $\therefore \frac{a}{m} : \frac{b}{m} :: c : d$;

Dividing the terms of the second fraction by m - - $\Big\}\ \frac{a}{b}=\frac{\frac{c}{m}}{\frac{d}{m}}$, $\therefore a : b :: \frac{c}{m} : \frac{d}{m}$;

Dividing the numerators by m - $\frac{\frac{a}{m}}{b}=\frac{\frac{c}{m}}{d}$, $\therefore \frac{a}{m} : b :: \frac{c}{m} : d$;

Dividing the denominators by m, $\frac{a}{\frac{b}{m}}=\frac{c}{\frac{d}{m}}$, $\therefore a : \frac{b}{m} :: c : \frac{d}{m}$.

368.—(6.) *If there be two sets of proportions having an antecedent and consequent in the one equal to an antecedent and consequent of the other, the remaining terms will be proportional.*

Let - - - - $a : b :: c : d$,

And - - - - $a : b :: m : n$:

By the first proportion - - $\frac{a}{b}=\frac{c}{d}$;

By the second - - - $\frac{a}{b}=\frac{m}{n}$;

Therefore - - - - - $\frac{c}{d}=\frac{m}{n}$, and $c : d :: m : n$.

369.—(7.) *If two homologous or two analogous terms be added to or subtracted from the two others, the proportion will be preserved.*

First. Let - - - $a : b :: c : d$,

Or - - - - - $\frac{a}{b}=\frac{c}{d}$:

Then (by Art. 358, th. 5), $\frac{a \pm c}{b \pm d}=\frac{a}{b}=\frac{c}{d}$;

Hence - - - $a+c : b+d :: a : b$, or *as* $c : d$,

And - - - $a-c : b-d :: a : b$, or *as* $c : d$.

Second. Inverting the order of the means, $a : c :: b : d$,

Then - - - - $\frac{a \pm b}{c \pm d}=\frac{a}{c}=\frac{b}{d}$;

Hence - - - $a+b : c+d :: a : c$, or *as* $b : d$,

And - - - $a-b : c-d :: a : c$, or *as* $b : d$.

Cor. 1. Since - $\frac{a \pm c}{b \pm d}=\frac{a}{b} \therefore \frac{a+c}{b+d}=\frac{a-c}{b-d}$;

Hence we have - $a+c : a-c :: b+d : b-d$.

Cor. 2. Since - $\frac{a \pm b}{c \pm d}=\frac{a}{b} \therefore \frac{a+b}{c+d}=\frac{a-b}{c-d}$;

Hence we have - $a+b : a-b :: c+d : c-d$.

370.—(8.) *If two sets of proportional numbers be multiplied, the products of the corresponding terms will be proportional.*

Let - - - $a : b :: c : d$, or $\frac{a}{b}=\frac{c}{d}$,

And - - - $m : n :: p : q$, or $\frac{m}{n}=\frac{p}{q}$:

Multiplying the two equations - $\frac{am}{bn}=\frac{cp}{dq}$;

Hence - - - - $am : bn :: cp : dq$.

371.—(9.) *If one set of proportional numbers be divided by the corresponding terms of another set, the quotients will be proportional.*

Let - - $a : b :: c : d$, or $\frac{a}{b}=\frac{c}{d}$,

And - $m : n :: p : q$, or $\frac{m}{n}=\frac{p}{q}$:

Dividing the first equation by the second - - - $\frac{a}{b} \div \frac{m}{n}=\frac{c}{d} \div \frac{p}{q}$,

Or - - - - - $\frac{\frac{a}{m}}{\frac{b}{n}}=\frac{\frac{c}{p}}{\frac{d}{q}}, \therefore \frac{a}{m} : \frac{b}{n} :: \frac{c}{p} : \frac{d}{q}$.

372.—(10.) *If four numbers be proportional, like powers or like roots of them will be proportional.*

Let - - $a : b :: c : d$:

Then - - $\frac{a}{b}=\frac{c}{d}$;

Involving - - $\frac{a^n}{b^n}=\frac{c^n}{d^n}$, $\therefore a^n : b^n :: c^n : d^n$;

Evolving - - $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{\sqrt[n]{c}}{\sqrt[n]{a}}$, $\therefore \sqrt[n]{a} : \sqrt[n]{b} :: \sqrt[n]{c} : \sqrt[n]{d}$.

PROBLEMS TO BE SOLVED BY PROPORTION.

1. There are two numbers whose difference is to the less as 100 is to the greater, and the same difference is to the greater as 4 to the less. What are the two numbers?

Let $x=$ the greater, and $y=$ the less:

Then - - - - - $x-y : y :: 100 : x$,
And - - - - - $x-y : x :: 4 : y$;
Multiplying the proportions, $(x-y)^2 : xy :: 400 : xy$;
Dividing consequents - $(x-y)^2 : 1 :: 400 : 1$;
Evolving - - - - $x-y : 1 :: 20 : 1$;
Converting into an equation - $x-y=20$;
Whence - - - - $x=20+y$;
Substituting this value of x in the first proportion $\}$ $20+y-y : y :: 100 : 20+y$,
Or - - - - $20 : y :: 100 : 20+y$;
Dividing antecedents - $1 : y :: 5 : 20+y$;
Converting into an equation - $5y=20+y$;
Whence - - - - $y=5$,
And - - - - - $x=20+5=25$.

2. The product of two numbers is 15, and the sum of their squares is to the difference of their squares as 17 to 8. What are the numbers?

Let $x=$ the greater, and $y=$ the less:

Then - - - - - $xy=15$,
And - - - - $x^2+y^2 : x^2-y^2 :: 17 : 8$;
Adding and subtracting - - $2x^2 : 2y^2 :: 25 : 9$;

Dividing first couplet by 2 - - $x^2 : y^2 :: 25 : 9$;
Evolving - - - - - $x : y :: 5 : 3$;
Whence - - - - $3x = 5y$;
Reducing - - - - $x = 5$, and $y = 3$.

3. What two numbers are those whose product is 320, and the difference of their cubes is to the cube of their difference as 61 to 1?

Let $x =$ the greater, and $y =$ the less:
Then - - - - - $xy = 320$,
And - - - - - $x^3 - y^3 : (x-y)^3 :: 61 : 1$;
Expanding 2d term, $x^3 - y^3 : x^3 - 3x^2y + 3xy^2 - y^3 :: 61 : 1$;
Subtracting consequents from antecedents - $3x^2y - 3xy^2 : (x-y)^3 :: 60 : 1$;
Dividing first ratio by $x-y$ - $3xy : (x-y)^2 :: 60 : 1$;
Dividing antecedents by 3 $xy : (x-y)^2 :: 20 : 1$;
Substituting value of xy - - $320 : (x-y)^2 :: 20 : 1$;
Dividing antecedents by 20 - $16 : (x-y)^2 :: 1 : 1$;
Evolving - - - - - $4 : x-y :: 1 : 1$;
Converting into an equation - $x - y = 4$;
Whence - - - - - - $x = 20$, and $y = 16$.

4. It is required to prove that $a : x :: \sqrt{2a-y} : \sqrt{y}$, on the supposition that $(a+x)^2 : (a-x)^2 :: x+y : x-y$.

Expanding first and 2d terms, $a^2 + 2ax + x^2 : a^2 - 2ax + x^2 :: x+y : x-y$;
Adding and subtracting, $2a^2 + 2x^2 : 4ax :: 2x : 2y$;
Dividing terms - - $a^2 + x^2 : 2ax :: x : y$;
Transferring the factor x, $a^2 + x^2 : 2a :: x^2 : y$;
Inverting means - - $a^2 + x^2 : x^2 :: 2a : y$;
Subtracting terms - - - $a^2 : x^2 :: 2a - y : y$;
Evolving - - - - - $a : x :: \sqrt{2a-y} : \sqrt{y}$.

5. It is required to prove that $dx = cy$, on the supposition that $x : y :: a^3 : b^3$, and $a : b :: \sqrt[3]{c+x} :: \sqrt[3]{d+y}$.

Inverting the order of the ratios in the first proportion - $a^3 : b^3 :: x : y$;
Involving second proportion - $a^3 : b^3 :: c+x : d+y$;
By equality of ratios - - $x : y :: c+x : d+y$;

Inverting means - - - $x : c+x :: y : d+y$;
Subtracting terms - - - $x : c :: y : d$;
Converting into an equation - $dx=cy$.

6. There are two numbers whose product is 24, and the difference of their cubes : the cube of their difference as 19 : 1. What are the numbers? *Ans.* 6 and 4.

7. The sum of two numbers is to their difference as 3 : 1, and the difference of their third powers is 56. What are the numbers? *Ans.* 4 and 2.

8. There are two numbers whose product is 135, and the difference of their squares is to the square of their difference as 4 to 1. What are the numbers? *Ans.* 15 and 9.

9. There are two numbers which are to each other in the duplicate ratio of 4 to 3, and 24 is a mean proportional between them. What are the numbers? *Ans.* 32 and 18.

10. There are two numbers which are to each other as 3 to 2. If 6 be added to the greater and subtracted from the less, the sum will be to the remainder as 3 to 1. What are the numbers? *Ans.* 24 and 16.

11. What number is that to which if 3, 8, and 17 be severally added, the first sum will be to the second as the second to the third? *Ans.* $3\frac{1}{4}$.

12. The sum of the third powers of two numbers is to the difference of the third powers as 559 to 127, and the square of the first, multiplied by the second, is equal to 294. What are the numbers? *Ans.* 7 and 6.

ARITHMETICAL PROGRESSION.

373. A series of numbers increasing or decreasing by a constant difference, is called an *arithmetical progression*, or *progression by difference.*

374. When the numbers increase by a common difference, they form an *ascending series*; when they decrease, a *descending series.*

Thus, the natural numbers,

1, 2, 3, 4, 5, 6, 7, 8, &c.,

form an ascending series.

Inverted, they form a descending series; as,

8, 7, 6, 5, 4, 3, 2, 1.

375. From the definition of arithmetical progression, it is evident that in an ascending series each term is found by *adding the common difference to the preceding term.*

Let $a=$ first term, $d=$ common difference, and $n=$ the number of terms:

Then the terms of the series will be

1	2	3	4	5		n
a,	$a+d$,	$a+2d$,	$a+3d$,	$a+4d$	- - - - - -	$a+(n-1)d$.

Hence, letting $l=$ the last term, we shall have,

1. - - - $l=a+(n-1)d$, the formula for the last term.

2. Transposing, &c., $a=l-(n-1)d$, the formula for the first term.

3. Transposing and dividing - - $d=\frac{l-a}{n-1}$, the formula for the common diff.

4. Transposing, &c., $n=\frac{l-a}{d}+1$, the formula for the number of terms.

These four formulas may be enunciated as follows;

1. *The last term is equal to the first term, plus the common difference multiplied by the number of terms less one.*

2. *The first term is equal to the last term, minus the common difference multiplied by the number of terms less one.*

3. *The common difference is equal to the difference between the extremes divided by the number of terms less one.*

4. *The number of terms is equal to the difference between the extremes divided by the common difference, the quotient increased by one.*

376. If the series is descending, the above formulas will evidently become,

1. - $l=a-(n-1)d$;

2. - $a=l+(n-1)d$;

3. - $d=\frac{a-l}{n-1}$;

4. - $n=\frac{a-l}{d}+1$.

377. If the common difference and first term are equal,

$$l=a+(n-1)d=a+(n-1)a=a+an-a=an,$$

Or, $l=a-(n-1)d=a-(n-1)a=a-an+a=2a-an.$

378. From the third formula, $d=\frac{l-a}{n-1}$, we may obtain a general method for finding any number of arithmetical means between two given numbers. To do this, it is only necessary to obtain, in addition to the given data, the common difference.

Let $m=$ the number of means. Then, since the whole number of terms consists of *two extremes*, plus the *means*, we shall have $m+2=n$.

Hence, substituting for n its value in the above formula,

$$d=\frac{l-a}{m+2-1}=\frac{l-a}{m+1}.$$

PROBLEMS FOR SOLUTION.

1. The first term of an arithmetical progression is 50, and the common difference 10. What is the 100th term?

Ans. $l=a+(n-1)d=50+(100-1).10=1040.$

2. The first term of an arithmetical series is 120, and the common difference 2. What is the 325th term?

Ans.

3. The first term of an arithmetical series is 2, the last term 1828, and the number of terms 42. What is the common difference? *Ans.*

4. The last term of an arithmetical series is 2680, the common difference 5, and the number of terms 30. What is the first term? *Ans.*

5. The first term of an arithmetical series is 8, the last term 1728, and the common difference 2. What is the number of terms? *Ans.*

6. The first term of a decreasing arithmetical series is 800, the number of terms 21, and the common difference 2. What is the last term? *Ans.*

7. Find 4 arithmetical means between 2 and 52.

Ans. $d=10$; and the series, 2, 12, 22, 32, 42, 52.

8. The first term of a descending arithmetical series is 480, the last term 12, and the number of terms 42. What is the common difference? *Ans.*

9. Find 8 arithmetical means between 12 and 52920.

Ans. $d=$, and the series .

10. The first term in a descending arithmetical series is 46450, the last term 10, and the common difference 2. What is the number of terms? *Ans.*

SUM OF THE SERIES.

379. The sum of the series may evidently be obtained by the addition of all the terms, nor will this sum be affected if the order of the terms be inverted. Thus,

$$S=[a]+[a+d]+[a+2d]+[a+3d]+\cdots+[a+(n-4)d]+[a+(n-3)d]+[a+(n-2)d]+[a+(n-1)d];$$

$$S=[a+(n-1)d]+[a+(n-2)d]+[a+(n-3)d]+[a+(n-4)d]+\cdots+[a+3d]+[a+2d]+[a+d]+[a].$$

Adding the two equations,

$$2S=[2a+(n-1)d]+[2a+(n-1)d]+[2a+(n-1)d]+[2a+(n-1)d]+\cdots+[2a+(n-1)d]+[2a+(n-1)d]+[2a+(n-1)d]+[2a+(n-1)d].$$

But, since there are n terms, and all the terms are equal,

$$2S=[2a+(n-1)d]n=.$$

Hence, by performing the necessary reductions,

1. $S=\frac{2a+(n-1)d}{2}\times n,$ the formula for the sum of the terms.

2. $a=\frac{2s-dn^2+dn}{2n}$, the formula for the first term.

3 $d=\frac{2s-2an}{n^2-n}$, the formula for the common difference.

4 $n=\frac{\sqrt{(2a-d)^2+8ds}-2a+d}{2d},$ the formula for the number of terms.

These four formulas may be enunciated in the form of general propositions or rules.

PROBLEMS TO BE SOLVED BY THE PRECEDING FORMULAS.

1. The first term of an arithmetical series is 5, the num-

ber of terms 30, and the common difference 3. What is the sum of all the terms? *Ans.* 1455.

2. The sum of the terms of an arithmetical series is 280, the first term 1, and the number of terms 32. What is the common difference? *Ans.* $\frac{1}{2}$.

3. The sum of the terms of an arithmetical series is 950, the common difference 3, and the number of terms 25. What is the first term? *Ans.* 2.

4. Suppose 100 balls be placed in a straight line, at the distance of a yard from each other; how far must a person, starting from the box, travel to bring them one by one to a box placed at the distance of a yard from the first ball?

Ans. 5 miles and 1300 yards.

5. In gathering up a certain number of balls, placed on the ground in a straight line, at the distance of 2 yards from each other, the first being placed 2 yards from the box in which they were deposited, a man, starting from the box, travelled 11 miles and 840 yards. How many balls were there? *Ans.* 100.

6. How many strokes do the clocks of Venice, which go on to 24 o'clock, strike in a day? *Ans.* 300.

7. In a descending arithmetical series the first term is 730, the common difference 2, and the last term 2. What is the number of terms? *Ans.* 365.

8. A speculator bought 47 house lots in a certain village, giving $10 for the first, $30 for the second, $50 for the third, and so on. What did he pay for the whole 47?

Ans. $22,090.

9. A man bought a certain number of acres of land, paying for the first \$$\frac{1}{3}$, for the second \$$\frac{2}{3}$, and so on. When he came to settle, he had to pay $3775. How many acres did he purchase, and how much did he give per acre?

Ans. 150 acres, at \$$25\frac{1}{6}$ per acre.

10. A wealthy gentleman offered to his daughter, on the evening of her marriage, $50,000 as her dowry; or he would give her on that evening $1, on the next $2, and so on to

the end of the year, 365 days, and also the balance of interest that might be found in her favour if she accepted the latter offer. The lady, being unskilled in mathematics, chose the first offer. Did she gain or lose by this choice?

Ans. She lost $16,795.

GEOMETRICAL PROGRESSION.

380. If a series of numbers increase or decrease by the continued multiplication or division by the same number, they are said to be in *Geometrical Progression.*

381. When the numbers increase by a common multiplier, they form an *ascending geometrical series;* and when they decrease by a common divisor, they form a *descending geometrical series.* The common multiplier or divisor is called the *ratio.*

382. In an ascending geometrical series, each succeeding term is found by multiplying the preceding term by the ratio.

383. The following symbols are used in geometrical progression, viz.: $a=$ first term, $l=$ last term, $n=$ number of terms, $r=$ ratio, and $S=$ sum of the terms.

Using the above symbols, we have the series,

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & & n-4. & n-3. & n-2. & n-1. \quad n. \\ a, & ar, & ar^2, & ar^3, & ar^4 & \cdots & ar^{n-5}, & ar^{n-4}, & ar^{n-3}, & ar^{n-2}, \; ar^{n-1}. \end{array}$$

Hence we shall have,*

1. - - - - - $l=ar^{n-1}$, the formula for the last term.

2. Dividing, &c. - - $a=\dfrac{l}{r^{n-1}}$, the formula for the first term.

3. Dividing, evolving, &c., $r=\left(\dfrac{l}{a}\right)^{\frac{1}{n-1}}$, the formula for the ratio.

These three formulas may be enunciated in the form of general propositions or rules.

384. If the series is descending, the above formulas may still be applied by taking $r=$ to the reciprocal of the com-

* The formula for the number of terms is solved by the aid of logarithms, and is, consequently, omitted in this place.

mon divisor; for, multiplying by the reciprocal of a number is the same as dividing by the number itself. In fact, whenever $r<1$, the series will be descending.

385. By the third formula, $r=\left(\frac{l}{a}\right)^{\frac{1}{n-1}}$, we may obtain a general method for finding any number of geometrical means between two given numbers. To do this, it is only necessary to obtain, in addition to the given data, the ratio.

Let $m=$ the number of means; then, since the whole number of terms in the series consists of *two extremes*, plus the means, $m+2=n$.

Hence - - $r=\left(\frac{l}{a}\right)^{\frac{1}{m+1}}$.

When the ratio is found, the means may be obtained by the continued multiplication of the first extreme.

PROBLEMS TO BE SOLVED BY THE PRECEDING FORMULAS.

1. The first term of a geometrical progression is 5, the ratio 4, and the number of terms 7. What is the last term?
Ans. 20480.

2. The last term of a geometrical series is 98415, the number of terms 11, and the ratio 5. What is the first term? *Ans.*

3. The first term of a geometrical series is 28, the last term 20872, and the number of terms 5. What is the ratio? *Ans.*

4. Find two geometrical means between 4 and 256.
Ans. 16 and 64.

5. The first term of a geometrical series is 2, the number of terms 8, and the ratio $\frac{1}{4}$. What is the last term.
Ans. $\frac{1}{8192}$.

6. Find three geometrical means between $\frac{1}{9}$ and 9.
Ans. $\frac{1}{3}$, 1, and 3.

7. A speculator wishes to purchase 8 house lots of a landholder, and agrees to pay for the 8 lots what the 8 would

come to if the first be valued at \$2, the second at \$6, &c. What did he pay? *Ans.* \$4374.

8. A man leased a plantation on condition of paying for the first month \$1, for the second \$2, and so on for 12 months. At the end of 10 months, finding he had made a bad bargain, he obtained a release from his engagement on condition of his paying what would have been the stipulated sum for that month. How much did he pay? *Ans.* \$512.

SUM OF THE SERIES.

386. The sum of the series may evidently be obtained by the addition of all the terms, but it is necessary to obtain a more expeditious method for finding it.

Using the same symbols as before, we have

$$S=a+ar+ar^2+ar^3+ \cdots +ar^{n-3}+ar^{n-2}+ar^{n-1}.$$

And, $rS= \quad ar+ar^2+ar^3+ \cdots +ar^{n-3}+ar^{n-2}+ar^{n-1}+ar^n.$

Subtracting the first equation from the second,

$$rS-S=ar^n-a;$$

Resolving into factors, $(r-1)S=ar^n-a$;

Dividing . . (1.) $S=\frac{ar^n-a}{r-1}$, the formula for the sum of the terms when the first term, the number of terms, and the ratio are given.

Or, since $ar^n=r\times ar^{n-1}=rl$, *i. e.*, the last term multiplied by the ratio,

(2.) $S=\frac{rl-a}{r-1}$, the formula for the sum of the terms when the first term, the last term, and the ratio are given.

These formulas may be enunciated in the form of general propositions or rules, and applied to the reduction of problems.

387. From the above formulas it appears that there are five things to be considered in geometrical progression, viz.:

1. The first term.
2. The last term.
3. The ratio.

4. The number of terms.

5. The sum of the terms.

Any *three* of these being given, the remaining *two*, excepting the number of terms, may be found.

388. If the ratio be less than 1, the progression is decreasing; we shall also have $rl<a$.

Hence the formula for the sum of the series may be put under the form

$$S=\frac{a-rl}{1-r}.$$

389. To obtain a formula for the sum of the terms of a decreasing series having an infinite number of terms,

Put the formula . $S=\frac{a-ar^n}{1-r}$,

Under the form . $S=\frac{a}{1-r}-\frac{ar^n}{1-r}$.

Now, since $r<1$, it must be a proper fraction, and r^n is a fraction which decreases as n increases. Therefore will $\frac{ar^n}{1-r}=\frac{a}{1-r}\times r^n$ decrease as n increases; and when n becomes greater than any assignable quantity, or when n becomes infinite, the fraction will become infinitely small, or $\frac{ar^n}{1-r}=0$, and the value of S be represented by $\frac{a}{1-r}$. Hence the formula for the sum of the terms of a decreasing geometrical series, in which the number of terms is infinite, is

$$S=\frac{a}{1-r}.$$

This is, properly speaking, the limit of the decreasing series, or the number to which the sum of the terms approaches as the number of terms increases; but it can never reach this number until an *infinite number* of terms be taken.

390. The above formula may also be applied to the summation of a circulating decimal series, as 3333, &c., *ad infin.*; for this series may be put under the form $\frac{3}{10}+\frac{3}{100}+$

$\frac{3}{1000}+\frac{3}{10000}$, &c. Hence, the first term, or $a=\frac{3}{10}$, the geometrical ratio, or $r=\frac{1}{10}$, and the sum

$$S=\frac{a}{1-r}=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{1}{3}, \text{ the limit of the series.}$$

2. The decimal series, 323232, &c., *ad infin.*, may be put under the form $\frac{32}{100}+\frac{32}{10000}$, &c. Hence, $a=\frac{32}{100}$, $r=\frac{1}{100}$, and $S=\frac{a}{1-r}=\frac{\frac{32}{100}}{1-\frac{1}{100}}=\frac{32}{99}$.

3. The decimal series, 713333, &c., *ad infin.*, may be put under the form $\frac{71}{100}+\frac{3}{1000}+\frac{3}{10000}$, &c. Hence, $\frac{71}{100}+S=\frac{71}{100}+\frac{a}{1-r}=\frac{71}{100}+\frac{\frac{3}{1000}}{1-\frac{1}{10}}=\frac{71}{100}+\frac{1}{300}=\frac{214}{300}$.

PROBLEMS TO BE SOLVED BY THE PRECEDING FORMULAS.

1. The first term of a geometrical series is 1, the number of terms 8, and the ratio 5. What is the sum of all the terms? *Ans.* 97656.

2. The first term of a geometrical series is 6, the last 1458, and the ratio 3. What is the sum of all the terms? *Ans.* 2184.

3. The last term of a geometrical series is $\frac{1}{128}$, the ratio $\frac{1}{2}$, and the sum of all the terms $7\frac{127}{128}$. What is the first term? *Ans.* 4.

4. What is the sum of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, &c., continued to an infinite number of terms? *Ans.* 2.

5. What is the sum of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, &c., continued to an infinite number of terms? *Ans.* $1\frac{1}{2}$.

6. The first term of a geometrical series is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the number of terms 5. What are the last term and the sum of the series? *Ans.* $l=\frac{1}{162}$, and $S=\frac{121}{162}$.

7. The first term of a geometrical series is 1, the ratio $\frac{2}{3}$, and the number of terms 10. What is the sum of all the terms? *Ans.* $\frac{174075}{59049}$.

8. A person being asked to dispose of a fine horse, said he would sell him on condition of having a cent for the first nail in his shoes, two for the second, four for the third, and

so on, doubling the price of every nail. There were 32 nails in his four shoes. What would the horse be sold for at that rate? *Ans.* $42949672,95.

9. A man failing in trade, found himself in debt to a certain amount, after he had given up all his property; but his creditors offered to employ him, giving him $1 for the first month's service, $3 for the second, and so on till the debt was paid. Having accepted the offer, he found that it required of him but 10 months' service to pay the debt. What was the debt, and what did he receive for his last month's services?

Ans. Debt $29,524, and he received for his last month's services $19,683.

