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ELEMENTS

OF

ANALYTICAL GEOMETRY,

AND THE

DIFFERENTIAL AND INTEGRAL

CALCULUS.

BY H. N. ROBINSON, A. M.

AUTHOR OF A COURSE OF MATHEMATICS, INCLUDING SURVEYING AND NAVIGATION, ASTRONOMY, AND NATURAL PHILOSOPHY.

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PREFACE.

Setting aside Astronomy and Natural Philosophy the following treatise is the sixth volume of a course of mathematics by the same author on the same general plan of familiar inductive instruction. All mathematical science is positive, sure, and simple, and it is capable of being set forth in a natural, clear, and comprehensive light; and to attain this end all our labors have been directed. Hence, we have aimed at a familiar, rather than a cold, scientific style, and have embraced every opportunity to give principles a practical application, and have in every way made exertions to reach the minds of those we hope to instruct.

We know, however, that our views will not be generally received, — we cannot at once convince people that science is truly simple, — they feel that whatever is not at once comprehended by them is necessarily vast and complex, and this is the greatest obstruction to mental progress the mind has to encounter. Those who can look at simple nature as she is, will learn with great rapidity; others will always be beclouded, and if at times they should here and there catch a momentary view of the simplicity of science, that very simplicity will only serve to perplex and confound them.

The impression is abroad that Analytical Geometry and the Differential and Integral Calculus are very abstruse and incomprehensible subjects, and so they are without other light than that which is furnished by many of the text books. Scarcely one of them explains to its readers the objects and the aims of its investigations, — they merely direct the learner to do thus and so, and he will find this and that result. Not a word of philosophical explanation — not a word in respect to the object of the pursuit, which would enable one to go forward, relying on his own knowledge and strength.

In this respect we hope to be unlike most others, — we have essayed to give to the learner the true object of the investigation before him, and have taken every occasion to enlighten his comprehension, requiring him to read an equation just as it is, and to give to it the most simple interpretation, and not wander away to the ends of the earth in search of intricasies which do not exist.

Those who have studied our elementary geometry will have less difficulty than others in analytical geometry, for that work is half analytical; trigonometry in that work is almost purely analytical; but the absolute analytical geometry is in the work before us. Algebra applied to geometry *is not* analytical geometry as at first view some might suppose, for that is only solving problems on principles already established; it is not investigating general principles, but applying those already known.

Analytical geometry is strictly what the term implies; it is a minute and careful investigation of a few obvious and well known truths in geometry which we combine and compare to discover what other geometrical truths inevitably flow from them, the language used being algebraic, with all its signs, symbols, and powers of combination.

Those who are natural algebraists will find very little difficulty in analytical geometry; but others, for a time are seriously troubled to interpret and comprehend the full import of algebraic equations geometrically applied. When the first chapter becomes well understood, there will be no serious difficulty in any subsequent part of the subject. When once the equation of a straight line in a plane is well understood, the whole theory of analytical geometry is before the mind, and the equations of all other lines, whether straight or curved, cannot be misunderstood. A person who really understands the equation *of a straight line*, can readily construct the line from the equation, and hence, every teacher should insist on such construction as long as he can find the least hesitation in the student to construct any equation that may be offered. According to this idea, we have given several practical examples, under various conditions—but as the teacher and the pupil can easily propose examples without number, we thought it not best to take up space in the book with many mere practical examples.

Analytical geometry is comparatively a modern science, and a few years ago it was not a subject of study even in our colleges,—hence it is that many teachers, and others of the old Euclidian school of geometry, do not well appreciate, and are in fact prejudiced against it. Let not this discourage the young and ambitious student; the modern analysis must now be learned by all who have any valid claims to science, nor does it impose on them any additional burdens.

The common simple truths of common geometry, will always be learned in the common way, until materials enough are gathered together to use the analysis—and then analysis should be used, because it affords the widest field for the exercise of judgment; it calls into exercise the inventive powers, and taxes the memory very little with unimportant particulars.

"It is in fact the only method by which the student can advance beyond the bare rudiments of science without an expense of time and labor wholly disproportioned to the ends attained, the only method which gives at once a progressive and a self-sustaining power." For these reasons we approximated to it as much as possible in our work on common geometry.

In this we have been as clear and elementary as possible, without diluting the subject in the least. In forms we have been as high toned as any other author, and in the extent and application of this science we surpass many others. We have illustrated every variety of curves, and have taken every opportunity to compare algebraic forms to geometrical lines. This will be seen in our geometrical solutions of geometrical equations, and in our delineations of the higher curves.

The calculus is a branch of analytical geometry, although that term might be applied to any thing admitting computation. The differential calculus takes into consideration small differences. The differential of a quantity is the difference between two quantities of the like kind, when one is very nearly equal to the other, and from this definition alone, the ingenious student might find the differential for himself.

In geometry, we can conceive a line to be formed by the motion of a point, a surface to be formed by the motion of a line, and a solid to be formed by the motion of a plane, either moving parallel with itself or by revolving about an axis.

Thus, when a point moved and formed a line, the point was said to *flow*, and a small amount of such motion was called by the English mathematician the *fluxion* of the line, — a very small surface formed by the flowing of a line which bounded any side of a surface, was called the *fluxion* of that surface, and so on. The fluxions of the English mathematicians is the same thing as the differential of the French — and of late all have adopted the differential technicalities of the French.

The differential calculus is generally regarded as a very abstruse and difficult science, but this is the fault of the text books used; when that science is really comprehended, it is found to be no more abstruse than any other of the mathematical sciences; indeed, it is but an extension of algebra and geometry, using no new and no other system of computation.

Any branch of science, however simple, would be perfectly dark and abstruse to us, provided we have no prior and proper apprehension of the object of pursuit, and this our text books have never given in respect to the differential and integral calculus. Authors on this subject have contented themselves with saying that in the investigations will be found two kinds of quantities, variables and constants; they then define what symbols denote the variables and what denote the constants, and then direct the student what to do.

We have taken great pains to remedy this deficiency, and we shall feel much disappointment if our efforts are pronounced abortive on these points.

Notwithstanding the great importance we attach to the illustrations of what the differential calculus is, they occupy but very little space, but two or three pages at the most, and they are chiefly to be found in the introduction.

The differential calculus may be applied to any thing susceptible of change. For instance, every one knows that the variation of the length of the shadow of any object on any fixed plane must correspond to the variation of the sun's altitude, and the variation of altitude depends on the latitude of the plane, the declination of the sun, and the time of day. In short, the differential or small change in the shadow of an object compared with the object itself, must have a corresponding variation in the time of day,* and any scientific computation between two such small corresponding differences is one application of the differential calculus. Thus, the differential calculus is the ratio of small corresponding differences.

The integral calculus is the converse of the differential, somewhat as the cube root is the converse operation of cubing the root. It is more difficult to find the cube root of a number than it is to cube the root, but still the one operation is the converse of the other, and the one is not directly obvious from the other.

In many cases the integral calculus is more difficult than its corresponding differential, but it is not so in every case. In short, if the student will look at nature in its true and simple light, all difficulties will quickly vanish, and his progress in science become pleasant and invaluable.

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* See 43d miscellaneous Example, near the end of the volume.

ANALYTICAL GEOMETRY.

SECTION I.—CHAPTER I.

CHAPTER II.—LINES OF THE SECOND ORDER.

CHAPTER III.—CONIC SECTIONS.

CHAPTER IV.—THE PARABOLA.

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SECTION II.—CHAPTER I.

CHAPTER II.

CHAPTER III.

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THE DIFFERENTIAL CALCULUS.

SECTION I.—CHAPTER I.

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CHAPTER IV.

ANALYTICAL GEOMETRY.

Introductory Remarks.

GEOMETRY, purely analytical, is the investigation of general geometrical truths by the aid of algebraic equations.

Many of the demonstrations in our elementary geometry, trigonometry, and conic sections, where algebra is used, are *partially* analytical, not purely so.

Algebra applied to geometry must not be mistaken for analytical geometry, because the operator in either case uses the same mathematical motive power, the science of algebra.

Algebra applied to geometry only contemplates the use of algebra in solving problems — and all the geometrical truths are supposed to be previously known.

Analytical geometry draws out algebraically, all the necessary results from given data.

To pursue this branch of science successfully, the student must perfectly comprehend the nature and import of algebraic expressions — must understand general proportion, and the common rules of plane trigonometry.

We shall adopt the same general notation as other writers on this subject.

With this brief introduction, we commence

CHAPTER I.

PROPOSITION I.—PROBLEM.

To find the equation of a straight line.

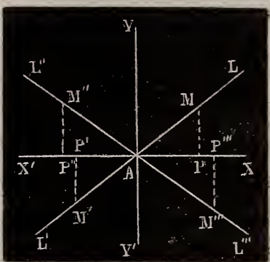
We now propose to show the equation which can be made to represent any straight line that can be drawn in a plane, and without a perfect comprehension of this, in both letter and spirit, it will be useless for the pupil to advance.

Draw a vertical and a horizontal line. On these two lines all measurements are to be made. The point of intersection of these two lines we shall call the *zero point*. *Horizontal measures* to the right from this point we shall call *plus*, to the left *minus*.

Vertical measures from the zero point upwards we call *plus*, downwards *minus*.

It has become the custom of all writers to denote unknown and indefinite distances along the horizontal by x, and along the vertical by y.

Hence the horizontal line of measure itself is called the axis of x, and the vertical line the axis of y, and they are marked as in the figure.



The point A in the adjoining figure is the zero point. Draw any line as $L'L$ through this point, and designate the natural tangent of the angle LAX by a, (the radius being unity.)

Then take any distance on AX as AP, and represent it by x, and the perpendicular distance PM put equal to y.

Then by trigonometry we have

$$\text{Rad.} : \tan. MAP :: AP : PM$$

$$1 : a :: x : y \qquad \text{or } y = ax \qquad (1)$$

Now this equation is general; that is, it applies to any point M on the line AL, because we can make x greater or less, and PM will be greater or less in like proportion, and M will move along on the line AL as we move P on the line AX. Because the point M will continue on the line AL through all changes of x and y, we say that $y=ax$, *is the equation of the line AL*.

Now let us diminish x to 0, and the equation reduces to $y=0$ in the same time, which brings M on to the point A.

Let x pass the line YY', it then becomes $-x$. AP' and the corresponding value of y will be $P'M'$, and being below the line $X'X$ will therefore be *minus.*

$$\text{Therefore } \pm y = \pm ax$$

is the general equation of the line LL', extending indefinitely in either direction.

If the tangent a becomes less, the line will incline more towards the line $X'X$. When $a=0$ the line will coincide with $X'X$, when *infinite*, it will coincide with YY'

Now let AP''' be $+x$, and a become $-a$, then $P'''M'''$ will correspond to y, and becomes *minus y*, because it is below the axis XX'. Or, algebraically $y=-ax$, indicating some point M''' below the horizontal axis.

Now we think it has been shown that $y=ax$ may represent any line as LL' passing through A *from the* 1*st into the* 3*d quadrant*, and $y=-ax$ may be made to represent any line as $L''L'''$ passing through A from the 2d into the 4th quadrant.

$$\text{Therefore} \qquad y = \pm ax$$

may be made to represent any straight line passing through the zero point.

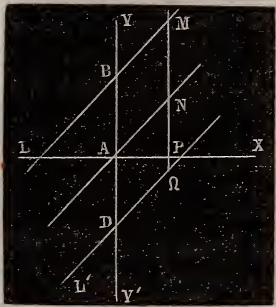
In case we have $-a$ and $-x$, that is, both a and x minus at the same time, their product will be $+ax$, showing that y must be *plus* by the rules of algebra.

We now request the learner to examine these geometrical lines and see whether they correspond.

When we have $-a$ we must draw the line from A to the right and *below* AX; then XAL''' is the angle whose natural tangent is $-a$. But the opposite angle $X'AL''$ is the same in value.

When we have $-x$ we must take the distance as AP'' to the left of the axis YY', and the corresponding line $P''M''$ is above XX', and therefore *plus*, as it ought to be.

But the equation of a straight line passing through the zero point is not sufficiently general for practical application; we will therefore suppose a line to pass in any direction across the axis YY', cutting it at the distance AB or AD $(\pm b)$ or b distance



above or below the zero point A, *and find its equation.*

Through the zero point A draw a line AN parallel to ML.

Take any point on the line AX and through P draw PM parallel to AY, then $ABMN$ will be a parallelogram.

Put $AP=x$. $PM=y$. The tangent of the angle $NAP=a$. Then will $NP=ax$.

To each of these equals add $NM=b$, then we shall have

$$y=ax+b$$

for the algebraic expression corresponding to the point M, and as M is any variable point on the line ML corresponding to the variations of x, this equation is said to be *the equation of the line ML.*

When b is *minus* the line is then QL', and cuts the axis YY' in D, a point as far below A as B is above A.

Hence we perceive that the equation

$$y=\pm ax\pm b$$

may represent the equation of any line in the plane YAX.

If we give to a, x, and b, their proper signs, in each case of application we may write

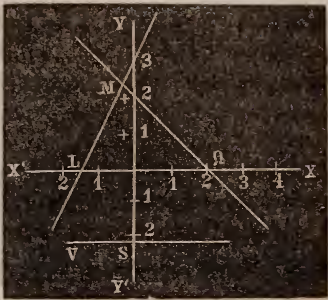
$$y=ax+b$$

for the equation of any straight line in a plane.

To fix in the minds of learners a complete comprehension of the equation of a straight line, we give the following practical

EXAMPLES.

1. Draw the line whose equation is $y=2x+3$. (1)
   Then draw the line represented by $y=-x+2$ (2)
and determine where these two lines intersect.



Draw YY' and XX' at right angles, and take any convenient space for the *unit of measure*, as 1, 2, 3, &c.

Equation (1) is true for all values of x and y. It is true then when $x=0$. But when $x=0$ the point on the line must be on the axis YY'.

When $x=0$. $y=3$.

This shows that the line sought for must cut YY' at the point $+3$.

The equation is equally true when $y=0$. But when $y=0$, the corresponding point on the line sought must be on the axis XX', and on the same supposition the equation becomes

$$0=2x+3, \quad \text{Or } x=-1\tfrac{1}{2}.$$

That is, midway between -1 and -2 is another point in the line which is represented by $y=2x+3$, but two points in any right line must define the line: therefore ML is the line sought.

Taking equation (2) and making $x=0$ will give $y=2$, and making $y=0$ will give $x=2$: therefore MQ must be the line whose equation is $y=-x+2$, and these two lines with the axis XX' form the triangle LMQ, whose base is $3\frac{1}{2}$ and altitude *about* $2\frac{1}{3}$.

But let the equations decide, (*not about*,) but exactly the position of the point M of intersection.

This point being in both lines, the co-ordinates x and y corresponding to this point are the same, therefore we may subtract one equation from the other, and the result will be a true equation, giving

$$3x+1=0. \quad \text{Or } x=-\tfrac{1}{3}.$$

Eliminating x from the two equations we find $y=2\frac{1}{3}$.

2. *For another example we require the projection of the line represented by the equation*

$$y=-\frac{x}{420}-2.$$

Making $x=0$, then $y=-2$. Making $y=0$, then $x=-840$.

Using the last figure, we perceive that the line sought for must pass through S two units below the zero point, and it must take such a direction SV as to meet the axis XX' at the distance of 840 units to the left of zero. Hence its *absolute* projection is practically impossible.

3. *Construct the line whose equation is* $y=2x+5$.

4. *Construct the line whose equation is* $y=-3x-3$.

PROPOSITION II.—PROBLEM.

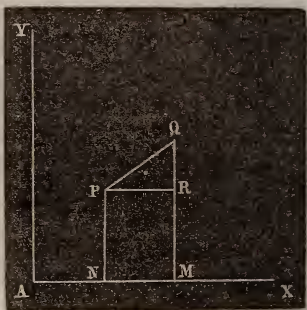
To find the distance between two given points in the plane of the co-ordinate axes. Also, to find the angle made by the line joining the two given points, and the axis of X.

DEFINITION.—A point is said to be given when its co-ordinates are known. Known co-ordinates are designated by x', y', — x'' y'' — x''', y''' ; which are read x' prime, x'' second, &c.

When the point designated by the co-ordinates is *no particular one,* we write simply x and y, to represent its co-ordinates.

Let the two given points be P and Q, and because the point P is said to be given, we know the two distances AN and NP.

$$AN=x'. \quad NP=y'.$$



And because the point Q is given we know the two distances

$AM=x''$ and $MQ=y''$.

$AM-AN=NM=PR=x''-x'$.

$MQ-MR=QR=y''-y'$.

In the right angled triangle PRQ we have

$$(PR)^2+(RQ)^2=(PQ)^2. \qquad \text{Put } D=PQ.$$

That is, $D^2=(x''-x')^2+(y''-y')^2$,

Or $D=\sqrt{(x''-x')^2+(y''-y')^2}$.

Thus we discover that the distance between any two given points is *equal to the square root of the sum of the squares of the difference of their abscisses and ordinates.*

If one of these points be the origin or zero point, then $x'=0$ or $y'=0$, and we have

$$D=\sqrt{(x'')^2+(y'')^2}.$$

a result obviously true.

To find the angle between PQ and AX.

PR is drawn parallel to AX, therefore the angle sought is the same in value as the angle QPR.

Designate the tangent of this angle by a, then by trigonometry we have

$$PR : RQ :: \text{radius} : \tan. QPR.$$

That is, $x''-x' : y''-y' :: 1 : a.$

Whence $$a=\frac{y''-y'}{x''-x'}.$$

In case $y''=y'$, PQ will coincide with PR, and be parallel to AX, and the tangent of the angle will then be 0, and this is shown by the equation, for then

$$a=\frac{0}{x''-x'}=0.$$

In case $x''=x'$, then PQ will coincide with RQ and be parallel to AY, and tangent a will be infinite, and this too the equation shows, for if we make $x''=x'$ or $x''-x'=0$, the equation will become

$$a=\frac{y''-y'}{0}=\infty$$

PROPOSITION III.

To find the equation of a line drawn through a given point.

Let P be the given point: The equation must be in the form

$$y=ax+b. \qquad (1)$$

Because the line must pass through the given point whose co-ordinates are x' and y', we must have

$$y'=ax'+b. \qquad (2)$$

Subtracting (2) from (1) we have

$$y-y'=a(x-x') \qquad (3)$$

for the equation sought.

In this equation a is indefinite, as it ought to be, because an infinite number of straight lines can be drawn through the point P.

We may give to y' and x' their numerical values, and give any value whatever to a, then we can construct a particular line that will run through the given point P.

Suppose $x'=2$, $y'=3$, and make $a=4$.
Then the equation will become

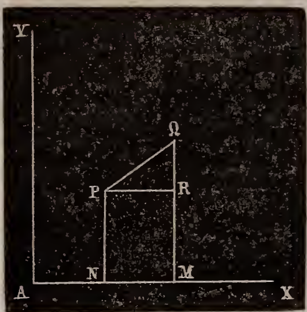
$$y-3=4(x-2).$$

Or $$y=4x-5.$$

This equation is obviously that of a straight line, hence (3) is of the required form.

PROPOSITION IV.

To find the equation of a line which passes through two given points.



More definitely, we say find the equation of the line which passes through the two given points P and Q.

As the equation is to be that of a line, it must correspond to

$$y=ax+b. \qquad (1)$$

As it must pass through the given point P, whose co-ordinates are x' and y', we must have

$$y'=ax'+b. \qquad (2)$$

Subtracting (2) from (1) we have

$$y-y'=a(x-x'). \qquad (3)$$

Because the line must also pass through the other point Q, we must have (Prop. II.)

$$a=\frac{y''-y'}{x''-x'}.$$

Substituting this value of a in (3) we have

$$y-y'=\left(\frac{y''-y'}{x''-x'}\right)(x-x').$$

the equation sought.

PROPOSITION V.

To find the equation of a straight line which shall pass through a given point and make a given angle with a given line.

The equation of the given line must be in the form

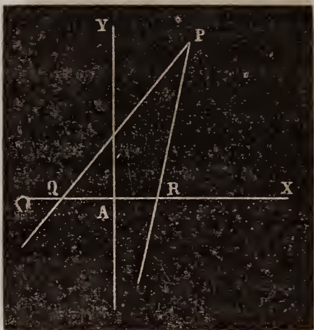
$$y=ax+b. \qquad (1)$$

Because the other line must pass through a given point its equation must be (Prop. III.)

$$y-y'=a'(x-x'). \qquad (2)$$

We have now to determine the value of a'.

When a and a' are equal, the two lines must be parallel, and the inclination of the two lines will be greater or less according to the relative values of a and a'.



Let PQ be the given line (the tangent of its angle with the axis of X equal a) and PR the other line which shall pass through the given point P and make a given angle QPR. The tangent of the angle $PRX=a'$.

Because $PRX=PQR+QPR$.

$$QPR=PRX-PQR.$$

$$\text{Tan.}\, QPR=\tan.(PRX-PQR.)$$

As the angle QPR is supposed to be known or given, we may put m to designate its tangent, and m is a known quantity.

Now by trigonometry we have

$$m=\tan.(PRX-PQR)=\frac{a'-a}{1+aa'}. \qquad (3)$$

Whence

$$a'=\frac{a+m}{1-ma}.$$

This value of a' put in (2) gives

$$y-y'=\left(\frac{a+m}{1-ma}\right)(x-x') \qquad (4)$$

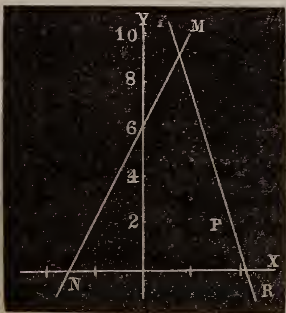
for the equation sought.

COROLLARY 1. When the given inclination is 90°, m its tangent is infinite, and then $a'=-\frac{1}{a}$. We decide this in the following manner:

An infinite quantity cannot be increased, therefore on that supposition $\frac{a+m}{1-ma}$ becomes $\frac{m}{-ma}$ or $-\frac{1}{a}$.

APPLICATION.—To make sure that we comprehend this proposition and its resulting equation, we give the following example:

The equation of a given line is $y=2x+6$.



Draw another line that will intersect this at an angle of 45° and pass through a given point P, whose co-ordinates are

$$x'=3\tfrac{1}{2}, \quad y'=2.$$

Draw the line MN corresponding to the equation $y=2x+6$. Locate the point P from its given co-ordinates.

Because the angle of intersection is to be 45°, $m=1$, $a=2$.

Substituting these values in (4) we have

$$y-2=-3(x-3\tfrac{1}{2}).$$

Or $$y=-3x+12\tfrac{1}{2}.$$

Constructing the line MR corresponding with this equation, we perceive it must pass through P and make the angle NMR 45°, as was required.

The teacher can propose any number of like examples.

COROLLARY 2. Equation (3) shows the tangent of the angle of the inclination of any two lines whose tangents are a and a'. That is, we have in general terms

$$m=\frac{a'-a}{1+aa'}.$$

In case the two lines are parallel $m=0$. Whence $a'=a$, an obvious result.

In case the two lines are perpendicular to each other, m must be infinite, and therefore we must put

$$1+aa'=0$$

to correspond with this hypothesis, and this gives

$$a'=-\frac{1}{a}$$

a result found in Cor. 1.

To show the practical value of this equation we require the angle of inclination of the two lines represented by the equations $y=3x-6$, and $y=-x+2$.

Here $a=3$ and $a'=-1$. Whence

$$m=\frac{4}{1-3}=-2.$$

This is the natural tangent of the angle sought, and if we have not a table of natural tangents at hand, we will take the log. of the number and add 10 to the index, then we shall have in the present example 10.301030 for the log. tangent which corresponds to 63° 26′ 6″ nearly.

The *minus* sign merely indicates the position of the angle, it is *below* the angular point.

PROPOSITION VI.

To find the co-ordinates which will locate the point of intersection of two straight lines given by their equations.

We have already done this in a particular example in Prop. I, and now we propose to show *general expressions* for the same thing.

Let $y=ax+b$ be the first line.

And $y=a'x+b'$ be the second line.

At their point of intersection y and x in both equations will represent the same point.

Therefore we may subtract one equation from the other, and the result will be a true equation.

For the sake of perspicuty, let x_1 and y_1 represent the co-ordinates of the point of intersection in each line, then by subtraction $(a-a')x_1+(b-b')=0$

Whence $$x_1=-\frac{(b-b')}{(a-a')} \text{ and } y_1=\frac{a'b-ab'}{a'-a}.$$

EXAMPLE.

At what point will the two following lines intersect:

$$y=-2x+1.$$

And $$y=5x+10.$$

Here $a=-2$, $a'=5$, $b=1$, $b'=10$. Whence $x=-\frac{9}{7}$, $y=-2\frac{1}{7}$.

If we take another line *not parallel* to either of these, the three will form a triangle.

Then if we *locate* the three points of intersection and join them, we shall have the triangle.

PROPOSITION VII.

To draw a perpendicular from a given point to a given straight line and to find its length.

Let $y=ax+b$ be the equation of the given straight line, and x', y', the co-ordinates of the given point.

The equation of the line which passes through the given point must take the form

$$y-y'=a'(x-x'). \quad \text{(Prop. III.)}$$

And as this must be perpendicular to the given line, we must have $a'=-\frac{1}{a}$. Therefore the equations for the two lines must be

$$y=ax+b \text{ for the given line.} \qquad (1)$$

And $$y-y'=-\frac{1}{a}(x-x') \text{ for the perpendicular line.}$$

Or $$y=-\frac{1}{a}x+\left(\frac{x'}{a}+y'\right) \text{ for the perpendicular.} \quad (2)$$

Let x_1 and y_1 represent the co-ordinates of the point of intersection of these two lines. Then by Prop. VI,

$$x_1=-\left(\frac{b-\frac{x'}{a}-y'}{a+\frac{1}{a}}\right) \quad \text{and } y_1=\frac{\frac{b}{a}+a\left(\frac{x'}{a}+y'\right)}{\frac{1}{a}+a}$$

Or $\quad x_1=-\left(\frac{ab-x'-ay'}{a^2+1}\right)$, and $y_1=\frac{b+ax'+a^2y'}{a^2+1}$

Or we may conceive x and y to represent the co-ordinates of the point of intersection, and eliminating y from (1) and (2) we shall find x as above, and afterwards we can eliminate y.

Now the length of the perpendicular is represented by

$$\sqrt{(x_1-x')^2+(y_1-y')^2}=D. \qquad \text{(Prop. II.)}$$

Whence $\sqrt{\left(\frac{-ab+ay'-a^2x'}{a^2+1}\right)^2+\left(\frac{b+ax'-y'}{a^2+1}\right)^2}$ = the perpendicular.

If we put $u=b+ax'-y'$, the quantities under the radical, will become

$$\sqrt{\frac{a^2u^2}{(a^2+1)^2}+\frac{u^2}{(a^2+1)^2}}=\sqrt{\frac{(a^2+1)u^2}{(a^2+1)^2}}=\pm\frac{u}{\sqrt{a^2+1}}.$$

Whence the perpendicular $=\pm\frac{b+ax'-y'}{\sqrt{a^2+1}}$.

EXAMPLES.

1. The equation of a given line is $y=3x-10$, and the co-ordinates of a given point are $x'=4$ and $y'=5$.

What is the length of the perpendicular from this given point to the given straight line? *Ans.* $\frac{1}{10}\sqrt{89}$.

2. The equation of a line is $y=-5x-15$, and the co-ordinates of a given point are $x'=4$ and $y'=5$.

What is the length of the perpendicular from the given point to the straight line? *Ans.* 7.84+.

PROPOSITION VIII.

To find the equation of a straight line which will bisect the angle contained by the inclination of two other straight lines.

Let $\qquad y=ax+b \qquad (1)$

And $\qquad y=a'x+b' \qquad (2)$

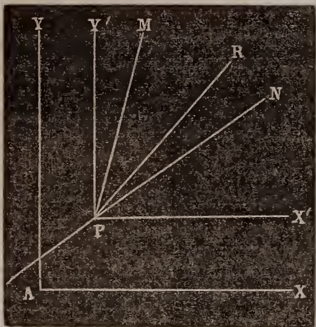
be the equations of two straight lines which intersect, and the co-ordinates of the point of intersection are

$$x_1=-\left(\frac{b-b'}{a-a'}\right) \qquad y_1=\frac{a'b-ab'}{a'-a} \qquad \text{(Prop. VI.)}$$

We now require a third line which shall pass through the same point of intersection and form an angle with the axis of X (the tangent of which may be represented by m) which will bisect the angle made by the inclination of the other two lines. Whence by (Prop. V.) the equation of the line sought must be

$$y-y_1=m(x-x_1) \qquad (3)$$

in case we can find the value of m.



Let PN represent the line corresponding to equation (1), PM the line whose equation is (2), and PR the line required.

Now the position or inclination of PN to AX depends entirely on the value of a, and the inclination of PM depends on a', and all are entirely independent of the position of the point P.

Now $RPN=RPX'-NPX'$ and $MPR=MPX'-RPX'$.

Whence by the application of a well known equation in plane trigonometry, (Equation (29), p. 143, in Robinson's Geometry,) we have

$$\tan. RPN=\tan.(RPX'-NPX')=\frac{m-a}{1+am}$$

And $$\tan. MPR=\tan.(MPX'-RPX')=\frac{a'-m}{1+a'm}$$

But by hypothesis these two angles RPN and MPR are to be equal to each other. Therefore

$$\frac{m-a}{1+am}=\frac{a'-m}{1+a'm}.$$

Whence $$m^2+\frac{2(1-aa')}{a'+a}m=1. \qquad (4)$$

This equation will give two values of m; one will correspond to the line PR, the other will be its supplement.

If the proper value of m be taken from this equation and put in (3); then (3) will be the equation required.

Practically we had better let the equations stand as they are, and substitute the values of a, a' x, and y, corresponding to any particular case.

To illustrate the foregoing proposition we propose the following

EXAMPLES.

Two lines intersect each other:

$$y=-2x+5 \text{ is the equation of one line.} \quad (1)$$

And

$$y=4x+6 \text{ is that of the other line.} \quad (2)$$

Find the equation of the line which bisects the angle contained by these two lines:

Here $a=-2,\quad a'=4,\quad b=5,\quad b'=6.$

Whence $x_1=-\frac{1}{6}$, and $y_1=\frac{16}{3}$.

Thus far (3) becomes

$$y-\tfrac{16}{3}=m(x+\tfrac{1}{6}).$$

And (4) becomes

$$m^2+9m=1.$$

Whence $m=0.1093$ or $m=-9.1095.$

$$y-\tfrac{16}{3}=0.1095(x+\tfrac{1}{6}). \quad (3)$$

Or

$$y-\tfrac{16}{3}=-9.1095(x+\tfrac{1}{6}). \quad (4)$$

Equation (4) is the line required; (3) is the line at right angles to the line required. All will be obvious if we construct lines (1), (2), (3), and (4).

For another example, find the equation of a line which bisects the angle contained by the two lines whose equations are

$$y=x+12, \qquad y=-20x+2.$$

Aus. $y-\frac{242}{21}=\frac{7\cdot32}{19}(x+\frac{10}{19})$, or $y-\frac{242}{21}=-\frac{49\cdot32}{19}(x+\frac{10}{19}.)$

Observation.—Two straight lines whose equations are

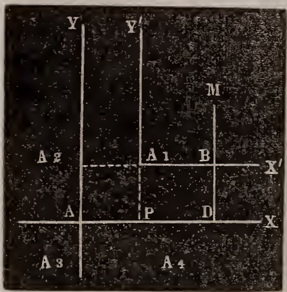
$$y=ax+b \quad \text{and} \quad y=a'x+b'$$

will always intersect at a point (unless $a=a'$) and with the axis of Y form a triangle. The area of such a triangle is expressed by

$$-\left(\frac{b-b'}{a-a'}\right)\times\left(\frac{b\backsim b'}{2}\right).$$

Transformation of Co-ordinates.

Let A be the zero point of the primitive system, and A' the zero point of this new system.



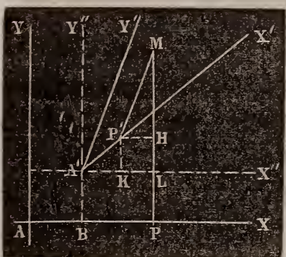
Let $AD=x$ and $DM=y$. Also, let $A'B=x'$ and $BM=y'$, we are to find the equation connecting x to x' and y to y'. The change of position from A to A' must be given in all cases.

Make $AP=a$ and $PA'=b$. It is now visible that $x=a+x'$ and $y=b+y'$.

We may transform the origin from A to A_2 or to A_3 or to A_4, as well as to A', by giving to a and b their proper corresponding values and signs.

PROPOSITION IX.

To find formulas for passing from a system of rectangular to a system of oblique co-ordinates from a different origin.



Let $AB=a$, $BA'=b$, $AP=x$, $PM=y$, $A'P'=x'$, $P'M=y'$ the angle $X'A'X''=m$, and the angle $Y'A'X''=n$. Now by trigonometry we have

$$A'K=x'\cos.m \quad KP'=LH=x'\sin.m$$

$$P'H=KL=y'\cos.n$$

And $$MH=y'\sin.n.$$

Whence $x=a+x'\cos.m+y'\cos.n$, $y=b+x'\sin.m+y'\sin.n$ the formulas required.

Scholium. In case the two systems have the same origin, we merely suppress a and b, and then the formulas required are

$$x=x'\cos.m+y'\cos.n \qquad y=x'\sin.m+y'\sin.n.$$

PROPOSITION X.

To find the formulas for passing from a system of oblique co-ordinates to a system of rectangular co-ordinates, the origin being the same.

Take the formulas of the last problem

$$x=x'\cos. m+y'\cos. n, \qquad y=x'\sin. m+y'\sin. n.$$

We now regard the oblique as the primitive axes, and require the corresponding values on the rectangular axes. That is, we require the values of x' and y'. If we multiply the first by $\sin. n$, and the second by $\cos. n$, and subtract their products, y' will be eliminated, and if x' be eliminated in a similar manner, we shall obtain

$$x'=\frac{x\sin. n-y\cos. n}{\sin.(n-m)} \qquad y'=\frac{y\cos. m-x\sin. m}{\sin.(n-m)}$$

SCHOLIUM. If the zero point be changed at the same time in reference to the oblique system, we shall have

$$x'=a+\frac{x\sin. n-y\cos. n}{\sin.(n-m)} \qquad y'=b+\frac{y\cos. m-x\sin. m}{\sin.(n-m)}$$

We close this subject by the following

EXAMPLE.

The equation of a line referred to rectangular co-ordinates is

$$y=a'x+b'.$$

Change it to a system of oblique co-ordinates having the same zero point.

Substituting for x and y their values as above, we have

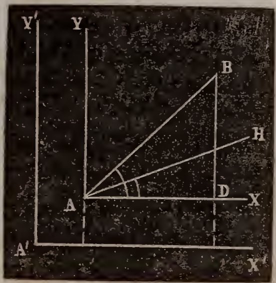
$$x'\sin. m+y'\sin. n=a'(x'\cos. m+y'\cos. n)+b'.$$

Reducing

$$y'=\frac{(a'\cos. m-\sin. m)x'}{\sin. n-a'\cos. m}+\frac{b'}{\sin. n-a'\cos. m}.$$

Polar Co-ordinates.

When a line is conceived to revolve round a point, that point is called a *pole*, and any other point in such a line referred to co-ordinates, is denominated *the system of polar co-ordinates.*



Conceive the line AB to revolve round the point A as a pole. Let $AB=r$. It may be a variable distance, and it is then called the *radius vector*.

Put the variable angle $BAD=v$, $AD=x$, $DB=y$, then by trigonometry

$$x=r\cos.v, \quad \text{and } y=r\sin.v.$$

Now from the first of these we have $r=\frac{x}{\cos.v}$, (v may revolve all the way round the pole), and as x and $\cos.v$ are both positive and both negative at the same time, that is, both positive in the first and fourth quadrants, and both negative in the second and third quadrants, therefore *r will always be positive.*

Consequently, should a negative radius appear in any equation, we *must infer* that some incompatible conditions have been admitted into the equation.

SCHOLIUM 1. If we change the origin now from A to A', writing x' and y' for the corresponding co-ordinates, we shall have

$$x'=a+r\cos.v$$

$$y'=b+r\sin.v.$$

SCHOLIUM 2. If in place of estimating the variable angle from the line AD the axis, we estimate it from the line AH which makes with the axis the given angle $HAD=m$, we shall have

$$x'=a+r\cos.(v+m).$$

$$y'=b+r\sin.(v+m).$$

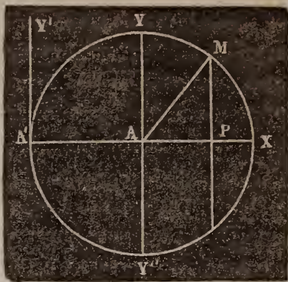
CHAPTER II.

Lines of the second order.

Straight lines can be represented by equations of the first degree, and they are therefore called lines of the first order. The circumference of a circle, and all the conic sections, are lines of the second order, because any point in them referred to co-ordinates requires equations of the second degree.

PROPOSITION I.

To find the equation of the circle.



Let the origin be the center of the circle. Draw AM to any point in the circumference, and let fall MP perpendicular to the axis of X. Put $AP=x$, $PM=y$ and $AM=R$.

Then the right angled triangle APM gives

$$x^2+y^2=R^2 \qquad (1)$$

and this is the equation of the circle when the zero point is the center.

When $y=0$, $x^2=R^2$, or $\pm x=R$, that is, P is at X or A'. When $x=0$, $y^2=R^2$, or $\pm y=R$, showing that M on the circumference is then at Y or Y''.

When x is positive, then P is on the right of the axis of Y, and when negative, P is on the left of that axis, or between A and A'.

When we make *radius unity*, as we often do in trigonometry, then $x^2+y^2=1$, and then giving to x or y any value *plus* or *minus* within the limit of unity, the equation will give us the corresponding value of the other letter.

In trigonometry y *is called the sine of the arc* XM, *and* x *its cosine.*

Hence in trigonometry we have $\sin.^2+\cos.^2=1$.

Now if we remove the origin to A' and call the distance $A'P=x$, then $AP=x-R$, and the triangle APM gives

$$(x-R)^2+y^2=R^2.$$

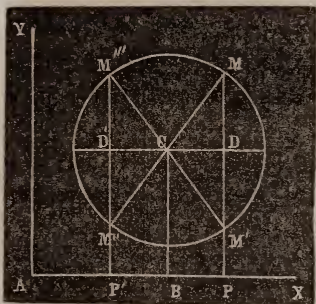
Whence $$y^2=2Rx-x^2. \qquad (2)$$

This is the equation of the circle, when the origin is on the circumference.

When $x=0$ $y=0$ at the same time. When x is greater than $2R$, *y becomes imaginary*, showing that *such an hypothesis is inconsistent with the existence of the circle.*

There is still a more general equation of the circle when the zero point is neither at the center nor in the circumference.

The figure in the margin will fully illustrate.



Let $AB=c$,* $BC=b$. Put $AP=x$, or $AP'=x$, and PM or $P'M'''=y$, CM, CM', &c. each $=R$.

In the circle we observe *four* equal right angled triangles. The *numerical* expression is the same for each. Signs only *indicate positions.*

Now in case CDM is the triangle we *fix upon,*

We put $AP=x$, then $BP=CD=(x-c)$,
$PM=y$, $MD=y-CB=(y-b)$.

Whence $$(x-c)^2+(y-b)^2=R^2 \qquad (1)$$

In case CDM' is the triangle, we put $AP=x$ and $PM'=y$.

Then $$(x-c)^2+(b-y)^2=R^2 \qquad (2)$$

In case $CD'M'''$ is the triangle, we put $AP'=x$, $P'M'''=y$.

Then $$(c-x)^2+(y-b)^2=R^2 \qquad (3)$$

If $CD'M''$ is the triangle, we put $P'M''=y$.

Then $$(c-x)^2+(b-y)^2=R^2 \qquad (4)$$

Equations (1), (2), (3), and (4), are in all respects numeri-

* We do not take a, because a, in this science is generally understood to represent the tangent of an angle.

cally the same in value, for $(c-x)^2=(x-c)^2$, and $(b-y)^2=(y-b)^2$. Hence we may take equation (1) to represent the general equation of the circle referred to rectangular coordinates.

The equation $$(x-c)^2+(y-b)^2=R^2 \qquad (1)$$

includes all the others by attributing proper values and signs to c and b.

If we suppose both c and b equal 0, it transfers the zero point to the center of the circle, and the equation becomes

$$x^2+y^2=R^2.$$

To find where the circle cuts the axis of X we must make $y=0$. This reduces the general equation (1) to

$$(x-c)^2+b^2=R^2.$$

Or $$(x-c)^2=R^2-b^2.$$

Now if b is numerically greater than R, the first member being *a square*, (and therefore positive,) must be equal to a negative quantity, which is impossible,—showing that in that case the circle does not meet or cut the axis of X, *and this is obvious in the figure.*

In case $b=R$ then $(x-c)^2=0$, or $x=c$, showing that the circle would then *touch* the axis of X at the point B.

To show where the circle cuts the axis of Y, make $x=0$: then (1) becomes

$$c^2+(y-b)^2=R^2.$$

Or $$(y-b)^2=R^2-c^2.$$

This equation shows that if c is greater than R, the circle does not cut the axis of Y, and this is also obvious from the figure.

If c be less than R, the second member is positive in value, and $$y=b\pm\sqrt{R^2-c^2},$$

showing that if it cut the axis at all, it must be in two points, as at M'', M'''.

PROPOSITION II.

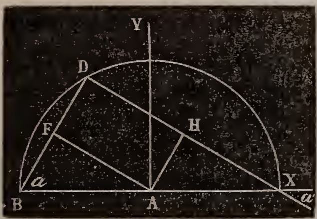
The supplementary chords in the circle are perpendicular to each other.

DEFINITION.—Two lines drawn through the two extremities of any diameter of a curve, and which intersect the curve in the same point, are called *supplementary chords.*

That is, the chord of an arc, and the chord of its supplement.

In common geometry this proposition is enunciated thus :

All angles in a semicircle are right angles.



The equation of a straight line which will pass through the given point B, must be of the form (Prop. III. Chap. I.)

$$y-y'=a(x-x'). \quad (1)$$

The equation of a straight line which will pass through the given point X, must be of the form

$$y-y'=a'(x-x'). \quad (2)$$

At the point B, $y'=0$, and $x'=-R$, or $-x'=R$. Therefore (1) becomes

$$y=a(x+R). \quad (3)$$

And for like reason (2) becomes

$$y=a'(x-R). \quad (4)$$

When these two lines intersect, y in (3) is the same as y in (4), and x in (3) is the same as x in (4), therefore these equations *for the point of intersection* may be regarded as two numerical equalities; hence we may multiply them together and obtain a true numerical equation, that is,

$$y^2=aa'(x^2-R^2). \quad (5)$$

But as the point of intersection must be on the *curve, by hypothesis*, therefore, x and y must conform to the following equation :

$$y^2+x^2=R^2. \quad \text{Or} \quad y^2=-1(x^2-R^2). \quad (6)$$

Whence $\quad aa'=-1, \quad$ or $\quad aa'+1=0.$

This last equation shows that the two lines are perpendicular to each other, as proved by (Cor. 2, Prop. V, Chap. I.)*

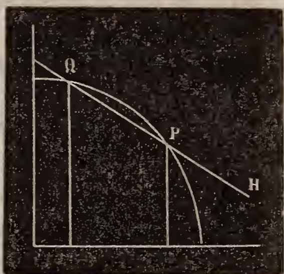
Because a and a' are indefinite, we conclude that an infinite number of supplemental chords may be drawn in the semicircle, which is obviously true.

SCHOLIUM. As BDX is a right angled triangle, and BX its hypotenuse, it follows *that the diameter is greater than any chord.* As one chord *increases.* its supplementary chord *decreases.*

From the center A let fall the perpendiculars AH, AF. Then the two triangles XAH and XBD are equiangular and similar; therefore, as A is the middle point of XB, H is the middle point of XD, and F is the middle point of BD. $AH=\frac{1}{2}(BD)$, and $AF=\frac{1}{2}(XD)$. That is, *the distance of any chord from the center is equal to half its supplementary chord.*

PROPOSITION III.

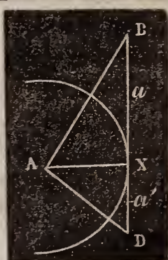
To find the equation of a straight line which shall be tangent to the circumference of a circle.



Draw a line cutting the curve in any two points, as P and Q. Designate the co-ordinates of the point P by x', y', and of the point Q by x'', y'', and of any other point in the line as H by x, y.

Now the equation of *any* line passing through point H may be expressed by

$$y=ax+b. \qquad (1)$$



* This condition of the perpendicularity of the two lines may be more satisfactory to some when they read the more direct demonstration.

Let AB be one line, and AD another line at right angles to it. Join BD, and from A draw AX perpendicular to BD, and conceive AX the axis. The tangent of the angle $BAX=a$, and $XAD=-a'$. $AX=1$, and it is the mean proportion between a and $-a'$. Therefore

$$a : 1 :: 1 : -a'.$$

Whence $\quad -aa'=1 \quad$ or $\quad aa'+1=0. \quad$ Q. E. D

If the same line passes through the point P, the equation for that point must be

$$y'=ax'+b. \qquad (2)$$

And the same line passing through Q, the equation for that point must be

$$y''=ax''+b. \qquad (3)$$

Subtracting (3) from (2) and we find

$$y'-y''=a(x-x'') \qquad (4)$$

for the equation which passes through the two points P and Q.

Subtracting (2) from (1) and we have

$$y-y'=a(x-x') \qquad (5)$$

for the equation of the line which passes through the two points P and H.

The line which passes through the three points Q, P, and H, is expressed in the two equations (4) and (5).

Conceive the line QPH to revolve on the point P, so as to make Q coincide with P, then the line will be a tangent at P.

We have now to determine the value of a, *when the line becomes a tangent at* P.

Because the two points P and Q are in the circumference, we have

$$x'^2+y'^2=R^2.$$
$$x''^2+y''^2=R^2.$$

Subtracting and factoring the remainder, gives us

$$(x'+x'')(x'-x'')+(y'+y'')(y'-y'')=0. \qquad (6)$$

The value of $(y'-y'')$ taken from (4) and substituted in (6), and then divided by $(x'-x'')$ will reduce (6) to

$$x'+x''+a(y'+y'')=0.$$

Whence $$a=-\left(\frac{x'+x''}{y'+y''}\right). \qquad (7)$$

This equation is true, however far or near P and Q may be from each other, provided they be on the curve; and when QPH becomes a tangent at P, $x'=x''$ and $y'=y''$, then (7) reduces to

$$a=-\frac{x'}{y'}. \qquad (8)$$

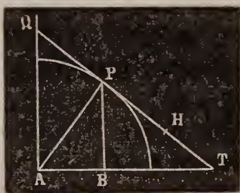
This value of a substituted in (5) gives

$$y-y'=-\frac{x'}{y'}(x-x'). \qquad (9)$$

$$yy'-y'^2=-xx'+x'^2.$$

$$yy'+xx'=y'^2+x'^2=R^2.$$

This is the general equation of a tangent line; x', y', are the co-ordinates of the tangent point, and x, y, the co-ordinates of any other point in the line.



