John Quincy Adams.

CYCLOMATHESIS:

OR AN
EASY INTRODUCTION
TO THE
SEVERAL BRANCHES
OF THE
MATHEMATICS;

Being principally deſigned for the
INSTRUCTION OF YOUNG STUDENTS,
Before they enter upon the more
ABSTRUSE and DIFFICULT PARTS thereof.

Scribere laus magna eſt ; ſed ſcriptis addere lucem
Hoc vero egregiæ dexteritatis opus. RUS. MED.

In TEN VOLUMES.

LONDON:
Printed for J. NOURSE, in the Strand;
Bookſeller in Ordinary to his MAJESTY.
MDCCLXX.

[illegible]

12

[illegible] 130

26.566

Hon Chas. F. Adams,

July 2 - 1891.

THE

GENERAL CONTENTS

OF THE

TEN VOLUMES

OF THE

CYCLOMATHESIS.

VOLUME I. Containing,

1. A General Introduction to the Cyclomatheſis.
2. A Treatiſe of Arithmetic.

VOL. II.

1. The Doctrine of Proportion, Arithmetical and Geometrical.
2. The Elements of Geometry.

Note, The above two Volumes may be bound in one.

VOL. III.

1. The Elements of Trigonometry, Plain and Spherical.
2. A Table of Natural Sines and Tangents.
3. A Table of Logarithmic Sines and Tangents.
4. A Table of Logarithms from 1 to 10,000.

VOL. IV.

A Treatiſe of Algebra, in Two Books.

VOL. V.

1. The Arithmetic of Inſinites, and the Differential Method.
2. Elements of the Conic Sections.
3. The Nature and Properties of Curve Lines.

CONTENTS.

VOL. VI.

The Elements of Optics and Perſpective.

VOL. VII.

1. Mechanics, or the Doctrine of Motion.
2. The Projection of the Sphere, Orthographic, Stereographic, and Gnomonical.
3. The Laws of Centripetal and Centrifugal Force.

VOL. VIII.

A Syſtem of Aſtronomy.

VOL. IX.

1. The Mathematical Principles of Geography.
2. The Theory of Navigation, Spherical and Spheroidical.
3. Dialling. Or the Art of Drawing Dials on all ſorts of Planes.

VOL. X.

1. The Doctrine of Combinations, Permutations, and Compoſitions of Quantities.
2. Chronology: Or the Art of Reckoning Time. With a Chronological Table.
3. Calculation, Libration, and Menſuration: Or the Art of Reckoning, Weighing, and Meaſuring.
4. The Art of Surveying, or Meaſuring of Land.

CYCLOMATHESIS:

OR AN

EASY INTRODUCTION

To the ſeveral Branches of the

MATHEMATICS.

Being principally deſigned for the Inſtruction
of young Students, before they enter upon
the more abſtruſe and difficult parts thereof.

Scribere laus magna eſt: ſed ſcriptis addere lucem,
Hoc verò egregiæ dexteritatis opus. Ruſ. Med.

LONDON,
Printed for J. Nourse, Bookſeller in Ordinary
to his MAJESTY.
MDCCLXIII

General Introduction

Concerning the

NATURE, USEFULNESS, and CERTAINTY

OF THE

MATHEMATICS.

AS man is endued with the noble faculty of reaſon, and likewiſe with a ſtrong innate deſire of knowledge; it is natural for him to exert this his diſtinguiſhing talent in the purſuit of knowledge. Truth alone is the object of knowledge; for it is impoſſible to know a falſe thing to be true: and evidence is the certain mark or criterion of truth; and this conſiſts in the perception of the agreement or diſagreement of our ideas in the mind, according as the things in nature agree or diſagree. As there is no ſtronger paſſion in the human ſoul than the love of truth, and no greater deſire for any thing than to find it out; ſo, when it is found, there is no greater pleaſure to the underſtanding, than the contemplation thereof in the ſeveral branches of ſcience; even when the ſearch of it is attended with the greateſt labour and pains. Truth is of ſuch a nature, as always to be conſiſtent with itſelf, and needs nothing to enforce or recommend it, but its own native evidence. It is but one ſimple, uniform, invariable thing; whilſt its oppoſite, falſhood, is infinitely various, inconſiſtent, and contradictory. As truth is what all men admire, and every one aims at; and error what every man hates, that is not blinded by ſelf-intereſt; it is neceſſary that we take care never to receive any thing for truth, which does not bring its proper evidence along with it. For it is evidence alone that can gain our aſſent, and remove all our doubts; and when that appears, the mind can neither expect nor deſire any thing further. By the help of this we are enabled to diſtinguiſh truth from falſhood, right from wrong; and we likewiſe have a power of ſuſpending our aſſent till that evidence appears; and when it does appear, it compels our aſſent, and carries abſolute conviction. Truth, when expreſſed in words, is the ſame thing as a true propoſition; and, as evidence is a neceſſary voucher for truth, we ought never to give aſſent to a doubtful or obſcure propoſition; but ſhould deny it as long as we can, and not give our judgment as long as we can withhold it, in ſuch things as we can have an evident knowledge of.

Now ſince truth is of ſo amiable a nature, and ſo deſirable to the underſtanding, it will be aſked where it is to be found, and how ſhall we come to the knowledge of it? I anſwer, it is to be found in the writings of the mathematicians, where the method of finding it is clearly explained. In the mathematical ſciences truth appears moſt conſpicuous, and ſhines in its greateſt luſtre. In other ſciences it is either ſelf evident, and then it affords little pleaſure to the mind; or elſe it appears with ſo much obſcurity, that falſhood is often miſtaken inſtead of it. The evidence for it is ſo dim, that it is only ſeen as in a miſt; and truth, ſeen through ſuch a dull medium, will hardly be known to be truth; the mind will be loſt in doubt and obſcurity, and will be unable to make any certain concluſion. But in the mathematics, all their demonſtrations are free from any obſcurity, every ſtep has a clear and intuitive evidence; and where that falls ſhort, the matter is thrown out as not deſerving a place among mathematical truths.

The manner whereby truth is found out, is by reaſoning, which is performed by firſt laying down, as a foundation, certain evident principles, or ſuch as cannot be denied; and then proceeding from theſe by ſeveral ſteps till they come at the concluſion; which ſteps are ſo to be linked with each other, and laid in ſuch order, that the underſtanding may perceive their connection and agreement; which being every where true and right, the concluſion muſt infallibly be true: for all the parts being locked together by truth; the laſt reſult, though never ſo long, muſt be equally true.

Thus mathematicians, from a few plain and ſimple principles, and a continued chain of reaſoning, proceed to the diſcovery and demonſtration of truths that appear at firſt ſight beyond human capacity. The art of finding proofs, and the admirable methods they have invented for finding out and laying in order, thoſe intermediate ideas that ſhew the connection of the ſeveral ſteps of the proof, or the ſeveral links of this chain of reaſoning; is that which has carried them ſo far, and produced ſuch wonderful and unexpected diſcoveries. In this ſcience there appears to be an inexhauſtible fund in the ſeveral branches thereof; any one of which a man may purſue as far as he pleaſes, and ſtill improve his knowledge further and further: and thus, by the help of truths already known, more and more may ſtill be found out *ad infinitum*.

When the mind works on mathematical ideas, it works ſecurely, which cannot be done in other things ſo truly; becauſe one cannot keep ſo ſtrictly to the definitions, or the meaning of words, in other ſubjects;

where the ideas are often confounded. But mathematicians take care not to confound theirs; for none ever miſtook the idea of a ſquare for that of a circle. Therefore mathematical demonſtrations are the moſt proper means to cleanſe the mind from errors, and to give it a reliſh of truth; which is the natural food and nouriſhment of the underſtanding.

Reaſoning, which is the exerciſe of reaſon, is beſt learned from the examples and practices of the mathematicians. It is certain, that no ſort of human knowledge can lay ſo juſt a claim to an unſhaken evidence and certainty, or boaſt ſo great a ſtrength of its demonſtrations, or produce ſuch a multitude of undeniable truths, as the mathematics. All that beautiful analogy, and that harmonious connection and conſiſtency, is quite loſt in other ſciences. Wherefore it is no wonder that greater improvements have been made in the mathematical ſciences, than in all the reſt put together. By following their methods, a habit of right reaſoning is obtained by frequent practice, like other things; and the cauſe why many people reaſon ſo badly is, for want of practice, due attention, and conſideration. They proceed in that tract which chance has put them into, being ignorant of true ſcience, and of thoſe univerſal invariable principles, upon which true reaſoning depends: as is evident from the many inſtances of falſe reaſoning and ignorance, wherewith the diſcourſes and writings of mankind abound.

In purſuance of our reaſoning in the mathematical way, we are often forced to draw diagrams, in order to repreſent the thing in queſtion; likewiſe to form ideas of the ſeveral parts, compound them, divide them, abſtract from them; to conſult the memory, to ſee what has been done and what is to do; to inſpect tables, books, inſtruments, *&c.* to call up all ſuch axioms, theorems, experiments, and obſervations, as are already known, and which can be uſeful to us. And then the mind examines, compares, methodizes, and alters them; till the ſeries be laid in a proper order, from the firſt principles to the laſt concluſion. For the principal thing required in ſtrict reaſoning is, to lay the ſeveral ſteps in due order, to ſee that they be firmly connected, and properly expreſſed, without any rhetorical flouriſhes, and to aim at truth by the ſhorteſt method. This indeed requires cool, ſedate, and ſober thinking; as alſo frequent application and practice, without which nothing can be done to the purpoſe. To which we may add, a fixt, conſtant, and firm reſolution to embrace truth wherever we find it; and to ſhun error and falſehood, when we find ourſelves in danger of falling into them.

There is but one method of true reaſoning, ſuch as has been deſcribed; but the grounds of falſe reaſoning are many, ſuch as theſe, want of faculties, want of learning, defects of memory, want of due reflection, not connecting the ſteps of the proof, truſting too much to the ſenſes, paſſions, appetites, prejudices, cuſtom, ſelf-intereſt, errors of education, wrong ſtating the queſtion, not underſtanding the terms, want of proofs, vulgar received opinions, weak authorities, precipitancy of judgment, *&c.* theſe will frequently diſturb us in our ſearch after truth, and are apt to biaſs the mind in reaſoning upon all other ſubjects; but few or none of them intrude in the mathematical ſciences. Mathematicians never attempt to reſolve any problems without proper *data.*

It muſt be owned that the progreſs of this ſort of knowledge is but ſlow, owing to the difficulty of the ſeveral branches that come under conſideration; but then it is ſure and certain; the acquiſition here gained is real knowledge. For this reaſon it is the work of ages to bring even a ſingle branch to perfection: and every ſucceeding age improves upon the foregoing.

And therefore it is no wonder if the ancients have, in many caſes, made uſe of round-about methods to encompaſs their ends, and given us long and tedious demonſtrations, and laid down many propoſitions, either of no uſe, or too ſimple and trifling to be taken notice of. Whence moſt of their inventions may be demonſtrated ſhorter, propounded eaſier, diſpoſed in a better method, and taught in a more compendious way.

But beſides the pleaſure a man finds in the ſearch and attaining of knowledge, and the agreeable ſurprize the mind is affected with, at the diſcovery of new and difficult truths; the advantages ariſing to mankind from theſe ſciences, in all the parts of human life, are endleſs. By help thereof we are able to keep our accounts regular and juſt, and manage all our tranſactions with one another; to caſt up and calculate immenſe ſums, for nothing lies without the power of numbers; to meaſure and divide lands and eſtates; and alſo all manner of ſurfaces or ſolids; to meaſure inacceſſible diſtances and altitudes, and find the hight of the clouds; to build houſes, caſtles, &c. by which we enjoy the principal delights of life, and ſecurity of health; to make fortifications to defend us from the enemy; to make guns and other inſtruments of war, and to ſhew how to uſe them in our defence; to reſolve all manner of pleaſant and ſubtle queſtions; to build ſhips, and by the help of wind and ſails, and the rules of art, to ſail upon the ſea, and find our way through it to diſtant countries, and traffick with foreign nations, whereby our wealth is increaſed; to contrive inſtruments to weigh and meaſure all ſorts of commodities, and give every man his juſt weight and meaſure; to make engines for raiſing and removing huge bodies; to invent innumerable machines, uſeful in private life, and neceſſary for our living commodiouſly, ſuch as clocks, watches, jacks, pumps, &c. to make dials and other inſtruments for keeping a rea regular account of time; to make ephemerides and chronological tables, to ſhew and account for the return of the various ſeaſons of the year, and to keep account of remarkable tranſactions and events; to deſcribe the ſeveral countries of the earth, and make maps and repreſentations thereof, and even to meaſure the whole earth and ſea; to account for the riſing and falling of the tides; to number the ſtars, and range them in their proper order; to meaſure the magnitude and diſtances of the planets, and explain the laws of their motion, and ſet bounds to their wandering courſes; to aſcertain the ſituation of all the great bodies of the univerſe, and ſhew the fabrick and conſtruction of the whole world; and to admire that wonderful power that contrived and framed it; to lead us through the dark mazes of nature, and through the intricate labyrinths and hidden ſecrets of philoſophy; to make proper inſtruments to improve the ſight, and even reſtore it in old age; and to magnify ſmall bodies, imperceptible to the naked eye, and make them become viſible; and to cauſe remote inviſible things to appear to us large and diſtinct; to give the true repreſentation or draught of any object, ſuch as towers, caſtles, trees, towns, *&c.* and to fix in the mind a method and habit of right reaſoning, a thing of the utmoſt conſequence, without which a man can hardly be called a rational creature.

The time would fail me in attempting to enumerate all the uſes and advantages of mathematical learning; and no words can fully expreſs the praiſes of that ſcience, which wanders through the heavens, the earth, and the ſeas: nor is it poſſible to ſet any bounds to ſo extenſive a ſcience. In this age, the number of its admirers and profeſſors are many, and daily increaſe more and more. Moſt people ſeem to be inſpired with the love of mathematical learning, and to be inamoured with its charms, and to court its

A lemma is a ſhort preparatory propoſition, laid down in order to ſhorten the demonſtration of the main propoſition which follows it.

A corollary, or *conſectary*, is a conſequence drawn from a propoſition already demonſtrated.

A ſcholium is a remark made on any propoſition, corollary, or other diſcourſe.

Principles are the firſt grounds, rules, or foundations, of any ſcience; as definitions, axioms, poſtulates, and hypotheſes.

A definition is the explication of any word or term, in any ſcience; every definition ought to be clear, and contain no word or term but what is perfectly underſtood.

An axiom, or *maxim*, is a ſelf-evident propoſition. Theſe appear to be true at firſt hearing, and no body can deny them, without contradicting common ſenſe and reaſon. Here nothing ought to be allowed for an axiom, but what is clear and ſelf-evident: as this, *the whole is greater than a part*. Out of an infinite number of ſelf-evident truths that occur to the mind, men ſelect ſuch as are general, and of moſt uſe in demonſtrating any ſcience, and lay them up in ſtore, to have recourſe to, as need requires. And though men in their reaſoning do not always mention ſuch and ſuch axioms; yet the mind perceives the force of them, and what they mean, without ſtopping to repeat the words, or name them.

A poſtulate, or *petition*, is ſomething required to be done, which is ſo eaſy, that no body will diſpute it.

An hypotheſis is a ſuppoſition aſſumed to be true, by which a man is to argue, and build his reaſoning upon.

Demonſtration is the collecting the ſeveral proofs and arguments, and laying them in ſuch order, as to ſhew the truth of the propoſition under conſideration. Theſe proofs are to be drawn only from firſt principles,

principles, and from propoſitions already demonſtrated. Here we muſt keep ſtrictly to one and the ſame ſenſe of each definition; and when nothing is admitted but definitions, and axioms, and ſuch poſtulates and hypotheſes as are agreeable to the nature of the thing; and the conſtruction of figures in geometrical ſubjects; and demonſtrated propoſitions; and when the ſeveral arguments, or ſteps, are rightly connected together, ſo as one is plainly ſeen to be directly inferred from another, through the whole ſeries or chain of reaſoning: the concluſion at laſt obtained muſt be certain and true. Thus one truth is drawn from another, and from theſe a third, and thus continuing to deduce truths from truths, through the whole train of truths, we come at laſt to the concluſion or truth ſought after.

A direct, *poſitive*, or *affirmative demonſtration*, is that which concludes with the certain and direct proof of the propoſition in hand. This kind of demonſtration is moſt ſatisfactory to the mind; and therefore is called an *oſtenſive demonſtration.*

A negative, or *indirect demonſtration*, is that which ſhews a propoſition to be true, by ſome abſurdity which would neceſſarily follow if the propoſition advanced ſhould be falſe: this is called *reductio* ~~and~~ ad *abſurdum*; and ſhews the abſurdity and falſhood of all ſuppoſitions, but that contained in the propoſition. This is frequently made uſe of for eaſe and brevity's ſake, and to avoid a long perplext *oſtenſive* demonſtration. But although this ſort equally convinces the mind, and forces aſſent, yet it does not equally enlighten it. For it does not ſo much demonſtrate the truth itſelf directly, as the conſequent abſurdity or impoſſibility of the oppoſite ſuppoſition; whence it follows certainly (though indirectly) that the propoſition is true. When, at the ſame time, the original reaſon of its truth, or by what intrinſic cauſe it comes to be ſo, remains quite obſcure and in the dark.

A geo-

A geometrical demonſtration, is that which depends on the principles of geometry.

It has been ſhewn, that when the firſt principles are all true, upon which the reaſoning relies; and all the ſteps truly and evidently connected together; that the concluſion we come to at laſt, muſt neceſſarily be true.

But if we lay down a falſe hypotheſis, and argue upon it as true, although we carry on our reaſoning ever ſo rightly, yet the concluſion will moſt certainly be falſe. For from falſe premiſes nothing but falſhood can follow. And therefore, on the contrary, when we argue from a precarious hypotheſis, and conduct our reaſoning with the greateſt rigour of truth, and at laſt come to a falſe concluſion; we may be aſſured, the hypotheſis we argued from is falſe. For there is no other poſſible cauſe for falling into a falſe concluſion. And this is the foundation of that way of reaſoning before mentioned, called *reductio ad abſurdum vel impoſſibile.* And this teaches us how to detect falſe hypotheſes.

Again, if our hypotheſis and other principles be all true; and we happen to reaſon wrong, either by giving a falſe meaning to any term, or making uſe of falſe propoſitions, in the courſe of our reaſoning; or not connecting the ſeveral ſteps rightly together; then falſhood and not truth muſt again be the concluſion; except it be by mere chance, that one error may correct another. And if our firſt principles and reaſoning be both falſe; it is a thouſand to one but the concluſion will be falſe, and truth here muſt have a poor chance for appearing.

Method is the art of diſpoſing a train of arguments, in a right order, either to find out the truth, or falſhood of a propoſition; or to demonſtrate it to others, when we have found it out. This is either analytical or ſynthetical.

Analyſis,

Analyſis, or *the analytic method*, is the art of finding out the truth of a propoſition, by ſuppoſing the thing to be done; and going back ſtep by ſtep, till we arrive at ſome known truth. This is called the *method of invention*, or *reſolution*, and is generally uſed in algebra.

Syntheſis, or *the ſynthetic method*, is the ſearching out truth, by firſt laying down ſome ſimple and eaſy principles, and purſuing the conſequences till we come at the concluſion. This method begins at the moſt ſimple and eaſy things, and proceeds to the more compounded and general. It is alſo called the *method of compoſition*, and is contrary to the analytic method; as this proceeds from known principles to an unknown concluſion; whilſt the other goes in a retrograde order from the thing ſought, as if it was known, to ſome known principle. And therefore when any truth has been found out by the analytic method; it may be demonſtrated in a backward order, by ſyntheſis.

Thus you have an account of the rules and methods, whereby the mathematicians manage this their ſcience, and handle their ſeveral ſubjects. Methods ſo clear and inſtructive, that they may juſtly challenge the world to produce any others, of equal perſpicuity, evidence, and certainty. And the ſtructures they erect thereby are equally ſtrong and impregnable, as well as admirable and ſurprizing. For in the firſt place, they premiſe ſome general principles to begin with, as definitions, axioms, *&c.* from theſe they derive ſome ſimple and eaſy propoſitions; and from theſe others are drawn ſtill harder; and then by degrees they arrive at the more difficult ones; what goes before being always helpful for finding out the following. Thus a chain of arguments is carried on in an uninterrupted ſeries, and their truth confirmed by infallible reaſoning. Then the moſt general and uſeful propoſitions are collected together, and drawn up in order, and put into a body or magazine, and reſerved for uſe, to be called forth, as occaſion requires, for the inveſtigation and demonſtration of others. Thus they form ſo many ſyſtems of mathematical truths, according to the various ſubjects they examine; which muſt ſtand as principles for finding out new ones, or as teſts for trying the truth of others. For any propoſition being once proved true, muſt eternally remain true, and can never vary: it being the nature and eſſence of truth to continue invariable.

Now theſe ſeveral ſyſtems, or branches of the mathematics, that is, the diviſion of the mathematical ſciences, have been differently made and reckoned up, by different men. But the principal branches or parts thereof, at leaſt thoſe of moſt uſe, may be reckoned to be theſe: arithmetic, geometry, proportion, trigonometry, projection of the ſphere, menſuration, ſurveying, guaging, dialling, gunnery, geography, conic ſections and curve lines, navigation, mechanics, optics, perſpective, chronology, algebra, centripetal forces, aſtronomy, fluxions, increments. I have already publiſhed ſeveral of theſe in ſeparate tracts; and from the regard I always had for theſe arts, and the great deſire I have of ſeeing them flouriſh; I intend from time to time, in the courſe of this work, to publiſh the reſt, as ſoon as they can be got ready for the preſs. Which done, I doubt not but the young ſtudent will be furniſhed with a compleat courſe of the mathematics, ſufficient to inſtruct him in his progreſs, through theſe difficult paths, and to make him fit and able to read larger, and more elaborate treatiſes.

◆

A

TREATISE

OF

ARITHMETIC,

CONTAINING

All the PRACTICAL PARTS thereof;

BOTH IN

WHOLE NUMBERS,
VULGAR FRACTIONS,
AND DECIMALS.

LIKEWISE

The THEORY of NUMBERS,

And their Principal Properties, demonſtrated in a plain and eaſy manner.

Doctores, elementa velint ut diſcere prima. HOR.

Cyclomathes [illegible] Vol. 1

THE

PREFACE.

HE that would make any conſiderable progreſs in the mathematics, muſt begin at the firſt principles, and proceed gradually forward from one branch of that ſcience to another; according as they are naturally connected together, and have a dependance upon one another. This will make the progreſs as eaſy, ſhort, and intelligible, as the nature of the thing will admit of. Whilſt he that takes a contrary courſe, will always be involved in difficulty, doubt, and obſcurity; the knowledge he gains will be imperfect; and for want of evidence, the mind will want that conviction which is neceſſary for eſtabliſhing truth.

Arithmetic may be juſtly ſaid to be the baſis of all the other parts of mathematics. All things of whatever kind they are, may be reduced to numbers, and their quantities and proportions, calculated by numbers. All other branches have need of arithmetic, ſome way or other; and would often be at a ſtand without it. Yet arithmetic has no need of them, but ſtands ſolely upon its own principles. In all parts of the mathematics, no problem of any ſort is deemed to be compleatly ſolved, till it be calculated arithmetically, and its value brought out in numbers. And ſince it is of ſuch conſequence, it is abſolutely neceſſary for the young ſtudent, who would lay a good foundation for attaining a competent knowledge in the mathematics, firſt of all to make himſelf ac-

quainted with all the parts of arithmetic, and the nature and properties of numbers: without which it would be in vain for him to attempt any thing.

And as it is of ſuch great uſe in the ſciences, ſo it is equally ſerviceable in human actions and employments. He muſt be very little verſed in the common affairs of life, that does not know the great uſefulneſs of arithmetic in every inſtance thereof. No buſineſs can be carried on without the help of numbers; no trade or commerce exerciſed without regular accounts: ſo that in all ſituations of life, arithmetic is a neceſſary accompliſhment.

As to the enſuing treatiſe, I have in the firſt book, fully, and yet very conciſely handled all the parts of common arithmetic; and have made all the rules thereof, as ſhort as poſſible, ſo as to be intelligible; and the reader cannot fail of underſtanding them, by means of the examples there given, which I ſuppoſe are ſufficient for that end, and no more. I have alſo endeavoured to give the reaſons for the ſeveral operations in the fundamental parts of this art, which cannot miſs pleaſing the reader, as he will have his judgment and underſtanding informed, at the ſame time he is learning the practice.

In the ſecond book, I have delivered the ſubſtance of what Euclid *and others have written about the properties of numbers, adding whatever I thought of any conſequence in the theory of numbers. And here I have for the moſt part demonſtrated the propoſitions of* Euclid *after a different manner from him, and often more generally. And though the theory ought to precede the practice, in any ſcience: yet here it was hardly poſſible to obſerve that rule. For there is not only frequent uſe made of multiplication, diviſion,* &c. *but there is a good deal of abſtract reaſoning about the properties of num-*

numbers, which could not well be underſtood, till the reader was well acquainted with the operations of arithmetic; which is the reaſon I have put it laſt. I know of nothing that is wanting in this treatiſe, except it be a greater variety of examples; and this would require more room; and the intelligent reader can eaſily ſupply theſe of himſelf; to whom I wiſh ſucceſs, anſwerable to his endeavours.

W. Emerſon.

THE
CONTENTS.

Page
DEFINITIONS — 1
Notation — — 3
Axioms, &c. — — 5

CHAP. I. WHOLE NUMBERS.

Numeration — — 8
Addition — — 10
Subtraction — — 15
Multiplication — — 17
Diviſion — — 24
Extraction of the ſquare root — 32
Extraction of the cube root — 36

CHAP. II. VULGAR FRACTIONS.

Reduction — — 42
Addition — — 53
Subtraction — — 56
Multiplication — — 60
Diviſion — — 64
Square root — — 67
Cube root — — 70

CHAP. III. DECIMAL FRACTIONS.

Notation — — 72
Addition — — 75
Subtraction — — 76
Multiplication — — 76
Diviſion — — 81
Reduction — — 85
Square root — — 91
Cube root — — 94

CHAP. IV. The PRACTICAL RULES of ARITHMETIC.

Page
Reduction — — 100
Single rule of three — — 104
Double rule of three — — 109
Rule of practice — — 115
Single rule of fellowſhip — — 122
Double rule of fellowſhip — 128
Alligation medial — — 132
Alligation alternate — — 134
Single rule of falſe — — 139
Double rule of falſe — — — 141
Rule of exchange — — 151
Solution of queſtions by logarithms — 153
Intereſt and annuities — — 161

CHAP. V. PRACTICAL QUESTIONS 180

BOOK II.

The THEORY of NUMBERS.

CHAP. I. Even and odd numbers; prime and compoſite numbers, ſquares, cubes, *&c.* — — 192
CHAP. II. Proportional numbers, mean proportionals, plane and ſolid numbers — — 207
CHAP. III. Particular numbers, diviſors, aliquot parts, circulating numbers — 218
CHAP. IV. Numerical problems — 228

ARITHMETIC.

DEFINITIONS.

1. *ARITHMETIC* is the art of computing by numbers; it is called *vulgar* or *common Arithmetic*, when it treats of whole numbers.

2. *Unity* is that by which every thing is called one; and a unit is the beginning of number.

3. *Number* is a multitude of units: by this every thing is reckoned.

4. An *integer* is any whole thing.

5. A *whole number* is a preciſe number without any parts annext.

6. A *mixt number* is a whole number with ſome part annext.

7. A *fraction* is a part or parts of an unit.

8. A *proper fraction* is leſs than a unit.

9. An *improper fraction* is greater than a unit.

10. An *aliquot part* is that which is contained a preciſe number of times in another.

Cor. Hence 1 is an aliquot part of any number: but a number cannot be called an aliquot part of itſelf.

11. An *aliquant part* is ſuch as is contained in another, ſome number of times, with ſome part or parts over.

12. One number is ſaid to be *multiple* of another, when it contains it a preciſe number of times.

13. One number is ſaid *to meaſure* another, when it is contained in the other a preciſe number of times, without a remainder. The ſaid meaſure is alſo a *diviſor.*

Cor. Any number is a meaſure to itſelf. And 1 is a meaſure to any number.

14. An *even number* is that whoſe half is a whole number.

15. An *odd number* is that which cannot be divided into two equal whole numbers.

Cor. The numbers one, two, three, four, *&c.* are alternately odd and even for ever.

16. A *prime number* is that which can only be meaſured by a unit.

17. *Numbers* are ſaid to be *prime to one another*, when only a unit meaſures both. Theſe are alſo called *coprimes.*

Cor. Therefore 1 is prime to every number.

18. A *compoſite number* is that produced by multiplying ſeveral other numbers together, called *factors* or *multipliers.* Alſo what is produced by ſuch multiplication, is called a *product.*

19. Numbers are ſaid to be *compoſed to one another*, when ſome number (greater than a unit) meaſures both.

20. A *plane number* is the number produced by multiplying two other numbers.

21. A *ſolid number* is the product of three numbers.

22. A *ſquare number* is the product of a number by itſelf.

23. A *cube number* is the product of a number, and its ſquare.

24. *Like or ſimilar plane or ſolid numbers*, are thoſe whoſe ſides or multipliers are proportional.

25. A *perfect number* is that which is equal to the ſum of all its aliquot parts.

26. The *power* of any number, ſignifies, that the number (called the *root*) ſhall be ſo often multiplied, as is denoted by the number (or index) expreſſing the power. Thus the 2d power of 5, is 5 multiplied by 5, or 25; the 3d power of 5, is 25 multiplied by 5, *&c.*

27. Four numbers are ſaid to be *proportional*, or in the *ſame proportion*, when comparing two and two; the firſt is the ſame multiple, or the ſame part or parts of the ſecond, as the third is of the fourth, thus: 6, 2, 9 and 3, are proportional; for 6 contains 2 thrice, and 9 contains 3 thrice. Alſo 4, 6, 10, 15, are proportional; for 6 is once and half 4, and 15 is once and half 10. And the ſeveral numbers are called the *terms* of the proportion; and the quotient ariſing, by dividing the former by the latter number, is called the *Ratio.*

28. Numbers are ſaid to be in *continual proportion*, or in *geometrical progreſſion*, when the firſt has the ſame proportion to the ſecond, as the ſecond to the third, and as the third to the fourth, and ſo on, thus: 2, 6, 18, 54, *&c.* are continual proportionals.

29. *Mean proportionals* are all the intermediate terms, between the *extremes*, in a geometrical progreſſion.

30. *Surds* are ſuch numbers as have no exact roots.

NOTATION.

1. The characters by which numbers are expreſſed, are theſe ten: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; 0 is called a *cypher*; and the reſt, or rather all of them, are called *figures*, or *digits*. The names and ſignification of theſe characters, and the origin or generation of the numbers they ſtand for, are here ſet down:

0 nothing.
1 one, or a ſingle thing called a unit.
then $1 + 1 = 2$ two.
$2 + 1 = 3$ three.
$3 + 1 = 4$ four.
$4 + 1 = 5$ five.
$5 + 1 = 6$ ſix.
$6 + 1 = 7$ ſeven.
$7 + 1 = 8$ eight.
$8 + 1 = 9$ nine.

then $9 + 1 =$ ten, which has no ſingle character; and thus by the continual addition of 1, all numbers are generated.

2. The value of any number depends not on the figure or figures alone, but upon the figures and places where they ſtand, jointly. And the order of places is backward from the right hand towards the left. The firſt place is called the place of units; the ſecond, tens; the third, hundreds; the fourth, thouſands; the fifth, ten thouſands; the ſixth, hundred thouſands; the ſeventh, millions; and ſo on. Thus in the number 765487654; 4 in the firſt place ſignifies only 4; 5 in the ſecond place ſignifies five tens or fifty; 6 in the 3d place ſignifies ſix hundred; 7 in the 4th place is ſeven thouſand; 8 in the 5th place is eighty thouſand; 4 in the 6th place is four hundred thouſand; 5 in the 7th place is five millions; and ſo on.

3. A cypher, though of no value by itſelf, yet it occupies a place, and advances the figures on the left hand into higher places, from whence they have a greater value. Thus 3 ſignifies only 3, but 30 ſignifies 3 tens or thirty, and 300 ſignifies 3 hundred.

4. The values of all figures increaſe in a tenfold proportion from the right hand towards the left, each following place being ten times greater than the foregoing. Thus in the number 33333333; 3 in the firſt place is three; in the ſecond, 30 thirty; in the third,

third, 300 three hundred; in the fourth, 3000 three thouſand; in the fifth, 30000 thirty thouſand, &c. And thus 1 ſignifies one, 10 ſignifies ten, 100 ſignifies a hundred, 1000 ſignifies a thouſand, and ſo on; and in general, ten units make 1 ten, ten tens make 1 hundred, ten hundred make 1 thouſand, &c.

5. Hence, placing 1, 2, 3, &c. cyphers on the right hand of any number, makes it ten, a hundred, a thouſand times, &c. greater than before. But placing cyphers on the left hand does not alter the value, becauſe every figure remains in the ſame place as before.

This method of expreſſing numbers, by the different values of the figures in different places, is an admirable invention; without which it had been neceſſary to have as many different characters, as there are numbers to be expreſſed; which would have been impoſſible.

AXIOMS.

1. If two numbers are equal to a third, they are equal to one another.

2. If equal numbers be added to equal numbers, the wholes will be equal.

3. If from equal numbers the ſame or equal numbers be taken away, the remainders will be equal.

4. Thoſe numbers are equal, which are the ſame multiple of equal numbers.

5. Thoſe numbers are equal, which are the ſame part of equal numbers.

6. The ſame powers, or the ſame roots of equal numbers, are equal.

7. Unity or 1 neither multiplies nor divides; that is, the product or quotient is ſtill the ſame number.

8. If a number be compoſed of two numbers, multiplied together; either of them meaſures it by the other.

9. If a number meaſures ſeveral other numbers; it

it likewiſe meaſures the ſum (or difference) of theſe numbers.

10. If a number meaſures another; it alſo meaſures every number which that other meaſures.

11. If a number meaſures the whole, and a part taken away; it alſo meaſures the reſidue.

The Signification of other Characters here uſed.

Characters.	Signification.
+	*more*, *and*, *to be added*, being an affirmative ſign. Thus $7+3$ ſignifies 3 added to 7; and $A+B$ denotes the ſum of A and B.
—	*leſs*, *leſſened by*, *abating*, being a negative ſign. Thus $7-3$ means 3 taken out of 7, and $A-B$ denotes the remainder, when B is ſubtracted from A.
×	*multiplied by*, as 7×3 ſignifies 7 times 3; alſo $A\times B$ or AB, is the product of A and B multiplied together. Where note, if letters ſtand to denote numbers, they are commonly ſet together, like letters in a word.
÷	*divided by*, thus $6\div 3$ ſignifies 6 divided by 3; alſo 3) 6 (ſignifies 6 divided by 3; alſo $\frac{6}{3}$ ſignifies 6 divided by 3; and in general $A\div B$, or B) A (, or $\frac{A}{B}$, is the quotient of A divided by B.
A^2	*the ſquare of* A, that is, AA.
A^3	*the cube of* A, that is, AAA.
A^n	*the* nth *power* of A, the index n being any number.
√	*the ſquare root*, thus $\sqrt{16}$ is the ſquare root of 16, and $\sqrt{A}$ is the ſquare root of A.

the

Characters.	Signification.
$\sqrt[3]{}$	*the cube root*, as $\sqrt[3]{8}$ is the cube root of 8, and $\sqrt[3]{A}$ is the cube root of A.
$=$	*equal to*, as $7+3=10$, 7 and 3 equal to 10.
$::$	*A note of proportion*, thus 2 : 3 :: 4 : 6, ſignifies 2 is to 3, as 4 to 6; and A : B : : a : b, A is to B, as a to b, ſometimes written thus, A—B—a—b.
∺	*continual proportionals*, A : B : C : D ∺, A, B, C, D are in continual proportion.
$\overline{A+B+C}$	the ſum of A, B, and C; a line drawn over ſeveral numbers, denotes the ſum of them.

BOOK I.

The Practice of Arithmetic.

CHAP. I.

The fundamental Rules of common or vulgar Arithmetic.

PROBLEM I.

To read or expreſs any Number written.

THIS is called *Numeration*, and is eaſily performed by help of the following table, which ſhews the names of the ſeveral places, and conſequently of the figures ſtanding there, as explained before in the Notation.

NUMERATION TABLE.

&c.	Tens of billions	Billions, or millions of millions	Hundred thouſands of millions	Ten thouſands of millions	Thouſands of millions	Hundreds of millions	Tens of millions	Millions	Hundreds of thouſands	Tens of thouſands	Thouſands	Hundreds	Tens	Units
	4	3	8	7	6	5	4	3	8	7	6	5	4	3

RULE.

RULE.

1. Begin at the units place, and divide, or rather diſtinguiſh your number into periods of 6 figures apiece, called *grand periods*, or *double periods*. The firſt period to the right is units, the ſecond millions, the third bi-millions, the fourth tri-millions, the 5th, 6th, *&c.* quadri-millions, quinti-millions, ſexti-millions, ſepti-millions, octi-millions, noni-millions, deci-millions, *&c.*

2. Likewiſe diſtinguiſh theſe grand periods into two parts, called *ſingle periods* of three figures apiece; in theſe write (or ſuppoſe to be written) units over the firſt place, tens over the ſecond place, and hundreds over the third place.

3. Begin to read at the left hand, expreſſing hundreds, tens, units, as you come to the reſpective places where theſe figures are; and at the end of each ſingle period (on the left hand) always pronounce thouſands; and at the end of the grand period, expreſs its title or ſurname belonging to it; proceeding thus to the right hand where the number ends.

Ex. 1.

Read the number 50765.

t u h t u
50 765

Having diſtinguiſhed the number into periods, and written u over units, t over tens, h over hundreds, it will be read thus: fifty thouſand, ſeven hundred and ſixty-five.

Ex. 2.

To read 43876543876543.

t u h t u h t u
43 876 543 876 543

Forty three bi-millions, eight hundred and ſeventy ſix thouſand, five hundred and forty three millions, eight hundred and ſeventy ſix thouſand, five hundred and forty three.

Ex.

Ex. 3.

Read this number 2418579643219004613254768096.

```
 htu      htu     htu     htu     htu
2418  579643  219004  613254  768096
   .       .       .       .       .
```

Two thouſand, four hundred and eighteen quadri-millions;
Five hundred ſeventy nine thouſand, ſix hundred forty three tri-millions;
Two hundred nineteen thouſand, and four bi-millions;
Six hundred thirteen thouſand, two hundred fifty four millions;
Seven hundred ſixty eight thouſand, and ninety ſix.

PROBLEM II.

To add whole numbers together.

Addition is the rule by which ſeveral numbers are put together, in order to find the ſum of them all.

RULE.

1. Place all the numbers ſo, that units may ſtand under units, tens under tens, hundreds under hundreds, *&c.* and draw a line underneath.

2. Begin at the units place, and reckon up all the figures in that place from the bottom to the top, and what overplus there is above even tens, ſet down, and carry ſo many to the next row as there were tens.

3. Reckon up all the figures in the place of tens, together with what you carried, and ſet down the overplus, carrying the tens to the next row; and ſo proceed to the laſt.

4. If you don't chooſe to reckon forward, you may make a prick when you have reckoned to ten or more, carrying on the overplus; and then add ſo many to the next row as you have pricks.

Ex. 1.

Let theſe numbers be added together:

```
 9482
  590
  307
   85
-----
10464
-----
```

Beginning

Beginning at 5, ſay the ſum of 5 and 7 is 12 and 2 is 14, ſet down 4 and carry 1. The ſum of 1 and 8 is 9 and 9 is 18 and 8 is 26, ſet down 6 and carry 2. Then 2 and 3 is 5 and 5 is 10 and 4 is 14, ſet down 4 and carry 1. Laſtly, 1 and 9 is 10, which being the laſt, ſet it down.

The reaſon of carrying the tens to the next place is plain; for the ſum of 5, 7 and 2 being 14, the 4 belongs to the units, and the 1 to the tens. Again, the ſum of 1, 8, 9 and 8 being 26, which are tens, the 6 belongs to the tens, and the 2 to the next ſuperior place, which is hundreds. Then the ſum of 2, 3, 5 and 4 being 14, *viz.* 14 hundreds, the 4 belongs to that place, and the 1 to the place above, which is thouſands. Laſtly, the ſum of 1 and 9 is 10, that is 10 thouſand, that is 0 in the place of thouſands, and 1 in the place of ten thouſands. In ſhort, thus:

The ſum of the row of units	14
The ſum of the row of tens	250
The ſum of the row of hundreds	1200
The ſum of the row of thouſands	9000
total	10464

Ex. 2.

Add theſe numbers together.

350709
31806500
339087
46011
2935

ſum 32545242

The *proof* of Addition is this: begin at the top, and add all the numbers downwards, by the ſame rule as you added them upwards before; then if the total ſums agree, the work is right.

PROB-

PROBLEM III.

To add numbers of ſeveral denominations together.

RULE.

1. Place the numbers ſo, that thoſe of the ſame denomination may ſtand directly under one another, then draw a line under them.

2. Begin at the loweſt denomination firſt, and reckon upwards till you get as many as makes one of the next denomination above; then make a prick, and carry the overplus, or exceſs, to the next figures; and ſo reckon forward, always pricking when you have as many as makes one of the next denomination. Proceed thus till that denomination is finiſhed, and ſet down the overplus at bottom.

3. Reckon your pricks in the denomination you have finiſhed, and carry ſo many, to be added to the next denomination, which muſt be added up by the ſame rule; and ſo of the reſt. In the laſt denomination, add them up as whole numbers.

Ex. 1. *Money.*

Add theſe ſums of money together.

	£.	*s.*	*d.*
	57	6	8·
	127	14·	0
	0	9	$6\frac{1}{2}$
	17	0	$3\frac{3}{4}$
ſum	202	10	$6\frac{1}{4}$

Note, 4 farthings make 1 penny, 12 pence 1 ſhilling, 20 ſhillings 1 pound.

Ex. 2. *Troy Weight.*

	oz.	*pwts.*	*grs.*
	207	13	19·
	81	0	11
	157	15·	6·
	31	9	20
total	477	19	8

Note. In Troy weight, 24 grains make a penny-weight, 20 penny-weights an ounce, 12 ounces a pound.

Ex. 3. *Apothecary's Weight.*

	oz.	*drs.*	*ſcr.*	*grs.*
	15	7·	2·	15·
	3	4·	0	12
	0	0	1	18·
	1	5	1·	3
total	21	2	0	8

Note, In Apothecary's weight, 20 grains make a ſcruple (℈), 3 ſcruples a dram (ʒ), 8 drams an ounce (℥), 12 ounces a pound (℔).

Ex. 4. *Averdupoize leſſer weight.*

	lb.	*oz.*	*dr.*
	15	11·	12·
	4	10	0
	12	0	13·
	0	15·	9
total	33	6	2

Note, 16 drams make an ounce, 16 ounces a pound.

Ex. 5. *Averdupoize greater weight.*

	tuns	*hunds.*	*ſto.*	*lb.*
	570	18·	6	11·
	38	7·	2·	0
	92	0	6	3
	12	15	0	10
total	714	1	7	10

Note, 14 pounds make a ſtone, 8 ſtone 1 hundred weight, 20 hundred weight 1 tun.

Ex. 6. *Long Meaſure.*

	yds.	*feet*	*inch.*
	37	2·	11·
	7	0	3
	8	1	10·
	4	2·	5
total	58	1	5

Note, 3 barley-corns make an inch, 12 inches a foot, 3 feet a yard; alſo 5½ yards make a pole, 22 yards a chain, 10 chains a furlong, 8 furlongs a mile.

Liquid Meaſure.

2 pints make a quart, 2 quarts a pottle, 2 pottles a gallon, 8½ gallons a firkin or anker, 6 firkins a hogſhead of ale, 63 gallons a hogſhead of wine.

Dry Meaſure.

2 pints make a quart, 2 quarts a pottle, 2 pottles a gallon, 2 gallons a peck, 4 pecks a buſhel, 8 buſhels a quarter, 4 quarters a chaldron, 10 quarters a laſt.

SCHOLIUM.

If a long liſt of numbers is to be added up, divide

vide it into ſeveral parcels, and add them ſeparately; and then add all theſe parcels together.

The *proof* of this rule is the ſame as the laſt; only in reckoning downward, make croſſes inſtead of pricks, to avoid confuſion.

PROBLEM IV.

To ſubtract one whole number from another.

Subtraction is the taking one number from another, to find their difference.

RULE.

1. Place the greater number uppermoſt, and the other under it, ſo as units may be under units, tens under tens, *&c.* and draw a line under them.

2. Begin at the right hand or place of units, and ſubtract the lower figure from the upper, and ſet down the difference underneath them; do the ſame with the reſt of the figures.

3. When the lower figure is greater, borrow 10, and add it to the upper number, from which ſubtract the lower, and ſet down the remainder; carry 1 to be added to the next lower figure, and ſubtract the ſum from the upper, and ſet down the remainder; and ſo on from one row to another.

Ex. 1.

	£.
from	270481467
take	31065363
rem.	239416104

The reaſon of this operation is plain, only when the lower number is leſs, 10 is added to the upper number, as here, 5 is leſs than 1, therefore 1 is borrowed from 8 to make 11, then 5 from 11 remains 6; then the next figure 6 ought in reality to be taken from 7, inſtead of 8; but the difference will be the ſame, whether you take 6 from 7, or add the 1 borrowed to 6, and take the ſum 7 out of 8, in either caſe 1 remains.

Ex. 2.

from	30076058972
take	17078032863
rem.	12998026109

Ex. 3.

One born in 1682, *how old is he in* 1763?

1763
1682

81 anſwer.

The *proof* of Subtraction is to add the remainder to the leſſer number, which ought to make up the greater, if the work be right.

PROBLEM V.

To ſubtract numbers of different denominations.

RULE.

1. Place the numbers, ſo that the greater may be uppermoſt, and that thoſe of the ſame denomination may ſtand directly under one another, and draw a line under them.

2. Begin at the loweſt denomination, and take the lower number from the upper one, and ſet down the difference, or remainder, underneath. Do the ſame with the next denomination, and ſo on till the laſt, which muſt be ſubtracted as whole numbers.

3. When the lower number in any denomination happens to be the greater, borrow 1, that is, add as many

many to the upper number as makes one of the next higher denomination, and then ſubtract the lower number, and ſet down the remainder. Then carry 1, and add it to the lower number of the next denomination, and then ſubtract as before.

Ex. 1. *Money.*

	£.	*s.*	*d.*
from	241	9	$6\frac{1}{4}$
take	82	6	3
rem.	159	3	$3\frac{1}{4}$

Ex. 2. *Money.*

from	3794	0	$3\frac{1}{2}$
take	129	5	$10\frac{3}{4}$
rem.	3664	14	$4\frac{3}{4}$

Ex. 3. *Troy Weight.*

	lb.	*oz.*	*pwts.*	*grs.*
from	19	12	15	18
take	13	11	17	7
rem.	6	0	18	11

PROBLEM VI.

To multiply one whole number by another.

Multiplication is taking the *multiplicand*, or number to be multiplied, ſo many times as there are units in the *multiplier*; and the reſult is called the *product.* Multiplication is a compendious method of addition, and is performed by help of the following table, which muſt be got by heart.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

The uſe of the table is this: find one figure on the ſide of the table, and the other at top; then in the angle of meeting is their product. Thus the product of 5 and 7 is 35; and the product of 9 times 8 is 72.

1. A GENERAL RULE.

1. Place the multiplier under the multiplicand, the units under units, *&c.* and draw a line under them.

2. You muſt multiply from the right hand to the left, thus: begin with the units or loweſt figure of the multiplier, by which multiply the loweſt figure of the multiplicand, and ſet down the overplus above the tens, and carry the tens. Then multiply the 2d figure of the multiplicand by the ſame, adding ſo many units, as you had tens to carry; and ſet down the overplus, and carry the tens as before. Do thus till

till you come to the laſt figure, whoſe product muſt be ſet down entire.

3. Then take the ſecond figure of the multiplier, and multiply by this as you did before; ſetting the firſt figure of the product under the figure you multiply with; do ſo with the reſt of the figures in the multiplier; ſetting the firſt figure of each product under, or in the ſame place as the figure you multiply by. Or, which is the ſame thing, ſetting each product ſo many places back towards the left hand, as the multiplying figure is diſtant from the firſt figure.

4. Laſtly, add all theſe products together, for the product of the two numbers given.

Note, you may eaſily multiply by 12 in one line, as if it was a ſingle figure, if you get by heart all the products of all the natural numbers by 12, as far as 9.

Ex. 1.

multiply 60735
by 7
product 425145

Explanation.

7 times 5 is 35; ſet down 5 and carry 3. 7 times 3 is 21 and 3 I carry is 24; ſet down 4 and carry 2. 7 times 7 is 49 and 2 carried is 51; ſet down 1 and carry 5. 7 times 0 is 0 but 5 is 5; ſet down 5 and carry 0. 7 times 6 is 42, which ſet down.

Ex. 2.

```
multiply 2760325
by         37072
         5520650
        19322275
       193222750
       8280975
product 102330768400
```

Demonſtration of the rule.

In Ex. 1. 7 multiplying 5 produces 35, the 5 will fall in the place of units, and the 3 belongs to the tens. Then 7 multiplying 3 in the 2d place, or place of tens, produces 21, of which 1 belongs to the tens, to which the 3 carried being alſo tens, muſt be added, which makes 4 tens; and the 2 belongs to the 3d place, or hundreds. Then 7 multiplying 7 in the third place, makes 49, the 9 belongs to the 3d place, to which add the 2, which alſo belongs to the 3d place, the ſum is 51; 1 belongs to the third place and 5 to the 4th place. Then 7 times 0 is 0, (in the 4th place) but 5 is 5. Laſtly, 7 times 6 is 42, the 2 belongs to the 5th place, and 4 to the 6th. Theſe particular products will ſtand thus:

$$\begin{array}{r} 60735 \\ 7 \\ \hline 35 \\ 21. \\ 49.. \\ 0... \\ 42.... \\ \hline 425145 \\ \hline \end{array}$$

And in Ex. 2. 2 multiplying 5 produces 10, the 0 is in the place of units, and ſo on. Again, 7 multiplying 5 makes 35, the 5 is in the 2d place, becauſe the multiplier is really 70. Again, 7 in the 4th place multiplying 5 makes 35, and the 5 will be in the 4th place, becauſe you really multiply by 7000, and ſo for all the reſt.

Ex. 3.

If 1 *hogſhead coſt* 13 *pound, what will* 18 *coſt?*

$$\begin{array}{rrl} & 18 & \\ \hline & 104 & \\ & 13 & \\ \hline \text{anſw.} & 234 & \text{pounds.} \\ \hline \end{array}$$

2 RULE.

When one or both the numbers end with cyphers, neglect the cyphers and multiply the remaining figures as before; and to the product, annex the cyphers that are in both numbers.

Ex. 4.

$$\begin{array}{lr} \text{multiply} & 507300 \\ \text{by} & 4020 \\ \hline & 10146 \\ & 20292 \\ \hline \text{product} & 2039346000 \\ \hline \end{array}$$

3 RULE.

When any number is to be multiplied by 10, 100, 1000, *&c.* annex ſo many cyphers at the end of the number, as there are in the multiplier.

Ex. 5.

Multiply 23079 by 100, the product is 2307900.

4 RULE.

In large multiplications, make a table of the multiplicand multiplied by all the 9 digits. Then you have no more to do, but to take out the reſpective product for each figure of the multiplier, and add them all together.

TABLE.

1	70500768
2	141001536
3	211502304
4	282003072
5	352503840
6	423004608
7	493505376
8	564006144
9	634506912

Ex. 6.

multiply 70500768
by 50431

70500768
211502304
282003072
352503840

product 3555424231008

The *proof* of Multiplication, is by making the multiplicand to be the multiplier; then if the product comes out the ſame as before, your work is right.

That two numbers will give the ſame product, whichever is the multiplier, will appear thus: ſuppoſe the numbers 4 and 36. Then 36 times 1 is the ſame with once 36; and therefore 36 times 1 + 1 + 1 + 1, or 36 times 4 is the ſame with 4 times 36; and ſo of others.

SCHOLIUM.

There is a way of proving multiplication by caſting away the nines, which though not infallible, ſerves to confirm the other, and is very expeditious. It is thus, ſee Ex. 4. make a croſs, and add all the figures or digits of the multiplicand together, as units, thus 5 + 7 + 3 = 15, throw away the nines, and ſet the remainder 6 on one ſide of the croſs. Do the ſame with the multiplier 4 + 2 = 6, ſet the remainder on the other ſide of the croſs. Do the like with the product, and ſet the remainder at top. Laſtly, multi-

multiply the figures on the ſides, and throw away the nines, and ſet the remainder at bottom, which muſt be the ſame with the top, if the work is right.

Ex. 6.

PROBLEM VII.

To multiply numbers of different denominations, by a given number.

1 RULE.

If the multiplier be a ſingle figure; begin at the loweſt denomination, and multiply it by the given number, and ſee how many of the next denomination is contained in the product; ſet down the odds, and carry ſo many to the next. Then multiply the next denomination, adding what you carried; and ſet down the odds. Proceed thus till all be multiplied.

This method is rather reckoning than multiplying.

Ex. 1. *Money.*

	£.	*s.*	*d.*
multiply	49	13	10
by			7
product	347	16	10

Ex. 2. *Weight.*

	c.	*ſt.*	*lb.*
multiply	11	2	13
by			6
product	68	1	8

2 RULE.

If the multiplier be a great number made up of ſeveral others multiplied together. Multiply ſucceſſively by the parts, inſtead of the whole.

Ex. 3.

	£.	s.	d.	
multiply	127	13	9	by 45.
			5	
	638	8	9	
			9	
product	5745	18	9	

3 RULE.

If the multiplier is not compoſed of others; find two or more numbers, whoſe product comes neareſt: then multiply as before, and add what is wanting, or ſubtract what is over.

Ex. 4.

	£.	s.	d.	
multiply	7	12	10	by 47.
			6	
	45	17	0	
			8	
	366	16	0	
ſubtract	7	12	10	
product	359	3	2	

PROBLEM VIII.

To divide one whole number by another.

Diviſion teaches to find how often one number, called the *diviſor*, is contained in another, called the *dividend*. Or it ſhews how to find ſuch a part of the dividend as the diviſor expreſſes. The number here ſought is called the *quotient*.

1. A GENERAL RULE.

1. Set down the dividend, and the diviſor on the left hand of it, within a crooked line; alſo make another crooked line on the right hand, for the quotient.

2. Enquire how oft the firſt figure of the diviſor is contained in the firſt figure of the dividend, or in the two firſt figures, when that of the diviſor is greater; and place the anſwer in the quotient.

3. Multiply the whole diviſor by the quotient figure, and ſet the product orderly under the dividend towards the left hand, and ſubtract it therefrom. But *note*, if this product be greater than that part of the dividend; a leſs figure muſt be placed in the quotient.

4. Make a prick under the next figure of the dividend to mark it, and bring it down, annexing it to the remainder; then this number is called the *dividual*.

5. Seek how oft the diviſor is contained in the dividual, and ſet the anſwer in the quotient; then multiply and ſubtract as before; and proceed thus till all the figures in the dividend are brought down one by one. And *note*, for every figure brought down, a figure (or a cypher) muſt be placed in the quotient.

Note, ſince there is a neceſſity of trial, to find out the true quotient figure; therefore, before it be ſet down, multiply 2 or 3 figures of the diviſor on the left hand, by that figure in mind, to ſee if it exceed the dividual.

Ex. 1.

Divide 14122 by 46.

46) 14122 (307 the quotient.

138··

322

322

·

Expla-

Explanation.

Firſt I aſk how oft 4 in 1, which is no times at all: then how oft 4 in 14, which is 3 times; then I place 3 in the quotient, and then multiply 46 by 3, and ſet the product 138 under 141, and ſubtracting there remains 3. Then I prick the 2 and bring it down to 3, which then is 32 for a dividual; then enquiring how oft 4 in 3, the anſwer is 0, which I place in the quotient. Then I prick, and bring down the next figure 2, and the dividual is now 322, then I aſk how oft 4 in 32, the anſwer would be 8; but then 46 multiplied by 8 would exceed 322, therefore I place 7 in the quotient, by which I multiply 46, and the product is 322; and that ſubtracted from 132, leaves nothing. Then 307 is the quotient.

Ex. 2.

Divide 18972584 by 6023.

6023) 18972584 (3150 the quotient.
18069 · · :

9035
6023

30128
30115

134 the remainder.

Demonſtration of the rule.

In Ex. 1. ſince 46 is contained 3 times in 141, therefore it is contained 300 times in 14122; that is, 3 muſt be in the third place.

Alſo ſince 46 is contained 7 times in the remainder 322; therefore 46 is contained in the whole dividend 307 times.

And

And in Ex. 2. ſince 6023 is contained 3 times in 18972; it is contained 3000 times in 18972584; and 100 times in the remainder 903584, and 50 times in the next remainder 301284; and 0 times in the laſt remainder 134. Therefore the diviſor is contained in the whole dividend, 3150 times.

2 RULE.

When the diviſor ends with cyphers, cut them off, and likewiſe cut off as many places of the dividend on the right hand; and perform the diviſion by the remaining figures. And when the diviſion is finiſhed, annex the figures cut off to the remainder.

Ex. 3.

Divide 745678 by 30400.

```
304|00) 7456|78 (24 quotient.
        608
        ----
        1376
        1216
        ----
         16078 remainder.
```

3 RULE.

To divide by 10, 100, 1000, *&c.* cut off from the dividend ſo many places as the diviſor has cyphers; and that will be the quotient; and the figures cut off the remainder.

Ex. 4.

Divide 78607 by 100.

The quotient is 786, and 07 remaining.

4 RULE.

When you have a large dividend, and your diviſor is often repeated; make a table of all the products

ducts of the diviſor and the nine digits; which is done by continually adding the diviſor. By this table diviſion may be wrought by inſpection, only by the help of addition and ſubtraction. For you have no more to do, but only to take out of the table the number always the next leſs than each dividual, and the quotient figure along with it; which numbers are to be continually ſubtracted from theſe dividuals, as in the general rule.

Ex. 5.

Divide 40377982057 by 35016.

TABLE.

1	35016
2	70032
3	105048
4	140064
5	175080
6	210096
7	245112
8	280128
9	315144
10	350160

35016) 40377982057 (1153129.
35016······
53619
35016
186038
175080
109582
105048
45340
35016
103245
70032
332137
315144
16993 remains.

5 RULE.

When you are to divide by a ſingle figure, you need not ſet down the operation at large, but perform it in mind; the ſame may be done with 12.

Ex. 6.

7) 30721
4388 quotient.
5 rem.

Thus 30721 divided by 7, the quotient is 4388, and 5 remaining.

Diviſion is proved by multiplying the diviſor and quotient together, and adding the remainder, when there is any; which muſt be equal to the dividend, when the work is right.

Or it may be proved by caſting away nines, as in multiplication. Caſt away the nines in the diviſor and quotient, and ſet the remainders on the ſides of the croſs. Do the ſame with the dividend, and ſet the remainder at top. Multiply the figures on the ſides, throw away the nines, and ſet the remainder at bottom, which muſt be equal to the top. See Ex. 1. *Note*, if there be a remainder, it muſt be added to the product, on the ſides of the croſs, and the nines thrown out as before.

PROBLEM IX.

To divide a number of different denominations by a given number.

1 RULE.

If the diviſor be a ſingle figure, begin at the higheſt denomination, which divide by the given diviſor, and ſet the anſwer in the quotient, and to be of the ſame denomination; what remains muſt be be multiplied by the number of parts in the next inferior denomination, and added to the given number of that denomination, and then divide as before. Proceed thus through all the denominations.

Ex. 1.

```
        £.  s.  d.
Divide 58  10   3 into 7 parts, what is 1 part?
         £.   s.   d. £.  s.  d.
      7) 58   10   3 (8   7   2.
         56
        ----
           2=40
            ----
          7) 50 (7
             49
            ----
              1=12
                ----
              7) 15 (2
                 14
                ----
                  1
                ----
```

Explanation.

Say how oft 7 in 58, 8 times; which ſet in the quotient, then 8 times 7 is 56, which ſubtracted from 58, leaves 2. But 2 pounds are 40 ſhillings, to which add 10, the ſum is 50. Then ſay how oft is 7 in 50, anſwer 7 times, which ſet in the quotient for ſhillings; then 7 times 7 is 49, which taken from 50 leaves 1 ſhilling, or 12 pence, to which add 3, the ſum is 15. Then ſay how oft 7 in 15, the anſwer is 2, which ſet in the quotient for pence, then 2 times 7 is 14, which taken from 15, 1 remains. So the anſwer is 8*l.* 7*s.* 2*d.*; and 1 penny remaining.

Ex. 2.

```
                             c.  ſt.  lb.
    What is the 6th part of 72   6   11?
      c.  ſt.  lb.  c.  ſt.  lb.
   6) 72   6   11 (12   0   15 the quotient.
      72
       7) 6 (0
          0
          6=84
         6) 95
            90
             5 remains.
```

2. RULE.

If the diviſor be a great number made up of ſeveral others by multiplication. Divide ſucceſſively by the parts, inſtead of the whole.

Ex. 3.

```
                    £.   s.   d.
           Divide 320   12    8  by 35.
    £.    s.   d. 7)           £.  s.  d.
 5) 320   12   8 (64   2   6¼ (9   3   2½ quotient.
    320              63
     0 = 0            1=20
 5)     12          7) 22 (3
        10             21
         2=24           1=12
      5) 32          7) 18 (2
         30             14
          2=8            4=16
       5) 8 (1       7) 17 (2
          5              14
          3 rem.          3 rem.
```

PRO-

PROBLEM X.

To extract the ſquare root.

1. A GENERAL RULE.

1. Begin at the units place, and point every other figure on the top, dividing it into ſeveral periods.

2. Find the greateſt ſquare that is contained in the firſt period, towards the left hand. Set the root in the quotient, and ſubtract the ſquare from the figures of that period.

3. To the remainder bring down the two figures under the next point, for a *reſolvend*. This is always to be repeated.

4. Double the quotient for a diviſor, and ſee how oft it is contained in the reſolvend (excepting the laſt figure); and ſet the anſwer in the quotient, and alſo after the diviſor. This muſt always be repeated; for a new diviſor muſt be found for every figure.

5. Then multiply this whole diviſor by that quotient figure, and ſubtract the product from the whole reſolvend; but if that product be greater, a leſs figure muſt be placed in the quotient. Proceed thus till all the figures or periods be brought down.

6. *Note*, inſtead of doubling the quotient every time for a diviſor, you may always add the laſt quotient figure to the laſt diviſor, for a new diviſor; and proceed as before.

Ex. 1.

Extract the ſquare root of 393129.

```
        .  .  .
        393129 (627 the root.
        36
        --
 122)   331
 +2     244
        ---
 1247)  8729
        8729
        ----
```

Explanation.

The neareſt ſquare to 39 the firſt pointing, is 36, whoſe root 6 I place in the quotient; and ſubtract the ſquare 36 from 39, the remainder is 3.

Then I bring down 31, the next point, and annex it to 3, and the reſolvend is 331. Then I double the quotient for a diviſor, which is 12; and I ſeek how oft 12 in 33, the anſwer is 2, which I place in the quotient, and alſo after 12; then the diviſor becomes 122; and 122 multiplied by 2 produces 244, which I ſubtract from 331, the remainder is 87.

Laſtly, I bring down 29, the next point, and the reſolvend is 8729. Then I either double the quotient 62, which is 124; or I add the quotient figure 2 to 122, the laſt diviſor, which is 124; and this is a new diviſor. Then I aſk how oft 124 in 872, the anſwer is 7 times. Then I multiply 1247 by 7, and ſubtract the product 8729 from 8729, and there remains 0. So the root is exactly 627.

Ex. 2.

Extract the root of 733120000.

```
        .  .  .  .  .
        733120000 (27076 the root.
        4  .  .  .
       ----
 47)    333
 +7     329
 ----  ----
 5407)   41200
   +7    37849
 -----  ------
 54146)   335100
          324876
          ------
           10224 rem.
          ------
```

Ex. 3.

What is the root of 3272869681?

```
       ·  ·  ·  ·  ·
       3272869681 (57209 root.
       25  ·  :  :  :
      ----
107)  772
  7   749
----  ----
1142)  2386
   2   2284
-----  -----
114409) 1029681
        1029681
        -------
        ......
        -------
```

2 RULE.

When more than half the figures of the root are found; all the reſt will be found as truly by plain diviſion; as is ſhewn more at large in the extraction of the roots of decimal fractions. But if common diviſion be uſed, you muſt bring down as many figures, as there were periods to come down, when you began with diviſion.

Ex.

Ex. 4.

Let 14876008357020684 be given.

```
              .  .  .  .  .  .  .  .  .
              14876008357020684 (121967243
              1  .  .  .  .....
              ---
      22) 48
      +2  44
      ------
      241) 476
       +1  241
      -------
      2429)23500
        +9 21861
      ----------
      24386) 163983
          +6 146316
      -------------
divifor 24392)  176675
                170744
                ------
                  59317
                  48784
                  -----
                  105330
                   97568
                  ------
                    77622
                    73176
                    -----
                     4446
                     ----
```

The proof is, to multiply the root by itſelf, and add the remainder; which muſt be equal to the number given to be extracted, if the work be right.

PROBLEM XI.

To extract the cube root.

RULE.

1. Begin at the units place, and point every third figure; that is, the 1ſt, 4th, 7th, *&c.* miſſing two places.

2. Find the neareſt leſs root of the figures of the firſt punctation on the left hand, ſubtract its cube from the number given; to the remainder annex the next figure, for the reſolvend.

3. Take $\frac{1}{3}$ of the reſolvend for a dividend.

4. And for a diviſor, take the ſquare of the root, added to half the root, (or rather added to the product of the root, and the next quotient figure, leaving out the laſt figure of the product).

5. Divide the ſaid dividend by that diviſor, the quotient is the ſecond figure of the root.

6. Begin the operation anew, *viz.* cube the two figures of the root, and ſubtract the cube from the given number, annexing another figure, for the reſolvend.

7. Take the third part of the reſolvend for a dividend, and the ſquare of the root added to half the root (or rather added to the product of the root, and next quotient figure, ſtriking off the laſt figure of the product) for a diviſor.

8. This diviſion gives another figure of the root, but the diviſion is to be continued on to two figures, by the contraction in diviſion of decimals, or otherwiſe.

9. Repeating the operation with 4 figures in the root, you will get 4 more by a new diviſion, which gives 8 figures in the root; and from 8 to 16, *&c.* always double.

10. *Note*, when the cube exceeds the number given, a leſs figure muſt be writ in the quotient. And obſerve every diviſion gives one figure, and the reſt are found by continuing the diviſion, and dropping a figure of the diviſor every time.

Ex. 1.

Extract the cube root of 7892485271.

```
                 .  .  .  .
                 7892485271 (19 = 1 root
                 1                                   19
                 ---                                 19
              3) 68 reſolvend                       ---
      diviſor 1) 22 (9                              171
             +1  18                                 19
             --  ---                                ---
 true diviſor 2   4                                 361
             -----------                             19
                                                   ----
                 .  .
                 78924 (                           3249
                 6859                              361
                 ----                              ----
              3) 10334 reſolvend                   6859
    diviſor 361) 3445 (91                          ----
            +17  3402
            ---  ----
true diviſor 378)  43        hence the root is 1991.
```

Then 1991 cubed is 7892485271, and therefore 1991 is exactly the root required.

Explanation.

1 being the greateſt cube contained in 7, the firſt point; ſubtract 1 there remains 6, to which annex 8, and the reſolvend is 68, the third part is 22 for a dividend. Then 1 the ſquare of the root being a diviſor, ſay how oft 1 in 22, the quotient would give more than 10, but ſince we can have no figure above 9, we will take 9 by gueſs for the quotient; then 9 times the root 1 is 9, which is very near 10, throw away the 0 and add 1 to the root 1, which makes 2 for

for the true diviſor; then to have the true quotient figure, ſay how oft 2 in 22, *anſ.* 9 times, for we can take no more; therefore 9 is rightly taken.

Then the root 19 being ſquared gives 361, and cubed is 6859. This cube ſubtracted from 78924 leaves 10334 the reſolvend, which divided by 3 gives 3445 for a dividend; and 361 is the diviſor, and the quotient is 9; then the root 19 multiplied by 9 gives 171, therefore add 17 to 361 gives 378 for the exact diviſor. Then by dividing you will get 91: and the root 1991.

Ex. 2.

To extract the cube root of 28373625.

```
     .  .  .
    28373625 (30 = 1 root
    27
    ---
 3) 13              900 = ſquare
 9)  4 (0         27000 = cube
 --------

    .  .
    283736              30 root
    27000                5 quotient
    ------              ---
   3) 13736             150
 900)  4579 (5
  15   4575   therefore 305 is the root, which
 ---   ----    cubed gives 28373625, exact.
diviſor 915       4
               ----
```

All the root might have been had at once by bringing down another figure, and that is becauſe the ſecond figure happens to be 0.

```
        .
Thus 2837
      27
    -----
  3) 137
  9) 4·5 (05
```

Ex. 3.

To extract the cube root of 8302348000000.

8302348000000 (202 = 1 root
8

3) 3
4) 1 (02
0

10

then 202 ſquared is 40804, and cubed is 8242408.

83023480
8242408

3) 599400
40804) 199800 (48
½ root 101 163620

40905 36180
32724

3456

therefore the root is 20248, or very near 20249.

Ex. 4.

Extr. the cube root of 118248245000000000000000.

118248 245000 000000 000000 (49
64

3) 542 reſolvend

16) 180 (9 4
3 171 9

diviſor 19 9 3|6

Then 49 ſquared is 2401, and cubed is 117649.

1182480
117649

3) 5990 reſolvend

diviſor 2401) 1996 (08; and the root is 4908.
1920

76

Then the ſquare of 4908 is 24088464, and its cube 118226181312, therefore proceed

118248245000
118226181312

3) 22063688 reſolvend
24088464) 73545626 (3052 4908
1472 72269808 3

diviſor 24089936 1275818 1472|4
1204496

71322
48180

23142, *&c.*

Therefore the root is 49083052, or very near 49083053.

The proof of your work is, to multiply the root by itſelf and the product by the root; which muſt equal, or nearly equal, the number given to be extracted.

CHAP.

CHAP. II.

VULGAR FRACTIONS.

DEFINITIONS.

1. A *FRACTION* is ſome part or parts of an integer or whole thing, repreſented by 1; as $\frac{3}{4}$ is a fraction denoting three fourth parts of an integer or 1. Every fraction conſiſts of two numbers, placed one above the other, with a line between them, as in this fraction $\frac{3}{4}$. The lower number 4 is called the *denominator*, and ſhows how many parts the integer is divided into; the upper number 3 is called the *numerator*, and expreſſes how many of theſe parts the fraction conſiſts of. And both numerator and denominator are called *terms* of the fraction.

2. A *proper fraction* is that where the numerator is leſs than the denominator, as $\frac{3}{4}$.

3. An *improper fraction* is that wherein the denominator is leſs than, or equal to, the numerator, as $\frac{4}{3}$ or $\frac{3}{3}$, &c.

4. A *ſingle fraction* is that which conſiſts of but one numerator and one denominator.

5. A *compound fraction*, or fraction of a fraction, is that whoſe parts are vulgar fractions, connected with the word *of*, as $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{4}{5}$.

6. A *mixt number* is a whole number with a fraction annexed, as $15\frac{2}{3}$.

7. *Denomination* is the name of any integer or thing. Thus pounds, ſhillings and pence are ſeveral denominations; where ſhillings are of a lower denomination than pounds, and higher than pence.

SCHOLIUM.

Any fraction, as $\frac{3}{4}$, may be conſidered either as $\frac{1}{4}$ of the number 3, or as $\frac{3}{4}$ of 1. For $\frac{1}{4}$ of 3 being thrice as much as $\frac{1}{4}$ of 1, and $\frac{3}{4}$ of 1 being alſo thrice as much as $\frac{1}{4}$ of 1; it follows, that $\frac{1}{4}$ of 3, and $\frac{3}{4}$ of 1 ſignify the ſame quantity.

Like-

Likewiſe in any fraction as $\frac{3}{4}$, the numerator 3 may be conſidered as a dividend, and the denominator 4 as a diviſor. For as $\frac{3}{4}$ ſignifies the fourth part of 3, it intimates a diviſion by 4; therefore 3 becomes a dividend and 4 a diviſor, by the nature of diviſion, and $\frac{3}{4}$ repreſents the quotient.

When an integer is divided into any number of parts (denoted by the denominator); the fewer or more parts taken, the leſs or greater is the fraction, that is, the leſs or greater the numerator, the leſs or greater is the fraction. And if the number of parts taken be the ſame as the integer is divided into, that is, if the numerator be equal to the denominator, then that fraction will be equal to the whole or integer. Thus 2 halfs, 3 thirds, *&c.* that is, $\frac{2}{2}$ or $\frac{3}{3}$ or $\frac{4}{4}$ *&c.* is equal the whole thing, or equal to 1 the integer. And therefore when the numerator is leſs or greater than the denominator, the fraction is leſs or greater than 1.

From what has been ſaid, if one fraction or mixt number as $18\frac{11}{14}$, be to be divided by another as $4\frac{3}{5}$, it may be written thus, $\frac{18\frac{11}{14}}{4\frac{3}{5}}$, and if any ſuch fractional quantity as this $\frac{18\frac{11}{14}}{14\frac{3}{5}}$ occur, it denotes a diviſion of the number $18\frac{11}{14}$ by $4\frac{3}{5}$.

PROBLEM I.

To reduce a fraction into another of equal value.

RULE.

Multiply (or divide) both terms of the fraction by one and the ſame number, and you will have a new fraction equivalent to the fraction given.

Example.

Let the fraction be $\frac{3}{5}$, multiply both terms by by 6 produces $\frac{18}{30}$ for the new fraction; that is, $\frac{3}{5} = \frac{3 \times 6}{5 \times 6} = \frac{18}{30}$. On the contrary, in the fraction $\frac{18}{30}$, divide both terms by 6, gives $\frac{3}{5}$, with is equivalent to $\frac{18}{30}$.

For in the fraction $\frac{3}{5}$, it is plain the 5th part of 3 is all one as the 10th part of 6, or the 15th part of 9, and ſo on; that is, the 5th part of 3, is the ſame as the $\overline{6 \times 5}$th part (30th part) of 6 × 3 or 18.

Or thus, in the improper fraction $\frac{4}{2}$, 4 contains 2 as oft as 3 times 4 (12), contains 3 times 2 (6); that is, $\frac{4}{2} = 2$ for the quotient, and $\frac{12}{6} = 2$ for the quotient, therefore $\frac{4}{2} = \frac{12}{6}$, *&c.*

In like manner it is evident that 3 pennies contain 1 penny, as oft as 3 groats contain 1 groat; or as oft as 3 ſhillings contain 1 ſhilling. That is, $\frac{3}{1} = \frac{3 \times 4}{1 \times 4} = \frac{3 \times 12}{1 \times 12}$, *&c.*

And the ſame holds equally true for diviſion, that is, $\frac{3 \times 12}{1 \times 12} = \frac{3}{1}$, *&c.*

PROBLEM II.

To reduce a whole number to the form of a fraction.

RULE.

Place 1 under it for a denominator.

Example.

Suppoſe 7 is the whole number, then it becomes $\frac{7}{1}$ for the fractional quantity required.

PROBLEM III.

To reduce a whole number to a fraction of a given denominator.

RULE.

Multiply the whole number by the given denominator, and under the product write the ſame denominator.

Example.

Suppoſe 7 to have the denominator 11.

$\begin{array}{r}7\\ \underline{11}\\ 77\end{array}$, then $\frac{7\times 11}{11}$ or $\frac{77}{11}$ is the fraction required.

For $\frac{7\times 11}{11}=\frac{7}{1}=7$.

PROBLEM IV.

To reduce a compound fraction into a ſingle one.

RULE.

Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator, of the ſingle fraction.

Ex. 1.

Let the fraction be $\frac{1}{2}$ of $\frac{3}{5}$ of $\frac{2}{7}$.

$\begin{array}{rr}2 & 7\\ \underline{3} & \underline{5}\\ 6 & 35\\ \underline{1} & \underline{2}\\ 6 & 70\end{array}$ then $\frac{1\times 3\times 2}{2\times 5\times 7}=\frac{6}{70}$ the ſingle fraction.

For $\frac{1}{5}$ of $\frac{2}{7}$ is the ſame as $\frac{2}{7}$ divided by 5, or $\frac{2}{5\times 7}$, therefore $\frac{3}{5}$ thereof will be 3 times as much or $\frac{3\times 2}{5\times 7}$. Laſtly, the whole fraction being now $\frac{3\times 2}{5\times 7}$, the

the $\frac{1}{2}$ of it is $\frac{3\times2}{5\times7}$ divided by 2, or $\frac{1\times3\times2}{2\times5\times7}=\frac{6}{70}$.

Ex. 2.

What fraction of a pound is $3\frac{1}{2}d.$?

$3\frac{1}{2}d. = \frac{7}{2}$ of $\frac{1}{12}$ of $\frac{1}{20}$ of a pound,

that is, $3\frac{1}{2}d. = \frac{7\times1\times1}{2\times12\times20} = \frac{7}{480}$ of a pound.

And thus $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of a pound is $\frac{24}{60}$ or $\frac{8}{20}$ of a pound or 20 ſhillings, that is, 8 ſhillings. For $\frac{4}{5}$ of a pound is 16 ſhillings, and $\frac{3}{4}$ of 16 ſhillings is 12 ſhillings, and $\frac{2}{3}$ of 12 ſhillings is 8 ſhillings.

PROBLEM V.

To reduce a mixt number into an improper fraction.

RULE.

Multiply the whole number by the denominator of the fraction, and to the product add the numerator; and the ſum is a new numerator, and the denominator the ſame as before.

Example.

The mixt number is $32\frac{5}{7}$.

$$\begin{array}{r}32\\ \underline{7}\\ 224\\ \underline{+5}\\ 229\end{array}$$

then $\frac{32\times7+5}{7}=\frac{229}{7}$ is the fraction required.

For 32 wholes or $\frac{32}{1}=\frac{23\times7}{7}=\frac{224}{7}$ or 224 ſevenths, to which if the other 5 ſevenths be added, the whole is 229 ſevenths or $\frac{229}{7}$.

PROBLEM VI.

To reduce an improper fraction into a whole or mixt number.

RULE.

Divide the numerator by the denominator, and the quotient is the whole number. Then what remainder there is, place it over the denominator, and annex this fraction to the quotient before found.

Example.

Let $\frac{631}{16}$ be propoſed; 631 divided by 16 gives 39 for the quotient, and 7 remaining, therefore $39\frac{7}{16} = \frac{631}{16}$ as required.

$$\begin{array}{l} 16)\ 631\ (39\tfrac{7}{16} \\ \quad\ \ 48 \\ \hline \quad\ \ 151 \\ \quad\ \ 144 \\ \hline \qquad\ 7 \\ \hline \end{array}$$

For the fraction $\frac{631}{16}$ ſignifying 631 ſixteenths, therefore every 16 makes 1, and therefore the quotient 39 ſhows how many ones are contained in the number, and the 7 ſixteenths which remains, muſt therefore be placed as a fraction.

PROBLEM VII.

To find the greateſt common diviſor for the numerator and denominator of a fraction, or for any two numbers.

1. RULE.

1 RULE.

Divide the greater by the leſſer, and the laſt diviſor by the remainder, and ſo on continually till nothing remain; then the laſt diviſor is that required.

Or in dividing take the neareſt quotient, and the difference between the dividend and that multiple, for the next diviſor, *&c.*

Ex. 1.

Let $\frac{252}{364}$ be propoſed; dividing according to rule, the laſt diviſor is 28, which is the greateſt number that will divide both numerator and denominator, without a remainder.

Note, if the laſt diviſor be 1, the 2 numbers are prime to one another.

252) 364 (1
252
112) 252 (2
224
28) 112 (4
112
..

For ſince 28 meaſures 112, it likewiſe meaſures twice 112, or 224; and therefore 28 meaſures 224 + 28, or 252.

Again, ſince 28 meaſures 112 and 252, therefore it meaſures 252 + 112, or 364; and ſo on. Therefore 28 meaſures both 252 and 364.

Now 28 is the greateſt common meaſure; for if there be a greater G, then ſince G meaſures 252 and 364, it alſo meaſures the remainder 112, and ſince G meaſures 112 and 252, it alſo meaſures the remainder 28, that is, the greater meaſures the leſs, which is abſurd.

2 RULE.

If the numbers given be mixt numbers, or fractions; reduce them to a common denominator; and take the two new numerators, and proceed as in the firſt rule to find their greateſt common meaſure; make it a numerator, under which put the common denominator; and that fraction will be the greateſt common meaſure ſought.

Ex. 2.

Let 9$\frac{3}{4}$ and 13 be propoſed.

Theſe reduced to a common denominator are $\frac{39}{4}$ and $\frac{52}{4}$, then 39) 52 (1

39

13) 39 (3 ſo $\frac{13}{4}$ is the greateſt

39 common meaſure of

0 9$\frac{3}{4}$ and 13.

PROBLEM VIII.

To reduce a fraction to its leaſt terms.

1. A GENERAL RULE.

Find the greateſt common meaſure, by which divide both terms of the fraction; the quotients will be the terms of the fraction required.

Ex. 1.

Let the fraction be $\frac{252}{364}$, whoſe greateſt common meaſure is 28, diviſion being performed, we have $\frac{9}{13}$, that is, $\frac{252}{364} = \frac{9}{13}$.

28) 252 (9 28) 364 (13 $\frac{9}{13}$ the fraction.
    252   28
    ———   ———
      .   84
             84
            ———
              .

PARTICULAR RULES.

2 RULE.

When the terms of the fraction are even numbers, divide them by 2 continually.

Ex. 2.

$\frac{48}{272}$, being continually halfed is $\frac{48}{272}\Big|\frac{24}{136}\Big|\frac{12}{68}\Big|\frac{6}{34}\Big|\frac{3}{17}$, therefore $\frac{48}{232} = \frac{3}{17}$.

3. When both terms end with 5; or one with 5, and the other with a cypher; divide both by 5.

Ex. 3.

As $\frac{225}{475}$; $5)\,\frac{225}{475}\,(\frac{45}{95}\,(\frac{9}{19}$.

4. When both terms end with cyphers, cut off equal cyphers in both.

Ex. 4.

As $\frac{10000}{25700}$, which becomes $\frac{100}{257}$.

5. If you can eſpy any number which will divide both terms, divide by that number.

Ex. 5.

As $\frac{21}{39}$, divide by $3\,)\,\frac{21}{39}\,(\frac{7}{13}$.

6. For expedition, try all numbers 2, 3, 4, 5, *&c.* till you find ſome that will divide both, if any there be.

Ex. 6.

As $\frac{119}{168}$; trying 2, 3, 4, 5, 6, none of them will do, but trying 7 it ſucceeds, $7)\ \frac{119}{168}\ (\frac{17}{24}$.

PROBLEM IX.

To reduce fractions of different denominators, to thoſe of equal value, having a common denominator.

1. A GENERAL RULE.

Multiply each numerator by all the denominators except its own, for a new numerator; then multiply all the denominators together for a new denominator.

Ex. 1.

$\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, become $\frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$.

$$\begin{array}{cccc} 2 & 3 & 4 & 3 \\ \underline{4} & \underline{3} & \underline{4} & \underline{4} \\ 8 & 9 & 16 & 12 \\ \underline{5} & \underline{5} & \underline{3} & \underline{5} \\ 40 & 45 & 48 & 60 \end{array}$$

For in each fraction, both terms are multiplied by the ſame number; and therefore its value is not altered.

PARTICULAR RULES.

2 RULE.

Divide the denominators by their greateſt common diviſor; and multiply both terms of each fraction, by all the other quotients, which will produce as many new fractions. This is the beſt rule for 2 fractions, as

Ex. 2.

$\frac{5}{12}, \frac{7}{18}$. Divide by $6\,\big)\,\underset{2}{\frac{5}{12}},\ \underset{3}{\frac{7}{18}}$, the quotients are 2, 3. Then $\frac{5\times3}{12\times3}=\frac{15}{36}$, and $\frac{7\times2}{18\times2}=\frac{14}{36}$.

3 RULE.

In ſeveral fractions, divide all the denominators by their greateſt common diviſor, ſetting the quotients underneath; then find the leaſt number which all theſe quotients can meaſure; and divide this number ſeverally by all theſe quotients, and ſet theſe new quotients underneath. Then multiply the terms of each fraction by its new quotient, gives the correſpondent fraction required, and all theſe will be in their leaſt terms.

Ex. 3.

$$\begin{array}{rccccc} 3\big) & \frac{13}{36} & \frac{1}{24} & \frac{11}{18} & \frac{7}{12} & \frac{4}{9}, \\ & 12 & 8 & 6 & 4 & 3, \\ & 2 & 3 & 4 & 6 & 8 \\ & \frac{26}{72} & \frac{3}{72} & \frac{44}{72} & \frac{42}{72} & \frac{32}{72} \end{array}$$

the greateſt com. diviſor is 3.
the leaſt number they meaſure is 24.
the fractions required.

It is evident each of theſe is of the ſame value as that given, having both its terms multiplied alike. And they will be in the leaſt terms, becauſe 24 is the leaſt number that the firſt quotients meaſure.

SCHOLIUM.

By this problem the greateſt of two or more fractions may be diſcovered.

PROBLEM X.

Several fractions being given; to find as many whole numbers, in the ſame proportion.

RULE.

Reduce the fractions to a common denominator, then the ſeveral numerators will be to one another as the fractions given.

Example.

Suppoſe $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$. Theſe are reduced to $\frac{6}{12}$, $\frac{4}{12}$, $\frac{3}{12}$, therefore the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, are as the numbers 6, 4, and 3.

PROBLEM XI.

To find the value of a vulgar fraction in known parts of the integer

RULE.

Multiply the numerator by the number of parts contained in the integer, and divide the product by the denominator, the quotient ſhews the known parts. If there be any remainder, multiply it by the next inferior denomination, and divide by the denominator as before: and continue this work till you come at the loweſt denomination.

Example.

What is $\frac{3}{17}$ of a pound ſterl.? Anſ. *3s. 6d.* 1$\frac{7}{17}$*f.*

3
20

17) 60 (3 ſhillings
51

9
12

18
9

17) 108 (6 pence
102

6
4

17) 24 (1$\frac{7}{17}$ farthings.
17

7

PROBLEM XII.

To reduce a fraction of one denomination to the fraction of another denomination.

RULE.

1. From a leſs to a greater denomination; multiply the *denominator* by all the denominations, from that given, to that ſought.

2. From a greater to a leſs denomination; multiply the *numerator* by all the denominations, from that given, to that ſought.

Ex. 1.

Given $\frac{3}{5}$ of a penny; what fraction of a pound is it?

Anſw. $\frac{3}{5 \times 12 \times 20} = \frac{3}{1200}$ of a pound.

Ex. 2.

$\frac{3}{5}$ of a pound, what is that of a penny?

Anſ. $\frac{3 \times 20 \times 12}{5} = \frac{720}{5}$ of a penny.

For $\frac{3}{5}$ of a penny is $\frac{3}{5}$ of $\frac{1}{12}$ of $\frac{1}{20} = \frac{3}{5 \times 12 \times 20}$.

And $\frac{3}{5}$ of a pound reduced to pence is $\frac{3}{5} \times 20 \times 12$.

PROBLEM XIII.

To add fractions together.

1. A GENERAL RULE.

Reduce compound fractions to ſingle ones; mixt numbers to improper fractions; and fractions of different denominators to a common denominator.

Then add the numerators, and ſubſcribe the common denominator.

Ex. 1.

What is the ſum of $\frac{2}{9}$ and $\frac{3}{9}$?

$$\begin{array}{ll} \text{to} & 2 \\ \text{add} & \underline{3} \\ & 5 \end{array} \quad \text{anſ. } \frac{5}{9}.$$

Ex. 2.

What is the ſum of $\frac{3}{4}$ and $\frac{3}{5}$?

When reduced to a common denominator they are $\frac{15}{20}$ and $\frac{12}{20}$,

$$\begin{array}{ll} \text{to} & 15 \\ \text{add} & \underline{12} \\ & 27 \end{array} \quad \text{the ſum } \frac{27}{20} \text{ or } 1\frac{7}{20}.$$

Ex. 3.

What is the ſum of $\frac{1}{3}$ of $\frac{1}{4}$, and $\frac{3}{8}$, and $1\frac{1}{4}$?

$\frac{1}{3}$ of $\frac{1}{4} = \frac{1}{12}$, alſo $1\frac{1}{4} = \frac{5}{4}$. Then $\frac{1}{12}$, $\frac{3}{8}$ and $\frac{5}{4}$, reduced to a common denominator, are $\frac{2}{24}$, $\frac{9}{24}$ and $\frac{30}{24}$.

$$\begin{array}{r} 2 \\ 9 \\ \underline{30} \\ 41 \end{array}$$

the ſum $\frac{41}{24}$ or $1\frac{17}{24}$.

PARTICULAR RULES.

2 RULE.

When many fractions are given, firſt add two of them, and to the ſum add a third, and to that ſum a fourth, and ſo on.

Ex. 4.

Add together $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$.

$\frac{2}{3}$ and $\frac{3}{4}$ are reduced to $\frac{8}{12}$ and $\frac{9}{12}$, whoſe ſum is $\frac{17}{12}$. Then $\frac{17}{12}$ and $\frac{4}{5}$ are reduced to $\frac{85}{60}$ and $\frac{48}{60}$, whoſe ſum is $\frac{133}{60}$. Then $\frac{133}{60}$ and $\frac{5}{6}$ are reduced to $\frac{133}{60}$ and $\frac{50}{60}$, whoſe ſum is $\frac{183}{60}$ or $3\frac{3}{60}$, the ſum of all the four fractions.

3 RULE.

When mixt numbers are to be added, firſt add the fractions to the fractions; and then the whole numbers by themſelves.

Ex. 5.

Let $3\frac{1}{2}$, $4\frac{1}{3}$, and $10\frac{3}{8}$ be added.

