The Algebraic Solution of Equations

OF ANY DEGREE

A Novel, Simple and Direct Method for the Solution of Equations of the Nth Degree

BY

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J. LEWIS ANDRÉ

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This method is not Algebraic, but
arithmetical, not easy to apply when the absolu
term is positive, not even for equation of 2nd
degree, almost inapplicable for negative roots,
and absolutely impracticable above 3d degree
in case of high degrees, that is a deformation of
the Newtons method for successive approximation
in case of imaginary roots, the scholar may search
indefinitely without being cared for. —

PREFACE.

In presenting the following pages we are attempting to introduce a method for the solution of equations of any degree which we believe to be novel and simple. A method that resorts to no artificial nor indirect processes for the completion of the given quantity to produce perfect squares; nor any mathematical substitution or jugglery* that may lead the student to fields not strictly algebraic, involving trigonometry, graphics, or the various processes sometimes resorted to in the solution of equations of a higher degree than the quadratic.*

* The booklet is full of jugglery.

* The booklet is an arithmetical exposition of these various processes.

We believe the method to be direct, simple and strictly in accordance with the common sense dictates of the given equation, to be solved; to illustrate, if $bxw=c$ and the value of x be required; we divide c by the coefficient of x and extract the n^{th} root of the unknown quantity and so on throughout for any given power or number of terms when the exponents are integers or can be readily transformed to integral powers.

The first chapter is devoted to a comprehensive review on the binomial formula upon which the solution of the method herein given is solely based, and the processes of involution and evolution. These topics have been dealt with thoroughly so far as it has been thought necessary and helpful to explain this method.

In the chapter on equations, we have added some of the the theory of equations, such as Descartes Rule, and the theory showing the relation between the coefficient of $f(x)$ and the roots of the equation $f(x)=O$: and as much more as was thought useful and necessary.

We have given the complete solution of quite a number of examples that differ from each other only by slight modifications of sign or degree; but experience has taught us that the average student learns more by following throughout an actual construction, than by the language of demonstration and rule; hence we have attempted to show all modifications of sign, degree and quantity that may come under general observation and practice.

It is hoped that the following method will be found useful to the engineer and man of practice as well as the student; for the ordinary methods of solving, even an equation of the third degree, are laborous and approximate, involving graphic constructions or the denser analytical methods of Sturm and Horner.

While rules for the application of the method are given in these pages, yet we have also attempted to make these rules, laws, or a deduction from a process of reasoning, and not merely statements to follow blindly.

We have consulted the works of many authors on algebra and used such demonstration from each as we considered useful.

Respectfully,

THE AUTHORS.

Some corrections

THE SOLUTION OF ALGEBRAIC EQUATIONS

OF ANY DEGREE.

CHAPTER I.

BINOMIAL FORMULA.

For the sake of a clear understanding of what follows, we shall develop the Binomial Formula upon which the theory of the following demonstration depends.

$$(x+a)\,(x+b)=x^2\begin{array}{r|}+a\\+b\end{array}\,x+ab$$

$$(x+a)\,(x+b)\,(x+c)=x^3\begin{array}{r|}+a\\+b\\+c\end{array}\,x^2\begin{array}{r|}+ab\\+ac\\+bc\end{array}\,x+abc$$

$$(x+a)(x+b)\,(x+c)(x+d)=x^4\begin{array}{r|}+a\\+b\\+c\\+d\\ \end{array}\,x^3\begin{array}{r|}+ab\\+ac\\+ad\\+bc\\+bd\end{array}\,x^2\begin{array}{r|}+abc\\+abd\\+acd\\+bcd\\ \end{array}\,x+abcd$$

From the above we observe the following laws :

FIRST—*The number of terms in the second member is one greater than the number of binomial factors in the first member.*

SECOND—*The exponent of* x *in the first term of the second member is equal to the number of binomial factors, and in each of the succeeding terms the exponent of* x *is one less than in the preceding term.*

THIRD—*The coefficient of the first term of the second member is unity, the coefficient of the second term is the sum of the second terms of the binomial factors; the coefficient of the third term is the sum of all the products of the second terms of the binomial factors, taken two at a time; the coefficient of the fourth term is the sum of all the products of the second terms of the binomial factors, taken three at a time, and so on; the last term is the product of all the second terms of the binomial factors.*

Let us assume it to be true for n binomials,

$(x+a)\ (x+b)\ (x+c)\ (x+d) \ldots\ldots\ldots\ldots (x+k)$ therefore; $(x+a)\ (x+b)\ (x+c)\ (x+d) \ldots\ldots (x+k)=x^n+P\,x^{n-1}+P_2\,x^{n-2}+P_3\,x^{n-3}+\ldots\ldots+P^n$, where $P=$ the sum of the terms $a, b, c, d, \ldots k$. (1)

$P_2=$the sum of the products of these terms, taken two at a time.
$P_3=$ " " " " " " " " " three " " "

$P^n=$the product of all these terms.

Multiply both members of (1) by $(x+l)$ and from the above laws this equals $(x+a)\ (x+b)\ (x+c)\ (x+d) \ldots\ldots\ldots\ldots (x+k)\ (x+l)=x^{n+1}+(P+l)x^n+(P_2+Pl)x^{n-1}+(P_3+P_2l)x^{n-2}\ldots\ldots\ldots P_nl$ (2.)

Now $P+l=a+b+c+d+\ldots\ldots\ldots+k+l$ according to section three of the foregoing rules, viz: the coefficient of the second term is the sum of the second terms of the binomial factors, and there are now $P+l$ factors.

$P_2+P\,l=P_2+(a+b+c+d+\ldots\ldots+k)\ l$ equals the sum of the products of all the terms $a, b, c, d, \ldots\ldots k, l$ taken two at a time by the foregoing rule and demonstration.

$P_3 + P_2 l = P_3 + (ab + ac + ad + bc + \ldots\ ak + bk)\, l$ equals the sum of the products of all the terms $a, b, c, d \ldots k, l$ taken three at a time by the rule.

$P_n l =$ products of all the terms $a, b, c, d, \ldots\ldots k, l.$

The law of the exponents in (2) is the same as in (1).

We have shown by former demonstration that the laws are true when the number of factors n is two, three, four, hence, they are true when the number of factors n is five and so on or when the number of factors is $(n+l)$.

The number of terms in P is obviously n; the number of terms in P_2 is equal to the number of combinations of n things, taken two at a time, that is $\frac{n\,(n-1)}{1 \times 2}$; the number of terms in P_3 is equal to the combinations of n things, taken three at a time, that is, $\frac{n\,(n-1)\,(n-2)}{1 \times 2 \times 3}$ and so on.

Now suppose $a = b = c = \ldots\ldots\ldots\ldots = k$; then $P = na$, $P_2 = \frac{n\,(n-1)}{1 \times 2}\, a^2$ because $a = b = c = \ldots\ldots$ $\ldots\ldots = k$ and so on, and $P_3 = \frac{n\,(n-1)\,(n-2)}{1 \times 2 \times 3}\, a^3$ for the same reason and so on.

Therefore under the hypothesis (1) becomes $(x + a)^n$

$$= x^n + n\,a\,x^{n-1} + \frac{n\,(n-1)}{1 \times 2}\, a^2\, x^{n-2} + \frac{n\,(n-1)\,(n-2)}{1 \times 2 \times 3}\, a^3\, x^{n-3} + \ldots\ldots + na^{n-1}\, x + a^n. (3)$$

We now have formulated and developed the binomial formula. The second member of (3) is called the expansion or the development of $(x + a)^n$.

Corillary (1)—The sum of the exponents of a and x in any term of the expansion of $(x + a)^n$ is equal to n.

To obtain the expansion of $(x - a)^n$ place $-a$ instead of $+a$ in the expansion of $(x + a)^n$. The terms which contain the odd powers of $-a$ will be negative and the terms which contain the even powers of $-a$ will be positive. Hence,........

$$(x - a)^n = x^n - nax^{n-1} + \frac{n(n-1)}{1 \times 2} a^2 x^{n-2} - \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a^3 x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4} a^4 x^{n-4} \ldots\ldots\ldots\ldots a^n.$$

Example.—Expand $(x + y)^5$. In this quantity $n = 5$.

$$x^5 + 5x^{5-1} y + \frac{5(5-1)}{1 \times 2} x^{5-2} y^2 + \frac{5(5-1)(5-2)}{1 \times 2 \times 3} x^{5-3} y^3 + \frac{5(5-1)(5-2)(5-3)}{1 \times 2 \times 3 \times 4} x^{5-4} y^4 + \frac{5(5-1)(5-2)(5-3)(5-4)}{1 \times 2 \times 3 \times 4 + 5} x^{5-5} y^5.$$

Simplifying this we get $x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5xy^4 + y^5$.

In the above expansion, the following order may be observed.

No. of terms=	1	2	3	4	5	6
The powers of x=	x^5	x^4	x^3	x^2	x^1	x^0
The powers of y=	y^0	y^1	y^2	y^3	y^4	y^5
Coefficients=	1	5	10	10	5	1

$$(x + y)^5 = x^5y^0 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5x^0$$

It will be observed that there is one more term in the expansion than the number of the given powers, viz: 5, or, according to the law, the number of terms is one greater than the number of binomial factors in the first member of

$(x + y)^5$, and the coefficient of the rth term from the beginning is equal to the coefficient of the rth term from the end.

The *exponent* of x decreases by one for each succeeding term of the second member, beginning with the exponent of the required power, and the exponent of y increases by one beginning with 0.

The *coefficient* of the first term is one, the *coefficient* of the second term is the same as the exponent of the given power; for the *coefficients* of the following terms, multiply the coefficient of the preceding term by the exponent of x for that preceding term and divide by the number of preceding terms, as: the coefficient of the third term is the coefficient of the second term, 5, multiplied by the exponent of x, which is 4 for that term, this equals 20, and divided by the number of preceding terms, which are two, giving 10 as the required coefficient, and so on.

This last method or form is a convenient method in which to place the expansion of any binomial.

Example.—Expand $(2a-3b)^4$. In this quantity $n=4$.

$$(2a - 3b)^4 = (2a)^4 - 4\,(2a)^3\, 3b + \frac{4\,(4-1)}{1 \times 2}(2a)^2\,(3b)^2 -$$

$$\frac{4\,(4-1)\,(4-2)}{1 \times 2 \times 3}(2a)^1\,(3b)^3 + \frac{4\,(4-1)\,(4-2)\,(4-3)}{1 \times 2 \times 3 \times 4}$$

$$(2a)^0\,(3b)^4 =$$

$$16a^4 - 96a^3 b + 216a^2 b^2 - 216a b^3 + 81b^4.$$

No. of terms=	1	2	3	4	5
Powers of a=	$(2a)^4$	$(2a)^3$	$(2a)^2$	$(2a)^1$	$(2a)^0$
Powers of b=	$-(3b)^0$	$-(3b)^1$	$-(3b)^2$	$-(3b)^3$	$-(3b)^4$
Coefficients=	1	4	6	4	1

Combining $1\,(2a)^4\,(3b)^0 - 4\,(2a)^3\,(3b)^1 + 6\,(2a)^2(3b)^2 -$ $4\,(2a)^1\,(3b)^3 + 1\,(2a)^0\,(3b)^4 =$

$16a^4 - 96a^3 b + 216a^2 b^2 - 216a b^3 + 81b^4$ as before.

Observe that all the odd powers of the minus quantity—$3b$ are negative.

Expand $(a+b+c+d)^3$. Let $(a+b)=m$ and $(c+d)=n$.

$(m+n)^3 = m^3 + 3m^2 n + 3m n^2 + n^3$. Substituting for m and n their values, $(a+b)$ and $(c+d)$, this gives, $(a+b)^3 + 3(a+b)^2(c+d) + 3(a+b)(c+d^2) + (c+d)^3 = a^3 + 3a^2 b + 3ab^2 + b^3 + 3(a^2 + 2ab + b^2)(c+d) + 3(a+b)(c^2 + 2cd + d^2) + c^3 + 3c^2 d + 3cd^2 + d^3$.

Example.—Expand $(45)^4$.

Let $45 = (x+y)$ or $40 = x$, and $5 = y$

$(x+y)^4 = x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$ substituting values.

$(40)^4 + 4 \times 40^3 \times 5 + 6 \times 40^2 \times 5^2 + 4 \times 40 \times 5^3 + 5^4 = 2560000 + 1280000 + 240000 + 20000 + 625 = 4100625$.

Let $45 = (x-y)$ or $(50-5)$,

Then $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = 50^4 - 4 \times 50^3 \times 5 + 6 \times 50^2 \times 5^2 - 4 \times 50 \times 5^3 + 5^4 =$

$6250000 - 2500000 + 375000 + 25000 + 625 = 4100625$.