SCHOLIUM. For the point in which the tangent line cuts the axis of X, we make $y'=0$, then

$$x=\frac{R^2}{x'}=AT.$$

For the point in which it meets the axis of Y, we make $x'=0$, and

$$y=\frac{R^2}{y'}=AQ.$$

DEFINITIONS.—A line is said to be *normal* to a curve when it is perpendicular to the tangent line at the point of contact.

Join AP, and if APT is a right angle, then AP is a *normal*, and AB, a portion of the axis of X *under it*, is called the *sub-normal*. The line BT *under* the tangent is called the *subtangent*.

Let us now discover whether APT is or is not a right angle.

Equation (8) shows us the tangent value of the inclination of the line PT with the axis of X.

Put $a'=$ the tangent of the angle PAT, then by trigonometry

$$a'=\frac{y'}{x'}.$$

But $$a=-\frac{x'}{y'}. \qquad \text{Eq. (8)}$$

Whence $$aa'=-1. \quad \text{Or} \quad a'=-\frac{1}{a}.$$

Therefore AP is at right angles to PT. (Prop. V. Chap. I.)

PROPOSITION IV.

To find the equation of a line which shall pass through a given point without the circle.

Let H be the given point, and x' and y' its given co-ordinates, and x and y the co-ordinates of the tangent point P.

The equation of the line passing through the two points H and P, must be of the form

$$y-y'=a(x-x'). \qquad (1)$$

And if PH is tangent at the point P, and x and y the co-ordinates of the point P, equation (8) of the last proposition gives us

$$a=-\frac{x}{y}.$$

This value of a put in (1) and we have

$$y-y'=-\frac{x}{y}(x-x')$$

for the equation sought.

This equation combined with that of the circle

$$x^2+y^2=R^2$$

will determine the values of x and y, and as there will be two values to each, *numerically* equal, it shows that two equal tangents can be drawn from H, or from any point without the circle, which is obviously true.

SCHOLIUM. We can find the value of the tangent PT by means of the similar triangles ABP, PBT, which give

$$x : R :: y : PT.$$

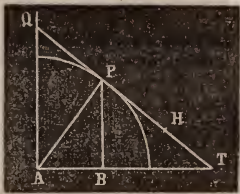
$$PT=R\frac{y}{x}.$$

More general and elegant formulas will be found in the calculus for the *normals, subnormals, tangents*, and *subtangents* applicable to all the conic sections.

NOTE TO PROPOSITIONS III AND IV OF THIS CHAPTER.—In the investigation of these propositions we followed in the footsteps of others, only hoping to be more *definite* and *clear*. But were we only in pursuit of results, we would have been more brief and practical.

In these propositions it is not assumed that the radius of the circle is at right angles to its tangent when drawn from the center to the point of contact, but we see no propriety in excluding this geometrical truth so well known in elementary geometry, especially when we consider that we have all along used the symbol a to represent the tangent of angles on the admission that the tangent of an angle was a line drawn at *right angles to the radius* from the extremity of the radius.

Using this truth we would not draw a line cutting the curve in *two* points, but would draw the tangent line PT at once, and *admit* that the angle APT was a right angle. Then it is clear that the angle $APB=$ the angle PTB.



Now to find the equation of the line, we let x' and y' represent the co-ordinates of the point P, and x and y the general co-ordinates of the line, and a the tangent of its angle with the axis of X, then by (Prop. III, Chap. I,) we have

$$y'-y=a(x'-x).$$

Now the triangle APB gives us the following expression for the tangent of the angle APB, or its equal PTB,

$$a=-\frac{x'}{y'}$$

This value of a put in the preceding equation, will give us

$$y'-y=-\frac{x'}{y'}(x'-x).$$

Or $$y'^2-yy'=-x'^2+xx'.$$

Whence $$yy'+xx'=R^2 \quad \text{the same as before.}$$

Of the Polar Equation of the Circle.

The polar equation of a curve is the equation for any point in the curve estimated from any fixed point called a pole. The variable distance from the pole to any point in the curve is called the *radius vector*, and the angle which the radius vector makes with a given straight line is called the *variable angle*.

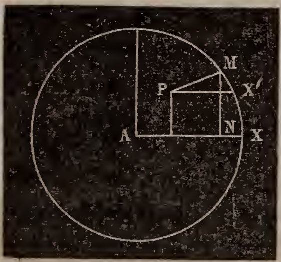
PROPOSITION V.

To find the polar equation of the circle.

When the center is *the pole* or the fixed point, the equation is

$$x^2+y^2=R^2 \qquad (1)$$

and the radius vector R is then *constant*.



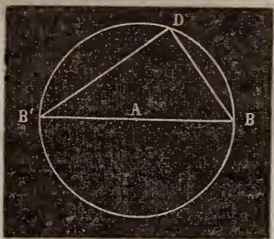
Now let P be the pole, and the co-ordinates of that point a and b. $PM=r$, and $MPX'=v$ the variable angle. $AN=x$ and $NM=y$. Then it is obvious that

$$x=a+r\cos.v, \text{ and } y=b+r\sin.v.$$

These values of x and y substituted in (1), (observing that $\cos.^2v+\sin.^2v=1$,) will give

$$r^2+2(a\cos.v+b\sin.v)r+a^2+b^2-R^2=0$$

which is the polar equation sought.



Scholium 1. P may be at any point on the plane. Suppose it at B'. Then $a=-R$ and $b=0$. Substituting these values in the equation, and it reduces to

$$r^2-2R\cos.vr=0.$$

As there is no absolute term, $r=0$ will satisfy one point in the curve, and this is true, as P is supposed to be in the curve. Dividing by r, and

$$r=2R\cos.v.$$

This value of r will be positive while $\cos.v$ is positive, and negative when $\cos.v$ is negative ; but r being a radius vector can never be negative, and the figure shows this, as r never passes to *the left of* B, but runs into zero at that point.

When $v=0$, $\cos.v=1$, then $r=BB'$. When $v=90$, $\cos.v=0$, and r becomes 0 at B, and the variations of v from 0 to 90, determine all the points in the semicircle BDB'.

Scholium 2. If the pole be placed at B, then $a=+R$ and $b=0$, which reduces the general equation to

$$r=-2R\cos.v.$$

Here it is necessary that $\cos.v$ should be negative to make r positive, therefore v must commence at 90° and vary to 270° ; that is, be on the left of the axis of Y drawn through B', and this corresponds with the figure.

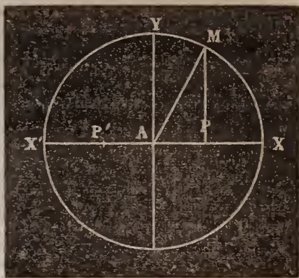
Application.—The polar equation of the circle in its most general form is

$$r^2+2(a\cos.v+b\sin.v)r+a^2+b^2=R^2. \qquad (1)$$

If we make $b=0$, it puts the polar point somewhere on the axis of X, and reduces the equation to

$$r^2+2a\cos.v.r+a^2=R^2. \qquad (2)$$

Now if we make $v=0$, then will $\cos. v=1$, and the lines represented by $\pm r$ would refer to the points X, X', in the circle.



This hypothesis reduces the last equation to

$$r^2+2ar=(R^2-a^2) \qquad (3)$$

and this equation is the same in form *as the common quadratic in algebra*, or in the same form as

$$x^2\pm px=q.$$

Whence $\quad x=r, \quad 2a=\pm p,$ and $R^2-a^2=q.$

$$a=\pm\tfrac{1}{2}p, \qquad R=\sqrt{q+a^2}=\sqrt{q+\tfrac{1}{4}p^2}.$$

These results show us that if we describe a circle with the radius $\sqrt{q+\frac{1}{4}p^2}$, and place P on the axis of X at a distance from the center equal to $\frac{1}{2}p$, then PX represents one value of x, and PX' the other. That is,

$$x=-\tfrac{1}{2}p+\sqrt{q+\tfrac{1}{4}p^2}=PX.$$

Or $\qquad x=-\frac{1}{2}p-\sqrt{x+\frac{1}{4}p^2}=PX',$

and this is the common solution.

When p is negative, the polar point is laid off to the left from the center at P'.

The operation refers to the right angled triangle APM.

$$AP=\tfrac{1}{2}p, \quad PM=\sqrt{q}, \quad \text{and} \quad AM=\sqrt{q+\tfrac{1}{4}p^2}.$$

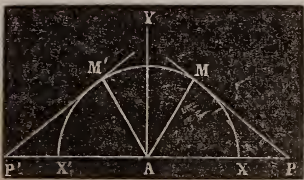
Let the form of the quadratic be

$$x^2\pm px=-q.$$

Then comparing this with the polar equation of the circle, we have

$$2a=\pm p. \qquad R^2-a^2=-q.$$

$$a=\pm\tfrac{1}{2}p. \qquad R=\pm\sqrt{\tfrac{1}{4}p^2-q}.$$



Take $AX=R$ and describe a semicircle. Take $AP=\frac{1}{2}p$ and $AP'=-\frac{1}{2}p$. From P and P' draw the lines PM, and $P'M'$ to touch the circle. Join AM, AM'.

Here AP is the hypotenuse of a right angled triangle. In the first case AP was a side.

In this figure as in the other, $PM=\sqrt{q}$; but here it is inclined to the axis of X; in the first figure it was perpendicular to it.

The figure thus drawn, we have PX for one value of x, and PX' is the other, which may be *determined geometrically*.

If $$x^2+px=-q$$

$$x=-\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2-q}=PX, \quad \text{or} \quad x=-\tfrac{1}{2}p-\sqrt{\tfrac{1}{4}p^2-q}=PX'.$$

Observe that the first part of the value of x, is *minus, corresponding to left position from P*.

If $$x^2-px=-q,$$

we take P' for one extremity of the line x.

$$x=\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2-q}=P'X, \quad \text{or} \quad x=\tfrac{1}{2}p-\sqrt{\tfrac{1}{4}p-q}=P'X'.$$

Here the first part of the value of x, $(\frac{1}{2}p)$, is *plus, because to the right of the point P'.*

Because $R=\sqrt{\frac{1}{4}p^2-q}$, R or AM becomes less and less as the numerical value of q approaches the value of $\frac{1}{4}p^2$. When these two are equal, $R=0$, and the circle becomes a point. When q is greater than $\frac{1}{4}p^2$, the circle has *more than vanished,* giving no real existence to any of these lines, and the values of x are said to be *imaginary*.

We have found another method of *geometrising* quadratic equations, which we consider well worthy of notice, although it is of no practical utility.

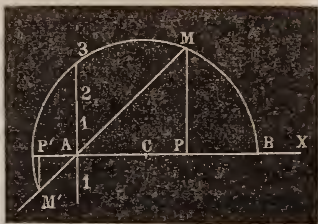
It will be remembered that the equation of a straight line passing through the origin of co-ordinates is

$$y=ax, \qquad (1)$$

and that the general equation of the circle is

$$(x\pm c)^2+(y\pm b)^2=R^2. \qquad (2)$$

If we make $b=0$, the center of the circle must be somewhere on the axis of X.



Let AM represent a line, the equation of which is $y=ax$, and if we take $a=1$, AM will incline 45° from either axis, as represented in the figure. Hence $y=x$, and making $b=0$, these two values substituted in (2), and that equation reduced, we shall find

$$y^2\pm cy=\frac{R^2-c^2}{2}. \qquad (3)$$

This equation has the common *quadratic form.*

Equation (1) responds to any point in the straight line $M'M$. Equation (2) responds to any point in the circumference BMM'.

Equations (1) and (2) combined must respond equally to the straight line and to the circle. Therefore equation (3) must *respond* to the points M and M', the points in which the circle cuts the line.

That is, PM and $P'M'$ are the two roots of equation (3), and when one is above the axis of X, as in this figure, it is the *positive* root, and $P'M'$ being below the axis of X, it is the *negative* root.

When both roots of equation (3) are positive, the circle will cut the line in two points above the axis of X. When the two roots are *minus*, the circle will cut the line in two points below the axis of X.

When the two roots of any equation in the form of (3) are equal and positive, the circle will *touch* the line above the axis of X. If the roots are *equal* and *negative*, the circle will touch the line below the axis of X. In case the roots of (3) are *imaginary*, the circle will not meet the line.

We give the following examples for illustration :

$$y^2-2y=5.$$

To determine the values of y by a geometrical construction of this kind, we must make

$$c=-2, \quad \text{and} \quad \frac{R^2-c^2}{2}=5.$$

Whence $R=3.74$, the radius of the circle. Take any distance on the axes for the unit of measure, and set off the distance c on the axis of X from the origin, for the center of the circle ;— to the right, if c is *negative*, and to the left, if c is positive.

Then from the center, with a radius equal to $R=\sqrt{2p+c^2}$, describe a circle cutting the line drawn midway between the two axes, as in the figure.

In this example the center of the circle is at C, the distance of two units from the origin A, to the right. Then, with the radius 3.74 we described the circle, cutting the line in M and M', and we find by measure (when the construction is accurate) that $MP=3.44$, the positive root, and $M'P'=-1.44$, the negative root.

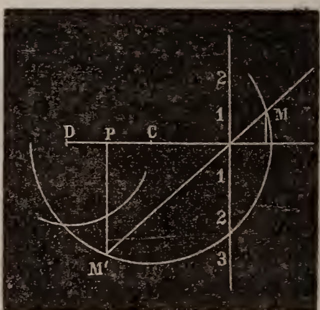
For another example we require the roots of the following equation by construction:

$$y^2+6y=27.$$

N. B. When the numerals are too large in any equation for convenience, we can always reduce them in the following manner:

Put $y=nz$, then the equation becomes

$$n^2z^2+6nz=27.$$



Or $$z^2+\frac{6}{n}z=\frac{27}{n^2}.$$

Now let $n=$ *any number whatever*. If $n=3$, then

$$z^2+2z=3.$$

Here $c=2$. $$\frac{R^2-c^2}{2}=3.$$

Whence $R=\sqrt{10}=3.16$.

At the distance of two units

to the left of the origin, is the center of the circle. We see by the figure that 1 is the positive root, and —3 the negative root.

But $y=nz$, $n=3$, $z=1$, $y=3$ or -9.

We give one more example.

Construct the equation

$$y^2+4y=-6.$$

Here $c=4$, and $\frac{R^2-c^2}{2}=-6$. Whence $R=2$.

Using the same figure as before, the center of the circle to this example is at D, and as the radius is only 2, the circumference does not cut the line $M'M$, showing that the equation has no *real roots.*

We have said that this method of finding the roots of a quadratic was of little practical value. The reason of this conclusion is based on the fact that it requires more labor to obtain the value of the radius of the circle than it does to find the roots themselves.

Nevertheless this method is interesting and instructive as an algebraic geometrical problem.

When we find the polar equation of the parabola, we shall then have another method of constructing the roots of quadratics which will not require the extraction of the square root.

CHAPTER III.

Conic Sections.

If we cut a cone by a plane through its vertex, the section will be a triangle. If we cut it by a plane at right angles with the axis of the cone, the section thus cut will be a circle. If we cut it on one side by a plane parallel to the other side, the section will be a *parabola.* If we cut it by a plane less *inclined* to the base than the sides of the cone, the section will be an *ellipse.* If by a plane more inclined to the base than to the sides of the cone, the section will be an *hyperbola.*

Hence, the triangle and the circle might be included in conic sections, — but custom has limited the term to the three curves, the *Ellipse*, the *Parabola*, and the *Hyperbola*.

We can and do examine the properties of a triangle and a circle without the least regard to a cone whatever. So also, can the cone be entirely dispensed with in discussing the properties of the ellipse, the parabola, and the hyperbola, and we shall dispense with it, commencing with

The Ellipse.

Definition 1.—An ellipse is a plane curve, confined by two fixed points, and the sum of the distances from any point in the curve to the fixed points, is constant.

2.—The two fixed points are called the *foci*.

3.—The center is midway in a straight line between the two *foci*.

4.—A diameter is a straight line passing through the center.

5.—The *major axis* is a diameter passing through the *foci*.

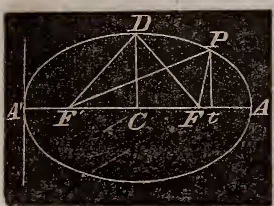
8.—The *minor axis* is at right angles to the *major axis*, passing through the center.

9.—The distance between the center and either *focus*, is called the *eccentricity* when the *semimajor* axis is unity.

10.—The *parameter* of an ellipse is the double ordinate passing through one of the *foci*.

PROPOSITION I.

To find the equation of the curve, the origin of the co-ordinates being in the center, the major axis being given, also the distance of the foci from the center.



The curve in the margin represents an ellipse.

Put $CF=c$, $CA=A$.

Take any point, as P, and let fall the perpendicular Pt.

By our conventional notation, put

$$Ct=x, \quad tP=y.$$

As $F'P+PF=2A$, we may put $F'P=A+z$, and $PF=A-z$. Then the two right angled triangles $F'Pt$, FPt, give us

$$(c+x)^2+y^2=(A+z)^2 \qquad (1)$$

$$(c-x)^2+y^2=(A-z)^2 \qquad (2)$$

For the points in the curve which cause t to fall between c and F, we would have

$$(x-c)^2+y^2=(A-z)^2 \qquad (3)$$

But when expanded, there is no difference between (2) and (3), and by giving proper values and signs to x and y, equations (1) and (2) will respond to *any point* in the curve as well as to the point P.

Substracting (2) from (1), and dividing by 4, we find

$$cx=Az, \quad \text{or} \quad z=\frac{cx}{A}. \qquad (4)$$

This last equation shows that $F'P$, the radius vector, varies as the abscissa x.

Add (1) and (2), and divide the sum by 2, and we have

$$c^2+x^2+y^2=A^2+z^2.$$

Substituting the value of z^2 from (4), and clearing of fractions, we have

$$c^2A^2+A^2x^2+A^2y^2=A^4+c^2x^2.$$

Or $$A^2y^2+(A^2-c^2)x^2=A^2(A^2-c^2). \qquad (5)$$

Now conceive the point P to move along describing the curve, and when it comes to the point D, so that DC makes a right angle with the axis of X, the two triangles DCF and DCF' are right angled and equal. DF and DF' each is equal to A, and as CF, CF', each is equal to c, we have

$$\overline{DC}^2=A^2-c^2.$$

It is customary to denote DC half the *minor* axis of the ellipse by B, as well as half the *major* axis by A, and adhering to this notation

$$B^2=A^2-c^2. \qquad (6)$$

Substituting this in (5) we have for the equation of the ellipse

$$A^2y^2+B^2x^2=A^2B^2,$$

referred to its center for the origin of co-ordinates.

If we wish to transfer the origin of co-ordinates from the center of the ellipse to the extremity A' of its major axis, we must put

$$x=-A+x', \quad \text{and} \quad y=y'.$$

Substituting these values of x and y in the last equation, and reducing, we have

$$y'^2=\frac{B^2}{A^2}(2Ax'-x'^2).$$

Or without the primes, we have

$$y^2=\frac{B^2}{A^2}(2Ax-x^2),$$

for the equation of the ellipse when the origin is at the extremity of the major axis.

Corollary 1. If it were possible for B to equal A, then c must equal 0, as shown by (6). Or, while c has a value, it is impossible for B to equal A.

If $B=A$, then $c=0$, and the equation becomes

$$A^2y^2+A^2x^2=A^2A^2.$$

Or $$y^2+x^2=A^2,$$

the equation of the circle. Therefore the circle may be called an ellipse, whose *eccentricity is zero*, or whose eccentricity is *infinitely small.*

Corollary 2. To find where the curve cuts the axis of X, make $y=0$ in the equation, then

$$x=\pm A,$$

showing that it extends to equal distances from the center.

To find where the curve cuts the axis of Y, make $x=0$, and then

$$y=\pm B.$$

Plus B refers to the point D, $-B$ indicates the point directly opposite to D, on the lower side of the axis of X.

Finally, let x equal any value whatever less than A, then

$$y=\pm\frac{B}{A}(A^2-x^2)^{\frac{1}{2}}.$$

an equation showing two values of y, numerically equal, indicating that the curve is symmetrical in respect to the axis of X.

If we give to y any value less than B, the general equation gives

$$x=\pm\frac{A}{B}(B^2-y^2)^{\frac{1}{2}}.$$

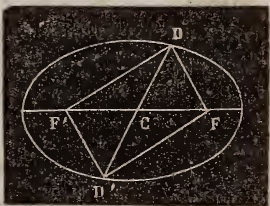
Showing that the curve is symmetrical in respect to the axis of Y.

SCHOLIUM. The ordinate which passes through one of the foci, corresponds to $x=c$. But $A^2-B^2=c^2$. Hence A^2-c^2 or $A^2-x^2=B^2$. Or $(A^2-x^2)^{\frac{1}{2}}=B$, and this value substituted in the last equation, gives $y=\pm\frac{B^2}{A}$. Whence $\frac{2B^2}{A}$ is the measure of the parameter of any ellipse, by Def. 10.

PROPOSITION II.

Every diameter is bisected in the center.

Let x, and y, be the co-ordinates of the point D, and x', y', the co-ordinates of the point D'.



Then by the equation of the curve

$$A^2y^2+B^2x^2=A^2B^2.$$

And $A^2y'^2+B^2x'^2=A^2B^2$.

The equation of a line passing through the center, must be of the form $y=ax$.

This equation combined with the equations of the curve, gives

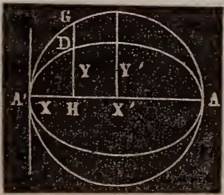
$$x=\frac{AB}{\sqrt{a^2A^2+B^2}}, \qquad y=\frac{aAB}{\sqrt{a^2A^2+B^2}}.$$

$$x'=-\frac{AB}{\sqrt{a^2A^2+B^2}}, \qquad y'=-\frac{aAB}{\sqrt{a^2A^2+B^2}}.$$

These equations show that the co-ordinates of the point D, are the same as those of the point D', except opposite in signs. Hence *DD' is bisected at the center.*

PROPOSITION III.

The squares of the ordinates are to one another as the rectangles of their corresponding abscissas.



Let y be any ordinate, and x its corresponding abscissa. Then, by the first proposition, we shall have

$$y^2 = \frac{B^2}{A^2}(2A - x)x.$$

Let y' be any other ordinate, and x' its corresponding abscissa, and by the same proposition we must have

$$y'^2 = \frac{B^2}{A^2}(2A - x')x'.$$

Dividing one of these equations by the other, omitting common factors in the numerator and denominator of the second member of the new equation, we shall have

$$\frac{y^2}{y'^2} = \frac{(2A - x)x}{(2A - x')x'}.$$

Hence $\quad y^2 : y'^2 = (2A - x)x : (2A - x')x'.$

By simply inspecting the figure, we cannot fail to perceive that $(2A - x)$, and x, are the abscissas corresponding to the ordinate y, and $(2A - x')$, and x', are the two corresponding to y'. Therefore, the squares of the ordinates, &c. Q. E. D.

SCHOLIUM. Suppose one of these ordinates, as y', to represent half the *minor axis*, that is, $y' = B$. Then the corresponding value of x' will be A, and $(2A - x')$ will be A, also. Whence the last proportion will become

$$y^2 : B^2 = (2A - x)x : A^2.$$

In respect to the third term we perceive that if $A'H$ is represented by x, AH will be $(2A - x)$, and if G is a point in the circle, whose diameter is $A'A$, and GH the ordinate, then

$$(2A - x)x = \overline{GH}^2,$$

and the proportion becomes

$$y^2 : B^2 = \overline{GH}^2 : A^2.$$

Or $\quad y : GH = B : A.$

Or $\quad A : B = GH : y = DH.$

PROPOSITION IV.

The area of an ellipse is the mean proportional between the areas of two circles, the diameter of one being the major axis, and the diameter of the other, the minor axis.

Conceive GH to be a *practical* as well as a mathematical line; or rather, conceive it be a *very narrow* parallelogrom.

Conceive also other lines $G'H'$, $G''H''$, &c, drawn so as to fill the whole space occupied by the semicircle and semi-ellipse.*

Then by scholium to Prop. III, we have

$$A : B = GH : DH.$$
$$= G'H' : D'H'.$$
$$= G''H'' : D''H''.$$
&c. &c.

But as the sums of proportionals have the same ratio as their like parts, (see proportion in algebra,) therefore

$$A : B : : (GH + G'H' + \&c.) : (DH + D'H' + \&c.)$$

But the sum of all the *narrow* parallelograms represented by $(GH + G'H' + \&c.)$ is the area of the *semicircle* on $A'A$: and the *sum* of all the parallelograms represented by $(DH + D'H' + \&c.)$ is the area of the *semi-ellipse.*

But wholes are in the same proportion as their halves, whence

$$A : B = \text{area circle} : \text{area ellipse}.$$

But the area of the circle on the major axis, is πA^2. Substituting this, and the proportion becomes

$$A : B = \pi A^2 : \text{area ellipse}.$$

Or $$\text{area ellipse} = \pi AB,$$

which is the mean proportional between (πA^2) and $(\pi B^2,)$ the expressions for the areas of the circles, one on the major axis, the other on the minor axis. Q. E. D.

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* These narrow parallelograms are called *differentials*, in the differential calculus—and the *sum* of them is called the *integral*, in the integral calculus.

SCHOLIUM. Hence the common rule in mensuration to find the area of an ellipse.

RULE—*Multiply the semi-major and semi-minor axes together, and multiply that product by* 3.1416.

PROPOSITION V.

To find the product of the tangents of two supplementary chords with the axis of X.

Let x, y, be the co-ordinates of any point, as P, and x', y', the co-ordinates of the point A'.

Then the equation of a line which passes through the two points A' and P, (Prop. III, Chap. I,) will be

$$y-y'=a(x-x'). \qquad (1)$$

The equation of the line which passes through the points A and P, will be of the form

$$y-y''=a'(x-x''). \qquad (2)$$

For the given point A', we have $y'=0$, and $x'=-A$.

Whence (1) becomes

$$y=a(x+A). \qquad (3)$$

For the given point A we have $y''=0$, and $x''=A$, which values substituted in (2) give

$$y=a'(x-A). \qquad (4)$$

As y and x are the co-ordinates of the same point P in both lines, we may combine (3) and (4) in any manner we please. Multiplying them, we have

$$y^2=aa'(x^2-A^2). \qquad (5)$$

Because P is a point in the ellipse, the equation of the curve gives

$$y^2=\frac{B^2}{A^2}(A^2-x^2)=-\frac{B^2}{A^2}(x^2-A^2). \qquad (6)$$

Comparing (5) and (6) we find

$$aa'=-\frac{B^2}{A^2} \text{ for the equation sought.}$$

Scholium 1. In case the ellipse becomes a circle, that is, in case $A=B$, $aa'+1=0$, showing that the angle $A'PA$ would then be a right angle, as it ought to be, by (Prop. II, Chap. II.)

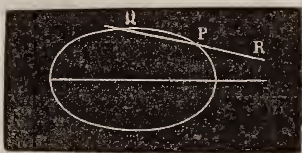
Because $\frac{B^2}{A^2}$ is less than *unity*, or aa' less than 1,* or *radius;* the two angles $PA'A$ and PAA' are together less than 90°; therefore the angle at P is obtuse, or greater than 90°.

Scholium 2. Since aa' has a constant value, the sum of the two, a, a', will be least when $a=a'$.

Hence the angle at P will be greatest when P is at the vertex of the minor axis, and the supplementary chords equal; and the angle at P will become nearer a right angle as P approaches A or A'.

PROPOSITION VI.

To find the equation of a straight line which shall be tangent to an ellipse.



Let x, y, be the co-ordinates of any indefinite point R, in a line cutting an ellipse; x', y', the co-ordinates of the point P, and x'', y'', the co-ordinates of the point Q. Also, let a be the tangent of the angle of inclination of the line PR with the axis of X. The object is to find the value of a when PR is tangent to the ellipse.

The equation of a line which passes through two points, as R and P, must be of the form

$$y-y'=a(x-x'). \quad (1) \quad \text{(Prop. III, Chap. I.)}$$

The equation for the *same line* passing through the two points R and Q, must be

$$y-y''=a(x-x''). \quad (2)$$

And the equation for the same line passing through the two points P and Q, must be

$$y'-y''=a(x'-x''). \quad (3)$$

* In trigonometry we learn that tan. x cot. $x=R^2=1$. That is, the product of two tangents the sum of whose arc is 90°, equals 1. When the sum is less than 90°, the product will be a fraction.

Because the points P and Q are in the *curve*, the co-ordinates of those points must correspond to the following equations :

$$A^2y'^2+B^2x'^2=A^2B^2.$$
$$A^2y''^2+B^2x''^2=A^2B^2.$$

By subtraction $\quad A^2(y'^2-y''^2)+B^2(x'^2-x''^2)=0.$

Or $\quad A^2(y'+y'')(y'-y'')=-B^2(x'+x'')(x'-x''). \quad (4)$

Dividing (4) by (3) we have

$$A^2(y'+y'')=-\frac{B^2}{a}(x'+x''). \qquad (5)$$

Now conceive the line to revolve on the point P until Q coincides with P, then PR will be tangent to the curve. But when Q coincides with P, we shall have

$$y'=y'' \text{ and } x'=x''.$$

Whence (5) becomes

$$2A^2y'=-\frac{2B^2}{a}x'.$$

Or $$a=-\frac{B^2x'}{A^2y'}.$$

This value of a put in (1) gives

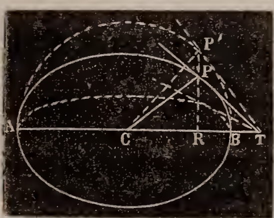
$$y-y'=-\frac{B^2x'}{A^2y'}(x-x').$$

Reducing $\quad A^2yy'+B^2xx'=A^2y'^2+B^2x'^2.$

Or $\quad A^2yy'+B^2xx'=A^2B^2.$

This is the equation sought, x and y being the general co-ordinates of the line.

Scholium 1. To find where the tangent meets the axis of X, we must make $y=0$.



This gives $x=\dfrac{A^2}{x'}=CT$.

In case the ellipse becomes a circle, $B=A$, and then the equation will become

$$yy'+xx'=A^2,$$

the equation for a tangent line to a

circle; and to find where this tangent meets the axis of X, we make $y=0$, and

$$x=\frac{A^2}{x'}=CT, \text{ as before.}$$

In short, as these results are all independent of B, the minor axis, it follows that the circle and all ellipses on the major axis AB can have tangents terminating at the same point T on the axis of X, if drawn from the same ordinate, as shown in the figure.

Scholium 2. To find the point in which the tangent to an ellipse meets the axis of Y, we make $x=0$, then the equation for the tangent becomes

$$y=\frac{B^2}{y'}.$$

As this equation is independent of A, it shows that all ellipses having the same *minor axis*, can have tangents terminating in the same point on the axis of Y, if drawn from the same abscissa.

Scholium 3. If from CT we subtract CR, we shall have RT, a common *subtangent* to a circle, and all ellipses which have $2A$ for a major diameter. That is

$$RT=\frac{A^2}{x'}-x'=\frac{A^2-x'^2}{x'}.$$

We can also find RT by the triangle PRT, as we have the tangent of the angle at T, $\left(-\frac{B^2x'}{A^2y'}\right)$ to the radius 1.

Whence we have the following proportion :

$$1 : -\frac{B^2x'}{A^2y'}=RT : y'$$

$$RT=-\frac{A^2y'^2}{B^2x'}.$$

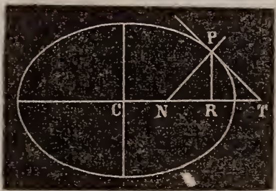
The *minus* sign indicates that the measure from T is towards the left.

PROPOSITION VII.

To find the equation of a normal line to the ellipse.

Since the normal passes through the point of tangency, its equation will be in the form

$$y-y'=a'(x-x'). \qquad (1)$$



Because PN is at right angles to the tangent,

$$aa'+1=0.$$

But by the last proposition

$$a=-\frac{B^2x'}{A^2y'}.$$

Whence $a'=\frac{A^2y'}{B^2x'}$, and this value of a' put in (1) gives

$$y-y'=\frac{A^2y'}{B^2x'}(x-x'),$$

for the equation sought.

Scholium 1. To find where the normal cuts the axis of X, we must make $y=0$, then we shall have

$$x=\left(\frac{A^2-B^2}{A^2}\right)x'=CN.$$

Application.—Meridians on the earth are ellipses; the semi-major axis through the equator is $A=3963$. miles, and the semi-minor axis from the center to the pole is $B=3949.5$.

A plumb line is everywhere at right angles to the surface, and of course its prolongation would be a normal line like PN. *In latitude* 42°, *what is the deviation of a plumb line from the center of the earth?* Or, how far from the center of the earth would a plumb line meet the plane of the equator? Or, what would be the value of CN?

As this ellipse is very near a circle, we may take CR for the cosine of 42°, which must be represented by x'. This being assumed, we have

$$x'=2940. \quad \left(\frac{A^2-B^2}{A^2}\right)2940.=23,+\text{ miles } CN. \quad \textit{Ans.}$$

SCHOLIUM 2. To find NR, the *subnormal*, we simply subtract CN from CR, whence

$$NR=x'-\left(\frac{A^2-B^2}{A^2}\right)x'=\frac{B^2x'}{A^2}.$$

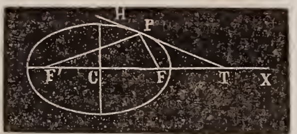
We can also find the *subnormal* from the proportional triangles PRT, PNR, thus:

$$TR : RP :: RP : RN.$$

$$-\frac{A^2y'^2}{B^2x'} : y' :: y' : -NR. \quad \text{Whence } NR=\frac{B^2x'}{A^2}.$$

PROPOSITION VIII.

Lines drawn from the foci to any point in the ellipse make equal angles with the tangent line drawn through the same point.



Let C be the center of the ellipse, PT the tangent line, and PF, PF', the two lines drawn to the foci.

Denote the distance $CF=\sqrt{A^2-B^2}$ by c, CF' by $-c$, the angle FPT by V, and the tangents of the angles PTX, PFT, by a and a'.

Now $$FPT=PTX-PFT.$$

By trigonometry, (Eq. 28, p. 143, Robinson's Geometry), we have

$$\text{Tan. } FPT=\text{tan. }(PTX-PFT). \text{ That is, tan. } V=\frac{a-a'}{1+aa'}. \quad (1)$$

Prop. VI, gives us $a=-\frac{B^2x'}{A^2y'}$, x', y', being the co-ordinates of the point P.

Let x, y, be the co-ordinates of the point F, then from Prop. IV, Chap. I, we have

$$a'=\frac{y'-y}{x'-x}.$$

But at the point F, $y=0$ and $x=c$.

Whence $$a'=\frac{y'}{x'-c}.$$

These values of a and a' substituted in (1) give

$$\text{Tan. } V = \frac{\frac{-B^2x'}{A^2y'} - \frac{y'}{x'-c}}{1 - \frac{B^2x'}{A^2(x'-c)}} = \frac{-B^2x'^2 + B^2cx' - A^2y'^2}{A^2y'(x'-c) - B^2x'y'}.$$

$$\text{Tan. } V = \frac{B^2cx' - A^2B^2}{(A^2-B^2)xy' - A^2cy'} = \frac{B^2(cx'-A^2)}{cy'(cx'-A^2)} = \frac{B^2}{cy'}.$$

Observing that $A^2y'^2 + B^2x'^2 = A^2B^2$, and $A^2 - B^2 = c^2$. The equation of the line PF will become the equation of the line PF' by simply changing $+c$ to $-c$, for then we shall have the co-ordinates of the other focus.

We now have

$$\text{tan. } FPT = \frac{B^2}{cy'}.$$

But if c is made $-c$, then

$$\text{tan. } F'PT = -\frac{B^2}{cy'}.$$

As these two tangents are *numerically* the same, differing only in signs, they must be equally inclined to the straight line from which they are measured, or be supplements of each other.

Whence $\quad FPT + F'PT = 180.$

But $\quad F'PH + F'PT = 180.$

Therefore $\quad FPT = F'PH. \qquad$ Q. E. D.

Corollary. The normal being perpendicular to the tangent, it must bisect the angle made by the two lines drawn from the tangent point to the foci.

Scholium. Any point in the curve may be considered as a point in a tangent to the curve at that point.

It is found by experiment that *light, heat,* and *sound,* after they approach to, are reflected off, from any reflecting surface at equal angles; that is, any and every single ray makes the angle of reflection equal to the angle of incidence.

Therefore, if a light be placed at one focus of an ellipse, and the sides a reflecting surface, the reflections will concentrate at the other focus. If the sides of a room be elliptical, and a stove is placed at one focus, it will concentrate heat at the other.

Whispering galleries are made on this principle, and all theaters and large assembly rooms should more or less approximate to this figure. The concentration of the rays of heat from one of these points to the other, is the reason why they are called the *foci*, or burning points.

OF THE ELLIPSE REFERRED TO ITS CONJUGATE DIAMETERS.

Two diameters drawn through the center of an ellipse so as to bisect two supplementary chords on the major axis, are said to be *conjugate*.

Hence, two conjugate diameters intersect one another by an angle equal to that of the two supplemental chords, which they are supposed to bisect, but by Prop. V, two supplemental chords intersect each other by an angle which must conform to the equation

$$aa' = -\frac{B^2}{A^2},$$

in which a is the tangent of the angle which one of the supplemental chords makes with the axis of X, and a' is the tangent of the angle made by the other chord.

Now let m be the angle whose tangent is a, and n be the angle whose tangent is a', then

$$a = \frac{\sin. m}{\cos. m}, \quad \text{and} \quad a' = \frac{\sin. n}{\cos. n}.$$

Substituting these values in the last equation, and reducing, we obtain

$$A^2 \sin. m \sin. n + B^2 \cos. m \cos. n = 0,$$

which expresses the relation which must exist between A, B, m, and n, to fix the position of any two conjugate diameters in respect to the major axis, and this equation is called *the equation of condition for conjugate diameters.*

In this equation of condition, m and n are undetermined, showing that an infinite number of conjugate diameters might be drawn, but whenever any value is assigned to one of these angles, that value must be put in the equation, and then a *deduction made* for the value of the other angle.

PROPOSITION IX.

To find the equation of the ellipse referred to its center and conjugate diameters.

The equation of the ellipse referred to its major and minor axes, is

$$A^2y^2+B^2x^2=A^2B^2.$$

The formulas for changing rectangular co-ordinates into oblique, the origin being the same, are (Prop. IX, Chap. I,)

$$x=x'\cos. m+y'\cos. n. \qquad y=x'\sin. m+y'\cos. n.$$

Squaring these, and substituting the values of x^2 and y^2 in the equation of the ellipse above, we have

$$\left\{\begin{matrix}(A^2\sin^2 n+B^2\cos^2 n)y'^2+(A^2\sin^2 m+B^2\cos^2 m)x'^2 \\ 2(A^2\sin. m\sin. n+B^2\cos. m\cos. n)y'x'\end{matrix}\right\}=A^2B^2$$

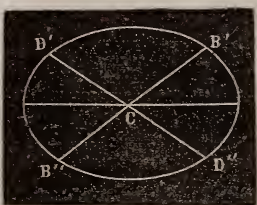
But if we now assume the condition that the new axes shall be conjugate diameters, then

$$A^2\sin. m\sin. n+B^2\cos. m\cos. n=0,$$

which reduces the preceding equation to [(F)

$$(A^2\sin.^2 n+B^2\cos.^2 n)y'^2+(A^2\sin.^2 m+B^2\cos.^2 m)x'^2=A^2B^2,$$

which is the equation required. But it can be simplified as follows:



The equation refers to the two diameters $B''B'$ and $D''D'$ as axes. For the point B' we must make $y'=0$, then

$$x'^2=\frac{A^2B^2}{A^2\sin.^2 m+B^2\cos.^2 m}=$$

$$(CB')^2=A'^2. \qquad \text{(P)}$$

Designating CB' by A', and CD' by B'.

For the point D' we must make $x'=0$. Then

$$y'^2=\frac{A^2B^2}{A^2\sin.^2 n+B^2\cos.^2 n}=(CD')^2=B'^2. \qquad \text{(Q)}$$

From (P) we have $(A^2\sin.^2 m+B^2\cos.^2 m)=\frac{A^2B^2}{A'^2}$.

From (Q) $(A^2\sin.^2 n+B^2\cos.^2 n)=\frac{A^2B^2}{B'^2}$.

These values put in (F) give

$$\frac{A^2B^2}{B'^2}y'^2+\frac{A^2B^2}{A'^2}x'^2=A^2B^2.$$

Whence $$A'^2y'^2+B'^2x'^2=A'^2B'^2.$$

We may omit the accents to x' and y', as they are general variables, and then we have

$$A'^2y^2+B'^2x^2=A'^2B'^2,$$

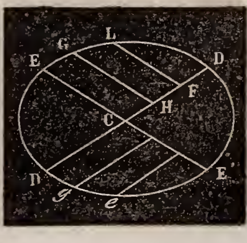
for the equation of the ellipse referred to its center and conjugate diameters.

Scholium. In this equation if we assign any value to x less than A', there will result two values of y, *numerically equal*, and to every assumed value of y less than B', there will result two corresponding values of x, numerically equal, differing only in signs, showing that the curve is symmetrical in respect to its conjugate axes, *and that each axis bisects all chords which are parallel to the other axis.*

Observation.—As this equation is of the same form as that of the general equation referred to rectangular co-ordinates on the *major* and *minor* axis, we may infer at once that we can find equations for ordinates, tangent lines, &c. referred to conjugate axes, which will be in the same form as those already found, which refer to the rectangular axes. But as a general thing it will not do to draw summary conclusions.

PROPOSITION X.

As the square of any diameter is to the square of its conjugate, so is the rectangle of any two segments of the diameter to the square of the corresponding ordinate; that, is, the ordinate drawn through the point of bisection.



Let CD be represented by A', and CE by $B_{\prime}$, CH by x, and GH by y, then by the last proposition we have

$$A'^2y^2+B'^2x^2.=A'^2B'^2.$$

Which may be put under the form

$$A'^2y^2=B'^2(A'^2-x^2).$$

Whence $A'^2 : B'^2 :: (A'^2 - x^2) : y^2$.

Or $(2A')^2 : (2B')^2 :: (A'+x)(A'-x) : y^2$.

Now $2A'$ and $2B'$ represent the conjugate diameters $D'D$, $E'E$, and since CH represents x, $A'+x=D'H$, and $A'-x=HD$. Also $y=GH$. Hence the above proportions correspond to

$$(D'D)^2 : (E'E)^2 :: D'H \times HD : (GH)^2. \quad \text{Q. E. D.}$$

SCHOLIUM. As x is no particular distance from C, CF may represent x, then LF will represent y, and the proportion then becomes

$$(D'D)^2 : (E'E)^2 :: D'F \times FD : (LF)^2.$$

Comparing the two proportions, we perceive that

$$D'H \cdot HD : D'F \cdot FD :: \overline{GH}^2 : \overline{LF}^2.$$

That is, *The rectangle of the abscissas are to one another as the squares of the corresponding ordinates.*

The same property as was demonstrated in respect to rectangular ordinates in Prop. III.

In the same manner we may prove that

$$Eh \cdot hE' : Ef \cdot fE' :: (hg)^2 : (fl)^2$$

PROPOSITION XI.

To find the equation of a tangent line to an ellipse referred to its conjugate diameters.

Conceive a line to cut the curve in two points, whose co-ordinates are x', y', and x'', y'', and x, y, the co-ordinates of any point on the line.

The equation of a line passing through two points, is of the form

$$y-y'=a(x-x'), \qquad (1)$$

an equation in which a is to be determined when the line touches the curve.

From the equation of the conjugate axes, we have

$$A'^2y'^2+B'^2x'^2=A'^2B'^2.$$

$$A'^2y''^2+B'^2x''^2=A'^2B'^2.$$

Subtracting one of these equations from the other, and operating as in Prop. VI, we shall find

$$a=-\frac{B'^2x'}{A'^2y'}.$$

This value of a put in (1) will give

$$y-y'=-\frac{B'^2x'}{A'^2y'}(x-x').$$

Reducing, and $A'^2y'y+B'^2x'x=A'^2B'^2$,
which is the equation sought, and it is in the same form as that in Prop. VI, agreeably to the observation made at the close of Prop. IX.

PROPOSITION XII.

To transform the equation of the ellipse in reference to conjugate diameters to an equivalent equation in reference to its rectangular axes.

The equation of the ellipse in reference to its conjugate diameter is

$$A'^2y'^2+B'^2x'^2=A'^2B'^2. \qquad (1)$$

And the formulas for passing from oblique to rectangular axes are (Prop, X, Chap. I.)

$$x'=\frac{x\sin.n-y\cos.n}{\sin.(n-m)}, \qquad y'=\frac{y\cos.m-x\sin.m}{\sin.(n-m)}.$$

These values of x' and y' substituted in (1) give

$$\left.\begin{array}{l}(A'^2\cos.^2m+B'^2\cos.^2n)y^2+(A'^2\sin.^2m+B'^2\sin.^2n)x^2\\ -2(A'^2\sin.m\cos.m+B'^2\sin.n\cos.n)xy\end{array}\right\}=$$
$$A'^2B'^2\sin.^2(n-m).$$

This equation must be true for any point in the curve, x being measured on the major axis, and y the corresponding ordinate at right angles.

This being the case, such values of A', B', m, and n, must be taken as will reduce the preceding equation to the well known form

$$A^2y^2+B^2x^2=A^2B^2.$$

Therefore we must assume

$$A'^2 \cos.^2 m + B'^2 \cos.^2 n = A^2. \quad (1)$$

$$A'^2 \sin.^2 m + B'^2 \sin.^2 n = B^2. \quad (2)$$

$$A'^2 \sin. m \cos. m + B'^2 \sin. n \cos. n = 0. \quad (3)$$

$$A'^2 B'^2 \sin.^2 (n-m) = A^2 B^2. \quad (4)$$

The values of m and n must be taken so as to respond to the following equation, because the rectangular axes are in fact *conjugate diameters.*

$$A^2 \sin. m \sin. n + B^2 \cos. m \cos. n = 0. \quad (5)$$

These equations unfold two very interesting properties.

Scholium 1. By adding (1) and (2)

$$A'^2 + B'^2 = A^2 + B^2.$$

Or $$4A'^2 + 4B'^2 = 4A^2 + 4B^2.$$

That is, the sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axes.

Scholium 2. Equation (3) or (5) will give us m when n is given; or give us n when m is given.

Scholium 3. The square root of (4) gives

$$A'B' \sin. (n-m) = AB,$$

which shows the equality of two *surfaces,* one of which is obviously the rectangle of the two axes.

Let us examine the other.

Let n represent the angle NCB, and m the angle PCB. Then the angle NCP will be represented by $(n-m)$.

Since the angle MNK is the supplement of NCP, the two angles have the same sine

$$NM = A'.$$

In the right angled triangle NKM, we have

$$1 : A' :: \sin.(n-m) : MK.$$

$$MK = A' \sin.(n - m).$$

But $$NC = B'.$$

Whence $MK \cdot NC = A'B' \sin.(n - m) =$ the parallelogram $NCPM$. Four times this parallelogram is the parallelogram ML, and four times the parallelogram $DCBH$ which is measured by AB, is equal to the parallelogram HF. Hence equation (4) reveals this general truth:

The rectangle which is formed by drawing tangent lines through the vertices of the axes is equivalent to any parallelogram which can be found by drawing tangents through the vertices of conjugate diameters.

NOTE.—The student had better test his knowledge in respect to the truths embraced in scholiums 1 and 3, by an example:

Suppose the semi-major axis of an ellipse is 10, *and the semi-minor axis* 6, *and the inclination of one of the conjugate diameters to the axis of* X *is taken at* 30° *and designated by* m.