$\frac{1}{2}$, $\frac{1}{3}$ and $\frac{3}{8}$ are reduced to $\frac{12}{24}$, $\frac{8}{24}$ and $\frac{9}{24}$,

$$\begin{array}{r}12\\8\\\underline{9}\\29\end{array}$$

$\frac{29}{24}$ or $1\frac{5}{24}$ is the ſum of the fractions,

to which add the whole numbers $\left\{\begin{array}{r}1\frac{5}{24}\\3\\4\\10\end{array}\right.$

$$\text{the ſum } \begin{array}{r}\hline 18\frac{5}{24}\\\hline\end{array}$$

4 RULE.

In fractions of different denominations, reduce them to thoſe of a common denomination, and then to a common denominator. Then add the numerators, and ſubſcribe the common denominator.

Ex. 6.

Add together

$\frac{3}{5}$ of a pound, $\frac{5}{10}$ of a ſhilling, and $\frac{7}{8}$ of a penny.

$\frac{5}{10}$ of a ſhilling is $\frac{5}{200}$ of a pound, and $\frac{7}{8}$ of a penny is $\frac{7}{1920}$ of a pound.

Then

$\frac{3}{5}$, $\frac{5}{200}$ and $\frac{7}{1920}$ are reduced to $\frac{5760}{9600}$, $\frac{240}{9600}$, $\frac{35}{9600}$.

$$\begin{array}{r} 5760 \\ 240 \\ 35 \\ \hline 6035 \end{array}$$

The ſum of the fractions is $\frac{6035}{9600}$ of a pound, or $\frac{1207}{1920}$ in leſs terms.

Or the fractions may be reduced to ſhillings, or pence.

PROBLEM XIV.

To ſubtract one fraction from another.

1. A GENERAL RULE.

Reduce compound fractions to ſingle ones; mixt numbers to improper fractions; and fractions of different denominations to thoſe of the ſame denomination; and laſtly, fractions of different denominators to a common denominator.

Then ſubtract the numerators, and ſubſcribe the common denominator.

Ex. 1.

From $\frac{4}{5}$ take $\frac{2}{5}$.

$$\begin{array}{lr} \text{from} & 4 \\ \text{take} & 2 \\ \hline & 2 \end{array}$$, the remainder is $\frac{2}{5}$.

Ex.

Ex. 2.

From $\frac{6}{13}$ take $\frac{3}{8}$.

Reduced to $\frac{48}{104}$, $\frac{39}{104}$.

$$\begin{array}{ll}\text{from} & 48\\ \text{take} & \underline{39}\\ & 9\text{, the rem.} = \frac{9}{104}.\end{array}$$

Ex. 3.

Take $\frac{2}{3}$ of $\frac{4}{5}$ from $\frac{2}{3}$.

$\frac{2}{3}$ of $\frac{4}{5}$ is reduced to $\frac{8}{15}$.

Then $\frac{8}{15}$ and $\frac{2}{3}$ are reduced to $\frac{8}{15}$ and $\frac{10}{15}$.

$$\begin{array}{r}10\\ \underline{8}\\ 2.\end{array} \quad \text{The remainder is } \frac{2}{15}.$$

Ex. 4.

From $25\frac{3}{8}$, take $21\frac{1}{4}$.

Reduced to $\frac{203}{8}$ and $\frac{85}{4}$.

$$\begin{array}{r}203\\ \underline{85}\\ 118\text{,}\end{array} \text{ the rem.} = \frac{118}{4} = 29\tfrac{2}{4}\text{, or } 29\tfrac{1}{2}.$$

Ex. 5.

From $\frac{1}{3}$ of a pound take $\frac{7}{9}$ of a ſhilling.

$\frac{1}{3}$ of a pound $= \frac{20}{3}$ of a ſhilling.

$$\begin{array}{r}20\\ \underline{7}\\ 13\text{,}\end{array} \text{ the rem.} = \frac{13}{3} \text{ of a ſhilling} = 4\tfrac{1}{3} \text{ ſhilling.}$$

Or $\frac{7}{9}$ of a ſhilling may be reduced to pounds, &c.

PARTICULAR RULES.

2 RULE.

In mixt numbers, take the fraction from the fraction, and the whole number from the whole number, remembering to reduce the fractions to a common denominator: and if the fraction to be ſubtracted is leſs, borrow 1.

Ex. 6.

Take $21\frac{1}{4}$ from $25\frac{3}{8}$.

$\frac{1}{4}$ is reduced to $\frac{2}{8}$. Then

from $25\frac{3}{8}$
take $21\frac{2}{8}$

remains $4\frac{1}{8}$

Ex. 7.

From $108\frac{3}{4}$ take $92\frac{5}{6}$.

$\frac{3}{4}$ and $\frac{5}{6}$ reduced to a com. denom. are $\frac{9}{12}$ and $\frac{10}{12}$.

from $108\frac{9}{12}$ or $107\frac{21}{12}$
take $92\frac{10}{12}$ $92\frac{10}{12}$

remains $15\frac{11}{12}$ $15\frac{11}{12}$

here as 10 is greater than 9; add 1, that is, $\frac{12}{12}$ to 9 makes $\frac{21}{12}$, then 10 from 21, remains 11 twelfths, then carry 1 to 2 makes 3; and 3 from 8, remains 5, 9 from 10 remains 1.

Ex. 8.

From $272\frac{7}{12}$ take 14.

$272\frac{7}{12}$
14

remains $258\frac{7}{12}$

Ex.

Ex. 9.

Take $59\frac{5}{9}$ from 120.

$$\begin{array}{rrr} & 120 & \text{or } 119\frac{9}{9} \\ & 59\frac{5}{9} & 59\frac{5}{9} \\ \hline \text{remains} & 60\frac{4}{9} & 60\frac{4}{9} \\ \hline \end{array}$$

3 RULE.

A fraction from 1 or an integer; ſubtract the numerator from the denominator, the remainder is the numerator to be placed over the given denominator.

Ex. 10.

Take $\frac{17}{23}$ from 1.

$$\begin{array}{r} 23 \\ 17 \\ \hline 6 \end{array}$$. Then the remainder is $\frac{6}{23}$.

4 RULE.

A proper fraction from any whole number; ſubtract the numerator from the denominator, for the numerator of the fraction, which is to be annext to the whole number leſſened by 1.

Ex. 11.

Take $\frac{17}{23}$ from 57, the remainder is $56\frac{6}{23}$.

$$\begin{array}{lr} \text{from} & 57 \\ \text{take} & 0\frac{17}{23} \\ \hline \text{rem.} & 56\frac{6}{23} \\ \hline \end{array}$$

The reaſon of the rules in addition and ſubtraction, is evident; for when fractions are reduced to the ſame denominator, they have the ſame name; therefore as 2 ſhillings and 3 ſhillings make 5 ſhillings, ſo ſo 2 twentieths and 3 twentieths, make 5 twentieths. And 2 twentieths from 3 twentieths leaves 1 twentieth. That is, $\frac{2}{20} + \frac{3}{20} = \frac{5}{20}$, and $\frac{3}{20} - \frac{2}{20} = \frac{1}{20}$. And for the ſame reaſon $\frac{2}{9}$ and $\frac{3}{9}$ make $\frac{5}{9}$. And $\frac{2}{5}$ from $\frac{4}{5}$, remains $\frac{2}{5}$, &c.

PROBLEM XV.

To multiply fractions together.

I. A GENERAL RULE.

Reduce mixt numbers to fractions; then multiply the numerators together for a new numerator, and the denominators together for a new denominator.

Ex. 1.

Multiply $\frac{2}{3}$ by $\frac{5}{7}$. The product is $\frac{2\times5}{3\times7} = \frac{10}{21}$.

Ex. 2.

Multiply $7\frac{1}{2}$ by $\frac{3}{4}$.

$7\frac{1}{2}$ is reduced to $\frac{15}{2}$; then the product is $\frac{15\times3}{2\times4} = \frac{45}{8}$, or $5\frac{5}{8}$.

Ex. 3.

Multiply $3\frac{4}{7}$ by 13.

Theſe are reduced to $\frac{25}{7}$ and $\frac{13}{1}$.

$$\begin{array}{rr} 25 & 7 \\ 13 & 1 \\ \hline 75 & 7 \\ 25 & \\ \hline 325 & \end{array}$$

the product is $\frac{325}{7}$, or $46\frac{3}{7}$.

PARTICULAR RULES.

2 RULE.

When the numerator of one and denominator of the other, can be divided by any number; take the quotients inſtead thereof.

Ex. 4.

Multiply $\frac{3}{8}$ by $\frac{4}{7}$.

Divide by 4. $4)\frac{3}{\underset{2}{8}}\times\frac{\overset{1}{4}}{7}$, then $\frac{3}{2}\times\frac{1}{7}=\frac{3}{14}$ the product.

Ex. 5.

Multiply $\frac{3}{8}$ by $\frac{4}{9}$.

$\begin{matrix}3)\\4)\end{matrix}\frac{3}{8}\times\frac{4}{9}=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$ the product.

3 RULE.

A mixt number or fraction, to multiply by a whole number; multiply the whole number by the whole number; and then multiply the numerator by the ſaid whole number, and divide by the denominator, and add this quotient to the former product.

Ex. 6.

Multiply $\frac{3}{4}$ by 9. Then $\frac{3\times9}{4}=\frac{27}{4}$ the product.

$$\begin{array}{rl} & 3 \\ & 9 \\ \hline 4) & 27\ (6\tfrac{3}{4} \text{ the product.} \\ & 24 \\ \hline & 3 \\ \hline \end{array}$$

Ex. 7.

Multiply $3\frac{4}{7}$ by 13.

$$\begin{array}{ccc} 3 & 13 & 39 \\ \underline{13} & \underline{4} & \underline{7\frac{3}{7}} \\ 39 & 7)\ 52\ (7\frac{3}{7} & 46\frac{3}{7}\ \text{the product.} \\ & \underline{49} & \\ & 3 & \end{array}$$

4 RULE.

When a fraction is to be multiplied by a number which happens to be the ſame with the denominator; take the numerator for the product.

Ex. 8.

Multiply $\frac{3}{5}$ by 5, the product is 3.

5 RULE.

When ſeveral fractions are to be multiplied; ſtrike out ſuch multipliers as are found both in the numerators and denominators.

Ex. 9.

Multiply theſe $\frac{2}{7}$, $\frac{14}{15}$, $\frac{5}{8}$.

That is, $\frac{2 \times 14 \times 5}{7 \times 15 \times 8}$.

This becomes $\frac{1 \times 2 \times 1}{1 \times 3 \times 4}$, or $\frac{1 \times 1 \times 1}{1 \times 3 \times 2} = \frac{1}{6}$.

For 2 and 8 become 1 and 4, 14 and 7 become 2 and 1, and 5 and 15 become 1 and 3; by dividing reſpectively by 2, 7, and 5

A fraction is multiplied by any number, by multiplying the numerator by that number, or dividing the denominator by it, when it can be done;

as to multiply $\frac{3}{4}$ by 9, the product is $\frac{27}{4}$. For ſince 3 of any denomination multiplied by 9 produces 27 of that denomination, therefore 3 fourths multiplied by 9 produces 27 fourths, or $\frac{27}{4}$. And ſince $\frac{3}{4} = \frac{3\times9}{4\times9} = \frac{27}{36}$, therefore if $\frac{3}{4}$ or $\frac{27}{36}$ be multiplied by 9, the product is $\frac{27\times9}{36}$, or $\frac{27\times9}{4\times9} = \frac{27}{4}$, the ſame as dividing 36 (the denominator of $\frac{27}{36}$) by 9.

The reaſon of the general rule is this; $\frac{2}{3}$ multiplied by $\frac{5}{7}$, makes $\frac{2\times5}{3\times7}$ or $\frac{10}{21}$. For to take $\frac{2}{3}$ once we ſhall have juſt $\frac{2}{3}$, but to take $\frac{2}{3}$ only $\frac{1}{7}$ of a time, we ſhall only have $\frac{2}{3\times7}$, or $\frac{2}{21}$, becauſe dividing any fraction by any number as 7, is but multiplying the denominator by that number 7. Again, taking $\frac{5}{7}$ of $\frac{2}{3}$ is taking 5 times as much as $\frac{1}{7}$, that is, 5 times $\frac{2}{21}$, and this will be $\frac{2\times5}{21}$, becauſe multiplying any fraction by any number 5, is the ſame as multiplying the numerator by that number 5; and therefore the product is $\frac{10}{21}$.

And in the particular contracted rules, ſince both numerator and denominator are divided by the ſame numbers, the fraction will be of the ſame value.

Multiplication of fractions is only reducing a compound fraction to a ſingle one, for to multiply $\frac{2}{3}$ by $\frac{5}{7}$, is no more than to take $\frac{5}{7}$ of $\frac{2}{3}$.

In multiplication of proper fractions, the product is leſs than either the multiplier or multiplicand. As if $\frac{2}{3}$ be multiplied by $\frac{5}{7}$; if $\frac{2}{3}$ be multiplied by 1, the product will be juſt $\frac{2}{3}$; but if $\frac{2}{3}$ be taken not ſo much as once, as only $\frac{5}{7}$ of a time, the product will be leſs than $\frac{2}{3}$. And for the ſame reaſon it will be leſs than $\frac{5}{7}$, if $\frac{2}{3}$ be the multiplier.

PROBLEM XVI.

To divide one fraction by another.

1 A GENERAL RULE.

Reduce compound fractions to ſingle ones, mixt numbers to improper fractions, and fractions of different denominations to thoſe of the ſame denomination. Then multiply the denominator of the diviſor by the numerator of the dividend, for a new numerator; alſo multiply the numerator of the diviſor by the denominator of the dividend, for a new denominator; the new fraction is the quotient.

Ex. 1.

Divide $\frac{5}{8}$ by $\frac{3}{7}$.

$$\frac{3}{7}\Big)\ \frac{5}{8}\ \Big(\frac{7\times5}{3\times8}=\frac{25}{24}=1\frac{1}{24}.$$

Ex. 2.

Divide $\frac{3}{5}$ of a pound by $\frac{8}{9}$ of a ſhilling.

$\frac{8}{9}$ of a ſhilling is reduced to $\frac{8}{180}$ of a pound $=\frac{2}{45}$ of a pound. $\frac{2}{45}\Big)\ \frac{8}{9}\ \Big(\frac{360}{18}=20.$

Ex. 3.

Divide $11\frac{2}{3}$ by $2\frac{3}{4}$.

Theſe are reduced to $\frac{35}{3}$ and $\frac{11}{4}$.

$$\frac{11}{4}\Big)\ \frac{35}{3}\ \Big(\frac{140}{33}=4\frac{8}{33}.$$

Ex. 4.

Divide 7 by $\frac{3}{5}$.

$$\frac{3}{5}\Big)\ \frac{7}{1}\ \Big(\frac{35}{3}=11\frac{2}{3}.$$

PARTICULAR RULES.

2 RULE.

When it can be done, divide the numerator of the dividend by the numerator of the diviſor, and the denominator by the denominator, for the quotient.

Ex. 5.

Divide $\frac{8}{15}$ by $\frac{2}{3}$.

$\frac{2}{3}\Big)\ \frac{8}{15}\ \Big(\frac{4}{5}$ the quotient.

3 RULE.

When the two numerators, or the two denominators, can be divided by any number; take the quotients inſtead thereof.

Ex. 6.

Divide $\frac{12}{27}$ by $\frac{8}{5}$.

$$\overset{2}{\frac{8}{5}}\Big)\ \overset{3}{\frac{12}{27}}\ \Big(\frac{15}{54}.$$

Ex. 7.

Divide $\frac{8}{9}$ by $\frac{2}{45}$.

$$\frac{\overset{1}{2}}{\underset{5}{45}}\Big)\ \frac{\overset{4}{8}}{\underset{1}{9}}\ \Big(\frac{20}{1}=20.$$

4 RULE.

A fraction by a whole number; multiply the denominator by the whole number.

Ex. 8.

Divide $\frac{13}{15}$ by 7, the quotient $\frac{13}{15\times7}=\frac{13}{105}$.

5 RULE.

If the denominators are equal, place the numerator of the dividend over the numerator of the diviſor, for the quotient.

Ex. 9.

Divide $\frac{8}{19}$ by $\frac{3}{19}$, the quotieni is $\frac{8}{3}$, or $2\frac{2}{3}$.

To demonſtrate that $\frac{5}{8}$ divided by $\frac{3}{7}$, gives $\frac{35}{24}$ in the quotient, let them be reduced to a common denominator, then $\frac{3}{7}=\frac{24}{56}$, and $\frac{5}{8}=\frac{35}{56}$; then it is plain $\frac{5}{8}$ divided by $\frac{3}{7}$ is the ſame as $\frac{35}{56}$ divided by $\frac{24}{56}$. But 35 fifty ſixths contain 24 fifty ſixths, as oft as 35 contains 24, therefore the quotient is $\frac{35}{24}$ or $\frac{7\times5}{3\times8}$, as by the rule.

Alſo a fraction is divided by a whole number by multiplying the denominator by that number. As if $\frac{13}{15}$ be divided by 7, the quotient is $\frac{13}{15\times7}=\frac{13}{105}$.

For

For $\frac{13}{15} = \frac{13\times7}{15\times7} = \frac{91}{105}$: now if we take the 7th part of $\frac{13}{15}$, or its equal $\frac{91}{105}$, this is the ſame as dividing 91 hundred and fifths by 7, and the quotient is 13 hundred and fifths, or $\frac{13}{105} = \frac{13}{15\times7}$. And hence a fraction is divided by a whole number, by dividing the numerator by that number, when it can be done; for $\frac{91}{105}$ divided by 7, gives $\frac{13}{105}$ for the quotient.

In diviſion of fractions, if the diviſor be a proper fraction, the quotient will always be greater than the dividend. For it is evident, when any quantity or dividend is to be divided by 1, the quotient will be equal to the dividend: therefore if it is divided by a proper fraction, which is leſs than 1, the quotient will then be greater than the dividend: for a leſs diviſor will be oftener contained in the dividend, than a greater diviſor.

PROBLEM XVII.

To extract the ſquare root of a fraction, &c.

RULE.

1. Reduce them to the leaſt terms; then extract the root of the numerator for a new numerator; and the root of the denominator for a new denominator.

2. When they have not exact roots, add an equal number of cyphers to both terms, and then extract: or

3. When neither numerator nor denominator has an exact root, multiply the numerator by the denominator, and extract the root of the product, for a numerator, and under it place the ſaid denominator.

4. To find the fractional part of the root of a whole number nearly, take the remainder for a numerator, and twice the root (+ 1 if you will) for a denominator, of the fractional part.

Or

Or more exactly, make twice the remainder a numerator; and add 1 to 4 times the root, for a denominator.

Ex. 1.

Extract the ſquare root of $\frac{50}{18}$.

Here $\frac{50}{18}=\frac{25}{9}$, and the root of 25 is 5, and the root of 9 is 3; therefore the root of $\frac{25}{9}$ is $\frac{5}{3}$, or $1\frac{2}{3}$.

Ex. 2.

Extract the root of $5\frac{3}{16}$.

$5\frac{3}{16}=\frac{83}{16}$, then the root is $\frac{\sqrt{83}}{4}=\frac{9}{4}$ nearly.

Or thus.

$\frac{83}{16}=\frac{83000}{16000}$, and the root of $\frac{83000}{16000}$ is $\frac{\sqrt{1328000000}}{16000}$ $=\frac{36441}{16000}=\frac{9110}{4000}=\frac{911}{400}$ near.

Ex. 3.

To extract the root of $\frac{2}{3}$.

Here $\frac{2}{3}=\frac{20000}{30000}$. But the root of 20000 is 141; and the root of 30000 is 173;

Therefore the root of $\frac{2}{3}$ is $\frac{141}{173}$.

Or thus.

$\frac{2}{3}=\frac{200}{300}$, and $200 \times 300 = 60000$, whoſe root is 245, then the root is $\frac{245}{300}=\frac{49}{60}$.

Ex. 4.

Extract the root of $27\frac{3}{5}$.

$27\frac{3}{5}=\frac{138}{5}$, and $138 \times 5 = 690$, and the root of 690 is 26, then the root is $\frac{26}{5}=5\frac{1}{5}$, nearly, but too ſmall.

Ex.

Ex. 5.

Extract the root of 22, or $\frac{22}{1}$.

$\dot{2}\dot{2}$ ($4\frac{6}{8}$, or $4\frac{6}{9}$ the root.
16
rem. 6

Or thus.

$2\dot{2}$ ($4\frac{12}{17}$ the root.
16
6 4
2 4
12 $16+1=17$.

Ex. 6.

To extract the root of 253.

$\dot{2}5\dot{3}$ ($15\frac{28}{30}$, or $15\frac{28}{31}$ the root.
1
25) 153
125 or more exactly $15\frac{56}{61}$ is the root.
28

Ex. 7.

Extract the root of $\frac{7}{8}$.

Here $8\times 7=56$. And the root of 56 is $7\frac{7}{14}$, or $7\frac{7}{15}$.

$5\dot{6}$ ($7\frac{7}{14}=7\frac{1}{2}$. And the root is $\frac{7\frac{1}{2}}{8}=\frac{15}{16}$.
49
7 or more exactly $\frac{7\frac{14}{29}}{8}$.

PROBLEM XVIII.

To extract the cube root of a fraction.

RULE.

1. Reduce the fraction to the leaſt terms; then extract the roots of the numerator and denominator, if they have any, for the numerator and denominator of the fraction.

2. If

2. If they have not exact roots, add an equal number of cyphers to both terms, and then extract: or

3. If neither of them have exact roots, multiply the numerator by the ſquare of the denominator, and extract the root of the product for a numerator, and under it place the ſaid denominator. And here you may add cyphers to both, before you begin, as before.

4. To find the fractional part of the cube root of a whole number; make the remainder a numerator, and thrice the ſquare of the root a denominator.

Or more exactly, make twice the remainder a numerator, and add 3 times the root to 6 times its ſquare, for a denominator.

But the moſt general method is to reduce the fraction to a decimal, and then extract the root, as hereafter.

Ex. 1.

Extract the cube root of $\frac{1}{27}$.

The root of 1 is 1, and the root of 27 is 3, then $\frac{1}{3}$ is the root.

Ex. 2.

To extract the root of $\frac{24}{375}$.

$\frac{24}{375}$ is reduced to $\frac{8}{125}$, whoſe root is $\frac{2}{5}$.

Ex. 3.

Extract the root of $\frac{2}{3}$.

$\frac{2}{3} = \frac{20000}{30000}$, the root of 20000 is 27, and the root of 30000 is 31, therefore the root of $\frac{2}{3}$ is $\frac{27}{31}$.

Or

Or thus.

$2 \times 3 \times 3 = 18$. And the root is $\frac{\sqrt[3]{18}}{3}$. But $\sqrt[3]{18} = 2\frac{5}{6}$.

$$\begin{array}{l} 18\ (2\frac{10}{12} = 2\frac{5}{6} \text{ the numerator.} \\ 8 \\ \hline 10 \\ \hline 2\times 3 = 6 \\ 4\times 6 = 24 \\ \hline 30 \end{array}$$

or rather $2\frac{20}{30} = 2\frac{2}{3}$ for the numerator, and the root is $\frac{2\frac{2}{3}}{3} = \frac{8}{9}$.

Ex. 4.

Extract the cube root of $13\frac{4}{7}$.

$13\frac{4}{7}$ is reduced to $\frac{95}{7}$, then $\frac{95}{7} = \frac{95000}{7000}$.

The root of 95000 is 45 the numerator.

And the root of 7000 is 19 the denominator.

And the root $\frac{45}{19} = 2\frac{7}{19}$.

Otherwiſe.

$95 \times 7 \times 7 = 4655$, whoſe root is 16 or 17; therefore the root is between $\frac{16}{7}$ and $\frac{17}{7}$.

Or thus.

$$\begin{array}{l} 4655\ (16 \\ 4096 \\ \hline \end{array}$$

rem. 559, and thrice the ſquare of $16 = 768$, and the root is $16\frac{559}{768} = 16\frac{8}{11}$ nearly, the numerator.

Therefore the root of $13\frac{4}{7}$ is $\frac{16\frac{8}{11}}{7} = 2\frac{30}{77}$.

CHAP. III.

DECIMAL FRACTIONS.

Notation.

A *DECIMAL FRACTION* is a fraction whoſe denominator is 1 with one or more cyphers; thus, $\frac{1}{10}$, $\frac{3}{10}$, $\frac{5}{100}$, $\frac{27}{100}$, $\frac{9}{1000}$, are decimal fractions.

Here 1, or the integer, is always ſuppoſed to be divided into 10, 100, 1000, &c. equal parts; or, which is the ſame thing, 1 is ſuppoſed to be divided into 10 equal parts, and each of theſe parts into 10 equal parts, and each of theſe into 10 parts more, and ſo on, by a continual ſubdiviſion.

A decimal fraction is expreſſed without the denominator, by writing only the numerator and prefixing a point on the left hand of it. And the number of places in the numerator is always equal to the number of cyphers in the denominator; thus .3 ſignifies $\frac{3}{10}$, .03 ſignifies $\frac{3}{100}$, .37 ſignifies $\frac{37}{100}$, and .004 ſignifies $\frac{4}{1000}$; therefore when the numerator hath not ſo many places as the denominator has cyphers, the void places muſt be filled up with cyphers towards the left hand. And from hence is diſcovered how many cyphers the denominator conſiſts of.

Cyphers on the right hand of a decimal do neither increaſe nor diminiſh the value; thus .3 and .30 and .300, &c. are all equal, becauſe $\frac{3}{10} = \frac{30}{100} = \frac{300}{1000}$, &c. as is plain from vulgar fractions: and therefore decimals are ſoon reduced to a common denominator, by annexing cyphers.

The *notation* of decimal fractions, will be plain from the following table.

tenth parts	hundred parts	thouſand parts	10 thouſand parts	100 thouſand parts	million parts	10 million parts	
.3	2	8	5	0	7	6	*&c.*

As in whole numbers, the 1ſt place contains units, the ſecond place to the left, tens; the third, hundreds; *&c.* So in decimals the order of places is contrary, for the firſt place in decimals is tenths; the 2d place to the right is hundred parts; the 3d, is thouſand parts; *&c.* And as whole numbers increaſe from the right hand to the left in decuple proportion, or decreaſe from the left to the right in a ſubdecuple proportion; ſo decimals alſo increaſe from the right to the left in a decuple proportion, and decreaſe from the left to the right in the ſame ſubdecuple proportion. Thus in the table above, 3 ſignifies $\frac{3}{10}$, 2 ſignifies $\frac{2}{100}$, 8 ſignifies $\frac{8}{1000}$.

But in reading any decimal, as .328, we do not ſay 3 tenths, 2 hundredths, 8 thouſands; but firſt reduce them all to the denominator of the greateſt; and call them all by that name. Thus $\frac{3}{10} = \frac{300}{1000}$, $\frac{2}{100} = \frac{20}{1000}$, and $\frac{8}{1000}$ remains the ſame; and collecting them together, we have $\frac{328}{1000}$, that is, three hundred

dred and twenty eight thouſand parts: for .300 + .020 + .008 = .328.

A *mixt number*, is made up of a whole number and a decimal, which are ſeparated from one another by a point. Thus 32.17 ſignifies $32\frac{17}{100}$. And 5.03 ſignifies $5\frac{3}{100}$.

Hence any mixt number, as 5.03, may be expreſſed thus, $\frac{503}{100}$, or $\frac{5030}{1000}$, or $\frac{50300}{10000}$, &c. and 32.17 $= \frac{32.17}{1} = \frac{321.7}{10} = \frac{3217}{100} = \frac{32170}{1000}$, &c.

Numeration, or the reading of decimals, is the very ſame as that of whole numbers, only adding the name of the parts ſignified by the decimal. Thus 328.328 ſignifies 328 thouſands, and 328 thouſand parts.

Since decimals as well as whole numbers decreaſe to the right hand in a ſubdecuple proportion, therefore decimals have the ſame properties as whole numbers, and are ſubject to the ſame rules of operation. For in any whole number, the ſeveral parts of it are, in effect, but decimal parts of one another.

PROBLEM I.

To add decimal fractions.

RULE.

Place all the points directly under each other, then tenths will be under tenths, and hundred parts under hundredths, &c. then add them together as if they were whole numbers; and laſtly, put a point under the other points, which will prick off the number of decimal places in the ſum.

Ex. 1.

.3527
62.013
.002
.5
ſum 62.8677

Ex. 2.

.0035
.02761
.81017
.22
.017
ſum 1.07828

Ex. 3.

32.
5.07
.81
.20571
.0035
ſum 38.08921

PROBLEM II.

To ſubtract one decimal from another.

RULE.

Place the greater number uppermoſt, the points under the points, tenths under tenths, *&c.* then ſubtract

ſubtract as in whole numbers; placing the point of ſeparation under the other points.

Ex. 1.

from	.4302
take	.257
rem.	.1732

Ex. 2.

from	17.203
take	.07542
rem.	17.12758

Ex. 3.

from	29.
take	.0545
rem.	28.9455

PROBLEM III.

To multiply decimals together.

1. A GENERAL RULE.

Multiply the decimals as if they were whole numbers; and from the product cut off as many decimal places, as there are in both numbers. If there be not ſo many places, make them out with cyphers on the left.

Ex. 1.

$$\begin{array}{lr} & .9087 \\ & .852 \\ \hline & 18174 \\ & 45435 \\ & 72696 \\ \hline \text{product} & .7742124 \\ \hline \end{array}$$

Ex. 2.

$$\begin{array}{lr} & 23.17 \\ & 2.016 \\ \hline & 13902 \\ & 2317 \\ & 4634 \\ \hline \text{product} & 46.71072 \\ \hline \end{array}$$

Ex. 3.

$$\begin{array}{lr} & .09047 \\ & .00125 \\ \hline & 45235 \\ & 18094 \\ & 9047 \\ \hline \text{product} & .0001130875 \\ \hline \end{array}$$

Ex. 4.

$$\begin{array}{lr} & .003479 \\ & 5081. \\ \hline & 3479 \\ & 27832 \\ & 17395 \\ \hline \text{product} & 17.676799 \\ \hline \end{array}$$

To prove the truth of the rule, let 9087 be multiplied by 852; theſe are equivalent to $\frac{9087}{10000}$ and $\frac{852}{1000}$, whence if the numerators be multiplied together, and the denominators alſo, the product will be $\frac{7742124}{10000000}$, that is, .7742124 conſiſting of as many decimal places as there are cyphers, that is, of as many places as are in both the numbers.

For the ſame reaſon $\frac{2717}{100}$ multiplied by $\frac{2016}{1000}$, produces $\frac{4671072}{100000}$, or 46.71072.

PARTICULAR RULES *for contracting the work.*

2 RULE.

In large decimals, you muſt multiply in a contrary order, thus: Begin with the left hand figure of the multiplier, by which multiply the whole multiplicand.

Then prick off the laſt figure of the multiplicand on the right, and multiply the reſt by the next figure of the multiplier on the left.

Then prick off another figure of the multiplicand, and multiply the reſt by the next figure of the multiplier. Go on thus with all the figures of the multiplier; always pricking off a figure in the multiplicand, at each multiplying. And obſerve what is to be carried from the preceding figure, when you begin each multiplication.

Set the firſt figure of each product directly in a line under one another, to be added together.

Laſtly, when you multiply by the units place, obſerve what place of the multiplicand it begins with; and cut off ſo many decimals, in the product.

Or, obſerve the places of any two decimals that begin the multiplication, and the ſum of them gives the number of decimal places in the product.

Note,

Note, inſtead of pricking off the figures gradually in the multiplicand; you may know where to begin to multiply every time thus: If the firſt figure on the left of the multiplier, begins with the firſt figure on the right of the multiplicand; then the 2d figure begins with the 2d; and the 3d with the 3d; and ſo on.

Ex. 1.

	.....
multiply	76.84375
by	8.21054
	61475000
	1536875
	76843
	3842
	307
product	630.92867

Ex. 2.

	.....
multiply	.3570643
by	.0210576
	7141286
	357064
	17853
	2499
	214
product	.007518916

Explanation.

In Ex. 1. 8 multiplying the whole multiplicand, gives 61475000 for the product. Then prick off 5, and multiply by 2, ſaying 2 times 5 is 10, carry 1, and 2 times 7 is 14 and 1 is 15, 2 times 3 is 6 and 1 is 7, &c. and the product is 1536875. Again, prick off 7, and ſay once 3 is 3, once 4 is 4, &c. and that product is 76843. Then prick off 3, and ſay 0 times 4 is 0; again, prick off 4, and ſay 5 times 4 is 20, carry 2, then 5 times 8 is 40, and 2 is 42, &c. and the product is 3842. Laſtly, prick off 8, and ſay 4 times 8 is 32, carry 3; then 4 times 6 is 24 and 3 is 27, 4 times 7 is 28, and 2 is 30, and that product is 307. And the ſum of all 630.92867. And ſince 8 the units begins with 5 in the 5th place, there muſt be 5 places of decimals.

And ſince 2 begins to multiply at 7, 1 at 3, 0 at 4, 5 at 8, and 4 at 6; it is plain the firſt figure of each product will be in the 5th place of decimals; becauſe the ſum of the places of the two multipliers always makes 5.

In the 2d Ex. 2 begins to multiply at 3, 1 at 4, 0 at 6, 5 at 0, 7 at 7, 6 at 5. Where the ſum of both places makes 9; therefore there are 9 places of decimals.

Ex. 3.

	. . . .
multiply	17.002576|830
by	.35608204
	51007730
	8501288
	1020154
	13602
	340
	7
product	6.0543121

3 RULE.

When any decimal is to be multiplied by 10, 100, 1000; &c. remove the ſeparating point ſo many

many places to the right hand, as there are cyphers.

Ex. 8.

multiply	32.075
by	10
product	320.75

Ex. 9.

multiply	25.7
by	1000
product	25700.

4 RULE.

In large multiplications, make a table of all the products of the multiplicand by the 9 digits; and then the ſeveral products, are eaſily taken out of the table and writ down, as directed in multiplication of whole numbers.

PROBLEM IV.

To divide one decimal by another.

1. A GENERAL RULE.

Divide as if they were whole numbers. Then cut off as many decimal places in the quotient, as the number of decimal places in the dividend exceeds the number in the diviſor; if there are not ſo many in the diviſor, prefix ſo many cyphers.

Or thus, the firſt figure of the quotient (or indeed any quotient figure) is of the ſame degree as that *figure* of the dividend, under which the units place of the product ſtands.

Annex cyphers to the dividend, when there are not places ſufficient. Likewiſe by continually annexing cyphers, the diviſion may be continued as far as you pleaſe.

Ex. 1.

Divide 13.4 by 3207.3

```
3207.3) 13.400000 (.00417
        128292 ··
        ---------
           57080 dividual.
           32073
           -----
           250070
           224511
           ------
            25559
            -----
```

Explanation.

As the dividend wants places, I add cyphers at pleaſure; and there being ſix places of decimals in the dividend, and 1 in the diviſor; there will be 5 in the quotient; therefore 2 cyphers muſt be prefixt before 417, and the quotient is .00417 as required.

Or thus, ſince 9 the units place (of the product of the diviſor by 4) ſtands under the third place of decimals, therefore 4 is in the third place of decimals.

Ex. 2.

Divide 271.5 by 5.746

```
5.746) 271.50000 (47.25
       22984...
       -----
        41660
        40222
        -----
         14380
         11492
         -----
          28880
          28730
          -----
            150 &c.
```

Ex. 3.

Divide .4368 by .0078

```
.0078) .4368 (56.
       390.
       ---
        468
        468
        ---
         ..
```

Ex. 4.

Divide .052701 by 36.

```
36) .052701 (.001463
     36...
     ---
     167
     144
     ---
      230
      216
      ---
       141
       108
       ---
        33
```

To prove the rule; ſince the number of decimals in the dividend is equal to the number in both diviſor and quotient; it follows that the quotient contains as many as the dividend exceeds the diviſor.

Again, the quotient contains as many decimals, as 12829 (the product of 3207. by 4) contains, (for there are none in 3207 the diviſor); and that is, as many as are in the dividend 13.400, under which it ſtands to be ſubtracted; therefore it follows, that the quotient figure 4 is of the ſame degree as 9, the product of the units place of the diviſor, or as (0) the figure above it in the dividend. Therefore 4 the quotient figure is in the 3d place of decimals.

2 RULE.

To contract the work in large diviſions, inſtead of pricking one down from the dividend, prick one figure off the diviſor each operation; and in multiplying leave out theſe figures prickt off, only you muſt have regard to what is to be carried from the figure laſt prickt off.

Note, if the firſt figure in the quotient begins to multiply at the firſt figure in the diviſor, then the 2d begins at the 2d, the 3d at the 3d, &c.

Ex. 5.

```
  ......
76.84375) 630.92878 (8.210541
          61475000
          --------
           1617878
           1536875
           -------
             81003
             76843
             -----
              4159
              3842
              ----
               317
               307
               ---
                10
                 7
                --
                 3
```

Explanation.

Here 8 is multiplied into 76.84375; then 2 is multiplied into 76.8437 (carrying 1); then 1 is multiplied into 76.843; the multiplication of 7684 by 0, is omitted; then 768 by 5; then 76 by 4, laſtly 7 by 1.

3 RULE.

To divide by 10, 100, 1000, *&c.* remove the ſeparating point, ſo many places to the left hand as there are cyphers.

Ex. 6.

Divide 32.075 by 10.
quotient 3.2075

Ex. 7.

Divide 25.7 by 1000.
quotient .0257

4 RULE.

In large diviſions, make a table of the products of the diviſor and all the 9 figures. And then diviſion will be wrought by inſpection; for the ſeveral products are eaſily taken out of the table, as you want them, according to the directions in diviſion of whole numbers.

PROBLEM V.

To reduce or change a vulgar fraction to a decimal fraction.

RULE.

Add cyphers at pleaſure to the numerator, repreſenting ſo many places of decimals; and then divide by the denominator, as far as you pleaſe.

Ex. 1.

Reduce $\frac{3}{4}$ to a decimal.

```
4) 3.0000 (.7500, or .75
   28 · · :
   ——
    20
    20
    ——
      .00
      ——
```

Ex. 2.

Reduce $13\frac{4}{7}$ to a decimal or mixt number.

```
7) 4.000000 (.571428
   35 · · · · ·
   ——
    50
    49   then 13 4/7 = 13.571428
    ——
     10
      7
     ——
      30
      28
      ——
       20
       14
       ——
        60
        56
        ——
         4 &c.
        ——
```

Ex.

Ex. 3.

To reduce $\frac{16}{3}$ to decimals.

```
3) 16.00000 (5,333 &c. = 16/3.
   15 ...
   --
    10
     9
    --
     10
      9
     --
      10
       9
      --
       1 &c.
      --
```

Ex. 4.

To change $\frac{1}{243}$ to a decimal.

```
243) 1.00000000 (.004115 = 1/243.
       972 .....
       ---
        280
        243
        ---
         370
         243
         ----
         1270
         1215
         ----
           55 &c.
           --
```

SCHOLIUM.

To reduce a decimal to a vulgar fraction, is no more than dividing by the greateſt common meaſure; the denominator of the decimal being 10, 100, 1000, *&c.*

PROBLEM VI.

To reduce the known part or parts of any integer to a decimal.

RULE.

Begin at the laſt part, and reduce it to a vulgar fraction, of the next ſuperior denomination, and ſo to a decimal. Then take that, and the next part, if there is any, which alſo reduce to a decimal of the next ſuperior denomination; and ſo on to the laſt.

Ex. 1.

What decimal of a ſhilling is three half-pence?

3 half-pence is $= 1\frac{1}{2}d. = 1.5d.$, then $\frac{1.5}{12}d. =$ the fraction of a ſhilling, by dividing, $\frac{1.5}{12} = .125$ the decimal of a ſhilling.

12) 1.500 (.125
12
——
30
24
——
60
60
——
.
——

Ex. 2.

Reduce 6*s.* $3\frac{1}{4}d.$ to the decimal of a pound.

Here $\frac{1}{4}$ of a penny $= .25$, and $3\frac{1}{4}$ or 3.25 divided by 12, that is, $\frac{3.25}{12} = .270833$ the fraction of a ſhilling; and 6*s.* $3\frac{1}{4}d.$ or 6.270833 divided by 20

$20 \left(\frac{6.270833}{20}\right)$ is = .31354166 the decimal of a pound.

Ex. 3.

What decimal of a hundred weight is 3 *ſt.* 7 *lb.* 9 *oz.*; at 14 *lb.* to the ſtone.

9 *oz.* = $\frac{9}{16}$ *lb.* = .5625 *lb.*, and $\frac{7.5625}{14}$ = .540178 *ſt.* and $\frac{3.540178}{8}$ = .442522 hundreds.

Hence the following decimal table is made.

Money. 1 *l.* the integer. 1 *s.* = .05 1 *d.* = .00416667 1 *f.* = .00104167	*Averdupoiſe weight.* 1 *lb.* the integer. 1 *oz.* = .0625 1 *dr.* = .00390625
Troy weight. 1 *lb.* the integer. 1 *oz.* = .0833333 1 *pwt.* = .0041666 1 *gr.* = .0001736	*Averdupoiſe weight.* 1 hundred the integer. 1 *qr.* = .25 1 *lb.* = .00892857 1 *oz.* = .00055803
Apothecary's weight. 1 *oz.* the integer. 1 *dr.* = .125 1 *ſcr.* = .0416666 1 *gr.* = .0020833	*Long meaſure.* A yard the integer. 1 *f.* = .3333333 1 *in.* = .0277777
Time. 1 day the integer. 1 *ho.* = .0416666 1 *min.* = .0006944 1 *ſec.* = .0000115	*Square and ſolid meaſure.* 1 *in.* = .006945, the decimal of a ſquare foot. 1 *in.* = .0005787, the decimal of a cubic foot.

PROBLEM VII.

To find the value of a decimal in known parts of the integer.

RULE.

Multiply the decimal by the number of parts contained in the next inferior denomination, gives the parts required: and if the decimal cut off be multiplied by the next lower denomination, you'll have the parts of that denomination; and ſo on.

Ex. 1.

How much money is .732 of a pound?

.732 *l.*
20
14.640 *s.* Anſ. 14 *s.* 7 *d.* $2\frac{7}{10}$ *f.*
12
7.680 *d.*
4
2.72 *f.*

Ex. 2.

What weight is 5.7305 *lb.* averdupoiſe?

5.7305 *lb.*
16 Anſ. 5 *lb.* 11 *oz.* 11 *dr.*
43830
7305
11.6880 *oz.*
16
4128
688
11.008 *dr.*

PROBLEM VIII.

To change a common diviſor into a common multiplier.

RULE.

Divide 1 by that diviſor, the quotient is a multiplier. If the diviſor be a vulgar fraction, invert it, making the numerator the denominator, &c.

Ex. 1.

If 2150.4 be a diviſor, what is the multiplier to effect the ſame thing?

```
2150.4) 1.000000000 (.00046503 the multiplier.
          86016····
          ──────
          139840
          129024
          ──────
           108160
           107520
           ──────
              64000
              64512
              ─────
```

Ex. 2.

If $\frac{5}{8}$ be a diviſor, what is the multiplier?

$\frac{5}{8}$) $\frac{1}{1}$ ($\frac{8}{5}$ the multiplier = 1.6

PROBLEM IX.

To extract the ſquare root of a decimal, or mixt number.

RULE.

Annex cyphers on the right hand as many as you pleaſe, and begin at the units place and point every other figure both to the left and right. Then proceed to extract in all reſpects as if it was a whole number; and cut off as many whole numbers in the root, as there are points in the whole number, and as many decimals, as points in the decimals. And the operation may be continued as far as you will, by adding pairs of cyphers.

Ex. 1.

Extract the root of 2211.8209

```
  ·  ·   ·  ·
  2211.8209 (47.03 the exact root.
  16 ·   ·  ·
87) 611
    609
9403)  28209
       28209
       · · · ·
```

Ex. 2.

What is the ſquare root of 10?

```
      ·    ·  ·
      10.0000 (3.16227 &c. the root.
       9   ·  ·

61)     100
+1      61

626)     3900
+6       3756

6322)     14400
  +2      12644

63242)     175600
   +2      126484

632447)     4911600
            4427129
             484471
```

Ex.

Ex. 3.

Extract the ſquare root of .001234

0.001234 (.0351283362 the root near.
9

65) 334
+5 325

701) 900
+1 701

7022) 19900
+2 14044

70248) 585600
···· 561984

23616
21074

2542
2107

435
421

14
14

Expla-

Ex. 4.

To extract the ſquare root of $\frac{7}{9}$.

$\frac{7}{9}$ reduced to a decimal is .777777, &c.

```
         .  .  .  .
        0.777777 (.8819171, &c. the root.
         64  .  .
        ----
  168)  1377
        1344
        -----
   1761) 3377
          1761
         ------
  17629) 161677
   ...   158661
         -------
            3016
            1763
           -----
            1253
            1233
           -----
              20
              17
             ---
               3
             ---
```

PROBLEM IX.

To extract the cube root of a decimal, or mixt number.

RULE.

Add cyphers at pleaſure on the right hand, that the decimals may conſiſt of 3, 6, 9, 12, &c. places; and begin at the units place and point every third figure

figure both to the left and right hand. Then extract the root as if it was a whole number; and the extraction may be continued as far as you will, by ſtill adding ternaries of cyphers. At laſt cut off as many places of whole numbers, as there are points in the whole numbers, and the like for decimals.

Note, if you deſire the laſt quotient to go true to more places of figures, do thus; add half the laſt quotient to the laſt root, and ſquare the ſum for a diviſor, and divide over again.

Ex. 1.

Extract the cube root of 146708.483

$14\dot{6}70\dot{8}.48\dot{3}00\dot{0}$ (52.74 the root.
125

3) 217
25) 72 (2
2) 54

27) 18

52 = 1ſt root.
2704 = ſquare.
140608 = cube.

$14\dot{6}70\dot{8}.4$
140608

3) 61004
2704) 20334 (74
$\frac{1}{2}$ the root 26 19110

2730 1224
1092

132

Ex. 2.

What is the cube root of 2 ?

```
       .   .   .
      2.000000 (1.259921 &c. the root.
      1
      —
  3)  10
  1)   3 (3 too much.
       3
       —
       0                        1 root  =  12
      ————                      ſquare  =  144
      .    .                    cube    =  1728
      2.000                                ————
      1.728
      —————
     3)2720                    2 root =1259
  144)  906 (60, too           ſquare =1585081
     7  906   much.            cube   =1995616979
  ————  ———
   151  ...
  ————————
      .    .    .    .
     2,0000000000
      1995616979
     ——————————
          3)  43830210
   1585081)   14610070 (92106
       1133   14275926
   ————————   ————————
   1586214)     334144
     :...       317243
                ——————
                 16901
                 15862
                —————
                  1039
                   951
                  ————
                    88
```

Ex.

Ex. 3.

What is the cube root of .0001357?

0.000135700000 (.05138 *&c.* the root.
125
3) 107
25) 35 (1
2 27
27 8
1357000
132651
3) 30490
2601) 10163 (38
15 7848
2616 2315
2093
222

1 root	51
ſquare	2601
cube	132651

Ex. 4.

Extract the cube root of 13⅔.

Reduce ⅔ to a decimal, and the number is 13.666666

13.666666 *&c.* (2.390 *&c.*
8
3) 56
4) 18
1 15
5 3
136666
12167
3) 14996
529) 4998 (90
21 4950
550 48

1 root	23
ſquare	529
cube	12167

Or thus.

$$23+\frac{.90}{2}=23.4\dot{5}\dot{0}$$

23.45

46900

7035

938

117

549.90

diviſor.

$\underset{\cdot}{5}\underset{\cdot}{4}\underset{\cdot}{9}$.9) 499.80 (9089

494 91 and the root 2.3908

489

439

50

Ex. 5.

What is the cube root of 171.46776406?

$17\dot{1}.46\dot{7}76\dot{4}06\dot{0}$ (5.5

125

3) 464

25.) 155 (5

2 135

27 20

1 root 5

ſquare 25

cube 125

$17\dot{1}.46\dot{7}7$

166 375

2 root 55

ſquare 3025

cube 166375

3) 50927

3025) 16975 (55

27 15260

3052) 1715

1526

189

and the root = 5.555

Or

Or thus.

3055·3|2) 169750 (55558
152767 and the root =
5·55555 &c. = 5$\frac{5}{9}$.
16983
15276
1707
1528
179
153
26

55·
+ ·275
55·2750
55·275
2763750
276375
11055
3869
276
3055·325
diviſor.

CHAP. IV.

Several Practical Rules in Arithmetic.

PROBLEM I.

To reſolve a queſtion in reduction.

R*Eduction deſcending* is when ſome integers of a greater denomination are to be reduced to thoſe of a leſs.

Reduction aſcending is when the leſſer denomination is to be reduced to the greater.

RULE.

In reduction deſcending, multiply continually by all the denominations from the given one to that ſought; adding to each product by the way, thoſe of the ſame denomination with itſelf, if ſuch there be.

In reduction aſcending, where the quantity is to be reduced to a higher denomination; divide continually by all the denominations from the given one to that ſought. Sometimes both rules muſt be uſed promiſcuouſly as occaſion requires.

Ex. 1.

In 415 pounds, how many pence?

$$\begin{array}{r} 415 \\ 20 \\ \hline 8300 \\ 12 \\ \hline 16600 \\ 8300 \\ \hline \end{array}$$

Anſwer 99600 pence.

Ex.

Ex. 2.

In 3076*l.* 13*s.* 11¼*d.* how many ſhillings, pence, and farthings?

```
   3076——13——11¼
     20
   ————
  61533 ſhillings adding 13
     12
 ——————
 123077
 61533
 ——————
 738407 pence adding 11
      4
———————
2953629 farthings adding 1
———————
```

Ex. 3.

In 354*lb.* 0*oz.* 16*dw.* 15*gr.* how many grains?

```
     12
    ———
    708
   354
   ————
   4248 ounces
     20
  —————
  84976 pennyweights
     24
 ——————
 339919
169952
———————
2039439 grains
———————
```

Ex. 4.

In 48067 ounces averdupoiſe, how many hundred weight?

```
               14)        8)
16) 48067 (3004lb. (214ſt. (26C. 6ſt. 8lb. 3oz.
    48···  28··     16·
    ─────  ────     ───
     067    20       54
      64    14       48
     ───   ────     ───
       3     64       6
       ─     56       ─
            ───
              8
              ─
```

Ex. 5.

In 11923 pence, how many pounds?

```
                  20)
12) 11923 (993 ſhillings (49 pounds.
    108··  80·
    ─────  ───
     112   193
     108   180
     ───   ───
      43    13
      36    ──
     ───
       7
       ─
```

Anſ. 49*l.* 13*s.* 7*d.*

Ex. 6.

In 207*l.* 15*s.* 6*d.* how many pieces (at 7*s.* $3\frac{1}{2}$*d.* per piece) gowlands (at 7 pieces per gowland) and ringlets (at 11 gowlands a ringlet)?

```
   7s. 3½d.       207l. 15s. 6d.
   12                 20
   --               ----
   87               4155
    2                 12
  ---               ----
  175               8316
halfpence          4155
                   -----
                   49866
                       2  7)          11)
                   -----
           175)    99732 (569 pieces (81 gowl. (7 ringl.
                   875··  56          77
                   ----   --          --
                   1223    9           4
                   1050    7          --
                   ----   --
                    1732   2
                    1575  --
                    ----
                     157
```

Ex. 7.

If 27 pounds be divided among 31 perſons, what is the ſhare of each?

```
                27l.
                 20
                ---
        31)     540 (17s. 5d. 0f. anſwer.
                31·
                ---
                230
                217
                ---
                 13
                 12
                ---
        31)     156 (5
                155
                ---
                  1
                  4
                 --
          31)     4 (0
```

Ex. 8.

In 8769 dollars, at 4*s.* 7*d.* per dollar, how many groats, ſhillings, crowns, and pounds?

4*s.* 7*d.*	8769
12	55
55 pence	43845
	43845
	4) 482295 (120573 groats.
	rem. 3 pence

5) 4)
3) 120573 (40191 ſhil. (8038 crowns (2009 pounds.
0 1 rem. 2 rem.

The proof of reduction is to work the queſtion backwards.

PROBLEM II.

To reſolve a queſtion in the rule of three.

Here are three numbers given to find a fourth in proportion. If a greater number requires a greater, or a leſs requires a leſs, it is called the *rule of three direct.*

But if a greater requires a leſs, or a leſs requires a greater number; it is called the *rule of three inverſe.*

1. A GENERAL RULE.

1. To ſtate the queſtion, place the three given terms ſo, that the firſt and third may be of one name, the third being that which aſks the queſtion. And the ſecond muſt be of the ſame name with the fourth term ſought. And let them be reduced to their loweſt denomination, where the firſt and third muſt be of the ſame.

2. Then ſay, if the firſt term give or require the ſecond, what does the third give or require. If *more* be required, mark the *leſſer* extreme; if *leſs* be required, mark the *greater* extreme, for a *diviſor.* Multiply the other two numbers together, and divide by

by this diviſor. The quotient is the anſwer, of the ſame denomination with the ſecond term.

3. What remains will either make a fractional part; or it muſt be reduced to a lower denomination, and divided as before.

Ex. 1.

If 18 lb. of Sugar coſt 12 ſhillings, what will 150 coſt?

```
        lb.      ſh.         lb.
        18   :   12   :  :   150.
```

Here, if 18 lb. coſt 12 ſhillings, 150 lb. muſt coſt more, therefore divide by 18 the leſſer extreme.

```
  * 18      12      150
                     12
                    ---
                    300
                   150
                   ----
              18) 1800 (100 ſhillings
                  18..
                  ----
                    00
        20) 100 (5 l. the anſwer.
            100
            ---
             ..
```

Ex. 2.

If 35 yards of cloth coſt 39 *l.* 7 *s.* 6 *d.* how many yards may be bought for 19 *l.* 2 *s.* 6 *d.*?

```
* 39 l. 7 s. 6 d. : 35 yds. : : 19 l. 2 s. 6 d.
    20                              20
   ---                             ---
   787                             382
    12                              12
  ----                            ----
  9450             : 35  : :      4590
                                    35
                                 -----
                                 22950
                                13770
                                ------
                        9450) 160650 (17 yds. anſw.
                               9450
                               -----
                               66150
                               66150
                               -----
                                 ..
```

Ex.

Ex. 3.

If $40\frac{1}{2}$ *lb.* of tobacco coſt 3 *l.* how much can I buy for 7 *l.* 15 *s.*?

* 3 *l.* : 40 *lb.* 8 *oz.* : : 7 *l.* 15 *s.*

20 16 20

60 : 648 : : 155

155

3240

3240

648

16)

60) 100440 (1674 ounces (104 *lb.* 10 *oz.*

16..

74

64

10

Or thus by vulgar fractions.

* 3 : $40\frac{1}{2}$: : $7\frac{3}{4}$:

that is 3 : $\frac{81}{2}$: : $\frac{31}{4}$:

81 4

31 2

81 8

243

$\frac{3}{1}$) $\frac{2511}{8}$ ($\frac{837}{8}$ = $104\frac{5}{8}$ *lb.*

Or

Or thus by decimals.

3*l.* : 40.5*lb.* : : 7.75*l.*
40.5
3875
3100
3) 313875 (104.625 = 104*l.* 10*oz.*
16
3750
625
10.000

Ex. 4.

If 6 men be 10 days in finiſhing a piece of work, how long will 8 men be?

6*m.* : 10*d.* : : 8*m.* *

Here 8 men will be leſs time than 6, therefore more requires leſs; and 8, the greater extreme, is the diviſor.

6 : 10 : 8 *
6
8) 60 ($7\frac{4}{8}$ = $7\frac{1}{2}$ days.
56
4

Ex. 5.

If I lend a perſon 300*l.* for a year, how long ought he to lend me 500*l.* to requite me?

300*l.* : 365*d.* : 500*l.* *
300
5|00) 1095|00 (219 days.

Here leſs time is required, and 500 the diviſor, by the inverſe rule.

Ex.

Ex. 6.

How many yards of cloth, a yard and a quarter broad, will line a piece of tapeſtry 10 yards long, and $3\frac{1}{2}$ broad?

$3\frac{1}{2}b.$: $10l.$: : $1\frac{1}{4}b.$ *

that is, $\frac{7}{2}$: 10 : : $\frac{5}{4}$ *.

$$\begin{array}{r} 7 \\ 10 \\ \hline 70 \end{array}$$

$$\frac{5}{4}\Big)\ \frac{70}{2}\ \Big(\frac{280}{10} = 28\ \textit{yds.}$$

2 Rule *for contracting the work.*

When the diviſor and either of the other terms, can be exactly divided by ſome common diviſor; then divide them, and take the quotients inſtead of theſe terms. And proceed thus as oft as you can.

Ex. 7.

If 63 gallons of brandy coſt $42l.$ what will 72 gallons coſt? Here 63 is the diviſor.

Divide by 9) * 63 : 42 : : 72
7) * 7 : 42 : : 8
* 1 : 6 : : 8 : $48l.$ anſ.

Ex. 8.

There is a paſture which will feed 18 horſes for 7 weeks; how long will it feed 42 horſes? Here 42 is the diviſor, and the rule inverſe.

7) 18 : 7 : : 42 *
6) 18 : 1 : : 6 *
3 : 1 : : 1 * : 3 weeks; anſwer.

$$\begin{array}{r} 3 \\ \hline 1)\ 3\ (3 \end{array}$$

Ex. 9.

If $\frac{3}{8}$ of a yard coſt 27 ſhillings; what will $\frac{7}{8}$ of a yard coſt?

$\frac{1}{8}$)* $\frac{3}{8}$: 27 : : $\frac{7}{8}$
3)* 3 : 27 : : 7
* 1 : 9 : : 7 : $63s.$ anſwer.

The

The proof is made by multiplying the quotient by the diviſor, adding the remainder; which muſt be equal to the product of the other two numbers.

PROBLEM III.

To reſolve a queſtion in the double rule or compound rule of three.

RULE.

1. Here, as in the ſingle rule of three, put that term into the ſecond place, which is of the ſame denomination with that ſought; and the terms of ſuppoſition one above another in the firſt place; alſo the terms of demand in the ſame order, one above another, in the third place. Then the firſt and third of every row will be of one name, and muſt be reduced to the ſame denomination, *viz.* the loweſt concerned.

2. Then proceed with each row as with ſo many ſeparate queſtions in the ſingle rule of three, in order to find out the ſeveral diviſors; uſing the ſecond term in common for each of them. That is, in any row, ſay, if the firſt term give the ſecond, does the third require more or leſs? if *more*, mark the *leſſer* extreme; if *leſs*, the *greater*, for a diviſor.

3. Multiply all theſe diviſors together for a diviſor; and all the reſt of the numbers together, for a dividend. The quotient is the anſwer, and of the ſame name with the ſecond term.

4. To contract the work, when the ſame numbers are concerned in both diviſor and dividend, throw them out of both. Or divide any numbers by their greateſt common diviſor, and take the quotients inſtead of them.

Ex.

Ex. 1.

If 16 horſes in 6 days eat up 9 buſhels of oats; how many horſes muſt there be to eat up 24 buſhels in 7 days?

* 9*b.*——16*h.*——24*b.*
6*d.*——————7*d.* *

9
7
—
63 diviſor

24
16
—
144
24
—
384
6
—
63) 2304 ($36\frac{12}{21}$ horſes; anſwer.
189
—
414
378
—
36
—

Explanation.

Say, if 9 buſhels ſerve 16 horſes, 24 buſhels will ſerve more horſes, therefore mark the leſſer extreme 9 for a diviſor.

Again, ſay if 6 days require 16 horſes to eat up any quantity, 7 days will require fewer horſes to eat them; ſo mark the greater extreme 7 for diviſor.

Then $9 \times 7 = 63$ for diviſor, and $16 \times 24 \times 6 = 2304$ for a dividend; and the quotient is $36\frac{12}{21}$ horſes $= 36\frac{4}{7}$.

Ex.

Ex. 2.

If 9 ſtudents ſpend 12 pounds in 8 months, how much will ſerve 24 ſtudents 16 months?

*9 *ſt.*——12 *l.*——24 *ſt.*
*8 *m.*————————16 *m.*
72 144
24
384
12
768
384
72) 4608 (64 pounds; anſwer.
432
288
288

Or thus by contraction.

3)*9 *ſt.*——12 *l.*——24 *ſt.*
8)*8 *m.*————————16 *m.*

And further.

3)*3——12——8
*1——————2
and then *1—— 4——8
*1——————2
diviſor 1 16
4
64 anſwer.

Ex. 3.

If 8 men be 6 days in digging 24 yards of earth; how many men muſt there be to dig 18 yards in 3 days?

d. *m.* *d.*
3) 6——8——3 *
6)*24*y.*————18*y.*

Contracted.

2*d.*——8*m.*——1*d.*
4)*4*y.*————3*y.*

Further contracted.

2*d.*——2*m.*——1*d.**
* 1*y.*————3*y.*
diviſor 1 2

6
2

12 men; anſwer.

Ex. 4.

If a garriſon of 6000 men may have each 15 ounces of bread to laſt 16 weeks, how much muſt 5000 men have a-piece to laſt 24 weeks?

1000) 6000*m.*——15*oz.*——5000*m.* *
8) 16*w.*————24*w.* *

Contracted.

3) 6——15——5 *
2————3 *

Further contracted.

1 2——3——1 * 3
1 2————1 * 2

1 diviſor 6
2

Anſwer 12 ounces.

Ex.

Ex. 5.

What principal will gain 20 pounds in 8 months, at 5 per cent. per annum?

12 *m.*——100 *l.*——8 *m.* *

* 5*g.*——————20*g.*

Here the principal is 100*l.* and the time 12 months.

$$\frac{\text{Dividend}}{\text{Diviſor}} = \frac{12\times100\times20}{8\times5} = \text{(by contraction)}\ \frac{3\times100\times4}{2\times1}$$

$$= \frac{3\times100\times2}{1\times1} =$$ 600*l.* principal, the anſwer.

Ex. 6.

If the carriage of 5 hundred weight coſt 3*l.* 7*s.* 6*d.* for 150 miles, what will the carriage of 7¾ hundred weight come to for 64 miles?

* 5*h.*——3*l.* 7*s.* 6*d.*——7*h.* 3*q.*

* 150*m.*——————64*m.*

reduced * 20——810*d.*——31*q.*

* 150*m.*——————64

$$\frac{\text{Dividend}}{\text{Diviſor}}\ \frac{810\times31\times64}{20\times150} = \frac{27\times31\times32}{10\times5}$$, by contraction.

```
10      27                          20
 5      31                  12) 535 (44 ſhill. (2 pnds.
——      ——                      48    40
50      27                      ——    ——
       81                       55     4
       ——                       48    ——
       837                      ——
        32                       7
      ————                       —
      1674                          Anſ. 2l. 4s. 7½d.
     2511
     —————
5|0) 2678|4 (535 pence
     ————
       34
        4
      ——
50)  136 (2
```

Ex. 7.

If the carriage of 150 feet of wood, that weighs 3 ſtone a foot, comes to 3l. for 40 miles, how much will the carriage of 54 feet of free ſtone, that weighs 8 ſtone a foot, coſt for 25 miles?

* 150*f.*——3*l.*——54*f.*
* 3*ſt* ———————— 8*ſt.*
* 40*m.* ———————— 25*m.*

$$\text{Dividend}\ \ \text{Diviſor}\ \ \frac{54\times 8\times 25\times 3}{150\times 3\times 40}=\frac{54\times 1\times 25\times 1}{150\times 1\times 5}=\frac{54\times 5}{150}$$

$$=\frac{54}{30}=\frac{18}{10}.$$

10) 18 (1*l.* 16*s.* anſwer.
10
—
8
20
—
10) 160 (16
160
—

Ex. 8.

If 248 men, in $5\frac{1}{2}$ days of 11 hours each, dig a trench of 7 degrees of hardneſs and $232\frac{1}{2}$ yards long, $3\frac{2}{3}$ wide, and $2\frac{1}{3}$ deep; in how many days of 9 hours will 24 men dig a trench, of 4 degrees of hardneſs, and $337\frac{1}{2}$ yards long, $5\frac{3}{5}$ wide, and $3\frac{1}{2}$ deep?

248*m.*——$5\frac{1}{2}$*d.*——24*m.* *
11*h.* ———————— 9*h.* *
7 *deg.* ———————— 4 *deg.* *
* $232\frac{1}{2}$*l.* ———————— $337\frac{1}{2}$*l.*
* $3\frac{2}{3}$*w.* ———————— $5\frac{3}{5}$*w.*
* $2\frac{1}{3}$ *deep* ———————— $3\frac{1}{2}$ *deep.*

$$\text{Dividend}\ \ \text{Diviſor}\ \ \frac{248.11.7.5\frac{1}{2}.337\frac{1}{2}.5\frac{3}{5}.3\frac{1}{2}}{232\frac{1}{2}.3\frac{2}{3}.2\frac{1}{3}.24.9.4}=\text{(by reducing)}$$

$$\frac{248.11.7.\frac{11}{2}.\frac{675}{2}.\frac{7}{2}.\frac{28}{5}}{\frac{465}{2}.\frac{11}{3}.\frac{7}{3}.24.9.4}=\frac{248.11.7.11.675.7.28}{465.11.7.24.9.4}\times\frac{2.}{8.}$$

=

$$= \frac{248.11.675.7.28.2}{465.24.4.8.5} = \frac{31.11.675.7.28}{465,24.2.5}$$

$$= \frac{31.11.135.7.7}{93.6.2.5} = \frac{11.27.7.7}{3.6.2} = \frac{11.9.7.7}{6.2} = \frac{11.3.7.7}{2.2}$$

$= \frac{1617}{4} = 404\frac{1}{4}$ days, the anſwer. All this by throwing equal quantities out of both numerator and denominator.

The proof of this rule is, by multiplying the quotient and all the diviſors together; whoſe product muſt be equal to the product of all the other numbers, when the work is right.

SCHOLIUM.

Any queſtion in the compound rule of three may alſo be reſolved at ſeveral operations, by the ſingle rule of three, but with more labour, thus:

The queſtion being rightly ſtated, take the three terms in the firſt row, and find a fourth term, by the ſingle rule. Make this the ſecond term in the ſecond row; from theſe three terms in the ſecond row find a fourth term. Proceed thus to the laſt.

As if the queſtion in Ex. 1. was propoſed, ſay, if 9 buſhels ſerve 16 horſes any time, how many horſes will 24 buſhels ſerve for the ſame time; they will ſerve more horſes, and therefore 9 is the diviſor, and the anſwer is $42\frac{2}{3}$ horſes.

Again ſay, if 6 days require $42\frac{2}{3}$ horſes to eat up any quantity, how many do 7 days require. Here fewer horſes are required, therefore 7 is diviſor, and the anſwer is $36\frac{4}{7}$ horſes.

PROBLEM IV.

To reſolve a queſtion by the rule of practice.

When a queſtion in the rule of three has 1 for the firſt term, it is more expeditiouſly reſolved, by taking

ing ſome aliquot part or parts of the thing propoſed: and this is called the *rule of practice.*

1. A GENERAL RULE.

Firſt value the integers, obſerving to multiply integers by integers; and for the inferior denominations take what aliquot part you can get, and for what is wanting take parts of that part, and ſo on. Then ſum up the whole.

Ex. 1.

What will 37*c.* 3*q.* 12*lb.* come to, at 5*l.* 15*s.* 7½*d.* the hundred weight?

	5*l.*	15*s.*	7½*d.*	
	37*c.*	3*q.*	12 *lb.*	
	185	0	0 .	37*c.* at 5*l.*
	18	10	0 .	37 at 10*s.*
	9	5	0 .	37 at 5*s.*
37*c.* at 1*s.* - - 1*l.* 17*s.*		18	6 .	37 at 6*d.*
		4	7½.	37 at 1½*d.*
	2	17	9¾.	price of ½*c.*
	1	8	11 .	pr. of 1 *q.*
price of $\frac{1}{7}$*q.* - - 4 $1\frac{4}{7}$		12	4½.	pr. of 12 *lb.*
tot.	218	17	2¾.	anſ.

Explanation.

Firſt I multiply 37 by 5 gives 185*l.* Then ſince 15*s.* is $\frac{3}{4}$ of a pound, or $\frac{1}{2}$ and $\frac{1}{2}$ of that. Therefore I take half 37 which 18*l.* 10*s.* and half of that which is 9*l.* 5*s.* and the fifth part of 9*l.* 5*s.* is 1*l.* 17*s.* the price at 1*s.* the hundred weight. Then becauſe 7½*d.* is the half of a ſhilling, and a fourth of that half. Therefore half of 1*l.* 17*s.* is 18*s.* 6*d.* and $\frac{1}{4}$ of that is 4*s.* 7½*d.*: ſo now the integers are valued. Then

Then $\frac{3}{4}$ of a hundred being a half and half of that half, I take half of 5*l.* 15*s.* $7\frac{1}{2}$*d.* which is 2*l.* 17*s.* $9\frac{3}{4}$*d.* and half that 1*l.* 8*s.* 11*d.* Laſtly, ſince 12*lb.* is $\frac{12}{28}$ or $\frac{3}{7}$ of a quarter, I take $\frac{1}{7}$ of 1*l.* 8*s.* 11*d.* which is 4*s.* $1\frac{4}{7}$*d.* and triple that is 12*s.* $4\frac{1}{2}$*d.* the price of 12 pounds. And the ſum of all theſe, is 218*l.* 17*s.* $2\frac{3}{4}$*d.*

PARTICULAR RULES.

2 RULE.

Sometimes the value may be eaſily found by reckoning the price ſome even number above what is given, which done, take ſome aliquot part for what it is above, and ſubtract it from the former.

Ex. 2.

If a pound of tobacco coſts 11*d.* what is a hundred weight?

	£.	s.	d.	
112*l.* (at 1*s.*)	5	12	0	
112*l.* (at 1*d.* is $\frac{1}{12}$) ſubt.	0	9	4	
	5	2	8	anſ.

3 RULE.

When the price is ſhillings, or pounds and ſhillings. Firſt multiply the quantity by the pounds, if there be any; then multiply by half the (even) number of ſhillings, obſerving to write double the product of the firſt figure for ſhillings, and the reſt of the product for pounds. And for an odd ſhilling take $\frac{1}{20}$ of the quantity.

Ex. 3.

What comes 413 yards to, at 2 ſhilling a yard?

413
1

Anſ. 41*l.* : 6*s.*

Ex. 4.

If an ounce coſts 12 ſhillings, what will 76 coſt?

76
6

Anſ. 45*l.* : 12*s.*

Ex. 5.

What is the price of 796 groſs, at 13*s.* the groſs?

796
6

477 : 12 at 12 ſhillings.
39 : 16 at 1 ſhilling.

Anſ. 517*l.* : 08*s.*

Ex. 6.

If a hundred weight coſt 2*l.* 17*s.* what will 238 coſt?

238
2 17

476 00 at 2*l.*
190 08 at 16*s.*
11 18 at 1*s.*

Anſ. 678 6

4 RULE.

When the price is pence, or ſhillings and pence. Multiply the quantity by the ſhillings, if there be any. Then for the pence take ſome aliquot part or parts of the quantity propoſed.

Ex. 7.

What comes 472 ounces to, at 8 *d.* an ounce?

3) 472 (157*s.* 4*d.* at 4*d.*
157 4 . at 4*d*,

20) 314*s.* 8*d.*
Anſ. 15*l.* 14*s.* 8*d.*

Ex. 8.

What will 74 yards of cloth coſt, at 13 *s.* 9*d.* the yard?

74
13 9

222
74

962 0 at 13*s.*
2) 74 —— 37 0 at 6*d.*
4) 74 —— 18 6 at 3*d.*

20 1017*l.* 6*d.*

An 50*l.* 17*s.* 6*d.*

Ex. 9.

What comes 150 hundred weight to, at 2 *l.* 11 *s.* 8½ *d.* or 51 *s.* 8 *d.* the hundred?

150
51 8

150
750

7650 *s.*
2) 150 —— 75
3) 75 —— 25

20) 7750 *s.*
Anſ. 387 *l.* 10 *s.*

5 RULE.

When the price is an aliquot part or parts of a pound; then take ſuch aliquot parts of the quantity propoſed.

Ex. 10.

What does 63 gallons come to, at 5 ſhillings a gallon?

l. *s.*
4) 63 (15 : 15 anſ.

Ex. 11.

If I gain 13 *s.* 4 *d.* for a dozen, what do I gain for 100 dozen?

3) 100——33 6 8 at 6 *s.* 8 *d.*
33 6 8

Anſ. 66 13 4

6 RULE.

If farthings be concerned in the price, take ſuch aliquot parts as you can find; or parts of aliquot parts.

Ex. 12.

What comes 371 gallons to, at 13½*d.* per gallon?

	s.		*d.*	
	371		0	at 1 ſhilling.
8) 371 ——	46		4½	at 1½*d.*
20)	417		4½	
Anſ.	20	17	4½	

Ex. 13.

How much money can I get for 347 French crowns, at 4*s.* 5¼*d.* a piece?

	347			
	4		5¼	
	1388		0	at 4*s.*
3) 347 ..(4)	115		8	at 4*d.*
4)	28		9	at 1*d.*
	7		2¼	at ¼*d*
20)	1539		7¼	
Anſ.	76	19	7¼	

The proof of this rule is to work the queſtion by different methods.

SCHOLIUM.

Other queſtions that may occur, are eaſily reſolved by the rules of compound multiplication.

When it happens that the firſt term is more than 1; work by the foregoing rules as if the firſt term was

was 1; and at laſt divide by that term, according to the rules of compound diviſion. But ſuch queſtions as theſe are beſt reſolved by the rule of three.

PROBLEM V.

To reſolve a queſtion in the ſingle rule of fellowſhip.

The ſingle rule of fellowſhip, is that which determines how much gain or loſs, is due to every partner concerned; by having the whole gain or loſs, and their particular ſtocks, given.

I. A GENERAL RULE.

Say by the rule of three, as the whole ſtock : is to the whole gain or loſs : : ſo is every man's particular ſtock : to his particular part of the gain or loſs.

Ex. 1.

Two partners A, and B, make a ſtock of 56 pounds; A puts in 24*l.*; and B 32*l.* They gain 7*l.* by trade. What is the gain of each?

$$\begin{array}{llllllll}
 & 24 & & & & & & \\
 & \underline{32} & & & & & & \\
(1) & 56 & : & 7 & :: & & 24 & \\
 & & & & & & \underline{7} & \\
 & & & & & 56) & 168 & (3l. = \text{A's gain.} \\
 & & & & & & \underline{168} & \\
 & & & & & & . & \\
(2) & 56 & : & 7 & :: & & 32 & \\
 & & & & & & \underline{7} & \\
 & & & & & 56) & 224 & (4l. = \text{B's gain.} \\
 & & & & & & \underline{224} & \\
 & & & & & & \underline{\cdots} &
\end{array}$$

Ex. 2.

Three men A, B, C, freight a ſhip with wine; A had 284 tuns; B, 140, and C, 64. By a ſtorm at ſea, they were obliged to caſt 100 tuns overboard. What loſs does each ſuſtain?

A 284
B 140
C 64
———
(1) 488 : 100 : : 284
100
———
488) 28400 ($58\frac{96}{488}$ tuns = A's loſs.
2440
———
4000
3904
———
96
———

(2) 488 : 100 : : 140
100
———
488) 14000 ($28\frac{336}{488}$ tuns = B's loſs.
976
———
4240
3904
———
336
———

(3) 488 : 100 : : 64
100
———
488) 6400 ($13\frac{56}{488}$.
488
———
1520
1464
———
56

2 RULE

2 RULE.

Where many partners are concerned; find the ſhare of 1 integer, by dividing the whole gain or loſs by the whole ſtock, and the quotient will be a common multiplier; by that multiply every man's part of the ſtock, and it will give his ſhare of the loſs or gain.

Ex. 3.

Four men trade together, A puts in 200*l.* B 150, C 85, D 70; and they gain 60*l.* What is the ſhare of each?

A 200	505) 60.0 (.11881 a common multiplier.
B 150	505
C 85	
D 70	950
	505
505	
	4450
	4040
	4100
	4040
	600

.11881	.11881	.11881	.11881
200	150	85	70
23.762|00	59405	59405	8.31670
20	11881	95048	20
15.24	17.82150	10.09885	6.334|00
12	20	20	12
2.88	16.43|00	1.977|00	4.008
	12	12	
	516	11.724	

A gains	23*l.*	15*s.*	2.9*d.*
B	17	16	5.2
C	10	1	11.8
D	8	6	4.1
	60	0	0

Ex.

Ex. 4.

Five captains plundered the enemy of 1200*l.* The firſt had 20 men, the ſecond 40, the third 55, the fourth 55, the fifth 70. What muſt each captain have in proportion to his number of ſoldiers?

1	20
2	40
3	55
4	55
5	70
	240

240) 1200 (5
1200

20	40	55	70
5	5	5	5
100*l.*	200*l.*	275*l.*	350

the firſt gets	100*l.*
the ſecond	200
the third	275
the fourth	275
the fifth	350
	1200

3 RULE.

When there are a great number of partners; the beſt way is to make a table, after this manner. Divide the gain or loſs by the whole ſtock, to find what is the gain or loſs of 1. Then by continual addition of this, make your table as far as 10; then by the continual addition of the gain or loſs of 10, continue the table through all the tens to 100: add in like manner, for all the hundreds to 1000, if there be occaſion. Then you have no more to do, but take every man's ſhare out of the table (at once or oftener) and write it down.

Ex. 5.

There is a certain townſhip, which is to raiſe a tax of 56*l.* 8*s.* 3*d.* To find what each much pay towards

towards it, the inhabitants being rated as in the following table.

Perſons.	Rent.	Perſ.	Rent.	Perſ.	Rent.
	l.		*l.* *s.*		*l.* *s.*
A	150	I	30	R	4
B	125	K	24	S	3–10
C	100	L	15–10	T	3
D	100	M	12	V	3
E	87	N	12	U	2
F	80	O	12	W	1–10
G	63	P	7	X	1–10
H	40	Q	5–10	Y	1
				Z	1
	745		118 0		
					20 10
					118 0
					745 0
					883 10

Here 883*l.* 10*s.* = 883.5*l.*
and 56*l.* 8*s.* 3*d.* = 56.4125*l.*

```
883.5) 56.4125 (.0638513
       53010             20
       -----     ----------
        34025    1.277026|0
        26505            12
        -----    ----------
         7520     3.324312
         7068             4
         ----    ----------
          452     1.297248
          441
          ---
          115
           88
          ---
           27
```

So

So 1*l.* is 1*s.* 3*d.* 1.297*f.* whence the following table is made.

£.	£.	s.	d.	f.
0½	0	0	7	2.648
1	0	1	3	1.297
2	0	2	6	2.594
3	0	3	9	3.891
4	0	5	1	1.188
5	0	6	4	2.485
6	0	7	7	3.782
7	0	8	11	1.079
8	0	10	2	2.376
9	0	11	5	3.673
10	0	12	9	0.97
20	1	5	6	1.94
30	1	18	3	2.91
40	2	11	0	3.88
50	3	3	10	0.85
60	3	16	7	1.82
70	4	9	4	2.79
80	5	2	1	3.76
90	5	14	11	0.73
100	6	7	8	1.7
200	12	15	4	3.4
300	19	3	1	1.1

&c.

	£.	s.	d.	f.
Hence the ſhare of A for 100 is	6	7	8	1.7
for 50 is	3	3	10	0.85
total ſhare of A	9	11	6	2.55

The ſhare of B

	£.	s.	d.	f.
for 100	6	7	8	1.7
20	1	5	6	1.94
5	0	6	4	2.48
whole ſhare of B	1	19	7	2.42

and ſo on with the reſt; whence we get the following bill.

	£.	s.	d.	f.		£.	s.	d.	f.
A	9	11	6	2.55	O	0	15	3	3.56
B	7	19	7	2.12	P	0	8	11	1.08
C	6	7	8	1.70	Q	0	7	0	1.13
D	6	7	8	1.70	R	0	5	1	1.19
E	5	11	1	0.84	S	0	4	5	2.54
F	5	2	1	3.76	T	0	3	9	3.89
G	4	0	5	1.71	V	0	3	9	3.89
H	2	11	0	3.88	U	0	2	6	2.59
I	1	18	3	2.91	W	0	1	10	3.95
K	1	10	7	3.13	X	0	1	10	3.95
L	0	19	9	2.11	Y	0	1	3	1.30
M	0	15	3	3.56	Z	0	1	3	1.30
N	0	15	3	3.56					
						2	17	5	2.37
	53	10	9	1.53		53	10	9	1.53
						56	8	2	3.9

true to the 10th part of a farth.

The proof is made, by adding together all the ſhares, which muſt be equal to the whole gain or loſs.

PROBLEM VI.

To reſolve a queſtion by the double rule of fellowſhip.

The double rule of fellowſhip, is that which determines how much gain or loſs is due to every partner concerned; by having the whole gain or loſs, and the particular ſtocks, and their times of continuance, given.

1 RULE.

Multiply every man's ſtock, by the time it is employed; then by the rule of three, ſay, as the ſum of theſe products : to the whole gain or loſs : : ſo each of theſe products : to each man's gain or loſs.

Ex. 1.

Three merchants, A, B, C, enter into partnerſhip. A puts in 65*l.* for 8 months; B 78*l.* for 12; and C 84 for 4 months, and 6*l. viz.* 90*l.* for 2 months. They gain 166*l.* 12*s.* What is each man's ſhare of the gain?

65	78	84	90	520
8	12	4	2	936
				516
A = 520	B = 936	336	180	1972
			336	
			C = 516	

1972 —— 166.6 —— 520
520

33320
8330

1972) 86632.0 (43*l.* 18*s.* 7½*d.* for A.
7888

7752
5916

1836
20

1972) 36720 (18
1972

17000
15776

1224
12

1972) 14688 ($7\frac{8}{19}$
13804

884

K

Again,

Again, 1972——166.6——936

936

9996
4998
14994

1972) 155937.6 ($79\frac{15}{197}$ = 79*l.* 1*s.* $6\frac{1}{4}$*d.* for B.
13804

17897
17748

149

Laſtly, 1972 — 166.6*l.* — 516 — 43*l.*:11*s.*:$10\frac{1}{4}$*d.* for C.

Ex. 2.

Four men, A, B, C, D, hold a paſture in common, for which they pay 60*l.* A had 24 oxen 32 days; B 12 oxen 48 days; C 16 oxen for 24 days; and B had 10 oxen for 30 days. What muſt each pay?

$24 \times 32 = 768$
$12 \times 48 = 576$
$16 \times 24 = 384$
$10 \times 30 = 300$

2028

Then 2028 : 60*l.* :: ſo each product : to its ſhare.
That is 169 : 5*l.* :: 768 : $22\frac{122}{169}$
and 169 : 5 :: 576 : $17\frac{7}{169}$
169 : 5 :: 384 : $11\frac{61}{169}$
169 : 5 :: 300 : $8\frac{148}{169}$

	£.	s.	d.
Hence there is paid by A,	22	14	$5\frac{1}{4}$
B,	17	0	10
C,	11	7	$2\frac{1}{2}$
D,	8	17	$6\frac{1}{4}$

2 RULE.

2 RULE.

When many people are concerned; divide the whole gain or loſs, by the firſt term or ſum of the products; the quotient is a common multiplier, by which multiplying the ſeveral products, you'll have the ſeveral ſhares.

Ex. 3.

Four merchants trade after this manner.

A puts in 100*l.* for 8 months.

B puts in 80*l.* for 5 months, and then puts in 40*l.* more for 3 months longer.

C puts in 176*l.* for 4 months, and then takes out 50*l.* for four months more.

D puts in 230*l.* for 6 months, and then takes out the whole.

They gained 212*l.* 10*s.*; then what is the gain of each merchant.

The ſeveral products of the ſtock and time will be as follows.

100 × 8 ——	800	for A.
80 × 5 ——	400	
120 × 3 ——	360	
	760	for B.
176 × 4 ——	704	
126 × 4 ſubt.	504	
	1208	for C.
230 × 6 ——	1380	for D.
ſum	4148	

4148) 212.50 (.05123 the ſhare of 1 pound being a common multiplier.

.05123	.05123	.05123	.05123
800	760	1208	1380
40.984	30738	40984	40984
for A.	35861	614760	15369
			5123
	38.9348	61.88584	70.6974
	for B.	for C.	for D.

	£.	s.	d.
Hence A's ſhare is	40	19	8
B's	38	18	8¼
C's	61	17	8½
D's	70	13	11¼

The proof is had, by adding all the parts of the gain or loſs together, which muſt be equal to the whole.

PROBLEM VII.

To reſolve a queſtion in the rule of alligation medial.

Alligation medial teaches how to find the mean rate of a mixture, when the particular quantities mixt, and their ſeveral rates are given.

RULE.

Multiply the quantities of the mixture by their reſpective prices, and divide the ſum of the products by the ſum of the quantities, gives the mean rate.

Ex. 1.

A man would mix 10 buſhels of wheat, at 4 ſhillings a buſhel, with 8 buſhels of rye at 2*s.* 8*d.* a buſhel.

a buſhel. At what price muſt the mixture be ſold?

£. d.
10 × 48 = 480 the wheat.
8 × 32 = 256 the rye.

18 736

18) 736 (40$\frac{8}{9}$, or 3*s.* 5*d.* a buſhel very near, the price of the miſlegin.
72
16
0
16

Ex. 2.

A vintner would mix 30 gallons of Malaga, at 7*s.* 6*d.* the gallon; with 18 gallons of Canary, at 6*s.* 9*d.*; and 27 gallons of white wine, at 4*s.* 3*d.* how muſt the mixture be ſold?

90 × 30 = 2700
81 × 18 = 1458
51 × 27 = 1377

75) 5535 (73$\frac{1}{5}$*d.* or 6*s.* 1$\frac{1}{5}$*d.* per gallon.
525
285
225
60

The proof is made, by finding the value of the whole mixture at the mean price; which muſt be equal to the total value of the ſeveral ingredients.

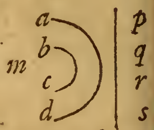
PROBLEM VIII.

To reſolve a queſtion in the rule of alligation alternate.

Alligation alternate ſhows how to find the particular quantities concerned in any mixture; when the particular rates of each ſort, and alſo the mean rate, are given.

Preparation.

Set down the ſeveral rates in order from the greateſt to the leaſt, as the letters *a*, *b*, *c*, *d*; and the mean price (*m*) behind in its due order.



Couple every two rates together by an arch, ſo as one rate may be greater and another leſs than the mean, till they be all coupled. Where *note*, that one rate may be coupled with ſeveral others one by one, as oft as you will.

Take the difference between each rate and the mean rate, and place it *alternately*, that is, againſt all its yoke-fellows. Do thus with all the rates; then the differences will ſtand as *p*, *q*, *r*, *s*. When ſeveral differences happen to ſtand againſt one rate, add them all together. Then,

1 RULE.

When no quantity is given of any of theſe ſorts; the numbers (or differences) ſtanding againſt the ſeveral rates, are the quantities required.

Ex. 1.

A man would mix wheat at 4*s.* a buſhel, with rye at 2*s.* 8*d.* a buſhel; to ſell it at 3*s.* 6*d.* per buſhel. How much of each muſt he take?

d.
42 48 | 10 buſhels of wheat } the anſwer.
32 | 6 buſhels of rye. }

Ex. 2.

A vintner would mix Malaga at 7*s.* 6*d.* per gallon, with Canary at 6*s.* 9*d.* and white wine at 4*s.* 3*d.*; to ſell it at 5*s.* 2*d.* per gallon. What quantity of each muſt he take?

	d.				
62	90	11	11	qrts. Malaga	anſwer.
	81	11	11	Canary	
	51	19.28	47	w. wine	

Explanation of Ex. 2.

The difference between 62 and 51 is 11, which I ſet againſt 81, and alſo againſt 90. The difference between 62 and 81 is 19, which I place againſt 51. The difference between 62 and 90 is 28, which I alſo ſet againſt 51. Then 19 added to 28 is 47. So the differences, to work by, will be 11, 11, 47.

2 RULE.

In alligation partial, where one of the quantities (to be mixed) is given. Say, by the rule of three,

As the difference ſtanding againſt the price of the given quantity :

To the given quantity : :

So are the ſeveral other differences :

To the reſpective quantities required.

Ex. 3.

I would mix 10 buſhels of wheat at 5*s.* with rye at 3*s.* 6*d.* and barley at 2*s.* 4*d.*; to be ſold at 4*s.* per buſhel. How much rye and barley muſt I take?

48	wheat	60	6.20	26
	rye	42	12	12
	barley	28	12	12

Then 26 : 10 : : 12 : $4\frac{8}{13}$ buſhels of rye and of barley.

Ex. 4.

How much Malaga at 7*s.* 6*d.* the gallon, ſherry at 5*s.* white wine at 4*s.* 3*d.* muſt be mixt with 24 gallons of Canary at 6*s.* 9*d.*; that the whole may be ſold for 6*s.* per gallon?

Or thus.

$$72\left\{\begin{array}{lr|r}\text{Malaga} & 90 & 12\\ \text{Canary} & 81 & 21\\ \text{ſherry} & 60 & 18\\ \text{w. wine} & 51 & 9\end{array}\right. \qquad 72\left\{\begin{array}{lr|r}\text{Malaga} & 90 & 21\\ \text{Canary} & 81 & 12\\ \text{ſherry} & 60 & 9\\ \text{w. wine} & 51 & 18\end{array}\right. \;\&c.$$

Then the quantity of Canary being given, ſay by the firſt method, 21 : 24 : : ſo is each difference : to its reſpective quantity; that is,

$$\text{As } 7 : 8 :: \left\{\begin{array}{rcll}12 & : & 13\frac{5}{7} & \text{gal. Malaga}\\ 18 & : & 20\frac{4}{7} & \quad\text{ſherry}\\ 9 & : & 10\frac{2}{7} & \quad\text{w. wine}\end{array}\right\}\text{anſwer.}$$

Or thus, by the latter method.

$$\begin{array}{l}\text{As } \quad 12 : 24 ::\\ \text{that is, } 1 : \;\; 2 ::\end{array}\left\{\begin{array}{rcll}21 & : & 42 & \text{gal. Malaga.}\\ 9 & : & 18 & \quad\text{ſherry.}\\ 18 & : & 36 & \quad\text{w. wine.}\end{array}\right.$$

3 RULE.

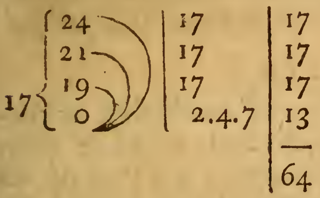
In alligation total, where the total ſum of the quantities (to be mixt) is given; add up all the differences together, then ſay by the rule of three,

As the ſum of the differences :
To the quantity given : :
So every particular difference :
To its reſpective quantity.