CHAPTER II.

THE N^{TH} ROOT OF QUANTITIES.

We have already shown the manner of developing or expanding a quantity by the Binomial Theorem. We shall show the converse of this process, or the extraction of the nth root of any quantity where n is any positive integer.

Any given quantity may be considered as the result of some expansion or the continued product of like factors of the required root, to find such factors is the result of the following section. The nth root of a quantity is one of the n equal factors of that quantity.

Rule for the extraction of the nth root of any quantity.

FIRST—*Place before you, or keep in mind the general form of the binomial theorem, viz:*

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1 \times 2} x^{n-2}y^2 \ldots . y^n$$

SECOND—*Arrange the expression according to the ascending or descending powers of some letter. Extract the* n*th root of the first term; the result will be the first term of the required root, or the equivalent of* x *in the general equation. Subtract the* n*th power of the first term of the root from the given polynomial.*

THIRD—*Divide the first term of the remainder by* n *times the* (n—1)*th power of the first term of the root, as a trial divisor; the quotient will be the second term of the root, or the equivalent of* y *in the general equation.*

Complete the divisor by adding the (n—1) *remaining terms of the general equation, with the terms of the root already found substituted in it, to the trial divisor, all* *divided by the second term of the root, and multiply this by the second term of the root. Subtract this product from the first remainder of the polynomial.*

Explanation—The student is cautioned to remember that the n referred to in this rule is the n of the required root or in other words the greatest exponent of x and y in the general formula

FOURTH—*Take* n *times the* (n—1)*th power of the sum of the first and second terms of the root for a second trial divisor. Divide the first term of the second remainder by the first term of the second trial divisor; the quotient will be the third term of the root, complete the divisor as before by adding the* (n—1) *remaining terms of the general equation, divided by the third term of the root, to the trial divisor and multiply this sum by the third term of the root.*

FIFTH—*Proceed in this manner until all the terms of the root have been found.*

Example.—Extract the square root of $a^2+b^2+c^2+2ab+2ac+2bc$.

$$a^2 + 2ab + 2ac + 2bc + b^2 + c^2 \mid a + b + c$$
$$a^2$$
$$2ab + 2ac + 2bc + b^2 + c^2 = 1^{st} \text{ remainder.}$$
$$2ab \qquad + b^2$$
$$2ac + 2bc \qquad + c^2 = 2^{nd} \quad \text{"}$$
$$2ac + 2bc \qquad + c^2$$

Trial divisor = $n(n—1)$ power of $a = 2a$ $\quad na^{n-1} = 2a$
Complete divisor = $2a + b$
" " $\times b$ the second term $\Big\}$ $(2a + b)b$.

Second trial divisor $n(n—1)$ power of $(a+b)^{n-1} = 2\,(a+b) = 2a+2b$
Second complete divisor $2a + 2b + c$
" " " $\times c$ the third term $\Big\}$ $(2a + 2b + c)\,c$

Explanation—

In this example $n=2$, and the general equation with 2 substituted for n equals $x^2+2xy+y^2$.

Three distinct terms will be found in the root, because there are three distinct letters in the given example. *

Arrange and proceed according to the rule (see example)

The square root of the first term a^2 is a, or the x of the general equation.

The second power of a is a^2. Subtract this from the given polynomial, leaving the first remainder as shown.

The first trial divisor is $n(n-1)$th power of a, or $2(2-1)$th power of a, which gives $2a$.

Divide the first term of the remainder by $2a$; this gives b the second term of the root, or the equivalent y of the general equation.

Complete the divisor by adding to the trial divisor the $(n-1)$ remaining terms, or the y^2 of the general equation, divided by y, which is y or b, this gives $2a+b$ as a complete divisor, and multiply this by b, which gives $2ab+b^2$. Subtract this quantity from the first remainder, leaving $2ac+2bc+c^2$ as a second remainder.

Now treat $(a+b)$ as one quantity or the x of the general equation, and according to rule (Fourth) take n or 2 times the $(n-1)$th power of $(a+b)$ or $2a+2b$, as a second trial divisor.

Divide the first term of the second remainder, $2ac$, by the first term of the second trial divisor $(2a)$; this gives c the third term of the root, or the y of the general equation, as before.

Complete the divisor, as before, by adding to the second trial divisor the $(n-1)$ remaining terms of the general equation (which in the square root is always one term) divided by the third term of the root, this gives $2a+2b+c$. Multiply this quantity by the third term c, and subtract from the second remainder, which leaves 0.

Therefore $(a+b+c)$ is the desired root and a, b, c are the separate terms of the root.

* Why? Not exact, except in especially prepared cases.

this supposes that an exact root is possible

Example. Extract the fourth root of the following:

$$
\begin{array}{l|l}
a^4+4a^3b+4a^3c+6a^2b^2+12a^2bc+6a^2c^2+4ab^3+4ac^3+12ab^2c+12abc^2+6b^2c^2+4b^3c+4bc^3+b^4+c^4 & a+b+c \\
a^4 & \\
\hline
\quad 4a^3b+4a^3c+6a^2b^2+12a^2bc+6a^2c^2+4ab^3+4ac^3+12ab^2c+12abc^2+6b^2c^2+4b^3c+4bc^3+b^4+c^4 & \\
\quad 4a^3b \qquad +6a^2b^2 \qquad\qquad +4ab^3 \qquad\qquad\qquad +b^4 & \\
\hline
\qquad 4a^3c+12a^2bc+6a^2c^2+4ac^3+12ab^2c+12abc^2+6b^2c^2+4b^3c+4bc^3+c^4 & \\
\qquad 4a^3c+12a^2bc+6a^2c^2+4ac^3+12ab^2c+12abc^2+6b^2c^2+4b^3c+4bc^3+c^4 & \\
\hline
\end{array}
$$

Explanation—

1st Trial Divisor $= n\ (n-1)^{th}$ power of $a = 4a^3$

$$
\left.\begin{array}{l}
\text{1st Complete Divisor} = 4a^3+6a^2b+4ab^2+b^3 = (n-1) \text{ remaining terms} \div b \\
\left.\begin{array}{l}\text{1st Complete Divisor} \times b \\ \quad \text{the second term}\end{array}\right\} = (4a^3+6a^2b+4ab^2+b^3)b = (n-1) \quad \text{"} \qquad \times b
\end{array}\right\}
$$

2nd Trial Divisor $= n\ (n-1)^{th}$ power of $(a+b) = 4\ (a+b)^3 =$

$$4a^3+12a^2b+12ab^2+4b^3$$

2nd Complete Divisor $= 4(a+b)^3+6(a+b)^2c+4(a+b)c^2+c^3 =$

$$4a^3+12a^2b+12ab^2+4b^3+6a^2c+12abc+6b^2c+4ac^2+4bc^2+c^3$$

$$\left.\begin{array}{l}\text{2nd Complete Divisor} \times c \\ \quad \text{the third term}\end{array}\right\} = (4a^3+12a^2b+12ab^2+4b^3+6a^2c+12abc+6b^2c+4ac^2+4bc^2+c^3)c$$

TO FIND THE N^{TH} ROOT OF A NUMBER.

The square of a number may contain twice as many figures as the given number, or twice as many less one.

The cube of a number may contain three times as many figures as the given number, or three times as many less one or two.

The nth power of a number may contain n times as many figures as the given number, or n times as many less $(n-1)$ or $(n-2)$ or $(n-3)$..........1.

Hence, if we separate the given number into periods of n figures each, beginning with the units, we shall determine the number of figures in the required root.

Let it be required to extract the square root of 50625, a perfect square. 50625 is evidently the square of some quantity of three periods or terms. Let a represent the value of the first figure of the root, b the second figure, and c the third figure. Hence, $(a+b+c)$ is a factor or root, and $(a+b+c)^2 = a^2+2ab+b^2+2ac+2bc+c^2$ equals 50625 or $50000 + 600 + 25$. Now follow the general rule already given, remembering that $n=2$.

$$
\begin{array}{lll}
 & \quad a \quad\; b \quad c & \\
5'06'25 & |\ 200+20+5 & \\
4\ 00\ 00 & & \\
\hline
1\ 06\ 25 = & \text{1st remainder} = & 2ab+b^2+2ac+2bc+c^2 \\
\quad 84\ 00 & & 2ab+b^2 \\
\hline
\quad 22\ 25 & \text{2d remainder} & 2ac+2bc+c^2 \\
\quad 22\ 25 & & 2ac+2bc+c^2 \\
\hline
\end{array}
$$

Let $a=200$ then $a^2=200^2=40000$

1st trial divisor$=n$ th $(n-1)$ power of $a=2a=400$

1st complete divisor$=(2a+b)= 400+20=420$

1st complete divisor $\times$ b the second term	$420 \times 20 = 8400$

Second trial divisor $= n \times (n-1)^{th}$ power of $(a+b) =$

$2\,(a+b) = 2a + 2b = 400 + 40 = 440$

2d complete divisor $= 2a + 2b + c = 440 + 5 = 445$

2d " " $\times c$ the third term	$445 \times 5 = 2225$

Therefore the square root of 50625 is $200 + 20 + 5 = 225$.

Explanation.

Since 600 nor 25 cannot be part of the square of the hundreds a must be the greatest multiple of the hundreds whose square is less than 50000 ; this multiple must be 200. Subtracting a^2 or 40000 from the given number, we have the first remainder, 10625, this equals $2ab + b^2 + 2ac + 2bc + c^2$. We already have the term a, therefore, if we double the term a or 200 the $n \times (n-1)^{th}$ power of the first term of the root we have 400, the trial divisor.

Divide the first remainder by 400 ; this gives 20, the second term, b. Complete the divisor by adding this term to the trial divisor and multiply by the second term ; this equals $(2a+b)\,b$ or $(400+20) \times 20 = 8400$. Subtract this from the first remainder leaving 2225 or $2ac + 2bc + c^2$. We now have terms a and b eliminated and desire term c.

Divide the second remainder by $n \times (n-1)^{th}$ power of the sum of the first and second terms as a trial divisor or $2a + 2b = 440$; this gives 5 for the third term c. Complete the divisor by adding this term to the trial divisor and multiply by the third term ; this equals $(2a + 2b + c)\,c$ or $(440 + 5) \times 5 = 2225$. Subtract this product from 2225 which leaves no remainder ; hence $200 + 20 + 5 = 225$ is the required square root.

ROOT CONTAINING DECIMALS.

In the extraction of a root containing decimals, care must be taken to divide the quantity into periods from the unit period as before, this, of course, makes the division of the periods from the left to the right beginning with the decimal point; and in a quantity containing whole numbers and decimals the periods should be spaced from the right to the left for the whole numbers and from the left to the right for the decimals. [See following example.]

Extract the square root of 275.029056

$$\begin{array}{l|l}
 & \;\;\; a \quad\; b \quad\; c \quad\;\; d \quad\;\; e \\
2'75.02'90'56 & \underline{14+2+.5+.08+.004} \\
\underline{1\ 96} & \\
\quad 79.02\ 90\ 56 = \text{1st remainder} & \\
\quad \underline{60} & \\
\quad 19.02\ 90\ 56 = \text{2d remainder} & \\
\quad \underline{16.25} & \\
\quad\;\; 2.77\ 90\ 56 = \text{3d remainder} & \\
\quad\;\; \underline{2.64\ 64} & \\
\quad\;\;\; .13\ 26\ 56 = \text{4th remainder} & \\
\quad\;\;\; \underline{.13\ 26\ 56} &
\end{array}$$

Let $a=14$ then $a^2=14^2=196$

1st trial divisor $= n \times (n-1)^{th}$ power of $a = 2 \times 14 = 28$

1st complete divisor $=$ 1st $T.\ D. + b = 28 + 2 = 30$

$$\left.\begin{array}{l}\text{1st complete divisor} \times b \\ \quad \text{the second term}\end{array}\right\} = 30 \times 2. = 60$$

2d trial divisor$=n\times(n-1)^{th}$ power of $(a+b)=2(a+b)=2\,(14+2)=32$

2d complete divisor=2d $T.\ D.+c=32+.5=32.5$

2d complete divisor $\times\ c$ } the third term } $32.5\times .5=16.25$

3d trial divisor $=n\times(n-1)^{th}$ power of $(a+b+c)=2(a+b+c)=2\,(14+2+.5)=33$

3d complete divisor=3d $T.\ D.+d=33+.08=33.08$

3d complete divisor $\times\ d$ } the fourth term } $33.08\times .08=2.6464$

4th trial divisor $n\times(n-1)^{th}$ power of $(a+b+c+d)=$

$2\,(a+b+c+d)=2(14+2+.5+.08)=33.16$

4th complete divisor=4th $T.\ D.+e=33.16+.004=33.164$

4th complete divisor $\times\ e$ } the fifth term } $33.164\times .004=.132656$

Therefore the square root of $275.029056=14+2+.5+.08+.004=16.584$.