10. Two couriers, A and B, set out at the same time to meet each other. A travels 6 miles the first hour, 8 the second, 10 the third, and so on, increasing at the rate of 2 miles every hour. B goes 3 miles the first hour, $4\frac{1}{2}$ the second, and $6\frac{3}{4}$ the third, travelling each hour $1\frac{1}{2}$ times as far as the preceding hour. They meet after six hours. What is the distance between the two places from which they set out? *Ans.* $128\frac{11}{32}$ miles.

11. Required the sum of the decimal series ,81343434, &c., *ad infin.* *Ans.* $\frac{8053}{9900}$.

12. Required the sum of the three following series, viz.:

,777777, &c., *ad infin.*
,232323, &c., *ad infin.*
,714141, &c., *ad infin.* *Ans.* $1\frac{617}{990}$.

SECTION IX.

*Theory of Logarithms, and Construction of Logarithmic Tables.**

391. LOGARITHMS are a series of exponents, computed and arranged into tables for the purpose of facilitating many difficult arithmetical calculations.

In forming a system of logarithms, some number, usually 10, is selected as the base of the system. Taking 10 as the base of the system, then the logarithm of any number is the exponent denoting the power to which 10 must be involved to produce that number.

Let a represent any known number, and x the unknown exponent denoting the power to which 10 must be involved in order that the power shall equal a; we shall then have

$$10^x = a.$$

To find the logarithm of a, then, requires the solution of this equation.

392. In order to unfold still farther the theory of logarithms, and a method by which the logarithm of any number may be calculated, let us take a geometrical progression whose first term is unity and the ratio 10; and also an arithmetical progression whose first term is 0, and whose common difference is unity.

The first series is geometrical, the second is arithmetical.

1,	10,	100,	1000,	10000,	100000,	1000000,	10000000,	&c.
0,	1,	2,	3,	4,	5,	6,	7,	&c.

Supposing the two series to be continued to any extent, the numbers in the arithmetical series are called the logarithms of the corresponding terms in the geometrical series; that is, they are the exponents showing the power to which 10 must be involved to produce the corresponding terms in the geometrical series. Thus (Art. 107), $10^0=1$, $10^1=10$, $10^2=100$, $10^3=1000$, &c.

* See Note D.

393. From the nature of logarithms, as exhibited above, it will be easy to verify the truth of the three following propositions:

1. *The sum of the logarithms of any two terms of the geometrical series is the logarithm of that term which is their product.*

For example, the sum of 2 and 5, the logarithms of 100 and 100000, is 7, which is the logarithm of $10000000 = 100 \times 100000$.

2. *The difference of the logarithms of any two terms of the series is the logarithm of that term which is the quotient of the greater divided by the less.*

For example, the difference between 7 and 4, the logarithms of 10000000 and 10000, is 3, which is the logarithm of $1000 = 10000000 \div 10000$.

3. *The arithmetical mean between the logarithms of any two terms in the series is the logarithm of the geometrical mean between those terms.*

For example, $(10 \times 1000)^{\frac{1}{2}} = \sqrt{10000} = 100$, which is the geometrical mean between 10 and 1000; also, $(1+3) \div 2 = 4 \div 2 = 2$, which is the arithmetical mean between the logarithms of 10 and 100, and is also the logarithm of 100.

394. If we take a decreasing geometrical series whose first term is unity, and whose ratio is also 10, we shall have

$$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \frac{1}{1000000}, \frac{1}{10000000}, \&c.,$$

Or, $1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}$, &c.

Hence the corresponding arithmetical series, or the logarithms, are

$$0, \ -1, \ -2, \ -3, \ -4, \ -5, \ -6, \ -7, \&c.$$

395. It is evident that the logarithms of 1, 10, 100, &c., being 0, 1, 2, &c., respectively, the logarithm of any number between 1 and 10 will be 0+ some decimal parts; that of a number between 10 and 100, 1+ some decimal parts; that of a number between 100 and 1000, 2+ some decimal parts, and so on for all the numbers falling between the successive terms of this progression.

It is also evident that the logarithms of $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$,

$\frac{1}{10000}$, &c., or .1, .01, .001, .0001, &c., being —1, —2, —3, —4, &c., the logarithm of any number between 1 and .1 will be —1+ some positive decimal parts; that of a number between .1 and .01, —2+ some decimal parts; that of a number between .01 and .001, —3+ some decimal parts, &c.

396. Between each two adjoining terms of both series in Art. 392 a term may be interpolated, and a new series of numbers and logarithms will be produced, each consisting of double the number of terms. This interpolation may be effected by finding the geometrical mean (Art. 365, Cor.), or taking the square root of the product of the two terms in the geometrical series, and the arithmetical mean, or half their sum, in the arithmetical series. The term interpolated between 1 and 10 in the geometrical series would be $\sqrt{1\times 10}=3,1622777$; between 10 and 100 would be $\sqrt{10\times 100}=31,62280$; the corresponding terms interpolated in the arithmetical series would be $(0+1)\div 2=,5$, and $(1+2)\div 2=1,5$.

The two series, then, would be

1,	3.162277,	10,	31.62280,	100, &c.
0,	.5,	1,	1.5,	2, &c.

These two series may again be interpolated as before, and so on continually. The number of terms in the two series will continually increase, and the differences between them continually decrease, with each succeeding interpolation.

397. To construct a table of logarithms, however, it is unnecessary to interpolate systematically throughout the series; for, if the logarithm of some few of the prime numbers be calculated, those of the composite numbers may be obtained by the process indicated in Art. 393. Indeed, these interpolations may be limited to any two adjoining terms in the series.

Hence, the logarithm of any number, whole or fractional, between any two terms of the series in Art. 392, may be calculated by the following general

RULE.

1. *Find a geometrical mean between* 1 *and* 10, 10 *and* 100, *or any other two adjacent terms of the series between which the number proposed lies. Also, between the mean thus found and the nearest extreme, observing that the proposed number shall fall between the mean found and that extreme, find another geometrical mean, as before; and so on, till you have arrived sufficiently near the number whose logarithm is sought.*

2. *Find as many arithmetical means between the corresponding terms of the arithmetical series* 0, 1, 2, 3, *&c., in the same order as the geometrical means were found, and the last of these will be the logarithm of the proposed number*

EXAMPLES.

1. Calculate the logarithm of 5.

Here the proposed number lies between 1 and 10.

First, then, the logarithm of 10 is 1, and the logarithm of 1 is 0.

Then, $(10\times 1)^{\frac{1}{2}}=3.162277$, which is the geometrical mean,

And, $(1+0)\div 2=\frac{1}{2}=.5$, which is the arithmetical mean.

Hence, the logarithm of 3.162277 is .5.

Secondly, the logarithm of 10 is 1, and the logarithm of 3.162277 is ,5.

Then, $(3.162277\times 10)^{\frac{1}{2}}=5.623413$, which is the geometrical mean,

And - $(1+,5)\div 2=0.75$, which is the arithmetical mean.

Hence, the logarithm of 5.623413 is 0.75.

Thirdly, the logarithm of 5.623413 is 0.75, and the logarithm of 3.162277 is 0.5.

Then, $(5.623413\times 3.162277)^{\frac{1}{2}}=4.216964$, which is the geometrical mean,

And - - $(0.5+0.75)\div 2=0.625$, which is the arithmetical mean.

Hence, the logarithm of 4.216964 is 0.625.

Fourthly, the logarithm of 5.623413 is 0.75, and the logarithm of 4.216964 is 0.625.

Then, $(5.623413 \times 4.216964)^{\frac{1}{2}} = 4.869674$, which is the geometrical mean,

And - $(0.75 + 0.625) \div 2 = 0.6875$, which is the arithmetical mean.

Hence, the logarithm of 4.869674 is 0.6875.

Fifthly, the logarithm of 5.623413 is 0.75, and the logarithm of 4.869674 is 0.6875.

Then, $(5.623413 \times 4.869674)^{\frac{1}{2}} = 5.232991$, which is the geometrical mean,

And - $(0.75 + 0.6875) \div 2 = 0.71875$, which is the arithmetical mean.

Hence, the logarithm of 5.232991 is 0.71875.

Proceeding in this way, the 22d geometrical mean will be found to agree with 5, as far, at least, as the sixth place of decimals; hence, for all practical purposes, they may be considered equal, and the 22d term in the corresponding arithmetical series be taken as the logarithm of 5.

These operations, and their results, may be expressed in the following table:

		Numbers.	Logarithms.
1.	$(10 \times 1)^{\frac{1}{2}}$	$= 3.162277$,	0.50000000.
2.	$(10 \times 3.162277)^{\frac{1}{2}}$	$= 5.623413$,	0.75000000.
3.	$(3.162277 \times 5.623413)^{\frac{1}{2}}$	$= 4.216964$,	0.62500000.
4.	$(5.623413 \times 4.216964)^{\frac{1}{2}}$	$= 4.869674$,	0.68750000.
5.	$(5.623413 \times 4.869674)^{\frac{1}{2}}$	$= 5.232991$,	0.71875000.
6.	$(4.869674 \times 5.232991)^{\frac{1}{2}}$	$= 5.048065$,	0.70312500.
7.	$(4.869674 \times 5.048065)^{\frac{1}{2}}$	$= 4.958069$,	0.69531250.
8.	$(5.048065 \times 4.958069)^{\frac{1}{2}}$	$= 5.002865$,	0.69921875.
9.	$(4.958069 \times 5.002865)^{\frac{1}{2}}$	$= 4.980416$,	0.69726562.
10.	$(5.002865 \times 4.980416)^{\frac{1}{2}}$	$= 4.991627$,	0.69824218.
11.	$(5.002865 \times 4.991627)^{\frac{1}{2}}$	$= 4.997240$,	0.69873046.
12.	$(5.002865 \times 4.997240)^{\frac{1}{2}}$	$= 5.000052$,	0.69897460.
13.	$(4.997240 \times 5.000052)^{\frac{1}{2}}$	$= 4.998647$,	0.69885254.
14.	$(5.000052 \times 4.998647)^{\frac{1}{2}}$	$= 4.999350$,	0.69891357.

		Numbers.	Logarithms.
15.	$(5.000052\times4.999350)^{\frac{1}{2}}=$	4.999701,	0.69894409.
16.	$(5.000052\times4.999701)^{\frac{1}{2}}=$	4.999876,	0.69895935.
17.	$(5.000052\times4.999876)^{\frac{1}{2}}=$	4.999963,	0.6989668.
18.	$(5.000052\times4.999963)^{\frac{1}{2}}=$	5.000008,	0.6989707.
19.	$(4.999963\times5.000008)^{\frac{1}{2}}=$	4.999984,	0.6989687.
20.	$(5.000008\times4.999984)^{\frac{1}{2}}=$	4.999997,	0.6989697.
21.	$(5.000008\times4.999997)^{\frac{1}{2}}=$	5.000003,	0.6989702.
22.	$(4.999997\times5.000003)^{\frac{1}{2}}=$	5.000000,	0.6989700.

Note 1.—A greater degree of exactness might be attained by carrying out the work to a greater number of decimal places, and continuing our interpolations.

Note 2.—Having thus obtained the logarithm of 5, and that of 10 being given, the logarithm of 2 can be readily found; for, since $10\div5=2$, logarithm of 10, minus logarithm of 5= logarithm of 2, or $1-0.6989700=0.3010300$, which is the logarithm of 2.

2. Required the logarithm of 3. *Ans.* 0.47712125.

3. Required the logarithm of 7. *Ans.* 0.84509804.

398. The great difficulty of constructing a table of logarithms is in finding the logarithms of the prime numbers. These were first computed by successive interpolations, as in the preceding examples. The logarithms of composite numbers are found by adding the logarithms of the factors whose product is equal to the composite number.

399. The computation of the logarithms of prime numbers, after the logarithm of 2 has been obtained, may be greatly abridged by the following general

RULE.*

When the logarithm of any number (n) is known, the logarithm of the next greater number may be readily found by substituting the numerical value of the letters in the following series, and then calculating a sufficient number of terms.

Let $n=$ the number whose logarithm is given, $n+1=$ the

* See Note E.

number whose logarithm is to be found, and M= the *modulus* of the system =0.4342944819, or 2M=0.8685889638. Then will

$$\text{Logarithm } (n+1) = \text{logarithm } n + 2M\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \frac{1}{7(2n+1)^7} + \frac{1}{9(2n+1)^9}\right), \text{ \&c.};$$

Or, letting A, B, C, D, &c., represent the terms immediately preceding those in which they are used,

$$\text{Logarithm } (n+1) = \text{logarithm } n + \frac{2M}{2n+1} + \frac{A}{3(2n+1)^2} + \frac{3B}{5(2n+1)^2} + \frac{5C}{7(2n+1)^2} + \frac{7D}{9(2n+1)^2}, \text{ \&c.}$$

EXAMPLES.

1. Required the logarithm of 3.

Here, since $n+1=3$, $n=2$, and $2n+1=5$, we shall have

$$\text{Logarithm } n = \text{logarithm } 2 = 0.301029995;$$

$$\frac{2M}{2n+1} = \frac{0.868588964}{5} = 0.173717793; \quad \text{(A.)}$$

$$\frac{A}{3(2n+1)^2} = \frac{0.173717793}{3\times 5^2} = 0.002316237; \quad \text{(B.)}$$

$$\frac{3B}{5(2n+1)^2} = \frac{3\times 0.002316237}{5\times 5^2} = 0.000055590; \quad \text{(C.)}$$

$$\frac{5C}{7(2n+1)^2} = \frac{5\times 0.000055590}{7\times 5^2} = 0.000001588; \quad \text{(D.)}$$

$$\frac{7D}{9(2n+1)^2} = \frac{7\times 0.000001588}{9\times 5^2} = 0.000000050; \quad \text{(E.)}$$

$$\frac{9E}{11(2n+1)^2} = \frac{9\times 0.000000050}{11\times 5^2} = 0.000000002; \quad \text{(F.)}$$

Whence the log. $(2+1)=$ log. 3 $=0.477121255$.

The above logarithm is correct as far as to the ninth place of decimals.

2. Required the logarithm of 11. *Ans.* 1.04139269.

400. The only numbers whose logarithms it will be found necessary to compute by the preceding formula, or by interpolating the series, are the prime numbers 3, 7, 11, 13,

17, 19, 23, 29, &c. The logarithms of composite numbers may be computed by the propositions verified in Art. 393.

COMPUTATION OF LOGARITHMIC TABLES.

401. The following table will exhibit the manner in which the logarithms of the natural series of numbers 1, 2, 3, 4, &c., to 30, may be computed:

Nos.	Method of Computation.	Logarithms.
1.	- - - - - - -	log. 1=0.00000000.
2.	Since 10÷5=2, log. 10− log. 5=	log. 2=0.30103000.
3.	Computed by formula in Art. 399,	log. 3=0.47712126.
4.	Since 2×2=4, log. 2+ log. 2 =	log. 4=0.60206000.
5.	Computed by interpolating the series in Art. 397 -	log. 5=0.69897000.
6.	Since 2×3=6, log. 2+ log. 3 =	log. 6=0.77815125.
7.	Computed by formula in Art. 399,	log. 7=0.84509804.
8.	Since 2×4=8, log. 2+ log. 4 =	log. 8=0.90308999.
9.	Since 3×3=9, log. 3+ log. 3 =	log. 9=0.95424251.
10.	- - - - - - -	log. 10=1.00000000.
11.	Computed by formula in Art. 399,	log. 11=1.04139269.
12.	Since 3×4=12, log. 3+ log. 4 =	log. 12=1.07918125.
13.	Computed by formula in Art. 396,	log. 13=1.11394335.
14.	Since 2×7=14, log. 2+ log. 7 =	log. 14=1.14612804.
15.	Since 3×5=15, log. 3+ log. 5 =	log. 15=1.17609126.
16.	Since 4×4=16, log. 4+ log. 4 =	log. 16=1.20411998.
17.	Computed by formula in Art. 399	log 17=1.23044892.
18.	Since 3×6=18, log. 3+ log. 6 =	log. 18=1.25527251.
19.	Computed by formula in Art. 399,	log. 19=1.27875360.
20.	Since 2×10=20, log. 2+ log. 10=	log. 20=1.30103000.
21.	Since 3× 7=21, log. 3+ log. 7=	log. 21=1.32221929.
22.	Since 2×11=22, log. 2+ log. 11=	log. 22=1.34242268.
23.	Computed by formula in Art. 399,	log. 23=1.36172784.
24.	Since 4× 6=24, log. 4+ log. 6=	log. 24=1.38021124.
25.	Since 5× 5=25, log. 5+ log. 5=	log. 25=1.39794001.
26.	Since 2×13=26, log. 2+ log. 13=	log. 26=1.41497335.
27.	Since 3× 9=27, log. 3+ log. 9=	log. 27=1.43136376.
28.	Since 4× 7=28, log. 4+ log. 7=	log. 28=1.44715803.

Nos.	Method of Computation.	Logarithms.
29.	Computed by formula in Art. 399,	log. 29 = 1.46239800.
30.	Since $3 \times 10 = 30$, log. 3 + log. 4	= log. 30 = 1.47712125.

The logarithm usually consists of two parts, the integral part, usually called the *index* or *characteristic*, and a decimal.

402. It will also be perceived that the multiplying or dividing of any number by 10, 100, 1000, &c., is performed by increasing or diminishing the integral part of its logarithm by 1, 2, 3, &c.; hence, all numbers which consist of the same figures, whether they be integers, decimals, or mixed numbers, will have for the *decimal part* of their logarithms the same positive number.

Thus, according to the tables now in common use, the logarithm of 3854 is 3.58591171.

$$\text{Log. } 3854 = 3.58591171;$$
$$\text{Log. } 38540 = \text{log.}(3854 \times 10) = \text{log. } 3854 + 1 = 4.58591171;$$
$$\text{Log. } 385{,}4 = \text{log.} \frac{3854}{10} = \text{log. } 3854 - 1 = 2.58591171;$$
$$\text{Log. } 38{,}54 = \text{log.} \frac{3854}{100} = \text{log. } 3854 - 2 = 1.58591171;$$
$$\text{Log. } 3{,}854 = \text{log.} \frac{3854}{1000} = \text{log. } 3854 - 3 = 0.58591171;$$
$$\text{Log. } {,}3854 = \text{log.} \frac{3854}{10000} = \text{log. } 3854 - 4 = \bar{1}.58591171;$$
$$\text{Log. } {,}03854 = \text{log.} \frac{3854}{100000} = \text{log. } 3854 - 5 = \bar{2}.58591171.$$

The number of units in the characteristic of a logarithm is one less than the number of digits in the natural number; and for decimals, the negative characteristic denotes how far the first significant figure is removed from the place of units. The decimal part of the logarithm is always positive.

APPLICATIONS OF LOGARITHMS.

403. The tables of logarithms in common use contain the logarithms of numbers from 1 to 10000. An explanation of these tables, and also of the methods of finding from them the *logarithm* of any number, or the *number* of any logarithm

whatever, usually accompanies them, so that such explanations are unnecessary here. The numbers and logarithms used in the following applications of logarithms are taken from these tables.

I. MULTIPLICATION AND DIVISION.

404. Since logarithms are a series of exponents denoting different powers of the common number 10, it is evident that the sum of the logarithms of any two numbers will be the logarithm of their product, and the difference of their logarithms will be the logarithm of the quotient produced by dividing the greater by the less. Hence,

I. To multiply by logarithms, *take the logarithms of the factors from the table, add them together, and then find the natural number corresponding to their sum; this will be the product required.*

1. Multiply 16 by 5, by logarithms.

Logarithm 16 $=1.20411998$;
Logarithm 5 $=0.69897000$;

Logarithm 80 $=1.90308998$. *Ans.* 80.

2. Multiply 37153 by 4086, by logarithms.

Logarithm 37153 $=4.5699939$;
Logarithm 408,6 $=2.6112984$;

Product, 15180715.8 . . 6.1812923.

3. Multiply 4675,12 by .03275, by logarithms.

Logarithm 4675.12 $=3.6697928$;
Logarithm 0.03275 $=\bar{2}.5152113$;

Product, 153,1102, &c. . 2.3850041.

II. To divide by logarithms, *subtract the logarithm of the divisor from the logarithm of the dividend, and the remainder will be the logarithm of the quotient.*

EXAMPLES.

1. Divide 72 by 24, by logarithms.

Logarithm 72 $=1.85733250$;
Logarithm 24 $=1.38021124$;

Quotient, 3 . . . . 0.47712126.

2. Divide 4768,2 by 36,954, by logarithms.

Logarithm 4768,2 $=3.6783545$;
Logarithm 36,954 $=1.5676615$;
Quotient, 129,032 . . . . 2.1106930.

3. Divide 46257 by ,17608, by logarithms.

Logarithm 46257 $=4.6651725$;
Logarithm ,17608 $=\bar{1}.2457100$;
Quotient, 262741 . . . . 5.4194625.

II. INVOLUTION AND EVOLUTION.

405. Involution is performed by multiplying the exponent of the number to be involved by the exponent denoting the power (Art. 157); and Evolution by dividing the exponent of the number by the exponent denoting the root to be taken (Art. 289). Hence,

I. To involve by logarithms, *multiply the logarithm of the number to be involved by the number denoting the power; the product will be the logarithm of the power.*

EXAMPLES.

1. Involve 9 to the second power, by logarithms.

Logarithm 9 $=0.95424251$;
Multiplying by 2, 2;
Square, 81 . . . . . . 1.90848502.

2. Involve 7.0851 to the third power, by logarithms.

Logarithm 7.0851 $=0.8503399$;
Multiplying by 3, 3;
Cube, 355,6475 . . . . 2.5510197.

3. Involve 0.9061 to the seventh power, by logarithms.

Logarithm 0.9061 $=\bar{1}.9571761304$;
Multiplying by 7, 7;
Power, 0.5015 . . . . $\bar{1}.7002329128$.

4. Involve 1.0045 to the 365th power, by logarithms

Logarithm 1.0045 $=0.0019499$;
Multiplying by 365, 365;
97495
116994
58497
Power, 5.148888 . . . 0.7117135.

II. To evolve by logarithms, *divide the logarithm of the given number by the number denoting the root to be taken ; the quotient will be the logarithm of the root.*

EXAMPLES.

1. Evolve 81 to the fourth root, by logarithms.

Logarithm 81	$=1.90848502$;
Dividing by 4,	$\div 4$;
Root, 3	0.47712120.

2. Evolve 7.0825 to the fifth root, by logarithms.

Logarithm 7.0825	$=0.8501866$;
Dividing by 5,	$\div 5$;
Root, 1.479235	0.1700373.

3. Evolve 1.045 to the 365th root, by logarithms.

Logarithm 1.045	$=0.0019116$;
Dividing by 365,	$\div 365$;
Root, 1.000121	0.0000524.

4. What is the 8th power of the 9th root of 654 ?

Logarithm 654	$=2.8155777483$;
Multiplying by 8,	8 ;
	22.5246219864 ;
Dividing by 9,	$\div 9$;
Root, 318.3	2.507357762.

III. EXPONENTIAL EQUATIONS.

406. Equations into which the unknown quantity enters in the form of an *index* are called *exponential equations.*

Such equations may be most readily solved by logarithms. Thus, $a^x=b$, but $a^x=(\text{log. } a)\times x$; therefore, $(\text{log. } a)\times x=$ log. b, or, dividing by log. a, $x=\frac{\text{log. } b}{\text{log. } a}$.

EXAMPLES.

1. Reduce the equation $5^x=100$.

As the two members are equal, their logarithms must also be equal ; therefore,

$$(\text{Log. }5)\times x = \text{log. }100;$$

Dividing - - $x=\dfrac{\text{log. }100}{\text{log. }5}=\dfrac{2.00000000}{0.69897000}=2.861.$

2. Reduce the equation $3^x=243$.

$$(\text{Log. }3)\times x = \text{log. }243;$$

Dividing - - $x=\dfrac{\text{log. }243}{\text{log. }3}=\dfrac{2.38561}{0.47712}=5.$

407. Another and a more difficult form of exponential equation is $a^{m^x}=b$. Here the exponent x is the exponent of the exponent m.

In this equation assume $m^x=y$, then $a^y=b$,

And - $(\text{log. }a)\times y=\text{log. }b;$

Dividing - - $y=\dfrac{\text{log. }b}{\text{log. }a};$

Hence - - $m^x=\dfrac{\text{log. }b}{\text{log. }a}$ (which let) $=c;$

Then - $(\text{log. }m)\times x=\text{log. }c;$

Dividing - - $x=\dfrac{\text{log. }c}{\text{log. }m}.$

EXAMPLE.

1. Reduce the equation $9^{3^x}=1000$.

$$(\text{Log. }9)\times 3^x=\text{log. }1000;$$

Dividing - - $3^x=\dfrac{\text{log. }1000}{\text{log. }9}=\dfrac{3.00000000}{0.95424251}=3.14;$

Then - - - $3^x=3.14 \therefore (\text{log. }3)\times x=\text{log. }3.14,$

And - - - $x=\dfrac{\text{log. }3.14}{\text{log. }3}=\dfrac{0.49692965}{0.47712126}=1.04.$

2. Reduce the equation $4^{3^x}=4096$.

Ans. $x=\dfrac{\text{log. }6}{\text{log. }3}=1.6309+.$

408. A third and a still more difficult form of the exponential equation is $x^x=b$.

Taking the logarithms of both sides, we have

$$(\text{Log. }x)\times x=\text{log. }b.$$

This equation may be solved by "*Trial* and *Error*." Thus, make two suppositions of the value of the unknown quantity, and find their errors; then institute the following proportion:

Diff. of the errors : diff. of the assumed numbers : : least error : to the correction required in the corresponding assumed number.

