GRAPHIC STATICS.

THE ELEMENTS

OF

GRAPHIC STATICS

By KARL VON OTT

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TRANSLATED FROM THE GERMAN

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TRANSLATOR'S PREFACE.

THE solution of problems of all kinds by purely graphic methods forms an important branch of study in the training of a Continental Engineer, and the publication of such considerable works as those of Reuleaux, Culmann, Bauschinger, and Levy, affords the best proof of the value attached to the subject.

In England, notwithstanding the valuable contributions to Graphic Statics made by Professor Clerk-Maxwell and the late Professor Rankine, the subject can hardly be said to have received the recognition it merits. It is true indeed that the power and facility conferred by certain isolated processes, such for instance as that of stress diagrams, are universally acknowledged; but these processes have for the most part been viewed as mere artifices for effecting special purposes, and not as applications of the principles of an important general method.

The present work, which has in Germany already gone through three editions, is for its size one of the most complete elementary treatises on the subject, while its essentially practical character and the extreme simplicity of the mathematics involved will, it is hoped, render it widely useful in an English form.

In carrying out the translation the Author's text has, as far as possible, been adhered to; but the peculiarities of German idiom cannot always be literally rendered, and in such cases a certain amount of freedom must necessarily be claimed.

It was thought advisable to omit entirely the first portion of the work treating of Graphic Arithmetic, which, though it certainly forms a useful and appropriate introduction to the study of Graphic procedure, has not been much used in practice.

Some few notes have been added where further explanation seemed desirable, or where, as in the case of the treatment of wind pressure on roofs, a divergence from English practice seemed to call for some remark. All such additional matter incorporated with the Author's text has been enclosed in brackets.

COOPERS HILL, *June*, 1876.

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GRAPHIC STATICS.

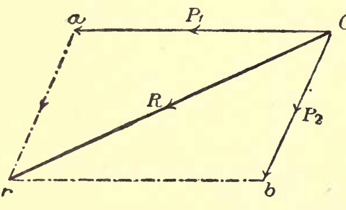
PART I.

COMPOSITION OF FORCES.

[In order that a line may be employed to represent a force, it is necessary, 1st. That its length should be proportional to the magnitude or intensity of the force. 2nd. That its position should correspond to the line of action of the force. 3rd. That an arrow should be attached showing the "sense" of the force, i.e. the direction along the line in which it acts.

In the following paragraphs the only theorem borrowed from Analytical Statics is that of the Parallelogram of Forces, which may be stated as follows.

Fig. 1.



If two forces P_1, P_2 whose directions and magnitudes are given by two lines aO, bO (Fig. 1) act at a point O, then the diagonal r O of the completed parallelogram a O b r gives the magnitude of the resultant R of those forces. Hence if starting from a point O two lines O a,

$a\,r$ are drawn parallel to and in the same direction as two forces P_1, P_2, then by joining r O a triangle $r\,a$ O is obtained of which the third side r O gives the magnitude and position of the resultant R of P_1, and P_2. The arrow indicating the direction of this resultant will point in the reverse direction round the triangle to the arrows indicating the direction of the forces. A force *maintaining equilibrium* with the two given forces will have the same magnitude as R, but its direction arrow will be that of R reversed.

Conversely, if lines parallel to the directions and proportional to the magnitudes of three forces acting at a point, form a triangle, the three forces are in equilibrium.

It should be observed that in such a triangle the three forces act in the same direction all round the triangle, and this is always the case. Hence if the direction of *one* of the forces forming the triangle is known, that of the others is known also.]

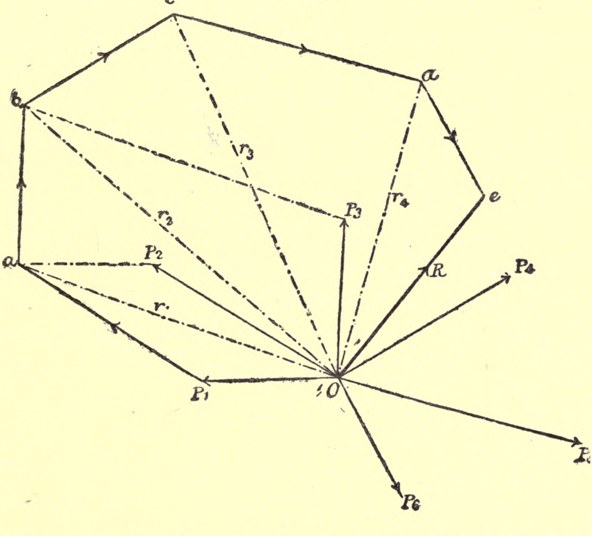
1. *Forces acting in the same Straight Line.*—If a number of forces whose magnitudes are expressed by lines representing them on any scale, act in the same straight line at a point, then (forces acting in one direction being reckoned positive and in the other direction negative) the algebraical sum of the lines gives the magnitude of the resultant and the sign (+ or −) of that sum indicates its direction or sense.

2. *Forces acting in any Directions at a Point.*—If any number of forces P_1, $P_2 \ldots P\,n$ act in any directions at a point, then the resultant of those forces can always be obtained graphically. Combine the forces P_1 and P_2 by means of the parallelogram of forces obtaining a resultant r_1, then combine r_1 and P_3 for

a new resultant r_2, and proceed in the same way till all the forces have been combined and the general resultant R obtained.

Thus in Fig. 2, P_1, P_2 .... P_6 are forces acting at a point O, the lengths P_1 O, P_2 O, ... etc., representing their respective magnitudes on any convenient scale. Starting from P_1 draw $P_1\,a$ equal and parallel to O P_2; then a O is the resultant r_1 of P_1 and P_2; draw $a\,b$ equal and parallel to P_3; then b O is the resultant r_2 of P_1, P_2, and P_3; proceeding in this way we finally obtain e O as the resultant R of P_1 .... P_6.

Fig. 2.



In practice the dotted lines a O, b O, etc., nee not be drawn, and we obtain therefore the following rule.

When any number of forces act in any direction at a *point, their resultant can be obtained by combining end to end the lines representing them in direction and magnitude; then the line closing the rectilinear polygon so obtained gives the resultant both in direction and magnitude.* The direction arrow of that resultant points in the *opposite* way round the polygon to the direction arrows of the forces.

The forces acting at O, Fig. 2, will be in equilibrium if a 7th force R equal in magnitude to their resultant but acting in the opposite direction to that indicated by the arrow is interposed. Hence,—

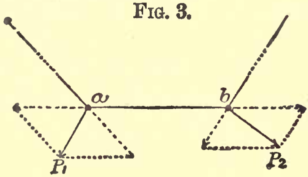
If any number of forces acting at a point are in equilibrium, lines drawn successively in the direction of the forces and proportional to their magnitudes must form a closed polygon.

This polygon is termed the Polygon of Forces, and the above theorem will hold *whatever is the order in which the forces are taken.*

3. *Forces acting on a Rigid Body.*—If forces P_1, P_2, P_3 .... act on a rigid body, the latter must be supposed to be replaced by a system of rigid rectilinear rods which intersect the directions of the forces and form a polygon. The several sides of this polygon must be capable of resisting the external forces (whether tensile or compressive) which are brought to bear upon them. Such a polygon replacing a rigid body is termed a *Funicular Polygon* if its sides are in tension, and a *Line of Resistance* or *Linear Arch* if its sides are compressed. In general it may be called a *Polygonal* or *Jointed Frame.* Its angular points are called *Joints* or *Nodes*, and the forces developed in the sides of the polygon by the exterior forces are termed *Interior Forces.*

In order to distinguish whether the force in any one of the sides of the polygon is tensile or compressive, resolve the exterior forces acting at both its end points in directions coinciding with that side and the adjacent sides. Then insert arrows showing the directions of the components of the resolved exterior forces; these arrows in Fig. 3 point outwards, and there evidently arises in the polygon sides a tensile stress; if however, the arrows point inwards, as in Fig. 4, the polygon side will be in compression.

FIG. 3.



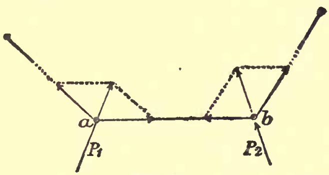


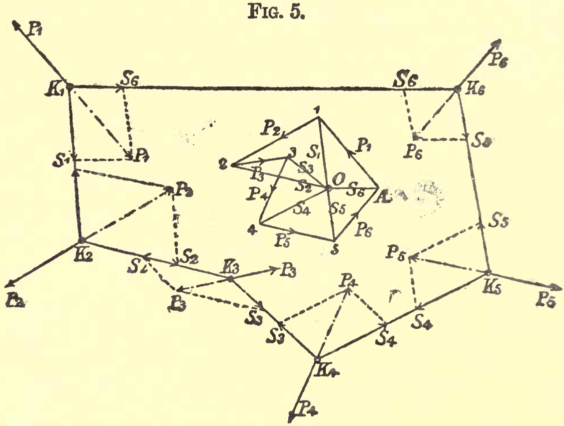
FIG. 4.

Since the interior forces act in directions opposite to the resolved components of the exterior forces, place at the ends *a* and *b* of the polygon side *a b* arrows in the opposite direction to those of the resolved components; thus we obtain the following arrow combinations:—

Interior forces	Compression	⟵	⟶
	Tension ..	⟶	⟵
Exterior forces	Compression	⟶	⟵
	Tension ..	⟵	⟶

4. *Equilibrium of the Forces acting on the Jointed Frame.*—Let the polygon $K_1, K_2, K_3 \ldots$, Fig. 5, be in equilibrium under the action of the exterior forces $P_1, P_2, P_3 \ldots$, then evidently the exterior force acting at any joint must be in equilibrium with the two interior forces or stresses acting in the two sides of the frame which meet at that joint.

FIG. 5.



Let $S_1, S_2, S_3 \ldots$ be the stresses in the several sides of the polygon, then as a necessary condition of equilibrium of the joint K_1, the three forces P_1, S_1 and S_6 acting at that joint must combine to form a triangle O 1 6. Similarly the forces P_2, S_1 and S_2 which are in equilibrium at K_2 must form a triangle O 1 2, which has a side O 1 ($= S_1$) common to the first triangle O 1 6. Similarly as a condition of equilibrium of the joint K_3, the three forces P_3, S_3 and S_2 must form a triangle O 2 3 having a side O 2 common to the triangle O 1 2, and so on. Hence, as a condition of equilibrium of the whole jointed frame, the successive triangles of forces must have one side in common.

Hence the theorem,—

If the forces $P_1, P_2, P_3 \ldots$ *acting on a jointed frame are in equilibrium, it must be possible to form them into a closed polygon* 1, 2, 3 . . ., *and the lines drawn from the angles* 1, 2, 3 . . . *of this polygon parallel to the sides* $S_1, S_2\ S_3 \ldots$ *of the jointed frame must meet in the same point* O, *termed the pole. Then, the lines* O1, O2, O3 . . . *radiating from this pole determine perfectly the magnitudes of the stresses* $S_1, S_2, S_3 \ldots$ *in the sides of the polygonal frame.*

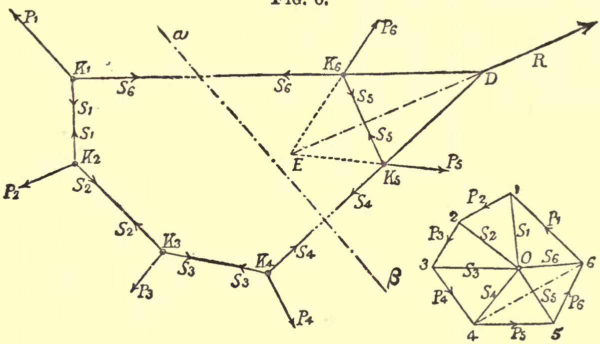
From the above theorem follows the important corollary that if the given exterior forces acting on a jointed frame are in equilibrium, then the assumption of any two consecutive sides of the frame determines all its other sides, and if the form of the frame and the directions of the exterior forces are given, then the magnitudes of all the exterior forces are determined if one of them is given or assumed.

It is also clear that if the forces acting on a polygonal frame are not in equilibrium, the closing side of the polygon of those forces determines their resultant in direction and magnitude.

[The polygon $K_1, K_2, K_3 \ldots$ is termed the Funicular Polygon of the Forces $P_1, P_2, P_3 \ldots$ with respect to the pole O. By taking different positions of this pole different funicular polygons are obtained. The form of the polygon $K_1, K_2, K_3 \ldots$ is evidently that which a flexible string suspended from two fixed points, K_1, K_6 and strained by the forces P_2, P_3, P_4, and P_5 would assume, and hence the term "funicular," which has, however, obtained a purely geometrical meaning. The general conditions of the equilibrium of a rigid body may be summed as follows:—1st. The polygon formed by the exterior forces must close. 2nd. Any funicular polygon of those forces with respect to any pole must also close.]

In the case of Fig. 5 all the sides of the frame are evidently in tension.

Fig. 6.



Suppose in a polygonal frame, Fig. 6, that two of its sides K_1 K_6 and K_4 K_5 are cut across, then evidently, in order that equilibrium may be maintained, forces having the same magnitude and direction as the stresses S_6 and S_4 must be applied at the points of section. Hence the resultant R of the stresses S_6 and S_4 holds in equilibrium all the exterior forces acting on the frame on the right or on the left of the section plane α β. The direction and magnitude of R is given by the diagonal 46 of the polygon of forces, since from what has been said, R must form a closed polygon both with the forces P_1, P_2, P_3, P_4, and also with P_5, P_6. Moreover, the resultant R of S_6 and S_4 must evidently pass through D, the intersection of the cut sides produced. R therefore acts on the one hand as the resultant of the forces

P_1, P_2, P_3 and P_4, and on the other hand as the resultant of P_5 and P_6.

Hence generally,

The resultant R *of all the exterior forces acting between any two sides of the funicular polygon passes through the intersection* D *of those sides produced, and the direction and magnitude of* R *is determined by the polygon of forces.*

This theorem plays an important part in the whole subject of Graphic Statics, and enables forces to be resolved and composed by the aid of the funicular polygon and polygon of forces. Thus, suppose the force R is to be resolved into two other forces P_5, P_6 having given directions. Produce R backwards in the funicular polygon, and from any point E on it draw $E K_5$, $E K_6$ parallel to the sides 45, 56 of the polygon of forces, then from K_5 draw P_5 equal and parallel to K_6, and from K_6 draw P_6 equal and parallel to 56. Then evidently the forces P_5 and P_6 replace the force R.

5. *Parallel Exterior Forces.*—If the exterior forces acting on the funicular polygon are parallel, then, in order that equilibrium may obtain, some of them must act in opposite directions, and the sum of the forces acting in one direction must be equal to that of the forces acting in the other direction. The polygon of forces will in this case be a straight line. Moreover, at any node of the funicular polygon the components of the stresses perpendicular to the directions of the exterior forces must be all equal. For, since the latter can only counterbalance those components which are parallel to themselves, therefore as a condition of equilibrium of the node, the components of the stresses at right angles to the direction of the exterior forces must be in equilibrium, which will be the case only when those components are equal in magnitude and opposite in sense.

Hence we have the following theorem,

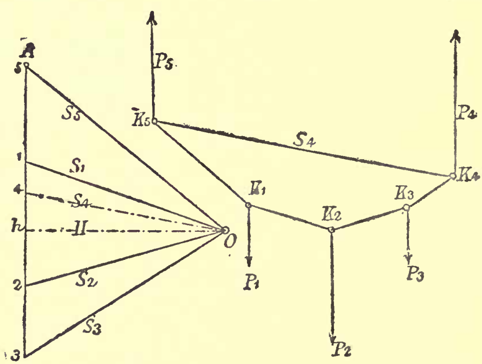
If the exterior forces acting at the angles of a funicular polygon are parallel, then the components of the stresses at right angles to the direction of the forces are equal in magnitude.

If the exterior forces are vertical, the constant horizontal component of the stresses is called the *horizontal thrust,* or *tension.*

In Fig. 7 the funicular and force polygons which constitute the conditions of equilibrium of the parallel forces $P_1, P_2, \ldots . P_5$ are shown. The mutual relations between these two polygons are the same as in those of Fig. 5, so that this figure may be considered as illustrating merely a special case of the preceding paragraph.

The constant horizontal thrust (H) is evidently given by the line O *h* drawn from O perpendicular to A 3. This line is termed the "polar distance."

FIG. 7.

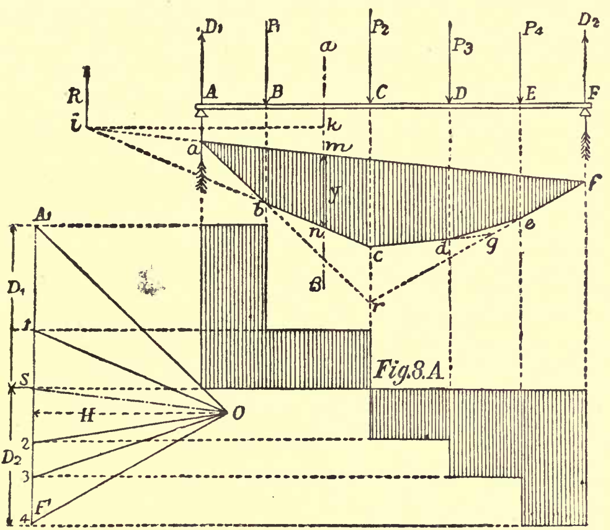


We will now proceed to the application of the results thus far obtained, applying them first to the case of parallel forces, which is one of the most common occurrence in practice.

EFFECT OF PARALLEL FORCES ACTING ON A SIMPLE BEAM.

6. *Determination of the Transverse Forces.*—Suppose the beam A F, Fig. 8, resting on two supports, to be loaded with weights P_1, P_2, P_3, P_4, we shall first determine D_1 and D_2, the pressures on the supports, and then the vertical or transverse stresses at any cross section of the beam.

FIG. 8.



Set the given forces P_1 P_2.... off in succession along a line A′ F′. The line A′ F′ is thus the polygon of the given forces, and F′ A′ its closing line, is their resultant. Take any point O as pole and draw the radii O A′, O 1, O 2, O 3, O 4. Then describe the funicular polygon *a*, *b*....*f* by drawing *a b* parallel to O A′, terminating in P_1 produced; *b c* parallel to O_1, terminating in the prolongation of P_2, and finally ef parallel to $F'O$ and terminating in the prolongation downwards of D_2.

The funicular polygon is now closed by the line fa, and a line OS is drawn through the pole O parallel to fa. Then, as a condition of equilibrium,

$$D_1 = A'S \text{ and } D_2 = SF'$$

Let the transverse or shearing forces in the several cross sections of the segments A B, B C, C D .... etc. be designated by V_1, V_2, V_3, etc. respectively.

Then,

$$V_1 = D_1 = SA'$$
$$V_2 = D_1 - P_1 = A'S - A'1 = S1$$
$$V_3 = D_1 - P_1 - P_2 = A'S - A'1 - 12 = S2$$

or *the shearing forces are equal to the distances of the various points of the polygon of forces from* S.

Accordingly in Fig. 8 A the shearing forces have been taken from the polygon of forces and used as ordinates of the segments of A F to which they correspond. [Thus the hatched figure is obtained, which is termed the "shearing force diagram," and the vertical ordinates of this diagram give the shearing force at any section of the beam A F.]

From the funicular polygon the resultant of two or more of the forces can be obtained. Thus g, the intersection of cd and fe produced, gives a point on the line of action of the resultant of P_3 and P_4. Further, r, the intersection of ab and fe produced, gives a point on the resultant of all the forces acting between a and f, i. e. of the loads P_1, P_2, P_3, P_4.

7. *Determination of the Bending Moments.*—Since the dimensions proper to the various cross sections of the beam depend more particularly upon the statical mo-

ments of the exterior forces, the determination of these moments is of the greatest importance in practice.

By statical or bending moment at any section $\alpha\beta$ of the beam A F, Fig. 8, is understood the product of the resultant R of all the forces acting on one or other side of the section into the perpendicular distance l of the line of action of R from $\alpha\beta$. In the case of the section $\alpha\beta$,

$$R = D_1 - P_1 = S1$$

and the point of application of R is i, the point in which those sides of the funicular polygon which are cut by $\alpha\beta$ meet.

Drop the perpendicular ik from i on $\alpha\beta$, this perpendicular is then equal to l, and the bending moment at the section $\alpha\beta$ is

$$M = R \, . \, l = S1 \, . \, ik$$

This product can readily be obtained graphically. Draw the constant horizontal thrust H, then the triangle O S 1 is similar to the triangle $i\,m\,n$, and hence

$$\frac{S1}{H} = \frac{mn}{l}$$

or if $mn = y$ and for S 1 its value R is substituted, we have

$$\frac{R}{H} = \frac{y}{l}$$

$$\text{and } M = R \, . \, l = H \, . \, y$$

The bending moment M *for any cross section is therefore directly proportional to the ordinate y of the funicular polygon at that cross section.*

If the constant horizontal thrust or "polar distance" H is taken as the unit of force, then

$$M = y$$

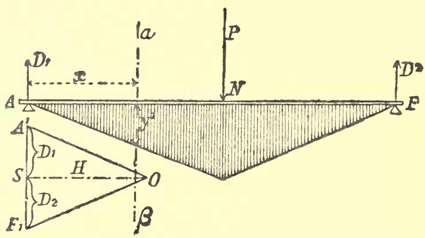
In this case the bending moments are directly given by the vertical ordinates of the funicular polygon.

It is evident from the figure that the maximum bending moment occurs always at one of the sections through which an exterior force acts, also that the above construction applies to *any* parallel forces whether vertical or not.

TRAVELLING LOAD.

8. *Effect of a Travelling Load on the Shearing Forces and Bending Moments.*—Since the effects of parallel forces are simply additive, any force P additional to the four forces P_1, P_2, P_3, P_4 (Fig. 8) entering within the limits of A F can be investigated separately in respect of its action on any particular cross section $\alpha\beta$ and the results obtained added to those previously found.

FIG. 9.



In Fig. 9 a construction similar to that of Fig. 8 is made for the load P acting on A F. Then the reactions of the supports at A and F (D_1 and D_2, Fig. 8) will be increased by $D_1 = A'\,S$ and $D_2 = F'\,S$ (Fig. 9), while the moment-ordinate y at the cross section $\alpha\beta$ will be increased by y' (Fig. 9).