Note the division of the periods from the decimal point to the left for whole numbers, and from the decimal point to the right for the decimals. This gives two periods of whole numbers and three periods for decimals, as shown. In this example proceed as before, remembering that the trial divisor is always n times the $(n-1)^{th}$ power of the sum of the terms of the root already found. The complete divisor is simply complying with the general equation of the expansion of $(x+y)^n$, and subtracting this expansion for a new remainder, if there be one.

CUBE ROOT.

The **cube root** of a number is one of the three equal **factors** of that number.

Example.—Extract the cube root of 2299968.

```
             a    b   c
2'299'968 |100+30+2
1 000 000
1 299 968=1st remainder.
1 197 000
  102 968=2d remainder.
  102 968
```

Let $a=100$ then $a^3=100^3=1000000$

1st trial divisor $=n(n-1)^{th}$ power of $a=3a^2=$
$3\times100^2=30000$

1st complete divisor $=$1st $T.\ D.+3ab+b^2=$
$30000+9000+900=39900$

1st complete divisor $\times\ b$ the second term $=39900\times30=1.197000$

2d trial divisor $=n(n-1)^{th}$ power of $(a+b)=3(a+b)^2$

Let $(a+b)=m$. Then $3(a+b)^2=3m^2=3\times130^2=50700$.

2d complete divisor$=$2d $T.\ D.+3mc+c^2=$
$50700+780+4=51484$

2d complete divisor $\times\ c$ the third team $=51484\times2=102968$

Explanation.

To extract the cube root of 2299968 we first divide the given number into periods of three figures each, beginning with the units; this gives the number of terms in the required root or factor. Hence, 2299968 may be considered as the cube of $(a+b+c)$ or $(a+b+c)^3$ when a, b, and c, or their numerical equivalents, are terms of the complete factor $(a+b+c)$.

Since 299000 nor 968 cannot be part of the cube of the hundreds, the term a must be the greatest multiple of hundreds whose cube is less than 2000000; this must be 100. Subtracting the cube of a, or $a^3 = 1000000$ from the given number, we have the first remainder 1299968.

We already have the first term a, therefore if we take three times the square of the first term or $n \times (n-1)^{th}$ power of the first term of the root we have 30000 as a trial divisor. Divide the first remainder by 30000; this gives 30, the second term of the root or b. Complete the divisor by adding 3 times the product of the first and second terms, plus the square of the second term already found. This equals 39900, and multiply this by the second term of the root b or 30, the product is 1197000. Subtract this from the first remainder, leaving 102968, or the second remainder.

Divide the second remainder by 3 times the square of the sum of the first and second terms as a second trial divisor, or $n \times (n-1)^{th}$ power of $(a+b) = 50700$; this gives 2 as the third term. Complete the divisor as before by adding to the trial divisor 3 times the product of the sum of the first and second terms by the third term, plus the square of the third term; this gives 51484, and multiply this total sum by the third term, giving 102968. Subtract this product from the second remainder, which leaves no remainder.

Hence, the terms of the root are $100+30+2=132$, the complete cube root of the number.

Extract the cube root of 264732.956461.

In this example a whole number and a decimal is given. Note that the number of figures in each place are three, and they are pointed off from the decimal point to the left and right for the whole number and decimal respectively.

$$
\begin{array}{r|l}
 & a\quad b\quad c\quad d \\
264'732.956'461 & 60+4+.2+.01 \\
216\ 000 & \\
\hline
48\ 732.956\ 461 & =\text{1st remainder} \\
46\ 144 & \\
\hline
2\ 588.956\ 461 & =\text{2d remainder} \\
2\ 465.288 & \\
\hline
123.668\ 461 & =\text{3d remainder} \\
123.668\ 461 & \\
\hline
\end{array}
$$

Let $a=60$ then $a^3=60^3=216000$

1st trial divisor $n\times(n-1)^{th}$ power of $a=3\,a^2=$
$3\times 60^2=10800$

1st complete divisor$=$1st $T.\ D.+3ab+b^2=$
$10800+720+16=11536$

$\left.\begin{array}{l}\text{1st complete divisor}\times b\\ \quad\text{the second term}\end{array}\right\}=11536\times 4=46144$

Let $(a+b)=m$

2d trial divisor$=n\times(n-1)^{th}$ power of $m=3m^2=$
$3\times 64^2=12288$

2d complete divisor$=$2d $T.D.+3mc+c^2=$
$12288+38.4+.04=12326.44$

$\left.\begin{array}{l}\text{2d complete divisor}\times c\\ \quad\text{the third term}\end{array}\right\}=12326.44\times .2=2465.288$

Let $(a+b+c)=n$

3d trial divisor$=n\times(n-1)^{th}$ power of $n=3n^2$
$3\times 64.2^2=12364.92$

3d complete divisor=3d $T.\ D.+3nd+d^2$
$12364.92+1.926+.0001=12366.8461$

$\left.\begin{array}{l}\text{3d complete divisor} \times d\\ \quad\text{the fourth term}\end{array}\right\}$ $12366.8461 \times .01=123.668461$

Therefore the cube root of $264732.956461=60+4+.2+.01=64.21.$

Extract the fifth root of 122298103125.

$$
\begin{array}{l|l}
 & \quad a \qquad b \qquad c \\
12'22981'03125 & 100+60+5 \\
\ \ 1\ 00000\ 00000 & \\ \hline
11\ 22981\ 03125 = \text{1st remainder} & \\
\ \ 9\ 48576\ 00000 & \\ \hline
\ \ 1\ 74405\ 03125 = \text{2d remainder} & \\
\ \ 1\ 74405\ 03125 & \\ \hline
\end{array}
$$

Let $a=100$ then $a^5=100^5=10000000000$

1st trial divisor$=n\times(n-1)^{th}$ power of $a=5a^4=$
$5\times100^4=500000000$

1st complete divisor=1st $T.D.+5(2a^3b+2a^2b^2+ab^3)+b^4=$
$500000000+600000000+360000000+108000000+12960000=1580960000$

$\left.\begin{array}{l}\text{1st complete divisor} \times b\\ \quad\text{the second term}\end{array}\right\}$ $1580960000 \times 60=94857600000$

Let $(a+b)=m$

2d trial divisor$=n\times(n-1)^{th}$ power of $m=5m^4=$
$5\times160^4=3276800000$

2d complete divisor=1st $T.D.+5(2m^3c+2m^2c^2+mc^3)+c^4=$
$3276800000+204800000+6400000+100000+625=3488100625$

$\left.\begin{array}{l}\text{2d complete divisor} \times c\\ \quad\text{the third term}\end{array}\right\}$ $3488100625 \times 5.=17440503125$

Explanation.

To extract the fifth root of 122298103125. Divide the given number into periods of five figures each, beginning with the units; this gives three terms or figures in the required root or factor. Hence, 122298103125 may be considered as the fifth power of $(a+b+c)$ when a, b, and c, or their numerical equivalents, are terms of the complete factor $(a+b+c.)$

The first term a of the fifth root of the given number 122298103125 must evidently be 100, since it is the greatest multiple of hundreds whose fifth power is less than 120000-000000. Subtracting the fifth power of the first term, or 10000000000 from the given number, we have the first remainder.

Divide the first remainder by the first trial divisor, or 5×100^4 or 500000000, we obtain 60 as the second term of the root. Complete the divisor by adding to the first term the terms indicated by the general formula, and multiply by the second term. Subtract and proceed as before.

Let us analyze this example and note its relation to the general binomial formula.

Let $(a+b)=m$. Therefore $a+b+c=(m+c)$. Expand $(m+c)^5$. This gives $m^5+5m^4c+10m^3c^2+10m^2c^3+5mc^4+c^5$. Substitute given values and expand further. $(a+b)^5+5(a+b)^4c+10(a+b)^3$ $c^2+10(a+b)^2c^3+5(a+b)c^4+c^5$

$$
\begin{array}{ll}
(a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5)= & m^5 \\
+5(a^4+4a^3b+6a^2b^2+4ab^3+b^4)c= & 5m^4c \\
+10(a^3+3a^2b+3ab^2+b^3)c^2= & 10m^3c^2 \\
+10(a^2+2ab+b^2)c^3= & 10m^2c^3 \\
+5(a+b)c^4= & 5mc^4 \\
+c^5= & c^5
\end{array}
$$

a equals 100, or the greatest multiple of hundreds whose fifth power is less than 120000000000. $a^5=10000000000$. Subtracting this from the number 122298103125 gives the first remainder 112298103125, or the remaining portion

$5a^4b+10a^3b^2+10a^2b^3+\ldots\ldots\ldots\ldots+c^5$. The trial divisor is $n\times(n-1)^{th}$ power of a equals $5a^4$ or $5\times(100)^4=$ 500000000. Divide the first remainder by this quantity. 500000000, which gives the second term $b=60$ of our root, We now have the $(a+b)$ terms of our root. Complete the divisor by adding to the trial divisor the remaining terms of $(a+b)^5$ divided by b, or $5a^4+10a^3b+10a^2b^2+5ab^3+b^4=$ 1580960000, and multiply this by $b=60$. This gives $5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$, which equals 94857600000, the complete fifth power of the first two terms of the root, or 160.

The second trial divisor is evidently ~~the~~ $n\times(n-1)^{th}$ power of $(a+b)$ or $5\ (a^4+4a^3b+6a^2b^2+4a'b^3+b^4)$ or letting $(a+b)=m$ as before to avoid lengthy terms the trial divisor $=5m^4$. $(a+b)=160$. $5m^4=160^4=3276800000$. Divide the second remainder by this quantity, this gives c or **5** the third term of the root. Complete the divisor as before by adding $10m^3c+10m^2c^2+5mc^3+c^4$ to the trial divisor. The complete divisor is now $5m^4+10m^3c+10m^2c^2+5mc^3+c^4$. Substituting the numerical values for m and c in this complete divisor gives 3488100625, and multiply this quantity by $c=5$ the third term, this gives 17440503125. Subtract this product from the second remainder which leaves no remainder. Hence, the complete root is the sum of the terms $a+b+c$ or $100+60+5=165$ as the fifth root of the given quantity.

CHAPTER III.

THEORY OF EQUATIONS.

It has been thought advisable to prelude the method for the solution of equations by some notes on their general theory for the guidance and simplicity in solution; as the following examples will show.

Every equation of the nth degree, containing one unknown quantity may be written under the form, $x^n+ax^{n-1}+bx^{n}\ ^{2}\ldots\ldots\ldots+kx+L=0$. This equation is called the general equation of the nth degree. The term L, or the absolute or independent term, may be considered the coefficient of x°.

A function of a quantity is an expression containing that quantity; thus ax^n+bx is a function of x.

The symbols $f(x)$, $F(x)$, are sometimes used for brevity to denote the function of x.

Any quantity, which substituted for x in $F(x)$ and causes $F(x)$ to vanish is a root of the equation $f(x)=0$; or in other words, a root of an equation is an expression, which, when substituted for the unknown quantity will satisfy the equation.

If $F(x)$ vanishes when $x=r$, the function is divisible by $(x-r)$. If $F(x)$ is divisible by $(x-r)$, then r is a root of the equation $f(x)=0$. When $F(x)$ is of the nth degree, it has n roots and no more.

THEOREM.—To find the relation between the coefficients of $f(x)$ and the roots of the equation $f(x)=0$.

Suppose the terms of $f(x)$ to be arranged according to the descending powers of x and that the coefficient of the first term is 1; then:

1. *The coefficient of the second term with its sign changed is equal to the sum of the roots.*

2. *The coefficient of the third term is equal to the sum of the products of the roots, taken two and two.*

3. *The coefficient of the fourth term with its sign changed is equal to the sum of the products of the roots, taken three and three and so on.*

4. *If the degree of the equation is even, the absolute term is equal to the product of all the roots. If the degree of the equation is odd, the absolute term with its sign changed is equal to the products of all the roots.*

Cor. 1. If the roots of $f(x)=0$ are all negative, each term of $f(x)$ is positive.

Cor. 2. If the roots of $f(x)=0$ are all positive, the signs of the terms of $f(x)$ will be alternately $+$ and $-$.

Cor. 3. If the second term of $f(x)$ does not appear, the sum of the roots of the equation $f(x)=0$ is equal to zero.