EXAMPLES.

1. Reduce the equation $x^x=256$.

Then - $(\log. x)\times x=\log. 256$;

Suppose - - $x=3.5$, or 3.6.

By first Supposition.		By second Supposition.	
$\text{Log. } x=\log. 3.5$	$=0.54406804$	$\text{Log. } x=\log. 3.6$	$=0.55630250$
Multiplying by	3 5	Multiplying by	3.6
$(\text{Log. } x)\times x$	$=1.90423814$	$(\text{Log. } x)\times x$	$=2.00263900$
But, log. 256	$=2$ 40823997	Log. 256	$=2.40823997$
Error* -	0.50400183	Error* -	0.39555097

Difference of the errors, 0.10844086.

Then, 0.10844086 : 0.1 : : 0.39555097 : 0.365 ; hence $x=3.965+$.

To correct this still farther, suppose $x=3.96$, or 4.01.

By first Supposition.		By second Supposition.	
$\text{Log. } x=\log. 3.96$	$=0.59769519$	$\text{Log. } x=\log. 4.01$	$=0.60314437$
Multiplying by	3 96	Multiplying by	4.01
$(\text{Log. } x)\times x$	$=2$ 36687295	$(\text{Log. } x)\times x$	$=2.41860892$
Log. 256	$=2.40823997$	Log. 256	$=2.40823997$
First error†	$=0.04136702$	Second error†	$=0.01036895$

Difference of the errors $=0$ 05173597.

Then, 0.05173597 : 0.05 : : 0.01036895 : 0.01.

Hence, $x=4.01-0.01=4$, which value for x satisfies the conditions of the equation; for

$$4^4=256.$$

2. Reduce the equation $4x^x=100x^3$.

* Both these suppositions are discovered to be less than the true number; hence the errors are *like* with reference to their signs.

† One of these suppositions is less, the other greater than the true value of x; hence the errors are *unlike* with reference to their signs.

Dividing by 4 - - - - $x^x=25x^3$;
Dividing by x^3 - - - - $x^{x-3}=25$;
Taking the logarithm, (log. x)$\times(x-3)=$ log. 25;
Suppose - - - - - $x=4$, or 6.

By first Supposition.		By second Supposition.	
Log. $x=$ log. 4	$=0.60205999$	Log. $x=$ log. 6	$=0.77815125$
Multiplying by $x-3$	$=$ 1	Multiplying by $x-3$	3
(Log. x)$\times(x-3)$	$=0.60205999$	(Log. x)$\times(x-3)$	$=2.33445375$
Log. 25	$=1.39794001$	Log. 25	$=1.39794001$
First error	$=0.79588002$	Second error	$=0.93651374$

Difference of the errors $=1.73239376$.
Then, 1.73239376 : 2 : : 0.79588002 : 0.092.
Hence, $x=4+0.92=4.92$.
Again, suppose $x=4.92$, or 4.93.

By first Supposition.		By second Supposition.	
Log. $x=$ log. 4.92	$=0.69196510$	Log. $x=$ log. 4.93	$=0.69284692$
Multiplying by $x-3$	$=$ 1.92	Multiplying by $x-3$	$=$ 1.93
(Log. x)$\times(x-3)$	$=1.32857299$	(Log. x)$\times(x-3)$	$=1.33719456$
Log. 25	$=1.39794001$	Log. 25	$=1.39794001$
First error	$=0.06936702$	Second error	$=0.06074545$

Difference of errors $=0.00851157$.
Then, 0.00851157 : 0.01 : : 0.06074545 : 0.07.
Hence, $x=4.93+0.07=5.00$, which value for x satisfies the conditions of the equation; for

$$4\times5^5=100\times5^3.$$

IV. GEOMETRICAL SERIES.

409. Logarithms are also very convenient in finding the last term, and also the sum of the series in Geometrical Progression, when n is not a very small number. The number of terms may also be obtained by the aid of logarithms.

I. The formula for the last term is (Art. 383),

$$l=ar^{n-1};$$

Taking the log., log. $l=$ log. $a+(\text{log. } r)\times(n-1)$.

Hence, to find the last term in a geometrical series by logarithms, *add the logarithm of the first term to the logarithm of the ratio multiplied by the number of terms less one; the sum will be the logarithm of the last term.*

EXAMPLE.

1. The first term of a geometrical series is 4, the ratio 5, and the number of terms 61. Required the last term.

$l = ar^{n-1} = 4 \times 5^{60}$;

Or, log. $l =$ log. 4+(log. 5)$\times$60=0.60205999+41.9382000 =42.54025999.

Hence, finding the natural number corresponding with 42.54025999,

l=346947939257793400974674442757034433170887 6.

2. The formula for the sum of the terms in a geometrical series is (Art. 386)

$$S = \frac{ar^n - a}{r-1}.$$

In this formula, if n is not a small number, it will be found convenient to find the value of ar^n by taking

$$\text{Log. } (ar^n) = \text{log. } a + (\text{log. } r) \times n.$$

Thus, we may find the value of ar^n in the same way that we found the *last term* in the preceding case.

3. To obtain a formula for the number of terms, let us resume the formula for the sum of the terms,

$$S = \frac{ar^n - a}{r-1};$$

Clearing of fractions, $rS - S = ar^n - a$;

Transposing - - $ar^n = rS - S + a$;

Dividing - - - $r^n = \frac{rS - S + a}{a}$;

Hence, - - (log. r)$\times n =$ log. $(rS - S + a) -$ log. a;

Dividing - - - - $n = \frac{\text{log. } (rS - S + a) - \text{log. } a}{\text{log. } r}$.

EXAMPLES.

1. The sum of a geometrical series is 6560, the first term 2, and the ratio 3. Required the number of terms.

Here - - S=6560, a=2, and r=3;

Hence - $n=\frac{\log.(rS-S+a)-\log.a}{\log.r}$;

$=\frac{\log.13122-\log.2}{\log.3}$.

$=\frac{3.8169700}{0.4771213}=8.$

2. The sum of a geometrical series is 1023, the first term 1, and the ratio 2. Required the number of terms.

Ans. 10.

3. The sum of a geometrical series is 640, the first term 4, and the ratio 1.01. Required the number of terms.

V. COMPOUND INTEREST.

410. By means of logarithms we may also determine the number of years it will take a given principal, at a given rate, compound interest, to gain a certain amount. Thus, in Art. 219, we have the formula

$$A=P(1+r)^n;$$

Taking the logarithms, $\log.A=\log.P+(\log.(1+r))\times n$;

Transposing, $(\log.(1+r))\times n=\log.A-\log.P$;

$$n=\frac{\log.A-\log.P}{\log.(1+r)}.$$

Note.—By means of this formula we may ascertain the number of years it would take a sum of money to double, triple, &c., or amount to m times itself, when put out at compound interest, at a given rate per cent.

EXAMPLES.

1. A man loans $1250, at 6 per cent. compound interest. In what time will it amount to $4008.92?

In this example, A=$4008.92, P=1250, and $r+1=,06+1=1.06$.

Hence - $n=\frac{\log.4008.92-\log.1250}{\log.1.06}$;

$=\frac{3.60302739-3.09691001}{0.02530587}=20$ nearly.

2. At 6 per cent. compound interest, in how many years will $1200 amount to $2149.19? *Ans.* 10 years.

3. At 6 per cent. compound interest, in how many years will money amount to double, triple, and quadruple the original sum? *Ans.* 11,9955, 18,8145, and 23,791 years.

APPENDIX.

The three following sections, as they contain those portions of algebraic analysis which are seldom pursued in academies and schools, but which are nevertheless essential to the successful prosecution of the higher branches of the mathematical course, have been thrown into the form of an appendix.

APPENDIX.

SECTION X.

Permutations, Arrangements, and Combinations.—Demonstration of the Binomial Theorem.—Continued Fractions.—Infinite Series.—Expansion of Infinite Series.—Indeterminate Coefficients.—Summation of Infinite Series.—Recurring Series.—Method of Differences.—Reversion of Series.

PERMUTATIONS, ARRANGEMENTS, AND COMBINATIONS.

I. PERMUTATIONS.

411. PERMUTATIONS are the results obtained by writing a given number of letters, one after the other, in every possible way, in such a manner that all the letters may enter into each result, and each letter enter but once.

Thus, the two letters a and b furnish the two permutations - - - - - - - - - $\begin{cases} ab \\ ba \end{cases}$

The three letters, a, b, and c, furnish six permutations, viz.:

$$\overbrace{abc,\ acb,\ cab,\ cba,\ bac,\ bca.}$$

Hence, *the permutations of three letters are equal to the permutations of two letters multiplied by three.*

412. In like manner, *the permutations of four letters will be found equal to the permutations of three letters multiplied by four.*

And, in general, *the permutations of any number whatever (n) of letters will be equal to the permutations of $n-1$ letters, multiplied by n, the number of letters employed.* Letting Q denote the number of permutations of $n-1$ letters, then the general formula for the permutations of n letters will be $Q\times n$.

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If $n=2$, the number of permutations will be $1\times2=2$.

If $n=3$, the number of permutations will be $1\times2\times3=6$.

If $n=4$, the number of permutations will be $1\times2\times3\times4=24$.

If $n=5$, the number of permutations will be $1\times2\times3\times4\times5=120$, &c.

Hence, for the permutation of any given number of letters or numbers, we infer the following general

RULE.

Multiply in order the natural numbers 1, 2, 3, 4, *&c., to the number denoting the letters employed inclusive; the result will be the permutations of the given number of letters.*

EXAMPLES.

1. How many permutations can be made of the first 6 letters of the alphabet? *Ans.* $1\times2\times3\times4\times5\times6=720$.

2. How many permutations can be made of the first 8 letters of the alphabet?

3. In how many different ways may 12 different persons be seated at the table?

II. ARRANGEMENTS.

413. *Arrangements* are the results obtained by writing a given number of letters in sets, 2 and 2, 3 and 3, &c., in every possible order.

Let it be required to arrange the three letters, a, b, and c, in sets of two each. Setting apart a, we write after it each one of the reserved letters, b and c, and thus form two of the arrangements sought, viz., ab and ac; next, setting apart b, we write after it each one of the reserved letters a and c, and form two more of the arrangements sought, viz., ba and bc; pursuing the same course with c, we obtain - - - - - - - - -

$$\begin{cases} ab \\ ac \\ \text{—}a \\ ba \\ bc \\ \text{—}b \\ ca \\ cb \\ \text{—}c \end{cases}$$

The arrangement of the same letters in sets of one each would give - - - - - -

$$\begin{cases} a \\ b \\ c \end{cases}$$

Hence, *the arrangement of three letters, taken two in a set, will* *be equal to the arrangement of the same letters taken one at a time, multiplied by the number of letters reserved.*

414. Let it be required, in the next place, to form the arrangement of four letters, a, b, c, and d, taken three in a set.

First arranging the letters two in a set, we shall have 12 arrangements, viz.:

$$\overbrace{ab,\ ac,\ ad,\ ba,\ bc,\ bd,\ ca,\ cb,\ cd,\ da,\ db,\ dc.}$$

Next, take one of the above sets, ab, for example, and write after it successively each one of the reserved letters c and d, and thus form two of the arrangements sought, viz., abc and abd. Proceeding in the same manner with the remaining sets, we shall obtain 24 arrangements, viz.:

abc	bac	cab	dab
abd	bad	cad	dac
acb	bca	cba	dba
acd	bcd	cbd	dbc
adb	bda	cda	dca
adc	bdc	cdb	dcb

Hence, *the arrangements of four letters, taken three in a set, will be equal to the arrangements of the same letters taken two in a set, multiplied by the number of letters reserved.*

415. In like manner, *we have the arrangements of any number* (m) *of letters taken n in a set, equal to the arrangements of the same letters, n—1 in a set, multiplied by the number of letters reserved.*

416. Let P represent the total number of *arrangements* of m letters taken $n-1$ in a set, supposing this number to be known; the reserved letters, when it is required to take n in a set, will be $m-(n-1)=m-n+1$, and the number of *arrangements* of m letters, taking n in a set, will be

$$P\times(m-n+1).$$

This is the general formula for arrangements. To apply it to particular cases, let $n=2$; then $m-n+1=m-1$, and P will represent the *arrangements* of m letters taken 1 at a time; whence $P=m$, and $m(m-1)$ will represent the arrangements of m letters taken two in a set.

Again, let $n=3$, then $m-n+1=m-2$, and $P=m(m-1)$; whence the formula becomes

$$m(m-1)(m-2).$$

Let $n=4$, then $m-n+1=m-3$, and $P=m(m-1)(m-2)$; whence the formula becomes

$$m(m-1)(m-2)(m-3).$$

Hence we infer the following general

RULE.

1. *From the number denoting the given letters, subtract successively the natural numbers* 1, 2, 3, *&c., to the number which denotes the letters to be taken at a time.*

2. *Multiply these several remainders and the number denoting the given letters together; the product will be the arrangements required.*

EXAMPLES.

1. How many arrangements may be made of the first six letters of the alphabet, taken three in a set?

In this example $m=6$, and $n=3$; then we have

$$m(m-1)(m-2)=6\times5\times4=120. \quad \textit{Ans.}$$

2. How many arrangements may be made of the 26 letters of the alphabet, taking 6 in a set?

Note.—It should be observed that, when $m=n$, the number of *arrangements* is the same as the number of *permutations*. Thus, if there be six letters, to be taken six in a set, we have

$$m(m-1)(m-2)(m-3)(m-4)(m-5)=6\times5\times4\times3\times2\times1=720.$$

III. COMBINATIONS.

417. *Combinations* are arrangements, any two of which will differ from each other by at least one of the letters which enter into them.

Let it be required to determine the number of combinations of which the three letters a, b, and c, taken two in a set, are susceptible. The arrangements of these letters, two in a set, are

$$\overbrace{\begin{matrix} ab & ac & bc \\ ba & ca & cb \end{matrix}}.$$

In these six arrangements we have but three combinations, viz., ab, ac, and bc, each one of which is repeated as many times as there are permutations of two letters.

Hence *the combinations of three letters, taken two in a set, will be equal to the arrangements of three letters, taken two in a set, divided by the permutations of two letters.*

418. In like manner, it may be shown that the combinations of four letters, taken three in a set, are equal to the arrangements of four letters, taken three in a set, divided by the permutations of three letters.

And, in general, the combinations of m letters, taken n in a set, will be equal to the arrangements of m letters, taken n at a time, divided by the permutations of n letters.

Hence we have the following general formula for combinations:

$$\frac{P\times(m-n+1)}{Q\times n}.$$

Letting $n=2$, the formula for the combinations of m letters, 2 in a set, becomes

$$\frac{m(m-1)}{1\times 2}.$$

Letting $n=3$, the formula for the combinations of m letters, taken 3 in a set, is

$$\frac{m(m-1)(m-2)}{1\times 2\times 3}.$$

And, if $n=4$, we shall have

$$\frac{m(m-1)(m-2)(m-3)}{1\times 2\times 3\times 4}.$$

Hence we infer the following general

RULE.

To find the combinations of m letters, taken n in a set, *divide the arrangements of m letters, taken n in a set, by the permutations of n letters.*

EXAMPLES.

1. How many combinations can be made of 10 letters, taking four in a set?

$$\frac{m(m-1)\,(m-2)\,(m-3)}{Q\times m}=\frac{10\times9\times8\times7}{1\times2\times3\times4}=210. \quad \textit{Ans.}$$

2. How many combinations can be made of the 26 letters of the alphabet, taking two in a set?

3. How many combinations can be made of 100 things, taking four in a set?

GENERAL DEMONSTRATION OF THE BINOMIAL THEOREM.

419. We have already exhibited the Binomial Theorem and its applications to numbers; but we propose now to give a more rigid, and, at the same time, a more general demonstration of it. It is easy to fix upon the law for the exponents, but that for the coefficients is not so obvious. If we observe, however, the manner in which the different terms that compose a power are formed, we shall perceive that the numerical coefficients are occasioned by the reduction of several similar terms into one, and that these similar terms arise from the equality of the factors, which compose a power. Hence, these reductions will not take place if the second terms of the binomial are different.

420. We will begin, therefore, by investigating the law for the formation of the product of any number of binomials, $x+a$, $x+b$, $x+c$, ... of which the first term is the same in each, and the second terms different. The coefficients of like powers of x are placed under each other, and separated from x, into which their sum is to be multiplied, by a vertical line.

$$\begin{array}{lll} & x+a & \\ & x+b & \\ \hline \text{1st product} \quad . \quad . & x^2+a & x+ab \\ & \quad\;\; +b & \end{array}$$

$$\begin{array}{llll} & x+c & & \\ \text{2d product . . .} & x^3+a & x^2+ab & x+abc \\ & \quad +b & \quad +ac & \\ & \quad +c & \quad +bc & \end{array}$$

$$\begin{array}{lllll} & x+d & & & \\ \text{3d product . . .} & x^4+a & x^3+ab & x^2+abc & x+abcd \\ & \quad +b & \quad +ac & \quad +abd & \\ & \quad +c & \quad +ad & \quad +acd & \\ & \quad +d & \quad +bc & \quad +bcd & \\ & & \quad +bd & & \\ & & \quad +cd & & \end{array}$$

421. From an inspection of the above products, which we have formed by the common rules of multiplication, it will be easy to infer the following laws:

1. *The exponent of x in the first term is the same as the number of binomial factors employed, and decreases by 1 in each of the following terms.*

2. *The coefficient of the first term is unity; the coefficient of the second term is equal to the sum of the second terms of the binomials; that of the third term is equal to the sum of the different combinations of the second terms of the binomials, taken two in a set; and that of the fourth is equal to the sum of the products of the second terms of the binomials, taken three in a set, and so on. The last term is equal to the product of the second terms of the binomials.*

422. In order to show that the same law will obtain whatever be the number of factors employed, it is only necessary to prove that if the law be true for the product of any number (m) of binomials, it will also be true for the product of $m+1$ binomials.

Let us represent the product of m binomial factors by

$$x^m+Ax^{n-1}+Bx^{m-2}+Cx^{n-3}+ \ldots\ldots\ldots\ldots U.$$

Multiplying this expression by a new factor, $x+K$, it becomes

$$\begin{array}{l|l|l|l} x^{m+1}+A & x^m+B & x^{m-1}+C & x^{m-2}+ \ldots\ldots UK. \\ \quad +K & \quad +AK & \quad +BK & \end{array}$$

423. Here the law of the exponents is evidently the same as before. With respect to the coefficients, it is evident,

1. That the coefficient of the first term is unity.

2. That A+K, the coefficient of the second term, is equal to the sum of the second terms of the $m+1$ binomials.

3. That, since B, by hypothesis, expresses the sum of the second terms of the m binomials, taken two in a set, and AK expresses the sum of the second terms of the m binomials, multiplied each by the new second term K, B+AK, the coefficient of the third term will be the sum of the products, two in a set, of the second terms of the $m+1$ binomials.

4. That C+BK is the sum of the products, taken three in a set, of the second terms of the $m+1$ binomials, and so on.

5. That the last term UK is the product of $m+1$ second terms.

424. The law laid down in Art. 421, being true for expressions of the fourth degree, will, from what has just been demonstrated, be true for those of the fifth; and, being true for expressions of the fifth degree, will also be true for those of the sixth, and so on indefinitely.

425. If, in the different products which we have formed, we make the second terms of all the binomials equal, *i. e.*, make $a=b=c=d$, &c., these products will be converted into powers of $x+a$. Thus,

$$\begin{array}{lll|l|l}
 & & x+a & & \\
 & & x+a & & \\
\hline
\text{1st product} & \ldots & x^2+a & x+a^2 & \\
 & & \quad +a & & \\
 & & x+a & & \\
\hline
\text{2d product} & \ldots & x^3+a & x^2+a^2 & x+a^3 \\
 & & \quad +a & \quad +a^2 & \\
 & & \quad +a & \quad +a^2 &
\end{array}$$

	$x+a$			
3d product . . .	x^4+a	x^3+a^2	x^2+a^3	$x+a^4$
	$+a$	$+a^2$	$+a^3$	
	$+a$	$+a^2$	$+a^3$	
	$+a$	$+a^2$	$+a^3$	
		$+a^2$		
		$+a^2$		

426. By comparing these results with the products from which they have been derived, we perceive,

1. That the multiplier of x in the second term has been converted into the first power of a, repeated as many times as there are units in the number of binomial factors used, or, which is the same thing, as there are units in the exponent denoting the power to which $x+a$ was to be involved.

2. That the multiplier of x in the third term has been converted into a^2, repeated as many times as there can be formed different products from a number of letters equal to the number of binomials employed, taken two in a set.

3. That the multiplier of the fourth term has been converted into a^3, repeated as many times as there can be formed different products from a number of letters, equal to the number of binomials employed, taken three in a set, and so on.

427. It is therefore evident that, whatever may be the power to which the binomial $x+a$ is to be raised, the formation of its power will be subject to the following laws, viz.:

1. *The exponent of x in the first term will be equal to the exponent of the power, and in the succeeding terms will decrease regularly by* 1 *to the last term, in which it will be* 0.

2. *The exponent of a in the first term will be* 0, *in the second* 1, *and that it will go on increasing by* 1 *until it becomes equal to the exponent of the power to which the binomial was to be involved.*

3. *That the numerical coefficient of x in the first term will be* 1; *in the second it will be equal to the exponent denoting the*

power to which the binomial was to be involved; in the third term it will be equal to the number of products, which may be formed from a number of letters, equal to the exponent denoting the power of the binomial, taken two in a set; in the fourth term it will be equal to the number of products which may be formed from the same number of letters taken three in a set, &c.

428. The above theorem, with reference to the coefficients, is too complicated for general use. In order to simplify it, let it be required to expand $(x+a)^m$. The first few and the last few terms, without the numerical coefficients, will be

$$x^m+ax^{m-1}+a^2x^{m-2}+a^3x^{m-3}+\ .\ .\ .\ .\ .\ +x^3a^{m-3}+x^2a^{m-2}+xa^{m-1}+a^m \quad \text{(A)}.$$

The numerical coefficient of the first term is 1; that of the second is m; that of the third is equal to the number of products which may be formed of m letters taken two in a set; this is expressed by the formula $\frac{m(m-1)}{1\times 2}$: the coefficient of the fourth term is $\frac{m(m-1)(m-2)}{1\times 2\times 3}$, &c.

By inspecting the above formulas for the numerical coefficients of x, it will be perceived that the coefficient of the third term is equal to the coefficient of the second term (m) multiplied by the exponent of x $(m-1)$ in that term, the product divided by the number (2) which marks the place of this term, counting from the left.

And, also, the coefficient of the fourth term is equal to the coefficient of the third term $\left(\frac{m(m-1)}{1\times 2}\right)$, multiplied by the exponent of x $(m-2)$ in that term, the product divided by the number (3) denoting the place of that term &c.

429. Again, since in the expression $(x+a)$, a may be substituted for x, and x for a, without altering its value, it follows that the same thing may be done in the development of it. Hence, if this development contains a term of the form $\text{K}a^nx^{m-n}$ (K representing the numerical coefficient), it

must have another equal to $Kx^n a^{m-n}$, or $Ka^{m-n}x^n$. These two terms are evidently at equal distances from the two extremes, for the number of terms which precede any term being indicated by the exponent of a in that term, it follows that the term $Ka^n x^{m-n}$ has n terms before it, and that the term $Ka^{m-n}x^n$ has $m-n$ terms before it, and, consequently, n terms after it, since the whole number of terms is denoted by $m+1$.

Therefore, *in the development of any power of a binomial, the coefficients at equal distances from the extremes are equal to each other.*

Hence, the numerical coefficients of the series A will be

$$1+m+\frac{m(m-1)}{1\times 2}+\frac{m(m-1)(m-2)}{1\times 2\times 3}+\ldots\ldots\frac{m(m-1)(m-2)}{1\times 2\times 3}+\frac{m(m-1)}{1\times 2}+m+1 \quad \text{(B)}.$$

Compounding the two series A and B, we have

$$(x+a)^m=x^m+max^{m-1}+\frac{m(m-1)}{1\times 2}a^2x^{m-2}+\frac{m(m-1)(m-2)}{1\times 2\times 3}a^3x^{m-3}+\ldots+\frac{m(m-1)(m-2)}{1\times 2\times 3}x^3a^{m-3}+\frac{m(m-1)}{1\times 2}x^2a^{m-2}+mxa^{m-1}+a^m.$$

430. The preceding operations give rise to the following simple theorem for obtaining the coefficients:

1. *The coefficient of the first term is* 1; *that of the second is equal to the number of units in the exponent, which denotes the power to which the binomial is to be raised.*

2. *And universally, if we multiply the numerical coefficient by the exponent of x in that term, and then divide the product by the number which marks the place of that term from the left, the quotient will be the coefficient of the succeeding term.*

3. *The terms in the last half of the series of coefficients will be found to correspond with those in the first half, placed in the inverse order.*

431. These results of the *Binomial Formula* are substantially the same as those obtained by a different process, and practically applied in Articles 259–272.

It should also be remarked that the same formula will apply whether m represent a positive or negative whole number or a fraction.

CONTINUED FRACTIONS.

432. A *continued fraction* is one which has 1 for its numerator, and for its denominator an entire number plus a fraction; which fraction also has 1 for its numerator, and for its denominator an entire number plus a fraction, and so on. Thus,

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+,\ \&c.,}}}}$$

is a continued fraction.