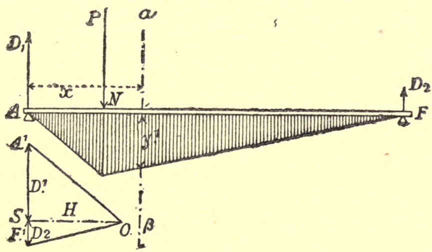
In order to deal with the effect of a single travelling load P on the shearing force V at the cross section $\alpha\beta$

(Fig. 9), distant x from A, let us investigate the two following cases.

1st. Suppose P to lie to the *right* of $\alpha\beta$, then the shearing force $V = D_1 = P.\frac{NF}{AF}$.

If we consider forces acting *upwards* as positive, then in the present case V is positive and will evidently be greater the greater N F is, that is the nearer the load P approaches the section $\alpha\beta$.

Fig. 10.



2nd. Suppose P to lie to the *left* of $\alpha\beta$, then the shearing force V at $\alpha\beta$ is (Fig. 10),

$$V = D_1 - P$$

but $$D_1 = P.\frac{NF}{AF};$$

therefore

$$V = P\left(\frac{NF}{AF} - 1\right) = -P\left(\frac{AP - FN}{AF}\right)$$

$$\text{or } V = -P.\frac{AN}{AF}.$$

In this case the shearing force is negative and will be numerically greater the nearer P approaches to the section $\alpha\beta$.

Hence generally,

Every single travelling load exerts a positive or negative shearing force according as it lies to the right or left of any particular section, and this shearing force increases in value as the load approaches the section.

Suppose the beam acted upon by a system of travelling loads, as is the case of a railway bridge when a train is passing over it. Then the pressure on every wheel axle to the right of any section $\alpha \beta$ will exert a positive shearing force, and that on every axle to the left of the section a negative shearing force. If therefore a positive shearing force only is brought to bear on a section $\alpha \beta$ by the train, the latter must evidently come on the bridge from the right abutment F and move up to the section $\alpha \beta$. If on the other hand the train comes from the left abutment and does not pass $\alpha \beta$ then a negative shearing force only is brought to bear on $\alpha \beta$ by the load on any wheel axle.

Hence generally,—

The greatest numerical value of the shearing force at any section is reached when a train coming from the further abutment arrives at that section, so that the leading locomotive axle is vertically over that section.

From Figs. 9 and 10 it appears that every load applied to the beam right or left of the section $\alpha \beta$ increases the ordinate y of the funicular polygon, and this increment is greater the nearer the load approaches to that section. Since the moment M at any section varies as the corresponding ordinate y of the funicular polygon it follows that—

The moment M *of the exterior forces* P_1, P_2, P_3, *etc., acting at any section,* $\alpha \beta$ (*Fig.* 8) *is increased by every force* P *interposed between the supports* A, F, *and this* *increment is greater the nearer* P *approaches to that section.*

Hence in bridges the moment of the exterior forces is a maximum at any section if the whole bridge is fully loaded and the greatest sinlge load is concentrated as near as possible to that section.

Since further, the greatest ordinate of a funicular polygon must always pass through one of its angles, the moment at any section must be a maximum when one of the greatest loads is at that section.

It can be ascertained in any particular case by means of the funicular polygon which load must be at any section so as to give rise to the maximum bending moment at that section.

9. *Example.*—The following example will serve to explain the method of operation.

Fig. 11 shows a bridge of 40′ span supporting an express engine and tender. The weights on the leading, driving, and trailing axles of the former are 9, 15, and 7 tons respectively; those on the three tender wheels are each 7 tons. The axle intervals of the engine are 8′ 0″, and of the tender 5′ 6″. The interval between the trailing axle of the locomotive and the leading axle of the tender is 8′.

The maxima shearing forces and bending moments due to the passing of the engine and tender will now be determined.

1st. *Determination of the Maxima Shearing Forces.*—Having selected a suitable scale for the weights, draw first the polygon of forces or load line A′, I .... B corresponding to the six given loads, and construct the funicular polygon I, II ... VI, relative to a pole O taken at a distance from A′ B′ equal to any convenient number on the scale of weights, say 30 tons.

FIG. 11.

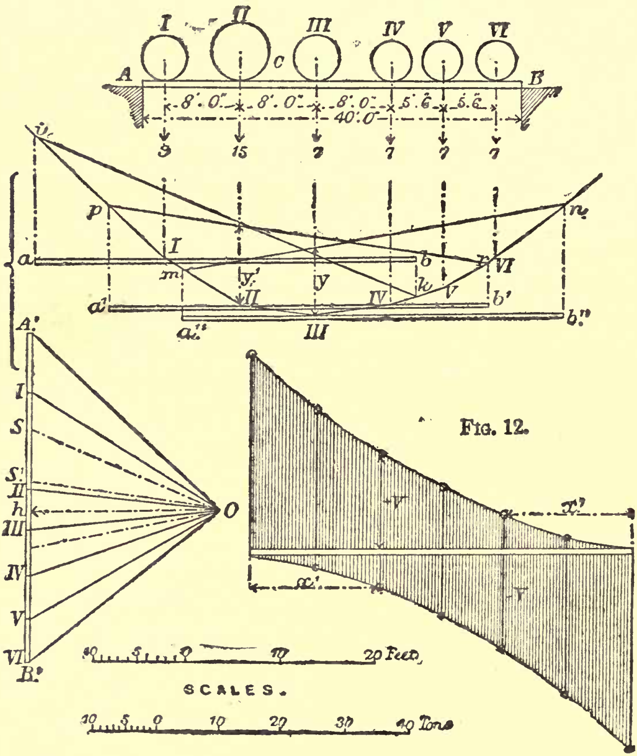


FIG. 12.

Now the maximum shearing force at a section C, whose distance from A is x, will be exerted when the train coming on the bridge at B (the farther abutment) arrives at C so that the leading axle I is vertically over C.

In order to obtain the shearing force at C when the load is in the above position, draw a horizontal line I *a* from the angle I of the funicular polygon, make I *a* = C A, and produce *a* I to *b* making *a b* equal to A B the span. Draw the vertical lines *a i*, *k b* cutting the extreme sides of the funicular polygon (produced if necessary) in *i* and *k*. Join *i k*, then *i k* is the closing line of the funicular polygon, and if O S is drawn in the polygon of forces parallel to this closing line *i k*, then according to para. 6 the length A′ S represents the magnitude of the shearing force V at the section C.

Now it is just possible that if the load II were largely in excess of I, the maximum shearing stress might arise when II was vertically over C. The diagram enables this to be tested instantly. Repeat the above construction for the point II of the funicular polygon, thus obtaining *p r* as the new closing line. Draw O S in the polygon of forces parallel to *p r*, then the new shearing force at C is A′ S′ − A′ I or S′ I, which is less than A′ S′.

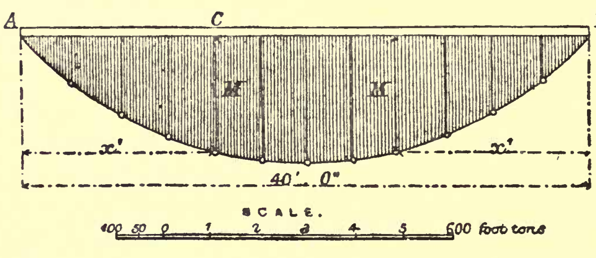
By proceeding in the way above described the shearing force at *any* section with *any* positions of the given loads can be obtained.

Since for every *positive* shearing force exerted during the passage of the load from the right to left of the bridge, there is a corresponding, numerically equal, *negative* shearing force exerted during its passage from left to right, at a point having the same distance from the centre of the bridge, we obtain a diagram of the form shown in Fig. 12, the ordinates of which give the maximum shearing forces at every section of the bridge.

2nd. *Determination of the Maxima Bending Moments.*—Since, from what has been said (para. 8) the maximum bending moment at any section is exerted when the bridge is fully loaded and one of the greatest loads is vertically over that section, hence the maximum bending moment at any section C at a distance x feet from A (Fig. 13) will be exerted when the train comes on the bridge at B and either the load III or II is vertically over C.

Suppose first that the load III is at C. From the angle III of the funicular polygon (Fig. 11) draw III a'' equal to x and produce a'' III to b'' making $a''b''$ equal to A B the span. Draw the vertical lines $a''m$, $b''n$ cutting the sides I II and V VI produced in m and n respectively. Join mn, then mn is the closing line of the funicular polygon when the train is in such a position that the load III is at C and (para. 6) the ordinate y multiplied by the polar distance, or constant horizontal tension H gives the bending moment at C.

FIG. 13.



It must now be ascertained whether the bending moment is not greater when II is at C.

Make II a' equal to x, and proceeding exactly as before we obtain pr as the new closing line and y' as the new ordinate.

Since y' is greater than y, the maximum bending moment at the section C is exerted when II is at C and is equal to $y' \times$ H, or $y' \times 30$ foot tons.

By treating a sufficient number of sections after the method above described we obtain a sufficient number of ordinates to enable the curve of maximum bending moment (Fig. 13) to be drawn. Since the curve is symmetrical about the centre line it will only be necessary to extend the construction to half of the bridge.

If now a scale is drawn one-thirtieth of the linear scale, then the ordinates of the curve read off on this scale give the maximum bending moment at any section of the bridge in foot tons.

[It will be seen that the principle of the above method consists in supposing the beam A B to be moved to the right or left, the loads remaining stationary. Thus the funicular polygon having been made once for all for the given weights at the given intervals, an alteration of the position of the beam relative to that of the loads merely affects the closing line of the polygon, and on this closing line both the shearing forces and bending moments depend.

It is clear that in the case above investigated the maxima shearing forces and moments obtained are those due to the passing of the given engine and tender only, and that different results might ensue if a second engine and tender were coupled to the first. In order to investigate the shearing forces and moments arising in the latter case, it would be necessary to draw the funicular polygon corresponding to the twelve axle pressures of the two engines and tenders and then to proceed to move the beam A B to the right or left as before. In fact the funicular polygon should be drawn in the first instance so as to correspond to a length of the heaviest portion of the heaviest train equal to twice the span of the bridge.

The curve of maxima bending moments (Fig. 13) has cusps which would be apparent if the scale to which it is drawn were larger.]

10. *Approximate Method of determining the Maxima Moments.*—Although the mode of procedure above described entails very little labour, the result can be obtained in a quicker way by the use of an approximate method published by Professor Dr. E. Winkler in the Austrian 'Ingenieur-und Architeckten-Verein' for 1870, part II., page 33, where it is stated that

In order that the moment at any section may be a maximum, the train must be in such a position that the loads on both sides of that section have nearly the same ratio to each other as the lengths into which the section divides the bridge; or, that the loads per unit of length on both sides of the section are nearly equal.

In conclusion, it remains to be said that in long railway bridges the greatest travelling load is usually taken to be a train of two or three of the heaviest locomotives fully equipped followed by such a number of the heaviest goods waggons loaded to their maximum that the train is of length sufficient to cover the whole bridge.

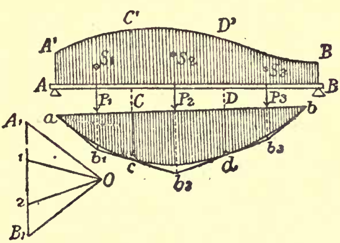
[In England it is usual to take as the heaviest travelling load for railway bridges, *a train of locomotives* of the heaviest class, fully equipped, sufficiently long to cover the whole bridge. For bridges of large span, however, a uniformly distributed, arbitrarily chosen load of from 1 to $1\frac{1}{4}$ tons per foot run for each line of rail is usually taken in place of the concentrated loads.]

Stationary Loads.

11. *Effect of a stationary Load having any fixed Distribution.*—A load distributed over the whole length of a beam can evidently be supposed to be split up into a number of single loads, so near to each other that the funicular polygon becomes a curve which follows the same laws as the polygon.

Suppose the partial load over each unit of length of the beam A B, Fig. 14, to be set up as an ordinate. Thus the figure A A′ C′ D′ B′ B is obtained. This figure is called the "loading area" of the beam, and evidently represents the load distribution.

Fig. 14.



The funicular curve corresponding to this load distribution must now be drawn.

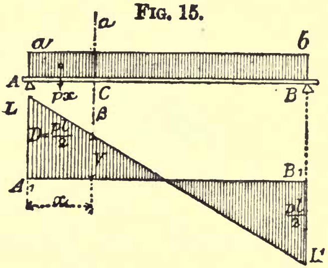
Suppose the loading area cut up into strips A C′, C D′ D B′, and that in place of the distributed load the concentrated loads P_1, P_2, P_3, acting at the centres of gravity S_1 S_2 S_3 of these strips, are substituted. Set off the weights P_1, P_2, P_3 on the load line A_1 B_1 and draw the funicular polygon $a\ b_1\ b_2\ b_3\ b$ relative to any pole O. The angles $b_1\ b_2\ b_3$ of the funicular polygon are vertically under S_1 S_2 S_3 the centres of gravity of the strips into which the loading area has been cut.

Now if the size of the strips is supposed to diminish indefinitely, the number of sides of the funicular polygon increases indefinitely, and this polygon becomes the funicular curve *a c d b*, to which curve the sides of the original polygon $a b_1$, $b_1 b_2$, $b_2 b_3$, $b_3 b$, are tangents at the points *a, c, d, b* vertically under the bounding lines A′ A, C′ C, D′ D, B′ B of the strips into which the loading area was originally divided.

Hence the loading curve having been first drawn, any required number of tangents to the corresponding funicular curve can be obtained as well as their points of contact with the curve. The curve can therefore be drawn, and by its means the bending moments and shearing forces at any cross sections of the beam can be determined as in paras. 6 and 7.

12. *Uniformly distributed Dead Load.*—If P is the whole load uniformly distributed over a beam A B of length *l*, then the load per unit of area is

FIG. 15.



$$p = \frac{P}{l}.$$

Set up *p* over A B as a constant ordinate, thus the rectangle A *a* B *b* (Fig. 15) is obtained as the loading area of the beam A B.

Since the whole load is uniformly distributed, the reactions of the two supports A and B are evidently equal, or

$$D = \frac{P}{2} = \frac{p.l}{2}.$$

The shearing force V at a section C for which A C $= x$ is given by the equation

$$V = D - p.x = \frac{p}{2}(l - 2x) \quad . . . (a)$$

and for

$$x = \frac{l}{2};\ V = 0.$$

Also V is a maximum when $x = 0$, for then

$$V = + \frac{p.l}{2}.$$

When

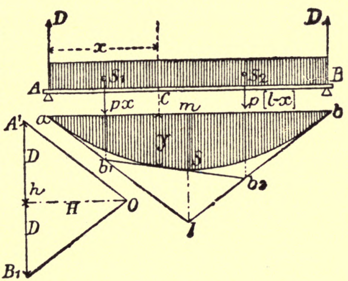
$$x = l\ V = - \frac{p.l}{2}.$$

Since by equation (a) V decreases as x increases and becomes zero when $x = \frac{l}{2}$; hence the shearing force diagram will be bounded by a straight line L L′ (Fig. 15) cutting the axis $A_1 B_1$ at its centre, and will have as ordinates

$$A_1 L = - B_1 L' = \frac{p.l}{2}$$

at the two points of support.

Fig. 16.



The bending moment at the section C (Fig. 16) is

$$M = D.x - \frac{p.x^2}{2} = \frac{p.x}{2}(l - x) . . . (\beta)$$

When $x = 0$, or $x = l$; $M = 0$; and M is a maximum when $x = \frac{l}{2}$ or

$$M = \frac{p.l^2}{8} \quad . . . . (\gamma)$$

Equation (β) shows that the curve of bending moments obtained for a uniformly distributed load is a parabola $a\ s\ b$ whose vertex s is vertically under the centre of $a\ b$ and at a distance from $a\ b$, $m\ s = \frac{p\,.\,l^2}{8}$.

This parabolic funicular curve can readily be drawn by means of its tangents, then the bending moments at the various sections are given by the ordinates of the curve.

In the polygon of forces make $A'\,B' = p\,.\,l = P$ and $O\,A' = O\,B'$. Then a I, I b parallel to O A′ and O B′ respectively are tangents to the funicular curve at a and b. Draw O h perpendicular to A′ B′ which it will bisect.

Then the triangle a I m is similar to the triangle O A′ h; hence

$$\frac{\mathrm{I}\,m}{A'\,h} = \frac{a\,m}{O\,h}$$

or

$$\frac{\mathrm{I}\,m}{\frac{p\,.\,l}{2}} = \frac{\frac{l}{2}}{H}$$

whence

$$\mathrm{I}\,m = \frac{p\,.\,l^2}{4\,H}$$

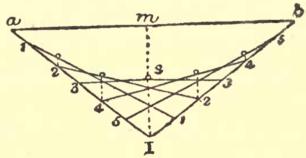
If the pole O is so taken that O h = H the unit on the scale of forces, then

$$\mathrm{I}\,m = \frac{p\,.\,l^2}{4}$$

Hence the vertex s of the parabola bisects I m.

In order to draw tangents at any point to the funicular curve under any section C, it must be remembered that by the preceding paragraph the intersections b_1 and b_2 of the required tangents with *a* I and *b* I must lie vertically under the centres of gravity of the loads on the segments A C and B C of the beam A B. In the present instance, since the load is uniformly distributed, these centres of gravity must lie at the middle points of those segments. We have therefore the following simple construction for obtaining the tangents. Divide *a* I, *b* I (Fig. 17) into an equal number of equal parts, and join the points of division as shown. Then the lines so obtained are tangents to the funicular curve, and moreover the points of contact of successive tangents bisect the distance between the points in which the tangents cut those adjacent to them on either side.

Fig. 17.



[In practice, if the scales are so arranged that the ordinate *m s* (Fig. 17) of the vertex is not greater than one-eighth of *a b*, the span, then a circle passing through *a*, *s*, *b* will sufficiently approximate to the required parabola.]

The polar distance or horizontal thrust H having been made equal to the unit on the scale of forces, then the vertical ordinates of the funicular curve give the bending moments at the sections corresponding to them. Thus if $H = 1$, the ordinate y (Fig. 16) is equal to the bending moment M at the section C. But if H is not unity, then $M = H \cdot y$.

13. *Reduction of concentrated Loads to a uniform Loading.*—Since by the preceding paragraph the determination of the bending moments at any section of a

uniformly loaded beam is extremely simple, it is not unusual in practice to reduce the concentrated loads to a uniform loading giving rise to *the same maximum bending moment.*

Suppose that by means of para. 9 the maximum moment M_m for the centre of the beam has been obtained, then a uniformly distributed load which would cause the same maximum bending moment at the centre of the beam is calculated. By the preceding paragraph the maximum bending moment due to a uniform load distribution is expressed by $\frac{p.l^2}{8}$, where p is the load per unit of length and l the clear span.

Hence putting

$$M_m = \frac{p.l^2}{8}$$

we obtain

$$p = \frac{8\,M_m}{l^2} \quad \ldots\ldots\ldots \quad (a)$$

as the required uniformly distributed load per unit of length.

For any section of the beam other than that at the centre however, the moment obtained on the hypothesis of a uniform distribution does *not* agree with that to which concentrated loads would give rise at that section.

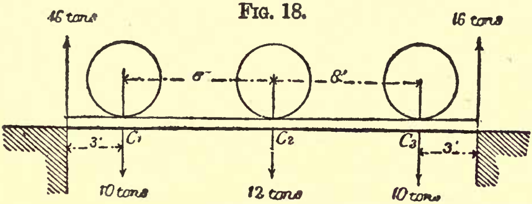
Still greater will be the error arising in the values of the shearing forces obtained on this hypothesis.

By proceeding on the hypothesis of a uniformly distributed load furnished by the above value for p, we introduce therefore a more or less considerable error in the determination of V and M. In fact, the values of M and V obtained on the supposition of this imaginary loading differ the more from their true values the greater the difference between the concentrated loads and the shorter the beam is.

14. *Example.*—The following simple example will serve to make the above clear.

Fig. 18 shows a bridge beam of 22 feet span carrying locomotive whose wheel base is 8 feet. The weights on the leading and trailing axles are taken as 10 tons each, that on the driving axle as 12 tons. The driving axle is over the centre of the beam.

FIG. 18.



The reaction of each of the supports will be

$$\frac{10 + 12 + 10}{2} = 16 \text{ tons.}$$

The bending moment at the centre C_2 of the beam will be a maximum.

Hence

$$M_m = D \times A C_2 - 10 \times 8 = 16 \times 11 - 10 \times 8 = 96 \text{ foot tons.}$$

And by equation (*a*) of the preceding paragraph,

$$p = \frac{8 \times 96}{22^2} = 1{\cdot}256 \text{ tons per foot run.}$$

We will now calculate the moment M at a section C_1 due to a uniform load distribution p.

From Fig. 19,

$$M = 3 \times D - 3p \times 1{\cdot}5 = 35{\cdot}796 \text{ foot tons.}$$

But for the moment M′ at C_1 due to the load distribution indicated in Fig. 18, we have

$$M' = 3D = 16 \times 3 = 48 \text{ foot tons,}$$

a result considerably in excess of that obtained on the hypothesis of a uniformly distributed load.

FIG. 19.

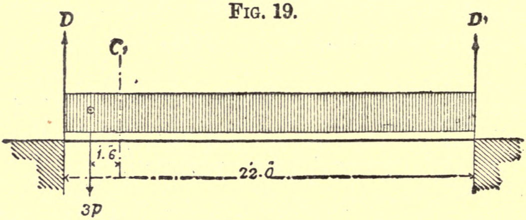
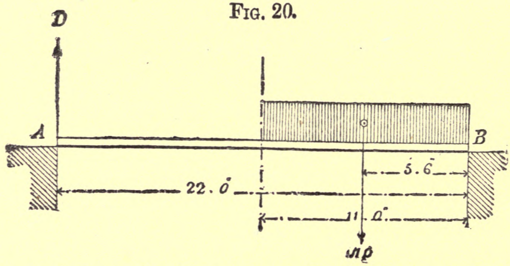


FIG. 20.