Cor. 4. If $f(x)$ has no absolute term, at least one of the roots of $f(x)=0$ is zero. Thus, one root of the equation $x^3-2x^2+3x=0$ is zero.

Cor. 5. The absolute term of $f(x)$ is divisible by each root of the equation $f(x)=0$.

Cor. 6. Let $a, b, c, d \ldots\ldots\ldots l$ denote the roots of the equation $x^n+Ax^{n-1}+Bx^{n-2}\ldots\ldots\ldots\ldots+Kx+L=0$;

$$\text{then } -A=a+b+c+d+\ldots\ldots\ldots+l$$

$$\text{and } \quad B=ab+ac+\ldots\ldots\ldots+bd+be+\ldots\ldots\ldots$$

$$\text{whence } \quad A^2-2B=a^2+b^2+c^2+d^2\ldots\ldots\ldots+l^2;$$

that is, A^2-2B is equal to the sum of the squares of the roots of the proposed equation. Hence if A^2-2B is negative, the roots of the equation cannot be all real. * Thus, the roots of the equation $x^5-4x^4+22x^3-25x-42=0$ are not all real for $(-4)^2-2\times 22$ is negative.

* This is true; but the condition is only one of the numerous conditions of reality of the roots; the number of these conditions is n reducible to $\frac{n}{2}$ or $\frac{n-1}{2}$ according as the

An equation whose coefficients are integers, that of its first term being unity, cannot have a root which is a rational fraction.

Let the equation be

$$x^n+Ax^{n-1}+Bx^{n-2}\ldots\ldots+Kx+L=0\ldots\ldots \quad (1)$$

in which the coefficients A, B, ... K, L are supposed to be integers.

Suppose, if possible that (1) has a rational fraction root which in its lowest terms is expressed by $\frac{a}{b}$. Substituting $\frac{a}{b}$ for x in (1), and multiplying the resulting equation by b^{n-1}, we obtain

$$\frac{a^n}{b}+Aa^{n-1}+Ba^{n-2}b+\ldots\ldots+Kab^{n-2}+Lb^{n-1}=0\ldots\ldots(2);$$

whence,

$$\frac{a^n}{b}=-(Aa^{n-1}+Ba^{n-2}b\ldots\ldots+Kab^{n-2}+Lb^{n-1})\ldots\ldots(3).$$

The second member of (3) is an integer, and its first member is an irreducible fraction. Hence $\frac{a}{b}$ cannot be a root of the proposed equation.

DESCARTES' RULE OF SIGNS.

A complete equation is one in which no power of x is wanting.

In any series of quantities a pair of consecutive like signs is called a permanence of signs and a pair of consecutive unlike signs is called a variation of signs. Thus in the expression $x^7-x^6-5x^5+4x^4+2x^3-3x^2-6x+7=0$ there are three permanences and four variations.

THEOREM OF DESCARTES'.—The number of real positive roots of the equation $f(x)=0$ cannot exceed the number of variations in the signs of its terms; and if the equation $f(x)=0$ is complete the number of real negative roots cannot exceed the number of permanences in the signs of its terms.

A complete equation whose signs are all positive can have no positive real root, for there is no variation of signs. When the signs are alternately positive and negative, there is no negative real root for there is no permanence of signs.

If $f(x)=0$ is of an odd degree, it has at least one real root; if of an even degree all the roots may be imaginary.

TRANSFORMATIONS OF EQUATIONS.

To transform an equation containing fractional coefficients into another in which the coefficients shall be integers, that of the first term being unity.

Let the proposed equation be

$$x^n + Ax^{n-1} + Bx^{n-2} \ldots\ldots + Kx + L = 0 \ldots\ldots (1),$$

in which some or all the coefficients A, B, C, ...K are supposed to be fractional.

Assume that $y=kx$, or $x=\frac{y}{k}$. Substituting $\frac{y}{k}$ for x in (1) and multiplying the resulting equation by k^n, we have

$$y^n + Aky^{n-1} + Bk^2y^{n-2} \ldots\ldots + Kk^{n-1}y + Lk^n = 0 \ldots (2).$$

Now since k is arbitrary, we may give it such a value as will make the coefficients Ak, Bk^2, ... Kk^{n-1}, Lk^n integers.

CHAPTER IV.

SOLUTIONS OF EQUATIONS OF HIGHER DEGREES.

In the previous chapters we have reviewed a comprehensive and detailed demonstration of the binomial formula, processes of involution and evolution, and the general theory of equations, for the purpose of our present chapter, namely, the solution of algebraic equations of higher degrees.

Let $Ax^n+Bx^{n-1}+Cx^{n-2}+Dx^{n-3}\ldots\ldots\ldots\ldots+K=0$ be a general equation of the nth degree, where x is the unknown quantity, and A, B, C, D, etc., coefficients of x, and K, the absolute term, which does not contain x.

In the above general equation the quantity sought is the value of x that will satisfy the above equation.

The above equation implies that the absolute term is A times the nth power of x, $+$ B times the $(n-1)^{th}$ power of x, $+$ C times the $(n-2)$ power of x, $+$ etc.; hence, if by some means we can simultaneously extract the nth root, the $(n-1)^{th}$ root, the $(n-2)^{th}$ root, of the various terms of the general equation, divided by the various coefficients A, B, C, etc., we shall obtain the desired result, and produce such factor or factors that will satisfy the above equation.*

Hence, to solve any equation of one unknown quantity, simultaneously extract the various roots, indicated by the various exponents, of the terms of the given equation, divided by the various coefficients.

RULE FOR THE SOLUTION OF EQUATIONS OF THE N^{TH} DEGREE.

$$Ax^n+Bx^{n-1}+Cx^{n-2}+Dx^{n-3}+\ldots\ldots\ldots+K=0 \qquad (1)$$

In the above equation let $x=(a+b)$ where a and b are terms of the desired root of the equation; that is, if $(a+b)$ were substituted, it would satisfy the equation.

* Heavy error: the root of a sum is not equal to any root of its parts.

*1 What is the expression of F(a)
*2 What is the expression of F'(a)

(1). Substitute this value into the given equation and expand according to the binomial theorem.

$$\left.\begin{array}{ll} A\,(a+b)^n = & \text{1st term} \\ +B\,(a+b)^{n-1} = & \text{2d term} \\ +C\,(a+b)^{n-2} = & \text{3d term} \\ +K = 0 & \text{Absolute term.} \end{array}\right\} \text{Equation (2).}$$

This expanded, equals:

$$\left.\begin{array}{lr} A(a^n + na^{n-1}b + \frac{n(n-1)}{1\times 2}a^{n-2}b^2 \ldots\ldots + b^n) = & \text{1st term} \\ +B(a^{n-1} + (n-1)a^{n-2}b + \frac{(n-1)\,(n-2)}{1\times 2}a^{n-3}b^2 + \ldots\ldots & \\ \quad b^{n-1}) = & \text{2d term} \\ +C\,(a^{n-2} + (n-2)a^{n-3}b + \ldots\ldots \text{etc.}) = & \text{3d term} \\ +K = 0 & \text{Absolute term} \end{array}\right\} (3).$$

(2). Separate and arrange the terms of (3) so that all the highest powers of a in each consecutive term, beginning with the highest term, are consecutive, as: $Aa^n + Ba^{n-1} + Ca^{n-2} + Da^{n-3} + \text{etc.}$ *1

(3). Arrange the terms so that all the $n(n-1)^{th}$ powers of the expanded consecutive terms in (3) are consecutive, as: $(Ana^{n-1} + B(n-1)a^{n-2} + C\,(n-2)a^{n-3} +$ etc.$)\,b$. These quantities are evidently the trial divisors for the extraction of the various roots indicated by their respective exponents of the various powers. *2

(4). Arrange the remaining terms so that they are placed according to their respective positions in the general equation, beginning with the highest remaining term of a.

The equation may now be arranged thus:

$$(Aa^n + Ba^{n-1} + Ca^{n-2} + Da^{n-3} + \ldots\ldots) = \text{1st term}$$
$$+[\,(Ana^{n-1} + B(n-1)a^{n-2} + C\,(n-2)a^{n-3} + D(n-3)^{n-4}$$
$$+ \ldots\ldots) = \text{2d term}$$

b.

$$+(A\frac{n(n-1)}{1\times 2}a^{n-2}b++\dots\dots B\frac{(n-1)(n-2)}{1\times 2}a^{n-3}b+++$$

$$C\frac{(n-1)(n-2)(n-3)}{1\times 2}a^{n-4}b+++)]b = \text{3d term}$$

$$+\dots\dots\dots K=O= \quad \text{Absolute term.} \quad =\text{Equation (4).}$$

(5). Substitute some trial value for a in the first term of (3) whose value is equal to, or less, than K preferable. *

(6). Subtract this value from K, which gives a first remainder. This evidently equals the remaining terms of equation (4).

(7). Divide the first term of the remainder by the trial divisor, or the second term of equation (4); the quotient is the second term of the root. *3 Complete the divisor by adding the remaining term of equation (4), and multiply this sum by the second term of the root. Subtract this product from the first remainder.

(8). Take the sum of the first and second terms of the root for the value of a^{1}, and substitute it into the second term of equation (4) for a second trial divisor. Divide the first term of the second remainder by the second trial divisor; the quotient is the third term of the root. Complete the divisor by adding the remaining terms of equation (4) and multiply this by the third term of the root. Subtract this product from the second remainder for a third remainder.

(9). Proceed in this manner until all the terms of the root have been found.

(**Note.**—The student may find some difficulty in finding a trial divisor as stated in (5), whose value substituted in the first term of equation (4) will be equal to or less than K. When the value equals K the equation is evidently solved. If the term substituted is greater than K, this will only give a minus quantity as a first remainder, but no way impair the solution of the given equation. By following the solution of the simpler numerical equations that follow, the process will be clearly shown.) (*)4

Except in case the equation contains imaginary roots; in such case each trial divisor gives a new result, without linking with the precedent one

* That is the ordinary research of the integral parts of the roots of a numerical equation to be made by grouping. F(a)

Example—

Solve the equation $x^2+x=650$

$x^2+x-650x^0=0$. In this equation we have one permanence and one variation, hence it contains one positive and one negative root according to Descartes' rule.

From **Cor.** 6, page (22), $A^2=1^2=$the sum of the squares of the coefficients, or the coefficient of the second term. *

The coefficient of the third term, $B\,x^0=-650$. Therefore$-2B=1300$.

$A^2-2B=1+1300$ a positive quantity; hence, all the roots are real. *

* Error: condition not sufficient, even for equation of second degree where the reduced condition is A^2-4B

$$x^2+x=650$$

$$\begin{array}{r|l} & \;a\quad b \\ 650 & 20+5 \\ \underline{420} & \\ 230 & [(2a+1)+b]b=\text{1st remainder.} \\ \underline{230} & \end{array}$$

Let $a=20$, then $a^2+a=20^2+20=420$

1st trial divisor$=2a+1=2\times20+1=41$

1st complete divisor$=$1st $T.\ D.+b=41+5=46$

$$\left.\begin{array}{l}\text{1st complete divisor }\times b\\ \quad\text{the second term}\end{array}\right\}=46\times5=230$$

$$20+5=25$$
$$x=25$$
$$x-25=0$$

$$\begin{array}{r|l} x^2+x-650 & x-25 \\ \underline{x^2-25x} & x+26 \\ 26x-650 & \\ \underline{26x-650} & \end{array}$$

Therefore the roots are 25 and -26, and the factors of $x^2+x-650=0$ are $(x-25)$ and $(x+26)$.

Explanation.

There is involved in this equation a second and a first power of x; hence, to determine the roots or factors of x we must evidently simultaneously extract the first and second roots of the absolute term divided by the coefficients of the several powers of x.

Apply the rule given on page (25) for the solution of equations.

Let x equal some quantity of two or more terms: (By terms are meant quantities whose algebraic sum is equal to the root of the given unknown quantity.) as $(a+b)$ or $(a+b+c)$. Now substitute this value in the given equation; therefore $x^2+x-650=0=(a+b)^2$ (1st term) $+(a+b)$ (2nd term) $-650=0$. (1). This expanded equals $(a^2+2ab+b^2)+(a+b)-650=0$. (2).

Separate and arrange according to (2) of the rule where a^2+a are the highest powers of a in the above consecutive terms. $(2a+1)$ are respectively the $n(n-1)$ powers of a^2 and a, the first terms of the expanded consecutive terms.

The equation may now be written in this form to comply with the rule. $(a^2+a)+[(2a+1)+b]b=650$ as per (4). (3).

Now assume a equals 20 as the first term of the root and substitute this in the first term of (3) according to (5) of the rule. $a^2+a=20^2+20=420$. Subtract this value from 650 which gives 230 as a first remainder; this (230) evidently equals $[(2a+1)+b]b$ for a is already known.