Ex. 5.

A goldſmith would mix gold of 24 carracts, with ſome of 21 carracts, and with ſome other of 19 carracts

racts fine, and with a due quantity of allay; ſo that 190 ounces might bear 17 carracts fine. How much of each ſort muſt he take?



here allay is to be reckoned 0 carracts.

$$\text{Then } 64 : 190 :: \left\{ \begin{array}{l} 17 : 50\tfrac{15}{32} \text{ of the 3 ſorts of gold.} \\ 13 : 38\tfrac{19}{32} \text{ of allay.} \end{array} \right.$$

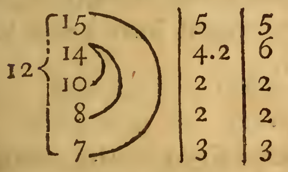
(oz.)

Ex. 6.

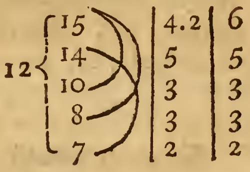
A mixture of wine is to be made up conſiſting of 130 quarts, from theſe five ſorts, whoſe prices are 7*d.*, 8*d.*, 10*d.*, 14*d.*, and 15*d.* a quart: and the whole is to be ſold at 12*d.* a quart. Quere, how much of each?

Here being 5 quantities concerned, they will admit of ſeveral alternations.

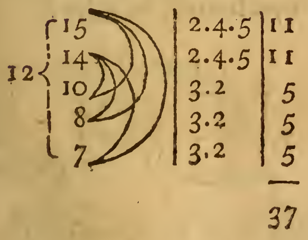
firſt way.



ſecond way.



third way.



&c.

The

The operation, by the laſt way, is thus.

$$37:130::\left\{\begin{array}{l} 11 : 38\tfrac{24}{37}\text{ qrts. of wine at }15d.\text{ and }14d. \\ 5 : 17\tfrac{21}{37}\text{ quarts, at }10d.,\ 8d.,\text{ and }7d. \end{array}\right.$$

SCHOLIUM.

Although the ſeveral ways of combining or coupling the rates, as before directed, afford ſo many different ſolutions to the queſtion; yet they do not give all the anſwers the queſtion is capable of. To remedy which, and to make the method more general; you may repeat any two alternate (or correſponding) differences as often as you will; and the like for any other two, &c. This will give a great variety of ſolutions, from which the eaſieſt, and moſt ſuitable may be ſelected. Or rather proceed by the following rule.

4 RULE, UNIVERSALLY.

Having coupled the rates as before directed; then inſtead of any couple of the differences, take any equimultiples thereof; that is, multiply them both by any number you will; do the like for any other couple, &c. By this means, you'll have a new ſet of differences, to work with.

Ex. 7.

A grocer would mix 12 *lb.* of ſugar at 10 *d.*, with two other ſorts of 8 *d.*, and 5 *d.*; ſo that the mixture may be ſold at 7 *d.* How much muſt he take?

common way.

$$7\left\{\begin{array}{l} 10 \\ 8 \\ 5 \end{array}\right. \begin{array}{|l|l} 2 & 2 \\ 2 & 2 \\ 1.3 & 4 \end{array}$$

general way.

$$7\left\{\begin{array}{l} 10 \\ 88 \\ 55 \end{array}\right. \begin{array}{|l|r} 2\times 2 & 4 \\ 2\times 3 & 6 \\ 1\times 2.3\times 3 & 11 \end{array}$$

Here the couple of differences againſt 10 and 5 being 2 and 1, I multiply them both by 2, and they become 4 and 2. Again, the couple againſt 8 and 5, being 2 and 3, I multiply them both by 3, and they become 6 and 9. Then you will have 4, 6, 11 for a new ſet of differences. Therefore

$$4 : 12 :: \begin{cases} 6 : 18\,\textit{lb.}\ \text{at}\ 8\,\textit{d.} \\ 11 : 33\,\textit{lb.}\ \text{at}\ 5\,\textit{d.} \end{cases}$$

Ex. 8.

A farmer would mix wheat at 4*s.* with rye at 3*s.* and barley at 2*s.* and oats at 1*s.* per buſhel; to have a quantity of 120 buſhels, to be ſold at 2*s.* 4*d.* the buſhel. How much of each muſt he take?

$$28\begin{cases} \text{wheat} & \overset{\textit{d.}}{48} \\ \text{rye} & 36 \\ \text{barley} & 24 \\ \text{oats} & 12 \end{cases} \left| \begin{array}{r} 16 \times 3 \\ 4 \times 5 \\ 8 \times 5 \\ 20 \times 3 \end{array} \right| \begin{array}{r} 48 \\ 20 \\ 40 \\ 60 \\ \hline 168 \end{array}$$

$$\text{Then } 168 : 120, \text{ or } 7 : 5 :: \begin{cases} 48 : 34\tfrac{2}{7}\ \text{buſh. wheat.} \\ 20 : 14\tfrac{2}{7}\ \text{rye.} \\ 40 : 28\tfrac{4}{7}\ \text{barley.} \\ 60 : 42\tfrac{6}{7}\ \text{oats.} \end{cases}$$

The proof is had by finding the value of the whole mixture at the mean rate; which muſt be equal to the total value of the ſeveral ſimples. And moreover, in *alligation total*, the ſum of the particulars, muſt agree with the ſum given.

PROBLEM IX.

To reſolve a queſtion in the ſingle rule of falſe.

This rule makes a ſingle ſuppoſition of ſome falſe number to reſolve the queſtion, by means whereof the true number or numbers are found out.

RULE.

RULE.

Suppoſe ſome fit number, and proceed with this according to the tenor of the queſtion. Then ſay by the rule of three,

As the falſe number reſulting :
To the true number given : :
So the whole or any part of the falſe number :
To the whole or reſpective part of the number ſought.

Ex. 1.

A man would divide 30 crowns among 3 perſons; ſo that the firſt ſhould have half; the ſecond, a third; and the third, a fourth part. To find each one's ſhare.

Take a number which is diviſible by 2, 3, 4; ſuppoſe 12, then 2) 12 (6 . 3) 12 (4 . 4) 12 (3.

$$\begin{array}{c|c|}1&6\\2&4\\3&3\\\hline&13.\end{array}\quad \text{Then } 13:30::\left\{\begin{array}{l}6:13\frac{11}{13}\text{ firſt ſhare.}\\4:\ 9\frac{3}{13}\text{ ſecond ſhare.}\\3:\ 6\frac{12}{13}\text{ third ſhare.}\end{array}\right.$$

Ex. 2:

A, B, and C buy a parcel of timber, which coſts 48 *l.* and it is agreed that B ſhall pay a third part more than A, and C a fourth more than B. What ſum muſt each pay?

Suppoſe A pays 3, then B pays 4, and C pays 5. But $3 + 4 + 5 = 12$, which ſhould be 48. Therefore ſay,

$$\text{As } 12:48, \text{ or as } 1:4::\left\{\begin{array}{l}3:12,\text{ A's ſhare.}\\4:16,\text{ B's ſhare.}\\5:20,\text{ C's ſhare.}\end{array}\right.$$

Ex. 3.

There are 3 cocks, A, B, C, belonging to a ciſtern; A can fill it in 1 hour, B in 2, and C in 3. In what time will they all fill it?

Suppoſe they fill it in half an hour; then ſay,

hour. *ciſtern.* *hour.*

As 1 —— 1 —— $\frac{1}{2}$ —— $\frac{1}{2}$ ciſtern for A.

2 —— 1 —— $\frac{1}{2}$ —— $\frac{1}{4}$ ciſtern for B.

3 —— 1 —— $\frac{1}{2}$ —— $\frac{1}{6}$ ciſtern for C.

But $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12}$ ciſtern, which ſhould be 1 ciſt.

Therefore $\frac{11}{12}$ ciſt. : 1 ciſt. : : $\frac{1}{2}$ hour : $\frac{6}{11}$ hour the time ſought.

The proof of this rule is made, by ſumming up the ſeveral parts, which muſt be equal to the whole.

PROBLEM X.

To reſolve a queſtion in the double rule of falſe.

This rule reſolves queſtions, by making two ſuppoſitions of falſe numbers; by means of which, the true number, which anſwers the queſtion, is found out.

1 RULE.

1. Take ſome number by gueſs, for a firſt ſuppoſition, and try if it will anſwer the queſtion. If not, ſet the error under it, and mark it with + if it exceeds the truth, or with — if it fall ſhort. Then make a ſecond ſuppoſition with another number, and proceed the ſame way with it. (It is uſual to ſet a croſs between them).

2. Multiply alternately the firſt number by the 2d error, and the 2d number by the 1ſt error. And divide the ſum of the products by the ſum of the errors, when the errors are of different kinds, (that is, when one is greater and the other leſs than the

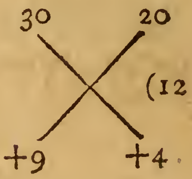
truth;) or the difference of the products by the difference of the errors, when both errors are of one kind; and the quotient is the true number ſought, for which the ſuppoſitions were made.

In ſhort thus, *addito diſſimiles, ſubtrahitoque pares.*

Ex. 1.

A workman agreed to thraſh 60 buſhels of corn, part of it wheat, and part oats; at the rate of 2*d.* per buſhel for the wheat, and $1\frac{1}{2}$*d.* for the oats. At laſt he received 8 ſhillings for his labour. How much of each did he thraſh?

1. Firſt, I ſuppoſe there are 30 buſhels of wheat; then there are alſo 30 buſhels of oats.



Price of the wheat 60 pence.
Price of the oats 45 pence.

too much 105
which ſhould be 8*s.* or 96

1 error +9

2. Again, I ſuppoſe 20 buſh. of wheat, the pr. 40*d.*
then there is 40 buſhels of oats, pr. 60

whole price, too much 100
96

2 error +4

Then

Then 30 20
4 9

120 180
120

9
4 5).60(12 buſhels the wheat.
60| 48 the oats.

5
60

Ex. 2.

A man hired a labourer for 40 days, on condition that he ſhould have 20 pence for every day he wrought, and forfeit 10 pence for every day he idled. At laſt he received 41 *s*. 8 *d*. for his labour. How many days did he work, and how many was he idle?

1. Suppoſe he wrought 24 days 480 pence.
then he idled 16 160

received but 320
inſtead of 41 *s*. 8 *d*. or 500

1 error ſhort —180

2. Suppoſe he wrt. 32 days 640 pence.
idled 8 80 24 32

ſhould receive 560
inſtead of 500 (30

2 error above +60| —180 +60

180

180 24
32 60

36 1440
54

180 5760
+60 +1440

240) 7200 (30 days he wrought, conſequently he idled 10 days.
720

..

Ex. 3.

Two merchants, A, B, lay out an equal ſum of money in trade. A gains 126*l.* and B loſes 87. And A's money is now double to B's. What did each lay out?

1. Suppoſe each lays out 200*l.*
then 200 200
126 87

A's money=326 113=B's money.
226 2

1 error +100 226

200 250

(300

+100 +50

100 25000 10000
50 10000

50) 15000 (300 £. the anſ.
150..

2. Suppoſe each lays out 250*l.*
then 250 250
126 87

A's money = 376 163 = B's money.
326 2

2 error + 50 326

Ex.

Ex. 4.

A perſon finding ſeveral beggars at his door, gave each of them 3 pence a-piece, and had 5 pence remaining. He would have given them 4 pence a-piece, but he wanted 7 pence to do it. How many beggars were there?

1. Suppoſe 14 beggars. 14

3 4

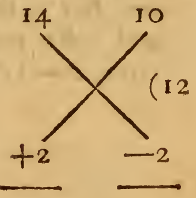
42 56

+5 −7

his money=47 49 his mon.

49 ——— alſo.

1 error + 2



2|28 20

2|20

4)48(12 beggars the anſwer,

2. Suppoſe 10 beggars. 10

3 4

30 40

+5 −7

35 33

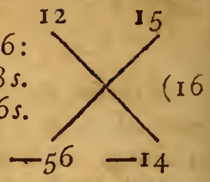
33

2 error —2

Ex. 5.

A and B play at cards; A ſtakes B 8*s.* to 6*s.* every game. After 28 games they leave off play, and find that neither of them are winners. How many games did each win?

1. Suppoſe A won 12, then B won 16: and A wins 72 *s.* and (B wins 128 *s.* or A) loſes 128 *s.* that is, he loſes 56 *s.* therefore 1ſt error = — 56.



2. Suppoſe A wins 15 games, and B 13, then A wins 90 *s.* and loſes 104: ſo the ſecond error is —14.

```
——       ——
56|840    168
14|168
——————
42)672(16 games
   42     for A.
   ———
   252
   252  and 12
   ———    for B.
```

2 RULE.

You muſt proceed as directed in the 1ſt rule, till you have found the errors, and their ſigns, then

1. Multiply the difference of the ſuppoſed numbers, by the leaſt error, and divide the product by the difference of the errors, if they are like; or by the ſum if unlike: The quotient is the correction of the number belonging to the leaſt error.

2. Obſerve whether this be the leſſer or greater number, as alſo whether the errors have like or unlike ſigns.

If it is the leſſer number, and like ſigns, ſubtract the correction; if unlike ſigns, add it.

If the greater number, and like ſigns, add the correction; if unlike ſigns, ſubtract it: ſo you'll have the true number required.

Or in other words,

If like ſigns, ſubtract from the leſſer, or add to the greater number.

Unlike ſigns, add to the leſſer, or ſubtract from the greater number; to get the true number.

Ex.

Ex. 6.

A certain man being aſked what was the age of his four ſons; anſwered, that his eldeſt was 4 years older than the ſecond, and the ſecond 5 years older than the third, and the third 6 years elder than the fourth, which was half the age of the eldeſt. How old was each?

1. Suppoſe 16 for the eldeſt, then
   the youngeſt is 1
   half the eldeſt 8
   1 error —7

2. Suppoſe 20 for the eldeſt,
   the youngeſt 5
   half the eldeſt 10
   2 error —5

16 20

20
—7 16 —5
7 4
5 5 (cor.
2) 20 (10 the
20
anſ. 30
the eldeſt.

Ex. 7.

Two perſons diſcourſing of their money; ſays A, if you will give me 25*l.* I ſhall have as much as you; ſays B, if you will give me 22*l.* I ſhall have twice as much as you. How much had each?

	1 *Sup.*	2 *Sup.*
A has	120	130
add	25	25
B has left	145	155
add	25	25
B had at firſt	170	180
add	22	22
B has now	192	202
A has left	98	108
double	196	216
1 error	+4	2 er. +14

120 130

+4 +14

130
—120

14 10
—4 4

10) 40 (4 cor.

120
—4

116 A's mon.

Ex. 8.

There is a crown weighing 60*lb.* which is made of gold, braſs, tin, and iron. The weight of the gold and the braſs together is 40*lb.* of the gold and tin, 45; of the gold and iron 36. Quere, how much gold was in it?

	1 *Sup.*	2 *Sup.*
Gold	35*lb.*	29*lb.*
Braſs	5	11
Tin	10	16
Iron	1	7
	51	63
	60	60
1 er.	—9.	2 er. +3

35 29

—9 +3

9 6
3 3

12) 18 (1½ = cor.
12

6

29
+1½

anſ. 30½ gold.

35
29

6

Ex.

Ex. 9.

A factor delivers 6 French crowns, and 2 dollars for 45 ſhillings. And at another time 9 French crowns, and 5 dollars for 76 ſhill. What is the value of each?

1. Suppoſe 5*s.* = 1 crown. 2. Suppoſe 7*s.* = 1 crown.

(1) $6\times5=30$ (45)

2 doll. = 15

1 doll. = $7\frac{1}{2}$

$9\times5=45$

$5\times7\frac{1}{2}=37\frac{1}{2}$

$82\frac{1}{2}$

76

1 error + $6\frac{1}{2}$

(2) $6\times7=42$ (45)

2 doll. 3

1 doll. = $1\frac{1}{2}$

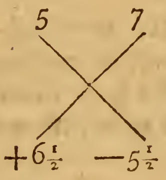
$9\times7=63$

$5\times1\frac{1}{2}=7\frac{1}{2}$

$70\frac{1}{2}$

76

2 er. $-5\frac{1}{2}$



7

5

2

$5\frac{1}{2}$

12)11($\frac{11}{12}$ = cor.

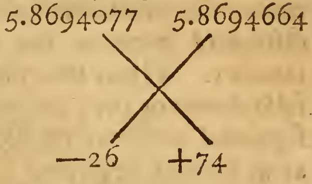
7

a crown = $6\frac{11}{12}$

a dollar = $4\frac{1}{4}$.

Ex. 10.

To find the logarithm of 740326.



1. I ſuppoſe 5.8694077 to be its log.; but by a table of logarithms, it proves only to be the logarithm of 740300.

740326

740300

1 error — 26

2. I ſuppoſe 5.8694664 for the log. but this by the table is the log. of 740400.

740400

740326

2 error +74

5.8694664

5.8694077

.0000587

26

3522

1174

74

26

100).0015262

(152

the

5.8694077
.0000152 cor.

the logarithm ſought 5.8694229

The proof of this rule is, by trying the number found, according to the conditions of the queſtion, in the ſame manner as you find out the errors. And if it agree, the work is right.

SCHOLIUM.

It will ſometimes ſhorten the work, by ſuppoſing one of the numbers 0, and you may ſuppoſe the other 1, if you pleaſe. A great many queſtions may be reſolved by this rule, which cannot be reſolved by any other rules in arithmetic. But there are many queſtions, where it cannot be certainly known, whether they can be reſolved by it or not, till they be tried.

The rule is founded upon this ſuppoſition, that the firſt error is to the ſecond; as the difference between the true and firſt ſuppoſed number, to the difference between the true and ſecond ſuppoſed number. When this does not happen, the rule of falſe does not give the exact anſwer, except the two ſuppoſed numbers be taken very near the true one: as in the laſt example.

In the rule of falſe, whatever operations the queſtion requires to be performed with the number ſought, and any given number or numbers; the ſame operations in every reſpect are to be made with the two ſuppoſed numbers, and the ſame given numbers. From the reſult of theſe three operations, are collected the errors, which are nothing elſe, but the differences between the true reſult, and each of the falſe reſults. Hence if the errors are unlike, the true number lies between the ſuppoſed numbers:

and if the errors are like, the true number lies without them both.

The rule of falſe, eſpecially the latter, will reſolve any the moſt difficult queſtion, by many trials; provided the queſtion can any way be proved, if the true reſolution was given. But then the ſuppoſed numbers muſt be taken near the truth. And after each operation is over, you muſt take the laſt reſult for one of the next ſuppoſed numbers; and the neareſt of the two former (or that with the leaſt error), for the other. And by repeating this proceſs, the anſwer will continually approximate to the true number, within any degree of exactneſs you pleaſe. For this reaſon it is of prodigious ſervice in the abſtruſer parts of the mathematics. For in many difficult problems, there is hardly any other way to come at a ſolution, but by this *method of trial and error.*

PROBLEM XI.

To reſolve a queſtion in the rule of exchange.

When ſeveral different ſorts of things are compared together, as to their value; this rule teaches to find, how many of one ſort is equal to a given number of another ſort.

RULE.

Place the terms in two perpendicular columns, ſo that there may not be found in either column, two terms of one kind. Then the numbers in the leſſer column muſt be multiplied for a diviſor; and the numbers in the greater column, where the odd term is, for a dividend. The quotient is the anſwer.

Note, to abridge the work, throw out any numbers that you can find in both columns.

Ex. 1.

If 6 *lb.* of ſugar be equal in value to 7 *lb.* of raiſons, 5 pound of raiſons to 4 yards of ribbon, 10 yards of ribbon to 40 nutmegs, and 7 nutmegs to 18 pence; what is 3 pound of ſugar worth?

6 ſug.	7 raiſ.	2100) 33600 (16 pence.
5 raiſ.	4 rib.	
10 rib.	40 nut.	
7 nut.	10 pnce.	
——	3 ſug.	
2100	——	
	33600	

Or thus.

$$\frac{7\times4\times40\times10\times3}{6\times5\times10\times7}=\frac{4\times40\times3}{6\times5}=\frac{4\times8}{2}=16.$$

Ex. 2.

If 3 pair of gloves be worth 2 yards of lace, 3 yards of lace equal to 7 dozen of buttons, 6 dozen of buttons to 2 penknives, and 21 penknives to 18 pair of buckles; how many pair of gloves is equal to 28 pair of buckles?

3 gloves	2 lace	
3 lace	7 buttons	504) 31752 (63.
6 buttons	2 pence	
21 pence	18 buckles	
28 buckles	——	
——	504	
31752		

Or thus.

$$\frac{3\times3\times6\times21\times28}{2\times7\times2\times18}=\frac{3\times21\times28}{2\times7\times2}=3\times21=63.$$

Ex.

Ex. 3.

If 9 ſhillings Engliſh be equal in value to 2 French crowns, and 1 French crown to 3 livrés, and 4 livrés to 3 guilders, and 9 guilders to 4 rix dollars, and 4 rix dollars to 3 Barcelona ducats; what is 5 Barcelona ducats worth in Engliſh money?

9 ſhil. Eng.	=	2 Fr. cr.
1 Fr. cr.		3 liv.
4 liv.		3 guil.
9 guil.		4 rix doll.
4 rix doll.		3 Barc. duc.
5 Barc. duc.		

$$\frac{9\times1\times4\times9\times4\times5}{2\times3\times3\times4\times3}$$

$$\frac{9\times4\times9\times5}{2\times3\times3\times3}$$

$$\frac{4\times3\times5}{2} = 2\times3\times5$$

$= 30$ ſhillings.

PROBLEM XII.

To reſolve a queſtion by help of a table of logarithms.

Logarithms are a certain ſet of artificial numbers, fitted to the ſeries of natural numbers, and formed into a table; whoſe property is ſuch, that they perform the ſame thing by addition and ſubtraction, which the natural numbers do by multiplication and diviſion.

A logarithm conſiſts of two parts, a decimal fraction and an integer. The decimal part is always affirmative, the integer may be either affirmative or negative, and is called the *characteriſtic.* It always ſhews how far the firſt figure of the abſolute number is diſtant from the units place. Thus when the characteriſtic is 0, 1, 2, 3, *&c.* the firſt figure of the correſponding number will be units, tens, hundreds, thouſands, *&c.* reſpectively. And if it be —1, —2, —3, *&c.* then the firſt figure of the number belonging, is in the firſt, ſecond, third, *&c.* place of decimals.

In

In many tables, the characteriſtic is not ſet down, becauſe it is eaſily ſupplied, for any given number, from the rule before mentioned; by only conſidering how many places of integers, *&c.*; the given number conſiſts of.

Though the decimal part of the log. is always affirmative, yet in ſome particular caſes, where the characteriſtic is negative, it is neceſſary to reduce it to another form, where the whole is negative. Thus the log. —2.3406424 which ſignifies the ſame as —2.+.3406424, is reduced to —1.—.6593576, or —1.6593576, where the whole is negative; which is done by ſubtracting the decimal from 1. But when the operation is over, it muſt be reduced to its original form. Or it may be otherways reduced ſo as to be expreſſed in two parts, without making the decimal negative, by adding equal numbers to both the negative and affirmative part. Thus —2.3406424 is equivalent to —3.+1.3406424, or = —4.+2.3406424 = —5.+3.3406424, *&c.* where the latter part is entirely affirmative: and this way is more commodious for ſome ſort of operations.

Having a number given to find its log. and the contrary. Look through the column of numbers, till you find the given number, againſt this is its logarithm. Or when the log. is given, look through the column of logarithms till you find it, or the neareſt thereto, and againſt it is the number. Thus if the number is 2191, the log. is 3.3406424. And if the log. be 2.8241900, the number is 667.1; and ſo of others. But if the number exceed the table, that is, if it conſiſts of more than 4 places, proceed as in Ex. 10. Prob. 10, to find the log. or the contrary.

The table of logarithms is too large for this book, its principal uſe being in trigonometrical operations, See my Trigonometry, Edit. 2.

RULE.

1 RULE.

After the queſtion is reſolved in form, and the numbers are ready for operation. To find the product of any numbers multiplied together. Set down all the numbers and their logarithms againſt them; then add all the logarithms together. When you come at the characteriſtics, add what you carried, to the affirmatives, and take the difference between the ſum of the affirmatives, and the ſum of the negatives, and ſet it down with the ſign of the greater. This is the characteriſtic of the product; whoſe number muſt be found in the table.

Ex. 1.

What is the product of 37×250?

37 - - - 1.5682017
250 - - - 2.3979400

prod. 9250 - - - 3.9661417

Ex. 2.

What is the product of 7×486×.0042?

7 - - - 0.8450980
486 - - - 2.6866363
.0042 - - —3.6232493

prod. near 14.29 - - +1.1549836

2 RULE.

When a quantity appears in form of a fraction, to find the quotient ariſing by dividing the numerator by the denominator. Subtract the log. of the denominator from the log. of the numerator. If you carry 1, add it to the lower charact. if +, or ſubtract it, if —; which done, if the charact. have unlike ſigns, add them with the ſign of the upper; if like ſigns,

ſigns, ſubtract with the ſame ſign; except the lower be the greater, and then with a contrary ſign.

If either numerator or denominator is any product of certain numbers, its log. muſt be found by Rule 1.

Ex. 3.

What is the value of $\frac{438}{73}$?

438 - - 2.6414741
73 - - 1.8633229

quotient 6 - - 60.7781512

Ex. 4.

Divide 125 by 3125.

125 - - 2.0969100
3125 - - 3.4948500

quotient .04 - - —2.6020600

Ex. 5.

Divide 342 by .035.

342 - - 2.5340261
.035 - - —2.5440680

quot. 9771 - - 3.9899581

Ex. 6.

What is the value of $\frac{.54 \times .0157}{48}$?

.54 - - —1.7323938
.0157 - - —2.1958996

product - - —3.9282934
48 - - 1.6812412

quot. 0001766 - - —4.2470522

3 RULE.

When a number is to be ſquared, cubed, *&c.* multiply its log. by the index of the power. Obſerving, when the characteriſtic is negative, to ſubtract what you carry thither. Then find the number anſwering.

Ex. 7.

What is the ſquare root of 2?

426 - - 2.6294096

2

ſquare 181500 - - 5.2588192

Ex. 8.

What is the cube of .405?

.405 - - —1.66074550

3

cube .06643 - - - —2.8223650

Ex. 9.

To find the 4th power of .09054.

.09054 - - —2.956840445

4th power .0000672 - - —5.8273620

4 RULE.

When any root is to be extracted; divide the log. of the number by the index of the root. Remembring to reduce the log. if the characteriſtic be negative, when there is occaſion.

Ex.

Ex. 10.

What is the ſquare root of 2?

2 - - - 2) 0.3010300
root 1.414 - - 0.1505150

Ex. 11.

Find the ſquare root of 4823.

4823 - - 2)3.6833173
root 69.45 - - 1.8416586

Ex. 12.

What is the cube root of .005832?

.005832 - - 3)—3.7658175
root .18 - - - —1.2552725

Ex. 13.

To find the cube root of .02456.

.02456 - - —2.3902284
reduced - - 3)—1.6097716 all neg.
reduce this back —0.5365905
root .2907 - - —1.4634095

Or thus.

The log. —2.3902284 is equal to —3.+1.3902284

3)—3.+1.3902284
root .2907 - - —1. 4634095

Ex. 14.

What is the 5th root of .004705?

.004705 - - —3.6725596
reduced to 5)—5+2.6725596
root .3424 - - —1.5345119

---

5 RULE.

When in the ſolution of a queſtion, you come at ſome compound quantity, conſiſting of products, powers, roots, &c. connected by the ſigns + and —; they muſt be wrought ſeparately by the foregoing rules, and the numbers found and collected, according to the ſigns.

Ex. 15.

To find the number expreſſed by this quantity.

$$\frac{350\times20\times11-108\times13^2}{11\times13\times15}.$$

This is the ſame as the two quantities $\frac{350\times20\times11}{11\times13\times15}$ $-\frac{108\times13\times13}{11\times13\times15}$. That is $\frac{350\times20}{13\times15}-\frac{108\times13}{11\times15}$.

350	2.5440680	13	1.1139433
20	1.3010300	15	1.1760913
	3.8450980		2.2900346
ſubt.	2.2900346		
35.90	1.5550634		

the firſt part.

108	2.0334238	11	1.0413927
13	1.1139433	15	1.1760913
	3.1473671		2.2174840
ſubt.	2.2174840		
8.509	0.9298831		

the ſecond part. Then from 35.900
take 8.509

the number ſought, 27.391

Ex. 16.

Suppoſe in a certain queſtion, I come to this concluſion for the number ſought, $\frac{12\times37\times20+\overline{25}^{3}}{37\times25-12\sqrt{37\times20}}$, what is the number?

n.	*log.*
12	1.0791812
37	1.5682017
20	1.3010300
	3.9484129
numb.	8880.

37	1.5682015
25	1.3979400
	2.9661417
numb.	925.0

n.	*log.*
25	1.3979400
	3
	4.1938200
numb.	15620

37	1.5682017
20	1.3010300
	2.8692317
half	1.4346158
12	1.0791812
	2.5137970
numb.	326.4

The ſolution becomes $\frac{8880+15620}{925.0-326.4}=$

$\frac{24500}{598.6}$

log. 4.3891661
log. 2.7771367

1.6120294
the numb. 40.93 anſwer.

PROBLEM XIII.

To reſolve the uſual queſtions about the intereſt of money, and annuities.

Intereſt is the money paid for the uſe or loan of any ſum or principal; and is generally eſtimated at ſo much per hundred for a year, as 4 per cent. 5 per cent.; *&c.* which is called the *rate of intereſt.*

Simple intereſt is that which is charged only upon the principal, for any length of time after it is due.

Compound intereſt, or intereſt upon intereſt, is that which ariſeth from both principal and intereſt; this ſuppoſes that the intereſt itſelf, ſhall alſo gain intereſt, after the time it becomes due.

Rebate is the abatement made by paying a ſum of money before it is due.

Amount is the quantity of money in arrear, conſiſting of the principal or annuity, together with its intereſt, forborn for ſome time after it is due.

Several queſtions in the buſineſs of intereſt being very difficult to reſolve ſolely by arithmetic; I have therefore inſerted the four following tables; by help of which all the common queſtions relating to intereſt and annuities may very ſpeedily be reſolved, for any numbers that come within the reach of theſe tables.

Their uſe is eaſy and evident at ſight: for the rate of intereſt being found at the top, and the time of continuance on the ſide; at the angle of meeting, you have the amount of 1 pound, (Tab. 1 and 3); or of 1 pound annuity (Tab. 2 and 4), at either ſimple or compound intereſt. But their uſefulneſs will more clearly appear from the following rules and examples.

1 RULE.

When the ſimple intereſt for days, is required; divide the rate by 100, to have the rate for 1 *l.* then multiply the principal, the rate for 1 pound, and the number of days, continually; and divide the product by 365; the quotient is the intereſt.

Ex. 1.

What is the intereſt of 160*l.* for 85 days, at 3 per cent.?

$\frac{3}{100} = .03$ the rate of 1*l.*

	£. s. d.
160	365) 408 (1 : 2 : 4¼ anſwer.
.03	365
4.80	43
85	20
240	860 (2
384	730
408.0	130
	12
	1560 (4.2
	1460
	100
	73

2 RULE.

To find the preſent worth of 1*l.* in money, due any number of years hence; or of 1*l.* annuity to continue any number of years, at a given rate either of ſimple or compound intereſt.

For 1*l. in money.* Look into Tab. I. for ſimple intereſt, or Tab. III. for compound intereſt, and under the given rate, and againſt the number of years, you'll find a number for a diviſor, by this divide 1, the quotient is the preſent worth.

For 1*l. annuity.* Conſult the Tables I. and II. for ſimple intereſt; or III. and IV. for compound intereſt. And under the given rate, and againſt the number of years, in both tables, you'll find two numbers, which take out, and divide the latter by the former, for the preſent worth.

Ex.

Ex. 2.

What is the preſent worth of 1 *l.* due 14 years hence, at 4 per cent. at ſimple or compound intereſt?

Num. Tab. I. - - 1.56) 1.000 (.64102 the preſ. worth at ſimp. inter.

936

640

624

160

156

400

Num. Tab. III. - - 1.73167) 1.000000 (.577476 the preſ. worth at comp. intereſt.

865835

134165

121217

12948

12122

826

693

133

121

12

Ex. 3.

What is the preſent worth of 1 *l.* annuity to continue 14 years, at 5 per cent. ſimple and compound intereſt?

$$\begin{array}{ll} \text{Tab. II. - -} & 18.55 \\ \text{Tab. I. - - -} & \overline{1.7} \end{array} = \text{preſent worth at ſimp. intereſt.}$$

That is, 1.7) 18.55 (10.91176 the preſent worth at ſimple intereſt.

$$\begin{array}{r} 17\,\cdot\,\cdot \\ \hline 155 \\ 153 \\ \hline 20 \\ 17 \\ \hline 30 \\ 17 \\ \hline 130 \\ 119 \\ \hline 11 \end{array}$$

$$\begin{array}{ll} \text{Tab. IV. - -} & 19.59863 \\ \text{Tab. III. - -} & \overline{1.97993} \end{array} = \text{preſent worth at comp. inter.}$$

That is, 1.97993) 19.59863 (9.89865 the preſ. worth at compound intereſt.

$$\begin{array}{r} 17\ 81937 \\ \hline 1\ 77926 \\ 1\ 58394 \\ \hline 19532 \\ 17819 \\ \hline 1713 \\ 1584 \\ \hline 129 \\ 118 \\ \hline 11 \\ \hline \end{array}$$

3 RULE.

Queſtions, where principal, annuity, amount, &c. are concerned, are likewiſe to be ſolved by the tables. For there are ſimilar numbers in the tables analogous to thoſe given; and therefore having three terms given, a proportion or analogy muſt be made by the rule of three, between the numbers given in the queſtion, and thoſe in the proper table, for the ſame rate and time, in order to find the 4th term, which is either the thing itſelf which is ſought, or it will ſhew it by the table. And as 1 is commonly a term in the proportion, the queſtion will generally be ſolved by multiplication or diviſion.

If any thing is wanting to make the proportion, or to carry on the proceſs, it muſt be found from what is given in the queſtion.

Ex. 4.

If 250*l.* be put out to intereſt, what will it amount to in 21 years, at 4*l.* per cent. ſimple or compound intereſt?

By Tab. I. the amount of 1*l.* for 21 years, at 4 per cent. is 1.84; therefore ſay, as 1 *(principal)* : 1.84 *(amount)* : : 250 *(principal)* : $1.84 \times 250 = 460$, the amount required, at ſimple intereſt.

Again, by Tab. III. the amount of 1*l.* is 2.27877; Therefore ſay, as 1 *(pr.)* : 2.27877 *(am.)* : : 250 *(pr.)* : $2.27877 \times 250 = 569.6925$*l.* the amount required, at compound intereſt.

Ex. 5.

What principal put out for 21 years will amount to 460*l.* at 4 per cent. ſimple intereſt?

By Tab. I. the amount of 1*l.* is 1.84 for the given time and rate; then ſay, 1.84 *am.*—1 *pr.*—460 *am.*— $\frac{460}{1.84} = 250$*l.* the principal ſought.

Ex. 6.

In what time will 250*l.* amount to 569.6925*l.* being put out at 4 per cent, compound intereſt?

Say, as 250 *pr.* : 569.6925 *am.* : : 1 *pr.* : $\frac{569.6925}{250}$ = 2.27877 the amount of 1 *l.* Seek this number in Tab. III. col. 4 per C. and you'll find it againſt 21 years, the time ſought.

Ex. 7.

At what rate of ſimple intereſt will 250*l.* amount to 460*l.* in 21 years?

By Tab. I. ſay, 250 *pr.*—460 *am.*—1 *pr.*—$\frac{460}{250}$ = 1.84, the amount of 1*l.*; which being ſought for againſt 21 years, will fall in col. 4 per C. the rate of intereſt required.

Ex. 8.

If 320*l.* yearly rent be forborn for 12 years, what will be in arrear at that time, at 4½ per cent. ſimple and compound intereſt?

By Tab. II. the amount of 1*l.* annuity for 12 years is 14.97; then ſay, 1 *an.*—14.97 *am.*—320 *an.* —14.97 × 320 = 4790.4*l.* the arrear ſought, at ſimple intereſt.

Again, by Tab. IV. the amount of 1*l.* annuity is 15.46403; therefore ſay, as 1 *rent*—15.46403 *am.* —320*r.*—15.46403 × 320 = 4948.49*l*, the amount, at compound intereſt.

Ex. 9.

What yearly rent being forborn 12 years, will amount to 4948.49, at 4½ per cent. comp. intereſt?

By Tab. IV. the amount of 1*l.* annuity is 15.46403; then ſay, as 15.46403 *am.*—1 *r.*—4948.49 *am.*— $\frac{4948.49}{15.46403}$ = 320*l.* the rent ſought

Ex. 10.

In what time will 320*l.* yearly rent, amount to 4790.4*l.* at $4\frac{1}{2}$ per cent. ſimple intereſt?

Say, 320 *rent* — 4790.4 *am.* — 1 *rent* — $\frac{4790.4}{320}$ = 14.97, the amount of 1 *l.* annuity; which being found in col. $4\frac{1}{2}$ per C. Tab. II. ſtands over-againſt 12 years, the time ſought.

Ex. 11.

At what rate of compound intereſt, does 320*l.* rent, amount to 4948.49*l.* in 12 years?

Say, as 320 *rent* — 4948.49 *am.* — 1 *rent* — $\frac{4948.49}{320}$ = 15.46403 the amount of 1 *l.* annual rent. Seek this number over-againſt 12 years in Tab. IV. and it is found under $4\frac{1}{2}$ per C. the rate ſought.

Ex. 12.

What is the preſent worth of 65*l.* a year, to continue 40 years, at 5 per cent. ſimple and compound intereſt?

By Rule 2, find the preſent worth of 1 *l.* annuity at ſimple intereſt, for the time and rate given, which is $\frac{79}{3}$; then ſay,

As 1 *an.* — $\frac{79}{3}$ *pr.* — 65 *an.* — $\frac{65\times79}{3}$ = 1711,66 the preſent worth ſought, at ſimple intereſt,

Again, by Rule 2, find the preſent worth of 1 *l.* annuity at compound intereſt, which is $\frac{120.79977}{7.03999}$; then ſay,

1 *an.* — $\frac{120.7}{7.0}$&c. *pr.* — 65 *an.* — $\frac{120.79977\times65}{7.03999}$ = 1115.34, the preſent worth ſought, at comp. intereſt.

Ex. 13.

What annuity to continue 40 years, will 1711.66*l.* ready money purchaſe, at 5 per cent. ſimple intereſt?

By Rule 2, find the preſent worth of 1 *l.* annuity, which is $\frac{79}{3}$; then ſay, $\frac{79}{3}$ *pr.*—1 *an.*—1711.66 *pr.*— $\frac{3\times1711.66}{79}$ = 65*l.* the annuity required.

Ex. 14.

How long may one have a leaſe of 65*l.* a year, for 1711.66*l.* ready money, at 5 per cent. ſimple intereſt?

Say, as 65 *rent* — 1711.66 *pr.* — 1 *rent* — $\frac{1711.66}{65}$ = 26.33, the preſent worth of 1 *l.* annuity, for an unknown time. Then,

Take ſome year by gueſs, and find the amount by Tab. II. and the preſent worth of that amount, by Tab. I. If this agrees not with 26.33, try again, and by a few eaſy trials you'll come to the truth.

In ſhort thus, ſet down the correſpondent numbers in Tab. II. and I. fractionwiſe, to approach continually to 26.33, which at laſt you'll obtain.

Suppoſe 30 years - - $\frac{51.75}{2.5}$ = 20. &c. too little.

38 years - - $\frac{73.15}{2.9}$ = 25.2 &c. too little.

40 years - - $\frac{79}{3}$ = 26.33 juſt. So 40 years is the time required.

Ex. 15.

If one give 1115.34*l.* ready money, for the purchaſe of an annuity of 65*l.* a year, to continue 40 years; what is the rate at compound intereſt?

Say,

Say, as 65 *an.* — 1115.34 *pr.* — 1 *an.* — $\frac{1115.34}{65}$ = 17.159, the preſent worth of 1 *l.* annuity, at an unknown rate.

Take ſome rate of intereſt by gueſs, and find the amount for 40 years by Tab. IV; and the preſent worth of that amount by Tab. III. repeat this work with other rates, till the reſult be 17.159.

Or in ſhort thus, ſet down the correſpondent numbers in Tab. IV. and III. fractionwiſe, and you will approach to the rate ſought by a few trials. Thus,

Suppoſe 3 per cent. - - $\frac{75.4}{3.2} = 23$, too great.

4 per cent. - - $\frac{95.0}{4.8} = 19.8$, too great.

5 per cent. - - $\frac{120.799}{7.0399} = 17.159$, juſt.

Therefore 5 per cent. is the rate required.

4 RULE.

When freehold eſtates are to be valued; divide 1 by the rate of 1 *l.* the quotient ſhows how many years purchaſe it is worth, at compound intereſt.

Or if the annuity or rent be required; multiply the purchaſe money by the rate of 1 *l.* for the annuity.

Ex. 16.

What is an eſtate at 30*l.* a year worth, at $3\frac{1}{2}$ per cent.?

Here $\frac{1}{.035} = 28.571$ years purchaſe.

Or 28.571 × 30 = 857.13 *l.* the purchaſe money.

Ex. 17.

What annuity can I buy for 857.13*l.* at $3\frac{1}{2}$ per cent.?

Here 857.13 × .035 = 29.999*l.* or 30*l.* the annuity.

5 RULE.

5 RULE.

When ſeveral ſums of money are out at ſimple intereſt, and are to be paid in, at different times; to find the time, when the whole may be paid in at once, without loſs to the debtor or creditor.

Multiply every ſum of money by the time it is to continue; and divide the ſum of the products, by the total ſum of all the money, the quotient will be the mean time of payment.

And the ſame rule holds true, very near; when ſeveral ſums of money are due at different times, only it makes the mean time a ſmall matter too big.

Ex. 18.

I have three ſums of money let out to intereſt, for different times; *viz.* 50*l.* continues for 2 years, 40*l.* for $3\frac{1}{2}$ years, and 20*l.* for $4\frac{1}{2}$ years. But it is now agreed, that they ſhall be all paid at once. The queſtion is, when muſt I receive the whole together?

50	40	20	50	100	
2	$3\frac{1}{2}$	$4\frac{1}{2}$	40	140	
			20	90	
100	120	80			
	20	10	110)	330	(3 years; anſwer.
				330	
	140	90			

Ex. 19.

A man has three ſeveral ſums of money due at different times, 50*l.* at the end of 5 months, 84*l.* at the end of 10 months, and 36*l.* a year and half hence. But he would receive them all at once; in what time ſhall he receive the whole ſum?

50	84	38	50	250
5	10	18	84	840
			36	684
50	840	304		
		38		
		684		

170)1774(10.43 months, nearly; the answer.
170
74
68
60

The proof, in all queſtions of intereſt, is to change e *data*, and work the queſtion backwards.

SCHOLIUM.

It is contrary to law to let out money at compound tereſt. Yet in the valuation of annuities, it is al-ays the cuſtom to allow compound intereſt; for by mple intereſt, they would be overvalued.

TAB. I.

A table of the amount of 1 pound for years, at ſimple intereſt.

Years.	3 per C.	3½ per C.	4 per C.	4½ per C.	5 per C.
1	1.03	1.035	1.04	1.045	1.05
2	1.06	1.070	1.08	1.090	1.10
3	1.09	1.105	1.12	1.135	1.15
4	1.12	1.140	1.16	1.180	1.20
5	1.15	1.175	1.20	1.225	1.25
6	1.18	1.210	1.24	1.270	1.30
7	1.21	1.245	1.28	1.315	1.35
8	1.24	1.280	1.32	1.360	1.40
9	1.27	1.315	1.36	1.405	1.45
10	1.30	1.350	1.40	1.450	1.50
11	1.33	1.385	1.44	1.495	1.55
12	1.36	1.420	1.48	1.540	1.60
13	1.39	1.455	1.52	1.585	1.65
14	1.42	1.490	1.56	1.630	1.70
15	1.45	1.525	1.60	1.675	1.75
16	1.48	1.560	1.64	1.720	1.80
17	1.51	1.595	1.68	1.765	1.85
18	1.54	1.630	1.72	1.810	1.90
19	1.57	1.665	1.76	1.855	1.95
20	1.60	1.700	1.80	1.900	2.00
21	1.63	1.735	1.84	1.945	2.05
22	1.66	1.770	1.88	1.990	2.10
23	1.69	1.805	1.92	2.035	2.15
24	1.72	1.840	1.96	2.080	2.20
25	1.75	1.875	2.00	2.125	2.25
26	1.78	1.910	2.04	2.170	2.30
27	1.81	1.945	2.08	2.215	2.35
28	1.84	1.980	2.12	2.260	2.40
29	1.87	2.015	2.16	2.305	2.45
30	1.90	2.050	2.20	2.350	2.50

TAB. I.

Years.	3 per C.	3½ per C.	4 per C.	4½ per C.	5 per C.
31	1.93	2.085	2.24	2.395	2.55
32	1.96	2.120	2.28	2.440	2.60
33	1.99	2.155	2.32	2.485	2.65
34	2.02	2.190	2.36	2.530	2.70
35	2.05	2.225	2.40	2.575	2.75
36	2.08	2.260	2.44	2.620	2.80
37	2.11	2.295	2.48	2.665	2.85
38	2.14	2.330	2.52	2.710	2.90
39	2.17	2.365	2.56	2.755	2.95
40	2.20	2.400	2.60	2.800	3.00
41	2.23	2.435	2.64	2.845	3.05
42	2.26	2.470	2.68	2.890	3.10
43	2.29	2.505	2.72	2.935	3.15
44	2.32	2.540	2.76	2.980	3.20
45	2.35	2.575	2.80	3.025	3.25
46	2.38	2.610	2.84	3.070	3.30
47	2.41	2.645	2.88	3.115	3.35
48	2.44	2.680	2.92	3.160	3.40
49	2.47	2.715	2.96	3.205	3.45
50	2.50	2.750	3.00	3.250	3.50
51	2.53	2.785	3.04	3.295	3.55
52	2.56	2.820	3.08	3.340	3.60
53	2.59	2.855	3.12	3.385	3.65
54	2.62	2.890	3.16	3.430	3.70
55	2.65	2.925	3.20	3.475	3.75
56	2.68	2.960	3.24	3.520	3.80
57	2.71	2.995	3.28	3.565	3.85
58	2.74	3.030	3.32	3.610	3.90
59	2.77	3.065	3.36	3.655	3.95
60	2.80	3.100	3.40	3.700	4.00

Tab. II.

A table of the amount of 1 pound annuity for years, at ſimple intereſt.

Years.	3 per C.	3½ per C.	4 per C.	4½ per C.	5 per C.
1	1.00	1.000	1.00	1.000	1.00
2	2.03	2.035	2.04	2.045	2.05
3	3.09	3.105	3.12	3.135	3.15
4	4.18	4.210	4.24	4.270	4.30
5	5.30	5.350	5.40	5.450	5.50
6	6.45	6.525	6.60	6.675	6.75
7	7.63	7.735	7.84	7.945	8.05
8	8.84	8.980	9.12	9.260	9.40
9	10.08	10.260	10.44	10.620	10.80
10	11.35	11.575	11.80	12.025	12.25
11	12.65	12.925	13.20	13.475	13.75
12	13.98	14.310	14.64	14.970	15.30
13	15.34	15.730	16.12	16.510	16.90
14	16.73	17.185	17.64	18.095	18.55
15	18.15	18.675	19.20	19.725	20.25
16	19.60	20.200	20.80	21.400	22.00
17	21.08	21.760	22.44	23.120	23.80
18	22.59	23.355	24.12	24.885	25.65
19	24.13	24.985	25.84	26.695	27.55
20	25.70	26.650	27.60	28.550	29.50
21	27.30	28.350	29.40	30.450	31.50
22	28.93	30.085	31.24	32.395	33.55
23	30.59	31.855	33.12	34.385	35.65
24	32.28	33.660	35.04	36.420	37.80
25	34.00	35.500	37.00	38.500	40.00
26	35.75	37.375	39.00	40.625	42.25
27	37.53	39.285	41.04	42.795	44.55
28	39.34	41.230	43.12	45.010	46.90
29	41.18	43.210	45.24	47.270	49.30
30	43.05	45.225	47.40	49.575	51.75

TAB. II.

Years.	3 per C.	3½ per C.	4 per C.	4½ per C.	5 per C.
31	44.95	47.275	49.60	51.925	54.25
32	46.88	49.360	51.84	54.320	56.80
33	48.84	51.480	54.12	56.760	59.40
34	50.83	53.635	56.44	59.245	62.05
35	52.85	55.825	58.80	61.775	64.75
36	54.90	58.050	61.20	64.350	67.50
37	56.98	60.310	63 64	66.970	70.30
38	59.09	62.605	66.12	69.635	73.15
39	61.23	64.935	68.64	72.345	76.05
40	63.40	67.300	71.20	75.100	79.00
41	65.60	69.700	73.80	77.900	82.00
42	67.83	72.135	76.44	80.745	85.05
43	70.09	74.605	79.12	83.635	88.15
44	72.38	77.110	81 84	86.570	91.30
45	74.70	79.650	84.60	89.550	94.50
46	77.05	82.225	87.40	92.575	97.75
47	79.43	84.835	90.24	95.645	101.05
48	81.84	87.480	93.12	98.760	104.40
49	84.28	90.160	96.04	101.920	107.80
50	86.75	92.875	99.00	105.125	111.25
51	89.25	95.625	102.00	108.375	114.75
52	91.78	98.410	105.04	111.670	118.30
53	94.34	101.230	108.12	115.010	121.90
54	96.93	104.085	111.24	118.395	125.55
55	99.55	106.975	114.40	121.825	129.25
56	102.20	109.900	117.60	125.300	133.00
57	104.88	112.860	120.84	128.820	136.80
58	107.59	115.855	124.12	132.385	140.65
59	110.33	118.885	127.44	135.995	144.55
60	113.10	121.950	130.80	139.650	148.50

Tab. III.

A table of the amount of 1 pound for years, at compound intereſt.

Years.	3 per C.	3½ per C.	4 per C.	4½ per C.	5 per C.
1	1.03000	1.03500	1.04000	1.04500	1.05000
2	1.06090	1.07122	1.08160	1.09202	1.10250
3	1.09273	1.10872	1.12486	1.14116	1.15762
4	1.12551	1.14752	1.16986	1.19252	1.21550
5	1.15927	1.18769	1.21665	1.24618	1.27628
6	1.19405	1.22925	1.26532	1.30226	1.34009
7	1.22987	1.27228	1.31593	1.36086	1.40710
8	1.26677	1.31681	1.36857	1.42210	1.47745
9	1.30477	1.36290	1.42331	1.48609	1.55132
10	1.34391	1.41060	1.48024	1.55297	1.62889
11	1.38423	1.45997	1.53945	1.62285	1.71034
12	1.42576	1.51107	1.60103	1.69588	1.79585
13	1.46853	1.56395	1.66507	1.77219	1.88565
14	1.51259	1.61869	1.73167	1.85194	1.97993
15	1.55797	1.67535	1.80094	1.93528	2.07893
16	1.60470	1.73398	1.87298	2.02237	2.18287
17	1.65285	1.79467	1.94790	2.11338	2.29202
18	1.70243	1.85749	2.02582	2.20848	2.40662
19	1.75350	1.92250	2.10685	2.30786	2.52695
20	1.80611	1.98979	2.19112	2.41171	2.65330
21	1.86029	2.05943	2.27877	2.52024	2.78596
22	1.91610	2.13151	2.36992	2.63365	2.92526
23	1.97359	2.20611	2.46471	2.75216	3.07152
24	2.03279	2.28333	2.56330	2.87601	3.22510
25	2.09378	2.36324	2.66583	3.00543	3.38635
26	2.15659	2.44596	2.77247	3.14068	3.55567
27	2.22129	2.53157	2.88337	3.28201	3.73345
28	2.28793	2.62017	2.99870	3.42970	3.92013
29	2.35656	2.71188	3.11865	3.58403	4.11613
30	2.42726	2.80679	3.24340	3.74532	4.32194

TAB. III.

Years.	3 per C.	3½ per C.	4 per C.	4½ per C.	5 per C.
31	2.50008	2.90503	3.37313	3.91386	4.53804
32	2.57508	3.00671	3.50806	4.08998	4.76494
33	2.65233	3.11194	3.64838	4.27403	5.00319
34	2.73190	3.22086	3.79431	4.46636	5.25335
35	2.81386	3.33359	3.94609	4.66735	5.51601
36	2.89828	3.45026	4.10393	4.87738	5.79181
37	2.98523	3.57102	4.26809	5.09686	6.08141
38	3.07478	3.69601	4.43881	5.32622	6.38548
39	3.16703	3.82537	4.61636	5.56590	6.70475
40	3.26204	3.95926	4.80102	5.81636	7.03999
41	3.35990	4.09783	4.99306	6.07810	7.39199
42	3.46069	4.24126	5.19278	6.35161	7.76159
43	3.56452	4.38970	5.40049	6.63744	8.14967
44	3.67145	4.54334	5.61651	6.93612	8.55715
45	3.78159	4.70236	5.84117	7.24825	8.98501
46	3.89504	4.86694	6.07482	7.57442	9.43426
47	4.01189	5.03728	6.31781	7.91527	9.90597
48	4.13225	5.21359	6.57053	8.27145	10.40127
49	4.25622	5.39606	6.83335	8.64367	10.92133
50	4.38390	5.58492	7.10668	9.03263	11.46740
51	4.51542	5.78040	7.39095	9.43910	12.04077
52	4.65088	5.98271	7.68659	9.86386	12.64281
53	4.79041	6.19211	7.99405	10.30774	13.27495
54	4.93412	6.40883	8.31381	10.77158	13.93869
55	5.08215	6.63314	8.64637	11.25631	14.63563
56	5.23461	6.86530	8.99222	11.76284	15.36741
57	5.39165	7.10558	9.35191	12.29217	16.13578
58	5.55340	7.35428	9.72599	12.84532	16.94257
59	5.72000	7.61168	10.11502	13.42335	17.78970
60	5.89160	7.87809	10.51963	14.02741	18.67918

TAB. IV.

A table of the amount of 1 pound annuity for years, at compound intereſt.

Years.	3 per C.	3½ per C.	4 per C.	4½ per C.	5 per C.
1	1.00000	1.00000	1.00000	1.00000	1.00000
2	2.03000	2.03500	2.04000	2.04500	2.05000
3	3.09090	3.10622	3.12160	3.13702	3.15250
4	4.18363	4.21494	4.24646	4.27819	4.31012
5	5.30913	5.36246	5.41632	5.47071	5.52563
6	6.46841	6.55015	6.63297	6.71689	6.80191
7	7.66242	7.77941	7.89829	8.01915	8.14201
8	8.89233	9.05169	9.21422	9.38001	9.54911
9	10.15910	10.36849	10.58279	10.80211	11.02656
10	11.46388	11.73139	12.00611	12.28821	12.57789
11	12.80779	13.14199	13.48635	13.84118	14.20679
12	14.19203	14.60196	15.02580	15.46403	15.91713
13	15.61779	16.11303	16.62684	17.15991	17.71298
14	17.08632	17.67698	18.29191	18.93211	19.59863
15	18.59891	19.29568	20.02359	20.78405	21.57856
16	20.15688	20.97103	21.82453	22.71934	23.65749
17	21.76159	22.70501	23.69751	24.74171	25.84036
18	23.41443	24.49969	25.64541	26.85508	28.13238
19	25.11687	26.35718	27.67123	29.06356	30.53900
20	26.87037	28.27968	29.77808	31.37142	33.06595
21	28.67648	30.26947	31.96920	33.78314	35.71925
22	30.53678	32.32890	34.24797	36.30338	38.50521
23	32.45288	34.46041	36.61789	38.93703	41.43047
24	34.42647	36.66653	39.08260	41.68919	44.50200
25	36.45926	38.94986	41.64591	44.56521	47.72710
26	38.55304	41.31310	44.31174	47.57064	51.11345
27	40.70963	43.75906	47.08421	50.71132	54.66912
28	42.93092	46.29063	49.96758	53.99333	58.40258
29	45.21885	48.91080	52.96628	57.42303	62.32271
30	47.57541	51.62268	56.08494	61.00707	66.43885

TAB. IV.

Yea.	3 per C.	3½ per C.	4 per C.	4½ per C.	5 per C.
31	50.00268	54.42947	59.32833	64.75239	70.76079
32	52.50276	57.33450	62.70147	68.66624	75.29883
33	55.07784	60.34121	66.20953	72.75622	80.06377
34	57.73018	63.45315	69.85791	77.03025	85.06696
35	60.46208	66.67401	73.65222	81.49662	90.32031
36	63.27594	70.00760	77.59831	86.16396	95.83632
37	66.17422	73.45787	81.70224	91.04134	101.62814
38	69.15945	77.02889	85.97033	96.13820	107.70954
39	72.23423	80.72490	90.40915	101.46442	114.09502
40	75.40126	84.55028	95.02551	107.03032	120.79977
41	78.66330	88.50954	99.82653	112.84669	127.83976
42	82.02320	92.60737	104.81960	118.92479	135.23175
43	85.48389	96.84863	110.01238	125.27640	142.99334
44	89.04841	101.23833	115.41288	131.91384	151.14300
45	92.71986	105.78167	121.02939	138.84996	159.70015
46	96.50146	110.48403	126.87057	146.09821	168.68516
47	100.39650	115.35097	132.94539	153.67263	178.11942
48	104.40839	120.38826	139.26320	161.58790	188.02539
49	108.54065	125.60184	145.83373	169.85936	198.42666
50	112.79687	130.99790	152.66708	178.50303	209.34799
51	117.18077	136.58284	159.77377	187.53566	220.81539
52	121.69620	142.36323	167.16472	196.97477	232.85616
53	126.34708	148.34595	174.85130	206.83863	245.49897
54	131.13749	154.53806	182.84536	217.14637	258.77392
55	136.07162	160.94689	191.15917	227.91796	272.71262
56	141.15377	167.58003	199.80554	239.17427	287.34825
57	146.38838	174.44533	208.79776	250.93710	302.71566
58	151.78003	181.55092	218.14967	263.22928	318.85144
59	157.33343	188.90519	227.87566	276.07460	335.79402
60	163.05344	196.51688	237.99068	289.49795	353.58372

CHAP. V.

A collection of queſtions to exerciſe the ſeveral rules of arithmetic.

Queſt. 1.

A Merchant buys 890*C.* 3*q.* groſs weight of goods, but tare is to be ſubtracted at the rate of 14*lb.* to the hundred of groſs weight, how much neat weight will remain?

Groſs weight is the weight of the goods, together with the cheſt, bag, *&c.*

Tare is the cheſt, bag, but, caſk, *&c.* which contains the goods.

Neat weight is the weight of the goods alone.

$890\frac{3}{4} \times 8 = 7126$ ſtone, and 14*lb.* $= 1$ ſtone, and 112*lb.* $= 8$*ſt.*

then 8*ſt.* : 1 *ta.* : : 7126*ſt.* : $\frac{7126}{8} = 89\frac{3}{4}$ ſtone, the tare.

from 7126
take $89\frac{3}{4}$
remains $7036\frac{1}{4}$ the neat weight.

Queſt. 2.

A merchant buys 235*lb.* weight of goods, but is to have an additional allowance of 4*lb.* tret for every 100*lb.* weight of goods. Then how much weight does he receive of all?

Tret is the allowance made to the buyer, of ſo much per hundred, *&c.* over and above. And *Cloſ* another allowance of the ſame kind.

to 100
add 4

Say as, 100 : 104 : : 235 : 244.4 *lb.* Anſwer.

Queſt. 3.

If 200 *lb.* weight of goods coſt 3 *l.* at what price muſt a pound be ſold, to gain 10 *l.* in the hundred laid out?

100
10

100 : 110 : : 3 : 3.3 advanced price.
200 : 3.3 : : 1 : .0165 *l.* the price of 1 *lb.*
but .0165 *l.* = 3,96 pence, near 4 *d.* a pound.

Queſt. 4.

How much ſugar, at 8 *d.* a pound, may be bought for 10 *C.* weight of tobacco, at 3 *l.* the *C.*?

1 *C.* : 3 *l.* : : 10 *C.* : 30 *l.* the value of the tobacco.

then, ſince 8 *d.* is $\frac{1}{3}$ of a pound,

$\frac{1}{30}$ *l.* : 1 *lb.* : : 30 *l.* : 30 × 30 = 900 *lb.* of ſugar.

Queſt. 5.

Two merchants, A and B, *barter* with one another thus, A has 43 yards of broad cloth, worth 9 *s.* 2 *d.* per yard, but in *barter* he will have 11 *s.* a yard. B has ſhaloon, worth 2 *s.* per yard, which he charges at 2 *s.* 6 *d.* How much ſhaloon muſt A receive for his cloth; and what does he gain or loſe by the bargain?

In this queſtion, firſt find what the cloth comes to at the advanced price; then how much ſhaloon, at its advanced price, may be bought for that money; and laſtly the true value of both.

1 *y.* : 11 *s.* : : 43 *y.* : 473 *s.* the price of the cloth.
$2\frac{1}{2}$ *s.* : 1 *y.* : : 473 *s.* $189\frac{1}{5}$ yards of the ſhaloon received.
then 1 *y.* : $9\frac{1}{6}$ *s.* : : 43 *y.* : $394\frac{1}{6}$ = 394 *s.*—2 *d.* the value of the cloth.
and 1 *y.* : 2 *s.* : : $189\frac{1}{5}$ *y.* : $378\frac{2}{5}$ = 378 *s.*—$4\frac{3}{4}$ *d.* the value of the ſhaloon.
diff. 15 *s.*—$9\frac{1}{4}$ *d.*

So A loſes 15 *s.*—$9\frac{1}{4}$ *d.* by the bargain.

Queſt. 6.

A hath 100 pieces of ſilk worth 3 *l.* a-piece; but he charges them at 4 *l.* a-piece, and barters them with B for wool worth 7 *l.*—10 *s.* the C weight. How much wool muſt A receive from B for the ſilk, that both may be equal gainers?

In this queſtion the price of B's wool muſt be advanced in the ſame proportion as A's ſilk.

3 *l.* : 4 *l.* : : $7\frac{1}{2}$ *l.* : 10 *l.* the advanced price of the wool.
then 100 *l.* × 4 = 400 *l.* the value of the ſilk.
10 *l.* : 1 *C.* : : 400 *l.* : 40 *C.* the quantity of wool.

Queſt. 7.

How many ducats, at 5 *s.*—6 *d.* may be had for 250 dollars, at 4 *s.*—3 *d.* a-piece?

66 *d.* = a ducat, 51 *d.* = 1 dollar.
250 × 51 = 12750 *d.* the value of 250 dollars.
$\frac{12750}{66}$ = $193\frac{2}{11}$ ducats.

Queſt.

Queſt. 8.

A man would exchange 200*l.* for dollars, at 54*d.* ducats at 68*d.* and crowns at 73*d.* and would have 2 ducats and 3 crowns for 1 dollar. How many of each muſt he have?

$$\begin{array}{rcrcl}
 & & 54 & = & 1 \text{ dollar} \\
2 \times 68 & = & 136 & = & 2 \text{ ducats} \\
3 \times 73 & = & \underline{219} & = & 3 \text{ crowns} \\
 & & 409 & = & \text{ſum.}
\end{array}
\qquad
\begin{array}{r}
200 \\
\underline{20} \\
4000 \\
\underline{12} \\
48000d.
\end{array}$$

Now it is plain, as oft as 409 is contained in 48000, ſo often 1 dollar, 2 ducats, and 3 crowns muſt be taken.

$$\frac{48000}{409} = 117\tfrac{147}{409} \text{ the dollars,}$$

$$234\tfrac{294}{409} \text{ the ducats,}$$

$$352\tfrac{32}{409} \text{ the crowns.}$$

Queſt. 9.

A man buys 120 ſtaves at 3 a penny, and afterwards 120 more for 2 a penny; how muſt he ſell them out to loſe nothing?

$$\begin{array}{rrcl}
3) & 120 & = & 40d. \text{ for the firſt bargain.} \\
2) & \underline{120} & = & \underline{60d.} \text{ for the ſecond bargain.} \\
 & \underline{240} & & \underline{100}
\end{array}$$

100*d.* : 240*ſt.* : : 1*d.* : $2\frac{2}{5}$*ſt.* per penny;
that is, 12 ſtaves for 5 pence.

Queſt. 10.

A tradeſman begins the world with 1000*l.* and finds that he can gain 1000*l.* in 5 years by land trade alone, and that he can gain 1000*l.* in 8 years by ſea trade alone; and likewiſe that he ſpends 1000*l.* in $2\frac{1}{2}$ years by gaming. How long will his eſtate laſt, if he follows all three?

$\frac{1000}{5} = 200$ his gain by land trade in 1 year.

$\frac{1000}{8} = 125$ his gain by ſea trade in 1 year.

325 his whole gain.

$\frac{1000}{2\frac{1}{2}} = 400$ his loſs by gaming in 1 year.

the difference 75 his loſs by all three in 1 year.

then 75*l.* : 1*y.* : : 1000*l.* : $13\frac{1}{3}$ years his eſtate will laſt.

Queſt. 11.

There were 25 coblers, 20 taylors, 18 weavers, and 12 combers, ſpent 133 ſhillings at a meeting; to which reckoning 5 coblers paid as much as 4 taylors, 12 taylors as much as 9 weavers, and 6 weavers as much as 8 combers; how much did each company pay?

Find 4 numbers by the rule of three to expreſs theſe proportions, as theſe,

coblers,	taylors,	weavers,	combers,
5	4	3	4

that is, 5 coblers paid as much as 4 taylors, or 3 weavers, or 4 combers. Suppoſe each company

paid

paid 1 ſhilling, then, by the ſingle rule of falſe,

1 man in each company will pay	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$
which multiply by the number of men	25	20	18	12
produces	5	5	6	3

whoſe ſum is 19; then it will be

$$19 : 133 :: \left\{\begin{array}{l} 5 : 35s. \text{ for the coblers.} \\ 5 : 35\text{————taylors.} \\ 6 : 42\text{————weavers.} \\ 3 : 21\text{————combers.} \end{array}\right.$$

Queſt. 12.

There is an iſland 72 miles about, and two footmen ſet out together to travel round it the ſame way. A travels 9 miles a day, and B 7. To find the time they will be together again.

It is plain A will overtake B when he leads him the circumference of the iſland.

$$\begin{array}{l} \text{A——}9 \\ \text{B——}\underline{7} \\ \quad\quad\;\; 2 \text{ miles gained by A in 1 day.} \end{array}$$

then 2*m.* : 1*d.* : : 72*m.* : 36 days, the Anſwer.

Queſt. 13.

There is an iſland 73 miles round, and 3 footmen all ſtart together, to travel the ſame way about it. A travels 5 miles a day, B 8, and C 10. When will they all come together again?

$$\begin{array}{l} \text{B——}8 \\ \text{A——}\underline{5} \\ \text{B gains } \underline{\underline{3}} \text{ miles a day of A.} \end{array}$$

C—10

C—10
A— 5

C gains 5 miles a day of A.

then $3m. : 1d. : : 73m. : 24\frac{1}{3}$ days when A and B [meet.
and $5 : 1 : : 73 : 14\frac{3}{5}$ days when A and C [meet.

Now $24\frac{1}{3}$ days being the period of B's meeting with A, and $14\frac{3}{5}$ days, the period of C's meeting with A; and they can never meet but at the end of theſe periods. Therefore B and C can never both meet with A, but when ſome number of B's periods is equal to ſome number of C's periods. Therefore find two whole numbers which are in the ſame proportion, as $24\frac{1}{3}$ to $14\frac{3}{5}$, which will be 365 and 219. Therefore after 365 of B's periods, or 219 of A's; all three men will meet again, and not before, as 365 and 219 are in their leaſt terms. Therefore the time of meeting is $219 \times 24\frac{1}{3} = 5329$ days.

Queſt. 14.

A clock hath two hands or pointers, the firſt, A, goes round once in 12 hours, the ſecond, B, once in an hour. Now, if they both ſet forward together, in what time will they meet again?

Here A goes only $\frac{1}{12}$ of the circumference in an hour.

And B goes the whole circumference in an hour.
So B gains $\frac{11}{12}$ of A in that time.

Therefore $\frac{11}{12}C : 1h. : : 1C : \frac{12}{11}h. = 1\frac{1}{11}h. = 1h. : 5\frac{5}{11}m.$ the Anſwer.

Queſt.

Queſt. 15.

A greyhound is courſing a hare, which is 100 f her leaps before him; and the hare takes 4 leaps r every 3 leaps of the greyhound; but 2 of the reyhound's leaps are equal to 3 of the hare's. How any leaps muſt he take before he catch her?

gr. : 3 *ha.* : : 3 *gr.* : 4½ hare's leaps = 3 of the greyhound's.

Therefore, for every 3 leaps of the greyhound, the re loſes ½ of one of hers. Therefore

h. : 3 *gr.* : : 100 *l.* : 600 of the greyhound's leaps; the Anſwer.

Queſt. 16.

Four merchants, A, B, C, D, gain 2000 *l.* by de, whereof ½ of A's ſhare is equal to ¾ of B's, of C's, and ⅚ of D's. What ſhare had each?

Take a number at pleaſure, and divide in proportion to their ſhares, then proceed by the ſingle le of falſe.

$$\begin{array}{ll} A & 120 \\ B & 80 \\ C & 75 \\ D & 72 \\ \hline & 347 : 2000 : : \end{array} \left\{ \begin{array}{lll} 120 : 691\frac{223}{347} & \text{for} & A. \\ 80 : 461\frac{33}{347} & & B. \\ 75 : 432\frac{96}{347} & & C. \\ 72 : 414\frac{342}{47} & & D. \end{array} \right.$$

Queſt. 17.

Two merchants together make up a ſtock of 600 *l.* s ſtock continued in company 9 months, and B's , they gain 200 *l.* which they divide equally. ow much did each put in?

Since

Since the gains are equal, A's ſtock multiplied b his time 9, is equal to B's ſtock multiplied by his tim 11; therefore A's ſtock is to B's ſtock as 11 to 9.

$$\begin{array}{l}11\\ \underline{\;\;9}\\ 20 : 600 :: \left\{\begin{array}{l}11 : 330 \text{ A's ſtock.}\\ 9 : 270 \text{ B's ſtock.}\end{array}\right.\end{array}$$

Queſt. 18.

An apothecary has ſeveral ſimples, A hot in degrees, B hot in 1, C temperate, D cold in and he intends to make up 17 drams, to be in degree of cold. How much of each muſt be take

Put 1, 2, 3, &c. for the 4th, 3d, 2d, &c. d gree of cold, and proceed by the rule of alligatio

$$4\left\{\begin{array}{l|l|l}8 & 1 & 1\\ 6 & 1 & 1\\ 5 & 1 & 1\\ 3 & 1.2.4 & \underline{7}\end{array}\right.$$

$$10 : 17 :: \left\{\begin{array}{l}1 : 1\tfrac{7}{10} \text{ of A, B,}\\ 7 : 11\tfrac{9}{10} \text{ of D.}\end{array}\right.$$

Queſt. 19.

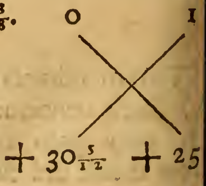
A factor delivers 6 French crowns and 4 doll for 53*s.*—6*d.* and at another time 4 French crow and 6 dollars for 49*s.*—10*d.* What was the val of each?

Suppoſe, by the double rule of falſe, there are French crowns;

then 4 doll. $= 53\frac{1}{2}$, 1 doll. $= 13\frac{3}{8}$.

and 4 cr. + 6 doll. $= 80\frac{1}{4}$

$$\begin{array}{r}\underline{49\frac{10}{12}}\\ 1 \text{ cr.} + 30\frac{5}{12}\end{array}$$



Aga

Again, ſuppoſe 1 crown, then 4 dollars = $47\frac{1}{2}$,
d 1 dollar = $11\frac{7}{8}$,
d 4 crowns + 6 dollars = $75\frac{1}{4}$

$$49\frac{10}{12}$$

$$2\ \text{cr.} + 25\frac{5}{12}$$

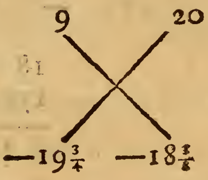
ff. cr. 5) $30\frac{5}{12}$ ($6\frac{1}{12}$ = 6*s.*—1*d.* the value of a crown.

d $4\frac{1}{4}$ or 4*s.*—3*d.* = a dollar.

Queſt. 20.

Three companies of ſoldiers paſſing by a ſhepherd, e firſt takes half his flock and half a ſheep, the cond takes half the remainder and half a ſheep, the ird takes half the laſt remainder and half a ſheep; ter which the ſhepherd had 20 ſheep remaining. ow many had he at firſt?

By the double rule of falſe, ppoſe two numbers, as follows.



ſup.	9	2 ſup.	20		20
	5		$10\frac{1}{2}$		9
rem.	4	1 rem.	$9\frac{1}{2}$		11
	$2\frac{1}{2}$		$4\frac{3}{4}$		$18\frac{3}{8}$
rem.	$1\frac{1}{2}$	2 rem.	$4\frac{1}{4}$		88
	$1\frac{1}{4}$		$2\frac{5}{8}$	$19\frac{3}{4}$	11
rem.	$\frac{1}{4}$	3 rem.	$1\frac{5}{8}$	$18\frac{3}{8}$	$4\frac{1}{8}$
	20		20	$1\frac{3}{8}$)	$202\frac{1}{8}$ (147
er.—	$19\frac{3}{4}$	2 er.—	$18\frac{3}{8}$		20
					167 ſheep, the Anſwer.

Queſt.

Queſt. 21.

There is a fiſh whoſe head is 9 inches in length, and his tail is as long as his head and half his body, and his body as long as his head and tail. How long was the fiſh?

1 ſup. body	0	2 ſup. body	1	0	1
head	9	head	9		
½ body	0	½ body	½		
tail	9	tail	9½	—18	—17½
body	18	body	18½		
	0		1		
1 er.	—18	2 er.	—17½		

18
17½
½) 18 (36 the body.
27
9
72 the whole fiſh.

9
18
tail 27

Queſt. 22.

There is an annuity of 75*l.* in *reverſion*, which is not to commence for ſeven years, and then it is to continue for 14 years; what is the preſent value of it at 4 per cent. compound intereſt?

Find

Find the preſent worth of the annuity of 1 *l.* for 4 years, and then the preſent worth of that ſum f money for 7 years, which multiply by the annuity.

By Tab. III. and IV. the preſent worth of 1 *l.* nnuity is $\frac{18.29191}{1.73167} = 10.56313$. Then by Tab. III. he preſent worth of 1 *l.* 7 years hence, is $\frac{1}{1.31593}$, his multiplied by 10.56313 gives $\frac{10.56313}{1.31593} =$ 8.02713, the preſent worth of 1 *l.* annuity in reverion; laſtly, 8.02713 × 75 = 602.035 *l.* the preſent alue required.

Queſt. 23.

There is a houſe rented at 25 *l.* a year for 21 years; ut the tenant is deſirous to pay 100 *l.* fine (or preſent oney). How much rent then muſt he pay, allowng 5 per cent. compound intereſt?

By Tab. III. and IV. the preſent worth of 1 *l.* nnuity for 21 years, is $\frac{35.71925}{2.78596}$; then ſay,

$\frac{5.71925}{2.78596}$ (*pr.*) : 1 *l.* (*an.*) : : 100 *l.* (*pr.*) : $\frac{278.596}{35.7192}$ = 7.7997 *l.* the rent anſwering the fine of 100 *l.*

hen from 25.0000

take 7.7997

remains 17.2003 the rent ſought.

BOOK II.

The Theory of Numbers.

CHAP. I.

Numbers produced by addition, ſubtraction, multiplication, and diviſion. Of odd and even numbers. Prime and compoſite numbers. Numbers that are prime to one another; and ſuch as meaſure others. Powers and products of ſquares, cubes, &c.

PROP. I.

If A *and* B *be two numbers; then* A *added to* B *is the ſame ſum as* B *added to* A.

FOR if both of them be reſolved into its units, and placed in a right line, they will count to the ſame number, begin at which end you will.

B, 5.	A, 3.
A, 3.	B, 5.
8	8

Cor. *Hence if ſeveral numbers are to be added together, they will amount to the ſame ſum, whatever order they are placed in. Or if ſeveral numbers are to be ſubtracted, it is the ſame thing, whether they be ſubtracted one after another, or all together.*

PROP. II.

If two numbers A, B, *are to be multiplied together; the product of* A *multiplied by* B, *is equal to the product of* B *multiplied by* A.

A, 3.	B, 5.
B, 5.	A, 3.
15	15

For A times 1 = to the units in A = 1ce A.

And A times B = B times that product, that is = B times A.

Cor. 1. *If ſeveral numbers are to be multiplied together; they will make the ſame product, in whatever order they are multiplied.*

Cor. 2. *If ſeveral numbers,* A, B, C, *are to be multiplied together; it is the ſame thing, whether* A *be multiplied by the product of the reſt* BC; *or* A *be multiplied firſt by* B, *and the product by* C; *and ſo on.*

For by either method the product will be ABC.

Cor. 3. *And on the contrary, if a number* ABC *is to be divided by another* BC; *it is the ſame thing whether* ABC *is divided by* BC *at once; or it be divided firſt by one factor* B, *and then the quotient by another factor* C, *and ſo on.*

For $\frac{ABC}{BC} = A$ (Ax. 8); and $\frac{ABC}{B} = AC$ (Ax. 8), and then $\frac{AC}{C} = A$ (Ax. 8), that is, $= \frac{ABC}{BC}$.

PROP. III.

If the number S, *be made up of the parts* A, B, C; *the product of* S, *by any number* M, *is equal to the ſum of the ſeveral products, made by multiplying ſeparately, each particular part* A, B, C, *by* M.

For $M \times S = M \times \overline{A+B+C}$ (Ax. 4) $= \overline{A+B+C} \times M$ (Pr. 2). But $\overline{A+B+C}$ times M is nothing elſe but taking M as oft as there are units in $A+B+C$; that is, as oft as there are units in A, and alſo as oft as there are units in B, and alſo in C; and that is, $AM + BM + CM$. Therefore $MS = AM + BM + CM =$ (Pr. 2) $MA + MB + MC$.

$$\begin{array}{rl} & A,\ \ B,\ \ C, \\ S, 13 = & 3+\ 4+\ 6 \\ M, 5 & \qquad\quad\ 5 \\ \hline 65 = & 15+20+30 \\ \hline \end{array}$$

Cor 1. *If* D *be the difference of two numbers* A *and* B; *then* D *multiplied by any number* M, *is equal to the difference of the products, of* A *by* M, *and* B *by* M.

$$\begin{array}{rl} & A,\ \ B, \\ D,\ 2 = & 9 - 7 \\ M,\ 5 & \qquad 5 \\ \hline 10 = & 45-35 \\ \hline \end{array}$$

Cor. 2. *If* $S = A+B+C$, *and* $M = F+G$; *then the product of the wholes*, $S \times M =$ *ſum of the products of all the parts of one, by all the parts of the other*, $FA+FB+FC+GA+GB+GC$.

For $SM = MA+MB+MC = \overline{F+G} \times A + \overline{F+G} \times B + \overline{F+G} \times C = FA + GA + FB + GB + FC + GC$.

PROP. IV.

The quotient ariſing by dividing the ſum of two or more numbers (A+B), *by any diviſor* D; *is equal to the ſum of the quotients ariſing by dividing the parts* A, B, *ſeparately by the ſame diviſor. That is,*

$$\frac{A+B}{D} = \frac{A}{D} + \frac{B}{D}. \qquad \frac{A+B}{D}.\ \frac{A}{D} + \frac{B}{D}$$

$$\frac{9}{3} = \frac{3}{3} + \frac{6}{3}.$$

For

For let the whole be called S, then ſince A + B = S, any part of A, together with the ſame part of B = the like part of S (Ax. 5); that is,

$$\frac{A}{D} + \frac{B}{D} = \frac{S}{D} = \frac{A+B}{D}.$$

PROP. V.

If any multitude of even numbers be added together, the ſum will be even.

For ſince an even number may be divided into two equal whole numbers, let theſe numbers be 2A, 2B, 2C, *&c.* then the ſum will be 2A + 2B + 2C, *&c.*; and the half is A + B + C, *&c.* a whole number (Def. 14).

Cor. *If an even number be taken from an even number, the remainder is even.*

PROP. VI.

If an even multitude of odd numbers be added together, their ſum is even.

For theſe odd numbers may be repreſented by 2A + 1, 2B + 1, *&c.* And the ſum of 2A and 2B, *&c.* is an even number (Pr. 5). And an even number of units, is an even number. Therefore their ſum is an even number.

$$\begin{array}{r} 9 \\ 7 \\ 5 \\ 3 \\ \hline 24 \\ \hline \end{array}$$

Cor. *An odd multitude of odd numbers added together makes an odd number.*

$$\begin{array}{r} 3 \\ 5 \\ 7 \\ \hline 15 \end{array}$$

PROP. VII.

If there be taken an even number from an odd number, or an odd number from an even number; the remainder is odd.

For let 2A be an even number, then ſince 2A taken from an even number, leaves an even number (Cor. Pr. 5); therefore 2A taken from that even number and 1 more, will leave 1 more; that is, an odd number will remain: and alſo 2A+1 (an odd number) taken from that even number, 1 leſs will remain; that is, an odd number remains.

$$\begin{array}{cc} 7 & 10 \\ 4 & 7 \\ \hline 3 & 3 \\ \hline \end{array}$$

Cor. *If an odd number be taken from an odd number, the remainder is even.*

PROP. VIII.

If an odd number be multiplied by an odd number, the product will be odd.

For the product conſiſts of an odd number taken an odd number of times, and therefore is odd (Cor. Pr. 6).

Cor. 1. *If an odd number be divided by an odd number, the quotient will be odd.*

Cor. 2. *Every number is odd, which meaſures an odd number. Or an even number cannot meaſure an odd number.*

PROP. IX.

If an even number be multiplied by any number, even or odd, the product will be even.

For the product conſiſts of the even number taken ſo many times as there are units in the multiplier, and therefore will be even (Pr. 5).

$$\begin{array}{cc} 6 & 6 \\ 2 & 3 \\ \hline 12 & 18 \end{array}$$

Cor. 1. *If an even number be divided by an odd number, the quotient will be even.*

Cor. 2. *If an odd number meaſures an even number, it ſhall alſo meaſure half of it.*

Cor. 3. *If an odd number* A, *be prime to any number* B, *it ſhall be prime to its double* 2B.

For no even number can meaſure A (Cor. 2. Pr. 8); and an odd number which meaſures 2B, will alſo meaſure B (Cor. 2); and then A and B would not be prime.

Cor. 4. *A number which is prime to any in a double progreſſion, is prime to them all.*

PROP. X.

If there be two numbers, A *the greater, and* B *the leſſer, and the leſſer* B *be continually taken from the greater* A; *and the remainder* C *from* B; *and the next remainder* D *from* C; *and the next remainder* E *from* D, *and ſo on, till nothing remains. I ſay, the laſt number* E *that remained, will be the greateſt common meaſure of the numbers* A *and* B.

For E meaſures D, ſince 0 remains; and it alſo meaſures C which is ſome multiple (once or oftener) of D with E over (Ax. 10, 11). For the ſame reaſon it meaſures B, which is a multiple of C with D over; and laſtly, it meaſures A, which is a multiple of B with C over. Therefore E is a common meaſure.

```
27)75(2
   54
   --
   21)27(1
      21
      --
       6)21(3
         18
         --
          3)6(2
            6
            -
            0
```

And it is the greateſt; for if there was one F greater than E, then ſince F is ſuppoſed to meaſure A and B, it alſo meaſures C (Ax. 11); and for the ſame reaſon ſince F meaſures both B and C, it alſo meaſures D; and ſince it meaſures both C and D, it alſo meaſures E, the greater the leſs; which is abſurd.

Cor. 1. *If there be two numbers given, and the greater be divided by the leſs; and then the leſſer divided by the remainder; and this remainder by the next remainder, and ſo on, ſtill making the laſt remainder a diviſor. By proceeding thus, if* 1 *remains at laſt, then the two given numbers are prime to one another.*

Ex. 28 *and* 19.

```
19)28(7
   19
  ---
    9)19(2
      18
      --
       1)9(9
         9
         -
         0
```

Cor. 2. *If a number* F *meaſures ſeveral numbers, it will alſo meaſure their greateſt common meaſure* E.

This is plain from the demonſtration of this prop. For if F meaſures A and B, it alſo meaſures E, the greateſt common meaſure of theſe two quantities. And if F meaſures E and a third number: it meaſures their greateſt common meaſure; that is, it meaſures the greateſt common meaſure of all the three numbers; and ſo on.

PROP. XI.

If the number N *be the leaſt, which ſeveral other numbers meaſure; theſe numbers ſhall only meaſure all the multiples of* N, *but no other number beſides.*

For ſince they meaſure N, they ſhall alſo meaſure 2N, 3N, *&c.* or in general rN (Ax. 10), r being any number.

But they can meaſure no other number as P; for take rN the neareſt multiple to P; then ſince they meaſure both rN and P, they will alſo meaſure their difference (Ax. 9). But that difference is leſs than N;

therefore N is not the leaſt number which they meaſure; contrary to the hypotheſis.

Cor. *If ſeveral numbers meaſure any number; the leaſt which they meaſure ſhall alſo meaſure the ſame number; that is, their leaſt common dividend, ſhall alſo meaſure it.*

PROP. XII.

If N *be the leaſt number (or the leaſt common dividend) that ſeveral prime numbers,* A, B, C, *meaſure: no other prime* D *ſhall meaſure the ſame.*

For if the prime D meaſures it, then D muſt be a factor in N, as well as A, B, C, are; and then N would not be the leaſt number, which A, B, C, meaſure.

PROP. XIII.

If two numbers, A, B, *be prime to one another; the number* C, *which meaſures one of them* A, *will be prime to the other* B.

A, 9. B, 4.
C, 3. D . .

For if C and B be not prime to one another, let D meaſure both. But becauſe D meaſures C, it alſo meaſures A (Ax. 10); conſequently A and B are not prime to one another: contrary to the hypotheſis.

PROP. XIV.

If two numbers, A, B, *be prime to any number* C, *their product* AB *will be prime to it.*

A, 5. C, 8.
B, 3.
AB, 15.

For no numbers can meaſure AB and C, but ſuch (prime) factors as A, B, and C, are made up of. But in A and C there are none that are common to both; becauſe A and C are prime to one another; nor in B and C for the ſame reaſon. Therefore let A be denoted by the factors P and Q; that is, let $A = PQ$, and $B = RS$; and alſo $C = EF$; then $AB = PQRS$. Now it is evident that PQRS and EF are prime to one another, becauſe there is no factor common to both, therefore their equals AB and C are prime to one another.

Cor. 1. *If ſeveral numbers, how many ſo ever,* A, B, C, D, *&c. be each of them prime to any number* F; *their product,* ABCD *&c. will alſo be prime to the ſame* F.

For (by this prop.) AB and C are both prime to F; therefore ABC is prime to F. Again, ABC and D are both prime to F; therefore ABCD is prime to F.

Cor. 2. *If one number* A *be prime to another* F; *its ſquare, cube, or any power* A^n, *ſhall alſo be prime to the ſame number* F.

This is evident from Cor. 1. by ſuppoſing A, B, C, D, *&c.* all equal.

PROP. XV.

If two numbers, A, B, *be prime to one number* C, *and alſo to another* D; *their products* AB *and* CD *ſhall alſo be prime to one another.*

For AB is prime to C, and alſo to D (Pr. 14); therefore AB is prime to CD.

Cor. 1. *If ſeveral numbers,* A, B, C, D, *&c. be prime to each of the numbers* F, G, H, I, *&c. then their products,* ABCD, *and* FGHI, *&c. will be prime to one another.*

For (by this prop.) AB is prime to FG, and ſince AB and C are prime to FG and H; therefore ABC is prime to FGH. Again, ſince ABC and D, are prime

prime to FGH and I, therefore ABCD is prime to FGHI, *&c.*

Cor. 2. *If two numbers,* A, F, *be prime to one another; then any power of one* A^m, *will be prime to any power of the other* F^n.

This follows from Cor. 1. by ſuppoſing B, C, D, *&c.* = A, and G, H, I, *&c.* = F.

PROP. XVI.

If two numbers, A, B, *be prime to one another, and each of them meaſures ſome number* D; *then their product* AB *ſhall meaſure the ſame number* D.

For ſince A and B are prime to one another, there is no factor common to both; and ſince they both of them meaſure D, therefore they both are factors in D. Therefore let D = ABF, then A and B meaſure ABF, and it appears that AB meaſures ABF or D.

Cor. *If ſeveral numbers* A, B, C, *&c. be prime to one another; and each of them meaſures another* D; *then their product* ABC, *&c. ſhall meaſure the ſame number* D.

PROP. XVII.

If two numbers, A, B, *be prime to one another; their ſum* A + B *will be prime to either of them.*

If you deny it, let D be the common meaſure of A and A + B, then it will meaſure the reſidue B (Ax. 11). Therefore A, B, are not prime: againſt the hypotheſis.

Cor. *If a number be prime to one of its parts; it is alſo prime to the remaining part.*

PROP. XVIII.

If the number A *be prime to* B; *then* A *ſhall meaſure no multiple of* B, *leſs than* A × B; *or whoſe multiplier is leſs than* A.