We will now obtain the maxima shearing forces at the centre of the beam, 1st, for the uniformly distributed load, and 2nd, for the real distribution.

1st. From Fig. 20 we have for the centre of the beam

$$V = D$$

and taking moments about B,

$$D \times 22 = 11p \times 5{\cdot}5.$$

Hence

$$V = 3{\cdot}454 \text{ tons.}$$

2nd. From Fig. 21 we obtain

$$V = D,$$

and taking moments about B we have

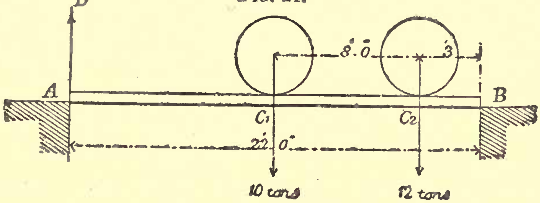
$$D \times 22 = 10 \times 11 + 12 \times 3.$$

Hence

$$V = 6 \cdot 64 \text{ tons nearly.}$$

The latter value of V differs therefore considerably from the former.

FIG. 21.



From the above example we gather that the determination of M and V on the hypothesis of an imaginary uniform load distribution involves considerable error. The proper mode of procedure is therefore that indicated in para. 9.

[*Note.*—If, however, *only one* concentrated load acts on the beam, the imaginary and the real load distribution give the same results.]

15. *Combined effect of the permanent and accidental loading of a Beam.*—The simultaneous action of the weight of a beam and of its accidental or temporary load (the former of which makes itself more especially felt in the case of bridges of long span) can evidently be dealt with by a combination of the methods above described. Suppose however, that for a first approximation to the calculation of the shearing forces and moments at any section of beam under consideration, the weight of the

beam is uniformly distributed along its length, then the results if obtained according to paras. 9 and 12 must be combined. To effect this the weight of the bridge may be obtained from empirical formulæ deduced from numerous structures of a similar class.

For instance, calling w the weight of the bridge, then, as an average for single line bridges,

$$w = 1763 \cdot 68 + 20 \cdot 16\, l,$$

where w is in lbs. and l the span in feet. [A better formula for deducing approximately the weight of a girder from its known load is given in Professor Unwin's 'Iron Bridges and Roofs.'

$$\left(W' = \frac{W\, l\, r}{C\, s - l\, r}\right).$$

Where

W = Total external distributed weight in tons (exclusive of girder).
W' = Weight of girder itself in tons.
l = Clear span in feet.
s = Average stress in tons per square inch of the gross section of the booms, at the centre, usually 4.
r = Ratio of depth to span.
C a coefficient depending on the description of girder.

VALUES OF C IN DIFFERENT BRIDGES.

Conway, tubular	1700
Britannia „	1461
Torksey „	1197
Cannon Street, box-girder	1540
„ „ plate-girder	1598
Charing Cross, lattice	1880
Crumlin, Warren	1820
Lough Ken, bowstring	1490
Small plate girders, 30 ft. to 60 ft.	1280]

Having ascertained the greatest shearing forces and bending moments at any sections due to the weight of the bridge (estimated by the above formula) and to the greatest temporary load, the dimensions of these sections must be calculated and from them the real weight of the bridge. Then the maxima shearing forces and bending moments for the various sections must be again determined on the basis of the corrected weight of the structure together with its temporary loading, after which the dimensions of the various sections should be redetermined.

In road bridges the maximum temporary load is that which arises from a crowd of people. Now from five to six persons is the maximum number which could be accommodated per square yard, and the average weight of a man does not amount to more than 155 lbs. Hence in road bridges the maximum temporary load will be from 775 to 930 lbs.

For bridges on turnpike roads the greater number can be considered as the limit.

[In English practice the weight of a crowd of people has been taken at 40 to 50 lbs. per square foot. The weight of a dense crowd may attain to 84 lbs. Generally, the load on the footways of bridges may be taken at 70 lbs. per square foot, while for bridges carrying road traffic from 80 to 120 lbs. per square foot of roadway may be allowed.]

Resolution of Forces.

16. *Resolution of a Force in two directions.*—A force P, Fig. 22, can be resolved into two components having given directions by means of the parallelogram of forces, i.e. by the application of the theorem stated on page 1. The direction arrow of P is reversed, and then, as in Fig. 22 A, P is made the closing line or

Fig. 22.

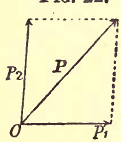
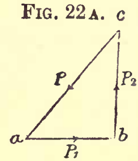


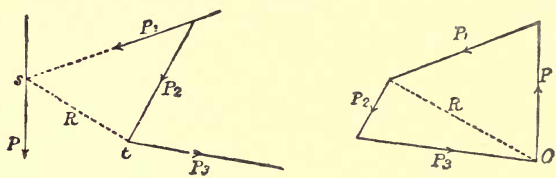
Fig. 22 A.



third side of the triangle of forces. The arrows will now point the same way round this triangle and will give the *sense* of the components.

17. *Resolution of a Force in three directions.*—Suppose that the force P, Fig. 23, is to be resolved into three components having the given directions P_1, P_2, P_3. Produce P to cut one of the given directions P_1 in *s*. Then, as in para. 16, resolve P in the direction of P_2 and of the line R, joining *s* with *t* the intersection of the other two given directions. Resolve R (again reversing the direction arrow) in the directions P_2 and P_3. Then the closed polygon P, P_1, P_2, P_3 gives the directions and magnitudes of the three components P_1, P_2, P_3 of the force P.

FIG. 23.



[The forces P_1, P_2, P_3 and the reversed force P form a system in equilibrium, hence the direction arrows of these forces point the same way all round the figure.]

If a force is to be resolved in *more than three* given directions, the problem is indeterminate.

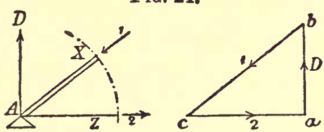
[The problem is also indeterminate if the three given directions are parallel to that of the given force, or if they meet in a point.]

Interior Forces or Stresses.

18. *Determination of the interior Forces or Stresses due to the exterior Forces.*—Since the exterior forces maintain equilibrium with, and act in opposite directions to the interior forces or stresses, the latter are equal in magnitude to the resolved components of the exterior forces to which they are due. In determining these components therefore, the direction arrows of the exterior forces must *not* be reversed, otherwise merely the resolved components of the exterior forces would be obtained.

For example, if the reaction D of the support A acting on the two bars A X, A Z, Fig. 24, is known, then the stresses 1 and 2 in those bars respectively are obtained by resolving D along their directions *without* reversing the direction arrow of D.

Fig. 24.



Thus D resolved in the directions of the bars A X and A Z gives *b c* and *c a* the required stresses in magnitude and sense.

Note.—If from the interior force S in a bar the corresponding stresses in two other bars meeting at the same joint are to be determined (i. e. if an interior force is to be resolved again into interior forces), then the process described in para. 16 must be carried out, and the direction arrow of S must be reversed.

PART II.

BRACED STRUCTURES.

19. *General Considerations.*—In the foregoing paragraphs it is established that a force fully determined can be resolved into two or three other components having given directions. If therefore girders made up of many parts are so put together that not more than three bars are cut across by any particular section plane, then the resultant of the exterior forces on one side of the section plane can be distributed in the directions of the cut bars without indeterminateness, and thus the stresses in those bars can be obtained.

If moreover, a bar is strained by a force acting along its axis, then this force, whether tensile or compressive, is uniformly distributed over the whole cross section of the bar.

On the other hand, if a bar is *bent* by the exterior forces, then evidently the stresses due to the bending are unequally distributed in the interior of the bar, and the stress over the area of any cross section is not uniform.

A good structure should therefore be made up as far as possible of members in which only longitudinal stresses arise. This is the case in braced beams.

A simple bracing in its most general form consists of two booms connected by bars forming a succession of triangles in such a way that the several members are *strained only in the direction of their length, i.e. they are either in direct tension or compression.*

It is necessary however, that the bars at their points of junction should be connected by a simple joint bolt, or that they should be "articulated," as it is termed. Then the rotation of the bars not being prevented, they are capable of placing themselves parallel to the directions of the forces forming the polygon of forces, and they thus form the corresponding polygonal frame.

In all following examples it will therefore be supposed that every pair of bars are connected at their intersection by a bolt. It will also be supposed that the joints form exclusively the loading points, as is the case in a properly constructed braced beam, in which the loads on the cross girders are transferred to the joints of the main girder.

If further, the weight of the whole structure is supposed to be equally distributed over the length of the frame, the weight between any two joints must be considered to belong half to each joint.

In constructing the diagram of forces for the determination of the stresses in the several members of the bridge, the following course of operation will be followed.

After the distribution of the load on all the joints is settled, one of the exterior forces (preferably the reaction of one of the supports) is resolved in the directions of the bars meeting at the end joint by para. 16. Then at the next joint the stress obtained is combined with the exterior forces acting at that joint, and the resultant is resolved in the directions of the new set of bars, and so on. The combination of the successive figures obtained forms what is termed the "stress diagram." For the sake of clearness the stresses denoted by the lines of the stress diagram are distinguished by the same numbers as the corresponding bars in the skeleton drawing or "frame diagram" of the structure. Moreover, tensile stresses are denoted by single, compressive stresses by double, and resultants by dotted lines. Bars in compression are termed "struts," those in tension "braces," or "tension bars."

Although after what has been said in para. 3 no doubt should arise as to the sense of the interior forces or stresses, it may be again stated that the direction arrow of an interior force as obtained from the stress diagram, is transferred to the bar to which it corresponds, being placed nearest to the joint at which the resolution was commenced. Then an arrow in the opposite direction is introduced near the other extremity of the bar, and according to para. 3 we obtain

$$\text{For interior forces} \begin{cases} \text{compression} & \longleftarrow \quad \longrightarrow \\ \text{tension} & \longrightarrow \quad \longleftarrow \end{cases}$$

while the reverse arrow combination obtains for the exterior forces.

Suppose the bar under consideration to be cut across, that part of it only remaining which lies nearest to the joint at which the resolution of the forces was made, and that in place of the portion cut away, the stress obtained from the stress diagram acts as an exterior force. Then if the direction arrow of the latter points outwards, i. e. away from the joint, the stress in the bar is tensile, if inwards, i. e. towards the joint, compressive.

20. *Equilibrium of the Forces in a braced Structure.* —If a braced structure is in equilibrium under the

action of the loads applied to it, then evidently the exterior and interior forces acting at each joint must be in equilibrium; it must therefore be possible to form them into a closed polygon.

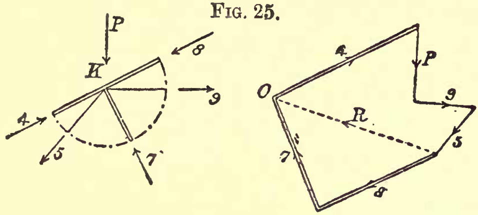
Hence we are able—

1st. To ascertain whether the requirements of the several bars meeting at a joint have been properly fulfilled.

2nd. With n bars meeting at a joint to ascertain the stresses of two of them, if the exterior forces acting at the joint and the stresses of $n-2$ of the bars are given in direction, sense, and magnitude.

For example, suppose that the forces P, 4, 5 and 9 acting at the joint K (Fig. 25) are known and 7 and 8 unknown. Combine the known forces 4, P, 9, 5 for a resultant R, reverse the direction arrow of R and resolve it into the two components 8 and 7 by drawing lines through its extremities parallel to their directions.

Fig. 25.



We will now proceed to the construction of stress diagrams for braced structures, dividing the latter into the two following classes, viz.:—

a. Braced beams subject to a constant load.

b. Braced beams subject to a travelling load.

In the first class are braced beams employed in roof construction, also lattice, suspension and combined lattice and suspension arrangements together with cranes, etc.; while to the second class belong bridge girders.

Braced Structures with Constant Loads.

21. *Roof Trusses.*—In making calculations for roof constructions a uniform vertical load is usually assumed to act upon the rafters. This is not strictly correct, as the wind pressure varies from the horizontal through an angle of about 10°. Since, moreover, greater safety will be ensured if the greatest *wind* pressure is assumed to act simultaneously with the greatest *snow* pressure, in practice it is usual to make a single calculation based on the above hypothesis, this course will be followed here for the sake of simplicity.

The loads on a roof will therefore consist of—

1. The dead weight of the structure, or the permanent load.
2. The weight of the greatest snowfall covering it.
3. The greatest wind pressure.

Dead Weight of Roofs.

22. The following table gives approximately the weights in kilos. per square metre and lbs. per square foot of various kinds of roof coverings :—

Wooden Roofs.

Nature of Covering.	Average Weight.	
	Kilos. per Sq. Metre.	Lbs. per Sq. Foot.
Single tiles	100	20
Double tiles	125	25
Ordinary slating	75	15
Asphalt on slabs	100	20
Tarred paper (Theerpappe)	30	6
Sheet iron or sheet zinc	40	8

IRON ROOFS.

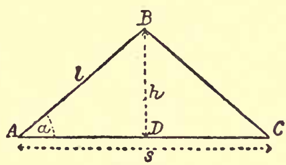
Nature of Covering.	Average Weight.	
	Kilos. per Sq. Metre.	Lbs. per Sq. Foot.
Slates on angle irons	50	10
Sheet iron on ditto	25	5
Corrugated iron on ditto	22	4·3
Corrugated zinc on ditto	24	4·7

SNOW PRESSURE.

23. The greatest depth of snowfall in Mid-Europe is about 0·625 metre, or 2 feet nearly. The specific gravity of snow is about one-eighth that of water.

Since 1 cubic metre of water weighs 1000 kilos., the snow pressure will amount to 78 kilos. per square metre, or 15·6 lbs. per square foot over the horizontal projection of the roof.

FIG. 26.



This pressure decreases per square foot in the ratio of the half span $\frac{s}{2}$ (Fig. 26) to the length of the rafter l.

The following table gives its value for the different values of the ratio $\frac{h}{s}$.

SNOW PRESSURE.

$\frac{h}{s}$	Kilos. per Sq. Metre.	Lbs. per Sq. Foot.	$\frac{h}{s}$	Kilos. per Sq. Metre.	Lbs. per Sq. Foot.
$\frac{1}{2}$	55	11	$\frac{1}{7}$	75	15
$\frac{1}{3}$	65	13	$\frac{1}{8}$	75·5	15·1
$\frac{1}{4}$	70	14	$\frac{1}{9}$	76	15·2
$\frac{1}{5}$	73	14·6	$\frac{1}{10}$	77	15·4
$\frac{1}{6}$	74	14·8			

[In England a snow pressure of from 5 to 6 lbs. per square foot of area covered may be taken as giving sufficient security.]

Wind Pressure.

24. The magnitude in kilogrammes per square metre of the pressure p which the wind exerts on a plane normal to its direction, is given by the empirical formula

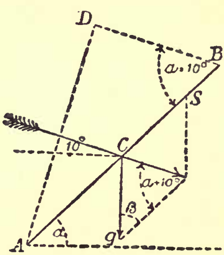
$$p = 0 \cdot 113\, v^2$$

where v is the velocity of the wind in metres per second; or if v is taken in feet,

$$p = 0 \cdot 00\,231\, v^2 \text{ lbs. per square foot.}$$

Since the direction of the wind usually makes an angle of about 10° with the horizontal, its direction will make an angle $a + 10°$ with a plane A B inclined a. Draw A D perpendicular to the direction of the wind meeting B D parallel to that direction in D, then the wind pressure acting on a surface of length A B and having a unit of breadth is

Fig. 27.



$$W = p . \text{A D},$$

or since

$$\text{A D} = \text{A B sin.}\ (a + 10°)$$

we have

$$W = p . \text{A B sin.}\ (a + 10°).$$

Hence the wind pressure *per unit of area* on A B is

$$w = \frac{W}{\text{AB}} = p . \text{sin.}\ (a + 10°) \quad . \quad . \quad . \quad 1.$$

Resolve w into a vertical component q and a component s acting along A B, then

$$\frac{q}{w} = \frac{\sin.\ (a + 10^\circ)}{\sin.\ \beta};$$

but

$$\beta = 90^\circ - a.$$

Hence

$$\frac{q}{w} = \frac{\sin.\ (a + 10^\circ)}{\cos.\ a}$$

and substituting the value of w obtained in I.,

$$q = \frac{p.\sin.^2\ (a + 10^\circ)}{\cos.\ a} \quad \ldots\ldots \quad \text{II.}$$

If $v = 31 \cdot 6$ metres is taken as the maximum velocity of the wind, then

$$p = 113 \text{ kilogrammes per square metre;}$$

taking $v = 100$ feet, we have

$$p = 23 \text{ lbs. per square foot.}$$

If h is the height of the roof and s the span, then

$$\tan.\ a = \frac{2\,h}{s}.$$

The following table gives the values per unit of area of the vertical component of the wind pressure on roof planes having various inclinations.

Vertical Component of the Wind Pressure.

$\frac{h}{s}$	Kilos. per Sq. Metre.	Lbs. per Sq. Foot.	$\frac{h}{s}$	Kilos. per Sq. Metre.	Lbs. per Sq. Foot.
$\frac{1}{2}$	107·20	21·44	$\frac{1}{7}$	22·45	4·49
$\frac{1}{3}$	64·75	12·95	$\frac{1}{8}$	19·30	3·86
$\frac{1}{4}$	44·65	8·93	$\frac{1}{9}$	17·00	3·40
$\frac{1}{5}$	33·90	6·78	$\frac{1}{10}$	15·02	3·04
$\frac{1}{6}$	27·10	5·42			

By adding together the permanent load, the maximum weight of snow, and the vertical component of the wind pressure, the total vertical load is obtained.

25. [*Note.*—The above treatment of the question of wind pressure cannot be regarded as satisfactory. It has been pointed out by Professor Unwin that wind pressure, like fluid pressures generally, acts normally to the roof surface instead of vertically, as is the case of the other loads to which a roof is subject. The usual *direction of the wind* is probably horizontal, and though it is quite possible that this direction may occasionally make a considerable angle with the horizontal, becoming, for example, normal to roofs of high pitch, it can very rarely, if ever, act vertically. Now a horizontal or normal wind can act on only one side of a roof, and it is evidently possible that this partial or unsymmetrical loading may produce a much greater distorting effect on the structure generally, and greater stresses in parts of the bracing than a uniformly distributed vertical load. Moreover, even on the supposition of a wind acting vertically, there will be a horizontal component which it would be unsafe to leave out of calculation.

It is therefore evidently necessary to ascertain what will be the effect of a horizontal or normal wind acting on one side of the roof, thus one stress diagram will not suffice.

The following formula deduced by Hutton from experiment gives the value of the normal pressure of the wind on any plane surface in terms of P the pressure on a plane surface perpendicular to its direction and *i* the angle of inclination of that direction to the plane of the surface.

$$\text{Normal pressure } P = P \sin. i^{1\cdot 84 \cos. i - 1}$$

The maximum force of the wind in England has been variously taken at 40 and 50 lbs. per square foot of surface perpendicular to its direction. Substituting either of these values of P in the above equation the normal pressure on the surface is obtained if the direction of the wind is known. Supposing the direction of the wind to be horizontal, *i* is equal to the inclination or pitch of the roof. The horizontal and vertical components of the wind's normal pressure can be obtained either by construction or by calculation.

The following table, taken from Professor Unwin's 'Iron Bridges and Roofs,' gives the values of the normal pressure (P_n), and of its horizontal and vertical components (P_h and P_v) for a horizontal wind acting with a force of 40 lbs. per square foot of vertical surface exposed to it, on roofs of various pitch.

Angle of Roof.	Lbs. per Sq. Foot of Surface.			Angle of Roof.	Lbs. per Sq. Foot of Surface.		
	P_n	P_v	P_h		P_n	P_v	P_h
° 5	5·0	4·9	0·4	° 50	38·1	24·5	29·2
10	9·7	9·6	1·7	60	40·0	20·0	34·0
20	18·1	17·0	6·2	70	41·0	14·0	38·5
30	26·4	22·8	13·2	80	40·4	7·0	39·8
40	33·3	25·5	21·4	90	40·0	0	40·0

To determine the stresses in the various members of a roof truss it will be necessary therefore—

1st. To draw a diagram corresponding to the dead or permanent load, including the weight of snow if it is thought necessary.

2nd. Assuming the direction of the wind to be horizontal, to draw either (*a*) a diagram corresponding to the normal pressure obtained, as in the above table, or (*b*) to draw two diagrams, one corresponding to the vertical and the other to the horizontal component of the wind pressure. The latter mode of procedure (*b*) will in some cases be simplest, though in other cases the diagram of the horizontal component may, from the coincidence of a large number of lines, give a bad figure.

Now if the wind instead of being horizontal is supposed to have a direction normal to the roof surface, it is evident that on the one hand the normal pressure diagram for a horizontal wind read off on a different scale will give the stresses due to a wind acting normally, while on the other hand the horizontal component diagram for a wind acting horizontally, read off from a new scale, will give the stresses due to the horizontal component of a wind acting normally. Similarly the vertical component diagram for a horizontal wind can be used to obtain the stresses due to the vertical component of a normal wind. It is usual to assume the direction of the wind to be horizontal, but it is possible that a normal wind may produce greater stresses on some bars.

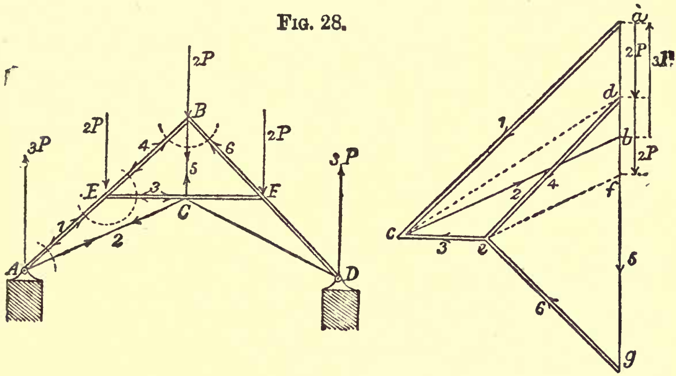
Having constructed the stress diagrams, it will be necessary to make three tables, one giving the stresses due to the dead load, another those due to the wind pressure, and a third giving the total stresses due to the wind and dead load together. The third table will then give the maximum stresses on each member of the roof.