Now divide the first remainder by the trial divisor $(2a+1)$ or $2\times20+1=41$. This gives 5, the second term of the root. Complete the divisor by adding the remaining terms of (3) or the $b=5$ just found, this gives $2a+1+b=41+5=46$ as a complete divisor and multiply by b or 5; $46\times5=230$. Subtract this product from the first remainder, leaving no remainder; hence the terms are $a=20$, $b=5$ and $(a+b)=25$ is the required root. Therefore $x=25$ and $x-25$ is one factor.

There are as many roots as the power of the highest exponent of the unknown quantity; hence two roots in this equation. Page 21.

If $x-25$ is a factor of $x^2+x-650=0$; $x^2+x-650$ is divisible by $x-25$. Dividing as shown gives $x+26$ as the other factor.

Therefore $x=\begin{cases}25\\-26\end{cases}$ and $(x-25)(x+26)=x^2+x-650=0$.

Let us suppose that in the same equation $x^2+x=650$; we had three terms in our desired factor or root, and solve under these conditions; then $x=(a+b+c)$ and x^2+x would equal to $(a+b+c)^2+(a+b+c)$. Expanding this equals $a^2+2ab+b^2+2ac+2bc+c^2+a+b+c$.

Separating the terms according to our rule equals $a^2+a+[(2a+1)+b]b+[(2a+2b+1)+c]c$.

$$
\begin{array}{l|l}
 & \;\;a \quad\;\; b \quad\;\; c \\
650 & 10+10+5 \\
\hline
\underline{110} & \\
540 & [(2a+1)+b]b+[(2a+2b+1)+c]c=\text{1st remainder.} \\
\underline{310} & [(2a+1)+b]b \\
230 & [(2a+2b+1)+c]c=\text{2d remainder.} \\
\underline{230} & \qquad\qquad (a+b+c)=25 \\
 & \qquad\qquad x=25 \text{ as before.}
\end{array}
$$

Let $a=10$, then $a^2+a=10^2+10=110$

1st trial divisor $=2a+1=2\times10+1=21$

1st complete divisor $=(2a+1)+b=21+10=31$

$\left.\begin{array}{l}\text{1st complete divisor}\times b\\ \quad\text{the second term}\end{array}\right\}=31\times10=310$

2d trial divisor $=(2a+2b+1)=20+20+1=41$

2d complete divisor $=$ 2d $T.\ D.+c=41+5=46$

$\left.\begin{array}{l}\text{2d complete divisor}\times c\\ \quad\text{the third term}\end{array}\right\}=46\times5=230$

Thus any number of terms may be used, and a or x may be represented by any numerical quantity if the requisite signs be given to them as we proceed; for example, let us assume $x=30$, which is greater than the desired factor, as our solution has shown.

		$a \quad b$
	650	$30-5$
Let $a=30$, then $a^2+a=30^2+30=$	930	
1st trial divisor $=2a+1=2\times 30+1=61$	-280	
1st complete divisor $=(2a+1+b)=$ $61+(-5)=61-5=56$	-280	
1st complete divisor $\times$ b the second term $=56\times -5=-280$		

Example.

Solve the equation $3x^2+4x=224$.

In this equation x^2 and x have coefficients 3 and 4 respectively, therefore 224 is 3 times the square of some quantity plus 4 times the first power of the same quantity. To solve this equation means to obtain the sum of the extraction of the square root plus the first root divided by their respective coefficients.

$3x^2+4x-224=0$. In this equation, we have one permanence and one variation : hence one negative root and one positive root.

* $A^2-2B=4-(-448)=16+448=464$, therefore all the roots are real. Cor. (6) page (22).

Let $x=(a+b)$, therefore $3x^2+4x-224=0$. $3(a+b)^2+4(a+b)-224=0$. Expanded $=3(a^2+2ab+b^2)+4(a+b)=224$. Arranged and separated according to the general rule, this equation equals $(3a^2+4a)+[(6a+4)+3b]b=224$, where $(3a^2+4a)$ correspond respectfully to the Aa^n and Ba^{n-1}

* Error.- Same remark as at page 28.

of the general equation : and $(6a+4)$ are respectfully the $n(n-1)^{th}$ powers of $3a^2$ and $4a$, or the first trial divisors of the given equation.

$$\begin{array}{r|l} & \overset{a}{3}+\overset{b}{4}+\overset{c}{1} \\ 224 & \\ \underline{39} & \\ 185 & \text{1st remainder.} \\ \underline{136} & \\ 49 & \text{2d remainder.} \\ \underline{49} & \end{array}$$

Let $a=3$, then $3a^2+4a=3\times 3^2+4\times 3=27+12=39$

1st trial divisor$=6a+4=6\times 3+4=22$

1st complete divisor$=$1st $T.\ D.+3b=22+12=34$

$\left.\begin{array}{l}\text{1st complete divisor} \times b \\ \text{the second term}\end{array}\right\}=34\times 4=136$

Let $(a+b)=m$

2d trial divisor$=6m+4=6\times 7+4=46$

2d complete divisor$=$2d $T.\ D.+3c=46+3=49$

$\left.\begin{array}{l}\text{2d complete divisor} \times c \\ \text{the third term}\end{array}\right\}=49\times 1=49$

$$3+4+1=8$$
$$x=8$$
$$x-8=0$$

$$\begin{array}{l|l} 3x^2+\ 4x-224 & x-8 \\ \underline{3x^2-24x} & \underline{3x+28} \\ \quad 28x-224 & \\ \quad \underline{28x-224} & \\ \quad 3x^2+4x=224. & \end{array}$$

Therefore $3x+28=0$. $3x=-28$. $x=-\frac{28}{3}=-9\frac{1}{3}$.

Explanation.

Referring to the example on page (31), assume that $a=3$ in the equation, then $(3a^2+4a)=39$. Subtract this from the given value 224, this gives the first remainder 185.

185 evidently equals $[(6a+4)+3b]b$. The value of a has been found; b, the second term, is now desired. $(6a+4)$ is our trial divisor. Substitute the value of $a=3$ in the trial divisor, this equals 22, and divide the first remainder, 185, by this quantity. Let the quotient be $4=b$, or the second term of the root. Complete the divisor by adding $3b$ or 12, and multiply by b or 4. $[(6a+4)+3b]b=[(6\times 3+4)+12]\times 4=136$.

Subtract this product from the first remainder, leaving 49 as a second remainder; hence, another term of the root is yet to be found.

Proceed as per (8) of the general rule. Take the sum of the first and second terms of the root, or 7 for the value of a^1 or m, and substitute this in the second term of the equation, $(6m+4)$, as a second trial divisor. $(6m+4)=6\times 7+4=46$, the second trial divisor.

Divide the second remainder by this new divisor. This gives 1 or c as the third term of the root. Complete the divisor as before and multiply by the third term c. This gives $[(6m+4)+3c]\times c=[(6\times 7+4)+3]\times 1=49$. Subtracting leaves no remainder; hence the root is $3+4+1$ or 8. Therefore $x=8$ or $x-8=0$.

Divide the given equation $3x^2+4x-224$ by this quantity; this gives $3x+28$ or $3x=-28$, therefore $x=-9\frac{1}{3}$ the other root.

Therefore $x=\begin{cases} 8 \\ -9\frac{1}{3} \end{cases}$ and the factors are $(x-8)(3x+28)$.

In this example we have three terms but have complied with our rule for the extraction of a root and taken for a second trial divisor n times the $(n-1)^{th}$ power of the sum of the first and second terms of the root viz., $(a+b)$ or $(3+4)$ and treated this as one term and proceeded as before.

Example.

Solve the equation $3x^2-2x=432.1468$.

In this equation we have a minus sign, coefficients and decimals to deal with. Hereafter, in all examples of more than two terms in the result or root, we shall consider the second and successive trial divisors as n times the $(n-1)$ power of the sum of the terms already found, to simplify the solution and to avoid the long demonstrations as given in the example on page (19).

The student should test this equation for positive, negative and imaginary roots as in the former equations.

The student can readily see that the first term of the required root may be given any value whatever; only the greatest value that can be readily seen to satisfy the equation is preferable, thus reducing the number of terms and the labor required to obtain the desired result.

It is probably prudent to take some number that is easily multiplied and divided as 10 or some factor of 10, when not too great, as the equivalent of a or the first term of the root.

$$\begin{array}{l|l}
 & \;\;a \quad\; b \quad\; c \quad\;\; d \\
4'32.14'68 & 10+2+.3+.04 \\
\hline
\underline{2\ 80} & \\
1\ 52.14\ 68 = \text{1st remainder.} & \\
\underline{1\ 28} & \\
\quad 24.14\ 68 = \text{2d remainder.} & \\
\quad \underline{21.27} & \\
\quad\; 2.87\ 68 = \text{3d remainder.} & \\
\quad\; \underline{2.87\ 68} &
\end{array}$$

Let $a=10$, then $3a^2-2a=300-20=280$

1st trial divisor$=6a-2=60-2=58$

1st complete divisor$=$1st $T.\ D.+3b=$

$58+6=64$

$$\left.\begin{array}{l}\text{1st complete divisor} \times b \\ \text{the second term}\end{array}\right\} = 64 \times 2 = 128$$

Let $(a+b)=m$

2d trial divisor$=6m-2=72-2=70$

2d complete divisor$=$2d $T.\ D.+3c=70+.9=70.9$

$$\left.\begin{array}{l}\text{2d \quad `` \quad `` } \times c \\ \text{the third term}\end{array}\right\} = 70.9 \times .3 = 21.27$$

Let $(a+b+c)=n$

3d trial divisor$=6n-2=73.8-2=71.8$

3d complete divisor$=$3d $T.\ D.+3d=71.8+.12=71.92$

$$\left.\begin{array}{l}\text{3d \quad `` \quad `` } \times d \\ \text{the fourth term}\end{array}\right\} = 71.92 \times .04 = 2.8768$$

$10+2+.3+.04=12.34$

$x=12.34$

$x-12.34=0$

$$\begin{array}{r|l} 3x^2-\quad 2x-432.1468 & x-12.34 \\ \cline{2-2} 3x^2-37.02x \qquad\qquad & 3x+35.02 \\ \cline{1-1} 35.02x-432.1468 & \\ 35.02x-432.1468 & \\ \cline{1-1} \end{array}$$

$$3x+35.02=0.\ 3x=-35.02.\ x=-\frac{35.02}{3}=-11.67333+$$

The solution of the above equation containing decimals is similar to the former solutions, care being taken to comply with the negative signs, where indicated, also to observe decimal quantities, remembering that each successive decimal term is given its proper decimal place as shown in the example given. No further detailed explanation is necessary if what has been done before is understood.

In this example $x = \begin{cases} 12.34 \\ -11.673+ \end{cases}$ and the factors are $(x-12.34)$, $(x+11.67333+)$.

CUBIC EQUATIONS.

A cubic equation is one of the third degree as $ax^3+bx^2+cx=k$.

Solve the equation $x^3+x^2+x-819=0$.

In this equation a third, a second and a first power of x is involved and the sum of the three terms is equal to 819. Hence to find the value of x which will satisfy the equation, the third, the second, and the first roots of 819, divided by their respective coefficients must simultaneously be extracted.