Let r be any number, and ſuppoſe r times B, or rB to be ſome multiple of B. Now the numbers A, B, being prime to one another, there is no factor common to both A and B: therefore if A meaſures rB, it muſt meaſure r alone. But it can never meaſure r leſs than itſelf: therefore r muſt be equal to A, or to ſome multiple of A.

Cor. *If* A, B, *be prime to one another*; *then* A *ſhall meaſure all the multiples of* AB, *and no other multiples of* B *beſides.*

PROP. XIX.

More prime numbers may be found, than any propoſed multitude, A, B, C.

Let N be the leaſt number which A, B, C, meaſure; then if $N + 1$ be a prime number, another prime is found. But if it is a compoſite number, then ſome other prime, as D, meaſures it, and ſo the prime D is found.

PROP. XX.

Let M *be any number,* 1, 2, 3, 4, *&c. then* $M \times 6 - 1$, *and* $M \times 6 + 1$, *conſtitute a ſeries, which contains all prime numbers above* 3.

For thoſe left out of the ſeries are no primes. For $6M + 2$, and $6M - 2$, are not primes, being diviſible by 2. Alſo $6M + 3$, and $6M - 3$, being diviſible by 3, are no primes. But theſe are all the numbers left out.

PROP.

PROP. XXI.

No number is a ſquare number, that conſiſts not of two equal factors; nor a cube, that conſiſts not of three equal factors: and ſo for higher powers.

This appears from the definition of ſquare and cube numbers; and other higher powers. For a ſquare requires to have two equal multipliers, or elſe a ſquare could not be produced; and a cube muſt have three. And ſo on.

Cor. 1. *There is no ſuch thing as the exact ſquare root of* 2, 3, 5, 6, 7, 8, 10, *&c.* *Nor the exact cube root of* 2, 3, 4, 5, 6, 7, 9, *&c.*

For there are no ſuch factors to be found in theſe numbers, and infinite others. For example, the two factors in 2, are 1 and 2; in 3, 1 and 3; in 6, 2 and 3, *&c.* and therefore they are no ſquares. Again, the three factors in 2, are 1, 1, and 2; in 3, are 1, 1, and 3; in 12, they are 2, 2, and 3, *&c.* which are no cubes.

Cor. 2. *All numbers are ſurds, whoſe roots are not ſome of the natural ſeries,* 1, 2, 3, 4, 5, 6, *&c.* ad infinitum.

PROP. XXII.

The ſum of two numbers differing by a unit, is equal to the difference of their ſquares.

Let N and $N+1$ be the numbers;

multiply - - $N+1$
by - - $N+1$

the ſquare of $N+1$ - - $NN+N+N+1$
the ſquare of N - - - NN ſubtract

remains - - - - $N+\overline{N+1}$,
the ſum of the two numbers.

Cor.

Cor. *The differences of the ſquares of* 0, 1, 2, 3, 4, *&c. proceed by the odd numbers,* 1, 3, 5, 7, *&c.*

PROP. XXIII.

The ſum of any number of terms (n), of the ſeries of odd numbers 1, 3, 5, 7, *&c. is equal to the ſquare (nn) of that number.*

Set down the ſeries of ſquares, and their differences, according to Cor. Pr. 21. and by adding them we ſhall have

$$\begin{array}{|cccccccc} 0 & 1^2 & 2^2 & 3^2 & 4^2 & 5^2 & 6^2 & 7^2. \\ & 1 & 3 & 5 & 7 & 9 & 11 & 13 \end{array}$$

$0+1$ or the ſum of 1 term $= 1^2$ or 1,
$1+3$ or the ſum of 2 terms $= 2^2$ or 4,
$4+5$ or the ſum of 3 terms $= 3^2$ or 9,
$9+7$ or the ſum of 4 terms $= 4^2$ or 16,
$16+9$ or the ſum of 5 terms $= 5^2$ or 25, and ſo on.

Whence it is plain, let n be what number you will, the ſum of n terms will be $= nn$.

PROP. XXIV.

The ſum of two numbers multiplied by their difference, is equal to the difference of their ſquares.

Let the numbers be A, E; which multiplied together will produce AA—EE (Prop. 3, and Cor. 1).

$$\begin{array}{lll} A+E & & \\ A-E & & \\ \hline AA+AE & & \\ & -AE & -EE \\ \hline AA & & -EE \\ \hline \end{array}$$

Cor. *The difference of the ſquares of two numbers, is diviſible, by either the ſum or difference of theſe numbers.*

PROP. XXV.

The ſum of two cube numbers is diviſible by the ſum of their roots. Or the ſum of any two numbers will meaſure the ſum of their cubes.

Let the numbers be A, E; multiply $AA - AE + EE$ by $A + E$

$$\begin{array}{r} AA - AE + EE \\ A + E \\ \hline A^3 - A^2E + AEE \qquad\qquad \\ + A^2E - AEE + E^3 \\ \hline A^3 \;\; - \;\; - \;\; - \;\; - \;\; + E^3 \\ \hline \end{array}$$

(by Pr. 3. and Cor.) product, $A^3 \; - \; - \; - \; - \; + E^3$

Therefore $A^3 + E^3$ is diviſible by $A + E$ (Ax. 8).

PROP. XXVI.

The difference of any two numbers will meaſure the difference of their cubes.

f A, E, be the umbers; mult. $AA + AE + EE$ by $A - E$

$$\begin{array}{r} AA + AE + EE \\ A - E \\ \hline A^3 + A^2E + AEE \qquad\qquad \\ - A^2E - AEE - E^3 \\ \hline A^3 \;\; - \;\; - \;\; - \;\; - E^3 \\ \hline \end{array}$$

the product (Pr. 3) $A^3 \; - \; - \; - \; - E^3$

Therefore the product $A^3 - E^3$ is diviſible by $A - E$ Ax. 8).

PROP. XXVII.

The product of two ſquare numbers, is a ſquare number; and of two cube numbers, a cube number: and ſo on.

For $AA \times BB = AABB = AB \times AB$, the ſquare f AB.

Alſo $A^3 \times B^3 = AAABBB = ABABAB$, the cube f AB, and ſo of others.

Cor.

Cor. *If a ſquare number divide or meaſure a ſquare number; or a cube number a cube number;* &c. *the quotient will be a ſquare, or cube number,* &c. *reſpectively.*

For $\frac{AABB}{BB} = AA$ (Ax. 8), the ſquare of A,

and $\frac{A^3B^3}{B^3} = A^3$, the cube of A; *&c.*

PROP. XXVIII.

Every power of a ſquare number is a ſquare number; and every power of a cube number is a cube number: and ſo on.

For AA or A^2 is the ſquare of A; and $\overline{AA}^2$ or A^4 is the ſquare of AA. $\overline{AA}^3$ or A^6 is the ſquare of A^3. $\overline{AA}^5$ or A^{10} is the ſquare of A^5, *&c.*

Again, $\overline{AAA}^2$ or A^6 is the cube of AA: and $\overline{AAA}^3$ or A^9 is the cube of A^3: alſo $\overline{AAA}^4$ or A^{12} is the cube of A^4, *&c.* and ſo of others.

* * *

CHAP. II.

f proportional numbers, and thoſe in geometrical progreſſion. Mean proportionals. Like plane and ſolid numbers.

PROP. XXIX.

f four quantities, A, B, C, D, *are proportional; the product of the means is equal to the product of the extremes,* AD = BC.

FOR ſince A : B : : C : D; then $\frac{A}{B} = \frac{C}{D} = r$ Def. 27); and $A = Br$, $C = Dr$ (Ax. 4, 5). Whence D = BrD, and BC = BDr (Ax. 4); therefore D = BC (Ax. 1).

Cor. 1. *The firſt is to the third, as the ſecond to the urth;* A : C : : B : D.

For ſince AD = BC, then $\frac{AD}{CD} = \frac{BC}{CD}$ (Ax. 5), at is, $\frac{A}{C} = \frac{B}{D}$, or A : C : : B : D.

Cor. 2. *The ſecond is to the firſt, as the fourth the third, or* B : A : : D : C.

For ſince BC = AD, $\frac{BC}{AC} = \frac{AD}{AC}$ (Ax. 5), that is, $= \frac{D}{C}$.

Cor. 3. A : B : : A + C : B + D : : A — : B — D.

For ſince $\frac{A}{B} = r$, and $A = Br$, $C = Dr$; then $+ C = Br + Dr = \overline{B + D} \times r$ (Ax. 2); there-e $\frac{A + C}{B + D} = r = \frac{A}{B}$ (Ax. 1).

In

In like mannner $A - C = Br - Dr = \overline{B - D} \times r$, and $\frac{A - C}{B - D} = r = \frac{A}{B}$, whence A : B : : A + C : B + D : : A — C : B — D (Def. 27).

Cor. 4. *If any like parts or multiples of* A *and* B *be denoted by* r, *then* A : B : : rA : rB.

For $\frac{rA}{A} = r = \frac{rB}{B}$; therefore rA : A : : rB : B (Def. 27); and rA : rB : : A : B (Cor. 1).

Cor 5. *If* A : B : : C : D; *then* D *can only be a whole number, when* A *meaſures the product* BC.

For AD = BC, and $D = \frac{BC}{A}$ (Ax. 5).

Cor. 6. *If three numbers*, A, B, C, *are in continual proportion; then the ſquare of the mean is equal to the product of the extremes*, BB = AC.

This is plain, by ſuppoſing the two middle terms to be equal; and then the fourth becomes the third.

PROP. XXX.

If two numbers, A, B, *are prime to one another, no other numbers can be found in that proportion, but what are ſome multiple of* A *and* B.

Let C, D be others in the ſame proportion, then ſince A : B : : C : D, then AD = BC (Pr. 29); and $D = \frac{BC}{A}$ (Ax. 5). Now D can only be a whole number, when A meaſures BC (Cor. 5. Pr. 29). But A being prime to B, there is no factor common to both; therefore if A meaſures BC, it muſt meaſure C alone; that is, C is ſome multiple of A, and conſequently D is ſome multiple of B.

A,	5.	B,	3.
C,	10.	D,	6.

Cor. 1. *Numbers*, A, B, *that are prime to one another, are the leaſt of all numbers in the ſame proportion.*

Cor. 2. *Numbers*, A, B, *that are the leaſt in a given proportion, are prime to one another.*

Fo

For if they are not prime, they may be reduced to leſs numbers in the ſame proportion.

PROP. XXXI.

If there be a ſeries of numbers, A, B, C, D, *(greater than* 1) *in continual proportion; and the extremes* A, D *prime to one another; there cannot be found another number in the ſame proportion.*

A . B : C : D : E
8 12 18 27

Let E be another term, if poſſible; then A : B : : D : E; and A : D : : B : E (Cor. 1. Pr. 29); but A, D, are prime to one another by ſuppoſition; therefore B, E are multiples of A and D (Pr. 30.); therefore A meaſures B. And ſince A meaſures B, therefore B meaſures C, and C meaſures D (Def. 27); therefore A meaſures D (Ax. 10). Therefore A and D are not prime to one another: contrary to the hypotheſis.

Cor. 1. *If two numbers (greater than* 1) *be prime to one another, there cannot be found a third number in the ſame proportion.*

PROP. XXXII.

If there be ſeveral numbers, A, B, C, D, *in continual proportion, and the extremes* A, D *prime to one another; then theſe numbers are the leaſt of all numbers in the ſame proportion. And the contrary.*

A : B : C : D
8 12 18 27
E F G H

For let E, F, G, H, be other numbers in the ſame proportion. Then ſince A : B : : E : F, therefore A : E : : B : F : : C : G : : D : H (Cor. 1. Pr. 29). And A : D : : E : H (ib.). But A and D are prime to one another, by ſuppoſition, and therefore the leaſt in that proportion (Cor. 1. Pr. 30.) therefore E, H are greater than A, D; and all of them, A, B, C, D, are leſs than E, F, G, H.

On the contrary, if A, B, C, D are the leaſt in that proportion, then A and D are prime to one another. For if you ſuppoſe E, H to be prime to one another, then E, F, G, H will be the leaſt in that proportion: contrary to the hypotheſis.

Cor. *If* A, B, C, D *be in continual proportion, and the extremes* A, D *prime to one another; then all other numbers,* E, F, G, H, *in the ſame proportion, muſt be ſome multiple of* A, B, C, D.

For it being A : D : : E : H, and A, D being prime to one another (this Prop.), E, H muſt be ſome multiple of A, D (Pr. 30). Therefore E, F, G, H are multiple of A, B, C, D.

PROP. XXXIII.

In a ſeries of numbers the leaſt in continual proportion; if there be three numbers, the extremes are ſquares; if four, cubes; and in general if there be n numbers, the extremes are the $\overline{n-1}^{\text{th}}$ *powers of two numbers, which are the leaſt in that proportion.*

For let A, B — A, 4 : B, 6 : C, 9.
be the leaſt in — A, 8 : B, 12 : C, 18 : D, 27.
that proportion,
then AA, AB, BB are continual proportionals, in the ſame proportion of A to B (Cor. 4. Pr. 29). And ſince A, B are prime to one another (Cor. 2. Pr. 30), AA and BB will be prime to one another (Cor. 2. Pr. 15); therefore AA, AB, and BB are the leaſt in the proportion of A to B (Pr. 28); where the extremes are ſquares.

For the ſame reaſon A^3, A^2B, AB^2, B^3 are the leaſt in continual proportion of A to B; where the extremes are the cubes of A and B. And ſo of others.

Cor. 1. *Between two ſquare numbers there is one mean proportional; between two cubes, two means. And in general, between two* n^{th} *powers, there are* $\overline{n-1}$ *means.*

For

For between AA and BB there is the mean AB, and between the cubes A^3 and B^3 are the means A^2B, AB^2. And ſo forward.

Cor. 2. *In a ſeries of numbers, the leaſt in continual proportion; two numbers, which are the leaſt in that proportion, meaſure all the means.*

For both A and B meaſure AB, the mean of three proportionals. Alſo both A and B meaſure A^2B and AB^2, the two means of four proportionals. And ſo on.

Cor. 3. *If there be three numbers the leaſt in continual proportion, the ſum of any two is prime to the other.*

For in the numbers AA, AB, BB no number can meaſure any one of them, and alſo the ſum of the other two.

PROP. XXXIV.

In a ſeries of numbers in continual proportion, if the firſt meaſure not the ſecond; neither ſhall any one meaſure any other.

A :	B :	C :	D :	E
16	24	36	54	81
	F	G	H	I

I ſay, for example, B does not meaſure E. For, as E is the fourth from B, take the four numbers, F, G, H, I, the leaſt in that proportion; then B : C : : F : G; therefore B : F : : C : G : : D : H : : E : I (Cor. 1. Pr. 29); and B : E : : F : I (ib.). But F, I are prime to one another (Pr. 32). Therefore F does not meaſure I (except F be 1), and conſequently B does not meaſure E.

Here F is not 1, for A : B : : F : G. If F was 1, F would meaſure G, and A meaſure B; contrary to the hypotheſis.

Cor. *If the firſt meaſure the laſt, it ſhall alſo meaſure the ſecond.*

For if you ſay it meaſures not the ſecond, then it ſhall not meaſure the laſt: againſt the hypotheſis.

PROP. XXXV.

If between two numbers there fall ſeveral mean proportionals; ſo many ſhall alſo fall between two other numbers, having the ſame proportion.

27 : 36 : 48 : 64
54 : 72 : 96 : 128

For ſuppoſe the four quantities, A^3, A^2B, AB^2, B^3, to be the leaſt in that proportion. Then, ſince A^3 and B^3 are prime to one another (Pr. 32), all other numbers, in that proportion, muſt be ſome multiple thereof (Cor. Prop. 32). Take any number, r, and let rA^3, rB^3 be the extremes; then rA^2B and rAB^2 will be the means (Cor. 4. Pr. 29). And the like for any other number of mean proportionals.

PROP. XXXVI.

If between two numbers, prime to one another, there fall ſeveral mean proportionals; ſo many ſhall alſo fall between either of them and a unit. And the contrary.

For in the four proportional numbers, A^3, A^2B, AB^2, B^3, there are two means, A^2B, AB^2, between A^3 and B^3, which ſuppoſe to be prime. Now put $A = 1$, then the four proportionals become 1, B, B^2, B^3; where B and BB are the two means. Again, put $B = 1$, then the four proportionals become A^3, A^2, A, 1; where A and AA are the two means.

And on the contrary, between A^3 and B^3 two mean proportionals fall (Cor. 1. Prop. 33). And ſo of others.

PROP. XXXVII.

If there be a ſeries of numbers continually proportional; and the firſt be a ſquare, the third ſhall be a ſquare. If the firſt be a cube, the fourth ſhall be a cube. If the firſt be a fourth power, the fifth ſhall be a fourth power.

Let

Let AA : B : C; then AAC = BB (Cor. 6. Pr. 29), and $C = \frac{BB}{AA}$; therefore C is a ſquare (Cor. Pr. 27).

Again, let A^3 : B : C : D; then BB = A^3C (Cor. 6. Pr. 29), and $B^3 = A^3BC$ (Ax. 4), and BC $= \frac{B^3}{A^3}$ (Ax. 5). Alſo $A^3D = BC$ (Pr. 29), and conſequently $A^3D = \frac{B^3}{A^3}$, and $D = \frac{B^3}{A^6}$; therefore D is a cube (Cor. Pr. 27).

Likewiſe if A^4 : B : C : D : E. Then $C = \frac{BB}{A^4}$, and $A^4E = BD = CC = \frac{B^4}{A^8}$, and $E = \frac{B^4}{A^{12}}$, a fourth power, whoſe root is $\frac{B}{A^3}$. And ſo on.

PROP. XXXVIII.

In a ſeries of numbers continually proportional, beginning at 1; *any prime number, that meaſures the laſt, ſhall meaſure all the reſt after the unit.*

Let the ſeries be 1 : A : AA : A^3 : A^4 : A^5; and let the prime P meaſure A^5; then if you deny that P meaſures A, then P is prime to A, and therefore it is prime to A^5 (Cor. 2. Pr. 14); contrary to the hypotheſis.

Cor. 1. *If any number meaſures the laſt and not the firſt (after the unit), it is a compoſite number.*

Cor. 2. *If the firſt term (after the unit) be a prime, no other prime ſhall meaſure the laſt.*

Cor. 3. *In a ſeries of continual proportionals from* 1, *if the term next* 1 *be a prime; no number ſhall meaſure the laſt, but thoſe in that ſeries.*

For A, A^2, A^3, *&c*, all meaſure A^5; and no others do, becauſe A is a prime number (Cor. 2. Pr. 14).

PROP. XXXIX.

If four numbers are proportional, and three of them ſquares, the fourth is a ſquare; and if three of them be cubes, the fourth is a cube; and ſo on.

Suppoſe AA : BB : : CC : D, then AAD $=$ BBCC (Pr. 29), and D $= \frac{BBCC}{AA}$ (Ax. 5); therefore D is a ſquare (Cor. Pr. 27).

Again, $A^3 : B^3 :: C^3 : D$; then $A^3D = B^3C^3$, and D $= \frac{B^3C^3}{A^3}$, and D is a cube (Cor. Pr. 27).

Cor. *Hence the proportion of a ſquare number to one not ſquare, cannot be expreſſed by two ſquare numbers; neither can the proportion of a cube number to one not cube, be expreſſed by two cube numbers.*

PROP. XL.

The product of two like plane numbers is a ſquare number; and of three like ſolid numbers, a cube; &c.

Let ab, AB be two like plane numbers; then ſince $a : A :: b : B$, we ſhall have $aB = Ab$ (Pr. 29). But $ab \times AB = aBbA = Ab \times bA$, or $aB \times aB$, a ſquare, whoſe root is aB or Ab.

Again, let abc, ABC, EFG, be three like cube numbers; then ſince $a : b :: A : B$, and $a : c :: E : G$; alſo $B : C :: F : G$; therefore $aB = bA$, $aG = cE$, and $CF = BG$; then $abc \times ABC \times EFG = a \times bA \times cE \times BG \times CF = a \times aB \times aG \times BG \times BG = a^3B^3G^3$, a cube, whoſe root is aBG or aCF, or bAG, or bCE, or cAF, or cBE.

Cor. 1. *If the product of two numbers be a ſquare; or of three numbers a cube; they are ſimilar plane or ſolid numbers.*

For if it is not $a : A :: b : B$, then it is not $aB = Ab$, but rather $aB = Db$, and then we ſhould not have $aB \times bA$, or $aB \times aB$, a ſquare number (but rather $aB \times bD$); contrary to the hypotheſis.

Cor.

Cor. 2. *Two diſſimilar plane numbers cannot produce a ſquare.*

For a ſquare is only produced from ſimilar numbers (Cor. 1).

Cor. 3. *If the ſquare of a number*, A, *be a cube, the number itſelf*, A, *is a cube.*

For A^3 is a cube by nature, and A^2 is a cube by ſuppoſition; therefore $\frac{A3}{A^2}$ or A is a cube (Cor. Pr. 27).

Cor. 4. *If any number meaſure or divide a ſquare number; the quotient will be a plane number, ſimilar to the diviſor.*

PROP. XLI.

Between two like plane numbers there is one mean proportional; between two like ſolid numbers there are two means; and ſo on.

Let ab, AB be two like plane numbers; then

theſe numbers are proportional $\left\{ \begin{array}{lll} a & : & A \\ & b & : \quad B. \end{array} \right.$

whence theſe are proportional $\Big\}$ $ab : Ab : AB$ (Cor. 4. Pr. 29).

Again, let abc, ABC be two ſimilar ſolid numbers; then

theſe numbers are proportional $\left\{ \begin{array}{lllll} a & : & A & & \\ & b & : & B & \\ & & c & : & C \end{array} \right.$

whence theſe are proportional $\Big\{$ $abc : Abc : ABc : ABC$ (Cor. 4. Pr. 29).

And ſo on for others.

Cor. 1. *Theſe are like plane numbers, that have one mean proportional between them; and like ſolid numbers, that have two means: And ſo on.*

For ſince $ab : Ab : AB$; therefore $abAB = AbAb$ (Pr. 29), and $aB = Ab$ (Ax. 5); alſo $\frac{aB}{AB} = \frac{Ab}{AB}$ (ib.) or $\frac{a}{A} = \frac{b}{B}$, therefore $a : A :: b : B$ (Def. 27).

Likewiſe $abc \times ABc = Abc \times Abc$, or $aB = Ab$, whence $a : A :: b : B$; alſo $abc \times ABC = Abc \times ABc$, or $aC = Ac$, whence $a : A :: c : C$. And ſo of others.

Cor. 2. *Between two nonſimilar numbers, one or more means cannot be found.*

For if there were any means, the numbers would be ſimilar (Cor. 1).

PROP. XLII.

Like plane numbers are to one another, as the ſquares of their ſimilar ſides or factors; and like ſolid numbers are as their cubes; and ſo on.

For if ab, AB be ſimilar planes, then $a : A :: b : B$, and $aB = Ab$; but $ab : AB :: aab : aAB$ or $AAb :: aa : AA$ (Cor. 4. Pr. 29).

Again, if abc, ABC are ſimilar cubes, then ſince $aB = Ab$, and $aC = Ac$, therefore $abc : ABC :: aa \times abc : aa \times ABC$ (Cor. 4. Pr. 29) $:: a^3 \times bc : A \times Ab \times Ac :: a^3 : A^3$ (Cor. 4. Pr. 29).

Cor. *No numbers prime to one another, except ſquares, are ſimilar plane numbers.*

For if they be ſimilar plane numbers, they are not prime; for if a be prime to A, yet b and B are ſome equal multiple of a, A; and therefore are not prime to one another (Pr. 30).

PROP. XLIII.

If a number of any power meaſures another number of the ſame power; then the root of the firſt will meaſure the root of the laſt. And the contrary.

For in the continual proportionals, A^3, A^2B, AB^2, B^3; ſince A^3 meaſures B^3, it alſo meaſures A^2B the ſecond term (Cor. Pr. 34) But ſince $A^3 : A^2B :: A : B$ (Cor. 4. Pr. 25); therefore if A^3 meaſures A^2B, A will meaſure B (Def. 27). On the contrary, if A meaſures B, A^3 will meaſure A^2B; and A^2B, AB^2; and $AB,^2$ B^3: therefore A^3 meaſures B^3, (Ax. 10).

Cor. *If the power does not meaſure the power, neither ſhall the root meaſure the root; and the contrary.*

For if you ſay A meaſures B, then ſhall A^3 meaſure B^3; contrary to the hypotheſis.

And if you ſay that A^3 meaſures B^3, then A will meaſure B; likewiſe againſt the hypotheſis.

* * *

CHAP. III.

The properties of particular numbers. Diviſors and aliquot parts. Circulating numbers, and ſuch as terminate, or run on ad infinitum *by diviſion.*

PROP. XLIV.

ALL the powers of any number, ending in 5, *will alſo end in* 5 : *and if a number ends in* 6, *all its powers end in* 6.

For 5 times 5 is 25. And 6 times 6 is 36.

PROP. XLV.

No number is a ſquare, that ends in 2, 3, 7, *or* 8.

This is plain by ſquaring all the natural numbers to 10.

PROP. XLVI.

Any even ſquare number is diviſible by 4.

The root is even (Pr. 9), therefore let $2n$ be the root, then $4nn$ is the ſquare of it; and 4 meaſures or divides $4nn$.

Cor. *A number conſiſting of two, three,* &c. *even ſquares, is diviſible by* 4.

PROP. XLVII.

An odd ſquare number, divided by 4, *leaves a remainder of* 1.

The root of an odd ſquare is odd (Pr. 8), therefore let $2n + 1$, be the root, which multiplied by itſelf, gives the ſquare $4nn + 4n + 1$, but 4 will meaſure $4nn + 4n$, and 1 will remain.

Cor. *If a number conſiſting of two odd ſquares, be divided by* 4, *it leaves a remainder of* 2; *of three odd ſquares, it leaves a remainder of* 3.

PROP. XLVIII.

In every ſquare number, the number of diviſors is odd; in nonquadrate numbers, even.

Let 36 (*aabb*) be a ſquare number; now ſince any diviſor and its quotient, are two diviſors; therefore if they be ſet down together, you will find them to proceed by couples, till you come to the ſquare root, where the diviſor and quotient are the ſame, and therefore that makes an odd one. But in a number not ſquare, there is no ſuch odd diviſor, for they proceed by couples to the laſt, and make an even number of diviſors.

1	36	1	*aabb*
2	18	*a*	*abb*
3	12	*b*	*aab*
4	9	*aa*	*bb*
6		*ab*	

Cor. *If the number of diviſors be odd, it is a ſquare number; if even, it is no ſquare.*

PROP. XLIX.

Any power of a prime number hath as many aliquot parts, as is the dimenſion of its power.

As if *a* be a prime, then any power as a^3 contains the 3 aliquot parts 1, *a*, *aa*. Alſo a^4 contains theſe, 1, *a*, *aa*, a^3, which are 4; and ſo on.

Cor. *The number of diviſors in any power of a prime number, is equal to the index of the next ſuperior power thereof.*

For it is 1 more than the number of aliquot parts.

PROP.

PROP. L.

In any number made up of different primes or their powers; the number of diviſors thereof, is equal to the continual product of the indices of the next ſuperior powers of theſe primes.

For the diviſors of a^3, are 1, a, aa, a^3 (Cor. Pr. 48); that is 4. And the diviſors of a^3b^2, are ſuch as are produced by multiplying 1, a, aa, a^3, by each of the diviſors in b^2, that is, by 1, b, bb, which will make 4 × 3 or 12 diviſors. Likewiſe the diviſors in a^3b^2c, are had by multiplying theſe twelve into 1, c, the two diviſors of c, which will be $4 \times 3 \times 2 = 24$; and ſo on.

$$\left.\begin{array}{cccc} 1, & a, & aa, & a^3 \\ b, & ba, & baa, & ba^3 \\ bb, & bba, & bbaa, & bba^3 \end{array}\right\} 12.$$

Cor. *If the powers of ſeveral different prime numbers be multiplied together; the number of diviſors in the product, is equal to the product made by the number of diviſors in each power, multiplied together.*

For the number of diviſors in a^3 is 4, in b^2 is 3, in c is 2; and in a^3b^2c is $4 \times 3 \times 2 = 24$.

PROP. LI.

Any number divided by 9, will leave the ſame remainder, as the ſum of its figures or digits divided by 9.

Let there be any number, as 7604; this ſeparated into ſeveral parcels becomes $7000 + 600 + 4$; but $7000 = 7 \times 1000 = 7 \times \overline{999 + 1} = 7 \times 999 + 7$. In like manner $600 = 6 \times 99 + 6$. Therefore $7604 = 7 \times 999 + 7 + 6 \times 99 + 6 + 4 = 7 \times 999 + 6 \times 99 + 7 + 6 + 4$. Therefore $\frac{7604}{9} = \frac{7 \times 999 + 6 \times 99}{9} + \frac{7+6+4}{9}$ (Ax. 5); but $7 \times 999 + 6 \times 99$ is evidently diviſible by 9, therefore 7604 divided by 9 leaves the

remainder

remainder $7+6+4$ to be divided by 9, which is nothing elſe but the ſum of the digits $7+6+0+4$. And the ſame holds for any other number.

Cor. 1. *If any number is diviſible by* 9, *the ſum of its figures or digits is diviſible by* 9. *And the contrary.*
For then the remainder will be nothing, in both of them.

Cor. 2. *Any number divided by* 9, *leaves the ſame remainder, as when all the figures of it are any way tranſpoſed, and then divided by* 9.
For the ſum of the digits ſtill remains the ſame.

PROP. LII.

Any number divided by 3, *will leave the ſame remainder, as the ſum of its figures or digits divided by* 3.

For ſuppoſe any number, as 7604, and proceeding as in the laſt Prop. we have $7604 = 7 \times 999 + 6 \times 99 + 7 + 6 + 4 = 7 \times 3 \times 333 + 6 \times 3 \times 33 + 7 + 6 + 4$, and $\frac{7604}{3} = \frac{21 \times 333 + 18 \times 33}{3} + \frac{7+6+4}{3}$. But it is evident $21 \times 333 + 18 \times 33$ is diviſibly by 3, conſequently there remains only $7+6+4$ to be divided by 3, which is the ſum of the digits, as was propoſed.

Cor. 1. *If any number is diviſible by* 3, *the ſum of its digits is alſo diviſible by* 3: *and the contrary.*
For in both caſes nothing will remain.

Cor. 2. *Any number divided by* 3, *leaves the ſame remainder as it would do, when its digits are tranſpoſed and placed in any other order.*
For the ſum of the digits remains the ſame in any poſition.

PROP. LIII.

If any two numbers are ſeparately divided by 9, *and the two remainders multiplied together, and that product divided by* 9, *this laſt remainder will be the ſame, as if you divide the product of the two firſt numbers by* 9.

For let $9A + a$, and $9B + b$, be two numbers; a, b, being the two remainders. Then the product of the two numbers is $9 \times 9AB + 9Ab + 9Ba + ab$. But $9 \times 9\ AB + 9Ab + 9Ba$ is diviſible by 9; therefore there is no remainder but what is had by dividing ab by 9.

Cor. *This Prop. holds equally true for the number* 3; *and is demonſtrated the ſame way.*

PROP. LIV.

If one number be divided by another prime to it, and the diviſion continued on indefinitely; the number of figures which circulate (or return again) in the quotient, will be always leſs than the number of units in the diviſor.

Suppoſe 6 divided by 7; here the diviſor being 7, the remainder muſt be always leſs than it, and muſt be either 1, 2, 3, 4, 5, or 6. So that in the 7th place, if not before, one of theſe remainders muſt needs return a ſecond time; and the ſame remainder returning, as before, a repetition of the ſame figures muſt return again in the quotient: and ſo forward. And it is evident the ſame will hold for any diviſor; the number of remainders, being always leſs than the number of units in it.

7) 6.0(857142,857142,8 &c.
56
—
40
35
—
50
49
—
10
7
—
30
28
—
20
14
—
6 &c.

PROP. LV.

If one number divide another prime to it, the quotient will end after a certain number of figures, when the diviſor is compounded of 2 or 5, or both: In all other caſes, the quotient will never end.

For ſince dividing by any power of 2 is equivalent to dividing, firſt by 2, and then the quotient by 2, and ſo on; alſo dividing by any power of 5 is the ſame as dividing firſt by 5, and then the quotient by 5, and ſo forward; and laſtly, ſince any number may be divided by 2 or 5, at moſt by adding a cypher: therefore it is plain, when the diviſor is a compoſite number made up of the powers of 2 and 5, if the diviſion be performed continually by the ſingle numbers 2, and 5, as often as they are involved; that ſo many ſeveral operations will end the diviſion, and the quotient be at an end.

On the contrary; any number P that is prime to 2 and 5, will be prime to 2 × 5 or 10 (Prop. 14). And the ſame being prime to 10, will be prime to 100, 1000, 10000, *&c. ad infinitum* (Cor. 2. Pr. 14); and therefore P can meaſure none in that ſeries. Likewiſe if Q be prime to P, then P will be prime to 10Q, 100Q, *&c.* (Pr. 14). So that P can ſtill meaſure none in this laſt ſeries. Whence if P divide any of theſe, the quotient will continue without end. Yet the numbers will at laſt circulate, according to Prop. 54.

PROP. LVI.

In any circulating number, the whole circulating or repeating part, running on for ever; is equal to a vulgar fraction whoſe numerator is the number repeating (or the repetend), and denominator as many 9's as there are figures in the repetend.

As

As in the number 24.35076 5076 5076 5076 *&c.* ad infinitum; 5076 5076 5076 *&c.* $= \frac{5076}{9999} = \frac{564}{1111}$, *in the leaſt terms.*

For let C = whole circulating part, R = repetend or repeating figures 5076; then from the whole circulating part, that is,

from .5076 5076 5076 5076 5076 *&c.* = C,

take .5076 5076 5076 5076 *&c.* $= \frac{1}{10000}$C,

---

rem. .5076 = R.

But this taking away from C the 10000th part of itſelf, is equivalent to multiplying C by $1 - \frac{1}{10000}$ or by $\frac{10000-1}{10000}$, that, is by $\frac{9999}{10000}$, where there are as many cyphers and 9's, as there are places of figures in the repetend. Therefore $\frac{9999}{10000}C = R = .5076$, and $C = \frac{10000 \times .5076}{9999} = \frac{5076.}{9999.} = .5076\ 5076\ 5076$ *&c. ad infinitum.* And it is evident from the proceſs, that it holds equally for any circulating number.

Cor. 1. *The circulation may be ſuppoſed to begin at any figure of the repetend, and therefore* 24.35076 5076 5076 *&c. for ever, is* $= 24.3\frac{5076}{9999} = 24.35\frac{0765}{9999} = 24.350\frac{7650}{9999} = 24.3507\frac{6507}{9999} = 24.35076\frac{5076}{9999}$ &c.

Cor. 2. *Hence if the repetend be divided by as many* 9*'s as it conſiſts of places; the quotient will be the whole circulating part, or the figures of the repetend, repeated over and over for ever.*

For $\frac{5076}{9999} = C$.

Cor. 3. *And if the whole circulating part be multiplied by a number conſiſting of as many* 9*'s, as there be places in the repetend (conſidered as a decimal); the product will be the repetend.*

For-

For 9999 C = 5076, and .9999 C = .5076, the firſt repetend.

Cor. 4. *If any circulating number be multiplied by any given number, the product will be a circulating number; and the repetend will conſiſt of the ſame number of figures as before.*

For in the circulating number 5076 5076 *&c.* every repetend 5076 being equally multiplied, muſt produce the ſame product. And if theſe products conſiſt of more places, the overplus in each being alike, is carried to the next, ſo that each product is equally increaſed, and therefore every four places continue alike. And the ſame holds for any other number. For example, 5076 × 13 = 65988, but the 6 belongs to the firſt place of the next repetend; which being every where added, the repetend now appears to be 5994.

$$\begin{array}{r} 65988 \\ 65988 \\ 659886 \\ \hline 6599459945994 \end{array}$$

But the ſame thing does not hold in diviſion.

Cor. 5. *If you take any prime number (except* 2 *and* 5) *for a diviſor; and by it divide* 1.0000 &c. *till* 1 *remains, or divide* .99999 &c. *till* 0 *remains; the number of cyphers or nines made uſe of, will be equal to the number of figures in the repetend; when the dividend is any number which is prime to the diviſor.*

For in dividing 1.00 *&c.* by any number, when 1 remains, the figures in the quotient begin then to repeat over again, as you had 1 at firſt to begin with. And ſince 999 *&c.* is leſs by 1 than 1000 *&c.* therefore 0 muſt remain here when the repeating figures are at their period. Whatever number of repeating figures we have when this dividend is 1; we ſhall have the ſame number of figures in the repetend, whatever the dividend be, by Cor. 4. Therefore altering the dividend at pleaſure, does not alter the number of places in the repetend, the diviſor continuing the ſame; provided the diviſor and dividend be prime to one another. For when the contrary happens, the quotient will circulate in fewer figures.

Cor. 6. *If a circulating decimal has a repetend of any number of figures, it may be conſidered as having a repetend of twice or thrice that number of figures, or any multiple thereof.*

Thus in the number 4.137,37,37, having the repetend 37 of 2 places; it may be conſidered as having the repetend 3737, or 373737; of 4 or 6 places, *&c.*

Cor. 7. *If two or more numbers be added together, that have repetends of equal places; the ſum will have a repetend of the ſame number of places.*

This appears from Cor. 1, and by the reaſoning in Cor. 4. For every column of periods or repetends amounts to the ſame ſum.

PROP. LVII.

If A, B, *be two numbers, prime to one another; and each of them divides a number prime to it, and gives in the quotients two repetends of* C *and* D *places: I ſay, the ſame number divided by the product* AB, *will give a repetend of ſo many places, as is denoted by the leaſt dividend of* C *and* D.

For let N be the leaſt number that C, D, divide; and let $a \times C = N = b \times D$. Now it is plain that a periods of C will end with b periods of D; and therefore they both terminate together after N places, if they begin together, as they may be ſuppoſed to do (Cor. 1. Pr. 56). And they do not end ſooner, becauſe N is the leaſt dividend. Therefore the repetend conſiſts of N places, and no more.

To make it plainer, ſuppoſe $\frac{1}{11 \times 37}$ or $\frac{1}{407}$ to be the fraction propoſed. Then ſince $\frac{1}{11} = 09$ *&c.*

repeats in 2 places, and $\frac{1}{37} = .027$, *&c.* repeats in three places. And the leaſt common dividend of 2 and 3 is 6, therefore we may ſuppoſe them both to repeat in 6 places (Cor. 6. Pr. 56). And ſince 99 is diviſible by 11; therefore 99,99,99 is alſo diviſible by 11; and ſince 999 is diviſible by 37, therefore 999,999, is alſo diviſible by 37. Therefore 999999 is diviſible both by 11 and 37; and therefore it is diviſible by 11×37 or 407 (Prop. 16). And therefore the repetend of $\frac{1}{407}$ will conſiſt of 6 places (Cor. 5. Pr. 56).

Cor. *If the ſeveral diviſors* A, B, C, &c. *be prime to one another, and repeat in* E, F, G, &c. *places, reſpectively. And if* N *be the leaſt dividend of* E, F, G, &c. *then if the product* ABC, &c. *be made a diviſor, the quotient will repeat in* N *places.*

This follows from Cor. Prop. 16, and the reaſoning in this Prop.

* * *

CHAP. IV.

Numerical Problems.

PROBLEM I.

To find the greateſt common meaſure of two or more numbers.

RULE.

TAKE two of the numbers, and divide the greater by the leſſer, and the leſſer by the remainder, and the laſt diviſor by the laſt remainder, and ſo on, till nothing remain: then the laſt diviſor is the greateſt common meaſure of theſe two numbers.

If there be more numbers, take the number laſt found and another of the given numbers, and find their greateſt common meaſure as before: then this is the greateſt common meaſure of the three given numbers. And ſo on. This proceſs is plain from Prop. 10.

Ex. 1.

Find the greateſt common meaſure of 72 *and* 60.

```
60)72(1
   60
   --
   12)60(5
      60
      --
```

So 12 is the greateſt common meaſure of 72 and 60.

Ex. 2.

To find the greateſt common meaſure of 72,60 *and* 28.

Find

Find 12 the greateſt common meaſure of 72 and 60; then find the greateſt common meaſure of 12 and 28.

12)28(2
24
4)12(3
12
·

So 4 is the greateſt common meaſure of 72,60, and 28.

PROBLEM II.

Two or more numbers being given, to find the leaſt numbers, that have the ſame proportion with them.

RULE.

Divide the ſeveral numbers by their greateſt common meaſure; and the quotients will be the numbers required. By Cor. 1. Pr. 30.

Ex. 1.

Let 12 and 18 be propoſed, then 6 is the greateſt common meaſure, found by Prob. 1.

6) 12 (2 6) 18 (3.

Then 2 and 3 are the numbers ſought.

Ex. 2.

Let 6, 4, and 8 be the numbers given; their greateſt common diviſor is 2.

2) 6 (3 2) 4 (2 2) 8 (4.

Then 3, 2, 4, are proportional to 6, 4, and 8, and the leaſt in that proportion.

PROBLEM III.

Two or more numbers being given, to find out their leaſt common dividend.

RULE.

Take two of the numbers, and divide their product by the greateſt common meaſure of theſe numbers; the quotient is the anſwer for theſe two numbers.

Then take a third number and the laſt quotient, and divide their product by their greateſt common meaſure; and the quotient is the leaſt number which theſe three numbers meaſure. And ſo on.

For let the two numbers be A, B: let P, Q, be the leaſt in that proportion, M their greateſt common meaſure; then $PM = A$, $QM = B$. Then AQ or $\frac{AB}{M}$ is the leaſt number A and B can divide or meaſure.

If you ſuppoſe F to be leſs; let $\frac{F}{A} = G$, $\frac{F}{B} = H$, or $F = AG$ or BH, then by proportion P : Q :: A : B :: AG or BH : BG :: H : G (Cor. 4. Pr. 29). But P meaſures H; and Q meaſures G (Prop. 30). And Q : G :: AQ : AG. And ſince Q meaſures G, therefore AQ or $\frac{AB}{M}$ meaſures AG or F; that is, the greater meaſures the leſs; which is abſurd.

And if there be three numbers A, B, C; let $D = \frac{AB}{M}$ be the leaſt dividend of A and B, and let E be the leaſt that C and D meaſure. Then E will be the leaſt that A, B, C, meaſure.

For if you ſay there is a leſs, as F; then ſince D is the leaſt that A, B, meaſure; therefore D meaſures F (Cor. Pr. 11); and ſince E is the leaſt that C, D meaſure; therefore E meaſures F, the greater the leſs: which is abſurd. *Ex.*

Ex. 1.

To find the leaſt number which 12 and 15 meaſure, or their leaſt dividend.

The greateſt common meaſure is 3.

12
15
―――
60
12
―――
3)180(60, anſ.
18·
―――
0

Ex. 2.

To find the leaſt number that 12, 15, and 24 meaſure.

60 is the leaſt dividend of 12 and 15. Then the greateſt common meaſure of 60 and 24 is 12.

24
60
―――
12)1440(120, the leaſt common dividend.

PROBLEM IV.

To find out the leaſt numbers continually proportional, as many as ſhall be required, in a given proportion.

RULE.

Find A, B, the leaſt numbers in the given proportion (Prob. 2); then A^2, AB, B^2, will be the three leaſt; and A^3, A^2B, AB^2, B^3, will be the four leaſt numbers. And in general if $n+1$ denote the number of terms required, then A^n, $A^{n-1}B$, $A^{n-2}B^2$, $A^{n-3}B^3$, *&c.* to B^n will be the numbers ſought.

This is plain from Prop. 33. and Cor. 1.

Ex. 1.

To find three the leaſt numbers in proportion as 8 to 12. Two the leaſt are 2 and 3, therefore the 3 numbers are 4 : 6 : 9.

Ex. 2.

To find the four leaſt numbers, as 4 to 6.

Anſ. 8 : 12 : 18 : 27.

Ex. 3.

To find five the leaſt numbers, as 2 to 3.

Anſ. 16 : 24 : 36 : 54 : 81.

PROBLEM V.

Several proportions being given in the leaſt terms; to find out the leaſt numbers that continue theſe proportions.

RULE.

Let A : B, C : D, E : F be the ſeveral proportions;

$$\begin{array}{llllllll} A & : & B & & & & & \\ & & C & : & D & & & \\ & & & & E & : & F & \\ \hline ACE & : & BCE & : & BDE & : & BDF. & \end{array}$$

The ſeveral proportions being placed as in the margin; multiply the two firſt terms A, B, by the leading terms of all the other proportions, C, E; this gives the two firſt terms.

Multiply the latter term D in the ſecond proportion, by ſuch factors as the firſt term C is multiplied by: this is the third term.

Multiply the latter term F in the third proportion, by ſuch factors as the former E is multiplied by, for the fourth term. And proceed thus through all the proportions.

Laſtly, divide all by their greateſt common meaſure, when there is any ſuch. By Cor. 4. Pr. 29.

Ex. 1.

Let the proportions be 6 : 5, and 10 : 9.

$$\begin{array}{rrcrcr} & 6 & : & 5 & & \\ & & & 10 & : & 9 \\ \hline \text{common diviſor } 5) & 60 & : & 50 & : & 45 \\ \text{anſwer} & 12 & : & 10 & : & 9 \end{array}$$

Ex. 2.

Suppoſe 6 : 5, and 4 : 3, and 2 : 7.

$$\begin{array}{rcccccccc} & 6 & : & 5 & & & & \\ & & & 4 & : & 3 & & \\ & & & & & 2 & : & 7 \\ \hline & 2\times4\times6 & : & 2\times4\times5 & : & 2\times5\times3 & : & 5\times3\times7 \\ \text{anſ. - -} & 48 & : & 40 & : & 30 & : & 105 \\ \hline \end{array}$$

PROBLEM VI.

To reſolve a number into all its component parts or factors.

RULE.

Divide the number by 2 as oft as you can, then by 3, then by 5, by 7, and all the ſmalleſt prime numbers, till you get a prime number in the quotient. Then you have all the compounding prime numbers, which being continually multiplied, produce the number given. Def. 18.

Ex. 1.

Let 60 be propoſed.

$$2)\ 60\ (\overset{2}{30}\ (\overset{3}{15}\ (5. \quad \text{then } 2\times2\times3\times5 = 60.$$

Ex. 2.

What are the component parts of 360?

2 2 3 3 5

2) 360 (180 (90 (45 (15 (5 (1.

Therefore 2 × 2 × 2 × 3 × 3 × 5 = 360.

PROBLEM VII.

To find all the juſt diviſors of a given number.

RULE.

Divide it and all the ſucceeding quotients by the ſmalleſt prime numbers in order, till the laſt quotient be 1. Then you have all the prime diviſors. Then multiply every two together, and every three, and every four, and ſo on. And thus you will have all the compound diviſors thereof.

This follows from Prop. 50.

Ex. 1.

What are all the diviſors of 48?

2 2 2 3

2) 48 (24 (12 (6 (3 (1. Then 1, 2, 2, 2, 2, 3, are all the prime diviſors, and 1×2, 1×3, 2×2, 2×3, and 2 × 2 × 2, 2 × 2 × 3, and 2 × 2 × 2 × 2, 2×2×2×3, and 2×2×2×2×3; that is, 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48, are all the diviſors.

Ex. 2.

What are all the diviſors of $abbc^3$?

The ſimple diviſors are 1, a, b, b, c, c, c. And all the diviſors will be 1, a, b, c, ab, ac, bc, abb, abc, acc, bb, cc, bbc, bcc, c^3, $abbc$, $abcc$, $bbcc$, ac^3, bc^3, $abbc^2$, abc^3, bbc^3, $abbc^3$.

PRO-

PROBLEM VIII.

'o find a number that ſhall have a given multitude of diviſors.

RULE.

Take the powers of as many prime numbers as is onvenient, ſo that their indices being each leſſened y 1, and then multiplied together, may be equal to e number of diviſors. I ſay, theſe powers all multiplied together is the number ſought. And the leſſer he primes, the leſſer the number will be.

This is plain by Prop. 50.

Example.

To find a number having 20 diviſors.

Here $0 = 10\times 2 = 5\times 4 = 5\times 2\times 2$. Then ake a, b, c, d, &c. and any of theſe a^{19}, a^9b, a^4b^3, ^4bc, will do. Let $a = 2$, $b = 3$, $c = 5$. Then ^{19}, $2^9\times 3$, $2^4 3^3$, $2^4\times 3\times 5$; that is, 524288, 1536, 32, 240, will any of them anſwer the queſtion.

SCHOLIUM.

The number of aliquot parts, being 1 leſs, is ound the ſame way. And by this operation it apears how to find all the different ways it can be deoted: which in this example are but four. But ny prime numbers may be uſed in each of theſe ays.

PROBLEM IX.

To reduce a given fraction, or a given ratio, to the leaſt terms; and as near as may be, of the ſame value.

1 RULE.

Let A, B, be the two numbers. Divide the latter B by the former A, and you will have 1 for A; and ſome number and a fraction annext, for B, call this C. Place theſe in the firſt ſtep.

Then ſubtract the fractional parts, from the denominator, and what remains put after C + 1, with a negative ſign. Then throw away the denominator, and place 1 and that laſt number in the ſecond ſtep. This is the foundation of all the reſt.

If the fractional parts in both be nearly equal, add theſe two ſteps together; if not, multiply the leſſer by ſuch a number as will make the fractional parts, in both, nearly equal, and then add. And this multiplier is found by dividing the greater fraction by the leſſer, ſo far as to get an integer quotient. When you add the ſteps together, you muſt ſubtract the fractional parts from one another, becauſe they have contrary ſigns.

The proceſs is to be continued on, the ſame way, adding the laſt ſtep, or its multiple, to a foregoing ſtep, *viz.* to that which has the leaſt fraction.

Note. The ratios thus found will be alternately greater and leſſer than the true one, but continually approaching nearer and nearer. And that is the neareſt in ſmall numbers, which precedes far larger numbers: and the exceſs or defect of any one is viſible in the operation.

Ex. 1.

To find the ratio of 10000 to 7854, in ſmall numbers.

	A	B	
1	1	0+.7854	
2	1	1—.2146,	firſt ratio.
	.2146)	.7854 (3	
3	3	3—.6438	
4	4	3+.1416,	2d ratio.
5	5	4—.0730,	3d ratio.
6	9	7+.0686,	4th ratio.
7	14	11—.0044,	5th ratio.
	.0044)	.0686 (15	
8	210	165—.0660	
9	219	172+.0026,	6th ratio.
0	233	183—.0018,	7th ratio.
1	452	355+.0008,	8th ratio.
	.0008)	0018 (2	
2	904	710+.0016	
3	1137	893—.0002,	9th ratio.
	.0002)	.0008 (4	
4	4548	3572—.0008	
5	5000	3927±.0000,	10th ratio.

Explanation.

The ratio of 10000 to 7854 is the ſame as 1 to +.7854 or 1 to 1—.2146; here 1 and 1 is the firſt atio. But 2146 being leſs than 7854, divide the atter by the former, and you get 3 in the quotient, hen multiply 1 and 1—.2146 by 3, produces 3 and —.6438 as in the 3d ſtep. This third ſtep added o the firſt ſtep produces 4 and 3 for the integers, nd ſubtracting the fractional parts, leaves .1416. So

So the 4th ſtep is 4 and 3 +.1416; and the integers 4 and 3 is the 2d ratio. In this manner it is continued to the end; and the ſeveral ratios approximating nearer and nearer, are $\frac{1}{1}$, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{9}{7}$, $\frac{14}{11}$, $\frac{219}{172}$, $\frac{233}{183}$, $\frac{452}{355}$, $\frac{1137}{893}$, and $\frac{5000}{3927}$. Here $\frac{14}{11}$ is the neareſt in ſmall numbers, the defect being only $\frac{44}{10000}$.

Ex. 2.

To find the ratio of 268.8 to 282 in the leaſt numbers.

$$2688)\ 2820\ (1\tfrac{132}{2688} = 2 - \frac{2556}{2688}.$$

2688

132

1	1	1 + 0132,	firſt ratio.
2	1	2 — 2556	
	132) 2556 (19		
3	19	19 + 2508	
4	20	21 — 48,	2d ratio.
	48) 132 (2		
5	40	42 — 96	
6	41	43 + 36,	3d ratio.
7	61	64 — 12,	4th ratio.
	12) 36 (3		
8	183	192 — 36	
9	224	235	, 5th ratio.

So the ſeveral ratios are $\frac{1}{1}$, $\frac{20}{21}$, $\frac{41}{43}$, $\frac{61}{64}$, $\frac{224}{235}$. And the defect or exceſs is plain by inſpection, *e. g.* $\frac{41}{43}$ differs from the truth only $\frac{36}{2688}$ parts; and $\frac{20}{21}$, but 48 ſuch parts.

The

The reaſon of this proceſs is evident from Cor. 3. r. 29. For if the terms of equal ratios be added ogether, the ſums will be in the ſame ratio.

2 RULE.

Divide the greater number by the leſſer, and the ivisor by the remainder, and the laſt diviſor by the aſt remainder, and ſo on till 0 remain. Then

1 divided by the firſt quotient, gives the firſt ratio.

And the terms of the firſt ratio multiplied by the cond quotient, and 1 added to the denominator, ives the ſecond ratio.

And in general, the terms of any ratio, multiplied y the next quotient, and the terms of the foregoing atio added, gives the next ſucceeding ratio.

Ex. 3.

Let the numbers be 10000 and 31416, or the ratio $\frac{0000}{1416}$.

```
10000) 31416 (3
       30000
       -----
        1416) 10000 (7
               9912
               ----
                 88) 1416 (16
                     88
                     ---
                     536
                     528
                     ---
                       8) 88 (11
                          88
                          --
                           0
```

Then

Then $\frac{1}{3}$ = firſt or leaſt ratio.

$\frac{1 \times 7}{3 \times 7 + 1}$ or $\frac{7}{22}$ = ſecond ratio.

$\frac{7 \times 16 + 1}{22 \times 16 + 3}$ or $\frac{113}{355}$ = third ratio.

$\frac{113 \times 11 + 7}{355 \times 11 + 22}$ or $\frac{1250}{3927}$ = fourth ratio.

Ex. 4.

The ratio of 268.8 to 282 is required.

```
2688)2820(1
     2688
     ----
      132)2688(20
          264
          ----
            48)132(2
                96
                --
                36)48(1
                   36
                   --
                   12)36(3
                      36
                      --
                       0
```

Then $\frac{1}{1}$ = firſt ratio.

$\frac{1 \times 20}{1 \times 20 + 1}$ or $\frac{20}{21}$ = 2d ratio.

$\frac{20 \times 2 + 1}{21 \times 2 + 1}$ or $\frac{41}{43}$ = 3d ratio.

$\frac{41 \times 1 + 20}{43 \times 1 + 21}$ or $\frac{61}{64}$ = 4th ratio.

$\frac{61 \times 3 + 41}{64 \times 3 + 43}$ or $\frac{224}{235}$ = 5th ratio.

To

To prove the truth of this rule, let $\frac{10000}{31416}$ be the ratio propoſed; this is reduced to $\frac{1}{3.1416}$. It is plain that $\frac{1}{3}$ is the firſt ratio, or that expreſſed in the leaſt erms. Now inſtead of 3 take $3\frac{1416}{10000}$ or $3\frac{1}{7}$, which s more exact than 3. Then inſtead of $\frac{1}{3}$ we ſhall ave $\frac{1}{3\frac{1}{7}}$ or $\frac{1\times7}{3\times7+1}=\frac{7}{22}$ for the 2d ratio. Now in-tead of 7 take $7\frac{88}{1416}$ or nearly $7\frac{1}{16}$, which is nearer han 7. Then $\frac{1\times7}{3\times7+1}$ becomes $\frac{1\times7\frac{1}{16}}{3\times7\frac{1}{16}+1}$ or $\frac{1\times7\times16+1}{3\times7\times16+16+3}=\frac{7\times16+1}{22\times16+3}$ for the third ratio, hich is equal to the 2d ratio multiplied by 16, + he 1ſt ratio. Again, for 16 take $16\frac{8}{88}$ or $16\frac{1}{11}$, hich will be more exact ſtill; then $\frac{7\times16+1}{22\times16+3}$ ecomes $\frac{7\times16\frac{1}{11}+1}{22\times16\frac{1}{11}+3}$ or $\frac{7\times16\times11+11+7}{22\times16\times11+3\times11+22}$ $=\frac{\overline{7\times16+1}\times11+7}{\overline{22\times16+3}\times11+22}$ for the 4th ratio, which is qual to the 3d ratio multiplied by 11, + the 2d atio. And ſo forward, if there were more.

PROBLEM X.

To reduce a decimal to a vulgar fraction.

RULE.

Place the decimal as a numerator over 1 and as any cyphers as there are figures, for a denominator. hen reduce it to the loweſt terms.

If the decimal circulate, place the figures of the epetend for a numerator, and as many 9's for a de-ominator: and reduce as before. This appears om Prop. 56.

Ex. 1.

Let .3065 be propoſed.

$.3065 = \frac{3065}{10000}$, divide by 5, then $\frac{613}{2000}$ is the fraction required.

Ex. 2.

To reduce 6.32309309309 &c. to the form of a vulgar fraction.

Here 6.32309309 &c. $= 6.32\frac{309}{999} = 6.32\frac{103}{333}$ $= 6\frac{32\frac{103}{333}}{100} = 6\frac{10759}{33300}$.

PROBLEM XI.

Having a vulgar fraction given in the loweſt terms, and the denominator a prime (neither 2 *nor* 5); *to find the number of figures that circulate, by dividing the numerator by the denominator.*

RULE.

Divide 9999 &c. by the denominator till 0 remains, then the number of 9's made uſe of, will be equal to the number of places in the repetend.

By Cor. 5. Prop. 56.

Ex. 1.

Suppoſe $\frac{287}{37}$, to be given.

```
37)99999(027.
   74
   ---
   259
   259
   ---
    .
   ---
```

Here are three nines uſed, therefore the repetend conſiſts of 3 places.

Ex.

Ex. 2.

Let $\frac{1}{11}$ be propoſed.

11)999(9. Here are 2 nines made uſe of, therefore the repetend has 2 places.
99
—
0
—

Ex. 3.

Let $\frac{2}{7}$ be given.

7)9999999(142857. Here are 6 nines, and the repetend conſiſts of 6 places.
••••••

PROBLEM XII.

Having a vulgar fraction in the loweſt terms, and the denominator made up of two or more different primes (neither 2 *nor* 5); *to find the number of figures circulating, by dividing thereby.*

RULE.

Find the number for each ſingle prime in the denominator, by Prob. 11. Then find the leaſt dividend of all theſe numbers, by Prob. 3. And that is the number of figures circulating.

This appears by Prop. 57. and Cor.

Ex. 1.

Let $\frac{13}{11 \times 37}$ be propoſed.

The repetend by 11 conſiſts of 2 places; and that by 37 of 3 places; and 6 is the leaſt number that 2 and 3 divide; therefore if 13 be divided by 407, the repetend in the quotient will conſiſt of 6 places.

Ex. 2.

Let the fraction be $\frac{1}{3\times7\times11\times37}$ or $\frac{1}{8547}$.

The repetend by 3, 7, 11, and 37, is 1, 6, 2, 3, reſpectively; and the leaſt number which 1, 6, 2, and 3 meaſure, is 6, for the number of places in the repetend.

SCHOLIUM.

It is not my deſign here to ſhew the ſeveral ways of working with circulating numbers, or repeating decimals. It is ſufficient for me to explain the general principles thereof; that the reader may have an idea of the nature of them. For almoſt all operations may be as ſpeedily performed by the ſhort rules delivered in multiplication and diviſion of decimals. They that would ſee more of it may conſult Mr. *Cun*'s treatiſe of circulating numbers.

FINIS.

ERRATA.

Page	*Line*	*Read*
79	8	*Ex.* 5.
	17	*Ex.* 6.
	27	In Ex. 5th.
80	18	In Ex. 6th.
	22	*Ex.* 7.

2

THE DOCTRINE OF PROPORTION, ARITHMETICAL AND GEOMETRICAL.

Together with a general Method of arguing
by proportional Quantities.

---

*Si Proportionis Doctrinam e Matheſi abſtuleris, nihil
fere præclarum aut egregium relinques.*
Wh. Tac. Eucl.

LONDON,
Printed for J. Nourse, Bookſeller in Ordinary
to his MAJESTY.
MDCCLXIII.

Cyclomathesis Vol. 2

THE

PREFACE.

*SINCE all manner of quantities require to be compared together, in mathematical computations, and their various relations ſearched
ut and determined; and as moſt of our knowledge
n mathematical ſubjects depends on the proportions
f ſeveral things to one another: ſo it is requiſite
hat the nature of proportion, and the methods of
eaſoning thereby, be diſtinctly laid down and well
nderſtood. It is a method of reaſoning ſo very
hort, ſubtle, ſolid, and certain, and likewiſe ſo
ſeful in all parts of the mathematics, that it is
mpoſſible to make the leaſt progreſs without it. It
s the marrow of the mathematics, and the very
ul of geometry and geometrical reaſoning. There-
re it is abſolutely neceſſary, that every one who
xpects to ſucceed in his mathematical ſtudies,
hould make himſelf acquainted with the nature
f reaſoning with proportional quantities, and be-
me ready and quick in the uſe thereof.*

*I had before, in the Treatiſe of Arithmetic,
monſtrated ſome few things relating to propor-
ons; but no more than I had then preſent occaſion
r, in treating of the properties of numbers.
ut in this ſmall tract, I have demonſtrated the
ctrine of proportions univerſally, for all quanti-
s whatſoever, as well as numbers.*

The method I have here followed is this: Sect. I. treats of arithmetical proportion and progreſſion. And Sect. II. of geometrical proportion. And herein I have taken the liberty to deviate from Euclid, *by giving a different definition of proportional quantities; his being abſtruſe and unintelligible, eſpecially to young ſtudents. This here laid down being evidently agreeable to, and deducible from, the firſt, ſimple, and natural idea we form of proportion. Neither have I followed his order of propoſitions, or method of demonſtration: but have omitted many of his propoſitions as of little uſe, and added ſeveral other more uſeful ones, which he had not. And theſe I have demonſtrated from that moſt ſimple idea of proportion before mentioned, with the greateſt eaſe and perſpicuity imaginable. And becauſe the method of arguing by a general proportion is vaſtly ſhorter and eaſier than the common way with four terms; therefore I have in Sect. III. demonſtrated the fundamental propoſitions it depends on; and has ſhewn and explained the way of proceeding, according to that method. And therefore I hope this will both inſtruct and delight the diligent reader.*

W. Emerſon.

AXIOMS.

1. The whole is equal to all the parts taken together.

2. If equal quantities be added to equal quantities; the ſums will be equal.

3. If equal quantities be taken from equal quantities; the remainders will be equal.

4. If equal quantities be equally multiplied; the products will be equal.

5. If equal quantities be divided by equal numbers; the quotients will be equal.

6. Equal quantities have the ſame proportion to any third quantity: and any quantity has the ſame ratio to equal quantities.

7. Thoſe quantities are equal, that have the ſame ratio to any third; or when a third has the ſame ratio to each of them.

8. Thoſe ratios or quantities, that are equal to a third, are equal to one another.

9. A greater quantity has a greater ratio to a third, than a leſſer quantity has. And that which has the greater ratio, is the greater quantity.

0. If there be two equal ratios, and one be greater than a third, the other will be greater; if leſs, the other will be leſs.

The ſignification of the Signs or Characters here uſed.

+	to be added.
—	to be ſubtracted.
×	multiplied by, or AB is A multiplied by B.
÷	divided by, or $\frac{A}{B}$ is A divided by B.
=	equal to.
::	geometrical proportion, as A : B :: C : D, ſignifies A is to B, as C is to D.
∝	is as; a mark of general proportion.
∺	continual proportion, or geometrical progreſſion. As A : B : C : D ∺, ſignifies that A is to B, as B to C, as C to D, *&c.*
∵	arithmetical proportion, as A . B ∵ C . D.
$\overset{\cdot\,\cdot}{\underset{\cdot}{-}}$	arithmetical progreſſion.
$\overset{\cdot\,\cdot}{\cdot\,\cdot\,\cdot}$	harmonic proportion.
$\overset{\cdot\,\cdot}{\underset{\cdot\,\cdot\,\cdot}{-}}$	harmonic progreſſion.

SECT. I.

Arithmetical Proportion.

DEFINITIONS.

1. ARITHMETIC *proportion*, is the relation that two quantities, of the ſame kind, have to one another, in reſpect of their difference. The former quantity is called the *antecedent*; and the latter, the *conſequent*. And theſe are called the *terms* of the proportion.

2. *Ratio* is the difference between the antecedent and conſequent. Therefore arithmetic ratio is of the ſame kind as the quantities themſelves. This is commonly called the *common difference*.

3. *Quantities arithmetically proportional*, are thoſe that have the ſame arithmetic ratio, when compared two and two; ſo that the antecedents, may be every where ſubtracted from the conſequents; or elſe the conſequents from the antecedents.

4. Continued proportion is when the firſt has the ſame proportion to the ſecond, as the ſecond to the third.

5. *Arithmetical progreſſion*, is when a ſeries of quantities are in the ſame arithmetical proportion. Or when they increaſe, or decreaſe by equal differences.

6. *Muſical proportion, and progreſſion*, is when there is a ſeries of quantities, where the numerators are the ſame, and the denominators in arithmetic progreſſion.

PROP. I.

If four quantities are arithmetically proportional, $A . B \because C . D$; *the ſum of the extremes is equal to the ſum of the means,* $A + D = B + C$.

For $A - B = C - D$ (Def. 3), and adding $B + D$, $A - B + B + D = C + B - D + D$ (Ax. 2); that is, $A + D = C + B$.

Cor. *If three quantities be in arithmetic progreſſion, the ſum of the extremes is double the mean.*

PROP. II.

If there be two ranks of quantities in arithmetic proportion; their ſums and differences ſhall alſo be in arithmetic proportion. If $A . B \because C . D$, *and* $P . Q \because R . S$; *then* $A + P . B + Q \because C + R . D + S$, *and* $A - P . B - Q \because C - R . D - S$.

For let $A - B = C - D = m$, and $Q - P = S - R = n$. Then $B = A - m$, $D = C - m$, $Q = P + n$, $S = R + n$. And $B + Q = A - m + P + n$, $D + S = C - m + R + n$. But $A + P . A - m + P - n \because C + R . C - m + R + n$ (Def. 3).

Again, $B - Q = A - m - P - n$, and $D - S = C - m - R - n$. But $A - P . A - m - P - n \because C - R . C - m - R - n$ (Def. 3).

PROP. III.

If three quantities are in arithmetic progreſſion; the rectangle of the extremes, together with the ſquare of the common difference, is equal to the ſquare of the middle term. If $A . B . C \div$, *then* $AC + \overline{B - A}^2 = BB$.

For let $D = B - A = C - B$, and $A = B - D$, $C = B + D$; then $AC = \overline{B - D} \times \overline{B + D} = BB + BD - BD - DD = BB - DD$. And $AC - DD = BB$ (Ax. 2).

Cor. *A ſet of arithmetical proportionals, whoſe common difference is exceeding ſmall, is nearly a ſet of geometrical proportionals.* See the next ſection.

PROP. IV.

In a ſeries of quantities in arithmetical progreſſion; he ſum of the extremes is equal to the ſum of any two means, equally diſtant from the extremes. If A . B . C . D . E . F . G ÷; *then* $A + G = B + F = C + E$, *&c.*

For ſince A . B ∵ F . G (Def. 5), therefore $A + G = B + F$ (Prop. I). And ſince B . C ∵ E . F, therefore $C + E = B + F = A + G$, *&c.*

Cor. *Hence the ſum of the extremes is double the ean, when the number of terms is odd.*

PROP. V.

If out of a ſeries of quantities in arithmetical progreſſion, there be taken any ſeries of equidiſtant terms; his ſeries will alſo be in arithmetic progreſſion.

If A . B . C . D . E . F . G . H . I . K . L . M ÷,
hen B E H L *are* ÷.

For $C - B = D - C = E - D = R$, and
dding all together, $E - B = 3R$.
lſo $F - E = G - F = H - G = R$, and
$H - E = 3R$.
gain, $I - H = K - I = L - K = R$, and
$L - H = 3R$, *&c.*
herefore, $E - B = H - E = L - H$ (Ax. 8).
nd B . E . H . L ÷ (Def. 5).

PROP.

PROP. VI.

In a ſeries of quantities in arithmetic progreſſion, A . B . C . D . E, *whoſe number is* n, *and common difference* x; *the laſt term* (E) $= A + \overline{n-1} \times x$ *in an increaſing progreſſion, or laſt term* (E) $= A - \overline{n-1} \times x$ *in a decreaſing one.*

For the difference between A and B, B and C, C and D, D and E, being x; the difference between A and E will be ſo many times x, as are the terms beyond A; that is, $\overline{n-1} \times x$. Whence $A - E$, or $E - A = \overline{n-1} \times x$. And $E = A + \overline{n-1} \times x$, or $= A - \overline{n-1} \times x$ (Ax. 2, 3).

Cor. *The common difference is equal to the difference of the extremes, divided by the number of terms leſs one.*

For $x = \dfrac{A - E \text{ or } E - A}{n-1}$ (Ax. 5).

PROP. VII.

The ſum, of a ſeries of quantities in arithmetic progreſſion, is equal to half the product, of the ſum of the extremes, multiplied by the number of terms.

If A . B . C . D . E $\div$, *then the ſum* $= \dfrac{\overline{A+E} \times n}{2}$, *$n$ being the number of terms.*

For $A + B + C + D + E =$ ſum
And $E + D + C + B + A =$ ſum
Adding, $A + E + B + D + C + C + B + D + A + E =$ twice the ſum.
That is, $A + E + A + E + A + E + A + E + A + E$ (Prop. IV).

Therefore twice the ſum is equal to as many times $A + E$, as there are terms, or the ſum $= \dfrac{A+E}{2} \times n$.

PROP. VIII.

In a ſeries of quantities in arithmetical progreſſion from o, their differences are equal; in their ſquares, the differences of the differences, or the ſecond differences, are equal; in their cubes, the third differences are equal; and ſo on.

Let the ſeries be $o, a, 2a, 3a, 4a, 5a, 6a,$ &c.

$$\begin{array}{llllllll}
\text{then } o, & aa & 4aa & 9aa & 16aa & 25aa & \&c. & \text{ſquares.} \\
 & aa & 3aa & 5aa & 7aa & 9aa & & \text{1 differences.} \\
 & & 2aa & 2aa & 2aa & 2aa & & \text{2 differences.}
\end{array}$$

$$\begin{array}{llllll}
\text{Again, } o & a^3 & 8a^3 & 27a^3 & 64a^3 & \text{cubes.} \\
 & a^3 & 7a^3 & 19a^3 & 37a^3 & \text{1 differences.} \\
 & & 6a^3 & 12a^3 & 18a^3 & \text{2 differences.} \\
 & & & 6a^3 & 6a^3 \ \&c. & \text{3 differences.}
\end{array}$$

And ſo for higher powers.

Cor. 1. *In the n^{th} powers, the $\overline{n+1}^{th}$ differences are o.*

Cor. 2. *The equal differences in the laterals, ſquares, cubes, biquadrates,* &c. *are* $1a$, $1 \times 2aa$, $1 \times 2 \times 3a^3$, $1 \times 2 \times 3 \times 4a^4$, &c. *reſpectively.*

* * *

SECT. II.

Geometrical Proportion.

DEFINITIONS.

1. GEOMETRICAL *proportion*, is the relation or reſpect, that two quantities, of the ſame kind, have to one another in regard to their bigneſs. The former quantity is called the *antecedent*; and the ſecond, the *conſequent*.

2. *Ratio* is the quotient ariſing by dividing the antecedent by the conſequent: Or it is the number which expreſſes how oft the antecedent contains the conſequent; which number may be either whole, fractional, or ſurd. When the antecedent and conſequent are equal; it is called a ratio of *equality*; if not, of *inequality*.

3. *Terms* of the ratio, are the antecedent and its conſequent.

4. *Proportional quantities* are thoſe that have the ſame ratio or proportion, when compared two and two together; that is, when the firſt is to the ſecond, as the third to the fourth; or when the firſt contains the ſecond, as oft as the third contains the fourth; and the contrary.

5. *Homologous or alternate terms*, are the antecedents of ſeveral ratios, or elſe the conſequents. And any antecedent and its conſequent, are called *analogous terms*.

6. *Direct proportion*, is when the ſame proportion holds from the firſt term to the ſecond, and

from the third to the fourth, as if A, B, C, D, be four quantities; then it is *directly* A : B :: C : D.

7. *Reciprocal or inverſe proportion*, is when one ſort of quantity increaſes, in the ſame proportion that another decreaſes.

8. *Diſcreet proportion*, is when out of four terms, the ſecond has not the ſame proportion to the third, which the firſt has to the ſecond, or the third to the fourth.

9. *Continual proportion*, is when the firſt term has the ſame proportion to the ſecond, as the ſecond to the third.

10. *Geometrical progreſſion*, is when a ſet of quantities are in continual proportion; or when the firſt has the ſame ratio to the ſecond, as the ſecond to the third, and as the third to the fourth, and the fourth to the fifth, *&c.*

11. *Extreme and mean ratio*, is when a quantity is ſo divided, that the leſſer part, the greater part, and the whole, are in continual proportion.

12. *Complicate ratio*, is that which ariſes by multiplying ſeveral other ratios together.

13. *Duplicate, triplicate, ratio,* &c. is the ſquare, cube, *&c.* of ſome given ratio.

14. *Harmonical ratio*, is when a quantity is divided into three parts, ſo that the whole is to one part, as the ſecond part to the third. And when the ſecond and third are equal; it is called *harmonic proportion continued.*

PROP.

PROP. I.

If ſeveral pairs of quantities are in the ſame proportion, A : B :: C : D :: E : F :: G : H; *then as any antecedent to its conſequent, ſo is any other antecedent to its conſequent,* A : B :: G : H.

For ſince $\frac{A}{B} = \frac{C}{D} = \frac{E}{F} = \frac{G}{H}$ (Def. 4), therefore $\frac{A}{B} = \frac{G}{H}$ (Ax. 8); whence A : B :: G : H (Def. 4).

PROP. II.

If four quantities are proportional, A : B :: C : D; *and if the firſt* A, *be greater than the ſecond* B; *then the third* C, *ſhall be greater than the fourth* D. *If equal, they ſhall be equal; if leſs, leſs.*

For ſince $\frac{A}{B} = \frac{C}{D}$ (Def. 4), by the nature of fractional quantities, if A be greater than B, the quotient or ratio will be more than 1, and therefore C greater than D. But if A be equal to B, $\frac{A}{B} = 1$, and C = D. But if A be leſs than B, the quotient is leſs than 1, and therefore C leſs than D.

PROP. III.

If four quantities are proportional, A : B :: C : D; *they ſhall alſo be proportional by reverſion; that is, the ſecond* B *is to the firſt* A; *as the fourth* D, *is to the third* C; *or* B : A :: D : C.

For let $\frac{A}{B} = \frac{C}{D} = r$ the ratio, then $A = Br$, and $C = Dr$ (Ax. 4); and $B = \frac{A}{r}$, and $D = \frac{C}{r}$ (Ax. 5); alſo $\frac{B}{A} = \frac{1}{r}$, and $\frac{D}{C} = \frac{1}{r}$ (ib.); whence $\frac{B}{A} = \frac{D}{C}$ (Ax. 8); therefore B : A :: D : C (Def. 4).

PROP. IV.

If four quantities of the ſame kind are proportional, A : B : : C : D; *they ſhall be proportional alter-*
ately or by permutation; that is, the firſt A, *ſhall be*
o the third C; *as the ſecond* B, *is to the fourth* D.

For let $\frac{A}{B} = \frac{C}{D} = r$, then $A = Br$, and $C =$
Dr (Ax. 4); then $\frac{A}{C} = \frac{Br}{Dr} = \frac{B}{D}$ (Ax. 5); there-
ore A : C : : B : D (Def. 4).

PROP. V.

Quantities are in the ſame ratio, as their equi-
ultiples; A : B : : nA : nB.

For let $\frac{A}{B} = r$, then $A = Br$ (Ax. 4); and nA
$= nBr$ (ib.); and $\frac{nA}{nB} = r$ (Ax. 5); therefore $\frac{A}{B} =$
$\frac{A}{B}$ (Ax. 8); therefore A : B : : nA : nB.

Cor. 1. *Quantities are in the ſame ratio, as their*
ke parts.

For nA : nB : : $\frac{nA}{n} : \frac{nB}{n}$: : A : B.

Cor. 2. *The like parts of two quantities, taken an*
ual number of times, are as the quantities themſelves.

PROP. VI.

If four quantities are proportional, A : B : : C
D; *and two homologous or analogous terms be both*
them equally multiplied, or divided; the four terms
ll ſtill be proportional.

For C : D : : nC : nD (Pr. V) : : $\frac{C}{n} : \frac{D}{n}$ (Pr.
Cor. 1); therefore A : B : : nC : nD : :
: $\frac{D}{n}$ (Prop. I).

Again,

Again, $A : C :: B : D$ (Prop. IV) $:: nB : nD :: \frac{B}{n} : \frac{D}{n}$ (Prop. VI). Therefore $A : nB :: C : nD$. And $A : \frac{B}{n} :: C : \frac{D}{n}$ (Prop. IV).

Cor. 1. *If two correſpondent terms be multiplied by one number, and the other two terms by another number; the reſulting terms will be proportional: If* $A : B :: C : D$, *then* $mA : mB :: nC : nD$; *or* $mA : nB :: mC : nD$.

Cor. 2. *And if two correſpondent terms be divided by one number, and the other two by another number; the quotients will be proportional.*

Cor. 3. *Hence, inſtead of any two correſpondent terms; two others, proportional to them, may be put in their room.*

PROP. VII.

If four quantities are proportional; and inſtead of two factors, in two analogous terms, if there be ſubſtituted two other quantities, in the ſame ratio; the four quantities will ſtill be proportional: If $A : B :: PQ : RS$; *and* $Q : S :: M : N$. *Then* $A : B :: PM : RN$.

For ſince $A : B :: PQ : RS$; by dividing the antecedents by P, and the conſequents by R, $\frac{A}{P} : \frac{B}{R} :: Q : S :: M : N$ (Prop. VI. Cor.); then multiplying the antecedents by P, and the conſequents by R, we have $A : B :: PM : RN$.

PROP. VIII.

If the parts taken away from two whole quantities, be as the wholes; then the remainders, ſhall be as the wholes. If A : C : : A + B : C + D; *then* B : D : : A + B : C + D.

For A : A + B : : C : C + D (Prop. IV); and A + B : A : : C + D : C (Prop. III); and $\frac{A+B}{A} = \frac{C+D}{C}$ (Def. 4); that is, $1 + \frac{B}{A} = 1 + \frac{D}{C}$, and $\frac{B}{A} = \frac{D}{C}$ (Ax. 3); therefore B : A : : D C, and B : D : : A : C (Prop. IV) : : A + B : C + D.

Cor. *The ſame things ſuppoſed, the remainders ſhall be as the parts taken away,* A : B : : C : D.

PROP. IX.

The ſum of the greateſt and leaſt, of four pro-ortional quantities, is greater than the ſum of the ther two.

Suppoſe A : B : : C : D, and let A be the reateſt term, then of conſequence D is the leaſt rop. II): then $\frac{A}{B} = \frac{C}{D} = r$. Now ſince A is reater than B, r is greater than 1, therefore put $= 1 + s$. Whence $A = rB = B + sB$, and C $rD = D + sD$ (Ax. 8). Then $A + D = B +$ $+ D$, and $B + C = B + D + sD$. But B is eater than D, and sB greater than sD; therefore $+ D + sB$ is greater than $B + D + sD$; or A D greater than B + C.

Cor. *The ſum of* A *and* D = *ſum of* B *and* C + $\overline{-1} \times \overline{B-D}$.

For one of theſe ſums exceeds the other by $s \times \overline{B - D}$.

PROP. X.

If ſeveral quantities are proportional; $A : B :: C : D :: E : F :: G : H$; *as one of the antecedents, to its conſequent; ſo is the ſum of all the antecedents, to the ſum of all the conſequents*; $A : B : A + C + E + G : B + D + F + H$.

For let $\frac{A}{B} = r$, or $A = Br$, $C = Dr$, $E = Fr$, $G = Hr$, and $A + C + E + G = Br + Dr + Fr + Hr = \overline{B + D + F + H} \times r$ (Ax. 8); therefore $\frac{A + C + E + G}{B + D + F + H} = \frac{\overline{B + D + F + H} \times r}{B + D + F + H} = r$; therefore $\frac{A}{B} = \frac{A + C + E + G}{B + D + F + H}$; therefore, &c.

PROP. XI.

If there be two ranks of proportional quantities, and the two means be the ſame in both; the extremes will be reciprocally proportional. If $A : B :: C : D$, *and* $E : B :: C : F$; *then* $A : E :: F : D$.

For let $\frac{A}{B} = \frac{C}{D} = r$, and ſince $B : E :: F : C$ (Pr. III); therefore let $\frac{B}{E} = \frac{F}{C} = s$. Then $rs = \frac{A}{B} \times \frac{B}{E} = \frac{C}{D} \times \frac{F}{C}$ (Ax. 4); that is, $\frac{A}{E} = \frac{F}{D}$; or $A : E :: F : D$.

Cor. *In two ranks of proportional quantities, if the extremes be the ſame in both; the means will be reciprocally proportional.*

For if B : A :: D : C, and B : E :: F : C; then by reverſion A : B :: C : D, and E : B :: C : F. Whence A : E :: F : D (Prop. XI).

PROP. XII.

If four quantities are proportional; A : B :: C : D; *the product of the extremes is equal to the product of the means*, AD = BC.

For let $\frac{A}{B} = \frac{C}{D} = r$, then A = B$r$, and C = D$r$ (Ax. 4); whence AD = BrD, and BC = BrD (Ax. 4); therefore AD = BC (Ax. 8).

Cor. 1. *If two products are equal*, AD = BC; *he ſides or factors are reciprocally proportional*, A : B :: C : D.

For let A : B : C : Q, then AQ = BC (Prop. XII) = AD (hyp.); therefore Q = D (Ax. 5); nd A : B :: C : D (Ax. 7).

Cor. 2. *If three quantities are continually proportional; the rectangle of the extremes is equal to the quare of the mean. And the contrary.*

Cor. 3. *In four proportional quantities, if one xtreme be multiplied by any number, and the other xtreme, divided by it; the quantities will ſtill be proportional. The ſame holds of the means. Conſequently ny two factors in the two extremes may change places; r in the two means.*

For if A : B :: C : D, then AD = BC, and AD = nBC (Ax. 4); then nA : B :: nC : D Cor. 1) :: C : $\frac{D}{n}$ (Cor. 1. Prop. 5).

SCHOLIUM.

It is ſuppoſed here that two analogous terms are numbers, or at leaſt, that they are repreſented by numbers.

PROP. XIII.

If four quantities are proportional, $A : B :: C : D$; *and if the analogous terms be compounded any way by addition or ſubtraction; ſo that both pairs be ordered alike; then they will ſtill be proportional.*

If $A : B :: C : D$.
Then $A : A + B :: C : C + D$.
$A : A - B :: C : C - D$.
$A : B - A :: C : D - C$.
$A + B : B :: C + D : D$.
$A - B : B :: C - D : D$.
$B - A : B :: D - C : D$.
$A + B : A - B :: C + D : C - D$.
$A + B : B - A :: C + D : D - C$.
$A : B :: A + C : B + D$.
$A : B :: A - C : B - D$, *&c.* and the reverſe thereof.

For in any caſe, the product of the means is equal to the product of the extremes.

Cor. *When the quantities are compounded after any of the foregoing ways, then it will be,* $A : B :: C : D$.

PROP. XIV.

If one quantity has the ſame proportion to ſeveral quantities ſeparately; as a ſecond quantity has to as many others: then the firſt has the ſame proportion to the ſum of the firſt ſet, as the ſecond has to the ſum of the laſt ſet.

$$\textit{If } A : \begin{cases} B \\ C \\ D \end{cases} :: F : \begin{cases} G \\ H \\ I \end{cases} \textit{ then } \begin{array}{l} A : B + C + D :: \\ F : G + H + I. \end{array}$$

For $\left.\begin{matrix}B\\C\\D\end{matrix}\right\} : A :: \left.\begin{matrix}G\\H\\I\end{matrix}\right\} : F$ (Prop. III), then
$= \frac{G}{F}$, $\frac{C}{A} = \frac{H}{F}$, $\frac{D}{A} = \frac{I}{F}$ (Def. 4). There-
re $\frac{B}{A} + \frac{C}{A} + \frac{D}{A}$ or $\frac{B+C+D}{A} = \frac{G}{F} + \frac{H}{F} +$
or $\frac{G+H+I}{F}$ (Ax. 2); therefore B + C + D
A :: G + H + I : F (Def. 4); and A : B +
+ D :: F : G + H + I (Prop. III).

Cor. 1. *If one quantity be ſeparately to two quan-*
ies; as a ſecond is to two others: the firſt will be
the difference of the firſt two; as the ſecond, is to
difference of the laſt two.

If A : $\left\{\begin{matrix}B\\C\end{matrix}\right. :: F : \left\{\begin{matrix}G\\H\end{matrix}\right.$. *Then* A : B — C ::
: G — H.

For then $\frac{B}{A} - \frac{C}{A} = \frac{G}{F} - \frac{H}{F}$ (Ax. 3); and
$\frac{- C}{A} = \frac{G - H}{F}$.

or. 2. *The ſame things ſuppoſed as in Cor.* 1, *then*
C :: G : H.

or $\frac{B}{A} = \frac{G}{F}$, and $\frac{C}{A} = \frac{H}{F}$, whence B : C ::
or $\frac{G}{F} : \frac{C}{A}$ or $\frac{H}{F}$ ::) G : H (Pr. V. and Cor. 1).

PROP. XV.

If there be two ranks of quantities; and it be, i theſe two ranks, as the firſt to the ſecond, ſo is th firſt to the ſecond; and as the ſecond to the third ſo the ſecond to the third; and ſo on: then will th firſt be to the laſt, as the firſt to the laſt, in the tw ranks. If A, B, C, D; *and* F, G, H, I, *are tw ranks; and it be,* A : B : : F : G, *and* B : C : : G : H, *and* C : D : : H : I; *then* A : D : : F : I.

For $\frac{A}{B} = \frac{F}{G}$, and $\frac{B}{C} = \frac{G}{H}$, and $\frac{C}{D} = \frac{H}{I}$ (Def. 4) therefore $\frac{ABC}{BCD} = \frac{FGH}{GHI}$ (Ax. 4), or $\frac{A}{D} = \frac{F}{I}$; tha is, A : D : : F : I.

PROP. XVI.

If two or more rows of quantities are reſpectivel proportional; the like terms are proportional, in an two rows.

If A : B : C : D : : P : Q : R : S. *Then* B D : : Q : S, *&c.*

Quantities are *reſpectively proportional*, when i the ſeveral rows, the firſt term is to the firſt, th ſecond to the ſecond, the third to the third, &c in the ſame proportion. And *like terms* are thoſ that are alike ſituated in all the rows; as the thir term and the third, the fourth and the fourth, *&c.*

For ſince B : C : : Q : R, and C : D : : R : S therefore B : D : : Q : S (Prop. XV); and ſo o others.

Or thus.

If theſe are reſpectively proportional,
A : B : C : D : E : :
F : G : H : I : K : :
L : M : N : O : P : :
Q : R : S : T : V : :

then A : D : : Q : T; and ſo o others.

Fo

For A : B : : Q : R, and B : C : : R : S, and C : D : : S : T. Therefore A : D : : Q : T. In like manner G : K : : R : V, and A : E : : L : P, and B : E : : R : V, *&c.* all the ways they can be thus compared.

PROP. XVII.

If there be two ſets of quantities; and if it be as the firſt to the ſecond (in the firſt ſet), ſo the laſt but one to the laſt (in the ſecond ſet); and as the ſecond to the third, ſo the laſt but two, to the laſt but one; and ſo on. Then the firſt will be to the laſt (in the firſt ſet), as the firſt to the laſt (in the ſecond ſet).

Firſt ſet A, B, C.
Second ſet F, G, H.

If A : B : : G : H, *and* B : C : F : G, *&c.* *then* A : C : : F : H.

For $\frac{A}{B} = \frac{G}{H}$, and $\frac{B}{C} = \frac{F}{G}$ (Def. 4, 2); therefore $\frac{AB}{BC} = \frac{GF}{HG}$ (Ax. 4), or $\frac{A}{C} = \frac{F}{H}$, and A : C : : F : H.

PROP. XVIII.

If there be four proportional quantities in one rank, and four more in another; and ſeveral ſuch ranks; then the products of the like terms will be proportional.

If A : B : : C : D,
and F : G : : H : I,
and P : Q : : R : S,
then AFP : BGQ : : CHR : DIS.

For $\frac{A}{B} = \frac{C}{D}$, and $\frac{F}{G} = \frac{H}{I}$, and $\frac{P}{Q} = \frac{R}{S}$ (Def. 4), therefore $\frac{AFP}{BGQ} = \frac{CHR}{DIS}$ (Ax. 4), or AFP : BGQ : : CHR : DIS.

Cor. 1. *If* $A : B :: C : D$,
and $B : P :: H : I$,
and $P : Q :: R : S$, *&c.*
then $A : Q :: CHR : DIS$.

For $ABP : BPQ :: A : Q :: CHR : DIS$.

Cor. 2. *The ſame things ſuppoſed with two ranks of proportionals, the quotients of the like terms will be proportional.* $\frac{A}{F} : \frac{B}{G} :: \frac{C}{H} : \frac{D}{I}$.

For $AD = BC$, and $FI = GH$ (Prop. XII); therefore $\frac{AD}{FI} = \frac{BC}{GH}$ (Ax. 5); therefore $\frac{A}{F} : \frac{B}{G} :: \frac{C}{H} : \frac{D}{I}$ (Cor. 1. Prop. XII).

Cor. 3. *The like powers, or the like roots of proportional quantities, will be proportional. If* $A : B :: C : D$, *then* $A^n : B^n :: C^n : D^n$, *and* $\sqrt[n]{A} : \sqrt[n]{B} :: \sqrt[n]{C} : \sqrt[n]{D}$: *n being any number.*

This is plain, by ſuppoſing A, F, P all equal; as alſo B, G, Q; and C, H, R; and alſo D, I, S.