In roofs provided with an arrangement permitting expansion at one of the supports, it will be necessary to draw diagrams to determine the wind pressure on each side of the roof separately, since only *one* of the supports can furnish the necessary reaction.]

Stress Diagrams.

26. *The German Truss,* Fig. 28.—The rafter A B is divided in this case into two equal segments A E and E B at the joint E. Suppose the load 2 P to act on each segment of the rafter, then P will act at the extremity of each segment. Hence on the middle joints E, B, F, there acts a load 2 P (since these joints are loaded on both sides), but at the extremities A and D of the rafters there is only the load P.

Fig. 28.



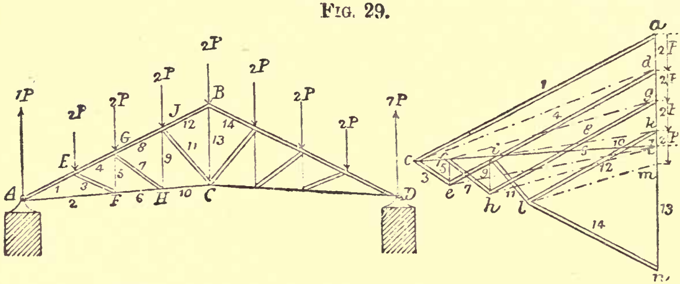
Each of the two supports has to supply a reaction equal to half the total load. Each reaction is therefore 4 P − P = 3 P, and this reaction must be considered to act as an exterior force on the adjacent bars.

In the stress diagram the reaction 3 P = ab is first resolved into the stresses 1 and 2 of the bars A E and A C. At the joint E there are now the forces 1, 2 P, 3 and 4, of which 1 and 2 P are known. Combine 2 P = ad with 1 for a resultant cd and resolve cd into the stresses 3 and 4 which are the required compressions. Proceeding to the joint B, combine 4 and

df for a resultant ef and resolve it into the forces 5 and 6. The remaining half of the diagram will, on account of the symmetry of the structure, be similar to that already drawn.

27. *The English Roof Truss*, Fig. 29.—Suppose each of the four segments of the rafter to be loaded with 2 P so that each of the two supports has to supply a reaction of (8 – 1) P

FIG. 29.

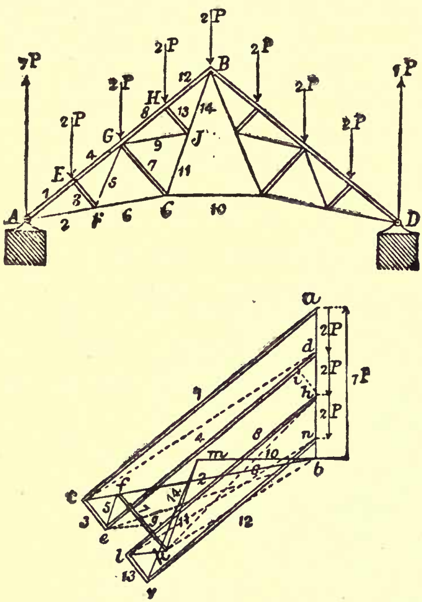


In the stress diagram, resolve the reaction $ab = 7$ P into $1 = ac$ parallel to A E and $2 = cb$ parallel to A F. Proceeding to the joint E, combine 1 with $ad = 2$ P for a resultant cd and resolve cd into $3 = ce$ parallel to E F and $4 = ed$ parallel to E G. Then passing to the joint F, combine 2 and 3 for a resultant eb and resolve eb into $5 = ef$ parallel to F G and $6 = bf$ parallel to F H. Proceed to G and combine 4 and 2 P $= dg$ with 5 for a resultant gf, resolve gf into 7 parallel to G H and $8 = hg$ parallel to G F. At the next joint H combine 6 and 7 and obtain hb, resolve hb into $9 = hi$ parallel to H J and $10 = ib$ parallel to H G. At J combine 8, 2 P, and 9 for a resultant ik and resolve ik into $11 = il$ parallel to J C and $12 = lk$ parallel to J B. Finally, at the joint B combine 12 and 2 P for a resultant $l m$ and resolve $l m$ into 13 = $m n$ parallel to B C and 14 = $l n$ parallel to B D.

From the symmetry of the structure the stresses in the corresponding bars on the other side of the centre line are identical with those already obtained.

28. *The Belgian, or French Roof Truss*, Fig. 30.—Suppose that each of the four segments of the rafters are loaded with a weight 2 P, so that the load distribution is the same as in the last case.

FIG. 30.



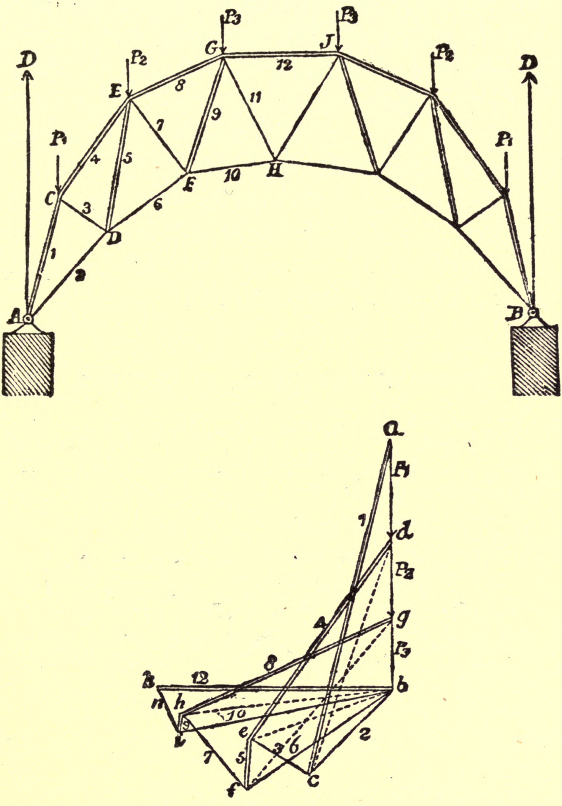
In the stress diagram, resolve the reaction 7 P = $a b$ into 1 = $a c$ parallel to A E and 2 = $c b$ parallel to A F. At the joint E combine 1 = $a c$ with 2 P = $a d$ for a resultant $c\,d$ and resolve $c\,d$ into $3=c\,e$ parallel to E F and $4=c\,d$ parallel to E G. Passing to F, $2=c\,b$ and $3=c\,e$ are combined for a resultant $b\,e$ and $b\,e$ is resolved into $5=e\,f$ parallel to P G and $6=f\,b$. At G, the resultant of the three known forces 4, 5 and 2 P must be resolved into three forces in the directions G C, G J and G H; and this resolution is indeterminate. We must therefore determine one of the three unknown forces for example 7. Now from the symmetry of the position of the two tension bars G F and G J with respect to G C the stresses 5 and 9 may be assumed to be equal since they resist 2 P in a similar way. The compression 7 on the strut G C must therefore be taken as the resultant of the two equal tensions 5 and 9 and of that component of 2 P which is parallel to G C. Hence $e\,g$ must be made equal to 5 and parallel to G J, and $g\,k$ equal to $h\,i$, then $f\,k$ is parallel to 7. Thus $h\,k$ is the resultant of 7, 5, 4 and 2 P, and $h\,k$ must therefore be resolved into two components $k\,l=9$ and $l\,h=8$.

The resultant $b\,k$ of $6=b\,f$ and $7=f\,k$ is resolved into $10=b\,m$ and $11=k\,m$, then $8=l\,h$ and $2\text{ P}=h\,n$ are combined and their resultant $l\,n$ is resolved into $12=v\,n$ and $13=v\,l$. Finally the resultant of 11, 9 and 13 is $v\,m=14$.

29. *The Bowstring Roof*, Fig. 31.—In this form of roof which is employed to cover a wider and higher space, the snow and wind pressure cannot be assumed to be the same per unit of area over the several segments A C, C E, E G, G J, and therefore the vertical component of the wind pressure and the weight of snow per unit of area of each individual segment must be separately determined from the tables given in pages 41 and 43. Only the weight of the structure, i.e. the

permanent load, can be considered as uniformly distributed over the several segments.

FIG. 31.



Suppose that P_1, P_2 and P_3 are the loads on the three upper joints C, E, G, then each reaction will be $D = P_1 + P_2 + P_3$.

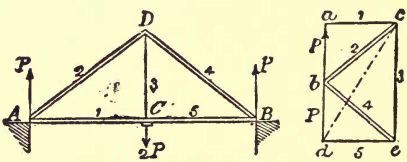
In the stress diagram the reaction $D = ab$ is resolved into $1 = ac$ parallel to A C and $2 = bc$ parallel to A D.

At the joint C, 1 and P_1 are combined for a resultant cd, and this resultant is resolved into $3 = ce$ parallel to C D and $4 = ed$ parallel to C E. Passing to the joint D combine the already determined forces 2 and 3 for a resultant be and resolve the latter into $5 = ef$ parallel to E D and $6 = fb$ parallel to D F. At the joint E, obtain fg the resultant of 4, 5, and P_2 and resolve fg into $7 = fh$ parallel to E F and $8 = hg$ parallel to E G. From 6 and 7 the resultant bh is obtained and resolved into $9 = hi$ parallel to G F and $10 = ib$ parallel to F H. Finally, 8, 9, and P_3 are combined and their resultant bi is resolved into $11 = ik$ parallel to G H and $12 = kb$ parallel to G J.

Beams supported at both Ends.

30. *The Simple Truss,* Fig. 32.—In this form of truss the beam A B is suspended at its centre C by an iron

Fig. 32.



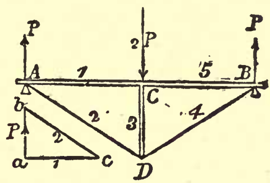
rod, or by a wooden king-post to the joint D of the rafters A D and B D, and supported at its ends. The half of the load 2 P concentrated at C falls on each of the supports since A C = C B. In the stress diagram make $ab = P$ and resolve ab into 1 parallel to A C and 2 parallel to A D. For the joint C combine $2P = ad$ with 1 for a resultant cd and resolve cd into 3 parallel to C D and

5 parallel to C B. Since finally the forces 2, 3 and 4 acting at D are in equilibrium, they must form a closed polygon, hence $b\,e = 4$.

From the symmetry of the figure 2 = 4 and 1 = 5. But since the tension bar C D must directly support the load 2 P, the triangle of forces *a b c* will suffice for the determination of all the stresses.

31. *The Simple Inverted Truss,* Fig. 33.—This is merely a simple truss inverted, and its bars will therefore have to resist stresses of the opposite kind to those of the simple truss.

FIG. 33.

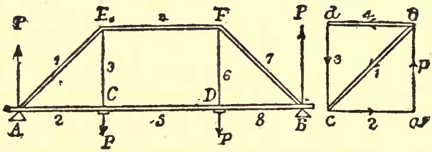


Now the strut C D must directly support the load 2 P and the forces 1 and 5 must from the symmetry of the positions of the bars be equal. Hence it is merely necessary to construct the triangle *a b c*.

N.B.—Suppose in the two preceding paragraphs that the load 2 P instead of being concentrated at C is uniformly distributed over the beams A B, then half of this load is borne on A C and half on C B. By reason of the load P over the segment A C, $\frac{P}{2}$ will act at A and also at C, and by reason of P over the segment C B $\frac{P}{2}$ will act at C and B. Thus at A and B there is a reaction $\frac{P}{2}$ and at C a load P. Hence the new stress diagram will give stresses only half the amount of those given above, i. e. a simple truss will bear a uniformly distributed load twice as great as that which it could support concentrated at its centre.

32. *The Queen-post Truss*, Fig. 34.—If the beam is suspended at *two* points, and loaded at those points

Fig. 34.

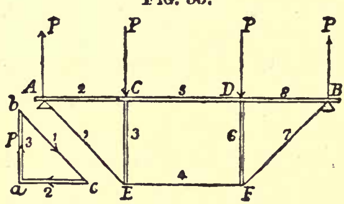


with a weight P, then by symmetry the reaction arising at each support will be P. Resolve the reaction P = *ab* into 1 = *bc* parallel to A E and 2 = *ca* parallel to A C. Resolve the interior force 1 (reversing its direction arrow) into 4 parallel to E F and 3 parallel to E C. Finally for the joint C draw the quadrilateral *abcd* made up of the forces P, 2, 3, and 5 acting at C; thus *db* = 5.

From the symmetry of the figure it is evident that 1 = 7; 2 = 8; and 3 = 6.

33. *The Inverted Queen-post Truss*, Fig. 35.—This is merely the queen-post truss of Fig. 34, inverted, consequently the similarly designated bars of both structures suffer stresses equal in magnitude but opposite in kind.

Fig. 35.



Since now by para. 32 the forces 2, 5 and 8 are equal and the force 4 is equal to 5 but of opposite kind. Hence (since 1 = 7 and 3 = 6) the triangle of forces *abc* suffices for the determination of the required stresses.

The numerous other forms of truss can be similarly treated, but since some of them can be brought under the head of "braced" or "lattice" girders we shall not make a separate reference to them here.

BEAMS FIXED AT ONE END, CANTILEVERS.

34. *The Simple Cantilever*, Fig. 36.—In this structure the beam B A is loaded at A, suspended by a tie-rod A C and made fast in the wall at B.

Resolving P into 1 and 2, we have 1 as the tensile stress in C A and 2 as the compressive stress in B A.

FIG. 36.

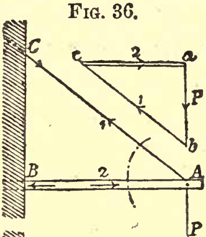
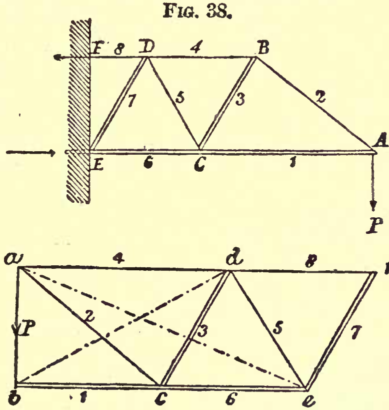


FIG. 38.



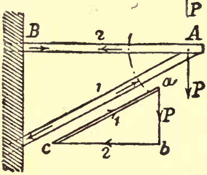


FIG. 37.

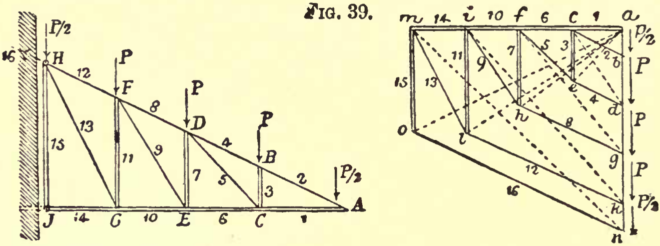
If (as in Fig. 37) the beam A B is supported at A by a strut C A from beneath, the above conditions of stress are merely reversed.

35. *The Braced Cantilever*, Fig. 38.—In this case the only exterior force is the load P suspended at A. For the stress diagram draw $a\,b$ = P and resolve it into 1 parallel to A C and 2 parallel to A B. Proceed to B and resolve 2 = $a\,c$ into 3 parallel to B C and 4 parallel

to B D. For the joint C combine 1 and 3 for a resultant *b d* and resolve it into 5 parallel to C D and 6 = *b e* parallel to C E. Finally for the joint D combine 4 and 5 for a resultant *e a* and resolve it into 8 = *a f* parallel to D F and 7 = *e f* parallel to D E.

36. *The "Perron" Roof*, Fig. 39.—This roof can evidently be treated as a braced cantilever. Suppose P to be the load on *each* of the four segments of the rafter A H, then at the extremities of the latter there acts a load $\frac{P}{2}$, and at each of the intermediate joints a load P.

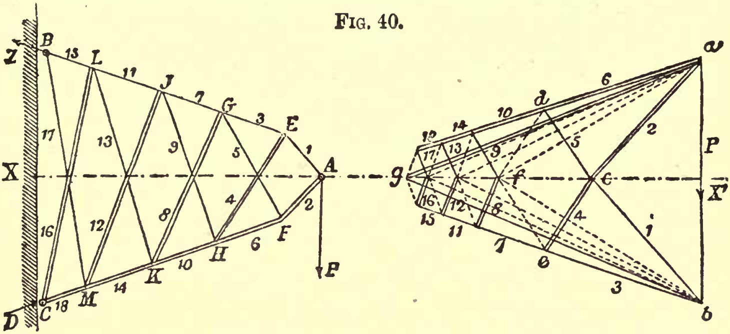
FIG. 39.



For the stress diagram draw $a b = \frac{P}{2}$ and resolve it into 1 = *a c* parallel to A C and 2 = *b c* parallel to A B. Proceeding to B combine 2 with P = *b d* for a resultant *d c* and resolve it into 3 = *c e* parallel to B C and 4 = *e d* parallel to D B. Now for the joint C combine 1 with 3 for a resultant *e a* and resolve *e a* into 6 = *a f* parallel to C E and 5 = *e f* parallel to C D. For the joint D combine 5, 4, and P = *d g* for a resultant *f g* and resolve *f g* into 7 = *f h* parallel to D E and 8 = *g h* parallel to D F. The forces meeting at the joints E, F, G, and H are similarly treated, and we obtain 9 = *h i*; 10 = *a i*; 11 = *i l*; 12 = *l k*; 13 = *l m*; 14 = *a m*; 15 = *m o*; 16 = *o n*.

37. *The Lattice Cantilever,* Fig. 40.—Suppose the structure loaded merely with a weight P suspended at A, to be fixed in the wall at B and C, and to be symmetrical about the horizontal line A X.

Fig. 40.

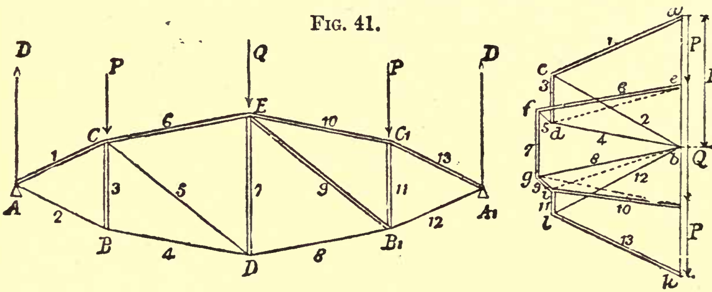


In the stress diagram begin by resolving the exterior force P = $a\,b$ into 1 and 2 respectively parallel to the directions of the bars A E, A F; the force 1 must now be resolved into the forces 3, 4 and the force 2 into 5, 6, since 1, 3, 4 and 2, 5, 6 meet in E and F respectively.

We thus arrive at the joint G, for which the two known forces 3 and 5 must be combined and their resultant resolved into 7 and 8, i. e. in the stress diagram 3 = $b\,e$ is combined with 5 = $e\,f$ and their resultant $f\,b$ is resolved into 7 parallel to G J and 8 parallel to G K, Proceeding similarly, the stresses in the bars meeting at the remaining joints H, J, K . . . are determined and the stress diagram obtained is symmetrical about the axis X X′; only one half of it therefore need be constructed. For instance, by reason of their symmetrical disposition with respect to P and to the axis X A, 1 = 2; 3 = 6; 4 = 5; and so on. These equal stresses will however, be opposite in kind; those indicated in the diagram by single lines being tensile, and by double lines compressive.

38. *Braced Structures of the most general form with fixed Load.*—In Fig. 41 such a structure is shown, having for the sake of simplicity of figure only four

FIG. 41.



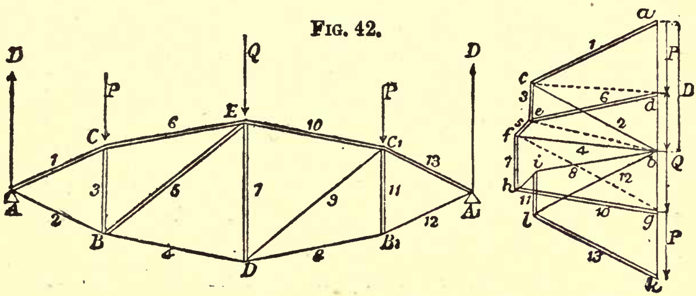
bays. It is formed of two polygonal booms A C E . . and A B D . . divided by verticals into unequal bays. It is symmetrical about the centre line E D, and has two diagonals inclined in the same direction.

The upper joints only are supposed loaded, and joints symmetrically situated about the centre line have equal loads. The reactions D at A and A′ are therefore each equal to $P + \frac{Q}{2}$.

For the stress diagram $ab = D$ is drawn first and resolved in the direction of the bars A C, A B, thus 1 and 2 are obtained as the stresses in those bars. Proceeding to B, $2 = bc$ is resolved into 3 parallel to B C and 4 parallel to B D. For the joint C combine 3, 1, and P for a resultant ed and resolve it into 5 parallel to C D and 6 parallel to C E. For D combine the known forces 4 and 5 for a resultant fb and resolve the latter into 7 parallel to D E and 8 parallel to D B. Similarly for E, obtain from the known forces Q, 6 and 7 the forces 9 and 10, and for B_1 obtain from 8 and 9 the forces 11 and 12. Finally 13 is the closing line of the four-sided figure formed of the forces 11, 10, P, 13, which act at C_1.

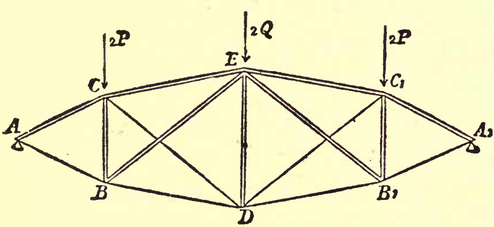
The structure shown in Fig. 42 differs from that in Fig. 41 only in that the diagonals have opposite posi-

Fig. 42.



tions. The corresponding stress diagram shows that the similarly figured diagonals differ only in being strained in the opposite directions. It follows therefore that if in a braced beam the diagonals have *opposite positions* to those of a similar beam, then the stresses in those diagonals will be of *opposite kinds;* i.e. if they are tensile in the one structure they will be compressive in the other, and *vice versâ.*

Fig. 43.