Let $x=(a+b)$. Therefore $x^3+x^2+x-819=(a+b)^3+(a+b)^2+(a+b)-819=0$. Expanding this equals

$$(a^3+3a^2b+3ab^2+b^3)=\text{first term}=A(a^n+na^{n-1}b+\frac{n(n-1)}{1\times 2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}b^3)$$

$$+(a^2+2ab+b^2)=\text{second term}=B(a^{n-1}+(n-1)a^{n-2}b+\frac{(n-1)(n-2)}{1\times 2}a^{n-3}b^2)$$

$$+(a+b)=\text{third term}=C(a^{n-2}+(n-2)a^{n-3}b).$$

Arranged and separated according to (4), the equation equals $(a^3+a^2+a)=$ first term $(Aa^n+Ba^{n-1}+Ca^{n-2})$ $+(3a^2+2a+1)b=$ second term $=[Ana^{n-1}+B(n-1)a^{n-2}+C(n-2)a^{n-3}]b$

$$+(3ab+b^2+b)b = \text{third term} = [A\frac{n(n-1)}{1\times 2}a^{n-2}b + A\frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}b^2 + B\frac{(n-1)(n-2)}{1\times 2}a^{n-3}b]b$$

The equation now assumes this form. The coefficients A, B, C in this equation equal 1. $(a^3+a^2+a)+[(3a^2+2a+1)+3ab+b^2+b]b=819$.

$$\begin{array}{l|l} & a\quad b \\ 819 & 5+4 \\ \underline{155} & \\ 664 = \text{1st remainder.} & \\ \underline{664} & \end{array}$$

Let $a=5$, then $a^3+a^2+a=5^3+5^2+5=155$

1st trial divisor $=3a^2+2a+1=75+10+1=86$

1st complete divisor $=$ 1st $T.$ $D.+3ab+b^2+b=$

$86+60+16+4=166$

$\left.\begin{array}{l}\text{1st complete divisor} \times b \\ \text{the second term}\end{array}\right\} = 166\times 4 = 664.$

$5+4=9$

$x=9$

$x-9=0$

$$\begin{array}{l|l} x^3+x^2+x-819 & x-9 \\ \underline{x^3-9x^2} & x^2+10x+91 \\ \quad 10x^2+x & \\ \quad \underline{10x^2-90x} & \\ \qquad 91x-819 & \\ \qquad \underline{91x-819} & \end{array}$$

$x^2+10x=-91$

$$\begin{array}{l|l} & a \qquad\qquad b \\ -91 & (-5) \pm\sqrt{-66} \\ \underline{-25} & \\ -66 & \\ \underline{-66} & \end{array}$$

Let $a=-5$, then $a^2+10a=25-50=-25$

1st trial divisor $2a+10=-10+10=0$

1st complete divisor $=$ 1st $T.\ D.+b$

$(0+b)=b$

$\left.\begin{array}{l}\text{1st complete divisor } \times b \\ \text{the second term}\end{array}\right\} =b\times b=b^2=-66$

Therefore $b=\pm\sqrt{-66}$

Explanation.

Let a equal some number whose value substituted in the above equation is less than 819.

Let $a=5$ as a trial value (5), therefore $a^3+a^2+a=5^3+5^2+5=155$. Subtract this quantity from the value of the given equation or 819; this gives 664 as a first remainder.

664 evidently equals $[(3a^2+2a+1)+3ab+b^2+b]b$.

The trial divisor for the cube root is $3a^2$, the trial divisor for the square root is $2a$, and the trial divisor for the first root is 1; (all these comply for the extraction of the n^{th} root, viz: divide the first term of the remainder by n times the $(n-1)^{th}$ power of the first term of the root, as a trial divisor). $3a^2$ is $n\ (n-1)^{th}$ power of a where $n=3$ or the extraction of the cube root; $2a$ is the $n\ (n-1)^{th}$ power of a where $n=2$ or in the extraction of the square root; 1 is the $n\ (n-1)^{th}$ power of a where $n=1$ or the extraction of the first root, which is simply the coefficient of a.

$3a^2+2a+1=3\times 5^2+2\times 5+1=86$ as the value of the trial divisor. Divide the first remainder 664 by this quantity. Let this equal 4 or b the second term of the root.

Complete the divisor by adding to the trial divisor the necessary quantities for the completion of the divisors in the extraction of the cube and square roots; that is, add $3ab+b^2=3\times 5\times 4+4\times 4$ for the completion of the cube root divisor, and b or 4 for the completion of the square root divisor, thus making the complete divisor, $[(3a^2+2a+1)+3ab+b^2+$

$b]=86+80=166$, and multiply this by 4; this gives 664. Subtract this from the first remainder, which leaves no remainder; hence the root is $5+4=9$ or $x=9$.

$x-9=0$ is therefore a factor of $x^3+x^2+x-819=0$.

Dividing $x^3+x^2+x-819$ by $(x-9)$ gives $x^2+10x+91$ as the other root of the equation $x^3+x^2+x-819=0=(x-9)(x^2+10x+91)$.

We must now comply with the method of solving the equation of the second degree to find the other factors.

In the equation $x^2+10x+91=0$, we have two permanences; hence the two roots are negative. A^2-2B is negative, hence both roots cannot be real.

Let $x=(a+b)$ as before and solve for the remaining roots.

$x^2+10x+91=(a+b)^2+10(a+b)+91=0=a^2+2ab+b^2+10a+10b+91=0=(a^2+10a)+[(2a+10)+b]\,b=-91.$

Assume $a=-5$, since the root is negative, therefore $a^2+10a=25-50=-25$. Subtract this from -91, leaving -66 the first remainder. $2a+10$ the first trial divisor equals $-10+10=0$.

-66 evidently equals $[(2a+10)+b]\,b$; but $2a+10$ equals 0; hence $(0+b)\,b$ or $b^2=-66$ or $b=\pm\sqrt{-66}$ an imaginary quantity as shown.

$$\text{Therefore } x=\begin{cases} 9 \\ -5+\sqrt{-66} \\ -5-\sqrt{-66} \end{cases}$$

By testing before solving the equations for positive and negative roots and imiginary quantities, according to corillary (6), the student will be somewhat guided in the selection of terms for a and b and avoid what may lead to tedious work and confusion.

Example.

Solve the cubic equation $x^3+6x^2-37x-210=0$.

The roots of this equation are three in number, two negative and one positive ; and all real. (The student is requested to test each equation for real and imaginary roots, also for positive and negative roots by the theory given on page 24).

Let $x=(a+b)$ as before.

(1). Therefore $x^3+6x^2-37x-210=0$, substituting $(a+b)$ equals

(2). $(a+b)^3+6(a+b)^2-37(a+b)-210=0$. Expanding equals $(a^3+3a^2b+3ab^2+b^3)+6(a^2+2ab+b^2)-37(a+b)-210=0$.

Separating and arranging according to (2), (3), and (4) of the rule.

(3). $(a^3+6a^2-37a)+[(3a^2+12a-37)+3ab+b^2+6b]\,b=210$.

$$x^3+6x^2-37x=210$$

$$\begin{array}{r|l} & a \quad b \\ 210 & 4+2 \\ \underline{12} & \\ 198 & =\text{1st remainder.} \\ \underline{198} & \end{array}$$

Let $a=4$, then $a^3+6a^2-37a=64+96-148=12$

1st trial divisor $=3a^2+6\times 2a-37=$
$48+48-37=59$

1st complete divisor $=$ 1st $T. D.+3ab+b^2+6b=$
$59+24+4+12=99$

$\left.\begin{array}{l}\text{1st complete divisor} \times b \\ \text{the second term}\end{array}\right\} =99\times 2=198$

$$4+2=6$$
$$x=6$$
$$x-6=0$$

$$\begin{array}{l|l}
x^3+6x^2-37x-210 & x-6 \\ \hline
x^3-6x^2 & x^2+12x+35 \\ \hline
\quad 12x^2-37x & \\
\quad 12x^2-72x & \\ \hline
\qquad\quad 35x-210 & \\
\qquad\quad 35x-210 & \\ \hline
\end{array}$$

$$x\ \ +12x=-35$$

$$\begin{array}{r|l}
 & \ \ a\quad b \\
-35 & -4-1 \\ \hline
-32 & \\ \hline
-\ 3 & \\
-\ 3 & \\ \hline
\end{array}$$

Let $a=-4$, then $a^2+12a=16-48=32$

1st trial divisor$=2a+12=-8+12=4$

1st complete divisor$=$1st $T.\ D.+b=4+(-1)=3$

1st " " $\times b$ / the second term $\Big\} =3\times(-1)=-3$

$$(-4)+(-1)=-5$$
$$x=-5$$
$$x+5=0$$

$$\begin{array}{l|l}
x^2+12x+35 & x+5 \\ \cline{2-2}
x^2+5x & x+7 \\ \hline
\quad 7x+35 & \\
\quad 7x+35 & \\ \hline
\end{array}$$

$$x+7=0$$
$$x=-7$$

Hence the factors equal $(x-6)\ (x+5)\ (x+7)$.

Explanation.

For a, substitute some numerical trial value, whose value substituted in the first term, (a^3+6a^2-37a), is less than 210. Assume a equals 4. (If 4 should satisfy the equation, we would have only one term in the factor, and this substituted in the equation would satisfy the equation at once; hence there would be no necessity for a second term b, and a^3+6a^2-37a would equal 210; but if a does not satisfy the equation we have still a second or perhaps a third term, and so on until we have reached a desired root, if it be a commensurable quantity). (a^3+6a^2-37a) therefore equals $(4^3+6\times 4^2-37\times 4)=12$.

Subtract this from the absolute term 210, leaving a remainder, 198. 198 evidently equals the remainder of the equation $[(3a^2+12a-37)+3ab-b^2+6b]\,b$, (3), after the first term a^3+6a^2-37a have been eliminated.

Now observe that $(3a^2+12a-37)$ is the first trial divisor.

$3a^2$ is the trial divisor for the cubic portion,
$6\times 2a$ " " " " " " quadratic portion,
$-37\times a^0$ " " " " " " first power portion;

hence the sum is the complete trial divisor, and all are the $n\,(n-1)^{th}$ powers of the highest powers of a according to rule.

$3a^2+6\times 2a-37=3\times 4^2+6\times 2\times 4-37=59$, the numerical trial divisor. Divide the first remainder by 59, let this equal $2=b$. (The student may try 2 by substituting its value in the complete remainder to see whether the quantity be too great, just as in the extraction of any root).

Complete the divisor by adding the remainder of (3) to the trial divisor and substitute the values of a and b in the equation and multiply by $b=(1\text{st } T.\ D.+3ab+b^2+6b)\,b=(59+3\times 4\times 2+2^2+6\times 2)\times 2=99\times 2=198$; this leaves no remainder.

Hence $x=6$ and $x-6=0$, one of the factors.

Divide $x^3+6x^2-37x-210$ by $(x-6)$ rule (3), this gives $x^2+12x+35$. (4).

Solve this equation (4) to determine the other roots or factors. Solving as shown gives $x = -5$ a negative root or $x+5=0$.

Divide $x^2+12x+35$ by $x+5$, this gives $x+7$, the other factor as shown.

We now have the three roots, $x=\begin{cases} 6 \\ -5 \\ -7 \end{cases}$ or two negative and one positive root.

The factors are $(x-6),(x+5),(x+7)$, therefore $(x-6)(x+5)(x+7)=x^3+6x^2-37x-210=0$.

Example.

Solve the equation $2x^3-650x=3000$.

This equation has no second power of x, hence it is not complete; the second term may be supplied thus, $2x^3 \pm 0x^2-650x=3000$.

The solution of this equation is solved like the former equations, only there being no term of the second degree in this equation there is of course no trial nor complete divisor for the second degree extraction; hence simply deal with the third and first degrees as shown. This example needs no further explanation.

$$2x^3 - 650x = 3000.$$

Let $x = (a+b)$, then $2x^3 - 650x = 2\,(a+b)^3 - 650\,(a+b) =$

$2a^3 + 6a^2b + 6ab^2 + 2b^3 - 650a - 650b = 2a^3 - 650a + [(6a^2 - 650) + 6ab + 2b^2]\,b$, arranged according to rule.

Let $a = 10$, then $2a^3 - 650a = 2000 - 6500 = -4500$

1st trial divisor $= 6a^2 - 650 = 600 - 650 = -50$

1st complete divisor $=$ 1st $T.\ D. + 6ab + 2b^2 =$

$$-50 + 600 + 200 = 750$$

$\left.\begin{array}{l}\text{1st complete divisor} \times b \\ \quad\text{the second term}\end{array}\right\} = 750 \times 10 = 7500$

$$\begin{array}{r|l} & \;a \quad b \\ 3000 & 10 + 10 \\ -4500 & \\ \hline 7500 & = \text{1st remainder.} \\ 7500 & \\ \hline \end{array}$$

$10 + 10 = 20$

$x = 20$

$x - 20 = 0 =$ one factor

$$\begin{array}{l|l} 2x^3 - 650x - 3000 & x - 20 \\ \cline{2-2} 2x^3 - 40x^2 & 2x^2 + 40x + 150 \\ \hline \quad 40x^2 - 650x & \\ \quad 40x^2 - 800x & \\ \hline \qquad 150x - 3000 & \\ \qquad 150x - 3000 & \\ \hline \end{array}$$

$$2x^2 + 40x = -150$$

Let $x = (a+b)$, then $2x^2 + 40x = 2\,(a+b)^2 + 40\,(a+b) = 2a^2 + 4ab + 2b^2 + 40a + 40b$
$= 2a^2 + 40a + [(4a+40) + 2b]\,b$

Let $a = -3$, then $2a^2 + 40a = 18 - 120 = -102$

1st trial divisor $= 4a + 40 = -12 + 40 = 28$
1st complete divisor $=$ 1st $T.D. + 2b = 28 - 4 = 24$
1st " " $\times b$ } the second term } $= 24 \times -2 = -48$.

$$\begin{array}{r|l} -150 & \overset{a}{-3} \quad \overset{b}{-2} \\ \hline -102 & \\ \hline -48 & = \text{1st remainder.} \\ -48 & \\ \hline \end{array}$$

$$\begin{array}{r|l} 2x^2 + 40x + 150 & x + 5 \\ \cline{2-2} 2x^2 + 10x & 2x + 30 \\ \hline 30x + 150 & \\ 30x + 150 & \\ \hline \end{array}$$

$x = -5$
$x + 5 = 0 =$ one factor.