I. To convert a vulgar fraction into a continued fraction.

RULE.

Apply to the two terms of the fraction the process of finding their greatest common divisor; continue the operation until 0 is obtained for a remainder; the reciprocals of the successive quotient will form the partial fractions, which constitute the continued fraction.

Note.—The above rule may be readily illustrated by applying it to a particular case. Take, for example, the fraction $\frac{351}{965}$; dividing both numerator and denominator by the numerator, we obtain

$$\frac{351}{965}=\cfrac{1}{2+\cfrac{263}{351}}.$$

Performing the same operation upon $\frac{263}{351}$, we obtain

$$\frac{263}{351}=\cfrac{1}{1+\cfrac{88}{263}}.$$

Again - - $\frac{88}{263}=\cfrac{1}{2+\cfrac{87}{88}}$,

And - - $\frac{87}{88}=\cfrac{1}{1+\cfrac{1}{87}}$;

Hence - - $\frac{351}{965}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{87}}}}}$.

Now, if we apply the ordinary rule for finding the greatest common measure of two numbers to the two terms of the fraction $\frac{351}{965}$, the successive quotients will be 2, 1, 2, 1, 87, and their reciprocals $\frac{1}{2}$, $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{1}$, and $\frac{1}{87}$, which are evidently the partial fractions which compose the above continued fraction.

EXAMPLES.

1. Transform $\frac{65}{149}$ into a continued fraction.

Ans. $\frac{65}{149}=\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}}}}$.

2. Transform $\frac{1769}{5537}$ into a continued fraction.

3. Transform $\frac{965}{351}$ into a continued fraction.

4. Transform $\frac{8786}{10948}$ into a continued fraction.

II. To find the equivalent vulgar fraction for a given continued fraction.

RULE.

1. *If there be any whole number prefixed to the fractional series, that will be the first approximate value; if there be no such whole number, then we know that the vulgar fraction sought is proper, and the symbol* $\frac{0}{1}$ *is used to express its first approximate value.*

2. *The second approximate value is obtained by taking the sum of the first approximate value and the first partial fraction.*

3. *To obtain the third approximate value, multiply the numerator and denominator of the second approximate value by the denominator of the next partial fraction, and to the respective products add the numerator and denominator of the first approximate value.*

4. *And universally, if we multiply the terms of the last approximate value by the denominator of the succeeding partial fraction, and to the products add the numerator and denominator of the preceding approximate value, the result will be the succeeding approximate value. Thus continue till the last partial fraction has been used.*

Note 1.—The preceding rule may be readily illustrated by applying it to a particular example. Thus, let it be required to find the equivalent vulgar fraction for the continued fraction

$$\frac{0}{1}+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}}}}.$$

Here the first approximate value is - - - $\frac{0}{1}$.

The second, omitting all after the first partial fraction, is - - - - - - $\frac{0}{1}+\frac{1}{2}=\frac{1}{2}$.

The third, omitting all after the first two partial fractions, is

$$\frac{0}{1}+\frac{1}{2+\frac{1}{3}}=\frac{0}{1}+\frac{1}{\frac{2\times3+1}{3}}, \text{ or } \frac{1\times3+0}{2\times3+1}=\frac{3}{7}.$$

The fourth is $\frac{0}{1}+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2}}} = \frac{3\times2+1}{7\times2+2}=\frac{7}{16}.$

The fifth is $\frac{0}{1}+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2}}}} = \frac{7\times2+3}{16\times2+7}=\frac{17}{39}.$

The sixth is $\frac{0}{1}+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1}}}}} = \frac{17\times1+7}{39\times1+16}=\frac{24}{55}$

The seventh is $\frac{0}{1}+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}}}} = \frac{24\times2+17}{55\times2+39}=\frac{65}{149}.$

Note 2.—The successive reductions, it will be perceived by inspecting the above results, are alternately less and greater than the whole continued fraction, and they approximate this fraction nearer and nearer. The first reduction is always less than the whole continued fraction. Hence *the reductions of an odd rank are always less than the whole continued fraction, and those of an even rank are greater.*

In the above reductions,

The second differs from the true value of the continued fraction by - - - $\frac{1}{2}-\frac{65}{149}=\frac{19}{298}.$

The third differs by - - $\frac{65}{149}-\frac{3}{7}=\frac{8}{1043}.$

The fourth differs by - - $\frac{7}{16}-\frac{65}{149}$ $=\frac{3}{2384}$.

The fifth differs by - - $\frac{65}{149}-\frac{17}{39}$ $=\frac{2}{5811}$.

The sixth differs by - - $\frac{24}{55}-\frac{65}{149}$ $=\frac{1}{8195}$.

The seventh differs by - - $\frac{65}{149}-\frac{65}{149}$ $= 0$.

Note 3.—If a vulgar fraction which is not expressed in its lowest terms be converted into a continued fraction, and all the reductions be formed to the last inclusive, the last reduction will not be the proposed fraction, but this fraction reduced to its lowest terms.

For example, let the fraction $\frac{348}{954}$ be converted into a continued fraction. Thus,

$$\frac{348}{954}=\frac{0}{1}+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{9}}}}}.$$

The reductions of this continued fraction are,

$$\frac{0}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{5},\ \frac{3}{8},\ \text{and}\ \frac{29}{77}.$$

The last reduction, $\frac{29}{77}$, is the same as $\frac{348}{954}$ reduced to its lowest terms.

EXAMPLES.

1. Required the vulgar fraction which is equivalent to the continued fraction

$$\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{6}}}}}.$$

Ans. $\frac{287}{992}$.

2. Required the approximate values of the continued fraction

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6+\cfrac{1}{7+\cfrac{1}{8+\cfrac{1}{9}}}}}}}}}.$$

3. Required the approximative values of the continued fraction

$$\cfrac{1}{9+\cfrac{1}{8+\cfrac{1}{7+\cfrac{1}{6+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2}}}}}}}}.$$

4. The ratio of the circumference of a circle to its diameter may be expressed by the fraction $\frac{314159}{100000}$; required some of the approximative values of this ratio.

Converting the given fraction into a continued fraction, we have

$$\frac{314159}{100000}=3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{25+\cfrac{1}{1+\cfrac{1}{7+\cfrac{1}{4}}}}}}}.$$

The successive reductions are,

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{9208}{2931}, \frac{9563}{3044}, \frac{76149}{24239}, \text{ and } \frac{314159}{100000}.$$

INFINITE SERIES.

433. An *Infinite Series* is a progression of numbers connected together by the signs + or −, proceeding onward without termination, but usually according to some regular law, which may be discovered by tracing a few of the leading terms.

A *Converging Series* is one whose successive terms decrease. Thus,

$$\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}+, \&c.,$$

and - - $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+, \&c.,$

are converging series, when $x>1$ in the first series.

A *Diverging Series* is one whose successive terms increase. Thus,

$$x+x^2+x^3+x^4+x^5+, \&c.,$$

and - $2+4+8+16+32+, \&c.,$

are diverging series, when $x<1$ in the first series.

I. EXPANSION OF INFINITE SERIES.

434. There are four general methods of converting algebraic expressions into an infinite series of equivalent value.

First. We have already seen that the division of algebraic quantities (Art. 110) will sometimes produce an infinite series. Also, a fraction may sometimes be expanded into an infinite series by dividing the numerator by the denominator.

EXAMPLES.

1. Divide $1+a$ by $1-a$.

$$\begin{array}{l|l} 1+a & 1-a \\ \hline 1-a & 1+2a+2a^2+2a^3+2a^4+, \text{ \&c., } ad\ infin. \\ \hline \quad 2a & \\ \quad 2a-2a^2 & \\ \hline \qquad 2a^2 & \\ \qquad 2a^2-2a^3 & \\ \hline \qquad\quad 2a^3 & \\ \qquad\quad 2a^3-2a^4 & \\ \hline \qquad\qquad 2a^4 & \end{array}$$

2. Reduce the fraction $\frac{1}{1-a}$ to an infinite series.

Since the value of a fraction is the quotient resulting from the division of the numerator by the denominator (Art. 125), the value of the above fraction will be obtained by dividing 1 by $1-a$.

$$\begin{array}{l|l} 1 & 1-a \\ \hline 1-a & 1+a+a^2+a^3+a^4+, \text{ \&c., } ad\ infin. \\ \hline \quad a & \\ \quad a-a^2 & \\ \hline \qquad a^2 & \\ \qquad a^2-a^3 & \\ \hline \qquad\quad a^3 & \\ \qquad\quad a^3-a^4 & \\ \hline \qquad\qquad a^4 & \end{array}$$

Note.—By observing that the value of a fraction is equal to the terms of the quotient + the fraction formed by placing the remainder over the denominator, we shall have

$$\frac{1}{1-a}=1+a+a^2+a^3+a^4+a^5+ \;.\;.\;.\;.\;.\; a^n+\frac{a^{n+1}}{1-a}.$$

3. Reduce the fraction $\frac{1}{1-\frac{1}{2}}$ to an infinite series.

Ans. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+$, &c., *ad infin.*

4. Reduce the fraction $\frac{h}{a-b}$ to an infinite series.

Ans. $\frac{h}{a}+\frac{bh}{a^2}+\frac{b^2h}{a^3}+$, &c

5. Reduce the fraction $\frac{1}{1+a}$ to an infinite series.

Ans. $1-a+a^2-a^3+a^4-a^5$, &c., *ad infin.*

6. Reduce the fraction $\frac{1}{1-2}$ to an infinite series.

Ans. $1+2+4+8+16+32+64+$, &c., *ad infin.*

Note.—The above result might, at first sight, seem absurd; but it should be remarked that, if we wish to stop at any term of the above series, we must add the fraction that remains to the terms taken. Thus, if we stop after taking seven terms of the quotient, we shall have

$$\frac{1}{1-2}=1+2+4+8+16+32+64+\frac{128}{1-2}=127+\frac{128}{-1}=-1.$$

7. Reduce the fraction $\frac{ax}{a-x}$ to an infinite series.

8. Reduce the fraction $\frac{1}{1-a+a^2}$ to an infinite series.

435. *Secondly.* An infinite series may be formed by extracting the root of a compound surd.

EXAMPLES.

1. Reduce $\sqrt{a^2+b^2}$ to an infinite series.

Extracting the square root, according to the rule given in Art. 291,

$$a^2+b^2\left(a+\frac{b^2}{2a}-\frac{b^4}{8a^3}+\frac{b^6}{16a^5}\right., \text{ \&c., } ad\ infin.$$

$$a^2$$

$$2a)\ \ast \quad b^2$$

$$a^2+b^2+\frac{b^4}{4a^2}$$

$$\ast-\frac{b^4}{4a^2}$$

2. Reduce $\sqrt{a^2-b^2}$ to an infinite series.

Ans. $a-\frac{b^2}{2a}-\frac{b^4}{8a^3}-\frac{b^6}{16a^5}$, &c., *ad infin.*

3. Reduce $\sqrt{1+x}$ to an infinite series.

Thirdly. We have already seen (Art. 295) that if a binomial which has a *negative* or *fractional* index be expanded by the Binomial Theorem, it will produce an infinite series.

This case has already been sufficiently explained and illustrated in the article referred to above.

436. *Fourthly.* An algebraic expression may also be expanded by assuming a series with *indeterminate coefficients*, and afterward finding the value of these coefficients.

To give some idea of this method of development, we will suppose it is required to expand $\frac{a}{c+bx}$ into a series arranged according to the ascending powers of x. This expression may evidently be expanded so as to answer these conditions; for $\frac{a}{c+bx}=a(c+bx)^{-1}$. Expanding this last expression by the binomial theorem, and representing the known terms and coefficients successively by A, B, C, &c., we shall have

$$\frac{a}{c+bx}=\mathrm{A}+\mathrm{B}x+\mathrm{C}x^2+\mathrm{D}x^3+\mathrm{E}x^4+, \text{ \&c., } \textit{ad infin.}$$

The above coefficients A, B, C, &c., being functions of a, b, and c, that is, dependant on them for their values, but independent of x, are called *indeterminate coefficients.*

It is now required to determine the value of these coefficients.

Multiplying both members of the equation by the denominator $c+bx$, and transposing a, we obtain

$$0=(\mathrm{A}c-a)+\left.\begin{matrix}\mathrm{A}b\\+\mathrm{B}c\end{matrix}\right|x+\left.\begin{matrix}\mathrm{B}b\\+\mathrm{C}c\end{matrix}\right|x^2+\left.\begin{matrix}\mathrm{C}b\\+\mathrm{D}c\end{matrix}\right|x^3+, \text{ \&c., } \textit{ad infin.}$$

Here it is evident that if Ac—a, Ab+Bc, Bb+Cc, &c., be made each equal to 0, the several terms of the second member will be reduced to 0, and, consequently, the member will equal 0, and thus the conditions of the equation may be satisfied. From the above assumption we derive the following values of the successive coefficients:

1st - $Ac-a=0$; hence $A=\frac{a}{c}$.

2d - $Ab+Bc=0$; hence $B=-\frac{Ab}{c}=-\frac{b}{c}\times\frac{a}{c}=-\frac{ab}{c^2}$.

3d - $Bb+Cc=0$; hence $C=-\frac{Bb}{c}=-\frac{b}{c}\times-\frac{ab}{c^2}=+\frac{ab^2}{c^3}$.

4th - $Cb+Dc=0$; hence $D=-\frac{Cb}{c}=-\frac{b}{c}\times\frac{ab^2}{c^3}=-\frac{ab^3}{c^4}$.

Hence we have

$$\frac{a}{c+bx}=\frac{a}{c}-\frac{ab}{c}x+\frac{ab^2}{c^3}x^2-\frac{ab^3}{c^4}x^3, \text{ \&c., } ad\ infin.$$

437. By inspecting the preceding operations, we shall perceive that each succeeding coefficient is equal to the preceding multiplied by $-\frac{b}{c}$; consequently, $-\frac{b}{c}$ is the ratio of the progression of the coefficients, and $-\frac{bx}{c}$ is the ratio of the progression of the series.

EXAMPLES.

1. Expand $\frac{d}{b-ax}$ into an infinite series.

Assume $\frac{d}{b-ax}=A+Bx+Cx^2+Dx^3+Ex^4+$, &c., *ad infin.*

Multiplying both members of the equation by $b-ax$, and transposing d, we have

$$0=(Ab-d)\begin{array}{c}-Aa\\+Bb\end{array}\Bigg| x\begin{array}{c}-Ba\\+Cb\end{array}\Bigg| x^2\begin{array}{c}-Ca\\+Db\end{array}\Bigg| x^3\begin{array}{c}-Da\\+Eb\end{array}\Bigg| x^4, \text{ \&c.}$$

Whence, making the several coefficients equal to 0, we have,

1st - $Ab-d=0$; hence $A=\frac{d}{b}$.

2d - $Bb-Aa=0$; hence $B=\frac{Aa}{b}=\frac{d}{b}\times\frac{a}{b}=\frac{ad}{b^2}$.

3d - $Cb-Ba=0$; hence $C=\frac{Ba}{b}=\frac{ad}{b^2}\times\frac{a}{b}=\frac{a^2d}{b^3}$.

4th - $Db-Ca=0$; hence $D=\frac{Ca}{b}=\frac{a^2d}{b^3}\times\frac{a}{b}=\frac{a^3d}{b^4}$.

5th - $Eb-Da=0$; hence $E=\frac{Da}{b}=\frac{a^3d}{b^3}\times\frac{a}{b}=\frac{a^4d}{b^5}$.

Hence we have

$$\frac{d}{b-ax}=\frac{d}{b}(1+\frac{ax}{b}+\frac{a^2x^2}{b^2}+\frac{a^3x^3}{b^3}+\frac{a^4x^4}{b^4}, \text{\&c.})$$

2. Expand $\frac{1+2x}{1-x-x^2}$ into an infinite series.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5$, &c.

3. Expand $\frac{1}{1-x-x^2+x^3}$ into an infinite series.

Ans. $1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7$, &c.

4. Expand $\frac{1-x}{1-2x-3x^2}$ into an infinite series.

Ans. $1+x+5x^2+13x^3+41x^4+121x^5+365x^6$, &c.

5. Expand $\frac{a+bx}{(1-dx)^2}$ into an infinite series.

REMARK.—The method of indeterminate coefficients requires that we should know the form of the development with reference to the exponents of x. The terms are generally supposed to be arranged according to the ascending powers of x, commencing with x^0. Sometimes, however, this form is not exact; in this case the calculus detects the error in the supposition.

For example, let it be required to expand the fraction $\frac{1}{3x-x^2}$.

Suppose $\frac{1}{3x-x^2}=A+Bx+Cx^2+Dx^3$, &c.

Multiplying both members by $3x-x^2$, and transposing 1, we have

$$0=-1+3Ax-A\left|x^2-B\right|x^3-C\left|x^4, \text{\&c.}\right.$$
$$\qquad\qquad\qquad +3B\ \ |\ \ +3C\ \ |\ \ +3D$$

Whence the conditions of the equation require that $-1=0$, which is absurd; hence the above form will not apply to the development of the expression $\frac{1}{3x-x^2}$.

II. SUMMATION OF INFINITE SERIES.

438. The summation of a series is the finding a finite expression equivalent to the series.

But as different series are often governed by very different laws, the methods of finding the sum which are applicable to one class of series, will not apply universally. Hence result different methods of summation.

I. FIRST METHOD.—If the series is a regular descending geometrical series, that is, if its terms decrease by a common divisor, the sum of the series may be obtained by the following formula: (Art. 389.)

$$S=\frac{a}{1-z},$$

As this formula has been explained and applied (see Articles 380 to 390) in Geometrical Progression, we need add nothing more concerning it in this place.

II. SECOND METHOD.—The summation of certain classes of infinite series may be effected by *subtraction.*

EXAMPLES.

1. Let it be required to find the sum of the infinite series

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\frac{1}{5.6.7}, \text{ \&c.} \qquad (1.)$$

By removing the last two factors from each of the denominators in the preceding series, let us form a new series whose value may be expressed by S; thus,

$$S=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}, \text{ \&c., } ad\ infin. \qquad (2.)$$

By transposition,

$$S-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}, \text{ \&c., } ad\ infin. \qquad (3.)$$

By subtracting the last equation (3) from the second (2),

$$1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}, \text{ \&c., } ad\ infin. \qquad (4.)$$

By transposition,

$$1-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}, \text{ \&c., } ad\ infin. \qquad (5.)$$

Whence, by subtracting this last equation (5) from the fourth (4), we have

$$\frac{1}{2}=\frac{4}{1.4.3}+\frac{6}{2.9.4}+\frac{8}{3.16.5}+\frac{10}{4.25.6}, \text{ \&c., } \textit{ad infin.}$$

Or, $\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}$, &c., *ad infin.*

Whence, dividing by 2,

$$\frac{1}{4}=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}, \text{ \&c., } \textit{ad infin.}$$

Hence the sum of the given series is $\frac{1}{4}$.

2. Required the sum of the infinite series

$$\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+\frac{1}{5.7}, \text{ \&c.}$$

Let - $S=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c., *ad infin.* (1.)

Or - $S=\frac{3}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c., *ad infin.* (2.)

By transposition,

$$S-\frac{3}{2}=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}, \text{ \&c., } \textit{ad infin.} \quad (3.)$$

Whence, subtracting the last equation (3) from the second (2), we shall have

$$\frac{3}{2}=\frac{2}{1.3}+\frac{2}{2.4}+\frac{2}{3.5}, \text{ \&c., } \textit{ad infin.}$$

Or - $\frac{3}{4}=\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}$, &c., *ad infin.*

Hence the sum of the given series is $\frac{3}{4}$.

3. Required the sum of the infinite series

$$\frac{1}{2.4.6}+\frac{1}{4.6.8}+\frac{1}{6.8.10}+\frac{1}{8.10.12}+\frac{1}{10.12.14}, \text{ \&c.}$$

III. Third Method.—The following method may sometimes be employed: Assume a decreasing series containing the powers of a variable quantity (x), whose sum is equal to S. Multiply both members of this equation by a compound factor, in which x and some constant quantity are contained; then give to x such a value that the compound factor shall be equal to 0. If one or more of the first terms be then transposed, these will be equal to the sum of the remaining series.

EXAMPLE.

Let - $S=1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5}$, &c., *ad infin.*

Multiplying both members by $x-1$, we have

$$S(x-1)=\left\{\begin{array}{r} x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5},\ \&c. \\ -1-\frac{x}{2}-\frac{x^2}{3}-\frac{x^3}{4}-\frac{x^4}{5}-\frac{x^5}{6},\ \&c. \end{array}\right\}$$

Reducing, $S(x-1)=-1+\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+\frac{x^5}{5.6}$, &c.

By making $x=1$, the equation becomes

$$0=-1+\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6},\ \&c.$$

Whence $1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}$, &c.

III. RECURRING SERIES.

439. A *recurring series* is one which is so constituted that a certain number of contiguous terms, taken in any part of the series, have a given relation to the term immediately succeeding. Thus, in the series

$$1+3x+4x^2+7x^3+11x^4+18x^5,\ \&c.,$$

the sum of the coefficients of any two contiguous terms is equal to the coefficient of the following term. If the series be expressed by

$$A+B+C+D+E+F,\ \&c.,\ \text{then}$$

The 1st term - - $A=1$;
The 2d term - - $B=3x$;
The 3d term - - $C=Bx+Ax^2=4x^2$;
The 4th term - - $D=Cx+Bx^2=7x^3$;

The 5th term - - $E=Dx+Cx^2=11x^4$;

The 6th term - - $F=Ex+Dx^2=18x^5$, &c.

That is, each of the terms after the second is equal to the one immediately preceding multiplied by x, plus the one next preceding multiplied by x^2. Hence all the terms after the first two are subject to a definite law.

440. The particular expression from which any term of the series may be found when the preceding terms are known is called the *scale of the series*, and that from which the coefficients may be formed the *scale of the coefficients*.

Recurring series are divided into orders, and the order is estimated by the number of terms contained in the scale.

In the expansion of $\frac{a}{c+bx}$ in Art. 436, we have a recurring series of the *first order*. Thus,

$$\frac{a}{c+bx}=\frac{a}{c}-\frac{abx}{c^2}+\frac{ab^2x^2}{c^3}-\frac{ab^3x^3}{c^4}, \text{ \&c.}$$

The scale of the coefficients here is $-\frac{b}{c}$; that of the terms is $-\frac{bx}{c}$. This is the simplest form of the recurring series.

441. In a recurring series of the *second order* the law of progression depends upon two terms, and, consequently, the scale consists of two parts. Let $m+n$ represent the scale of the series, and

$$A+B+C+D+E+F, \text{ \&c.},$$

represent the recurring series. Then

The 3d term - - $C=Bmx+Anx^2$;

The 4th term - - $D=Cmx+Bnx^2$;

The 5th term - - $E=Dmx+Cnx^2$, &c.

Taking the last two terms in the above expression, we have the two equations

$$\left.\begin{array}{l} D=Cmx+Bnx^2 \\ E=Dmx+Cnx^2 \end{array}\right\} \text{ to find the values of } m \text{ and } n.$$

Since the scale of the series is the same, whatever be the value of x, the reduction may be rendered more simple by making $x=1$. The equations then become

$$D=Cm+Bn,$$
$$E=Dm+Cn.$$

These, reduced, give

$$m=\frac{DC-BE}{CC-BD}. \quad\Big|\quad n=\frac{CE-DD}{CC-BD}.$$

In the series $1+3x+5x^2+7x^3+9x^4+11x^5$, &c.,

$$A=1,\ B=3x,\ C=5x^2,\ D=7x^3,\ E=9x^4.$$

Then, making $x=1$, we have

$$m=\frac{7\times5-3\times9}{5\times5-3\times7}=2. \quad\Big|\quad n=\frac{5\times9-7\times7}{5\times5-3\times7}=-1.$$

442. In a recurring series of the *third order* the law of progression depends upon three contiguous terms. Letting $m+n+r$. represent the scale of series, and

$A+B+C+D+E+F$, &c., the series, then

The 4th term - - $D=Cmx+Bnx^2+Arx^3$;
The 5th term - - $E=Dmx+Cnx^2+Brx^3$;
The 6th term - - $F=Emx+Dnx^2+Crx^3$, &c.

In a similar manner, we may obtain the succeeding terms in the higher orders of the recurring series.

443. To ascertain whether the law of progression depends on two, or three, or more terms, we may first make trial of two terms; and if the scale of the series thus found does not correspond with the series, we may try three or more terms. If we begin with a number of terms greater than is necessary, one or more of the values found will be 0, and the others will constitute the true scale of valuation.

444. When the scale of a decreasing series is known, the sum of the terms may be found.

Let - $a+bx+cx^2+dx^3+ex^4+fx^5$, &c.,

be a recurring series, whose scale of relation is $m+n$. Then

The 1st term - - $=A$;
The 2d term - - $=B$;
The 3d term - - $C=B\times mx+A\times nx^2$;
The 4th term - - $D=C\times mx+B\times nx^2$;
The 5th term - - $E=D\times mx+C\times nx^2$, &c.

If the series be infinitely extended, the last term may be neglected as of no comparative value; and if S= the sum of the terms, we shall have

$$S=A+B+mx\times(B+C+D, \&c.)+nx^2\times(A+B+C, \&c.).$$

But $B+C+D$, &c., $=S-A$, and $A+B+C$, &c., $=S$.

Hence, by substitution,

$$S=A+B+mx(S-A)+nx^2\times S,$$

Or - $S=A+B+Smx-Amx+Snx^2$.

