XIX. *On a New Method of Approximation applicable to Elliptic and Ultra-elliptic Functions.*—Second Memoir*. *By* Charles W. Merrifield. *Communicated by* W. Spottiswoode, *Esq.*, *F.R.S.*

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Since my first memoir on this subject was read before the Society, Mr. Sylvester has published a method, more general than mine, of applying rational approximation to facilitate the computation of the integrals of irrational functions. This method, at which he had arrived independently, included, *à majori*, the one which was the subject of my memoir. Aided by his papers, my subsequent studies have enabled me to view the method with more generality, as well as with more precision and completeness of detail, and I am now able to present it in a sufficiently finished and practical form for the immediate use of the computer. I have also computed auxiliary Tables, to render its application easier in certain cases.

Any rational formula, which gives approximately the value of a function to be integrated, may be integrated in lieu of it, and the result will in general be an approximate value of the integral sought. But for such a process to be of any practical utility, the convergence of the formula must be excessive, for the complexity of the integral forms is so great that the labour would be enormous, unless the terms were very few in number. In the discovery of formulæ sufficiently convergent for the purpose, lies the success of the method.

We are by no means restricted to functions under a square root, or even to pure radical forms at all. The principle applies with equal generality to functions which are given implicitly as roots of equations, and thus to a class of differential equations; and Mr. Sylvester has well remarked that these formulæ not only afford facilities for computation, as by a method of quadratures, but also enable us to assign superior and inferior limits to an integral, without losing its generality of form.

I shall begin with the approximation to the square root, giving it in its general form, and explaining its exact analytical signification. I shall then show its application to Elliptic Functions, and how, in the ordinary cases, certain simple reductions can be effected, which greatly lessen the labour of computation; and I shall give these reductions for the cases more commonly occurring, with some examples and working formulæ. I shall then add a short account of the extension of the method.

The paragraphs in the first two sections of this paper bear a consecutive number for convenience of reference.

* For the First Memoir, see the Philosophical Transactions for 1860, p. 223.

SECTION I.—*Approximants to the Square Root.*

1. Mr. SYLVESTER gives, for the approximants to the square root, the following statement:—

"Let r be an approximate value of $\sqrt{N}$; then by that mode of application of NEWTON'S method of approximation to the equation $x^2=N$, which is equivalent to the use of continued fractions, we may easily establish the following theorem, viz., that

$$\frac{r^2+N}{2r},\ \frac{r^3+3rN}{3r^2+N},\ \frac{r^4+6r^2N+N^2}{4r^3+4rN},\ \frac{r^5+10r^3N+5rN^2}{5r^4+10r^2N+N^2},$$

will be successive approximations to $\sqrt{N}$."

2. Their general form is

$$y=\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i}\sqrt{N}, \quad \ldots \ldots \ldots \quad (1.)$$

which is always rational. In this form the approximation to $\sqrt{N}$ as i increases is obvious. The method of my previous memoir is simply the particular case of $i=2^k$.

3. If we wish to approximate to $N^{-\frac{1}{2}}$, we may take the reciprocal of (1.), or, what is simpler, we may divide (1.) by N, thus obtaining

$$z=\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i}\frac{1}{\sqrt{N}}. \quad \ldots \ldots \ldots \quad (2.)$$

Before we can integrate these formulæ, we must reduce them by means of the method of rational fractions; the simplest and most general way is as follows:—

4. Let ϱ be an ith root of unity; then, obviously,

$$\log(1-x^i)=\log(1-\varrho x)+\log(1-\varrho^2x)+\ldots\ldots+\log(1-\varrho^ix).$$

Multiplying the differential coefficient of this by $(-x)$, we obtain

$$\frac{ix^i}{1-x^i}=\frac{\varrho x}{1-\varrho x}+\frac{\varrho^2x}{1-\varrho^2x}+\frac{\varrho^3x}{1-\varrho^3x}+\ldots\ldots+\frac{\varrho^ix}{1-\varrho^ix};$$

and since $\frac{1}{1-x^i}=1+\frac{x^i}{1-x^i}$ and $\frac{1+x^i}{1-x^i}=1+\frac{2x^i}{1-x^i}$,

$$\frac{i}{1-x^i}=\frac{1}{1-\varrho x}+\frac{1}{1-\varrho^2x}+\frac{1}{1-\varrho^3x}+\ldots\ldots+\frac{1}{1-\varrho^ix},$$

$$i\frac{1+x^i}{1-x^i}=\frac{1+\varrho x}{1-\varrho x}+\frac{1+\varrho^2x}{1-\varrho^2x}+\frac{1+\varrho^3x}{1-\varrho^3x}+\ldots\ldots+\frac{1+\varrho^ix}{1-\varrho^ix}.$$

Making $x=\frac{r-\sqrt{N}}{r+\sqrt{N}}$, we may thus divide $N^{\pm\frac{1}{2}}$ into i fractions, each of the form

$$\frac{1}{i}\frac{(r+\sqrt{N})+\varrho^k(r-\sqrt{N})}{(r+\sqrt{N})-\varrho^k(r-\sqrt{N})}N^{\pm\frac{1}{2}},$$

k being any integer not exceeding i.

5. If we add the pairs k and $i-k$, we obtain for the sum of the pair,

$$\frac{2(r+\sqrt{N})^2-2(r-\sqrt{N})^2}{(r+\sqrt{N})^2+(r-\sqrt{N})^2-(\varrho^k+\varrho^{i-k})(r^2-N)}N^{\pm\frac{1}{2}}$$

$$=\frac{8rN}{2(r^2+N)-(\varrho^k+\varrho^{i-k})(r^2-N)} \text{ or } \frac{8r}{\text{same denominator}},$$

according to whether the upper or lower sign be taken. Now, because ϱ is an ith root of unity, $\varrho^k + \varrho^{i-k} = 2\cos\frac{2k\pi}{i}$, and the sum of the pair reduces itself, for $\sqrt{N}$, to

$$\left.\begin{aligned} y_k &= \frac{1}{i}\,\frac{4rN}{(r^2+N)-\cos\frac{2k\pi}{i}(r^2-N)} = \frac{1}{i}\,\frac{2rN}{r^2\sin^2\frac{k\pi}{i}+N\cos^2\frac{k\pi}{i}} \\ &= \frac{1}{i}\,\frac{2rN}{N+\sin^2\frac{k\pi}{i}(r^2-N)} = \frac{1}{i}\,\frac{2rN}{r^2-\cos^2\frac{k\pi}{i}(r^2-N)}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; (3.)$$

For $N^{-\frac{1}{2}}$ we have the simpler forms,

$$\left.\begin{aligned} z_k &= \frac{1}{i}\,\frac{4r}{(r^2+N)-\cos\frac{2k\pi}{i}(r^2-N)} = \frac{1}{i}\,\frac{2r}{r^2\sin^2\frac{k\pi}{i}+N\cos^2\frac{k\pi}{i}} \\ &= \frac{1}{i}\,\frac{2r}{N+\sin^2\frac{k\pi}{i}(r^2-N)} = \frac{1}{i}\,\frac{2r}{r^2-\cos^2\frac{k\pi}{i}(r^2-N)}. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; (4.)$$

All that remains is to integrate these terms, and sum them.