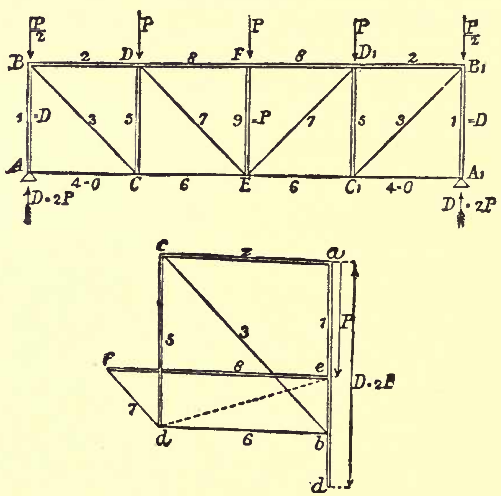
39. *The Combined Braced Beam,* Fig. 43.—Suppose that the two single braced beams shown in Figs. 41 and 42 are placed one upon another and combined so that

the combined braced beam (Fig. 43) results. Then it may be assumed that the combined braced beam will bear as much load as the two single beams together.

If the stresses in the members of the combined beam (Fig. 43) are to be ascertained, suppose the latter to be resolved into the two single beams (Figs. 41 and 42), and determine the stresses for each single beam on the supposition that it bears only one half of the total load to be borne by the combined beam. Now, suppose the two beams recombined, and obtain finally the stresses of the boom segments and of the verticals in the combined beam by adding their values as obtained for the single beams.

A braced beam with any number of bays can be similarly treated.

FIG. 44.



40. *Braced Beam with parallel Booms and fixed Load,* Fig. 44.—The structure shown is symmetrical about its

centre line E F and has to sustain over each bay a uniformly distributed load P, so that at each of the middle joints D, F, D_1, there is a load P and at each of the end joints B, B_1 a load $\frac{P}{2}$. For the reactions at A and A_1

$$D = 2P.$$

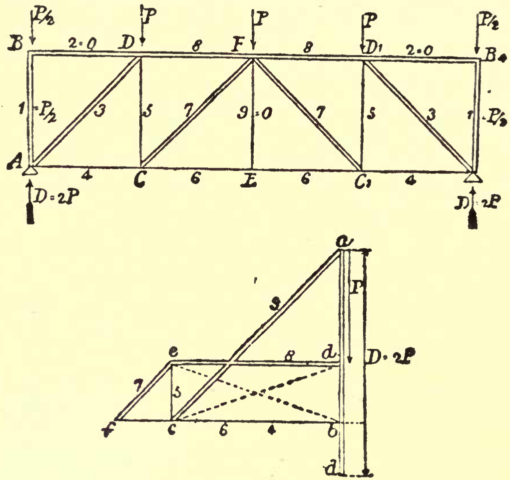
The vertical end pillar A B has evidently to support the reaction D *directly*, hence the bar A C appears to be unstrained. The force acting at A can therefore be removed to B without disturbing equilibrium, and consequently the resolution of D is commenced at B.

For the stress diagram make $a\,d = D$, and since part of the reaction is balanced by the load $\frac{P}{2}$ at B, make $a\,b$ equal to the remainder $D - \frac{P}{2} = \frac{3}{2}P$, and resolve $\frac{3}{2}P = a\,b$ into 2 parallel to B D and 3 parallel to B C. For the joint C resolve 3 into 5 parallel to C D and 6 parallel to C E. Passing to the joint D combine 5 and 2 with P for a resultant $e\,d$ and resolve it into 7 parallel to D E and 8 parallel to D F. The centre strut F E has to bear at its top a load P only, and thus in F E there is a compressive stress $9 = P$. Since the beam is symmetrical about F E the stresses of the bars forming its right half are identical with those of the similarly figured bars of the left half.

The beam shown in Fig. 45 differs from that of Fig. 44 only in the opposite position of its diagonals, the diagonals of Fig. 45 therefore, undergo the opposite stresses to the corresponding diagonals of Fig. 44, as the stress diagram shows.

The two end pillars A B, $A_1 B_1$ each suffer stresses of $\frac{P}{2}$ only, since portions of the reactions equal to $D - \frac{P}{2}$,

FIG. 45.

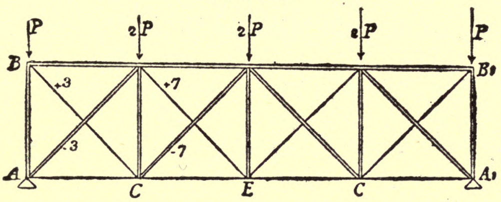


or $\frac{3}{2}$ P are directly supported by the bars A D and $A_1 D_1$. Moreover the bars B D, $D_1 B_1$ appear to be unloaded, as is also the case with the centre vertical bar E F, the load P at F being directly supported by the bars F C and F C_1.

41. *The Braced Beams of Figs.* 44 *and* 45 *combined.*—Suppose the two single beams, Figs. 44 and 45, superposed and combined. We thus obtain the form of structure shown in Fig. 46, in which the stresses of those members which coincide must be added while those of the diagonals remain unchanged. Since the vertical struts in the two single beams, with the exception of the two end pillars, suffer opposite stresses,

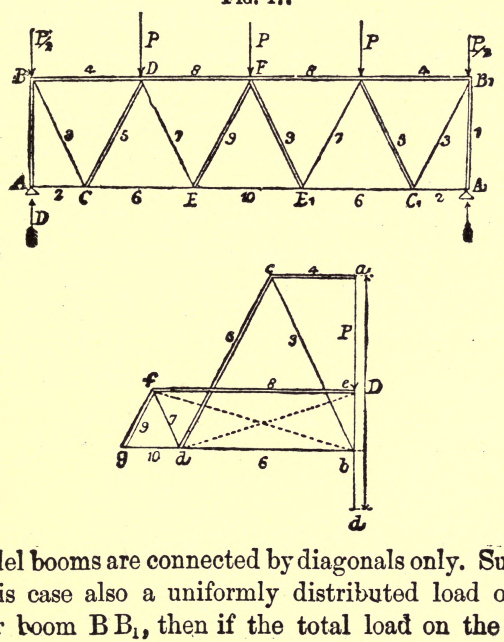
hence for each of the centre verticals of the combined beam we have only the compression P, while for the end pillars we have 2 P.

FIG. 46.



42. *Braced Beams with parallel Booms and without Verticals.*—In the single braced beam, Fig. 47, the

FIG. 47.

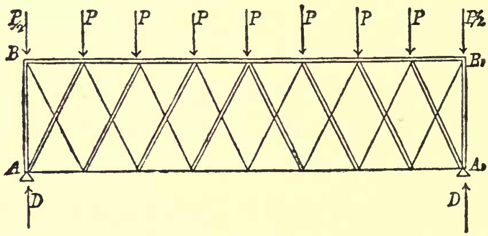


parallel booms are connected by diagonals only. Suppose in this case also a uniformly distributed load on the upper boom $B B_1$, then if the total load on the beam together with its weight is 4 P, we shall have the load distribution indicated in the figure, and the reactions D = 2 P. A portion of these reactions equal to $\frac{P}{2}$ will evidently be directly balanced by the loads $\frac{P}{2}$ acting at B and B_1, while the remainder equal to $\frac{3}{2}$ P will be transmitted by the end pillars AB, A_1B_1 to B and B_1, and thence to the bars 3 and 4. Hence in A C and A_1C_1 there exist no stresses.

For the stress diagram, the partial reaction $\frac{3}{2}$ P transmitted to B is resolved into the bar-stresses 3 parallel to B C and 4 parallel to BD. From 3 the stresses 5 = *c d* and 6 = *d b* are obtained, the resultant *d e* of 5, 4 and P is then resolved into 7 = *f d* and 8 = *e f*. Finally 6 and 7 are combined for a resultant *f b* and *f b* is resolved into 9 = *f g* and 10 = *g b*.

Suppose now that a second braced beam, having an equal load and differing solely in the reversed positions

FIG. 48.



of its diagonals, is superposed upon, and combined with the first. Then the booms will coincide, and we obtain the lattice girder shown in Fig. 48, which will sustain twice the load of the single girder (Fig. 47).

The stresses on the members of this structure are obtained by adding the stresses of the coinciding booms and leaving those of the diagonals unaltered.

[Lattice girders and girders with parallel booms generally, are more usually treated by calculation than by the graphic method.]

Braced Beams with permanent and travelling Load. (Bridge Girders.)

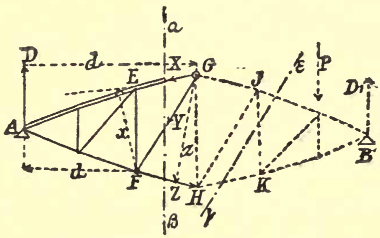
43. *Effect of a travelling Load on braced Structures.*—Bridge girders have, besides their own dead weight, which is constant, to sustain a load over the whole bridge due to the heaviest passing train. The weight of the bridge may be supposed uniformly distributed over its whole length, and hence by para. 12 we can very easily obtain the bending moments and shearing forces at any section, due to it. It is however, different in the case of the travelling load. It is necessary in this case to determine for what position of the load the stresses on the various constructional parts of the girder attain a maximum. For this purpose we shall investigate first the effect produced by a concentrated load on the bars to the right and left of it, and then proceed to apply the results obtained.

[*Note.*—In the following figures those portions of a girder which are left out of consideration are dotted.]

44. *Maximum Stress of the Booms.*—In order to determine the effect which a concentrated load P applied on the right of the section $\alpha\beta$ (Fig. 49) exerts on the boom E G, i. e. to ascertain whether the stress in E G is tensile or compressive, suppose the girder cut into two parts by a section plane $\alpha\beta$ and that imaginary exterior forces X, Y, Z, are applied at the points of

section, capable of maintaining equilibrium. Now on the left portion A E F of the girder there are acting only the three forces X, Y, Z, and the reaction D at A due to the concentrated load P. These four forces must

FIG. 49.



therefore be in equilibrium, and hence the algebraic sum of their statical moments about any point in their plane must be nil. In order to eliminate the forces Y and Z, take moments about F the intersection of the directions of those forces. Thus if d and x are the perpendiculars dropped from F on the directions of D and X,

$$D.d - X.x = 0; \text{ or } X = \frac{D.d}{x}.$$

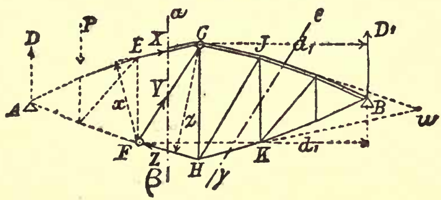
The direction arrow of X indicates that the segment E G of the upper boom is in compression under the action of P, and hence *every load applied on the right of the section plane* $a\ \beta$ *exerts a compressive stress in that segment of the upper boom which is cut by the plane.*

The effect produced by a load P applied to the *left* of the section plane $a\ \beta$ (Fig. 50) can be similarly investigated. Taking moments about F,

$$-D_1.d_1 + X.x = 0; \text{ or } X = \frac{D_1.d_1}{x}.$$

The direction arrow indicates that E G is again in compression; thus, *every load applied to the left of the segment* E G *exerts in* E G *a compressive stress.*

FIG. 50.



The above reasoning will of course hold good for *any* section between E and G.

Hence generally—

The upper boom suffers compressive stress only, and this stress attains a maximum when the whole girder is fully loaded.

In a similar way the condition of maximum stress in the lower boom can be investigated. Suppose a concentrated load P to be applied to the right of $a\,\beta$ (Fig. 49). Take moments about G and call d and z the perpendiculars dropped on the directions of D and Z respectively. Then—

$$D.d - Z.z = 0; \text{ or } Z = \frac{D.d}{z}.$$

The direction arrow of Z indicates a tensile stress in the segment F H of the lower boom.

If the load P is applied to the left of F H (Fig. 50), then taking moments about G,

$$-D_1\, d_1 + Z.z = 0; \text{ or } Z = \frac{D_1 . d_1}{z}.$$

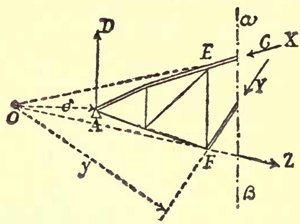
The stress in F H is therefore again tensile, and hence generally—

The lower boom is subject to tension only, and this tension is a maximum when the girder is fully loaded.

45. *Maximum Stresses of the Bracing Bars.*—Under the term "bracing bar" are included those members of the structure which serve to unite the upper and lower booms, e. g. the vertical and diagonal bars in Fig. 49.

In order to obtain the stress Y in a diagonal bar F G (Fig. 49), suppose a concentrated load P applied to the right of the section plane $\alpha \beta$, and obtain the condition of equilibrium for the left portion (Fig. 51) of the structure. Taking moments about O the point in which the boom segments E G and F H meet if produced. Then if y and δ are the respective perpendiculars dropped from O on the directions of Y and of D,

Fig. 51.



$$-D.\delta + Y.y = 0;\ \text{or}\ Y = \frac{D.\delta}{\gamma}.$$

The direction arrow of Y indicates that *any load applied to the right of the bar* F G *exerts a compressive stress in* F G.

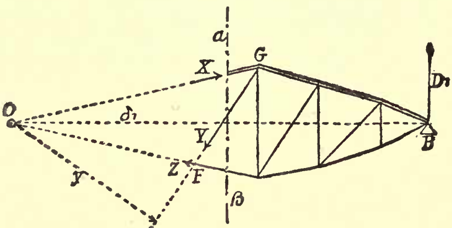
Suppose the load P to be applied on the left of the section $\alpha \beta$ (Fig. 52), then taking moments about O and calling y and δ_1 the respective perpendiculars dropped from O on Y and on D_1 the reaction of the support due to P,

$$-D_1.\delta_1 + Y y = 0;\ \text{or}\ Y = \frac{D_1.\delta_1}{y}$$

The direction arrow of Y indicates tensile stress in F G. Hence—

Every load applied to the left of the bracing bar F G *will exert in that bar a tensile stress.*

FIG. 52.



Since then, loads applied on opposite sides of any diagonal bar produce opposite stresses, hence—

A bracing bar is in the condition of maximum stress if the girder on one side of it is fully loaded and on the other is unloaded.

The same rule holds in the case of the vertical bars. To ascertain the maximum stress in J K (Figs. 49 and 50), suppose the girder cut by a section plane $\gamma\ \epsilon$, so that only three bars including J K are divided. Let a force P be now applied, 1st, to the right of the section plane $\gamma\ \epsilon$, and 2nd, to the left of that plane, and obtain the condition of equilibrium of the two other bars by taking moments about u, their intersection when produced. It will then be seen that in the girder (Figs. 49 and 50) any concentrated load P applied to the right of J K produces in J K a tensile stress, while a load applied to the left produces a compressive stress.

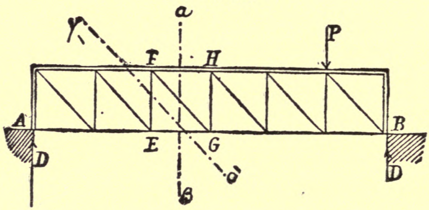
Hence generally in a braced girder—

Each bracing bar is in the condition of maximum stress when every joint of the girder either on the right or on the left of the bar is fully loaded.

46. *Girders with parallel Booms.*—In the case of braced girders with parallel booms the above method of investigating the greatest stresses in the several parts can equally be employed.

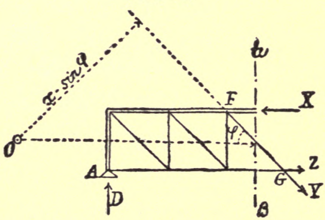
In order, for example, to determine the effect produced by a concentrated load applied to the right of the section plane $\alpha\, \beta$ (Fig. 53). Suppose as before,

FIG. 53.



that by the application of the forces X, Y, Z at the points of section the equilibrium of the left portion (Fig. 54) is maintained. Resolve vertically, then

FIG. 54.



$$Y \cos. (90° - \phi) = D;$$

or, $$Y = \frac{D}{\sin. \phi}.$$

In this case Y tends to produce a right-handed rotation, and the corresponding direction arrow evidently indicates that the stress in F G due to P is tensile.

Treating the case of a concentrated load applied to the left of $\alpha\, \beta$ (Fig. 55) in a precisely similar way, we obtain

$$Y = \frac{D_1}{\sin. \phi}.$$

In this case the direction arrow of Y indicates a compressive stress in the bar F G.

Hence generally—

Every concentrated load applied to the right of the diagonal bar F G produces tension in it, while a load applied to the left produces compression. *Any diagonal of a braced girder with parallel booms is consequently in the condition of greatest stress when the girder is loaded to its maximum on one side of that diagonal.*

FIG. 55.

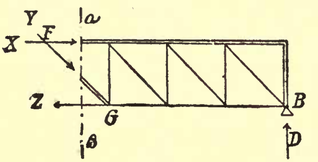
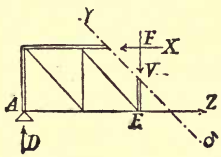


FIG. 56.



The same rule holds also for the vertical bars.

For instance, in order to determine the greatest stress in the third vertical E F (Fig. 53) draw the section line $\gamma\,\delta$ and investigate the effect of a concentrated load P applied to the right of this section plane, and producing a reaction D at A (Fig. 56). Apply the forces X, V, Z, at the sectional parts and resolve vertically, thus

$$V = D.$$

Hence the bar F E is in compression.

Again if the load P is applied to the left of $\gamma\,\delta$ and produces a reaction D_1 at B (Fig. 57), then resolving vertically,

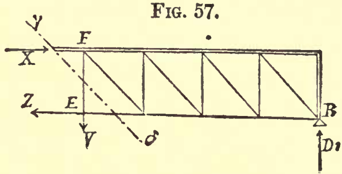
$$V = D_1.$$

And the direction arrow of V indicates a tensile stress in E F.

Hence generally—

The vertical bars suffer opposite stresses when the load is applied on the right and on the left of them, and their greatest stress occurs when the girder is loaded to its maximum extent on one side of them.

FIG. 57.



In conclusion with regard to the greatest values of the stresses X and Z of the two horizontal segments of the booms (F H and E G, Fig. 53), these stresses will as in all braced girders be a maximum when the whole structure is fully loaded. Moreover in the case of parallel booms X and Z are the only *horizontal* forces which enter into the case, therefore since no variation in a horizontal direction can arise,

$$X = Z.$$

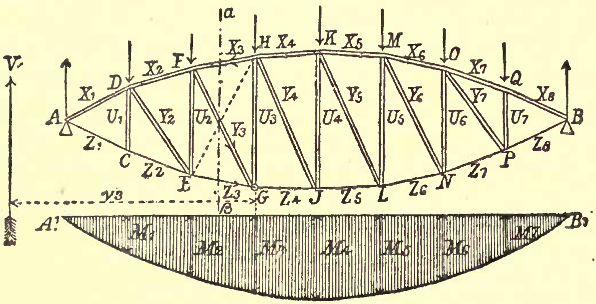
BRACED GIRDERS FOR RAILWAY BRIDGES.

47. *General Case.*—By means of the results obtained in the foregoing paragraphs it will not be difficult to determine the greatest stresses on the various constructional parts of a braced girder.

It must first be ascertained what amount of the dead load and of the greatest live load on the whole bridge falls on a single girder. If the total load is borne by *n* equally strong girders, then the dead load together with the aggregate axle pressures of the heaviest trains covering the bridge, must be divided by *n*. Then by construction obtain, as in paras. 9 and 12, the greatest shearing forces (V) and moments (M) at the various joints of a single girder due to the dead and to the live loads.

This having been effected we pass to the determination of the stresses in the booms.

Fig. 58.



Stresses of the Booms.—Fig. 58 represents a braced girder with single diagonals and M_1, M_2, M_3 . . . . the maximum total moments at its successive joints obtained by adding the values of the moments given by the dead and live loads. In order to determine the stress in any segment F H of the boom, suppose the girder cut into two parts by a section plane $\alpha\beta$, and the right half having been removed apply the forces X_3, Y_3, Z_3 at the points of section. Let V_3 be the shearing force due to the exterior forces acting on the left portion of the girder, and in order to obtain X_3

take moments about G the intersection of the directions of Y_3 and Z_3, thus

$$V_3 . v_3 - X_3 . x_3 = 0 ; \text{ or } X_3 = \frac{V_3 . v_3}{x_3}.$$

But the product $V_3 . v_3$ is the moment M_3 of the exterior forces acting on the left portion of the girder. Hence

$$X_3 = \frac{M_3}{x_3}.$$

Similarly the stresses of all the remaining segments of the boom can be obtained. The segments A C, A D however, which meet at the support, can be most easily treated by a direct resolution of the maximum support reaction along their directions. This maximum reaction will evidently arise when the train moves from B to A and the leading locomotive axle is over A.

Stresses of the diagonal Bars.—The maximum stress of the bracing bars arises when the moving load covers one side of the girder and when the larger portion of the girder; i. e. the portion between the bar in question and the farther abutment, is covered by the moving load. But since the dead load is constant, it will be best to determine first the stresses caused in the several bracing bars by this dead load and then to pass to the determination of the maximum stresses due to the moving load. By adding the stresses thus obtained (having due regard to their sign) we obtain the maximum total stress on the various bracing bars. The stress due to the dead load can be treated as has already been shown for the case of braced structures with a fixed load. Hence it will be unnecessary to treat it further, and we can therefore proceed to the determination of the stresses in the bracing bars due to the moving load.

For example, the maximum stress Y_3 in the diagonal F G (Fig. 58) will by para. 45 occur when the train, with two of the heaviest locomotives at its head, coming from the right abutment B, arrives at the section plane $\alpha\,\beta$. Obtain, as in para. 9, the shearing force V_3 corresponding to this position of the load, then V_3 will (since only one portion of the girder is loaded) be the reaction at the abutment A.