Transposing, $S-Smx-Snx^2=A+B-Amx$,

Or - $S(1-mx-nx^2)=A+B-Amx$.

Dividing, $S=\dfrac{A+B-Amx}{1-mx-nx^2}$.

EXAMPLES.

1. Required the sum of the infinite series

$$1+6x+12x^2+48x^3+120x^4, \&c.$$

$$A=1,\ B=6x,\ C=12x^2,\ D=48x^3,\ E=120x^4, \&c.$$

Then, making $x=1$, we have

$m=\dfrac{12\times48-6\times120}{12\times12-6\times48}=1.$ | $n=\dfrac{12\times120-48\times48}{12\times12-6\times48}=6.$

Substituting the values of A, B, m, and n in the formula,

$$S=\frac{A+B-Amx}{1-mx-nx^2},$$

We shall have - $S=\dfrac{1+6x-x}{1-x-6x^2}$,

Or - - - $S=\dfrac{1+5x}{1-x-6x^2}$.

2. Required the sum of the infinite series

$$1+2x+8x^2+28x^3+100x^4, \&c.$$

Substituting, as before,

$m=\dfrac{8\times28-2\times100}{8\times8-2\times28}=3.$ | $n=\dfrac{8\times100-28\times28}{8-8-2\times28}=2.$

$$S=\frac{1+2x-3x}{1-3x-2x^2}=\frac{1-x}{1-3x-2x^2}.$$

3. Required the sum of the infinite series

$$1+3x+4x^2+7x^3+11x^4+18x^5+29x^6, \&c.$$

4. Required the sum of the infinite series

$$1+2x+3x^2+4x^3+5x^4+6x^5, \&c.$$

5. Required the sum of the infinite series

$$1+3x+5x^2+7x^3+9x^4+11x^5+13x^6, \&c.$$

IV. METHOD OF DIFFERENCES.

445. We will now proceed to point out another process by which the summation of various kinds of series to a limited number of terms may be obtained. This is termed *Method of Differences*, as it depends on finding the several orders of differences belonging to the series.

1. *Orders of Differences.*

1. If we take the first term from the second, the second from the third, the third from the fourth, &c., in the given series, the remainders will form a new series, which is called the *first order of differences.*

2. If we proceed with this new series in the same manner as with the given series, we shall obtain the *second order of differences.*

3. In the same manner we may obtain the *third*, *fourth*, *fifth*, &c., orders of differences.

446. It should be observed, however, that when the several terms of the series increase, the differences will all be positive; but when they decrease, the differences will be negative and positive alternately.

EXAMPLES.

1. Required the several orders of differences in the series

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2,\ 6^2, \&c.$$

The proposed series -	1,	4,	9,	16,	25,	36,	&c.
1st order of difference -	-	3,	5,	7,	9,	11,	&c.
2d order of difference -	-		2,	2,	2,	2,	&c.
3d order of difference -	-		-	0,	0,	0,	&c.

2. Required the several orders of differences in the series

	1,	6,	20,	50,	105,	196,	&c.
1st order of difference -		5,	14,	30,	55,	91,	&c.
2d order of difference -		-	9,	16,	25,	36,	&c.

3d order of difference - - - 7, 9, 11, &c.
4th order of difference - - - - 2, 2, &c.

3. Required the several orders of differences in the series

$\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, &c.

1st order of difference - $-\frac{1}{4}$, $-\frac{1}{8}$, $-\frac{1}{16}$, $-\frac{1}{32}$, &c.
2d order of difference - - $+\frac{1}{8}$, $+\frac{1}{16}$, $+\frac{1}{32}$, &c.
3d order of difference - - - $-\frac{1}{16}$, $-\frac{1}{32}$, &c.
4th order of difference - - - - $+\frac{1}{32}$, &c.

2. *Law of the Coefficients.*

447. Letting a, b, c, d, &c., represent a series, and proceeding with this series in the same manner as with the preceding, we shall likewise obtain the several orders of differences.

Proposed series, a, b, c, d, e, f, &c.
1st order of differ., $b-a$, $c-b$, $d-c$, $e-d$, $f-e$, &c.
2d differ., $c-2b+a$, $d-2c+b$, $e-2d+c$, $f-2e+d$, &c.
3d diff., $d-3c+3b-a$, $e-3d+3c-b$, $f-3e+3d-c$, &c.
4th differ., $e-4d+6c-4b+a$, $f-4e+6d-4c+b$, &c.
5th difference - $f-5e+10d-10c+5b-a$, &c.

448. In these expressions, each difference in the several orders, whether simple or compound, is called a *term*. By inspecting the *first terms* in the preceding orders of differences and the first term of the series, we shall find the coefficients to be as follows:

1st term of the series - - - 1.
1st order of difference - - - 1, 1.
2d order of difference - - - 1, 2, 1.
3d order of difference - - 1, 3, 3, 1.
4th order of difference - - 1, 4, 6, 4, 1.
5th order of difference - - 1, 5, 10, 10, 5, 1;

which are the same as the coefficients in the powers of binomials (Art. 265). Therefore, the coefficients of the first term in the nth order of differences are (Art. 429.)

$$1,\ n,\ n\times\frac{n-1}{2},\ n\times\frac{n-1}{2}\times\frac{n-2}{3}\ \ldots\ldots\ \frac{n-2}{3}\times\frac{n-1}{2}\times n,$$

$$\frac{n-1}{2}\times n,\ n,\ 1.$$

3. *To find any Term in the Series.*

449. In order to obtain a general expression for any term of the series a, b, c, d, &c., when the differences of any order become at last equal to each other, let d', d'', d''', &c., be the first terms in the first, second, third, &c., orders of differences. Then

$$d'=b-a;$$
$$d''=c-2b+a;$$
$$d'''=d-3c+3b-a;$$
$$d''''=e-4d+6c-4b+a, \text{ \&c.}$$

Transposing and reducing these several equations, we obtain the following expressions for the terms of the original series:

2d term - $b=a+d'$;
3d term - $c=a+2d'+d''$;
4th term - $d=a+3d'+3d''+d'''$;
5th term - $e=a+4d'+6d''+4d'''+d''''$, &c.

450. By inspecting the above, we shall discover that the coefficients observe the same law as in the powers of a binomial, with this difference, that the coefficients of the nth term of the series are the coefficients of the $(n-1)$th power of a binomial. Substituting, then, $n-1$ for n in the formula for the coefficients of an involved binomial (Art. 448), and applying the coefficients thus obtained to d', d'', d''', d'''', &c., as in the preceding equations, we have the following general expression for the nth term of the series, a, b, c, d, &c.:

$$n\text{th term}=a+\frac{n-1}{1}\cdot d'+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot d''+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}\cdot d'''+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}\cdot\frac{n-4}{4}\cdot d'''', \text{ \&c.}$$

Note.—When the differences, after a few of the first orders, become 0, any term of the series is easily found.

EXAMPLES.

1. Required the 12th term of the series 2, 6, 12, 20, 30, &c.

Proposed series - - - 2, 6, 12, 20, 30, &c.

1st order of difference - - -	4,	6,	8,	10,	&c.
2d order of difference - - -		2,	2,	2,	&c.
3d order of difference - - - -			0,	0,	&c.

Here $d'=4$, $d''=2$, and $n=12$; and as $d'''=0$, it will be necessary to use only the first three terms of the formula.

Hence, $a+\frac{n-1}{1}.d'+\frac{n-1}{1}.\frac{n-2}{2}.d''=2+\frac{12-1}{1}.4+\frac{12-1}{1}.\frac{12-2}{2}.2=2+44+110=156.$ *Ans.* 12th term $=156$.

2. Required the 20th term of the series 1, 8, 27, 64, 125, &c.

Proposed series - - -	1,	8,	27,	64,	125,	&c.
1st order of difference - -		7,	19,	37,	61,	&c.
2d order of difference - - -			12,	18,	24,	&c.
3d order of difference - - - -				6,	6,	&c.
4th order of difference - - - - -					0,	&c.

Here $n=20$, $d'=7$, $d''=12$, $d'''=6$, and $d''''=0$; therefore only four terms of the formula will be required.

Hence, $a+\frac{n-1}{1}.d'+\frac{n-1}{1}.\frac{n-2}{2}.d''+\frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}.d'''=1+\frac{20-1}{1}.7+\frac{20-1}{1}.\frac{20-2}{2}.12+\frac{20-1}{1}.\frac{20-2}{2}.\frac{20-3}{3}.6=1+133+2052+5814=8000.$ *Ans.* 20th term $=8000$.

3. Required the 15th term of the series 1, 4, 9, 16, 25, 36, &c. *Ans.* 255.

4. Required the 50th term of the series 1, 3, 6, 10, 15, 21, &c. *Ans.* 1275.

5. Required the 30th term of the series 1, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{15}$, $\frac{1}{21}$, &c. *Ans.* $\frac{1}{465}$.

4. *To find the Sum of* n *Terms.*

In order to find the sum of n terms of the series a, b, c, d, &c., when the differences of any order become at last equal to each other, let one, two, three, &c., terms be successively added together, so as to form a new series, as

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d,\ \&c.$$

Taking the differences in this series, we have

1st difference - - - a, b, c, d, &c.
2d difference - - $b-a$, $c-b$, $d-c$, $e-f$, &c.
3d differ., $c-2b+a$, $d-2c+b$, $e-2d+c$, $f-2e+d$, &c.
4th diff., $d-3c+3b-a$, $e-3d+3c-b$, $f-3e+3d-c$, &c.

Here it will be observed that the first order of differences in the new series is the same as the original series, and the second order of differences is the same as the first order in the original series a, b, c, d, &c.; and, generally, that the $(n+1)$th order in the new series is the same as the nth order in the original series.

In this case,

0 = 1st term; a = 1st order of difference;
d' = 2d order of difference; d'' = 3d order of difference;
d''' = 4th order of difference; d'''' = 5th order of difference.

Resuming now the formula (Art. 450)

$$a+\frac{n-1}{1}\cdot d'+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot d''+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}\cdot d''', \text{ \&c.,}$$

which is the general expression for the nth term of a series whose first term is a; applying it to the new series, in which the first term is 0, and substituting $n+1$ for n, we have

$$0+na+\frac{n}{1}\cdot\frac{n-1}{2}\cdot d'+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot d''+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdot d''', \text{ \&c.,}$$

which is a general expression for the $(n+1)$th term of the series

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d, \text{ \&c.;}$$

Or the nth term of the series

$$a,\ a+b,\ a+b+c,\ a+b+c+d, \text{ \&c.}$$

But the nth term of the latter series is evidently the *sum* of n terms of the series a, b, c, d, &c.

Hence, *the general formula for the sum of n terms, a series of which a is the first term, is*

$$na+\frac{n}{1}\cdot\frac{n-1}{2}\cdot d'+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot d''+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdot d''', \text{ \&c.}$$

EXAMPLES.

1. Required the sum of n terms of the series 1, 2, 3, 4, 5, 6, &c.

Proposed series - - -	1, 2, 3, 4, 5, 6, &c.
1st order of difference - - -	1, 1, 1, 1, 1, &c.
2d order of difference - - -	0, 0, 0, 0, &c.

Here $a=1$, $d'=1$, and $d''=0$; therefore,

$$na+\frac{n}{1}\cdot\frac{n-1}{2}\cdot d'=n+\frac{n}{1}\cdot\frac{n-1}{2}=n+\frac{n^2-n}{2}=\frac{n^2+n}{2}=\text{ sum of } n$$
terms.

In the above example let $n=20$; then

$$\frac{n^2-n}{2}=\frac{400-20}{2}=210. \quad \textit{Ans.}$$

2. Required the nth term of the series of odd numbers 1, 3, 5, 7, 9, &c.

Proposed series - - -	1, 3, 5, 7, 9, &c.
1st order of difference - - -	2, 2, 2, 2, &c.
2d order of difference - - -	0, 0, 0, &c.

Here $a=1$, $d'=2$, and $d''=0$; therefore,

$$na+\frac{n}{1}\cdot\frac{n-1}{2}\cdot d'=+\frac{n^2-n}{2}\cdot 2=n^2.$$

Hence, *the sum of the terms is equal to the square of the number of terms.*

3. Required the sum of n terms of the series 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, &c., or 1, 4, 9, 16, 25, 36, 49, &c. Also, the sum of 20 terms.

Proposed series -	1, 4, 9, 16, 25, 36, 49, &c.
1st order of difference -	3, 5, 7, 9, 11, 13, &c.
2d order of difference -	2, 2, 2, 2, 2, &c.
3d order of difference - -	0, 0, 0, 0, &c.

Here $a=1$, $d'=3$, $d''=2$, and $d'''=0$; therefore,

$$na+\frac{n}{1}\cdot\frac{n-1}{2}\cdot d'+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot d''=n+\frac{3n^2-3n}{2}+$$
$$\frac{2n^3-6n^2+4n}{6}=\frac{6n}{6}+\frac{9n^2-9n}{6}+\frac{2n^3-6n^2+4n}{6}=\frac{1}{6}n(2n^2+$$
$$3n+1)=\text{ the sum of } n \text{ terms.}$$

Or, if $n=20, \frac{1}{6}n(2n^2+3n+1)=\frac{1}{6}.20(2.20^2+3.20+1)=2870.$

4. Required the sum of n, and also of 50 terms of the series 1^3, 2^3, 3^3, 4^3, 5^3, &c.

Ans. n terms$=\frac{n^4 4 2n^3+n^2}{4}$.

50 terms$=1625625$.

5. Required the sum of n, and also of 12 terms of the series 1^4, 2^4, 3^4, 4^4, 5^4, 6^4, &c.

Ans. n terms$=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$.

12 terms$=60710$.

6. Required the sum of n terms of the series 1^5, 2^5, 3^5, 4^5, 5^5, &c.

Ans. $\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}$.

7. Required the sum of n terms of the series 2, 6, 12, 20, 30, &c.

Ans. n terms$=\frac{n(n+1)(n+2)}{3}$.

8. Required the sum of n, and also of 20 terms of the series 1, 3, 6, 10, 15, &c.

Ans. n terms$=\frac{n(n+1)(n+2)}{1.2.3}$.

20 terms$=1540$.

9. Required the sum of n terms of the series 1, 4, 10, 20, 35, &c.

Ans. $\frac{n(n+1)(n+2)(n+3)}{1.2.3.4}$.

V. REVERSION OF SERIES.

451. To revert a series is to express the value of the unknown quantity in it by means of another series involving the powers of some other quantity.

Let x and y represent two indeterminate quantities, and let the value of y be expressed by a series composed of the powers of x; thus,

$$y=ax+bx^2+cx^3+dx^4+, \text{ \&c.},$$

in which a, b, c, d, &c., are known quantities; then, to revert the series is to express the value of x in a series containing only y, and the known quantities a, b, c, d, &c.

Then, in order to express the value of x in terms of y, assume $x=Ay+By^2+Cy^3+Dy^4+$, &c.

Substituting this value for x in the proposed series, and transposing y, it will become

$$\begin{array}{r|l|l|l|l} 0=aA & y+aB & y^2+aC & y^3+aD & y^4+, \&c. \\ -1 & \quad +bA^2 & \quad +2bAB & \quad +2bAC & \\ & & \quad +cA^3 & \quad +bB^2 & \\ & & & \quad +3cA^2B & \\ & & & \quad +dA^4 & \end{array}$$

Whence we have

$aA-1$ - - - - $=0$, and $A=\frac{1}{a}$;

$aB+bA^2$ - - - $=0$, and $B=-\frac{b}{a^3}$;

$aC+2bAB+cA^2$ - - $=0$, and $C=\frac{2b^2-ac}{a^5}$;

$aD+2bAC+bB^2+3cA^2B+dA^4=0$, and $D=-\frac{5b^3-5abc+a^2d}{a^7}$,

&c.

Substituting these values of A, B, C, D, &c., we have

$$x=\frac{1}{a}\times y-\frac{b}{a^3}\times y^2+\frac{2b^2-ac}{a^5}\times y^3-\frac{5b^3-5abc+a^2d}{a^7}\times y^4+, \&c. \quad (1)$$

If the series be of the form

$$y=ax+bx^3+cx^5+, \&c.,$$

in which the even powers of x are not contained, then we shall obtain instead of the above formula (1)

$$x=\frac{1}{a}\times y-\frac{b}{a^4}\times y^3+\frac{3b^2-ac}{a^7}\times y^5-\frac{12b^3+a^2d-8abc}{a^{10}}\times y^7-, \&c. (2)$$

EXAMPLES.

1. It is required to revert the series $y=x+x^2+x^3+$, &c.

Here $a=1$, $b=1$, and $c=1$; therefore,

$\frac{1}{a}=1$, $-\frac{b}{a^3}=-1$, $\frac{2b^2-ac}{a^5}=1$, and $-\frac{5b^3-5abc+a^2d}{a^7}=-1$.

Hence $x=y-y^2+y^3-y^4$, &c.

2. Revert the series $x=2y+3y^3+4y^5+5y^7+$, &c.

Here $a=2$, $b=3$, $c=4$, $d=5$, &c.; therefore,

$\frac{1}{a}=\frac{1}{2}$, $-\frac{b}{a^4}=-\frac{3}{16}$, $\frac{3b^2-ac}{a^7}=\frac{19}{128}$, and $-\frac{12b^3+a^2d-8abc}{a^{10}}=-\frac{152}{1024}$.

Hence $-y=\frac{1}{2}x-\frac{3}{16}x^3+\frac{19}{128}x^5-\frac{152}{1024}x^7$, &c.

3. Revert the series $y=x-\frac{1}{2}x^2+\frac{1}{4}x^3-\frac{1}{8}x^4+$, &c.

Ans. $x=y+\frac{1}{2}y^2+\frac{1}{4}y^3+\frac{1}{8}y^4+$, &c.

4. Revert the series $y=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7$, &c.

Ans. $x=y+\frac{1}{3}y^3+\frac{2}{15}y^5+\frac{17}{351}y^7+$, &c.

SECTION XI.

GENERAL THEORY OF EQUATIONS.

General Properties of Equations.—Composition of Equations.—Transformation of Equations.

GENERAL PROPERTIES OF EQUATIONS.

452. Every complete equation of the nth degree, n being a positive whole number, if it involves but one unknown quantity, may by reduction be put under the form

$$Ax^n+Bx^{n-1}+Cx^{n-2}+\ \ldots\ +Tx+U=0.$$

If this equation be divided by A, and the coefficients $\frac{B}{A}$, $\frac{C}{A}$, ... $\frac{T}{A}$, and $\frac{U}{A}$ be represented by b, c, ... t, and u, we shall have

$$x^n+bx^{n-1}+cx^{n-2}+\ \ldots\ldots\ +tx+u=0.$$

Note.—Any real or imaginary algebraic expression which, being substituted for x in an equation, satisfies its conditions, is called a *root* of that equation.

453. THEOREM.—If a represent any root of the equation $x^n+bx^{n-1}+cx^{n-2}+\ .\ .\ .\ .\ .\ +tx+n=0$, the first member of this equation is divisible by $x-a$.

DEMONSTRATION.—By supposition, $x=a$; then, substituting a for x, we have

$$a^n+ba^{n-1}+ca^{n-2}+\ .\ .\ .\ .\ .\ +ta+u=0.$$

Or, by transposition,

$$u=-a^n-ba^{n-1}-ca^{n-2}-\ .\ .\ .\ .\ .\ -ta.$$

Substituting this value for u in the original equation, we have

$$\left.\begin{array}{r} x^n+bx^{n-1}+cx^{n-2}+\ .\ .\ .\ .\ .\ +tx \\ -a^n-ba^{n-1}-ca^{n-2}-\ .\ .\ .\ .\ .\ -ta \end{array}\right\}=0.$$

Or, by uniting the corresponding terms, the equation becomes

$$(x^n-a^n)+b(x^{n-1}-a^{n-1})+c(x^{n-2}-a^{n-2})+\ .\ .\ +t(x-a)=0.$$

In this equation

$$\begin{array}{l} x^n\ -a^n, \\ x^{n-1}-a^{n-1}, \\ x^{n-2}-a^{n-2}, \\ .\ \ .\ \ . \\ .\ \ .\ \ . \\ .\ \ .\ \ . \\ .\ \ .\ \ . \\ x\ \ -a, \end{array}$$

are each divisible by $x-a$ (Art. 203, th. 7); therefore, the first member of the original equation is also divisible by $x-a$.

EXAMPLES.

Suppose 2 to be a root of the equation

$$x^3-16x^2+56=0.$$

By the *theorem* just demonstrated, the first member of this equation must be divisible by $x-2$. Thus,

$$
\begin{array}{l|l}
x^3-16x^2+56 & x-2 \\
x^3-\ 2x^2 & x^2-14x-28 \\
\hline
\quad -14x^2+56 & \\
\quad -14x^2+28x & \\
\hline
\qquad\quad -28x+56 & \\
\qquad\quad -28x+56 & \\
\hline
\qquad\qquad\quad 0 &
\end{array}
$$

COROLLARY 1. If we divide the general equation

$$x^n+bx^{n-1}+cx^{n-2}+\ .\ .\ .\ +tx+u=0, \qquad (1.)$$

by $x—a$, there will result a new equation one degree less than the given equation, which may be put under the general form

$$x^{n-1}+b'x^{n-2}+c'x^{n-3}+, \text{ \&c.}=0. \qquad (2.)$$

Hence, the original equation may be transformed into the following equivalent expression:

$$(x-a)\ (x^{n-1}+b'x^{n-2}+c'x^{n-3}+, \text{ \&c.})=0.$$

The conditions of this equation are satisfied on the supposition that

$$x=a.$$

COR. 2. The result (2) obtained in the preceding corollary may evidently be divided by $x—a'$, if a' represent a *root* of that equation; then,

$$x^{n-2}+b''x^{n-3}+c''x^{n-4}+, \text{ \&c.},=0. \qquad (3.)$$

Hence we shall have

$$x^{n-1}+b'x^{n-2}+c'x^{n-3}+, \text{ \&c.} = (x-a')\ (x^{n-2}+b''x^{n-3}+c''x^{n-4}+, \text{ \&c.})=0,$$

and the original equation becomes

$$(x-a)\ (x-a')\ (x^{n-2}+b''x^{n-3}+c''x^{n-4}+, \text{ \&c.})=0.$$

The conditions of this equation are satisfied on either of the following suppositions, viz.:

$$x=a,$$

Or - - - - $x=a'$.

Proceeding in the same way to find the remaining roots, a'', a''', &c., the original equation will eventually assume the form

$$(x-a)\ (x-a')\ (x-a'')\ (x-a''')\ (x-a''''), \text{ \&c.},=0. \qquad (4.)$$

COR. 3. In the above (4) equation there are evidently n factors. Hence, *the number of roots of an equation is denoted by the degree of the equation.* Thus,

An equation of the second degree has *two roots ;*

An equation of the third degree has *three roots ;*

An equation of the fourth degree has *four roots*, &c.

SCHOLIUM. If any of the factors into which the first member of the equation may be resolved are equal, the number of *unequal* roots will evidently be less than the number of units in the exponent expressing their degree.

EXAMPLE.

The equation $(x-a)^4\,(x-a')^3\,(x-a'')^2\,(x-a''')=0$ has but *four different* roots, although it is an equation of the 10th degree.

COR. 4. If one root of a cubic equation be found, and the equation be divided by the simple equation containing that root, the quotient will be an equation of the second degree containing the other roots.

EXAMPLES.

1 If one root of the cubic equation $x^3-7x^2+36=0$ is 3, what are the other two roots ?

By the conditions of the problem, $x=3 \therefore x-3=0$.

Dividing the given equation by this, we have

$$\begin{array}{l|l}
x^3-7x^2+36 & x-3 \\
\cline{2-2}
x^3-3x^2 & x^2-4x-12 \\
\hline
\quad -4x^2+36 & \\
\quad -4x^2+12x & \\
\hline
\qquad\qquad -12x+36 & \\
\qquad\qquad -12x+36 & \\
\hline
\qquad\qquad\qquad\quad 0 &
\end{array}$$

Hence we have the quadratic equation

$$x^2-4x-12=0,$$

which, reduced, gives $x=6$, or -2, the other two roots of the cubic equation. Hence, the three roots of the proposed equation are 3, 6, and -2.

2. If one root of the equation $x^3+x^2-16x+20=0$, is -5, what are the other two roots? *Ans.* 2 and 2.

3. If one root of the equation $x^3+3x^2-10y=0$, is 2, what are the other two roots? *Ans.* -5 and 0.

Cor. 5. If two roots of an equation of the fourth degree be given, the remaining two may also be found; and so of the higher equations.

EXAMPLE.

1. Two roots of the equation $x^4-3x^3-14x^2+48x-32=0$, are 1 and 2; what are the other roots? *Ans.* 4 and -4.

Cor. 6. Equations of the form

$$x^n=a$$

would appear to have but one root; but, from the preceding reasoning, it must have n roots.

EXAMPLES.

1. What are the two roots of the equation $x^2=4$?

Ans. 2 and -2.

2. What are the roots of the equation $x^3=1$?

Extracting the cube root, we obtain

$$x=1.$$

Consequently, 1 is one of the roots; then, to ascertain if it has any more roots, we may put the equation under the form

$$x^3-1=0.$$

This equation must be divisible by $x-1$; therefore,

$$x^3-1=(x-1)(x^2+x+1)=0,$$

Or - - - - - $x^2+x+1=0.$

The roots of this last equation are $\frac{1}{2}(-1+\sqrt{-3})$, and $\frac{1}{2}(-1-\sqrt{-3})$. Hence, the three roots of the equation $x^3=1$ are 1, $\frac{1}{2}(-1+\sqrt{-3})$, and $\frac{1}{2}(-1-\sqrt{-3})$.