We are required to find A'^2 and B'^2, which together should equal $A^2 + B^2$, or 136, and the area $NCPM$, which should equal AB, or 60, if the foregoing theory is true.

Equation (5) will give us the value of n as follows:

$$100 \cdot \tfrac{1}{2} \tan. n + 36 \tfrac{1}{2} \sqrt{3} = 0.$$

Or $$\tan. n = -\frac{36\sqrt{3}}{100}.$$

Log. $36 + \frac{1}{2}$ log. 3—log. 100. Plus 10 added to the index to correspond with the tables, gives 9.794863 for the log. tangent of the angle n, which gives 31° 56′ 42″, and the sign being negative, shows that 31° 56′ 42″ must be taken below the axis of X, or we must take the supplement of it, NCB. for n, whence $n = 148° \; 3' \; 18''$, and $(n - m) = 118° \; 3' \; 18''$.

To find A'^2 and B'^2, we take the formulas from Proposition IX.

$$A'^2 = \frac{A^2 B^2}{A^2 \sin.^2 30 + B^2 \cos. 30} = \frac{100 \cdot 36}{100 \cdot \frac{1}{4} + 36 \cdot \frac{3}{4}} = \frac{3600}{52} = 69.23.$$

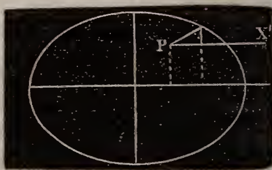
$$B'^2 = \frac{A^2 B^2}{A^2 \sin.^2 31° \, 56' \, 42'' + B^2 \cos.^2 (31° 56' 42'')} = \frac{3600}{27 \cdot 99 + 25 \cdot 92} = \frac{66 \cdot 77}{136.00}.$$

This agrees with scholium 1.

As radius		10.000000
Is to	$A'\frac{1}{2}(\log. 69.23)$	0.920147
So is sine $(n-m)$ 61° 56′ 42″		9.945713
	log. $MK=$	0.865860
Log. $B'=\frac{1}{2}$ log. (66.77)		0.912290
$AB=60$.	log. 60=	1.778150

PROPOSITION XIII.

To find the general polar equation of an ellipse.



If we designate the co-ordinates of the pole P, by a and b, and estimate the angles v from the line PX' parallel to the transverse axis, we shall have the following formulas:

$x=a+r\cos.v$. $\quad y=b+r\sin.v$.

These values of x and y substituted in the general equation

$$A^2y^2+B^2x^2=A^2B^2,$$

will produce

$$\left.\begin{array}{r} A^2\sin.^2v \\ B^2\cos.^2v \end{array}\right| r^2 \left.\begin{array}{c} +2A^2b\sin.v \\ +2B^2a\cos.v \end{array}\right| r+A^2b^2+B^2a^2=A^2B^2,$$

for the general polar equation of the ellipse.

SCHOLIUM 1. When P is at the center, $a=0$, and $b=0$, and then the general polar equation reduces to

$$r^2=\frac{A^2B^2}{A^2\sin.^2v+B^2\cos.^2v}.$$

a result corresponding to equations (P) and (Q) in Prop. IX.

SCHOLIUM 2. When P is on the curve $A^2b^2+B^2a^2=A^2B^2$, therefore

$$\left.\begin{array}{r} A^2\sin.^2v \\ B^2\cos.^2v \end{array}\right| r^2 \left.\begin{array}{c} +2A^2b\sin.n \\ +2B^2a\cos.v \end{array}\right| r=0.$$

This equation will give two values of r, one of them is 0, as it should be. The other value will correspond to a chord, according to the values assigned to a, b, and v. Dividing the last equation by the equation $r=0$, and we have

$$\left.\begin{array}{l|l} A^2 \sin.^2 v & r+2A^2 b \sin.v \\ B^2 \cos.^2 v & +2B^2 a \cos.v \end{array}\right|=0.$$

The value of r in this equation is the value of a chord.

When the chord becomes 0, the value of r in the last equation becomes 0 also, and then

$$A^2 b \sin.v+B^2 a \cos.v=0.$$

Or
$$\tan.v=-\frac{B^2 a}{A^2 b},$$

a result corresponding to Prop. VI, as it ought to do, because the *radius vector* then becomes tangent to the curve.

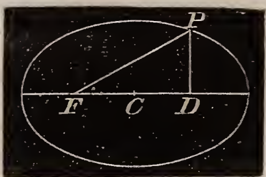
SCHOLIUM 3. When P is placed at the extremity of the major axis on the right, then $\sin.v=0$, $\cos.v=1$, $a=A$, and $b=0$. These values substituted in the general equation will reduce it to
$$B^2 r^2+2B^2 Ar=0,$$
which gives $r=0$, and $r=-2A$, obviously true results.

When P is placed at either foci, then $a=\sqrt{A^2-B^2}=c$, and $b=0$. These values substituted, and we shall have

$$(A^2 \sin.^2 v+B^2 \cos.^2 v)r^2+2B^2 a \cos.v \cdot r=B^4.$$

It is difficult to deduce the values of r from this equation. Therefore we adopt a more simple method.

Let F be the focus, and FP any radius, and put the angle $PFD=v$.



By Prop. I, of the ellipse, we learn that

$$FP=r=A+\frac{cx}{A}, \qquad (1)$$

an equation in which $c=\sqrt{A^2-B^2}$, and x any variable distance CD.

Take the triangle PDF, and by trigonometry we have

$$1 : r :: \cos.v : c+x.$$

Whence
$$x=r \cos.v-c.$$

This value of x placed in (1), will give

$$r=A+\frac{cr \cos.v-c^2}{A}.$$

Whence $(A-c\cos.v)r=A^2-c^2$

Or $r=\dfrac{A^2-c^2}{A-c\cos.v}$.

This equation will correspond to all points in the curve by giving to cos.v all possible values from 1 to -1. Hence, the greatest value of r is $(A+c)$, and the least value $(A-c)$, obvious results when the polar point is at F.

The above equation may be simplified a little by introducing the *eccentricity*. The eccentricity of an ellipse is the distance from the center to either focus, when the semi-major axis is taken as unity. Designate the eccentricity by e, then

$$1 : e=A : c.$$

Whence $c=eA$.

Substituting this value of c in the preceding equation, we have

$$r=\frac{A^2-e^2A^2}{A-eA\cos.v}=\frac{A(1-e^2)}{1-e\cos.v}.$$

This equation is much used in astronomy.

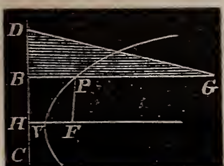
CHAPTER IV.

The Parabola.

DEFINITION.—1. A *parabola* is a plane curve, every point of which is equally distant from a fixed point and a given straight line.

2. The given point is called the *focus*, and the given line is called the *directrix*.

To describe a parabola.



Let CD be the given line, and F a given point. Take a square, as DBG, and to one side of it, GB, attach a thread, and let the thread be of the *same length as the side* GB *of the square.* Fasten one end of the thread at the point G, the other end at F.

Put the other side of the square against the given line, CD, and with a pencil, P, in the thread, bring the thread up to the side of the square. Slide one side of the square along the line CD, and at the same time keep the thread close against the other side, permitting the thread to slide round the pencil P. As the side of the square, BD, is moved along the line CD, the pencil will describe the curve represented as passing through the points V and P.

$$GP+PF= \text{the thread.}$$
$$GP+PB= \text{the thread.}$$

By subtraction $PF-PB=0$, or $PF=PB$.

This result is true at any and every position of the point P; that is, it is true for every point on the curve corresponding to definition 1. Hence, $FV=VH$.

If the square be turned over and moved in the opposite direction, the other part of the parabola, the other side of the line FH may be described.

3. A *diameter* to a parabola is a straight line drawn through any point of the curve *perpendicular to the directrix*. Thus, the line HF is a diameter; also, BG is a diameter; and all diameters are parallel to one another.

4. The point in which the diameter cuts the curve, is called the *vertex*.

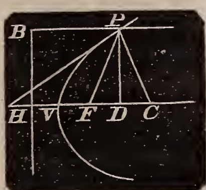
5. The axis of the parabola is the diameter which passes through the focus.

6. The parameter to any diameter is the double ordinate which passes through the focus.

7. The parameter to the principal diameter is sometimes called the *latus-rectum*.

PROPOSITION I.

To find the equation of the curve.



The vertex of the parabola is the zero point, or the origin of the co-ordinates.

The distance of the focus F, in the direction perpedicular to BH, is called p, a constant quantity, and when this constant is large, we have a parabola on a *large scale*, and when small, we have a parabola on a *small scale*.

By the definition of the curve, V is midway between F and the line BH, and $PF=PB$.

Put $VD=x$ and $PD=y$, and operate on the right angled triangle PDF.

$$FD=x-\tfrac{1}{2}p, \qquad PB=x+\tfrac{1}{2}p=PF.$$

$$(FD)^2+(PD)^2=(PF)^2.$$

That is, $$(x-\tfrac{1}{2}p)^2+y^2=(x+\tfrac{1}{2}p)^2.$$

Whence $$y^2=2px, \text{ the equation sought.}$$

Corollary 1. If we make $x=0$, we have $y=0$ at the same time, showing that the curve passes through the point V, corresponding to the definition of the curve.

As $y=\pm\sqrt{2px}$, it follows that for every value of x there are two values of y, *numerically equal*, one $+$, the other $-$, which shows that the curve is symmetrical in respect to the axis of X.

Corollary 2. If we convert the equation $(y^2=2px)$ into a proportion, we shall have

$$x : y :: y : 2p,$$

a proportion showing that *the parameter of the axis is a third proportional to any abscissa and its corresponding ordinate.*

PROPOSITION II.

The squares of ordinates to the axis are to one another as their corresponding abscissas.

Let x, y, be the co-ordinates of any point P, and x', y', the co-ordinates of any other point in the curve.

Then by the equation of the curve we must have

$$y^2=2px. \qquad (1)$$

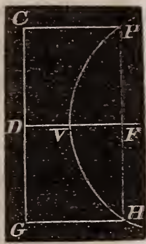
$$y'^2=2px', \qquad (2)$$

By division $$\frac{y^2}{y'^2}=\frac{x}{x'}.$$

Whence $$y^2 : y'^2 :: x : x'. \qquad \text{Q. E. D.}$$

PROPOSITION III.

The latus-rectum is four times the distance from the focus to the vertex.



Let PVH be a parabola, F the focus, and V the principal vertex. PH, at right angles to DF, through the point F, is the *latus-rectum.*

We are to prove that $PH=4FV$.

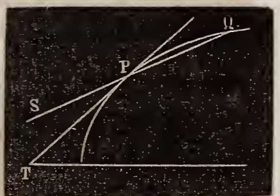
In the equation of the curve, $(y^2=2px)$ for the point P, we must necessarily make $x=\frac{1}{2}p$, then the equation becomes $y=p$. That is,

$PF=FD=2VF$, or $PH=4VF$. Q. E. D.

COROLLARY. It will be observed that CF and DH are squares, and the line DF or its equal PF is the quantity represented by p. It is the same for the same parabola, but different in different parabolas.

PROPOSITION IV.

To find the equation of a tangent line to the parabola.



Let the line SPQ cut the parabola in two points P and Q.

Let x, y, be the general co-ordinates of any point in the line as S; x', y' the co-ordinates of the point P; and x'', y'', the co-ordinates of the point Q.

The equation of a straight line which passes through the two points, S and P, must be of the form

$$y-y'=a(x-x'). \qquad (1)$$

We require the value of a when SP is tangent to the curve.

If the same line passes through the two points S and Q, we must have

$$y-y''=a(x-x''). \qquad (2)$$

And the same line passing through the two points P and Q will require the equation

$$y'-y''=a(x'-x''). \qquad (3)$$

The two points P and Q being in the curve, also require

$$y'^2=2px', \qquad (4)$$

And $$y''^2=2px''. \qquad (5)$$

By subtraction $$y'^2-y''^2=2p(x'-x'').$$

Or $$(y'-y'')(y'+y'')=2p(x'-x'') \qquad (6)$$

Dividing (6) by (3) will give

$$y'+y''=\frac{2p}{a}. \qquad (7)$$

Now conceive the line SQ to turn on the point P as a center until Q *flows** into P, then we shall have

$$y'=y''.$$

Put this value of y'' in (7), and we find

$$a=\frac{p}{y'}. \qquad (8)$$

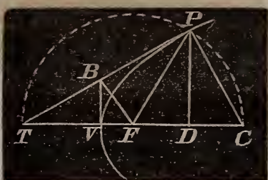
This value of a put in (1) will reduce that equation to

$$yy'-y'^2=px-px'.$$

But $$y'^2=2px'$$

By addition $$yy'=p(x+x')$$

and this is the equation sought, x, y, are the co-ordinates of any point in the line, and x', y', the co-ordinates of the tangent point in the curve.



Corollary. To find the point in which the tangent meets the axis of X, we must make $y=0$, this makes

$$p(x+x')=0.$$

Or $$x'=-x.$$

That is, $VD=VT$, *or the sub-tangent is bisected by the vertex.*

Hence, to draw a tangent line from any given point, as P, we draw the ordinate PD, then make $TV=VD$, and from the point T draw the line TP, and it will be tangent at P, as required.

**Flows.* These conceptions of motion, to make two quantities equal — or one to flow out a little in excess of the other, caused Newton to adopt the name of Fluxions.

PROPOSITION V.

To find the equation of a normal line to the parabola.

The equation of a straight line passing through the point P is

$$y-y'=a(x-x'). \quad (1) \quad \text{(Prop. IV, Eq. (1).)}$$

Let x_1, y_1, be the general co-ordinates of another line passing through the same point, and a' the tangent of its angle, its equation will then be

$$y_1-y'=a'(x_1-x'). \qquad (2)$$

But if these two lines are perpendicular to each other, we must have

$$aa'=-1. \qquad (3)$$

The first line being a tangent, makes

$$a=\frac{p}{y'}.$$

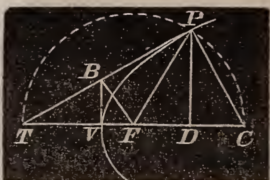
This value substituted in (3) gives

$$a'=-\frac{y'}{p}.$$

And this value put in (2) will give

$$y_1-y'=-\frac{y'}{p}(x_1-x')$$

for the equation required.



Corollary 1. To find the point in which the normal meets the axis of X, we must make $y_1=0$. Then by a little reduction we shall have

$$p=x_1-x'.$$

But $VC=x_1$, and $VD=x'$. Therefore $DC=p$, that is, *The sub-normal is a constant quantity, double the distance between the vertex and focus.*

Corollary 2. As $TV=VD$, and $VF=\frac{1}{2}DC$. $TF=FC$. Therefore, if the point F be the center of a circle and radius FC, that circle will pass through the point P, because TPC is a right angle. Hence the triangle PFT is isosceles.

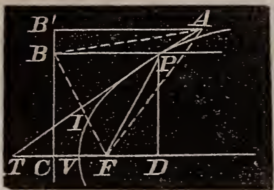
Now as V bisects TD, and VB is parallel to PD, the point B bisects TP. Join FB, and that line bisects the base of an isosceles triangle, it is therefore perpendicular to that base. Hence, we have this general truth,

If from the focus of a parabola a perpendicular be drawn to any tangent, it will meet the tangent on the axis of Y.

Also, from the two similar right angled triangles, we have

$$TF : FB :: FB : FV. \qquad \overline{BF}^2 = TF \cdot FV.$$

But FV *is constant, therefore* $(BF)^2$ *varies as* TF, *or as its equal* PF.

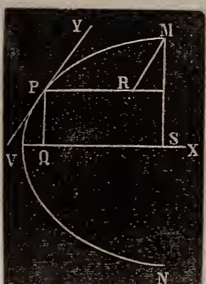


SCHOLIUM. Conceive a line drawn parallel to the axis to meet the curve at P; that line will make an angle with the tangent equal to the angle FTP, but the angle FTP is equal to the angle TPF. Therefore, conceiving this line to be a line of light, its reflection from the point P will take the direction PF, and this will be true for every other point in the curve; hence, if a reflecting mirror have a parabolic surface, all the rays of light that meet it parallel with the axis, will be reflected to the focus; and for this reason many attempts have been made to form perfect parabolic mirrors for reflecting telescopes

If a light be placed at the focus of such a mirror, it will reflect all its rays in one direction; hence, in certain situations, parabolic mirrors have been made for lighthouses, for the purpose of throwing all the light seaward.

PROPOSITION VI.

To find the equation of the parabola referred to a tangent line, and the diameter passing through the point of contact, the origin being the tangent point.



Let V be the vertex of the parabola, VX the axis, and P the origin of the coordinates.

Let $VS=x$, $SM=y$. Then

$$y^2=2px. \qquad (1)$$

Put $VQ=c$, $QP=b$, $PR=x'$, $RM=y'$, and the angle $MRS=m$.

According to this notation we have

$$VS=x=c+x'+y'\cos.m.$$

$$SM=y=b+y'\sin.m.$$

These values of x and y substituted in (1) will give

$$b^2+2by'\sin.m+y'^2\sin.^2m=2pc+2px'+2py'\cos.m. \quad (2)$$

Because P is on the curve, $b^2=2pc$, and because RM is parallel to the tangent PY, we must have (Prop. IV, Eq. (8).)

$$\frac{\sin.m}{\cos.m}=\frac{p}{b}.$$

Whence $$2by'\sin.m=2py'\cos.m.$$

This equation subtracted from (2) and $b^2=2pc$; also subtracted from (2) will reduce (2) to

$$y'^2\sin.^2m=2px'.$$

Or $$y'^2=\frac{2p}{\sin.^2m}x'.$$

If we put $\frac{2p}{\sin.^2m}=2p'$, we shall have for the equation of the curve referred to the origin P, and the oblique axes PX, PY,

$$y'^2=2p'x',$$

an equation of the same form as that referred to the vertex and rectangular axes.

Corollary 1. As the equation gives $y'=\pm\sqrt{2p'x'}$, that is, for every value of y' two values of x', numerically equal, it follows that the axis PX bisects all diameters parallel to PY.

Observe that $2p'$ may be called the parameter of the axis PX.

Corollary 2. *The squares of the ordinates of any diameter are to each other as their corresponding abscissas.*

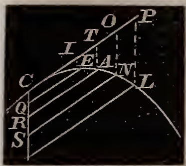
Let x, y, and x', y', be the co-ordinates of any two points on the curve, then

$$y^2=2p'x.$$

$$y'^2=2p'x'.$$

Whence $\frac{y^2}{y'^2}=\frac{x}{x'}$, or $y^2 : y'^2 :: x : x'$. Q. E. D.

SCHOLIUM. *Projectiles, if not disturbed by the resistance of the atmosphere, would describe parabolas.*



Let C be the origin of a projectile thrown in any direction as CP. Undisturbed by the atmosphere and by gravity it would continue in that line, describing equal spaces in equal times. But gravity causes bodies to fall in proportion to the squares of the times.

Hence draw IE, TA, ON, proportional to the squares of CI, CT, CO, or in proportion to the squares of their equals QE, RA, &c.

Let $CQ=IE=x$. $CR=TA=x'$. $QE=y$. $RA=y'$.

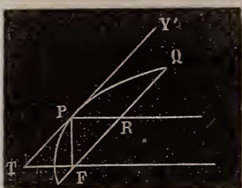
Then by the construction

$$x : x'=y^2 : y'^2.$$

But this is the property of the parabola, therefore the curve made by a projectile is a parabola.

PROPOSITION VII.

The parameter of any diameter is four times the distance from the vertex of that diameter to the focus.



We are to prove that $2p'=4PF$.

Let the angle $YPR=m$ as before.

Then by (Prop. IV,)

$$\frac{\sin. m}{\cos. m}=\frac{p}{b}. \qquad (1)$$

The co-ordinates of the point P being c, b, as in the last proposition, whence

$$b^2=2pc. \qquad (2)$$

From (1) $\quad b^2\sin.^2 m=p^2\cos.^2 m.$

$$=p^2(1-\sin.^2 m)=p^2-p^2\sin.^2 m.$$

Or $\quad \sin.^2 m=\dfrac{p^2}{b^2+p^2}=\dfrac{p^2}{2pc+p^2}=\dfrac{p}{2c+p}.$

But in the last proposition $\dfrac{2p}{\sin.^2 m}=2p'$. Whence $\sin.^2 m=\dfrac{p}{p'}$.

Therefore $\quad p'=2c+p.$

Or $\quad 2p'=4\left(c+\dfrac{p}{2}\right)$

But $\left(c+\frac{p}{2}\right)=PF$. (Prop. I.) Hence $2p'$, the parameter of the diameter PR, is *four times the distance of the origin from the focus.*

SCHOLIUM. Through the focus F draw a line parallel to the tangent PY. Designate PR by x, and RQ by y, then by (Prop. VI,)

$$y^2=2p'x.$$

But $PF=FT$. (Prop. V, Cor. 2.) And $PR=TF$, because $TFRP$ is a parallelogram. Whence $PR=PF$. But $PR=x$, and $PF=c+\frac{p}{2}$.

$$x=\left(c+\frac{p}{2}\right)$$

Therefore $\quad 4x=4\left(c+\frac{p}{2}\right)=2p'$, or $x=\frac{p'}{2}$.

This value of x put in the equation of the curve gives

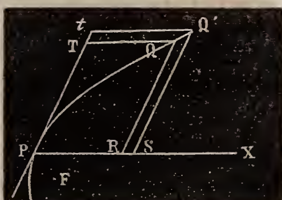
$$y=p', \quad \text{or} \quad 2y=2p'.$$

That is, the quantity $2p'$, which has been called the parameter of the diameter PR, is equal to the double ordinate passing through the focus, corresponding to Def. 6.

PROPOSITION VIII.

The area of any segment of a parabola made by co-ordinates, (whether right angled or oblique,) is equal to two-thirds of the parallelogram formed by the co-ordinates and their parallels.

Let PX be any diameter, PT a tangent, and QR an ordinate parallel to it.



Let the angle TPX, or its equal $QRX=m$.

Put $PR=x$, and $RQ=y$. Then, by the equation of the curve we have

$$y^2=2p'x. \qquad (1)$$

Now let $PS=x+h$, x is increased by RS a space which we designate by h. In consequence of the increase of x, y must be increased by Tt, which we will designate by k. Then $SQ'=y+k$.

Now the equation of the curve at Q' is

$$(y+k)^2=2p'(x+h). \qquad (2)$$

Expanding (2) and subtracting (1), and we have

$$2ky+k^2=2p'h. \qquad (3)$$

Divide by k, then

$$2y+k=2p'\left(\frac{h}{k}\right).$$

This equation is true whatever may be the values of h and k, and we can take k as *small as we please*. If we take it *extremely* small, we may omit k in the first member without any appreciable error; then

$$2y=\frac{2p'h}{k}.$$

Dividing this equation by (1) and

$$\frac{2y}{y^2}=\frac{h}{kx}.$$

Whence $\qquad 2kx=hy.$

Multiply each member by sin.m, then we shall have

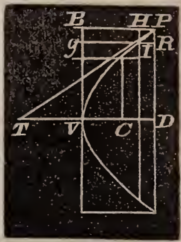
$$2x \cdot k \sin.m = h \cdot y \sin.m.$$

Now observe that $k\sin.m$ represents the perpendicular distance between the lines TQ and tQ', and that $y\sin.m$ represents the perpendicular altitude of the parallelogram $RSQQ'$, and as h is the base, $h \cdot y\sin.m$ is the area of that parallelogram, and it is equal to two parallelograms whose base is x, and perpendicular $k\sin.m$.

Now the curve space TQR may be considered as made up of a great number of parallelograms on the ordinate y, each equal to two corresponding parallelograms on the base x. Or any parallelogram external to the curve is half of a corresponding parallelogram in the curve, therefore the area of the curve is double the corresponding external space, or, the area of the segment of the curve is *two-thirds* of the parallelogram formed by the co-ordinates of the curve.

The expression for the area of the segment is

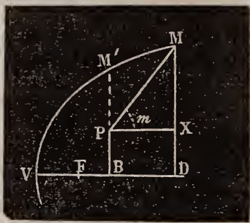
$$\tfrac{2}{3}xy\sin.m.$$



COROLLARY. When the diameter is the axis of the parabola, then $m=90°$, and $\sin.m=1$, and the expression for the area becomes $\tfrac{2}{3}xy$. That is, *every segment of a parabola at right angles with the axis is two-thirds its circumscribing rectangle.*

PROPOSITION IX.

To find the general polar equation of the parabola.



Let P be the polar point whose co-ordinates are c and b. Put $VD=x$, and $DM=y$, then by the equation of the curve we have

$$y^2=2px. \qquad (1)$$

Put $PM=R$, the angle $MPX=m$, then we shall have

$$VD=x=c+R\cos.m.$$
$$DM=y=b+R\sin.m.$$

These values of x and y substituted in (1) will give

$$(b+R\sin.m)^2=2p(c+R\cos.m). \qquad (2)$$

Expanding and reducing, (R being the unknown and variable quantity), will give us

$$R^2\sin.^2m+2R(b\sin.m-p\cos.m)=2pc-b^2$$

for the general polar equation of the parabola required.

COROLLARY 1. When P is on the curve, then $b^2=2pc$, and the general equation becomes

$$R^2\sin.^2m+2R(b\sin.m-p\cos.m)=0.$$

Here one value of R is 0, as it should be, and the other value is

$$R=\frac{2(p\cos.m-b\sin.m)}{\sin.^2m}$$

When $m=90$, $\cos.m=0$, and $\sin.m=1$. Then this last equation becomes

$$R=-2b, \quad \text{a result obviously true.}$$

COROLLARY 2. When the pole is at the focus F, then $b=0$, and $c=\frac{p}{2}$, and these values reduce the general equation to

$$R^2 \sin.^2 m-2Rp\cos.m=p^2.$$

But $$\sin.^2 m=1-\cos.^2 m.$$

Whence $$R^2-R^2\cos.^2 m-2Rp\cos.m=p^2.$$

Or $$R^2=p^2+2Rp\cos.m+R^2\cos.^2 m.$$

Or $$R=p+R\cos.m.$$

Whence $$R=\frac{p}{1-\cos.m},$$

and this is the polar equation *when the focus is the pole.*

When $m=0$, $\cos.m=1$, and then the equation becomes

$$R=\frac{p}{1-1}, \quad \text{or} \quad R=\frac{p}{0}=\text{ infinite,}$$

showing that there is no termination of the curve at the right of the focus on the axis.

When $m=90°$, $\cos.m=0$, then $R=p$, as it ought to be, because p is the ordinate passing through the focus.

When $m=180°$, $\cos.m=0$, then $R=\frac{1}{2}p$, that is, the distance from the focus to the vertex is $\frac{1}{2}p$.

As m can be taken both above and below the axis and the $\cos.m$ is the same to the same arc above and below, it follows that the curve must be symmetrical above and below the axis.

SCHOLIUM. If we take p for the unit of measure, that is, assume $p=1$, then the general polar equation will become

$$R^2 \sin.^2 m+2R(b\sin.m-\cos.m)=2c-b^2.$$

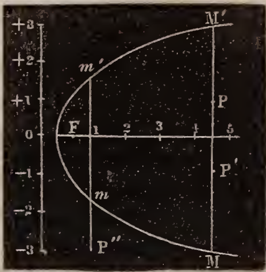
Now if we suppose $m=90°$, then $\sin.m=1$, $\cos.m=0$, and R would be represented by the line PM', and the equation would become

$$R^2+2bR=(2c-b^2),$$

and this equation is in the common form of a quadratic.

Hence, a parabola in which $p=1$ will solve any quadratic equation by making $c=VB$, $BP=b$, then PM' will give one value of the unknown quantity.

To apply this to the solution of equations, we construct a parabola as here represented.



Now suppose we require the value of y, by construction, in the following equation

$$y^2+2y=8.$$

Here $2b=2$, and $2c-b^2=8$.

Whence $b=1$, and $c=4.5$.

Lay off c on the axis, and from the extremity lay off b at right angles above the axis if b is *plus*, and below if *minus*.

This being done, we find P is the polar point corresponding to this example, and $PM'=2$ is the *plus* value of y, and $PM=-4$ is the *minus* value.

Had the equation been

$$y^2-2y=8,$$

then P' would have been the polar point, and $P'M'=4$ the plus value, and $P'M=-2$ the minus value.

For another example let us construct the roots of the following equation:

$$y^2-6y=-7.$$

Here $b=-3$, and $2c-b^2=-7$. Whence $c=1$.

From 1 on the axis take 3 downward, to find the polar point P''. Now the roots are $P''m$ and $P''m'$, both *plus*. $P''m=1.58$, and $P''m'=4.414$.

Equations having two *minus* roots will have their polar points above the curve.

When c comes out negative, the ordinates cannot meet the curve, showing that the roots would not be *real*, but *imaginary*.

The roots of equations having large numerals cannot be constructed unless the numerals are first reduced.

To reduce the numeral in any equation, as

$$y^2+72y=146,$$

we proceed as follows:

Put $y=nz$, then

$$n^2z^2+72nz=146$$

$$z^2+\frac{72}{n}z=\frac{146}{n^2}.$$

Now we can assign any value to n that we please. Suppose $n=10$, then the equation becomes

$$z^2+7{\cdot}2z=1.46.$$

The roots of this equation can be *constructed*, and the values of y are *ten* times those of z.

When the square is completed to a quadratic equation, *that square may be considered the square of an ordinate to a parabola.*

CHAPTER V.

The Hyperbola.

Definitions.—1. The *hyperbola* is a plane curve, confined by two fixed points called the *foci*, and the difference of the distances of each and every point in the curve from the two fixed points, is constantly equal to a *given line*.

Remark.—The distance between the foci, is also supposed to be known; and the *given line* must be less than the distance between the *foci*.

2. The line joining the *foci*, and produced, if necessary, is called the axis of the hyperbola.

3. The middle point of the straight line which joins the *foci*, is called the *center* of the hyperbola.

4. The *eccentricity*, is the distance from the center to either focus, divided by half the given line.

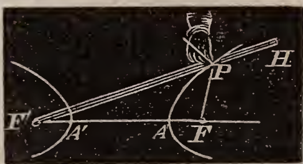
5. A diameter is any straight line passing through the center and terminated by two opposite hyperbolas.

6. The extremities of a diameter are called its *vertices*.

According to these definitions F', F, are the foci, C the center of the hyperbola, A', A, the given line, and $D'D$ a diameter.

The *parameter* is a double ordinate, passing through the focus. The *principal parameter* passes through the focus at right angles to the axis.

The definition of this curve suggests the following method of describing it mechanically:



Take a ruler $F'H$, and fasten one end at the point F', on which the ruler may turn as a hinge. At the other end of the ruler attach a thread, and let it be less than the ruler by the given line $A'A$. Fasten the other end of the thread at F.

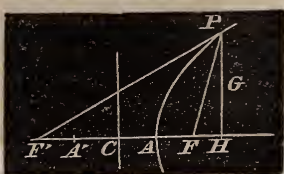
With a pencil, P, press the thread against the ruler and keep it at equal tension between the points H and F. Let the ruler turn on the point F', keeping the pencil close to the ruler and letting the thread slide round the pencil; the pencil will thus describe a curve on the paper.

If the ruler be changed and made to revolve about the other focus as a fixed point, the opposite branch of the curve can be described.

In all positions of P, except when at A or A', PF' and PF will be two sides of a triangle, and the difference of these two sides is constantly equal to the difference between the ruler and the thread; but that difference was made equal to the given line $A'A$; hence, by Def. 1, the curve thus described must be an hyperbola.

PROPOSITION I.

To find the equation of the curve in relation to the center and axis.



Let C be the zero point. Put $CA=A$. $CF=c$. $CH=x$, and $PH=y$. (P being any point in the curve). Join PF and PF'. Put $PF=r$, and $PF'=r'$.

Now we have two right angled triangles, PHF and PHF'. By the definition of the curve we have

$$r'-r=2A. \qquad (1)$$

The right angled $\triangle$ PHF gives

$$r^2=(x-c)^2+y^2. \qquad (2)$$

The right angled $\triangle$ PHF' gives

$$r'^2=(x+c)^2+y^2. \qquad (3)$$

Subtracting (2) from (3) produces

$$r'^2-r^2=4cx. \qquad (4)$$

Dividing (4) by (1), and we have

$$r'+r=\frac{2cx}{A}. \qquad (5)$$

Combining (1) and (5), we find

$$r'=A+\frac{cx}{A}, \quad \text{and} \quad r=-A+\frac{cx}{A}.$$

This value of r substituted in (2) gives

$$A^2-2cx+\frac{c^2x^2}{A^2}=x^2-2cx+c^2+y^2.$$

Reducing, we find

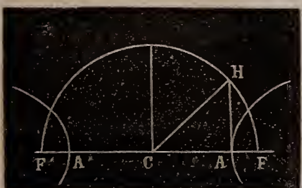
$$A^2y^2+(A^2-c^2)x^2=A^2(A^2-c^2),$$

for the equation sought.

Scholium. As c is greater than A, it follows that (A^2-c^2) must be negative, therefore we may assume this value equal to $-B^2$. Then the equation becomes

$$A^2y^2-B^2x^2=-A^2B^2.$$

This form is preferred to the former one on account of its similarity to the equation of the ellipse,—the difference is only in the negative value of B^2. Because $A^2-c^2=-B^2$, $A^2+B^2=c^2$.



Now to show the geometrical magnitude of B, take C as a center, and CF radius, and describe the circle FHF'. From A draw AH at right angles to CF. Now $CH=c$, $CA=A$, and if we put $AH=B$, we shall have $A^2+B^2=c^2$, as above. Whence AH must equal B.

PROPOSITION II.

To determine the figure of the hyperbola from its equation.

Resuming the equation

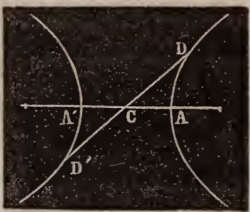
$$A^2y^2-B^2x^2=-A^2B^2,$$

From which we find

$$y=\pm\frac{B}{A}\sqrt{x^2-A^2}.$$

If we make $x=0$, or assign to it any value less than A, the corresponding value of y will be imaginary, showing that the curve does not exist above or below the line $A'A$.

If we make $x=A$, then $y=\pm 0$, showing two points in the curve, one at A, the other at A'.



If we give to x any value greater than A, we shall have two values of y, numerically equal, showing that the curve is symmetrical above and below the axis $A'A$ produced

If we now assign the same value to x taken *negatively*, that is, make x_1 $(-x)$, we shall have two other values of y, the same as before, corresponding to the left branch of the curve. Therefore, *the two branches of the curve are equal in magnitude, and are in all respects symmetrical, except opposite in position.*

Hence, every diameter as DD′ *is bisected in the center, for any other hypothesis would be absurd.*

Scholium 1. If through the center C, we draw CD, CD', at right angles to $A'A$, and each equal to B, we can have two opposite hyperbolas passing through D and D' *above* and below C, as the two others which pass through the points A' and A, at the right and left of C.

The hyperbolas which pass through D and D', are said to be *conjugate* to those which pass through A and A', or the *two pair are conjugate to each other.*

DD' is the conjugate diameter to $A'A$, and DD' may be *less, equal*, or *greater* than $A'A$, according to the relative values of c and A, in Proposition I.

When B is numerically equal to A, the equation of the curve becomes

$$y^2-x^2=-A^2,$$

and $DD'=AA'$. In this case the hyperbola is said to be *equilateral*.

Scholium 2. To find the value of the *parameter*, that is, the double ordinate which passes through the focus, we must take the equation of the curve

$$A^2y^2-B^2x^2=-A^2B^2,$$

and make $x=c$, then

$$A^2y^2=B^2(c^2-A^2).$$

But we have shown that $A^2+B^2=c^2$, or $B^2=c^2-A^2$.

Whence $$A^2y^2=B^4.$$

Or $$Ay=B^2, \quad \text{or} \quad 2y=\frac{2B^2}{A}.$$

That is, $$2A : 2B :: 2B : 2y,$$

Showing *that the parameter is a third proportional to the transverse and conjugate axes.*

Scholium 3. To find the equation for the conjugate hyperbolas which pass through the points D, D', we take the general equation

$$A^2y^2-B^2x^2=-A^2B^2,$$

and change A into B, and x into y, the equation then becomes

$$B^2x^2-A^2y^2=-A^2B^2,$$

which is the equation for conjugate hyperbolas.

PROPOSITION III.

To find the equation of the hyperbola when the origin is at the vertex of the transverse axis.

When the origin is at the center, the equation is

$$A^2y^2-B^2x^2=-A^2B^2.$$

And now if we move the origin to the vertex at the right, we must put

$$x=A+x'.$$

Substituting this value of x in the equation of the center, we have

$$A^2y^2-B^2x'^2-2B^2Ax'=0.$$

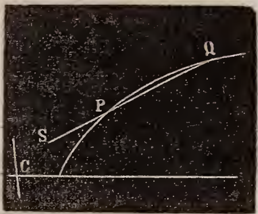
We may now omit the accents, and put the equation under the following form,

$$y^2=\frac{B^2}{A^2}(x^2+2Ax),$$

which is the equation of the hyperbola when the origin is the vertex and the co-ordinates rectangular.

PROPOSITION IV.

To find the equation of a tangent line to the hyperbola, the origin being the center.

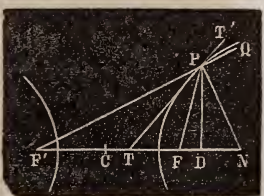


In the first place conceive a line cutting the curve in two points, P and Q. Let x and y be co-ordinates of any point on the line, as S, x' and y' co-ordinates of the point P on the curve, and x'' and y'' the co-ordinates of the point Q on the curve.

The student can now work through the proposition in precisely the same manner as Proposition VI, of the ellipse was worked, except using the equation for the hyperbola in place of that of the ellipse, and in conclusion we shall find

$$A^2yy'-B^2xx'=-A^2B^2,$$

for the equation sought.



Corollary. To find the point in which a tangent line cuts the axis of X, we must make $y=0$, in the equation for the tangent; then

$$x=\frac{A^2}{x'}=CT.$$

If we subtract this from CD, (x') we shall have

$$TD=x'-\frac{A^2}{x'}=\frac{x'^2-A^2}{x'}.$$

PROPOSITION V.

To find the equation of a normal to the hyperbola.

Let a be the trigonometrical tangent of the line TP, (see last figure,) and a' the trigonometrical tangent to the line PN. Then if PN is a normal, it must be at right angles to PT, and hence we must have

$$aa'+1=0. \qquad (1)$$

Let x' and y' be the co-ordinates of the point P on the curve, and x, y, the general co-ordinates of any point on the line PN, then we must have

$$y-y'=a'(x-x'). \qquad (2)$$

In working the last proposition, for the tangent line PT we should have found

$$a=\frac{B^2x'}{A^2y'}.$$

This value of a put in (1) will show us that

$$a'=-\frac{A^2y'}{B^2x'}.$$

And this value of a' put in (2) will give us

$$y-y'=-\frac{A^2y'}{B^2x'}(x-x'),$$

for the equation of the normal required.

Corollary. To find the point in which the normal cuts the axis of X, we must make $y=0$.

This reduces the equation to

$$1=\frac{A^2}{B^2x'}(x-x').$$

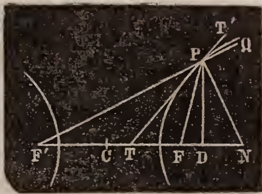
Whence $$x=\left(\frac{A^2+B^2}{A^2}\right)x'=CN.$$

If we subtract CD, (x'), from CN, we shall have DN, the *sub-normal.*

That is, $$\left(\frac{A^2+B^2}{A^2}\right)x'-x'=\frac{B^2x'}{A^2},$$ the *sub-normal.*

PROPOSITION VI.

A tangent to the hyperbola bisects the angle contained by lines drawn from the point of contact to the foci.



If we can prove that

$$F'P : PF :: F'T : TF, \quad (1)$$

it will then follow (Theorem 25, Book II, Geometry,) that the angle $F'PT=$ the angle TPF.

In Prop. I, of the hyperbola, we find that

$$F'P=r'=A+\frac{cx}{A}, \text{ and } FP=r=-A+\frac{cx}{A}.$$

$$F'T=F'C+CT=c+\frac{A^2}{x}, \text{ and } TF=c-\frac{A^2}{x}.$$

We will now assume the proportion

$$\left(A+\frac{cx}{A}\right) : \left(-A+\frac{cx}{A}\right) :: \left(c+\frac{A^2}{x}\right) : z. \quad (2)$$

Multiply the first couplet by A, and the last couplet by x, then we shall have

$$(A^2+cx) : (-A^2+cx) :: (cx+A^2) : xz.$$

Observing that the first and third terms are equal, therefore

$$xz=cx-A^2.$$

Or $$z=c-\frac{A^2}{x}=TF.$$

Now the first three terms of proportion (2) were taken equal to the first three terms of proportion (1), and we have proved that the fourth term of (2) must be equal to the fourth term of (1), therefore proportion (1) is true, and consequently

$$F'PT=TPF.$$

COROLLARY 1. As TT' is a tangent, and PN its normal, it follows that the angle $TPN=$ the angle $T'PN$, for each is a right angle. From these equals take away the equals TPF, $T'PQ$, and the remainder FPN must equal the remainder QPN. That is, *the normal line bisects the exterior angle formed by two lines drawn from the foci to any point in the curve.*

Corollary 2. The value of CT we have found to be $\frac{A^2}{x}$, and the value of CD is x, and it is obvious that

$$\frac{A^2}{x} : A :: A : x,$$

is a true proportion. *Therefore* (A) *is a mean proportional between* CT and CD.

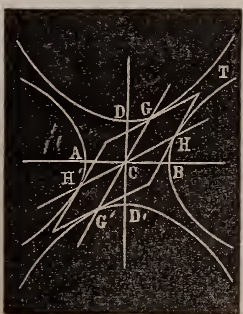
A tangent line can never meet the axis in the center, because the above proportion must always exist, and to make the first term *zero in value*, we must suppose x to be *infinite*. *Therefore a tangent line passing through the center cannot meet the hyperbola short of an infinite distance therefrom.*

Such a line is called an *asymptote.*

On Conjugate Diameters.

Definition.—"*Two diameters of an hyperbola are said to be conjugate to one another when each is parallel to a tangent line drawn through the vertex of the other.*"

According to this definition, GG' and HH' in the adjoining figure are conjugate diameters.



Explanation 1.—The tangent line which passes through the point H is parallel to CG. Hence CG makes the same angle with the axis as that tangent line does.

If we designate the co-ordinates of the point H, in reference to the center and axis by x' and y', and a the tangent of the angle made by the inclination of CG with the axis, then in the investigation (Proposition IV,) we find

$$a=\frac{B^2x'}{A^2y'}. \qquad (1)$$

Now if we designate the tangent of the angle which CH makes with the axis by a, the equation of CH must be of the form

$$y'=a'x',$$

because the line passes through the center.

Whence $$a'=\frac{y'}{x'}. \qquad (2)$$

Multiplying (1) and (2) together, and we find

$$aa'=\frac{B^2}{A^2},$$

to which equation *all conjugate diameters must correspond.*

EXPLANATION 2.—If we designate the angle GCB by n, and HCB by m, we shall have

$$\frac{\sin. m}{\cos. m}=a', \qquad \frac{\sin. n}{\cos. n}=a.$$

And $$\tan. m.\tan. n=\frac{B^2}{A^2}.$$

PROPOSITION VII.

To find the equation of the hyperbola referred to its center and conjugate diameters.

The equation for the center and axis is

$$A^2y^2-B^2x^2=-A^2B^2.$$

Now to change rectangular co-ordinates into oblique, the origin being the same, we must put

$$\left.\begin{aligned} x&=x'\cos.m+y'\cos.n. \\ \text{And} \quad y&=x'\sin.m+y'\sin.n. \end{aligned}\right\} \text{Chap. I, Prop. X.}$$

These values of x and y substituted in the above general equation, will produce

$$\left.\begin{aligned} (\sin.^2nA^2-\cos.^2nB^2)y'^2+(\sin.^2mA^2-\cos.^2mB^2)x'^2 \\ 2(\sin.m\sin.nA^2-\cos.m\cos.nB^2)x'y' \end{aligned}\right\} = -A^2B^2 \qquad (1)$$

Because the diameters are conjugate, we must have

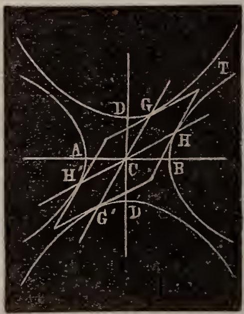
$$\frac{\sin.m}{\cos.m}\cdot\frac{\sin.n}{\cos.n}=\frac{B^2}{A^2}.$$

Whence $(\sin.m \sin.nA^2 - \cos.m \cos.nB^2) = 0.$ (k)

This last equation reduces (1) to [(2)

$$(\sin.^2 nA^2 - \cos.^2 n)y'^2 + (\sin.^2 mA^2 - \cos.^2 mB^2)x'^2 = -A^2B^2,$$

which is the equation of the hyperbola referred to the center and conjugate diameters.



If we make $y'=0$, we shall have

$$x'^2 = \frac{-A^2B^2}{(\sin.^2 mA^2 - \cos.^2 mB^2)} = \overline{CH}^2 \quad (3)$$

If we make $x'=0$, we shall have

$$y'^2 = \frac{-A^2B^2}{(\sin.^2 nA^2 - \cos.^2 nB^2)} = \overline{CG}^2 \quad (4)$$

If we put A'^2 to represent $\overline{CH}^2$, and regard it as *positive*, the denominator of (3) must be negative, the numerator being negative. That is, $\sin.^2 mA^2$ must be less than $\cos.^2 mB^2$.

That is, $\sin.^2 mA^2 < \cos.^2 mB^2.$

Or $\tan.m < \frac{B}{A}.$

But $\tan.m \tan.n = \frac{B^2}{A^2}.$

Whence $\tan.n > \frac{B}{A},$ or $\sin.^2 nA^2 > B^2 \cos.^2 n.$

Therefore the denominator in (4) is positive, but the numerator being negative, therefore $\overline{CH}^2$ must be negative. Put it equal to $-B'^2$.

Now equations (3) and (4) become

$$A'^2 = \frac{-A^2B^2}{(\sin.^2 mA^2 - \cos.^2 mB^2)}, \quad -B'^2 = \frac{-A^2B^2}{(\sin.^2 nA^2 - \cos.^2 nB^2)}.$$

Or $(\sin.^2 mA^2 - \cos.^2 mB^2) = \frac{-A^2B^2}{A'^2},$

$$(\sin.^2 nA^2 - \cos.^2 nB^2) = \frac{A^2B^2}{B'^2}.$$

Comparing these equations with equation (2) we perceive that equation (2) may be written thus :

$$\frac{A^2B^2}{B'^2}y'^2-\frac{A^2B^2}{A'^2}x'^2=-A^2B^2.$$

Whence $$A'^2y'^2-B'^2x'^2=-A'^2B'^2.$$

Omitting the accents of x' and y', since they are general variables, we have

$$A'^2y^2-B'^2x^2=-A'^2B'^2,$$

for the equation of the hyperbola referred to its center and *conjugate diameters.*

Scholium 1. As this equation is precisely similar to the general equation referred to the center and rectangular co-ordinates, it follows that all results hitherto determined in respect to the center and rectangular co-ordinates will apply to conjugate diameters by changing A to A', and B to B'.