6. Our grouping the terms in pairs has limited the value of k to range from i to $\frac{1}{2}(i-1)$ when i is odd. There is an odd term which, however, presents no difficulty, being simply $\frac{r}{i}$ in the case of $\sqrt{N}$, and $\frac{r}{iN}$ in the case of $N^{-\frac{1}{2}}$. When i is even, k is limited to range from 1 to $\frac{1}{2}i-1$, and the odd term becomes $\frac{r^2+N}{ir}$ in the case of $\sqrt{N}$, and $\frac{r^2+N}{iNr}$ in the case of $N^{-\frac{1}{2}}$. It is important to bear in mind that the term just mentioned is an odd term, and therefore not affected with the coefficient 2, which appears in the terms composed of pairs corresponding to imaginary roots.

7. The value of i, which I consider to be most useful for general purposes, is $i=8$: in this case the odd term becomes $\frac{r^2+N}{8r}$ or $\frac{r^2+N}{8Nr}$, and the other values of $\frac{k\pi}{i}$ are three in number, viz. 22° 30′, 45°, 67° 30′. With proper precautions $i=8$ will almost always give seven or more figures correct.

8. If we now give infinite values to k and i and pass from the summation to the definite integral, we have $\left(\text{putting } \lambda=\frac{k}{i}\right)$

$$z = N^{-\frac{1}{2}} = \int_0^{\frac{1}{2}\lambda} \frac{2r\,d\lambda}{r^2+(N-r^2)\cos^2\lambda\pi} = \frac{2}{\pi}\frac{r}{N}\int_0^{\frac{1}{2}\pi} \frac{d\phi}{1+\frac{r^2-N}{N}\sin^2\phi};$$

and since

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{1+p\,.\sin^2\phi} = \frac{\frac{1}{2}\pi}{\sqrt{1+p}},$$

this is an identical equation, as it ought to be.

9. This use of approximants, therefore, is simply the application of the method of

quadratures to a definite integral, which we substitute for the surd proposed for evaluation.

10. It would appear at first sight that a full application of the method of quadratures in the ordinary way, with the help of differences, would give better results than the mere summation of the ordinates. But this is not the case; for the differences diverge immediately. If we use differential coefficients for the quadrature, instead of differences, we have an opposite anomaly, namely that the correction of the summation appears to be absolutely *nil*, inasmuch as the differential coefficients which appear in the series are all of odd order, and the numerator of each of them contains the factor $\sin\phi\cos\phi$, which vanishes at both the limits 0 and $\frac{1}{2}\pi$. LEGENDRE has discussed this point. See the Appendix to the second volume of his 'Fonctions Elliptiques,' p. 578.

11. The application of the method to integrations, then, lies in the substitution for

$$\int_0^t \frac{M}{\sqrt{N}}\,dt \text{ of } \int_0^t\int_0^{\frac{1}{2}} \frac{2Mr\,.\,d\lambda\,.\,dt}{r^2+(N-r^2)\cos^2\lambda\pi},$$

in which, since λ and t are perfectly independent of each other, we may change the order of integration, thus obtaining

$$\int_0^{\frac{1}{2}}\left\{\int_0^t \frac{2Mr\,.\,dt}{r^2+(N-r^2)\cos^2\lambda\pi}\right\}d\lambda\,;$$

and the rest of the operation depends upon our being able to perform the integration in $\}$ generally, and then to determine the integral in λ by quadratures. The great advantage of the method turns upon the easy application of the method of quadratures, in consequence of our not requiring to *difference* the ordinates.

12. One way of exhibiting generally the degree of convergence is as follows: $N^{\pm\frac{1}{2}}$ always lies between

$$N^{\pm\frac{1}{2}}\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i} \text{ and } N^{\pm\frac{1}{2}}\frac{(r+\sqrt{N})^i-(r-\sqrt{N})^i}{(r+\sqrt{N})^i+(r-\sqrt{N})^i},$$

and the error of either is therefore always less than their difference,

$$N^{\pm\frac{1}{2}}\frac{4(r^2-N)^i}{(r+\sqrt{N})^{2i}-(r-\sqrt{N})^{2i}}.$$

13. There is another mode, by which, in any given case, we may see how far it is necessary to carry our work in order to obtain a given number of decimals correctly in the result. Let θ_m be determined by the equation

$$\int_0^{\theta_m}\frac{d\theta_m}{\cos\theta_m}=m\int_0^{\theta_1}\frac{d\theta_1}{\cos\theta_1},$$

and let $\sin\theta_1=\frac{\sqrt{N}}{r}$ or $\frac{r}{\sqrt{N}}$, whichever may be less than unity; then the mth approximant will be $\frac{\sqrt{N}}{\sin\theta_m}$. This is easily seen from the general term of the approximant, since

$$\int\frac{d\theta}{\cos\theta}=\frac{1}{2}\log_\epsilon\left(\frac{1+\sin\theta}{1-\sin\theta}\right).$$

14. A table of meridional parts, such as is given in the books on Navigation, if carried far enough, would solve this equation. I have calculated an auxiliary Table for the purpose, as follows:—

Let $\operatorname{cosec}\varphi-1=z$, $\log_\epsilon \tan\left(\frac{1}{4}\pi+\frac{1}{2}\varphi\right)=y$, then

$$y=\frac{1}{2}\log_\epsilon \frac{2+z}{z}=\frac{1}{2}\log 2-\frac{1}{2}\log z+\frac{1}{2}\log\left(1+\frac{1}{2}z\right)$$

$$=\frac{1}{2}\log 2-\frac{1}{2}\log z+\frac{z}{4}-\frac{z^2}{16}+\frac{z^3}{48}-\frac{z^4}{128}+\ldots\ldots\ldots$$

To bring this formula to the same unit as the common Table of meridional parts, we must multiply it by the number of minutes in the arc equal to unity, or by L=3437·74677 07849 4, whence we have $\frac{1}{2}$ L $\log_\epsilon 2$=1191·43224 08243 2, and $\frac{1}{2}$ L $\log_\epsilon 10$=3958·85223 39129 100. These data give the following Table, the argument being the common logarithm of z with its sign changed; that is, the number of places which are correct:

$-\log z$.	y.	$-\log z$.	y.	$-\log z$.	y.
1	5234·14859	6	24944·54650	11	44738·80681
2	9117·70966	7	28903·39796	12	48697·65905
3	13068·84816	8	32862·25012	13	52656·51128
4	17026·92712	9	36821·10235	14	56615·36352
5	20985·70200	10	40779·95458	15	60574·21575

15. As a simple example, let N=3, $r=2$; $\therefore \frac{\sqrt{N}}{r}=\sin 60°$: the meridional parts for 60°=4527; and in order that the error may not exceed unity in the tenth place of figures, we must have m or $i=\frac{40780}{4527}=9$; so that we must make $i=9$ at least, for the 10th figure to be correct.

16. These methods of course only exhibit the degree of approximation on the surd itself. The proportionate approximation is generally greater on the integral than on the simple surd, because the first approximant is usually so chosen as to be identical with the surd at one of the limits, and it is only near the other limit that the discrepancy tells.

Section II.—*Details of Reduction and Computation.*

17. The chief assistance, which can be provided *à priori* for the computer, consists in the exhibition and discussion, for the ordinary forms, of the integral $\int_0^t \frac{2\mathrm{M}r\,dt}{r^2+(\mathrm{N}-r^2)\cos^2\lambda\pi}$ and of the auxiliary functions which present themselves in its reduction.

18. In applying these methods to elliptic integrals, the radical and the first approximant r must both be of a simple form, and it is advisable that $r^2-\mathrm{N}$ or $\mathrm{N}-r^2$ should be

of a square form. For the common form of the elliptic radical $\sqrt{(1-\sin^2\theta\,.\,\sin^2\phi)}$, our choice is practically limited to

(1) $r=1$, (2) $r=\cos\phi$, (3) $r=\sin\theta\,.\,\cos\phi$,
(4) $r=\cos\theta$, (5) $r=\cos\theta\,.\,\sin\phi$.