Hence the factors are $(x-20)\,(x+5)\,(2x+30)$ or $2\,(x-20)\,(x+5)\,(x+15)$.

If the second term of function x does not appear, the sum of the roots of the equation $f(x)=0$ is zero. Thus the sum of the roots of the above equation equals 0: viz. $-20+5+15$ equals 0 from (3) page (22).

Example.

Solve the equation $3x^3-4x^2+6x-82068.904448=0$.

In this equation we have negative and positive signs and decimals. The solved example will show all that is necessary without any further explanation.

$3x^3-4x^2+6x=82068.904448$

Let $x=(a+b)$, then $3x^3-4x^2+6x=3(a+b)^3-4(a+b)^2+6(a+b)=$
$3(a^3+3a^2b+3ab^2+b^3)-4(a^2+2ab+b^2)+6(a+b)=$
$3a^3-4a^2+6a+[(9a^2-8a+6)+9ab+3b^2-4b]b$

$$\begin{array}{l|l} & \;\; a \quad\; b \quad\;\; c \\ 82068.904448 & 20+5.8+4.76 \\ 22520 & \\ \hline 59548.904448 = \text{1st remainder.} & \\ 26492.776 & \\ \hline 33056.128448 = \text{2d remainder} & \\ 33056.128448 & \end{array}$$

Let $a=20$, then $3a^3-4a^2+6a=24000-1600+120=22520$

1st trial divisor $=9a^2-8a+6=$
$3600-160+6=3446$

1st complete divisor $=$ 1st *T. D.* $+9ab+3b^2-4b=$
$3446+1044+100.92-23.2=4567.72$

$\left.\begin{array}{l}\text{1st complete divisor} \times b \\ \text{the second term}\end{array}\right\} = 4567.72\times 5.8=26492.776$

Let $(a+b)=m$

2d trial divisor $=9m^2-8m+6=5990.76-206.4+6=5790.36$

2d complete divisor $=$ 2d *T. D.* $+9mc+3c^2-4c=$
$5790.36+1105.272+67.9728-19.04=6944.5648$

$\left.\begin{array}{l}\text{2d complete divisor} \times c \\ \text{the third term}\end{array}\right\} = 6944.5648\times 4.76=33056.128448$

Therefore $x=20+5.8+4.76=30.56$.

The student is requested to test and solve for the remaining roots. There are three.

Example.

Solve the equation $x^4+11x^3-42x^2-680x-1600=0$.

In this equation there are three permanences of signs and one variation, hence by Descarte's rule of signs there will be one positive root and three negative roots.

The roots of the equation are all real because A^2-2B is positive. $121-(-84)$ equals a positive quantity. *

Let $x=(a+b)$ as in former equations. Therefore $x^4+11x^3-42x^2-680x-1600=(a+b)^4+11(a+b)^3-42(a+b)^2-680(a+b)-1600=0$

$(a^4+4a^3b+6a^2b^2\times 4ab^3+b^4)+11(a^3+3a^2b+3ab^2+b^3)-42(a^2+2ab+b^2)-680(a+b)-1600=0.$

Arrange and separate the terms so that all the powers of a beginning with the highest power are consecutive: so that all the terms used in the trial divisor of the various powers, beginning with the highest power, are consecutive: so that the remaining terms arranged according to their respective positions in the general equation according to rule.

$(a^4+11a^3+42a^2-680a)$ (1.)

$+[(4a^3+33a^2-84a-680)$ (2.)

$+6a^2b+4ab^2+b^3+33ab+11b^2-42b]b$ (3.)

We will solve for the positive root first: since there are one positive and three negative roots: let $x=4=a$ the first term of the positive root, therefore $a^4+11a^3-42a^2-680a=-2432$ as shown. Subtract -2432 from 1600: this gives 4032 the first remainder (2) and (3) above.

Take the algebraic sum of the various trial divisors of the various powers of a as the complete trial divisor shown in (2) above: each term of which is equal to the $n(n-1)th$ power of a, multiplied by their respective coefficients: viz., $4a^3+33a^2-84a-680=256+528-336-680=-232$ the first trial divisor.

* Heavy error: two conditions are necessary to know if the roots of an equation of fourth degree are real: these conditions are determined by the rule of Sturm.

Divide 4032 by the trial divisor; this equals b or 4, the second term of the root. Complete the divisor by adding to the trial divisor, this value for b or 4 substituted in (3) above, this gives 1008; multiply this by 4, which gives 4032.

Subtract this from the first remainder leaving 0: therefore $4+4=8$ is a root of the equation or $x=8$. The factor is therefore $x-8$ or $x-8=0$.

The remaining roots are found as shown by the example and are of course similar to former cubic and quadratic equations.

The student should test each equation for positive and negative roots, and for real and imaginary roots, to become familiar with the use of the given theory.

Example.

$$x^4+11x^3-42x^2-680x=1600.$$

Let $a=4$, then $a^4+11a^3-42a^2-680a=$
$256+704-672-2720=-2432$

1st trial divisor $=4a^3+33a^2-84a-680=$
$256+528-336-680=-232$

1st complete divisor $=$ 1st $T.\ D.+6a^2b+4\ ab^2+b^3+$
$33ab+11b^2-42b=$
$-232+384+256+64+528+176-168=1008$

$\left.\begin{array}{l}\text{1st complete divisor } \times b \\ \text{the second term}\end{array}\right\}=1008\times 4=4032$

$$\begin{array}{rl}
 & \;\,a \;\;\; b \\
1600 & \big|\, 4+4 \\
-2432 & \\
\hline
4032 & \text{1st remainder.} \\
4032 & \\
\hline
\end{array}$$

$4+4=8$

$x=8$

$x-8=0$

$$\begin{array}{l|l}
x^4+11x^3-42x^2-680x-1600 & x-8 \\
\cline{2-2}
x^4-\;\;8x^3 & x^3+19x^2+110x+200 \\
\cline{1-1}
\qquad 19x^3-\;\;42x^2 & \\
\qquad 19x^3-152x^2 & \\
\cline{1-1}
\qquad\qquad 110x^2-680x & \\
\qquad\qquad 110x^2-880x & \\
\cline{1-1}
\qquad\qquad\qquad 200x-1600 & \\
\qquad\qquad\qquad 200x-1600 & \\
\cline{1-1}
\end{array}$$

$$\begin{array}{r|l} & a \quad b \\ -200 & -6-4 \\ -192 & \\ \hline -\ \ 8 & \\ -\ \ 8 & \\ \hline \end{array}$$

Let $a = -6$, then $a^3 + 19a^2 + 110a = -216 + 684 - 660 = -192$

1st trial divisor $= 3a^2 + 38a + 110 = 108 - 228 + 110 = -10$

1st complete divisor $=$ 1st *T. D.* $+ 3ab + b^2 + 19b =$

$-10 + 72 + 16 - 76 = 2$

1st complete divisor $\times\ b$ / the second term $\Big\} = 2 \times -4 = -8$

$(-6) + (-4) = -10$

$x = -10$

$x + 10 = 0$

$$\begin{array}{l|l} x^3 + 19x^2 + 110x + 200 & x + 10 \\ \cline{2-2} x^3 + 10x^2 & x^2 + 9x + 20 \\ \cline{1-1} \quad 9x^2 + 110x & \\ \quad 9x^2 + \ \ 90x & \\ \cline{1-1} \qquad\quad 20x + 200 & \\ \qquad\quad 20x + 200 & \\ \end{array}$$

$$x^2 + 9x = -20$$

$$\begin{array}{r|l} & a \quad b \\ -20 & -3-1 \\ -18 & \\ \hline -\ 2 & \\ -\ 2 & \\ \hline \end{array}$$

Let $a = -3$, then $a^2 + 9a = 9 - 27 = -18$

1st trial divisor $= 2a + 9 = -6 + 9 = 3$

1st complete divisor $=$ 1st *T. D.* $+ b = 3 - 1 = 2$

1st " " $\times\ b$ / the second term $\Big\} = 2 \times -1 = -2$

$(-3) + (-1) = -4$

$x = -4$

$x + 4 = 0$

$$\begin{array}{l|l} x^2 + 9x + 20 & x + 4 \\ \cline{2-2} x^2 + 4x & x + 5 \\ \cline{1-1} \quad 5x + 20 & \\ \quad 5x + 20 & \\ \end{array}$$

Example.

Solve the equation $2x^5+3x^4-220x^3+84x^2+4880x-9600=0$.

In the above example we have an equation of the fifth degree; hence five roots or factors are available. By Descarte's rule there are three positive and two negative roots. A^2-2B is a positive quantity hence all the roots are real.

Arrange and separate the terms so that all the powers of a beginning with the highest power are consecutive; so that all the terms used in the trial divisors of the various powers beginning with the highest power are consecutive; so that all the remaining terms are arranged according with their respective positions in the general equation.

Let $x=(a+b)$. Substitute this value in the given equation.

(1). $2(a+b)^5+3(a+b)^4-220(a+b)^3+84(a+b)^2-4880(a+b)=9600$.

(2). $2(a^5+5a^4b+10a^3b_2+10a^2b^3+5ab^4+b^5)+3(a^4+4a^3b+6a^2b^2+4ab^3+b^4)-220(a^3+3a^2b+3ab^2+b^3)+84(a^2+2ab+b^2)-4880(a+b)=9600$.

Arranged and separated according to rule, this equation assumes this form, viz.:

(3). $(2a^5+3a^4-220a^3+84a^2-4880a)=$ first term.
$+[(10a^4+12a^3-660a^2+168a-4880)=$ second term
$+(20a^3b+20a^2b^2+10ab^3+2b^4+18a^2b+12ab^2+3b^3-660ab-220b^2+84b)]b=$third term
$=9600=$absolute term.

The first term of (3) consists, evidently of the highest powers of a in each consecutive term of the expansion (2). The second term of (3) is the complete trial divisor, or the $n(n-1)^{th}$ powers of the consecutive terms of (1). The third term is the remainder of the above expanded equation, or the quantity added to form the complete divisor and the sum multiplied by the second term of the root b, as shown.

The solution is the same as in former problems.

In this example we have solved for one of the positive roots first which equals 4 as shown; and the factor is $x-4$.

Divide the first equation by this factor which gives $2x^4+11x^3-176x^2-620x+2400=0$.

Solve as before for the next root. This gives the negative root -6 and the second factor equals $(x+6)$.

Divide the equation $2x^4+11x^3-176x^2-620x+2400=0$ by the factor $(x+6)$, this gives the cubic equation or portion of our required solution. Proceed as before until all the desired roots and factors are obtained.