For instance, the equation for a tangent line in respect to the center and axes has been found to be

$$A^2yy'-B^2xx'=-A^2B^2.$$

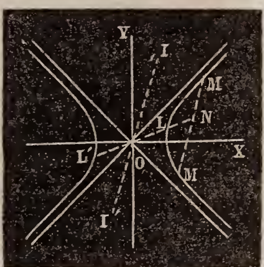
Therefore in respect to conjugate diameters it must be

$$A'^2yy'-B'^2xx'=-A'^2B'^2,$$

and so on, for normals, sub-normals, tangents, and sub-tangents.

Scholium 2. If we take the equation

$$A'^2y^2-B'^2x^2=-A'^2B'^2,$$



and resolve it in relation to y, we shall find that for every value of x greater than A', we shall find two values of y numerically equal, which shows that ON bisects MM and every line drawn parallel to MM, or parallel to a tangent drawn through L, the vertex of the diameter A'.

Observation.—Let the student observe that these several geometrical truths were discovered by changing rectangular to oblique co-ordinates. We will now take the reverse operation, in the hope of discovering other geometrical truths.

Hence the following :

PROPOSITION VIII.

To change the equation of the hyperbola in reference to oblique co-ordinates, to an equivalent equation in reference to rectangular co-ordinates.

The equation for the hyperbola in respect to oblique co-ordinates is

$$A'^2y'^2 - B'^2x'^2 = -A'^2B'^2.$$

To change oblique to rectangular co-ordinates, the formulas are (Chap. I, Prop. X.)

$$x' = \frac{x\sin.n - y\cos.n}{\sin.(n-m)}, \qquad y' = \frac{y\cos.m - x\sin.m}{\sin.(n-m)}.$$

Substituting these values of x' and y' in the equation, we shall have

$$\frac{A'^2(y\cos.m - x\sin.m)^2}{\sin.^2(n-m)} - \frac{B'^2(x\sin.n - y\cos.n)^2}{\sin.^2(n-m)} = -A'^2B'^2$$

By expanding and reducing, we shall have

$$\left.\begin{array}{l}(A'^2\cos.^2m - B'^2\cos.^2n)y^2 + (A'^2\sin.^2m - B'^2\sin.^2n)x^2 \\ \quad 2(-A'^2\sin.m\cos.m + B'^2\sin.n\cos.n)xy\end{array}\right\}$$
$$= -A'^2B'^2\sin.^2(n-m),$$

which must be a true equation of the hyperbola corresponding to the center and rectangular axes. Therefore it must take the well known form

$$A^2y^2 - B^2x^2 = -A^2B^2.$$

Or in other words, these two equations must be, *in fact*, identical, and we must have

$$A'^2\cos.^2m - B'^2\cos.^2n = A^2. \qquad (1)$$

$$A'^2\sin.^2m - B'^2\sin.^2n = -B^2. \qquad (2)$$

$$-A'^2\sin.m\cos.m + B'^2\sin.n\cos.n = 0. \qquad (3)$$

$$-A'^2B'^2\sin.^2(n-m) = -A^2B^2. \qquad (4)$$

By adding (1) and (2), observing that $(\cos.^2 m+\sin.^2 m)=1$, we shall have

$$A'^2-B'^2=A^2-B^2,$$

Or $$4A'^2-4B'^2=4A^2-4B^2,$$

which equation shows this general *geometrical truth:*

That the difference of the squares of any two conjugate diameters is equal to the difference of the squares of the axes.

Hence, there can be no equal conjugate diameters unless $A=B$, and then every diameter *will be equal to its conjugate:* that is $A'=B'$.

Equation (3) corresponds to $\tan.m\tan.n=\frac{B'^2}{A'^2}$, the equation of condition for conjugate axes.

Equation (4) reduces to

$$A'B'\sin.(n-m)=AB.$$

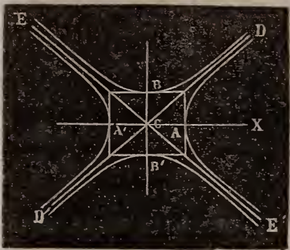
The first member is the trigonometrical measure of the parallelogram $GCHT$, and it being equal to AB, shows this geometrical truth:

That the parallelogram formed by drawing tangent lines through the vertices of conjugate diameters, is equivalent to the rectangle formed by drawing tangent lines through the vertices of the axes.

Remark.—The reader should observe that this proposition is similar to (Prop. XI,) of the ellipse, and the general equation here found, and the incidental equations (1), (2), (3), and (4), might have been directly deduced from the ellipse by changing B into $B\sqrt{-1}$, and B' into $B'\sqrt{-1}$. But learners would generally demur at results so summarily obtained.

On the Asymptotes of the Hyperbola.

Definition.—If tangents to four conjugate hyperbolas be drawn through the vertices of the axes, the diagonals of the rectangle so formed and produced indefinitely, are called *asymptotes* of the hyperbola.



Let AA', BB', be the axes of four conjugate hyperbolas, and through the vertices A, A', B, B', let tangents to the curves be drawn forming the rectangle, as seen in the figure. The diagonals of this rectangle produced, that is, DD' and EE', are the *asymptotes* to the curve corresponding to the definition.

If we represent the angle DCX by m, $E'CX$ will be m also, for these two angles are equal because $CB=CB'$.

It is obvious that

$$\tan.m=\frac{B}{A}.$$

Whence

$$\frac{\sin.^2 m}{\cos.^2 m}=\frac{B^2}{A^2}.$$

But $\cos.^2 m=1-\sin.^2 m$. Therefore

$$\frac{\sin.^2 m}{1-\sin.^2 m}=\frac{B^2}{A^2}.$$

Consequently $\sin.^2 m=\dfrac{B^2}{A^2+B^2}$, and $\cos.^2 m=\dfrac{A^2}{A^2+B^2}$,

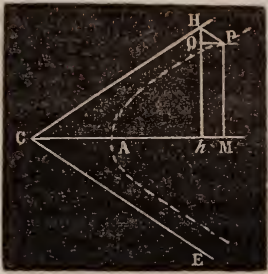
which equations furnish the value of the angle which the asymptotes form with the transverse axis.

PROPOSITION IX.

To find the general equation of the hyperbola, referred to its center and asymptotes.

Let $CM=x$, and $PM=y$. Then the equation of the curve referred to its center and axes is

$$A^2y^2-B^2x^2=-A^2B^2. \qquad (1)$$



From P draw PH parallel to CE, and PQ parallel to CM. Let $CH=x'$, and $HP=y'$.

Now the object of this proposition is to find the values of x and y in terms of x' and y', to substitute in (1), and then the equation reduced to its most simple form will be the equation sought.

The angle HCM is designated by m, and because HP is parallel to CE, and PQ parallel to CM, the angle $HPQ=m$ also.

Now in the right angled triangle CHh we have $Hh=x'\sin.m.$ and $Ch=x'\cos.m.$

In the right angled triangle PQH we have $HQ=y'\sin.m.$ and $PQ=y'\cos.m.$

Whence $Hh-HQ=Qh=PM=y=x'\sin.m-y'\sin.m.$

Or $$y=(x'-y')\sin.m. \qquad (2)$$

$$Ch+QP=CM=x=x'\cos.m+y'\cos.m.$$

Or $$x=(x'+y')\cos.m. \qquad (3)$$

These values of y and x found in (2) and (3) substituted in (1) will give

$$A^2(x'-y')^2\sin.^2m-B^2(x'+y')^2\cos.^2m=-A^2B^2.$$

Taking the values of $\sin.^2m$ and $\cos.^2m$, previously determined, we have

$$\frac{A^2B^2}{A^2+B^2}(x'-y')^2-\frac{A^2B^2}{A^2+B^2}(x'+y')^2=-A^2B^2.$$

Dividing by A^2B^2, and multiplying by (A^2+B^2), will give

$$(x'-y')^2-(x'+y')^2=-(A^2+B^2).$$

Or $$-4x'y'=-(A^2+B^2).$$

Or $$x'y'=\frac{A^2+B^2}{4},$$

which is the equation of the hyperbola referred to its center and asymptotes.

Corollary. As x' and y' are general variables, we may omit the accents, and as the second member is a constant quantity, we may represent it by M^2. Then

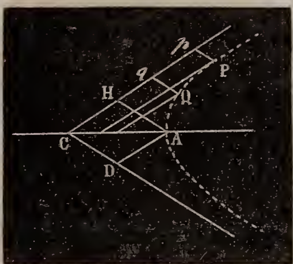
$$xy=M^2, \quad \text{or} \quad x=\frac{M^2}{y}.$$

This last equation shows that x increases as y decreases; that is, *the same curve approaches nearer and nearer the asymptote as the distance from the center becomes greater and greater.*

But x can never become infinite until y becomes 0; that is, *the asymptote meets the curve at an infinite distance,* corresponding to Cor. 2, Prop. VI.

PROPOSITION X.

All parallelograms between the asymptotes and the curve are equal, and each equal to $\frac{1}{2}$AB.



Let x and y be the co-ordinates corresponding to any point in the curve, as P. Then by the equation of the curve in relation to the center and asymptotes, we have

$$xy=M^2. \qquad (1)$$

Also let x' represent Cq, and y' qQ, that is, x', y', co-ordinates of the point Q. Then

$$x'y'=M^2. \qquad (2)$$

The angle pCD between the asymptotes we will represent by $2m$. Now multiply equations (1) and (2) by sin.$2m$.

Then we shall have

$$xy \sin.2m=M^2 \sin.2m. \qquad (3)$$

$$x'y' \sin.2m=M^2 \sin.2m. \qquad (4)$$

The first member of (3) represents the parallelogram CP, and the first member of (4) represents the parallelogram CQ; and as each of these parallelograms is equal to the same constant quantity, *they are equal to each other.*

Now A is another point in the curve, and therefore the parallelogram $AHCD$ is equal to (M^2 sin.$2m$), and therefore equal to

CQ, or *CP*. Hence all parallelograms bounded by the asymptotes and terminating in a point in the curve, are equal to one another, and each equal to the parallelogram *AHCD*, which has for one of its diagonals half of the transverse axis of *A*.

We have now to show the analytical expression for this parallelogram.

The angle $HCA=m$, $ACD=m$, and because *AH* is parallel to *CD*, $CAH=m$. Hence, the triangle *CAH* is isosceles, and $CH=HA$. The angle $AHq=2m$. Now by trigonometry

$$\sin.2m : A :: \sin.m : CH.$$

But $\sin.2m=2\sin.m\cos.m$. Whence

$$2\sin.m\cos.m : A :: \sin.m : CH.$$

$$CH=\frac{A}{2\cos.m}.$$

Multiply each member of this equation by $CA=A$ and $\sin.m$, then

$$A.(CH)\sin.m=\frac{A^2}{2}\,\frac{\sin.n}{\cos.m}=\frac{A^2}{2}\tan.m.$$

The first member of this equation represents the area of the parallelogram *CHAD*, and the $\tan.m=\frac{B}{A}$. Hence the parallelogram is equal $\frac{A^2}{2}\cdot\frac{B}{A}=\frac{1}{2}AB$, which is the value also of all the other parallelograms, as *CQ*, *CP*, &c.

Scholium. When the asymptotes and any point in the curve are given, other points may be determined by the equation.

For instance, let the asymptotes be given in position, and *Q* a given point in the curve whose co-ordinates are x' and y', then

$$x'y'=M^2.$$

Let *Cp*, any assumed distance, be represented by *b*, and *pP* by *y*, then

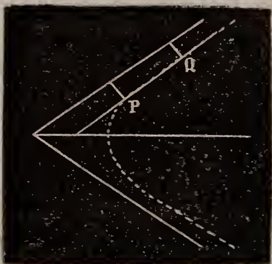
$$by=M^2=x'y'.$$

Or $$y=\frac{x'y'}{b}.$$

That is, let the numerical value of pP be equal to $\frac{x'y'}{b}$, then P will be a point in the curve — and thus any other point may be found when the distance along the asymptote is given.

PROPOSITION XI.

To find the equation of a tangent line to the hyperbola referred to its center and asymptotes.



Let x, y, be the general co-ordinates of a straight line passing through the two points P and Q.

Then the equation of the line must be of the form

$$y=ax+b. \qquad (1)$$

The same line passing through the point P, whose co-ordinates are x', y', must be

$$y'=ax'+b. \qquad (2)$$

And the same line passing through the point Q, whose co-ordinates are x'', y'', must be

$$y''=ax''+b. \qquad (3)$$

Subtracting (2) from (1), and

$$y-y'=a(x-x'). \qquad (4)$$

Subtracting (3) from (2), and

$$y'-y''=a(x'-x''). \qquad (5)$$

Now the object is to find the value of a when the line becomes a tangent at P.

From (5) we have

$$a=\frac{y'-y''}{x'-x''}.$$

Which value of a substituted in (4) gives

$$y-y'=\frac{y'-y''}{x'-x''}(x-x'). \qquad (6)$$

But because P and Q are points in the curve, we have

$$x'y'=x''y''.$$

From each member of this last equation subtract $x'y''$, then

$$x'y'-x'y''=x''y''-x'y''.$$

Or $$x'(y'-y'')=-y''(x'-x'').$$

Whence $$\frac{y'-y''}{x'-x''}=-\frac{y''}{x'}.$$

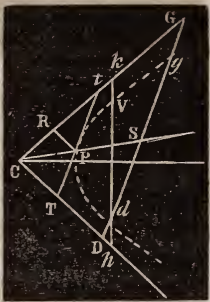
This value of the tangent angle put in (6) gives

$$y-y'=-\frac{y''}{x'}(x-x'). \qquad (7)$$

Now if we suppose the line to revolve on the point P as a center until Q coincides with P, then the line will be a tangent, and $x'=x''$, and $y'=y''$, and (7), will become

$$y-y'=-\frac{y'}{x'}(x-x'),$$

which is the equation sought.



Corollary. To find the point in which the tangent line meets the axis of X, we must make $y=0$; then

$$x=2x'.$$

That is, Ct is double CR, and as RP and CT are parallel, $tP=PT$.

A tangent line included between the asymptotes is bisected by the point of tangency.

Scholium. From any point, as D, draw DG parallel to Tt, and from C draw CP, and produce it to S.

By Scholium 2, to Prop. VII, we learn that CP produced will bisect all lines parallel to tT and within the curve; hence gd is bisected in S.

But as CP bisects tT, it bisects all lines parallel to tT within the asymptotes, and DG is also bisected in S; hence $dD=Gg$.

In the same manner we might prove $dh=kv$, because hk is parallel to some tangent which might be drawn to the curve, the same as DG is parallel to the *particular* tangent tT.

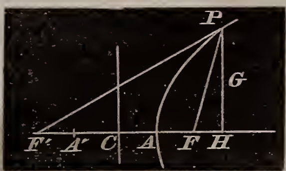
Hence, *If any line be drawn cutting the hyperbola, the parts between the asymptotes and the curve are equal.*

This property enables us to describe the hyperbola by points, when the asymptotes and one point in the curve are given.

Through the given point d, draw any line, as DG, and from G set off $Dg=dD$, and then g will be a point in the curve. Draw any other line, as hk, and set off $kv=dh$, then v is another point in the curve. And thus we might find other points between v and g, or on either side of v and g.

PROPOSITION XII.

To find the polar equation of the hyperbola, the pole being at either focus.



Take any point P in the hyperbola, and let its distance from the nearest focus be represented by r, and its distance from the other focus be represented by r'.

Put $CH=x$, $CF=c$, and $CA=A$. Then by Prop. I, we have

$$r=-A+\frac{cx}{A}, \quad (1) \qquad r'=A+\frac{cx}{A}. \quad (2)$$

Now the problem requires us to remove the symbol x, and replace its value by some quantity expressing the value of the sine or cosine which r and r' make with the transverse axis.

1st. In the right angled triangle PFH, if we designate the angle PFH by v, we shall have

$$1 : r :: \cos.v : FH=r\cos.v.$$

$$CH=CF+FH. \quad \text{That is,} \quad x=c+r\cos.v.$$

The value of x put in (1), gives

$$r=-A+\frac{c^2+cr\cos.v}{A}.$$

Whence
$$r=\frac{c^2-A^2}{A-c\cos.v}. \qquad (3)$$

2d. In the right angled triangle $F'PH$, if we designate the angle $PF'H$ by v', we shall have

$$1 : r' :: \cos.v' : F'H=r'\cos.v'.$$

But $F'H=F'C+CH$. That is, $r'\cos.v'=c+x$.

Or $x=r'\cos.v'-c$, and this value of x put in (2) gives

$$r'=A+\frac{cr'\cos.v'-c^2}{A}.$$

Whence $$r'=\frac{A^2-c^2}{A-c\cos.v'}. \qquad (4)$$

Equations (3) and (4) are the polar equations required.

Let us examine (3). Suppose $v=0$, then $\cos.v=1$, and

$$r=\frac{c^2-A^2}{A-c}=-A-c.$$

But a radius vector can never be a *minus* quantity, therefore there is no portion of the curve in the direction of the axis to the right of F.

To find the length of r, when it first strikes the curve, we find the value of the denominator when its value first becomes positive, which must be when A becomes equal or greater than $c\cos.v$; that is, when the denominator is 0, the value of r will be real and infinite.

If $$A-c\cos.v=0,$$

Then $$\cos.v=\frac{A}{c}.$$

This equation shows that when r first meets the curve, it is parallel to the asymptote, and infinite.

When $v=90°$, $\cos.v=0$, and then r is perpendicular at the point F, and equal to $\frac{c^2-A^2}{A}$, or $\frac{B^2}{A}$, half the parameter of the curve, as it ought to be.

When $r=180°$, then $\cos.v=-1$, and $-c\cos.v=c$; then

$$r=\frac{c^2-A^2}{c+A}=c-A=FA,$$ a result obviously true.

Now let us examine equation (4). If we make $v'=0$, then

$$r'=\frac{A^2-c^2}{A-c}=A+c=F'A,$$ as it ought to be.

To find when r' will have the greatest possible value, we must put

$$A - c\cos.v' = 0.$$

Whence $$\cos.v' = \frac{A}{c}.$$

Showing, that v' is then of such a value as to make r' parallel to the *asymptote*, and infinite in length. If we increase the value of v' from this point, the denominator will become positive, while the numerator is negative, which shows that then r' will become negative, indicating that it will not meet the curve.

General Remarks.

When the origin of co-ordinates is at the circumference of a circle, its equation is

$$y^2 = 2Rx - x^2.$$

When the origin of a parabola is at its vertex, its equation is

$$y^2 = 2px.$$

When the origin of co-ordinates of the ellipse is at the vertex of the major axes, the equation of the curve is

$$y^2 = \frac{B^2}{A^2}(2Ax - x^2).$$

When the origin of co-ordinates is on the vertex of the hyperbola, the equation for that curve is

$$y^2 = \frac{B^2}{A^2}(2Ax + x^2).$$

But all of these are comprised in the general equation

$$y^2 = 2px + qx^2.$$

In the circle and the ellipse q is negative; in the hyperbola it is positive, and in the parabola it is 0.

SECTION II.

CHAPTER I.

On the geometrical representation of Equations of the second degree between two variables.

It has been shown in Chap. I, Sec. 1, that every equation of the first degree between two variables may be represented by a straight line.

It has also been shown that the equation of the circle, the equation of the ellipse, the equation of the parabola, and the equation of the hyperbola, each and all correspond to equations of the second degree between two variables; and hence, we might naturally infer that a general equation of the second degree *must represent one or the other of these curves.*

Within the limits designed for this work, there is not space to demonstrate this truth rigorously, but we will illustrate it and bring it to the comprehension of the learner, partly by general theory, and partly by examples.

An equation of the second degree, in its most comprehensive form, is represented as follows:

$$Ay^2+Bxy+Cx^2+Dy+Ex+F=0.$$

Observe that this equation contains the first and second powers of each of the variables, their product, and an absolute term, F.

The co-efficients A, B, C, &c. may be *plus*, *minus*, or *zero*, although they are represented above as plus.

Resolving this equation in relation to y, we obtain [(1)

$$y=-\frac{1}{2A}(Bx+D)\pm\frac{1}{2A}\sqrt{\left.\begin{matrix}+B^2\\-4AC\end{matrix}\right| x^2+\left.\begin{matrix}2BD\\-4AE\end{matrix}\right| x+\begin{matrix}D^2\\-4AF\end{matrix}}$$

Now whatever value may be assigned to x, the equation will give the corresponding value of y, and if we assume x to be of such a value as to make the quantity under the radical equal to 0, we shall have

$$\left.\begin{matrix}+B^2\\-4AC\end{matrix}\right|x^2\left.\begin{matrix}+2BD\\-4AE\end{matrix}\right|x\left.\begin{matrix}+D^2\\-4AF\end{matrix}\right|=0. \qquad (2)$$

And $$y=-\frac{B}{2A}x-\frac{D}{2A}. \qquad (3)^*$$

Equation (3) is the equation of a straight line which can easily be constructed, and this line will be the same, whatever value may be assigned to x.

Equation (2) is an equation of the second degree, and therefore it may represent a curve; *hence equation* (1), *which is the sum of* (3) *and* (2), *will represent a curve branching out of a straight line.*

We will illustrate this general equation by the following particular example :

Find or construct the curve represented by the equation

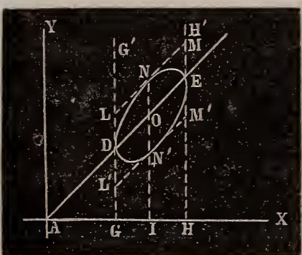
$$y^2-2xy+2x^2-3x+2=0.$$

Here $A=1$, $B=-2$, $C=2$, $D=0$, $E=-3$, $F=2$.

These values substituted in (1) give

$$y=x\pm\tfrac{1}{2}\sqrt{-4x^2+12x-8}.$$

Or $$y=x\pm\sqrt{-x^2+3x-2}. \qquad (1)$$



If we put the part under the radical equal to zero we shall have

$$y=x$$

And $$-x^2+3x-2=0.$$

The first of these represents the straight line AE, passing through the origin A at an angle of 45° with the axis of X.

* When A and B in the original equation, have the same sign, the tangent of the angle which the line makes with the axis of X, is *minus* ; when they have unlike signs, that tangent is *plus*.

The second part resolved, gives $x=1$, or $x=2$. Taking $x=1$, equation (1) becomes

$$y=1\pm\sqrt{0}.$$

And taking $x=2$, the same equation becomes

$$y=2\pm\sqrt{0}.$$

The first result corresponds to the point D, the second to the point E, and DE is the diameter of the curve.

To find its *conjugate* diameter NN', we must make x correspond to the point I, the middle point between $AG=1$ and $AH=2$. Hence we must make $x=1\frac{1}{2}$, and substituting this value in equation (1) we have

$$y=1\tfrac{1}{2}\pm\sqrt{-\tfrac{9}{4}+\tfrac{9}{2}-2}.$$

Whence $$y=1\tfrac{1}{2}\pm\tfrac{1}{2}=2 \text{ or } 1.$$

The first result is $IN=2$, the second is $IN'=1$, and therefore $N'N=1$, the conjugate diameter sought.

It is obvious from the figure and the values of lines already discovered, that $DE=\sqrt{2}$.

If we assign to x a value greater than 0, and less than 1, the value of the expression under the radical $(-x^2+3x-2)$ will be negative, and hence its square root is impossible, or imaginary, and the corresponding value of y imaginary, showing that the ordinate would not in that case meet the curve. Again, if we take x greater than 2, we shall find a like result. Hence the curve must be between the parallels GG' and HH'.

The curve must also be within the parallels LM and $L'M'$. Hence it is an ellipse within the parallelogram $LL'M'M$, and DE and $N'N$ are its conjugate diameters, and their angle of inclination as shown in this example is 45°.

Now by the well known properties of the ellipse we can find the rectangular axes and their inclination from these *conjugate axes.*

If we simply wish to determine whether the curve or line cuts either co-ordinate, we take the equation

$$y^2-2xy+2x^2-3x+2=0,$$

and make $x=0$, then $y^2=-2$, which makes y *imaginary*, showing that the curve does not cut the axis of Y.

Now if we make $y=0$ in the equation, we have

$$2x^2-3x+2=0.$$

Whence x is *imaginary*, showing also that the curve does not cut the axis of X.

It appears from the preceding example that the equation

$$y=-\tfrac{1}{2}A(Bx+D)\pm\tfrac{1}{2}A\sqrt{\left.\begin{matrix}+B^2\\-4AC\end{matrix}\right| x^2 \left.\begin{matrix}+2BD\\-4AE\end{matrix}\right| x \left.\begin{matrix}+D^2\\-4AF\end{matrix}\right|}$$

represents a *curve on a straight line.*

We now attempt to show *the natural and possible variations of the curve.*

The part under the radical may be represented as follows:

$$\sqrt{Mx^2+Nx+P}. \qquad (1)$$

The equation of the circle when the origin is at the circumference, is

$$y=\sqrt{2Rx-x^2}.$$

Now as x is a variable quantity it is certainly possible that

$$\sqrt{Mx^2+Nx+P}=\sqrt{2Rx-x^2}.$$

This gives $\qquad (M+1)x^2+(N-2R)x+P=0,$

a quadratic in which there is nothing impossible or absurd.

Hence, *it is possible that the curve indicated by the quantity under the radical may be a circle.*

When the origin of the co-ordinates is at the vertex of the major axis of an ellipse, the equation for that curve is

$$y=\pm\sqrt{\frac{2B^2}{A}x-\frac{B^2}{A^2}x^2}.$$

Now it is possible that

$$Mx^2+Nx+P=\frac{2B^2}{A}x-\frac{B^2}{A^2}x^2.$$

Hence, it is possible *that the curve under consideration may be an ellipse.*

The equation of the hyperbola, when the origin is at the vertex of the curve, is

$$y=\pm\sqrt{\frac{2B^2}{A}x+\frac{B^2}{A^2}x^2}.$$

But it is possible that

$$Mx^2+Nx+P=\frac{2B^2}{A}x+\frac{B^2}{A^2}x^2.$$

That is, *it is possible that the curve may be an hyperbola.*

The equation for the parabola is

$$y=\pm\sqrt{2px}.$$

And it is possible that

$$Mx^2+Nx+P=2px,$$

and therefore, *it is possible that the curve may be a parabola.*

It is possible that

$$Mx^2+Nx+P=0.$$

Then the curve may still be a circle, provided

$$2Rx-x^2=0, \text{ also.}$$

The same consideration may be applied to the ellipse, the hyperbola, and the parabola.

Lastly, in the equation

$$Mx^2+Nx+P=0,$$

The values of x may be *imaginary,* and in that case no lines can represent it, and *the curve itself will be imaginary.*

In short, the equation

$$y=-\tfrac{1}{2}A(Bx+D)\pm\tfrac{1}{2}A\sqrt{Mx^2+Nx+P},$$

represents a curve on the straight line, and that curve may be a circle, an ellipse, an hyperbola, or a parabola, or the curve may be reduced to a point, and then the equation will represent a straight line only, or two parallel lines.

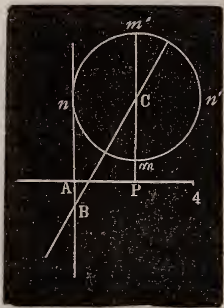
When Mx^2 is *affirmative,* the curve is an *hyperbola;* when Mx^2 is *negative,* the curve is an *ellipse,* or a *circle;* and when that term is absent, or zero, the curve is a *parabola.*

From the preceding *summary* we learn that the equation

$$y=2x-1\pm\sqrt{-x^2+4x},$$

must represent a circle on the straight line, whose equation is

$$y=2x-1.$$



1st. *Construct that straight line BC.*

2d. Put $\pm\sqrt{-x^2+4x}=0.$

Whence $x=0$, or 4.

That is, the curve extends from the axis of Y to the distance of plus 4, on the axis of X.

Now take P, the middle point between A and 4, and make $AP=x$, then we shall have $x=2$, which substituted in the equation, gives

$$y=4-1\pm2=5 \text{ or } 1.$$

That is, $Pm'=5$, and $Pm=1$, showing that $mm'=4$. But $nn'=4$, therefore the curve is a circle.

OTHER EXAMPLES.

1. *Find or construct the curve represented by the equation*

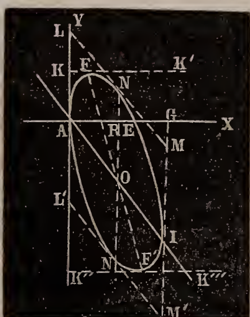
$$y^2+2xy+3x^2-4x=0.$$

Whence $y=-x\pm\sqrt{-2x(x-2)}.$

If we make $x=0$, we shall find $y=0$ at the same time, therefore the curve passes through the origin A.

In the original equation, if we make $y=0$ we shall have

$$3x^2-4x=0.$$



Whence $x=0$, or $x=1\frac{1}{3}$.

Hence, the point E in the curve is $1\frac{1}{3}$ units distance from A.

If we put

$$\sqrt{-2x(x-2)}=0,$$

We shall have $x=0$, and $x=2$.

Take $AG=2$, and through G draw GIM' parallel to the axis of Y, the point I is in the curve at its extreme distance in the direction of the axis of X. AI is one diameter of the curve.

As $AG=2$, take $AR=1$, for the value of x, and substitute that value in the equation, and we shall have

$$y=-1\pm\sqrt{2}=+0.41 \text{ or } -2.41.$$

From R draw $RN=0.41$ and $RN'=-2.41$, and through the two points N and N', draw lines parallel to the diameter AI. The curve then must be an *ellipse* described in the parallelogram $LL'M'M$, and NN', $-AI$, are its *conjugate diameters.*

2. *Determine what curve corresponds to the equation*

$$y^2+2xy+x^2-6y+9=0.$$

Resolving in relation to y, we find

$$y=-x+3\pm\sqrt{-6x}.$$

This last equation shows that the curve is a *parabola,* because the quantity under the radical does not contain x^2.

By making $x=0$, we find $y=3$, showing that the curve meets the axis of Y *three* units above the origin.

Because the sign under the radical is *minus,* we must take x *negative,* to render the product *positive,* and hence we decide that the parabola must extend in the direction of x *negative.*

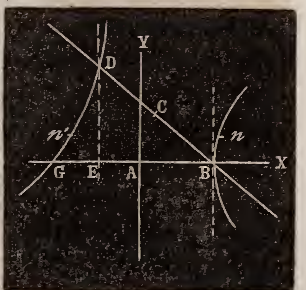
3. *Determine what curve is represented by the equation*

$$y^2+2xy-2x^2-4y-x+10=0. \quad (1)$$

From whence we deduce

$$y=-x+2\pm\sqrt{3x^2-3x-6}. \quad (2)$$

Put the quantity under the radical equal to 0, and the corresponding values of x are -1, and $+2$.



Construct the line BD corresponding to the equation

$$y'=-x+2.$$

This line is the *diameter* of the curve.

Make $x=0$ in (1), and we shall have

$$y^2-4y+10=0.$$

In this equation y is *imaginary,* showing that no point of the curve is in the axis of Y.

Because $x=-1$ or $+2$, under the radical, if we take $AB=2$ and $AE=-1$, and through B and E draw the dotted lines parallel to the axis of Y, we shall have the limits of the curve, and as BD is a diameter of the curve, one point in the curve must be at B, and the other at D. Hence the curve has *two branches*, and it is an *hyperbola*.

We might have determined this before, *because* the co-efficient of the second power of x under the cardinal, is *positive*. Hence we can have two positive values for the quantity under the radical, one corresponding to x taken as *positive*, and another corresponding to x taken as *negative*.

The positive value corresponds to the right branch of the curve ; the negative value corresponds to the left branch of the curve, and BD is one of its *conjugate diameters*.

If we make $y=0$ in either (1) or (2), the corresponding values of x will be $+2$ and $-2\frac{1}{2}$, showing that one branch of the curve passes through B, and the other through G.

If we make $y=1$, the corresponding values of x will be $+2.14$ or -1.64, defining the points n and n', and thus other points may be defined.

4. *Determine the curve represented by the equation*

$$y^2+6xy+9x^2-2y-6x-15=0.$$

Resolving the equation in relation to y, we find

$$y+3x-1=\pm 4.$$

Whence $\quad y+3x-5=0, \quad$ or $\quad y+3x+3=0.$

Showing *no curve*, but two parallel lines at the distance of 8 units from each other, measured on the axis of Y.

5. *Determine the curve represented by the equation*

$$y^2-4xy+5x^2-2y+5=0.$$

On resolving this equation in relation to y, we shall find that

$$(y-2x-1)^2+(x-2)^2=0^2.$$

This last equation will be recognized as the equation of a circle whose radius is *zero;* that is, the curve is diminished down to a *point*.

6. *Determine the curve represented by the equation*

$$y^2-2xy+2x^2-2x+4=0.$$

Resolving, we find

$$(y-x)^2+(x-1)^2+3=0.$$

This is the equation of a circle whose radius is $\sqrt{-3}$, but that is impossible. Such a radius is *imaginary*, and the curve *imaginary*.

7. *What kind of a curve corresponds to the equation*

$$y^2-2xy+x^2+x=0\ ?$$

Ans. It is a parabola passing through the origin and extending in the direction of *minus* x and *minus* y.

8, *What kind of a curve corresponds to the equation*

$$y^2-2xy+x^2-2y-1=0\ ?$$

Ans. It is a parabola, cutting the axis of X at the distance of -1 and $+1$, from the origin, and extending in the direction of *plus* x and *plus* y.

9. *What kind of a curve corresponds to the equation*

$$y^2-4xy+4x^2=0\ ?$$

Ans. It is a straight line passing through the origin, making an angle of 30° with the axis of Y.

10. *What kind of a curve corresponds with the equation*

$$y^2-2xy+2x^2-2y+2x=0\ ?$$

Ans. It is an ellipse limited by parallels to the axis of Y drawn through the points -1, and $+1$, on the axis of X.

11. *What kind of a curve corresponds with the equation*

$$y^2-2xy+x^2+2y-2x+1=0\ ?$$

Ans. It is a straight line cutting the axis of X at an angle of 45°, at the point $+1$ from the origin.

8

12. *What kind of a curve corresponds to the equation*

$$y^2-2xy-x^2-2y+2x+3=0\ ?$$

Ans. It is an *hyperbola*. The axis of Y is midway between the two branches. One branch of the curve cuts the axis of X at the point -1; the other branch cuts the same axis at the point $+3$.

CHAPTER II.

On Curves and Lines corresponding to Equations.

We have seen that the equation of a straight line is

$$y=tx+c,$$

And that the general equation of a circle is

$$(x\pm a)^2+(y\pm b)^2=R^2.$$

The first is a simple, the second a quadratic equation, and if we eliminate x in the first equation, and substitute its value for x in the second, we shall have a resulting equation of the second degree, which cannot correspond to every point in the straight line, nor to every point in the circle, but it will correspond to the two points in which the straight line cuts the circle, and to those points only.

And if the straight line should not cut the circle, the values of y in the resulting equation *must necessarily become imaginary.* All this has been shown in the application of the polar equation of the circle, in Chap. II, Sec. I.

We are now about to extend this principle another step. The equation of the parabola is

$$y^2=2px,$$

an equation of the second degree, and the equation of a circle is

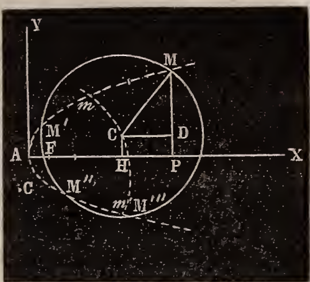
$$(x\pm a)^2+(y\pm b)^2=R^2,$$

also an equation of the second degree. But when two equations of the second degree are combined, they will produce an equation of the fourth degree.

But this resulting equation of the fourth degree cannot correspond to all points in the parabola, nor to all points in the circle, but it must correspond equally to both; hence, it will correspond to the points of intersection, and if the two curves do not intersect, the combination of their equations will produce an equation whose roots are *imaginary*.

Let us take the equation $y^2=2px$, and take p for the *unit* of measure, (that is, the distance from the divertrix to the focus is unity,) then $x=\frac{y^2}{2}$, and this value of x substituted in the equation of the circle, will give

$$\left(\frac{y^2}{2}\pm a\right)^2+(y\pm b)^2=R^2.$$



Let the vertex of the parabola be the origin of rectangular coordinates.

Take $AP=x$, and let it refer to either the parabola or the circle, and let $PM=y$, $AF=\frac{1}{2}$, $AH=a$, $HC=b$, and $CM=R$.

Now in the right angled triangle CMD, we have

$HP=CD=x-a$, $MD=y-b$,

and corresponding to this *particular* figure, we shall have in lieu of the equation above

$$\left(\frac{y^2}{2}-a\right)^2+(y-b)^2=R^2.$$

Whence $\quad y^4+(4-4a)y^2-8by=4(R^2-a^2-b^2)$. (F)

This equation is of the fourth degree, hence it must have *four* roots, and this corresponds with the figure, for the circle cuts the parabola in *four* points, M, M', M'', and M'''.

The second term of the equation is wanting, that is, the coefficient to y^3 is 0, and hence it follows from the theory of equations, that the sum of the *four* roots must be *zero*.

The sum of two of them, which are above the axis of AX, (the two *plus* roots,) must be equal to the sum of the two *minus* roots corresponding to the points M'' and M'''.

The values of a and b and R may be such as to place the center C in such a position that the circle can cut the parabola in only two points, and then the resulting equation will be such as to give two *real* and two *imaginary* roots.

Indeed, a circle referring to the same unit of measure and to the same co-ordinates, might not cut the parabola at all, and in that case the resulting equation would have only *imaginary* roots.

In case the circle touches the parabola, the equation will have two equal roots.

Now it is plain that if we can construct a figure that will truly represent any equation in this form, that figure will be a solution to the equation. For instance, a figure correctly drawn will show the magnitude of PM, one of the roots of the equation.

We will illustrate by the few following

EXAMPLES.

1. *Find the roots of the equation*

$$y^4-11.14y^2-6.74y+9.9225=0.$$

This equation is the same in form as our theoretical equation (F), and therefore we can solve it *geometrically*, as follows:

Draw rectangular co-ordinates, as in the figure, and take $AF=\frac{1}{2}$, and construct the *parabola*.

To find the center of the circle, and the radius, we put

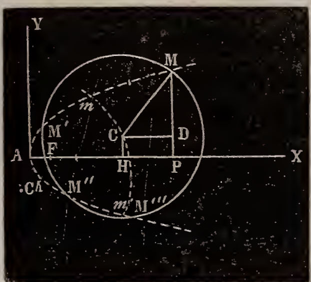
$$4-4a=-11.14, \quad (1) \qquad -8b=-6.74, \quad (2)$$

And $$4(R^2-a^2-b^2)=-9.9225. \quad (3)$$

From (1) $a=3.78$. From (2) $b=0.88$.

And these values of a and b substituted in (3) give

$$R=3.34, \text{ nearly.}$$



Take from the scale which corresponds to $AF=\frac{1}{2}$, $AH=a=3.78$, $HC=0.88$, and from C as a center, with a radius equal to 3.34, describe the circle cutting the parabola in the four points M, M', M'', and M'''. The distance of M from the axis of X is $+3.5$, of M' it is $+0.7$, of M''

it is —1.5, and of M''' it is —2.7, and these are the four roots of the equation.

Their sum is 0, as it ought to be, because the equation contains no third power of y.

2. *Find the roots of the equation*

$$y^4+y^3+6y^2+12y-72=0.$$

This equation contains the third power of y, therefore this geometrical solution will not apply until that term is removed.

But we can remove that term by putting

$$y=z-\tfrac{1}{4}.$$

(See theory of transforming equations in algebra.)

This value of y substituted in the equation, it becomes

$$z^4+5\tfrac{5}{8}z^2+9\tfrac{1}{8}z=74\tfrac{163}{256},$$

and this equation is in the proper form.

Now put $4-4a=5\frac{5}{8}$, $-8b=9\frac{1}{8}$, and $4(R^2-a^2-b^2)=74\frac{163}{256}$.

Whence $a=-\frac{13}{32}$, $b=-\frac{73}{64}$, and $R=4.485$.

These values of a and b designate the point C' for the center of the circle. From this center, with a radius $=4.485$, we strike the circle cutting the parabola in the two points m and m'. The point m is $2\frac{1}{4}$ units above the axis AX, and the point m' is $-2\frac{3}{4}$ units from the same line, and these are the two roots of the equation. *The other two roots are imaginary*, shown by the fact that *this* circle can cut the parabola in two points only.

If we conceive a circle to pass through the vertex of the parabola A, then will

$$a^2+b^2=R^2,$$

and this supposition reduces the general equation (F) to

$$y^4+(4-4a)y^2-8by=0.$$

Here $y=\pm 0$ will satisfy the equation, and this is as it should be, for the circle actually cuts the parabola on the axis of X.

Now divide this last equation by this value of y, and we have

$$y^3+(4-4a)y=8b. \qquad \text{(G)}$$

Here is an equation of the third degree, referring to a parabola and a circle; the circle cutting the parabola at its vertex for one point, and if it cuts the parabola in any other point, that other point will designate another root in equation (G).

It is possible for a circle to touch one side of the parabola within, and cut at the vertex A, and at some other point. Therefore, it is possible for an equation in the form of (G) to have three real roots, and two of them equal.

Most circles, however, can cut the parabola in A, and in one other point, showing one real root and two *imaginary roots.*

The theoretical equation (G) can be used to effect a mechanical solution of all numerical equations of the third degree, in that form.*

We will illustrate this by one or two

EXAMPLES.

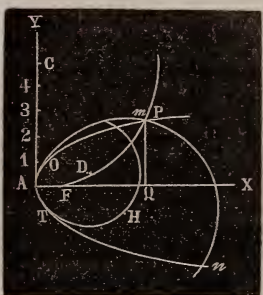
1. *Given* $y^3+4y=39$, *to find the value of* y *by construction.*

Put $4-4a=4$, and $8b=39$. Whence $a=0$, and $b=4\frac{7}{8}$.

These values of a and b designate the point C on the axis of Y for the center of the circle, $CA=4\frac{7}{8}$, the radius.

The circle again cuts the parabola in P, and PQ measures three units, *the only real root of the equation.*

2. *Given* $y^3-75y=250$, *to find the values of* y *by construction.*



When the co-efficients are large, a large figure is required; but to avoid this inconvenience, we reduce the co-efficients, as shown in Chap. II, Sec. I.

Thus put $y=nz$.

Then the equation becomes

$$n^3z^3-75nz=250.$$

$$z^3-\frac{75}{n^2}z=\frac{250}{n^3}.$$

* Observe that the second term or y^2 in a regular cubic is wanting. Hence if any example contains that term it must be removed before a geometrical solution can be given.

Now take $n=5$, then we have

$$z^3-3z=2.$$

In this last equation the co-efficients are sufficiently small to apply to a construction.

Put $4-4a=-3$, and $8b=2$.

Whence $a=1\frac{3}{4}$ and $b=\frac{1}{4}$.

These values of a and b designate the point D for the center of the circle. DA is the radius.

The circle cuts the parabola in t, and touches it in T, showing that one root of the equation is $+2$, and two others each equal to -1.

But $y=nz$. That is, $y=5\cdot 2$, or -5, -5.

Or the roots of the original equation are $+10$, -5, -5.

When an equation contains the second power of the unknown quantity, it must be removed by transformation before this method of solution can be applied.

3. *Given* $y^3-48y=128$ *to find the values of* y *by construction.*

Ans. $+8$, -4, -4.

4. *Given* $y^3-13y=-12$, *to find the values of* y *by construction.*

Ans. $+1$, $+3$, and -4.

Conversely we can describe a parabola, and take any point as H, at haphazard, and with HA as radius, describe a circle and find the equation to which it belongs.

This circle cuts the parabola in the points m, n, and o, indicating an equation whose roots are $+1$, $+2.4$, and -3.4.

We may also find the particular equation from the general equation

$$y^3+(4-4a)a=8b,$$

observing the locality of H, which corresponds to $a=3\cdot 3$ and $b=-1$, and taking these values of a and b, we have

$$y^3-9.2y=-8,$$

for the equation sought.

Remarks and Observations on the general interpretation of Equations.

In every science, it is important to take an occasional retrospective view of first principles, and none demand this more imperatively than geometry, and this conviction will excuse us for reconsidering the following truths so often in substance, (if not in words,) called to mind before.

An equation, geometrically considered, whatever may be its degree, is but the equation of a point, and can only designate a point.

Thus, the equation $y=ax+b$ designates a point, which point is found by measuring any assumed value which may be given to x from the origin of co-ordinates on the axis of X, and from that extremity measuring a distance represented by $(ax+b)$ on a line parallel to the axis of Y.

The extremity of the last measure *is the point designated by the equation.* If we assume another value for x, and measure again in the same way, we shall find the point which now corresponds to the value of x. Again, assume another value for x, and find the designated point.

Lastly, if we connect these several points, we shall find them all in the same *right line*, and in this sense the equation of the first degree $y=ax+b$,
is the general equation of a right line, but the right line is found by finding points in the line and connecting them.

In like manner the equation of the second degree

$$y=\pm\sqrt{2Rx-x^2},$$

only designates a point when we assume any value for x, (not inconsistent with the existence of the equation,) and take the *plus* sign. It will also designate another point when we take the *minus* sign. Taking another value of x, and thus finding two other points, we shall have four points, — still another value of x and we can find two other points, and so on we might find any number of points. Lastly, on comparing these points we shall find *that they are all in the circumference of the same circle*, and hence we say that the preceding equation is the equation of a *circle*. Yet it can designate only one, or at most two points at a time.

If we assume different values for y, and find the corresponding values of x, the result will be the same circle, because the x and y mutually depend upon each other.

Now let us take the last practical example

$$y^3-13y=-12,$$

and for the sake of perspicuity change y into x, then we shall have

$$x^3-13x+12=0.$$

Now we can suppose $y=0$ to be another equation; then will

$$y=x^3-13x+12, \qquad \text{(A)}$$

be an independent equation between two variables, and of the third degree.

The particular hypothesis that $y=0$ gives three values to x, ($+1$, $+3$, and -4,) that is, *three points* are designated, the first at the distance of one unit to the right of the axis of Y; the second at the distance of three units on the same side of the axis of Y; and the third point four units on the opposite side of the same axis, and *this is all the equation can show until we make another hypothesis.*

Again, let us assume $y=5$, then equation (A) becomes

$$5=x^3-13x+12, \quad \text{or} \quad x^3-13x+7=0,$$

and this is in effect changing the origin five units on the axis of Y. A solution of this last equation designates three other points from the axis of Y.

Again, let us assume $y=10$, then equation (A) becomes

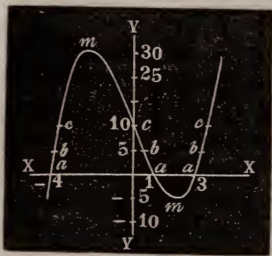
$$x^3-13x+2=0,$$

and a solution of this equation gives three other points.

And thus we may proceed, assigning different values to y, and deducing the corresponding values of x, as appears in the following table, commencing at the origin of the co-ordinates, where $y=0$, and varying each way.