PROP. XIX.

If between any two quantities propoſed, there be interpoſed any number of terms; the proportion of the firſt to the laſt, is compounded of the firſt to the ſecond, the ſecond to the third, and ſo on to the laſt. Suppoſe A, B, C, D, E, F.

The proportion of A *to* F, *is compounded of* A *to* B, B *to* C, C *to* D, D *to* E, *and* E *to* F.

For $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} \times \frac{D}{E} \times \frac{E}{F}$ or $\frac{ABCDE}{BCDEF} = \frac{A}{F}$, all the intermediate terms deſtroying one another, in the dividend and diviſor.

PROP. XX.

In a ſeries of quantities in geometrical progreſſion, A : B : C : D : E : F : G $\div\!\div$; *the product of the extremes is equal to the product of any two means, equally diſtant from the extremes:* $AG = BF = CE$, &c.

For ſince A : B : : F : G (Def. 10); therefore $AG = BF$ (Prop. XII). And ſince B : C : : E : F; therefore $CE = (BF =)$ AG, and ſo on,

Cor. *Hence the product of the extremes, is equal to the ſquare of the middle term; when the number of terms is odd.*

PROP. XXI.

If, out of a ſeries of quantities in geometrical progreſſion, there be taken any ſeries of equidiſtant terms; that ſeries will alſo be in geometrical progreſſion.

If A : B : C : D : E : F : G : H : I : K : L : M, *in* ∺,
then B : E : H : L *are alſo* ∺.

For $\frac{B}{C} = \frac{C}{D} = \frac{D}{E} = r$, and $\frac{BCD}{CDE} = r^3 = \frac{B}{E}$ (Ax. 4). Alſo $\frac{E}{F} = \frac{F}{G} = \frac{G}{H} = r$, and $\frac{EFG}{FGH}$ or $\frac{E}{H} = r^3$; alſo $\frac{H}{I} = \frac{I}{K} = \frac{K}{L} = r$, and $\frac{HIK}{IKL}$ or $\frac{H}{L} = r^3$, *&c.* Therefore $\frac{B}{E} = \frac{E}{H} = \frac{H}{L}$ *&c.* (Ax. 8); and B : E : H : L *&c.* are ∺ (Def. 10).

PROP. XXII.

If there be a ſeries of quantities in geometrical progreſſion, A : B : C : D : E : F, &c. ∺; *their differences will alſo be in the ſame geometrical progreſſion,* A : B : : A — B : B — C : C — D, *&c.*

For ſince A : B : : B : C : : C : D, *&c.* (Def. 10); therefore A : A — B : : B : B — C : : C : C — D *&c.* (Prop. XIII). And A : B : : A — B : B — C, and B : C : : B — C : C — D (Prop. IV). That is, A : B : C, *&c.* : : A — B : B — C : C — D, *&c.*

Cor. *The ſecond, third, fourth differences,* &c. *ſhall alſo be in the ſame geometrical progreſſion.*

PROP. XXIII.

If there be a ſeries of quantities in geometrical progreſſion; the ratio of the firſt, to the ſecond, third, fourth, &c. *is in the ſimple, duplicate, triplicate,* &c. *ratio of the firſt to the ſecond, reſpectively. If* A : B : C : D : E, &c. *then* $\frac{A}{B}=\frac{A}{B}$, $\frac{A}{C}=\frac{AA}{BB}$, $\frac{A}{D}=\frac{A^3}{B^3}$, $\frac{A}{E}=\frac{A^4}{B^4}$, &c.

For $\frac{A}{B}=\frac{B}{C}=\frac{C}{D}=\frac{D}{E}$, *&c.* (Def. 10). And $\frac{A}{C}=\frac{A}{B}\times\frac{B}{C}=\frac{AA}{BB}$ (Def. 13), $\frac{A}{D}=\frac{A}{B}\times\frac{B}{C}\times\frac{C}{D}=\frac{A3}{B^3}$; alſo $\frac{A}{E}=\frac{A}{B}\times\frac{B}{C}\times\frac{C}{D}\times\frac{D}{E}=\frac{A^4}{B^4}$; *&c.*

PROP. XXIV.

If A, B, C, D, E, *&c. be a ſet of quantities in geometrical progreſſion, whoſe differences are infinitely ſmall; and n any number; then it will be,* $A^n : A^n - B^n :: A : n\times\overline{A-B}$.

Since the differences are infinitely ſmall, they will be (nearly) equal, $A-B=B-C=C-D$, *&c.* and $A-C=\overline{A-B}+\overline{B-C}=2\times\overline{A-B}$; $A-D=3\times\overline{A-B}$; $A-E=4\times\overline{A-B}$, *&c.* But $A^2 : B^2 :: A : C$, and $A^3 : B^3 :: A : D$, *&c.* (Prop. XXIII); then

$A^2 : A^2 - B^2 :: A : A - C = 2\times\overline{A-B}$ (Pr. XII).
alſo $A^3 : A^3 - B^3 :: A : A - D = 3\times\overline{A-B}$.
and $A^n : A^n - B :: A^n : n\times\overline{A-B}$.

PROP.

PROP. XXV.

In a rank of quantities in geometrical progreſſion, $A : B : C : D : E$, *whoſe number is* n; *and the ratio* $r = \frac{A}{B}$; *the laſt term* $(E) = \frac{A}{r^{n-1}}$ or $\overline{\frac{B}{A}}\Big|^{n-1} \times A$.

For $\frac{A}{B} = r$, or $A = Br$, $B = Cr$, $C = Dr$, $D = Er$.

And $A = Br = Crr = Dr^3 = Er^4$.

Therefore $B = \frac{A}{r}$ the 2d term.

$C = \frac{A}{rr}$ the 3d term.

$D = \frac{A}{r^3}$ the 4th term.

$E = \frac{A}{r^4}$ the 5th term.

And in general the n^{th} term $= \frac{A}{r^{n-1}}$.

PROP. XXVI.

In a rank of quantities in geometrical progreſſion, $A : B : C : D : E$, *whoſe number is* n, *and common ratio* $r = \frac{A}{B}$; *the ſum of all the terms is,* $\frac{AA - BE}{A - B} = \frac{BE - AA}{B - A}$.

For $A : B :: B : C :: C : D :: D : E$ (Def. 10). And $A : B ::$ ($r : 1 ::$) $A + B + C + D : B + C + D + E$, (Prop. X); that is, (putting $S =$ ſum), $A : B :: S - E : S - A$. Therefore $SA - AA = BS - BE$ (Prop. XII); and $SA - SB = AA - BE$, or $SB - AS = BE - AA$ (Ax. 2, 3); therefore $S = \frac{AA - BE}{A - B}$ or $\frac{BE - AA}{B - A}$ (Ax. 5).

Cor.

Cor. 1. *The ſum of the terms* $= A + \frac{A - E}{A - B}B$, or $A + \frac{E - A}{B - A}B$.

For $A + \frac{A - E}{A - B}B = \frac{AA - AB + AB - BE}{A - B} = \frac{AA - BE}{A - B}$, &c.

Cor. 2. *In a decreaſing geometrical progreſſion, the ſum of all the terms* $= \frac{rA - E}{r - 1}$.

For ſince $r : 1 :: S - E : S - A$. Therefore $S - E = rS - rA$ (Prop. XII); and $rS - S = rA - E$ (Ax. 2, 3); whence $S = \frac{rA - E}{r - 1}$ (Ax. 5).

Cor. 3. *In an increaſing geometrical progreſſion; put* $R = \frac{B}{A}$, *then the ſum of the terms* $= \frac{R^n - 1}{R - 1}A$.

For $B = RA$, $C = RB = R^2A$, $D = RC = R^3A$, $E = rD = r^4A$, or $E = r^{n-1}A$. But $1 : R :: S - E : S - A$, and $S - A = RS - RE$ (Prop. XII), and $RS - S = RE - A = r^nA - A$ (Ax. 2, 3); whence $S = \frac{r^nA - A}{r - 1}$ (Ax. 5).

PROP. XXVII.

In an infinite decreaſing geometrical progreſſion, $A : B : C : D : E \div\div$ &c. *Put the ratio* $\frac{A}{B} = \frac{m}{n}$; *then the ſum of all the terms* ad infinitum $= \frac{AA}{A - B}$ *or* $\frac{mA}{m - n}$.

For the ſum $= \frac{AA - BE}{A - B}$ (Prop. XXVI); but when the progreſſion is infinitely continued, the laſt term E is *o*, and then the ſum becomes $\frac{AA}{A - B}$. Alſo (by Cor. 2. Prop. XXVI), the ſum $= \frac{rA - E}{r - 1}$ becomes $\frac{rA}{r - 1} = \frac{\frac{m}{n}A}{\frac{m}{n} - 1} = \frac{mA}{m - n}$.

SECT. III.

General Proportions.

Definition and Notation.

IF A, B, C, D, *&c.* be any variable quantities, and a, b, c, d, &c. other values thereof; and if they be ſo dependent on one another, that when A is increaſed or diminiſhed to a; B, C, D, *&c.* become b, c, d, &c.

Then $A \propto B$, ſignifies that A is directly as B, or that $A : a :: B : b$.

Likewiſe $A \propto \frac{1}{C}$, denotes that A is reciprocally as C, or that $A : a :: \frac{1}{C} : \frac{1}{c}$.

Alſo $A \propto \frac{BC}{D}$, ſignifies that A is directly as B and C, and reciprocally as D, or that $A : a :: \frac{BC}{D} : \frac{bc}{d}$.

And if $AB \propto \frac{C}{D}$, the product of A, B is directly s C, and reciprocally as D; or $AB : ab :: \frac{C}{D} : \frac{c}{d}$.

And on the contrary, if $A : a :: B : b$, then A $\propto$ B, *&c.*

PROP.

PROP. I.

If one quantity A *is as a ſecond* B; *then, on the contrary, the ſecond* B *is as the firſt* A. *If* A $\propto$ B, *then* B $\propto$ A.

For A : *a* : : B : *b* (Def.).
Therefore B : *b* : : A : *a*; that is, B $\propto$ A (Def.).

PROP. II.

If one quantity A *is as a ſecond* B, *and the ſecond* B *as the third* C, *and the third* C *as a fourth* D, &c. *then the firſt* A *is as the laſt* D. *If* A $\propto$ B $\propto$ C $\propto$ D, *then* A $\propto$ D.

For A : *a* : : B : *b*,
and B : *b* : : C : *c*,
and C : *c* : : D : *d* (Def.)
therefore A : *a* : : D : *d* (Prop. I. Sect. II).
therefore A $\propto$ D (Def.).

Cor. *If one quantity* A *is as a ſecond* B, *and the ſecond* B *reciprocally as a third* C. *Then the firſt* A *is reciprocally as the third* C. *If* A $\propto$ B $\propto \frac{1}{C}$, *then* A $\propto \frac{1}{C}$.

For A : *a* : : B : *b* : $\frac{1}{C} : \frac{1}{c}$; and A $\propto \frac{1}{C}$ (Def.).

PROP. III.

If one quantity A *be as a ſecond* B, *and alſo as a third* C; *then the firſt* A *will be as the ſum or difference of the ſecond and third,* C *and* D. *If* $A \propto B \propto C$, *then* $A \propto B + C$, *or* $A \propto B - C$.

For $A : a :: B : b :: C : c$. Therefore $A : a :: B + C : b + c$, or $A : a :: B - C : b - c$ (Prop. X. Sect. II). And $A \propto B \pm C$.

PROP. IV.

Either ſide of a general proportion, may be mul-iplied or divided by any given quantity. If $A \propto B$, *hen* $A \propto nB$, *or* $A \propto \frac{B}{n}$.

For $A : a :: B : b :: nB : nb$ (Prop. V. Sect. II) $: \frac{B}{n} : \frac{b}{n}$ (Cor. 1. ibid.).

PROP. V.

If both ſides of a general proportion be multiplied or ivided by any variable quantity, they will ſtill be pro-ortional. If $A \propto B$, *and* C *a variable quantity, en* $AC \propto BC$.

For $A : a :: B : b$ (Def.). And $CA : ca :: CB :$ (Prop. VI. Cor. 1); that is, $CA \propto CB$.

Alſo $A : a :: B : b$; and $\frac{A}{C} : \frac{a}{c} :: \frac{B}{C} : \frac{b}{c}$ (Cor. 2. rop. VI); that is, $\frac{A}{C} \propto \frac{B}{C}$.

Cor. 1. *If* $Q \propto BC$, *then* $\frac{Q}{B} \propto C$, *and* $\frac{Q}{BC}$ *a given quantity, or always the ſame.*

For $\frac{Q}{BC}$ is as 1, an invariable quantity.

Cor. 2. *If* $A \propto \frac{1}{B}$, *then* $B \propto \frac{1}{A}$.

For $AB \propto 1$ (Prop. V), $\frac{AB}{A}$ or $B \propto \frac{1}{A}$ (ibid.).

PROP. VI.

Inſtead of any quantity in one ſide of a general proportion, one may ſubſtitute any other quantity proportional thereto. If $A \propto BC$, *and* $C \propto D$; *the* $A \propto BD$.

For ſince $C \propto D$, $BC \propto BD$ (Prop. V) whence $A \propto BD$ (Prop. II).

PROP. VII.

If the two ſides of one general proportion, be multiplie or divided by the two ſides of another general propor tion; they will ſtill be proportional. If $A \propto B$, *an* $C \propto D$; *then* $AC \propto BD$, *and* $\frac{A}{C} \propto \frac{B}{D}$.

For $A : a :: B : b$, and $C : c :: D : d$, therefo $AC : ac :: BD : bd$ (Prop. XVIII. Sect. II); th is, $AC \propto BD$.

And $\frac{A}{C} : \frac{a}{c} :: \frac{B}{D} : \frac{b}{d}$ (ibid. Cor. 1); that is, $\frac{A}{C} \propto \frac{B}{D}$.

Cor. 1. *The equal powers or roots of both ſides any general proportion, will ſtill be proportional.*

$A \propto B$, *then* $A^2 \propto B^2$, $A^3 \propto B^3$, $\sqrt[2]{A} \propto \sqrt[2]{}$ *&c.*

This is plain by putting C = A, and D = B, *&c.*

Cor. 2. *If* $A \propto B \propto C$, *then* $AA \propto BC$.

PROP. VIII.

If any quantity Q *be as the product of ſeveral others* A, B, *&c. then if* B, *&c. be given,* $Q \propto A$; *and if* A, &c. *be given,* $Q \propto B$.

For by Prop. IV. ſince $Q \propto AB$, and B given, $Q \propto A$. And if A be given, $Q \propto B$ (ibid.).

Cor. *If any variable quantity* Q *depends on ſeveral others* A, B; *and if* $Q \propto A$, *when* B *is invariable*; *and* $Q \propto B$, *when* A *is invariable*; *then* $Q \propto AB$, *when all are variable.*

PROP. IX.

Any general proportion may be turned into an equation, by multiplying one ſide by a proper homologous quantity.

If $A \propto BC$, *then* $A = n \times BC$. *n being ſome given quantity.*

For ſince $A \propto BC$, therefore $A : a :: BC : bc$ (Def.); and $A \times bc = a \times BC$ (Prop. XII. Sect. II); and $A = \frac{a}{bc} \times BC$, therefore $n = \frac{a}{bc}$ the quantity aſſumed for a multiplier.

Or if $mA = BC$, it will be found that $\frac{bc}{a} \times A = BC$, or that $m = \frac{bc}{a}$.

PROP.

PROP. X. *Problem.*

Any general proportion being given, $A \propto \frac{B^2C}{D}$; *to find the proportion any one has to the reſt.*

This is done by help of the foregoing propoſitions.

Since $A \propto \frac{B^2C}{D}$;

Multiply by D, then $AD \propto B^2C$ (Prop. V).

Divide by A, then $D \propto \frac{B^2C}{A}$ (Prop. V).

Divide ($AD \propto B^2C$) by B^2, and then $C \propto \frac{AD}{B^2}$ (Prop. V).

Divide ($AD \propto B^2C$) by C, and then $B^2 \propto \frac{AD}{C}$ (Prop. V).

Extract the ſquare root, $B \propto \sqrt{\frac{AD}{C}}$.

And the ſame may be done by aſſuming a given quantity m, and making $mA = \frac{B^2C}{D}$; and the foregoing proceſs is the ſame as in the reduction of algebraic equations.

FINIS.

THE

LEMENTS

OF

GEOMETRY.

IN WHICH,

e principal Propoſitions of EUCLID,
ARCHIMEDES, and others, are demonſtrated after the moſt eaſy manner.

To which is added,

Collection of uſeful Geometrical Problems.

---

rveniri ad ſummum, niſi ex principiis, non poteſt.
Quint.

LONDON,
Pted for J. NOURSE, Bookſeller in Ordinary
to his MAJESTY.
MDCCLXIII.

THE

PREFACE.

HAVING in the firſt volume treated of arithmetic, which is one of the main pillars of the mathematics; I come now geometry, which is the other pillar, on which eſe ſciences are ſupported. On theſe two founda- ns, all the other branches are built; and from m they derive their whole ſtrength and evidence. nd theſe two ſciences are eſſentially different; former conſiders numbers, without any regard extenſion; the latter conſiders extenſion, with- any regard to numbers. And both of them at their particular ſubjects in the moſt abſtract nner.

Geometry is of ſo excellent a nature and of ſuch enſive uſe, that it lays the foundation of all the es to work by, in the common affairs of life, hout which we could do nothing. For inſtance, diſtances of places, or remote objects, and their ation in reſpect of one another; cannot be had hout meaſuring, and the rules of geometry. drawing of maps or charts can only be done by metry. The meaſuring and dividing of lands, t give every man his due ſhare, cannot be per- med, without meaſuring certain figures, and ing their contents. Houſes and towns cannot uilt without knowing the figures and dimenſions

*thereof. Without this art, no place can be fortifie
to reſiſt the attacks of an enemy. Tradeſmen mu
be acquainted with the meaſures of length an
capacity. Joiners, maſons,* &c. *muſt underſtan
how to form their materials into proper figur
where there will be frequent occaſion for parallel a
perpendicular lines. And the figures they have pe
petually to deal with, are triangles, ſquares, para
lelograms, circles,* &c. *and ſuch ſolids as pyram
cones, cubes, priſms, ſpheres,* &c. *the nature
which can only be known from geometry. T
dimenſions and areas of plane figures, the conte
of ſolid bodies, cannot be had without it. So th
geometry gives life and ſpirit to all arts.*

*Geometry examines the nature of all figur
compares them together, and finds out their prope
ties. It is a key to all the other branches. T
elements of plane geometry, are likewiſe the fou
dation of the higher geometry, relating to all ſo
of curve lines, their nature and properties; a
is a neceſſary introduction to the knowledge of the*

*Geometry is a ſcience inexhauſtible, and whi
knows no bounds. In it there is always room l
for the diſcovery of new theorems. It is alſo
moſt excellent logic, teaches men how to rea
truly, and accuſtoms the mind to a habit of cl
and ſtrict reaſoning.*

*The ſcience of geometry is certainly very ol
for look as far back as we will, we ſhall alw
find men who have been profeſſors and encourag
of geometry, and the value the ancients ſet u
it, may be known from this famous motto of* Pl
ſet over the door of his academy, οὐδεὶς ἀγεωμέτρη

εισίτ

εισίτω. *Some of the principal among them who ſtudied it were,* Thales, Pythagoras, Plato, Ariſtotle, Euclid, Archimedes, Appollonius, Ptolomy, *and many more. But we are not to ſuppoſe that in theſe ancient times, this ſcience was any thing near the perfection it is now in: but in ſucceeding ages, men of great genius, by their ſtudy and induſtry, by degrees added new improvements; till at laſt it arrived at the pitch we now ſee it. So that we need not wonder that* Euclid, *or even* Archimedes, *have taken round-about methods in demonſtrating many of their propoſitions, which are now done vaſtly ſhorter and clearer. For it cannot be denied, that* Euclid's *elements abound with a great many trifling propoſitions, which are of no other uſe but to demonſtrate, in his way, the propoſitions that follow after. But they are diſpoſed in no proper order or method. For he frequently treats of different ſubjects, promiſcuouſly together, in the ſame place; without any regard to the nature of things, or their connection with one another. And as often, has the ſame ſubject to conſider in different places; which can breed nothing but confuſion. But there are likewiſe a great many propoſitions in the preſent ſyſtem of geometry, which theſe ancient mathematicians knew nothing of; and which are equally uſeful with thoſe of* Euclid.

One method of demonſtration, which Euclid *and the ancients frequently make uſe of, is* reductio ad abſurdum, *which is generally ſhorter than the direct method, and equally certain. For it is an axiom in logic, that* that ſuppoſition muſt needs be be true, which deſtroys the contrary ſuppoſition. *But though it be equally true, yet it gives not that ſatisfaction to the mind, which a poſitive proof gives.*

It is a common practice among geometers, after a propoſition is proved, for them to prove the reverſe of it. But this in many caſes is needleſs and impertinent. For where the eſſential property of a ſubject is found; there, moſt certainly, you will find that ſubject, without farther inquiry. For example, when it is proved to be the property of parallel lines, when cut by a third line, to make the alternate angles equal; or the ſum of the internal angles equal to two right angles: it is ſuperfluous to prove, that when the alternate angles are equal, or the ſum of the internal angles equal to two right ones, that theſe lines are parallel; becauſe it was proved before to be the abſolute right and property of parallel lines. Likewiſe when it is proved to be the diſtinguiſhing property of a right-angled triangle, that the ſquare of the hypothenuſe is equal to the ſum of the ſquares of the two ſides. It need not be proved, that when theſe ſquares are equal, the angle is right. In ſuch caſes, there needs, at moſt, nothing but an illuſtration, and then this method (reductio ad abſurdum) *is very properly applied.*

There are alſo many propoſitions in geometry, which are convertible; that is, where the property or predicate may become the ſubject; and the ſubject, the predicate, being of equal extent. And here a deal of labour might be ſaved in demonſtrating the propoſition both ways. For inſtance, when the

the two ſides of a triangle are equal, it may be proved, that the two oppoſite angles are equal. Or when the two angles of a triangle are equal, it may be proved, that the oppoſite ſides are equal. But it need not be proved both back and forward. And here can want nothing but the application of the former rule (reductio ad abſurdum), *to illuſtrate the reverſe. But mathematicians had rather prove too much than too little; they had rather have ſomething* ex abundanti, *than be defective. Though for my own part, I have often ſaved myſelf that ſuperfluous labour.*

To give ſome account of the method wherein I have handled this ſubject; it is in ſhort this. The firſt book treats of right lines. The ſecond of triangles. The third of polygons. The fourth of the circle. The fifth of planes. The ſixth of ſolids. The ſeventh of the ſphere. The eighth is geometrical problems. This is the method I have choſen to digeſt theſe things in, as being agreeable to the nature of the ſubject, and the mutual dependance of the ſeveral parts upon one another. The laſt book contains a collection of the moſt uſeful geometrical problems. I have ſpent but little time in demonſtrating them, as moſt of them do not need it, being perſuaded that they who underſtand the elements, will eaſily perceive their evidence, without any more words. They that would ſee more problems of this kind, may conſult the writers of practical geometry.

W. Emerſon.

THE
CONTENTS.

Page

Definitions, &c. — — — — — — 1

Book I. *Of angles, and right lines, and their rectangles* — — — — — — — 5

Book II. *Of triangles* — — — — 12

Book III. *Of quadrangles, and polygons* — 36

Book IV. *Of the circle, and its inſcribed, and circumſcribed figures* — — — — 50

Book V. *Of planes, and ſolid angles* — 85

Book VI. *Of ſolids* — — — — — 97

Book VII. *Of the ſphere, and its inſcribed and circumſcribed bodies* — — — — 125

Book VIII. *The conſtruction of geometrical problems* — — — — — — — 156

THE

ELEMENTS

OF

GEOMETRY.

DEFINITIONS.

GEOMETRY is a ſcience which teaches FIG.
and demonſtrates the properties, affecti-
ons, and meaſures of all ſorts of mag-
nitude.

2. *Magnitude* is continued quantity, or any thing
at is extended ; as a line, ſurface, or ſolid.

3. A *point* is that which has no parts. •

4. A *line* is a length without breadth or thick-
ſs.

Cor. *The extremes of a line are points.* 

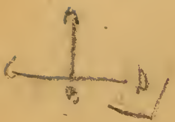
5. A *right line* is that which lies evenly, or in
e ſame direction, between two points A, B. A 1.
ve line continually changes its direction.

Cor. *Hence there can only be one ſpecies of*
ht lines, but there is infinite variety in the ſpecies
curves.

6. *Parallel lines* are thoſe which are always at the
e diſtance from one another, as AB, CD.  2.

7. An *angle* is the inclination of two lines, to
e another, meeting in a point, called the *angular*
nt. When it is formed by two right lines, it is
plain angle, as A ; if by curve lines, it is a *cur-* 3.
ineal angle.

FIG. 8. A *right angle* is that which is made b
one right line AB falling upon another CD, an
4. making the angles on each ſide equal, ABC =
ABD ; ſo that AB does not incline more to on
ſide than another : AB is called a *perpendicula*
All other angles are called *oblique angles.*



9. An *obtuſe angle* is greater than a right angl
5. as R.

10. An *acute angle* is leſs than a right angl as S.

11. *Contiguous angles*, are thoſe made by one lin falling upon another, and joining to one anothe as R, S.

12. *Oppoſite angles*, are thoſe made on contrar
ſides of two lines interſecting one another, a
6. A, B.

13. A *ſurface* is that magnitude which hath onl length and breadth.

Cor. *The extremes or limits of a ſurface are lines.*

14. A *plane* is that ſurface which lies perfectl even between its extremes ; or in which, right line any way drawn, do coincide.

15. A *plain figure*, is a plain ſurface, bounde on all ſides by one or more lines.

16. A *right-lined figure*, is a plain figure, bounded with right lines only.

Cor. *Every right-lined figure has as many angl as ſides.*

17. A *ſolid* is a magnitude extended every way or which has length, breadth, and depth.

Cor. *The terms or extremes of a ſolid, ar ſurfaces.*

18. The *ſquare of a right line* is the ſpace included by four right lines equal to it, ſet perpendicular to one another.

19. The *rectangle of two lines* is the ſpace included by four lines equal to them, ſet perpendicular to one another, the oppoſite ones being equal.

20. *Com

20. *Commenſurable magnitudes*, are ſuch as may e meaſured by one and the ſame meaſure.

21. *Incommenſurable magnitudes*, are ſuch as have o common meaſure.

22. *Rational magnitudes*, are thoſe that are ex-reſſed by a rational number, or by one that in-ludes not a ſurd.

23. *Irrational magnitudes*, are ſuch as are denoted y a ſurd, as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, &c.

AXIOMS or MAXIMS.

1. Things equal to the ſame thing, are equal one another.

2. The whole is equal to all its parts taken to-ther.

3. If equal things be added to equal things, the holes will be equal.

4. If equal things be taken away from equal ings, the remainders will be equal.

5. If equal things be equally multiplied, the oducts will be equal.

6. If equal things be equally divided, the quo-nts will be equal.

7. All right angles are equal to one another.

8. Thoſe magnitudes are equal, which being plied, exactly agree or coincide with one another.

POSTULATES.

1. Between any two points a right line may be awn.

2. That a right line or plane may be produced far as we pleaſe.

3. That a circle may be deſcribed from any cen- at any diſtance. See Book IV. Def. 1.

4. That any magnitude being given, an equal gnitude may be made.

5. That any magnitude may be ſo often multiplied, as to exceed any magnitude of the ſame kind.

6. That any magnitude may be divided into a many equal parts as we pleaſe.

Explanation of Characters.

$+$	added to, being the ſign of addition.
$-$	ſubtracted from, the ſign of ſubtraction-
$\times$	multiplied by.
$\div$	divided by.
$=$	equal to.
$::$	the mark of proportion.
$\div\!\!\div$	geometrical progreſſion.
$\backsim$	difference.
$\square$	ſquare.
▭	rectangle.
$\sqrt{}$	ſquare root.
$\sqrt[3]{}$	cube root.
A^2	A ſquared; alſo $\overline{AB}^2$ is AB ſquared.
A^3	A cubed; and AB^3 is AB cubed.
$\angle$	an angle.
$\parallel$	parallel.
$\perp$	perdendicular.

Sometimes one letter denotes a line; but if a lin is expreſſed by two letters, as AB, then the letter A, B denote the extreme points of that line.

When one letter denotes an angle, it is ſuppoſe to ſtand at the angular point; but if three letter expreſs the angle, the middle one is at the angul point; the other two in the ſides.

When three letters ſtand for a rectangle, as ABC it ſignifies AB × BC; where AB, BC are the ſide Or when four letters ſtand for a rectangle, a AB × CD; AB and CD are the ſides.

The citations are thus to be underſtood; the fir number denotes the Prop. the ſecond the Boo When proportion is referred to, it ſignifies ge metrical proportion.

BOO

BOOK I.

f Angles, and Right Lines, and their Rectangles.

PROP. I.

FIG. 7. *If to any point* C *in a right line* AB, *ſeveral other ht lines* DC, EC *are drawn on the ſame ſide; all e angles formed at the point* C, *taken together, are ual to two right angles*, ACD + DCE + ECB *two right angles.*

FOR ſuppoſe PC to be perpendicular to AB, then ſince ACP and PCB = two right angles, ef. 8); and theſe angles ACD, DCE, ECB cupy the ſame angular ſpace; therefore they are equal to two right angles (Ax. 2).

Cor. 1. *All the angles made about one point in a ne, being taken together, are equal to four right gles.*

Cor. 2. *If all the angles at* C, *on one ſide of the e* AB, *happen to be equal to two right angles; then* CB *is a ſtraight line.*

PROP. II.

8. *If two right lines*, AB, CD, *cut one another; the poſite angles* E *and* G *will be equal.*

For AEC + E = two right angles (Prop. I), d AEC + G = two right angles (ibid.); there-re AEC + E = AEC + G (Ax. 1), and E = G x. 4). After the ſame manner AGC = BGD.

FIG. 8. Cor. *If* AB *is a right line, and* CEB *happen to b* *equal to* AGD, *or* E *equal to* G; *then* CD *is a righ* *line.*

PROP. III.

9. *A right line* AB, *which is perpendicular to one o* *two parallels* FH, *is perpendicular to the other* DC.

For ſuppoſe the end HC of the figure CBAH, b raiſed up, and turned over the line AB, ſo tha HC may fall towards FD, the line AB remainin fixed. Then ſince the ∠BAH = BAF (Ax. 7) therefore the line AH will fall upon AF, and le the line BC fall on the line B*d*. Draw the line *d*D perpendicular to HF. Now ſince FH, DC ar parallels; therefore the diſtances BA, DF, and *d* (or CH) are all equal (Def. 6); therefore the point D, *d* muſt coincide; and therefore the line B*d* coincides with BD. Therefore ∠ABC = ABD = right angle (Def. 8).

Cor. 1. *Hence two lines* FH, DC, *perpendicular t* *the ſame line* AB, *are parallel.*

Cor. 2. *Hence the ſegments of two parallels, intercepted between two perpendiculars* AB, HC, *ar* *equal*, AH = BC.

For ſince the angles at A, H, B, C are right therefore the two lines AB, HC, interſecting AH and being both perpendicular thereto, are paralle (Cor. 1); and therefore AH = BC (Def. 6).

PROP. IV.

10. *If a right line* CG, *interſect two parallels* AD FH; *the alternate angles*, ABE, *and* BEH, *will b* *equal.*

Let AE, BH be perpendicular to AD, and FH Then ſince AE = BH (Def. 6), and AB = EH (Prop. III. Cor. 2), and the angles at A and H right

right; therefore if the figure EHB be laid upon FIG. BAE, the ∠H upon A, and HE upon AB, and 10. conſequently HB will fall upon AE; and the whole figure EHB coincides with the figure BAE, and the angle HEB with EBA, and conſequently theſe angles are equal. Likewiſe the angles DBE and FEB will be equal, being the remainders to two right angles (Ax. 4).

Cor. 1. *The external angle* CBD, *is equal to the internal angle on the ſame ſide* BEH.

For CBD = ABE = BEH (Prop. 2).

Cor. 2. *The two internal angles on the ſame ſide are equal to two right angles*; DBE + BEH = *two right angles.*

For EBA = BEH (Prop. IV), and DBE + EBA = two right angles, = DBE + BEH.

Cor. 3. *If the angles* CBD *and* BEH *are equal*; *or* ABE *and* BEH *equal*; *or* DBE + BEH *be equal to two right angles*; *the lines* AD, FH *are parallel.*

For if any angle is greater than is here mentioned, it deſtroys the parallelism of the lines AD, FH.

PROP. V.

Two lines drawn between two parallels AB, CD, 11. *making equal angles with either of them, will be equal,* AC = BD.

Draw CF, DH perpendicular to FB, then ſince ∠ACD = BDI; alſo FCD and HDI right angles (Prop. III), the remainders FCA and HDB are equal; and the angles at F and H being right, and FC = HD (Def. 6); therefore if HD be laid on FC, the line DB will fall on CA, and HB on FA, and B on A; therefore DB = CA.

FIG. Cor. 1. *If the lines* AC *and* BD *are equal, then the angles* ACD *and* BDI *are equal.*

11. For if one angle was greater, it would make the lines AC, BD unequal.

Cor. 2. *The parts intercepted are equal,* AB = CD.

For FA = HB; and adding AH, AH + HB, or AB = FA + AH, or FH = CD (Cor. 2. Prop. III).

Cor. 3. *If two equal and parallel lines* AB, CD, *be joined by two others* AC, BD; *they ſhall alſo be equal and parallel.*

PROP. VI.

12. *Right lines* AB, CD, *parallel to the ſame right line* EF, *are parallel to one another.*

Let GI cut the three lines, then ſince AB, EF are parallel, AGI = EHI (Cor. 1. Prop. IV); and becauſe EF and CD are parallel, ∠EHI = DIG (Prop. IV). Therefore AGI = DIG (Ax. 1), whence AB, CD, are parallel (Cor. 3. Prop. IV).

PROP. VII.

13. *If two lines* AB, BD, *which cut one another, be parallel to two other lines* EC, CH, *which alſo cut one another; they ſhall contain equal angles* ABD = ECH.

For produce EC to interſect BD in F; then by reaſon of the parallels AB, EF, ∠ABD = EFD (Cor. 1. Prop. IV); and ſince BD and CH are parallel, EFD = ECH (ibid.); therefore ABD = ECH.

PROP.

PROP. VIII.

FIG.

Two right lines AF, AB *being given; and one of them* AB *be divided into ſeveral parts; the rectangle under the two whole lines, will be equal to all the rectangles contained under the whole line, and the ſeveral ſegments of the other;* $ABGF = ADHF + DEIH + EBGI$. 14.

For let AF be perpendicular to AB, and DH, EI, BG equal to AF, and alſo perpendicular to AB. Then $AD \times AF =$ rectangle ADHF, and $HD \times DE$, or $FA \times DE =$ rectangle DEIH, and $IE \times EB$, or $AF \times EB =$ rectangle EBGI (Def. 19); but the ſum of theſe rectangles fill the ſame ſpace as ABGF, and therefore they are equal (Ax. 8).

Cor. 1. *If both lines be divided into parts, the ſum of the rectangles of all the parts, is equal to the rectangle of the wholes.*

Cor. 2. *If the two given lines be equal; the ſum of the rectangles under the whole and the parts, is equal to the ſquare of the whole.*

PROP. IX.

If a line AC *be divided into two parts at* B; *the rectangle under the whole, and one of the ſegments,* $AC \times BC$, *is equal to the rectangle of the ſegments and the ſquare of the ſaid ſegment,* $AB \times BC + BC^2$. 15.

Suppoſe AF, BE, CD all equal to BC, and perpendicular to AC; then the rectangle ACDF $= AC \times CD = AC \times BC$ (Def. 19); alſo $AB \times BC = AB \times BE =$ rectangle ABEF, and $BC \times CD$ or $BC^2 = BCDF$ (Def. 18). But $ABEF \times BCDE$ fill the rectangle ACDF, and therefore they are equal (Ax. 8).

PROP. X.

FIG. 16. *If a right line* AC *be divided into two parts* AB, BC; *the ſquare of the whole line is equal to the ſquares of both the parts, and twice the rectangle of the parts,* $AC^2 = AB^2 + BC^2 + 2AB \times BC$.

Let AG, BH, CI be equal to AC, and perpendicular thereto, and AD, BE, CF equal to AB; then FI = BC, *&c.* then ABED is the ſquare of AB (Def. 18), and EFIH is the ſquare of BC; and the figures BF and EG, are the rectangles of BC and BE, and DG and DE; or of AB and BC twice taken (Def. 19). But all theſe fill the ſquare AI, and therefore are equal to it (Ax. 8).

PROP. XI.

16. *The ſquare of the difference of two lines* AC, BC, *is equal to the ſum of their ſquares, wanting twice their rectangle,* $AB^2 = AC^2 + BC^2 - 2AC \times BC$.

For the ſquare AI contains the ſquare AE, the rectangle CH, and rectangle DH; that is, $AC^2 = AB^2 + CH + DH$; and adding FH, $AC^2 + FH = AB^2 + CH + DI$; that is, $AC^2 + BC^2 = AB^2 + 2ACB$, and AB^2 or $\overline{AC - BC}^2 = AC^2 + BC^2 - 2ACB$.

PROP. XII.

16. *The rectangle of the ſum and difference of two lines* AC, AB, *is equal to the difference of their ſquares,* $\overline{AC + AB} \times BC = AC^2 - AB^2$.

For the difference of the ſquares AI and AE is the rectangles $CH + HD = \overline{BH + HG} \times BC = \overline{AC + AB} \times BC$.

PROP.

PROP. XIII.

The ſquare of the ſum, together with the ſquare of the difference of two lines, is equal to twice the ſum of their ſquares.

Let the lines be A, E. Then
the ſquare of $A + E = A^2 + E^2 + 2AE$ (Prop. X).
the ſquare of $A - E = A^2 + E^2 - 2AE$ (Prop. XI).
then $\overline{A+E}^2 + \overline{A-E}^2 = 2A^2 + 2E^2$ (Ax. 3).

PROP. XIV.

The difference of the ſquares, made of the ſum and difference of two right lines, is equal to four times their rectangle.

For if A, E be the lines, then
$\overline{A+E}^2 = A^2 + E^2 + 2AE.$
$\overline{A-E}^2 = A^2 + E^2 - 2AE.$
difference $=$ $4AE.$

Cor. *The ſquare of the ſum is equal to the ſquare of the difference, together with four times their rectangle.*

BOOK II.

Of Triangles.

DEFINITIONS.

1. A *Triangle* is a plain figure bounded by three right lines, called the *ſides* of the triangle.

2. An *equilateral triangle* is that which has three equal ſides.

3. An *equiangular triangle* is that which has three equal angles.

4. An *iſoſceles triangle* is that having two ſides equal.

5. A *right-angled triangle* is that which has a right angle. The ſide oppoſite to the right angle is called the *hypothenuſe.*

6. An *oblique triangle* is that having oblique angles.

7. An *obtuſe angled triangle* has one obtuſe angle.

8. An *acute angled triangle* has three acute angles.

9. A *ſcalenous triangle* has three unequal ſides.

10. *Similar triangles* are thoſe whoſe angles are reſpectively equal, each to each. And *homologous ſides* are thoſe lying between equal angles.

11. *Baſe* of a triangle, is the ſide on which a perpendicular is drawn from the oppoſite angle called the *vertex*; the two ſides, proceeding from the vertex, are called the *legs.*

PROP.

PROP. I. FIG.

In any triangle ABC, *if one ſide* BC *be drawn out; the external angle* ACD *will be equal to the two internal oppoſite angles* A, B. 17.

Draw CE parallel to AB, then the $\angle A = ACE$ (4. 1); alſo the $\angle B = ECD$ (Cor. 1. ibid.); therefore $A + B = ACE + ECD = ACD$ (Ax. 3).

PROP. II.

In any triangle ABC, *the ſum of the three angles is equal to two right angles*, $A + B + C =$ *two right angles.* 17.

For $A + B = ACD$ (Prop. I), and $A + B + C = ACD + ACB$ (Ax. 3) = two right angles (1. I).

Cor. 1. *If two angles in one triangle, be equal to two angles, in another; the third will alſo be equal to the third.*
Cor. 2. *If one angle of a triangle be a right angle, the ſum of the other two will be equal to a right angle.*
Cor. 3. *There can only be one perpendicular drawn, to any line, from a given point.*

PROP. III.

The angles at the baſe of an iſoſceles triangle, are equal $\angle C = B$. 18.

For let AD biſect the angle BAC; then if the triangle DAC be laid upon the triangle DAB; then by reaſon of the equal angles at A, and $AC = AB$, AC will coincide with AB, and C with B, and CD with BD; and therefore $\angle ACD = ABD$.

Cor. 1. *If the angles* B, C *at the baſe be equal, the ſides* AB, AC *are equal.*

Cor. 2. *An equilateral triangle is alſo equiangular; and the contrary.*

Cor.

FIG. Cor. 3. *The line which is perpendicular to the baſe of an iſoſceles triangle, biſects it and the vertical angle.*

Cor. 4. *Only two equal lines can be drawn from a given point to a right line.*

For if AB = AD = AC; then ∠B as well as ∠D = ∠C, which is abſurd (Prop. I).

PROP. IV.

In any triangle, the greateſt ſide is oppoſite to the greateſt angle, and the leaſt to the leaſt.

19. Let AC be the greateſt ſide, and ſuppoſe AD = AB, then the ∠ADB = ABD (Prop. III), but ADB = DBC + DCB (Prop. I); therefore ADB is greater than C; whence ABD is greater than C, therefore much more is ABC greater than C. After the ſame manner it is proved, that ABC is greater than A.

And if AB be the leaſt ſide, C is leſs than ABC; and may be proved in like manner to be leſs than A.

PROP. V.

20. *In any triangle* ABC, *the ſum of any two ſides* BA, AC, *is greater than the third* BC.

Produce the ſide BA, and let AD = AC, and draw DC; then ſince ∠ACD = D (Prop. III); therefore BCD is greater than D; and therefore the oppoſite ſide BD is greater than BC, that is, BA + AC is greater than BC.

Cor. 1. *A right line is the ſhorteſt diſtance between any two points.*

21. Cor. 2. *The ſum of two lines* BD, DC, *drawn from two angles to any point within the triangle, is leſs than the two ſides of the triangle*; BD + DC *is leſs than* BA + AC, *but contain a greater angle.*

For drawing BDE, then, in the triangle BAE, BE is leſs than BA + AE (Prop. V), add EC, then BE + EC is leſs than BA + AC. And in the triangle DEC, DC is leſs than DE + EC; add BD, and BD + DC is leſs than BE + EC, and much leſs than BA + AC. FIG.

Alſo ∠BDC is greater than DEC, which is greater than A (Prop. I).

PROP. VI.

If two triangles ABC, *abc*, *have two ſides and the included angle equal in each; theſe triangles, and their correſpondent parts, ſhall be equal.* 22.

For ſince the ∠A = *a*, and AB = *ab*, alſo AC = *ac*, therefore if A be laid upon *a*, ſo that AB fall upon *ab*, then AC will fall upon *ac*, the point B will coincide with *b*, and C with *c*; therefore the whole triangles coincide. Whence the baſe CB = *cb*, ∠B = *b*, and C = *c*. And the whole triangles are equal.

Cor. *If two triangles* ABC, *abc*, *have two ſides reſpectively equal; that which has the greater baſe, has the greater oppoſite angle; and the contrary.*

For if the ſides CA, BA intercept a greater baſe BC, the angle at A will be ſo much the wider or greater; and as the angle increaſes, the more of the baſe it intercepts, as is evident.

PROP. VII.

If two triangles ABC *and abc*, *have two angles and a ſide equal, each to each; the remaining parts ſhall be equal, and the whole triangles equal.* 22.

For ſince two angles are equal, the third will be equal (Cor. 1. Prop. II); therefore if the equal ſides BC and *bc* be laid one upon another, then, by reaſon of the equal angles B and *b*, C and *c*, the

FIG. the ſides BA and *ba* will coincide, as alſo CA and *ca*, and A will fall on *a*; whence all the parts will be equal (Ax. 8).

PROP. VIII.

If two triangles have all their ſides reſpectively equal; all the angles will be equal, and the wholes equal.

23. For if the baſe of one be laid upon the baſe of the other, the other two ſides will coincide, provided the correſpondent ones lie the ſame way. For if you ſay they don't coincide, let one triangle be ABC, the other ABD: then ſince AB, AC are equal to AB, AD (hyp.), and the angle BAD greater than BAC, therefore BD is greater than BC (Cor. Prop. VI); contrary to the hypotheſis.

Cor. 1. *From two points in a right line, as* A *and* B, *two lines equal to* AC, BC *cannot be drawn to any other point* D.

Cor. 2. *Triangles mutually equilateral, are mutually equiangular.*

PROP. IX.

24. *If in two triangles* ABC, *abc*; *two ſides* AC, CB, *of the one, be equal to ac, cb of the other; and an oppoſite angle* A *equal to the correſpondent oppoſite angle a; and the other oppoſite angles* B, *b*, *either both acute or both obtuſe; the remaining parts of the triangles will be equal.*

For if *cab* be laid upon CAB, ſo that *ca* fall upon CA; then ſince the $\angle a =$ A, *ab* will fall upon ABD. And as *c* falls upon C; *cb* will fall upon either CB or CD (Cor. 4. Prop. III); which here will be CB, as the angle at *b* is obtuſe. Therefore the triangles coincide, and all the parts are equal.

PROP. X.

FIG. 25.

Triangles BCA, BCF, *ſtanding upon the ſame baſe, and between the ſame parallels, are equal.*

Let CD be parallel to BA, and BE to CF. Then the triangle CBA = ADC (Prop. VI); for BA = CD (5. 1); and CB = AD (Cor. 2. ibid.), and ∠B = D (4. 1). Therefore the triangle BCA = half of BCDA. For the ſame reaſon BCF = BEF = half of CBEF.

Again, the triangles BAE, CDF are equal, having two ſides and the contained angle equal; add the figure BCDE, and then BCDA = BCFE, and their halves BCA = BCF.

Cor. 1. *Triangles of equal baſes and hights are equal.*

For if their baſes be laid upon one another, the angular points of both (by reaſon of their equal hight) will fall in the ſame parallel; and are therefore equal (Prop. X).

Cor. 2. *Every triangle is equal to half the rectangle of its baſe and hight.*

For ſuppoſe CBA to be a right angle, then it was proved that the triangle CBA is half of the rectangle CBAD; and CBF (equal to it), is therefore equal to half that rectangle.

PROP. XI.

26.

Triangles ABC, ABD, *of the ſame hight, are in proportion to one another as their baſes* BC, *and* BD.

Divide BC into any number of equal parts BF, FG, GH, HC; and BD into ſome number of the ſame equal parts, BI, IK, KD. The triangles ABF, AFG, *&c.* and ABI, AIK, *&c.* are all equal (Cor. 1. Prop. X); and the triangle ABC contains

FIG. 26. ABF as oft as BC contains BF; alſo ABD contains ABI or ABF as oft as BD contains BI or BF; whence ABF : BF : : ABC : BC : : ABD : BD (Def. 4. Proportion and Cor. 2. Prop. XIV. ibid.).

Cor. 1. *Hence triangles are to one another as their baſes and altitudes.*

It follows from this Propoſition, and Cor. 2. Prop. X. therefore,

Cor. 2. *Triangles of equal baſes, are as their hights.*

PROP. XII.

27. *If a line* DE *be drawn parallel to one ſide* BC, *of a triangle; the ſegments of the other ſides will be proportional*; AD : DB : : AE : EC.

For draw BE, DC; then the triangle DEB = triangle DEC (Prop. X); and triangle ADE : BDE : : AD : BD (Prop. XI); and triangle ADE : CDE : : AE : CE (ibid.); therefore AD : DB : : AE : EC (Prop. I. Proportion).

Cor. 1. *If the ſegments be proportional,* AD : DB : : AE : EC; *then the line* DE *is parallel to the ſide* BC.

For if theſe lines were not parallel, the triangles DEB and DEC would not be equal (Cor. 2. Prop. X); and the ſegments would not be proportional.

Cor. 2. *If ſeveral lines be drawn parallel to one ſide of a triangle, all the ſegments will be proportional.*

Cor. 3. *A line, drawn parallel to any ſide of a triangle; cuts off a triangle ſimilar to the whole.*

For ∠D = B, and ∠E = C (Cor. 1. Prop. IV. I); therefore they are ſimilar (Def. 10).

Cor. 4. *The whole ſides are as the ſegments*; AB : DB : : AC : EC.

For

For it is AD : DB : : AE : EC (Prop. XII), FIG.
vhence AD + DB (AB) : DB : : AE + EC (AC) 27.
EC (Prop. XIII. Proportion).

PROP. XIII.

In ſimilar triangles, the homologous ſides are proortional; AB : AC : : DE : DF. 28.

In the longer ſide AC make Af = DF, the longer ide. And in the ſhorter ſide AB, make the ſhorter ide DE = Ae; and draw ef; then the ∠ A being uppoſed = to D, and the comprehending ſides qual, ∠ Aef = E, and Afe = F (Prop. VI). herefore Aef = B, and Afe = C; conſequently f is parallel to BC (Cor. 1. Prop. 4. I); thereore AB : eB : : AC : fC (Cor. 4. Prop. XII); and B : AB — eB (Ae) : : AC : AC — fC (Af), Prop. III. Proportion). That is, AB : DE : : AC : DF, r AB : AC : : DE : DF (Prop. IV. Proportion).

And if a triangle was made at the ∠ C equal to FE; it will appear the ſame way, that AC : CB : DF : FE. Whence AB : CB : : DE : EF Prop. XV. Proportion).

Cor. *A line* AE *drawn from the oppoſite angle* A, *cuts wo parallel lines proportionally*; BE : EC : : DI : IF. 29.

For BE : DI : : AE : AI : : EC : IF.

PROP. XIV.

If two triangles have one angle equal to one, and the des about the equal angles proportional; theſe tringles are ſimilar. 28.

For let ∠ D = A, and let the triangle DEF be laid pon ABC; then, by reaſon of the equal angles, the ides DE, DF will fall upon AB, AC, the points E, upon e and f. Then ſince DE (Ae) : DF (Af) : : B : AC, or Ae : AB : : Af : AC, therefore Ae :

FIG. 28. *e*B : : A*f* : *f*C (Prop. XIII. Proportion) ; whence *ef* is parallel to BC, (Cor. 1. Prop. XII) ; and ∠*e* or E = B, as alſo *f* or F = C (Cor. 1. Prop. IV. I). Whence the triangles DEF, ABC are ſimilar (Def. 10).

PROP. XV.

30. *If two triangles have all their ſides reſpectively proportional, theſe triangles are ſimilar*; AB : DE : : BC : EF : : AC : DF.

Let the ∠FEG = B, and EFG = C, then G = A (Cor. 1. Prop. II) ; whence GE : EF : : AB : BC (Prop. XIII) : : DE : EF (hyp.); therefore GF = DE, (Ax. 7. Proportion). Likewiſe GF : EF : : AC : BC : : DF : EF ; therefore GF = DF (Ax. 7. Proportion). Whence the triangles DEF, GEF have all their ſides reſpectively equal ; and are therefore equiangular ; therefore G = D = A, DEF = GEF = B, and GFE = DFE = C.

PROP. XVI.

If two triangles have one angle in each, equal ; *and the ſides about the ſecond angles proportional* ; *and the third angles both of one kind, acute or obtuſe* ; *theſe triangles are ſimilar.*

31. Let ∠A = D, and AB : BC : : DE : EF. Make ∠ABG = DEF, then ∠G = F (Cor. 1. Prop. II.); whence AB : BG : : DE : EF (Prop. XIII.) : : AB : BC, therefore BG = BC, and BCG is an iſoſceles triangle, and AGB is obtuſe, of the ſame kind with DFE ; and ACB is acute, the ſame as DIE ; whence the angles F and G, or I and C, muſt be of the ſame kind, to have the triangles ſimilar.

SCHOLIUM.

This does not always hold good, if the angles B and E are required to be of the ſame kind, inſtead of

of G and F. For if ABC be acute, ABG will alſo FIG.
be acute; but ABG is not ſimilar to DEI, nor ABC 31.
to DEF; though ABC be ſimilar to DEI, and ABG
to DEF.

PROP. XVII.

Equal triangles, that have one angle equal, have the ſides about the equal angles reciprocally proportional.

Let the oppoſite angles at B be the equal angles, 32.
and ABC, DBE, the two equal triangles; then
AB : BE : : DB : BC (hyp.).

Draw CE, then AB : BE : : triangle ABC or DBE : triangle CBE (Prop. XI) : : DB : BC.

Cor. 1. *Thoſe triangles are equal, that have the ſides about the equal angles, reciprocally proportional.*

For triangle ABC : CBE : : AB : BE (Prop. XI) : DB : BC (hyp.) : : triangle DBE : CBE; therefore triangle ABC = DBE (Ax. 7. Proportion).

Cor. 2. *Equal triangles have their baſes and hights reciprocally proportional.*

For each triangle is equal to a right-angled triangle of the ſame baſe and hight (Prop. X); and then the ſides about the right angles, are reciprocally proportional (Prop. XVII).

PROP. XVIII.

Like triangles ABC *and* DEF *are in the duplicate* 33.
ratio, or as the ſquares of, their homologous ſides, BC,
EF.

Let there be taken BG, ſo that BC : EF : : EF : BG, and draw AG. Then ſince AB : DE : BC : EF (Prop. XIII) : : EF : BG (Conſtruct.); therefore the triangle ABG = DEF. But ABC ABG or DEF : : BC : BG (Prop. XI) : : BC^2 EF^2 (Prop. XXIII. Proportion).

FIG.

PROP. XIX.

34. *Triangles that have one angle equal to one, are to one another in the complicate ratio of the ſides about the equal angles*; $ABC : EBD :: AB \times BC : EB \times BD$.

Draw CE, then CD, AE being ſtraight lines, the angles at B are equal (Prop. II. 1). Then triangle $ABC : CBE :: AB : BE$ (Prop. XI), and $CBE : EBD :: CB : BD$ (ibid.); therefore $ABC : EBD :: AB \times CB : BE \times BD$ (Cor. 1. Prop. XVIII. Proportion).

PROP. XX.

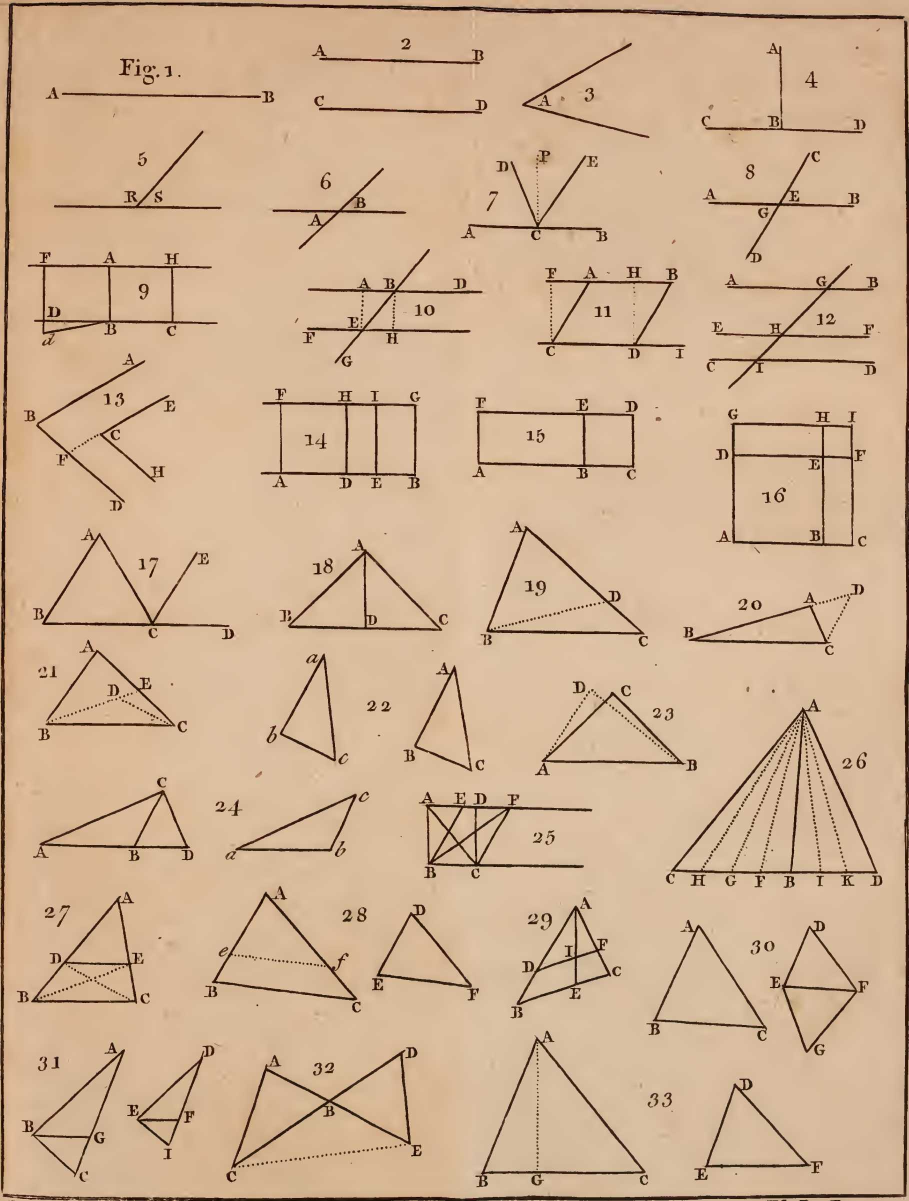
35. *In a right-angled triangle* BAC, *if a perpendicular be let fall from the right angle upon the hypothenuſe, it will divide it into two triangles ſimilar to one another and to the whole*, ABD, ADC.

For in the triangles ABD, ABC, the angle B is common to both, and angles D and BAC are right ones; therefore the remaining angles BAD and BCA are equal; therefore the triangles ABD and ABC are ſimilar.

Again, in the triangles ACD and ACB, $\angle C$ is common, $\angle D = CAB$, and therefore $\angle DAC = B$, therefore ACD and ABC are ſimilar; and conſequently ABD and ADC.

Cor. 1. *The rectangle of the hypothenuſe and either ſegment is equal to the ſquare of the adjoining ſide.*
For $BD : BA :: BA : BC$ (Prop. XIII), and $CD : CA :: CA : CB$ (ibid.); whence $BD \times BC = BA^2$, and $CD \times CB = CA^2$ (Prop. XII. Proportion).

Cor. 2. *The rectangle of the hypothenuſe and perpendicular, is equal to the rectangle of the legs.*
For $BC : AB :: AC : AD$ (Prop. XIII), and $AB \times AC = BC \times AD$ (Prop. XII. Proportion).



Geometry. Pl.I. Pa. 22.

Cor. 3. *The perpendicular is a mean proportional between the ſegments of the hypothenuſe.* FIG. 35.

For $BD : DA : DC$, and $BD \times DC = DA^2$.

Cor. 4. *The ſegments of the hypothenuſe are as the ſquares of the adjoining ſides.*

For by this Prop. $BD : DA :: BA : AC$ (Prop. XIII), and $BD^2 : DA^2 :: BA^2 : AC^2$ (Cor. 3 Prop. XVIII. Proportion). And by Cor. 3. (and Prop. XXIII. Proportion) $BD : DC :: BD^2 : DA^2 :: BA^2 : AC^2$.

Cor. 5. *As the perpendicular, to the hypothenuſe; ſo the rectangle of the ſegments, to the rectangle of the legs.*

For $AD : AB :: CD : CA$, by the ſim. triangles BAD, DAC.

And $BA : BC :: BD : BA$ by the ſim. triangles BAC, BAD.

Therefore $AD : BC :: BDC : BAC$ (Cor. 1. Prop. XVIII. Proportion).

Cor. 6. *The diſtance of the right angle, from the middle of the hypothenuſe, is equal to half the hypothenuſe.*

For let $Bo = oC$, and draw *on*, *or* parallel to AC, AB; and draw A*o*. Then $Bn = nA$, and $Cr = rA$ (Prop. XII); and the angles at *n* and *r* are right (Cor. 1. Prop. IV. I). Then the triangles B*on*, A*on*, as alſo the triangles C*or*, A*or*, have two ſides, and the included angle, equal; therefore $Bo = Ao = Co$ (Prop. VI).

PROP. XXI.

In a right-angled triangle BAC, *the ſquare of the hypothenuſe* BC, *is equal to the ſum of the ſquares of the two ſides*, BA, AC. 36.

FIG. 36. For let BG be the ſquare deſcribed on BC, and draw ADF perpendicular to BC, or parallel to CG or BE. Then $BA^2 =$ rectangle of BD and BC or BE (Cor. 1. Prop. XX), = rectangle BF. Alſo the ſquare of AC = rectangle of CD and CB = rectangle CF (ibid.) : but rectangle BF + CF = ſquare BG (Ax. 8); therefore BG or the ſquare of $BC = BA^2 + AC^2$.

Cor. 1. *The ſquare of either ſide is equal to the difference between the ſquares of the hypothenuſe and the other ſide*; $BA^2 = BC^2 - AC^2$, *and* CA^2 $BC^2 - BA^2$.

Cor. 2. *The rectangle of the ſum and difference of the hypothenuſe and one of the ſides, is equal to the ſquare of the other ſide.*

For $BA^2 = BC^2 - AC^2$ (Cor. 1) $= \overline{BC + AC} \times \overline{BC - AC}$ (Prop. XII. I).

Cor. 3. *If the ſquare of one ſide of a triangle be equal to the ſum of the ſquares of the other two ſides; then the angle comprehended by them is a right angle.*

For if it was greater or leſs than a right angle, the oppoſite ſide would be greater or leſs than the hypothenuſe of a right-angled triangle (Cor. Prop. VI); and its ſquare greater or leſs than the ſquares of the other ſides.

Cor. 4. *A perpendicular* CA *is the neareſt diſtance of a point* C, *from a right line* BA.

Cor. 5. *In any triangle* ACB, *if a perpendicular be let fall from the oppoſite angle* A, *on the baſe* CB. *The difference of the ſquares of the ſides, is equal to the difference of the ſquares of the ſegments,* $AB^2 - AC^2 = BD^2 - CD^2$.

For $AB^2 - BD^2 = AD^2 = AC^2 - CD^2$ (Cor. 1. Prop. XXI). And $AB^2 - AC^2 = BD^2 - CD^2$ (Ax. 3, 4).

PROP.

PROP. XXII.

FIG. 37.

In an obtuſe angled triangle ABC, *if a perpendicular be let fall upon the baſe; or one ſide adjoining to the obtuſe angle* B; *then the ſquare of the ſide oppoſite to that obtuſe angle is equal to the ſum of the ſquares of the two leſſer ſides, together with twice the rectangle of the baſe and the diſtance of the perpendicular from the obtuſe angle:* $AC^2 = AB^2 + CB^2 + 2CBD$.

For $AC^2 = AD^2 + CD^2$ (Prop. XXI) $= AD^2 + CB^2 + BD^2 + 2CBD$ (10. I) $= AB^2 + CB^2 + 2CBD$ (Prop. XXI).

Cor. *The diſtance of the perpendicular from the btuſe angle*, $BD = \frac{AC^2 - AB^2 - CB^2}{2CB}$.

PROP. XXIII.

FIG. 38. 39.

If a perpendicular be let fall upon the baſe, or ſide djoining to an acute angle B, *of any triangle. Then,*

The ſquare of the ſide oppoſite to that acute angle, toether with twice the rectangle, of the baſe, and the iſtance of the perpendicular from the acute angle; is qual to the ſum of the ſquares of the two other ſides: $C^2 + 2CBD = AB^2 + BC^2$.

For $AC^2 = AD^2 + DC^2$ (Prop. XXI) $= AD^2 + BC^2 + BD^2 - 2BD \times BC$ (Prop. XI. I) $= AB^2 + BC^2 - 2CBD$ (Prop. XXI). And $AC^2 + 2CBD = AB^2 + BC^2$ (Ax. 3).

Cor. *The diſtance of the perpendicular from the acute gle* B *is* $= \frac{AB^2 + BC^2 - AC^2}{2CB}$.

PROP. XXIV.

FIG. 40. 41.

In any triangle ABC, *let fall a perpendicular* AD *the baſe* BC, *and make* DF = DB. *Then*

As the baſe, CB :
to ſum of the ſides, AC + AB : :

So

FIG. *So difference of the ſides,* $AC - AB$:
40. *to difference of the ſegments of the baſe,* } CF.
41. *or the alternate baſe*

For $CA^2 - AB^2 = CD^2 - DB^2$ (Cor. 5 Prop. XXI); that is,
$\overline{CA + AB} \times \overline{CA - AB} = CF \times CB$ (Prop. XII. I)
whence $CB : CA + AB :: CA - AB : CF$ (Cor. 1 Prop. XII. Proportion).

Cor. *The difference of the ſquares of the ſides, i equal to twice the rectangle of the baſe, and the diſtanc of the perpendicular from the middle of the baſ* $CA^2 - AB^2 = 2CB \times oD$.

For if $Co = oB$, then $CA^2 - AB^2 = CF \times CB = \frac{1}{2}CF \times 2CB$; but $\frac{1}{2}CF = Do$; for (Fig. 40 $CF = 2Bo - FB$, and $\frac{1}{2}CF = Bo - BD = Do$ And (Fig. 41) $CF = 2Bo + FB$, and $\frac{1}{2}CF = B$ $+ BD = Do$.

PROP. XXV.

42. *If an angle* A *of a triangle* BAC *be biſected by a right line* AD, *which cuts the baſe; the ſegments o the baſe will be proportional to the adjoining ſides of th triangle;* $BD : DC :: AB : AC$.

Produce BA, and make $AE = AC$, and draw th line CE; becauſe $AE = AC$, the $\angle ACE = E$ (Prop. II) $= \frac{1}{2}BAC$ (Prop. I) $= BAD$ (hyp.) Therefore DA, CE are parallels (Cor. 3. Prop. IV) Therefore $BA : AE$ or $AC :: BD : DC$ (Prop. XII)

Cor. 1 *If the ſides be as the ſegments of the baſe the line* AD, *biſects the angle* A.

For ſince $BA : AC$ or $AE :: BD : DC$, DA and CE are parallels (Cor. 1. Prop. XII); an $BAD = \angle E$, and $DAC = ACE = E$ (Prop. III) Whence $BAD = DAC$, and A is biſected by AD.

43. Cor. 2. *If a line biſecting the vertical angle of triangle cuts the baſe, it will be*

As the ſum of the ſides, $BA + AC$: FIG.
to their difference, $BA - AC$: : 43.
So the baſe, BC :
to difference of the ſegments $BD - DC$.

For $BA : AC :: BD : DC$ (Prop. XXV), and $BA + AC : BA - AC :: BD + DC$ (BC) $: BD - DC$ or $2DO$ (Prop. XIII. Proportion); where O is the middle point of the baſe BC.

PROP. XXVI.

43. *If an angle* A *of a triangle* ABC, *be biſected by a right line* AD, *which cuts the baſe*; *the ſquare of the biſecting line, together with the rectangle of the ſegments, is equal to the rectangle of the ſides*; $AD^2 + BDC = BAC$.

Produce AD and make the $\angle DBP = DAC$. Then the three triangles CDA, PDB, and PBA are ſimilar. For $AD = PBD = PAB$, $CDA = PDB$ (2. I), whence $C = P$, and $ADC = ABP$ (Cor. 1. Prop. II). Therefore $CD : DA :: PD : BD$ (Prop. XIII), whence $DA \times PD = CD \times BD$ (12. Proportion). Again, $CA : DA ::$ AP or $AD + DP : AB$ (Prop. XIII), therefore $CA \times AB = AD^2 + DA \times DP$ (12. Proportion) $= AD^2 + CD \times BD$ (Ax. 3).

PROP. XXVII.

44. *In an iſoſceles triangle* ABC, *if a line be drawn from the vertex to cut the baſe*; *the ſquare of that line, together with the rectangle of the ſegments of the baſe, is equal to the ſquare of the ſide*; $BE^2 + AEC = BA^2$.

Let BD be perpendicular to the baſe, then $BA^2 = BD^2 + AD^2$ (Prop. XXI) $= BD^2 + \overline{AE + ED}^2 = BD^2 + AE^2 + ED^2 + 2AED$ (Prop. X. I.) $= BE^2 + AE^2 + 2AED$ (Prop. XXI) $= BE^2$

FIG. $= BE^2 + AE \times \overline{AE + 2ED}$ (Prop. IX. I)
44. $= BE^2 + AE \times EC$, becauſe $AE + 2ED = EC$. For $2AE + 2ED = AC$, therefore taking away AE, $AE + 2ED = EC$.

PROP. XXVIII.

45. *In any triangle* BAC, *if a line* AD *be drawn from the vertex to the middle of the baſe. The ſum of the ſquares of the ſides, is equal to twice the ſquare of half the baſe, together with twice the ſquare of the line that biſects the baſe*; $AB^2 + AC^2 = 2AD^2 + 2DC^2$.

For $AC^2 + 2CDP = AD^2 + DC^2$ (Prop. XXIII), and $DC = DB$ (hyp.),
therefore $AC^2 = AD^2 + DC^2 - 2CDP$ (Ax. 4);
and $AB^2 = AD^2 + DB^2 + 2CDP$ (Prop. 22),
therefore $AB^2 + AC^2 = 2AD^2 + 2DC^2$ (Ax. 3).

Cor. $AB^2 - AC^2 = (4CDP =)\ 2BC \times DP$.

PROP. XXIX.

46. *If through any point* E, *within a triangle* ABC, *three lines* TQ, VR, PS, *be drawn parallel to the three ſides of the triangle*; *the product or ſolid made by the alternate ſegments of theſe lines, will be equal.* $TE \times PE \times RE = QE \times SE \times VE$.

The triangles TEV, PEQ, SER, and ABC are all ſimilar (7. I), whence

$TE : VE :: AC : BC$ (Prop. XIII).
$PE : QE :: AB : AC$.
$RE : SE :: BC : AB$.

whence $TE \times PE \times RE : VE \times QE \times SE :: AC \times AB \times BC : BC \times AC \times AB$ (Prop. XVIII. Proportion). But the two laſt terms are equal, therefore $TE \times PE \times RE = VE \times QE \times SE$ (Prop. II. Proportion).

PROP.

PROP. XXX. FIG.

If three lines AF, BG, CD, *be drawn through any point* E, *within a triangle* ABC, *to the oppoſite ſides; the products of the alternate ſegments of the ſides are equal; that is,* $AG \times CF \times BD = CG \times BF \times AD$. 46.

For drawing TQ, VR, PS parallel to the ſides of the triangle, then

$AG : GC :: TE : QE$ (Cor. Prop. XIII).
$CF : BF :: RE : VE$.
$BD : AD :: PE : SE$.

whence $AG \times CF \times BD : GC \times BF \times AD :: TE \times RE \times PE : QE \times VE \times SE$ (Prop. XVIII. Proportion), but the two laſt are equal (Prop. XXIX); therefore $AG \times CF \times BD = GC \times BF \times AD$ (Prop. II. Proportion).

PROP. XXXI.

Three lines drawn from the three angles of a triangle to the middle of the oppoſite ſides, all meet in one point.

Let BD, AE biſect the oppoſite ſides AC, BC; and through the point of interſection G, draw CGK, and EL, DI parallel to it. 47.

Now ſince $BE = EC$, and $AD = DC$, we have $BL = LK$, and $AI = IK$ (Prop. XII). Alſo ſince $BE = \frac{1}{2}BC$, and $AD = \frac{1}{2}AC$, it will be $EH = \frac{1}{2}CG = DF$ (Prop. XIII). Therefore the triangles DGF, HGE, having all the angles equal (4. I), are ſimilar and equal (Prop. VII); whence $FG = GE$, and conſequently $IK = KL$ (Cor. 2. Prop. XII), therefore $AI = IK = KL = LB = \frac{1}{4}AB$. And $AK = KB$. And therefore if the line CK be drawn through the middle point K, it will paſs through G; otherwiſe the line paſſing through G, would make AK greater or leſſer than KB. This may alſo be demonſtrated from Prop. XXX.

FIG. 47. Cor. *Hence the diſtance of the point of interſection* G, *from any angle, is twice the diſtance from the oppoſite ſide,* BG = 2GD, *&c.*

For ſince BK = 2KI, and AK = 2KL, therefore BG = 2GD, and AG = 2GE. Alſo ſince DI = DF + FI = 3HL or 3FI, therefore 2FI = DF = GK = EH = ½CG.

PROP. XXXII.

Three perpendicular lines erected on the middle of the three ſides of any triangle, all meet in one point.

48. Let E, F be the middle points of AB, CB; FO, EO two perpendiculars. From O draw OD perpendicular to AC. The right-angled triangles COF, BOF are ſimilar and equal, and CO = OB (Prop. VI); alſo the right-angled triangles BOE, AOE, are ſimilar and equal, whence BO = OA (ibid.); therefore CO = AO; therefore in the iſoſceles triangle AOC, the perpendicular OD biſects the baſe AC (Cor. 3. Prop. III): and if it biſects the baſe, it paſſes through O.

Cor. *The point of interſection* O, *of the three perpendiculars, will be equally diſtant from the three angles.*

For the triangles COF, BOF, are ſimilar and equal (Prop. VI), and OB = OC. Alſo the triangles COD, AOD, are ſimilar and equal (ibid.), and CO = AO = BO.

PROP. XXXIII.

49. *If two right-angled triangles* BID, BED, *be deſcribed upon one hypothenuſe* BD, *lying on different ſides thereof, and the line* EI *drawn to the oppoſite angles; I ſay, the angles* DBI *and* DEI *are equal, which ſtand upon the ſame ſide* DI.

Make

Make $BC = CD$; draw ECF and CI. Then FIG.
CD, CI, CB, and CE are all equal (Cor. 6. Prop. 49.
XX). The external angle $ICD = CIB + CBI$ (Prop. I) $= 2CBI$ (Prop. III). Alſo the external angle $ICF = EIC + IEC = 2IEC$ (ibid.). Alſo $FCD = CDE + CED = 2CED$ (ibid.). Therefore by addition $ICF + FCD$, that is, $ICD = 2AED = 2CBI$, and $AED = CBI$, or $IED = IBD$.

PROP. XXXIV.

Three perpendiculars drawn from the three angles of a triangle, upon the oppoſite ſides, all meet in one oint.

Let AI, CE be perpendicular to CB, AB; and 50.
hrough the point of interſection D draw BDF; raw CK perpendicular to CA, alſo draw EI.

The oppoſite angles IDC and EDA are equal 2. I), and the angles at E and I are right, thereore the triangles ADE and CDI are ſimilar, whence AD : ED :: CD : DI (Prop. XIII); therefore the iangles ADC, and EDI are ſimilar (Prop. XIV), nd angle $DEI = DAC = ICK$ (Prop. XX). But ie triangles DBE, DBI are right-angled at E and whence $\angle DEI = DBI$ (Prop. XXXIII); thereore DBI or $FBC = ICK$, and therefore BF is pallel to CK (Cor. 3. Prop. IV), or perpendicular AC. And if BF be perpendicular to AC, it ill paſs through D.

PROP. XXXV.

Three lines biſecting the three angles of a triangle, l meet in one point.

For let CDF and ADE biſect the angles C, A; 51.
d through D, the point of interſection, draw G. Then BC : CG :: BD : DG :: BA : AG rop. XXV); and BC : BA :: CG : AG (Prop. . Proportion); whence BDG biſects the angle B (Cor.

FIG. 51. (Cor. 1. Prop. XXV), therefore the line biſecting the ∠B, paſſes through D.

Cor. 1. *If two lines biſect two angles of a triangle, the point of interſection* D, *is equally diſtant from the three ſides.*

Let D*n*, D*o*, D*p* be perpendicular on the three ſides. Then the triangles BD*n*, BD*o* have one ſide and all the angles equal, therefore D*n* = D*o* (Prop. VII); alſo the triangles AD*o*, AD*p*, have one ſide and all the angles equal; therefore D*o* = D*p* (ibid.) = D*n*.

Cor. 2. *Segment* A*p* + *the oppoſite ſide* BC = *half the ſum of the ſides.*

For half the ſum of the ſides = 2A*p* + 2C*n* + 2B*n*.

PROP. XXXVI.

52. *If the three angles of a triangle be biſected by the lines* AC, BC, DC, *and any one* BC *continued to the oppoſite ſide, and* CP *be drawn perpendicular to that ſide,* AD; *I ſay, the angle* ACE = DCP, *or* ACP = DCE.

For ſince ∠A + B + D = two right angles (Prop. II), therefore CAB + CBA + CDP = a right angle = DCP + CDP (Cor. 2. Prop. II); therefore CAB + CBA or ACE (Prop. I) = DCP.

PROP. XXXVII.

53. *The area of a right-angled triangle* ABC, *is equal to the rectangle under half the perimeter, and its exceſs above the hypothenuſe.*

The perimeter or circumference is the ſum of the three ſides. Now ſince the triangle ABC is right-angled at C, the area = $\frac{AC \times CB}{2}$ (Cor. 2. Prop. X); and $AB^2 = AC^2 + CB^2$ (Prop. XXI), or $AC^2 + CB^2 - AB^2 = 0$. Hence four times the

the area $= 2AC \times CB = AC^2 + CB^2 + 2ACB - AB^2 = \overline{AC + CB}^2 - AB^2$ (10. I) $= \overline{AC + CB + AB} \times \overline{AC + CB - AB}$ (12. I). FIG. 53.

And the area $= \frac{AC+CB+AB}{2} \times \frac{AC+CB-AB}{2}$.

But $\frac{AC+CB-AB}{2} = \frac{AC+CB+AB}{2} - AB$.

Cor. *The area of a right-angled triangle, is equal to the rectangle under the two exceſſes, of half the perimeter above each ſide*; $\frac{AC+CB+AB}{2} - BC$, *and* $\frac{AC+CB+AB}{2} - AC$.

For $\frac{AC+CB+AB}{2} - CB = \frac{AB+AC-BC}{2}$, and $\frac{AC+CB+AB}{2} - AC = \frac{AB+CB-AC}{2}$, and $\frac{AB+AC-BC}{2} \times \frac{AB+BC-AC}{2} = \frac{AB+\overline{AC-CB}}{2} \times \frac{AB-\overline{AC-CB}}{2} = \frac{AB^2-\overline{AC-CB}^2}{4} = \frac{AB^2-AC^2-CB^2+2ACB}{4}$ (Prop. XXI) $= \frac{ACB}{2}$ = area (Cor. 2. Prop. X).

PROP. XXXVIII.

In any triangle ABC; *add the three ſides together nto one ſum; and likewiſe from the ſum of every two des, ſubtract the third; and you will have three remainders. Then take the product of the ſaid ſum, nd one of the remainders; and likewiſe the product f the other two remainders.* 54.

Then I ſay, four times the area of the triangle, is mean proportional, between theſe two products.

Take AE, and AF, equal to AC, and draw CF, CE; alſo draw CD perpendicular to AB. Then

FIG. $AB \times CD =$ twice the area (Cor. 2. Prop. X).
54. And the angle FCE is a right angle; for $AFC = ACF$ (Prop. III), and $AEC = ACE$ (ibid.); therefore $AFC + AEC = ACF + ACE = FCE$ (Ax. 3) = a right angle (Cor. 2. Prop. II). And $AD = \frac{AB^2 + AC^2 - CB^2}{2AB}$ (Cor. Prop. XXIII).

Now $DE = AE - AD = AC - AD = \frac{AC \times 2AB - AB^2 - AC^2 + CB^2}{2AB} = \frac{\overline{CB + AB - AC} \times \overline{CB + AC - AB}}{2AB}$ (11. I).

Alſo $FD = FE - DE = 2AC - DE = \frac{2AC \times 2AB}{2AB} - DE = \frac{2AC \times 2AB - AC \times 2AB + AB^2 + AC^2 - CB^2}{2AB} = \frac{AB^2 + AC^2 + 2AC \times AB - CB^2}{2AB} = \frac{\overline{AB + AC + BC} \times \overline{AB + AC - BC}}{2AB}$ (12. I); but DC is a mean between DE and DF (Cor. 3. Prop. XX), therefore $DC \times 2AB$ is a mean between $DE \times 2AB$ and $DF \times 2AB$ (Prop. V. Proportion); that is, four times the area of the triangle ABC, is a mean proportional, between $\overline{CB + AB - AC} \times \overline{CB + AC - AB}$, and $\overline{AB + AC + BC} \times \overline{AB + AC - BC}$.

55. Cor. 1. *From half the ſum of the three ſides of any triangle* ABC, *ſubtract each ſide ſeparately. Then take the product of that half ſum and one remainder; and alſo the product of the other two remainders.*

Then I ſay, the area of the triangle is a mean proportional between theſe two products.

For $\frac{CB + AB - AC}{2} \times \frac{CB + AC - AB}{2}$: area ABC : $\frac{AB + AC + BC}{2} \times \frac{AB + AC - BC}{2}$

(Cor.

(Cor. 1. Prop.V. Proportion) are in continual proportion (Prop. XXXVIII). FIG. 55.

But $\frac{CB + AB - AC}{2} = \frac{CB + AB + AC}{2} - AC.$

and $\frac{CB + AC - AB}{2} = \frac{CB + AB + AC}{2} - AB.$

and $\frac{AB + AC - BC}{2} = \frac{CB + AB + AC}{2} - BC.$

therefore, *&c.*

Cor. 2. *Let* $S = AC + BC$, $D = AC - BC$, *then the area* ABC *is a mean proportional between* $\frac{1}{4} \times \overline{SS - AB^2}$, *and* $\frac{1}{4} \times \overline{AB^2 - DD}$. 56.

For $\frac{1}{4} \times \overline{SS - AB^2} = \frac{S + AB}{2} \times \frac{S - AB}{2} = \frac{AC + BC + AB}{2} \times \frac{AC + BC - AB}{2}$, and $\frac{1}{4} \times \overline{AB^2 - DD} = \frac{AB + D}{2} \times \frac{AB - D}{2} = \frac{AB + AC - BC}{2} \times \frac{AB + BC - AC}{2}$, which is the ame, as Cor. 1. ſuppoſing two terms in the extremes o change places, by Cor. 3. Prop. XII. Proportion.

PROP. XXXIX.

The ſquare of the ſide of an equilateral triangle, to the area; as 4 *to* √3.

Let CD be perpendicular to AB, then $AD = B = \frac{1}{2}AB$. Then $CD^2 = CA^2 - AD^2$ (Cor. 1. rop. XXI) $= AB^2 - \frac{1}{4}AB^2 = \frac{3}{4}AB^2$. And $CD = \frac{3AB^2}{4} = \frac{AB}{2}\sqrt{3}$. But the area of the triangle is $B \times \frac{1}{2}CD = AB \times \frac{AB}{4}\sqrt{3}$, and $4 \times \text{area} = AB^2 \times \sqrt{3}$ or. 2. Prop. X); whence AB² : area :: 4 : √3. 57.

Cor. *The ſquare of the perpendicular is equal to the ſquare of the ſide*; $CD^2 = \frac{3}{4}CA^2$.

For $CD^2 = CA^2 - AD^2$ (Cor. XXI) $= CA^2 - A^2 = \frac{3}{4}CA^2$.

BOOK III.

Of Quadrangles and Polygons.

DEFINITIONS.

FIG. 1. A *Quadrangle or quadrilateral*, is a plane figure bounded by four right lines.

58. 2. A *parallelogram* is a quadrangle whoſe oppoſite ſides are parallel, as AGBH. The line AB drawn to the oppoſite corners is called the *diameter* o *diagonal*. And if two lines be drawn parallel to th two ſides, through any point of the diagonal; the divide it into ſeveral others, and then C, D ar called *parallelograms about the diameter*: and E, *the complements*: and the figure EDF a *gnomon*.

3. A *rectangle* is a parallelogram whoſe ſides ar perpendicular to one another.

4. A *ſquare* is a rectangle of four equal ſides.

59. 5. A *rhombus* is a parallelogram, whoſe ſides ar equal, and angles oblique.

58. 6. A *rhomboides* is a parallelogram, whoſe ſide are unequal, and angles oblique.

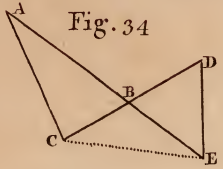
60. 7. A *trapezoid* is a quadrangle, having only tw ſides parallel.

61. 8. A *trapezium* is a quadrangle, that has n two ſides parallel.

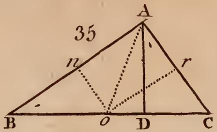
62. 9. A *polygon* is a plane figure encloſed by man right lines. If all the ſides and angles are equa it is called a *regular polygon*, and denominated a cording to the number of ſides, as a *pentagon* ſides, a *hexagon* 6, a *heptagon* 7, &c.

10. T

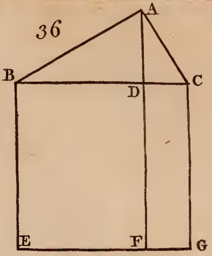
Fig. 34



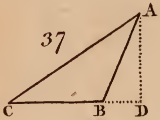
35



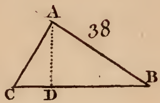
36



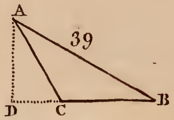
37



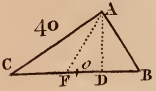
38



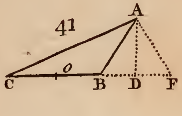
39



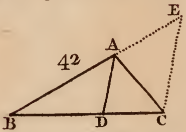
40



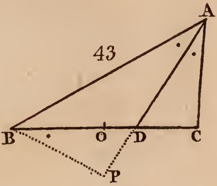
41



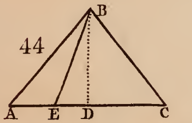
42



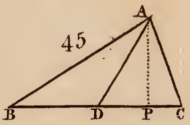
43



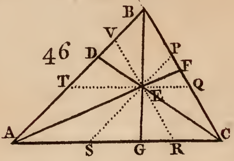
44



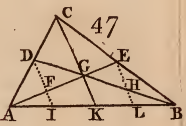
45



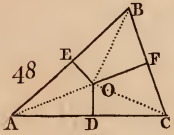
46



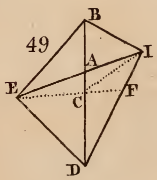
47



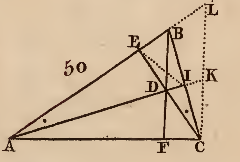
48



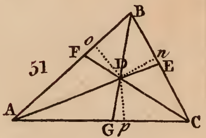
49



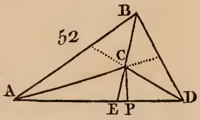
50



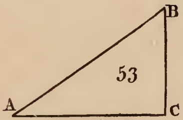
51



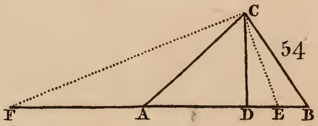
52



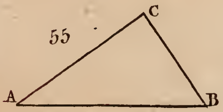
53



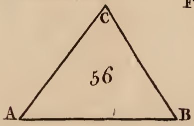
54



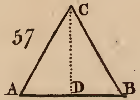
55



56



57



Pl.II. *Pa.36.*

10. The *diagonal* of a quadrangle or polygon, is a line drawn between any two oppoſite corners of the figure, as AB. FIG. 62.

11. The *hight* of a figure is a line drawn from the top, perpendicular to the *baſe*, or oppoſite ſide, on which it ſtands.

12. *Like or ſimilar figures*, are thoſe whoſe ſeveral angles are equal to one another; and the ſides about the equal angles, proportional.

13. *Homologous ſides* of two figures, are thoſe between two angles, reſpectively equal.

14. The *perimeter* or *circumference* of a figure, is the compaſs of it, or ſum of all the lines that incloſe it.

15. The *internal angles* of a figure, are thoſe on the inſide, made by thoſe lines that bound the figure, ADC. 76.

16. The *external angle* of a figure, is the angle made by one ſide of a figure, and the adjoining ſide drawn out, BAF. 77.

PROP. I.

In any parallelogram the oppoſite ſides, and angles, are equal; and the diagonal divides it into two equal triangles: $AB = CD$, $AC = BD$, *and triangle* $ABD = ADC$, *&c.* 63.

For ſince AB, and CD are parallel (Def. 2), $\angle BAD = ADC$ (4. I): alſo, becauſe AC and BD are parallel, $BDA = CAD$ (ibid.). Therefore the triangles ABD and DCA, are equal in all reſpects (7. II).

PROP. II.

The diagonals of a parallelogram, interſect each other in the middle.

In the triangles APC, BPD, $\angle CAP = BDP$, and $ACP = DBP$ (4. I), and $\angle BPD = APC$ (2. I), and $AC = BD$ (Prop. I); therefore $AP = PD$, and $CP = PB$ (7. II). 64.

FIG.

PROP. III.

Any line BC *paſſing through the middle of the diagonal of a parallelogram* P, *divides the area into two equal parts.*

65. For in the triangles ABP, and DCP, AP = PD (Prop. II); and all the angles are equal (4. I). Therefore the triangle ABP = DCP (7. II); and BP = PC (ibid.). And ſince triangle AED = AFD (Prop. I); the remainders BPDE and CPAF are equal; therefore BPDE + PDC = CPAF + APB, that is, EBCD = BAFC.

Cor. *Any right line* BC *drawn through the middle point* P *of the diagonal of a parallelogram, is biſected in that point*; BP = PC.

PROP. IV.

66. *In any parallelogram* ABDC, *the complements* CI, *and* IB, *are equal.*

For triangle ADC = ABD (Prop. I), and AHI = AGI, and IED = IFD (ibid.); therefore parallelogram HE = parallelogram GF (Ax. 4).

PROP. V.

66. *The parallelograms* HG, EF, *which are about the diameter* AD, *of any parallelogram* CB, *are ſimilar to the whole* CB, *and to one another.*

The parallelograms HG, EF are equiangular to the whole CB (4. I), and to one another. The triangles ACD, ABD, as alſo AHI, AGI, and IED, IFD, are ſimilar and equal (Prop. I). Therefore AH : HI or AG : : AC : CD or AB : : IE : ED or IF, therefore the parallelograms are like (Def. 12).