And on these suppositions I now proceed to the integration of the general form of the reduced approximant for $\int_0^\phi(1-\sin^2\theta.\sin^2\phi)^{-\frac{1}{2}}d\phi=\int zd\phi$. I omit mention of the constants of integration, because very slight changes in the function may alter them. The first of our three cases require, as they stand, no constant, and these are the most useful cases.

(1) $r=1,\ r^2-\mathrm{N}=\sin^2\theta\,.\,\sin^2\phi,$

$$z_k=\frac{2}{1-\sin^2\theta\,.\,\cos^2\frac{k\pi}{i}\,.\,\sin^2\phi},$$

$$\int z_k d\phi=2\left(1-\sin^2\theta\,.\,\cos^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\tan^{-1}\left\{\left(1-\sin^2\theta\,.\,\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}\tan\phi\right\}.$$

(2) $r=\cos\phi,\ r^2-\mathrm{N}=-\cos^2\theta\,.\,\sin^2\phi,$

$$z_k=\frac{2\cos\phi}{\cos^2\phi+\cos^2\theta\,.\,\cos^2\frac{k\pi}{i}\,.\,\sin^2\phi}=\frac{2\cos\phi}{1-\left(1-\cos^2\theta\,.\,\cos^2\frac{k\pi}{i}\right).\,\sin^2\phi},$$

$$\int_k zd\phi=\left(1-\cos^2\theta\,.\,\cos^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\log_e\left\{\frac{1+\sin\phi\left(1-\cos^2\theta\,.\,\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}{1-\sin\phi\left(1-\cos^2\theta\,.\,\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}\right\}.$$

(3) $r=\sin\theta\,.\,\cos\phi,\ r^2-\mathrm{N}=-\cos^2\theta,$

$$z_k=\frac{2\sin\theta\,.\,\cos\phi}{\left(1-\cos^2\theta\,.\,\sin^2\frac{k\pi}{i}\right)-\sin^2\theta\,.\,\sin^2\phi},$$

$$\int z_k d\phi=\left(1-\cos^2\theta\,.\,\sin^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\log\left\{\frac{\left(1-\cos^2\theta\,.\,\sin^2\frac{k\pi}{i}\right)^{\frac{1}{2}}+\sin\theta\,.\,\sin\phi}{\left(1-\cos^2\theta\,.\,\sin^2\frac{k\pi}{i}\right)^{\frac{1}{2}}-\sin\theta\,.\,\sin\phi}\right\}.$$

(4) $r=\cos\theta,\ r^2-\mathrm{N}=-\sin^2\theta\cos^2\phi,$

$$z_k=\frac{2}{\cos\theta}\,\frac{1}{1+\tan^2\theta\,.\,\cos^2\frac{k\pi}{i}\,.\,\cos^2\phi},$$

$$\int z_k d\phi=2\left(1-\sin^2\theta\,.\,\sin^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\tan^{-1}\left\{\cos\theta\,.\,\tan\phi\left(1-\sin^2\theta\,.\,\sin^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\right\}.$$

(5) $r=\cos\theta\sin\phi,\ r^2-\mathrm{N}=-\cos^2\phi,$

$$z_k=\frac{2\cos\theta\,.\,\sin\phi}{\cos^2\theta\,.\,\sin^2\phi+\cos^2\frac{k\pi}{i}\,.\,\cos^2\phi},$$

$$\int z_k d\phi = 2\left(\cos^2\frac{k\pi}{i} - \cos^2\theta\right)^{-\frac{1}{2}} \tan^{-1}\left\{\frac{\cos\theta}{\cos\phi}\left(\cos^2\frac{k\pi}{i} - \cos^2\theta\right)^{-\frac{1}{2}}\right\}$$

$$= \left(\cos^2\theta - \cos^2\theta\frac{k\pi}{i}\right)^{-\frac{1}{2}} \log_\epsilon \left\{\frac{1-\frac{\cos\phi}{\cos\theta}\left(\cos^2\theta-\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}{1+\frac{\cos\phi}{\cos\theta}\left(\cos^2\theta-\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}\right\}.$$

(6) If we make $t=\tan\frac{1}{2}\phi$, we obtain

$$(1-\sin^2\theta\,.\,\sin^2\phi)^{-\frac{1}{2}}\,d\phi = 2(1-2\cos 2\theta\,.\,t^2+t^4)^{-\frac{1}{2}}\,dt.$$

Taking $r=1-t^2$, the terms which we have to integrate are of the form

$$\int\frac{4(1-t^2)\,dt}{(1-t^2)^2+4\cos^2\theta\,.\,\cos^2\frac{k\pi}{i}\,.\,t^2}.$$

Putting $q^2=1-\cos^2\theta\,.\,\cos^2\frac{k\pi}{i}$, we have

$$\int z_k d\phi = \frac{1}{q}\log_\epsilon\left(\frac{t^2+2qt+1}{t^2-2qt+1}\right).$$

The same expression serves for the integral

$$\int\frac{2dt}{\sqrt{(1+2\cos 2\theta\,.\,t^2+t^4)}},$$

if we put $q^2=1-\sin^2\theta\,.\,\cos^2\frac{k\pi}{i}$.

19. It will be observed that the first four cases, and the sixth, depend upon a radical of the form $\sqrt{(1-\sin^2 A\,.\,\sin^2\omega)}$, where ω is restricted to the selected values of $\frac{k\pi}{i}$. Assuming the modulus sin A not to vary, it would therefore in general be better to begin by computing the radical for the selected values. I have computed, and I append to this paper, a Table of this radical, the selected values of $\frac{k\pi}{i}$ being 22° 30′, 45°, and 67° 30′, while A ranges by whole degrees from 1° to 90° inclusive. Every entry but the last in the 2nd, 3rd, and 4th columns of the Table was computed by myself in duplicate with VEGA's ten-figure logarithms, by the help of two or more of the following formulæ, some of which are from LEGENDRE.

20. Putting Δ for $\sqrt{(1-\sin^2 A\,.\,\sin^2\omega)}$,

(1) Make $\sin A\,.\,\sin\omega=\sin M$; then $\Delta=\cos M$,

$\log\sin M=\log\sin A+\log\sin\omega$, $\log\Delta=\log\cos M$; or else

(2) Make $\tan A\,.\,\cos\omega=\tan M$; then $\Delta=\cos A\,.\,\sec M$,

$\log\tan M=\log\tan A+\log\cos\omega$, $\log\Delta=\log\cos A+\text{ar. co.}\log\cos M$.