3. What are the roots of the equation $x^4=1$?

Ans. 1, -1, $\sqrt{-1}$, and $-\sqrt{-1}$.

4. What are the roots of the equation $x^5=1$?

Ans. 1, and $\frac{1}{4}\left(-1-\sqrt{5}\pm\sqrt{-10\pm2\sqrt{5}}\right)$.

COMPOSITION OF EQUATIONS.

454. From what has been said, it will be readily inferred

that equations of any degree higher than the first may be produced by the successive multiplication of equations of the first degree.

$$\begin{array}{lll}
\text{Let} & - & x-2=0, \\
\text{And} & - & x-3=0; \\ \hline
\text{Multiplying} & - & x^2-5x+6=0. \\
\text{Again, let} & - & x-4 \quad =0, \\ \hline
\text{Multiplying} & - & x^3-9x^2+26x-24=0. \\
\text{Again, let} & - & x-5 \qquad =0, \\ \hline
\text{Multiplying} & - & x^4-14x^3+71x^2-154x+120=0, \text{ \&c.}
\end{array}$$

Hence, *the product of two equations of the first degree is an equation of the second degree; the product of three equations of the first degree is an equation of the third degree*, &c.

The above equation of the fourth degree has evidently *four* roots, viz., 2, 3, 4, and 5.

455. The law by which the coefficients are governed may be seen by inspecting the results obtained by the actual multiplication of the factors.

Let a, a', a'', a''', a'''', &c., represent the roots of the general equation

$$x^n+bx^{n-1}+cx^{n-2}+, \text{ \&c.},=0.$$

Then we shall have (by Art. 453, Cor. 2)

$$x^n+bx^{n-1}+cx^{n-2}+, \text{ \&c.},=(x-a)\,(x-a')\,(x-a'')\,(x-a'''), \text{\&c.}, =0.$$

Or, multiplying the factors and writing the coefficients of the same power of x under each other, we have

$$1.\ (x-a)(x-a') \quad = \begin{array}{r|l} x^2-a & x+aa'=0. \\ -a' & \end{array}$$

$$2.\ (x-a)(x-a')(x-a'') \quad = \begin{array}{r|l|l} x^3-a & x^2+aa' & x-aa'a''=0. \\ -a' & \ \ +aa'' & \\ -a'' & \ \ +a'a'' & \end{array}$$

$$3.\ (x-a)(x-a')(x-a'')(x-a''')= \begin{array}{r|l|l|l} x^4-a & x^3+aa' & x^2-aa'a'' & x+aa'a''a'''=0 \\ -a' & \ \ +aa'' & \ \ -aa'a''' & \\ -a'' & \ \ +aa''' & \ \ -aa''a''' & \\ -a''' & \ \ +a'a'' & \ \ -a'a''a''' & \\ & \ \ +a'a''' & & \\ & \ \ +a''a''' & & \end{array}$$

456. By attending carefully to the above results, we shall discover the following properties:

1. *The coefficient of x in the first term is always* 1.

2. *The coefficient of x in the second term is the sum of all the roots of the equation taken with contrary signs.*

Thus, the roots of the equation of the second degree are a and a'; the coefficients of x in the second term are $-a$, and $-a'$. In the cubic equation the roots are a, a', and a''; the coefficients are $-a$, $-a'$, and $-a''$. In the equation of the fourth degree the roots are a, a', a'', and a'''; the coefficients are $-a$, $-a'$, $-a''$, and $-a'''$.

3. *The coefficient of x in the third term is the sum of all the products of the roots taken two and two*, and so on.

Thus, in the equation of the fourth degree, the roots are a, a', a'', and a'''; and the coefficients in the third term are aa', aa'', aa''', $a'a''$, $a'a'''$, $a''a'''$.

4. *The last term, which is independent of x, is the product formed from all the roots of the equation after the signs are changed.*

Thus, in the cubic equation, the last term $-aa'a''=-a\times -a'\times -a''$; and in the biquadratic equation, the last term $+aa'a''a'''=-a\times -a'\times -a''\times -a'''$.

Cor. 1. If the roots are all negative, the terms of the equation to which they belong will all be positive.

For, letting - - $x=-a$,

$$x=-a'$$

$$x=-a'', \text{ \&c.}$$

By transposition, we have $x+a=0$, $x+a'=0$, $x+a''=0$, &c. Consequently,

$$(x+a)\,(x+a')\,(x+a''), \text{ \&c.},=0.$$

Cor. 2. If part of the roots are positive and part negative, part of the terms of the equation to which they belong will be positive and part negative.

TRANSFORMATION OF EQUATIONS.

457. The transformation of an equation consists in changing its form without destroying the equality of its members.

458. THEOREM.—*Any proposed equation may be transformed into another, the roots of which shall be any multiples or submultiples of those of the former.*

First. In order to demonstrate the above, let us resume the general equation

$$x^n+bx^{n-1}+cx^{n-2}+\ldots\ldots+tx+u=0.$$

Let y represent the unknown quantity of a new equation, of which the roots are a times greater than those of the proposed equation; then

$$y=ax, \text{ and } x=\frac{y}{a}.$$

Substituting this value for x in the general equation,

$$\frac{y^n}{a^n}+b\frac{y^{n-1}}{a^{n-1}}+c\frac{y^{n-2}}{a^{n-2}}+\ldots\ldots+t\frac{y}{a}+u=0.$$

Multiplying by a^n,

$$y^n+bay^{n-1}+ca^2y^{n-2}+\ldots\ldots+ta^{n-1}y+a^nu=0. \qquad (1.)$$

This last equation will evidently fulfil the conditions required, since $y=ax$

Secondly. Let $y=\frac{x}{a}$; then $x=ay$.

Substituting and reducing, as before, we shall obtain

$$y^n+\frac{b}{a}y^{n-1}+\frac{c}{a^2}y^{n-2}+\ldots\ldots+\frac{t}{a^{n-1}}y+\frac{u}{a^n}=0. \qquad (2.)$$

COROLLARY. Since the coefficients in the preceding equation (1) are multiples of the coefficients in the general equation, it is evident that any equation having fractional coefficients may be transformed into another, in which all the terms shall be entire numbers, and the coefficient of whose first term shall be unity.

EXAMPLE.

Transform the equation $x^3+\frac{1}{2}x^2+\frac{2}{3}x+\frac{3}{4}=0$.

Multiplying this equation by 12, the least common multiple of the denominators,

$$12x^3+6x^2+8x+9=0.$$

In this equation all the terms are entire numbers, but the coefficient of the first term is greater than unity.

Then, let $y=12x$, and $x=\frac{y}{12}$.

Whence, by substitution,

$$\frac{y^3}{12^2}+\frac{6y^2}{12^2}+\frac{8y}{12}+\quad 9=0.$$

Multiplying by 12^2, $y^3+6y^2+96y+1296=0$.

If the value of y in this equation be found, that of x can be readily obtained, since $x=\frac{y}{12}$.

459. THEOREM.—*An equation may be transformed into another, the roots of which shall be greater or less than those of the former by a given number.*

Let us resume the general equation

$$x^n+bx^{n-1}+cx^{n-2}+\ldots\ldots+tx+u=0,$$

and suppose it were required to transform it into another, whose roots (y) shall be less or greater than those of the given equation by e.

First. Let $x=y+e$.

By substituting $y+e$ for x in the general equation, we shall obtain

$$(y+e)^n+b(y+e)^{n-1}+c(y+e)^{n-2}+\ldots\ldots+t(y+e)+u=0.$$

Or, expanding,

$$\left.\begin{array}{l|l|l} y^n+ne & y^{n-1}+\frac{n(n-1)}{2}e^2 & y^{n-2}+\ldots\ldots+e^n \\ \quad +b & \quad +(n-1)be & \qquad\qquad +be^{n-1} \\ & \quad + \qquad c & \qquad\qquad +ce^{n-2} \\ & & \qquad\qquad \ldots\ldots \\ & & \qquad\qquad \ldots\ldots \\ & & \qquad\qquad +te \\ & & \qquad\qquad +u \end{array}\right\}=0. \quad (1.)$$

This equation will evidently fulfil the conditions required, since $y=x-e$.

Secondly. Let $x=y-e$; then,

Substituting, as before, we have

$$(y-e)^n+b(y-e)^{n-1}+c(y-e)^{n-2}+\ldots\ldots+t(y-e)+u=0.$$

Or, expanding,

$$\left.\begin{array}{r} y^n - ne \\ +b \end{array}\right| \begin{array}{c} y^{n-1} + \frac{(n-1)n}{2}e^2 \\ -(n-1)be \\ + \quad c \end{array} \left| \; y^{n-2} +, \text{ \&c.}, =0. \right. \qquad (2.)$$

Which equation also fulfils the required conditions, since $y=x+e$.

COR. 1. Since the truth demonstrated in the above theorem does not depend upon any particular value of e, or since e is indeterminate, it may be taken to satisfy any proposed condition. Letting it, then, be taken of such value that the coefficient of y^{n-1} may be equal to zero, or $-ne+b=0$, in the above equation (1), the second term vanishes, and the equation becomes

$$\begin{array}{c} y^n + \frac{n(n-1)}{2}e^2 \\ -(n-1)be \\ + \quad c \end{array} \left| \; y^{n-2} +, \text{ \&c.}, =0. \right. \qquad (3.)$$

Or, representing the coefficients by c', d', &c., we shall have

$$y^n + c'y^{n-2} + d'y^{n-3} + \dots + u = 0.$$

This condition is represented by the equation

$$b+ne=0,$$

which gives - - - $e=-\frac{b}{n}$;

whence *the second term of an equation may be removed by substituting for the unknown quantity some other unknown quantity, together with such a part of the coefficient of the second term, taken with a contrary sign, as is denoted by the index of the highest power of the equation.*

EXAMPLE.

Transform the equation $x^3-9x^2+7x+12=0$ into one which shall want the second term.

Let - - - $x=y+3$;

Then - - -
$$\left.\begin{array}{rl} x^3 = & y^3+9y^2+27y+27 \\ -9x^2 = & \quad -9y^2-54y-81 \\ +7x = & \qquad\quad +7y+21 \\ +12 = & \qquad\qquad\quad +12 \end{array}\right\} =0.$$

Whence $x^3-9x^2+7x+12=y^3 \quad -20y-21=0.$

COR. 2. In the same manner, the third term may be removed from an equation by taking the value of e, such that

$$\frac{n(n-1)}{2}e^2-(n-1)be+c=0.$$

For, transposing c, and dividing by the coefficient of e^2, we obtain

$$e^2-\frac{2b}{n}e=-\frac{2c}{n(n-1)}.$$

Reducing - - $e=\frac{b}{n}\pm\sqrt{-\frac{2c}{n(n-1)}+\frac{b^2}{n^2}}.$

EXAMPLE.

Transform the equation $x^3-6x^2+9x-1=0$ into one in which the third term shall be wanting.

Here $e=\frac{6}{3}\pm\sqrt{-\frac{2\times 9}{3(3-1)}+\frac{36}{9}}=2\pm\sqrt{-\frac{18}{6}+\frac{36}{9}}=2\pm\sqrt{-3+4}=3$, or 1.

Then, letting - $x=y+3$,

we shall have -

$$\left.\begin{aligned} x^3&=y^3+9y^2+27y+27\\ -6x^2&=\quad -6y^2-36y-54\\ +9x&=\qquad\qquad +9y+27\\ -1&=\qquad\qquad\qquad -1 \end{aligned}\right\}=0.$$

Whence $x^3-6x^2+9x-1=y^3+3y^2 \quad -1=0.$

Or, taking $e=1$, and proceeding in the same manner, we shall obtain $y^3-3y^2+3=0$.

COR. 3. Since n, in the general equation, is indeterminate, an equation of any degree may be transformed into another from which the second or third term shall be removed.

EXAMPLE.

Transform the equation $x^4-8x^3+5x^2-10x+4=0$ into another that shall want the second term.

Let - $x=y+\frac{b}{n}=y+2.$

Then -

$$\left.\begin{aligned} x^4&=y^4+8y^3+24y^2+32y+16\\ -8x^3&=\quad -8y^3-48y^2-96y-64\\ +5x^2&=\qquad\qquad +5y^2+20y+20\\ -10x&=\qquad\qquad\qquad -10y-20\\ +4&=\qquad\qquad\qquad\qquad +4 \end{aligned}\right\}=0.$$

Whence $y^4 \quad -19y^2-54y-44=0.$

COR. 4. If, in the general equation,

$$x^n+bx^{n-1}+cx^{n-2}+\ldots\ldots+tx+u=0,$$

we let $x=y+e$, and this value be substituted, the equation becomes

$$(y+e)^n+b(y+e)^{n-1}+c(y+e)^{n-2}\ldots\ldots s(y+e)^2+t(y+e)+u=0.$$

Expanding each binomial term of this equation separately

$$\left.\begin{array}{ll}
(1). & (y+e)^n=y^n+ny^{n-1}e+\frac{n(n-1)}{2}y^{n-2}e^2\ldots\ldots\frac{n(n-1)}{2} \\
& y^2e^{n-2}+nye^{n-1}+e^n \\
(2). & b(y+e)^{n-1}=by^{n-1}+(n-1)by^{n-2}e+\frac{(n-1)(n-2)}{2} \\
& by^{n-3}e^2\ldots\ldots(n-1)bye^{n-2}+be^{n-1} \\
(3). & c(y+e)^{n-2}=cy^{n-2}+(n-2)cy^{n-3}e+\frac{(n-2)(n-3)}{2} \\
& cy^{n-4}e^2\ldots\ldots ce^{n-2} \\
(4). & d(y+e)^{n-3}=dy^{n-3}+(n-3)dy^{n-4}e+\frac{(n-3)(n-4)}{2} \\
& dy^{n-5}e^2\ldots\ldots \\
\ldots & \ldots\ldots\ldots\ldots \\
(n-2). & r(y+e)^3=ry^3+3ry^2e+3rye^2+re^3 \\
(n-1). & s(y+e)^2=sy^2+2sye+se^2 \\
(n). & t(y+e)=ty+te \\
(n+1). & u=u
\end{array}\right\}=0.$$

By inspecting the above results, it will be perceived that the exponents of e form an ascending series,

$$0,\ 1,\ 2\ \ldots\ldots\ldots\ldots n-2,\ n-1,\ n.$$

Then, putting V= to the given equation, W= to the sum of the coefficients of e^0, X= to the sum of the coefficients of e', $\frac{Y}{2}$= to the sum of the coefficients of e^2, $\frac{Z}{2.3}$= to the sum of the coefficients of e^3, &c., we shall have

$$V=x^n+bx^{n-1}+cx^{n-2}+dx^{n-3}\ldots\ldots rx^3+sx^2+tx+u=0;$$
$$W=y^n+by^{n-1}+cy^{n-2}+dy^{n-3}\ldots\ldots ry^3+sy^2+ty+u;$$
$$X=ny^{n-1}+(n-1)by^{n-2}+(n-2)cy^{n-3}\ .3ry^2+2sy+t;$$
$$Y=n(n-1)y^{n-2}+(n-1)(n-2)by^{n-3}.6ry+2s;$$
$$Z=n(n-1)(n-2)y^{n-3}+(n-1)(n-2)$$
$$(n-3)by^{n-4}\ldots\ldots\ldots\ldots 6r,\ \&c.$$

The above expressions are called *derived polynomials*, and by examining them we may readily discover the manner in which they are derived. Thus,

1. W is derived from V, by simply changing x into y.

2. X is derived from W, by multiplying each of the terms of W by the exponent of y in that term, and diminishing this exponent by 1.

The above law will be found useful in the transformation of the higher equations. To illustrate its application, we will subjoin a few examples.

EXAMPLES.

1. Transform the equation $x^4-12x^3+17x^2-9x+7=0$ into another which shall want the second term.

Let $x=y+\frac{12}{4}=y+3$, or $3+y$.

Substituting this for x in the given equation,

$$(3+y)^4-12(3+y)^3+17(3+y)^2-9(3+y)+7=0.$$

This will give the transformed equation of the 4th degree, and of the form

$$W+Xy+\frac{Y}{2}y^2+\frac{Z}{2\,.\,3}y^3+y^4=0,$$

and the operation will be reduced to finding the values of these coefficients.

Now it follows, from the preceding law, that

$$W = (3)^4-12\,.\,(3)^3+17\,.\,(3)^2-9\,.\,(3)^1+7=-110;$$
$$X = 4\,.\,(3)^3-36\,.\,(3)^2+34\,.\,(3)^1-9 \quad . \quad . \quad =-123;$$
$$\frac{Y}{2} = 6\,.\,(3)^2-36\,.\,(3)^1+17 \quad . \quad . \quad . \quad . \quad . \quad =-\ 37;$$
$$\frac{Z}{2\,.\,3} = 4\,.\,(3)^1-12 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad = \quad 0.$$

Therefore, the transformed equation becomes

$$y^4-37y^2-123y-110=0.$$

2. Transform the equation $4x^3-5x^2+7x-9=0$ into another, the roots of which shall exceed the roots of the given equation by unity.

Let $y=x+1$, then $x=-1+y$, which gives the transformed equation

$$W+Xy+\frac{Y}{2}y^2+4y^3=0.$$

But,

$W = 4.(-1)^3-5.(-1)^2+7.(-1)^1-9 \ldots = -25$;
$X = 12.(-1)^2-10.(-1)^1+7 \ldots = +29$;
$\frac{Y}{2} = 12.(-1)-5 \ldots = -17$;
$\frac{Z}{2.3} = 4 \ldots = +4$.

Whence the transformed equation becomes

$$4y^3-17y^2+29y-25=0.$$

3. Transform the equation $x^5-10x^4+7x^3+4x-9=0$ into another which shall want the second term.

Let $x=y-\frac{10}{5}=y-2$, or $-2+y$; then the transformed equation becomes

$$W+Xy+\frac{Y}{2}y^2+\frac{Z}{2.3}y^3+\frac{Z^1}{2.3.4}y^4+y^5=0.$$

But,

$W = (2)^5-10.(2)^4+7.(2)^3+4.(2)^1-9 = -73$;
$X = 5.(2)^4-40.(2)^3+21.(2)^2+4 \ldots = -152$;
$\frac{Y}{2} = 10.(2)^3-60.(2)^2+21.(2)^1 \ldots = -118$;
$\frac{Z}{2.3} = 10.(2)^2-40.(2)^1+7 \ldots = -33$;
$\frac{Z^1}{2.3.4} = 5.(2)^1-10 \ldots = 0$.

Hence, the transformed polynomial is

$$y^5-33y^3-118y^2-152y-73=0.$$

4. Transform the equation $3x^3+15x^2+25x-3=0$ into an equation wanting the second term.

Divide the equation by 3, and proceed as before.

Ans. $3y^3-\frac{152}{27}=0$.

5. Transform the equation $3x^4-13x^3+7x^2-8x-9=0$ into an equation in which the roots shall be less than the roots of the given equation by $\frac{1}{3}$.

Ans. $3y^4-9y^3-4y^2-\frac{65}{9}y-\frac{102}{9}=0$.

Cor. 5. If a, a', a'' a'' . . . . a^m represent the n roots of the general equation $V=0$, or $x^n+bx^{n-1}+$, &c., $=0$, we shall have, by Art. 453, Cor. 2.

$x^n+bx^{n-1}+cx^{n-2}+$, &c., $=(x-a)(x-a')(x-a'')\ldots(x-a^m)$.

Now if we put $x=y+e$, and substitute this value for x in the equation, it becomes

$(y+e)^n+b(y+e)^{n-1}+$, &c., $=(y+e-a)(y+e-a')\ldots(y+e-a^m)$,

Or,

$(y+e)^n+b(y+e)^{n-1}+$, &c., $=(e+\overline{y-a})(e+\overline{y-a'})\ldots(e+\overline{y-a^m})$.

The first member, by Cor. 4, equals

$$Xe+\frac{Y}{2}e^2 \ldots e^n.$$

With respect to the second member, it follows, from the preceding theorem,

1. *The part involving* e, *or the last term, is equal to the product* $(y-a)(y-a')\ldots(y-a^m)$ *of the factors of the proposed equation;* hence,

$$W=(y-a)(y-a')\ldots(y-a^n).$$

2. *The coefficient of* e *is equal to the sum of the products of these n factors, taken n—1 and n—1, or equal to the sum of all the quotients that can be obtained by dividing* W *by each of the n factors of the first degree in the given equation;* hence,

$$X=\frac{W}{y-a}+\frac{W}{y-a'}+\frac{W}{y-a''}\ldots\ldots\frac{W}{y-a}.$$

3. The coefficient of e^2 is equal to the sum of the products of these n factors, taken $n-2$ and $n-2$, or equal to the sum of the quotients that can be obtained bydividing W by each of the factors of the second degree; hence,

$$\frac{Y}{2}=\frac{W}{(y-a)(y-a')}+\frac{W}{(y-a)(y-a'')}\ldots\ldots\frac{W}{(y-a^{m-1})(y-a^m)}.$$

Cor. 6. If two or more of the roots of the given equation are equal to each other; that is,

$$a=a'=a'', \text{ \&c.},$$

the derived polynomial, which is the sum of the products n factors, taken $n-1$ and $n-1$, contains a factor in its different parts, which is two or more times a factor of the proposed equation.

Hence, *if the equation contain equal roots, there must be a common divisor between the first member of the proposed equation and its first derived polynomial.*

460. PROBLEM.—*Having given an equation, it is required to discover whether it has equal roots, and to discover the method of determining these roots.*

Resume the general equation; or, since the polynomial W differs from V only by the substitution of y for x,

$$y^n+by^{n-1}+cy^{n-2} \dots\dots ty+u=0.$$

Then, supposing the equation to contain m factors equal to $y-a'$, &c., and also to contain the simple factors $y-p$, $y-q$, &c., then will

$$W=(y-a)^m(y-a')^{m'}(y-a'')^{m''}\dots(y-p)(y-q)(y-r)\dots$$

Whence, by the preceding corollary,

$$X=\frac{mW}{y-a}+\frac{m'W}{y-a'}+\frac{m''W}{y-a''}\dots\dots\frac{W}{y-p}+\frac{W}{y-q}+\frac{W}{y-r}\dots$$

Now $(y-a)^{m-1}$, $(y-a')^{m'-1}$, &c., are factors common to all the terms of the above polynomial; hence their product

$$(y-a)^{m-1}\times(y-a')^{m'-1}\times(y-a'')^{m''-1}\dots\dots$$

is the greatest common divisor of the polynomials W and X; or,

$$D=(y-a)^{m-1}\times(y-a')^{m'-1}\times(y-a'')^{m''-1}\dots\dots;$$

that is, the greatest common divisor is composed of the product of those factors which enter two or more times in the given equation, each being raised to a power less by unity than in the given equation.

Hence, *to discover whether an equation* $W=0$ *contains any equal roots, form* X, *or the derived polynomial of* W; *then seek for the greatest common divisor between* W *and* X; *if one cannot be obtained, the equation has no equal factors, and, consequently, no equal roots.*

461. Again, if the greatest common divisor (D) is of the first degree, or of the form $y-a$, make $y-a=0$, whence $y=a$; and we may conclude that the equation has two roots

equal to a; if it is of the form $(y-a)^n$, we may conclude that the equation has $n+1$ roots, each equal to a.

If the greatest common measure (D) is of the form

$$y^2+by+c=0,$$

we must find the two values of y. Let a and a' represent those values, then the equation will have two roots each equal to a, and two each equal to a'.

Hence, *the equal roots of an equation may be obtained by finding the greatest common divisor of its first member and its derived polynomial, and solving the equation obtained by putting this common divisor equal to* 0.

EXAMPLE.

Has the equation $x^3-7x^2+16x-12=0$ equal roots? if so, how many, and what are they?

The derived polynomial of this equation is

$$3x^2-14x+16.$$

Performing upon this and the first member of the given equation the operations indicated in Art. 137 to find the greatest common divisor, we obtain

$$x-2.$$

Then - - - $x-2=0$,

And - - - $x=2$.

Therefore, we conclude the equation has two roots equal to 2.

Now, since the equation has two roots equal to 2, it must (Art. 460) be divisible by

$$(x-2)^2=x^2-4x+4.$$

Whence - $x^3-7x^2+16x-12=(x-2)^2(x-3)=0$,

And - - - - $x-3=0$, or $x=3$,

which is the other root of the equation.

462. To show the application of the preceding principles, we will subjoin a few equations with equal roots.

EXAMPLES.

1. Reduce the equation

$$2x^4-12x^3+19x^2-6x+9=0,$$

which has eoual roots.

The derived polynomial is

$$8x^3-36x+38x-6.$$

Whence - - $D=x-3=0,$

And - - - - $x=3.$

Therefore, the equation has two roots equal to 3.

Dividing its first member by $(x-3)^2=x^2-6x+9$, we obtain

$$2x^2+1=0, \text{ or } x=\pm\sqrt{-\tfrac{1}{2}}.$$

Hence, the four roots of the equation are

$$3,\ 3,\ \sqrt{-\tfrac{1}{2}}, \text{ and } -\sqrt{-\tfrac{1}{2}}.$$

2. Reduce the equation

$$x^5-2x^4+3x^3-7x^2+8x-3=0,$$

which has equal roots.

The first derived polynomial is

$$5x^4-8x^3+9x^2-14x+8.$$

Whence - $D=x^2-2x+1$, or $(x-1)^2$;

And the given equation has three roots equal to 1.