PROP.

PROP. VI.

FIG. 67.

Parallelograms ABCD, *and* EBCF, *ſtanding upon the ſame baſe and between the ſame parallels, are equal.*

For AD = BC = EF (Prop. I); add DE, then AE = DF, and AB = DC (Prop. I), and ∠A = CDF (Cor. 1. 4. I). Therefore triangle ABE = DCF (6. II); ſubtract DGE; then the figure ABGD = EGCF; add BGC, then ABCD = BEFC.

Cor. 1. *Parallelograms of equal baſes and hights, are equal.*

For if their baſes be laid upon one another, the tops of both will fall in the ſame parallel, being of equal hight; and therefore they are equal (this prop.).

Cor. 2. *Every parallelogram is equal to the rectangle of its baſe and hight.*

Cor. 3 *Figures of the ſame area, may have their compaſs vaſtly different. And figures of equal compaſs may contain very different areas.*

PROP. VII.

A parallelogram is double to a triangle of the ſame or an equal baſe and hight.

For the triangle ACD = ABD (Prop. I), that is, the triangle ACD, on the baſe CD, is half the parallelogram ACDB on the ſame baſe CD, and between the ſame parallels. And ſince any triangle of an equal baſe and hight is = ACD, and any parallelogram of the ſame or an equal baſe and hight = ACDB. Therefore any triangle is half the parallelogram of the ſame or equal baſes and hights. 63.

FIG.

PROP. VIII.

68. *Parallelograms of the ſame hight, are to one another as their baſes*; DC : GF : : BC : GH.

Draw the diameters BA, EH. Then the triangles BCA, GHE, of the ſame hight, are as their baſes BC, GH (11. II). Therefore 2BCA : 2GHE : : BC : GH (Prop. V. Proportion): that is, parallelogram BCAD : parallelogram GHFE : : baſe BC : baſe GH.

Cor. 1. *Parallelograms of equal baſes, are as their hights.*
By Cor. 2. Prop. VI. as likewiſe
Cor. 2. *Parallelograms are to one another, as their baſes and hights.*

PROP. IX.

69. *Equal parallelograms having one angle equal to one; have the ſides about the equal angles reciprocally proportional. If* ABCD = EFGH, *then* AB : BG : : BE : BC.

Let the oppoſite angles at B be equal; produce DC and FG to H. Then AB : BG : : BD : BH (Prop. VIII) : : BF : BH (Ax. 6. Proportion) : : BE : BC (Prop. VIII).

Cor. 1. *Thoſe parallelograms are equal; which have one angle equal to one; and the ſides about the equal angles, reciprocally proportional.*
For BD : BH : : AB : BG (Prop. VIII) : : BE : BC (hyp.) : : BF : BH (Prop. VIII). Therefore parallelogram BD = parallelogram BF.
Cor. 2. *Equal parallelograms, have their baſes and hights, reciprocally proportional.*
Cor. 3. *If four lines are proportional; the rectangle of the means, is equal to the rectangle of the extremes.*

PROP.

PROP. X. FIG.

Equiangular parallelograms AC, EG, *are in the complicate ratio of their homologous ſides,* ABC, EBG. 70.

Produce DC, FG to H. Then parallelogram AC : BH : : AB : BG (Prop. VIII), and parallelogram BH : BF : : CB : BE (ibid.). Therefore parallelogram AC : parallelogram BF : : $AB \times CB$: $BG \times BE$ (Cor. 1. Prop. XVIII. Proportion).

Cor. 1. *Parallelograms are to one another, in the complicate ratio of their baſes and hights.*

Cor. 2. *The rectangle of two lines, is a mean proportional between their ſquares.*

For ſuppoſing AC, EG, to be ſquares; then AC : BH : : (AB : BG : : BC : BE : :) BH : BF.

PROP. XI.

In any parallelogram AD, *the ſum of the ſquares of the diagonals, is equal to the ſum of the ſquares of all the ſides:* $AD^2 + CB^2 = CA^2 + AB^2 + BD^2 + DC^2$. 71.

For CE = EB, and AE = ED (Prop. II). Alſo $CD^2 + DB^2 = 2DE^2 + 2CE^2$ (28. II). And $2CD^2 + 2DB^2 = 4DE^2 + 4CE^2$, that is, $CD^2 + AB^2 + DB^2 + CA^2 = DA^2 + CB^2$.

PROP. XII.

If from any point O, *in the rectangle* AD, *lines be drawn to all the angles; the ſum of the ſquares of the lines drawn to the oppoſite corners, will be equal:* $AO^2 + OD^2 = BO^2 + OC^2$. 72.

Draw AD, BC, to interſect in P, then AD = CB (6. II), and their halfs, AP = PC = PD. Then $CO^2 + OB^2 = 2CP^2 + 2OP^2$ (28. II) $= 2AP^2 + 2OP^2 = AO^2 + OD^2$ (28. II).

PROP.

FIG.

PROP. XIII.

73. *In any trapezium* ABDC, *let* E, F *be the middle points of the diagonals*, AD, BC. *Then the ſum of the ſquares of the ſides, is equal to the ſum of the ſquares of the diagonals, together with four times the ſquare of the diſtance, between the middle points of the diagonals:* $AB^2 + BD^2 + CD^2 + CA^2 = AD^2 + CB^2 + 4EF^2$.

For $AE^2 + ED^2 = 2AF^2 + 2EF^2$ (28. II). Alſo $AB^2 + AC^2 = 2CE^2 + 2AE^2$ (ibid.); alſo $BD^2 + DC^2 = 2CE^2 + 2DE^2$. And adding the two laſt equations, $AB^2 + BD^2 = DC^2 + CA^2 = 4CE^2 + 2AE^2 + 2ED^2 = CB^2 + 4AF^2 + 4EF^2 = CB^2 + AD^2 + 4EF^2$.

PROP. XIV.

74. *In any trapezium* ADBC, *let* E, F, *be the middle points of two oppoſite ſides. Then the ſum of the ſquares of the other two ſides, together with the ſquares of the diagonals, is equal to the ſum of the ſquares of the biſected ſides, together with four times the ſquare of the diſtance of theſe middle points:* $AC^2 + DB^2 + AB^2 + CD^2 = AD^2 + CB^2 + 4EF^2$.

Draw AE, ED. Then $AE^2 + ED^2 = 2AF^2 + 2EF^2$ (28. II), and $AB^2 + AC^2 = 2CE^2 + 2AE^2$ (ibid.), and $DB^2 + DC^2 = 2CE^2 + 2DE^2$ (ibid.). Add the two laſt equations, $AB^2 + AC^2 + DB^2 + DC^2 = 4CE^2 + 2AE^2 + 2ED^2 = CB^2 + 4AF^2 + 4EF^2 = CB^2 + AD^2 + 4EF^2$.

PROP. XV.

75. *In any trapezium* ADBC, *if lines be drawn to the middle of the oppoſite ſides; the ſum of the ſquares of the diagonals, is equal to twice the ſum of the ſquares of the biſecting lines:* $AB^2 + CD^2 = 2EF^2 + 2PQ^2$.

For

For $AB^2 + DC^2 + BD^2 + CA^2 = AD^2 + CB^2 + 4EF^2$ (Prop. XIV). FIG. 75.

And $AB^2 + DC^2 + BC^2 + DA^2 = AC^2 + DB^2 + 4PQ^2$ (ibid.).

and adding theſe equations,
$2AB^2 + 2DC^2 + BD^2 + CA^2 + BC^2 + DA^2 = AD^2 + CB^2 + AC^2 + DB^2 + 4EF^2 + 4PQ^2$, and ſubtracting what is common, $2AB^2 + 2DC^2 = 4EF^2 + 4PQ^2$, and $AB^2 + DC^2 = 2EF^2 + 2PQ^2$.

PROP. XVI.

The ſum of the four internal angles of any quadrilateral figure, is equal to four right angles.

Draw the diagonal AC; then the ſum of all the angles in the triangle ABC, or ADC, is two right angles (2 II); therefore the ſum of both is four right angles. 76.

Cor *If two angles of a quadrangle be right angles; the ſum of the other two amounts to two right angles.*

PROP. XVII.

The ſum of all the internal angles of a polygon, makes twice as many right angles, abating four, as the polygon has ſides.

For drawing lines from all the angles, to a point O within the figure, it comes to be divided into as many triangles, as the figure has ſides or angles. And each triangle contains two right angles (2. II), ſo theſe amount to twice as many right angles, as the figure has ſides; but the angles at O are to be abated, and theſe amount to four right angles (Cor. 1. Prop. 1. I). 77.

Cor. *Hence all right-lined figures, of the ſame number of ſides, have the ſum of all the internal angles equal.*

PROP.

FIG.

PROP. XVIII.

The ſum of the external angles of any polygon, is equal to four right angles.

77. For all the internal angles, together with the external angles at the points A, B, C, *&c.* make twice as many right angles, as the figure has ſides (1. I); and the ſum of all the angles of the triangles ABO, BCO, *&c.* amounts to the ſame (2. II). Take away all the angles, EAB, ABC, *&c.* and there remains all the external angles A, B, C, *&c.* equal to all the angles at O, that is, four right angles (Cor. 1. (Prop. 1. I).

+ Cor. *All right-lined figures, have the ſum of their external angles equal.*

SCHOLIUM.

78. If any of the angles be greater than two right angles, as A; the external angle will run into the figure, and muſt be ſubtracted from the ſum of the reſt.

PROP. XIX.

79. *In two ſimilar figures* AC, PR; *if two lines* BE, QT, *be drawn after a like manner, as ſuppoſe, to make the angle* CBE = RQT; *then theſe lines have the ſame proportion, as any two homologous ſides of the figure,* BC *to* QR, *&c.*

Since ∠CBE = RQT, and R = C (hyp.); therefore BE : QT : : BC : QR (13. II) : : BA : QP (Def. 12) : : AD : PS (ibid.) : : DC : SR. Alſo BC : CE : : QR : RT; and BC : BE : : QR : QT, *&c.*

Cor. 1. *Hence all ſimilar figures are made up of ſimilar triangles.*

Draw

Draw BD, QS; and AC, PR; then BE : QT : : BC : QR (this prop.) : : CD : RS (Def. 12) : : CE : RT (this prop.) : : DE : ST (Prop. VIII. Proportion); therefore the triangles BCE and QRT are ſimilar; and BED and QTS are ſimilar. FIG. 79.

Again, the $\angle A = P$, and AB : AD : : PQ : PS (Def. 12); therefore BAD, QPS are ſimilar (14. II). Alſo $\angle B = Q$, and AB : BC : : PQ : QR, therefore ABC and PQR are ſimilar (14. II). Laſtly, $\angle D = S$, and AD : DC : : PS : SR (Def. 12); therefore ADC, PSR are ſimilar (14. II).

Cor. 2. *Hence it may be laid down, as a diſtinguiſhing property of ſimilar figures, that they are made up of ſimilar triangles, placed in the ſame order.*

PROP. XX.

All ſimilar figures are to one another as the ſquares of their homologous ſides.

Let AD, PS be ſimilar polygons; draw AC, AD, PR, PS, which will divide the figures into triangles (Cor. 1. Prop. XIX). 80.

Becauſe AB : PQ : : AC : PR : : AD : PS (13. II); therefore

$AB^2 : PQ^2$: : triangle ABC : PQR (18. II).
and $AB^2 : PQ^2 : : AC^2 : PR^2$: : triangle ACD : PRS (ibid.).
and $AB^2 : PQ^2 : : AD^2 : PS^2$: : triangle ADE : PST (ibid.).
therefore $AB^2 : PQ^2$: : triangle ABC + ACD + ADE : triangle PQR + PRS + PST (Prop. X. Proportion) : : figure ABCDE : figure PQRST.

Cor. *If three lines* A, B, C *be in continual proportion; then as the firſt to the third, ſo any figure deſcribed on the firſt, to a ſimilar one upon the ſecond.*

For $A : C : : A^2 : B^2$ (Prop. XXIII. Proportion) : : figure upon A : figure upon B (this prop.).

PROP.

FIG.

PROP. XXI.

81. *If four lines be proportional,* AB : DE :: GH : LM ; *ſimilar figures, alike deſcribed upon, two and two, ſhall alſo be proportional:* ABC : DEF :: GHIK : LMNO.

And if four figures be proportional, and two and two be ſimilar ; their like ſides ſhall be proportional.

For ſince AB : DE :: GH : LM (hyp.),
therefore $AB^2 : DE^2 :: GH^2 : LM^2$ (Cor. 3. (Prop. XVIII. Proportion).
whence ABC : DEF : GHI : LMN (Prop. XX).
Again, if the figures be ſimilar,
and ABC : DEF :: GHIK : LMNO (hyp.).
then $AB^2 : DE^2 :: GH^2 : LM^2$ (Prop. XX).
whence AB : DE :: GH : LM (Cor. 3. 18. Proportion).

PROP. XXII.

82. *Any figure deſcribed on the hypothenuſe of a right-angled triangle, is equal to two ſimilar figures deſcribed the ſame way upon the two ſides:* BFC = ALC + AGB.

For fig. BCF : CAL :: $BC^2 : CA^2$ (Prop. XX).
BAG AB^2
therefore, BCF : CAL + BAG :: $BC^2 : CA^2 + AB^2$ (14. Proportion).

But $BC^2 = CA^2 + AB^2$ (21. II); therefore BCF = CAL + BAG (Prop. II. Proportion).

PROP. XXIII.

83. *The area of a trapezoid* ABCD, *is equal to the rectangle of half the ſum of the parallel ſides, and the perpendicular between them:* $\frac{BA + CD}{2} \times BP$.

Draw

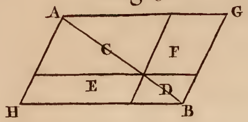
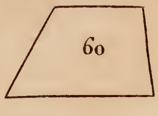
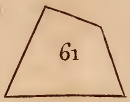
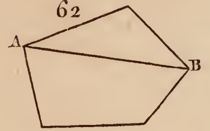


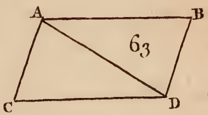
Fig: 58

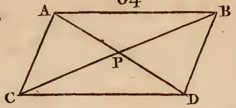

59

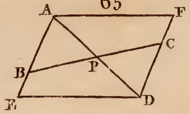

60

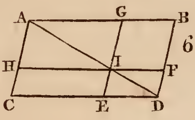

61

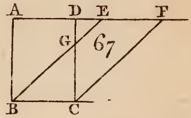

62

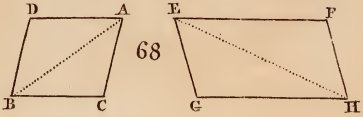

63

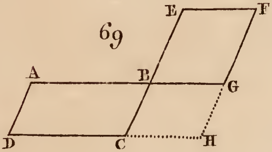

64

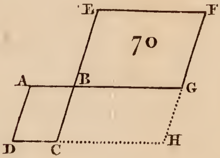

65

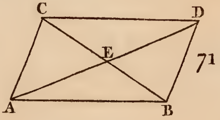

66

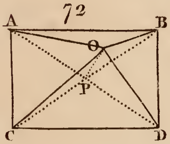

67

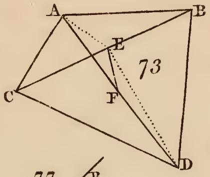

68

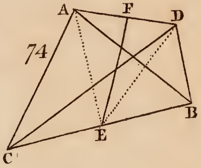

69

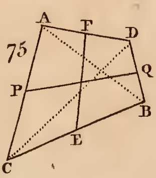

70

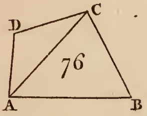

71

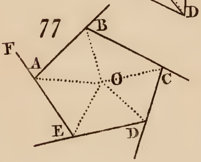

72

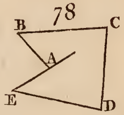

73

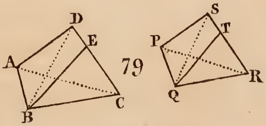

74

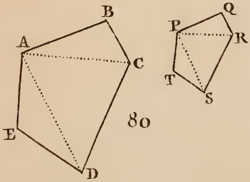

75

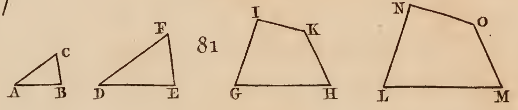

76


77


78


79


80


81

Pl. III. *Pa. 46.*

Draw AF parallel to BD, and BP perpendicular to CD. Then the area ABDF = AB × BP or FD × BP (Cor. 2. Prop. VI) $= \frac{AB + FD}{2} \times BP$. And the area of the triangle ACF = $\frac{1}{2}$CF × BP (Cor. 2. (Prop. X. II). Therefore CAF + AFDB or the trapezoid CABD $= \frac{AB + DF + CF}{2} \times BP = \frac{AB + CD}{2} \times BP$. FIG. 83.

PROP. XXIV.

The area of a trapezium ABDF, *is equal to half the rectangle under the diagonal* AD, *and the ſum of the perpendiculars falling thereon from the oppoſite angles:* $AD \times \frac{BC + EF}{2}$. 84.

For the triangle ABD $= \frac{AD \times BC}{2}$ (Cor. 2. 10. II); and the triangle AFD $= \frac{AD \times FE}{2}$ (ibid.): therefore ABD + AFD or the trapezium ABDF $= AD \times \frac{BC + FE}{2}$.

PROP. XXV.

Any regular figure ABCDE, *is equal to a triangle, whoſe baſe is the perimeter* ABCDEA; *and hight, the perpendicular* OP, *drawn from the center, perpendicular to one ſide.* 85.

Two perpendiculars, as PO, ſtanding on the middle of two ſides, meet in the center, O (9. II). Or two angles A, B biſected by two right lines, meet in the center, O (Cor. 1. 3. II): whence all the lines OA, OB, OC are equal; and all perpendiculars drawn from O, upon AB, BC, CD, &c. are equal. And all the triangles AOB, BOC, &c. are

FIG. 85. equal and ſimilar. The ſum of all the triangles make up the figure, that is, $\frac{AB \times OP}{2}$, $\frac{BC \times OP}{2}$, *&c.* or $\frac{AB + BC + CD + DE + EA}{2} \times OP =$ figure, or a triangle whoſe baſe is ABCDEA, and hight OP = the figure.

Cor. *The area of a regular polygon is equal to the rectangle of one ſide into the perpendicular from the center upon that ſide, and that multiplied by half the number of ſides.*

SCHOLIUM.

Any polygon, regular or irregular, may be divided in as many triangles, leſs 2, as the figure has ſides; by drawing diagonal lines.

PROP. XXVI.

Only three ſorts of regular figures can fill up a plane ſurface; and theſe are ſix triangles, four ſquares, and three hexagons.

It is required to place ſome number of theſe figures, with their angles upon one point, ſo that being joined cloſe together, they may fill the whole ſpace around it, and leave no vacancy.

It is plain the angles about one point are four right angles (Cor. 1. 11. I), which want to be filled up. Now if the angles of the ſeveral figures be computed by Prop. XVII, they will be found as follows.

86. A triangle $\frac{2}{3}$ of a right angle = A.
A ſquare 1 right angle = B.
A pentagon $1\frac{1}{5}$ right angle.
A hexagon $1\frac{1}{3}$ a right angle = C.
&c.

Now $\frac{2}{3}$ of a right angle 6 times repeated, makes 4 right angles, and therefore fills all the ſpace; that is, 6 angles of an equilateral triangle fills it.

Alſo

FIG. 86.

Alſo 4 angles of a ſquare (or 4×1), makes 4 right angles.

But 3 angles of a pentagon (or $3 \times 1\frac{1}{5}$) falls ſhort; and 4 angles (or $4 \times 1\frac{1}{5}$) exceeds.

Alſo 3 angles of a hexagon (or $3 \times 1\frac{1}{3}$) makes 4 right angles. And theſe are all; for

The angle of a heptagon (and other figures) is bigger, and therefore 3 angles will exceed 4 right ones. And to have two angles, each muſt be right angles, which is abſurd.

BOOK IV.

Of the Circle, and inſcribed and circumſcribed Figures.

DEFINITIONS.

FIG. 87. 1. A *Circle* is a plane figure deſcribed by a right line moving about a fixt point, ABD. Or it is a figure bounded by one line equidiſtant from a fixt point.

2. The *center* of a circle, is the fixt point about which the line moves, C.

3. The *radius*, is the line that deſcribes the circle, CA.

Cor. *All the radii of a circle, are equal.*

4. The *circumference* is the line deſcribed by the extreme end of the moving line, ABDA.

88. 5. The *diameter*, is a line drawn through the center, from one ſide to the other, AD.

6. A *ſemicircle*, is half the circle, cut off by the diameter, as ABD.

7. A *quadrant*, or quarter, is the part between two radii perpendicular to one another, as CDE.

89. 8. An *arch* is any part of the circumference, AB.

9. A *ſector*, is a part bounded by two radii, and the arch between them, ACB.

10. A *ſegment*, a part cut off by a right line, DEF, or DABF.

11. A *cord*, a right line drawn through the circle, as DF. FIG. 89.

12. *Angle at the center*, is that whoſe angular point is at the center ACB.

13. *Angle at the circumference*, is when the angular point is in the circumference, BAD. 90.

14. *Angle in a ſegment*, is the angle made by two lines drawn from ſome point of the arch of that ſegment, to the ends of the baſe; as BCD is an angle in the ſegment BCD.

15. *Angle upon a ſegment*, is the angle made in the oppoſite ſegment, whoſe ſides ſtand upon the baſe of the firſt; as BAD, which ſtands upon the ſegment BCD.

16. A *tangent* is a line touching a circle, which produced, does not cut it, as GAF.

17. Circles are ſaid *to touch* one another, which meet, but do not cut one another.

18. *Similar arches*, or *ſimilar ſectors*, are thoſe bounded by radii that make the ſame angle.

19. *Similar ſegments* are thoſe which contain ſimilar triangles, alike placed.

20. A *figure* is ſaid to be *inſcribed in a circle*, or a *circle circumſcribed about a figure*; when all the angular points of the figure are in the circumference of the circle.

21. A *circle* is ſaid to be *inſcribed* in a figure, or a *figure circumſcribed about* a circle; when the circle touches all the ſides of the figure.

22. One *figure* is *inſcribed* in another, when all the angles of the inſcribed figure, are in the ſides of the other.

PROP. I.

The cord of any arch AB, *falls intirely within the ircle.* 91.

For draw CA, CB; and CD to any point of the ord; then $\angle A = B$ (3. II). And $\angle$ CDB =

FIG. 91. A + ACD (1. II) = B + ACD, therefore CDB is greater than B, conſequently CB is greater than CD, (4. II); therefore D is within the circle.

PROP. II.

92. *The radius* CR, *biſects any cord at right angles, which paſſes not through the center, as* AB.

For draw AC, BC, and if AF = FB, then ſince AC = CB, and CF common; therefore CFA = CFB (8. II) = a right angle; and angle ACF = BCF.

Or if AFC = CFB and A = B, then ACF = BCF; and CF being common, AF = FB. This prop. follows from Cor. 3. Prop. III. Book II.

Cor. 1. *If a line biſects a cord at right angles, it paſſes through the center of the circle.*

Cor. 2. *The radius that biſects the cord, alſo biſects the arch.*

For ſince ACR = RCB. If CBR be laid upon CAR, the point B will fall upon A, and therefore RB = RA.

Cor. 3. *If two right lines do not both paſs through the center, they cannot both be biſected by each other.*

For if they could, they muſt both make right angles with the radius.

PROP. III.

93. *In a circle, equal cords* AB, GD, *are equally diſtant from the center,* C.

For let CE, CF be perp. to the cords; and draw CD, CA; then in the triangles, ACF, DCE, AC = CD, AF = DE being half the cords (Prop. II); and angles at F, E right; and the angles at C, both acute, therefore CF = CE (9. II).

Cor. *If ſeveral lines be drawn through a circle, the greateſt is the diameter, and thoſe that are nearer the center* HI, *are greater than thoſe that are farther off*, DG. FIG. 93.

For draw CH, then CH is greater than OH (4. II), and therefore 2CH or the diameter is greater than HI. And ſince ∠HCI is greater than DCG, HI is greater than DG (Cor. 6. II).

PROP. IV.

If from a point G, *out of the center, ſeveral lines* GD, GE, *&c. be drawn, the greateſt is that* GF *which paſſes through the center, and thoſe nearer to* GF *are greater than thoſe further off.* 94.

Alſo GH *(the remainder to* GF) *is the leaſt, and thoſe nearer to it, as* GA, *are leſs than thoſe further off*, GB.

Draw CE, CD, CA, CB, from the center C. Then GC + CE or GF is greater than GE (5. II). Alſo in the triangles GCE, GCD; GC, EC are equal to GC, DC; but ∠ECG is greater than DCG; therefore EG is greater than DG.

Alſo CG + GD is greater than CD or CH, take away CG, and GD is greater than GH. After the ſame manner GA is greater than GH; and GB greater than GA.

Cor. 1. *Only two lines drawn from* G *to the circumference can be equal; and lie on different ſides of the diameter* HF.

For no two lines on the ſame ſide can be equal.

Cor. 2. *If from any point, three equal right lines can be drawn to the circumference; that point is the center*, C.

Cor. 3. *No circle can cut another in more than two points.*

FIG. 94. For then three equal lines might be drawn from a point out of the center to the circumference; which is abſurd.

PROP. V.

95. *If from a point* G *without a circle, ſeveral right lines be drawn to cut it. Of thoſe that paſs to the concave part, the greateſt is that* GF *which paſſes through the center, and thoſe nearer to* GF *are greater than thoſe further off.*

But of thoſe that go to the convex part, the leaſt is that GH, *which continued would paſs through the center, and thoſe nearer to that, as* GA, *are leſs than thoſe further off,* GD.

For in the triangle GCE, GC + CE or GF is greater than GE. And in the triangles GCE, GCB; GC, CE are equal to GC, CB, and ∠GCE greater than GCB, therefore GE is greater than GB.

Alſo in the triangle CGA, CA + AG is greater than CG or CH + HG (5. II); take away CA = CH, and AG is greater than HG. And in the triangles CAG, CDG; CG, CA are equal to CG, CD; and angle GCA leſs than GCD; therefore GA is leſs than GD (Cor. 6. II).

Cor. 1. *There can only two equal lines be drawn from the point* G *to the circumference of the circle.*

For no two are equal on one ſide of GF.

Cor. 2. *The greateſt to the convex part, or the leaſt to the concave part, is the tangent to the circle.*

PROP. VI.

96. *In any circle, if ſeveral radii be drawn making equal angles; the arches and ſectors comprehended thereby will be equal, if* ACB = BCD; *then, arch* AB = *arch* BD; *and ſector* ACB = BCD.

For

For ſince ∠ACB = BCD, and CA = CD; FIG. 96. therefore if the angle DCB be laid upon BCA, DC will fall upon CA, and D upon A, and conſequently the arch DB will coincide with AB, as well as the ſector DBC with ABC, conſequently arch DB = AB, and ſector DBC = ABC (Ax. 8).

Cor. 1. *In equal circles, the radii making equal angles, comprehend equal arches, and ſectors.*

Cor. 2. *In the ſame or equal circles, the radii making equal angles, comprehend equal cords* AB, BD.

For theſe will coincide with one another. It alſo follows from Prop. VI. II.

Cor. 3. *Equal cords cut off equal arches, and equal ſegments, in the ſame circle.*

For if laid upon one another, they perfectly coincide, as has been proved.

PROP. VII.

In the ſame or equal circles, the arches, and alſo the ſectors, are proportional to the angles intercepted by the radii. 97.

Take any arch AB as ſmall as you will, and let AB = BC, *&c.* alſo AB = QR = RS, *&c.* and drawing CA, CB, CD, *&c.* and PQ, PR, PS, *&c.* then all the angles ACB, BCD, QPR, RPS, *&c.* are equal (Cor. 1. Prop. VI). Whence AF is as multiple of AB, as the angle ACF is of ACB. Therefore AB : AF : : ACB : ACF (Prop. V. Proportion). Alſo QV is as multiple of QR or AB, as QPV is of QPR or ACB, whence AB : QV : : ACB : QPV (ibid.); whence AF : QV : : ACF : QPV (Cor. 2. 14. Proportion).

The ſame reaſoning holds in the ſectors, for ſect. ACF is as multiple of ACB; as ∠ACF is of the ∠ACB. And ſect. QPV is as multiple of QPR or ABC;

FIG. 97. ABC; as $\angle$ QPV is of ACB. Therefore ſect. ACF : ſect. QPV : : angle ACF : $\angle$ QPV.

Cor. *The angle* ACF *is to* 4 *right angles; as the arch* AF, *is to the whole circumference.*

PROP. VIII.

98. *In all circles, ſimilar arches are as the radii of the circles.*

Let the circles AFG and *afg* be both deſcribed from the ſame center, C. Draw the radii CA, CF; then the arches AF, *af* are ſimilar (Def. 18). Draw CB extremely near CA. Then the figures or ſectors C*ab*, CAB, approach very near to iſoſceles triangles, which are ſimilar to one another, becauſe the $\angle$ at C is common (3. II). Therefore C*a* : *ab* : : CA : AB (13. II); and C*a* : CA : : *ab* : AB (4. Proportion). Now if you ſuppoſe BF divided into more arches, equal to AB; and more radii CB drawn; *bf* will then contain as many arches equal to *ab*. Therefore *af* is as multiple of *ab*, as AF is of AB; therefore *ab* : AB : : *af* : AF (5. Proportion); whence C*a* : CA : : *af* : AF (1. Proportion).

PROP. IX.

98. *The circumferences of circles are to one another, as their diameters.*

For AF : circumference AFGA : $\angle$ ACF : 4 right angles (Cor. 7) : : $\angle$ *a*C*f* : 4 right angles : : *af* : circumference *afga*. And AFGA : *afga* : : AF : *af* (4. Proportion) : : CA : *ca* (Prop. VIII) : : 2CA : 2 C*a* (5. Proportion).

Cor. *The circumferences of circles are as their radii.*

PROP.

PROP. X.

FIG. 88.

A right line AG, *perpendicular to the diameter* AD *of a circle, at the extreme point* A, *touches the circle in that point; and lies wholly without the circle.*

To any point O in the line GAF, draw the line CO from the center. Then the hypothenuſe OC is greater than the ſide AC (4. II). Therefore O is without the circle. And ſo it is for any point beſides A; therefore the line GF is entirely out of the circle.

Cor. 1. *Hence a right line touches a circle only in one point.*

Cor. 2. *If a right line touches a circle in one point, it is perpendicular to the diameter in that point.*

Cor. 3. *All circles, whoſe centers are in the line* AD, *and whoſe circumferences paſs through the point* A, *touch one another, and the line* GAF, *in the ſame point* A.

Cor. 4. *Hence, if two circles touch one another, either inwardly or outwardly; the line paſſing through their centers,* C, B, D, *ſhall alſo paſs through the point of contact,* A. 99.

Otherwiſe a line, touching both circles in that point, could not be perpendicular to both diameters.

Cor. 5. *Two circles, can only touch in one point.*

From the centers B, D, draw BO, DO, to a point O in the exterior circle. Then in the triangle BOD; DB + BO is greater than DO or DA or DB + BA (5. II). Whence BO is greater than BA; therefore the point O, is without the circle AE. In like manner, drawing CO; DO + CO is greater than DA + CA, and CO greater than CA, therefore O falls without the circle AI.

PROP.

FIG.

PROP. XI.

100. *The angle of contact between a right line and a circle* DAI, *is leſs than any right-lined angle whatever*, DAL.

Draw BE perpendicular to AL, then the ſide BA oppoſite to the right angle BEA, is greater than the ſide BE oppoſite to the acute angle BAE (4. II). Therefore the point E, and ſo the whole line AEL, falls within the circle.

Cor. 1. *Hence the angle of a ſemicircle* BAI *is greater than any acute angle whatever.*

Cor. 2. *The angle of contact* DAI, *is infinitely leſs than a right angle.*

For if it was in a finite proportion to a right angle, then an acute angle might be found equal to it.

Cor. 3. *If any other circle be deſcribed through* A, *with any radius greater than* AB, *it will fall entirely between the tangent* AD *and the circle* AL, *and make the angle of contact leſs. And circles may be deſcribed* ad infinitum, *which ſhall only touch one another in* A; *their centers being all in the line* AB *produced.*

All this appears by Cor. 5. Prop. X. compared with this prop.

PROP. XII.

101.
102. *In a circle, the angle at the center is double the angle at the circumference, ſtanding upon the ſame arch*; $BDC = 2BAC$.

Caſe 1. When one ſide AF paſſes through the center; in the iſoſceles triangle ADC, $\angle DAC = DCA$ (3. II), and the $\angle FDC = DAC + DCA$ (1. II) $= 2FAC$.

Caſe 2. If the center of the circle be within the angle BAC; draw ADF, then by Caſe 1, $FDC = 2FAC$, 2FAC, and FDB = 2FAB, therefore the whole BDC = 2BAC. FIG. 101.

Caſe 3. If the center of the circle be without the angle, BAC; draw ADF, then by Caſe 1, FDB = 2FAB, and FDC = 2FAC, therefore the remainder BDC = 2BAC (Ax. 4). 102.

Cor. 1. *The angle at the circumference ſtanding upon any arch, is equal to half the angle at the center, upon the ſame arch; or to the angle at the center upon half the arch.*

Cor. 2. *In the ſame or equal circles, the angles at the circumference, are equal, which ſtand upon equal arches or equal cords.*

This is plain from Cor. 1, 2. Prop. VI.

PROP. XIII.

All angles in the ſame ſegment of a circle, are equal, DAC = DBC, *and* DGC = DHC. 103. ×

For ∠DGC and DHC are each equal to the angle at the center, on half the arch DABC. And DAC, DBC are each of them equal to the angle at the center, on half the arch AGHC.

Or thus.

The ∠DGC = $\frac{1}{2}$DOC = DHC (Prop. XII). Again, ∠DFC = DAF + ADF (1. II) = DBC + BCF (ibid.), but ADF = BCF (Prop. XII); therefore DAF or DAC = DBC (Ax. 4).

Cor. *If the extremities of two equal arches* DA, BC, *be joined by right lines*, DC, AB; *they will be parallel.*

For ∠BAC = DCA (Cor. 2. 12), therefore AB, CD are parallel (Cor. 3. 4. I).

PROP.

PROP. XIV.

FIG. 104. *The angle* ABC *in a ſemicircle is a right angle.*

For draw BD to the center, then BDA, BDC are two iſoſceles triangles, therefore DAB = DBA, and DCB = DBC (3. II). And DAB + DCB = DBA + DBC = ABC (Ax. 3) = half of two right angles (2. II) = a right angle.

Cor. 1. *The angle* ABG, *in a greater ſegment* ABFG, *is leſs than a right angle; and the angle* ABF, *in a leſs ſegment* ABF, *is greater than a right angle.*

This is evident by inſpecting the figure.

Cor. 2. *If a line be drawn from the middle of the hypothenuſe (of a right-angled triangle), to the right angle; it cuts the triangle into two iſoſceles triangles.*

PROP. XV.

105. *If two lines cutting a circle, interſect one another in* A; *and there be made at the center,* $\angle$ECF = BAD;

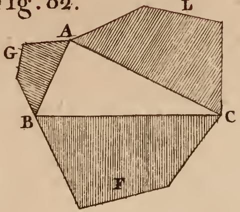
Then arch BD + GH = 2EF, *if* A *is within the circle; or arch* BD — GH = 2EF, *if* A *is without.*

For draw HI parallel to GD, then DI = GH (Cor. 13); and angle BHI = BAD = ECF (4. I). Therefore EF = $\frac{1}{2}$BI (Cor. 1. 12); and 2EF = BI = BD + GH, when A is within, but = BD — GH, when A is without the circle.

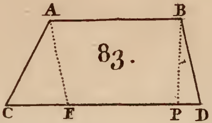
Cor. 1. *If from a point without, two lines touch a circle; the angle made by them is equal to the angle at the center, ſtanding on half the difference, of theſe two parts of the circumference.*

This

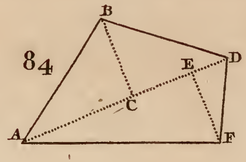
Fig. 82.



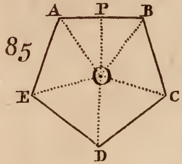
83.

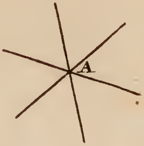


84

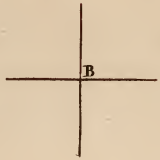


85

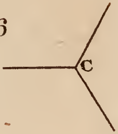




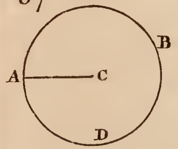
86



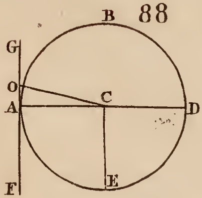
86



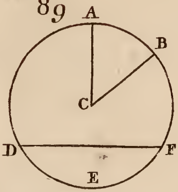
87



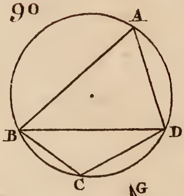
88



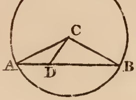
89



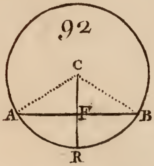
90



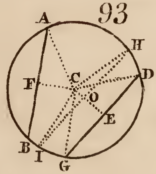
91



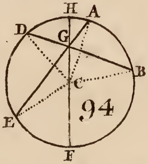
92



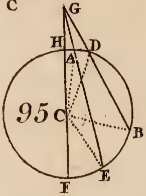
93



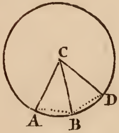
94



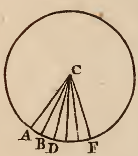
95

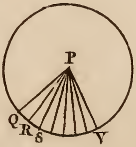


96

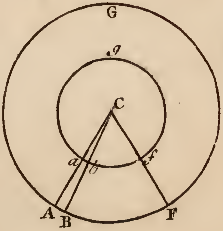


97

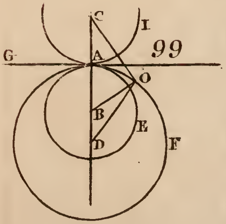




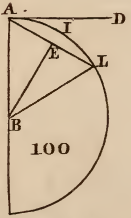
98



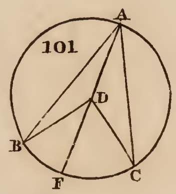
99



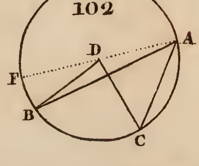
100



101



102



This is plain, by ſuppoſing B, H, and G, D to coincide in the periphery, then half their difference will be = EF. FIG. 105.

Cor. 2. *The angle* A = ∠BHD + HDG, *when* A *is within*; *or* A = BHD − HDG, *when* A *is without the circle* (1. II).

PROP. XVI.

In a circle, the angle made at the point of contact between the tangent and any cord, is equal to the angle in the alternate ſegment; ECF = EBC, *and* ECA = EGC. 106.

Through the center O, draw the diameter COD, which is ⊥ to CF (Cor. 2. 10). The ∠CED is right (Prop. XIV); therefore ∠D + DCE = a right angle (Cor. 2. 2. II) = DCE + ECF; therefore D = ECF, or EBC = ECF (Prop. XIII). Again, CEG + ECG + G = two right angles (2. II) = GCF + ECG + ECA (1. I), CEG + G = GCF + ECA, but CEG = GCF (this prop.), therefore G = ECA (Ax. 4).

Cor. *A tangent to the middle point of an arch, is parallel to the cord of it.*

For if arch CB = CE, then cord CB = cord CE (Prop. VI. and Cor. 2); whence ∠E = B = ECF (this prop.), whence BE, CF are parallel (Cor. 3. 4. I).

PROP. XVII.

If from any point B *in a ſemicircle, a perpendicular* BD *be let fall upon the diameter, it will be a mean proportional between the ſegments of the diameter:* AD : DB : : DB : DC. 107.

For drawing AB, BC, the triangles ABC, ABD, DBC are ſimilar, for ∠ABC is right (Prop. XIV),

FIG. 107. and angles at D are right, and BAD = BAC, ABD = BCD, and therefore DBC = BAD. Therefore AD : DB : : DB : DC (13. II).

Cor. *The cord is a mean proportional between the adjoining ſegment, and the diameter*; AD : AB : AC. *And* CD : CB : CA ∺.

This is evident from the ſimilarity of the triangles.

PROP. XVIII.

108. *In a circle, if the diameter* AD *be drawn, and from the ends of the cords* AB, AC, *perpendiculars be drawn upon the diameter*; *the ſquares of the cords will be as the ſegments of the diameter*; AE : AF : : AB^2 : AC^2.

For AE × AD = AB^2 (Cor. 17), and AF × AD = AC^2 (ibid.); therefore AB^2 : AC^2 : : AE × AD : : AF × AD : : AE : AF (Cor. 1. 5. Proportion).

PROP. XIX.

109. *If two circles touch one another in* P, *and the line* PDE *be drawn through their centers*; *and any line* PAB *is drawn through that point, to cut the circles, that line will be divided in proportion to the diameters*; PA : PB : : PD : PE.

For drawing AD, BE; the triangles PAD, PBE, are right-angled at A, B (14), and conſequently ſimilar, therefore PD : PE : : PA : PB (13. II).

Cor. *The arches* AD, BE *are ſimilar*; *as alſo the arches* PA, PB; *and theſe arches are as the whole circumferences of the circles, or as the diameters*; AD : BE : : PA : PB : : PD : PE, *&c.*

They are ſimilar by Def. 18. and proportional by Prop. VIII.

PROP.

PROP. XX.

FIG. 110.

If through any point F *in the diameter of a circle, any cord* CFD *is drawn; the rectangle of the ſegments of the cord, is equal to the rectangle of the ſegments of the diameter;* CFD = AFB.

Draw AC, BD; then the triangles CAF, BDF are ſimilar, for the angle F is common, and CAF = BDF, and ACF = DBF (Cor. 2. 12); therefore AF : FC : : FD : FB (13. II), and AF × FB = CF × FD (12. Proportion).

Cor. 1. *Let* O *be the center; then the rectangle* CFD, *is equal to radius ſquare — the ſquare of the diſtance from the center;* CFD = $AO^2 - OF^2$.

For AF × FB = $\overline{AO + OF} \times \overline{AO - OF} = AO^2 - OF^2$ (12. I).

Cor. 2. *If ſeveral cords* CD, EG, *be drawn through the ſame point* F, *the rectangles of their ſegments will be all equal to one another;* CFD = EFG.

For they are all equal to the rectangle AFB.

PROP. XXI.

111.

If through any point F *out of the circle in the diameter* BA *produced, any line* FCD *is drawn through the circle; the rectangle of the whole line and the external part, is equal to the rectangle of the whole line paſſing through the center, and the external part;* DFC = AFB.

For drawing DA, CB, the triangles DFA and BFC are ſimilar; for ∠FDA = FBC, and F is common; therefore AF : FD : : CF : FB (13. II); and AF × FB = CF × FD.

Cor.

FIG. 111. Cor. 1. *Let* O *be the center, then the rectangle* CFD *is equal to the ſquare of the diſtance from the center — radius ſquare*; $CFD = FO^2 - AO^2$.

For $AF \times FB = \overline{FO - AO} \times \overline{FO + AO} = FO^2 - AO^2$ (12. I).

Cor. 2. *Let* HF *be a tangent at* H; *then the rect-angle* CFD = *ſquare of the tangent* FH.

For $FO^2 - AO^2 = FO^2 - OH^2 = FH^2$ (Cor. 1. 21. II).

Cor. 3. *If ſeveral lines* FD, FG, *are drawn from the ſame point* F; *the rectangles of the whole and external ſegment, will be all equal to one another*; $CFD = EFG$.

For they are all equal to the rectangle AFB.

Cor. 4. *If from the ſame point* F, *two tangents b drawn to the circle, they will be equal*; $FH = FI$.

For the ſquare of either of them is equal to the rectangle AFB.

PROP. XXII.

112. *If a line* PFC *be drawn perpendicular to the diame-ter* AD *of a circle*; *and any line drawn from* A *t cut the circle and perpendicular*; *then the rectangle o the diſtances of the ſections from* A, *will be equal to th rectangle of the diameter and the diſtance of the per-pendicular from* A; $AB \times AC = AP \times AD$.

For draw BD, and the triangles ABD, AP are ſimilar, for ∠ at A is common, and ∠P and are right (14); therefore AD : AB :: AC : A (13. II), and $AD \times AP = AB \times AC$ (12. Pro-portion).

Cor. 1. *If* PF *cuts the circle in* K, *then* $AB \times A$ $= AK^2$.

Co

Cor. 2. *If more lines* AEF *be drawn, all the rectangles* EAF, BAC *are equal.* FIG. 112.

For they are all equal to the rectangle PAD.

PROP. XXIII.

113. *In a circle* EDF *whoſe center is* C, *and radius* CE, *f the points* B, A, *be ſo placed in the diameter produced, hat* CB, CE, CA *be in continual proportion, then wo lines* BD, AD *drawn from theſe points, to any oint in the circumference of the circle, will always be n the given ratio of* BE *to* AE.

Draw DP perpendicular to the diameter EF, then $DP^2 = EP \times PF$ (17) $= 2CE \times EP - EP^2$, whence $AD^2 = \overline{AE + EP}^2 + PD^2$ (21. II) $= AE^2 + EP^2 + 2AEP + 2CEP - EP^2$ (10. I) $= AE^2 + 2CE \times EP + 2AE \times EP$. Alſo $BD^2 = \overline{BE - EP}^2 + PD^2$ (21. II) $= BE^2 - 2BEP + EP^2 + 2CEP - EP^2$ (11. I) $= BE^2 + 2CE \times EP - 2BE \times EP$. And ſince CA : CE : CB ∺, therefore AE. CE :: EB : CB (13. Proportion), or AE : EB : CE : CB (4. Proportion). Alſo $AE^2 : EB^2 :: E^2 : CB^2 :: CA : CB$ (23. Proportion) $:: CE + AE : CE - EB :: 2CE \times EP + 2AE \times EP$ $2CE \times EP - 2EB \times EP$ (5. Proportion). And $E^2 : EB^2 :: AE^2 + 2CE \times EP + 2AE \times EP : B^2 + 2CE \times EP - 2EB \times EP$ (10. Proportion) $: AD^2 : BD^2$. And AE : EB :: AD : BD Cor. 3. 18. Proportion).

PROP. XXIV.

114. *If* D, C *be two points in the diameter of a circle, uidiſtant from the center* O; *and if two lines be rawn from thence to any point* E, *in the circumference, e ſum of their ſquares will be equal to the ſum of the uares of the ſegments of the diameter*; $DE^2 + CE^2 = AC^2 + CB^2$.

FIG. 114. For draw EO to the center O, then $DE^2 + CE^2 = 2DO^2 + 2OE^2$ (28. II) $= 2AO^2 + 2OC^2$. But $AC^2 + CB^2 = \overline{AO + OC}^2 + \overline{AO - OC}^2 = AO^2 + OC^2 + 2AOC + AO^2 + OC^2 - 2AOC$ (10. II) $= 2AO^2 + 2OC^2 = DE^2 + CE^2$.

Cor. 1. *Hence the ſum of the ſquares of* DE, CE *is equal to twice the ſquare of the radius* + *twice the ſquare of the diſtance of one of the points from the center*; $DE^2 + CE^2 = 2AO^2 + 2OC^2$.

Cor. 2. *The ſum of the ſquares of any two correſpondent ones will be equal.*

For they are all equal to the ſame given quantity.

PROP. XXV.

115. *If any cord* PQ *be drawn parallel to the diameter* AB, *of a circle*; *and from a given point* C *in that diameter, the lines* CP, CQ *be drawn to the two ends of the cord*; *I ſay the ſum of their ſquares is equal to the ſum of the ſquares of the ſegments of the diameter*; $CP^2 + CQ^2 = AC^2 + CB^2$.

For draw PS, QR ⊥ to the diameter AB, then PS^2 or $QR^2 = PC^2 - SC^2 = QC^2 - RC^2$ (21. II); that is, $PC^2 - \overline{SO + OC}^2 = QC^2 - \overline{SO - OC}^2$; or $PC^2 - SO^2 - 2SOC - OC^2$ (10. I) $= QC^2 - SO^2 + 2SOC - OC^2$, becauſe $OQ = OS$. Therefore $PC^2 = QC^2 + 4SOC$, but $AC^2 + CB^2 = \overline{AO + OC}^2 + \overline{AO - OC}^2 = 2AO^2 \times 2OC^2$ (10, 11. I). But $PC^2 = AO^2 + OC^2 + 2SOC$ (22. II) $= QC^2 + 4SOC$. Therefore

$$QC^2 = AO^2 + OC^2 - 2SOC$$
$$PC^2 = AO^2 + OC^2 + 2SOC$$

therefore $PC2 + QC^2 = 2AO^2 + 2OC^2 = AC^2 + CB^2$.

Cor.

Cor. 1. *The ſum of their ſquares*, $PC^2 + QC^2 = 2AO^2 + 2OC^2$. FIG. 115.

Cor. 2. *The difference of their ſquares*, $PC^2 - QC^2 = 4SOC$.

Cor. 3. *All theſe things hold good, if the point* C *is taken without the circle.*

PROP. XXVI.

In a circle, if a perp. DB *be let fall from any point* D, *upon the diameter* CI, *and the tangent* DO *drawn from* D; *then* AB, AC, AO, *will be continually proportional.* 116.

Draw the radius DA, then the triangles ABD, ADO, are ſimilar, for the angles at B and D are right (Cor. 2. 10), and angle A common; whence AB : AD : : AD : AO; that is, AB : AC : AO $\div\div$.

PROP. XXVII.

If a triangle ADC *be inſcribed in a circle; and if* BC *be drawn parallel to the tangent* AT; *then* AB, AC, AD, *are continually proportional.* 117.

For the triangle ABC, is ſimilar to ACD; for $\angle$ D = TAC (16) = ACB (4. 11), and A is common; therefore AB : AC : : AC : AD (13. 11).

Cor. AD : DC : : AC : CB.

PROP. XXVIII.

If a triangle BDF *be inſcribed in a circle, and a perpendicular* DP *let fall from* D *on the oppoſite ſide* BF, *and the diameter* DA *drawn; then as the perpendicular, is to one ſide including the angle* D; *ſo the other ſide, to the diameter of the circle*; DP : DB : : DF : DA. 118.

FIG. 118. For drawing AF, the triangles BDP, and ADF are ſimilar; for ∠A = B (13), and angles at P and F are right (14); therefore DP : DB : : DF : DA (13. II).

Cor. *The rectangles of the ſides of an inſcribed triangle; is equal to the rectangle of the diameter, and the perp. on the third ſide.*

PROP. XXIX.

119. *If a triangle* BAC *be inſcribed in a circle, and the angle* A *biſected by the right line* AED; *then as one ſide, to the ſegment of the biſecting line, within the triangle; ſo the whole biſecting line, to the other ſide;* AB : AE : : AD : AC.

Draw BD, then the triangles ABD, ACE are ſimilar; for ∠D = C (Cor. 2. 12), and BAD = EAC (hyp.); therefore AB : AD : : AE : AC (13. II); and AB : AE : : AD : AC (4. Proportion).

Cor. *If an angle of a triangle (inſcribed in a circle) be biſected; the rectangle of the ſides, is equal to the rectangle of the whole biſecting line within the circle, and the ſegment within the triangle:* BAC = DAE.

PROP. XXX.

120. *If a circle be inſcribed in a triangle* ABC, *and lines be drawn from the center* D, *to the points of contact* E, F, G; *then any ſegment* BF *or* BE *joining to the angle* B, *is equal to half the ſum of the three ſides — the oppoſite ſide* AC.

For the triangles BDF, BDE are ſimilar and equal (9. II); for ∠F = ∠E a right one (10), and DE = DF, and BD common; whence BF = BE. In like manner CF = CG, and AE = AG. Then ſince the ſum of the ſides is BC + CA + AB =

2BF + 2CG + 2AG, therefore half the ſum = BF + CG + AG = BF + AC, therefore BF = $\frac{1}{2}$ſum — AC. FIG. 120.

Cor. *The area of the triangle* BAC, *is equal to the rectangle of the radius* DF, *and half the ſum of the three ſides.*

For the triangle ABC is made up of the three triangles ADB, BDC, CDA, whoſe common hight is the radius DF.

PROP. XXXI.

121. *If a quadrilateral* ABCD *be inſcribed in a circle, the ſum of two oppoſite angles is equal to two right angles*; ADC + ABC = *two right angles.*

Draw AC, BD, and produce AB to E; then the external angle CBE = BCA + BAC (1. II) = BDA + BDC (13) = ADC; therefore CBE + CBA = ADC + CBA = 2 right angles (1. I).

Cor. *If one ſide of a quadrangle (inſcribed in a circle) be produced, the external angle* EBC *is equal to the internal oppoſite angle* ADC.

PROP. XXXII.

122. *If a quadrangle be inſcribed in a circle; the rectangle of the diagonals, is equal to the ſum of the rectangles of the oppoſite ſides*; AC × BD = AB × CD + AD × BC.

Make the angle ABF = CBD, then ABD = CBF; and ſince the ∠CDB = FAB (13), the triangles FAB, and CDB are ſimilar, whence DC : DB : : AF : AB (13. II), and CD × AB = BD × AF (12. Proportion). Alſo ſince ∠BCF = BDA (13), the triangles CBF and DBA are ſimilar; whence CB : CF : : DB : DA (13. II), and CB

FIG. 122. $\times DA = BD \times CF$ (12. Proportion). Therefore $CD \times AB + CB \times DA = BD \times AF + BD \times CF = BD \times AC$ (Ax. 3).

PROP. XXXIII.

A circle is equal to a triangle whoſe baſe is the circumference of the circle; and hight, its radius.

123. Let AB be equal to the length of the circumference, and let the circle touch it in I; draw CI, and CD extremely near it. Then by reaſon of the extreme ſmallneſs of the arch DI, the ſector CD coincides with the triangle CDI, and the arch with a portion of the right line. Now ſince the circle DEGF may be ſuppoſed to be made up of ſuch ſectors CDI, and the triangle ACB of as many triangles CDI equal to the ſector CDI; it follows that all theſe ſectors are equal to all the triangles, or the circle DEGF = the triangle ABC.

This is alſo evident by the 25. III. for a circle may be conſidered as a regular polygon of an infinite number of ſides, whoſe hight is the radius of the circle.

Cor. *The ſector of a circle is equal to a triangle, whoſe baſe is the arch, and hight the radius.*

PROP. XXXIV.

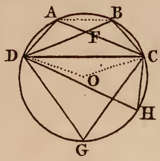
123. *The area of a circle is equal to the rectangle of half the circumference and half the diameter.*

For the circle EGF is equal to the triangle ABC (33), and that triangle is equal to the rectangle of half the baſe AB and hight CI, that is, of half the circumference DFGED, and half the diameter CI (Cor. 2. 10. II).

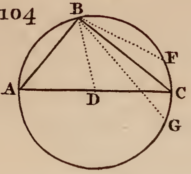
Cor. 1. *The ſector of a circle is equal to the rectangle of half the arch and the radius.*

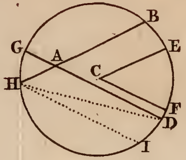
Cor.

Fig. 103

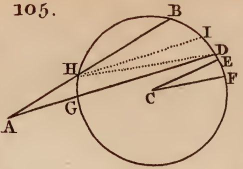


104

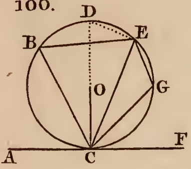




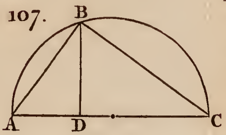
105.



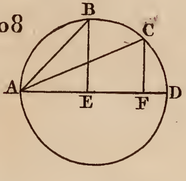
106.



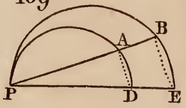
107.



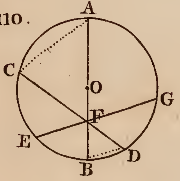
108



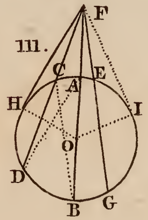
109

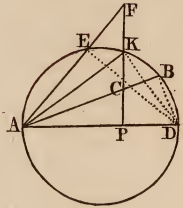


110.

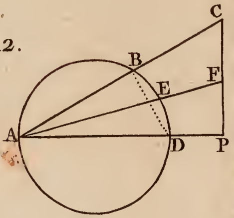


111.

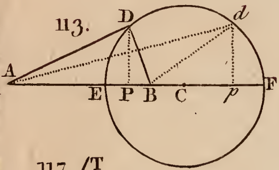




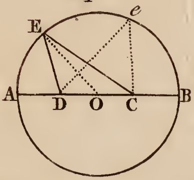
112.



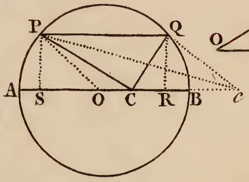
113.



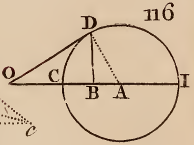
114.



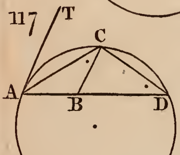
115.



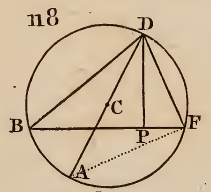
116



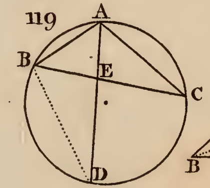
117



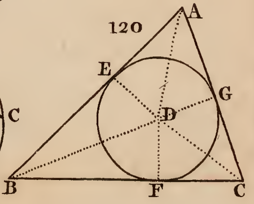
118



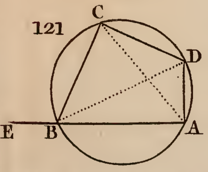
119



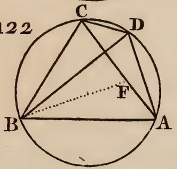
120



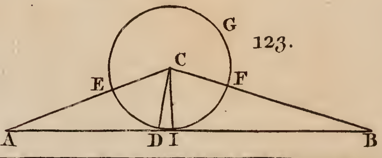
121



122



123.



Pl. V. *Pa. 70.*

Cor. 2. *Sectors are to one another in the complicate ratio of their arches and radii.* FIG. 123.

For triangles, to which they are equal, are in that ratio (Cor. 1. 11. II).

PROP. XXXV.

Circles (that is, their areas) are to one another as the ſquares of their diameters.

For circumference EFGE : cir. IHKI : : AB : CD (9); and the areas of the circles EFG, and IHK are $\frac{EFG \times AB}{4}$, and $\frac{IHK \times CD}{4}$ (34); therefore circle EF : circle IH : : EFGE × AB : IHKI × CD (5. Proportion) : : $AB^2 : CD^2$ (7. Proportion). 124.

Cor. 1. *Circles are to one another as the ſquares of the radii, or as the ſquares of the circumferences.*

For theſe are all in the ſame ratio (Prop. IX).

Cor. 2. *A circle made on the hypothenuſe, is equal to two circles made alike on the two ſides, of a right-angled triangle.*

PROP. XXXVI.

Similar polygons deſcribed in circles, are to one another, as the circles wherein they are inſcribed.

Draw CK, AG, then becauſe ſimilar polygons may be reſolved into ſimilar triangles (Cor. 2. 19. III), therefore AF : AG : : CH : CK, and AG : AB : : CK : CD (13. II), therefore AF : AB : : CH : CD. Or at once, AF : AB : : CH : CD (19. III). And polygon EF : polygon IH : : $AF^2 : CH^2$: : $AB^2 : CD^2$ (20. III) : : circle EF : circle IH (35). 124.

Cor.

FIG. 124. Cor. 1. *Like polygons inſcribed in circles, are as the ſquares of the diameters.*

Cor. 2. *The peripheries of like polygons inſcribed in circles, are as the diameters of the circles.*

For AF : CH : : FG : HR : : GB : KD : : BE : DI : : EA : IC (Def. 12), therefore AFGBEA : CHKDIC : : AF : CH (10. Proportion) : : AB : CD (19. III).

P R O P. XXXVII.

A circle is to any circumſcribed rectilineal figure; as the circle's periphery, to the periphery of the figure.

125. From O, the center of the inſcribed circle, draw OP perp. to the ſide AD. Then the figure AC conſiſts of the triangles ABO, BCO, CDO, DAO, whoſe common hight is the radius OP. Therefore its area $= \frac{ABCDA}{2} \times OP$; and the circle = circumference $\frac{PRQ}{2} \times PO$ (34); but $\frac{ABCDA \times OP}{2} : \frac{PRQ \times OP}{2}$: : ABCDA : circumference PQR (5. Proportion).

Cor. 1. *Any polygon circumſcribed about a circle, is equal to a triangle whoſe hight is the radius, and baſe the circumference of the polygon.*

Cor. 2. *Of figures of equal compaſs, the circle is the biggeſt or moſt capacious.*

For if the ſides of the figure be ſuppoſed to touch the circle, they will be greater than the circumference of the circle, contrary to the ſuppoſition. Therefore they will fall within the circle, and then the perpendicular upon them will be ſhorter than the radius. And therefore the polygon will be leſs than the circle, becauſe the triangle (to which it is equal)

has the ſame baſe, and a leſs hight, than the triangle FIG.
to which the circle is equal. 125.

Cor. 3. *All figures circumſcribing the ſame circle, are to one another as their circumferences.*

PROP. XXXVIII.

The area of a crown, ring, or annulus ABC *(contained between the circumferences of two circles), is equal to the rectangle under the breadth* RF, *and half the ſum of the perimeters.* 126.

Let C, c be the circumferences of the greater and leſſer circles, then KF : c : : KR : C (9), and KF : KR : : c : C (4. Proportion), and KF : RF : : c : C — c (13. Proportion), whence $KF \times \overline{C - c} = RF \times c$.

But the annulus = difference of the circles =

$$\frac{RK \times C}{2} - \frac{KF \times c}{2} = \frac{RF \times C + FK \times C - FK \times c}{2}$$

$$= \frac{RF \times C + FK \times \overline{C - c}}{2} = \frac{RF \times C + RF \times c}{2}$$

$$= \frac{C + c}{2} \times RF.$$

Cor. *If* FG *be perp. to* RP, *and a mean proportional between the two radii; then the circle deſcribed, with the radius* FG, *is equal to the annulus* ABC.

For $FG^2 = KG^2 - KF^2$ (Cor. 1. 21. III); therefore the circle whoſe radius is FG is equal to the difference of the circles whoſe radii are KG or KR and KF (22. III).

PROP. XXXIX.

Let ABCD *be a trapezium inſcribed in a circle, and put* R = *the radius,* P = AB × BC + CD × DA, Q = AB × CD + BC × AD, T = AB × AD + BC × CD. *Then the area of the trapezium will be* $= \frac{\sqrt{PQT}}{4R}$. 127.

For

FIG. 127. For $\frac{AB \times AD}{2R}$ = perp. from A upon BD (28), and $\frac{BC \times CD}{2R}$ = perp. from C on BD (ibid.), and $\frac{AB \times AD + BC \times CD}{2R} = \frac{T}{2R}$ is the ſum of the perpendiculars.

Therefore $\frac{T \times BD}{2R}$ = twice the area of the trapezium (24. III); and in like manner $\frac{AB \times BC + AD \times DC}{2R} \times AC$ or $\frac{P \times AC}{2R}$ = twice the ſame area. Therefore $T \times BD = P \times AC$, and $AC = \frac{T \times BD}{P}$. But $AC \times BD = AB \times CD + AD \times CB$ (32) $= Q = \frac{T \times BD^2}{P}$, and $BD^2 = \frac{PQ}{T}$, and $BD = \sqrt{\frac{PQ}{T}}$. Whence $\frac{T \times BD}{2R} = \frac{T}{2R}\sqrt{\frac{PQ}{T}} = \frac{\sqrt{PQT}}{2R}$ = twice the area; and the area $= \frac{\sqrt{PQT}}{4R}$.

Cor. 1. $BD^2 = \frac{PQ}{T}$, and $AC^2 = \frac{QT}{P}$.

Cor. 2. $BD : AC :: P : T :: AB \times BC + CD \times DA : AB \times AD + BC \times CD$.

PROP. XL.

127. *If* ABCD *be a trapezium inſcribed in a circle, and each ſide be ſubtracted from the ſum of the other three, there will be four remainders; then take the product of two of theſe remainders, and likewiſe the product of the other two; I ſay,* 4 *times the area of the trapezium will be a mean proportional, between theſe two products.*

From

FIG. 127.

From A let fall the perpendiculars AF, AP upon CB, CD. Then ſince ∠ABF = ADC (Cor. 31); the firſt perp. falls without, and the ſecond within the figure. Alſo the angles B, P, being right, the triangles ABF, APD, are ſimilar. Therefore AB : BF : : AD : DP $= \frac{AD\times BF}{AB}$ (13. II), and AF $= \sqrt{AB^2 - FB^2}$ (21. II), alſo AB : AF : : AD : AP $= \frac{AD}{AB}\sqrt{AB^2 - FB^2}$. Draw AC, then $AB^2 + BC^2 + 2BC\times BF = AC^2 = AD^2 + CD^2 - 2CD\times\frac{AD\times BF}{AB}$ (22, 23. II). Whence $2BC\times BF + \frac{2CD\times AD}{AB}\times BF = AD^2 + CD^2 - AB^2 - BC^2$, and $\frac{BF}{AB} = \frac{AD^2 + CD^2 - AB^2 - BC^2}{2AB\times BC + 2CD\times DA}$ $= \frac{AD^2 + CD^2 - AB^2 - BC^2}{2P}$ (putting P = AB × BC + CD × DA). And

$$\frac{AB + BF}{AB} = \frac{CD^2 + 2CD\times AD + AD^2 - AB^2 + 2AB\times BC - BC^2}{2P}$$

$$= \frac{\overline{CD + AD}^2 - \overline{AB - BC}^2}{2P},$$ and $$\frac{AB - BF}{AB} = \frac{AB^2 + 2AB\times BC + BC^2 - CD^2 + 2CD\times AD - AD^2}{2P}$$

$$= \frac{\overline{AB + BC}^2 - \overline{CD - AD}^2}{2P}.$$

But twice the area = AF × BC + AP × CD (24. III) $= BC\sqrt{AB^2 - BF^2} + \frac{CD\times AD}{AB}\sqrt{AB^2 - BF^2}$ $= \frac{AB\times BC + CD\times AD}{AB}\sqrt{AB^2 - BF^2} =$ $\frac{P}{AB}\sqrt{AB^2 - BF^2}$; and 4 × area ſquare = $PP\times\frac{AB^2 - BF^2}{AB^2} = PP\times\frac{AB + BF}{AB}\times\frac{AB - BF}{AB}$

$=$

FIG. 127. $= PP \times \frac{\overline{CD + AD}^2 - \overline{AB - BC}^2}{2P} \times \frac{\overline{AB + BC}^2 - \overline{CD - AD}^2}{2P}$, and $16 \times \overline{area}^2 = \overline{\overline{CD+AD}^2 - \overline{AB-BC}^2} \times \overline{\overline{AB+BC}^2 - \overline{CD-AD}^2} = \overline{CD+AD+AB-BC} \times \overline{CD+AD-AB+BC} \times \overline{AB+BC+CD-AD} \times \overline{AB+BC+AD-CD}$ (12. I).

Cor. *If* S = *half the ſum of the four ſides, then the area is a mean proportional between theſe rectangles* $\overline{S - AB} \times \overline{S - BC}$, *and* $\overline{S - CD} \times \overline{S - DA}$.

For $area^2 = \frac{CD + AD + AB - BC}{2} \times$ *&c.* but $S - AB = \frac{CD + AD + AB + BC}{2} - AB = \frac{CD + AD - AB + BC}{2}$, and ſo of the reſt.

PROP. XLI.

128. *If an equilateral triangle* ABC *be inſcribed in a circle; the ſquare of the ſide thereof, is equal to three times the ſquare of the radius:* $AB^2 = 3AD^2$.

Draw the diameter AE, and the cord BE. Then the triangle BDE is equiangular (Cor. 1. 2. II), for $\angle BDE = BAC$ (Cor. 1. 12) $= BCA = BED$, and $BE = DB$ (Cor. 1. 3. II). Then $AB^2 + BE^2 = AE^2$ (21. II) $= 4DB^2 = 4BE^2$, and $AB^2 = 3BE^2 = 3BD^2$.

Cor. 1. $AB^2 : AF^2 :: 4 : 3$.

For $AB^2 : AF^2 :: AE : AF$ (23. Proportion) $:: 4 : 3$.

Cor. 2. DF = *half* DE.

Cor. 3. *The ſide* BC *of the equilateral triangle, cuts off a fourth part of the diameter.*

PROP. XLII.

FIG. 129. *A ſquare inſcribed in a circle, is equal to twice the ſquare of the radius*; $AB^2 = 2BO^2$.

For $AB^2 = AO^2 + OB^2$ (21. II) $= 2AO^2$.

Cor. *The circumſcribed ſquare* EG *is double the inſcribed ſquare*, AC.

For EG is the ſquare of the diameter or 4 ſquares of the radius, and therefore equal to two of the inſcribed ſquares, ABCD.

PROP. XLIII.

130. *If two diagonals* BD, EC *be drawn to cut one another, in an inſcribed regular pentagon. The greater ſegments* EF, BF, *will be equal to the ſide of the pentagon*, AB.

For ſince the arch AE = BC, and AB = ED, therefore EC is parallel to AB, and BD parallel to AE (Cor. 13); therefore ABFE is a parallelogram, and EF = AB = AE = BF (1. III).

Cor. 1. *The diagonals* BD, CE *cut one another in extreme and mean proportion*; BD : BF : : BF : FD.

For ∠DCF = CDF = CBD (Cor. 2. 12); therefore the triangles CDF, CDB are ſimilar, BD : DC : : DC : DF (13. II); that is, BD : BF : : BF : FD.

Cor. 2. *The diagonal* CE *is parallel to* AB, *and* BD *to* AE.

Cor. 3. *The ſide of the pentagon* BC, *is to the diagonal* BD, *as* 1 *to* $\frac{1 + \surd 5}{2}$.

For

FIG. 130. For $BD \times FD$ or $BD \times \overline{BD - BC} = BF^2$ (Cor. 1); that is, $BD^2 - BD \times BC = BC^2$; add $\frac{1}{4}BC^2$, then $BD^2 - BD \times BC + \frac{1}{4}BC^2 = \frac{5}{4}BC^2$; that is, $\overline{BD - \frac{1}{2}BC}^2 = 5 \times \frac{BC^2}{4}$, and the root is $BD - \frac{1}{2}BC = \frac{BC}{2}\sqrt{5}$, and $BD = \frac{BC}{2} + \frac{BC}{2}\sqrt{5} = BC \times \frac{1 + \sqrt{5}}{2}$.

Cor. 4. *The angle* BCF *is double to the angle* CBF.

For it ſtands on double the arch.

PROP. XLIV.

130. *If a regular pentagon be inſcribed in a circle; the ſquare of the radius* AH, *is to the ſquare of its ſide*, AB; *as* 2 *to* $5 - \sqrt{5}$.

Let HG biſect AB in I, and make $IO = IG$. Then the angles AIO, AIG are right (Cor. 3. 3. II). And put $R =$ radius AH. The $\angle GHA = \frac{2}{5}$ of a right angle (1. I), and $IAH = \frac{3}{5}$ of a right angle (Cor. 2. 2. II); but IAG or $IAO = \frac{1}{2}GHA$ (12) $= \frac{1}{5}$ of a right angle, therefore $OAH = \frac{2}{5}$ of a right angle (Ax. 4) $= OHA$, whence $HO = OA = AG$. $2R \times GI = GA^2$ (Cor. 17) $= HO^2 = \overline{R - 2GI}^2 = RR - 4R \times GI + 4GI^2$ (11. I), and $4GI^2 - 6R \times GI + RR = 0$ (Ax. 3. 4), and $GI^2 - \frac{3}{2}R \times GI + \frac{1}{4}RR = 0$ (Ax. 6); add $\frac{5}{16}RR$, then $\frac{9}{16}RR - \frac{3}{2}R \times GI + GI^2 = \frac{5}{16}RR$; that is, $\overline{\frac{3}{4}R - GI}^2 = 5 \times \frac{RR}{16}$, whence the root $\frac{3}{4}R - GI = \frac{R}{4}\sqrt{5}$, and $GI = \frac{3 - \sqrt{5}}{4}R$. But $\frac{3}{2}R \times GI - GI^2 = \frac{1}{4}RR$. And $AI^2 = 2R \times GI - GI^2$ (17) $= \frac{1}{2}R \times GI + \frac{3}{2}R \times GI - GI^2 = \frac{1}{2}R \times GI + \frac{1}{4}RR = \frac{1}{2}R \times \frac{3 - \sqrt{5}}{4}R + \frac{RR}{4} =$

$RR \times \frac{5-\sqrt{5}}{8}$, and $4AI^2$ or $AB^2 = RR \times \frac{5-\sqrt{5}}{2}$. FIG. 130.

Cor. 1. *The ſquare of the perpendicular* HI, *upon the ſide of the pentagon, is equal to* $\frac{3+\sqrt{5}}{8}RR$.

For $HI^2 = R^2 - AI^2 = \frac{8R^2 - 5R^2 + \sqrt{5} \, . \, RR}{8} = \frac{3+\sqrt{5}}{8}RR$.

Cor. 2. *The ſquare of radius* AH, *is to the ſquare of the diagonal* BD, *as* 1 *to* $\frac{5+\sqrt{5}}{2}$.

For BC or $AB = \frac{2BD}{1+\sqrt{5}}$ (Cor. 3. 43), and $AB^2 = \frac{4BD^2}{\overline{1+\sqrt{5}}^2} = \frac{4BD^2}{1+5+2\sqrt{5}}$ (10. I) $= \frac{4BD^2}{6+2\sqrt{5}}$, and $AB^2 = RR \times \frac{5-\sqrt{5}}{2}$ (44), therefore $\frac{4BD^2}{6+2\sqrt{5}}$ or $\frac{2BD^2}{3+\sqrt{5}} = RR \times \frac{5-\sqrt{5}}{2}$, and $2BD^2 = RR \times \frac{5-\sqrt{5}}{2} \times \overline{3+\sqrt{5}} = RR \times \frac{15+5\sqrt{5}-3\sqrt{5}-5}{2}$ (Cor. 1. 8. I) $= RR \times \frac{10+2\sqrt{5}}{2}$; that is, $BD^2 = RR \times \frac{5+\sqrt{5}}{2}$.

Cor. 3. *If* CE *be the diagonal of the pentagon, and* OLD *be drawn; then* $DL = R \times \frac{5-\sqrt{5}}{4}$. 134.

For $DL = \frac{DE^2}{DF}$ (Cor. 17) $= RR \times \frac{5-\sqrt{5}}{4R} = R \times \frac{5-\sqrt{5}}{4}$ (44).

PROP.

FIG.

PROP. XLV.

131\. *The ſide of a regular hexagon inſcribed in a circle, is equal to the radius of the circle:* BE = BD.

For ∠BDE = $\frac{1}{6}$ of four right angles (Cor. 1. I) = $\frac{1}{3}$ of two right angles. And the angles B and E together = $\frac{2}{3}$ of two right angles (2. II), whence BED = $\frac{1}{3}$ of two right angles (3. II) = BDE; therefore BE = BD (Cor. 1. 3. II).

PROP. XLVI.

132\. *The ſquare of the ſide of a regular octagon, inſcribed in a circle; is equal to the ſquare of half the ſide of the inſcribed ſquare, together with the ſquare of the difference of that half ſide and the radius;* $AB^2 = AP^2 + \overline{OB - AP}^2$.

For $AB^2 = AP^2 + PB^2$ (21. II), but ∠PAO = POA (Cor. 1. 12); therefore PO = AP, and BP = OB — OP = OB — AP; and $AB^2 = \overline{OB - AP}^2 + AP^2$.

Cor. *The ſquare of radius, is to the ſquare of the ſide of the octagon; as* 1 *to* $2 - \sqrt{2}$.

For $AP = \sqrt{\frac{1}{2}AO^2}$, and $PB^2 = \overline{BO - AP}^2 = \overline{BO - AO\sqrt{\frac{1}{2}}}^2 = BO^2 - \frac{2BO \times AO}{\sqrt{2}} + \frac{AO^2}{2} = 1\frac{1}{2}AO^2 - AO^2\sqrt{2}$, add $AP^2 = \frac{1}{2}AO^2$, and $AP^2 + BP^2$ or $AB^2 = 2AO^2 - AO^2\sqrt{2}$.

PROP. XLVII.

133\. *The radius of a circle is a mean proportional, between the ſide of an inſcribed regular decagon, and the ſum of that ſide and the radius;* AB : DA :: DA : DA + AB.

FIG. 133.

Produce AB to F, ſo that BF may be = BD; and draw DF, DB. Then ∠ADB = $\frac{1}{5}$ of two right angles, and therefore DAB and DBA together = $\frac{4}{5}$ of two right angles (2. II), and ABD = $\frac{2}{5}$ of two right angles (3. II) = BDF + BFD (1. II) = 2BDF (3. II); therefore BDF or BFD = $\frac{1}{5}$ of a right angle = ADB. Therefore the triangles ADB, and ADF are ſimilar, for F = ADB; and A is common, whence AF or AB + BD : AD : : AD : AB.

Cor. 1. *If the radius be cut in extreme and mean atio, the greater ſegment is the ſide of the decagon,* AB.

For ſince AB + AD : AD : : AD : AB. Therefore AB : AD : : AD — AB : AB (13. Proportion), or AD : AB : : AB : AD — AB, therefore AD is cut in extreme and mean proportion Def. 11. Proportion).

Cor. 2. *The radius is to the ſide of the decagon; as to* $\sqrt{5} - 1$.

For $AB^2 + AB \times AD = AD^2$ (12. Proportion), dd $\frac{1}{4}AD^2$, then $AB^2 + AB \times AD + \frac{1}{4}AD^2 = AD^2$ (Ax. 3), and $AB + \frac{1}{2}AD = \frac{AD}{2}\sqrt{5}$, and $B = \frac{AD}{2}\sqrt{5} - \frac{AD}{2}$, and $2AB = AD \times \overline{\sqrt{5} : - 1}$.

Cor. 3. *The ſquare of the perpendicular upon the de of a decagon, is* $\frac{5 + \sqrt{5}}{8}$ × *the ſquare of the adius.*

For $\frac{1}{2}AB$ = rad. × $\frac{\sqrt{5} - 1}{4}$, and its ſquare =

FIG. $AD^2 \times \frac{3 - \sqrt{5}}{8}$, and the ſquare of the perpend.
133. $= AD^2 - AD^2 \times \frac{3 - \sqrt{5}}{8} = AD^2 \times \frac{5 + \sqrt{5}}{8}$.

PROP. XLVIII.

134. *The ſquare of the ſide of a regular pentagon inſcribed in a circle, is equal to the ſum of the ſquares of the radius, and of the ſide of a regular decagon, inſcribed in the ſame circle;* $AB^2 = FA^2 + AO^2$.

Draw OG perpendicular to the cord FA, to cut it in G, and draw FH. The triangles ABO, HBO are ſimilar; for $\angle AOB = \frac{1}{5}$ of 4 right angles, or $\frac{2}{5}$ of 2 right angles (1. I), alſo BAO and ABO are together $= \frac{3}{5}$ of 2 right angles (2. II), and therefore $BAO = \frac{3}{10}$ of 2 right angles, but BOG ($= \frac{3}{4}$ AOB) $= \frac{3}{4} \times \frac{2}{5}$ of 2 right angles $= \frac{3}{10}$ of 2 right angles, therefore BAO = BOG, and B is common; whence AB : BO : : BO : BH $= \frac{BO^2}{AB}$.

Again, the triangles AFH and ABF are ſimilar, for $\angle A = AFH$ (3. II), and $\angle A = B$ (Cor. 2. 12), therefore BA : AF : : AF : AH $= \frac{AF^2}{AB}$; therefore $AB = AH + BH = \frac{AF^2 + BO^2}{AB}$, or $AB^2 = BO^2 + AF^2$.

Cor. *The perpendicular* OI *upon the ſide of the pentagon, is equal to half the ſum of the radius and ſide of the decagon;* $OI = \frac{AO + AF}{2}$.

For $OI = OF - FI = \frac{2OF^2 - 2OF \times FI}{2OF} = \frac{2OF^2 - FA^2}{2OF}$ (Cor. 17). And ſince $AO^2 = FA^2 +$

AO

AO × AF (47), therefore $AO^2 - FA^2 = AO \times$ AF, and $2FO^2 - FA^2 = FO^2 + AF \times FO$, and $\frac{AF + FO}{2} = \frac{2OF - FA^2}{2OF} = OI$. FIG. 134.

PROP. XLIX.

The ſide of a regular dodecagon inſcribed in a circle, is a mean proportional between the radius, and the exceſs of the diameter above the ſide of the inſcribed equilateral triangle. 135.

Let AB be a ſide of the dodecagon, and draw CB, CF, and DF the ſide of the triangle, and FR perpendicular to AC. Then ACF = $\frac{1}{3}$ of 2 right angles = CAF = CFA (2. II), therefore ACF is an equilateral triangle, and AO = $\frac{1}{2}$AC, and CO = $\sqrt{\frac{3}{4}AC^2}$ (Cor. 39. II), and BO = CA — CO = CA — $\sqrt{\frac{3}{4}CA^2}$, and $BO^2 = CA^2 + \frac{3}{4}CA^2 - 2CA \times \sqrt{\frac{3}{4}CA^2} = 1\frac{3}{4}CA^2 - CA^2\sqrt{3}$ (11. I), and $AB^2 = AO^2 + OB^2$ (21. III) $= \frac{1}{4}AC^2 + 1\frac{3}{4}AC^2 - CA^2\sqrt{3} = 2AC^2 - CA^2\sqrt{3}$. Therefore CA : AB :: AB : $2CA - \overline{CA\sqrt{3}}$. But 2CA is the diameter, and CA × $\sqrt{3}$ = ſide DF of the equilateral triangle (41).

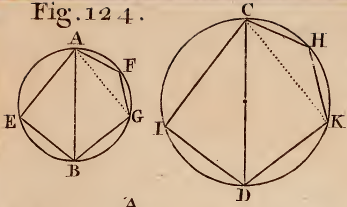
Cor. *The ſide of the dodecagon,* AB = CA × $\sqrt{2 - \sqrt{3}}$.

PROP. L.

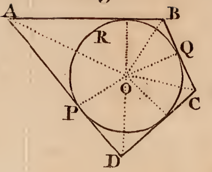
If ABH *be an equilateral triangle, and* APDFG *an equilateral pentagon, inſcribed in a circle, both placed with their angles at* A; *then the cord* BD *will be the ſide of an equilateral quindecagon; and* BI *will be half the difference of the ſides of the triangle and pentagon, and* DI (*perp. to* BH) *is the difference of the perpendiculars in the two figures; and* $BD^2 = BI^2 + DI^2$. 136.

FIG. 136. For arch AP $= \frac{1}{5} = \frac{3}{15}$ the circumference, and APD $= \frac{6}{15}$ of it; alſo APB $= \frac{1}{3} = \frac{5}{15}$ the circumference; therefore APD — APB $= \frac{1}{15}$ the circumference; and the cord BD, the ſide of the quindecagon; moreover DI = CO — CE, is known from Cor. Prop. XLIV. and Cor. 2. Prop. XLI. Alſo BI $= \frac{BH - DF}{2}$, which will be known by Prop. XLI, and Prop. XLIV; and thence $BD^2 = BI^2 + DI^2$, will be known.

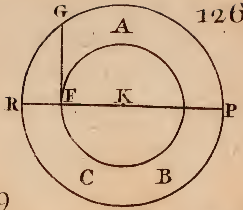
Fig. 124.



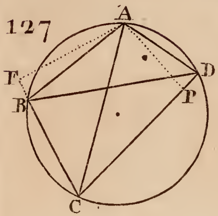
125



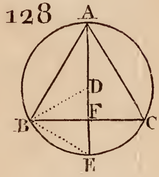
126



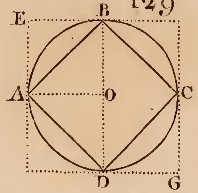
127



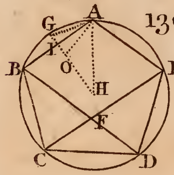
128



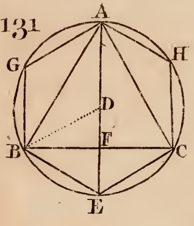
129



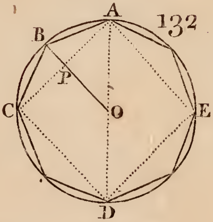
130



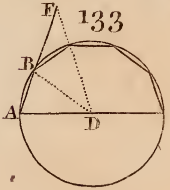
131



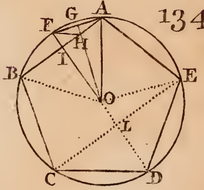
132



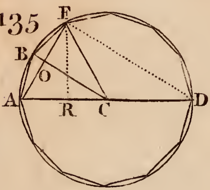
133



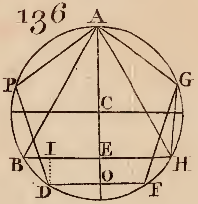
134



135



136



BOOK V.

Of Planes, and ſolid Angles.

DEFINITIONS.

FIG.

1. A *Plane* is a ſurface which lies even between the extremes; or in which all right lines coincide.

2. A *curve ſurface*, is that whoſe parts lie not even between the extremes; but gradually riſe or fall.

3. A *convex ſurface*, is that which ſwells or riſes up towards the middle.

4. A *concave ſurface*, is that whoſe middle parts are hollow, or fall lower than the extremes.

5. A *right line*, is ſaid to be *perpendicular* to a plane, when it is perpendicular to all lines, drawn to the foot of it, as AB. 140.

6. The *common ſection* of two planes, is the line made by two planes cuting each other, as EF. 139.

7. One *plane* is ſaid to be *perpendicular* to another, when it paſſes through a right line which is perpendicular to the other, as CD. 144.

8. The *inclination* of a *line* to a plane, or the angle it makes with it, is the angle that line makes with a line drawn from the foot of it to the point where a perpendicular, let fall from the top, cuts the plane, as FCI. 142.

9. The *inclination* of two planes, is the angle made by two right lines; both drawn perpendicular to the common ſection, from any point therein; 149.

FIG. 149. in; as ∠BFD is the inclination of the planes, AB, CD.

This is the angle which the planes make with one another.

145. 10. *Parallel planes* are thoſe which are every where at the ſame diſtance from each other, as AC, EG.

150. 11. A *ſolid angle* is a ſpace bounded by ſeveral plane angles, meeting in one point, called the *angular point or vertex*, as A.

PROP. I.

137. *If a right line in a plane be produced, it will ſtill be in that plane.*

For produce BC in the plane AD, directly to E; and if BCF be alſo a right line, then BCE and BCF are both right lines; and you have two right lines BCF, BCE, with the part BC the ſame in both; contrary to the nature of a right line (Def. 5. I).

Cor. 2. *If two diſtant points of a right line be in a plane, all the line is ſo.*

PROP. II.

138. *If two right lines* GH, IK, *interſect one another, they are in the ſame plane.*

For imagine A, and B, to be in one plane with C; then the line IACK, as alſo HBCG, and all the points between A and B, are in one plane (Prop. I. and Cor.). And the like for all other lines as AB, drawn between GCH, and ICK.

Cor. *Every part of a triangle is in the ſame plane.*

PRO-

PROP. III.

FIG. 139.

If two planes AB, CD, *cut one another; their common ſection* EF *is a right line.*

For draw the line EF in the plane AB, then any point G of that right line, is in the plane AB (1); but becauſe the line EF (drawn between the two points E, F, in the plane CD) is alſo in the plane CD; any point G of that line will be in the plane CD. Therefore G being in both planes, will be in their common ſection; and their common ſection EGF is conſequently a right line.

PROP. IV.

140. *If a right line* AB *be perpendicular to two lines* IK, GH, *at the point of interſection* B; *then is the line* AB *perpendicular to the plane* FD *paſſing through them.*

Let BI = BG = BK = BH, and draw GI, HK, LM, AI, AL, AG, AH, AM, AR. The triangles ABI, ABG, ABH, ABK, are all equal (6. II), and AI = AG = AH = AK. Alſo the triangles GBI, HBK are equal (6. II), and GI = KH, ∠G = ∠H. Alſo the triangles LBG, MBH are equal (7. II), and BL = BM, and GL = HM. And the triangles AGI, AKH are equal (8. II); and conſequently ∠AGI = ∠AHM; whence the triangles AGL, AHM are equal (6. II), and AL = AM. Alſo the triangles ABL, ABM are equal (8. II), and ∠ABL = ABM = a right angle. And therefore AB is perp. to LM.

Cor. *If a right line* AB *be perpendicular to ſeveral lines meeting in* B, *as* IB, LB, GB; *theſe lines are all in one plane.*

For

FIG. 140. For if any of them was out of the plane, AB would make an angle with it, greater or leſſer than a right angle.

PROP. V.

141. *Two right lines,* AB, CD, *perpendicular to a plane, are parallel.*

Make BDI a right angle, and DI = AB, and draw BI, AI, AD; AB is ⊥ to BD (Def. 5); and ∠ ABD = BDI; therefore the triangles ABD, DBI are equal (6. II), and AD = BI. Then the triangles ADI and ABI are equal (8. II); and ABI = ADI = a right angle. Therefore ID is ⊥ to DC, DA, DB; and therefore all three are in one plane (Cor. 4). Therefore AB, CD are in the ſame plane (Cor. Prop. II), and are likewiſe parallel (Cor. 3. 4. I).

Cor. 1. *Two parallel lines* AB, CD, *are in the ſame plane.*

Cor. 2. *A line drawn from one parallel to another, is in the ſame plane with them.*
By Cor. 1. and Cor. to Prop. I.

Cor. 3. *Through one point, there can be drawn but one line perpendicular to a plane.*

PROP. VI.