Moreover, let L be the tabular angle nearest to the angle M: it is not necessary to obtain the value of M: so that we have simultaneously,

$\log\sin M=\log\sin L\pm s$,

$\log\tan M=\log\tan L\pm t$,

$\log\cos M=\log\cos L\mp c$;

then we shall also have, and with great approximation,

$$\log s = \log (t \,.\, \cos^2 \mathrm{L}) \mp (t - t \,.\, \cos^2 \mathrm{L})$$
$$= \log (c \,.\, \cot^2 \mathrm{L}) \mp (c + c \,.\, \cot^2 \mathrm{L}),$$
$$\log c = \log (s \,.\, \tan^2 \mathrm{L}) \pm (s + s \,.\, \tan^2 \mathrm{L})$$
$$= \log (t \,.\, \sin^2 \mathrm{L}) \pm (t - t \,.\, \sin^2 \mathrm{L}),$$
$$\log t = \log (s \,.\, \sec^2 \mathrm{L}) \mp (s - s \,.\, \sec^2 \mathrm{L})$$
$$= \log (c \,.\, \mathrm{cosec}^2 \mathrm{L}) \mp (c - c\, \mathrm{cosec}^2 \mathrm{L}).$$

I have given the whole set of six, but my Table was computed with the pair for log c. By way of example, I add a specimen copy of one of my working sheets. The use of so many as ten figures is not altogether unnecessary, because otherwise, when Δ is nearly equal to unity, the value of $\log (1-\Delta)$ or of $\log \frac{1-\Delta}{1+\Delta}$ cannot be had with exactness.

21. The following formulæ will also be found in many cases preferable, both for exactness and facility, to the ordinary use of logarithmic tables by means of differences. These formulæ, as well as those of the previous paragraph, are but applications of TAYLOR'S theorem, reduced to a shape fit for the computer. Even where only seven figures are required their application is frequently much easier, and gives more exact results, than interpolation by differences. In what follows, x is supposed to be the nearest tabular entry.

22. *To find* $\log y$ *from* $\log \tan y$.—Let us assume simultaneously

$$\log y = \log x \pm l, \quad \log \tan y = \log \tan x \pm t.$$

Putting $u = \log x$, $z = \log \tan x$, we have

$$\frac{du}{dz} = \frac{\sin 2x}{2x} \text{ and } \frac{d^2u}{dz^2} = \mathrm{M} \frac{\sin 2x}{2x}\left(\cos 2x - \frac{\sin 2x}{2x}\right),$$

M being the modulus of the logarithms.

Hence, by TAYLOR'S theorem,

$$l = t\,\frac{\sin 2x}{2x}\left\{1 \mp \mathrm{M}t\left(\frac{\sin 2x}{2x} - \cos 2x\right)\right\} \text{ nearly.}$$

Taking the logarithm, this becomes

$$\log l = \log\left(t\,\frac{\sin 2x}{2x}\right) \mp t\left(\frac{\sin 2x}{2x} - \cos 2x\right)$$
$$= \log\left(\frac{t \,.\, \sin x \,.\, \cos x}{x}\right) \mp \frac{t \,.\, \sin x \,.\, \cos x}{x} \pm t \mp 2t \sin^2 x.$$

The latter is the better shape for a working formula, because $\log \sin x$ and $\log \cos x$ are found in the same page and line as $\log \tan x$, while $\log \sin 2x$ must be looked for elsewhere. The first term alone is sufficient when x is small; but when x much exceeds 45°, $\cos 2x$ changes its sign, and even the entire formula is insufficient. The maximum value of the coefficient of t in the second term is 1·0631, corresponding to $x = 78° \; 33' \; 26''\cdot 5$. In many cases, where the first term alone is insufficient, a rough interpolation, made at sight from the following Table, will answer the purpose; it is a Table of the value of $\left(\frac{\sin 2x}{2x}-\cos 2x\right)$ and of its logarithm, from $x=45°$ to $x=90°$.

45°	0·63662	9·80387	70°	1·02910	0·01246
50	0·73816	9·86815	75	1·05658	0·02390
55	0·83149	9·91986	80	1·06216	0·02619
60	0·91349	9·96070	85	1·04334	0·01843
65	0·98041	9·99141	90	1·	0·

The Table shows that, past 45°, the formula

$$\log l=\log\left(\frac{t\sin x\cos x}{x}\right)\mp t$$

is a better approximation than when the $\mp t$ is omitted. It is to be remarked that t is at its minimum for $x=45°$, and increases both towards $x=0$ and $x=90°$. Near the latter limit, where great accuracy is required, we must proceed as follows.

Find the correction for the logarithm of the complement of the arc by the above process, and then find $\log\left(\frac{1}{2}\pi-y\right)$ from $\log y$. For this purpose, I observe that $\log y=\log x\pm l$ is equivalent to $y=x\,.\,10^{\pm l}$, hence

$$\tfrac{1}{2}\pi-y=\tfrac{1}{2}\pi-x\,.\,10^{\pm l}=\left(\tfrac{1}{2}\pi-x\right)-x(10^{\pm l}-1).$$

Now, let $\pm A=10^{\pm l}-1$, whence

$$\log(\pm mA)=\log(\pm l)\pm\tfrac{1}{2}l-\frac{1}{12}Ml^2 \text{ nearly, and also}$$

$$\log\left(\tfrac{1}{2}\pi-y\right)=\log\left(\tfrac{1}{2}\pi-x\right)-\left(\frac{\pm mAx}{\frac{1}{2}\pi-x}\right)-\tfrac{1}{2}M\left(\frac{\pm mAx}{\frac{1}{2}\pi-x}\right)^2-\cdots$$

It is not often that the third term of either formula will be required.

I have gone into all this detail, because the inverse tangent is continually presenting itself in all these integrations, and because no book that I know shows the proper way of handling it.

23. The following constants are needed for these and similar formulæ:—

$$10+\log m=9{\cdot}63778\ 43113\ 00537, \qquad \log M=0{\cdot}36221\ 56886\ 99463,$$
$$10+\log 1°=8{\cdot}24187\ 73675\ 90828, \qquad -\log 1°=1{\cdot}75812\ 26324\ 09172,$$
$$10+\log 1'=6{\cdot}46372\ 61172\ 07184, \qquad -\log 1'=3{\cdot}53627\ 38827\ 92816,$$
$$10+\log 1''=4{\cdot}68557\ 48668\ 23541, \qquad -\log 1''=5{\cdot}31442\ 51331\ 76459.$$

24. As an example of finding the inverse tangent, let it be required to find $\log y$ and $\log\left(\frac{1}{2}\pi-y\right)$ from

$$\log\tan y=9{\cdot}02313\ 50437. \quad \text{Here we must take}$$
$$\log\tan x=9{\cdot}02303\ 57359$$
$$+t=\qquad 9\ 93078; \quad \therefore\ x=6°\,1'\,10''=21670''$$
$$\tfrac{1}{2}\pi-x=302330''$$

$\log\sin x = 9{\cdot}0206346$	$\log\sin^2 x = 8{\cdot}0413$	$\log 21670 = 4{\cdot}33585\ 89113$
$\log\cos x = 9{\cdot}9975988$	$\log 2 = 0{\cdot}3010$	$\log 1'' = 4{\cdot}68557\ 48668$
$\log t = 5{\cdot}9969834$	$\log t = 5{\cdot}9970$	$\log x = 9{\cdot}02143\ 37781$
ar. co. $\log x = 0{\cdot}9785662$	$\log(2\sin^2 x) = 4{\cdot}3393$	$l = +9\ 85786$
$\log\frac{t\sin 2x}{2x} = 5{\cdot}9937830$	$2\sin^2 x = 21{,}800$	$\log y = 9{\cdot}02153\ 23567$
2nd correction -14	$\frac{t\sin 2x}{2x} = 985{,}8$	
$\log l = 5{\cdot}9937816$	$1007{,}6$	
$\frac{1}{2}l\ \ +493$	$-t = -993{,}1$	$\log 302330 = 5{\cdot}48048\ 12441$
$\log mA = 5{\cdot}9938309$	2nd correction $= 14{,}5$	$\log 1'' = 4{\cdot}68557\ 48668$
$\log x = 9{\cdot}0214338$	The comma cuts off the eighth decimal.	$\log(\frac{1}{2}\pi - x) = 0{\cdot}16605\ 61109$
ar. co. $\log(\frac{1}{2}\pi - x) = 9{\cdot}8339439$		correction $= -70666$
log correction $= 4{\cdot}8492086$		$\log(\frac{1}{2}\pi - y) = 0{\cdot}16604\ 90443$

Verification.—The numbers corresponding to these logarithms of y and of $\frac{1}{2}\pi - y$ are 0·10508 29743 and 1·46571 33525, the sum of which, to the very last figure, is exactly $\frac{1}{2}\pi$.