$y=30.0388$	$x=-2.0814$	$+4.1628$	-2.0814
$y=25.$	$x=-1.1$	$+4.03$	-2.91
$y=20.$	$x=-0.40$	$+3.80$	-3.41
$y=15.$	$x=-0.20$	$+3.70$	-3.50
$y=10.$	$x=+0.14$	$+3.52$	-3.66
$y=5.$	$x=+0.55$	$+3.3$	-3.85

When $y=0$.	then will	$x=+1.$	$+3.$	$-4.$
$y=-5$		$x=+1.66$	$+2.477$	-4.14
$y=-6.0388$		$x=+2.0814$	$+2.0814$	-4.1628



Taking $y=0$, a solution of the equation $y=x^3-13x+12$, gives the three points a, a, a, on the axis of X.

Then taking $y=5$, and a solution gives three points b, b, b, on a line parallel to the axis of X, and at the distance of 5 units above said axis.

Again, taking $y=10$, and another solution gives the three points c, c, c. Now joining the three points $a\ b\ c$, $a\ b\ c$, and $a\ b\ c$, we shall have apparently *three* curves corresponding to the equation of the *third* degree, and thus, if we were hasty in drawing conclusions, we might assume that every equation of the third degree might give *three curves*, and every equation of the fourth degree *four curves*, &c. &c. *but this is not true.*

If we continue finding points as before, we shall find that the three curves $(a, b, c,)$ $(a, b, c,)$ and $(a, b, c,)$ are but different portions of the *same curve*, and we can now venture to draw this general conclusion :

That an equation involving y, *the ordinate to the first power, and the abscissa* x *to the third power, the axis of* X, *or lines parallel to that axis, may cut the curve in three points.*

From analogy, we also infer that an equation involving x to the *fourth* power, the axis of X, or its parallels, will cut the curve in *four* points; and an equation involving x to the *fifth* power, that axis or its parallels will cut the curve in *five* points, and so on.

In the equation under consideration, $(y=x^3-13x+12)$, if we assume y greater than 30.0388, or less than -6.0388, we shall find that two values of x in each case will become imaginary, and on each side of these limits the parallels to X will cut the curve only *in one point.*

Two points vanish at a time, and this corresponds with the truth demonstrated in algebra, "that *imaginary roots* enter equations in pairs."

The points m, m, the turning points in the curve, are called *maximum* points, and can be found only by approximation, using the ordinary processes of computation, but the peculiar operation of the *calculus* gives these points at once, and we mention the fact here, to show the student the practical importance of that *higher branch* of analytical geometry.

To find the points in the *curve* we might have assumed different values of x in succession, and deduced the corresponding values of y, but this would have given but one point for each assumption; and to define the curve with sufficient accuracy, many assumptions must be made with very small variations to x. We solved the equations approximately and with great rapidity by means of the *circle* and *parabola* as previously shown.

We conclude this subject by the following example:

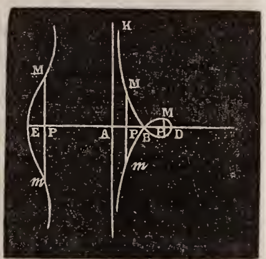
Let the equation of a curve be

$$(a^2-x^2)(x-b)^2=x^2y^2,$$

from which we are required to give a geometrical delineation of the curve. From the equation we have

$$y=\pm\frac{\sqrt{(a^2-x^2)(x-b)^2}}{x}.$$

The following figure represents the curve which will be recognized as corresponding to the equation, after a little explanation.



If $x=0$, then y becomes infinite, and therefore the ordinate at A is an *asymptote* to the curve. If $AB=b$, and P be taken between A and B, then PM and Pm will be equal, and lie on different sides of the abscissa AP. If $x=b$, then the two values of y vanish, because $x-b=0$; and consequently, the curve passes through B, and has there a *duplex* point. If AP be taken greater than AB, then there will be two values of y, as before, having contrary signs, that value which was positive before, now becomes negative, and the negative value becomes positive. But if AD be taken $=a$, and P comes to D, then the two values of y vanish, because $\sqrt{a^2-x^2}=0$. And if AP is taken greater than AD, then a^2-x^2 becomes negative, and the value of y *impossible;* and therefore, the curve does not extend beyond D.

If x now be supposed negative, we shall find

$$y=\pm\sqrt{a^2-x^2}\times b+x\div x.$$

If x vanish, both these values of y become infinite, and consequently, the curve has two infinite arcs on each side of the *asymptote* AK. If x increase, it is plain y diminishes, and if x becomes $=a$, y vanishes, and consequently the curve passes through E, if AE be taken $=AD$, on the opposite side. If x be supposed greater than a, then y becomes *impossible;* and no part of the curve can be found beyond E. This curve is the *conchoid* of the ancients.

CHAPTER II.

Straight Lines in Space.

Straight lines in one and the same plane are referred to *two* co-ordinates in that plane,—but straight lines in space require *three* co-ordinates, made by the intersection of *three planes.*

To take the most simple and practical view of the subject, conceive a *horizontal* plane cut by a *meridian* plane, and by a *perpendicular east* and *west* plane.

The common point of intersection we shall call the *zero point,* and we might conceive this point to be the center of a sphere, and from it will be eight quadrangular spaces corresponding to the eight quadrants of a sphere, which extended, would comprise *all space.*

Horizontally, *east and west,* we shall call the axis of X. Horizontally in the direction of the *meridian,* the axis of Y; and perpendicularly in the plane of the meridian, the axis of Z. From the zero point horizontally to the right we shall designate as *plus,* to the left *minus.*

Along the axis of Y and parallel thereto towards us from the zero point, we shall call *plus*; from the opposite direction will therefore be *minus.* Perpendicularly from the horizontal plane upwards is taken as *plus,* downward *minus.*

The horizontal plane is called the plane of xy, the meridian plane is designated as the plane of yz, and the perpendicular east and west plane the plane of xz.

Now let it be observed that x will be *plus* or *minus*, according to its direction from the plane of yz, y will be *plus* or *minus*, according to its direction from the plane xz, and z will be *plus* or *minus*, according as it is above or below the horizontal xy.

PROPOSITION I.

To find the equations of a straight line in space.

Conceive a straight line passing in any direction through space, and conceive a plane coinciding with it, and perpendicular to the plane xz. The intersection of this plane with the plane xz, will form a line on the plane xz, and this is said to be the projection of the line on the plane xz, and the equation of this projected line will be in the form

$$x=az+\alpha. \qquad \text{(Chap. I, Prop. 1.)}$$

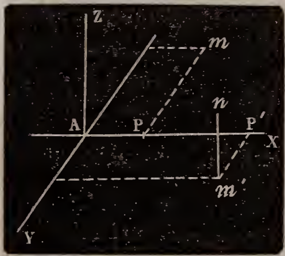
Conceive another plane coinciding with the proposed line, and perpendicular to the plane yz, its intersection with the plane yz is said to be the projection of the line on the plane yz, and the equation of this projected line is in the form

$$y=bz+\beta.$$

These two equations taken together are said to be equations of the line, because the first equation is a general equation for all lines that can be drawn in the first projecting plane, and the second equation is a general equation for all lines that can be drawn in the second projecting plane; therefore taken together, they express the intersection of the two planes, which is the line itself.

For illustration, we give the following example: Construct the line whose equations are

$$\left.\begin{aligned} x&=2z+1 \\ y&=3z-2 \end{aligned}\right\}$$



Make $z=0$, then $x=1$, and $y=-2$. Now take $AP=1$, and draw Pm parallel to the axis of Y, making $Pm=-2$; then m is the point in the plane xy, through which the line *must pass*.

Now take z equal to any number at pleasure, say 1, then we shall have $x=3$ and $y=1$.

Take $AP'=3$, $P'm'=+1$, and from the point m' in the plane xy erect $m'n$ perpendicular to the plane xy, and make it equal to 1, because we took $z=1$, then n is *another* point in the line. Join nm and produce it, and it will be the line designated by the equations.

PROPOSITION II.

To find the equations of a straight line which shall pass through a given point.

Let the co-ordinates of the given point be represented by x', y', z'.

The equations sought must satisfy the general equations

$$\left.\begin{array}{l} x=az+\alpha. \\ y=bz+\beta. \end{array}\right\} \qquad (1)$$

The equations corresponding to the given point are

$$x'=az'+\alpha. \qquad y'=bz'+\beta.$$

Subtracting (1) from these, we have

$$x'-x=a(z'-z), \quad \text{and} \quad y'-y=b(z'-z),$$

the equations required.

PROPOSITION III.

To find the equations of a straight line which shall pass through two given points.

Let the co-ordinates of the second point be x'', y'', z''. Now by the second proposition, the equation of the line which passes through the two points, will be

$$x''-x'=a(z''-z').$$

Whence $$a=\frac{x''-x'}{z''-z'}.$$

And $$y''-y'=b(z''-z'), \qquad b=\frac{y''-y'}{z''-z'}.$$

Substituting the values of a and b in the resulting equations of Prop. II, we have

$$x'-x=\left(\frac{x''-x'}{z''-z'}\right)(z'-z). \qquad y'-y=\left(\frac{y''-y'}{z''-z'}\right)(z'-z),$$

for the equations required.

PROPOSITION IV.

To find the condition under which two straight lines intersect in space, and the co-ordinates of the point of intersection.

Let the equation of the lines be

$$x=az+\alpha. \qquad y=bz+\beta.$$
$$x=a'z+\alpha'. \qquad y=b'z+\beta'.$$

If the two lines intersect, (as they do by hypothesis,) then x and y may represent the co-ordinate of the point of intersection; therefore by subtraction, we have

$$(a-a')z+\alpha-\alpha'=0, \qquad (b-b')z+\beta-\beta'=0.$$

Whence, by eliminating z, we find

$$\frac{\alpha-\alpha'}{a-a'}=\frac{\beta-\beta'}{b-b'},$$

which is the condition under which two lines intersect.

Now $z=\frac{\alpha'-\alpha}{a-a'}$, and this value of z being substituted in the first equations, we obtain

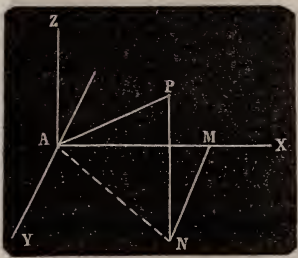
$$x=\frac{a\alpha'-a'\alpha}{a-a'} \quad \text{and} \quad y=\frac{b\beta-b'\beta}{a-a'},$$

for the value of the co-ordinates of the point of intersection.

COROLLARY. If $a=a'$, the denominators in the second member will become 0, making x and y infinite; that is, the point of intersection is at an infinite distance from the origin, and the lines are therefore parallel.

PROPOSITION V.—PROBLEM.

To express analytically the distance of a given point from the origin.



Let P be the given point in space; it is perpendicular over the point N, which is in the plane xy.

The angle $AMN=90°$. Also, the angle $ANP=90°$.

Let $AM=x$, $MN=y$, $NP=z$.

Then $\overline{AN}^2=x^2+y^2$.

But $\overline{AP}^2=\overline{AN}^2+\overline{NP}^2=x^2+y^2+z^2$.

Now if we designate AP by r, we shall have

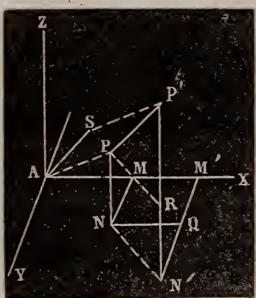
$$r^2=x^2+y^2+z^2,$$

for the expression required.

PROPOSITION VI.—PROBLEM.

To express analytically the length of a line in space.

N. B.—The only difficulty a learner can experience will arise from a want of the proper perception of the figure projected on a plane. Hence, teachers should construct the proper *pasteboard* figures, which will give the *real* and simple representation.



Let $PP'=D$ be the line in question.

Let the co-ordinates of the point P be x, y, z, and of the point P' be x', y', z'.

Now $MM'=x'-x=NQ$.

$$QN'=y'-y.$$

$$\overline{NN'}^2=(x'-x)^2+(y'-y)^2=\overline{PR}^2.$$

$$P'R=z'-z.$$

In the triangle PRP' we have

$$\overline{PP'}^2=\overline{PR}^2+\overline{P'R}^2=(x'-x)^2+(y'-y)^2+(z'-z)^2.$$

Or $$D^2=(x'-x)^2+(y'-y)^2+(z'-z)^2, \qquad (1)$$

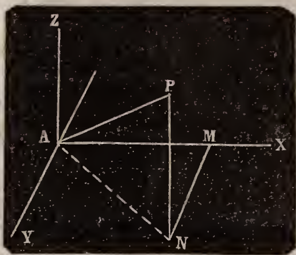
which is the expression required.

SCHOLIUM. If through one extremity of the line, as P, we draw PA to the origin, and from the other extremity P', we draw $P'S$ parallel and equal to PA, and join AS, it will be parallel to PP', and equal to it, and this virtually reduces this proposition to the previous one. This also may be drawn from the equation, for if A is one extremity of the line, its co-ordinates x, y, and z, are each equal to zero, and

$$D^2=x'^2+y'^2+z'^2.$$

PROPOSITION VII.—PROBLEM.

To find the inclination of any line in space to the three axes.



From the origin draw a line parallel to the given line, and the inclination of this line to the axes will be the same as that of the given line.

The equations for the line passing from the origin are

$$x=az, \quad \text{and} \quad y=bz, \qquad (1)$$

Let X represent the inclination of this line with the axis of x, Y its inclination with the axis of y, and Z the inclination with the axis of z.

The three points P, N, M, are in a plane which is parallel to the plane zy, and AM is a perpendicular between the two planes. AMP is a right angled triangle, the right angle at M.

Let $AP=r$ and $AM=x$. Then, by trigonometry, we have

As $r : \sin.90° :: x : \cos.X$. Whence $x=r\cos.X$.

Also, as $r : \sin.90° :: y : \cos.Y$. Whence $y=r\cos.Y$.

Also, as $r : \sin.90° :: z : \cos.Z$. Whence $z=r\cos.Z$.

From Prop. V, we have

$$r^2=x^2+y^2+z^2. \qquad (2)$$

Substituting the values of x, y, and z, as above, we have

$$r^2=r^2\cos.^2X+r^2\cos.^2Y+r^2\cos.^2Z.$$

Dividing by r^2 will give

$$\cos.^2X+\cos.^2Y+\cos.^2Z=1, \qquad (3)$$

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an equation which is easily called to mind, and one that is useful in the higher mathematics.

If in (2) we substitute the values of x^2 and y^2 taken from (1), we shall have

$$r^2=a^2z^2+b^2z^2+z^2. \qquad (4)$$

But we have three other values of r^2, as follows:

$$r^2=\frac{x^2}{\cos.^2 X}. \qquad r^2=\frac{y^2}{\cos.^2 Y}. \qquad \text{and} \qquad r^2=\frac{z^2}{\cos.^2 Z}.$$

Whence $$\frac{x}{\cos. X}=\pm z\sqrt{1+a^2+b^2}. \qquad (5)$$

$$\frac{y}{\cos. Y}=\pm z\sqrt{1+a^2+b^2}. \qquad (6)$$

And $$\frac{1}{\cos. Z}=\pm\sqrt{1+a^2+b^2}. \qquad (7)$$

In (5) put the value of x drawn from (1), and in (6) the value of y from (1), and reduce, and we shall obtain

$$\left.\begin{aligned}\cos. X&=\frac{a}{\pm\sqrt{1+a^2+b^2}}\\ \cos. Y&=\frac{b}{\pm\sqrt{1+a^2+b^2}}\\ \cos. Z&=\frac{1}{\pm\sqrt{1+a^2+b^2}}\end{aligned}\right\}$$ The analytical expressions for the inclination of a line in space to the three co-ordinates.

The double sign shows two angles supplemental to each other, the plus sign corresponds to the acute angle, the minus sign to the obtuse angle.

PROPOSITION VIII.

To find the inclination of two lines in terms of their separate inclinations to the axes.

Through the origin draw two lines respectively parallel to the given lines. An expression for the angle between these two lines is the quantity sought.

Let AP be parallel to one of the given lines, and AQ parallel to the other. The angle PAQ is the angle sought.

Let the equations of one of these lines be

$$x=az, \qquad y=bz,$$

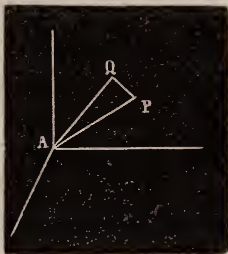
And for the other

$$x'=a'z', \qquad y'=b'z'.$$

Let $AP=r$, $AQ=r'$, $PQ=D$, and the angle $PAQ=V$.

Now in plane trigonometry (Prop. 8, page 150 Geometry,) we have

$$\cos. V=\frac{r^2+r'^2-D^2}{2rr'}. \qquad (1)$$



From Prop. VI, we have

$D^2=(x'-x)^2+(y'-y)^2+(z'-z)^2.$

Expanding, and

$D^2=(x'^2+y'^2+z'^2)+(x^2+y^2+z^2)-2x'x-2y'y-2z'z.$

But from Prop. V, we learn that

$$x^2+y^2+z^2=r^2$$

And $\qquad x'^2+y'^2+z'^2=r'^2.$

Whence $\qquad 2x'x+2y'y+2z'z=r^2+r'^2-D^2.$

This equation applied to (1) reduces it to

$$\cos. V=\frac{x'x+y'y+z'z}{rr'}.$$

But r and r' may be any values taken at pleasure, their lengths will have no effect on the angle V, therefore for convenience, we take each of them equal to unity.

Whence $\qquad \cos. V=x'x+y'y+z'z. \qquad (2)$

But Prop. VII, shows that $x=r\cos.X$, $y=r\cos.Y$, &c. and that $x'=r'\cos.X'$, $y'=r'\cos.Y'$, &c.

But we have taken $r=1$, and $r'=1$, therefore $x=\cos.X$, &c. and $x'=\cos.X'$, &c. Therefore

$$\cos. V=\cos.X\cos.X'+\cos.Y\cos.Y'+\cos.Z\cos.Z'. \qquad (3)$$

But by Prop. VII, we have

$$\cos.X=\frac{a}{\pm\sqrt{1+a^2+b^2}} \quad \text{and} \quad \cos.X'=\frac{a'}{\pm\sqrt{1+a'^2+b'^2}}, \text{ \&c.}$$

Substituting these values in (3) we have

$$\cos. V = \frac{1+aa'+bb'}{\pm(\sqrt{1+a^2+b^2})(\sqrt{1+a'^2+b'^2})},$$

for the expression required.

The cos. V will be plus or minus, according as we take the signs of the radicals in the denominator alike or unlike. The plus sign corresponds to an acute angle, the minus sign to its supplement.

Corollary 1. If we make $V=90°$, then cos. $V=0$, and the equation becomes

$$1+aa'+bb'=0,$$

which is the equation of condition to make two lines at right angles in space.

Corollary 2. If we make $V=0$, the two straight lines will become parallel, and the equation will become

$$\pm 1 = \frac{1+aa'+bb'}{\sqrt{1+a^2+b^2}\ \sqrt{1+a'^2+b'^2}}$$

Squaring, clearing of fractions, and reducing, we shall find

$$(a'-a)^2+(b'-b)^2+(ab'-a'b)^2=0.$$

Each term being a square, will be positive, and therefore the equation can only be satisfied by making each term separately equal to 0.

Whence $a'=a$, $b'=b$, and $ab'=a'b$.

The third condition is in consequence of the first two.

CHAPTER IV.

On the Equation of a Plane.

An equation which can represent any point in a line is said to be the equation of the line.

Similarly, an equation which can represent or indicate any point in a plane, is, in the language of analytical geometry, the equation of the plane.

The co-ordinates *AZ*, and *AX*, designate a plane which we call the plane of *xz*. The equation for any line in this plane, as *M*, is in the form $z=ax+b$.

This equation represents points in the line *M*, but if we assign to *b* any other value, as *b′*, we shall have points in another line parallel to the line *M*. In short, if in place of the constant *b* we write a *numerical variable w*, we shall have

$$z=ax+w, \qquad (1)$$

an equation which will not only represent points in the line *M*, but points also in all lines which can be drawn parallel to *M* in the plane *xz*: that is, it is an equation which can represent any point in the plane *xz*; therefore, *it is the equation of that plane.*

Like considerations will give us

$$z=by+w\,, \qquad (2)$$

for the equation of the plane *yz*, and

$$x=b'y+w'', \qquad (3)$$

for the equation of the plane *xy*.

On inspecting either one of the equations (1), (2), or (3), we shall perceive that the equation of a plane *must be an equation of the first degree between three variables*, and if either one of the variables becomes constant, or is suppressed, the equation *will become that of a straight line.*

Now if the plane in question, is the plane of *xz*, or parallel thereto, the equation of the plane must contain the two variables *x*, *z*, and one other; and similarly for each of the other planes. But if we have a plane which neither coincides nor is parallel to either plane *xz*, *yz*, or *xy*, but intersects all of them, its equation must contain the three variables, *x*, *y*, and *z*, and be of the first degree. Therefore it must be of the form

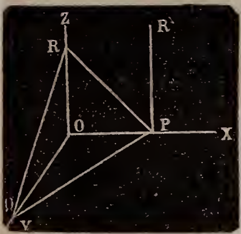
$$Ax+By+Cz+D=0,$$

which is recognized at once by all mathematicians as the most general and symmetrical equation of a plane.

SCHOLIUM. This notation being adopted, we can at once draw from it the following general truths:

1st. If we suppose a plane to pass through the origin of the co-ordinates, the equation for that point requires that $x=0$, $y=0$, and $z=0$, and these values substituted in the equation of the plane, will give $D=0$ also. Therefore, when a plane passes through the origin of co-ordinates, the general equation for the plane reduces to

$$Ax+By+Cz=0.$$



2d. To find the points in which the plane cuts the axes, we reason thus:

The equation of the plane must respond to each and every point in the plane; the point P, therefore, in which the plane cuts the axis of X, must correspond to $y=0$ and $z=0$, and these values substituted in the equation, reduce it to $Ax+D=0$.

Or $$x=-\frac{D}{A}=OP.$$

For the point Q we must take $x=0$ and $z=0$.

And $$y=-\frac{D}{B}=OQ.$$

For the point R, $$z=-\frac{D}{C}=OR.$$

3d. If we suppose the plane to be perpendicular to the plane XY, PR' a *trace* in it may be drawn parallel to OZ, and the plane will meet the axis of Z at the distance *infinity*. That is, OR, or its equal $\left(-\frac{D}{C}\right)$ must be infinite, which requires that $C=0$, which reduces the general equation of the plane to

$$Ax+By+D=0,$$

which is the equation of the *trace* or line PQ on the plane XY. If the plane were perpendicular to the plane ZX, the plane OQ, or its equal $\left(-\frac{D}{B}\right)$, must be *infinite*, which requires that $B=0$, and this reduces the general equation to

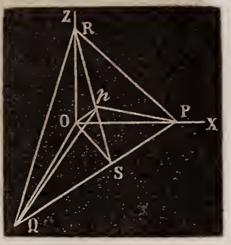
$$Ax+Cz+D=0,$$

which is the equation for the *trace* PR, and hence we may conclude in general terms,

That when a plane is perpendicular to any one of the co-ordinate planes, its equation is that of its trace on the same plane.

PROPOSITION IX.—PROBLEM.

To find the length of a perpendicular drawn from the origin to a plane, and to find its inclination with the three rectangular co-ordinates.



Let RPQ be the plane, and from the origin O draw Op perpendicular to the plane; this line will be at right angles to every line drawn in the plane from the point p.

Whence $OpQ=90°$, $OpR=90°$, $OpP=90°$. Let $Op=p$.

Designate the angle pOP by X, pOQ by Y, and pOR by Z.

By the *preceding scholium* we learn that

$$OP=-\frac{D}{A}, \quad OQ=-\frac{D}{B}, \quad \text{and} \quad OR=-\frac{D}{C},$$

A, B, C, and D, being the constants in the equation of a plane.

Now in the right angled triangle OpP, we have

$$OP : 1 :: Op : \cos.X.$$

That is, $$-\frac{D}{A} : 1 :: p : \cos.X. \qquad (1)$$

The right angled triangle OpQ gives

$$-\frac{D}{B} : 1 :: p : \cos.Y. \qquad (2)$$

The right angled triangle OpR gives

$$-\frac{D}{C} : 1 :: p : \cos.Z. \qquad (3)$$

Proportion (1) gives us

$$\cos.^2 X=\frac{p^2}{D^2}A^2, \qquad (4)$$

(2) gives $$\cos.^2 Y=\frac{p^2}{D^2}B^2, \qquad (5)$$

and (3) gives $$\cos.^2 Z=\frac{p^2}{D^2}C^2. \qquad (6)$$

Adding these three equations, and observing that the sum of the first members is *unity*, (Prop. VII, Chap. I, Sec. II,) and we have

$$\frac{p^2}{D^2}(A^2+B^2+C^2)=1.$$

Whence $$p=\pm\frac{D}{\sqrt{A^2+B^2+C^2}}. \qquad (7)$$

This value of p placed in (4), (5), and (6), and reduced, will give

$$\cos.X=\pm\frac{A}{\sqrt{A^2+B^2+C^2}}. \qquad (8)$$

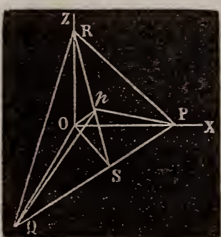
$$\cos.Y=\pm\frac{B}{\sqrt{A^2+B^2+C^2}}. \qquad (9)$$

$$\cos.Z=\pm\frac{C}{\sqrt{A^2+B^2+C^2}}. \qquad (10)$$

Expressions (7), (8), (9), and (10), are those sought.

PROPOSITION X.—PROBLEM.

To find the analytical expressions for the inclination of a plane to the three co-ordinate planes respectively.



Let $Ax+By+Cz+D=0$ be the equation of the plane, and let PQ represent its *trace* or line of intersection with the co-ordinate plane (xy).

From the origin O draw OS perpendicular to the trace PQ. Join pS. OpS is a right angled triangle, right angled at p, and the angle OSp measures the inclination of the plane with the horizontal plane (xy). Our object is to find the angle OSp.

In the right angled triangle POQ we have found

$$OP=-\frac{D}{A}, \qquad OQ=-\frac{D}{B}.$$

Whence $$PQ=\frac{D}{AB}\sqrt{A^2+B^2}.$$

Now PS, a segment of the hypotenuse made by the perpendicular OS, is a third proportional to QP and PO. Therefore

$$\frac{D}{AB}\sqrt{A^2+B^2} : -\frac{D}{A} :: -\frac{D}{A} : PS.$$

Or $$\sqrt{A^2+B^2} : -B :: -\frac{D}{A} : PS=\frac{BD}{A\sqrt{A^2+B^2}}.$$

The other segment QS is a third proportional to PQ and OQ. Therefore

$$\frac{D}{AB}\sqrt{A^2+B^2} : -\frac{D}{B} :: -\frac{D}{B} : QS.$$

Or $$\sqrt{A^2+B^2} : -A :: -\frac{D}{B} : QS=\frac{AD}{B\sqrt{A^2+B^2}}.$$

But the perpendicular OS is a *mean proportional* between these two segments. Therefore we have

$$OS=\frac{D}{\sqrt{A^2+B^2}}.$$

Now by simple permutation we may conclude that the perpendicular from the origin O to the *trace* PR, is

$$\frac{D}{\sqrt{A^2+C^2}},$$

and that to the *trace* QR is

$$\frac{D}{\sqrt{B^2+C^2}}.$$

We shall designate the angle which the plane makes with the plane of (xy) by (xy), and the angle it makes with (xz) by (xz), and that with (yz) by (yz).

Now the triangle OpS gives

$$OS : \sin.90^\circ :: Op : \sin. OSp.$$

That is $\frac{D}{\sqrt{A^2+B^2}} : 1 :: \frac{D}{\sqrt{A^2+B^2+C^2}} : \sin. OSp$

Whence $\sin.^2 OSp = \sin.^2(xy) = \frac{A^2+B^2}{A^2+B^2+C^2}.$

Similarly, $\sin.^2(xz) = \frac{A^2+C^2}{A^2+B^2+C^2}.$

And $\sin.^2(yz) = \frac{B^2+C^2}{A^2+B^2+C^2}.$

But by trigonometry we know that $\cos.^2 = 1 - \sin.^2$.

Whence $\cos.^2(xy) = 1 - \frac{A^2+B^2}{A^2+B^2+C^2} = \frac{C^2}{A^2+B^2+C^2}$, &c.

Whence
$$\left.\begin{aligned} \cos.(xy) &= \frac{\pm C}{\sqrt{A^2+B^2+C^2}} \\ \cos.(xz) &= \frac{\pm B}{\sqrt{A^2+B^2+C^2}} \\ \cos.(yz) &= \frac{\pm A}{\sqrt{A^2+B^2+C^2}} \end{aligned}\right\} \text{Expressions sought}$$

Squaring, and adding the last three equations, we find

$$\cos.^2(xy) + \cos.^2(xz) + \cos.^2(yz) = 1.$$

That is, *the sum of the squares of the cosines of the three angles which a plane forms with the three co-ordinate planes, is equal to radius square, or unity.*

PROPOSITION XI.—PROBLEM.

To find the equation of the intersection of two planes.

Let
$$Ax + By + Cz + D = 0, \qquad (1)$$
$$A'x + B'y + C'z + D' = 0, \qquad (2)$$
be the equations of the two planes.

If the two planes intersect, the values of x, y, and z, will be the same for any point in the line of intersection. Hence, we may combine the equations for that line.

Multiply (1) by C', and (2) by C, and subtract the products and we shall have

$$(AC'-A'C)x+(BC'-B'C)y+(DC'-D'C)=0,$$

for the equation of the line of intersection on the plane (xy). If we eliminate y in a similar manner, we shall have the equation of the line of intersection on the plane (xz); and eliminating x will give us the equation of the line of intersection on the plane $(yz.)$

PROPOSITION XII.—PROBLEM.

To find the equation to a perpendicular let fall from a given point x′, y′, z′) *upon a given line.*

As the perpendicular is to pass through a given point, its equations must be of the form

$$x-x'=a(z-z'), \qquad (1)$$

$$y-y'=b(z-z'), \qquad (2)$$

in which a and b are to be determined.

The equation of the plane is

$$Ax+By+Cz+D=0.$$

The line and the plane being perpendicular to each other, by hypothesis, their projections on any one of the co-ordinate planes will be perpendicular to each other.

The given plane then projected on the planes (xz) and (yz), will give $Az+Cz+D=0$ for the equation of the trace on (yz).

From the former $$x=-\frac{C}{A}z-\frac{D}{A}. \qquad (3)$$

From the latter $$y=-\frac{C}{B}z-\frac{D}{B}. \qquad (4)$$

Now equations (1) and (3) represent lines which are at right angles with each other.

Also (2) and (4) represent lines at right angles with each other.

But when two lines are at right angles, (Prop. V, Sec. I, Chap. I,) and a and a', their trigonometrical tangents, we must have $(aa'+1=0)$.

That is, $-a\frac{C}{A}+1=0,$ or $a=\frac{A}{C}.$

Like reasoning gives us $b=\frac{B}{C}$, and these values put in (1) and (2) give

$$\left.\begin{aligned} x-x'&=\frac{A}{C}(z-z') \\ y-y'&=\frac{B}{C}(z-z') \end{aligned}\right\} \text{ for the equations sought.}$$

PROPOSITION XIII.—PROBLEM.

To find the angle included by two planes given by their equations.

Let $Ax+By+Bz+D=0, \qquad (1)$

And $A'x+B'y+C'z+D'=0, \qquad (2)$

be the equations of the planes.

Conceive lines drawn from the origin perpendicular to each of the planes. Then it is obvious that the angle contained between these two lines is the *supplement* of the inclination of the planes. But an angle and its supplement have numerically the same trigonometrical expression.

Designate the angle between the two planes by V, then Proposition VIII, in the last chapter gives

$$\cos. V=\frac{1+aa'+bb'}{\pm\sqrt{1+a^2+b^2}\sqrt{1+a'^2+b'^2}}. \qquad (3)$$

The equations of the two perpendicular lines from the origin must be in the form

$$x=az, \qquad y=bz,$$
$$x=a'z \qquad y=b'z.$$

But because the first line is perpendicular to the first plane, we must have

$$a=\frac{A}{C}, \quad \text{and} \quad b=\frac{B}{C}, \quad \text{(Prop. XII.)}$$

And the second line perpendicular to the second plane requires that

$$a'=\frac{A'}{C'}, \quad \text{and} \quad b'=\frac{B'}{C'}.$$

These values of a, b, and a', b', substituted in (3) and reduced, will give

$$\cos. V=\pm\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}},$$

for the equation required.

Corollary. When two planes are at right angles, $\cos. V=0$, which will make

$$AA'+BB'+CC'=0$$

PROPOSITION XIV.—PROBLEM.

To find the inclination of a line to a plane.

Let MN be the plane given by its equation

$$Ax+By+Cz+D=0,$$

and let PQ be the line given by its equations

$$x=az+\alpha.$$
$$y=bz+\beta.$$

Take any point P in the given line, and let fall PR, the perpendicular, upon the plane; RQ is its projection on the plane, and PQR is obviously the least angle made between the line and the plane, and *it is the angle sought.*

Let $x=a'z+\alpha'$, and $y=b'z+\beta'$,

be the equation of the perpendicular PR, and because it is perpendicular to the plane, we must have (by the last proposition)

$$a'=\frac{A}{C}, \qquad \text{and} \qquad b'=\frac{B}{C}.$$

Because PQ and PR are two lines in space, if we designate the angle included by V, we shall have

$$\cos. V=\pm\frac{1+aa'+bb'}{\sqrt{1+a^2+b^2}\sqrt{1+a'^2+b'^2}}. \qquad \text{(Prop. VIII.)}$$

But the $\cos. V$ is the same as the $\sin. PQR$, or $\sin. V$, as the two angles are complements to each other.

Making this change, and substituting the values of a and b', we have

$$\sin. V = \pm \frac{Aa+Bb+C}{\sqrt{1+a^2+b^2}\sqrt{C^2+B^2+A^2}},$$

for the required result.

Corollary. When $V=0$, $\sin. V=0$, and this hypothesis gives

$$Aa+Bb+C=0,$$

for the equation of a line when it is parallel to the plane.

We now conclude this branch of our subject with a few practical examples, by which a student can test his knowledge of the two preceding chapters.

EXAMPLES.

1. *What is the distance between two points in space of which the co-ordinates are*

$$x=3, \quad y=5, \quad z=-2, \quad x'=-2, \quad y'=-1, \quad z'=6.$$

Ans. 11.180+.

2. *Of which the co-ordinates are*

$$x=1, \quad y=-5, \quad z=-3, \quad x'=4, \quad y'=-4, \quad z'=1.$$

Ans. $5\frac{1}{10}$ nearly.

3. *The equations of the projections of a straight line on the co-ordinate planes* (xz), (yz), *are*

$$x=2z+1, \qquad y=\tfrac{1}{3}z-2,$$

required the equation of projection on the plane (xy).

Ans. $y=\frac{1}{6}x-2\frac{1}{6}$.

4. *The equations of projections of a line on the co-ordinate planes* (xy) *and* (yz) *are*

$$2y=x-5, \qquad \text{and} \qquad 2y=z-4,$$

required the projection on the plane (xz.)

Ans. $x=z+1$.

5. *Required the equations of the three projections of a straight line which passes through two points whose co-ordinates are*

$x'=2, \quad y'=1, \quad z'=0, \quad$ and $\quad x''=-3, \quad y''=0, \quad z''=-1.$

What are the projections on the planes (xz) and (yz)?

Ans. $x=5z+2, \quad y=z+1.$

And from these equations we find the projection on the plane (xy), that is, $5y=x+3$.

(See Prop. III, Sec. II, Chap. II.)

6. *Required the angle included between two lines whose equations are*

$\left.\begin{matrix} x=3z+1 \\ y=2z+6 \end{matrix}\right\}$ of the 1st, and $\left.\begin{matrix} x=z+2 \\ y=-z+1 \end{matrix}\right\}$ of the 2d.

Ans. $V=72° \; 1' \; 29''$.

(See Eq. (3), Prop. XIII.

7. *Find the angles made by the lines designated in the preceding example, with the co-ordinate axes.* (See Prop. VII.)

Ans. The 1st line $\begin{cases} 36° \; 42' & \text{with } X, \\ 57° \; 41' \; 20'' & Y, \\ 74° \; 29' \; 5'' & Z, \end{cases}$ 2d line $\begin{cases} 54° \; 44' & \text{with } X, \\ 125° \; 16' & Y, \\ 54° \; 44' & Z. \end{cases}$

8. *Having given the equation of two straight lines in space, as*

$\left.\begin{matrix} x=3z+1 \\ y=2z+6 \end{matrix}\right\}$ of the first, and $\left.\begin{matrix} x=z+2 \\ y=-z+\beta' \end{matrix}\right\}$ of the second,

to find the value of β', *so that the lines shall actually intersect, and to find the co-ordinates of the point of intersection.*

Ans. $\begin{cases} \beta'=7\tfrac{1}{2}, & y=4\tfrac{1}{2}, \\ x'=1, & z=-\tfrac{1}{2}. \end{cases}$

(See Prop. IV. Sec. II, Chap. I.)

9. *Given the equation of a plane*

$$8x-3y+z-4=0,$$

to find its intersection with the three axes, and the perpendicular distance of the origin to the plane. (Prop. IX.)

Ans. It cuts the axis of X at the distance of $\frac{1}{2}$ from the origin; the axis of Y at $-1\frac{1}{3}$; and the axis of Z at $+4$.

The origin is $\frac{2\,9}{4\,3}$ of unity below the plane.

10. *Find the equations for the intersections of the two planes*

(Prop. XI.) $\quad 3x-4y+2z-1=0,$

$7x-3y-z+5=0.$

Ans. On the plane (xy) $\quad 17x-10y+9=0.$

On the plane (xz) $\quad 12x-10z+23=0.$

11, *Find the inclination of these two planes.* (Prop. XIII.)

Ans. 41° 27′ 30″.

12. *The equations of a line in space are*

$$x=-2z+1, \quad \text{and} \quad y=3z+2.$$

Find the inclination of this line to the plane represented by the equation (Prop. XIV.)

$$8x-3y+z-4=0.$$

Ans. 48° 13′.

13. *Find the angles made by the plane whose equation is*

$$8x-3y+z-4=0,$$

with the co-ordinate planes. (Prop. X.)

Ans. $\begin{cases} 84^\circ\ 9'\ 40'' \text{ with } (xy). \\ 110^\circ\ 24'\ 40'' \text{ with } (xz). \\ 21^\circ\ 34' \text{ with } (yz). \end{cases}$

14. *Find the equation of a plane being*

$$Ax+By+Cz+D=0,$$

Required the equation of a parallel plane whose perpendicular distance is (a) *from the given plane.*

Ans. Because the planes are to be parallel, they must have the same co-efficients, A, B, and C.

In Prop. IX, we learn that the perpendicular distance of the origin from the given plane may be represented by

$$p=\pm\frac{D}{\sqrt{A^2+B^2+C^2}}$$

Now, as the planes are to be a distances asunder, the distance of the origin from the required plane must be

$$\frac{D}{\sqrt{A^2+B^2+C^2}}+a \quad \text{or} \quad \frac{D+a\sqrt{A^2+B^2+C^2}}{\sqrt{A^2+B^2+C^2}}.$$

Whence the equation required is

$$Ax+By+Cz+\left(\frac{D+a\sqrt{A^2+B^2+C^2}}{\sqrt{A^2+B^2+C^2}}\right)=0.$$

THE DIFFERENTIAL CALCULUS.

SECTION I.

CHAPTER I.

Definitions and Illustrations.

The differential calculus may be considered a branch of analytical geometry ; more literally, it is a science for computing the *ratio* of small differences.

Newton and his followers called this science *fluxions*, because magnitudes were conceived to *flow*, thus making an *increase* or *decrease*, and the amount of increase or decrease was the *fluxion* of the particular magnitude or algebraic quantity. But the French, and the moderns who have adopted the French phraseology, call this quantity the differential of the given magnitude.

In some instances the old English method of illustrating this science is most simple, and we shall not entirely disregard it. They conceived a line to be extended by the *motion of a point* at its extremity, — a surface to be extended by the *motion of a line*, and a solid to be extended by the *motion of a surface*.

To illustrate and explain the object of the calculus, we adduce the following questions :

If a side of a square be increased by a very small quantity, what effect will that have on the square itself ?

If a side of a cube be increased by a very small quantity, how much will the cube itself be increased ?

If the arc of a circle be increased by a very small quantity what effect will this have on the sine and cosine of that arc? What on its tangent and co-tangent?

If the base of a known right angled triangle be increased by a very small quantity, what effect will that have on the hypotenuse and the acute angles?

The sun's motion in longitude along the elliptic has a corresponding motion in declination; the *ratio* of these two motions at each and every point is a problem in the differential calculus.

The calls of astronomy gave birth to this science.

It is not necessary that every part of a magnitude should increase or decrease, and therefore we must have *variables* and *constants*. For instance, one side of a right angled triangle may increase while the other side remains constant, and the hypotenuse increase in consequence of the increase of one side. Or the two sides may vary at the same time, the one increase, the other decrease, and the hypotenuse remain constant.

Constant quantities are generally represented by the first letters of the alphabet, a, b, c, &c. and the variable quantities by the final letters as u, v, x, y, &c.

In any equation, as $y=ax$, if an increment is given to x, y will have a corresponding increase, and in that case x is said to be the *independent variable*, and y the *dependent variable*. That is, the variation of y certainly *depends* on the variation of x.

Thus far we have merely been giving an idea of what the differential calculus is.

(Art. 1.) When two or more variables enter into an equation one is said to be a *function* of the other.

Thus $y=a+3x$, $y=3a^2x+x^2$, $y=\sqrt{1-x^3}$, are three different equations, and here are three different functions of x. That is, y expresses three different functions of x, and might express ten thousand other functions as well as these.

If the foregoing equations be resolved in relation to x, so that x stands alone as one member, then we might say that x is a function of y, hence x and y are functions of each other.

When we wish to express functions in a brief and comprehensive manner without designating *any particular* equation, we write $y=f(x)$, which means that y equals some algebraic expression in which x, as a variable, is contained, and it is read y equals a function of x. The letters f, f_1, F, F_1, stand in the place of the word function. Each indicates a different function from the other.

Thus, $f(x.y)=0$. $F(u.x.y)=0$, &c.

The first of these is a symbol for an equation containing x and y as variables, and every quantity in the first member of the equation. The second is also a general symbol for some equation containing u, x, and y, as variables, and all in the first member of the equation.

If we had an equation in which u was the first member, and equal to some algebraic expression containing x, y, and z, as variables, and if it were not necessary to write out the *explicit* functions of x, y, and z, we would indicate it generally.

Thus $u=f(x.y.z)$.

Or we may write $f(u.x.y.z)=0$.

If we wished to indicate another equation containing these same letters, which might exist in the same time, we would write

$$F(u.x.y.z)=0.$$

If still another, we would write

$$F_1(u.x.y.z)=0.$$

Functions are either *algebraic*, *circular*, or *exponential*.

(Art. 2.) We now commence to form rules to *differentiate* all classes of algebraic quantities.

For example, we have the equation

$$y=ax+b.$$

What will be the effect on y, provided x becomes $(x+h)$?

Let y' represent what y then becomes, and the equation will become

$$y'=ax+ah+b$$

But $y=ax+b$

Therefore $y'-y=ah$

The first member of this equation is obviously the *increment* of y, whatever be the value of h, and when it is *extremely* small in relation to x, (we will not say *infinitely* small, as that word puzzles, and it is unnecessary) then $(y'-y)$ is extremely small in relation to y, and in that case we write dy in place of $(y'-y)$, and dx in the place of h.

The learner must be particular not to regard d in dy or dx as a numeral or a co-efficient. It is a *symbol*, and is read the *differential* of y, the differential of x, &c., as the case may be.

Thus, in general dy means an *extremely* small increment or decrement of y, and dP would denote that P was a variable and dP the amount of the variation.

In the old English fluxions dx, dy, &c. are represented by $\dot{x}$, $\dot{y}$, &c. the variable with a *point over it*. The terms *fluxion*, *differential*, and *derivative*, all mean the same thing. There are cases, as in *derived polynomials* in algebra, that it would not do to call the derivative a differential, as the increment might be too large.

For another example. If

$$u=a+bx+cy+z,$$

and if we suppose x becomes $(x+h)$, y becomes $(y+k)$, and z becomes $(z+l)$, what effect will this have on the value of u?

In consequence of these *increments* to the variables x, y, and z, the *dependent* variable u becomes u', and the equation becomes

$$u'=a+bx+bh+cy+ck+z+l.$$

From this, if we subtract the primitive equation, we shall have

$$u'-u=bh+ck+l.$$

If we now suppose h, k, and l, to be extremely minute quantities, dx must be written for h, dy for k, dz for l, and du for $(u'-u.)$

Then $$du=bdx+cdy+dz,$$

and this is the differential of the primitive equation.

Comparing this result with the given equation, we draw the following rule for differentiating an equation, or any quantity involving only the first power of the variables :

Rule 1. *Change each variable into its differential by simply writing* dx *in place of* x, dy *in place of* y, *and so on for any other* *variable, preserving the same constant co-efficients that belong to the variable, and drop all constants which stand alone, or such as have no variable factor.*

We give the following examples under this rule:

1. Differentiate $u=a^3+3a^2x+b^2y+4z.$

$$Ans.\ du=3a^2dx+b^2dy+4dz.$$

2. Differentiate $u=\frac{x}{a}+\frac{y}{3b}+1.$

$$Ans.\ du=\frac{dx}{a}+\frac{dy}{3b}.$$

3. Differentiate $3u=\sqrt{ax}+4a^{\bar{3}}y+\frac{1}{c}.$

$$Ans.\ 3du=\sqrt{a}.dx+4a^3dy.$$

(Art. 3.) Let us now investigate and draw out a rule to determine the differential of the product of two variables.

Let $u=xy$, and now suppose that x becomes $(x+h)$, and y becomes $(y+k)$, and in consequence of these increments, u becomes u', and the equation becomes

$$u'=(x+h)(y+k)=xy+yh+xk+hk.$$

Subtracting the original equation, and we have

$$u'-u=yh+xh+hk.$$

If now we suppose h and k to be extremely minute quantities, their product hk will be still less, and therefore may be omitted when h takes the form dx and k becomes dy. This supposition reduces the last equation to

$$du=ydx+xdy.$$

Comparing this equation with the original one ($u=xy$), will show the truth of the following rule to differentiate a product:

Rule 2. *Multiply each variable by the differential of the other variable, and add their products.*

We may extend this rule to apply to any number of variables. For instance, let

$$P=xyz.$$

Also, let $u=xy$, as in the former equation, then

$$P=uz.$$

Taking the differential of this last equation by the rule just formed, and we have

$$dP=udz+zdu.$$

In this equation, for u, write its value (xy), and for du write its value $(ydx+xdy)$. Then

$$dP=yzdx+xzdy+xydz.$$

This equation furnishes a rule for the differential of the product of three variables, which principle being extended, gives the following general rule, which will apply to the product of any number of variables:

RULE 3. *Take the differential of each variable and multiply it into the product of all the other variables, and add the several products together.*

Differentiate the following examples under this rule:

1. $u=xy+xyz$.

   Ans. $du=ydx+xdy+yzdx+xzdy+xydz$.

2. $u=ty-3xy+tx$.

   Ans. $du=ydt+tdy-3ydx-3xdy+tdx+xdt$.

3. $u=vxyz$.

   Ans. $du=xyzdv+vyzdx+vxzdy+vxydz$.