Dividing the first member by $(x-1)^3$, or x^3-3x^2+3x-1, we have

$$x^2+x+3=0, \text{ or } x=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-11}.$$

The five roots are,

$$1,\ 1,\ 1,\ -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-11}, \text{ and } -\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-11}.$$

3. Reduce the equation

$$x^7+5x^6+6x^5-6x^4-15x^3-3x^2+8x+4=0,$$

which has equal roots.

$$W= x^7+5x^6+6x^5-6x^4-15x^3-3x^2+8x+4;$$
$$X=7x^6+30x^5+30x^4-24x^3-45x^2-6x+8;$$
$$D= x^4+3x^3+x^2-3x-2.$$

Since D surpasses the second degree, we must apply to it the same process we have to W.

Its first derived polynomial is

$$4x^3+9x^2+2x-3,$$

And the greatest common divisor; or,

$$D'=x+1.$$

Hence, D has two equal roots equal to -1; and, dividing it by

$$(x+1)^2, \text{ or } x^2+2x+1,$$

we have - $x^2+x-2=0$; or $x=1$, or -2.

Therefore,

D, or $x^4+3x^3+x^2-3x-2=(x+1)^2(x-1)(x+2)$,

And - - - - $W=(x+1)^3(x-1)^2(x+2)^2$.

The roots of the equation, then, are

1, 1, —1, —1, —1, —2, and —2.

4. Required the equal roots of the equation

$$x^8-8x^7+26x^6-45x^5+45x^4-21x^3-10x^2+20x-8=0.$$

Ans. 1 and 2.

SECTION XII.

RESOLUTION OF THE HIGHER EQUATIONS.

Resolution of the Cubic Equations by Cardan's Rule.—Young's Method.—Des Cartes' Method of Resolving Biquadratic Equations.—Newton's Method of Approximation.—Resolution of Higher Equations by Trial and Error.

CARDAN'S METHOD OF RESOLVING CUBIC EQUATIONS.

463. We will now proceed to investigate the methods by which affected equations of the third degree may be solved. Equations of this nature may all be exhibited under the three following forms, in which p, p', and q may be either + or —:

(1.) $x^3+px=q$;

(2.) $x^3+p'x^2=q$;

(3.) $x^3+p'x^2+px=q$.

Note.—The known quantities p, p', and q are here used in their most general sense, and may be entire or fractional, positive or negative quantities.

First Form.

464. In order to deduce a general formula for the reduction of cubic equations of the first form, let us take

$$x^3+px=q.$$

Let - $y+z=x$, and $3yz=-p$;

Then - - $x^3=(y+z)^3=y^3+3y^2z+3yz^2+z^3$;

Resolving into factors - $x^3=y^3+3yz(y+z)+z^3$;

Substituting x for $y+z$ - $x^3=y^3+3yzx+z^3$;

Substituting this value of x^3 in the 1st equation - $y^3+z^3+3yzx+px=q$;

Resolving into factors - $y^3+z^3+(3yz+p)x=q$,

Or - - - - - $y^3+z^3+(-p+p)x=q$;

Whence - - - - $y^3+z^3 \quad =q$.

To determine the values of y and z, we have the two equations,

$$y^3+z^3=q; \quad (1.)$$

$$3yz=-p; \quad (2.)$$

Dividing the 2d 3 - - $yz=-\frac{1}{3}p$; (3.)

Cubing - - - - $y^3z^3=-\frac{1}{27}p^3$; (4.)

Squaring the 1st equation - $y^6+2y^3z^3+z^6=q^2$; (5.)

Multiplying the 4th by 4 - $4y^3z^3=-\frac{4}{27}p^3$; (6.)

Subtracting the 6th from the 5th - $y^6-2y^3z^3+z^6=q^2+\frac{4}{27}p^3$; (7.)

Extracting the square root, $y^3-z^3=\pm\sqrt{q^2+\frac{4}{27}p^3}$, (8.)

Or - - - - - - $y^3-z^3=\pm 2\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}$. (9.)

Adding the 9th to the 1st - $2y^3=q\pm 2\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}$;

Dividing and evolving - $y=\sqrt[3]{\frac{1}{2}q\pm\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}}$;

Subtracting the 9th from the 1st - - - - $2z^3=q\mp 2\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}$;

Dividing and evolving - $z=\sqrt[3]{\frac{1}{2}q\mp\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}}$;

Consequently, we take - $y=\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}}$,

And - - - - - $z=\sqrt[3]{\frac{1}{2}q-\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}}$.

Adding the last two equations, and observing that $y+z=x$,

$$x=\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}}+\sqrt[3]{\frac{1}{2}q-\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}} \quad (A.)$$

Or, since $z=-\frac{\frac{1}{3}p}{y}$, and $x=y+z$,

$$x=\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}}-\frac{\frac{1}{3}p}{\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}q^2+\frac{1}{27}p^3}}}. \quad (A'.)$$

Again, taking the equation $x^3-px=q$, and letting $y+z=x$, and $3yz=+p$, and proceeding as before, we shall obtain*

$$x=\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}q^2-\tfrac{1}{27}p^3}}+\sqrt[3]{\tfrac{1}{2}q-\sqrt{\tfrac{1}{4}q^2-\tfrac{1}{27}p^3}}. \quad \text{(B.)}$$

Or, since $z=\frac{\frac{1}{3}p}{y}$, and $x=y+z$,

$$x=\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}q^2-\tfrac{1}{27}p^3}}+\frac{\frac{1}{3}p}{\sqrt[3]{\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}q^2-\tfrac{1}{27}p^3}}}. \quad \text{(B}'\text{.)}$$

By the above formulas we may obtain the exact or approximate roots of cubic equations of the first form.†

EXAMPLES.

1. Find the value of x in the equation $x^3+6x=2$.

Substituting 6 for p, and 2 for q, in formula A, we have

$$x=\sqrt[3]{1+\sqrt{1+8}}+\sqrt[3]{1-\sqrt{1+8}}=\sqrt[3]{1+\sqrt{9}}+\sqrt[3]{1-\sqrt{9}}.$$

Whence

$$x=\sqrt[3]{4}+\sqrt[3]{-2}=1{,}587401-1{,}259921=,32748+. \quad \textit{Ans.}$$

2. Find the value of x in the equation $x^3-2x=-4$.

By formula B, we have

$$x=\sqrt[3]{\tfrac{-4}{2}+\sqrt{\tfrac{16}{4}-\tfrac{8}{27}}}+\sqrt[3]{\tfrac{-4}{2}-\sqrt{\tfrac{16}{4}-\tfrac{8}{27}}};$$

$$=\sqrt[3]{-2+\tfrac{10}{9}\sqrt{3}}+\sqrt[3]{-2-\tfrac{10}{9}\sqrt{3}};$$

$$=\sqrt[3]{-2+1{,}9245}+\sqrt[3]{-2-1{,}9245};$$

$$=\sqrt[3]{-,0755}-\sqrt[3]{3{,}9245}=-,41226-1{,}5773;$$

$$=-1{,}9999+, \text{ or } -2. \quad \textit{Ans.}$$

* In formulas B and B′, it is evident that, if $\frac{1}{27}p^3>\frac{1}{4}q^2$, the equation cannot be reduced, since it involves the extraction of the square root of a negative quantity; hence, the value of x can only be obtained by imaginary quantities, and the conditions of the question are incompatible with each other.

† These formulas are substantially what is known under the cognomen "Carden's Rule for Cubic Equations." The invention of the rule, however, is due to Nicholas Tartalea and to Scipio Ferreus, who found it independently of each other; but Carden first published it to the world.—See *Ed. Encycl., Art. Alg.*

3. Find the value of x in the equation $x^3-6x=12$.

By formula B′, we have

$$x=\sqrt[3]{\frac{12}{2}+\sqrt{\frac{12^2}{4}+\frac{-6^3}{27}}}+\frac{2}{\sqrt[3]{\frac{12}{2}+\sqrt{\frac{12^2}{4}+\frac{-6^3}{27}}}}=3,1392. \quad \textit{Ans.}$$

4. Find the value of x in the equation $x^3-15x=4$.

Ans. $x=4$.

5. Find the value of x in the equation $x^3+9x=584$.

Ans. $x=8$.

Second Form.

$$x^3+p'x^2=q.$$

465. If the second term be made to disappear from a cubic equation of this form, there will result a cubic equation of the first form (Art. 459, Cor. 1).

Hence, to reduce equations of the second form, we have the following general

RULE.

Transform the given equation into one of the first form, and then reduce as before.

EXAMPLES.

1. Find the value of x in the equation $x^3+3x^2=54$.

Let - - - - - $x=z-\frac{3}{3}=z-1$;

Then - - - - - $x^3=z^3-3z^2+3z-1$,

And - - - - - $3x^2= \quad +3z^2-6z+3$;

Adding the two equations, $x^3+3x^2=z^3-3z+2$;

Hence - - - $z^3-3z+2=54$;

Transposing - - - $z^3-3z=52$.

Applying formula B, we have

$$z=\sqrt[3]{\frac{52}{2}+\sqrt{\frac{52^2}{4}-\frac{3^3}{27}}}+\sqrt[3]{\frac{52}{2}-\sqrt{\frac{52^2}{4}-\frac{3^3}{27}}};$$

$$=\sqrt[3]{26+\sqrt{\frac{2704}{4}-\frac{27}{27}}}+\sqrt[3]{26-\sqrt{\frac{2704}{4}-\frac{27}{27}}};$$

$$=\sqrt[3]{26+\sqrt{676-1}}+\sqrt[3]{26-\sqrt{676-1}};$$

$$=\sqrt[3]{26+25,980761921}+\sqrt[3]{26-25,980761921};$$

$=\sqrt[3]{51,980761921}+\sqrt[3]{,019238079}$;

$=3,732+,267=3,999.$

Hence $x=3,999-1=2,999+$, or 3. *Ans.*

2. Find the value of x in the equation $x^3-3x^2=16$.

Let - - $x=z+\frac{3}{3}=z+1$;

Then - $z^3-\frac{3^2}{3}z=16+\frac{2}{27}\times 3^3$, or $z^3-3z=18$.

Applying formula B′ to this last equation,

$$z=\sqrt[3]{\frac{18}{2}+\sqrt{\frac{18^2}{4}-\frac{3^3}{27}}}+\frac{\frac{3}{3}}{\sqrt[3]{\frac{18}{2}+\sqrt{\frac{18^2}{4}-\frac{3^3}{27}}}}=3.$$

Hence $x=z+1=3+1=4$. *Ans.*

3. Find the value of x in the equation $x^3+6x^2=1600$.

Ans. 10.

4. Find the value of x in the equation $x^3-,3x^2=-,004$.

Ans. ,2.

Third Form.

$$x^3+p'x^2+px=q.$$

466. Making the second term disappear, we shall have, as before, an equation of the first form; hence, the method of reducing an equation of the second form will be the same as that for the second form.

EXAMPLES.

1. Find the value of x in the equation $x^3-6x^2+18x=22$.

Let $x=z+\frac{6}{3}=z+2$; then we shall have (Art. 459)

$$z^3+6z=2.$$

Applying formula A, we shall find $z=\sqrt[3]{4}-\sqrt[3]{2}$.

Whence

$x=z+2=\sqrt[3]{4}-\sqrt[3]{2}+2=1,5874-1,2599+2=2,3274$. *Ans.*

2. Find the value of x in the equation $x^3+8x^2-4x=32$.

Ans. $x=2$.

3. Find the value of x in the equation $x^3-10x^2+10x=100$.

Ans. $x=10$.

YOUNG'S METHOD OF RESOLVING CUBIC EQUATIONS.

467. Every cubic equation may be transformed so as to appear under the form

$$x^3+bx^2+cx=\text{N}. \qquad \text{(A.)}$$

468. Now, suppose that two consecutive numbers in either of the series

1, 2, 3, &c., or 10, 20, 30, &c., or 0.1, 0.2, 0.3, &c., are found such, that, substituting the first for x in the above equation, the result shall be less than N, and, by substituting the second, the result shall be greater than N; then the first of these numbers will be the first figure of one of the roots of the equation. Let this figure be represented by r, and the other succeeding figures of the same root by s, t, u, &c.; then, substituting for x the first figure (r) of its root in the equation (A), we shall have

$$r^3+br^2+cr=\text{N}; \qquad \text{(B.)}$$

Whence - - - $$r=\frac{\text{N}}{r^2+br+c}. \qquad \text{(C.)}$$

469. Let the remaining figures of the root equal y, then $x=r+y$: substituting this value for x in the first equation (A), we have

$$\left.\begin{array}{r} cy+cr=cx \\ by^2+2bry+br^2=bx^2 \\ y^3+3ry^2+3r^2y+r^3=x^3 \end{array}\right\}=\text{N}.$$

Adding, $y^3+(3r+b)y^2+(3r^2+2br+c)y+(r^3+br^2+cr)=\text{N}. \qquad$ (D.)

But, if - $b'=3r+b, \qquad$ (1.)

$c'=3r^2+2br+c, \qquad$ (2.)

$\text{N}'=\text{N}-r^3-br^2-cr, \qquad$ (3.)

the above equation becomes

$$y^3+b'y^2+c'y=\text{N}'. \qquad \text{(E.)}$$

470. This equation is in all respects similar to the first (A); and, since s is the first figure of the root y of this equation, substituting as before,

$$s^3+b's^2+c's=\text{N}';$$

Whence - - $$s=\frac{\text{N}'}{s^2+b's+c'}. \qquad \text{(F.)}$$

Supposing the value of s found, and putting t, u, &c., equal to z, or $y=z+s$, we have

$$c'z+c\,s=c'y\,;$$

$$b'z^2+2b'sz+b's^2=b'y^2\,;$$

$$z^3+3sz^2+3s^2z+s^3=y^3\,;$$

Adding,

$$z^3+(3s+b')z^2+(3s^2+2b's+c')z+(s^3+b's^2+c's)=N'. \quad \text{(G.)}$$

But, if - $b''=3s+b'$, (4.)

$$c''=3s^2+2b's+c', \quad (5.)$$

$$N''=N'-s^3-b's^2-c's, \quad (6.)$$

the above equation becomes

$$z^3+b''z^2+c''z=N'', \quad \text{(H.)}$$

an equation which is in all respects similar to the first. Hence we may proceed in the same way to find the first figure t, in the root z, and so on till we have found all the figures in the root x of the proposed equation.

471. Now, by observing the formation of the coefficients b', c' in the equation marked (F), and recollecting that r, being the first figure of the root, must be greater than s, it will appear obvious that c' must form a part of the divisor $s^2+b's+c'$, and if r be already known, the value of c' will become known (2), which may, therefore, be used as a trial divisor for finding s; the same may be observed of the next and the succeeding divisors; but these trial divisors, c'', c''', &c., will continually approach nearer the true divisors.

472. Now, if the first figure of the root r be found by trial, and $r+b$ be multiplied by it, and the product added to c, the sum will be the first divisor; thus,

$$r(r+b)=r^2+br$$

$$r^2+br+c= \text{1st divisor.} \quad (7.)$$

Hence - $$r=\frac{N}{r^2+br+c}=\frac{N}{r(r+b)+c}.$$

If under these two expressions we write r^2, and add up the three, we shall obtain c'; thus,

$$\begin{array}{l} r^2+br, \\ r^2+br+c, \\ r^2 \\ \hline 3r^2+2br+c=c'. \end{array} \qquad (8.)$$

Having obtained c', we have a *trial* divisor of N′ that will enable us to determine more easily the next figure s of the root.

When s is found, the second divisor may be computed; thus,

$$\begin{array}{r} s(s+3r+b)=s^2+b's, \\ +c', \\ \hline s^2+b's+c'= \text{2d divisor.} \end{array}$$

Hence - - $s=\dfrac{N'}{s^2+b's+c'}=\dfrac{N'}{s(s+3r+b)+3r^2+2br+c}$.

By a similar process we shall obtain

$$t^2+b''t+c''= \text{3d divisor.}$$

Hence, $t=\dfrac{N''}{t^2+b''t+c''}=\dfrac{N''}{t(t+3(r+s)+b)+3s^2+2b's+c'}$.

Also, $u^2+b'''u+c'''=$ 4th divisor.

Hence, $u=\dfrac{N'''}{u^2+b'''u+c'''}=\dfrac{N'''}{u(u+3(r+s+t)+b)+3t^2+2b''t+c''}$,

&c., &c., &c.

The above formulas may be readily applied to the reduction of cubic equations. By a careful inspection of them, we may obtain the following general

RULE.

1. *Put down* c, *the coefficient of* x, *and a little to the right place the absolute number, which is to be considered as a dividend, the figures of the root forming the quotient.*

2. *Place the first figure of the root, found by trial, in the quotient, above which write the coefficient of* x^2, *observing that its unit's place be over the unit's place of the quotient.*

3. *Multiply the value of the quotient figure, taking in those*

above by that value; add the product to c, *and the sum is the first divisor.*

4. *Write the square of the quotient figure just found under the first divisor, add it to the two sums immediately above, and the result will be the trial divisor for finding the next figure of the root.*

5. *Find now the next figure of the root, and to its value (including those above it) prefix three times the preceding, taking the value of the figure above it; multiply the result by the last found figure; add the product to the trial divisor, and we shall have the true divisor; and in the same manner are the succeeding divisors to be obtained.*

EXAMPLES

1. Reduce the equation $x^3+8x^2+6x=75.9$.

$$\begin{array}{cccc} x^3+ & 8x^2+ & 6x= & 75.9. \\ & \shortparallel & \shortparallel & \shortparallel \\ & b & c & N \end{array}$$

	$c =$	6
$r(r+b)=2(2+8)$	$=$	20
1st divisor $=r(r+b)+c$	$=$	26
r^2	$=$	4
1st trial divisor $=3r^2+2br+c=c$	$=$	50
$s(s+3r+b)$	$=$	5.76
2d divisor $=s(s+3r+b)+c'$	$=$	55.76
s^2	$=$	.16
2d trial divisor $=3s^2+2b's+c'=c''$	$=$	61.68
$t(t+3(r+s)+b)$	$=$	.3044
3d divisor $=t(t+3(r+s)+b)+c''$	$=$	61.9844
t^2	$=$	0.0004
3d trial divisor $=3t^2+2b''t+c''=c'''$	$=$	62.2892
$u(u+3(r+s+t)+b)$	$=$	0.076325
4th divisor $=u(u+3(r+s+t)+b)+c'''$	$=$	62.365525,
&c.		

$N=75.9\,|\,2.425+$	$=x$, or one of the roots.
52	$=r^3+br^2+cr=$ 1st subtrahend.
23.9	$=N'=$ 1st dividend.
22.304	$=s^3+b's^2+c's=$ 2d subtrahend.
1.596	$=N''=$ 2d dividend.
1.239688	$=t^3+b''t^2+c''t=$ 3d subtrah'd.
0.356312	$=N'''=$ 3d dividend.
0.311827625	$=u^3+b'''u^2+c'''u=$ 4th subt'd.
0.044484375	$=N''''=$ 4th dividend,
&c.	

Hence, one of the roots of the given equation is 2.425+.

Note.—By inspecting the preceding example, we shall observe that if, after obtaining three places of decimals in the right-hand column, we had continued to reject the remaining decimals, we should have had the root equally correct to three places of decimals. Now, in order that the number of decimals in the last column may not exceed three, it is obvious that the divisor corresponding to the first decimal in the root must contain but two decimals, that corresponding to the next decimals of the root but one, and that for every succeeding decimal in the root the right-hand digit of the corresponding divisor must be cut off. It should, however, be observed, that whatever would have been carried had the complete multiplication been performed, is still to be carried for the increase of the next figure; and, indeed, if the figure cut off exceed 5, one is to be carried to the next figure.

Hence the work of the above example may be rendered more concise, and will stand as follows, the figures cut off being placed a little to the right:

6		75.9	2.4257+
20		52	
26		23.9	
4		22.304	
50		1.596	
5.76		1.240	688
55.76		.356	312
.16		.312	827625
61.68		.044	484375
.30	44		
61.9	844		
0	004		
62.3	892		
1	76325		
62.4			

2. Reduce the equation $x^3+x^2=500$.

This equation is the same as $x^3+x^2+0x=500$; hence $b=1$, $c=0$, and $N=500$.

The first figure of the root is 7.

<table>
<tr><td>0</td><td>500 |7.61727975, &c., $=x$.</td></tr>
<tr><td>56</td><td>392</td></tr>
<tr><td>56</td><td>108</td></tr>
<tr><td>49</td><td>104.736</td></tr>
<tr><td>161</td><td>3.264</td></tr>
<tr><td>13.56</td><td>1.887181</td></tr>
<tr><td>174.56</td><td>1.376819</td></tr>
<tr><td>36</td><td>1.323862</td></tr>
<tr><td>188.48</td><td>52957</td></tr>
<tr><td>.2381</td><td>37859</td></tr>
<tr><td>188.7181</td><td>15098</td></tr>
<tr><td>1</td><td>13251</td></tr>
<tr><td>188.9563</td><td>1847</td></tr>
<tr><td>1669</td><td>1704</td></tr>
<tr><td>189.123|2</td><td>143</td></tr>
<tr><td></td><td>133</td></tr>
<tr><td>189.290</td><td></td></tr>
<tr><td>5</td><td>10</td></tr>
<tr><td>1|8|9.|2|9|5</td><td>9</td></tr>
</table>

3. Reduce the equation $x^3-17x^2+54x=350$.

Ans. $x=14.954$, &c.

4. Reduce the equation $x^3+2x^2+3x=13089030$.

Ans. $x=235$.

5. Reduce the equation $x^3+2x^2-23x=70$.

Ans. $x=5.1345$, &c.

6. Reduce the equation $x^3-2x=5$.

Ans. $x=2.09455148154232659 17$, &c.

DES CARTES' METHOD OF RESOLVING EQUATIONS OF THE FOURTH DEGREE.

473. Every equation of the fourth degree may be reduced to the form

$$x^4+bx^3+cx^2+dx+e=0. \qquad \text{(A.)}$$

This equation may also be transformed into another which shall want the second term (Art. 459); thus,

$$x^4+c'x^2+d'x+e'=0. \qquad \text{(B.)}$$

474. Now if we can arrive to a solution of the equation in this form, in which the roots sustain a given relation to the original equation (A), the complete solution of that equation may be effected.

Now, suppose B to be formed by the product of

$$x^2+px+q=0, \qquad (1.)$$
$$x^2+rx+s=0, \qquad (2.)$$

two equations in which p, q, r, and s are unknown quantities, and we shall obtain by the actual multiplication of the factors (1) (2), and, taking the sum of the coefficients of the equal powers of x,

$$x^4+(p+r)x^3+(s+q+pr)x^2+(ps+qr)x+qs=0. \qquad \text{(C.)}$$

Whence - $p+r=0$, or $r=-p$; (3.)

$$s+q+pr=c'; \qquad (4.)$$
$$ps+qr=d'; \qquad (5.)$$
$$qs=e'. \qquad (6.)$$

Or, substituting $-p$ for r in (4) and (5), and transposing, they will become

$$s+q=c'+p^2; \qquad (7.)$$
$$s-q=\frac{d'}{p}. \qquad (8.)$$

And, by subtracting the square of (8) from the square of (7), we have

$$c'^2+2c'p^2+p^4-\frac{d'^2}{p^2}=4qs, \text{ or } 4e'.$$

Or, clearing of fractions, and arranging the terms with reference to the highest power of p, we have

$$p^6+2c'p^4+(c'^2-4e')p^2-d'^2=0. \qquad \text{(D.)}$$

If $p^2=z$, this equation will become

$$z^3+2c'z^2+(c'^2-4e')z-d'^2=0. \qquad (E.)$$

Now, if we add and subtract equations (7) and (8), and divide the result by 2, we shall have

$$s=\tfrac{1}{2}c'+\tfrac{1}{2}p^2+\frac{d'}{2p}; \qquad (9.)$$

$$q=\tfrac{1}{2}c'+\tfrac{1}{2}p^2-\frac{d'}{2p}. \qquad (10.)$$

475. From these two formulas (9) and (10), p being known from equation (E), s and q can be obtained.

Hence, substituting $-p$ for r in equation (2), and reducing the two equations (1) and (2), we shall have

$$x=-\tfrac{1}{2}p\pm\sqrt{\tfrac{1}{4}p^2-q}; \qquad (11.)$$
$$x=+\tfrac{1}{2}p\pm\sqrt{\tfrac{1}{4}p^2-s}. \qquad (12.)$$

These equations (11) and (12) give the four roots of the biquadratic equation (B).

COR. 1. The cubic equation (E) gives three roots; but the same values of x will be obtained, whichever of the roots be used.

COR. 2. If the roots of the cubic equation (E) are all *real*, the roots of the biquadratic equation (B) will be *real* also.

If only *one* root of the cubic equation (E) be *real*, then the proposed biquadratic (B) will have *two real* and *two imaginary* roots.

476. The above formulas may be readily applied to the reduction of equations of the fourth degree.

EXAMPLES.