25. *To find* $\log\frac{y+1}{y-1}$ *from* $\log y$.—Let $\log\frac{y+1}{y-1} = \log\frac{x+1}{x-1} \mp p$, and $\log y = \log x \pm q$; then $\log p = \log\left(\frac{2qx}{x^2-1}\right) \mp \frac{1}{2}q \mp \frac{q}{x^2-1}$, nearly. This formula obviously fails where y is near unity; in this case $\log\frac{y+1}{y-1}$ cannot be had with great accuracy, unless y itself be given absolutely. All the cases of $\int\frac{dy}{y^2-a^2}$ may be included in the above formula by giving proper signs to p and q. It may save trouble to remark that x must not always be taken to the extreme limit of the Table, because $\log(x+1)$ and $\log(x-1)$ have also to be taken out. As an example, let

$\log y = 0{\cdot}36290\ 63835$		
$\log x = 0{\cdot}36285\ 93030$		$x = 2{\cdot}306,\ x+1 = 3{\cdot}306,\ x-1 = 1{\cdot}306$
$q = +4\ 70805$		$\log(x+1) = 0{\cdot}51930\ 28492$
$\log q = 5{\cdot}6728411$		$\log(x-1) = 0{\cdot}11594\ 31769$
ar. co. $\log(x^2-1) = 9{\cdot}3647540$		sum $= 0{\cdot}63524\ 60261$
$5{\cdot}0375951$	$\frac{q}{x^2-1} = 0{\cdot}0001090$	difference $= 0{\cdot}40335\ 96723$
$\log x = 0{\cdot}3628593$	$\frac{1}{2}q = 135$	$-p = -5\ 02761$
$\log 2 = 0{\cdot}3010300$	2nd corr$^{n} = 0{\cdot}0001225$	$\log\frac{y+1}{y-1} = 0{\cdot}40330\ 93962$
$5{\cdot}7014844$		
2nd correction $= -1225$		
$\log p = 5{\cdot}7013619$		

This example has been so chosen as to admit of easy verification. In fact $y=2{\cdot}30625$, and $\log\frac{y+1}{y-1}=\log\frac{529}{209}=0{\cdot}40330\ 93959\ 24$. The error is therefore only of three units in the tenth decimal place, where there was no reason to expect accuracy.

26. The only other formulæ which I shall give are the following, for finding the logarithm of a number, and *vice versâ*. They are indispensable where more than seven figures are required.

Let $\log(x\pm h)=\log x\pm k$, then

$$\log k=\log\left(\frac{mh}{x}\right)\mp\frac{1}{2}\frac{mh}{x}\text{ nearly,}$$

$$\log h=\log(\mathrm{M}xk)\pm\frac{1}{2}k\text{ nearly.}$$

The values of $\log m$ and $\log \mathrm{M}$ have been given in paragraph 23.

27. As an example of the application of the method to the evaluation of elliptic integrals of the third class, let us take the integral

$$\int_0^\varphi\frac{d\varphi}{(1-\sin^2\alpha.\sin^2\varphi)\,(1-\sin^2\theta.\sin^2\varphi)^{\frac{1}{2}}}$$

for the values $\alpha=45°$, $\theta=30°$, $\varphi=60°$.

I have selected these values because they can be obtained without reduction or interpolation from the Table of $\Delta(\theta,\varphi)$ which I have given, and also because $\sin^2\alpha=\sin\theta$, and therefore the integral can be reduced to one of the first class, *plus* an inverse tangent, thus admitting of easy verification. For this case

$$z_k=\frac{2}{(1-\sin^2\alpha.\sin^2\varphi)\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}.\sin^2\varphi\right)},$$

$$\int z_k\,d\varphi=\frac{2\sin^2\alpha}{\sin^2\alpha-\sin^2\theta.\cos^2\frac{k\pi}{i}}\cdot\frac{1}{\cos\alpha}\tan^{-1}(\cos\alpha.\tan\varphi)$$

$$-\frac{2\sin^2\theta\cos^2\frac{k\pi}{i}}{\sin^2\alpha-\sin^2\theta.\cos^2\frac{k\pi}{i}}\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\tan^{-1}\left\{\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}\tan\varphi\right\}.$$

Making $\frac{k\pi}{i}$ successively 22° 30′, 45°, 67° 30′, and, for the odd term, 90°, we find, after a few obvious reductions, that eight times the value of the integral is

$$\left\{\frac{17}{3}+\frac{2}{\Delta^2(45°,67\tfrac{1}{2})}+\frac{2}{\Delta^2(45°,22\tfrac{1}{2})}\right\}\frac{1}{\cos 45°}\tan^{-1}\{\cos 45°.\tan 60°\}$$

$$-\frac{1}{\cos 30°}\tan^{-1}\{\cos 30°.\tan 60°\}-\frac{2}{3}\frac{1}{\Delta(30°,45°)}\tan^{-1}\{\Delta(30°,45°).\tan 60°\}$$

$$-\frac{\tan^{-1}\{\Delta(30°,67\tfrac{1}{2}).\tan 60°\}.\cos^2 22\tfrac{1}{2}}{\Delta^2(45°,67\tfrac{1}{2}).\Delta(30°,67\tfrac{1}{2})}-\frac{\tan^{-1}\{\Delta(30°,22\tfrac{1}{2}).\tan 60°\}.\cos^2 67\tfrac{1}{2}}{\Delta^2(45°,22\tfrac{1}{2}).\Delta(30°,22\tfrac{1}{2})}.$$

As these inverse tangents range generally from 45° to 60°, I computed them by the shortened formula of paragraph 22, namely $\log\left(\frac{t.\sin x.\cos x}{x}\right)\pm t$; this being sufficient to give eight figures of decimals accurately. I found

$$\log\tan^{-1}\{\cos 45°\tan 60°\} = 9\cdot 94747\ 15296,$$
$$\log\tan^{-1}\{\cos 30°\tan 60°\} = 9\cdot 99246\ 23739,$$
$$\log\tan^{-1}\{\Delta(30°, 45°)\tan 60°\} = 0\cdot 00766\ 92607,$$
$$\log\tan^{-1}\{\Delta(30°, 67\tfrac{1}{2}).\tan 60°\} = 9\cdot 99727\ 33807,$$
$$\log\tan^{-1}\{\Delta(30°, 22\tfrac{1}{2}).\tan 60°\} = 0\cdot 01665\ 09657.$$

I hence obtained the following values:—

For the positive terms.	For the negative terms.
7·10091 3039	1·13483 2441
4·37212 6152	0·72539 4027
2·70421 6251	1·66839 9478
14·17725 5442	0·16728 4032
3·69590 9978	3·69590 9978
8)10·48134 5464	
1·31016 8183 value required	

A more exact value of the integral, otherwise obtained, is

$$\tfrac{1}{2}F(30°, 60°)+\tan^{-1}\left(\frac{4\sin 60°}{\sqrt{13}}\right)=1\cdot 31016\ 8161,$$

which differs from the previous value by 2 units in the eighth decimal place.