(Art. 4.) If in this last equation we suppose v, y, z, each equal to x, then will $u=x^4$, and dv, dy, dz, must each equal dx, and each one of the four products in the answer will be x^3dx.

Consequently $du=4x^3dx$, and the differential of x^4 must be $4x^3dx$.

Let us now test this by another course of operation.

Let $$u=x^4.$$

Now suppose that x becomes $(x+h)$, and in consequence of this u becomes u', then

$$u'=(x+h)^4=x^4+4x^3h+6x^2h^2+4xh^3+h^4.$$

Subtracting the original equation and dividing by h, will give us

$$\frac{u'-u}{h}=4x^3+6x^2h+4xh^2+h^3.$$

Now in case h is taken for an *extremely* small quantity in relation to x, all the terms that contain h in the second member are comparatively *valueless* in respect to $(4x^3)$ the first term.

But in case of an *extreme* small quantity for h we write (dx) for h, and du for $(u'-u)$, therefore

$$\frac{du}{dx}=4x^3. \qquad (1)$$

The same result as before deduced from the consideration of products.

In case h is absolutely zero, dx becomes 0, and du also becomes 0, and equation (1) becomes

$$\frac{0}{0}=4x^3.$$

But there is nothing absurd in this, as we learn by algebra that 0 divided by 0 can be any quantity whatever.

Now $(u'-u)$ represents the increment of the function u; and h that of the variable x, and therefore $\frac{u'-u}{h}$ is the *ratio*, and this ratio diminishes as h diminishes, and comes to a *limit* when h equals 0.

Therefore $\frac{0}{0}$ is the *limiting ratio* between a function and its variable. In this example $4x^3$ is that *ratio*, it is also called the *differential co-efficient*.

For $\qquad \frac{du}{dx}=4x^3, \quad$ or $\quad du=4x^3dx.$

Here it is obvious that $4x^3$ is a co-efficient to dx, the differential of the variable.

For another example, let

$$u=x^m.$$

Now as before let x become $(x+dx)$, then u becomes $u+du$, and the equation becomes

$$u+du=(x+dx)^m=x^m+mx^{m-1}dx+m\frac{m-1}{2}\,x^{m-2}(dx)^2+\&c.$$

Subtracting the original equation, and dividing by (dx), we have

$$\frac{du}{dx}=mx^{m-1}+m\frac{m-1}{2}x^{m-2}dx+ \&c.$$

Now let us suppose $dx=0$, that is, *pass to the limit,* and

$$\frac{du}{dx}=mx^{m-1}.$$

From this equation we can draw the following general rule to find the *differential co-efficient* of any quantity in the form x^m, that is, any power of the variable:

RULE 4. *Multiply by the exponent, and diminish the exponent by unity.*

By the first example in this article the learner will perceive the truth of this rule when the exponent m is a *whole positive number,* such as x^3, x^4, &c. &c., and yet not convinced of its application when m is fractional or negative.

But we learn in algebra that $(x+dx)^m$ expands in the same form, whatever be the value of m, whole or fractional, positive or negative, therefore *the rule must be generally applicable, whatever be the value of* m.

For example, what is the *differential co-efficient* in the following equation, in which m is taken both negative and fractional:

$$u=x^{\frac{-s}{t}}?$$

By the rule we have at once

$$\frac{du}{dx}=-\frac{s}{t}x^{\frac{-s}{t}-1}$$

Or the differential of the function u is

$$-\frac{s}{t}x^{\frac{-s}{t}-1}dx.$$

Suppose now that we distrust the rule, and require the result by a more elementary and obvious process, and if we arrive at the same conclusion it would be very unphilosophical to distrust it in any future case.

Resuming $u=x^{\frac{-s}{t}}$. This is the same as

$$u=\frac{1}{x^{\frac{s}{t}}} \quad \text{or} \quad ux^{\frac{s}{t}}=1.$$

Raising each member to the power t, then

$$u^t x^s=1 \qquad (1)$$

Put $P=u^t$ and $Q=x^s$, then $PQ=1$.

Now we can differentiate this equation by (Rule 2,) which gives

$$PdQ+QdP=0. \qquad (2)$$

As t and s are whole positive numbers,

$$dP=tu^{t-1}du, \quad \text{and} \quad dQ=sx^{s-1}dx.$$

Substituting these quantities in (2), and retaining the original values of P and Q, equation (2) becomes

$$su^t x^{s-1}dx+tx^s u^{t-1}du=0. \qquad (3)$$

Multiply (3) by ux, which gives

$$su^t x^s udx+tx^s u^t xdu=0. \qquad (4)$$

But equation (1) gives $u^t x^s=1$. Therefore (4) reduces to

$$sudx+txdu=0. \qquad (5)$$

Or $$\frac{du}{dx}=-\frac{su}{tx}.$$

Substituting the value of u in the second member taken from the original equation, and we have

$$\frac{du}{dx}=-\frac{s}{t}x^{-\frac{s}{t}-1}$$

The same value as given by the rule, and thus the rule could be verified in every possible case.

Rule 4, can be made of very extensive application in the calculus, as the following examples will show:

$$u=(1+x)^{\frac{1}{2}}.$$

Put $(1+x)=z$, then $u=z^{\frac{1}{2}}$, an equation to which the rule will apply, giving

$$\frac{du}{dz}=\tfrac{1}{2}z^{\frac{1}{2}-1}=\tfrac{1}{2}z^{-\frac{1}{2}}=\frac{1}{2\sqrt{z}}.$$

Because $1+x=z$, $dx=dz$. Therefore

$$\frac{du}{dx}=\frac{1}{2(1+x)^{\frac{1}{2}}}.$$

Or
$$du=\frac{dx}{2(1+x)^{\frac{1}{2}}}.$$

For another example, take

$$u=\sqrt{1+x-x^2}.$$

As before, put $1+x-x^2=z$. Then $dx-2xdx=dz$, and $u=z^{\frac{1}{2}}$.

By the rule,
$$\frac{du}{dz}=\tfrac{1}{2}z^{-\frac{1}{2}}=\frac{1}{2\sqrt{z}}.$$

Or
$$du=\frac{dz}{2\sqrt{z}}=\frac{(1-2x)dx}{2\sqrt{1+x-x^2}}.$$

From these examples we may draw the following rule to differentiate the *square root* of any quantity:

RULE 5. *Differentiate the quantity under the radical, and divide it by twice the radical.*

The last equation may be differentiated without substitution, thus

$$u=\sqrt{1+x-x^2}.$$

Square both members, and

$$u^2=1+x-x^2.$$

Now apply the rule to each member, and

$$2udu=dx-2xdx.$$

Whence
$$du=\frac{(1-2x)dx}{2\sqrt{1+x-x^2}}.$$

The preceding rule may be made general, as will appear by the following example:

$$u=(a+bx-cx^2)^{\frac{1}{n}}.$$

Take the nth power of each member, then

$$u^n=a+bx-cx^2.$$

Taking the differential of each member, gives us

$$nu^{n-1}du=(b-2cx)dx.$$

Multiply this by the given equation, and we have

$$nu^n du=(b-2cx)dx(a+bx-cx^2)^{\frac{1}{n}}$$

Whence $$du=\frac{(b-2cx)dx(a+bx-cx^2)^{\frac{1}{n}}}{n(a+bx-cx^2)}$$

From this equation we draw the following general rule to differentiate any radical quantity:

Rule 6. *Take the differential of the quantity under the radical, multiply it by the radical, and divide the product by* n *times the quantity under the radical,* n *being the index of the root.*

For an example under this rule we give the following equation:

$$u=(1+2x-x^2)^{\frac{1}{6}}.$$

By the rule $$du=\frac{(2-2x)dx(1+2x-x^2)^{\frac{1}{6}}}{6(1+2x-x^2)}.$$

Results under this rule are always reducible, as we have the same quantity in the numerator and denominator of the second member, with different exponents. By subtracting one exponent from the other, and dividing numerator and denominator by 2, we get the following reduced result:

$$du=\frac{(1-x)dx}{3(1+2x-x^2)^{\frac{5}{6}}}.$$

(Art. 5.) Sometimes we have $u=f(y)$, and $y=F(x)$, and require the differential co-efficient between the function u and the variable x.

For that purpose we first find $\frac{du}{dy}$ from one equation, and then $\frac{dy}{dx}$ from the other, and multiply them together, and we have $\frac{du}{dx}$ as required.

For example, suppose $u=1+2y+y^2$, and $y=a+x^3$, and we require the *ratio* $\frac{du}{dx}$, we obtain it thus :

From the first $\frac{du}{dy}=2+2y.$

From the second $\frac{dy}{dx}=3x^2.$

Multiplying these together and we obtain the final result

$$\frac{du}{dx}=6(1+y)x^2.$$

We might have taken the value of $y, (a+x^2)$ in the second equation and substituted it in the first, and then have taken the differential, but this would have been more troublesome.

(Art. 6.) We have but one more rule to advance to enable the learner to differentiate all kinds of *algebraic quantities*. That is, a rule to differentiate a fraction.

Let it be required to differentiate the fraction $\frac{x}{y}$. Put $u=\frac{x}{y}$. Then the value of du will be the differential of the fraction.

Clear of fractions and $uy=x.$

The first member is a product of two variables, therefore differentiate it by Rule 2.

Whence $udy+ydu=dx$

Restoring the value of u, this last equation becomes

$$\frac{xdy}{y}+ydu=dx.$$

Or $xdy+y^2du=ydx.$

Whence $du=\frac{ydx-xdy}{y^2}.$

From this equation we can draw the following rule to differentiate a fraction :

Rule 7. *From the differential of the numerator multiplied by the denominator, subtract the differential of the denominator multiplied into the numerator, and divide the difference by the square of the denominator.*

We can obtain the same result, independent of a product, as follows :—The fraction to be differentiated is $\frac{x}{y}$. Let x become $x+dx$ and y become $y+dy$.

Then the fraction becomes $\frac{x+dx}{y+dy}$, and from this subtract the original fraction, and we shall have the difference, which is the differential in case both dx and dy are extremely small.

$$\frac{x+dx}{y+dy}-\frac{x}{y}=\frac{xy+ydx-xy-xdy}{y^2+ydy}.$$

The *difference* is $\frac{ydx-xdy}{y^2+ydy}$, *whatever be the values* of dx and dy; but when dy is *extremely* small, (not to say infinitely small), then y^2 is not sensibly augmented by the addition of ydy, and therefore the *differential* of the fraction is $\frac{ydx-xdy}{y^2}$, as before found.

The preceding rules and *combinations* of them will serve to differentiate any algebraical expression that can be given. Yet there are cases that might appear inapplicable to the rules, at first view, and in these operations there is room to exercise algebraical tact and skill.

We give the following examples :

1. *Differentiate the equation*

$$u=(a+bx^3)^m.$$

As explained in (Art. 5,) put $y=a+bx^3$, then $u=y^m$. Now from this last equation

$$\frac{du}{dy}=my^{m-1}. \qquad \text{(Rule 4.)}$$

Also $$\frac{dy}{dx}=3bx^2. \qquad \text{(Rule 4.)}$$

By multiplication $$\frac{du}{dx}=3bmx^2y^{m-1}.$$

Substituting the value of y^{m-1}, and multiplying by dx, and

$$du=3bmx^2(a+bx^3)^{m-1}dx.$$

2. *Differentiate the equation*

$$u=x(a^2+x^2)\sqrt{a^2-x^2}. \qquad (1)$$

This requires the application of Rules 3 and 5, but to show independence and tact, put

$$y=\sqrt{a^2-x^2}. \qquad (2)$$

Multiply (1) and (2), and we obtain

$$uy=a^4x-x^5.$$

Taking the differential of each member,

$$udy+ydu=a^4dx-5x^4dx. \qquad (3)$$

From (2) we find

$$dy=\frac{-xdx}{\sqrt{a^2-x^2}}. \qquad (4)$$

The product of (4) and (1) is

$$udy=-(a^2x^2+x^4)dx. \qquad (5)$$

Subtracting (5) from (3), and we have

$$ydu=(a^4+a^2x^2-4x^4)dx.$$

Finally, $$du=\frac{(a^4+a^2x^2-4x^4)}{\sqrt{a^2-x^2}}dx.$$

Some expressions may be reduced or changed in form to advantage before attempting to take the differential. The following is an example of the kind:

3. *Given* $u=\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$ *to find the differential of* u.

Reduce the second member by multiplying numerator and denominator of the fraction by the numerator, then

$$u=\frac{1+\sqrt{1-x^2}}{x}.$$

Whence $$du=\frac{\frac{-x^2\,dx}{\sqrt{1-x^2}}-dx(1+\sqrt{1-x^2})}{x^2} \qquad \text{(Rule 7.)}$$

Dividing by dx, and changing signs, we have

$$\frac{-du}{dx}=\frac{1}{\sqrt{1+x^2}}+\frac{1+\sqrt{1-x^2}}{x^2}.$$

$$=\frac{x^2+\sqrt{1-x^2}+1-x^2}{x^2\sqrt{1-x^2}}=\frac{1+\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}.$$

Whence $$du=-\left(\frac{1+\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}\right)dx.$$

We add the following unwrought examples to exercise the powers of the learner:

4. $u=\frac{-a}{x^3}$ .... $du=\frac{3adx}{x^4}$.

5. $u=\frac{a}{(x)^{\frac{1}{3}}}$ .... $du=\frac{-adx}{3x^{\frac{4}{3}}}$.

6. $u=x^2y^2$ .... $du=2x^2ydy+2y^2xdx$.

7. $u=\sqrt{2ax+x^2}$...(Rule 6, reduced.)..$du=\frac{(a+x)dx}{\sqrt{2ax+x^2}}$.

8. $u=\frac{x}{x+\sqrt{1-x^2}}$ .... $du=\frac{dx}{(x+\sqrt{1-x^2})^2\sqrt{1-x^2}}$.

9. $u=\frac{1}{\sqrt{1-x^2}}$ .... $du=\frac{xdx}{(1-x^2)^{\frac{3}{2}}}$.

10. $u=\frac{a}{(a-x)^3}$ .... $du=\frac{3adx}{(a-x)^4}$

11. $u=\frac{1+x^2}{1-x^2}$ .... $du=\frac{4xdx}{(1-x^2)^2}$.

12. $u=\frac{x^n}{(1+x)^n}$ .... $du=\frac{nx^{n-1}dx}{(1+x)^{n+1}}$.

13. $u=(1+x)\sqrt{1-x}$ . . . . $du=\frac{(1-3x)dx}{2\sqrt{1-x}}$.

14. $u=(\sqrt{1-x})(\sqrt{1+x^2})$ . . . . . $du=\frac{-(1-2x+3x^2)dx}{2\sqrt{1-x}\cdot\sqrt{1+x^2}}$.

15. $u=\frac{3}{\sqrt{x^2-y^2}}$ . . . . $du=\frac{3(ydy-xdx)}{(x^2-y^2)^{\frac{3}{2}}}$.

16. $u=2x\sqrt{a^2+x^2}$ . . . . $du=\frac{(2a^2+4x^2)dx}{\sqrt{a^2+x^2}}$.

17. $u=\left(a+\sqrt{b-\frac{c}{x^2}}\right)^4$ . . . . . . $du=\frac{\frac{4c}{x^3}\left(a+\sqrt{b-\frac{c}{x^2}}\right)^3 dx}{\sqrt{b-\frac{c}{x^2}}}$.

18. $y=(a+\sqrt{x})^3$ . . . . $dy=\frac{3(a+\sqrt{x})^2 dx}{2\sqrt{x}}$.

19. $y=\frac{1}{(a+\sqrt{x})^3}$ . . . . $dy=\frac{-3dx}{2\sqrt{x}(a+\sqrt{x})^4}$.

20. $P=2xy^2$ . . . . $dP=2y^2dx+4xydy$.

21. $P=\frac{1}{2xy^2}$ . . . . $dP=\frac{-(2y^2dx+4xydy)}{4x^2y^4}$.

22. $z=(\sqrt{a^2+x^2})(\sqrt{b^2+y^2})$ . . $dz=\frac{(b^2+y^2)xdx+(a^2+x^2)ydx}{(\sqrt{a^2+x^2})(\sqrt{b^2+y^2})}$.

23. Find the differential co-efficient of $(a+x^3)(b+3x^2)$.

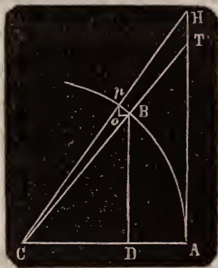
Ans. $15x^4+3x^2b+6ax$.

N. B. Put $u=(a+x^3)(b+3x^2)$, and find the value of $\frac{du}{dx}$.

23. Find the differential coefficient of a cube whose side is x. The function is then $u=x^3$, and $du=(3x^2)dx$, and $3x^2$ is the *coefficient required*, showing that the differential of the side must be multiplied by the coefficient $3x^2$, to obtain the *differential of the volume*, and this will explain the general application of differential coefficients.

CHAPTER II.

The Differential of Circular Functions.



(Art. 7.) Let AB be a circular arc designated by x, and let it receive an increment h, as represented by Bp, and from this we are to determine the value of (op), (oB), and TH.

When Bp is so small that the *chord* and the *arc* may each be considered a right line, then h becomes (dx), the differential of the arc (op) is the differential of the sine x, (oB) is the differential of the cosine x, and TH is the differential of the tangent x.

The two triangles poB and CDB are equiangular and similar. Each has a right angle, one at D, the other at o.

The angle pBC is a right angle, so is oBD; from each to take the common angle oBC, and we shall perceive that the angle pBo is equal the angle CBD. Whence the angle opB is equal the angle BCD, and the two triangles give the following proportions:

$$pB : Bo :: CB : BD. \qquad (1)$$

$$pB : po :: CB : CD. \qquad (2)$$

If the radius CB is taken equal to *unity*, and pB sufficiently small to call it dx, then proportion (1) becomes

$$dx : -d.\cos.x :: 1 : \sin.x. \qquad (3)$$

and (2) becomes $dx : d\sin.x :: 1 : \cos.x. \qquad (4)$

$$\left.\begin{array}{ll}\text{Whence} & d.\cos.x = -\sin.x.dx. \\ \text{And} & d\sin.x = \cos.xdx.\end{array}\right\} \quad (A)$$

It now remains to find the value of TH, the differential of $\tan.x$. For this purpose we will resolve the triangle CTH trigonometrically, thus:

$$\sin.CHT : CT :: \sin.HCT : HT.$$

Now let it be observed that the sine of the angle CHT is very nearly the same as the sine of the angle CTA, which is equal to the cosine of the angle TCA or cos.x.

Also, as the angle HCT is an extremely small angle, (by hypothesis,) and as the sine of a *very* small arc is the same as the arc itself, therefore $\sin.HCT=pB=dx$.

Whence the preceding proportion becomes

$$\cos.x : CT :: dx : d\tan.x.$$

Now, in the similar triangles CDB, CAT, we have

$$CD : CB :: CA : CT.$$

That is, $\cos.x : 1 :: 1 : CT=\frac{1}{\cos.x}$.

Therefore, $\cos.x : \frac{1}{\cos.x} :: dx : d\tan.x$.

Or $d\tan.x=\frac{dx}{\cos.^2x}$.

(Art. 8.) All this can be drawn more readily from the trigonometrical formula, but we gave the preceding article because we deemed it essential for a learner to have a geometrical view of the subject.

Now we will show the same as follows:

Let $x=$ the arc AB, and $h=$ the arc Bp, then by trigonometry

$$\sin.(x+h)=\sin.x\cos.h+\cos.x\sin.h. \quad (1)$$

$$\cos.(x+h)=\cos.x\cos.h-\sin.x\sin.h. \quad (2)$$

Now if we suppose h represents an *extremely small* arc, then $\cos.h=1$, and $\sin.h=h$, and (1) becomes

$$\sin.(x+h)-\sin.x=\cos.xh.$$

But under this supposition the first member of this last equation becomes $d.\sin.x$, and $h=dx$, and the equation itself becomes

$$d.\sin.x=\cos.x\,dx. \quad (3)$$

By parity of reasoning, equation (2) becomes

$$d\cos.x=-\sin.x\,dx. \quad (4)$$

We perceive that (3) and (4) are the same as (A) in Art. 7, as they ought to be.

To obtain the differential of a tangent we observe that

$$\tan.x = \frac{\sin.x}{\cos.x}.$$

The differential of the first member of this equation is simply $(d\tan.x)$, and the second member must be differentiated by the rule applicable to fractions.

Thus $$d\tan.x = \frac{d\sin.x\cos.x - d\cos.x\sin.x}{\cos.^2 x},$$

Substituting the values of $d\sin.x$, and of $d\cos.x$, taken from (3) and (4), we have

$$d\tan.x = \frac{(\cos.^2 x + \sin.^2 x)dx}{\cos.^2 x} = \frac{dx}{\cos.^2 x}.$$

By trigonometry, $\text{secant}.x = \frac{1}{\cos.x}$. (Radius unity.)

Whence $$d.\sec.x = \frac{\sin.x dx}{\cos.^2 x} = \frac{\tan.x\, dx}{\cos.x}.$$

Also, $$\cot.x = \frac{1}{\tan.x}.$$

Whence $$d.\cot.x = \frac{-d\tan.x}{\tan.^2 x} = \frac{-dx}{\cos.^2 x\tan.^2 x}.$$

But $$\tan.x\cos.x = \sin.x.$$

Therefore $$d\cot.x = \frac{-dx}{\sin.^2 x}.$$

By trigonometry $$\text{cosec}.x = \frac{1}{\sin.x}.$$

Therefore $$d.\text{cosec}.x = \frac{-\cos.x\, dx}{\sin.^2 x}.$$

But $\frac{\cos.x}{\sin.x} = \frac{1}{\tan.x}$. Whence $d\,\text{cosec}.x = \frac{-dx}{\tan.x\sin.x}$.

It is obvious that the differential of the versed sine of an arc is the same as the differential of the cosine, differing only in their signs.

For the sake of reference we collect the preceding results, showing the differential expressions for all trigonometrical lines.

$d\sin.x=\cos.x\,dx.$ (a) $d\cot.x=-\dfrac{dx}{\sin.^2 x}.$ (e)

$d\cos.x=-\sin.x\,dx.$ (b) $d\sec.x=\dfrac{\tan.x\,dx}{\cos.x}.$ (f)

$d\,\text{ver.}\sin.x=\sin.x\,dx.$ (c) $d\,\text{cosec}.x=\dfrac{-dx}{\tan.x\sin.x}.$ (g)

$d\tan.x=\dfrac{dx}{\cos.^2 x}.$ (d)

To differentiate any power of a sine or cosine, we proceed as follows:

Let $u=\sin.^n x.$ (1) Put $y=\sin.x.$ (2)

Then $u=y^n.$ And $du=ny^{n-1}dy.$ (3)

But from (2) we find $dy=\cos.x\,dx.$ Now substituting the value of y, and dy, in (3), we have

$$du=n\sin.^{n-1}x\cos.x\,dx.$$

From this we perceive that the differential of $u=\sin.^4 x$ must be $du=4\sin.^3 x\cos.x\,dx.$

If we have $u=\cos.^n x$, a similar process will give

$$du=-n\cos.^{n-1}x\sin.x\,dx.$$

The practical uses of these equations will be shown in future portions of this work.

(Art. 9.) Hitherto we have shown the differential of sines, cosines, tangents, &c. considered as the function of an arc; we now propose to show the differential of an arc regarded as a function of its sine, cosine, tangent, &c.

When we represent a sine by x, the arc to which it corresponds is designated thus

$$\text{arc}(\sin.=x).$$

If we represent a cosine by x, its corresponding arc would be designated thus

$$\text{arc}(\cos.=x),$$

which is read, an arc whose cosine equals x.

This notation not being satisfactory nor convenient, modern mathematicians have adopted the following:

$$\sin.^{-1}x, \qquad \cos.^{-1}x, \qquad \tan.^{-1}x, \qquad \&c.\ \&c.$$

Thus $u=\sin.^{-1}x$, indicates that x is the *numerical value* of a sine, and u is the *numerical value* of the corresponding arc, therefore the equation may be written $\sin.u=x$.

Similarly $\cos.^{-1}x=u$, *is the same as* $\cos.u=x$.

If x represents the sine of an arc, $\sqrt{1-x^2}$ must represent its cosine, and $\frac{x}{\sqrt{1-x^2}}$ must represent the tangent of the same arc.

Take $$\sin.u=x.$$

We differentiate the first member as a sine, and the second member as an algebraical quantity; therefore

$$\cos.u\,du=dx.$$

Whence $$du=\frac{dx}{\sqrt{1-x^2}}. \qquad (1)$$

Let t be a tangent, and u its corresponding arc to radius unity, then we may write

$$u=\tan.^{-1}t.$$

Or $$\tan.u=t.$$

Now by equation (d) of (Art. 8,) we have

$$\frac{du}{\cos.^2 u}=dt.$$

$du=dt(\cos.^2u)=dt\frac{1}{1+t^2}=\frac{dt}{1+t^2}$, (2), because $\frac{1}{1+t^2}$, is the cosine of an arc when radius is unity, and t the tangent.

Again, let $$\cos.u=y.$$

Then $$-\sin.u\,du=dy.$$

Or $$du=\frac{-dy}{\sqrt{1-y^2}}. \qquad (3)$$

Equations (1), (2), and (3), of this article will be frequently referred to and applied in the integral calculus.

We give the following examples to discipline the learner:

1. *Given* $\sin.(mx)=u$, *to find* du.

$$\cos.(mx)d(mx)=du.$$

Or

$$mdx\cos.mx=du.$$

2. *Given* $u=\sin.^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, or $\sin.u=\frac{1-x^2}{1+x^2}$, *to find* du.

$$\cos.u\,du=\frac{-2xdx(1+x^2)-2xdx(1-x^2)}{(1+x^2)^2}.$$

But if the sine of an arc is $\frac{1-x^2}{1+x^2}$, the cosine of the same arc is $\frac{2x}{1+x^2}$, because $\sin.^2+\cos.^2=1$. Therefore

$$\frac{2x}{1+x^2}du=\frac{-2xdx(1+x^2)-2xdx(1-x^2)}{(1+x^2)^2}.$$

Whence by reduction $du=-\frac{2dx}{1+x^2}$

3. *Given* $u=\sin.^{-1}\sqrt{\left(\frac{1-x}{2}\right)}$, or $\sin.u=\sqrt{\left(\frac{1-x}{2}\right)}$ *to find* du.

Ans. $du=\frac{-dx}{2\sqrt{1-x^2}}$.

4. *Given* $\cos.u=4x^3$, *to find the value of* du.

Ans. $du=\frac{-12x^2\,dx}{\sqrt{1-16x^6}}$.

5. *Given* $\sin.u=\frac{x}{\sqrt{1+x^2}}$ *to find* du.

Ans. $du=\frac{dx}{1+x^2}$.

6. *Given* $\sin.z=2u\sqrt{1-u^2}$ *to find* dz.

Ans. $dz=\frac{2du}{\sqrt{1-u^2}}$.

CHAPTER III.

On the Differential of Exponential Quantities and Logarithms.

(Art. 10.) Hitherto the variable quantities have either been algebraic or circular, but we may have an equation in the form

$$y=a^x. \qquad (1)$$

In this equation the exponent x is variable, and if it becomes $(x+h)$ we are to show what effect that will have on the value of y.

As in our previous notation, if x becomes $(x+h)$, let y become y', then

$$y'=a^{x+h}=a^x a^h. \qquad (2)$$

Subtracting the original equation, we have

$$y'-y=a^x a^h-a^x. \qquad (3)$$

That is, $$\frac{y'-y}{a^x}=a^h-1. \qquad (4)$$

Or $$\frac{y'-y}{y}+1=a^h. \qquad (5)$$

If we put $a=1+b$, we can expand the second member of (5) by the binomial theorem thus:

$$a^h=(1+b)^h=1+hb+h\frac{h-1}{2}b^2+h\frac{h-1}{2}\frac{h-2}{3}b^3+$$

$$h.\frac{h-1}{2}\frac{h-2}{3}b^4+, \text{ \&c.}$$

This substituted in (5) and one dropped for each member, and dividing by h, we shall have

$$\frac{y'-y}{yh}=b+\left(\frac{h-1}{2}\right)b^2+\left(\frac{h-1}{2}\right)\left(\frac{h-2}{3}\right)b^3+$$

$$\left(\frac{h-1}{2}\right)\left(\frac{h-2}{3}\right)\left(\frac{h-3}{4}\right)b^4+ \text{ \&c.}$$

This last equation is true for all values of h; it is true then when h has a value *inconceivably* small, but in that case $y'-y$

becomes dy, and h becomes dx. On this supposition the preceding equation becomes

$$\frac{dy}{ydx}=b-\frac{b^2}{2}+\frac{b^3}{3}-\frac{b^4}{4}+\frac{b^5}{5}-\frac{b^6}{6}, \text{ \&c.} \qquad (6)$$

The second member of (6) is a series of constant terms, and is always the same, while a in equation (1) remains the same. Let the sum of this series be represented by A, then (6) becomes

$$\frac{dy}{y}=Adx. \qquad (7)$$

If we take $A=\frac{1}{m}$. Then (7) becomes

$$dx=m\frac{dy}{y}. \qquad (8)$$

If we observe equation (1) $y=a^x$, we must recognize a logarithmetic equation, x is the logarithm of the number y, and the base of the system is a.

Equation (8) gives us a general rule to differentiate a logarithm :

RULE. *The differential of a logarithm is equal to the differential of the number divided by the number, multiplied by the modulus of the system.*

When the base is so taken as to make $A=1$, then will $m=1$, and we shall have the hyperbolic or Naperian system. For convenience merely, Lord Naper the first investigator of logarithms, assumed $A=1$. This system is still used in mathematical operations, and the results changed into the common system, if need be, by applying the factor m. When $m=1$ equation (8) gives the following rule to differentiate a logarithm :

RULE. *Take the differential of the quantity and divide it by the quantity.*

Practical application is made of equation (8) in the author's treatise on Surveying and Navigation, and we will give a few examples here by way of illustration.

1. *The logarithm of* 10452 *is* 4.01919941, *what is the logarithm of* 10452.12, *the modulus of the system being* 0.43429448?*

Here $y=10452$, $dy=0.12$.

Therefore $dx=\frac{.43429448.12}{10452}$.

To..................4.01919941
Add.............$dx=0.00000498$

Log. 10452.12..........$=4.01920439$

2. *The lagarithm of* 104521.2 *is* 5.01920439, *what is the logarithm of* 104520.7?

Here $y=104521.2$. $dy=-0.7$.

$$dy=\frac{.43429448(-0.7)}{104521.2}.$$

To..............5.01920439
Subtract...$dx=-0.00000028$

Log. 104520.7......$=5.01920411$

Thus we might give examples without end.

(Art. 11.) Logarithms are exponents, therefore the addition of two logarithms corresponds to the logarithm of the product of the two members.

Thus log. 2^2=log.aa=log.a+log.a=2 log.a.

The log. of a is half as much as the log. of a^2.

In the common system the log. of 100 is 2, the square root of 100 is 10, and its log. 1. The square root of 10 is 3.16227766, therefore the log. of this number is 0.50000. Thus we may go on extracting square root for succeeding numbers, and halving the log. for the corresponding log.

* We will soon show how this number may be found.

The following table shows some of these results:

Numbers.	Logarithms.
10.00000	1.00000
3.1622776630	0.50000
1.7782793430	0.25000
1.3335214070	0.12500
1.1547819700	0.06250
1.0746077770	0.03125
1.0366328673	0.015625
1.0181521828	0.0078125
1.0090352733	0.00390625
1.0045074297	0.001953125
1.0022511809	0.0009765625
1.0011249572	0.00048828125
1.0005623151	0.000244140625
1.0002811174	0.0001220703125
1.0001405488	0.00006103515625
1.0000702719	0.000030517578125
1.0000351353	0.0000152587890625
1.00001756752	0.00000762939453125

Thus we might go on, but we have gone far enough to illustrate the possibility of finding the value of m independently of the Naperian system.

The log. of 1 is 0 in every system. Our last log. just found corresponds to a number a little greater than 1, but the decimal is so small in relation to 1, that it may be taken for the differential of 1.

Equation (8) gives us $dx=m\frac{dy}{y}$. Whence $m=\frac{ydx}{dy}$.

But if $y=1$, $dy=0.00001756752$, and $dx=$ the last log.

Then $m=\frac{0.00000762939453125}{0.00001756752}=0.434294+$.

To fix these principles in mind we give the following examples to differentiate. The word logarithm is indicated by log., and indicates the hyperbolic or Naperian log. unless otherwise expressed.

1. *Given* $u=\log.\left(\frac{x}{\sqrt{a^2+x^2}}\right)$ *to find* du.

We first take the differential of the second member as an algebraic quantity, thus :

$$\frac{dx\sqrt{a^2+x^2}\;\frac{-x^2dx}{\sqrt{a^2+x^2}}}{a^2+x^2}=\frac{a^2dx}{(a^2+x^2)\sqrt{a^2+x^2}}.$$

Whence $du=\frac{a^2dx}{(a^2+x^2)\sqrt{a^2+x^2}}\times\frac{\sqrt{a^2+x^2}}{x}=\frac{a^2dx}{x(a^2+x^2)}$, *Ans.*

2. *Given* $u=\log.(x+\sqrt{1+x^2})$ *to find* du.

Ans. $du=\frac{dx}{\sqrt{1+x^2}}$.

3. *Given* $u=\frac{1}{\sqrt{-1}}\log.(x\sqrt{-1}+\sqrt{1-x^2})$, *to find* du.

N. B. Put $a=\sqrt{-1}$ for the sake of perspicuity.

Ans. $du=\frac{dx}{\sqrt{1-x^2}}$.

4. *Given* $u=\log.(\sqrt{1-x^2})$ *to find* du.

Ans. $du=\frac{-xdx}{(1-x^2)}$.

5. *Given* $u=\log.(3x^2+x)$ *to find* du.

Ans. $du=\left(\frac{6x+1}{3x^2+x}\right)dx$.

N. B. We can if we please use logarithms to differentiate common algebraic quantities. To show this, we take example 11 from (Art. 6,) of this volume.

6. *Given* $u=\frac{1+x^2}{1-x^2}$, *to find* du.

Take the log. of each member, then

$$\log.u=\log.(1+x^2)-\log.(1-x^2).$$

Now differentiate each member by the logarithmetic rule, and we find

$$\frac{du}{u}=\frac{2xdx}{1+x^2}+\frac{2xdx}{1-x^2}=\frac{4xdx}{(1+x^2)(1-x^2)}.$$

Or $$du=\frac{4xdx}{(1+x^2)(1-x^2)}\times\frac{1+x^2}{1-x^2}=\frac{4xdx}{(1-x^2)^2},\ \textit{Ans.}$$

7. *Given* $u=\dfrac{x^n}{(1+x)^n}$ *to find* du. (From Art. 6.)

$$\log.u=n\log.x-n\log.(1+x).$$

$$\frac{du}{u}=\frac{ndx}{x}-\frac{ndx}{(1+x)}=\frac{ndx}{x(1+x)}.$$

$$du=\frac{ndx}{x(1+x)}\left(\frac{x^n}{(1+x)^n}\right)=\frac{nx^{n-1}dx}{(1+x)^{n+1}}.\ \textit{Ans.}$$

8. *Given* $u=2x\sqrt{a^2+x^2}$ *to find* du, *making use of logarithms.*

Ans. $du=\dfrac{(2a^2+4x^2)dx}{\sqrt{a^2+x^2}}.$

9. *Given* $u=\log.(2x\sqrt{a^2+x^2})$ *to find* du.

The differential of the quantity under the log. was found in the last example, hence the answer to this is found simply by dividing that answer by $2x\sqrt{a^2+x^2}$.

Ans. $du=\dfrac{(a^2+2x^2)dx}{x(a^2+x^2)}.$

The following examples come nearer the practical uses of these principles:

(Art. 12.)

10. *Given* $u=\log.\sin.x$ *to find* du *; that is, given the log. sine of any arc to find its differential, or its rate of increase or decrease at that point.*

$$du=\frac{\cos.x\,dx}{\sin.x}=\cot.x\,dx.$$

This result corresponds to the *modulus* of unity : for the modulus of our common system we must multiply by $0.43429448=m$.

For example, if we assume $x=25^\circ$, and also assume $dx=1'$, the differential, or the difference between the log. sine of 25° and the log. sine of 25° 1′ is expressed by $m \cot.25^\circ \times 1'$.

Log. m	—1.637784
cot. 25°	0.331327
Log. sine 1′, less 10	—4.463726
.0002709	—4.432837

To the log. sine of 25°	9.625948
Add the differential	.000271
Log. sine of 25° 1′=	9.626219

We might assume $dx=2'$ as well as 1′, without error as far as six places of decimals; but it would not do to assume $dx=$ any large number of minutes; hence the differential calculus must be applied with judgment.

11. *Given* $u=\log.\cos.x$ *to find* du.

Ans. $du=-\tan.x\,dx$.

To apply this, we demand the variation of the log. cosine of 34°, corresponding to 1′ increase of arc.

Log. m	—1.637784
tan. 34°, (less 10)	—1.828987
Log. sine 1′, (less 10)	—4.463726
$du=0.00008521$	—5.930497

To log. cos. 34°	9.918574
Subtract du	0.000085
Log. cos. 34° 1′	9.918489

12. *Given* $u=\log.\tan.x$ *to find* du.

$$du=\frac{dx}{\cos.^2x\tan.x}=\frac{dx}{\cos.x\sin.x}=\frac{2dx}{\sin.2x}.$$

What is the variation corresponding to 1′ to the logarithmic tangent of 40°?

Log. m	—1.637784
Log. sin. 1′+log.2	—4.764756
Sin.$2x$=sin.80° complement......	.006649
du=0.00025653	—4.409189

To log. tan. 40°	9.923813
Add du	257
Log. tan. 40° 1′	9.924070

13. *Given* $u=\log.(\cot.x)$ *to find* du.

$$du=\frac{-dx}{\sin.^2x\cot.x}=\frac{-dx}{\cos.x\sin.x}=\frac{-2dx}{\sin.2x}.$$

This last result shows that the log. tangent and log. cotangent vary alike in amount, the first *positive*, the last *negative*.

(Art. 13.) The use of logarithms is very essential in differentiating examples like the following:

It is *customary* to represent the base of the Naperian system by e, and the log. of the base of any system is 1; hence, if we have any equation in the form $u=e^x$ and take the log. of each member, we shall have $\log.u=x$ simply, and if $u=e^xy$, $\log.u=x+\log.y$, &c. &c.

14. *Given* $u=e^x(x-1)$ *to find the value of* du.

$$\text{Log}.u=x+\log.(x-1).$$

$$\frac{du}{u}=dx+\frac{dx}{x-1}=\frac{xdx}{x-1}.$$

$$du=\frac{xudx}{x-1}=e^xxdx.$$

15. *Given* $u=e^x(x^3-3x^2+6x-6)$ *to find* du.

$$\text{Log}.u=x+\log.(x^3-3x^2+6x-6).$$

$$\frac{du}{u}=dx+\frac{(3x^2-6x+6)\,dx}{x^3-3x^2+6x-6}=\frac{x^3dx}{x^3-3x^2+6x-6}.$$

$$\frac{du}{dx}=\frac{x^3u}{x^3-3x^2+6x-6}=e^xx^3.\qquad \text{Or } du=e^xx^3dx.$$

16. *Given* $u=\frac{e^x x}{1-x}$ *to find* du.

$$Ans.\ du=\frac{(1+x-x^2)e^x dx}{(1-x)^2}.$$

17. *Given* $u=e^x \log.x$ *to find* du.

$$Ans.\ du=\left(\frac{x\log.x+1}{x}\right)e^x dx.$$

18. *Given* $u=\frac{e^x-1}{e^x+1}$ *to find* du.

$$Ans.\ du=\frac{2e^x dx}{(e^x+1^2)}.$$

19. *Given* $u=x^m(\log.x)^n$ *to find* $\frac{du}{dx}$.

Put $z=\log.x$, then $u=x^m z^n$.

$$Ans.\ \frac{du}{dx}=(m\log.x+n)x^{m-1}(\log.x)^{n-1}.$$

20. *Given* $u=\frac{x^4(\log.x)^2}{4}-\frac{x^4(\log.x)}{8}+\frac{x^4}{32}$ *to find* du.

$$Ans.\ du=x^3(\log.x)^2 dx.$$

21. *Given* $u=x^y$ *to find the value of* du.

$$\text{Log.}u=y\log.x.$$

$$\frac{du}{u}=\log.x.dy+y\frac{dx}{x}.$$

$$du=x^y\log.xdy+yx^{y-1}dx.$$

22. *Given* $u=\log.(\cos.x+\sqrt{-1}\sin.x)$ *to find* du.

$$Ans.\ du=\sqrt{-1}dx.$$

23. *Given* $u=\frac{\log.x}{x}$ *to find* du.

$$Ans.\ du=\left(\frac{1-\log.x}{x^2}\right)dx.$$

CHAPTER IV.

Successive Differentials.

TAYLOR'S THEOREM.

(Art. 13.) When we have an equation $u=f(x)$, its differential coefficient is generally another function of x, symbolically represented thus:

$$\frac{du}{dx}=f_1(x).$$

This new function of x can again be differentiated and divided by dx, giving still another function of x, then we shall have

$$\frac{d^2u}{dx^2}=f_{\prime\prime}(x), \text{ \&c. \&c.}$$

until the last differential becomes constant or valueless, as the case may be.

For example, let $u=x^3$.

The 1st diff. coefficient is $\frac{du}{dx}=3x^2$.

The 2d is $\frac{d^2u}{dx^2}=6x$.

The 3d is $\frac{d^3u}{dx^3}=6$.

Here the operation must stop, as the second member is constant. By this we perceive that if $u=x^n$ after n differentiations, the second member will become constant and terminate the operation.

We sometimes write

$$\frac{du}{dx}=p, \qquad \frac{dp}{dx}=q, \qquad \frac{dq}{dx}=r, \qquad \text{\&c.}$$

Then will $\frac{du}{dx}=p, \quad \frac{d^2u}{dx^2}=q, \quad \frac{d^3u}{dx^3}=r,$ &c.

du is the differential of u;
d^2u is the second differential;
d^3u is the third differential; &c.

It should be remembered, that the exponent which accompanies the characteristic d, indicates the repetition of an operation, and not a power of the letter d, which is never considered as a quantity, but merely as a sign.

The expression dx^2 signifies the square of dx, or $(dx)^2$, and not the differential of x^2, which is usually denoted by $d.x^2$ or $d(x^2)$; again, dx^3 signifies the cube of dx or $(dx)^3$, and so on.

(Art. 14.) *If we have a function of the sum or difference of two variables, the differential coefficient will be the same whichever be supposed to vary, the other being constant.*

For example, let $u=(x\pm y)^4$.

Take the differential coefficient on the supposition that x is variable and y constant, then

$$\frac{du}{dx}=4(x\pm y)^3. \qquad (1)$$

Now on the supposition that y is variable and x constant, and

$$\frac{du}{dy}=4(x\pm y)^3 \qquad (2)$$

Comparing equations (1) and (2), we perceive that

$$\frac{du}{dx}=\frac{du}{dy}.$$

Taking the differential coefficient of (1) in relation to x, and

$$\frac{d^2u}{dx^2}=12(x\pm y)^2. \qquad (3)$$

And the differential coefficient of (2) in relation to y, is

$$\frac{d^2u}{dy^2}=12(x\pm y)^2. \qquad (4)$$

Comparing (3) and (4), we perceive that

$$\frac{d^2u}{dx^2}=\frac{d^2u}{dy^2}.$$

Thus we might show that

$$\frac{d^3u}{dx^3}=\frac{d^3u}{dy^3}, \quad \text{and} \quad \frac{d^nu}{dx^n}=\frac{d^nu}{dy^n}.$$

(Art. 15.) If we have $u=fx$ and suppose that x becomes $(x+h)$, and in consequence of this u becomes u', then

$$u'=u+Ah+Bh^2+Ch^3+Dh^4, \text{ \&c.} \qquad (1)$$

To give the learner a clear comprehension of this, we will illustrate it by one or two particular examples.

Let $$u=ax^4.$$

Now suppose x becomes $x+h$, and u becomes u', then

$$u'=a(x+h)^4=ax^4+4ax^3h+6ax^2h^2+4axh^3+ah^4. \quad (2)$$

Now it is visible that the first term of the second member is u, and if we put $4ax^3=A$, $6ax^2=B$, &c. we have

$u'=u+Ah+Bh^2+Ch^3+Dh^4$, &c. which we proposed to show.

Again, let $u=a+bx+cx^2$, and suppose x becomes $x+h$, &c.

Then $$u'=a+b(x+h)+c(x+h)^2.$$

Or $$u'=(a+bx+cx^2)+(b+2cx+ch)h.$$

Here again we have

$$u'=u+Ah+Bh^2+Ch^3+ \text{ \&c.}$$

according to the degree of the variable x in the original function.

Resume the equation $u=ax^4$, and take its successive differential coefficients thus:

$$\left.\begin{aligned} &\frac{du}{dx}=4ax^3=A \\ &\frac{d^2u}{dx^2}=12ax^2=2B. \\ &\frac{d^3u}{dx^3}=24ax=2.3.C. \\ &\frac{d^4u}{dx^4}=24a=2.3.4D. \end{aligned}\right\} \text{Comparing these results with equation (2).}$$

Substituting these values of A, B, C, &c. in equation (1), we have

$$u'=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{2}+\frac{d^3u}{dx^3}\frac{h^3}{2.3}+\frac{d^4u}{dx^4}\frac{h^4}{2.3.4}, \text{ \&c.}$$

a general expression for the development $f(x+h)=u'$, and this is Taylor's Theorem, from Dr. Brook Taylor, an English mathematician who discovered it about the year 1715.

(Art. 16.) In the preceding article we drew out Taylor's theorem by inspection. Let that be well understood, and the learner is prepared to appreciate the following general demonstration :

Let $$u=f(x).$$

Now suppose x to take an increase y, and in consequence of this, u becomes u', then

$$u'=f(x+y)=u+Ay+By^2+Cy^3+Dy^4, \text{ \&c.} \quad (1)$$

In the second member, u, A, B, &c. contain functions of x, and the constants that enter into x.

Take the differential of each member in relation to x as a variable, and divide each term by dx, then we shall have

$$\frac{du'}{dx}=\frac{du}{dx}+\frac{dA}{dx}y+\frac{dB}{dx}y^2+\frac{dC}{dx}y^3+\frac{dD}{dx}y^4+\text{ \&c.} \quad (2)$$

Take the differential coefficient of (1) again, regarding x as constant and y variable, and we shall have

$$\frac{du'}{dy}=A+2By+3Cy^2+4Dy^3+\text{ \&c.} \quad (3)$$

Now the first members of (2) and (3) are equal by (Art. 14), therefore the second members are equal, and the terms containing like powers of y are equal. (Algebra Art. 128.)

Therefore $\frac{du}{dx}=A$, $\frac{dA}{dx}=2B$, $\frac{dB}{dx}=3C$, $\frac{dC}{dx}=4D$, &c.

Because $\frac{du}{dx}=A$, $\frac{d^2u}{dx^2}=\frac{dA}{dx}=2B$, $\frac{d^3u}{2\,dx^3}=\frac{dB}{dx}=3C$, &c.

Whence $A=\frac{du}{dx}$, $B=\frac{d^2u}{dx^2 2}$, $C=\frac{d^3u}{dx^3\, 2.3}$, $D=\frac{d^4u}{dx^4\, 2.3.4}$, &c.

