Pb-66.1

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**Problem Statement:**

(i) Verify that the polynomial in (1) on 66-2 is a Laguerre polynomial, except for a minus sign.
(ii) Find the expressions of $P_N(u)$ for various values of $N$, and compare them to a table of Laguerre polynomials.
(iii) Plot the associated Laguerre polynomials for $n = 3$, and $\alpha = 0, -1, -2, -3, -4$.

**Solution:**

**i.)**

The given equation is:

$$P_N(u) = u + \sum_{n=2}^{N} (-1)^{n-1} \frac{(N-1)(N-2)\ldots(N-n+1)}{n!\,(n-1)!} u^n \quad (Eq. 1)$$

$$N \geq 2 \quad (Eq. 2)$$

(Source: https://docs.google.com/file/d/0B9xP2cHngVLiT2o5NGN6LXdtcjA/edit?pli=1)

Some nomenclature is needed:

The term: $(N-1)(N-2)\ldots(N-n+1)$ can be expressed by a pochammer symbol for the falling factorial. The equation below describes this notation.

$$(\mathrm{N})_{\mathrm{n}} = N(N-1)(N-2)\ldots(N-n+1) \quad (Eq. 3)$$

(Source: https://en.wikipedia.org/wiki/Pochhammer_symbol)

This can be related to a binomial coefficient by:

$$\frac{(N)_n}{n!} = \binom{N}{n} \quad (Eq. 4)$$

(Source: https://en.wikipedia.org/wiki/Pochhammer_symbol)

Which has the explicit form:

$$\binom{N}{n} = \frac{N!}{n!\,(N-n)!} \; for\; 0 \leq N \leq n \quad (Eq. 5)$$

(Source: https://en.wikipedia.org/wiki/Binomial_coefficient)

If (Eq.4) is substituted into (Eq. 1):

$$P_N(u) = u + \sum_{n=2}^{N} \binom{N}{n} \frac{(-1)^{n-1}}{N!} u^n \quad (Eq. 6)$$

The closed form expression for the Laguerre polynomials is:

$$L_N(u) = \sum_{n=0}^{N} \binom{N}{n} \frac{(-1)^{n}}{N!} u^n \quad (Eq. 7)$$

(Source: https://en.wikipedia.org/wiki/Laguerre_polynomials)

(Eq. 6) is almost off by only a negative sign, but it is also missing the first term in the series (the constant term).

If I take the first two terms of (Eq. 7) outside of the sum:

$$L_N(u) = 1 - u + \sum_{n=2}^{N} \binom{N}{n} \frac{(-1)^{n}}{N!} u^n \quad (Eq. 8)$$

If I multiply (Eq. 6) by -1:

$$P_N(u) = -u + \sum_{n=2}^{N} \binom{N}{n} \frac{(-1)^{n}}{N!} u^n \quad (Eq. 6)$$

I come to the conclusion that not only is the sum given in the problem statement off by a negative sign, but it is also missing the constant term.