28. In order to find how many places ought to have been accurately obtained, I observe that the method followed gives $N=\frac{13}{16}$, $r=1$, whence

$$\log\left(\frac{1}{r}\sqrt{N}\right)=9\cdot 95491=\log\sin 64°\ 20'\ 30''.$$

The corresponding meridional parts are 5086·5, which must be multiplied by $i=8$, giving 40692·0. Referring to the Table in paragraph 14, I find that this nearly corresponds to *ten* places correct, and therefore that the integral ought to be correct to at least that extent. That it is not so, is due to my having curtailed the formula for finding the logarithms of the inverse tangents. But my object was only to give seven decimals correct, and my going beyond that was simply because, with a ten-figure Table, putting down the additional figures gave me less trouble (once I had to use more than seven) than abbreviation would have done. This remark may at first sight seem strange to any one who has not had some practice in using large Tables. But the logarithmic corrections are given in the shape of arithmetical complements: with reference to the 10th figure, therefore, considered as an integer, the index is right as it stands, and we need not bestow thought on the proper placing of the correction, as we must if we use any other number of figures.

29. If we had been content with five decimals, the calculation would have been very easy, for in that case we might have used six-figure logarithms, and have made $i=4$, thus omitting the terms containing $22°\frac{1}{2}$ and $67°\frac{1}{2}$. We should get

7·10091 3039		1·13483 2441
1·86022 6468		0·72539 4027
4)5·24068 6571		1·86022 6468
1·31017 1643	value required.	

30. It is worth while to notice a case which will sometimes occur, namely (using the notation of the last example), that the values may be so selected as to give, for one of the values of k, $\sin\alpha=\sin\theta.\cos\frac{k\pi}{i}$, and thus each of the terms into which $\int z_k d\phi$ was divided would become infinite. Of course the difficulty is only apparent; for in this case the proper value is $\int z_k d\phi=\int\frac{2d\phi}{(1-\sin^2\alpha.\sin^2\phi)^2}$, of which the integral may be at once found by differentiating the expression $\frac{\sin\phi.\cos\phi}{1-\sin^2\alpha.\sin^2\phi}$.

SECTION III.—*Extension of the Method.*

In respect of rapid approximation and precision of limit, the foregoing processes leave nothing to be desired, as far as concerns the radical of the square root; but they do not go beyond that. Mr. SYLVESTER has given an elegant extension of the method to radicals of a higher index, by means of symmetric functions*.

The more general problem before us is that of approximating to the integrals of irrational functions by means of rational substitutions.

Let ϕ and ψ be functional symbols, and y a function of z; then, that $\phi(z).y_m$ and $\phi(z):y_m$ should both be approximations to $\phi(z)$, depends upon y_m approaching unity as m increases. Assuming that y_m and y_1 are connected by the equation $y_m=\psi(m, y_1)$, our problem is to choose ψ so that, in the first place, the approximation shall be exceedingly rapid, and, in the next place, that $\phi(z).y_m$ and $\phi(z):y_m$ shall both (or at least one of them) be thoroughly manageable, and easily integrable. In the case of the approximants already given, the equation $y_m=\psi(m_1y_1)$ has been made $\int\frac{dy_m}{1-y_m^2}=m\int\frac{dy_1}{1-y_1^2}$.

I am acquainted with three general methods which effect the object more or less. The first is the obvious one afforded by the Newtonian approximation to the roots of an equation; viz., let a be a first approximate solution, obtained by trial, of the equation $fx=0$, and call $f'x$ the differential coefficient of fx; then a second approximation is

* See the Philosophical Magazine for December 1860, Supplementary Number, vol. xx. p. 525, note A.

$a-\frac{fa}{f'a}=b$; a third approximation will evidently be $b-\frac{fb}{f'b}=c$, and so forth. If we apply this method to the pure equation $x^n=p$, the convergent terms which we obtain are as follows:—

$$b=\frac{(n-1)a^n+p}{na^{n-1}},$$

$$c=\frac{(n-1)\{(n-1)a^n+p\}^n+n^n.p.a^{n(n-1)}}{n^2a^{n-1}\{(n-1)a^n+p\}^{n-1}}, \text{ \&c.}$$

The second method is that of the reversion of series; it is sufficiently discussed by ARBOGAST*.

The third method was suggested to me by Mr. CAYLEY's remark that Mr. SYLVESTER's third approximation is a particular case, for $n=2$, of the common form (of the books on the binomial theorem) $\sqrt[n]{\mathrm{N}}=\frac{(n+1)\mathrm{N}+(n-1)a^n}{(n-1)\mathrm{N}+(n+1)a^n}\cdot a$, approximately, a being a first approximation. In order to gain generality, and thereby symmetry, I shall pass from the particular form $\sqrt[n]{\mathrm{N}}$ to the more general $\varphi^{-1}\mathrm{N}$ by the following Lemma:—

Let $$\mathrm{N}=a_0+a_1x+a_2x^2+a_3x^3+\dots\dots, \qquad (1.)$$

and let $x_1, x_2, x_3, x_4 \dots$ be determined by the system of equations,

$$\left.\begin{aligned}\mathrm{N}&=a_0\left(1+\frac{a_1}{a_0}x_1\right)\\&=a_0\left(1+\frac{a_1}{a_0}x_2\left(1+\frac{a_2}{a_1}x_1\right)\right)\\&=a_0\left(1+\frac{a_1}{a_0}x_3\left(1+\frac{a_2}{a_1}x_2\left(1+\frac{a_3}{a_2}x_1\right)\right)\right),\end{aligned}\right\} \qquad (2.)$$

and so forth; also let $\mathrm{N}-a_0=\mu$, and

$$1=(\mu-a_1x-a_2x^2-a_3x^3-\dots)(\lambda_0+\lambda_1x+\lambda_2x^2+\lambda_3x^3+\dots), \qquad (3.)$$

then
$$x_1=\frac{\lambda_0}{\lambda_1},\ x_2=\frac{\lambda_1}{\lambda_2},\ x_3=\frac{\lambda_2}{\lambda_3}\dots x_n=\frac{\lambda_{n-1}}{\lambda_n}.$$

For, if we substitute these values in the equations (2.) after placing them in the following form,

$$\left.\begin{aligned}\mu=\mathrm{N}-a_0&=a_1x_1\\&=a_1x_2+a_2x_2x_1\\&=a_1x_3+a_2x_3x_2+a_3x_3x_2x_1,\end{aligned}\right\} \qquad (4.)$$

and so forth, we obtain

$$\mu=\frac{a_1\lambda_0}{\lambda_1}=\frac{a_1\lambda_1+a_2\lambda_0}{\lambda_2}=\frac{a_1\lambda_2+a_2\lambda_1+a_3\lambda_0}{\lambda_3}=\text{\&c.}, \qquad (5.)$$

which are the same equations as we should get by multiplying the two series in (3.) and equating to zero the coefficients of x and of its powers. The coefficient $\lambda_0=\frac{1}{\mu}$, obviously.

* Calcul des Dérivations, pp. 288–296.