Substituting these values of A, B, C, &c. in (1), we have

$$u'=u+\frac{du}{dx}y+\frac{d^2u}{dx^2}\frac{y^2}{2}+\frac{d^3u}{dx^3}\frac{y^3}{2.3}+\frac{d^4u}{dx^4}\frac{y^4}{2.3.4}+\text{ \&c.}$$

N. B. The expressions u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, $\frac{d^3u}{dx^3}$, &c. are the same as X, X', X'', X''', &c. in Robinson's Algebra, (Art. 171,) page 273, and are there called *derived polynomials*.

MACLAURIN'S THEOREM.

(Art. 17.) Maclaurin, a Scotch mathematician, has given us a theorem very similar to that of Taylor, which demonstrates the binomial theorem, and enables us to develop any function of a single variable, *provided the development is susceptible of containing the ascending powers of the variable.*

But the theorem does not apply to other forms of development.

Let $u=(a+x)^n=(a)+Bx+Cx^2+Dx^3+Ex^4$, &c. (1)

Here we are sure the first term of the development (a) does not contain x, and B, C, D, &c. are each independent of the value of x.

Take the successive differential coefficients of (1), and the results will be as follows:

$$\frac{du}{dx}=B+2Cx+3Dx^2+4Ex^3+ \text{ \&c.} \qquad (2)$$

$$\frac{d^2u}{dx^2}=2C+2.3Dx+3.4Ex^2+ \text{ \&c.} \qquad (3)$$

$$\frac{d^3u}{dx^3}=2.3D+2.3.4Ex+ \text{ \&c.} \qquad (4)$$

Equations (1), (2), (3), &c. are all true for all values of x; they are therefore true when $x=0$. Making this supposition, they become

$$u=a^n=(a)$$

$$B=\frac{du}{dx}.$$

$$C=\frac{d^2u}{dx^2\,2}.$$

$$D=\frac{d^3u}{dx^3\,2.3}, \text{ \&c.}$$

Substituting these values of B, C, D, &c. in (1) and we have

$$u=(a)+\left(\frac{du}{dx}\right)x+\left(\frac{d^2u}{dx^2}\right)\frac{x^2}{2}+\left(\frac{d^3u}{dx^3}\right)\frac{x^3}{2.3}+ \text{ \&c.}$$

The first term (a) is whatever u becomes when the variable is made equal to 0 in the primitive function.

APPLICATION.

Let us now apply this theorem to this very example; that is, develop $(a+x)^n$ by it.

$$\frac{du}{dx}=n(a+x)^{n-1}.$$

$$\frac{d^2u}{dx^2}=n(n-1)(a+x)^{n-2}.$$

$$\frac{d^3u}{dx^3}=n(n-1)(n-2)(a+x)^{n-3}.$$

$$\frac{d^4u}{dx^4}=n(n-1)(n-2)(n-3)(a+x)^{n-4}. \text{ \&c.}$$

Making $x=0$, these equations become

$$\frac{du}{dx}=na^{n-1}, \quad \frac{d^2u}{dx^2}=n(n-1)a^{n-2}, \quad \frac{d^3u}{dx^2}=n(n-1)(n-2)a^{n-3}$$

Hence, by substituting these values in the formula, we obtain

$$u=(a+x)^n=a^n+na^{n-1}x+n\left(\frac{n-1}{2}\right)a^{n-2}x^2+$$
$$n\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right)a^{n-3}x^3+ \text{ \&c.}$$

This is the same result as would arise from the direct application of the binomial theorem, and this formula can be used to develop binomials generally — but there would be no advantage in using it for common cases, for the direct application of the binomial theorem is less circuitous and more brief.

But this theorem is *more powerful* than the binomial theorem, and will apply to *other functions* as well as to simple algebraic binomials, — hence its utility.

To show the power of this theorem, and at the same time draw out useful mathematical truths, we give the following

EXAMPLES.

1. *Develop* a^x *into a series containing the ascending powers of* x, *if possible.*

Making $x=0$ the function becomes 1, a rational finite quantity therefore the development is possible—as the demonstration was general under that hypothesis,

$$u=a^x=1+\left(\frac{du}{dx}\right)x+\left(\frac{d^2u}{dx^2}\right)\frac{x^2}{2}+\left(\frac{d^3u}{dx^3}\right)\frac{x^3}{2.3}+\text{ \&c. } (l)$$

In equation (7), (Art. 10,) we find $\frac{du}{dx}=Aa^x$. As A is a constant quantity, and $u=a^x$, a second differentiation will give

$$\frac{d^2u}{dx^2}=A^2a^x.$$

A third $$\frac{d^3u}{dx^3}=A^3a^x.$$

A fourth $$\frac{d^4u}{dx^4}=A^4a^x. \text{ \&c.}$$

Making $x=0$ in these equations, as in the previous example, and we have

$$\frac{du}{dx}=A, \quad \frac{d^2u}{dx^2}=A^2, \quad \frac{d^3u}{dx^3}=A^3, \quad \text{\&c. \&c.}$$

Therefore equation (l) becomes

$$a^x=1+Ax+\frac{A^2x^2}{2}+\frac{A^3x^3}{2.3}+\frac{A^4x^4}{2.3.4}+\frac{A^5x^5}{2.3.4.5}+\text{ \&c.}$$

As x is unrestricted in value, we may make $x=\frac{1}{A}$.

Then $a^{\frac{1}{A}}=1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\frac{1}{2.3.4.5}+\text{ \&c.}=e.$

The value of e taken to seven decimal places, it is 2.7182818. This is the base of the Naperian system of logarithms, and it is much used in analysis.

From the last equation we find $a=e^A$.

Taking the logarithms we have $\log.a=A\log.e$.

Or $$A=\frac{\log.a}{\log.e}.$$

Now since a and e are known, A is known. If $\log.a=1$, $A=\frac{1}{\log.e}$, and if $\log.e=1$, $A=\log.a$. That is, A is equal to the Naperian logarithm of 10. We shall soon discover that the value of this logarithm is 2.302585093.

The modulus of the common system of logarithms is the reciprocal of A designated by m in (Art. 10); therefore $m=.434294482$, corresponding to the result approximately obtained in (Art. 11).

(Art. 18.) To show the distinction between the theorems of Taylor and Maclaurin, we will now apply the former to the development of this same function

Let $$u=a^x. \quad (1)$$

Then $$u'=a^{x+h} \quad (2)$$

And by the theorem

$$u'=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{2}+\frac{d^3u}{dx^3}\frac{h^3}{2.3}+\text{ \&c.} \quad (3)$$

Taking the successive differential coefficient of (1) we find

$$\frac{du}{dx}=Aa^x, \quad \frac{d^2u}{dx^2}=A^2a^x, \quad \frac{d^3u}{dx^3}=A^3a^x, \quad \text{\&c. \&c.}$$

Substituting these quantities in the formula, and

$$a^{x+h}=a^x+Aa^xh+\frac{A^2a^xh^2}{2}+\frac{A^3a^xh^3}{2.3}+\frac{A^4a^xh^4}{2.3.4}.$$

Divide each side by a^x, and

$$a^h=1+Ah+\frac{A^2h^2}{2}+\frac{A^3h^3}{2.3}+\frac{A^4h^4}{2.3.4}+\text{ \&c.}$$

As h is arbitrary, we may put $h=\frac{1}{A}$, then

$$a^{\frac{1}{A}}=1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}, \text{ \&c. }=e.$$

Or $$a=e^A, \text{ as before.}$$

In Maclaurin's theorem the differential coefficients $\left(\frac{du}{dx}\right)$, $\left(\frac{d^2u}{dx^2}\right)$, &c. correspond to the variable $x=0$ in the second member, and they are put in parenthesis to distinguish this theorem from that of Taylor.

2. *For a second example, let* $u=\sin.x$, *and* $u'=\sin.(x+h)$.

Then (Art. 7,) $\frac{du}{dx}=\cos.x$, $\frac{d^2u}{dx^2}=-\sin.x$, $\frac{d^3u}{dx^3}=-\cos.x$, $\frac{d^4u}{dx^4}=\sin.x$, $\frac{d^5u}{dx^5}=\cos.x$, &c.

Substituting these values in Taylor's formula, we find

$$\sin.(x+h)=\sin.x+\cos.x\frac{h}{1}-\sin.x\frac{h^2}{1.2}-\cos.x\frac{h^3}{2.3}+\sin.x\frac{h^4}{2.3.4}.$$

This is true for all values of x;—it is true then when $x=0$, and this supposition gives $\sin.x=0$, and $\cos.x=1$, and the result becomes

$$\sin.h=h-\frac{h^3}{2.3}+\frac{h^5}{2.3.4.5}-\frac{h^7}{2.3.4.5.6.7}+\text{ \&c.}$$

REMARK.—This operation compared with that in Geometry, pages 221, 222, 223, for the same object, shows the superior power of the calculus over common geometry in a very clear light.

3. *For a third example, let* $u=\cos.x$, *then* $u'=\cos.(x+h)$.

Taking u, and the successive differential coefficients of u, and substituting them in the formula, we shall find

$$\cos.h=1-\frac{h^2}{2}+\frac{h^4}{2.3.4}-\frac{h^6}{2.3.4.5.6}+\text{ \&c.}$$

These results may also be deduced from the theorem of Maclaurin.

These formula are used to compute the sines and cosines of small arcs when the arcs are known.

(Art. 19.) By Taylor's theorem we can easily develop a logarithmic function into a series.

Let $$u=\log.x.$$
$$u'=\log.(x+h).$$

Taking the successive differential coefficients of u, we find

$$\frac{du}{dx}=\frac{1}{x},\qquad \frac{d^2u}{dx^2}=-\frac{1}{x^2},\qquad \frac{d^3u}{dx^3}=\frac{2}{x^3},$$
$$\frac{d^4u}{dx^4}=-\frac{3}{x^4},\qquad \frac{d^5u}{dx^5}=\frac{3.4}{x^5},\ \text{\&c.}$$

on the supposition that the *modulus is unity.*

These values substituted in the formula give

$$\log.(x+h)=\log.x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\frac{h^4}{4x^4}+\frac{h^5}{5x^5}-\ \&c.\quad (1)$$

If h be made *minus*, the result of the formula will be

$$\log.(x-h)=\log.x-\frac{h}{x}-\frac{h^2}{2x^2}-\frac{h^3}{3x^3}-\frac{h^4}{4x^4}-\frac{h^5}{5x^5}-\ \&c.\quad (2)$$

Subtracting the latter from the former, we have

$$\log.(x+h)-\log.(x-h)=\frac{2h}{x}+\frac{2h^3}{3x^3}+\frac{2h^5}{5x^5}+\frac{2h^7}{7x^7}+\ \&c.$$

Or $$\log.\left(\frac{x+h}{x-h}\right)=2\left(\frac{h}{x}+\frac{h^3}{3x^3}+\frac{h^5}{5x^5}+\frac{h^7}{7x^7}+\right)\ \&c.\quad (3)$$

If we make $x+h=2$ and $x-h=1$, then $x=\frac{3}{2}$, and $h=\frac{1}{2}$. Also $\frac{h}{x}=\frac{1}{3}$, and (3) becomes

$$\log.2=2\left(\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{27}+\frac{1}{5}\cdot\frac{1}{243}+\frac{1}{7}\cdot\frac{1}{2187}+\right)\&c.$$

This gives the hyperbolic logarithm of $2=0.69$, and so on.

As formula (3) is not convenient for all numbers, we will modify it. It is obvious that the first member is greater than 1, therefore we may assume

$$\frac{x+h}{x-h}=1+\frac{1}{z}.$$

This gives $\frac{h}{x}=\frac{1}{2z+1}$, and these values substituted in the formula, give

$$\log.\left(\frac{z+1}{z}\right)=2\left(\frac{1}{(2z+1)}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\right)\ \&c.$$

Or $$\log.(z+1)=\log.z+2\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\right)$$

This formula gives the hyperbolic logarithm of $(z+1)$ when the log. of z is known.

When $z=1$, $\log.z=0$, and the formula gives the hyperbolic log. of 2. Because $2^3=8$, three times the log. of 2 will give the hyperbolic log. of 8.

Now making $z=8$, an application of the equation will give the hyperbolic log. of 9. Then making $z=9$, another application will give the hyperbolic log. of 10, which is 2.302585093, and it is represented by A, or log.a in (Art. 17.)

In (Art. 10,) we represented the *modulus* of the common system by m, and $m=\frac{1}{A}$.

Hence $$m=\frac{1}{2.302585093}=0.43429448.$$

Therefore the formula for common logarithms is

$$\log(z+1)=\log.z+.86858896\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}\right)+\&c.$$

ANOTHER METHOD OF DEVELOPING LOGARITHMIC AND CIRCULAR FUNCTIONS.

(Art. 20.) We can best illustrate this method by taking the preceding example, and comparing the results step by step.

Let $$u'=\log.(x+h).$$

Conceive x to be variable and h constant, then

$$\frac{du'}{dx}=\frac{1}{x+h}=\frac{1}{h+x}.$$

The second member of this equation can be developed by division, or by the binomial theorem. When so developed, we shall have

$$\frac{du'}{dx}=\frac{1}{h}-\frac{x}{h^2}+\frac{x^2}{h^3}-\frac{x^3}{h^4}+\frac{x^4}{h^5}-\ \&c. \quad (1)$$

This equation shows us that u' expands into a series containing *all* the ascending powers of x, and possibly there is a term not containing the variable x.

Therefore we may assume

$$u'=A+Bx+Cx^2+Dx^3+\ \&c. \quad (2)$$

From this equation we find

$$\frac{du'}{dx}=B+2Cx+3Dx^2+4Ex^3,\ \&c. \quad (3)$$

The first members of (1) and (3) are the same, therefore the second members are equal, and the terms containing equal powers of x are equal. Therefore

$$B=\frac{1}{h},\quad C=-\frac{1}{2h^2},\quad D=\frac{1}{3h^3},\quad E=-\frac{1}{4h^4},\ \&c.$$

These values of B, C, D, &c. substituted in (2) give

$$u'=\log.(x+h)=A+\frac{x}{h}-\frac{x^2}{2h^2}+\frac{x^3}{3h^3}-\frac{x^4}{4h^4}+\frac{x^5}{5h^5}-\ \&c. \quad (4)$$

This equation must be true for all values of x. It must be true then when $x=0$. Making that supposition, and

$$\log.h=A.$$

Substituting this value of A in (4), and the development is complete.

$$\log.(x+h)=\log.h+\frac{x}{h}-\frac{x^2}{2h^2}+\frac{x^3}{3h^3}-\frac{x^4}{4h^4}+\ \&c. \quad (5)$$

This equation is the same as (1) in (Art. 19,) when we change x to h, and h to x. This arose from our expanding $\frac{1}{h+x}$ in place of $\frac{1}{x+h}$.

Putting $u'=\log.(x+h)$, and the result of the operation will give a similar equation to (2) of (Art. 19.)

This principle of operation is best adapted to the development of circular functions.

EXAMPLES.

Let x represent an arc of a circle whose radius is unity, and y its sine. Then the cosine must be

$$\sqrt{1-y^2}.$$

1. *It is required to find the value of the arc in terms of its sine.*

$$\sin.x=y.$$

Whence $$\cos.x dx=dy.$$

And $$\frac{dx}{dy}=\frac{1}{\cos.x}=\frac{1}{\sqrt{1-y^2}}.$$

$$\frac{1}{\sqrt{1-y^2}}=1+\frac{y^2}{2}+\frac{3y^4}{2.4}+\frac{3.5.y^6}{2.4.6}+\frac{3.5.7.y^8}{2.4.6.8} \text{ \&c.}$$

$$\frac{dx}{dy}=1+\frac{y^2}{2}+\frac{3y^4}{2.4}+\frac{3.5.y^6}{2.4.6}+\frac{3.5.7y^8}{2.4.6.8}. \qquad (1)$$

Here we perceive that dx develops into a series containing the *even* powers of y, therefore x itself must develop into a series containing the *odd* powers of y. *As each term containing an odd power of* y *will have the power of* y *diminished by unity after differentiation,* therefore we may assume

$$x=Ay+By^3+Cy^5+Dy^7+Ey^9, \text{ \&c.} \qquad (2)$$

Whence $\frac{dx}{dy}=A+3By^2+5Cy^4+7Dy^6+9Ey^8$, &c. (3)

Comparing (1) and (3), we find

$$A=1, \quad B=\frac{1}{2.3}, \quad C=\frac{3}{2.4.5}, \quad D=\frac{3.5}{2.4.6.7}, \quad E=\frac{3.5.7}{2.4.6.8.9},$$

These values of A, B, C, D, &c. substituted in (2), give

$$x=y+\frac{y^3}{2.3}+\frac{3y^5}{2.4.5}+\frac{3.5y^7}{2.4.6.7}+\frac{3.5.7y^9}{2.4.6.8.9}, \text{ \&c.} \qquad (8)$$

the development required. Knowing the sine y of any definite arc, this equation will give the value of x, the arc itself, to any required degree of exactness.

When the arc is 30°, the sine is $\frac{1}{2}$, and this value given to y in the equation, there results

Arc of 30° =

$$\frac{1}{2}+\frac{1}{2.3.8}+\frac{3}{2.4.5.32}+\frac{3.5}{2.4.6.7.128}+\frac{3.5.7}{2.4.6.8.9.512}+\text{ \&c.}$$

Taking the sum of seven terms, we find

Arc of 30°=0.523597....

And multiplying by 6, $180°=\pi=3.14159....$

2. *It is required to find the value of an arc of a circle in terms of its cosine.*

Let $x=$ the arc and z its cosine.

That is, $\cos.x=z$.

$$\frac{dx}{dz}=-\frac{1}{\sin.x}=-\frac{1}{\sqrt{1-z^2}}.$$

This is the same form as the former example, except the sign. Hence the development must be the same as (8) of the first example, except changing y to z and changing signs.

In equation (8) the arc and its sine commence at the same point and increase together from 0 to 90°; hence, when we make $x=0$ in (8), y becomes 0 also, and both sides of the equation vanish together and the equation is complete.

Not so with the cosine, for when the arc *increases* the cosine *decreases*, and when the arc is 0, the cosine is *radius* or 1.

Therefore we cannot develop an arc in terms of its cosine independent of the corresponding sine.

If z is the cosine of the arc x, it must be the sine of the arc $(90°-x)$. Now by example 1, equation (8), the

$$\text{Arc } (90°-x)=z+\frac{z^3}{2.3}+\frac{3z^5}{2.4.5}+\frac{3.5z^7}{2.4.6.7}+ \ \&c.$$

Transposing 90°, and changing signs, we have

$$\text{Arc } x=\text{arc } 90°-z-\frac{z^3}{2.3}-\frac{3z^5}{2.4.5}-\frac{3.5z^7}{2.4.6.7}- \ \&c.$$

Here the arc x is developed in terms of the cosine z, as required, but the result necessarily includes the arc of 90°, and the value of this depends on the development of the sine.

To find the value of the *arc* of 90°, we again turn to equation (8), making $y=1$; then that equation becomes

$$\text{Arc } 90°=1+\frac{1}{2.3}+\frac{3}{2.4.5}+ \ \&c.$$

and this value of the arc of 90° substituted in the preceding equation, and we shall have the value of x complete, as was required.

3. *Let* x *be an arc and* t *its corresponding tangent; required the value of* x *in terms of* t.

$$\tan.x=t.$$

Then

$$\frac{dx}{\cos.^2x}=dt.$$

Or $$\frac{dx}{dt}=\cos.^2x.$$

Now we can develop this by Maclaurin's theorem, or as follows: In any arc we have the following proportion :

$$\cos.x : 1 : : 1 : \sec.x, \quad \text{or} \quad \cos.x=\frac{1}{\sec.x}.$$

Whence $$\cos.^2x=\frac{1}{\sec.^2x}=\frac{1}{1+t^2}=(1+t^2)^{-1}.$$

Therefore $$\frac{dx}{dt}=(1+t^2)^{-1}=1-t^2+t^4-t^6+t^8, \text{ \&c.} \quad (1)$$

Here we perceive that the second member contains only the even powers of t, therefore the development of x before differentiation must contain only the odd powers of t. Consequently we will assume

$$x=At+Bt^3+Ct^5+Dt^7+ \text{ \&c.} \quad (2)$$

$$\frac{dx}{dt}=A+3Bt^2+5Ct^4+7Dt^6+9Et^8, \text{ \&c.} \quad (3)$$

Comparing (1) and (3) we find that

$$A=1, \quad B=-\tfrac{1}{3}. \quad C=\tfrac{1}{5}, \quad D=-\tfrac{1}{7}, \quad E=\tfrac{1}{9}, \quad \text{\&c.}$$

These values of A, B, C, &c. substituted in (2), give

$$x=t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\frac{t^9}{9}-\frac{t^{11}}{11}, \text{ \&c.} \quad (4)$$

This formula will give the arc x, provided we know t, any corresponding tangent. We learn in geometry that the tangent of 45° is equal to the radius $=1$, and the tangent of 30° is equal to $\frac{1}{\sqrt{3}}$, therefore

$$\text{arc } 45^\circ=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}, \text{ \&c.}$$

$$=(1-\tfrac{1}{3})+(\tfrac{1}{5}-\tfrac{1}{7})+(\tfrac{1}{9}-\tfrac{1}{11})+(\tfrac{1}{13}-\tfrac{1}{15}), \text{ \&c.}$$

$$=\frac{2}{1.3}+\frac{2}{5.7}+\frac{2}{9.11}+\frac{2}{13.15}+\frac{2}{17.19}, \text{ \&c.}$$

But the series is not sufficiently convergent to be satisfactory, and therefore we will take the value of t corresponding to 30°.

That is $t=\frac{1}{\sqrt{3}}$, $t^3=\frac{1}{3\sqrt{3}}$, &c.

$$\text{arc } 30^\circ=\frac{1}{\sqrt{3}}-\frac{1}{3.3\sqrt{3}}+\frac{1}{5.3^2\sqrt{3}}-\frac{1}{7.3^3\sqrt{3}}+$$

$$\frac{1}{9.3^4\sqrt{3}}-\frac{1}{11.3^5\sqrt{3}}+ \text{ \&c.}$$

$$=\frac{1}{\sqrt{3}}\left(1-\frac{1}{3.3}+\frac{1}{5.3^2}-\frac{1}{7.4^3}+\frac{1}{9.3^4}-\frac{1}{11.3^5}+\frac{1}{13.3^6}-\right)\text{\&c.}$$

To sum up this series we take the first term in parenthesis, 1, and divide it by 3, that quotient again by 3, and so on; this will give us a series of decimals, the second of which divided by 3, the third by 5, the fourth by 7, &c. and we shall have the series of terms within the vinculum.

		+ Terms.	— Terms.
	1.000000000	1.000000000	
3)	.333333333(		.111111111
5)	111111111(	.022222222	
7)	37037037(		.005291005
9)	12345679(	1371742	
11)	4115226(		374111
13)	1371742(	105518	
15)	457247(		30483
17)	152416(	8965	
19)	50805(		2673
21)	16935(	806	
23)	5645(		245
25)	1881(	75	
27)	627(		23
29)	209(	7	
		1.023709235	.116809651
		.116809651	
		.906899584	

$$\text{arc } 30° = \frac{.906899584}{\sqrt{3}} = \frac{.906899584}{1.7320508} = .5235987.$$

$$\text{arc } 30°.6 = \text{arc } 180° = \pi = .5235987.6 = 3.1415922.$$

The utility of the calculus will be apparent on comparing this operation and its result with the like problem in common geometry.

REMARKS ON MACLAURIN'S THEOREM.

The learner must bear in mind that Maclaurin's theorem will only apply to such functions as expand into a series according to the ascending powers of the variable. Hence, when we attempt to apply it to other functions, and it fails to produce the desired result, the failure should not be called a failure of the theorem.

For example: Suppose we have the function $\left(a+\frac{1}{x}\right)^n$, and attempt to expand it into a series by Maclaurin's theorem, *we should fail* to produce the proper result because this function does not expand according to the ascending powers of the variable x. It expands in the form $A+Bx^{-1}+Cx^{-2}+$ &c., which is a series containing the *descending* powers of the variable, and the formula was not framed to meet such cases.

This formula requires that the variable, in the primitive function and in the second differential equations, be made equal to 0, and produce finite results.

But if we make $x=0$ in the function $\left(a+\frac{1}{x}\right)^n$ we shall have $\left(a+\frac{1}{0}\right)^n$, a result mathematically infinite, and we shall have the same indefinite and incomprehensible result in each of the differential equations under the same hypothesis.

We can, however, expand the function $\left(a+\frac{1}{x}\right)^n$ by a modification of Maclaurin's theorem, *which is to make* x *infinite where that theorem requires us to make* $x=0$.

Make $y=\frac{1}{x}$, then $\left(a+\frac{1}{x}\right)^n$ becomes $(a+y)^n$, and this can be expanded by Maclaurin's series, making $y=0$ for the first term, but $y=0$ is the same as making x *infinite*, because $y=\frac{1}{x}$.

This, and other artifices may apply to other functions, and in short, these remarks are made to show the learner that he must rely on his judgment in the application of this theorem.

REMARKS ON TAYLOR'S THEOREM.

Taylor's theorem is designed to apply to the development of a function, whatever value be assigned to the variable. But there are some functions which *change their form* when some *particular value* is given to the variable. To such functions the theorem will not apply when that particular value is taken, but for all other values of the variable the theorem will apply. We illustrate this by the following example :

Let $$u=\sqrt{a+x}. \qquad (1)$$

Then $$u'=\sqrt{a+x+y}. \qquad (2)$$

Here x is the variable and y the increment.

$$u'=u+\frac{du}{dx}y+\frac{d^2u}{dx^2}\frac{y^2}{2}+ \text{ \&c. is the formula.}$$

$$u'=\sqrt{a+x}+\frac{1}{2\sqrt{a+x}}y-\frac{1}{8(a+x)^{\frac{3}{2}}}y^2, \text{ \&c.} \qquad (3)$$

Now let $x=-a$, and this development becomes

$$\sqrt{y}=0+\frac{1}{2\sqrt{0}}y-\frac{1}{8(0)^{\frac{3}{2}}}y^2, \text{ \&c.}$$

Here the finite quantity $\sqrt{y}$ is equal to a series consisting of mathematical infinites, alternately *plus* and *minus*, which is *indeterminate*, if not *absurd*. Hence, for this *particular value of x* the theorem is said to *fail*, but for all other values of x the development is rational and true.

Now in (2) make $(a+x)=A$, then $u'=\sqrt{A+y}$.

The second member expanded by the *binomial theorem*, gives

$$u'=A^{\frac{1}{2}}+\frac{y}{2A^{\frac{1}{2}}}-\frac{y^2}{8A^{\frac{3}{2}}}, \text{ \&c.}$$

Now on the supposition that $x=-a$, A becomes 0, and

$$\sqrt{y}=0+\frac{y}{2(0)^{\frac{1}{2}}}-\frac{y^2}{8(0)^{\frac{3}{2}}}, \text{ \&c.}$$

Here we observe that whatever value is given to the variable y, *Taylor's theorem,* and the *binomial theorem,* will give the same result.

Hence, when one of these theorems *fail* the other *fails,* but we never apply the word fail to the binomial theorem, and it is not clear to us that such an expression should ever be applied to Taylor's theorem.

The *failure* is in the hypothesis and not in the theorems. In the present example the hypothesis that x equals minus a destroys the *binomial* form of the function $\sqrt{(a+x)+y}$, and makes it $\sqrt{y}$ a monomial, and Taylor's theorem is not designed to apply to a monomial.

CHAPTER V.

The general development of functions containing two or more variables.

(Art. 21.) We have thus for examined the development of functions containing only one independent variable; it is now proposed to extend the same principles to any number of independent variables.

Let $$u=f(x.y),$$

and if x and y are entirely independent of each other, y may be regarded (for a moment) as constant, and then if x becomes $(x+h)$, Taylor's theorem gives

$$f(x+h.y)=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\frac{h^3}{2.3}+ \text{ \&c.} \quad (1)$$

Now if we suppose y to become $(y+k)$, every term of the second member of (1) must receive an increment.

That is, u becomes

$$u+\frac{du}{dy}k+\frac{d^2u}{dy^2}\frac{k^2}{1.2}+\frac{d^3u}{dy^3}\frac{k^3}{2.3}+\text{ \&c.} \quad (2)$$

$$\frac{du}{dx} \text{ becomes } \frac{du}{dx}+\frac{d\left(\frac{du}{dx}\right)k}{dy}+\frac{d^2\left(\frac{du}{dx}\right)k^2}{dy^2\ 1.2}+\text{ \&c.}$$

$$\text{Or } \frac{du}{dx} \text{ becomes } \frac{du}{dx}+\frac{d^2u}{dxdy}k+\frac{d^3u}{dxdy^2}\frac{k^2}{1.2}+\text{ \&c.} \quad (3)$$

In the same manner we find that

$$\frac{d^2u}{dx^2} \text{ becomes } \frac{d^2u}{dx^2}+\frac{d^3u}{dx^2dy}k+\frac{d^4u}{dx^2dy^2}\frac{k^2}{1.2}+\text{ \&c.} \quad (4)$$

$$\frac{d^3u}{dx^3} \text{ becomes } \frac{d^3u}{dx^3}+\frac{d^4u}{dx^3dy}k+\text{ \&c.} \quad (5)$$

The developments in (2), (3), (4), and (5), substituted in the second member of (1) will give the following result, which is the development of the second state of a function containing two variables:

$$\left.\begin{aligned} f(x+h,\ y+k)=u&+\frac{du}{dy}k+\frac{d^2u}{dy^2}\frac{k^2}{1.2}+\frac{d^3u}{dy^3}\frac{k^3}{1.2.3}+\text{ \&c.}\\ &+\frac{du}{dx}h+\frac{d^2u}{dxdy}hk+\frac{d^3u}{dxdy^2}\frac{hk^2}{1.2}+\text{ \&c.}\\ &+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^2dy}\frac{h^2k}{1.2}+\text{ \&c.}\\ &\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\text{ \&c.}\end{aligned}\right\} (6)$$

This formula is Taylor's theorem extended, and it is true for all values of h and k. When h and k are *extremely* small, the terms containing h^2, k^2, and hk, may be omitted, and then dx may be written for h, and dy for k. This supposition will reduce the formula to

$$f(x+h,\ y+k)-f(x,y)=\frac{du}{dy}dy+\frac{du}{dx}dx.$$

$$\text{Or} \qquad du=\frac{du}{dy}dy+\frac{du}{dx}dx. \quad (7)$$

The expression $\frac{du}{dy}dy$ represents the differential of the variable y in any function u, on the supposition that all else is constant, and it is called a *partial differential.*

Also $\frac{du}{dx}dx$ represents the *partial differential* of the function u in respect to x.

In using this formula it is important to preserve the forms $\frac{du}{dy}dy$, $\frac{du}{dx}dx$, &c. otherwise we might confound these *partial* differentials with the *total* differential du in the first member.

Formula (7) is the same as (Rule 1,) (Art. 2,) and from that rule we infer at once that if u is a function of any number of variables, as $f(x.y.z.t)$, then

$$du=\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{du}{dz}dz+\frac{du}{dt}dt. \qquad (8)$$

Art. 22.) Formulas (6), (7), and (8), should not be regarded as equations of magnitude; *they are simply equivalent forms or symbols.*

Let us now examine formula (6). It can be put in this form,

$$f(x+h,\ y+k)-f(xy)=\left(\frac{du}{dx}h+\frac{du}{dy}k\right)+\frac{1}{1.2}\left(\frac{d^2u}{dx^2}h^2+\frac{2d^2u}{dxdy}hk+\frac{d^2u}{dy^2}k^2\right)+\frac{1}{1.2.3}\left(\frac{d^3u}{dx^3}h^3+\frac{3d^3u}{dx^2dy}h^2k+\frac{3d^3u}{dxdy^2}hk^2+\frac{d^3u}{dy^3}k^3\right) \text{ \&c. \&c.}$$

If we conceive h and k to be *extremely* small, as we are at liberty to do, and then write dx for h, and dy for k, the preceding formula becomes

$$du=\left(\frac{du}{dx}dx+\frac{du}{dy}dy\right)+\frac{1}{1.2}\left(\frac{d^2u}{dx^2}dx^2+\frac{2d^2u}{dxdy}dx\,dy+\frac{d^2u}{dy^2}dy^2\right)+\frac{1}{1.2.3}\left(\frac{d^3u}{dx^3}dx^3+\frac{3d^3u}{dx^2dy}dx^2dy+\frac{3d^3u}{dxdy^2}dx\,dy^2+\frac{d^3u}{dy^3}dy^3\right)\text{\&c.}$$

Observe that the expression $\frac{d^2u}{dxdy}$ indicates that the function u is to be differentiated *twice*, once in respect to x, and once in respect to y. *It is immaterial which differential is taken first*, for $\frac{d^2u}{dxdy}$, and $\frac{d^2u}{dydx}$, are identical.

The general expression $\frac{d^{m+n}u}{dx^m dy^n}$, indicates that u must be differentiated $(m+n)$ times, m times in respect to x, and n times in respect to y.

Observe the last formula. Take the first parenthesis in the second member,

$$\frac{du}{dx}dx+\frac{du}{dy}dy.$$

Differentiate each term *twice*, once in respect to x, and once in respect to y, and add the results together, and we shall have the term in the second parenthesis.

Thus $$d\left(\frac{du}{dx}dx\right)=\frac{d^2u}{dx^2}dx^2+\frac{d^2u}{dxdy}dxdy.$$

$$d\left(\frac{du}{dy}dy\right)=\frac{d^2u}{du^2}dy^2+\frac{d^2u}{dxdy}dxdy.$$

By addition $$\frac{1}{1.2}\left(\frac{d^2u}{dx^2}dx^2+\frac{2d^2u}{dxdy}dxdy+\frac{d^2u}{dy^3}dy^3\right)$$

Which is the second term of the formula taken as a whole.

The differential of this again will give the next term, and thus we might go on indefinitely.

Observe that the quantities in parenthesis take the form of an *expanded binomial*, and such in fact they are in a certain sense.

(Art. 23.) Again let us inspect formula (6), for it is a very general formula including several rules and theorems. We may use it to develop the function of any two variables, however great the increments h and k.

If we suppose both x and h equal nothing, we have

$$f(y+k)=u+\frac{du}{dy}k+\frac{d^2u}{dy^2}\frac{k^2}{2} \text{ \&c.}$$

and this is Taylor's theorem.

If we suppose x, h, and y, each equal nothing, and represent by A, A_1, A_2, A_3, &c. what u, $\frac{du}{dy}$, $\frac{d^2u}{dy^2}$, $\frac{d^3u}{dy^3}$, become under his supposition, we shall have

$$f(k)=A+A_1k+A_2\frac{k^2}{1.2}+A_3\frac{k^3}{1.2.3}+ \text{ \&c.}$$

and this is Maclaurin's theorem.

To illustrate these principles, we now give the following

EXAMPLES.

1. Expand x^3y^4 by the differential formula (6) on the supposition that x becomes $(x+h)$ and y becomes $(y+k)$.

Let $u=x^3y^4$.

Take the first horizontal column in (6).

$$\left.\begin{array}{ll}
\text{The first term is} \ldots\ldots & x^3y^4 \\
\frac{du}{dy}k \ldots\ldots & 4x^3y^3k \\
\frac{d^2u}{dy^2}\frac{k^2}{1.2} \ldots\ldots & 6x^3y^2k^2 \\
\frac{d^3u}{dy^8}\frac{k^3}{1.2.3} \ldots\ldots & 4x^3yk^3 \\
\frac{d^4u}{dy^4}\frac{k^4}{1.2.3.4} \ldots\ldots & x^3k^4
\end{array}\right\} (1)$$

Here the first column runs out, the last result x^3k^4 no longer contains y to admit of another differential in respect to that letter.

Now we will run along the next horizontal column, taking the successive differentials in relation to y.

$$\left.\begin{array}{ll}
\frac{du}{dx}h \cdots\cdots\cdots & 3x^2y^4h \\
\frac{d^2u}{dxdy}hk \cdots\cdots\cdots & 12x^2y^3hk \\
\frac{d^3u}{dxdy^2}\frac{hk^2}{1.2} \cdots\cdots\cdots & 18x^2y^2hk^2 \\
\frac{d^4u}{dxdy^3}\frac{hk^3}{1.2.3} \cdots\cdots\cdots & 12x^2yhk^3 \\
\frac{d^5u}{dxdy^4}\frac{hk^4}{1.2.3.4} \cdots\cdots\cdots & =3x^2hk^4
\end{array}\right\} (2)$$

Here the second horizontal column runs out.

$$\left.\begin{array}{ll}
\frac{d^2u}{dx^2}\frac{h^2}{1.2} \cdots\cdots\cdots & 3xy^4h^2 \\
\left(\frac{d^3u}{dx^2dy}\frac{h^2}{2}\right)\frac{k}{2} \cdots\cdots\cdots & 12xy^3h^2k \\
\left(\frac{d^4u}{dx^2dy^2}\frac{h^2}{2}\right)\frac{k^2}{2} \cdots\cdots\cdots & 18xy^2h^2k^2 \\
\left(\frac{d^5u}{dx^2dy^3}\frac{h^2}{2}\right)\frac{k^3}{2.3} \cdots\cdots\cdots & 12xyh^2k^3 \\
\left(\frac{d^6u}{dx^2dy^4}\frac{h^2}{2}\right)\frac{k^4}{2.3.4} \cdots\cdots\cdots & =3xh^2k^4
\end{array}\right\} (3)$$

Here the third horizontal column runs out.

$$\left.\begin{array}{ll}
\frac{d^2u}{dx^3}\frac{h^3}{1.2.3} \cdots\cdots\cdots & =y^4h^3 \\
\left(\frac{d^3u}{dx^3dy}\frac{h^3}{1.2.3}\right)k \cdots\cdots\cdots & =4y^3h^3k \\
\text{2d term} \cdots\cdots\cdots & =6y^2h^3k^2 \\
\text{3d term} \cdots\cdots\cdots & =4yh^3k^3 \\
\text{4th term} \cdots\cdots\cdots & =h^3k^4
\end{array}\right\} (4)$$

Here the process ends; it is very easy when one is familiar with the forms.

We will now do the same by common algebra.

$$(y+k)^4=y^4+4y^3k+6y^2k^2+4yk^3+k^4$$
$$(x+h)^3=x^3+3x^2h+3xh^2+h^3$$

$$x^3y^4+4x^3y^3k+6x^3y^2k^2+4x^3yk^3+x^3k^4 \quad \text{column (1)}$$
$$3x^2y^4h+12x^2y^3hk+18x^2y^2hk^2+12x^2yhk^3+$$
$$3x^2hk^4+3xy^4h^2+\ \&c. \qquad (2)$$

These several columns are generally indicated by the corresponding columns in formula (6).

We may use formula (7) to differentiate examples like the following, but the rules in Chap. I, are less formal and consequently more brief.

2. Let $$u=\frac{x}{y}.$$

Then $$\frac{du}{dx}dx=\frac{dx}{y}, \quad \text{and} \quad \frac{du}{dy}dy=-\frac{xdy}{y^2},$$

Whence, by adding these results, we have

$$du=\frac{ydx-xdy}{y^2}.$$

3. Let $$u=\frac{x}{\sqrt{1-y^2}}.$$

$$\frac{du}{dx}dx=\frac{dx}{\sqrt{1-y^2}}. \qquad \frac{du}{dy}dy=\frac{xydy}{(1-y^2)^{\frac{3}{2}}}.$$

Whence $$du=\frac{dx}{(1-y^2)^{\frac{1}{2}}}+\frac{xydy}{(1-y^2)^{\frac{3}{2}}}.$$

4. Let $$u=\tan.^{-1}\frac{x}{y}, \quad \text{or} \quad \tan.u=\frac{x}{y}.$$

$$\frac{du}{dx}dx=\left(\frac{du}{\cos.^2u}\right)=\frac{dx}{y}, \quad \frac{du}{dy}dy=\left(\frac{du}{\cos.^2u}\right)=-\frac{xdy}{y^2},$$

$$du=\left(\frac{du}{\cos.^2u}\right)=\frac{ydx-xdy}{y^2}.$$

When $\frac{x}{y}$ represents the tangent of an arc, the cosine of the same arc must be $\frac{y^2}{x^2+y^2}$. Whence $du=\frac{ydx-xdy}{x^2+y^2}$.

CHAPTER VI.

Application of the Differential Calculus to discover some of the properties of Plane Curves.

It is said by some, that the investigation of the properties of curves led to the consideration of *flowing* and *vanishing* quantities, and from thence came *fluxions,* now called the differential calculus.

Whether this be true or not, the following general problems will show the *geometrical power* of the calculus better than any thing thus far advanced.

(Art. 24.) The theory we are now about to present to the reader is general, and therefore we shall refer to no particular curve until we apply the theory. We now propose to show analytical expressions for *tangents, sub-tangents, normals,* and *sub-normals,* to curves in general.

A tangent is a line drawn to touch the curve, and it is terminated by the point of contact and the *ordinate.*

A normal is a line drawn perpendicular to the tangent from its point of contact, and it may be within or without the curve, according to the nature of the curve and the position of the ordinate.

A line drawn from the point where the tangent meets the curve perpendicular to the *ordinate,* will divide the ordinate into two parts ; the part lying under the tangent is called the *sub-tangent,* and the part under the normal is called the *sub-normal.*

According to these definitions, the reader will observe that in each of the adjoining figures, 1 and 2, MN is a *normal,* PN is a *sub-normal,* MT is a *tangent,* and PT a *sub-tangent.*

Fig. 1.

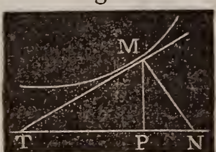


Fig. 2.

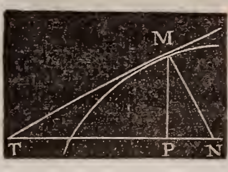
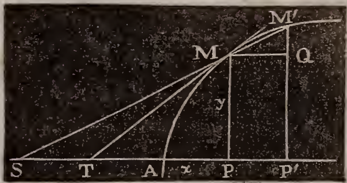


Fig. 3.



In every known curve some relation must be given between AP and PM, (fig. 3,) the co-ordinates of the curve.

If we represent AP by x, and PM by y, then $y=f(x)$.

The similar triangles SPM, MQM', give us the following proportion:

$$SP : PM :: MQ : QM'.$$

Now as we diminish PP' or h, the point M' becomes nearer and nearer to M, and the line $M'MS$ revolving on the point M will bring S nearer and nearer to T, and when M' comes to M, then S will be at T, and the line MS become MT.

But when h becomes *extremely* small, we call it dx, and then $M'Q$ becomes dy, and corresponding to this the line MS becomes *extremely* near MT, so near that we may call it MT, then the preceding proportion becomes

$$PT : y :: dx : dy$$

Whence $$PT=\frac{ydx}{dy}=\textit{sub-tangent}.$$

(Art. 25.) In the triangle TPM we have

$$(MT)^2=(TP)^2+(PM)^2.$$

That is, $$(MT)^2=\frac{y^2dx^2}{dy^2}+y^2.$$

Or $$MT=y\sqrt{\frac{dx^2}{dy^2}+1}=\textit{the tangent}.$$

The two triangles (Fig. 1), TMP, MPN, are rectangular and similar.

Whence $$TP : PM :: PM : PN$$

That is, $$\frac{ydx}{dy} : y :: y : PN.$$

Or $$PN=\frac{ydy}{dx}=\textit{ the sub-normal}.$$

Also, $TP : TM :: MP : MN.$

That is, $\frac{ydx}{dy} : y\sqrt{\frac{dx^2}{dy^2}+1} :: y : MN.$

$$\frac{dx^2}{dy^2} : \left(\frac{dx^2}{dy^2}+1\right) :: y^2 : (MN)^2.$$

Whence $MN=\frac{y}{dx}\sqrt{dx^2+dy^2}=$ *the normal.*

It is obvious from the triangle MQM', that the differential of an arc is $\sqrt{dx^2+dy^2}$, and calling the arc s, we have

$$ds=\sqrt{dx^2+dy^2}.$$

We will now collect these important expressions for future use, taken in the order of their development.

$$\text{sub-tan.} =\frac{ydx}{dy}. \qquad (1)$$

$$\text{tangent} =y\sqrt{\frac{dx^2}{dy^2}+1}. \qquad (2)$$

$$\text{sub-normal} =\frac{ydy}{dx}. \qquad (3)$$

$$\text{normal}=\frac{y}{dx}\sqrt{dx^2+dy^2} \qquad (4)$$

$$\text{Differential of an arc} =\sqrt{dx^2+dy^2}.$$

APPLICATION OF THESE EXPRESSIONS.

(Art. 26.) To apply these expressions to any curve, we must know the equation of the curve, otherwise we could not find dx and dy.

1. *Find the sub-tangent, tangent, sub-normal and normal to the parabola, the equation being* $y^2=2px$.

For the sub-tangent we must differentiate the equation, and reduce to the form $\frac{ydx}{dy}$, and so on for the other lines.

$$ydy=pdx.$$

$$\frac{y}{p}=\frac{dx}{dy}. \qquad (a)$$

$$\frac{y^2}{p}=\frac{2px}{p}=2x=\frac{ydx}{dy}.$$

Thus we discover that the sub-tangent of any parabola is $(2x)$, twice the *abscissa*, a result corresponding to (Prop. V.) page 252, Robinson's Geometry.*

For the tangent we square (a), and $\frac{y^2}{p^2}=\frac{dx^2}{dy^2}$.

Or $\frac{2px}{p^2}+1=\frac{dx^2}{dy^2}+1.$ $y\sqrt{\frac{2x}{p}+1}=y\sqrt{\frac{dx^2}{dy^2}+1}.$

Whence the tangent $=y\sqrt{\frac{2x}{p}+1}.$

Sub-normal $=p$. Normal $=\sqrt{y^2+p^2}$.

2. *The equation of the ellipse, (the origin of the axes being the center,) is* $A^2y^2+B^2x^2=A^2B^2$.

What is the value of the sub-tangent?

Ans. $-\frac{A^2}{B^2}\frac{y^2}{x}.$

What is the value of the sub-normal?

Ans. $-\frac{B^2x}{A^2}.$

3. *The equation of the circle is* $x^2+y^2=R^2$, *find the sub-tangent, tangent, sub-normal, and the normal.*

Ans. Sub-tangent $=-\frac{y^2}{x}$. The minus sign indicates that the sub-tangent *decreases* as x *increases.*

Tangent $=\frac{Ry}{x}$. Sub-normal $=-x$. Normal $=R$.

We shall apply these formulas to other curves, as occasion may require.

The student will perceive that these results are here obtained far more easily than in analytical geometry, but we are indebted to analytical geometry for the primary *equation of the curve.*

* It is important that the student should observe that this portion of the calculus is pure analytical geometry.

SECTION II.

CHAPTER I.

Maxima and Minima.

(Art. 27.) The differential calculus embraces mathematical functions and geometrical magnitudes which admit of variation, whether increasing or decreasing in value.

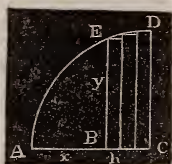
A differential of a quantity is an expression for a minute increase or decrease of the quantity.

But when a quantity has increased to its maximum value, a further increase is impossible, and the expression of such an increase *must therefore be zero.*

A decreasing quantity can of course have a differential, but when it has decreased to its smallest possible or minimum value, a further decrease is impossible, and the expression for it *must therefore be zero.*

Hence the differential expressions for any function at its *maximum* or *minimum* points must equal nothing.