141. *If one* AB, *of two parallels, be perpendicular to a plane; the other will alſo be perpendicular to it.*

Suppoſe as in the laſt Prop. then the angles IDA, IDB, are right. Therefore DI is ⊥ to the plane ADB, in which AB, CD are (Cor. 1. 5); there-

therefore ID is $\perp$ to CD; but CDB = a right angle. FIG.
Therefore CD is $\perp$ to the plane EG. 141.

PROP. VII.

If FI *be perpendicular to the plane* DE, *and* FC 142.
perpendicular to a line AB, *drawn in that plane*; *then the line* CI *joining their ſections, is alſo perpendicular to the line* AB.

For firſt, ſuppoſe CB $\perp$ to CI, draw IG parallel to CB, then IG being $\perp$ to CI and FI, is $\perp$ to the plane CFI (4); and ACB is alſo $\perp$ to the plane FCI (6); therefore BC is $\perp$ to CI and to CF. And on the contrary being $\perp$ to CF, it is alſo $\perp$ to CI; otherwiſe it could not be $\perp$ to the plane FCI; nor its parallel GI.

PROP. VIII.

Right lines AH, CI, *parallel to the ſame right* 143.
line EG, *though not in the ſame plane, are parallel to one another.*

In the plane of the parallels AH, EG, let HG be $\perp$ to EG. Alſo in the plane of the parallels EG, CI, draw GI $\perp$ to EG. Therefore EG is $\perp$ to the plane HGI; therefore AH, CI are alſo $\perp$ to the ſame plane HGI (6), whence AH and CI are parallels (5).

PROP. IX.

If two planes AB, CD *be perpendicular to one an-* 144.
other; *and from any point* P *in one, a perpendicular* PN, *be let fall to the other*; *it ſhall fall upon the common ſection* PI.

For the line PN $\perp$ to the common ſection, is $\perp$ to the plane AB (Def. 7), and if another perp. could be drawn which falls not upon the common

ſection,

FIG. 144. ſection; then two perpendiculars might be let fall from one point, which is abſurd (Cor. 3. 5).

Cor. *A line* NP *in one plane, perpendicular to the common ſection of two perp. planes, will be perp. to the other plane.*

PROP. X.

145. *Thoſe planes* AC, EG, *are parallel, when the ſame right line* IK, *is perpendicular to both.*

Draw DL parallel to IK, and draw ID, KL; then ſince the angles LKI = KID (hyp.) = IDL (6) = a right angle, therefore KLD is a right angle (16. III); therefore ID is parallel to KL (Cor. 3. 4. I); whence IKLD is a parallelogram, and IK = DL, therefore AC is parallel to EG (Def. 10).

Cor. *If a right line is perpendicular to one of two parallel planes, it is perpendicular to the other.*

PROP. XI.

145. *If two parallel planes* AC, EG, *be cut by a third* IL; *their common ſections are parallel*; ID, *and* KL.

For it was proved in the laſt prop. that IDLK is a parallelogram, and that ID and KL are parallel.

Or thus.

Let the plane IBED cut the parallel planes AC, EG, in the ſections ID, BE. Now if ID, BE be not parallel, or equidiſtant, they will meet ſome way; and conſequently the planes wherein they are placed, muſt meet, which is abſurd.

Cor. *If a line* ID *be parallel to the plane* EG; *all planes drawn through this line* ID, *ſhall interſect the plane* EG *in lines parallel to* ID, *and to one another.*

For

For KL is parallel to ID, and BE is parallel to ID, and therefore KL, BE are parallel to one another (8). FIG. 145.

PROP. XII.

Right lines AQ, BR, *cut by parallel planes*, G, H, I, *are cut proportionally*; AC : CE : : BD : DF. 146.

Draw AB, EF; and BE to cut the plane H in P. Then in the planes, BEF, EAB, the ſections PD, EF, as alſo CP, AB, will be parallel (11); therefore in the triangles BEF, EAB; AC : CE : : BP : PE : : BD : DF (12. II).

Cor. *The ſegments of parallel lines, cut off by parallel planes, are equal.*

PROP. XIII.

If two lines AB, AC, *cuting one another; be parallel to two other right lines*, ED, DF, *cuting one another, though not in the ſame plane; theſe lines will make equal angles*; BAC = EAD. 147.

Let AB = DE, AC = DF, and draw BE, AD, CF, and alſo BC, EF. Since AB, DE are parallel and equal, therefore BE, AD are equal and parallel (Cor. 3. 5. I). For the ſame reaſon CF, AD, are equal and parallel. Therefore BE, FC are parallel and equal (Prop. VIII. and Ax. 1). Therefore BC is equal and parallel to EF (Cor. 3. 5. I). The triangles BAC, EDF, have all their ſides equal, therefore ∠BAC = EDF (8. II).

PROP. XIV.

If two lines AB, AC, *which meet one another, be parallel to two other lines* DE, DF, *that alſo meet one another, though not in the ſame plane; the planes* BC, EF, *drawn through them, will be parallel.* 148.

Let

FIG. 148. Let AG be perpendicular to the plane EF, and GH, GI parallel to DE, DF; then GH, GI will be parallel to AB, AC. And ſince IGA, HGA are right angles, CAG, BAG, will be right angles (4. I); therefore GA is ⊥ to the plane BC, and ſince it is ⊥ to the plane EF (conſtruct.), therefore the planes BC, EF are parallel (10).

PROP. XV.

149. *If two planes* AB, CD, *which cut one another, be both of them perpendicular to a third plane* GH; *their common ſection* EF, *ſhall alſo be perpendicular to the third plane,* GH.

For a perpendicular to the plane GH, at the point F (in the common ſection of the planes AB, GH), muſt be ſomewhere in the plane AB (Def. 7). Alſo a perpendicular at F (in the common ſection of the planes CD, GH), muſt be ſomewhere in the plane CD (ibid.); therefore it muſt be in their common ſection; that is, the common ſection EF is ⊥ to the plane GH.

Cor. *The common ſection* EF *will be perpendicular to* FD, *or* FB, *the ſection of either plane with the third.*

PROP. XVI.

150. *In a ſolid angle* A, *contained under three plane ones,* BAD, DAC, BAC; *any two of them is greater than the third.*

Let BAC be the greateſt, and let ∠BAE = BAD, and AD = AE. And draw BEC, BD, DC. The triangle BAE = BAD, for BA, AE are equal to BA, AD, and ∠BAE = BAD, therefore BE = BD, and AE = AD (6. II). But BD + DC is greater than BC (5. II), and DC greater than EC. And ſince AD = AE, and AC common,

common, $\angle$CAD is greater than CAE (Cor. 6. II). FIG.
Therefore BAD + CAD is greater than BAC. 150.

PROP. XVII.

Every ſolid angle is contained under leſs plane angles than four right angles. 151.

Suppoſe a plane to cut the ſides of the angle, and to make a polygon BCDE, to conſiſt of as many triangles, as there are to make up the ſolid angle A.

Let X = ſum of all the external angles of the polygon B, C, D, &c. Y = ſum of all the angles at the baſes of the triangles compoſing the ſolid angle, EBA, ABC, &c. Then will X + 4 right angles = Y + A (2. II). But ſince EBA + ABC is greater than B (16), &c. therefore Y is greater than X, and conſequently A is leſs than 4 right angles.

PROP. XVIII.

Theſe ſolid angles are equal A, G; *which are contained under the ſame number of plane angles, alike ſituated, and having the ſame inclinations, reſpectively.* 151. 152.

For ſince $\angle$BAC = HGI; CAD = IGK, &c. therefore if HGI be laid upon BAC, they will coincide, and GI will fall upon AC. Alſo if IGK be laid upon CAD, they will likewiſe coincide. And moreover, ſince the inclination of the planes HGI and KGI is the ſame as BAC and DAC; therefore the ſolid under HGIK will exactly coincide with that under BACD. For the ſame reaſon the ſolid, under the planes IGKL and CADE, will likewiſe concide; and alſo the ſolid under KGLH and DAEB will coincide; and thoſe under LGHI, and EABC, will coincide; and ſo the whole ſolid angle G will coincide with the whole ſolid angle A, and conſequently they are equal (Ax. 8).

PROP. XIX.

FIG. 153. 154. *If two ſolid angles* A, B, *be contained under three plane angles reſpectively equal, and alike ſituated; the like planes have the ſame inclination to one another.*

Let $\angle KAD = MBG$, $KAE = MBH$, and $EAD = HBG$; the $\angle$ made by KAD and KAL, will be equal to that made by MBG and MBN. For make $BM = AK$, and let KD, KL be $\perp$ to AK, and MG, MN $\perp$ to BM. Draw LD, NG; in the triangles KAD, MBG, $\angle$ $KAD = MBG$, and K, M right, and $AK = BM$; therefore $KD = MG$, and $AD = BG$ (7. II). For the ſame reaſon, in the triangles KAL, MBN; $KL = MN$, and $AL = BN$. And in the triangles LAD, NBG; LA, AD are equal to NB, BG, and $\angle A = B$, therefore $LD = NG$ (6. II). In the triangles KLD, MNG; the three ſides are equal; therefore $\angle DKL = \angle GMN$, which are the inclination of the planes. And the ſame way it is demonſtrated for the other planes.

Cor. *Theſe ſolid angles are equal, which are contained under three plane angles, reſpectively equal.*

For the planes of theſe angles will have the ſame inclination to one another reſpectively (19); and conſequently the ſolid angles, contained thereby, will be equal (18).

SCHOLIUM.

It is evident from hence, that a ſolid angle, conſiſting of 3 planes, is determined from the quantity
153. of the 3 plane angles it conſiſts of. For (fig. 153), the triangle KLD, which is its baſe, is determined from the three ſides, KL, LD, KD, being given. And if the point A be alſo given; the planes AKL, ALD, AKD, are capable of no alteration in their poſition;

poſition; and ſo the ſolid angle A is determined. FIG.
But although a ſolid angle of 3 plane angles is determined from the quantity of the angles alone; yet when 4 or more planes are concerned, the quantity of their angles is not ſufficient. This will be plain by inſpecting fig. 155. Where the baſe of the ſolid 155.
angle A, is the trapezium BCDI. For the 4 ſides of the trapezium alone are not ſufficient to determine its figure; and by altering its figure, the poſition of the planes is altered (though the ſeveral angles are not), and conſequently the quantity of the ſolid angle A, is altered. So that the ſolid angle can no more be determined, from the plane angles given; than a trapezium can, by having all its ſides given; and much leſs can it be ſo in polygonal angles and baſes.

PROP. XX.

If there be two ſolid angles A, G, *and the ſides of one,* 151.
AB, AC, AD, AE, *be reſpectively parallel to the ſides* 152.
GH, GI, GK, GL, *of the other; theſe ſolid angles will be equal.*

For ſince AB, AC, are parallel to GH, GI; ∠BAC = HGI (13); for the ſame reaſon ∠CAD = IGK, DAE = KGL, EAB = LGH. Moreover, as AB, AD are parallel to GH, GK; ∠BAD = HGK, therefore the ſolid angle made by the three planes BAC, CAD, BAD, is equal to that made by the three planes HGI, IGK and HGK (Cor. 19). For the ſame reaſon the ſolid angle made by the three planes CAD, DAE, CAE is equal to that made by IGK, KGL, IGL. And for the ſame reaſon the ſolid angle A made by DAE, EAB = ſolid angle G made by KGL, LGH. And ſolid angle made by EAB, BAC = ſolid angle made by LGH, HGI. Whence all the parts of the ſolid angles A, G, being mutually equal, and having a like

FIG. like ſituation; the whole angle A, muſt be equal to
151. the whole angle B.
152.

Cor. *In two ſolid angles* A, G, *whoſe planes* BAC, CAD, *&c. are reſpectively parallel to the planes* HGI, IGK, *&c. theſe ſolid angles will be equal.*

For it comes to the ſame thing, whether the lines AB, GH, be parallel, or the planes BAC, HGI, &c. (14).

⁂

BOOK VI.

Of Solids.

DEFINITIONS.

FIG. 155. 1. A *Pyramid*, is a ſolid ABD, made by the motion of a line as AB, along the circumference BCDIB of the plane figure BD, the other end at A, remaining fixt. The figure BCDA is called the *baſe* of the pyramid. The fixt point A is the *vertex*. If the baſe be a triangle, it is a *triangular pyramid* 150.; if a polygon, a *multangular pyramid*.

156. 2. A *cone* is a ſolid generated by a line AB moving about the circle BCD, the end A remaining fixt. The *vertex* is the fixt point A. The *axis* is the line AO drawn from the vertex to the center O of the circle. The *baſe* is the circle BCD. The *ſide* is AB or AD. It is called a *right cone*, if the axis is perpendicular to the baſe; otherwiſe an *oblique* or *ſcalene cone*. An *equilateral cone*, is a right cone whoſe ſide is equal to the diameter of the baſe.

157. 3. A *cylinder* is a ſolid, formed by a line FB moving about two equal and parallel circles, ſo as that the moving line always keep parallel to the line PO joining their centers. The circle FG or BD is called the *baſe*. The line PO, drawn between the centers of the circles, is the *axis*. If the axis is perpendicular to the baſe, it is a *right cylinder*; if not, an *oblique one*. FB or GD is the *ſide*. If the ſide of a right cylinder be equal to the diameter of the baſe, it is called an *equilateral cylinder*.

FIG. 4. A *priſm* is a ſolid, as ACEH, whoſe ends are
159. two ſimilar equal plane figures, and parallel to one another; and the ſides, are parallelograms. The *baſe* is the plane figure at either end ABCD or HGEF. If all the ſides are perpendicular to the baſe, it is a *right priſm*; otherwiſe an *oblique one.*
158. If the baſe is a triangle, it is a *triangular priſm*; if a polygon, a *multangular priſm.*

Cor. *A cylinder is a priſm of an infinite number of ſides.*

160. 5. A *parallelopipedon* is a priſm contained under ſix plane figures, whoſe baſes, and oppoſite ſides are parallel, as ABD. If the ſides are all perpendicular to the baſes, it is an *upright parallelopipedon*; if not, an *oblique one.*

161. 6. A *cube* is a ſolid contained under ſix equal ſquares, ſet perpendicular to one another, as AB.

7. A *polyedron*, is a ſolid contained under ſeveral rectilineal figures.

8. A *regular ſolid* or *body*, is a ſolid contained under ſome number of equal and regular plane figures of the ſame ſort; otherwiſe, they are *irregular bodies.*

160. 9. *Hight*, of a ſolid, is the perpendicular falling from the vertex or top, upon the baſe, as BP.

10. *Fruſtum*, of a ſolid, is the lower part, cut off by a plane parallel to the baſe.

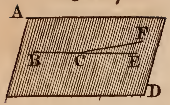
11. *Similar pyramids*, are thoſe contained under ſimilar plane figures, equal in number, and alike placed.

12. *Similar ſolids* are thoſe which are made up of an equal number of ſimilar pyramids, alike placed: or which may be reſolved into ſuch.

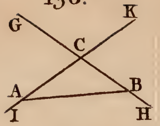
13. *Area*, is the quantity of the ſuperficies of any plane figure.

14. *Bodies* are ſaid to *touch* one another, when they meet, but do not cut or enter into one another.

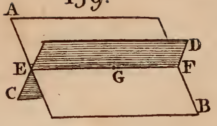
Fig. 137.



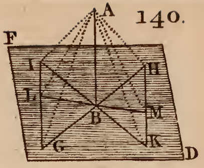
138.



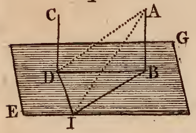
139.



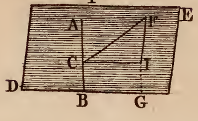
140.



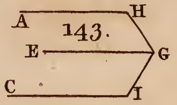
141.



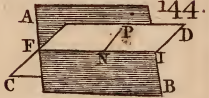
142.



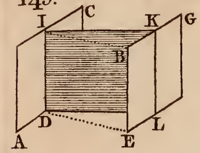
143.



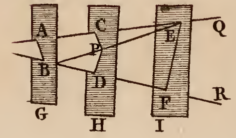
144.



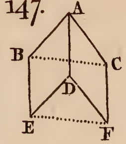
145.



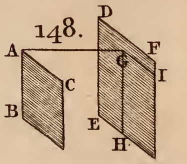
146.



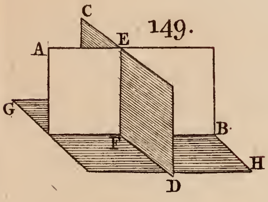
147.



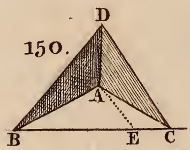
148.



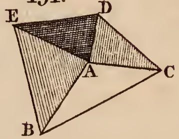
149.



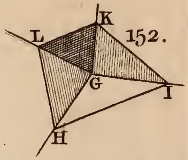
150.



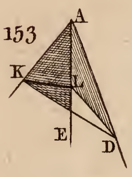
151.



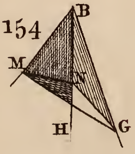
152.



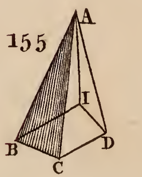
153



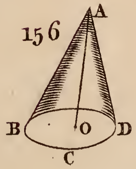
154



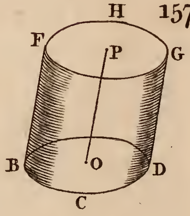
155



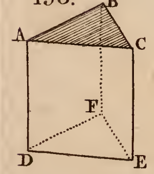
156



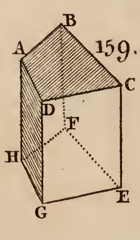
157.



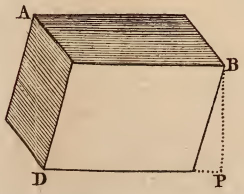
158.



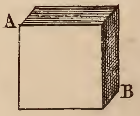
159.



160.



161.



PROP. I.

FIG. 162. *In any parallelopipedon* EH, *the oppoſite planes* AE, HB, *are ſimilar and equal parallelograms.*

The plane AC, cuting the parallel planes AG, DB, make the ſections AH, DC parallel (11. V). And the ſame plane AC, cuting the parallel planes AE, HB, make the ſections AD, HC parallels (ibid.); therefore ADCH is a parallelogram. By the ſame reaſoning, all the other planes are parallelograms. Therefore BG = CH = DA = EF (1. III). And ſince DA, AF are parallel to CH, HG; therefore ∠DAF = CHG (13. V); therefore the parallelogram AE = HB (Ax. 8), having equal ſides and angles. And the ſame way it is ſhewn of the other oppoſite planes.

PROP. II.

163. *If a priſm* HC, *be cut by a plane parallel to the baſe* AC; *the ſection* EG, *will be ſimilar and equal to the baſe.*

Since AE, BF, CG, DI are parallel (Def. 4), and the plane ABFE is cut by the parallel planes AC, EG, the ſections AB, EF will be parallel, therefore ABFE is a parallelogram, and EF = AB (1. III). For the ſame reaſon FG = BC, GI = CD, and EI = AD. And ſince AB, BC are parallel to EF, FG; ∠ABC = EFG (13. 5). After the ſame manner ∠C = G, and I = D, and E = A. Whence the figure EFGI is ſimilar and equal to ABCD (Ax. 8), having the ſides and angles all equal.

Cor. *If a parallelopipedon be cut by a plane parallel to any ſide; the ſection will be ſimilar and equal to that ſide.*

FIG. 163. For in that ſolid, any ſide may be taken for the baſe (1).

PROP. III.

The ſurface of any polyedron, is equal to the ſum of the areas of all the figures that incloſe it.

For all theſe figures make up the ſurface, therefore the ſum of their areas is equal to the area of the whole (Ax. 2).

Cor. 1. *The ſurface of a pyramid is equal to the ſum of all the triangles incloſing it, together with the baſe.*

159. Cor. 2. *The ſurface of an upright priſm* AE, *is equal to the rectangle of its hight*, CE, *and the circumference of its baſe*, GEFH.

For all the ſides are rectangles of the ſame hight, all which are equal to a rectangle, whoſe baſe is the ſum of all theſe, and hight the ſame (8. III).

Cor. 3. *The ſurface of any regular body is equal to the area of one of the faces, multiplied by the number of them.*

PROP. IV.

164. *The curve ſurface of a right cylinder* AD, *is equal to the rectangle of its hight, and the circumference of the baſe:* BD × CKDC.

Suppoſe FK, OI to be drawn upon its ſurface parallel to the axis, and extremely near together. Then the part of the ſurface OK is equal to the ſmall parallelogram OIKF, or OI × IK (Cor. 2. 6. III). In like manner the whole ſurface may be divided into ſuch parallelograms, the ſum of all which, will be = the ſum of all the IK's × OI; that is, the curve ſurface will be = the circumference CKDC × OI.

Cor:

Cor. 1. *The curve ſurface of a right cylinder, is equal to a circle whoſe radius is a mean proportional between the ſide* AB, *and diameter of the baſe* BD. FIG. 164.

For let R, C be the radius and circumference of this circle, A its area. Then AC : R : : R : CD (hyp.), and AC : $\frac{1}{2}$R : : 2R : CD (Cor. 3. 12. Proportion) : : C : circumference CKDC (9. IV). Therefore AC × CKDC = $\frac{1}{2}$RC; that is, the ſurface of the cylinder = A (34. IV).

Cor. 2. *As half the radius of the baſe : to the ſide : : ſo the baſe of the cylinder : to its curve ſurface.*

For the baſe = $\frac{CD \times CKDC}{4}$ (34. IV), and $\frac{CD \times CKDC}{4}$: ſurface AC × CKDC : : $\frac{CD}{4}$: AC (Cor. 1. 5. Proportion).

Cor. 3. *The curve ſurfaces of right cylinders, are in the complicate ratio of the hights, and diameters of the baſes.*

For their equal rectangles are in that ratio (Cor. 2. 8. III), and the diameters are as the circumferences (9. IV).

PROP. V.

The curve ſurface of a right cone, is equal to the area of a triangle, whoſe hight is the ſide AB, *and baſe the circumference of the cone's baſe*, BKDB. 165.

Take the very ſmall arch IK, and draw AI, AK. Then the part of the ſurface AIK coincides with the ſmall iſoſceles triangle AIK, whoſe baſe is IK, and hight AI. In like manner the whole curve ſurface of the cone, may be ſuppoſed to conſiſt of ſuch triangles, whoſe common hight is AI, and baſes ſo many KI's. All which triangles are equal to the triangle whoſe

FIG. 165. hight is AI; and baſe, the ſum of all the IK's, or the circumference BKDB (Ax. 2).

Cor. 1. *The curve ſurface of a right cone is equal to half the rectangle, of the ſide* AB, *and circumference of the baſe*, BKDB.

For half of that rectangle is equal to a triangle of the ſame hight and baſe (7. III).

Cor. 2. *The curve ſurface of a right cone is equal to a circle, whoſe radius is a mean proportional between the ſide* AB, *and the radius of the baſe* BC.

For the conic ſurface $= \frac{AB \times BKDB}{2}$, and the area of the baſe BD $= \frac{BC \times BKDB}{2}$ (34, IV). Let the radius $R = \sqrt{AB \times BC}$, its area $= A$. Then conic ſurface : circle BD : : $\frac{AB \times BKDB}{2} : \frac{BC \times BKDB}{2}$: : AB : BC (Cor. 1. 5 Proportion) : : $AB \times BC : BC^2$ (5. ibid.).

And circle BD : circle A : : $BC^2 : R^2$ or $AB \times BC$ (Cor. 35. IV). Therefore conic ſurface : circle A : : $AB \times BC : AB \times BC$ (15 Proportion). Therefore the conic ſurface = circle A, whoſe radius is $\sqrt{AB \times BC}$ (Ax. 7. Proportion).

Cor. 3. *In a right cone, as the radius of the baſe* BC : *to the ſide* AB : : *ſo the area of the baſe* BD : *to the curve ſurface of the cone* ABD.

For it is $\frac{BC \times BKDB}{2} : \frac{AB \times BKDB}{2}$: : BC : AB.

Cor. 4. *The curve ſurfaces of right cones, are in the complicate ratio of the ſides and diameters of the baſes.*

For

For the equal triangles are in that ratio (Cor. 1. 11. II), and the diameters are as the circumferences (9. IV). FIG. 165.

Cor. 5. *The curve ſurface of a right cylinder, is to the curve ſurface of a right cone, on the ſame baſe; as the ſide of the cylinder, to half the ſide of the cone.*

PROP. VI.

166. *The curve ſurface of the fruſtum of a right cone* PD, *is equal to half the rectangle under the ſide* PB, *and the ſum of the circumferences of the baſes*, PE, BD.

Produce BP, DE to A, and compleat the cone; then from A draw OI, FK exceeding near one another, then the ſmall part of the curve ſurface OK, falls in with the ſmall trapezoid OFKI, whoſe area is $\frac{\overline{OF + IK}}{2} \times OI$ (23. III). And as all the ſurface of the fruſtum may be divided into ſuch trapezoids, therefore its ſurface is = ſum of all the trapezoids = ſum of all the $\frac{OF + IK}{2} \times OI = \frac{BKDB + POEP}{2} \times OI$.

Cor. *The curve ſurface of the fruſtum of a right cone, is equal to a circle, whoſe radius is a mean proportional between the ſide* PB, *and the ſum of the radii of the baſes*, BC + PH.

For let R = radius, C = $\frac{1}{2}$ circumference, of the circle equal to the ſurface of the cone ABD. And r = radius, $c = \frac{1}{2}$ circumference of the circle equal to the ſurface of the cone APE. And ſince R : C :: $r : c$ (9. IV), let $\frac{C}{R} = \frac{c}{r} = n$, or $C = Rn$, and $c = rn$. The triangles APH, ABC are ſimilar, and BC : PH :: BA : PA (13. II), and BC — PH

: PH

FIG.
166. : PH : : PB : PA = $\frac{PH \times PB}{BC - PH}$ (13. Proportion); but ſurface of the cone ABD = RC = nRR = $n \times AB \times BC$ (Cor. 2. 5) = $n \times \overline{AP + PB} \times BC = n \times \frac{PH \times PB}{BC - PH} + PB : \times BC = n \times \frac{PH \times PB + BC \times PB - PH \times PB}{BC - PH} \times BC = n \times \frac{BC \times PB}{BC - PH} \times BC$.

Alſo ſurface of the cone APE = $rc = nrr = n \times AP \times PH$ (Cor 2. 5) = $n \times \frac{PH \times PB \times PH}{BC - PH}$. Therefore their difference, or the ſurface of the fruſtum is $n \times \frac{BC^2 \times PB}{BC - PH} - n \times \frac{PH^2 \times PB}{BC - PH} = n \times PB \times \frac{BC^2 - PH^2}{BC - PH} = n \times PB \times \overline{BC + PH}$ (12. I) = the circle whoſe radius is $\sqrt{PB \times \overline{BC + PH}}$, and circumference $n \times \sqrt{PB \times \overline{BC + PH}}$.

PROP. VII.

167. *The ſurfaces of ſimilar ſolids* AD, PS, *are as the ſquares of their homologous ſides*, AB² *and* PQ².

Draw the diagonals AC, PR. Then ſince the bodies are reſolvable into ſimilar pyramids (Def. 12), which are contained under ſimilar plane figures (Def. 11). Let the planes incloſing them, be ABC, PQR, and AGC, PIR, &c. which being ſimilar, it is AB : PQ : : AC : PR : : AG : PI : : GE : IT, &c. (13. II); and ſince $AB^2 : PQ^2$: : triangle ABC : PQR (18. II), and $AC^2 : PR^2$: : ACG : PRI; and $AG^2 : PI^2$: : trapezium AE : PT (20. III); and $GE^2 : IT^2$: : GD : IS, &c. therefore $AB^2 : PQ^2$: : ABC : PQR : : ACG : PRI : : AE : PT : : GD : IS &c. whence $AB^2 : PQ^2$: : BG + AE + GD &c. : QI + PI + IS &c. (10. Proportion) : : ſurface of AD : ſurface of PS.

Cor.

Cor. *Similar parts of the ſurfaces of ſimilar ſolids, are as the ſquares of the homologous ſides.* FIG. 167.

PROP. VIII.

A right triangular priſm ABCFHE *is equal to an oblique one* APIGHD, *of the ſame length* AH, *contained within the ſame three parallel lines* EP, HA, FI, *or the planes paſſing through them.* 168.

For AH = PD = BE = IG = CF (1. III), whence PB = DE, IC = GF, and AP, AB, AC, AI being parallel to HD, HE, HF, HG (Cor. 3. 5. I), the ſolid angle A = ſolid angle H (20. V). For the ſame reaſon the ſolid angles at P, B, C, I, are reſpectively equal to thoſe at D, E, F, G. And ſince the ſides are all equal, each to each, therefore the two ſolids APBCI and HDEFG will exactly coincide, and be equal the one to the other (Ax. 8); and therefore the rectangled priſm HEFCAB = the oblique one HDGIAP.

Cor. 1. *If a parallelopipedon* AB, *be cut by a plane paſſing through the diagonals of the oppoſite planes; it ſhall be cut into two equal parts.* 169.

For the triangle CGF = CBF, and DAE = DHE (1, III); and the length AG = length BH; therefore if the two priſms CFA, and CFH be laid ſo, that H may coincide with A, and EHB with DAG, their planes will concide, and each of them being oblique, is equal to a right one of the ſame length (8).

Cor. 2. *Hence any priſmatic ſolid cut obliquely by parallel planes, is equal to the ſame cut off at right angles, and of the ſame length.*

For any ſuch ſolid may be divided into triangular priſms, by planes paſſing through both ends of the ſolid.

FIG. 169. ſolid. And each triangular priſm cut obliquely, is equal to one of the ſame length, cut at right angles (8).

PROP. IX.

170. *If a parallelopipedon* AS, *be cut by a plane, paſſing through* O *the middle of the diameter* CQ; *the plane biſects it.*

Let the diagonals AD, BC cut each other in F; and RQ, PS, in I. Draw the axis FI, which cuts CQ in O, becauſe BCRQ is a parallelogram (2. and Cor. 3. III); and FO = OI. Let the plane EHOVX be parallel to ABDC. Then the parallelopip. AX = half AS. Let any plane GTOLN paſs through O. And let the ſolid be cut by the two planes ADSP, and CBQR, into four triangular priſms.

The two oppoſite ſolids OTGEH and OLNXV, are equal; for the ſides are parallel (11. V), and equal (Cor. 3. III). And therefore the ſolid angles, at the correſpondent points, are equal (20. V); therefore the ſolid EOG = XON. Therefore in the oppoſite priſms ACI, and BDI, the ſolids contained between the planes EVXH and GTLN, are equal. And it is proved the ſame way, that the ſolids, in the oppoſite priſms ABI, and DCI, contained between the planes EVXH and GLNT, are equal. And therefore ſince AX is half the parallelop. the plane GTNL cuts off half the parallelop. or divides it into two equal parts.

Cor. *The axis* FI, *and diagonal* CQ, *biſect each other.*

For they are both in the parallelogram BCRQ (Cor. 3. III).

PROP. X.

FIG. 171. *Parallelopipedons upon the ſame baſe* CDFI, *and between the ſame parallel planes*, CIFD *and* BHVOLA, *are equal.*

The triangles LAI and KEF are equal and ſimilar (6. II); and the priſm KEFDQH = LAICBG; ſubtract the common ſolid E*r*LQ*s*G, and the ſolid AI*r*EBC*s*Q = L*r*FKG*s*DH; add the priſm I*r*FC*s*D, and the ſolid paral. CDFIQEAB = CDFIHKLG upon the ſame baſe ID.

Again, the triangles FVK and DMH are ſimilar and equal (6. II), and the priſm FVKIOL = DMHCLG; ſubtract the common priſm M*t*KP*n*L, and the ſolid FVM*t*IOP*n* = D*t*KHC*n*LG; add the priſm DF*t*CI*n*, and the ſolid par. FVMDIOPC = DFRHCILG = CDFIQEAB.

PROP. XI.

Parallelopipedons of equal baſes and hights are equal. 172.

Let the parallelogram AGIC be the baſe of the parallelopiped. Draw BH, DF parallel to AG, AC. The ſolid pip. upon the baſe AGI = that on the baſe ACI (Cor. 1. 8); and ſolids on ABE and EFI, are = thoſe on ADE, and EHI (ibid.). Take the two laſt from the firſt, and there remains the ſolid on DH = ſolid on BF. But parallelogram DH = BF (4. III). Therefore ſolid pips. on equal baſes and hights are equal, when the angle at E is the ſame. Moreover, the pip. on the baſe BCEF is equal to that on the baſe EOPF, and the ſame hight (Cor. 2. 8); reckoning OP or EF the length of the ſolid. Whence the parallelopip. on the baſe DH, is equal to the pip. on the baſe EP, and hight the ſame.

PROP.

FIG.

PROP. XII.

173. *Parallelopipedons of the ſame hight are in proportion as their baſes.*

Let BN be the baſe of a parlopip. divide the baſe into any number of equal parts at D, E, F, G, *&c.* and draw planes || to ABC; then the parlopips. ſtanding upon CD, DE, EF, *&c.* will be all equal (11); whence the pip. AK is as multiple of AD, as the baſe BK is of the baſe BD, alſo the pip. LN is as multiple of AD, as the baſe ON is of BD. Whence it will be as pip. AK : pip. LN : : baſe BK : baſe ON (Def. 4. Proportion). Moreover, let the baſe PQ be = ON, and hight QR = AB, then the pip. PR = LN (11), whence pip. AK : pip. PR : : baſe BK : baſe PQ.

Cor. 1. *Parallelopipedons of equal baſes are as their hights.*

For in rectangled ones, any ſide or face may be taken for the baſe; and rightangled ones are equal to oblique ones, between the ſame parallel planes (10).

Cor. 2. *Parallelopipedons are to one another, in the complicate ratio of their baſes and hights.*

PROP. XIII.

175. *If two parallelopipedons*, AD, FI, *be equal; their baſes and hights are reciprocally proportional*; AC : FH : : HI : CD.

Suppoſe the ſides CD, HI perpendicular to the baſes, and make HM = CD. Then baſe AC : baſe FH : : ſolid AD or FI : ſolid FM (12) : : HI : HM or CD (Cor. 1, 12). And if the pips. be oblique, inſtead of ſuppoſing CD, HI to be the ſides, let

let them be the hights, and then oblique pips. FIG. being equal to upright ones (10); the proportion 175. continues the ſame.

Cor. *If the baſes and hights of two parallelopipedons be reciprocally proportional, they are equal.*

For ſince baſe AC : baſe FH : : HI : CD (hyp.), therefore AC × CD = FH × HI (12. Proportion), and ſolid BE : ſolid FI : : AC × CD : FH × HI (Cor. 2. 12). Therefore ſolid BE = ſolid FI (Ax. 7. Proportion).

PROP. XIV.

All priſms whatſoever, ABD, PSR, *of equal baſes and hights, are equal.* 174.

For any polygonal baſe BD may be divided into triangles, by diagonal lines; and the polygonal priſm may likewiſe be divided into triangular priſms, by planes paſſing through theſe diagonals; each of which triangular priſms is equal to half a parallelopipedon ſtanding on double the baſe (9); and as all theſe triangular priſms make up the polygonal priſm, this priſm muſt be equal to a parallelopip. of the ſame baſe and hight; and that equal to the priſm PRS of an equal baſe and hight (Cor. 1. 12).

Cor. 1. *Priſms of equal baſes are as their hights; and of equal hights, are as their baſes.*

For they may be divided into triangular priſms, which are half of parlopips. on double the baſe, and theſe pips. are as their hights, when the baſe is the ſame; or as the baſes, when the hight is the ſame. (Cor. 2. 12).

Cor. 2. *All priſms are to one another in the complicate ratio of their baſes and hights.*

Cor

FIG. 174. Cor. 3. *Bodies of equal ſurfaces may be very different in ſolidity. And equal ſolids may have ſurfaces vaſtly different.*

Cor. 4. *In equal priſms, the baſes and hights, are reciprocally proportional; and the contrary.*

PROP. XV.

The ſolidity of any priſm is equal to the product of the baſe and hight.

For a priſm is equal to a right-angled parallelopip. of the ſame baſe and hight; and that is equal to the product of its baſe and hight; or (which is the ſame) it is equal to the ſolid ſpace contained under the planes of the upright parallelopipedon (Def. 5).

PROP. XVI.

176. *Equiangular parallelopipedons* AB, CD, *are in the complicate ratio of their homologous ſides*, FG, GI, GB, *and* OE, EH, ED.

Let FP, OK be ⊥ upon the baſes IB, HD. Then by reaſon of the equal angles at G and E, the triangles GFP, EOK will be ſimilar; and FP : OK : : FG : OE (13. II). The parallelograms IB and HD being equiangular at G and E, are to one another as IG × GB, to HE × ED (10. III). The parlepip. AB : CD : : baſe IB × FP : baſe HD × OK (Cor. 2. 12) : : IG × GB × FP : HE × ED × OK : : GI × GB × GF : HE × ED × EO (7. Proportion).

PROP. XVII.

177. *Pyramids upon the ſame baſe, and of equal attitudes, are equal:* ACF = HCF.

Draw

FIG. 177. Draw the plane AH, through the tops of the pyramids, which will be parallel to CF. Alſo through any points of the pyramids, draw the plane BE, alſo parallel to CF; then by ſimilar triangles, CF : BD : : AC : AB (13. II) : : HC : HL (12. II) : : CF : LE (13. II); therefore BD = LE. And by the ſame reaſoning, BO = LI, and DO = EI. Whence the ſection BOD = LIE (8. II). Therefore if another plane NP be drawn very near, and parallel to BE, the ſegments of the pyramids, ND, PL, comprehended between theſe planes, will be equal (14). And therefore if never ſo many ſuch planes be drawn, the parts intercepted will always be equal. Therefore the ſum of all the parts of one pyramid, will be equal to the ſum of all the parts of the other; or the pyramid ACGF = pyramid HCGF (Ax. 2).

Cor. 1. *If a pyramid is cut by a plane parallel to the baſe, the ſection will be ſimilar to the baſe.*

For by ſimilar triangles, it is AC : AB : : CG : BO : : GF : OD : : CF : BD.

Cor. 2. *If a cone be cut by a plane parallel to the baſe; the ſection will be a circle.*

For a cone may be conſidered as a pyramid of an infinite number of ſides.

PROP. XVIII.

Every priſm is three times the pyramid of the ſame baſe and hight. 178.

Let AFC be a triangular priſm, draw AC, CF, FD, the diagonals of the three parallelograms. The triangle ACB = ACD (1. III); therefore pyramid ACBF = ACDF, their vertexes being in F (17); likewiſe triangle DFA = DFE (1. III), and pyramid DFAC = DFEC, their vertexes being

FIG. 178. ing in C (17). But ACDF and DFAC are one and the ſame pyramid. Therefore the three pyramids, that make up the priſm, are equal to one another, ACBF = ACDF = DFEC; and each of them is $\frac{1}{3}$ the priſm.

And ſince any polygonal priſm may be reſolved into triangular ones; and the pyramid, upon the ſame baſe, into triangular pyramids. Then all the triangular priſms will be triple to all the triangular pyramids; and conſequently the whole priſm triple to the whole pyramid.

Cor. 1. *Pyramids of the ſame hight, are to one another as their baſes.*

For priſms, which are triple of them, are in that ratio (Cor. 1. 14). Whence,

Cor. 2. *Pyramids of the ſame or equal baſes are as the hights.*

Cor. 3. *Pyramids are to one another in the complicate ratio of their baſes and hights.*

Cor. 4. *Pyramids of equal baſes and hights are equal.*

Cor. 5. *In equal pyramids the baſes and hights are reciprocally proportional; and the contrary.*

For priſms are in that ratio (14. and Cor.).

PROP. XIX.

Cylinders of equal baſes and hights are equal.

For cylinders are nothing but priſms, whoſe baſes are polygons of an infinite number of ſides. And theſe priſms are equal (14).

Cor. 1. *Cylinders of equal baſes are as the hights.*

Cor. 2. *Cylinders of equal hights are as the baſes.*

Cor.

FIG. 178. Cor. 3. *Cylinders are to one another in the complicate ratio of their baſes and hights.*

Cor. 4. *In equal cylinders, the baſes and hights are reciprocally proportional: and the contrary.*
All this follows from Prop. 13, 14, and Corol.

PROP. XX.

Every cone is the third part of a cylinder of the ſame baſe and hight.

For cones and cylinders may be conſidered as pyramids, and priſms, whoſe baſes are regular polygons of an infinite number of ſides. And conſequently the cone = $\frac{1}{3}$ the cylinder (18).

Cor. 1. *Cones of equal baſes, are as their hights.*

Cor. 2. *Cones of equal altitudes, are as the baſes.*

Cor. 3. *Cones are to one another in the complicate ratio of the baſes and hights.*

Cor. 4. *In equal cones, the baſes and hights are reciprocally proportional.*
All theſe things appear by Prop. 13 and 14, and 19. For the cylinders are in that ratio, and the cone is $\frac{1}{3}$ the cylinder.

PROP. XXI.

The fruſtum of a pyramid or cone BG, *is equal to the third part of a parallelopipedon, of the ſame hight, and its baſe equal to the ſum of the baſes of the fruſtum* BOD + EFG, *together with a mean proportional between theſe baſes.* 179.

Draw EB, GD to meet in A, the top of the whole ſolid, and let ACP be ⊥ to the baſe. Draw the diameters BD, EG; then the two baſes BOD,

FIG. 179. EFG will be ſimilar (Cor. 1, 2. 17). Whence, baſe BOD : baſe EFG :: $BD^2 : EG^2$ (20. III). Therefore ſuppoſe $\frac{\text{baſe BOD}}{BD^2} = \frac{\text{baſe EFG}}{EG^2} = n$, or baſe BOD $= n \times BD^2$, and baſe EFG $= n \times EG^2$. By ſimilar triangles, EG : BD :: (AE : AB ::) AP : AC (13. II), and EG — BD : BD :: CP : $AC = \frac{BD \times CP}{EG - BD}$. Then the whole pyramid or cone $=$ baſe EFG $\times \frac{1}{3}AP$ (18, 20) $= \frac{n \times EG^2}{3} \times \overline{CP + AC} = \frac{n \times EG^2}{3} \times CP + \frac{n \times EG^2}{3} \times \frac{BD \times CP}{EG - BD}$ $= \frac{n \times EG^3 \times CP - n \times EG^2 \times BD \times CP + n \times EG^2 \times BD \times CP}{3 \times EG - BD}$ $= \frac{n \times EG^3 \times CP}{3 \times EG - BD}$. And the top part ABD $= \frac{\text{baſe BOD}}{3} \times AC$ (18, 20) $= \frac{n \times BD^3 \times CP}{3 \times EG - BD}$; this taken from the whole, leaves $\frac{n \times CP}{3} \times \frac{EG^3 - BD^3}{EG - BD}$ for the fruſtum $= \frac{CP}{3} \times \overline{n \times EG^2 + n \times EG \times BD + n \times BD^2}$, becauſe $\overline{EG^2 + EG \times BD + BD^2} \times \overline{EG - BD} = EG^3 - BD^3$ (Cor. 1. 8. I), and $n \times EG^2 =$ baſe EFG, $n \times BD^2 =$ baſe BOD, and $n \times EG \times BD$ is a mean between them (Cor. 2. 12. Proportion).

Cor. *If* $n = \frac{\text{baſe EFG}}{EG^2}$, *the fruſtum* $= \frac{n \times CP}{3} \times \frac{EG^3 - BD^3}{EG - BD}$.

PROP. XXII.

180. *In ſimilar ſolids*, AD, PS, *the homologous ſides are*
181. *proportional*; AB : AF :: PQ : PV.

Through the diagonals AC, FG, GD, and PR, FIG.
VI, IS, let planes be drawn to divide the ſolids 180.
into pyramids. Then ſince theſe pyramids are ſi- 181.
milar (Def. 12), and their planes ſimilar figures (Def. 11); therefore if ABC, PQR, and ACG, PRI, and AGF, PIV, &c. be ſimilar planes belonging to the ſimilar pyramids; it will be AB : PQ (: : AC : PR : : AG : PI) : : AF : PV. Alſo AF : PV : : (FG : VI : :) FE : VT, &c.

Cor. *The like planes or ſurfaces, which incloſe ſimilar ſolids, are proportional.*

For ſince AB : PQ : : AF : PV; $AB^2 : PQ^2$: : $AF^2 : PV^2$ (Cor. 3. 18. Proportion); that is, ABCG : PQRI : : AGEF : PITV (20. III).

PROP. XXIII.

Similar triangular pyramids ABCD, PQRS *are as* 182.
the cubes of their homologous ſides, AB^3 *and* PQ^3. 183.

Suppoſe CE, BF drawn parallel to AD, and RT, QV, ∥ to PS; and the planes DFE, SVT, ∥ to ABC, and PQR; and ſo the priſms AF, and PV, compleated.

Then ſince the pyramid ABCD = $\frac{1}{3}$ priſm, AF; and pyramid PQRS = $\frac{1}{3}$ priſm PV; therefore pyramid ABCD : pyramid PQRS : : priſm AF : priſm PV (5. Proportion) : : AB × AC × AD : PQ × PR × PS (16).

But AB : PQ : : AB : PQ,
AB : PQ : : AC : PR (22),
AB : PQ : : AD : PS (22).

Therefore $AB^3 : PQ^3$: : AB × AC × AD : PQ × PR × PS (18. Proportion) : : pyramid ABCD : pyramid PQRS.

Cor. *Any ſimilar pyramids are as the cubes of the homologous ſides.*

FIG. For they may be divided into ſimilar triangular
182. pyramids, all which are in that proportion, and
183. their ſums in the ſame proportion (10. Proportion).

PROP. XXIV.

180. *All ſimilar ſolids,* AD, PS, *are to one another,*
181. *as the cubes of their homologous ſides,* AB, *and* PQ.

Let the planes AC, PQ, and FG, VI, and GD, IS, *&c.* divide the bodies into ſimilar pyramids. Then ſince AB : PQ : : AG : PI : : EG : TI, *&c.* (22). Therefore

$AB^3 : PQ^3 :: \text{pyr. } ABC : \text{pyr. } PQR$ (23),

and $AB^3 : PQ^3 :: AG^3 : PI^3 :: \text{pyr. } AGC : \text{pyr. } PIR :: \text{pyr. } AGF : \text{pyr. } PIV.$

and $AB^3 : PQ^3 :: EG^3 : TI^3 :: \text{pyr. } FGE : \text{pyr. } VIT :: \text{pyr. } EGD : \text{pyr. } TIS,$ *&c.*

Therefore

$AB^3 : PQ^3 :: \text{pyr. } ABC + AGC + AGF + FGE + EGD,$ *&c.* $: \text{pyr. } PQR + PIR + PIV + VIT + TIS,$ *&c.* $:: \text{ſolid } AD : \text{ſolid } PS.$

Cor. *If four lines* A, B, C, D *be in continual proportion; then as the firſt* A *to the fourth* D; *ſo any ſolid deſcribed on the firſt* A, *to a ſimilar one, on the ſecond* B.

For $A : D :: A^3 : B^3$ (23. Proportion) : : ſolid upon A : ſolid upon B (24).

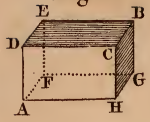
PROP. XXV.

184. *If four lines be proportional,* AB : CD : : GH : LM; *ſimilar ſolids, alike deſcribed, upon two and two, ſhall alſo be proportional:* ABE : CDF : : GHK : LMN.

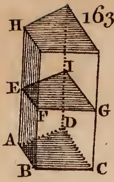
And if four figures be proportional, and two and two ſimilar; their homologous ſides ſhall be proportional.

For

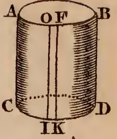
Fig. 162



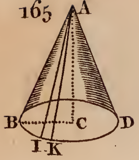
163.



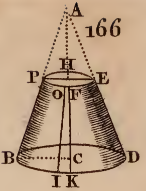
164



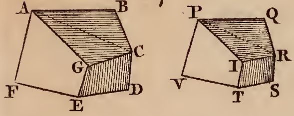
165

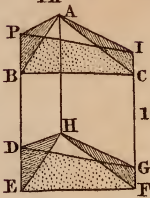


166



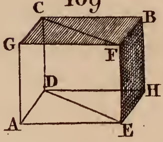
167



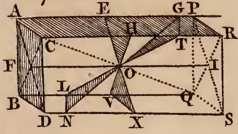


168

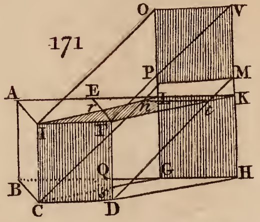
169

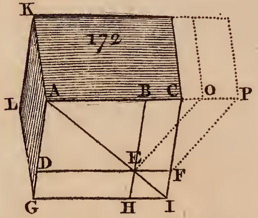


170

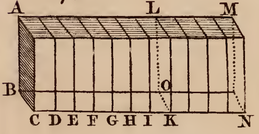


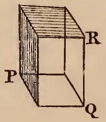
171

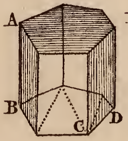




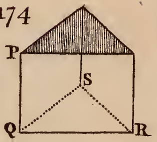
173

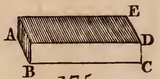




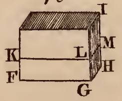


174

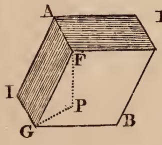


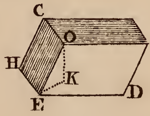


175

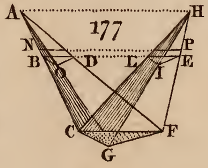


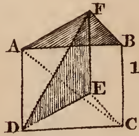
176





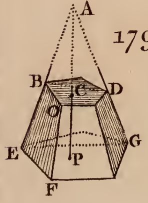
177

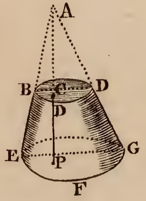




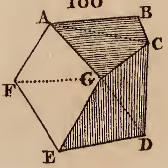
178

179

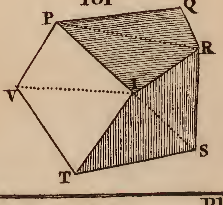




180



181



Pl. VIII. *Pa. 116.*

FIG. 184.

For ſince $AB : CD :: GH : LM$ (hyp.),
therefore $AB^3 : CD^3 :: GH^3 : LM^3$ (Cor. 3. 18. Proportion),
whence $ABE : CDF :: GHK : LMN$ (24).

Again, if the ſolids be ſimilar,
and $ABE : CDF :: GHK : LMN$ (hyp.),
then $AB^3 : CD^3 :: GH^3 : LM^3$ (24),
whence $AB : CD :: GH : LM$ (Cor. 3. 18. Proportion).

PROP. XXVI.

None other but three ſorts of regular plane figures, joined together, can make a ſolid angle; and theſe are, 3, 4, or 5 triangles, 3 ſquares, and 3 pentagons:

And therefore there can only be five regular bodies, the pyramid, cube, octaedron, dodecaedron, and icoſaedron.

Three plane angles at leaſt, are required to make a ſolid angle. One angle of the triangle $= \frac{2}{3}$ of a right angle (2. II), therefore 3 of them put together make two right angles. Alſo 4 of them make $2\frac{2}{3}$ right angles. And 5 make $3\frac{1}{3}$ right angles; all which are leſs than 4 right angles. But 6 of them make 4 right angles, and therefore cannot make a ſolid angle (17. V).

Again, one angle of the ſquare is a right angle, and 3 of them make 3 right angles. But 4 make 4 right angles, and therefore can make no ſolid angle (17. V).

Alſo one angle of the pentagon is $1\frac{1}{5}$ right angle (17. III). And 3 angles make $3\frac{3}{5}$. But 4 of them make $4\frac{4}{5}$, which exceeds 4 right angles.

Laſtly, one angle of the hexagon is $\frac{4}{3}$ of a right angle, therefore 3 angles make 4 right angles; but no ſolid angle. And the angle of a heptagon, octagon, &c. being greater; 3 of them will exceed 4 right angles; and conſequently, there can be no

FIG. 184. more than 3 triangles, 1 ſquare, and 1 pentagon, to conſtitute a ſolid angle.

Hence there can only be 5 regular bodies, to anſwer the 5 combinations of triangles, ſquares, and pentagons. Three faces of the triangle make the *pyramid*; 4 make the *octaedron*; and 5 make the *icoſihedron*; alſo 6 faces of the ſquare make the *cube*; and 12 faces of the pentagon, make the *dodecaedron*.

SCHOLIUM.

185. 186. 187. 188. 189. In order to get a clear idea of the five regular bodies, you may cut out all their faces in paſteboard, as repreſented in the figures, and fold them up, ſo that the creaſes may be in the black lines; and their edges being put cloſe together, you'll have the figure of theſe bodies. Fig. 185 is the pyramid, 186 the cube, 187 the octaedron, 188 the dodecaedron, and 189 the icoſihedron.

PROP. XXVII.

No other but only one ſort, of the five regular bodies, joined at their angles, can compleatly fill a ſolid ſpace; and that is eight cubes.

To demonſtrate this, we muſt obſerve that among other properties, this is abſolutely neceſſary, that the inclination of two adjoining planes in the body, be ſuch; that being taken a certain number of times, they will compleatly make up four right angles. For when the bodies are put together, the faces of every two adjoining bodies muſt coincide; and one edge or ſide of all the bodies muſt coincide with the ſide of the firſt; which will be as an axis, round which theſe bodies are placed; and therefore they muſt compleatly fill up the ſpace quite round, which is four right angles. And the

the angle of each (that is, the inclination of two adjoining planes), muſt be a certain part of 4 right angles. Therefore what we have to do, is to compute the inclination of their planes, and alſo to enquire what inclination is requiſite in the ſeveral bodies, to have this effect.

1. To begin with pyramids. It is plain, the baſe of the ſolid, being an equilateral triangle, the angle at any point is $\frac{2}{3}$ of a right angle; but the inclination of the planes is greater; for it is contained by two perpendiculars let fall on the common ſection of two planes, which perpendiculars are leſs than the ſides of the triangle (Cor. 4. 21. II); and ſtanding on the ſame baſe, muſt contain a greater angle (Cor. 2. 5. II). To 190. find the inclination of the planes; let CPH, CPA, and CDH be three of the equilateral triangles conſtituting a pyramid. Draw AG, DI $\perp$ to CP, CH. Let the plane CHP be fixt, whilſt the planes CAP, CDH, are raiſed up, (moving about the fixt lines CP, CH,) till the points A and D meet ſomewhere. It is plain a perpendicular dropt from A (elevated on high), upon the plane CPH, will always be ſomewhere in the line AG. And a like perpendicular from D will be ſomewhere in the line DI. Therefore when A and D meet, the perp. will be at the interſection O, in the middle of the triangle; and GO $= \frac{1}{3}$GH (Cor. 31. II) $= \frac{1}{3}$GA. Therefore, if you make the ſeparate right-angled triangle GAO, ſo that the hyp. GA may be treble the baſe GO, the $\angle$AGO is the angle of the pyramid, (that is, of its planes CAP, CHP), which was required. Now if EG be $\perp$ to GK, alſo if GBK be an equilateral triangle, then the baſe GF, will be half the hypothenuſe GB (Cor. 3. 3. II), and $\angle$BGK $= \frac{2}{3}$ a right angle (2. II). Then its plain, 4 times $\angle$ AGK will be leſs than 4 right angles, becauſe 4 times EGK make but 4 right angles;

FIG. gles; therefore more than 4 times AGK is required
190. to compleat 4 right angles. Likewiſe, ſince 6 times
BGK make 4 right angles, 6 times AGK will be
too much; and of conſequence we muſt either
have 5 times AGK, to make 4 right angles, or
134. nothing. Then to find whether that will anſwer
exactly or not; draw the diagonal EC of the pentagon, and OLD ⊥ to it; then 5 times the angle
EOL = 4 right angles. But DL = $\frac{5-\sqrt{5}}{4}$R
(Cor. 3. 44. IV), and OL = R − R × $\frac{5-\sqrt{5}}{4}$ =
190. 134. $\frac{\sqrt{5}-1}{4}$ × R. But GO (fig. 190) = $\frac{1}{3}$ the hypothenuſe AG or R, and $\frac{1}{3}$ is greater than $\frac{\sqrt{5}-1}{4}$, that is, GO is greater than OL, and conſequently the angle AGO is leſſer than EOL, which it ſhould be equal to; therefore 5 times AGO falls ſhort of 4 right angles; whence it is clear, that no combination of regular pyramids can compleatly fill all ſpace.

2. And it is as clear that 4 cubes ſet together will make up 4 right angles, each cube containing one. And therefore 8 cubes, joined at their angular points, will quite fill all ſpace on all ſides.

191. 3. Next for the octaedron. As half the octaedron ABE ſtands on a ſquare baſe BCED, the angles at the baſe, as BCE, are right, and then 4 of theſe would be 4 right ones; but the inclination of the planes ACB, ACE, are greater than right angles (for the ſame reaſon as in the pyramid), being made by a plane ⊥ to their common ſection AC; therefore 4 of theſe angles will be too much, and conſequently 3 or none of theſe angles of inclination muſt be equal to 4 right angles; or, which is the ſame thing, 6 halfs of the ∠ of inclination muſt be = 4 right angles. Now to try this, draw AG ⊥ to BC,

BC, and AO $\perp$ to the baſe BE, alſo draw GO. Then FIG. 192. hyp. AG = $\frac{AB}{2}\sqrt{3}$ (Cor. 39. II), and baſe GO = $\frac{1}{2}$BD = $\frac{1}{2}$AB. Therefore AG : GO :: $\frac{AB}{2}\sqrt{3}$: $\frac{AB}{2}$ :: $\sqrt{3}$: 1 :: 3 : $\sqrt{3}$ :: 1 : $\frac{\sqrt{3}}{3}$ :: AG : $\frac{AG}{3}\sqrt{3}$; then GO = $\frac{AG}{3}\sqrt{3}$. And as $\angle$AGO = half the angle of inclination, 6 of theſe muſt make up 4 right angles. And therefore $\angle$AGO muſt be = $\angle$BDE (fig. 131), if this ſucceed. For 6 of theſe make up 4 right angles. But in this caſe, DF = $\frac{1}{2}$DB, whence if DB (fig. 131) = AG (fig. 191), then GO = $\frac{DB}{3}\sqrt{3}$. But $\frac{1}{3}\sqrt{3}$ is greater than $\frac{1}{2}$ (as is eaſily known by ſquaring them); that is, GO is greater than DF, and conſequently $\angle$AGO is leſs than BDF. Therefore 6 of theſe, or 3 whole angles of inclination, fall ſhort of 4 right angles. So theſe bodies cannot entirely fill all ſpace.

4. Next comes the dodecaedron. As the angle of inclination of the planes of this body exceeds a right angle; therefore 4 ſuch angles will exceed 4 right angles; therefore only three of theſe bodies can be laid together; in which caſe the angle of inclination muſt be juſt $1\frac{1}{3}$ right angle For $3 \times 1\frac{1}{3} = 4$. If the $\angle$ be leſs, the third body will leave a vacuity; if greater, it cannot come in. Let 192. BPC, PCH, DGH, be 3 pentagons joining upon one another. Draw AG, DI $\perp$ to PC, HC, continued. Then let the plane PCH, be fixt, whilſt ABP, DEH, are raiſed up, and moved round the lines PC, HC, till the points A, D, meet. It is evident a perpend. dropt from A upon the plane PCH, will always fall on the line AG. And a like perpend. from D, will fall upon DI. And when A and D meet, it will fall on the interſection O.

Let

FIG. 192. Let R ſtand for a right angle. Then ſince CE is ∥ to HN (Cor. 2. 43. IV), ∠ECH + CHN = 2R (Cor. 2. 4. I) = ECH + PCH, therefore PCE is a right line (1. I). For the ſame reaſon BCH is a right line. Since ∠DCH = $\frac{6}{5}$R (17. III), DCE = $\frac{2}{5}$R, DCP = $\frac{8}{5}$R, take away ACP = $\frac{6}{5}$R, then ACD = $\frac{2}{5}$R. In the iſoſceles triangle ACD, COF biſects the ∠C and baſe AD (Cor. 3. 3. II), and ∠ACF = $\frac{1}{5}$R = DCF, and CDA = $\frac{4}{5}$R; and ſince CDE = $\frac{6}{5}$R, therefore CDA + CDE = 2R, and EDA is a ſtraight line (1. I). In the right-angled triangle ACF, ∠ACF or ACO = $\frac{1}{5}$R; and in the right-angled triangle ACG, ſince ACE = ACD + DCE = ACG = $\frac{4}{5}$R, CAG = $\frac{1}{5}$R = ACO, or CAO = ACO, and AO = OC (Cor. 1. 3. II). Therefore OG is leſs than OC or OA (5. II), and OG is leſs than half of AG. Make a right angle triangle ſeparately, as AGO, where the hypothenuſe is AG, and baſe OG, of a due length, and AGR is one of the angles of the dodecaedron. Where the ∠AGZ or GAO ought to be $\frac{1}{3}$R, that 3 dodecaedrons laid together may fill up 4 right angles. Now to ſee how this agrees, we find (in fig. 128), that EF = $\frac{1}{2}$DE, or DF = $\frac{1}{2}$DB (Cor. 3. 41. IV), and ∠ABF or BAC = $\frac{2}{3}$R = BDF (Cor. 1. 12. IV), and conſequently DBF = $\frac{1}{3}$R (Cor. 2. 2. II). Therefore if you make the baſe GQ = $\frac{1}{2}$ the hypothenuſe GM, then the ∠GMQ or MGZ is = $\frac{1}{3}$R. Therefore, ſince GO is leſs than $\frac{1}{2}$GA, the ∠AGZ is leſs than $\frac{1}{3}$R, and MGR leſs than 1$\frac{1}{3}$R, to which it ſhould have been equal; and conſequently 3 times MGR falls ſhort of 4 right angles: therefore the dodecaedrons cannot fill a ſolid ſpace.

This might be otherwiſe ſolved, by ſuppoſing one of its ſolid angles to ſtand upon an equilateral triangle, whoſe ſide is the diagonal of the pentagon.

5. Laſtly,

5. Laſtly, the icoſaedron has 5 triangles ſtanding upon a pentagonal baſe ABCDE. Draw the diagonal AC of the pentagon, and BQ the diameter of the circumſcribing circle. And let the plane AFC be drawn at right angles to BO, the common ſection of the two faces of the ſolid ABO, CBO. Draw FP, which will be ⊥ to AC. Then we are to find the quantity of the ∠AFC, the inclination of the planes, or rather, of its half AFP. Call $\frac{1}{2}$BQ, the radius of the circle, R; then AP^2 (Cor. 2. 44. IV) $= \frac{5+\sqrt{5}}{8}RR$. Alſo $AB^2 = RR \times \frac{5-\sqrt{5}}{2}$ (44. IV), and $AF^2 = \frac{3}{4}AB^2$ (Cor. 39. II) $= \frac{3}{4}RR \times \frac{5-\sqrt{5}}{2}$. Therefore $AF^2 : AP^2 :: \frac{3}{4}RR \times \frac{5-\sqrt{5}}{2} : \frac{5+\sqrt{5}}{8}RR :: 15 - 3\sqrt{5} : 5 + \sqrt{5}$. And $AF^2 : AF^2 - AP^2$ or $FP^2 :: 3 \times \overline{5-\sqrt{5}} : 10-4\sqrt{5} :: 3 : \frac{10-4\sqrt{5}}{5-\sqrt{5}} :: 3 : \frac{\overline{10-4\sqrt{5}}\times\overline{5+\sqrt{5}}}{\overline{5-5\sqrt{5}}\times\overline{5+\sqrt{5}}} :: 3 : \frac{50+10\sqrt{5}-20\sqrt{5}-20}{25-5=20}$ (Cor. 1. 8. I) $:: 3 : \frac{30-10\sqrt{5}}{20} :: 3 : \frac{2-\sqrt{5}}{2} :: 1 : \frac{3-\sqrt{5}}{6} :: 1 : .12732 :: AF^2 : .12732AF^2$. And by extracting the root, it is $AF : FP :: AF : .3568 \times AF = FP$. Now if three icoſaedrons laid together can fill up the whole ſpace, then three times the angle AFC, or ſix times the ∠AFP, muſt make four right angles; and in that caſe AFP muſt be $\frac{2}{3}$ of a right angle. But (fig. 128) the ſide DF muſt be half the hypothenuſe DB, when the ∠ between them BDF is $\frac{2}{3}$ of a right angle (Cor. 3. 41. IV): for ∠BDF = BAC in the equilateral triangle BAC (Cor. 1. 12. IV) = $\frac{2}{3}$ of a right angle (2. II). But here the ſide FP is leſs than half AF or $.5 \times AF$; therefore the ∠FAP will be leſs, and AFP greater, than it ſhould be; that is,

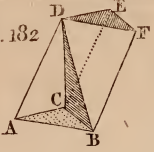
FIG. 193.

FIG. 193. is, AFP is more than $\frac{2}{3}$ of a right angle; and 6 times AFP, more than 4 right angles; and therefore 3 icoſihedrons cannot find room.

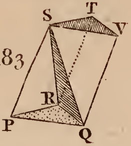
Thus I have demonſtrated from pure geometrical principles, that no combination of regular bodies of the ſame ſort (except cubes), can adequately fill up all the ſpace round about. The calculations of all theſe caſes are extremely eaſy, by working with the rules of trigonometry; but that was not my buſineſs here.

* * *

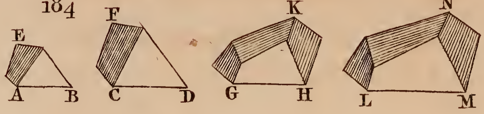
Fig. 182



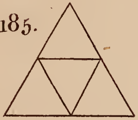
183



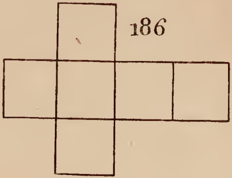
184



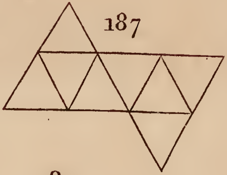
185.



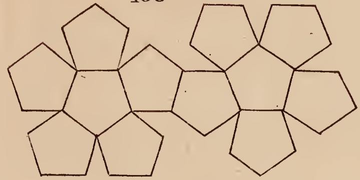
186



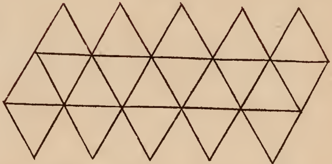
187



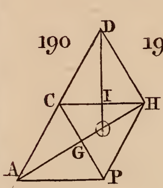
188



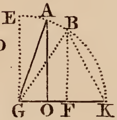
189.



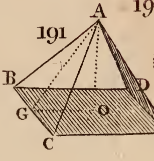
190



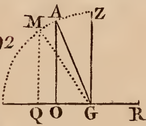
190



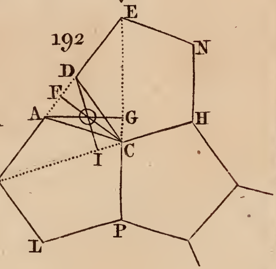
191



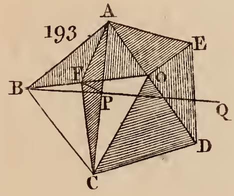
192



192



193



Pl.IX. *Pa.124*

BOOK VII.

Of the ſphere, and its inſcribed and circumſcribed bodies.

DEFINITIONS.

1. A *Sphere or globe*, is a ſolid made by a ſemicircle ABD, moving round about its diameter AD, which remains fixt; and is called the *axis* of the ſphere; and the point A, the *vertex*. FIG. 194.

2. The *center* of the ſphere is the center C of the ſemicircle ABD.

3. The *radius* of the ſphere, is a line drawn from the center to the ſurface of the ſphere.

Cor. *All the radius of a ſphere are equal to one another.*

4. The *diameter* of a ſphere, is a right line drawn from one ſide to the other, through the center.

5. A *ſector* of a ſphere, CFDG, is a part of the ſphere made by the circular ſector FCD, moving round the radius CD. 193.

6. *Segment* of a ſphere, is a part of a ſphere, as FIGD, cut off by a plane FIG. If the plane paſs through the center, that ſegment is a *hemiſphere.*

7. A *zone*, is a part of a ſphere intercepted between two parallel planes.

8. The *middle zone*, is the part between two parallel planes which are equally diſtant from the center.

9. A

FIG. 193. 9. A ſolid is ſaid to be *inſcribed* in a ſphere, or a ſphere *circumſcribed* about a ſolid; when all the angles of the ſolid touch the ſurface of the ſphere.

10. A ſphere is ſaid to be *inſcribed* in a ſolid, or a ſolid *circumſcribed* about a ſphere; when the ſphere touches all the planes of the ſolid.

PROP. I.

194. *If a ſphere be cut by a plane* FOG; *the ſection will be a circle.*

Let the two planes CFDG and COD be ⊥ to the cuting plane FOG; then the common ſection CI is ⊥ to the plane FOG (15. V). Draw the line FIG. Then in the triangles CFI, COI, CGI, the ſides CF, CO, CG are equal (Cor. Def. 3), and CI common, and the angles at I right; therefore IF = IO = IG (9. II). Therefore FDG is a circle whoſe center is I (Cor. Def. 3. IV).

PROP. II.

195. *If a ſphere* ABDI *touch a plane* HGL; *a right line* CD, *drawn from the center to the point of contact* D, *is perpendicular to the ſaid plane.*

Let the planes ADB, ADF, cut the touching plane in the lines DH, DG. Then ſince HD, GD, touch the circles BD, FD, (whoſe center is C,) in D, therefore CD is ⊥ to HD, GD (Cor. 2, 10. IV); and therefore CD is ⊥ to the plane HGL (4. V).

PROP. III.

196. *The ſurface of a ſphere is equal to the curve ſurface of its circumſcribing cylinder.*

Let BAP be a hemiſphere, and BHOP a cylinder on the ſame baſe, BTP, and of the ſame altitude.

FIG. 196.

tude. Take IL, an extremely ſmall part of the quadrant BLA, and through L and I, ſuppoſe two planes MLEVQ, and NIFSR to be drawn ⊥ to AC. Through L and I draw the line ILD, and through S and V the line SVG. Then becauſe IL and VS are extremely ſmall; the right lines and arches LI, VS nearly concide. And if the figure DISG be turned about the radius AC, it will generate the fruſtum of a cone; and the ſmall parts of its ſurface ILVS will concide with the portion of the ſpherical ſurface, and be equal thereto (Ax. 8). But the ſurface of the fruſtum ILVS is = IL × half the ſum of the circumferences whoſe diameters are LV and IS (6. VI), that is = IL × circumference of LV or IS, they being nearly equal. Let C = circumference whoſe radius is BC, and c = circumference whoſe radius is LE or IF, then the ſurface ILVS = c × IL, and the cylindric ſurface NMQR = C × MN (4. VI). But the triangles ILK, and LCE are ſimilar; for ∠ILC (Cor. 2. 10. IV) = KLE = a right angle; take away KLC, then ∠ILK = CLE; alſo ∠IKL = LEC = a right angle. Therefore LC or BC : LE :: LI : LK (13. II). But C : c :: BC or ME : LE (Cor. 9. IV) :: LI : LK or MN. Whence C × MN = c × IL (12. Proportion); that is, the cylindric ſurface NMQR = ſpherical ſurface ILVS. Therefore if more parallel planes, as MLVQ, be drawn, exceeding near to one another, the ſmall parts of the cylindric ſurface will be equal to the correſpondent parts of the ſpherical ſurface, and therefore the ſum of all the parts of the cylindric ſurface, equal to the ſum of all the parts of the ſpherical ſuface (Ax. 2); that is, the ſurface of the hemiſphere is equal to the ſurface of the cylinder BO, and the ſurface of the whole ſphere = ſurface of its circumſcribing cylinder.

FIG. 196. Cor. 1. *If the ſphere and its circumſcribing cylinder be cut by two planes parallel to the baſe, the intercepted parts of the ſurfaces of the ſphere and cylinder, will be equal.*

For ſurface MR = ſurface LS, and all the MR = all the LS.

Cor. 2. *The ſurface of the hemiſphere* BAP, *is double the baſe* BTP.

For the ſurface of the cylinder = C × AC (4. VI); and the area of the baſe = $\frac{C \times BC}{2}$, or $\frac{C \times AC}{2}$ (34. IV).

Cor. 3. *The ſurface of the whole ſphere is equal to four great circles of the ſame ſphere; or to the rectangle of the circumference and diameter.*

Cor. 4. *The areas of ſpherical ſurfaces cut off by parallel planes, are as the ſegments of the diameter, perpendicular thereto.*

For theſe areas are equal to the correſponding cylindrical ſurfaces, which are as the hights (Cor. 3. 4. VI).

Cor. 5. *The ſurface of any ſegment of the ſame ſphere, is as the hight of the ſegment.*

Cor. 6. *The ſurface of the ſphere is* $\frac{2}{3}$ *the whole ſurface of the circumſcribing cylinder.*

For the two baſes of the cylinder is half its curve ſurface (Cor. 3).

PROP. IV.

197. 198. *The ſurface of the ſegment* BAD, *of a ſphere; is equal to the area of a circle, whoſe radius is the cord* AB, *drawn from the vertex to the baſe.*

Let

Let C = circumference of the radius AB, and FIG. 200. ABED = circumference of the ſphere. Then ſince the circumferences are as the radii (Cor. 9. IV), let $\frac{ABED}{AC} = \frac{C}{AB} = n$, or ABED = $n \times$ AC; and C = $n \times$ AB. Then the ſurface BAD = AF × ABED (Cor. 1. III) = AF × n × AC = $\frac{n \times AF \times AE}{2} = \frac{n \times AB^2}{2}$ (Cor. 17. IV) = $\frac{AB \times C}{2}$ = area of a circle whoſe radius is AB (34. IV).

Cor. *The ſurface of the whole ſphere, is equal to the area of a circle, whoſe radius is the diameter* AE.

PROP. V.

The ſurface of a ſphere is double the curve ſurface of the inſcribed ſquare (or equilateral) cylinder EB. 201.

Draw the diameter ECB, then ED = DB. And ſince $EB^2 = ED^2 + DB^2$ (21. II) = $2ED^2$; therefore circle EB = 2 circles ED (22. III). But ſurface of the ſphere = 4 circles EB (Cor. 3. III) = 8 circles ED. And $\frac{1}{4}$ED : AE or ED : : circle ED : curve ſurface AD (Cor. 2. 4. VI) = 4 circles ED. But 8 circles ED = twice 4 circles ED, or the ſurface of the ſphere = twice the curve ſurface of the cylinder.

Cor. 1. *The whole ſurface of the inſcribed cylinder is* $\frac{3}{4}$ *the ſurface of the ſphere.*

For the two baſes AB, ED = 2 circles ED, and the whole ſurface AD = 6 circles ED.

Cor. 2. *The curve ſurface of a cylinder, circumſcribing the ſphere, is double the curve ſurface of the inſcribed equilateral one. And the whole ſurface, is double to the whole ſurface.*

FIG. 201. For the ſurface of the ſphere = ſurface of the circumſcribing cylinder (3). And the ſurface of the ſphere = twice the ſurface of the inſcribed one (5).

Again, the ſurface of the ſphere = $\frac{2}{3}$ the whole ſurface of the circumſcribing cylinder (Cor. 6. 3). And the ſurface of the ſphere is = $\frac{4}{3}$ the whole ſurface of the inſcribed cylinder (Cor. 1).

PROP. VI.

202. *The ſurface of any ſegment of a ſphere* ABDC : *is to the curve ſurface of its inſcribed cone* ABC :: *as the ſide of the cone* AB : *to the radius of the baſe* AO.

For if $n \times AB$ = circumference of the radius AB, and $n \times AO$ = circumference of the radius AO (Cor. 9. IV), then the circle AB = $\frac{AB^2 \times n}{2}$ (34. IV), and conic ſurface ABC = $\frac{AB \times n \times AO}{2}$ (Cor. 1. 5. VI). And the ſurface of the ſegment ABDC = circle AB (4); therefore ſurface of the ſegment ABDC : conic ſurface ABC :: $\frac{AB^2 \times n}{2}$: $\frac{AB \times n \times AO}{2}$ :: AB : AO (5. Proportion).

Cor. 1. *The ſurface of a hemiſphere, is to the curve ſurface of its inſcribed cone; as the diagonal of a ſquare, to the ſide.*

For then AO, BO become radii of the ſphere, and AB the diagonal.

Cor. 2. *If* ABC *be an equilateral cone, then the ſurface of the ſegment* ABDC *is twice the curve ſurface of the cone* ABC.

For then AB = AC = 2AO.

PROP. VII.

FIG. 203.

Let the cone DAE *be right-angled at* A. *Then the ſurface of the hemiſphere* BGE, *is to the curve ſurface of the right-angled circumſcribing cone* DAE; *as the ſide of a ſquare* AD, *is to the diagonal* DE.

Draw AC from the vertex of the cone A, to the center C; and CF ∥ to AE, or ⊥ to AD. Then $AF = FD = FC = BC$, and $CD^2 = CF^2 + FD^2 = 2BC^2$ (21. II). And the circle whoſe radius is CD = twice the circle whoſe radius is CB (Cor. 2. 35. IV) = ſurface of the hemiſphere BGE (Cor. 2. 3). Therefore the ſurface of the hemiſphere, or the circle whoſe radius is CD : ſurface of the cone DAE : : CD : AD (Cor. 3. 5. VI) : : AD : DE (20. II).

Cor. *The ſurface of a right-angled cone circumſcribing a hemiſphere, is double the ſurface of one inſcribed; taking either the curve ſurfaces, or the whole ſurfaces.*

For $\sqrt{2} \times$ ſurface of the inſcribed cone = ſurface of the hemiſphere (Cor. 1. 6) $= \frac{1}{\sqrt{2}} \times$ ſurface of the circumſcribing cone (7). Therefore the latter is = twice the former. And the baſe of the latter is likewiſe = twice the baſe of the former (by the demonſtration of this Prop.), therefore the whole is double to the whole.

PROP. VIII.

FIG. 204.

The ſurface of the ſphere, is to the curve ſurface of an equilateral inſcribed cone BAD; *as* 8, *to* 3.

For ſince $EF = \frac{1}{4}AF$ (Cor. 3. 41. IV), therefore ſurface BFD $= \frac{1}{4}$, and ſurface BAGD $= \frac{3}{4}$, the ſurface of the ſphere (Cor. 4. 3); = 2 curve

FIG. 204. ſurfaces of the cone BAD (Cor. 2. 6); or the ſurface of the cone $= \frac{3}{8}$ the ſurface of the ſphere.

Cor. *The whole ſurface of an equilateral cone* BAD, *inſcribed in a ſphere, is* $\frac{9}{16}$ *of the ſphere's ſurface.*

For $3BC^2 = BD^2$ (41. IV) $= 4BE^2$, and $BE^2 = \frac{3}{4}BC^2$, whence circle BD $= \frac{3}{4}$ circle BDG (35. IV) $= \frac{3}{16}$ the ſurface of the ſphere (Cor. 3. 3); add this to the curve ſurface of the cone; then the whole ſurface of the cone $= \frac{3}{8} + \frac{3}{16}$ the ſphere's ſurface $= \frac{9}{16}$ the ſurface of the ſphere.

PROP. IX.

205. *The curve ſurface of an equilateral cone* ABD, *is to the ſurface of its inſcribed ſphere; as* 3 *to* 2.

Draw AE, CF ⊥ to BD, BA; then by ſimilar triangles AEB, AFC; $AE^2 : EB^2 :: AF^2 : FC^2$. But $AE^2 = \frac{3}{4}AB^2$ (39. II) $= 3AF^2$. Therefore $3AF^2$ (AE^2) : $AF^2 :: EB^2 : FC^2$ (4. Proportion) :: circle BD : circle FEG. But BE :: BA or 2BE :: circle BD : curve ſurface of the cone BAD (Cor. 3. 5. VI) = 2 circles BD; and circle FEG $= \frac{1}{4}$ ſurface of the ſphere (Cor. 3. 3). Whence $3 : 1 :: 3AF^2 : AF^2 :: \frac{1}{2}$ ſurface of the cone : $\frac{1}{4}$ ſurface of the ſphere. Therefore the ſurface of the ſphere $= \frac{2}{3}$ the curve ſurface of the cone.

Cor. 1. *The ſurface of the ſphere is* $\frac{4}{9}$ *the whole ſurface of the circumſcribing equilateral cone.*

For the baſe BD $= \frac{1}{2}$ curve ſurface of the cone $= \frac{3}{4}$ ſurface of the ſphere. Add this to the curve ſurface, which is $= \frac{3}{2}$ ſurface of the ſphere; then the whole ſurface of the cone $= \frac{3}{2} + \frac{3}{4}$ the ſurface of the ſphere $= \frac{9}{4}$ the ſurface of the ſphere, or $\frac{4}{9}$ the whole ſurface of the cone = the ſurface of the ſphere.

Cor.

Cor. 2. *The curve ſurface of an equilateral cone inſcribed in a ſphere is* = $\frac{1}{4}$ *the curve ſurface of the circumſcribing equilateral one. And the whole ſurface of one* = $\frac{1}{4}$ *the whole ſurface of the other.* FIG. 205.

For $\frac{8}{3}$ the ſurface of the inſcribed cone = ſurface of the ſphere (8) = $\frac{2}{3}$ ſurface of the circumſcribed cone (9). Therefore the ſurface of the inſcribed = $\frac{1}{4}$ the ſurface of the circumſcribed one.

Alſo $\frac{4}{9}$ the whole ſurface of the circumſcribing one = ſurface of the ſphere (Cor. 1. 9) = $\frac{16}{9}$ the whole ſurface of the inſcribed cone (Cor. 8). Therefore the ſurface of the inſcribed cone = $\frac{1}{4}$ the ſurface of the circumſcribed cone.

Cor. 3. *The ſurfaces of a cylinder and equilateral cone, both circumſcribed about a ſphere, are as* 2 *to* 3; *both their curve ſurfaces and whole ſurfaces.*

For $\frac{2}{3}$ the curve ſurface of the cone = ſurface of the ſphere (9) = ſurface of the cylinder (3). Surface of the cylinder : ſurface of the cone : : 2 : 3.

Alſo $\frac{4}{9}$ the whole ſurface of the cone = ſurface of the ſphere (Cor. 1. 9) = $\frac{2}{3}$ the whole ſurface of the cylinder (Cor. 6. 3). Therefore, whole ſurface of the cylinder : whole ſurface of the cone : : $\frac{4}{9}$: $\frac{2}{3}$ or $\frac{6}{9}$: : 2 : 3.

SCHOLIUM.

From the foregoing propoſitions are deduced, the proportion of the ſphere's ſurface, to the ſurfaces of the inſcribed and circumſcribed equilateral cylinder and cone, as follows:

FIG. 205.

Surface of the ſphere —— ——	16
Inſcribed cylinder's curve ſurface —	8
—— —— whole ſurface —	12
Circumſcribed cylinder's curve ſurface	16
—— —— whole ſurface —	24
Inſcribed cone's curve ſurface ——	6
—— —— whole ſurface —	9
Circumſcribing cone's curve ſurface —	24
—— —— whole ſurface —	36.

PROP. X.

206. *A ſphere is equal to a cone whoſe hight is the radius* AC, *and baſe the ſurface of the ſphere* AEF.

Take three points in the ſurface of the ſphere, as A, B, D, extremely near together, forming the ſmall triangle ABD, on the ſurface of the ſphere. Let a plane paſs through theſe three points A, B, D; the ſmall portion of which ABD will coincide with a portion of the ſpherical ſurface ABD, extremely near. And the radius CA will be $\perp$ thereto (2). Therefore the portion of the ſphere CABD is nothing but the pyramid whoſe baſe is ABD, a ſmall part of the ſphere's ſurface, and hight the radius CA. In like manner the whole ſphere may be divided into ſmall pyramids, ſuch as CABD, whoſe baſe is a ſmall portion of the ſpherical ſurface; and common altitude, the radius CA. Therefore the ſum of all theſe pyramids CABD, make up the ſphere; and the ſum of all the baſes ABD, make up the ſpherical ſurface. That is, the ſphere is equal to the ſum of all theſe pyramids, whoſe baſes are all the parts of the ſurface, of the ſphere, and common altitude the radius CA; and that is equal to one pyramid or cone, whoſe baſe is the ſurface of the ſphere, and hight the radius (Ax. 2).

Cor,

Cor. 1. *A ſphere is equal to a cone, whoſe hight is the radius, and baſe equal to four great circles of the ſphere.* FIG. 206.

For the ſurface of the ſphere is equal to four great circles (Cor. 3. 3).

Cor. 2. *A ſphere is equal to a cone whoſe hight is twice the diameter, and baſe, a great circle of the ſphere.*

By Cor. 4. 20. VI.

Cor. 3. A *hemiſphere is double its inſcribed cone.*

For a hemiſphere = a cone whoſe baſe is a great circle, and hight equal to the diameter (Cor. 2); and that is double to a cone of the ſame baſe, and half the hight (Cor. 1. 20. VI).

PROP. XI.

Any ſphere BANR, *is* $\frac{2}{3}$ *its circumſcribing cylinder,* DM. 207.

Let AC be the axis of the hemiſphere BAN. From the center C, draw the diagonal CD; and draw PL ⊥ to AC, and OH parallel to it, and exceeding near it. Then if the figure ADBC revolve round the axis AC; then ADBC will generate the cylinder BDGN; the quadrant BVA, the hemiſphere BAN; and ADC, the cone ADCG. Then $VC^2 = VL^2 + LC^2$ (21. II); that is, $PL^2 = VL^2 + KL^2$ (for DA = AC, and KL = LC (13. II). Therefore the circle deſcribed by LP = the two circles deſcribed by LV and LK (Cor. 2. 35. IV). Take away the circle deſcribed by LV, from both, and there remains the annulus or ring deſcribed by VP = circle deſcribed by LK. For the ſame reaſon the annulus deſcribed by OI = circle deſcribed by FH. Therefore the ſmall priſmatic ſolid contained between PN and OI, quite round the figure = cone fruſtum contained between KL and FH, round the figure

FIG. 207. figure (12. VI). In like manner every part of the figure BDAVB = correſpondent part of DACG. Therefore the total ſum of the firſt = total ſum of the laſt, that is, the ſolid BDAGNAVB = cone DCG (Ax. 2) = $\frac{1}{3}$ the cylinder DBNG (20. VI). Therefore the remaining part, or the hemiſphere BAN = the remaining $\frac{2}{3}$ of the cylinder BDGN. Whence the double thereof, or the whole ſphere ABRN = $\frac{2}{3}$ of the whole cylinder EG.

Otherwiſe.

The cone whoſe baſe is BN, and hight CA, or the cone DCG = half the hemiſphere BAN (Cor. 3. 10). And the ſame cone DCG = $\frac{1}{3}$ the cylinder BDGN. (20. VI). Therefore $\frac{1}{2}$ hemiſphere = $\frac{1}{3}$ cylinder, and the hemiſphere = $\frac{2}{3}$ cylinder BG. Whence the whole ſphere = $\frac{2}{3}$ the cylinder EG.

Cor. 1. *The concave ſolid* BFADBER &c. = $\frac{1}{2}$ *the ſphere* BANR.

208. Cor. 2. *A right cone, ſphere, and cylinder, all of the ſame diameter and hight, are as* 1, 2, 3 *reſpectively; or* ABD : AHGI : EBDF : : 1 : 2 : 3.

PROP. XII.

206. *The ſector of a ſphere* CGAH, *is equal to a cone whoſe hight is the radius; and baſe, the ſurface of the ſector* GAH.

This is demonſtrated as Prop. X. For if the ſector be divided into a multitude of extremely ſmall ſectors CABD, the baſe of each will be a ſmall portion of the ſpherical ſurface ABD. And as all the pyramids make up the ſector, and are the elements thereof; ſo all the baſes are the elements of the ſurface GAH, and make it up. And as the hights of all the pyramids is the ſame, they are all equal to one pyra-

pyramid of the ſame hight, and baſe the ſum of all the baſes (Cor. 1. 18. VI). That is, the ſector CGAH = a pyramid or cone whoſe hight is the radius, and baſe the ſurface GAH. FIG. 206.

Cor. 1. *The ſector of a ſphere,* CGAH = *a cone, whoſe hight is the radius* AC; *and baſe a circle whoſe radius is* AG. *And the ſector* CGBH = *a cone whoſe radius is* CB, *and baſe a circle whoſe radius is* BG.

For the ſurface GAH = a circle whoſe radius is AG (4); and the ſurface GBH = a circle whoſe radius is BG (ibid.).

Cor. 2. *Sectors of ſpheres, are to one another, in the complicate ratio of their ſurfaces and radii.*

For the cones, equal thereto, are as the baſes and hights (Cor. 3. 20. VI).

PROP. XIII.

If it be made, as BD : BA : : *radius* CA : CF; *then the cone* GFH *is equal to the ſegment of the ſphere,* GAH. 210.

Draw CG, BG and FCB; then CA : CF : : BD : BA (hyp.) : : $BD^2 : BG^2$ (Cor. 1. 20. II) : : $GD^2 : GA^2$ (20. II) : : circle GD (or circle whoſe radius is GD) : circle GA (35. IV). Therefore the cone whoſe hight is CF, and baſe the circle GD = cone whoſe hight is CA, and baſe the circle GA (Cor. 4. 20. VI) = ſector CGAH (Cor. 1. XII). Subtract, or add the cone GCH, on the ſame baſe GH, and then the cone GFH = ſegment GAH.

Cor. 1. *If* BD : DA : : *radius* CA : AF. *Then the cone* GFH = *ſegment* GAH.

For ſince BD : BA : : CA : CF, therefore BD : BA — BD : : CA : CF — CA (13. Proportion); that is, BD : DA : : CA : AF.

Cor. 2. *The ſegment* GAH, *is to the inſcribed cone* GAH; *as* FD *to* AD. Cor.

FIG. 210. Cor. 3. *The ſegment* GAH : *ſegment* GBH : : $\overline{GC + DB} \times AD^2 : \overline{GC + AD} \times DB^2$.

For the hight of the cone, equal to the ſegment GAH, that is, $DE = \frac{GC}{DB} \times DA + DA$ (Cor. 1) $= \frac{GC + DB}{DB} \times DA$. And in like manner, the hight of the cone equal to the ſegment GBH, is $\frac{GC + DA}{DA} \times DB$. And theſe cones are as the altitudes (Cor. 1. 20. VI); that is, as $\frac{GC + DB}{DB} \times DA$, and $\frac{GC + DA}{DA} \times DB$, or as $\overline{GC + DB} \times DA^2 : \overline{GC + DA} \times DB^2$.