The coefficients λ may now be found in a variety of ways; by solving equations (4.) or (5.), by simple division, or by ARBOGAST'S processes *. The object of the preceding lemma is to connect the quantities x_n with the coefficients of division and of recurring series. Our results in any way are,

$$x_1=\frac{\mu}{a_1},\ x_2=\frac{a_1\mu}{a_2\mu+a_1^2},$$

$$x_3=\frac{a_2\mu^2+a_1^2\mu}{a_3\mu^2+2a_2a_1\mu+a_1^3},$$

$$x_4=\frac{a_3\mu^3+2a_2a_1\mu^2+a_1^3\mu}{a_4\mu^3+(2a_3a_1+a_2^2)\mu^2+3a_2a_1^2\mu+a_1^4}.$$

If for a_0, a_1, &c. we substitute the coefficients of the binomial theorem, so as to make $N=(a+x)^n$, we obtain

$$a+x_1=\frac{(n-1)a^n+N}{na^n}\cdot a,$$

$$a+x_2=\frac{(n-1)a^n+(n+1)N}{(n+1)a^n+(n-1)N}\cdot a,$$

$$a+x_3=\frac{(n^2-1)a^{2n}+(4n^2+2)a^nN+(n^2-1)N^2}{(n+1)(n+2)a^{2n}+4(n^2-1)a^nN+(n-1)(n-2)N^2}\cdot a.$$

Making $n=2$, we obtain Mr. SYLVESTER'S approximants to the square root, and λ_n is then the coefficient of x^n in the development by ascending powers of

$$\frac{1}{(N-a^2)-2ax-x^2};$$

and so far the method agrees with the Newtonian approximation by continued fractions; but from this point the two methods diverge. For $n=3$, λ_n is the coefficient of x^n in the development of

$$\frac{1}{(N-a^3)-3a^2x-3ax^2-x^3};$$

and the successive approximants are

$$\frac{2a^3+N}{3a^3}\cdot a,\ \frac{a^3+2N}{2a^3+N}\cdot a,\ \frac{4a^6+19a^3N+4N^2}{10a^6+16a^3N+\ N^2}\cdot a,\ \frac{5a^9+45a^6N+30a^3N^2+N^3}{15a^9+51a^6N+15a^3N^2}\cdot a,\ \&c.;$$

while the second approximant obtained by successive substitution is

$$\frac{16a^9+51a^6N+12a^6N^2+2N^3}{36a^9+36a^6N+9a^3N^2}\cdot a.$$

What these methods all effect is simply a rational approximation to the value of y in the equation $\phi(y,z)=0$. Then, making $y=\frac{du}{dz}$, we have only to integrate in order to find the value of u. They thus constitute a means of approximately solving, in respect of u, differential equations of the form $\phi\left(z,\frac{du}{dz}\right)=0$; but they do not effect the solu-

* See his 'Calcul des Dérivations,' pp. 26, 29; or DE MORGAN, 'Diff. Calc.' p. 331.

tion of this equation in respect of z, and still less do they solve the more general form $\phi\left(u, z, \frac{du}{dz}\right)=0$.

It may suggest processes of reduction in some cases to remark, that there are many other functions of y_m and $\phi(z)$, which will approximate to $\phi(z)$ as m increases, besides the simple product or quotient of $\phi(z)$ by y_m.

There is one point about these higher approximants, of which a solution, even if accompanied with considerable restrictions, would be extremely desirable,—I mean the resolution of the denominators into factors. I do not suppose that the problem, in its perfectly general form, admits of a compact solution; but any class of cases, of even moderate generality, for which it could be elegantly solved, would probably have very useful applications. The criterion of convergence and the measure of approximation would also have their interest.

Specimen sheet of work for the Table. Arc of 60°.

log sin $22\frac{1}{2}$=	9·58283	96605	83*	log tan² L=	9·0913166	log cos L=	9·97473	27132
log sin A =	9·93753	06316	96	log s=	5·3160144	c=	+	25545
log sin M =	9·52037	02922	79		4·4073310	log Δ=	9·97473	52677
log sin L =	9·52039	09944		s+ / s tan² L=	−233			
s =	−2	07021						
s tan² L =		25546		log c=	4·4073077			

log sin 45°=	9·84948	50021	68*	log tan² L=	9·7781471	log cos L=	9·89794	07883
log sin A =	9·93753	06316	96	log s=	4·1137763	c=	−	7797
log sin M =	9·78701	56338	64		3·8919234	log Δ=	9·89794	00086
log sin L =	9·78701	43344		s+ / s tan² L=	+ 21			
s =	+	12995						
s tan² L =		7821		log c=	3·8919255			

log sin $67\frac{1}{2}$=	9·96561	53459	21*	log tan² L=	10·2501550	log cos L=	9·77806	24352
log sin A =	9·93753	06316	96	log s=	4·7816118	c=	− 1	07593
log sin M =	9·90314	59776	17		5·0317668	log Δ=	9·77805	16759
log sin L =	9·90313	99296		s+ / s tan² L=	+168			
s =	+	60480						
s tan² L =	1	07589		log c=	5·0317836			

log tan $22\frac{1}{2}$=	9·61722	43146	62*	log sin² L=	8·6140815	log cos $22\frac{1}{2}$=	9·96561	53459 21*
log cos A =	9·69897	00043	36	log t=	5·5451559	log sec L =	0·00911	84784
log tan M =	9·31619	43190	08		4·1591374		9·97473	38243
log tan L =	9·31615	92213		t− / t sin² L=	+327	c=	+	14430
t =	+ 3	50877				log Δ=	9·97473	52673
t sin² L =		14018		log c=	4·1591701			

log tan 45 =	10·00000	00000	00*	log sin² L=	9·3009948	log cos 45 =	9·84948	50021 68*
log cos A =				log t=	5·3429218	log sec L =	0·04845	06017
log tan M =	9·69897	00043	36		4·6439166		9·89793	56039
log tan L =	9·69894	79790		t− / t sin² L=	+176	c=	+	44049
t =	+2	20253				log Δ=	9·89794	00082
t sin² L =		44047		log c=	4·6439342			

log tan $67\frac{1}{2}$=	10·38277	56853	38*	log sin² L=	9·7730722	log cos $67\frac{1}{2}$=	9·58283	96605 83*
log cos A =	9·69897	00043	36	log t=	4·7719327	log sec L =	0·19521	55228
log tan M =	10·08174	56896	74		4·5450049		9·77805	51834
log tan L =	10·08175	16044		t− / t sin² L=	− 24	c=	−	35075
t =	−	59147				log Δ=	9·77805	16759
t sin² L =		35075		log c=	4·5450025			

Note.—The entries marked *, and the whole of the letter-press, were printed on the sheets. The letter A on this page corresponds to θ in the Table.

Table of the value of the function log $\Delta(\theta, \omega)$ or log $\sqrt{(1-\sin^2\theta \,.\, \sin^2\omega)}$ for four values of ω, viz. 22° 30′, 45°, 67° 30′, and 90°.