1. Reduce the equation $x^4-3x^2+6x+8=0$.

Comparing this equation with formula (B), we shall have $c'=-3$, $d'=6$, and $e'=8$; and substituting these values for c', d', and e' in formula (E), it becomes

$$z^3-6z^2-23z-36=0.$$

Reducing this equation, $z=9$; hence $p=\sqrt{z}=\sqrt{9}=\pm 3$. Substituting $+3$ for p in formulas (9) and (10), we have

$$s=\tfrac{1}{2}c'+\tfrac{1}{2}p^2+\frac{d'}{2p}=-\tfrac{3}{2}+\tfrac{9}{2}+\tfrac{6}{6}=4;$$

$$q=\tfrac{1}{2}c'+\tfrac{1}{2}p^2-\frac{d'}{2p}=-\tfrac{3}{2}+\tfrac{9}{2}-\tfrac{6}{6}=2.$$

Substituting these 3, 4, and 2 for p, s, and q in formulas (11) and (12), we have

$$x=-\tfrac{1}{2}p\pm\sqrt{\tfrac{1}{4}p^2-q}=-\tfrac{3}{2}\pm\sqrt{\tfrac{9}{4}-2}=-\tfrac{3}{2}\pm\tfrac{1}{2}=-1, \text{ or } -2;$$

$$x=+\tfrac{1}{2}p\pm\sqrt{\tfrac{1}{4}p^2-s}=+\tfrac{3}{2}\pm\sqrt{\tfrac{9}{4}-4}=+\tfrac{3}{2}\pm\sqrt{-\tfrac{7}{4}}=\tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{-7}.$$

Hence the four roots of the biquadratic equation are -1, -2, $\tfrac{3}{2}+\tfrac{1}{2}\sqrt{-7}$, and $\tfrac{3}{2}-\sqrt{-7}$.

2. Reduce the equation $x^4-4x^3-8x+32=0$.

Ans. 4, 2, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$.

3. Reduce the equation $x^4-9x^3+30x^2-46x+24=0$.

Ans. 1, 4, $2+\sqrt{-2}$, and $2-\sqrt{-2}$.

4. Reduce the equation $x^4+16x^3+99x^2+228x+144=0$.

Ans. -1, -3, $-6+2\sqrt{-3}$, and $-6-2\sqrt{-3}$.

NEWTON'S METHOD OF APPROXIMATION.

477. This is an expeditious method of finding the approximate root of an equation, when its near root is given or has been ascertained by trial, and is equally applicable, whatever be the degree of the equation.

478. Let us resume the general equation

$$x^n+bx^{n-1}+cx^{n-2}+\ldots\ldots\ldots sx^2+tx+u=0. \quad \text{(A.)}$$

Then let a represent the near root of the equation which is known, and z represent the part to be added to make the root complete; then

$$x=a+z. \quad (1.)$$

Substituting this value for x in the first equation (A), we have

$$(a+z)^n+b(a+z)^{n-1}+c(a+z)^{n-2}+\ldots\ldots s(a+z)^2+t(a+z)+u=0. \quad \text{(B.)}$$

Then, transforming, as in Art. 459,

$$W+Xz+\frac{Y}{2}z^2+\frac{Z}{2\,.\,3}z^3\ldots\ldots\ldots z^n=0. \quad \text{(C.)}$$

479. Now, since z, by hypothesis, is a proper fraction, the terms that involve z^2, z^3, &c., being less than z, may be re-

jected from the equation without departing far from rigid exactness. The equation (C) will then become

$$W+Xz=0. \qquad (2.)$$

Whence - - $z=-\frac{W}{X}$. (3.)

But, comparing with the transformations in Art. 459, we shall find

$$W= a^n \ +ba^{n-1} \qquad +ca^{n-2} \ldots\ldots\ sa^2+ta+u; \ (4.)$$

$$X =na^{n-1}+(n-1)ba^{n-2}+(n-2)ca^{n-3} \ldots 2sa +t. \qquad (5.)$$

Substituting these values for W and X in equation (3), we have

$$z=-\frac{a^n+ba'^{n-1}+ca^{n-2} \ \ldots\ldots\ sa^2+ta+u}{na^{n-1}+(n-1)ba^{n-2}+(n-2)ca^{n-3} \ \ldots\ldots\ 2sa+t}. \ (D.)$$

The numeral value of this expression should be calculated to within one or two places of decimals, and added to the root (a) found by trial. Let the resulting approximate root be represented by a', then $a'=a+z$; and if z' represent the part still to be added, we shall have

$$z'=-\frac{a'^n+ba^{n-1}+ca'^{n-2} \ \ldots\ldots\ sa'^2+ta'+u}{na'^{n-1}+(n-1)ba'^{n-2}+(n-2)ca'^{n-3} \ \ldots\ldots\ 2sa'+t}. \ (D'.)$$

Letting a'' represent the third approximate root, we shall have

$$a''=a'+z'.$$

480. Proceeding in this manner, the approximation may be carried to any assigned degree of exactness.

EXAMPLES.

1. Reduce the equation $x^3+2x^2-8x=24$.

By making trial of 1, 2, 3, and 4, we shall find that the root of the equation is between 3 and 4, and very nearly equal to 3.

Then $n=3$, $a=3$, b or $s=2$, c or $t=-8$, $u=-24$, and $x=a+z$.

By substituting these values in formula (D), we have

$$z=-\frac{3^3+2.3^2-8.3-24}{3.3^2+4.3-8}=\tfrac{3}{31}=0.09; \text{ hence } x=3.09, \text{nearly.}$$

Again, if 3.09 be substituted for a' in formula (D′), we shall have

$z'=-\frac{(3.09)^3+2.(3.09)^2-8.(3.09)-24}{3.(3.09)^2+4(3.09)-8}=.00364$; hence $x=3.09364$.

2. Reduce the equation $x^3+x^2+x=90$.

Here $n=3$, b or $s=1$, c or $t=1$, $n=-90$, and a will be found $=4$; hence $x=4+z$, and we shall have

$z=-\frac{4^3+4^2+4-90}{3.4^2+2.4+1}=\frac{6}{57}=0.1$; hence $x=4.1$, nearly.

Again,

$z'=-\frac{(4.1)^3+(4.1)^2+(4.1)-90}{3.(4.1)+2.(4.1)+1}=0.00283$; hence $x=4.10283$.

3. Reduce the equation $x^4-38x^3+210x^2+538x+289=0$.
Ans. $x=30.535653$.

4. Reduce the equation $x^5+6x^4-10x^3-112x^2-207x+110=0$. *Ans.* $x=4.46410161$.

RESOLUTION OF HIGHER EQUATIONS BY TRIAL AND ERROR.

481. The roots of cubic equations may also be found to a sufficient degree of exactness by successive approximations. From the laws of the coefficients, as stated in Art. 455, it is evident that the roots must be such that, when their signs are changed, their *product* shall be equal to the last term of the equation, and their *sum* equal to the coefficient of the second term. By considering this law, some estimate may be formed of the values of the roots, and a trial may then be made, by substituting in the place of the unknown letter its supposed value. If this proves too small or too great, it may be increased or diminished, and the trials repeated till one is found which will nearly satisfy the conditions of the equations.

482. Now, since the errors in the results will be very nearly proportioned to the errors in the assumed numbers, after we have assumed two approximate values, and calculated the errors which result from them, we may obtain a more exact correction of the root by the following proportion:

The difference of the errors : to the difference of the assumed numbers : : the least error : to the correction required.

For, letting N and $n=$ the assumed numbers, S and $s=$ the errors of these numbers, and R and $r=$ the errors of the results, we shall have

$$R : r :: S : s \text{ very nearly.}$$

Hence, by Art. 369, $R-r : S-s :: r : s$.

483. If the value which is first found is not sufficiently correct, this may be taken as one of the numbers for a second trial; and the process may be repeated till the error is diminished as much as is required.

There will generally be an advantage in assuming two numbers whose difference is .1, or .01, or .001, &c.

EXAMPLES.

1. Reduce the equation $x^3-8x^2+17x-10=0$.

Here the signs are alternately positive and negative, therefore (Art. 455) the roots must all be positive; their product $=10$, and their sum $=8$.

Suppose $x=5.1$ or 5.2; then,

By 1st supposition,

$$(5.1)^3-8.(5.1)^2+17.(5.1)-10=1.271.$$
By 2d, $$(5.2)^3-8.(5.2)^2+17.(5.2)-10=2.688.$$ } errors.

Difference of errors - - - - 1.417

Then - - $1.4 : 0.1 :: 1.27 : 0.09$.

Hence - - $x=5.1-0.09=5.01$, nearly.

To correct this farther, suppose $x=5.01$, or 5.02; then,

By 1st supposition,

$$(5.01)^3-8.(5.01)^2+17.(5.01)-10=0.121$$
By 2d, $$(5.02)^3-8.(5.02)^2+17.(5.02)-10=0.246$$ } errors.

Difference of errors - - - - 0.125

Then - - $0.125 : 0.01 :: 0.121 : 0.01$.

Hence - - $x=5.01-0.01=5$.

This value of x satisfies the conditions of the equation; for,

$$5^3-8\times5^2+17\times5-10=0.$$

Therefore, one of the roots of the equation is 5.

To find the other two roots, let the first member be divided by $x-5$ (Art. 453), and the quotient put equal to 0.

$$\begin{array}{l|l} x^3-8x^2+17x-10 & x-5 \\ x^3-5x^2 & \overline{x^2-3x+2} \\ \hline \quad -3x^2-17x-10 & \\ \quad -3x^2-15x & \\ \hline \qquad\quad 2x-10 & \\ \qquad\quad 2x-10 & \\ \hline \qquad\qquad 0 & \end{array}$$

Hence - - $x^2-3x+2=0.$

Reducing - - $x=2$ or 1.

The three roots of the given equation, then, are 5, 2, and 1.

2. Reduce the equation $x^3-8x^2+4x+48=0$.

Let $x=4.1$, or 4.2.

Substituting successively these values for x in the equation, we have,

$$\left.\begin{array}{ll} \text{1st,} & (4.1)^3-8.(4.1)^2+4.(4.1)+48=-1.159 \\ \text{2d,} & (4.2)^3-8.(4.2)^2+4.(4.1)+48=-2.282 \end{array}\right\} \text{errors.}$$

Difference of errors - - - -1.123

Then - - $-1.1:0.1::-1.1:0.1.$

Hence - - $x=4.1-0.1=4.$

This value of x satisfies the conditions of the equation; for,

$$4^3-8.4^2+4.4+48=0.$$

Therefore, one of the roots is 4.

Dividing the first member of the given equation by $x-4$, the quotient is

$$x^2-4x-12=0.$$

Reducing - $x=6$, or -2.

The roots of the equation are -2, 4, and 6

3. Reduce the equation $x^3+16x^2+65x-50=0$.

Ans. 1, 5, and 10.

4. Reduce the equation $2x^4-16x^3+40x^2-30x=-1$.

Ans. $x=1.2847$.

5. Reduce the equation $x^5+2x^4+3x^3+4x^2+5x=5.4321$.

Ans. $x=8.414455$.

YOUNG'S METHOD OF RESOLVING HIGHER EQUATIONS.

484. The method of solving cubic equations in Art. 472 is obviously adapted to equations of any higher degree; and, by carefully inspecting the properties of equations, and the mode of reduction there employed, we shall be able to deduce, for the reduction of equations of the nth degree, the following general

RULE.

1. *Arrange the coefficients of the given equation in a row, commencing with that of the first term then find by trial the first figure of the root.*

2. *Add the product of the first root figure and the first coefficient to the second coefficient; the product of this sum and the same figure to the third coefficient, and so on to the last coefficient, and the last sum will be the* DIVISOR. *Multiply this by the first figure of the root, and subtract the product from the term constituting the right-hand member of the equation; the remainder will form the first* DIVIDEND.

3. *Repeat this process with the first coefficient and these sums, and the number under the last sum will be the* TRIAL DIVISOR *for the next figure.*

4. *Perform a similar process with the first coefficient and these second sums, stopping under the $n-1$th coefficient. Again, perform a similar process with the same first coefficient and these last sums, stopping under the $n-2$th coefficient, and so on till the last sum falls under the second coefficient.*

5. *Find now, from the trial divisor and the first dividend, the next figure of the root, and proceed with the last set of sums and this new figure exactly the same as with the original coefficients and the first figure in finding the preceding divisor, and the* SECOND DIVISOR *will be obtained. Then proceed, as before, to find the* SECOND DIVIDEND, *and so on till the work has been carried to a sufficient degree of exactness.*

Note.—The work may be contracted by cutting off decimals as before

EXAMPLES.

1. Reduce the equation $x^4 - 3x^2 + 75x = 10000$.

Operation.

0	—3	75	10000	|9 . 8860027, &c., $=x$.
9	81	702	6993	
9	78	777	3007	
9	162	2160	2677 . 5616	
18	240	2937	329 . 4384	
9	243	409 . 952	306 . 1662	
27	483	3346 . 952	23 . 2722	
9	29 . 44	434 . 016	23 . 2616	
36 . 8	512 . 44	3780 . 968	106	
. 8	30 . 08	46 . 110	78	
37 . 6	542 . 52	3827 . 07|8	28	
. 8	30 . 72	46 . 36	27	
38 . 4	573 . 24	3873 . 44	1	
. 8	3 . 14	3 . 50		
3|9 . |2	576 . . 3|8	3876 . 9|4		
	3 . 1	3 . 5		
	579 . |5	3880 . 4		
	3			
	5|8|3			

Note.—By bringing down one period of decimals, we have found the root to eight places of figures. If another period, or eight decimals, had been brought down, the root might have been found to twelve places of figures, or $x = 9 . 88600270094$.

2. Reduce the equation $x^5 + 6x^4 - 10x^3 - 112x^2 - 207x = 110$.

Operation.

1	6	—10	—112	—207	110 |4.46410161.
	4	40	120	32	—700
	10	30	8	—175	810
	4	56	344	1408	667.05984
	14	86	352	1233	142.94016
	4	72	632	434.6496	133.46395
	18	158	984	1667.6496	9.47621
	4	88	102.624	477.4144	9.24089
	22	246	1086.624	2145.0640	23532
	4	10.56	106.912	79.3352	23158
	26.4	256.56	1193.536	2224.399|2	374
	.4	10.72	111.264	80.389	232
	26.8	267.28	1304.800	2304.788	142
	4	10.88	17.453	5.434	139
	27.2	278.16	1322.25|3	2310.22|2	3
	4	11.04	17.56	5.44	2
	27.6	289.20	1339.8|1	2315.66	
	4	1.68	17.6	14	
	2|8.|0	290.8|8	1357.4	2315.8|0	
		1.7	1.2	1	
		292.|6	1358.|6	2|3|1|5.9	
		1	1		
		2|9|4	1|3|5|9		

3. Reduce the equation $x^5+2x^4+3x^3+4x^2+5x=54321$.
Ans. $x=8.41445475$, &c.

4. Reduce the equation $x^6+2x^5+3x^4+4x^3+5x^2+6x=654321$.
Ans. $x=8.95697957$, &c.

5. Reduce the equation $x^7-3x^6-2.5x^5+10x^4+x^3-9x^2+2x=2$.
Ans. $x=2.62599736$, &c.

6. Reduce the equation $x^8+10x^7+21x^6-55x^5-100x^4+525x^3+804x^2-630x=216$.
Ans. $x=.79128785$, &c.

NOTES.

Note A, *page* 13, *Art.* 2.

The term Quantity seems to be used by writers on Mathematics with a great degree of vagueness, and the definitions of it are liable to many objections. For instance: "Quantity is a general term, embracing everything which admits of increase or diminution."* Now, it is with perfect consistency that the natural philosopher speaks of "increasing or diminishing" heat: so mental power or energy may be increased or diminished; and so, also, passion, resentment, anger, benevolence, or love may be increased or diminished. Hence, by the above definition, they are included under the term Quantity, and are, consequently, objects of mathematical investigation. The incorrectness of this definition needs no farther illustration. With regard to Number, we believe it cannot properly be included under Quantity. Dugald Stewart† remarks: "As to number and proportion, it might be easily shown that neither of them fall under the definition of Quantity, in any sense of that word." Believing the term Quantity incorrectly applied in most treatises on Algebra, we have endeavoured to substitute the word *number* in its place.

Dr. Reid‡ suggests a distinction of Quantity into *proper* and *improper*. *Proper Quantity* is that which is measured by its own kind, such as extension and duration. *Improper Quantity* is that which cannot be measured by its own kind, but to which we assign a measure in some proper quantity that is related to it. Velocity, density, elasticity, &c., may be considered as examples of this kind of quantity.

Note B, *page* 16, *Art.* 20.

The origin of Algebra, like that of other sciences of ancient date and gradual progress, is not easily ascertained. We, however, have derived it from the Arabians, among whom it was cultivated at a very early period. The most ancient treatise on Algebra now extant is that of Diophantus, a Greek author of Alexandria, who flourished about A.D. 350, and wrote thirteen books, six of which are now extant. The following is a list of some of the early writers on Algebra: Pisanus, 1400; Lucas de Burgo, 1476; Scipio Ferreus, 1505; Nicholas Tartalea, 1539; Cardan, 1545; Xylander, 1575; Bachet, 1621; M. Fermat, 1670. Of later date, writers have been abundant; and among them may be ranked some of

* Davies' Bourdon. † Works, vol. ii., p. 364. ‡ Essay on Quantity.

the most distinguished mathematicians and philosophers, such as Newton, Euler, Des Cartes, and a host of others.

Note C, *page* 136, *Art.* 244.

The theory of indetermination is productive of several important consequences.

The remark here made upon indetermination will aid the student in his analysis of several curious problems. For instance, we might cite the process by which two is made to appear to be equal to one; thus,

Let - - - - - - $a=1$, and $x=1$,
Then - - - - - - - $x=a$;
Multiplying by x - - - - - $x^2=ax$;
Subtracting a^2 from both members, $x^2-a^2=ax-a^2$;
Resolving into factors - $(x+a)(x-a)=a(x-a)$;
Dividing by $x-a$ - - - $x+a=a$;
Restoring values of x and a - - $1+1=1$, or $2=1$.

The fallacy in the above reasoning will be easily detected; for, expressing the division, we have

$$x+a=\frac{a(x-a)}{x-a},$$

which, since $x=a$, becomes

$$x+a=\frac{a\times 0}{0}=\frac{0}{0}.$$

The above case is not singular. Take the identical equation

$$10=10:$$

Resolving into terms - - - - $8+2=8+2$;
Transposing - - - - - - $2-2=8-8$;
Resolving 2d member into factors $2-2=4(2-2)$;
Dividing by $2-2$ - - - - - - $1=4$.

Note D, *page* 257, *Art.* 391.

The invention of Logarithms is undoubtedly due to John Napier, Baron of Merchiston, in Scotland, who gave it to the world in a book written in Latin, and entitled, "*Mirifici Logarithmorum Canonis Descriptio, ejusque usus in utraque Trigonometria, ut etiam in omni Logistica Mathematica, Amplissimi, Facilimi, et Expeditissimi Explicatio. Auctore ac inventore Joanne Nepero, Baronne Merchistonii.*

Note E, *page* 262, *Art.* 399.

Formula: logarithm $(n+1)=$log. $n+2\text{ M}\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}\right.$
$\left.+\frac{1}{7(2n+1)^7}+\frac{1}{9(2n+1)^9}+\frac{1}{11(2n+1)^{11}},\ \&c.\right\}$

The reduction of the above formula was not introduced into the body of the work, on account of its length and complexity.

Let a represent the base of the system, $v+1$ be any number in the common arithmetical scale, and x its logarithm; then (Art.) $a^x=v+1$. Again, let $a=1+b$, then $(1+b)^x=1+v$; and to find the log. of $1+v$, we must solve this equation, in which x is the unknown quantity.

Involving both members to the power m, we have

$$(1+b)^{mx}=(1+v)^m;$$

Expanding, $$1+mxb+\frac{mx(mx-1)}{2}\times b^2+\frac{mx(mx-1)(mx-2)}{6}\times b^3+, \&c.$$

$$=1+mv+\frac{m(m-1)}{2}\times v^2+\frac{m(m-1)(m-2)}{6}\times v^3+, \&c.$$

Rejecting 1 from each member, and dividing by m,

$$xb+\frac{x(mx-1)}{2}b^2+\frac{x(mx-1)(mx-2)}{6}b^3+, \&c., =v+\frac{m-1}{2}v^2+\frac{(m-1)(m-2)}{6}v^3+, \&c.$$

Or - $$x(b+\frac{(mx-1)}{2}b^2+\frac{(mx-1)(mx-2)}{6}b^3+, \&c.,)=v+\frac{m-1}{2}v^2+\frac{(m-1)(m-2)}{6}v^3+, \&c.$$

Now, let $m=0$, and we shall have

$$x(b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\frac{b^5}{5}-, \&c.,)=v-\frac{v^2}{2}+\frac{v^3}{3}-\frac{v^4}{4}+\frac{v^5}{5}-, \&c.$$

Dividing - - $$x=\frac{v-\frac{1}{2}v^2+\frac{1}{3}v^3-\frac{1}{4}v^4+\frac{1}{5}v^5-, \&c.}{b-\frac{1}{2}b^2+\frac{1}{3}b^3-\frac{1}{4}b^4+\frac{1}{5}b^5-, \&c.}$$

But $x=$ log. $(v+1)$; therefore,

$$\text{log. }(v+1)=\frac{v-\frac{1}{2}v^2+\frac{1}{3}v^3-\frac{1}{4}v^4+\frac{1}{5}v^5-, \&c.}{b-\frac{1}{2}b^2+\frac{1}{3}b^3-\frac{1}{4}b^4+\frac{1}{5}b^5-, \&c.}$$

Or, since $a=b+1$,

$$\text{log. }(v+1)=\frac{v-\frac{1}{2}v^2+\frac{1}{3}v^3-\frac{1}{4}v^4+\frac{1}{5}v^5-, \&c.}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\frac{1}{5}(a-1)^5-, \&c.}$$

Or, if we take $M=\frac{1}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-, \&c.}$,

$$\text{log. }(v+1)=M(v-\frac{1}{2}v^2+\frac{1}{3}v^3-\frac{1}{4}v^4+\frac{1}{5}v^5-, \&c.) \quad \text{(A.)}$$

Since a is a constant quantity, M, which is termed the *modulus* of system, must also be a constant quantity.

$$M=\frac{1}{2{,}30258509}=.434294482.$$

Or, more correctly,

$$M=.4342944819032518276511289189166.$$

This is the series (A) earliest known for the calculation of logarithms But the difficulty with it is that it will either diverge, or not converge so quickly as to make the summation of a few terms of it a sufficient approximation to the value of x or log. $(v+1)$, unless v be a proper fraction sufficiently small. If v be nearly equal to 1, the series converges too slowly to be of any use; and if v be greater than 1, the series diverges, and is, consequently, useless.

We may, however, transform this series (A) into others, so as to obtain a series that will apply in every possible case. For if $1-v$ instead of $1+v$ be used, we shall obtain

$$\log.(1-v)=M(-v-\frac{v^2}{2}-\frac{v^3}{3}-\tfrac{1}{4}v^4-\tfrac{1}{5}v^5-, \&c.)$$
$$=-M(v+\tfrac{1}{2}v^2+\tfrac{1}{3}v^3+\tfrac{1}{4}v^4+\tfrac{1}{5}v^5+, \&c.)$$

But $\log.(1-v)=-\log.\frac{1}{1-v}$; therefore,

$$\log.\frac{1}{1-v}=M(v+\tfrac{1}{2}v^2+\tfrac{1}{3}v^3+\tfrac{1}{4}v^4+\tfrac{1}{5}v^5+, \&c.) \qquad \text{(B.)}$$

By adding together the formulas (A) and (B), and observing that $\log.(1+v)+\log.\frac{1}{1-v}=\log.\frac{1+v}{1-v}$, we have

$$\log.\frac{1+v}{1-v}=M(2v+\tfrac{2}{3}v^3+\tfrac{2}{5}v^5+\tfrac{2}{7}v^7+, \&c.)$$
$$=2M(v+\tfrac{1}{3}v^3+\tfrac{1}{5}v^5+\tfrac{1}{7}v^7+, \&c.) \qquad \text{(C.)}$$

Again, let us put $\frac{1+v}{1-v}=\frac{u}{t}$, then $v=\frac{u-t}{u+t}$, and substituting in the above formula, it becomes

$$\log.\frac{u}{t}=2M\left(\frac{u-t}{u+t}+\tfrac{1}{3}\left(\frac{u-t}{u+t}\right)^3+\tfrac{1}{5}\left(\frac{u-t}{u+t}\right)^5+\tfrac{1}{7}\left(\frac{u-t}{u+t}\right)^7+, \&c.\right)$$

Or - $\log.u-\log.t=2M\left(\frac{u-t}{u+t}+\tfrac{1}{3}\left(\frac{u-t}{u+t}\right)^3+\tfrac{1}{5}\left(\frac{u-t}{u+t}\right)^5+\tfrac{1}{7}\left(\frac{u-t}{u+t}\right)^7+, \&c.\right)$

Transposing - $\log.u=\log.t+2M\left(\frac{u-t}{u+t}+\tfrac{1}{3}\left(\frac{u-t}{u+t}\right)^3+\tfrac{1}{5}\left(\frac{u-t}{u+t}\right)^5+\tfrac{1}{7}\left(\frac{u-t}{u+t}\right)^7+, \&c.\right)$

Now, letting $u=n+1$ and $t=n$, we shall have $u-t=1$, and $u+t=2n+1$; substituting these values in the preceding formula, it becomes

$$\log.(n+1)=\log.n+2M\left(\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\frac{1}{7(2n+1)^7}+, \&c.\right)$$

This series evidently converges very rapidly, even when $n=1$; but converges more rapidly as n increases. Hence, having found the logarithm of any number, we may easily find the logarithm of the next higher in the natural series of numbers by the application of this formula. *See Edinburgh* and *Rees's Encyclopedias.*