PROP. XIV.

210. *The ſegment of a ſphere* GAH, *is equal to a cone, whoſe hight is* AD, *the hight of the ſegment; and baſe,* $\frac{3}{2}$ *the baſe of the ſegment* GH, *together with* $\frac{1}{2}$ *a circle whoſe radius is the hight of the ſegment* AD.

Let $\ominus AG$ denote the circle whoſe radius is AG, and ſo of the reſt. Then ſegment GAH = ſector CGAH ∓ cone GCH (fig. 1, 2) $= \frac{1}{3}AC \times \ominus AG \mp \frac{1}{3}CD \times \ominus GD$ (Cor. 1. 12); and 3 ſegments GAH $= AC \times \ominus AG + \overline{AD + AC} \times \ominus GD = AC \times \overline{\ominus AG - \ominus GD} + AD \times \ominus GD = AC \times \ominus AD + AD \times \ominus GD$ (Cor. 2. 35. IV).

But $AD : AB : : AD^2 : AG^2$ (Cor. 1. 20. II) $: : AD^2 : AD^2 + DG^2 : : \ominus AD : \ominus AD + \ominus DG$ (Cor. 2. 35. IV), therefore $AD \times \overline{\ominus AD + \ominus DG} = AB \times \ominus AD = 2AC \times \ominus AD$, and $AD \times \overline{3\ominus DG + \ominus AD} = 2AC \times \ominus AD + 2AD \times \ominus GD$. And $AD \times \overline{1\frac{1}{2}\ominus GD + \frac{1}{2}\ominus AD} = AC \times \ominus AD + AD \times \ominus GD = 3$ ſegments GAH.

Corol-

Corollary. *The ſegment* GAH $= \frac{1}{6}AD \times \overline{3\odot GD + \odot AD}$. FIG. 210.

PROP. XV.

The fruſtum or middle zone of a ſphere ZGHF, *is equal to a cone whoſe hight is the hight of the zone* CD; *and baſe, two great circles* ZF, *together with the leſſer baſe* GH. 211.

For the zone ZH = hemiſphere ZAF — the ſector CGAH + the cone GCH $= AC \times \frac{2}{3}\odot ZC$ (11) — $AC \times \frac{1}{3}\odot AG$ (Cor. 1. 12) + $CD \times \frac{1}{3}\odot GD$ (20. VI) $= AD \times \frac{2}{3}\odot ZC + DC \times \frac{2}{3}\odot ZC - AC \times \frac{1}{3}\odot AG + CD \times \frac{1}{3}\odot GD$. But $AD : AC :: AG^2 : AZ^2$ (18. IV) $:: AG^2 : 2AC^2$ (21. II) $:: \odot AG : 2\odot ZC$ (35. IV). Therefore $AD \times 2\odot ZC = AC \times \odot AG$. And $AD \times \frac{2}{3}\odot ZC = AC \times \frac{1}{3}\odot AG$. Therefore the zone ZH $= DC \times \frac{2}{3}\odot ZC + DC \times \frac{1}{3}\odot GD = \frac{1}{3}DC \times \overline{2\odot ZC + \odot GD}$.

Cor. *The zone* ZH *is equal to* $\frac{1}{3}DC \times$ *twice the circle* ZF + *the circle* GH.

PROP. XVI.

An orb or hollow ſphere is equal to the fruſtum of a cone, whoſe greater baſe is the ſurface of the greater ſphere; and leſſer baſe, the ſurface of the leſſer: and hight, the difference of the radii.

For the orb is equal to the difference of the two ſpheres; that is, to the difference of two cones whoſe hights are the radii of the ſpheres, and baſes the ſurfaces (10).

PROP.

FIG.

PROP. XVII.

212. *The ſurfaces of ſpheres* GH, IK, *are as the ſquares*
213. *of the diameters*, AB, DF.

For the ſurface of the ſphere GH = 4 circles AGBH, and the ſurface of the ſphere IK = 4 circles IDKF (Cor. 3. III). But 4 circles AGBH : 4 circles DIFK : : circle AGBH : circle DIFK (Cor. 1. 5. Proportion) : : $AB^2 : DF^2$ (35. IV).

PROP. XVIII.

Spheres GH, IK, *are to one another, as the cubes of their diameters*, AB, DF.

For the ſphere GH = $\frac{2}{3}$ the cylinder, whoſe baſe is AGBH, and hight AB. And the ſphere IK = $\frac{2}{3}$ the cylinder, whoſe baſe is DIFK, and hight DF (11). Therefore ſphere GH : ſphere IK : : $\frac{2}{3}AGBH \times AB : \frac{2}{3}DIFK \times DF$ (Cor. 3. 19. VI) : : $AGBH \times AB : DIFK \times DF$ (5. Proportion) : : $AB^2 \times AB : DF^2 \times DF$ (35. IV. and 7. Proportion) : : $AB^3 : DF^3$.

PROP. XIX.

214. *Similar ſolids inſcribed in ſpheres* GH, IK, *are as the*
215. *cubes of the diameters of the ſpheres* AB : DF.

From any two equal and correſpondent angles A, D, draw the diameters AB, DF. Then ſince the ſolids are inſcribed after a ſimilar manner in reſpect of the diameters AB, DF. It will be AG : DI : : AB : DF (19. III). But ſolid AE : ſolid DL : : $AG^3 : DI^3$ (24. VI) : : $AB^3 : DF^3$ (Cor. 3. 18. Proportion).

Cor. 1. *Similar ſolids inſcribed in ſpheres, are as the ſpheres.*

For ſpheres are alſo as the cubes of their diameters (18).

Cor.

Cor. 2. *The ſurfaces of ſimilar ſolids inſcribed in ſpheres, are as the ſquares of the diameters of the ſpheres.* FIG. 214. 215.

For ſurface of AE : ſurface of DL : : $AG^2 : DI^2$ (7. VI) : : $AB^2 : DF^2$.

Cor. 3. *The ſurfaces of ſimilar ſolids inſcribed in ſpheres, are as the ſurfaces of the ſpheres.*

For they are both as the ſquares of the diameters (17).

PROP. XX.

A ſphere, is to any circumſcribing ſolid BF, *(all whoſe planes touch the ſphere) ; as the ſurface of the ſphere, to the ſurface of the ſolid.* 216.

Since all the planes touch the ſphere, the radius drawn to all the points of contact, will be ⊥ to each plane (2). Therefore if planes be drawn through the center C of the ſphere, and through all the ſides of the body ; then the body will be divided into pyramids, BCAE, BCAD, *&c.* whoſe baſes are the planes BAE, BAD, *&c.*; and their common altitude CP, the radius of the ſphere. And the ſum of all theſe pyramids, or the whole ſolid, is equal to a pyramid or cone, whoſe baſe is the ſum of all the plane figures, and hight the radius CP (Cor. 1. 18. and Cor. 2. 20. VI). But the ſphere is alſo equal to a cone or pyramid whoſe baſe is the ſurface of the ſphere, and hight the ſame radius CP (10). And this laſt cone : former cone : : baſe of the latter : baſe of the former (Cor. 2. 20. VI.) ; that is, the ſphere : circumſcribing ſolid : : ſurface of the ſphere : ſurface of the ſolid.

Cor. 1. *All circumſcribing cylinders, cones,* &c. *are to the ſphere, as their ſurfaces are.*

For

FIG. 216. For any cylinder, or cone, may be conceived to be made up of an infinite number of ſmall planes, all of which touch the ſphere.

Cor. 2. *All bodies circumſcribing the ſame ſphere, are to one another as their ſurfaces.*

Cor. 3. *The ſphere is the greateſt or moſt capacious of all bodies of equal ſurface.*

For if the planes be ſuppoſed to touch the ſphere, their areas will be greater than the ſurface of the ſphere, which is contrary to the hypotheſis; therefore the planes muſt fall within the ſphere; and then the perpendicular upon them will be ſhorter than the radius, and therefore the body will be leſs than the ſphere, as having the ſame baſe, and a leſs hight.

PROP. XXI.

210. *Any ſegment of a ſphere* GAH, *is to its inſcribed cone; as* BC + BD, *to* BD.

For if AF $= \frac{AD}{DB} \times AC$, then the ſegment GAH = cone GFH (Cor. 1. 13). Therefore FD $= \frac{AD}{DB} \times AC + AD$. And this cone GFH : cone GAH :: DF : DA (Cor. 1. 20. VI) :: $\frac{AC}{BD} \times AD + AD : AD :: \frac{AC+BD}{BD} \times AD : AD ::$ AC + BD : BD (5. Proportion).

Cor. 1. *A hemiſphere is double the inſcribed cone.*

For then BD = AC or BC.

Cor. 2. *The ſegment containing an equilateral cone, is equal to 3 times the cone.*

For then BD $= \frac{1}{2}$BC (Cor. 3. 41. IV).

PROP.

PROP. XXII.

FIG. 203.

If the cone DAE *circumſcribing a hemiſphere be right-angled at* A; *that cone* DAE *is to the inſcribed hemiſphere*; *as* $\sqrt{2}$ *to* 1.

For let Θ ſtand for circle, then ſuppoſing the ſame conſtruction as in Prop. VII, then we have $CD^2 = 2BC^2$, and $CD = BC\sqrt{2} = DF\sqrt{2}$, and $CD : DF :: \sqrt{2} : 1$ (Cor. 1. 12. Proportion); alſo $\Theta CD = 2\Theta CB$, and $AC = CD$. The cone DAE $= \Theta CD \times \frac{1}{3}AC$ (20. VI) $= 2\Theta CB \times \frac{1}{3}CD$. Alſo the hemiſphere $= \frac{2}{3}\Theta CB \times GC$ (11) $= 2\Theta CB \times \frac{BC}{3}$. Therefore the cone : hemiſphere :: $2\Theta CB \times \frac{1}{3}CD : 2\Theta CB \times \frac{1}{3}BC :: CD : CB$ or $DF :: \sqrt{2} : 1$.

Cor. *A right-angled cone, circumſcribing a hemiſphere, is to the inſcribed cone*; *as* $2\sqrt{2}$ *to* 1.

For the circumſcribed cone : hemiſphere :: $\sqrt{2} : 1 :: 2\sqrt{2} : 2$ (22).

And hemiſphere : inſcribed cone :: $2 : 1$ (Cor. 1. 21).

Therefore circumſ. cone : inſ. cone :: $2\sqrt{2} : 1$ (15. Proportion).

PROP. XXIII.

A ſphere is to its inſcribed equilateral cylinder AD, *as* $4\sqrt{2}$ *to* 3. 201.

Draw the diameter BE, then $BE^2 = DE^2 + DB^2$ (21. II) $= 2DE^2$, and circle AEDB = 2 circles BD (35. IV); alſo $BE = DE\sqrt{2} = BD\sqrt{2}$. Now The ſphere $= \frac{2}{3}AEDB \times BE$ (11) $= \frac{2}{3}AEDB \times BD\sqrt{2}$, the cylind. = circle $ED \times BD = \frac{1}{2}AEDB \times BD$. Then ſphere : cylinder :: $\frac{2}{3}AEDB \times BD\sqrt{2} : \frac{1}{2}AEDB \times BD :: \frac{2}{3}\sqrt{2} : \frac{1}{2} :: 4\sqrt{2} : 3$.

FIG. 201. Cor. *The circumſcribed equilateral cylinder, is to the inſcribed equilateral cylinder; as* $2\sqrt{2}$ *to* 1.

For $\frac{2}{3}$ the circumſcr. cylinder $=$ ſphere (11) $= \frac{4\sqrt{2}}{3} \times$ the inſcr. cylinder. Therefore the circumſ. cylinder $= 2\sqrt{2} \times$ inſcr. cylinder.

PROP. XXIV.

204. *The ſphere is to the inſcribed equilateral cone* BAD, *as* 32 *to* 9.

Let ΘBE denote the circle whoſe radius is BE, *&c.* then BD^2 or $4BE^2 = 3BC^2$ (41. IV), and $BE^2 = \frac{3}{4}BC^2$, and $\Theta BE = \frac{3}{4}\Theta BC$ (Cor. 1. 35. IV). Alſo $AE = \frac{3}{2}AC$ (Cor. 3. 41. IV). Then the ſphere $= \frac{2}{3}\Theta BC \times 2AC$ (11). And cone $= \Theta BE \times \frac{1}{3}AE$ (20. VI) $= \frac{3}{4}\Theta BC \times \frac{1}{3} \times \frac{3}{2}AC$. Therefore, ſphere : cone : : $\frac{2}{3}\Theta BC \times 2AC : \frac{3}{4}\Theta BC \times \frac{1}{2}AC : : \frac{4}{3} : \frac{3}{8} : :$ 32 : 9.

PROP. XXV.

205. *A ſphere is to its circumſcribed equilateral cone* ABD, *as* 4 *to* 9.

The conſtruction of Prop. IX. remaining; let ΘFC denote the circle whoſe radius is FC, *&c.* Then $EB^2 = 3FC^2$, and $\Theta BE = 3\Theta FC$ (35. IV), and CF or $CE = \frac{1}{2}CA$ (Cor. 31. II), and $AE = 3CF$.

The ſphere $= \frac{2}{3}\Theta CF \times 2CF$ (11).

The cone $= \Theta BE \times \frac{1}{3} AE$ (20. VI) $= 3\Theta FC \times FC$.

Therefore ſphere : cone : : $\frac{2}{3}\Theta CF \times 2CF : 3\Theta CF \times CF : : \frac{4}{3} : 3 : : 4 : 9$.

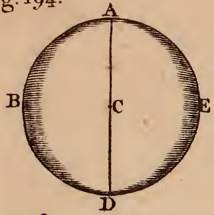
Cor. 1. *The circumſcribed equilateral cone is eight times the inſcribed equilateral cone.*

For the circumſcr. cone : ſphere : : 9 : 4.

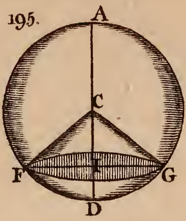
And ſphere : inſcr. cone : : 32 : 9 (24).

Therefore circumſcr. cone : inſcr. cone : : 32 : 4 (15. Proportion) : : 8 : 1.

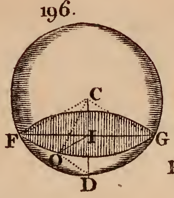
Fig. 194.



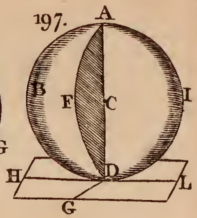
195.



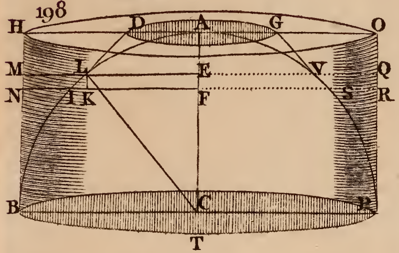
196.



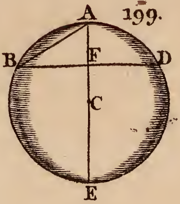
197.



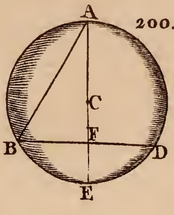
198



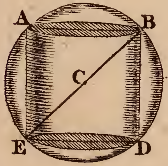
199.



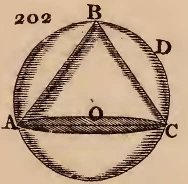
200.



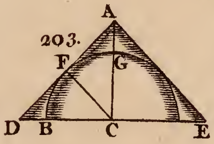
201.



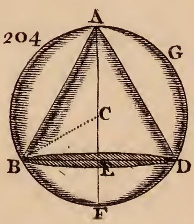
202



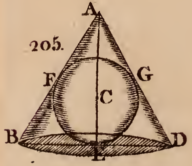
203.



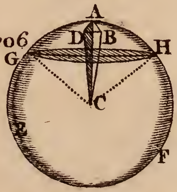
204



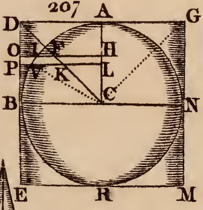
205.



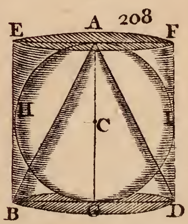
206



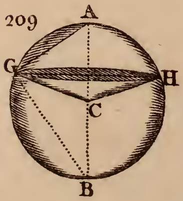
207



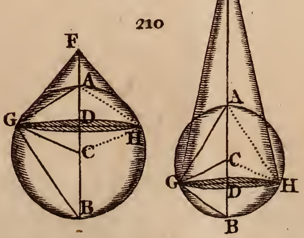
208



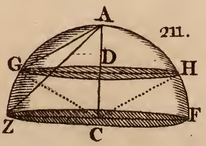
209



210



211.



Pl. X *Pa.144.*

Cor. 2. *The circumſcribed cylinder is $\frac{2}{3}$ the circumſcribed equilateral cone.* FIG. 205.

For $\frac{2}{3}$ the cylinder = ſphere (11) = $\frac{4}{9}$ the cone; and the cylinder = $\frac{2}{3}$ the cone.

Cor. 3. *The ſphere* EF *is to the circumſcribing right cylinder* BC, *and this cylinder to the circumſcribing equilateral cone* ADG, *as* 2 *to* 3; *both in reſpect of their whole ſurfaces and ſolidities.* 217.

This appears from Cor. 6. 3. and Cor. 3. 9. and Prop. 11. and Cor. 2. 25.

Cor. 4. *The circumſcribing right cylinder, and equilateral cone, are to one another as* 2 *to* 3; *both in regard to their curve ſurfaces, their whole ſurfaces, ſolidities, baſes, and hights.*

As to the ſurfaces it appears by Cor. 3. 9; and the ſolidities, by Cor. 3. of this. As to the baſes, ſince $\ominus BE = 3\ominus FC$ (fig. 205), or $\ominus FC = \frac{1}{3}\ominus BE$, therefore $2\ominus FC = \frac{2}{3}\ominus BE$, or the two baſes of the cylinder = $\frac{2}{3}$ the baſe of the cone.

And for the hight, $AE = 3CF$, or $2CF = \frac{2}{3}AE$; that is, the hight of the cylinder = $\frac{2}{3}$ the hight of the cone.

SCHOLIUM.

From the foregoing propoſitions, is eaſily deduced the proportion which the ſphere has to the inſcribed and circumſcribed equilateral cylinders and cones, as follows:

Solidity of the ſphere	32
———— inſcribed cone	9
———— inſcribed cylinder	$12\sqrt{2}$
———— circumſcribed cylinder	48
———— circumſcribed cone	72.

PROP. XXVI.

FIG. 218. *The ſquare of the ſide of a regular pyramid inſcribed in a ſphere, is* $\frac{2}{3}$ *the ſquare of the diameter :* $AE^2 = \frac{2}{3}EF^2$.

For drawing ECF ⊥ to the baſe ABD, $3AC^2 = AB^2$ (41. IV) $= AE^2 = AC^2 + CE^2$ (21. II); and $2AC^2 = CE^2$, or $\frac{1}{2}CE^2 = AC^2 = EC \times CF$ (17. IV), therefore $CF = \frac{1}{2}CE$, and $EF = \frac{3}{2}CE$, or $CE = \frac{2}{3}EF$, and $AC^2 = \frac{1}{2}CE^2 = \frac{2}{9}EF^2$. Therefore $AE^2 = AC^2 + CE^2$ (21. II) $= \frac{2}{9}EF^2 + \frac{4}{9}EF^2 = \frac{6}{9}EF^2 = \frac{2}{3}EF^2$.

Cor. 1. *The hight of the pyramid is* $\frac{2}{3}$ *the diameter of the ſphere,* $EC = \frac{2}{3}EF$.

Cor. 2. *The diameter of the ſphere : diameter of the circle comprehending the baſe of pyramid : : as* $3 : \sqrt{8}$.

For $AC^2 = \frac{2}{9}EF^2$, and $4AC^2 = \frac{8}{9}EF^2$.

Cor. 3. *The area of the baſe* $ADB = EF^2 \times \frac{\sqrt{3}}{6}$.

For the area $ADB = \frac{AB^2}{4}\sqrt{3}$ (39. II). And AB^2 or $AE^2 = \frac{2}{3}EF^2$. Therefore $ADB = \frac{1}{6}EF^2\sqrt{3}$.

Cor. 4. *The radius of the inſcribed ſphere* $= \frac{1}{6}EF$.

For it is $= EC - \frac{1}{2}EF = \frac{1}{6}EF$.

PROP. XXVII.

218. *The ſolidity of a regular pyramid inſcribed in a ſphere, is* $\frac{1}{27}EF^3\sqrt{3}$.

For the ſolidity $= \frac{1}{3}EC \times$ baſe ABD (18. VI) $= \frac{2}{9}EF \times ABD$ (Cor. 1. 26) $= \frac{2}{9}EF \times \frac{1}{6}EF^2\sqrt{3}$ (Cor. 3. 26) $= \frac{\sqrt{3}}{27} \times EF^3$.

PROP.

PROP. XXVIII.

FIG. 219.

The ſquare of the diameter of a ſphere, is thrice the ſquare of the ſide of its inſcribed cube: $FA^2 = 3FD^2$.

Through the oppoſite ſides AG, DF, ſuppoſe the plane FDAG to be drawn; and through two oppoſite angles A, F, draw the diameter of the ſphere AF. Then $DA^2 = DB^2 + BA^2 = 2DB^2 = 2DF^2$ (21. II). Alſo $FA^2 = FD^2 + DA^2 = FD^2 + 2FD^2 = 3FD^2$ (ibid.).

Cor. 1. *The ſide of the cube* $DF = \frac{1}{3}FA\sqrt{3}$.

Cor. 2. *The diameter of the ſphere* AF, *is to the diameter* DA *of the circle comprehending one face of the cube; as* 1 *to* $\frac{1}{3}\sqrt{6}$.

For $FA = FD \times \sqrt{3}$, and $DA = FD\sqrt{2}$; and $FD\sqrt{3} : FD\sqrt{2} :: 1 : \sqrt{\frac{2}{3}}$, or $1 : \frac{1}{3}\sqrt{6}$.

Cor. 3. *The area of one face of the cube* DBAI *is equal to* $\frac{1}{3}FA^2$.

Cor. 4. *The ſum of the ſquares of the ſides of the inſcribed pyramid and cube, is equal to the ſquare of the diameter.*

For the former is $\frac{2}{3}$, and the latter $\frac{1}{3}$, of the ſquare of the diameter (26 and 28).

Cor. 5. *The diameter of the circle containing one face of the cube* DA, *is equal to the ſide of the pyramid.*

For $DA^2 = 2DF^2 = \frac{2}{3}FA^2$ (28) = ſquare of the ſide of the pyramid (26).

Cor. 6. *The radius of the inſcribed ſphere is* $\frac{1}{2}$ *the ſide* FD.

FIG.

PROP. XXIX.

219. *The ſolidity of a cube inſcribed in a ſphere, is* $\frac{\sqrt{3}}{9}$ *multiplied into the cube of the diameter of the ſphere:* $\frac{\sqrt{3}}{9} \times AF^3$.

For $DF = FA\sqrt{\frac{1}{3}}$, and $DF^3 = FA^3 \times \frac{1}{3}\sqrt{\frac{1}{3}} = FA^3 \times \frac{1}{9}\sqrt{3}$ (28).

Cor. *The inſcribed cube is thrice the inſcribed pyramid.*

PROP. XXX.

220. *The ſquare of the diameter of a ſphere is double to the ſquare of the ſide of an inſcribed regular octaedron* ABFDEG: $AG^2 = 2AB^2$.

Through two oppoſite angles A, G, draw the diameter AG; then the angle ABG is right (14. 4); therefore $AG^2 = AB^2 + BG^2 = 2AB^2$ (21. II).

Cor. 1. *The ſquare of the diameter of a circle comprehending a triangle of the octaedron, is* $\frac{2}{3}$ *the ſquare of the diameter of the ſphere.*

For AB^2 = thrice the ſquare of the radius (41. IV) = $\frac{3}{4}$ the ſquare of the diameter, and $AB^2 = \frac{1}{2}AG^2$ (30); therefore the diameter ſquare $= \frac{2}{3}AG^2$.

Cor. 2. *The diameter of a circle containing the triangle of the octaedron, is equal to the ſide of the pyramid.*

Cor. 3. *The ſame circle comprehends both the ſquare of the cube, and the triangle of an octaedron, inſcribed in the ſame ſphere.*

For

For the former diameter is $\frac{1}{3}\sqrt{6}$, and the latter $\sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{6}$. FIG. 220.

Cor. 4. *The area of one of the faces of the octaedron, as* ABE, *is* $\frac{\sqrt{3}}{8}$ *multiplied into the ſquare of the diameter of the ſphere*; $= \frac{\sqrt{3}}{8} \times AG^2$.

For the triangle ABE $= \frac{AB^2}{4}\sqrt{3}$ (39. II) $= \frac{AG}{8}\sqrt{3}$.

Cor. 5. *The radius of the inſcribed circle is* $\frac{1}{3}$ AB $\sqrt{6}$.

For it is the perpen. from C upon ABE, ſuppoſe it $= P$, $D =$ diameter of the circle encompaſſing ABE. Then $PP = \frac{1}{4}AG^2 - \frac{1}{4}DD$; and $4PP = AG^2 - D^2 = AG^2 - \frac{2}{3}AG^2$ (Cor. 1. 30) $= \frac{1}{3}AG^2 = \frac{2}{3}AB^2$, and $2P = AB\sqrt{\frac{2}{3}} = \frac{1}{3}AB\sqrt{6}$. and $P = \frac{1}{6}AB\sqrt{6}$.

PROP. XXXI.

The ſolidity of an octaedron BD, *inſcribed in a ſphere, is* $\frac{1}{6}$ *the cube of the diameter of the ſphere* AG.

For the body conſiſts of two pyramids BEDFA and BEDFG, ſtanding on the ſquare baſe BEDF, 220. Therefore the ſolidity $= DE^2 \times \overline{\tfrac{1}{3}AC + \tfrac{1}{3}CG} = \frac{1}{2}BD^2 \times \frac{AG}{3} = \frac{1}{6}AG^3$.

Cor. *A ſphere, is to the inſcribed octaedron; as the circumference of the ſphere, to its diameter.*

For the ſphere is $= \frac{2}{3}$ the circle ABGD $\times$ AG (11) $= \frac{2}{3}AG \times$ circumference ABGD $\times \frac{1}{4}AG$ (34. IV). Therefore ſphere : octaedron : : ABGD $\times \frac{1}{6}AG^2 : \frac{1}{6}AG^3$: : ABGD : AG.

FIG.

PROP. XXXII.

221. *The ſquare of the diameter of a ſphere, is to the ſquare of the ſide of its inſcribed regular dodecaedron* DA; *as* 6 *to* 3 — √5; *or as* 9 + 3√5, *to* 2.

Let A be a ſolid angle of the dodecaedron; AG, AI, AL, three pentagons forming the ∠ A. Draw the diagonals, BD, BF, DF. And on the plane BDF let fall the perp. AC, and draw DC. Then $DF^2 = 3DC^2$ (41. IV), and $DC^2 = \frac{1}{3}DF^2$, and $CA^2 = DA^2 - DC^2$ (Cor. 1. 21. II) $= DA^2 - \frac{1}{3}DF^2 = DA^2 - \frac{1}{3}DA^2 \times \frac{3 + \sqrt{5}}{2}$ (Cor. 3. 43. IV) $= \frac{3 - \sqrt{5}}{6}DA^2$, therefore $CA = \frac{\sqrt{3 - \sqrt{5}}}{\sqrt{6}}DA$. But $\frac{DA^2}{CA}$ = diameter of the ſphere (Cor. 17. IV), or the diameter $= \frac{DA^2 \times \sqrt{6}}{DA \times \sqrt{3 - \sqrt{5}}} = \frac{\sqrt{6}}{\sqrt{3 - \sqrt{5}}} \times DA = S$; and diameter ſquare, $SS = \frac{6DA^2}{3 - \sqrt{5}}$; and $DA^2 = \frac{3 - \sqrt{5}}{6}SS = \frac{2SS}{9 + 3\sqrt{5}}$.

Cor. 1. *The ſquare of the diameter of the ſphere, is to the ſquare of the diameter of the circle containing one face of the dodecaedron* AL; *as* 15 *to* 10 — 2√5.

Let S = diameter of the ſphere, R = radius of the circle circumſcribing the pentagon, then $AD^2 = \frac{3 - \sqrt{5}}{6}SS$ (32); and $RR = \frac{2AD^2}{5 - \sqrt{5}}$ (44. IV) $= \frac{2}{5 - \sqrt{5}} \times \frac{3 - \sqrt{5}}{6} SS = \frac{1}{3}SS \times \frac{3 - \sqrt{5}}{5 - \sqrt{5}} = \frac{1}{3}SS \times \frac{3 - \sqrt{5}}{5 - \sqrt{5}} \times \frac{5 + \sqrt{5}}{5 + \sqrt{5}} = \frac{1}{3}SS \times$

15 +

$\frac{15+3\sqrt{5}-5\sqrt{5}-5}{25-5} = \frac{1}{3}SS \times \frac{10-2\sqrt{5}}{20} =$ FIG. 221.
$\frac{5-\sqrt{5}}{30}SS$, and the ſquare of the diameter of that circle or $4RR = \frac{10-2\sqrt{5}}{15}SS$.

Cor. 2. *The area of one pentagon of the dodecaedron, is equal to* $\frac{5}{12}\sqrt{\frac{5-\sqrt{5}}{10}}$ *multiplied by the ſquare of the diameter of the ſphere.*

For let O be the center of the circle circumſcribing the pentagon AI; and OP ⊥ to FI. Then $OP^2 = \frac{3+\sqrt{5}}{8} \times RR$ (Cor. 1. 44. IV) $= \frac{3+\sqrt{5}}{8} \times \frac{5-\sqrt{5}}{30}SS$; and the area FOI $= \frac{1}{2}OP \times FI = \frac{SS}{2}\sqrt{\frac{3+\sqrt{5}}{8} \times \frac{5-\sqrt{5}}{30} \times \frac{3-\sqrt{5}}{6}} = \frac{SS}{2}\sqrt{\frac{4}{48} \times \frac{5-\sqrt{5}}{30}}$; and ſince there are 5 ſuch triangles in the pentagon, the pentagon $= \frac{5}{2}SS\sqrt{\frac{1}{12} \times \frac{5-\sqrt{5}}{30}} = \frac{5SS}{12}\sqrt{\frac{5-\sqrt{5}}{10}}$.

Cor. 3. *The ſide of the cube is equal to the diagonal* DF, *of the pentagon of a dodecaedron inſcribed in the ſame ſphere.*

For $DA^2 = \frac{3-\sqrt{5}}{6}SS$ (32), and $DF = \frac{1+\sqrt{5}}{2}DA$ (Cor. 3. 43. IV); and $DF^2 = \frac{6+2\sqrt{5}}{4}DA^2 = \frac{3+\sqrt{5}}{2}DA^2 = \frac{3+\sqrt{5}}{2} \times \frac{3-\sqrt{5}}{6}SS = \frac{9-5}{2 \times 6}SS = \frac{4}{2 \times 6}SS = \frac{1}{3}SS$. But the ſquare of the ſide of the inſcribed cube is alſo $= \frac{1}{3}SS$ (28).

FIG. 221. Therefore the diagonal in the pentagon = ſide of the cube.

PROP. XXXIII.

The cube of the diameter of a ſphere, is to the ſolidity of the inſcribed regular dodecaedron; as 1, *to* $\frac{5}{6}\sqrt{\frac{3+\sqrt{5}}{30}}$.

Let S = diameter of the ſphere, R = radius of the circle encompaſſing the pentagon, P = perpendicular from the center of the ſphere upon the pentagon, then $RR = \frac{5-\sqrt{5}}{30}SS$ (Cor. 1. 32). Then $PP = \frac{1}{4}SS - RR$ (Cor. 1. 21. II) $= \frac{15-10+2\sqrt{5}}{60}SS = \frac{5+\sqrt{5}}{60}SS$, and $P = S\sqrt{\frac{5+2\sqrt{5}}{60}}$, and the area of the pentagon $= \frac{5}{12}SS\sqrt{\frac{5-\sqrt{5}}{10}}$ (Cor. 2. 32). Therefore the pyramid whoſe baſe is the pentagon, and vertex at the center of the ſphere, is $= \frac{1}{3}S^3 \times \frac{5}{12}\sqrt{\frac{5+2\sqrt{5}}{60}} \times \frac{5-\sqrt{5}}{10}$ (18. VI) $= \frac{5}{36}S^3\sqrt{\frac{25+10\sqrt{5}-5\sqrt{5}-10}{600}} = \frac{5}{36}S^3\sqrt{\frac{15+5\sqrt{5}}{600}} = \frac{5}{36}S^3\sqrt{\frac{3+\sqrt{5}}{120}}$; but as there are 12 ſuch pyramids in the body, therefore the dodecaedron $= \frac{5}{3}S^3\sqrt{\frac{3+\sqrt{5}}{120}} = \frac{5}{6}S^3\sqrt{\frac{3+\sqrt{5}}{30}}$.

Cor. *The radius of the ſphere inſcribed in the dodecaedron, is* $DA\sqrt{\frac{25+11\sqrt{5}}{40}}$; DA *being the ſide of the dodecaedron.*

For

For that radius is $= P = S\sqrt{\frac{5+2\sqrt{5}}{60}} =$ FIG. 221. $DA\sqrt{\frac{5+2\sqrt{5}}{60}} \times \frac{9+3\sqrt{5}}{2}$ (32) $= DA\sqrt{\frac{75+33\sqrt{5}}{120}}$ $= DA\sqrt{\frac{25+11\sqrt{5}}{40}}$.

PROP. XXXIV.

222. *The ſquare of the diameter of a ſphere, is to the ſquare of the ſide of its inſcribed regular icoſihedron; as* 10 *to* $5-\sqrt{5}$; *or as* $5+\sqrt{5}$ *to* 2.

Let BDEFG be the pentagonal baſe of the ſolid angle A, made by 5 triangles of the icoſiedron; let AC be perp. to it, and draw DC. Then $DC^2 = \frac{2}{5-\sqrt{5}}DE^2$ (44. IV) $= \frac{2}{5-\sqrt{5}}AD^2 = AD^2 \times \frac{2}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}} = \frac{10+2\sqrt{5}}{25-5}AD^2 = \frac{5+\sqrt{5}}{10}AD^2$. And $AC^2 = AD^2 - DC^2$ (Cor. 1. 21. II) $= AD^2 - \frac{5+\sqrt{5}}{10}AD^2 = \frac{5-\sqrt{5}}{10}AD^2$, and $AC = AD\sqrt{\frac{5-\sqrt{5}}{10}}$. But the diameter of the ſphere $= \frac{AD^2}{AC} = \frac{AD^2}{AD\sqrt{\frac{5-\sqrt{5}}{10}}} = AD\sqrt{\frac{10}{5-\sqrt{5}}}$, and the ſquare of the diameter $= AD^2 \times \frac{10}{5-\sqrt{5}} = SS$; and $AD^2 = \frac{5-\sqrt{5}}{10}SS = \frac{2SS}{5+\sqrt{5}}$.

Cor. 1. *The diameter of the ſphere, is to the diameter of the circle comprehending five ſides of the icoſiedron; as* $\sqrt{5}$ *to* 2.

For if S = diameter of the ſphere, then $SS = AD^2 \times \frac{10}{5-\sqrt{5}}$, and $DC^2 = AD^2 \times \frac{2}{5-\sqrt{5}}$, and

FIG. 222. $4DC^2 = AD^2 \times \frac{8}{5-\sqrt{5}}$; therefore $SS : 4DC^2 :: 10 : 8 :: 5 : 4$. And $S : 2DC :: \sqrt{5} : 2$.

Cor. 2. *The ſquare of the diameter of the ſphere, is to the ſquare of the diameter of the circle containing one triangle of the icoſiedron; as* 15, *to* $10 - 2\sqrt{5}$.

For let R = radius of the circle circumſcribing the triangle ADB; then $AD^2 = 3RR$ (41. IV), and $AD^2 = \frac{5-\sqrt{5}}{10}SS$ (34); therefore $\frac{5-\sqrt{5}}{10}SS = 3RR$, and $\frac{5-\sqrt{5}}{30} \times SS = RR$, and $\frac{10-2\sqrt{5}}{15}SS = 4RR$.

Cor. 3. *The ſame circle comprehends both the pentagon of a dodecaedron, and the triangle of an icoſiedron, inſcribed in the ſame ſphere.*

Cor. 4. *The area of a triangle* ADB *of the icoſiedron, is equal to* $\frac{5\sqrt{3} - \sqrt{15}}{40}$ × *ſquare of the diameter of the ſphere.*

For the area $= \frac{DA^2}{4}\sqrt{3}$ (39. II) $= \frac{SS}{4} \times \frac{5-\sqrt{5}}{10}\sqrt{3}$ (34) $= SS \times \frac{5-\sqrt{5}}{40}\sqrt{3} = SS \times \frac{5\sqrt{3} - \sqrt{15}}{40}$.

PROP. XXXV.

222. *The cube of the diameter of a ſphere, is to the ſolidity of the inſcribed regular icoſihedron; as* 6 *to* $\sqrt{\frac{5+\sqrt{5}}{2}}$.

Let P = the perpendicular from the center of the ſphere, upon the triangle ADB of the icoſiedron. R = radius of the circle encompaſſing the triangle. Then $RR = \frac{5-\sqrt{5}}{30}SS$ (Cor. 1. 34).

Then

FIG. 222.

Then $PP = \frac{1}{4}SS - RR = \frac{1}{4}SS - \frac{5-\sqrt{5}}{30}SS = \frac{5+2\sqrt{5}}{60} \times SS$, and $P = S\sqrt{\frac{5+2\sqrt{5}}{60}}$. And area of the triangle $ADB = \frac{5-\sqrt{5}}{40}\sqrt{3} \times SS$ (Cor. 4. 34). Therefore the pyramid whoſe baſe is ADB, and vertex at the center of the ſphere is $= \frac{1}{3}P \times ADB$ (18. VI) $= \frac{SS\sqrt{3}}{3} \times \frac{5-\sqrt{5}}{40} \times S\sqrt{\frac{5+2\sqrt{5}}{60}}$ (dividing by $\sqrt{3}$) $= \frac{S^3}{3 \times 40} \times \overline{5-\sqrt{5}} \times \sqrt{\frac{5+2\sqrt{5}}{20}}$ (ſquaring $5-\sqrt{5}$) $= \frac{S^3}{3\times40} \times \sqrt{\overline{30-10\sqrt{5}} \times \frac{5+2\sqrt{5}}{20}}$ $= \frac{S^3}{3 \times 40}\sqrt{\frac{50+10\sqrt{5}}{20}} = \frac{S^3}{3 \times 40}\sqrt{\frac{5+\sqrt{5}}{2}}$. And 20 ſuch pyramids, or the icoſiedron $= \frac{S^3}{6}\sqrt{\frac{5+\sqrt{5}}{2}}$.

Cor. *The radius of the ſphere inſcribed in the icoſihedron, is* $DA\sqrt{\frac{7+3\sqrt{5}}{24}}$, DA *being the ſide of the icoſihedron.*

For that radius is $= P = S\sqrt{\frac{5+2\sqrt{5}}{60}} = DA\sqrt{\frac{5+2\sqrt{5}}{60} \times \frac{5+\sqrt{5}}{2}}$ (34) $= DA\sqrt{\frac{35+15\sqrt{5}}{120}}$ $= DA\sqrt{\frac{7+3\sqrt{5}}{24}}$.

SCHOLIUM.

A ſphere may be inſcribed or circumſcribed to any regular body, or to any triangular pyramid.

BOOK VIII.

The conſtruction of geometrical problems.

DEFINITION.

FIG. A Problem is ſaid to be *conſtructed geometrically*, when it is done by the help only of a ſtraight ruler, and a pair of compaſſes.

PROB. I.

223. *To draw a ſtraight line from one point* A, *to another* B, *upon a plane.*

Set one foot of the compaſſes in the point A, and apply the edge of one end of the ruler to it; keep it cloſe there, whilſt you turn the other end of the ruler about, till the edge of it fall upon the other point B; then draw a line by the edge of the ruler, which will go from one point to the other.

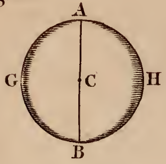
PROB. II.

224. *To produce a line* AB, *that is too ſhort.*

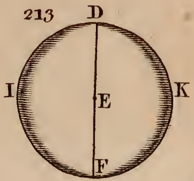
Lay the edge of one end of the ruler againſt the foot of the compaſſes, placed at one end of the line A; and turn the other end about it, till the edge fall upon the other end of the line B. Then through B, draw a line by the edge of the ruler, from B to F.

Other-

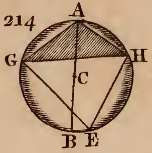
Fig. 212.



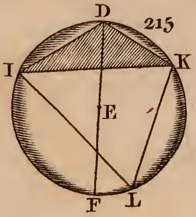
213



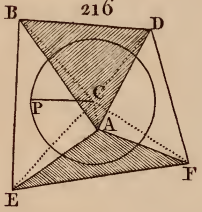
214



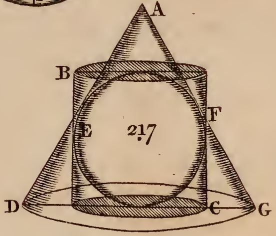
215



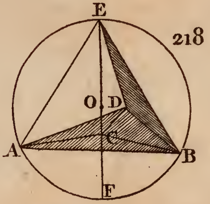
216



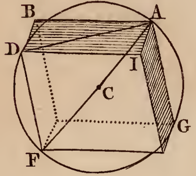
217



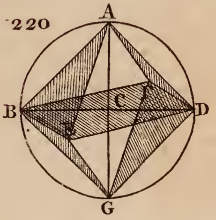
218



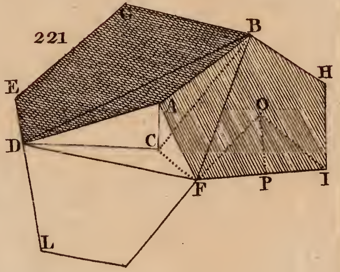
219



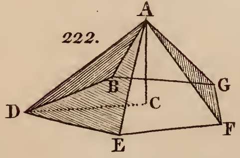
220



221



222.



Otherwiſe.

FIG.

Place one foot of the compaſſes in the end A, and through the other end, draw the obſcure arch CBD, with the other foot of the compaſſes, opened to the diſtance AB. In that arch take BC equal to BD; then with any opening of the compaſſes, ſeting one foot in C and D, deſcribe two obſcure arches to interſect in E; then draw BEF. 224.

For if the lines AC, AD, CE, DE, CD be ſuppoſed to be drawn; the line CD will be ⊥ to AB, and to BF (Cor. 3. 3. II); and ABF, a right line (1. I).

PROB. III.

From a given point C, *to draw a line equal to a given line* AB. 225.

Draw the line CD, ſufficiently long; then take the extent AB in your compaſſes, and ſeting one foot in C, ſtrike the obſcure arch, F. Then CF = AB.

PROB. IV.

To find the ſum and difference of two given lines AB, BD. 226.

Draw any line DA ſufficiently long, then take the ſhorter line AB in your compaſſes, and ſeting one foot in B, deſcribe two arches to cut AD in A and F; then will DA = BD + BA, and DF = BD — BA.

PROB. V.

To divide a given angle ACB *into two equal parts.* 227.

From the angular point C deſcribe any arch AB, to cut CA, CB; then with any extent, ſeting one foot in A and B, deſcribe two obſcure arches, to cut

FIG. 227. cut each other in D; then draw CD; and ∠ACD = DCB.

For if AD, BD be ſuppoſed to be drawn; the ∠DCA = DCB (8. II).

PROB. VI.

228. *To divide a given right line* AB *into two equal parts.*

From the ends A, B, with the ſame extent, deſcribe two arches, to cut one another in C, and D. Draw CD to cut AB in I. Then AI = IB.

For if AC, AD, BC, BD be ſuppoſed to be drawn, ACBD will be a rhombus; and AI = IB (2. III).

PROB. VII.

229. *To make an angle* B, *equal to a given angle* A.

Upon the angular point A as a center deſcribe an arch FG. Draw any line BC, and from B with the ſame extent as before, deſcribe the arch CD. Make the arch CD = FG, and draw BD. Then ∠CBD = ∠FAG.

PROB. VIII.

230. *Through a given point* A, *to draw a line* AB *parallel to another* CD.

Take the neareſt diſtance of the point A from CD; and ſeting one foot in ſome point of the line CD, deſcribe an occult arch O. Then through A draw a line AB to touch the arch O; which will be ∥ to CD.

Otherwiſe.

231. From ſome point O in the line CD as a center, with the diſtance OA, deſcribe a ſemicircle CABD paſſing through A; then make the arch DB

DB = arch CA; and draw AB, which will be ∥ to CD (Cor. 13. IV). FIG. 231.

Or thus.

With any extent, and one foot in A, deſcribe an arch to cut CD in ſome point C. And with the ſame extent, and one foot in ſome point as D, in the line CD, deſcribe an arch B to cut AB. Then with the extent CD, and one foot in A, croſs the laſt arch in B; then draw AB, which is parallel to CD (1. III). 232.

Or thus.

From a point D taken at pleaſure in the line DC, deſcribe through A, the arch AC; and from A, with the ſame extent, the arch DB. Make DB = AC. And draw AB, which will be ∥ to DC (Cor. 2. 4. I). 233.

PROB. IX.

From a given point P *in a right line* AB, *to raiſe a perpendicular.* 234.

Make PC equal to PB, and from C and B, with a convenient extent, deſcribe two arches to cut each other at D; draw DP, which will be ⊥ to CB (8. II).

Or thus.

With any diſtance PF, and one foot in P, deſcribe the circle FCD, and ſet FP from F to C, and from C to D; from the points C, D, with any extent, deſcribe two arches to interſect at O, then draw OP, which is ⊥ to AB. 235.

For FC is the third part of a ſemicircle (45. IV), and CD is biſected by OP (Cor. 3. 3. II), and alſo the arch CD (Cor. 2. 2. IV), and therefore ∠FPO = OPB = a right angle.

PROB.

FIG.

PROB. X.

236. *To raiſe a perpendicular on the end* A, *of a line given*, AB.

Set one foot in A, and extend the other to any point C out of the line AB. From C as a center deſcribe the ſemicircle PAF, to cut AB in F. Through F and C draw FCP, to cut the ſemicircle in P. Then draw PA, which will be $\perp$ to AB (14. IV).

Otherwiſe.

237. From the center A, at any diſtance AF, deſcribe the arch FG; ſet AF from F to G. And from G with the ſame extent deſcribe an arch P. Through F and G, draw the line FGP, to cut the arch in P. Then draw PA, which is perpendicular to AB.

For if AG be drawn, $\angle$FAG $= \frac{2}{3}$ of a right angle (Cor. 2. 3. II) $=$ AGF $=$ 2GAP (1. II). Therefore GAP $= \frac{1}{3}$ a right angle; and the whole FAG $+$ GAP $= \frac{2}{3} + \frac{1}{3} =$ 1 whole right angle.

Or thus.

238. Take any length in your compaſſes, as AC; and ſet it 5 times along the line AB, to C, E, D, I, K; take 3 parts AD in your compaſſes, and with one foot in A deſcribe an arch P; then with extent AK (or 5 parts), and one foot in I, croſs the arch P; then from the point of interſection P to A draw PA, which is $\perp$ to AB (Cor. 3. 21. II).

It will be the ſame thing, if you ſet AI from A to P, and AK from D to P.

PROB. XI.

FIG.

From a given point P, *to let fall a perpendicular upon a given line* AB. 239.

From the center P deſcribe an arch to cut AB in E and F. From E and F, with a proper diſtance, deſcribe two obſcure arches to interſect in I, then through P and I, draw PC; which is perp. to AB (Cor. 3. 3. II).

Or thus.

From a point A in the line AB, with diſtance AP, deſcribe the arch PI; likewiſe from another point D, in AB, with diſtance DP, deſcribe the arch PI to cut the former in I. Draw PCI, and PC is ⊥ to AB. 240.

For if AP, AI be drawn, then PC = CI, and AC ⊥ PI (Cor. 3. 3. II. and 8. II).

PROB. XII.

To divide the given line AB *into any number of equal parts.* 241.

Draw any indefinite line AP, on which ſet the equal parts AL, LM, MN, NP. Draw PB, and through L, M, N, draw LD, ME, NF ∥ to PB. Then AD = DE = EF = FB (12. II).

Otherwiſe.

From the ends A, B, of the given line, draw two lines AP, BK as long as you will, parallel to one another. Then ſet any equal parts from A towards P, and likewiſe from B towards K. Then draw lines between the correſpondent points, NG, MH, LI, which will divide AB into the equal parts AD, DE, EF, FB (12. II). 242.

FIG.

Or thus.

243. Let AB be given to be divided; draw CP ∥ to AB. Set any equal parts, from C to L, L to M, M to N, and from N to P. Draw CA and PB to interſect in G; and draw GL, GM, GN, to cut AB in D, E, F. Then AD, DE, EF, FB are all equal (Cor. 13. II).

PROB. XIII.

244. *To divide a given line* AB, *in proportion as another line* AC *is divided in* D *and* E.

Let AB and AC be joined at A, making the angle BAC; draw CB; and through D, E, draw DF, EG ∥ to CB. Then will AF : AD : : FG : DE : : GB : EC (Cor. 2. 12. II).

PROB. XIV.

245. *To find a third proportional to two given lines*, AB, AD.

Join AB, AD at A, ſo as to make an angle BAD. Produce AD, and make AC = AD, and draw CE ∥ to BD; then AE is the third proportional. For AB : AD : : AC or AD : AE (13. II).

PROB. XV.

246. *To find a fourth proportional to three given lines*, AB, AC, AD.

Let AB, AC make any angle at A, apply the third line from A to D. Draw BC, and DE ∥ to BC; then AE is the fourth proportional. For AB : AC : : AD : AE (13. II).

PROB.

PROB. XVI.

FIG.

To find a mean proportional between two given lines AB, BD. 247.

Let AD be the ſum of the two lines AB, BD (4); biſect AD in C. With center C, and radius CA or CD, deſcribe the ſemicircle AED. From B erect BE $\perp$ to AD, to cut the circle in E; then BE is the mean proportional ſought.

For $AB : BE :: BE : BD$ (17. IV).

Or thus.

Let BA be the greater, biſect it in C, and from the center C, with radius CA or CB, deſcribe the ſemicircle BEA. Let BD be the leſſer given line. Erect DE $\perp$ to BA (9), to cut the circle in E, draw BE, which is a mean between BD and BA (Cor. 17. IV). 248.

PROB. XVII.

To divide the given line AB *in extreme and mean proportion.* 249.

Draw EAF $\perp$ to AB, and make $AE = \frac{1}{2}AB$, and draw EB, and make $EF = EB$, and $AG = AF$. And G is the point of diviſion.

For $AF = EF - EA$ (Conſt.), that is, $AG = EB - EA$, and $AG + AE = EB$ (Ax. 3), that is, $AG + \frac{1}{2}AB = EB$; and $AG^2 + AG \times AB + \frac{1}{4}AB^2 = EB^2$ (10. I), and $AG^2 = EB^2 - AG \times AB - EA^2$ (becauſe $\frac{1}{4}AB^2 = EA^2$) $= AB^2 - AB \times AG$ (Cor. 21. II) $= AB \times \overline{AB - AG} = AB \times BG$, therefore AB is cut in G, in extreme and mean proportion (Def. 11. Proportion).

FIG. 249. *Cor.* $AG = AB \times \frac{\sqrt{5} - 1}{2}$, *and* $BG = AB \times \frac{3 - \sqrt{5}}{2}$.

For EB or $EF = \sqrt{\frac{5}{4}AB^2} = \frac{AB}{2}\sqrt{5}$, and AF or $AG = EF - \frac{1}{2}AB = AB \times \frac{\sqrt{5} - 1}{2}$.

Alſo $BG = AB - AG = AB \times \frac{2 - \sqrt{5} + 1}{2} = AB \times \frac{3 - \sqrt{5}}{2}$.

PROB. XVIII.

250. *In any triangle* ABC, *to draw a perpendicular from any angle* A *to its oppoſite ſide* CB.

About either of the other ſides AB, deſcribe a ſemicircle ADB, to cut the ſide CB in D. Draw AD, which will be $\perp$ to CB (14. IV).

PROB. XIX.

251. *Upon a given line* AB, *to make an equilateral triangle.*

Take AB in your compaſſes, and with one foot in A and B, deſcribe two arches to croſs each other at C. Draw AC, BC; and ABC is the triangle.

PROB. XX.

252. *To make a triangle of three given lines* A, B, C; *of which any two muſt be greater than the third.*

Draw DE = the line A; then take B in your compaſſes, and with one foot in E deſcribe an occult arch F. Then take C in your compaſſes, and with one foot in D, croſs the former arch at F; draw DF, EF; and DEF is the triangle required.

Cor.

Cor. *After the ſame manner, a triangle is made* FIG.
equal to a given triangle. 252.

PROB. XXI.

To make an iſoſceles triangle ABD, *whoſe ſide is the* 253.
given line AB; *and angle at the baſe* B *or* D, *double*
to that at the top A.

Let AC be the greater part of the line AB divided in extreme and mean ratio (17). From the center A through B, deſcribe the circle BD; and with extent CA, and one foot in B, croſs the circle in D; and draw AD. Then ABD is the triangle ſought.

For draw CD; then ſince AB : AC :: AC : CB (Def. 11. Proportion), that is, AB : BD :: BD : BC; therefore the triangles ABD, BDC are ſimilar (14. II), and $BD = DC = CA$. Whence the $\angle B$ or $BCD = \angle A + CDA$ (1. II) $= 2\angle A$ (3. II).

Cor. *The angle* A *is* $\frac{2}{5}$ *of a right angle.*

PROB. XXII.

A triangle ABC *being given; to reduce it to ano-* 254.
ther of a different baſe, AED.

Let AE be the baſe propoſed, being in the ſame line AB. Draw the line CE, from the top of the given triangle, to the point E propoſed. And through $\angle B$ of the given triangle, draw $BD \parallel$ to CE; draw the line DE. Then the triangle $ADE = ACB$.

For triangle $DBE =$ triangle DBC (10. II). Add ADB, then $ADB + BDE$ or $ADE = ADB + BDC$ or ABC.

Cor. *Thus a triangle may be reduced to another of a different hight.*

FIG.

PROB. XXIII.

255. *To divide a triangle* ABC, *in any proportion, by a line drawn from an angle* A.

Divide the baſe, or oppoſite ſide BC, in D, in the proportion given (13); to D, draw the line AD; which divides the triangle ABC, in the ſame given ratio (11. II).

PROB. XXIV.

256. *To reduce a polygon* ABCDE *to fewer ſides.*

To take away the angle B; produce the next ſide EA, then draw the diagonal CA, and from B, draw BG ∥ to CA, to cut EA in G; and draw CG, which takes in the triangle CAG, inſtead of its equal CAB (10. II). So the polygon becomes CGED.

Cor. *By thus taking away one angle after another; any polygon may, at laſt, be reduced to a triangle.*

PROB. XXV.

257. *Upon a given line* A, *to make a ſquare.*

Draw BC = A, take A or BC in your compaſſes, and with one foot in C, deſcribe the arch BED; and with one foot in B, the arch CEF. Set the ſame extent from the interſection E to F; draw CF to cut BE in G; make ED and EI = EG; and draw BI, ID, DC, and BIDC is the ſquare required.

For if CE, BE, BF be drawn, ∠BCE = $\frac{2}{3}$ a right angle (Cor. 2. 3. II) = CBE = EBF, and ∠ECF = $\frac{1}{3}$ a right one (12. IV), therefore ECD = $\frac{1}{4}$ a right angle, and BCD = a right angle.

Or

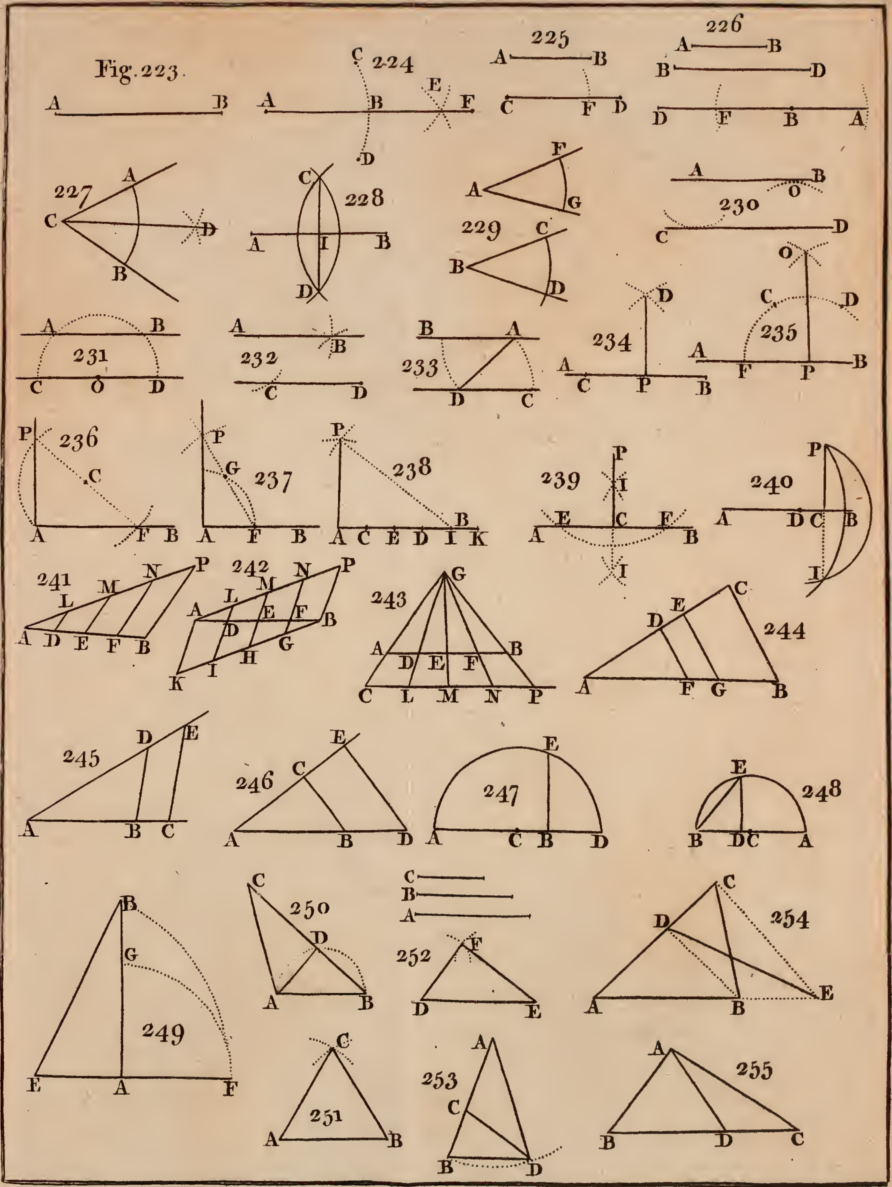


Fig. 223. A B
224 C A B E F D
225 A B C F D
226 A B B D D F B A
227 A C D B
228 C A I B D
229 F A G C B D
230 A B O C D
231 A B C O D
232 A B C D
233 B A D C
234 D A C P B
235 O C D A F P B
236 P C A F B
237 P G A F B
238 P B A C E D I K
239 P I E C F A B I
240 P A D C B I
241 N P L M A D E F B
242 N P L M A E F B D G H I K
243 G A B D E F C L M N P
244 C E D A F G B
245 D E A B C
246 E C A B D
247 E A C B D
248 E B D C A
249 B G E A F
250 C D A B
C B A
252 F D E
254 C D A B E
251 C A B
253 A C B D
255 A B D C

Pl. XII. *Pa.* 166.

Or thus. FIG.

Make BC = A, draw CD ⊥ and = CB (9); 257. with extent BC, and one foot in B, deſcribe an arch I; with the ſame extent and one foot in D, croſs the arch at I; draw BI, ID; then BIDC is the ſquare.

PROB. XXVI.

With two given lines A, B, *to make a rectangle.* 258.

Make the baſe CD = B, draw CF perp. to CD (9), and = A; with the extent B, and one foot in F, deſcribe an arch E; and with extent A, and one foot in D, croſs the arch at E; draw FE, ED; and CFED is the rectangle ſought.

PROB. XXVII.

To make a ſquare equal to a given rectangle ABCD. 259.

Produce the ſides AD, CD, and make DF = DC; biſect AF in I, and with radius IA or IF, deſcribe the ſemicircle AEF to cut CE in E. On DE make the ſquare DH, which will be equal to the rectangle AC or AD × DF (17. IV).

PROB. XXVIII.

To make a parallelogram equal to a triangle given 260. ABC; *and having an angle, equal to a given angle* D.

Through A draw AG ‖ to BC, and make the ∠BCG = D; biſect BC in E, and draw EF ‖ to CG; then the parallelogram EG = triangle ABC (7. III).

FIG.

PROB. XXIX.

261. *Upon a given right line* A, *to make a parallelogram equal to a given triangle* B; *having an angle, equal to a given one* C.

Make a parallelogram GE = triangle B (28), whoſe angle G = C; produce DG, EF, DE, GF; and make FH = A; through H, draw IL ∥ to EF, to meet DE in I; draw IFK, to cut DG in K; through K draw KL ∥ to GH, meeting EF and IH in M and L. Then the parallelogram MH = B.

For parallelogram MH = GE (4. III) = B (Conſtr.).

Or thus.

262. Let B be the given triangle; produce the baſe, and draw EG, parallel thereto; make the ∠DCG = C, and CI = baſe of the triangle B. Then triangle CGI = B (10. II); make CD = A, and make triangle CKD = CGI (22); biſect CK in H, draw HL, DL ∥ to CD, CH; then CL is the parallelogram ſought.

For CHLD = triangle CKD (7. III) = CGI (Conſtr.) = B.

PROB. XXX.

263. *Upon a given right line* FG, *to make a parallelogram equal to a given polygon* BACD, *having an angle equal to a given one* E.

Divide the polygon into triangles CAB, CBD. Make the angle GFK = E; and make the parallelogram GI = triangle CAB, and HK = CBD (29). Then parallelogram FL = CABD.

Cor.

Cor. 1. *Hence a ſquare may be made equal to any* FIG.
given polygon; by making a rectangle equal to the polygon, and then a ſquare equal to the rectangle (27). 263.

Cor. 2. *Thus a parallelogram may be made equal to the ſum or difference of two given polygons.*

PROB. XXXI.

To make a ſquare equal to the ſum of two ſquares. 264.

Make FBD a right angle; make BA = ſide of one given ſquare; BC = ſide of the other ſquare, draw AC; then the ſquare made on AC, is equal to the ſum of the ſquares made upon AB, and BC (21. II).

Cor. *After the ſame manner a ſquare may be found equal to three or more ſquares. For draw* OC ⊥ *to* AC, *and equal to the ſide of a third ſquare, and draw* AO. *Then* $AO^2 = AC^2 + CO^2 = AB^2 + BC^2 + CO$ (21. II); *and ſo on.*

PROB. XXXII.

To make a ſquare equal to the difference of two ſquares. 264.

Make the right angle FBD; ſet the ſide of the leſſer ſquare from B to A; take the ſide of the greater in your compaſſes, and ſeting one foot in A, with the other croſs the line BD, in C. Then CB is the ſide of the ſquare equal to the difference of the ſquares (Cor. 21. II).

PROB. XXXIII.

To find the proportion of one polygon A *to another* B. 265.

Find two ſquares equal to the two polygons A, B (Cor. 1. 30); let CF be the ſide of the

FIG. 265. firſt, and draw FE $\perp$ to it, and equal to the ſide of the ſecond. Draw the hypothenuſe CE; from F, let fall the perpendicular FD upon it: then CD : DE : : polygon A : polygon B.

For CD : DE : : CF^2 : FE^2 (Cor. 4. 20. II) : : A : B (Conſtr.).

PROB. XXXIV.

266. *To make a triangle equal and ſimilar to a given triangle* ABC.

Draw any line DE, and make DE = AB; then with extent AC, and one foot in D, deſcribe an occult arch F. And with extent BC, and one foot in E, croſs the arch at F; draw DF, EF; and DEF is the triangle required (8. II).

Or thus.

Make the $\angle$EDF = BAC, and DE = AB, and DF = AC, and draw EF. And DEF is the triangle ſought (6. II).

PROB. XXXV.

267. *To make a plane figure equal and ſimilar to another* ABCDEF.

In any line AF, take two marks or points M, N. Alſo in the line *af*, take *mn* = MN. With the diſtances from M to B, C, D, &c, and one foot in *m*, deſcribe as many arches; then with the diſtances from N to B, C, D, &c, and one foot in *n*, croſs them in *b*, *c*, *d*, *e*, &c. make *ma* = MA, *nf* = NF; and draw the lines *ab*, *bc*, *cd*, *de*, *ef*, in like manner as the correſpondent lines are drawn in the other figure; and it is done.

Or thus. FIG.

Let the given figure ABCDE be divided into 268.
the triangles BAC, CAD, DAE. Then make triangle GFH = BAC, HFI = CAD, and IFK = DAE (34). And the polygon GK will be equal and ſimilar to BE.

PROB. XXXVI.

To make a polygon ſimilar to another ABCDE, 269.
and in the given ratio of AF *to* AB.

Find AG a mean proportional between AF and AB. Draw the diagonals AC, AD. Then from G, draw GH, HI, IK parallel to BC, CD, DE. And AGHIK is the polygon.

For the correſpondent triangles in both being ſimilar, the polygons are ſimilar (Cor. 2. 19. III). Alſo AF : AG : : AG : AB (Conſtr.), and AF : AB : : AG^2 : AB^2 (23. Proportion) : : polygon GI : polygon BD (20. III).

Otherwiſe thus.

Make PQ = AG; alſo make the angles QPR, 269.
RPS, SPT, reſpectively equal to BAC, CAD, DAE. 270.
And make the angles Q, R, S, T ſucceſſively = B, C, D, E. And the polygon PQRST is that ſought.

For each of the triangles in one figure is ſimilar to each in the other (Def. 10. II); and therefore the polygons are ſimilar (Cor. 1. 19. III).

Cor. *And thus a polygon is made upon a given line* AG *or* PQ, *ſimilar to another polygon* ABCDE.

PROB.

FIG.

PROB. XXXVII.

270. *To make a polygon* TQ *equal to a given one* F, *and*
271. *ſimilar to another* ACDEB.
272.

Upon BA make the rectangle BM = BACDE (30); and upon BH make the rectangle BI = F (30). Take PQ a mean proportional between BA and BR (16); and upon PQ, make the polygon PQRST ſimilar to BACDE (Cor. 36); and TQ is the polygon ſought = F.

For polygon BD : polygon PS : : BA^2 : PQ^2 (20. III) : : BA : BR (23. Proportion) : : BM or polygon BD : BI or polygon F (8. III). Therefore polygon PS = F (Ax. 7. Proportion).

Cor. *If the polygon* TQ *was to be to* F, *in the given ratio of* R *to* S; *it is done the ſame way; only the parallelogram* BI *muſt be made* $= \frac{R}{S} \times F$.

PROB. XXXVIII.

To find the center of a circle ADF.

273. Draw any line AD, and biſect it in B; through B draw GBF ⊥ to AD. Biſect GF in C, for the center (Cor. 1. 2. IV).

Cor. *By the ſame rule, the arch* AGD *is biſected* (Cor. 1. 2. IV).

PROB. XXXIX.

274. *Through three given points* A, B, F, *to deſcribe a circle.*

Draw AB, BF, and biſect them in D, E. Thro' D and E, draw the perpendiculars DC, EC, to meet in C. C is the center (Cor. 1. 2. IV).

Cor.

FIG. Cor. *If an arch of a circle be given; the center may be found, by taking three points in that arch. And then the circle may be compleated.* 274.

PROB. XL.

To draw a tangent to a circle from a given point A. 275.

From the point A to the center C, draw the line AC, biſect AC in D. With the radius DA or DC, deſcribe a ſemicircle to cut the circle in B. Draw AB, which will touch the circle in B (10 and 14. IV).

PROB. XLI.

Upon a right line AB, *to deſcribe the ſegment of a circle, which ſhall contain an angle* AIB, *equal to a given angle* C. 276.

Make the angle BAD = C. Through A draw AE ⊥ to AD. At the other end B, make the ∠ ABF = BAF, to cut AE in F. From the center F, with radius FA or FB, deſcribe the ſegment of a circle AIEB. Then ∠ AIB = C.

For AF = FB (Cor. 1. 3. II); and ∠ AIB = BAD (16. IV) = C.

Or thus.

Cut out a piece of wood, *&c.* with two ſtraight ſides, making an angle equal to C. And placing it between the fixt points A, B; move the angular point about, while the ſides move cloſe by the points A, B; then the angular point will deſcribe the arch AIEB.

Cor. *In the ſame manner a ſegment is cut off from a circle, to contain a given angle; by drawing the tangent* AD *at* A, *and making the angle* BAD = C. *Then* AIEB *is the ſegment.*

PROB.

FIG.

PROB. XLII.

277. *In a circle* AEC, *to inſcribe a triangle ſimilar to a triangle given*, DFG.

Draw LK to touch the circle at A; make ∠KAC = F, and ∠LAB = G. Draw BC, and the triangle BAC is ſimilar to FDG (16. IV).

PROB. XLIII.

278. *In a given triangle* ABC, *to inſcribe a circle.*

Biſect two angles B, C, with the lines BD, CD, meeting in D. Let fall DF ⊥ to BC. With radius DF, and center D, deſcribe the circle FEG, which will touch all the ſides of the triangle ABC (Cor. 1. 35. II).

PROB. XLIV.

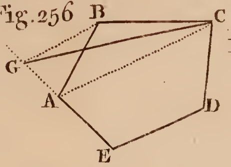
279. *About a given circle* ABC, *to deſcribe a triangle ſimilar to a triangle given*, DEF.

Produce the ſide EF both ways, to G and H. At the center I, make ∠AIB = GED, and BIC = DFH. Then to the points B, A, C, draw three tangents to the circle, to interſect in the points L, M, N. Then the triangle LMN, is ſimilar to EFD.

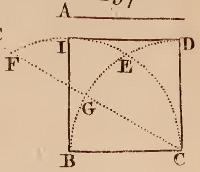
For ſince the ∠s at A, B, C are right angles; ∠L + AIB = 2 right angles (Cor. 16. III) = GED + DEF, and taking away the equal angles AIB, and GED; then ∠L = DEF. For the ſame reaſon M = DFE, conſequently N = D.

PROB.

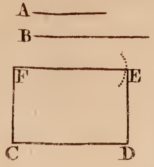
Fig. 256



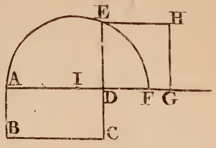
257



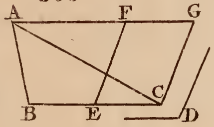
258



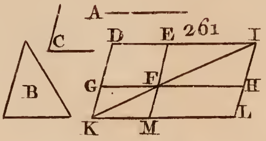
259



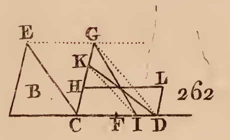
260



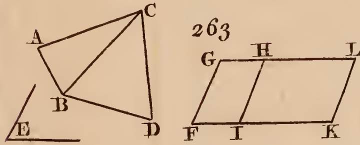
261



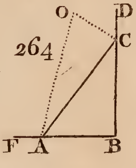
262



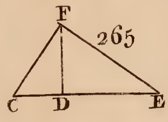
263



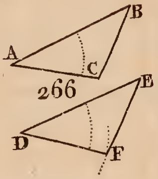
264



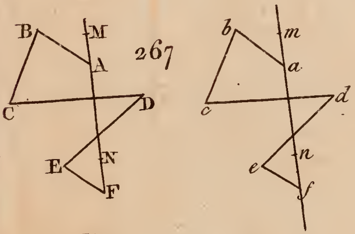
265



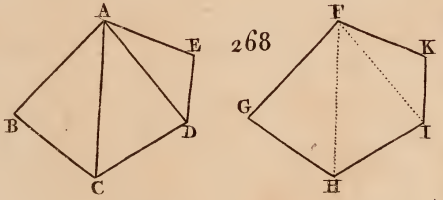
266



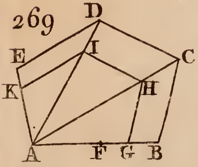
267



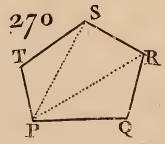
268



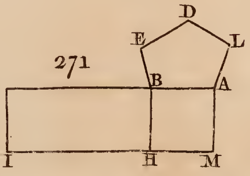
269



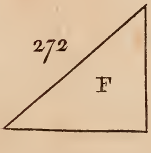
270



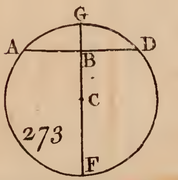
271



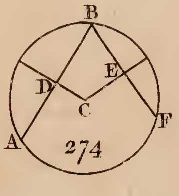
272



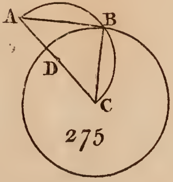
273



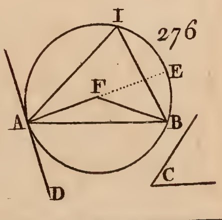
274



275



276



F I G.

277. *In a circle*
a triangle give

Draw LK
∠KAC = F,
the triangle

278. *In a give*

Biſect two
meeting in
radius DF, an
which will tou
(Cor. 1. 35.

279. *About a gi*
ſimilar to a tri

Produce th
At the center
= DFH. T
three tangent
points L, M,
ſimilar to EF
For ſince t
∠L + AIB
GED + DEF,
AIB, and GE
ſame reaſon M

PROB. XLV.

FIG. 280.

About a triangle ABC, *to deſcribe a circle.*

Biſect any two ſides, AB, BC, in D and E. Raiſe the perpendiculars DF, EF, to interſect in F. From F as a center deſcribe a circle through B, which will paſs through A, C (Cor. 32. II).

Cor. *In an acute-angled triangle, the center is within the triangle; in an obtuſe one, without* (Cor. 1. 14. IV).

PROB. XLVI.

In a given circle FCD, *to inſcribe an equilateral triangle.* 281.

Draw the diameter FB. With the radius BA and center B, deſcribe two arches C, D, to cut the circle in C and D. Draw the lines CD, DF, FC. And CFD is an equilateral triangle.

For arch CB or BD $= \frac{1}{6}$ the circumference (45. IV); therefore CBD $= \frac{1}{3} =$ CF $=$ FD.

PROB. XLVII.

In a given circle ABCD, *to inſcribe a ſquare, or regular octagon.* 282.

Draw the diameters AC, BD at right angles to one another, cuting the circle in A, B, C, D. Draw the lines AB, BC, CD, DA; and ABCD is a ſquare (Cor. 2. 6. IV).

If the diameters FG, HI be drawn, biſecting the arches AFB, AHD, DGC, CIB. Then AF, or FB, &c. will be the ſide of the octagon.

Cor. *If* AF, FB, &c. *be biſected, a polygon of* 16 *ſides, will be inſcribed; and ſo on.*

FIG.

PROB. XLVIII.

283. *In a given circle* ADBG, *to inſcribe a regular pentagon, or decagon.*

Draw the diameter AB; from the center C draw CD $\perp$ to AB; biſect CB in E; and make EF = ED, and draw DF, which will be the ſide of the pentagon; therefore if DH, HG, *&c.* be made = DF, DHGIK will be the pentagon required. Alſo FC is the ſide of the decagon; therefore if DL, LK, *&c.* be made = CF; a regular decagon will be inſcribed.

For $DF^2 = DE^2 + EF^2 - 2FE \times EC$ (23. II) $= 2DE^2 - 2DE \times EC = 2DE^2 - DE \times DC$. But $DE^2 = DC^2 + CE^2$ (21. II) $= \frac{5}{4}DC^2$, and $DE = \frac{1}{2}DC\sqrt{5}$. Therefore $DF^2 = \frac{5}{2}DC^2 - \frac{DC^2}{2}\sqrt{5} = DC^2 \times \frac{5-\sqrt{5}}{2}$. Therefore DF is the ſide of a pentagon (44. IV). And FC is the ſide of a decagon (48. IV).

PROB. XLIX.

284. *In a given circle* ACE, *to inſcribe a regular hexagon.*

Make AB, BC, CD, DE, EF and FA, all equal to the radius AG: and drawing the lines, the figure ABCDEF is a hexagon (45. IV).

Cor. *If the arch* AB *be biſected, you will have the ſide of a regular dodecagon.*

PROB.

PROB. L.

FIG. 285.

About a given circle ABC, *to deſcribe a regular polygon.*

Either inſcribe a polygon of the ſame ſort, or divide the circle into ſo many equal parts AB, BC, *&c.* as the polygon has ſides. To the points of diviſion, draw the radii GA, GB, GC, *&c.* To theſe lines at A, B, C, *&c.* draw tangents to the circle, KD, DE, EF, *&c.* to interſect in D, E, F, *&c.* then DEFHIK is the polygon required.

For if GD, GK be ſuppoſed to be drawn, AK = AD, and ∠K = ∠D (7. II). Alſo DA = DB (Cor. 4. 21. IV), whence KD = DE = EF, *&c.*

PROB. LI.

To inſcribe a circle in any regular polygon; or deſcribe a circle about it. 285.

Biſect any two adjoining angles D, K, with the lines DG, KG, and they will meet in the center G. Or biſect any two adjoining ſides, DK, DE, with the perpendiculars AG, BG, which will meet in the center G. Take GA the neareſt diſtance to any ſide, and from G deſcribe the circle ABC, which will touch all the ſides of the polygon DH.

Likewiſe biſect any two angles A, B, with the lines AG, BG, which will meet in the center G. Or biſect any two ſides CD, DE, with two perpendiculars meeting in G, the center. Then from A with diſtance GA deſcribe a circle ABCE, which will paſs through all the angles of the figure. 284.

Cor. *A circle may be inſcribed, or circumſcribed, to any regular polygon.*

FIG.

PROB. LII.

286. 287. *To deſcribe a polygon in one circle* ABDE, *which ſhall be ſimilar to a polygon* FGI, *deſcribed in another*, GIK; *regular or irregular.*

Draw lines from the center P, to all the angles of the polygon, as PF, PK, PI, *&c.* Then at the center O, of the other circle, make the angles AOE, EOD, DOC, COB, BOA, reſpectively equal to FPK, KPI, IPH, HPG, GPF. Draw lines between the points A, E, D, *&c.* Then ABCDE is ſimilar to FGHIK (Cor. 1. 19. III).

Cor. *After the ſame manner, a polygon may be deſcribed about one circle, ſimilar to a polygon deſcribed about another circle.*

PROB. LIII.

288. *From a given point* A *on high*; *to let fall a perpendicular to a plane* BC.

In the plane BC draw any line DE. From A draw AF ⊥ to DE. Through F, draw FH ⊥ alſo to DE. Then let fall AI perp. to FH. Then AI is ⊥ to the plane BC.

For DE is ⊥ to the plane AFI (4. V). And if KL be ∥ to DE, then KL is ⊥ to the plane AFI (6. V). Therefore AI is ⊥ to the plane HIL or BC (4. V).

Otherwiſe thus.

289. Deſcribe a circle BFD, from the point A, upon the plane, with a pair of compaſſes or a ſtring. Then find the center C of that circle (37, 38. VIII); and AC is ⊥ to the plane. In practice you need only extend from A, to three points of the plane, B, D, F.

PROB.

PROB. LIV. FIG.

From a given point A, *in a plane* BC, *to raiſe a perpendicular.* 290.

From ſome point D, above the plane, draw DE ⊥ to the plane (52). Draw AE, and draw AF ∥ to ED. Then AF is perp. to the plane BC (6. V).

Both this and the laſt Prob. may eaſily be done with two ſquares: ſetting them croſs one another, and both of them cloſe to the point A.

PROB. LV.

To draw one plane parallel to another DE, *at a given diſtance.* 291.

Take three points A, B, C in the plane DE, but not in a right line. At theſe points erect three perpendiculars AI, BK, CL, to the plane DE (53); and of equal lengths, the ſame as the given diſtance. Through I, K, L, draw the plane FG, and it will be parallel to DE.

PROB. LVI.

To draw a plane perpendicular to a right line AB, *at* B. 292.

Draw two lines CD, EF perp. to AB at B. Through C, E, D, draw the plane CEDF, which is ⊥ to AB (4. V).

PROB. LVII.

FIG. 293. *Through any two lines* AB, CD, *inclined to one another, which do not interſect; to draw two planes perpendicular to one another.*

Through any point E of the line AB, draw EF ∥ to CD. Through the lines AEF, let the plane AEBF be drawn. From any point C, in the line CD, let fall the perp. CI, upon the plane AFB. Draw IH ∥ to FE, to interſect AB in H. At H let fall HG ⊥ to CD. Then the plane CIHG will be perp. to the plane AFH.

For CD is ∥ to IH (8. V). And ſince CI is perp. to IH, it is alſo ⊥ to CG (3. I). Therefore CI, HG are parallels (Cor. 3. 4. I); and HG ⊥ to the plane AFB (6. V). Therefore the plane DCIH is perp. to the plane AFB (7. Def. V).

Cor. 1. *The right line* GH *is perpendicular to both lines* AB, CD.

For it is ⊥ to CD (Conſtr.), and it is ⊥ to the plane AHI, and therefore to AHB.

Cor. 2. GH *is the neareſt diſtance between the two lines* AB *and* CD.

For the point H is nearer G, than any other point in the line AB (Cor. 4. 21. II). And G is nearer H than any point in CD.

Cor. 3. *Hence no two lines can poſſibly be drawn; but another line may be drawn, which is perp. to them both.*

Cor. 4. *And no two lines can be drawn, but two planes may be drawn through them, perpendicular to one another.*

Cor.

Cor. 5. *The given line* CD, *is parallel to the plane* AFB, *paſſing through the other line* AB. FIG. 293.

For it is parallel to HI.

PROB. LVIII.

Through any two inclined lines, which cut not one another, AB, CD; *to draw two parallel planes through them.* 293.

Draw the plane HICD and BIFA perp. to one another, and paſſing through the two given lines AB, CD (56). Then through CG at the diſtance GH, draw a plane ⊥ to GH (54), and it will be parallel to the plane ABF (Def. 10. V).

Cor. 1: *The line* GH, *(which is perpendicular to both the given lines,* AB, CD*), is the diſtance of the two parallel planes.*

Cor. 2. *No two lines can be drawn, but there may be two planes drawn through them, parallel to one another.*

PROB. LIX.

To make a ſolid angle BAD, *of three given plane angles, whoſe ſum is leſs than four right angles, and any two greater than the third.* 294.

There is no more to do than to join all their ſides AB, AC, AD, together; ſo that the vertices or angular points may all meet together in A; then A is the ſolid angle required (Cor. 19. V).

PROB.

FIG.

PROB. LX.

295. *To make a ſolid angle, equal to any ſolid angle given,* A.

Cut off the given ſolid angle A, by a plane BCDE; and from the given planes, make the angles QPR, RPS, SPT, and TPQ reſpectively equal to BAC, CAD, DAE, and EAB; alſo make PQ, PR, PS, PT reſpectively equal to AB, AC, AD, AE. Then the plane triangles in one, will be equal to the triangles in the other. Then place the ſides PR, PS, *&c.* together as in the other ſolid angle A, ſo that all their angular points may meet in P; and likewiſe ſo that the angles Q, R, S, T, may be reſpectively equal to B, C, D, E. And then the ſolid angle P will be equal to the ſolid angle A.

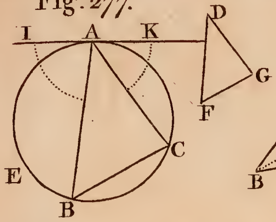
For all the 3 angles at Q, being equal to thoſe at B; and all the three angles at R, equal to thoſe at C, *&c.* The ſolid angles at B, C, D, E, will be equal to thoſe at Q, R, S, T (Cor. 19. V). And conſequently ∠P muſt be equal to A.

PROB. LXI.

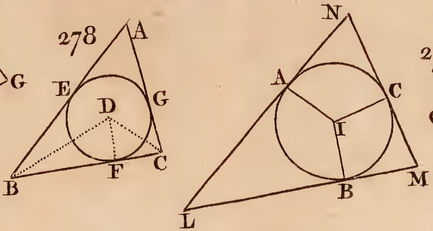
296. *Upon a given line* AB, *to deſcribe a parallelopipedon, ſimilar to a given parallelopipedon* CD.

Make the ſolid angle A equal to the ſolid angle C (59); alſo make as CF : CE : : AB : AH; and CF : CG : : AB : AI. Then finiſh the parallelopipedon AK, by drawing the planes KI, KH, and KB, parallel

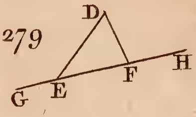
Fig. 277.



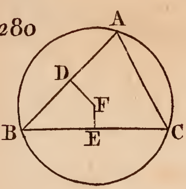
278



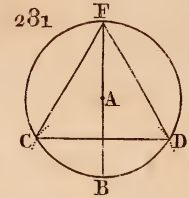
279



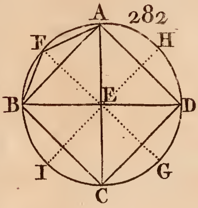
280



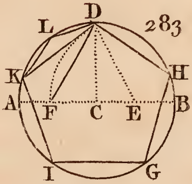
281



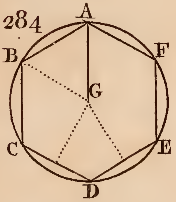
282



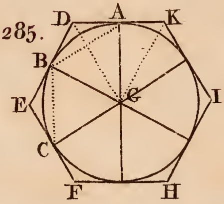
283



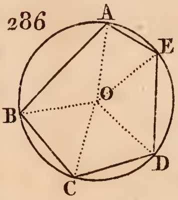
284



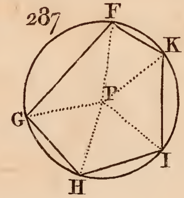
285.



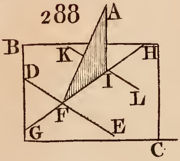
286



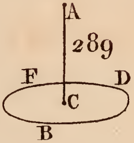
287



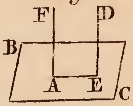
288



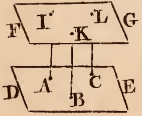
289



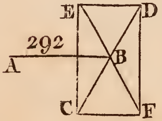
290



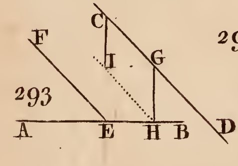
291



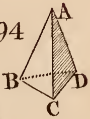
292



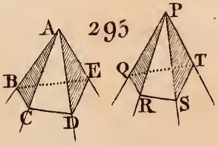
293



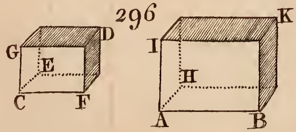
294



295



296



Pl. XIV. *Pa. last*

FIG.

295. *To*
given

C
BC
angl
equa
mak
AC,
will
plac
ſolic
may
Q,
An
ang
I
at
tho
D,
(C
equ

296.
lel
Cl

C
Cl
pe

parallel to the oppoſite ones BH, BI, and IH. Then IB is ſimilar to GF. FIG. 297.

For their ſolid angles, are equal, and the ſides proportional, and therefore they are ſimilar (22. VI).

FINIS.

ERRATA.

Page	*Line*	*Read*
107	16	DFKHCILG =
121	2	Fig. 191.
	25	DCH, be 3 ——Fig. 192.
125	18	Fig. 195.
126	2, 8 19, 29	Fig. 195, 196, 197, 198.
127	2	Fig. 198.
128	2	Fig. 198.
	28, 9	Fig. 199, 200.
129	1, 2	Fig. 199, 200.
137	5	Fig. 209.

The following BOOKS were written by the late Mr. THOMAS SIMPSON, F. R. S. and printed for J. NOURSE.

I. ESSAYS ON SEVERAL CURIOUS AND USEFUL SUBJECTS, IN SPECULATIVE AND MIXED MATHEMATICKS; in which the moſt difficult Problems of the firſt and ſecond Books of *Newton's Principia* are explained; in 4to.

II. MATHEMATICAL DISSERTATIONS on a variety of Phyſical and Analytical Subjects, in 4to.

III. MISCELLANEOUS TRACTS on ſome curious and very intereſting Subjects in Mechanicks, Phyſical-Aſtronomy, and Speculative Mathematicks, in 4to, 1757.

IV. THE DOCTRINE OF ANNUITIES AND REVERSIONS, deduced from general and evident Principles; with uſeful Tables, ſhewing the Values of ſingle and joint Lives, *&c.* in 8vo.

V. A TREATISE OF ALGEBRA; wherein the fundamental Principles are fully and clearly demonſtrated, and applied to the Solution of a great Variety of Problems, and to a Number of other uſeful Enquiries. Second Edition, in 8vo.

VI. THE DOCTRINE AND APPLICATION OF FLUXIONS; containing (beſides what is common on the Subject) a Number of new Improvements in the Theory, and the Solution of a Variety of new and very intereſting Problems in different Branches of the Mathematicks. 2 Vols. 8vo.

VII. TRIGONOMETRY, PLANE AND SPHERICAL, with the Conſtruction and Application of Logarithms, in 8vo.

VIII. SELECT EXERCISES for young Proficients in the Mathematicks; containing, beſides a choice Collection of Problems, both algebraical and geometrical, the whole Theory of Gunnery; a very accurate and ſuccinct Demonſtration of the firſt Principles of Fluxions; and a Set of Tables for the Valuation of Annuities and Reverſions, more comprehenſive than any extant. 8vo.

IX. ELEMENTS OF GEOMETRY; with their Application to the Menſuration of Superficies and Solids, to the Determination of the Maxima and Minima of Geometrical Quantities, and to the Conſtruction of a great Variety of Geometrical Problems. The Second Edition, with large Alterations and Additions.