θ.	log Δ (θ, 22° 30′).	log Δ (θ, 45°).	log Δ (θ, 67° 30′).	log cos θ.	θ.
1°	9·99999 03138	9·99996 69274	9·99994 35386	9·99993 38497	1°
2	9·99996 12643	9·99986 77197	9·99977 41349	9·99973 53589	2
3	9·99991 28792	9·99970 24074	9·99949 17311	9·99940 44063	3
4	9·99984 52048	9·99947 10407	9·99909 62308	9·99894 07898	4
5	9·99975 83051	9·99917 36910	9·99858 74990	9·99834 42260	5
6	9·99965 22633	9·99881 04507	9·99796 53618	9·99761 43489	6
7	9·99952 71805	9·99838 14325	9·99722 96070	9·99675 07098	7
8	9·99938 31764	9·99788 67716	9·99637 99831	9·99575 27754	8
9	9·99922 03891	9·99732 66254	9·99541 62004	9·99461 99270	9
10	9·99903 89748	9·99670 11740	9·99433 79300	9·99335 14589	10
11	9·99883 91084	9·99601 06211	9·99313 48042	9·99194 65764	11
12	9·99862 09825	9·99525 51957	9·99183 64168	9·99040 43940	12
13	9·99838 48090	9·99443 51515	9·99041 23213	9·98872 39328	13
14	9·99813 08170	9·99355 07689	9·98887 20337	9·98690 41185	14
15	9·99785 92542	9·99260 23560	9·98721 50300	9·98494 37781	15
16	9·99757 03868	9·99159 02501	9·98544 07479	9·98284 16370	16
17	9·99726 44986	9·99051 48183	9·98354 85857	9·98059 63156	17
18	9·99694 18915	9·98937 64594	9·98153 79027	9·97820 63255	18
19	9·99660 28857	9·98817 56060	9·97940 80200	9·97567 00654	19
20	9·99624 78189	9·98691 27246	9·97715 82198	9·97298 58164	20
21	9·99587 70469	9·98558 83197	9·97478 77462	9·97015 17377	21
22	9·99549 09429	9·98420 29331	9·97229 58056	9·96716 58605	22
23	9·99508 98979	9·98275 71478	9·96968 15661	9·96402 60827	23
24	9·99467 43204	9·98125 15899	9·96694 41603	9·96073 01625	24
25	9·99424 46358	9·97968 69297	9·96408 26837	9·95727 57115	25
26	9·99380 12870	9·97806 38852	9·96109 61968	9·95366 01869	26
27	9·99334 47337	9·97638 32245	9·95798 37258	9·94988 08840	27
28	9·99287 54524	9·97464 57677	9·95474 42643	9·94593 49269	28
29	9·99239 39363	9·97285 23905	9·95137 67741	9·94181 92587	29
30	9·99190 06948	9·97100 40265	9·94788 01877	9·93753 06317	30
31	9·99139 62526	9·96910 16705	9·94425 34100	9·93306 55951	31
32	9·99088 11517	9·96714 63813	9·94049 53211	9·92842 04835	32
33	9·99035 59484	9·96513 92852	9·93660 47788	9·92359 14023	33
34	9·98982 12144	9·96308 15797	9·93258 06231	9·91857 42135	34
35	9·98927 75363	9·96097 45359	9·92842 16784	9·91336 45194	35
36	9·98872 55150	9·95881 95031	9·92412 67607	9·90795 76446	36
37	9·98816 57654	9·95661 79121	9·91969 46797	9·90234 86165	37
38	9·98759 89157	9·95437 12781	9·91512 42488	9·89653 21441	38
39	9·98702 56072	9·95208 12065	9·91041 42888	9·89050 25944	39
40	9·98644 64943	9·94974 93945	9·90556 36388	9·88425 39665	40
41	9·98586 22425	9·94737 76364	9·90057 11640	9·87777 98629	41
42	9·98527 35294	9·94496 78273	9·89543 57674	9·87107 34581	42
43	9·98468 10429	9·94252 19663	9·89015 64024	9·86412 74638	43
44	9·98408 54812	9·94004 21611	9·88473 20869	9·85693 40901	44
45	9·98348 75524	9·93753 06317	9·87916 19193	9·84948 50022	45

Table (*continued*).

θ.	log Δ (θ, 22° 30′).	log Δ (θ, 45°).	log Δ (θ, 67° 30′).	log cos θ.	θ.
46°	9·98288 79722	9·93498 97136	9·87344 50980	9·84177 12731	46°
47	9·98228 74657	9·93242 18620	9·86758 09423	9·83378 33303	47
48	9·98168 67640	9·92982 96543	9·86156 89166	9·82551 08951	48
49	9·98108 66053	9·92721 57944	9·85540 86587	9·81694 29168	49
50	9·98048 77326	9·92458 31150	9·84910 00105	9·80806 74967	50
51	9·97989 08943	9·92194 45794	9·84264 30543	9·79887 18039	51
52	9·97929 68415	9·91927 32846	9·83603 81532	9·78934 19787	52
53	9·97870 63285	9·91660 24627	9·82928 59976	9·77946 30249	53
54	9·97812 01111	9·91392 54820	9·82238 76555	9·76921 86852	54
55	9·97753 89457	9·91124 58470	9·81534 46333	9·75859 13013	55
56	9·97696 35878	9·90856 71996	9·80815 89399	9·74756 16513	56
57	9·97639 47918	9·90589 33160	9·80083 31625	9·73610 87645	57
58	9·97583 33090	9·90322 81075	9·79337 05490	9·72420 97077	58
59	9·97527 98869	9·90057 56154	9·78577 51006	9·71183 93361	59
60	9·97473 52675	9·89794 00084	9·77805 16759	9·69897 00043	60
61	9·97420 01874	9·89532 55788	9·77020 61055	9·68557 12291	61
62	9·97367 53742	9·89273 67330	9·76224 53190	9·67160 92909	62
63	9·97316 15478	9·89017 79885	9·75417 74828	9·65704 67649	63
64	9·97265 94174	9·88765 39622	9·74601 21530	9·64184 19615	64
65	9·97216 96810	9·88516 93618	9·73776 04387	9·62594 82593	65
66	9·97169 30239	9·88272 89739	9·72943 51756	9·60931 32999	66
67	9·97123 01175	9·88033 76506	9·72105 11125	9·59187 80116	67
68	9·97078 16179	9·87800 02961	9·71262 51046	9·57357 54170	68
69	9·97034 81644	9·87572 18497	9·70417 63081	9·55432 91617	69
70	9·96993 03790	9·87350 72689	9·69572 63771	9·53405 16846	70
71	9·96952 88643	9·87136 15099	9·68729 96519	9·51264 19176	71
72	9·96914 42028	9·86928 95088	9·67892 33280	9·48998 23640	72
73	9·96877 69551	9·86729 61579	9·67062 76041	9·46593 53400	73
74	9·96842 76594	9·86538 62846	9·66244 57824	9·44033 80750	74
75	9·96809 68303	9·86356 46269	9·65441 43168	9·41299 62305	75
76	9·96778 49569	9·86183 58088	9·64657 27859	9·38367 51767	76
77	9·96749 25025	9·86020 43157	9·63896 37732	9·35208 80330	77
78	9·96721 99032	9·85867 44678	9·63163 26324	9·31787 89102	78
79	9·96696 75670	9·85725 03958	9·62462 71226	9·28059 88450	79
80	9·96673 58731	9·85593 60134	9·61799 68925	9·23967 02300	80
81	9·96652 51705	9·85473 49940	9·61179 28070	9·19433 24413	81
82	9·96633 57778	9·85365 07456	9·60606 61106	9·14355 53039	82
83	9·96616 79819	9·85268 63874	9·60086 74357	9·08589 44712	83
84	9·96602 20377	9·85184 47281	9·59624 56795	9·01923 45656	84
85	9·96589 81672	9·85112 82461	9·59224 67793	8·94029 60083	85
86	9·96579 65594	9·85053 90708	9·58891 24439	8·84358 45184	86
87	9·96571 73697	9·85007 89667	9·58627 88989	8·71880 01636	87
88	9·96566 07187	9·84974 93212	9·58437 57166	8·54281 91639	88
89	9·96562 66934	9·84955 11322	9·58322 48116	8·24185 53184	89
90	9·96561 53459	9·84948 50022	9·58283 96696	—log. infin.	90