If then the portion $\alpha\,A\,\beta$ (Fig. 59) is in equilibrium under the forces X_3, Y_3, Z_3 and V_3 acting on it, the algebraic sum of the statical moments of these forces about any point in their plane must be nil. Hence taking moments about O the intersection of the directions of X_3 and Z_3,—

$$-V_3 . v_3 + Y_3 . y_3 = 0; \text{ or } Y_3 = \frac{V_3 . v_3}{y_3}.$$

FIG. 59.

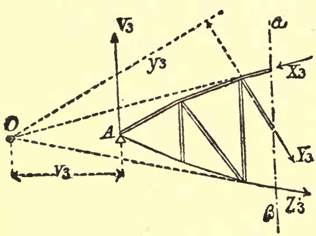
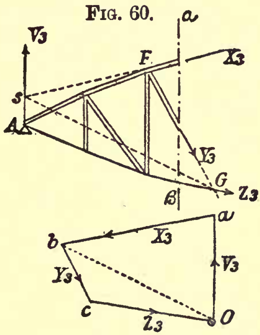


FIG. 60.

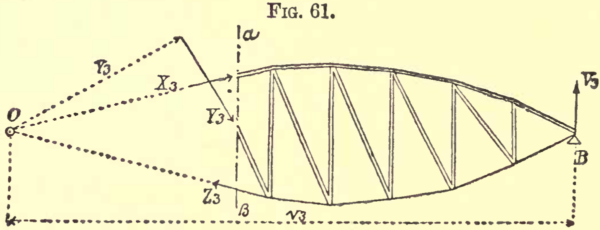


Since V_3 tends to cause left-handed rotation Y_3 will tend to cause right-handed rotation, and hence the direction arrow of Y_3 indicates that the greatest stress in F G is tensile.

Note.—If the intersection O of the directions of the orces X_3 and Z_3 is unavailable, the determination of Y_3 can be dealt with according to para. 17. Resolve the shearing force V_3 acting at A, Fig. 60, in the direction of the three other forces X_3, Y_3 and Z_3. Produce one of them X_3 to meet V_3 in *s*, then the resultant of V_3 and X_3 must be equal to the resultant of Y_3 and Z_3 and have G*s* as its direction. Make $O a = V_3$ draw from the extremities of O*a* lines parallel to X_3 and G*s*, and resolve O*b* the resultant of V_3 and X_3 into $Y_3 = b c$ and $Z_3 = c O$.

It has been shown above that the maximum stress in the diagonal bar F G (Fig. 58) is tensile and arises when the train coming from the *right* abutment arrives at the section plane $\alpha \beta$. According to para. 45 the same diagonal will suffer the greatest compressive stress when the train coming from the *left* arrives at $\alpha \beta$.

Fig. 61.



Let V_3 be the shearing force corresponding to this position of the load. Then considering the portion lying to the right of the section plane $\alpha \beta$, Fig. 61, on which, besides the forces X_3, Y_3, Z_3, there acts only the reaction V_3, and taking moments about O,—

$$-V_3 . v_3 + Y_3 . y_3 = 0; \text{ or } Y = \frac{V_3 . v_3}{y_3}.$$

The diagonal F G will consequently be in the condition of maximum *tensile* or *compressive* stress according as the train arrives at $\alpha\beta$ from the *right* or from the *left.*

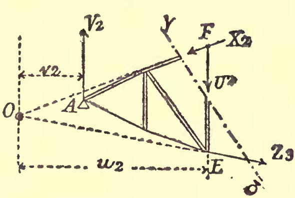
Similar conditions obtain in the case of all the diagonals of the left half-girder.

In the *right* half-girder opposite conditions of stress obtain on account of the reversed position of the diagonals with respect to the centre line; i. e. the diagonals of the right half-girder are in the state of maximum *tension* if all the bays to the *left* of them are loaded, and in that of maximum *compression* if all the bays to the *right* are loaded.

If in any bay, e. g. the third, a diagonal bar E H, Fig. 58, leaning to the right is substituted for F G leaning to the left, then this bar will evidently suffer stress of opposite kind to that in F G, and it will in fact suffer the same stress as M N to which its position is symmetrical.

Stresses of the vertical Bars.—By para. 45 the vertical bars also, are in the condition of maximum stress when the structure on *one* side of them is fully loaded.

Fig. 62.



Suppose that it is required to determine U_2 the maximum stress of the second vertical bar E F, Fig. 62, then the train coming from the *right* abutment must have arrived at E F. Obtain as in para. 9 the shearing force V_2 corresponding to this position of the wheels, then V_2 will act as a reaction at the abutment A. Suppose that the portion $\gamma\,\delta$ A is in equilibrium under the action of the forces X_2, U_2, Z_2, and V_2, then taking moments about O,—

$$- V_2 . v_2 + U_2 . u_2 = 0 ; \text{ or } U_2 = \frac{V_2 . v_2}{u_2}.$$

Since in this case U_2^- tends to produce right-handed rotation, its direction arrow indicates that the maximum stress in E F is compressive.

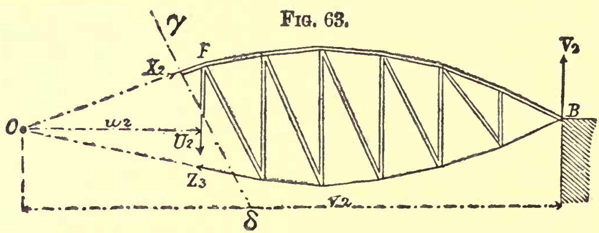
If the train comes from the *left* abutment and arrives at E F, then E F is in the condition of maximum tension.

Let V_2 be the shearing force due to this position of the load, then as a condition of equilibrium of the portion γ B δ, taking moments about O,—

$$- V_2 . v_2 + U_2 . u_2 = 0 ; \text{ or } U_2 = \frac{V_2 . v_2}{u_2},$$

and the direction arrow of U_2 now indicates a tensile stress.

FIG. 63.

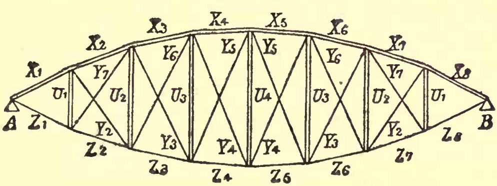


All the other vertical bars of the *left half-girder* are subject to similar conditions; that is to say, they are in the condition of maximum *compression* if the train covers all the bays to the *right* of them, and in that of maximum *tension* if the train occupies all the bays to the *left* of them.

In the right half-girder, it results from the reversed positions of the diagonals with respect to the centre line, that the vertical bars also, are strained in the opposite way; i.e. the verticals of the *right half-girder* are in a condition of maximum *compression* or *tension* according as the train occupies the bays on the *left* or *right.*

48. *Crossed, or redundant Diagonals.*—It has been shown that in a braced girder of the general form given in Fig. 58, the bracing bars are alternately in tension and compression. Since however, it is not easy to arrange and connect bars capable of resisting both tension and compression, it is usual to introduce crossed diagonals in every bay in which the single diagonals would be subject to both tension and compression. If both these diagonals are constructed as tension bars incapable of resisting compression, then each bar will be subjected to stress only by that load disposition which causes tension in it. In the braced girder with redundant diagonals (Fig. 64,) therefore,

FIG. 64.



only the maxima *tensile* stresses, determined in the case of the single braced girder, Fig. 58, need be taken into consideration. For instance, in the third bay there comes into play in the diagonal F G the tension Y_3 (max.) and in E H the tension Y_6. In the same way the stresses of the crossed diagonals of the remaining bays can be obtained directly from Fig. 58.

If the redundant diagonals are constructed solely as tension bars incapable of resisting compression, then evidently the vertical bars must be capable of resisting the maximum compressions arising in them. These maximum compressions (U) can be determined as in the case of the girder with single diagonals, Fig. 62.

In conclusion it should be noticed that on account of the symmetrical arrangement with respect to the centre line of the various bars of the girder with redundant bracing (Fig. 64), the symmetrically situated bars will undergo equal stresses. Thus—

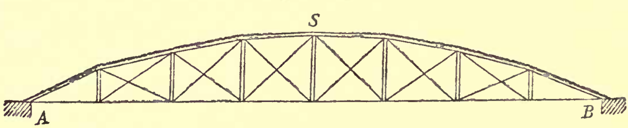
$$X_1 = X_8;\ X_2 = X_7;\ X_3 = X_6;\ X_4 = X_5;$$
$$Z_1 = Z_8;\ Z_2 = Z_7;\ Z_3 = Z_6;\ Z_4 = Z_5$$

and so on.

Note.—If the crossed diagonals are so constructed as to be capable of resisting compression only (as is the case in wooden structures), then the diagonals or struts will take up only the greatest *compressions*, and the verticals only the greatest *tensions* of the corresponding singly braced girder.

49. *Special Cases.*—The determination of the stresses in the "Pauli" and parabolic bowstring girders can be proceeded with in the same way as in the case of the bowstring suspension shown in Fig. 64.

FIG. 65.



The former has virtually the same form as the bowstring suspension, while the latter is of the form shown in Fig. 65. The upper joints lie on a parabola whose vertex is at the centre joint *s* and the lower joints are in a straight line. More rarely the reverse arrangement is met with.

In the "Schwedler" girder, Fig. 66, in the girder with parallel booms, Fig. 67, and in the half parabolic girder, Fig. 68, the crossed or redundant diagonal braces are required only in the respective centre bays, since it is in these bays only, that the diagonals are subject to both tension and compression. For further information with respect to the systems above named the reader is referred to the 'Baumechanik,' published by Dominicus of Prague.

Fig. 66.

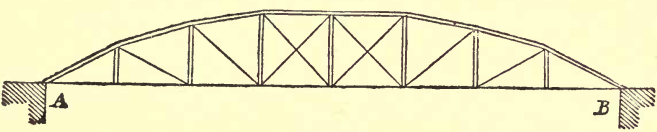


Fig. 67.

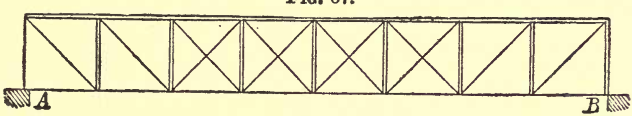
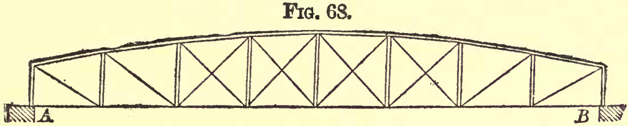


Fig. 68.



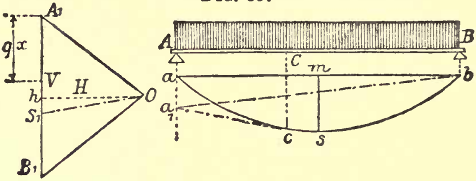
50. *Fixed Load in place of travelling Load.*—The graphic determination of the stresses in bridge girders becomes still simpler if (as has hitherto been the almost universal practice) a corresponding fixed load is taken into consideration in place of the real travelling load. This mode of procedure leads, however, to results which are far from correct, more especially in the case of

railway bridges of short span; it is however, permissible in the case of road bridges, since for them the worst case of loading is usually taken as a crowd of people.

The maximum bending moments are obtained as in para. 12 by means of a parabola, while the shearing forces corresponding to a load covering the bridge on *one* side of any section can be determined by means of the funicular curve and polygon of forces.

Since the determination of the maximum bending moments has been fully explained in the paragraph above named, it will be necessary only to deal with that of the shearing forces due to a travelling load partially covering a beam.

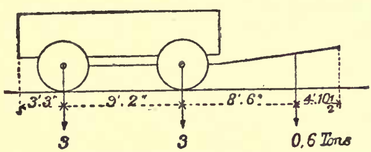
Fig. 69.



Let q be travelling load per unit of length of a beam A B, Fig. 69, and l the length of the beam. Then the total travelling load is $q . l$ and the reactions (if the load covers the beam) are both equal to $\frac{q . l}{2}$. The funicular polygon corresponding to this position of the load becomes a parabola whose vertex s is obtained by making $m\,s = \frac{q . l^2}{8\,H}$, where H is the polar distance in the corresponding polygon of forces. In this polygon of forces $A_1\,B_1 = q . l$ and $A_1\,h = h\,B_1 = \frac{q . l}{2}$. Suppose that the maximum shearing force due to the travelling load is to be determined for any cross section C distant x from A, then the load must cover B C the greater portion (or the distance of C from the farther abutment) only, and the portion of the load $q \cdot x$ on the smaller portion A C must be supposed removed. This portion of the load can (as in para. 9) be immediately cut away by drawing a tangent to the parabolic funicular curve at c (vertically under C) and producing it to cut a vertical dropped from the nearest support in a_1. Then $a_1 b$ is the new closing line of the funicular polygon corresponding to the partial loading of the beam. In the polygon of forces draw O S_1 parallel to $a_1 b$ through O the pole and set off A_1 V on the line of loads equal to $q \cdot x$, then S_1V will be the new reaction at the support A and S_1V is therefore the required shearing force at C.

51. *Concentrated Loads on small Spans.*—In bridges of small span the stress due to a heavily laden waggon may be greater than that due to a crowd of men. In Figs. 70, 71, and 72, three different load distributions are given. That of Fig. 70 serves for bridges in ordinary country roads, that of Fig. 71 for bridges in turnpike roads, and that of Fig. 72 for bridges in large

Fig. 70.



towns, where not unfrequently from the transport of heavy machines, boilers, railway waggons, and dismounted locomotives, very considerable axle pressures have to be provided for.

The weight of a pair of horses has been taken in each case as 0·6 ton. Bridges constructed for the load indicated in Fig. 72 can also be used for tramways.

The determination of the shearing forces and bending moments can be proceeded with as in para. 9.

Fig. 71.

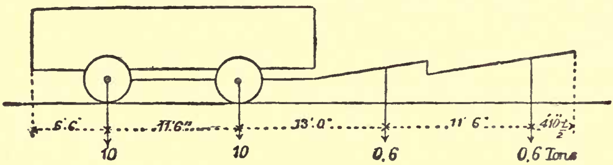
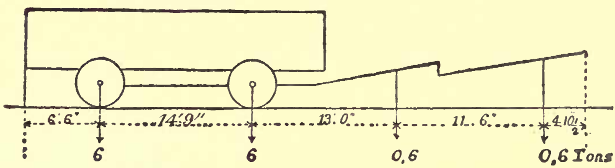


Fig. 72.



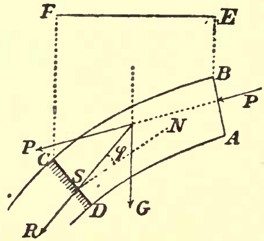
Construction of the Line of Resistance of an Arch.

52. In conclusion it is proposed to determine the conditions of stability of a symmetrically formed and symmetrically loaded arch, by the construction of its line of resistance.

Let A B C D (Fig. 73) be any portion of an arch whose *thickness* perpendicular to its face will in the following investigations be considered as unity. Two forces G and P act on this fragment, of which the first is the weight of the portion A B C D of the arch-ring together with its superincumbent load B E F C, while the second is the pressure transmitted from the adjacent portion of the arch. The resultant of G and P must be balanced by the reaction of C D.

If then the portion A B C D rests in stable equilibrium on the skewback C D under the action of these forces. Then—

Fig. 73.



1. In order that rotation about the edges C and D may not take place, the resultant R of P and G must cut the plane C D somewhere *between* those edges.

2. In order that no sliding may take place along C D, the angle ϕ made by R with the normal N S must be *less* than the angle of friction of the stone of which the arch is built, which angle is about 30°.

3. The greatest pressure per unit of area on any part of the plane C D must not *exceed* the safe resistance of the material.

In order therefore that every portion of the arch may be in stable equilibrium, these three conditions must hold at every joint.

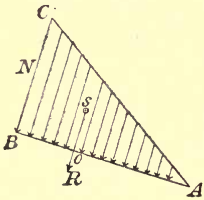
By joining the successive points in which the resultant of the exterior forces cuts the successive joints a continuous line, termed the "Line of Resistance," is obtained, which line is clearly the funicular polygon of the exterior forces acting on the several portions of the arch.

The best distribution of pressure in the arch-ring would therefore occur if this line of resistance coincided with the curve passing through the centre point of each joint, for then the pressure would be uniformly distributed over the plane of every joint.

In practice however, such a high degree of stability is seldom attained and it is usual only to provide that the stress on every joint should nowhere change its sign; i. e. become a tension instead of a compression.

Let A B (Fig. 74) be any joint, then the limiting condition of the compression on A B is that this compression should be nil at A and increase uniformly from A to B. The resultant compression on A B will therefore be represented by a triangle, and if A B = b and B C (the maximum compression at B) = N,

Fig. 74.



$$R = \frac{b . N}{2} \quad . \; . \; . \; . \; . \; . \quad (\alpha)$$

Since under these conditions R must pass through the centre of gravity of the triangle B C A, the point O in which R cuts A B is determined by making $B O = \frac{1}{3} A B.$

Hence, it follows that in a properly constructed arch the line of resistance should cut every joint within the centre third of the length of that joint.

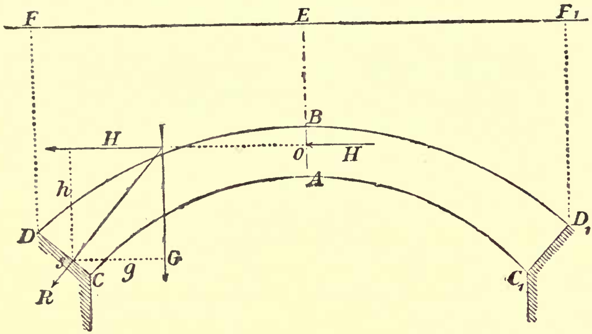
From equation (α)

$$N = \frac{2 . R}{b} \quad . \; . \; . \; . \; . \; . \quad (\beta)$$

and this equation therefore gives the value per unit of area of the maximum compression N, which maximum compression is double as great as it would be if R passed through the centre point of A B.

Considering the symmetrically formed and symmetrically loaded arch in Fig. 75, it is evident that the two halves A B C D and A B C_1 D_1 produce equal but opposite moments relatively to the skewbacks D C and D_1 C_1 exerting at the crown a mutual horizontal thrust H. The same conditions of equilibrium must evidently obtain in the case of both half arches, hence the half A B C D alone need be taken into considera-

Fig. 75.



tion, the other half being supposed replaced by the horizontal thrust H.

The question arises, how to determine the height above A of the point of application of H?

In order that the arch may have the highest degree of stability, the line of resistance should pass through the middle point of A B; this would therefore be the point of application of H. If however, it is merely intended that the designed arch should possess a degree of stability which comes within the prescribed limits, i.e. that the line of resistance should nowhere cut a joint outside the *middle third* of its length, then the point of application of H will be within the centre third of A B. According to the theory of least resistance, the horizontal thrust on A B will be the minimum consistent with the maintenance of equilibrium. Suppose provisionally that the arch is stable and let G be the weight of the half arch together with its load, then the resultant of G and H ought to cut the joint C D within its middle third. Then taking moments about the point s,

$$H.h = G.g; \text{ or } H = \frac{G.g}{h} \quad . \quad . \quad (\gamma)$$

Thus H is greater the greater h is and the less g is, and in the limiting case $BO = \frac{AB}{3}$ and $Cs = \frac{CD}{3}$.

In order to simplify the construction of the line of resistance, the load on the arch is reduced to such an equivalent mass of the arch material, as will exert the same pressure as the actual load.

Thus if γ is the weight of a cube foot of the arch material, γ_1 the weight of a cube foot of the superincumbent material, and h its height above any point on the extrados of the arch, then the height y of the corresponding equivalent load is given by the equation—

$$\gamma_1 . h = \gamma . y; \text{ whence } y = \frac{\gamma_1 . h}{\gamma} \quad . \quad . \quad . \quad (\delta)$$

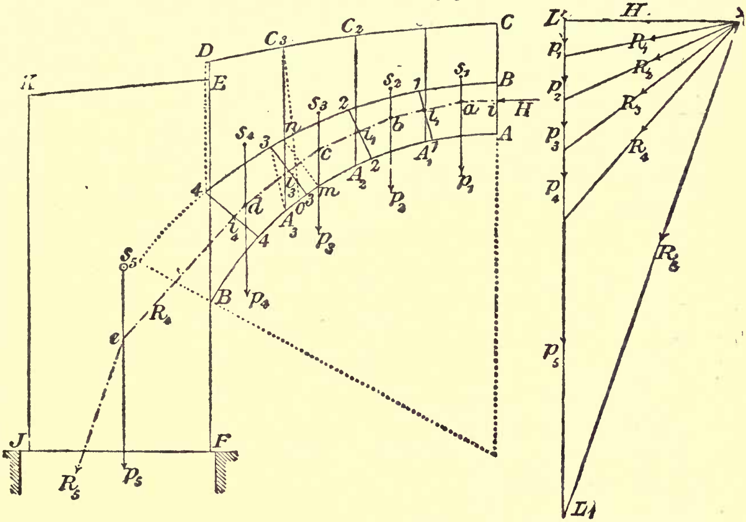
If the arch is a bridge arch the live load must be taken into account and must also be represented by an equivalent mass of arch material. The live load per square foot of roadway may be taken in street bridges at 80 lbs. and in railway bridges at 290 lbs. Thus if g is the weight per square foot of the live load on a bridge and γ the weight of a cubic foot of arch material, then the height (y_1) of homogeneous material equivalent to the live load is given by the equation—

$$g = \gamma . y_1; \text{ or } y_1 = \frac{g}{\gamma} \quad . \quad . \quad . \quad . \quad (\epsilon)$$

[In English practice it is usual to allow about 70 lbs. per square foot as the weight of a crowd of people, and for railway bridges about 2 tons per foot for each line of rail.]

In Fig. 76 an arch with its load line deduced from equations (δ) and (ε) is shown. The line of resistance of this arch is to be drawn.

FIG. 76.



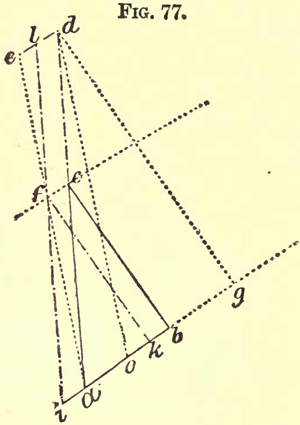
Assuming the *thickness* of the arch perpendicular to the plane of the paper to be 1 foot and the weight of a cube foot of the arch material to be taken as the unit of the forces acting on the arch, then the *areas* $A\,C\,C_1\,A_1$, $A_1\,C_1\,C_2\,A_2$, etc., are evidently proportional to the *weights* of the several vertical strips and may therefore be taken to represent those weights.