We find the factors to be $(x-4)(x+6)(x-8)(2x-5)(x+10)$.

$$\text{The roots are } \begin{cases} x=4 \\ x=-6 \\ x=8 \\ x=-10 \\ 2x=5 \end{cases}$$

For equations of higher degree proceed as in the example given, care being taken for the systematic arrangement of terms, trial divisors, etc.

$$2x^5+3x^4-220x^3+84x^2+4880x-9600=0.$$

Let $a=3$, then $2a^5+3a^4-220a^3+84a^2+4880a=$	$\begin{array}{r\|l} & a \quad b \\ 9600 & 3+1 \end{array}$
$486+243-5940+756+14640=10185$	10185
1st trial divisor $=10a^4+12a^3-660a^2+168a+4880=$	$-585=$ 1st remainder.
$810+324-5940+504+4880=578$	-585

1st complete divisor $=$ 1st $T.D.$ $+20a^3b+20a^2b^2+10ab^3+$
$2b^4+18a^2b+12ab^2+3b^3-660ab-220b^2+84b=$

$578+540+180+50+2+162+36+3-1980-220+84=-585$

$\left.\begin{array}{l}\text{1st complete divisor} \times b \\ \quad \text{the second term}\end{array}\right\} = -585\times 1=-585$

$$\begin{array}{l|l}
2x^5+3x^4-220x^3+84x^2+4880x-9600 & x-4 \\ \hline
2x^5-8x^4 & 2x^4+11x^3-176x^2-620x+2400 \\ \hline
\quad 11x^4-220x^3 & \\
\quad 11x^4-44x^3 & \\ \hline
\qquad -176x^3+84x^2 & \\
\qquad -176x^3+704x^2 & \\ \hline
\qquad\quad -620x^2+4880x & \\
\qquad\quad -620x^2+2480x & \\ \hline
\qquad\qquad 2400x-9600 & \\
\qquad\qquad 2400x-9600 & \\ \hline
\end{array}$$

$$2x^4+11x^3-176x^2-620x=-2400$$

Let $a=-4$, then $2a^4+11a^3-176a^2-620a=$
$512-704-2816+2480=-528$

1st trial divisor $=8a^3+33a^2-352a-620=$
$-512+528+1408-620=804$

1st complete divisor $=$ 1st *T. D.* $+12a^2b+8ab^2+2b^3+$
$33ab+11b^2-176b=$
$804-384-128-16+264+44+352=936$

$$\left.\begin{array}{l}\text{1st complete divisor}\times b\\ \quad\text{the second term}\end{array}\right\}=936\times-2=-1872$$

$$\begin{array}{r|l} & a\quad b\\ -2400 & -4-2\\ -\ \ 528 & \\ \hline -1872 & =\text{1st remainder.}\\ -1872 & \\ \hline \end{array}$$

$(-4)+(-2)=-6$
$x=-6$
$x+6=0$

$$\begin{array}{l|l}
2x^4+11x^3-176x^2-620x+2400 & x+6\\ \hline
2x^4+12x^3 & 2x^3-x^2-170x+400\\ \hline
\quad -x^3-176x^2 & \\
\quad -x^3-\ \ \ 6x^2 & \\ \hline
\qquad -170x^2-\ \ 620x & \\
\qquad -170x^2-1020x & \\ \hline
\qquad\qquad 400x+2400 & \\
\qquad\qquad 400x+2400 & \\ \hline
\end{array}$$

$$2x^3 - x^2 - 170x = -400$$

$$\begin{array}{r|l} & a \quad b \\ -400 & 2 + .5 \\ \hline -328 & \\ \hline -72 & = \text{1st remainder.} \\ -72 & \\ \hline \end{array}$$

Let $a = 2$, then $2a^3 - a^2 - 170a = 16 - 4 - 340 = -328$

1st trial divisor $= 6a^2 - 2a - 170 = 24 - 4 - 170 = -150$

1st complete divisor $=$ 1st $T.D. + 6ab + 2b^2 - b =$

$$-150 + 6 + .5 - .5 = -144$$

$\left.\begin{array}{l}\text{1st complete divisor } \times b \\ \quad \text{the second term}\end{array}\right\} = -144 \times .5 = -72$

$$2 + .5 = 2.5$$
$$x = 2.5$$
$$2x = 5$$
$$2x - 5 = 0$$

$$\begin{array}{l|l} 2x^3 - x^2 - 170x + 400 & 2x - 5 \\ \cline{2-2} 2x^3 - 5x^2 & x^2 + 2x - 80 \\ \cline{1-1} \quad 4x^2 - 170x & \\ \quad 4x^2 - 10x & \\ \cline{1-1} \qquad -160x + 400 & \\ \qquad -160x + 400 & \\ \cline{1-1} \end{array}$$

$$x^2 + 2x = 80.$$

$$\begin{array}{l|l} & a \quad b \\ 80 & 6+2 \\ 48 & \\ \hline 32 = \text{1st remainder} & \\ 32 & \\ \hline \end{array}$$

Let $a = 6$, then $a^2 + 2a = 36 + 12 = 48$

1st trial divisor $= 2a + 2 = 12 + 2 = 14$

1st complete divisor $=$ 1st $T.D. + b = 14 + 2 = 16$

1st " " $\times b$ the second term $\Big\} = 16 \times 2 = 32$

$6 + 2 = 8$

$x = 8$

$x - 8 = 0.$

$$\begin{array}{r|l} x^2 + 2x - 80 & x - 8 \\ \cline{2-2} x^2 - 8x & x + 10 \\ \hline 10x - 80 & \\ 10x - 80 & \\ \hline \end{array}$$

Therefore the roots $= \begin{cases} 4 \\ -6 \\ 2.5 \\ 8 \\ -10 \end{cases}$ and the factors are $(x-4)(x+6)(2x-5)(x-8)(x+10)$.

Example.

Find the roots of x in the equation $6x^3+12mx^2-9nx^2-18mnx+4kx^2+8mkx+6nkx-12mnk=0$.

The above equation is an equation of the third degree, or a cubic equation, where 6, $12m$, $9n$, $18mn$, etc., are coefficients of x corresponding to the coefficients A, B, C, D, etc., of the general equation (1). The exponent is three and the absolute term is $12mnk$; each respectfully corresponding to the exponent n and the absolute term K of the general equation (1).

The roots of an algebraic equation are functions of its coefficients; hence the roots of this equation are evidently m, n and k combined with the numerical terms of the given equation. *

In the above equation let $x=(a+b)$, where a and b are terms of the desired root. Substitute this value in the given equation and expand according to (1) of the rule.

(1) $6x^3+12mx^2-9nx^2-18mnx+4kx^2+8mkx+6nkx-12mnk=0$.

(2) $6x^3+(12m-9n+4k)x^2+(8mk-18mn-6nk)x-12mnk=0$.

(3) $6(a+b)^3+(12m-9n+4k)(a+b)^2+(8mk-18mn-6nk)(a+b)=12mnk$.

The expanded form equals

$$\left.\begin{array}{ll} 6(a^3+3a^2b+3ab^2+b^3) & =\text{1st term} \\ +(12m-9n+4k)(a^2+2ab+b^2) & =\text{2d term} \\ +(8mk-18mn-6nk)(a+b) & =\text{3d term} \end{array}\right\} =12mnk=(4)$$

Separating and arranging (4) according to the rule (4) equals

$$\left.\begin{array}{ll} (6a^3+a^2(12m-9n+4k)+a(8mk-18mn & \\ \quad -6nk) & =\text{1st term} \\ +[(18a^2+2a(12m-9n+4k)+(8mk- & \\ \quad 18mn-6nk)]b & =\text{2d term} \\ +[18ab+6b^2+b(12m-9n+4k)]b & =\text{3d term} \end{array}\right\} =12mnk=(5)$$

* Not true in ordinary cases, the linking between the roots and the coefficients may be by surd or imaginary expressions and all appearance of the coefficients disappear,

$6a^3+a^2(12m-9n+4k)+a(8mk-18mn-6nk)$ are the terms separated and arranged so that all the highest powers of a in each consecutive term of the expanded form (4) are consecutive. These terms correspond to the $Aa^n+Ba^{n-1}+Ca^{n-2}$, etc., of the general equation.

$18a^2+2a(12m-9n+4k)+(8mk-18mn-6nk)$ are each respectfully the n times the $(n-1)$ power of the descending powers of a and correspond to the $Ana^{n-1}+B(n-1)a^{n-2}+C(n-2)a^{n-3}$ of the general equation; that is $3a^2$ is the $(n-1)^{th}$ power of a, this times its coefficient 6 equals $18a^2$; $2a$ is the n $(n-1)^{th}$ power of a, this times its coefficient equals $2a$ $(12m-9n+4k)$; a^1 or 1 is the $n(n-1)^{th}$ power of a, this times its coefficient equals $(8mk-18mn-6nk)$; the sum of these is the complete trial divisor for the finding of the remaining terms of the desired root and each is respectfully the trial divisor for the simultaneous extraction of the cubic, quadratic and the first power root as indicated by the equation.

The remaining terms, $18ab+6b^2+b(12m-9n+4k)$, evidently must be added to the trial divisor to form the complete divisor after the second term b of the root has been found. This complete divisor multiplied by the term b completes the extraction of the required root if there be no remainder. If there be a remainder let the sum of $(a+b)=a^1$ and proceed as before.

By referring to the example given and solved we have let $a=-m$ and substituted it in the first term of (5) which equals the quantity shown. Subtracting this from the absolute term $12mnk$ equals the first remainder.

The trial divisor is equal to the second term of equation (5) with the value of $-m$ substituted for a, which gives the quantity shown and marked as first trial divisor.

Divide the first term of the first remainder or the remaining portion of the absolute term by the term of the trial divisor that contain the same factors; that is, follow the rule of algebraic division. The quotient is the second term of the root or b. ($6mnk$ divided by $-6nk=-m$, the second term of the root.)

Complete the divisor by adding to the trial divisor the third term of equation (5), where $b=-m$, and multiply the complete divisor by b or its value $-m$ as shown. This quantity subtracted from the first remainder leaves 0; hence the equation is solved for one of the roots which equals $-2m$ and the factor is $(x+2m)$.

Divide the given equation by this factor; this gives $6x^2-9nx+4kx-6nk=0$. In this equation proceed as before and extract the root n as shown. The factor is $(x-\frac{3}{2}n)$, or $(2x-3n)$.

Divide the equation $6x^2-9nx+4kx-6nk$ by $(2x-3n)$; this gives $3x+2k$ the third factor.

Therefore $6x^3+12mx^2-9nx^2+4kx^2-18mnx+8mkx+6nkx-12mnk$ equals $(x+2m)(2x-3n)(3x+2k)$.

$$\text{The roots are} \begin{cases} -2m \\ \frac{3}{2}n \\ -\frac{2}{3}k \end{cases}$$

$12mnk.$

$$6mnk+6m^3+9m^2n-4m^2k \quad \Big| \; -m-m$$

$6mnk-6m^3-9m^2n+4m^2k$ = 1st remainder

$$6mnk-6m^3-9m^2n+4m^2k$$

Let $a=-m$, then $6a^3+a^2(12m-9n+4k)+a(8mk-18mn-6nk)=$

$$6mnk-6m^3+12m^3-9m^2n+4m^2k-8m^2k+18m^2n=$$

$$6mnk+6m^3+9m^2n-4m^2k$$

1st trial divisor $=18a^2+2a(12m-9n+4k)+(8mk-18mn-6nk)=$

$$18m^2-24m^2+18mn-8mk+8mk-18mn-6nk=$$

$$-6nk-6m^2$$

1st complete divisor = 1st $T.\ D.+18ab+6b^2+b(12m-9n+4k)=$ $\qquad (-m)+(-m)=-2m$

$$-6nk-6m^2+18m^2+6m^2-12m^2+9mn-4mk= \qquad x=-2m$$

$$-6nk+6m^2+9mn-4mk \qquad x+2m=0$$

1st complete divisor $\times\, b$ the second term $\Big\}=(-6nk+6m^2+9mn-4mk)-m$

$$6x^3+12mx^2-9nx^2-18mnx+4kx^2+8mkx-6nkx-12mnk \;\Big|\; x+2m$$

$$6x^3+12mx^2 \qquad\qquad 6x^2-9nx+4kx-6nk$$

$$-9nx^2-18mnx$$

$$-9nx^2-18mnx$$

$$4kx^2+8mkx$$

$$4kx^2+8mkx$$

$$-6nkx-12mnk$$

$$-6nkx-12mnk$$

$$6x^2 - 9nx + 4kx = 6nk$$

Let $x = (a+b)$ then the equation $= 6(a+b)^2 + (4k - 9n)(a+b) = 6nk =$

$$6(a^2 + 2ab + b^2) + (4k - 9n)(a+b) = 6nk =$$

$$6a^2 + a(4k - 9n) + [(12a + \{4k - 9n\}) + 6b]b$$

$$\begin{array}{l|l} & a \quad b \\ & n + \frac{n}{2} \end{array}$$

$$\begin{array}{l} 6nk \\ \underline{4nk - 3n^2} \\ 2nk + 3n^2 = \text{1st remainder} \\ \underline{2nk + 3n^2} \end{array}$$

$$n + \frac{n}{2} = \frac{3}{2}n$$

$$x = \frac{3}{2}n$$

$$2x = 3n$$

$$2x - 3n = 0$$

Let $a = n$, then $6a^2 + a(4k - 9n) =$

$$6n^2 + 4kn - 9n^2 = 4kn - 3n^2$$

1st trial divisor . . . . . $= 12a + (4k - 9n) = 12n + 4k - 9n =$ $4k + 3n$

1st complete divisor $=$ 1st $T.\ D. + 6b = 4k + 3n + 3n =$ $4k + 6n$

1st " " $\times b$ the second term $\Big\} = (4k + 6n) + \frac{n}{2}$

$$\begin{array}{l|l} 6x^2 - 9nx + 4kx - 6nk & 2x - 3n \\ \underline{6x^2 - 9nx} & \overline{3x + 2k} \\ \quad\quad 4kx - 6nk & \\ \quad\quad \underline{4kx - 6nk} & \end{array}$$

Therefore the factors are $(x + 2m)(2x - 3n)(3x + 2k)$