The weight of the vertical strip above the skewback is borne entirely by the abutment and does not therefore enter into the question of the determination of the stability of the arch.

The breadths of the strips should be so small that they are nearly trapeziums. The areas of these trapeziums will then be equal to the product of their breadths into their mean heights, and their centres of gravity will be approximately in their centre lines.

In supposing that at the joint *m n* the weight $A A_3 C_3 C$ of the first three trapeziums acts, there is an error, since the real weight acting at that joint will be *less* than $A A_3 C_3 C$ by the triangle $A_3 m n$. Hence the joint at which the weight $A A_3 C_3 C$ acts must be somewhere a little lower down. To obtain this joint bisect $A_3 m$ (the base of the triangle of error), join *o* the point of bisection with C_3 and draw A_3 3 parallel to $o C_3$. Then the joint passing through 3 is the one required, and on this joint act the combined weights of the three first strips.

FIG. 77.



This correction of the joint amounts to reducing a triangle *a b c* to a parallelogram *adli* (Fig. 77) having as one of its sides the hypothenuse *a c* of the triangle produced to a fixed point *d*.

Now $ao = ob$, and *cf* is parallel to *o b* and *d e*, hence the area *a d l i* or *b c d l f k* is equal to *a b c*. For—

$$\Delta\, abc = ao.bc$$

and

$$bcdlfk = adli = fc.dg,$$

but

$$fc : ed = bc : dg$$

or

$$fc.dg = bc.ed = ao.bc.$$

Therefore

$$\Delta\, abc = \text{area } bcdlfk.$$

By means of the above construction obtain a new series of joints on which the weights of the strips $A\,C_1$, $A\,C_2$, $A\,C_3$ act. Then determine the horizontal thrust H by equation (γ) or more correctly from

$$H = r.z, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\zeta)$$

where r is the radius of curvature in feet at the crown of the arch and z the height of the point C above the crown A.*

In order to investigate the stability of the arch with respect to its several joints, the line of resistance must be drawn, i. e. the horizontal thrust H must be combined successively with the weights of the portions $A\,C_1$, $A\,C_2$, $A\,C_3$ ... and the intersections i_1, i_2, i_3, ... of the resultants so obtained with the corresponding joints 11, 22, 33 .... will then give points on the required line of resistance.

The construction can be made in the following way. Draw the diagram of forces as shown in Fig. 76, making O L equal to the horizontal thrust H. Then set off p_1, $p_1 + p_2$, $p_1 + p_2 + p_3$ .... (the weights of the strips $A\,C_1$, $A\,C_2$, $A\,C_3$ ....) from L on $L\,L_1$ the line of loads, thus obtaining the resultants R_1, R_2, R_3 .... In order to obtain the lengths representing p_1, p_2, p_3, ... scale off the breadth and mean height of each strip from the drawing, and obtain the areas of all the strips in square feet. Then take off the numbers so obtained from *any* convenient scale and set them off from L along $L\,L_1$. The thrust H (obtained from equations γ or ζ) must of course be taken off from the same scale.

Through a the intersection of H with p_1 draw a line

* The proof of this equation is given in Professor von Ott's 'Baumechanik,' 1st part, p. 56.

parallel to the ray R_1, then i_1, the point in which $a\,b$ cuts the joint 11, is a point on the line of resistance. From b draw $b\,c$ parallel to R_2, thus obtaining i_2 in the joint line 22, which is a second point on the line of resistance. The points i_3 and i_4 in the joints 33 and 44 respectively are similarly obtained.

If now the line of resistance passes within the centre third of each joint, the arch possesses the degree of stability against rotation necessary in practice. If moreover, the line of resistance makes with the normal to each joint an angle less than the angle of friction of the arch material (about 30°) *the arch possesses sufficient stability against sliding. If finally the maximum normal pressure* $N\left(=\frac{2\,R}{b}\right)$ *brought to bear on the joints is less than the safe resistance of the arch material to crushing, (using a factor of safety of* 30 *for a bridge of about* 100 *feet span,) the arch is safe against crushing.*

It is evident that in flat arches the line of resistance $i\,i_1\,i_2\,i_3\,i_4$ will approximately coincide with the line of pressure $a\,b\,c\,d\,e$, and hence in such arches the degree of stability can be inferred merely from the line of pressure.

In order to determine whether the abutment is strong enough, combine the resultant of H and the weight of the portion A B D C, which resultant is given by the ray R_4, with the weight of the abutment p_5 obtaining a new resultant R_5, and see whether R_5 cuts the ground surface F J in such a way as to fulfil the three above conditions.

The moving load has been taken only as covering the arch and not the abutment, and hence the reduced load line C D E K is in Fig. 76 a broken and not a continuous line.

PART III.

ELEMENTS OF THE THEORY OF STRENGTH OF MATERIALS.

53. *Strength of a Prismatic Bar.*—The tensile or compressive strength of a prismatic bar comes into play when P, the resultant of the exterior forces, coincides in direction with the axis of the bar and is exerted either as a tension or compression according as P tends to produce an elongation, or compression between two neighbouring cross sections. Hence, if in addition to the external loading, the weight of the bar is taken into account, the bar must be placed vertically in order that its normal strength may be exerted. In short bars however, such as those which usually occur in braced structures, the effect of the weight of the bar may be left out of consideration, since it is very small relatively to the exterior forces.

The tensile or compressive strength of a bar, i. e. its resistance to tension or compression in the direction of its length, is directly proportional to the cross section, since evidently the greater that cross section is the more fibres there are to resist tearing or crushing.

That weight which is just capable of tearing or crushing a prism whose cross section is a unit of area is termed the "coefficient," or "modulus" of resistance to tension or compression.

If K is the modulus of resistance to tension,
K_1 „ „ „ „ compression,
F the area of the cross section of a bar,
P the force which will tear or crush it.

Then P is given by the equations,

$$\left.\begin{array}{l} P = K \,.\, F \\ P = K_1 \,.\, F \end{array}\right\} \ldots\ldots\ldots\ldots (a)$$

The following table gives the mean value of the moduli of resistance of the most important materials, the pound being taken as unit of weight and the square inch as unit of area.

Material.	Modulus of Resistance.	
	Tension.	Compression.
Wrought iron..	57,000	57,000
Ordinary steel	85,000	85,000
Cast steel	114,000	142,000
Cast iron	18,000	105,000
Wood	11,400	8,500
Stone	—	4,300

In actual practice we have to deal less with the force by which a prism can be torn asunder or crushed than with the load which that prism can *safely support* for the time intended, and it is therefore usual to load each unit of section with a fixed fraction of the modulus of resistance only, this fractional portion of the modulus of resistance being called the "safe" or "working load." If the working load is $\frac{1}{n}$th of the modulus of resistance, n is termed the "factor of safety."

If therefore n is the factor of safety, k the working load for tension and k_1 that for compression, then

$$\left.\begin{aligned} k &= \frac{K}{n} \\ k_1 &= \frac{K_1}{n} \end{aligned}\right\} \ldots\ldots\ldots\ldots (b)$$

Hence the safe load of a bar of cross section F is

$$\left.\begin{aligned} P &= k\,.\,F \\ \text{or} \quad P &= k_1\,.\,F \end{aligned}\right\} \ldots\ldots\ldots (c)$$

If however, the actual and the working loads are given, then the cross section is determined by

$$\left.\begin{aligned} F &= \frac{P}{k} \\ \text{or} \quad F &= \frac{P}{k_1} \end{aligned}\right\} \ldots\ldots\ldots\ldots (d)$$

The proper degree of security to be assumed, i. e. the value of the factor of safety n, depends not only on the nature of the material but also on the purpose for which the bar in question is intended. In general n should be greater for a travelling than for a fixed load and n should moreover be greater the greater the vibration to which the bar is subjected by the travelling load is. For instance, in calculating the dimensions of the structural parts of a wrought-iron bridge, n is taken at 5 for the main girders and 6 for the cross girders which are subject to greater vibration; while in building where the loads are mostly stationary n may be taken at 3 or 4.

The following table gives the usual values assigned to n for different materials.

Material.	Fixed Load.	Travelling Load.	
		Temporary Structures.	Permanent Structures.
Wrought iron, Steel, Cast steel	4	3	5—6
Cast iron	5	4	6—8
Wood	10	6	12
Stone	20	—	30

Hence referring back to the first table we have the following as the mean value of the working load, i. e. the permissible load per unit of sectional area, taking as before the pound as the unit of weight and the square inch as the unit of area.

Material.	Dead Load.		Travelling Load.			
			Temporary Structures.		Permanent Structures.	
	Tension.	Compression.	Tension.	Compression.	Tension.	Compression.
Wrought iron	11,400	11,400	14,000	14,000	9,900— 8,500	9,900— 8,500
Ordinary steel	21,000	21,000	—	—	17,000—14,000	17,000—14,000
Cast steel	28,000	28,000	—	—	23,000—18,000	23,000—18,000
Cast iron	3,700	21,000	4,000	26,000	2,800— 2,300	17,000—12,800
Wood	1,100	800	1,800	1,400	800	
Stone	—	200	—	—	—	

Example.—What must be the diameter of a wrought-iron rod capable of sustaining a load of 10,000 lbs. taking $\frac{1}{6}$ of the breaking weight as the safe load?

By equation (d) $F = \frac{P}{k}$. Let d be the diameter of the bar, then its cross section is $\frac{\pi}{4} d^2$, and since $P = 10000$ and $k = \frac{57000}{5} = 11400$. Hence

$$\frac{\pi}{4} d^2 = \frac{10000}{11400},$$

whence $d = 1 \cdot 05$ inches nearly.

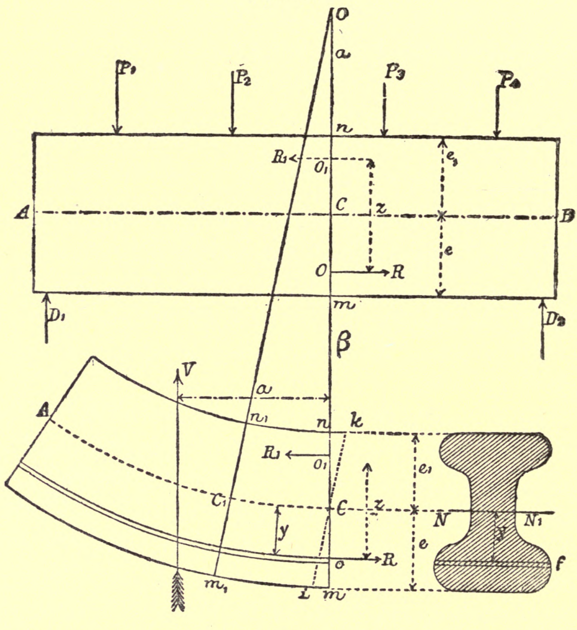
Resistance to Bending of a simple straight Beam.

54. *General Considerations.*—A body will be subject to bending stress when the exterior forces are perpendicular to its axis, i. e. perpendicular to the line joining the centres of gravity of all cross sections, and when the forces lie in one plane passing through the axis of the body.

The forces may be partly concentrated loads and partly loads distributed over the whole length, or over particular portions of the length of the beam. The reactions of the supports moreover, belong to the exterior forces. Through the action of all the exterior forces, the beam, which may be conceived to be a bundle of fibres running parallel to its axis and fast bound together, undergoes a bending which is supposed however, to be so slight that the limit of elasticity is not passed; i. e. that on the removal of the load the elasticity of the material will restore it to its original form. Consider any portion A C of a beam A B (Fig. 78) in equilibrium, and let C, C_1 be two sections indefinitely near and parallel to each other, then after bending has taken place these sections are no longer parallel but are normal to the bent fibres, and cut each other if produced in a straight line O, Fig. 78.

Since after the bending the two sections C and C_1 converge, it follows that the fibres of different layers

FIG. 78.



intercepted between them have different lengths. Thus a lengthening, i. e. a tension of the fibres lying on the under, or convex side of the beam has taken place and a shortening, i. e. a compression of the fibres on the upper, or concave side.

Hence between the fibres in tension and those in compression there must be a layer of fibres which are neither in tension nor compression and which retain their original length in spite of their curvature. This layer is called the *neutral surface;* its intersection with any cross section the *neutral axis* of that section, and the curve in which the plane of forces cuts the neutral surface is termed the *line of mean fibre.*

From the neutral surface outwards the extension and compression of the fibres gradually increase towards the exterior fibres and attain their greatest value at those exterior fibres. This lengthening and shortening of the fibres may be supposed to be such that the two contiguous cross sections remain perpendicular to the axis after bending has taken place.

55. *Determination of the Stresses at any particular Cross Section.*—By the term stress is understood the interior force per unit of area of section.

Suppose a beam A B, Fig. 78, acted upon by the external forces D_1, $P_1 \ldots D_2$ and in equilibrium, to be cut into two portions A C and B C by a section plane $\alpha\beta$. Now the exterior forces acting on *one* portion of the beam cannot maintain equilibrium unless new exterior forces are applied at the plane of section. These new exterior forces restoring equilibrium must evidently be equal to the interior forces acting in the beam at the position of the section plane $\alpha\beta$ before separation took place.

Consider the portion A C and in place of the exterior forces acting on this portion, take into account their resultant, i. e. the shearing force V, Fig. 78, corresponding to the section C and obtained as in para. 6. Thus in order that equilibrium may be maintained an exterior force must be applied at the point of section of every fibre, coinciding in direction with the length of the fibre, and equal to the stress which existed in that fibre previous to the section. The resistances to stress of individual fibres can be considered as horizontal forces if the bending, as has been already premised, does not exceed the elastic limit of the material and is in fact very small.

Besides the horizontal forces replacing the fibre stresses there must also be applied at the section C a vertical force acting downwards maintaining equilibrium with the shearing force V and therefore having a magnitude $-$ V.

Hence at the section C there exists besides the resistances in the direction of the length of the fibres, a resistance to shearing acting along the section plane $\alpha \beta$ equal in magnitude but opposite in direction to V.

In ascertaining the fibre stresses acting at the section C, it must be remembered that by reason of the flexure which is supposed to have taken place, the originally straight and equal elementary fibres lying between two indefinitely near and originally parallel cross sections C and C_1 have become curved, so that after flexure they take the form of small circular arcs of different lengths and different radii having a common centre in O.

To obtain the amount of extension and compression of the fibres, draw $i\,k$ through C parallel to the plane $m_1\,n_1$, then the portions of arcs intercepted between C i and C m give the extensions and those intercepted between C n and C k the compressions which the several elementary fibres at different distances from the neutral surface and of the original length C C_1 have undergone during the flexure of the beam A B.

These extensions and compressions are evidently proportional to the distances of the fibres from the neutral surface $C C_1$ and by Hooke's law the alterations of length are also proportional to the stresses, hence *the stress in any fibre is proportional to its distance from the neutral surface.*

Let s be the stress of a fibre at a unit of distance from the neutral axis $N N_1$, then the stress in a fibre distant y from the neutral axis is $s . y$.

In order to ascertain the fibre stresses at different points on the plane of section, suppose the area of the section to be cut up into indefinitely small strips parallel to the neutral axis $N N_1$. Then, if f is the area of an elementary strip distant y from the neutral axis, the stress in the strip in question is $s . y . f$.

This stress is a tension, or positive stress if the elementary strip in question lies *below* the neutral axis and a compression, or negative stress if *above* the neutral axis.

If now the portion A C of the beam is in equilibrium, then, since no part of the horizontal stresses can be balanced by the vertical shearing force V, the algebraic sum of these horizontal stresses must be nil. Hence employing the symbol of summation Σ,

$$\Sigma (s . y . f) = o;$$

or, since s is a common factor to every term of the series and cannot be zero,

$$\Sigma (y . f) = o.$$

Now $f . y$ is the statical moment of an elementary area about the neutral axis $N N_1$, hence by the last equation the sum of the moments of all the elementary areas about the neutral axis is zero. By the properties of the centre of gravity of a plane area the product of the whole area into the distance of its centre of gravity from any point is equal to the sum of the products of the elementary areas into their distances from the same point.

Hence the equation $\Sigma (y . f) = o$ shows that *the neutral axis passes through the centre of gravity of the cross sections.*

Again, since the portion A C of the beam is in equilibrium and no rotation takes place, the algebraic sum of the statical moments of all the forces acting on A C about any axis, e. g. the neutral axis, must be zero. Hence M the moment of the shearing force V which tends to cause left-handed rotation, must be equal to the sum of the statical moments of all the fibre resistances of the elementary areas, each of which resistances tends to cause right-handed rotation in A C.

The statical moment of the fibre resistances of an elementary area distant y from the neutral axis is

$$s . y . f . y ; \text{ or } s . f . y^2.$$

Hence as a condition of equilibrium

$$\text{M} = \Sigma (s . f . y^2) ;$$

or, taking out the common factor s,

$$\text{M} = s . \Sigma (f . y^2).$$

Now $\Sigma (f . y^2)$ the sum of the products of the elementary areas into the squares of their distances from the neutral axis is termed the *moment of inertia* of the whole sectional area about the neutral axis. Putting I for the moment of inertia,

$$\text{M} = s . \text{I} \quad . \; . \; . \; . \; . \; . \; . \; . \; (a)$$

It is necessary to bring in a more practically useful expression instead of s the stress per unit of area of the fibres at a unit of distance from the neutral axis. The exterior fibres which are at the greatest distance from the neutral axis suffer the greatest stress, which stress, if the beam is not strained beyond its elastic limit, must not exceed a certain fixed value depending on the strength or elasticity of the material, a proper proportion of which fixed value can be substituted in equation (a) in place of s.

Let e and e_1 be the respective distances of the extreme extended and compressed fibres, and suppose k and k_1 to be the greatest permissible extension and compression per unit of area of those fibres. Then, since the stresses of the fibres are proportional to the distances of the latter from the neutral axis,

$$\frac{s}{k} = \frac{1}{e}; \text{ and } \frac{s}{k_1} = \frac{1}{e_1},$$

whence

$$s = \frac{k}{e}; \text{ or } s = \frac{k_1}{e_1}.$$

Substituting these values in equation (a)

$$M = \frac{k}{e} I; \text{ or } M = \frac{k_1}{e_1} . I \quad . . . . (\beta)$$

The expression $\frac{k}{e} . I$ or $\frac{k_1}{e_1} I$ is called the *moment of resistance* of the cross section. Hence—

The moment of resistance of any cross section is equal to the bending moment M, *or the moment of the shearing force* V *at that section.*

From equations (β)

$$\frac{k}{e} = \frac{k_1}{e_1}.$$

Hence a well-constructed beam should be so designed that the greatest permissible tensile and compressive stresses in the exterior fibres are simultaneously brought into action.

For steel, wrought iron, and wood $k = k_1$ nearly, in beams formed of these materials therefore, the cross sections will be symmetrical about their neutral axis.

For cast iron, however, the resistance to compression is at least twice as great as the resistance to tension, a cast-iron beam should therefore be so designed that the extreme compressed fibres are at least twice as far from the neutral axis as the extreme extended fibres.

Moreover, since the material nearest to the neutral axis is least strained, it follows that in a well-designed beam the material will be disposed as far as possible from the neutral surface.

The equation $\frac{k}{e} = \frac{k_1}{e_1}$ shows that it is indifferent in practice whether the equation $M = \frac{k}{e} . I$, or $M = \frac{k_1}{e_1} . I$ is made use of; it is more usual, however, to employ the former.

56. *Practical Applications.*—The equation $M = \frac{k}{e} . I$ is most important in the case of beams subject to flexure, since on the one hand it serves to determine the sectional dimension proper to any part of the beam so as to offer an adequate resistance to the known loading, while on the other hand it serves to determine the stresses of the most strained portions of the beam, due to an ascertained load distribution.

With regard to the former application of the equation it has to be decided whether the beam is to have a uniform, or a varying section. If the former, (which is allowable only in beams of small length on account of the greater simplicity of manufacture,) it is evident that the uniform section must be that calculated for the particular section of the beam at which the bending moment M of the exterior forces is a maximum, i. e. for the *breaking*, or *dangerous section*.

In the case of long beams however, not only would unnecessary material be used by assuming a uniform section, but the supporting power of the beam would be diminished. For long beams therefore, a varying cross section corresponding as closely as possible to the bending moment at various positions along the beam should be adopted. A beam so designed evidently combines an equal resisting power along its whole length with the least waste of material.

57. *Determination of the Moment of Resistance.*—The equation $M = \frac{k}{e} I$ involves four quantities, hence if one of them is to be determined the remaining three must be known.

Usually the three known quantities are the length l of the beam, the material of which it is to be formed and the amount and disposition of the load.

Having given the length l and the disposition and amount of the load, the bending moment M can be determined by the methods previously indicated.

Since however, k is settled by the choice of the material, it will usually be necessary only to determine the moment of inertia $I = \Sigma (f.y^2)$ by which in general the form of the section is given and its height, which in beams of uniform section amounts to from $\frac{1}{8}$ to $\frac{1}{16}$ of the length l.

The moment of inertia of various forms of section can be obtained both graphically and analytically. Since the determination of moments of inertia in simple ways, i.e. without the use of the calculus is very circuitous, and moreover the moments of inertia of sections of simple form are to be found in every text-book of Mechanics or of the Strength of Materials, the graphic method alone is given here.

Suppose the portion A C of the beam, Fig. 78, to be bent by the action of the shearing force V to such an extent that the stresses per unit of area of the outermost fibres have attained their greatest permissible values k and k_1. Determine R and R_1 the respective resultants of the horizontal tensile and compressive stresses acting at the cross section $m\ n$ as well as o and o_1 their points of application whose distance apart is z, then as a condition of equilibrium

$$R - R_1 = o; \text{ or } R = R_1;$$

and in order that rotation may not take place the algebraic sum of the moments of the forces acting on the portion A C with respect to any point in the plane of those forces must be zero.

Thus taking moments about o_1

$$V.a - R.\overline{o\,o_1} = o; \text{ or } V.a = R.\overline{o\,o_1}$$

Putting M for $V.a$ and z for $\overline{o\,o_1}$

$$M = R.z.$$

But by equation (β)

$$M = \frac{k}{e} I.$$

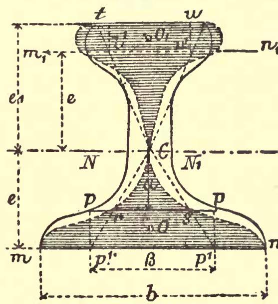
Hence the moment of inertia is given by the equation

$$I = \frac{e}{k} . R . z.$$

We shall now proceed to determine the moments of resistance and of inertia for the section given in Fig. 79.

Let b be the breadth of the undermost layer of fibres $m\,n$ and k the stress per unit of area in it. Then $b.k$ is the resistance of this layer of fibres, or if k is taken as representing unity on the scale of resistances b will be the resistance of the layer.

FIG. 79.



It is required to determine ρ the resistance of any layer of fibres $p\,p$ at a distance a from the neutral axis $N\,N_1$ and having a breadth β. By para. 55 the stress s per unit of area of the fibres at a unit of distance from $N\,N_1$ is equal to $\frac{k}{e}$, or if $k = 1$, $s = \frac{1}{e}$, hence the resistance of the layer $p\,p$ is given by the equation

$$\rho = s . \beta . a = \frac{\beta . a}{e}.$$

Thus ρ is a fourth proportional to the three magnitudes e, a, β, and can therefore be easily obtained graphically.

Project $p\,p$ perpendicularly on to the base line $m\,n$ and join both the points $p'\,p'$ to C the centre of gravity of the section, then the lines p' C, p' C cut off on $p\,p$ the required resistance $r\,s = \rho$. For the triangle $C\,p'\,p'$ is similar to the triangle C r s and in similar triangles the bases are as the heights, hence

$$\beta : \rho :: e : a \text{ or } \rho = \frac{a . \beta}{e}.$$

Similarly the resistances of other fibre-layers of the lower half of the section are obtained and their end points are jointed by a continuous curve, the included area being cross lined in the figure.

Suppose that the stress k of the undermost layer of fibres acts uniformly over the whole of the lined area C r m n s C; k having been taken as the unit on the scale of resistances. Then R the total resistance to tension of the lower half section is obtained by multiplying F the area of C r. . . . C, by k, or

$$R = F . k$$

and the point of application of R is O the centre of gravity of the plane figure C r. . . . C, which figure may be termed the "resistance area" (Widerstandsfläche).

[The term *resistance area* is applied to an area which expresses in magnitude and distribution the direct stresses on any cross section of a beam.]

Proceeding in the same way the total resistance to compression R_1 (= R) of the upper portion of the section lying above the neutral axis N N_1 is obtained. If, however, the uppermost layer t u is not at the same distance from N N_1 as m n, then the reduction of the resistances must be made relatively to the layer m_1 n_1 situated at a distance from N N_1 equal to that of m n the resistance of which (k per unit of area) has been taken as unity on the scale of resistances.

If the section is symmetrical about the neutral axis, it will be necessary only to obtain R and the point O, then R_1 is equal to R and O_1 will be at the same distance from $N N_1$ as O.

Having determined R and the distance $O O_1 = z$, then the moment of resistance of whole section $M = R.z$ and the moment of inertia $I = \frac{e}{k}.R.z$.

But $R = F.k$; hence $I = e.F.z$, where F is the resistance area.

Note.—In determining the moments of inertia and resistance of a section by means of the above method it is necessary to be able to obtain the centres of gravity of irregular plane areas.

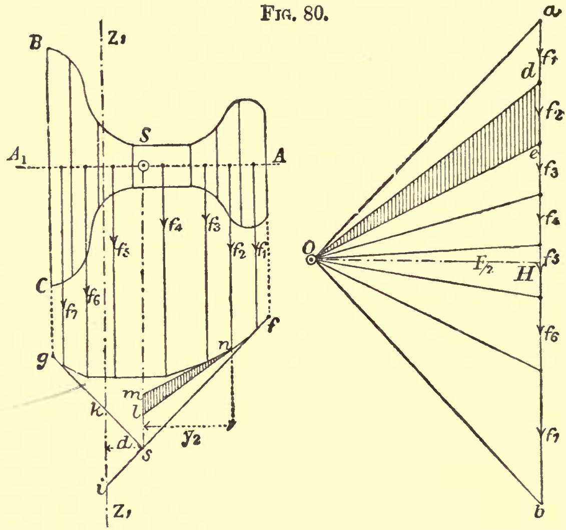
In the following paragraph it is briefly shown how such centres of gravity can be found and at the same time another method is indicated for determining the moment of inertia of a plane area by means of the funicular and force polygons.

58. *Graphic determination of the Centre of Gravity and Moment of Inertia of a Plane Area.*—Let A B C, Fig. 80, be the given figure of area F, whose centre of gravity and moment of inertia about an axis passing through the centre of gravity and parallel to $Z Z_1$ are to be determined.

Draw lines parallel to $Z Z_1$ cutting the area A B C into small strips $f_1, f_2, f_3, \ldots$, whose breadths are so small that each strip may be considered to be a trapezium whose area f is equal to the product of the breadth into the mean height. Draw the polygon of forces corresponding to the elements $f_1, f_2, f_3 \ldots$ and also a funicular polygon, making the polar distance O H equal to $\frac{F}{2}$ and bisecting $a b$ at right angles so that the angle $a O b$ is a right angle.

Produce the outermost sides of the funicular polygon to meet in s, then (para. 6) the resultant of $f_1, f_2 \dots$ passes through s, hence a line through s parallel to $Z_1 Z_1$ passes through the centre of gravity of the whole area

FIG. 80.



A B C. Hence if A B C is symmetrical about the axis $A_1 A$, the point S is its centre of gravity. If the figure is not symmetrical, then the above construction must be repeated and a second line passing through the centre of gravity and parallel to another axis $Z_2 Z_2$ must be obtained. Then the intersection of this second line with the line through s parallel to $Z_1 Z_1$ will be the required centre of gravity.

(If the section is a very irregular figure, its centre of gravity can be obtained by drawing the figure on card or stiff paper and cutting it out. The cut out figure is suspended vertically by a pin passing through any point near its edge, then a plumb line suspended from the same pin traces on the figure a line passing through its centre of gravity. This line must be marked on the figure, and the process having been repeated by suspending the figure about another point a second line is obtained which will intersect the first in the required centre of gravity.)

Having determined the line S *s* and having calculated F_1 the area of the figure *f s g k* enclosed by the funicular polygon and its produced sides *f s* and *g s*, then the moment of inertia I of the figure A B C about an axis S *s* is given by the equation

$$I = F . F_1 \, . \, . \, . \, . \, . \, . \, . \, . \quad (a)$$

And if I_1 is the moment of inertia about $Z_1 Z_1$,

$$I_1 = F \, (F_1 + F_2) \, . \, . \, . \, . \, . \, . \quad (b)$$

where F_2 is the area of the triangle *i k s* enclosed by the axis $Z_1 Z_1$ and the two produced sides of the funicular polygon *g s* and *f s*.

To prove the truth of the above, produce any two adjacent sides of the funicular polygon to cut the line S *s* in *m* and *l*. Then the triangle *l m n* is similar to the triangle *d* O *e* in the polygon of forces. Let y_1, y_2, y_3, etc., be the respective distances from S *s* of the lines parallel to S *s* passing through the centres of gravity of $f_1, f_2, f_3 \ldots$ then since in similar triangles the bases are as the heights,

$$\frac{m\,l}{y_2} = \frac{d\,e}{O\,H}.$$

But

$$d\,e = f_2 \text{ and } O\,H = \frac{F}{2}$$

therefore
$$\frac{m\,l}{y_2} = \frac{f_2}{\frac{F}{2}}$$

Hence
$$m\,l\,.\,\frac{y_2}{2} = \frac{f_2\,y_2^{\,2}}{F} = \text{triangle } l\,m\,n\,.\;\;.\;\;.\;\;.\;\;(c)$$

But F_1 the area of the figure $f\,s\,g\,k$ is made up of the sum of the areas of all the other triangles corresponding to $l\,m\,n$.

Thus
$$F_1 = \frac{1}{F}\Sigma\,(f.y^2)$$

and $\Sigma\,(f.y^2)$ is the moment of inertia of the plane figure A B C about an axis S s. Hence
$$I = \Sigma\,(f.y^2) = F.F_1.$$

Again, if I_1 is the moment of inertia of A B C about an axis $Z_1\,Z_1$ parallel to and at a distance d from S s; $I_1 = I + F.d^2$, but since the triangles $i\,k\,s$ and $a\,b\,O$ are similar,
$$\frac{i\,k}{d} = \frac{F}{\frac{F}{2}} = 2;$$

hence $i\,k = 2\,d$.

The area of the triangle $i\,k\,s$ is d^2, hence
$$I_1 = F\,(F_1 + F_2).$$

59. *Moments of Inertia of Simple Sections.*—In the case of sectional areas composed of rectangles one side of which is parallel to the axis about which the moment of inertia is to be obtained, the latter is determined more readily and accurately by analytical processes.

Thus the moment of inertia of the rectangle A B C D

Fig. 81, about an axis Z Z passing through its centre of gravity can be obtained by means of the formula $I = e\,F.z$ (para. 57) and is

$$I = \tfrac{1}{12}\, b.h^3 \quad \ldots\ldots\ (1)$$

Fig. 81.

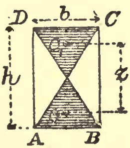


Fig. 82.

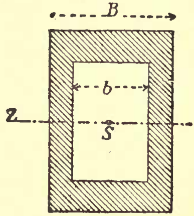
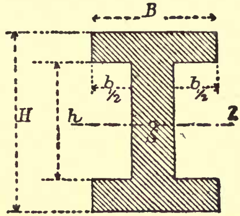


Fig. 83.



By making use of this fundamental formula the moments of inertia of the five following forms of section can be obtained.

I. The sections given in Figs. 82 and 83 can be regarded as the difference of the two rectangles B.H and $b.h$, hence their moments of inertia about the axis $z\,z$ are

$$I = \frac{B.H^3 - b.h^3}{12} \quad \ldots\ldots\ (2)$$

II. The section, Fig. 84, can be regarded as the sum of the two rectangles $b.h$ and $b_1.h_1$, hence for this section,

$$I = \frac{b.h^3 + b_1.h_1^3}{12} \quad \ldots\ldots\ (3)$$

III. For the section, Fig. 85,

$$I = \frac{B.H^3 - b.h^3 - b_1.h_1^3}{12} \quad \ldots\ (4)$$

IV. For the section, Fig. 86,

$$I = \frac{B.H^3 - b.h^3 - b_1.h_1^3 - b_2.h_2^3}{12}. \quad (5)$$

V. For a circular section, Fig. 87, the moment of inertia about a diameter as axis is

$$I = \frac{\pi}{64} D^4 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

VI. Hence for the annular section, Fig. 88,

$$I = \frac{\pi}{64}(D^4 - d^4) \quad . \; . \; . \; . \; . \quad (7)$$

Fig. 84.

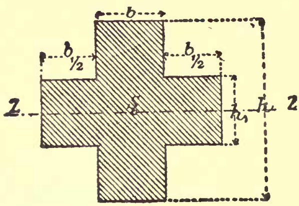


Fig. 85.

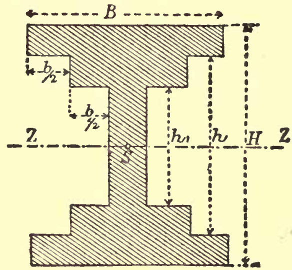


Fig. 86.

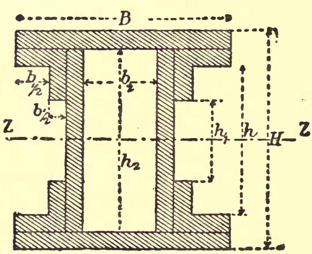


Fig. 87.

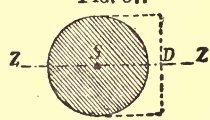
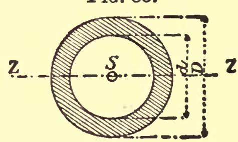


Fig. 88.



60. *Examples in Resistance to Flexure.*—I. A fir baulk of rectangular section (8″ × 12″) length 20′ 0″ is *sup-*

ported at both ends. What uniformly distributed load over its whole length can it support taking $\frac{1}{10}$ of the breaking weight as the safe load?

The bending moment is greatest at the centre of the baulk, and if p is the load per unit of length,

$$M_{max} = \frac{p.l^2}{8}.$$

But

$$M = \frac{k}{e} I; \text{ where } k = \frac{8500}{10}; \; e = 6'';$$

$$I = \frac{1}{12} \times 8 \times 12^3 = 1152; \text{ and } l = 240 \text{ inches}$$

Hence

$$p.l = \frac{8 \times 8500 \times 1152}{240 \times 6 \times 10} = 5440 \text{ lbs.}$$

and

$$p = 272 \text{ lbs. per foot run.}$$

The weight W of the baulk, supposing that of a cube foot of fir to be 30 lbs., is

$$W = \frac{8 \times 12}{144} \times 20 \times 30 = 400 \text{ lbs.}$$

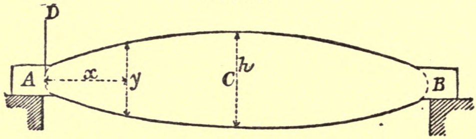
The total load which the baulk can support therefore is

$$p.l - W = 5040 \text{ lbs.}$$

II. What must be the form of a beam A B, Fig. 89, of variable height h and constant breadth b, so that it may offer an equal resistance to flexure at every section, when strained by a concentrated load P acting at its centre?

The reaction of the supports is $D = \frac{P}{2}$; hence the bending moment at the centre of the beam is $M = \frac{P}{2} \cdot \frac{l}{2}$ and at a section distant x from A the moment $M_1 = \frac{P}{2} . x$.

FIG. 89.



But $M = \frac{k}{e}$ I, and in the present case $e = \frac{y}{2}$ and $I = \frac{1}{12} . b y^3$. Hence

$$M_1 : M = \frac{P}{2} . x : \frac{P}{2} \cdot \frac{l}{2} = \frac{1}{6} k . b . y^2 : \frac{1}{6} k . b . h^2$$

or

$$x : \frac{l}{2} = y^2 : h^2;$$

therefore

$$y^2 = \frac{2 h^2}{l} . x \quad . \; . \; . \; . \; . \; . \; . \quad (a)$$

This is the equation to a parabola with its vertex at A and parameter $\frac{2 h^2}{l}$. The requisite form of the beam therefore is that of two parabolas whose vertices are at the supports A B and whose common ordinate is h; and since

$$\frac{P}{2} \cdot \frac{l}{2} = \frac{1}{6} k . b . h^2,$$

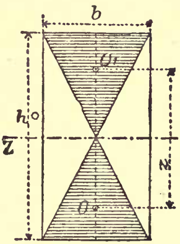
therefore

$$h = \frac{1}{2} \sqrt{\frac{6 P . l}{b . k}}.$$

A beam of the above form is said to have *uniform strength* or to offer *uniform resistance* to flexure.

For $x = 0$, the equation (a) gives $y = 0$, and the section at the points of support would be nil. Since however, the reactions $\frac{P}{2}$ act at these supports as shearing forces, the height h_0 of the sections at A and B must be made to depend on these shearing forces. Now if b is the breadth of the section, $b = \frac{V}{\sigma z}$,* where $V = \frac{P}{2}$; σ is the resistance to shearing per unit of sectional area which in this case is $0 \cdot 6\, k$ and for the rectangular section, Fig. 90, $z = \frac{2}{3} h_0$; hence

Fig. 90.



$$b = \frac{\frac{P}{2}}{0 \cdot 6\, k . \frac{2}{3} h_0};$$

or

$$h_0 = \frac{P}{0 \cdot 8 b . k}.$$

Note.—If the beam A B were uniformly loaded with a load p per unit of length, then the bending moment M for the middle of the beam is equal to $\frac{p . l^2}{8}$ and for a section distance x from A the moment

$$M_1 = \frac{p . l}{2} x - \frac{p . x^2}{2} = \frac{p . x}{2} (l - x).$$

But

$$M = \frac{k}{e} I, \text{ where } e = \frac{y}{2}, \text{ and } I = \frac{1}{12} b y^3;$$

* The *rationale* of this formula is given in Prof. von Ott's 'Baumechanik,' Part I., page 78.

therefore

$$M_1 : M = \frac{p.x}{2}(l - x) : \frac{p.l^2}{8} = y^2 : h^2;$$

hence

$$y = \frac{2h}{l}\sqrt{x(l - x)}.$$

This equation is that of an ellipse whose major axis A B $= l$ and whose minor axis h is given by the equation

$$\frac{p.l^2}{8} = \frac{1}{6}k.b.h^2,$$

from which

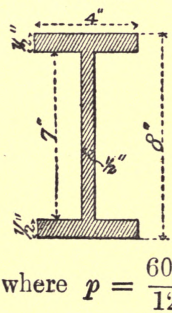
$$h = \frac{1}{2}\sqrt{\frac{3.p}{b.k}}.$$

The height h_0 of the section of the beam at the points of support is obtained in the same way as before and

$$h_0 = \frac{p.l}{0{\cdot}8\,b.k}.$$

III. What is the length of a wrought-iron beam of the section shown in Fig. 91 capable of supporting a load of 600 lbs. per foot run including its own weight? The beam to be supported at both ends and the greatest possible stress 11200 lbs. per square inch.

FIG. 91.



Taking an inch as the unit of length, the length l can be found by means of the formula

$$M_{max} = \frac{p.l^2}{8} = \frac{k}{e}I;$$

where $p = \frac{600}{12}$ lbs.; $k = 11200$; $e = \frac{8}{2} = 4$; and

$$I = \frac{4 \times 8^3 - 3{\cdot}5 \times 7^3}{12} = 70{\cdot}625;$$

therefore $$\frac{50 . l^2}{8} = \frac{11200}{4} \times 70 \cdot 625.$$

$$l^2 = 31640 \text{ inches.}$$

$$l = 178 \text{ inches nearly} = 14' \cdot 10''.$$

61. *Resistance to Crippling.*—If the length L of a compression bar A B, Fig. 92, is from five to ten times the middle sectional dimensions D, then by continuing the axial loading the bar will not be more compressed, but it will be bent and finally broken across by the load. A bar so broken is said to fail by crippling. Resistance to crippling is therefore a combination of resistance to crushing and flexure. Since the theory of resistance to crippling would take up too much space and moreover this theory does not give results in accordance with the rules deduced by Hodgkinson from experiment, we shall merely give two empirical formulæ for the resistance of a bar liable to failure by crippling, which agree with experiment.

FIG. 92.

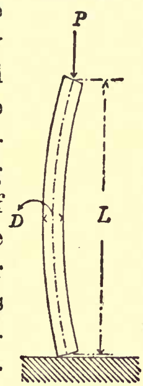
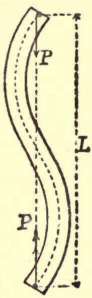


FIG. 93.



For a bar free at both ends the resistance P is given by the formula

$$P = \frac{k . F . I}{I + C F . L^2} \quad \ldots\ldots (\alpha)$$

$$P = \frac{k_1 . F . I}{C . F . L^2 - I} \quad \ldots\ldots (\beta)$$

Where F is the sectional area.

I the moment of inertia of the cross section about an axis passing the angle the centre of gravity and perpendicular to the bending plane.

L the length.

k and k_1 the safe loads which the material can bear in compression, and tension respectively.

C an experimental coefficient depending on the material.

The values of C and k and k_1 for different materials are given in the following table.

In the case of *cast-iron* struts P should be calculated from both the above formulæ, and the *least* of the two values should be taken. For *steel, wrought-iron,* and *wooden* struts, the formula (a) only need be used.

Material.	C	k in lbs. per square inch.	k_1 in lbs. per square inch.
Ordinary steel	0·00009	14000	14000
Wrought iron	0·00009	8500	8500
Cast iron	0·00027	11300	2850
Wood	0·00022	700	800

For example, to determine the load which a wooden bar of square section ($F = D^2$) can safely support, we have $I = \frac{1}{12} D^4$, and hence

$$P = \frac{700\ D^4}{D^2 + 0\cdot00264\ L^2}$$

where D and L are both in inches.

If $\frac{L}{D}$ =	1	10	20	30	40	50	100
$\frac{P}{D_2}$ =	110 23	87·43	53·91	32·96	21·28	14·66	4·08

The formulæ (a) and (β) are applicable only to bars whose extremities are free; if however the ends of a bar are fixed, the bar undergoes the change of form shown in Fig. 93, and the resistance can be found by putting $\frac{L}{2}$ for L in formulæ (a) and (β).

Hence in the case of a bar whose ends are *fixed* (Fig. 93),

$$P = \frac{4\,k.F.I}{4\,I + C.F.L^2} \quad \ldots\ldots \quad (\gamma)$$

or

$$P = \frac{4\,k_1.F.I}{C.F.L^2 - 4\,I} \quad \ldots\ldots \quad (\delta)$$

Example.—What is the resistance to crippling of a wrought-iron bar of rectangular section (2″ × 4″) length 10′, fixed at both extremities?

Here $k = 8500$, $C = 0 \cdot 00009$, $F = 2'' \times 4'' = 8$ square inches. $L = 120''$. $I = \frac{1}{12} h\,b^3 = 2 \cdot 66$.

Substituting these values in equation (γ)

$$P = \frac{4 \times 8500 \times 8 \times 2 \cdot 66}{4 \times 2 \cdot 66 + 0 \cdot 00009 \times 8 \times (120)^2} = 34440 \text{ lbs.}$$

nearly.

The safe compression per square inch of section is therefore

$$\frac{34440}{8} = 4305 \text{ lbs.}$$

For a *cast-iron* strut calculate P from (γ) and (δ), and take the *least* of the values obtained, while for a *steel, wrought-iron*, or *wooden* strut P need be calculated from (γ) only.

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