To geometrise this principle *and make the idea visible,* we present the following figure.



Let AED be a curve, and for the sake of perspicuity we will suppose it to be a circle. Let AC be one semi-diameter, and CD another, at right angles to it.

Let us commence computation from the point A, and put $AB=x$, and $BE=y$.

Now the magnitude of y will depend on that of x, or in other words, y is a function of x, and it may be written

$$y=f(x).$$

If x is increased by h, y must be increased by k, and by inspecting the figure we perceive that for equal increments of x

by h, the increments of y become less and less as E approaches D, and when BE becomes CD, the increment k becomes nothing. That is, *the differential of* y *is zero when* y *itself becomes a maximum.*

(Art. 28.) The differential of an increasing quantity is *positive* before arriving at the maximum *zero*, and *negative* afterwards as we perceive by merely inspecting the figure, and *this is a general principle.*

In like manner the differential of a decreasing quantity is minus before it attains its minimum point, it is zero at that point, and positive after passing that point.

Hence, if the second differential of a function is *minus*, it indicates that the first differential corresponds to a *maximum*, and if *plus* it indicates a *minimum*.

We will now work the example represented in the figure, which is this:

What is the relation between the sine and versed sine of an arc when the sine is a maximum?

Let R represent the radius of the circle.

Then by trigonometry

$$y^2=(2R-x)x.$$

$$2ydy=2Rdx-2xdx.$$

The first member contains dy as a factor; and because y is to be a maximum, $dy=0$ and makes the first member 0.

Therefore $(R-x)dx=0.$

This equation will be verified either by making

$$dx=0, \quad \text{or} \quad R-x=0.$$

That is, $x=0, \quad \text{or} \quad x=R.$

The first corresponds to the point A, where y is a minimum, and the last corresponds to CD where y is a maximum.

If we differentiate the equation $(R-x)dx=0$, regarding dx as constant, we shall have $-dx^2=0$ for the second differential of the function, and it being *minus*, it indicates that the first differential, or *this factor of it*, corresponds to a maximum.

(Art. 29.) The foregoing illustrations are too plain and practical to meet the entire approbation of some minds; therefore, we give the following as more general and abstract.

Let $y=f(x)$, and if y is a maximum it is greater than its corresponding value when we make $x=x-h$, or $x=x+h$. Let y' correspond to $(x-h)$, and y'' correspond to $(x+h)$.

Then, by Taylor's theorem, we have

$$y'-y=-\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}-\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\text{ \&c.} \qquad (1)$$

And $$y''-y=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{1.2.3},\text{ \&c.} \qquad (2)$$

Divide (1) and (2) by h, and

$$\frac{y'-y}{h}=-\frac{dy}{dx}+\frac{d^2y}{dx^2}\frac{h}{2}-\text{ \&c.} \qquad (3)$$

$$\frac{y''-y}{h}=\frac{dy}{dx}+\frac{d^2y}{dx^2}\frac{h}{2}+\text{ \&c.} \qquad (4)$$

Now h can be taken so small that $\left(\frac{dy}{dx}\right)$ in (3) and (4) will be greater than all the following terms, and we make this supposition.

Then if y is greater than both y' and y'' the sign of the first members of (3) and (4) are both minus. Therefore the sign of the second members must be *both essentially minus.*

But this cannot be *unless* $\frac{dy}{dx}=0$, and therefore this condition must exist.

Now suppose that y be less than either y' or y'', then the sign of the first members of (3) and (4) must both be *plus*, and hence the sign of the second members must both be *plus*, but this cannot be *unless* $\frac{dy}{dx}=0$.

Therefore when y, a function of x, is a maximum or a minimum, $\frac{dy}{dx}=0$, or $dy=0$.

MISCELLANEOUS PROBLEMS IN MAXIMA AND MINIMA.

From the nature of the question or problem, find a general algebraical expression for the quantity that is to be a maximum or minimum, and pronounce it such. Then its differential can be put equal to 0, and a solution of this equation will answer the question proposed. We must find as many independent equations as the problem contains variables.

1. *Divide a line or any given numerical quantity* (a) *into two such parts that their product will be a maximum.*

Let $x=$ one part, then $a-x=$ the other part.

The problem demands that $(ax-x^2)$ shall be a maximum, and this is the same as to demand that its differential shall be $=0$.

Whence $(a-2x)dx=0$, or $x=\frac{1}{2}a$, *Ans.*

2. *Divide a given quantity* (a) *into two such parts that the square of one part multiplied by the other part shall be a maximum.*

Ans. The part to be squared is $(\frac{2}{3}a)$.

3. *Divide the number* 80 *into such parts,* (x) *and* (y), *that* $2x^2+xy+3y^2$ *may be a maximum or minimum.*

Ans. $x=50$, $y=30$, for a minimum.
For a maximum $y=80$, and $x=0$.

Here we have two equations

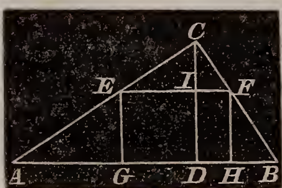
$$x+y=80, \quad (1)$$

And $$2x^2+xy+3y^2= \text{maximum or minimum.} \quad (2)$$

The differential of the first equation is zero, *because it is constant;* the second is zero, because it is a minimum.

4. *Find the greatest rectangle that can be inscribed in a given triangle.*

Ans. The altitude of the rectangle is one half the altitude of the triangle.



Let ABC be the triangle.
$AB=b$, $DC=a$. Take $CI=x$.
Then $x : EF :: a : b$.

$$EF=\frac{bx}{a}. \quad ID=a-x=FH.$$

$$\frac{bx}{a}(a-x)=\text{maximum.}$$

The differential of this expression will contain $\left(\frac{b}{a}\right)$ as a common factor. The product made by it and another factor must equal zero. Therefore the other factor alone must equal zero, for $\left(\frac{b}{a}\right)$, known constant values, cannot equal 0.

Hence we may have $x(a-x)=\text{maximum}$.

That is, *constant factors to the whole member, expressing a maximum or a minimum, may be omitted before differentiation.*

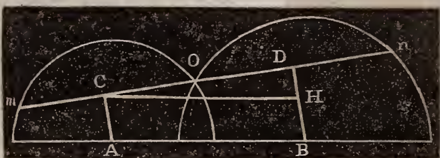
5. *Find the greatest rectangle that can be inscribed in the quadrant of a given circle.*

Ans. The rectangle is a square.

Remark.—In these philosophical mathematical problems the operator should not consider himself restricted to any mere rules; he is at liberty to apply general principles in their widest sense; the following example is an illustration of this remark,—observe its solution.

6. *If two given circles cut each other, find the greatest line that can be drawn in them passing through either point of intersection.**

Ans. The line is parallel to that joining the centers.



Let $BD=x$, and put R to represent the radius of the circle.

Let $AC=y$, and put r to represent the radius of that circle.

Then $OD=\sqrt{R^2-x^2}$, and $CO=\sqrt{r^2-y^2}$.

Because $mC=CO$ and $OD=Dn$,

$$2\sqrt{R^2-x^2}+2\sqrt{r^2-y^2}=\text{max.}$$

Or $$\sqrt{R^2-x^2}+\sqrt{r^2-y^2}=\text{max.}$$

Let AB, the distance between the two centers be represented

* This is problem 9 page 207, Mathematical Operations.

by a, then $CH=a$, and $DH=x-y$, and the right angled triangle CDH gives

$$(\sqrt{R^2-x^2}+\sqrt{r^2-y^2})^2+(x-y)^2=a^2.$$

In every equation the differential of one member taken as a whole is equal to the differential of the other.

But the first term of this last equation is a *maximum*, therefore its differential is 0, and the differential of the second member is 0, because it is invariable. Therefore

$$d.(x-y)^2=0.$$

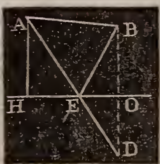
Or $$2(x-y)(dx-dy)=0.$$

Whence $$x-y=0, \quad \text{or} \quad (dx-dy)=0.$$

From either of these $x=y$, which shows that the line through O must be drawn *parallel to* AB.

7. *From two given points on the same side of a line given in position, draw two lines to meet in the line given in position, whose sum shall be less the sum of any other two lines drawn from the same points to the same line.*

{*Ans.* The two lines make equal angles with the line given in position.



Let A and B be the two points and HO the line given in position.

From A and B drop the two perpendiculars BO and AH, and these lines are given because the position of the two points are given; and for the same reason OH is given.

Make $OE=x$, and $EH=y$, $BO=a$. $AH=b$, and $HO=c$.

Then $$\sqrt{a^2+x^2}=BE, \quad \text{and} \quad \sqrt{b^2+y^2}=AE.$$

Now the problem requires that

$$\sqrt{a^2+x^2}+\sqrt{b^2+y^2}=\text{minimum}.$$

Whence $$\frac{xdx}{\sqrt{a^2+x^2}}+\frac{ydy}{\sqrt{b^2+y^2}}=0. \qquad (1)$$

But $x+y=c$. Hence $dx+dy=0$. (2)

From (2) we have $dx=-dy$, and this value of dx put in (1), and that equation reduced, we find

$$\frac{x}{\sqrt{a^2+x^2}}=\frac{y}{\sqrt{b^2+y^2}}.$$

By inspecting the figure we find

$$\frac{OE}{EB}=\frac{EH}{EA}.$$

That is, $EB : OE :: EA : EH$,

showing that the two triangles BEO and AEH are equi-angular, and the angle $BEO=$ to the angle AEH.

The angle AEH is equal to the vertical angle OED, and BO produced will make $OD=OB$.

Hence to find the point required, produce BO, making $OD=OB$, and join AD cutting OH in E. Join BE, and we have AE and EB, the two lines required.

Let OH be a plane mirror, B an object, and A the eye of an observer, the object B would be seen below the mirror at D. Hence, rays of light reflected from a surface take the *shortest possible* distance in passing to and from the reflecting surface.

This problem shows us the truth of the definition *that the calculus is a branch of analytical geometry.*

8. *Required the greatest possible rectangle that can be inscribed in a given parabola.*

Ans. The altitude of the rectangle is $\frac{2}{3}$ the altitude of the parabola.

9. *Required the same in a given semi-ellipse.*

Let $y=$ the altitude of the rectangle, and B half the shorter axis of the ellipse, then we shall find that $y=\frac{B}{\sqrt{2}}$, the result.

10. *Required the maximum cone that can be inscribed in a given sphere.*

Ans. The altitude of the cone is $\frac{4}{3}$ the radius of the sphere.

11. *Required the relation between the diameter and altitude of a cylindrical cup to hold a given quantity* (a) *of water, and to contain the least possible surface.*

Ans. The radius = the altitude.

12. *Required the maximum parabola that can be cut from a given right cone.*

Ans. The axis of the parabola is equal to $\frac{3}{4}$ the side of the cone.

13. *Required the maximum cylinder that can be cut from a given right cone.*

Ans. The altitude of the cylinder must be $\frac{1}{3}$ the altitude of the cone.

14. *On a horizontal plane stands a tower* 60 *feet high, and on the tower stands a spire* 20 *feet high; how far from the foot of the tower will the spire appear under the greatest possible angle, and what will that angle be?*

Let $x=$ the distance from the foot of the tower.

Put $a=60$ feet, $b=20$.

Then $x=\sqrt{(a+b)a}=40\sqrt{3}=69.28$ feet, *Ans.*

This example is a very simple one, but we solve it to explain one important expedient that may often be resorted to in working questions in maxima and minima.

Let $x=$ the distance on the plane from the tower, and put A to represent the whole angle at that point between the foot of the tower and the top of the spire.

Put B to represent the angle from the foot of the tower to its top. Then $(A-B)$ is the angle to be a maximum.

By trigonometry,

$$1 : \tan.A :: x : a+b, \qquad \tan.A=\frac{a+b}{x}. \qquad (1)$$

$$1 : \tan.B :: x : a, \qquad \tan.B=\frac{a}{x}. \qquad (2)$$

But $\tan.(A-B)=\frac{\tan.A-\tan.B}{1+\tan.A\tan.B}=\frac{\frac{b}{x}}{1+\frac{(a+b)a}{x^2}}$.

Or $$\tan.(A-B)=\frac{bx}{x^2+(a+b)a}. \qquad (3)$$

The arc $(A-B)$ is to be a maximum, not the $\tan.(A-B)$.

But if we differentiate the first member of the equation, it will contain $d(A-B)$ as a factor, and as this factor must be 0, the product of all the factors will be 0, and therefore the differential of the second member must be 0.

That is, $$\frac{bdx(x^2+(a+b)a)-2bx^2dx}{(x^2+(a+b)a)^2}=0.$$

Whence $$x=\sqrt{(a+b)a}.$$

To find the magnitude of the angle $(A-B)$, we substitute this value of x in equation (3),

$$\text{Tan.}(A-B)=\frac{b\sqrt{(a+b)a}}{2(a+b)a}=\frac{b}{2\sqrt{(a+b)a}}=\frac{20}{2.40\sqrt{3}}=\frac{1}{4\sqrt{3}}.$$

This result corresponds to radius unity; multiply it by R, the radius of our tables, (as follows:)

R	log.................	10.000000
$4\sqrt{3}$	log.................	0.840620
Tan.$(A-B)=8°\ 12'\ 47''$	....	9.159380

15. *An architect was required to give the relative length, breadth, and hight of a rectangular building, to contain a given cubical space* (a) *to be enclosed, sides, top and bottom, by the least possible surface.*

Ans. The building must be a cube.

16. *Divide a given number* (a) *into three parts so that their continual product may be a maximum.*

Ans. The parts must be equal.

17. *Find the minimum value of* y *in the equation* $y=x^x$.

Ans. $y^{10}=\frac{1}{10}$.

N. B. Take log. of each member, then differentiate and place $dy=0$, and we shall find $\log.x+1=0$.

18. *Two roads, one exactly north and south, the other exactly east and west, intersect each other. One traveler ten miles north of the intersection, starts and travels south at the rate of four miles per hour. Another traveler six miles west of the intersection, starts at the same moment and travels east at the rate of three miles per hour. How long after starting will they be at the minimum distance from each other, and what will that distance be, and what will be the locality of each?*

Ans. The time will be $2\frac{32}{100}$ hours. The one traveling south will be $\frac{72}{100}$ of a mile north of the intersection; the one going east will be $\frac{96}{100}$ of a mile east of the intersection, and their distance asunder will be $1\frac{2}{10}$ miles.

19. *Now suppose two roads to intersect as before, and one traveler to start from* a *miles north of the intersection and travel south at the rate of* m *miles per hour, and the other traveler at the same time to start from* b *miles west of the intersection and travel east at the rate of* n *miles per hour, what time must elapse before they arrive at a minimum distance, and what will that distance be?*

Ans. Let $t=$ the time. Then $t=\frac{am+bn}{m^2+n^2}$

Distance $=\sqrt{(a-mt)^2+(b-nt)^2}$.

Any number of numerical examples can be formed from this one by giving different values to a, b, m, and n.

20. *The difference of arc between the sun's right ascension and its longitude gives rise to one part of the equation of time. What is the sun's right ascension when this part of the equation is a maximum, and what is the maximum value?*

**Ans.*	Sun's Long.	46° 14′ 10″
	R. A.	43° 45′ 50″
	Diff.	2° 28′ 20″=9*m* 54.6 *s*.

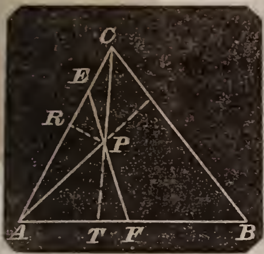
21. *What must be the inclination of the roof of a building to make the water run off in the least possible time?*

Ans. 45°.

* Examples 20, 21, and 22, are solved in the author's Mathematical Operations.

22. *Within a triangle is a given point* P, *the distance to the nearest angle* A *is given, and the line* AP *divides the angle* A *into two angles* m *and* n, *of which* m *is greater than* n.

It is required to find the line EF *drawn through the point* P, *so that the triangle* AEF *shall be the least possible.*



Let $AP=a$, $AF=x$, $AE=y$. The angle $PAF=m$, $PAE=n$.

The area of the $\triangle\ APF=ax \sin.m$.

The area of the $\triangle\ APE=ay \sin.n$.

By the conditions,

$$ax \sin.m+ay \sin.n= \text{minimum}.$$

Also, by the conditions,

$$xy \sin.(m+n)= \text{minimum}.$$

Ans. The line EF must be drawn so as to make the triangle $APE=$ to the triangle APF.

23. *What is the altitude of the maximum cylinder which can be inscribed in a given paraboloid?*

Ans. Half the axis of the paraboloid.

24. *Conceive an ellipse to revolve on its longer axis, thus forming an ellipseoid. Find the maximum cylinder which can be cut from this ellipseoid?*

Ans. The diameter of the cylinder is $\frac{4B}{\sqrt{6}}$.

Its solidity is $\frac{2(3.1416)A.B^2}{3\sqrt{3}}$.

A represents the major semi-axis of the ellipse, and B the minor semi-axis.

25. *Find the least triangle that can be made to enclose the quadrant of a given circle.*

Ans. The point of contact is at the middle of the arc.

26. *There is a perpendicular chimney; width of cavity* b *inches, hight of the jamb above the floor* a *inches. Required the longest inflexible pole that can be put up the cavity.*

Ans. $\sqrt{\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^3}$

27. *It is required to determine the size of a ball which being let fall into a conical glass full of water, shall expel the most water possible from the glass; its depth being* 6, *and its diameter* 5 *inches.*

Let ABC represent the conic section of the glass, and DHE the ball, touching the sides in the points D and E, the center of the ball being at some point F in the axis GC of the cone.

Let $FD=FE=x$, the radius of the sphere, then find an expression for the magnitude of the segment of the sphere immersed in the water, and this segment must be a maximum.

Ans. $x=2\frac{11}{92}$ inches.

An equation may have more than one maximum or minimum, according to the degree of the equation, as the following example will show.

Let $x^4-8x^3+22x^2-24x+12=$ a maxima or minima.

Then $4x^3-24x^2+44x-24=0$.

Or $x^3-6x^2+11x-6=0$.

Whence $x=1$, or 2, or 3. Substituting 1 in the equation, and we have 3, a minimum. Substituting 2, and the value of the equation is 4, a maximum. Again, substituting 3, and the equation produces 3, a minimum.

28. *Required the least triangle that can be drawn about a given parabola.*

Ans. The sub-tangent on the axis is two-thirds of the given axis.

29. *Required the same about a given semi-ellipse.*

N. B. In solving this, we use the sub-tangent taken from (Art. 26,) which is $-\frac{A^2y^2}{B^2x}$, but we change its sign, for the signs in geometry refer only to direction, and not to numerical values.

The zero point being the center of the ellipse, if we commence at the left or longer axis produced, and reckon towards the right, our distances will be *plus* all along the base of the triangle, because that direction is plus along horizontal lines as the upward direction is plus on perpendicular lines.

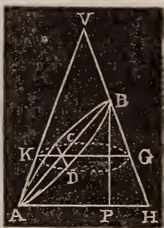
If we put z to represent the altitude of the triangle, x and y being co-ordinates of the tangent point, we shall have

$$\frac{A^2y^2}{B^2x} : y :: \frac{A^2y^2}{B^2x}+x : z=\frac{B^2}{y}.$$

In conclusion we shall find $y=\frac{B}{\sqrt{2}}$, and $x=\frac{A}{\sqrt{2}}$.

If we compare examples 29 and 9, we shall find that the interior *maximum* triangle and the exterior *minimum* triangle of the ellipse meet the curve in the same point. The same is true in respect to the circle and the parabola.

30. *It is required to cut the greatest possible ellipse from a given right cone.*



Let $AH=2a$, the base of the cone, and V, the vertex of the cone, be b distance above the base.

Let AB be the greater axis of the ellipse. Let fall BP, the perpendicular, on the base, and take G, the middle point between B and H, and pass a horizontal plane through the cone parallel to the base of the cone.

This plane will cut the plane of the ellipse at the center C, and CD will be the *minor axis*.

As C is the middle point between A and B, and G the middle point between B and H, it follows that CG is the half of AH.

Put $HP=x$, $PB=2y$. Imagine a perpendicular from V to AH, which is b. Then we have the following proportion:

$$x : 2y :: a : b. \quad \text{Or} \quad x=\frac{2ay}{b}. \qquad (1)$$

Also, $\overline{AP}^2+\overline{PB}^2=\overline{AB}^2$. That is, $(2a-x)^2+4y^2=\overline{AB}^2$.

The greater axis of the ellipse $=\sqrt{4a^2-4ax+x^2+4y^2}$. (2)

We now require the value of GK. The perpendicular from V to AH is b. From V to GK is $b-y$. Therefore we have

$$b-y : GK :: b : 2a. \qquad GK=2a-\frac{2ay}{b}.$$

From this take GC (a), and we have $KC=a-\frac{2ay}{b}$.

But $KC.CG=\overline{CD}^2$. That is, $\sqrt{a^2-\frac{2a^2y}{b}}=CD$, minor axis.

But the product of the *major* and *minor* axes of an ellipse determines its area. When that product is greater, the area is proportionally greater, and when less, less.

Therefore $$(\sqrt{4a^2-4ax+x^2+4y^2})\sqrt{a^2-\frac{2a^2y}{b}}=\text{max.}$$

Or $$(4a^2-4ax+x^2+4y^2)\left(1-\frac{2y}{b}\right)=\text{maximum.}$$

Taking the value of x from (1), and substituting it in the above, we have

$$\left(4a^2-\frac{8a^2y}{b}+\frac{4a^2y^2}{b^2}+4y^2\right)\left(\frac{b-2y}{b}\right)=\text{maximum.}$$

Whence $$\left(\frac{-8a^2}{b}dy+\frac{8a^2y}{b^2}dy+8ydy\right)\left(\frac{b-2y}{b}\right)$$
$$=\frac{2}{b}dy\left(4a^2-\frac{8a^2y}{b}+\frac{4a^2y^2}{b^2}+4y^2\right)$$

Dividing each side by $\frac{2dy}{b}$, and we have

$$\left(\frac{-4a^2}{b}+\frac{4a^2}{b^2}y+4y\right)(b-2y)=4a^2-\frac{8a^2y}{b}+\frac{4a^2y^2}{b^3}+4y^2.$$

$$-4a^2+\frac{4a^2y}{b}+4by+\frac{8a^2y}{b}-\frac{8a^2y^2}{b^2}-8y^2=4a^2-\frac{8a^2y}{b}+\frac{4a^2}{b^2}y^2$$

$$+4y^2. \qquad -12y^2-\frac{12a^2}{b^2}y^2+\frac{20a^2}{b}y+4by=8a^2.$$

Dividing by -4 produces

$$3y^2+\frac{3a^2}{b^3}y^2+\left(\frac{5a^2}{b}-b\right)y=-2a^2.$$

$$(3b^2+3a^2)y^2-(5a^2b+b^3)y=-2a^2b^2.$$

$$y^2-\left(\frac{5a^2b+b^3}{3b^2+3a^2}\right)y=-\frac{2a^2b^2}{3b^2+3a^2}. \qquad (3)$$

REMARKS.*—The shape of a cone depends on the relative values of a and b, b must be greater than a, or y will be imaginary in the above equation, *showing* that the oblique elliptic surface will not be greater than the horizontal base of the cone. And to render a maximum ellipse possible, the relative values of a and b must be so taken in the last equation that y will have a positive value.

To be sure of obtaining real values of y, *the square of half the coefficient of* y *must be numerically greater than the second member, or at least equal to it.*

That is, we must have $\frac{(5a^2+b^2)^2b^2}{4(3b^2+3a^2)^2}=\frac{2a^2b^2}{3b^2+3a^2}$, at least.

Or $$\frac{(5a^2+b^2)^2}{3b^2+3a^2}=8a^2.$$

Or $$25a^4+10a^2b^2+b^4=24a^2b^2+24a^4.$$

Or $$a^4+b^4=14a^2b^2.$$

Add $2a^2b^2$ to each side, to make complete squares.

Then $$a^4+2a^2b^2+b^4=16a^2b^2.$$

Square root $$a^2+b^2=4ab.$$

If we put $b=ma$, this last equation reduces to $1+m^2=4m$, and this resolved, gives $m=2\pm\sqrt{3}=3.732$.

This shows that b must be greater than (3.732) times a, otherwise y will be imaginary in equation (3), and the *circular base will be greater than any ellipse that can be cut from the cone, and in that case no maximum ellipse will be possible.*

We may therefore take b of any value greater than (3.732), we will then assume $b=4a$, and this reduces (3) to

$$51a^2y^2-84a^3y=-2a^2.16a^2.$$

Whence $y=1.048a$ nearly, or $\frac{2}{3}a$ nearly.

---

*These remarks show that this whole subject is one of analytical geometry.

It may be that if a plane be passed through to the opposite extremity of the base of the cone from each of these points, the elliptic surfaces will be the same, and greater than any other above, below, or between, then and there are *two maximums.*

31. *It is required to find that fraction which exceeds its square by the greatest possible quantity.*

Ans. $+\frac{1}{2}$.

32. *It is required to find that fraction which exceeds its cube by the greatest possible quantity.*

Ans. $+\frac{1}{\sqrt{3}}$.

CHAPTER II.

On the signification of Differential Coefficients as applicable to Curves.

(Art. 30.) In analytical geometry a curve is traced by connecting different points found by an equation — the equation to the particular curve in question.

The nearer a curve is to a right line, the less will be the value of the second and third differential coefficients — and when the curve becomes a right line, the first differential coefficient is constant, and the second, third, and all the following differential coefficients are zero.

For example, the equation of a straight line is

$$y=ax+b.$$

Whence $\frac{dy}{dx}=a,$ and $\frac{d^2y}{dx^2}=0,$ &c. &c.

The first of these differential equations is *the differential equation of a right line,* and let the reader observe that it is the trigonometrical tangent of the angle which the line makes with the *abscissas.*

But before we proceed with our theoretical investigations we will draw out and arrange the following *differential equations.*

In *analytical geometry* we found

1. $x^2+y^2=R^2$, to be the equation of the circle.

From which we find

$$\frac{dy}{dx}=-\frac{x}{y}$$

for the *differential equation of the circle.*

2. From $(A^2y^2+B^2x^2=A^2B^2)$, the equation of the ellipse, we derive

$$\frac{dy}{dx}=-\frac{B^2x}{A^2y},$$

the differential equation of the ellipse.

3. From $(y^2=2px)$, the equation of the parabola, we derive

$$\frac{dy}{dx}=-\frac{p}{y},$$

for the differential equation of the parabola.

4. From $(A^2y^2-B^2x^2+A^2B^2=0)$, the equation of the hyperbola, we derive

$$\frac{dy}{dx}=\frac{B^2x}{A^2y},$$

for the differential equation of the hyperbola.

5. From $(xy=M)$, the equation of the hyperbola referred to its center and *asymptotes,* we derive

$$\frac{dy}{dx}=-\frac{y}{x},$$

for the *corresponding differential equation* of the hyperbola.

Thus, every curve that can be indicated by an equation has its *corresponding differential equation.*

When $\frac{dy}{dx}$ is applied to a right line, it is the trigonometrical tangent of the angle included between the line and the axis of *abscissas;* therefore we can use its equal for that quantity in analytical geometry.

Conceive a right line touching an ellipse in a single point, the co-ordinates of which are x', y', then as above

$$\frac{dy'}{dx'}=-\frac{B^2x'}{A^2y'}.$$

But a line passing through a given point is represented by the equation $$y-y'=a(x-x'),$$
as is well known by all readers of analytical geometry.

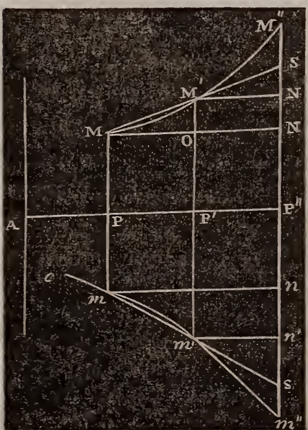
But $$\frac{dy'}{dx'}=a=-\frac{B^2x}{A^2y}$$

Therefore $$y-y'=-\frac{B^2x'}{A^2y'}(x-x').$$

Reduced $$A^2yy'+B^2xx'=A^2B^2.$$

Which is the same equation for the tangent of the ellipse as may be found in analytical geometry, x and y being the general co-ordinates of the line, and x' y', the co-ordinate of the particular point touching the ellipse.

Thus we may find the equations for the tangent lines to all known curves.



Examples like this serve to impress upon the minds of learners the connection between this analysis and analytical geometry.

(Art. 31.) We shall now attempt to show the analytical expressions for the deviation of curves from a right line in the vicinity of a given point, and also what uses can be made of such expressions.

Let $AP=x$, and $PM=y$.

Then $y=f(x)$.

Put $PP'=h$, and $PP''=2h$.

Then $P'M'=f(x+h)=y+\left(\frac{dy}{dx}\right)h+\left(\frac{d^2y}{dx^2}\right)\frac{h^2}{1.2}+$ &c.

And $P''M''=f(x+2h)=y+\left(\frac{dy}{dx}\right)2h+\left(\frac{d^2y}{dx^2}\right)\frac{4h^2}{1.2}+$ &c.

Whence $P'M'-y=OM'=\left(\frac{dy}{dx}\right)h+\left(\frac{d^2y}{dx^2}\right)\frac{h^2}{1.2}+$ &c. (1)

And $P''M''-y=NM''=\left(\frac{dy}{dx}\right)2h+\left(\frac{d^2y}{dx^2}\right)\frac{4h^2}{1.2}+$ &c. (2)

Because $MN=2MO$, $NS=2OM'$.

That is, $NS=2OM'=\frac{dy}{dx}2h+\frac{d^2y}{dx^2}\frac{2h^2}{1.2}+$ &c. (3)

From (2) subtract (3) and we shall have

$$NM''-NS=\frac{d^2y}{dx^2}h^2+\text{ \&c.} \qquad (4)$$

Now if h be taken sufficiently small, the sign of the first term will be the sign of the sum of all the terms.

The first member of (4) is obviously *positive*, and the curve being above the axis of X, all the ordinates are positive.

Hence, *when a curve is convex towards the axis of abscissas and the ordinates positive, or the curve above the axis of X, the ordinate and second differential coefficients will have the plus sign.*

Let us now examine the curve below the axis of X, which is also convex towards the axis of abscissas.

Here $AP=x$, $Pm=-y$, $PP'=h$, $PP''=2h$.

Whence $P'm'=-f(y+h)=-y-\left(\frac{dy}{dx}\right)h-\left(\frac{d^2y}{dx^2}\right)\frac{h^2}{1.2}-$ &c.

And $P''m''=-f(y+2h)=-y-\left(\frac{dy}{dx}\right)2h-\left(\frac{d^2y}{dx^2}\right)\frac{4h^2}{1.2}-$ &c.

$$P'm'+y=nn'=-\left(\frac{dy}{dx}\right)h-\left(\frac{d^2y}{dx^2}\right)\frac{h^2}{1.2}-\text{ \&c.} \quad (1)$$

$$P''m''+y=nm''=-\left(\frac{dy}{dx}\right)2h-\left(\frac{d^2y}{dx^2}\right)\frac{4h^2}{1.2}-\text{ \&c.} \quad (2)$$

As ns is double nn', we have

$$ns=2nn'=-\left(\frac{dy}{dx}\right)\frac{2h}{1}-\left(\frac{d^2y}{dx^2}\right)\frac{2h^2}{1.2}. \quad (3)$$

Subtracting (3) from (2), and we obtain

$$sm''=nm''-ns=-\left(\frac{d^2y}{dx^2}\right)h^2-\text{ \&c.}$$

It is obvious that sm'' is the deviation of the curve from the right line, and it is *minus* downward, as the ordinates are.

Hence, *when the curve is convex toward the axis of abscissas, and the ordinates minus, or the curve below the axis of X, the ordinates and the second differential coefficient will have the minus sign.*

More generally, let the curve be above or below the axis of X, and convex towards that axis.

Then the ordinates and the second differential coefficient will have the same sign.

(Art. 32.) Now let us determine what the result must be when the curve is concave towards the axis of the abscissas.

Let $AP=x$ as before, and $PM=y$, $PP'=h$, and $PP''=2h$.

$$PM=y=f(x).$$

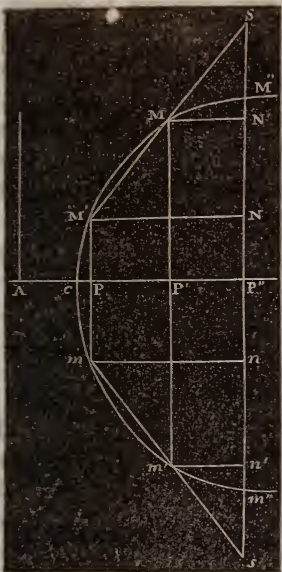
$$P'M'=f(x+h)=y+\left(\frac{dy}{dx}\right)h+\left(\frac{d^2y}{dx^2}\right)\frac{h^2}{1.2}+\text{ \&c.}$$

And $P''M''=f(x+2h)=y+\left(\frac{dy}{dx}\right)2h+\left(\frac{d^2y}{dx^2}\right)\frac{4h^2}{1.2}+$ &c.

$$P'M'-PM=NN'=\left(\frac{dy}{dx}\right)h+\left(\frac{d^2y}{dx^2}\right)\frac{h^2}{1.2}+\text{ \&c.} \quad (1)$$

But $NS=2NN'=\left(\frac{dy}{dx}\right)2h+\left(\frac{d^2y}{dx^2}\right)\frac{2h^2}{1.2}+$ &c. (2)

$$P''M''-PM=NM''=\left(\frac{dy}{dx}\right)2h+\left(\frac{d^2y}{dx^2}\right)\frac{4h^2}{1.2}-\text{ \&c.} \quad (3)$$



Subtracting (3) from (2), and we obtain

$$NS - NM'' = M''S = -\frac{d^2y}{dx^2}h^2, \text{ \&c.}$$

Now h may be taken so small as to make the first term in the second member numerically greater than all the terms that followed, and as we have its square, the sign cannot be affected by the essential sign of h, hence the second member of the equation is *negative* — which is also shown by the figure, the point M'' in the curve is below the line MS.

But in this case the ordinates are *positive*, hence the ordinates and the second differential coefficient have different signs. On a like examination of the points of the curve below the axis of X we shall find the same result.

Hence, *if a curve is concave towards the axis of abscissas the ordinates and second differential coefficient will have contrary signs.*

For an example to apply this theory, let it be required to determine whether the parabola is convex or concave towards the axis of abscissas.

$$y^2 = 2px, \quad \frac{dy}{dx} = \frac{p}{y}, \quad \frac{d^2y}{dx^2} = -\frac{pdy}{y^2dx} = -\frac{p^2}{y^3}$$

The last equation gives a clear response, for the quantity $-\frac{p^2}{y^3}$ is obviously negative when y is positive, and positive when y is negative, therefore the curve is concave towards the axis of abscissas.

Thus we might determine the position of the concavity of any other curve.

(Art. 33.) A *point of inflection* is a point at which a curve changes from convex to concave, or from concave to convex, towards the same line.

When a curve is convex towards the axis of abscissas, the ordinates and second differential coefficient have the same sign, and when concave towards the same axis, those two quantities have different signs. Therefore if a curve changes its position of convexity, the second differential coefficient must change sign at the point of *inflection.*

But when a quantity changes sign it must pass through *zero* or *infinity;* hence,

$$\frac{d^2y}{dx^2}=0, \qquad \text{or} \qquad \frac{d^2y}{dx^2}=\infty$$

will give the abscissas of the point of inflection.

To find a point of inflection we will therefore put the second differential coefficient equal to 0 or infinity, and determine the value of x, which value we will increase and diminish by a small quantity h, and if we find contrary signs for these new values of x, we must conclude that here is in *fact* a point of inflection.

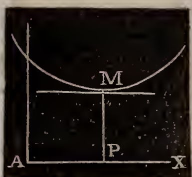
The following general equations represent an interesting class of curves which serve to illustrate this theory.

$$y=b+(x-a)^m. \qquad (1)$$

Whence $$\frac{dy}{dx}=m(x-a)^{m-1}. \qquad (2)$$

$$\frac{d^2y}{dx^2}=m(m-1)(x-a)^{m-2}. \qquad (3)$$

1st. If we assume that $\frac{dy}{dx}=0$, it follows that $x=a$, and this



value put in (1) will give $y=b$.

But to assume that $\frac{dy}{dx}=0$, is the same as to assume that the tangent line through the point M in the curve is parallel to the axis of X, as represented in the figure.

If m represents an *entire* and *even* number, then $(m-2)$ will be even, and all values of x, except $x=a$, will give y and $\frac{d^2y}{dx^2}$ positive, for suppose $x=a\pm h$, then $(x-a)=\pm h$, and substituting this in (3) we obtain

$$\frac{d^2y}{dx^2}=m(m-1)(\pm h)^{m-2}.$$

Here as $(m-2)$ is even, the power of h will be positive, whichever sign we give to h, and as m is even, the whole product will be positive, hence this curve is convex towards the axis of X. (Art. 31.)

2d. *Now let* m *be an entire and odd number.*

Then, as before, when $x=a$, $\frac{dy}{dx}=0$, and the second differential will also equal 0.

But since $(m-2)$ must be odd, every value of x *less than* a will make the second differential coefficient *negative*, and every value of x *greater than* a will make it *positive:* hence, for all values of x less than a, which give y positive, the curve is concave towards the axis of X, and for all values of x greater than a, it is convex, as in the figure adjoining.

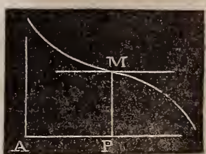
Therefore at the point M, the co-ordinates of which are $x=a$, $y=b$, the curve changes from being concave, and becomes convex towards the axis of X.

If the last term of equation (1) be negative, that is, if

$$y=b-(x-a)^m,$$

the reverse position will correspond with the curve, as in the next figure.

At the point M whose co-ordinates are $x=a$, $y=b$, there is a change of *convexity* to *concavity* towards the axis of X.



Such points are by some called *singular points,*—by others they are denominated points of *inflection.*

In both cases the tangent line at the point of inflection is parallel to the axis of X, and it *also cuts the curve.*

3d. *Let* m *be a fraction, the numerator and denominator of which are odd, as* $\frac{3}{5}$.

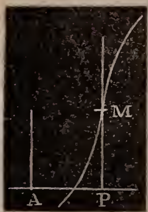
Then $$y=b+(x-a)^{\frac{3}{5}}.$$

$$\frac{dy}{dx}=\frac{3}{5(x-a)^{\frac{2}{5}}}, \qquad \frac{d^2y}{dx^2}=-\frac{6}{25(x-a)^{\frac{7}{5}}},$$

And if we now take $x=a$, we shall have $y=b$,

$$\frac{dy}{dx}=\text{ infinity, and } \frac{d^2y}{dx^2}=\text{ infinity, \&c.}$$

Now if we suppose x less than a, $\frac{d^2y}{dx^2}$ will be *positive,* and if greater than a, *negative.*

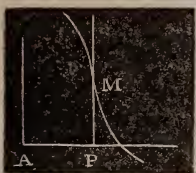


Hence for all values of x less than a, which give y positive, the curve will be *convex,* and for all values of x greater than a, it will be *concave* towards the axis of X, as shown in this figure.

But if the binomial term be negative, that is, if we have

$$y=b-(x-a)^{\frac{3}{5}},$$

the second differential coefficient will be positive, and the reverse will be the case as represented in the next figure.



The point M, whose co-ordinates are $x=a$ and $y=b$, in both cases is a point of *inflection* at which the tangent line is perpendicular to the axis of X. Whence we may say, *a point of inflection is one at which as the abscissa increases, a curve changes from concave to convex, or the reverse, towards any right line not passing through the point.*

4th. *Let* m *be a fraction with an even numerator, as* $\frac{2}{3}$, *then*

$$y=b+(x-a)^{\frac{2}{3}}.$$

$$\frac{dy}{dx}=\frac{2}{3(x-a)^{\frac{1}{3}}}, \qquad \frac{d^2y}{dx^2}=-\frac{2}{9(x-a)^{\frac{4}{3}}}.$$

If $x=a$, $y=b$. $\frac{dy}{dx}=$ infinity, and $\frac{d^2y}{dx^2}=$ infinity.

If x is *less* than a, $\frac{dx}{dy}$ will be negative, and if x is greater than a, it will be positive. Hence, at the point whose co-ordinates are $x=a$ and $y=b$, $\frac{dy}{dx}$ must change its sign from *minus* to *plus*, which change indicates a minimum ordinate.

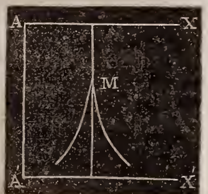
If the sign before $(x-a)$ be negative, the reverse will be the case, and there will be a change from *plus* to *minus*, indicating a maximum ordinate.

In the first case the second differential coefficient is negative for all values of x, and the ordinate positive, the curve is therefore concave towards the axis of X, as represented in the adjoining figure.

In the second case, that is,

$$y=b-(x-a)^{\frac{2}{3}},$$

the second differential coefficient is always positive for all values of x, (except $x=a$). Then for all positive values of y, the curve will be *convex*, and for all negative values of y, *concave* towards the axis of X, as this last figure illustrates.



The tangent at the point M, in both cases, is perpendicular to the axis of X.

The point M is singular and is called a CUSP *of the first order*.

It is a point at which apparently two curves unite, but it is really the same curve, as one equation represents any point in either branch.

5th. *Let* m *be a fraction of an even denominator, as* $\frac{3}{2}$.

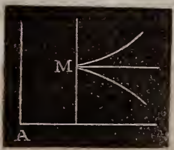
Since the denominator of the fraction denotes square root, the double sign must be placed before $(x-a)^{\frac{3}{2}}$, and we have

$$y=b\pm(x-a)^{\frac{3}{2}}.$$

$$\frac{dy}{dx}=\pm\tfrac{3}{2}(x-a)^{\frac{1}{2}}. \qquad \frac{d^2y}{dx^2}=\pm\frac{3}{4\sqrt{x-a}}.$$

$$x=a \text{ gives } y=b,\ \frac{dy}{dx}=0, \text{ and } \frac{d^2y}{dx^2}=\text{ infinity}.$$

If x is taken less than a, y will be imaginary, showing that no ordinate from a point nearer to the origin than a, can meet the curve. If x be taken greater than a, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ will be real quantities with the double sign $\pm$, showing two branches of the curve, as the figure represents.



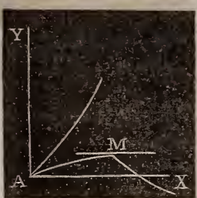
The point M is a *cusp*, and the tangent at the point M is parallel to the axis of X.

(Art. 34.) To draw out a little more light on the theory of curves, which is considered by mathematicians as one of the most beautiful features of the calculus, we will take the equation

$$y=x^2\pm x^{\frac{5}{2}}. \qquad (1)$$

$$\frac{dy}{dx}=2x\pm\tfrac{5}{2}x^{\frac{3}{2}}. \quad (2) \qquad \frac{d^2y}{dx^2}=2\pm\tfrac{15}{4}x^{\frac{1}{2}}. \quad (3)$$

When $x=0$, $y=0$, hence the curve will pass through the origin. If x be negative, y will be *imaginary*, because the equations would then demand the square root of negative quantities which have no existence, hence no part of the curve is on the negative side of the axis of Y. We also perceive that for every positive value of x there are two real values of y, both of which are positive as long as x^2 is greater than $x^{\frac{5}{2}}$; after which one is positive and the other negative.



When $x=0$, $\frac{dy}{dx}=0$. Also, the first differential is 0, when

$$2x\pm\tfrac{5}{2}x^{\frac{3}{2}}=0.$$

Whence $x=0$, or $x=\frac{16}{25}$, indicating that the axis of X is tangent to the curve at the origin, and the tangent to the lower branch must be parallel to that axis at a point whose abscissa is $\frac{16}{25}$.

The first value of $\frac{d^2y}{dx^2}$ belongs to the upper branch of the curve, and it is always positive. The second value is also positive as long as 2 is greater than $\frac{15}{4}\sqrt{x}$. Hence, the point that corresponds to

$$\frac{15}{4}\sqrt{x}=2,$$

must be a point of *inflection* whose abscissa is $\frac{64}{225}$.

Hence, the preceding figure represents this curve, and its origin is a *cusp* of the second order.

(Art. 35.) In analytical geometry (page 118) we delineated the curve corresponding to the equation

$$y=x^3-18x+12,$$

and there gave the maximum and minimum points corresponding to y. But the determination of those points of course depended on the calculus, which the reader was not then supposed to understand, and we now notice the fact to show that the subject of curves requires the calculus to be complete.

By taking the values of $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$, in connection with their signs, we can determine the concavity of the curve at any assumed point.

Thus $\frac{dy}{dx}=3x^2-13.$ (1) $\frac{d^2y}{dx^2}=6x.$ (2)

If we put $3x^2-13=0$, we shall find $x=\pm 2.0814$, showing two points at which the tangent is parallel to the axis of X. If in the equation

$$\frac{dy}{dx}=3x^2-13, \qquad (3)$$

we assume $x=0$ we shall have $\frac{dy}{dx}=-13$, showing that a tangent to the curve at the point where it cuts the axis of Y is -13, or 94° 24′ with the axis of X.

If in (3) we make $x=4$, we shall have

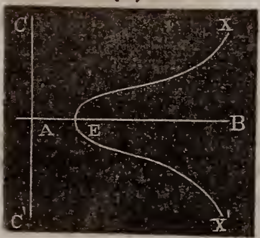
$$\frac{dy}{dx}=35,$$

showing that at that point the natural tangent with the axis of X is 35 to radius unity, or 88° 21′ 49″.

If we make x negative in (2) while y is positive, the curve will be concave towards the axis of X. (Art. 32.)

If we make x positive in (2) while y is positive, the curve at the corresponding point will be convex towards the axis of X, as is already shown by the construction of the figure.

(a)



(Art. 36.) Curves are sometimes accompanied by *insulated points*, as the following equation will illustrate.

$$a^2y^2=x^3-bx^2. \qquad (1)$$

Or
$$y=\pm\frac{x\sqrt{x-b}}{a}. \qquad (2)$$

In either (1) or (2), if we make $x=0$, we shall have $y=0$, therefore the origin is a *point in the curve.*

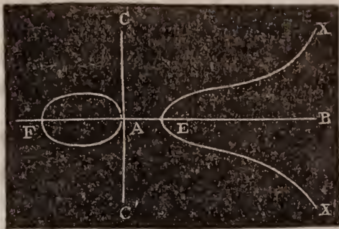
But on inspecting (2) it is obvious that y must be imaginary until x becomes greater than b, after which there will be two branches of the curve, as shown by the double sign, alike situated above and below the axis of X. Hence figure (a) will represent this curve, and the origin A will be an *insulated point* of the curve, because it is comprised in the equation as well as the various points in the two extended branches.

The equation $a^2y^2-x^3+(b-c)x^2+bcx=0$, (3)
is the same as (1) when we make $c=0$. From (3) we obtain

$$y=\pm\left(\frac{x(x-b)(x+c)}{a}\right)^{\frac{1}{2}}, \qquad \frac{dy}{dx}=\pm\frac{3x^2-2x(b-c)-bc}{2\sqrt{ax(x-b)(x+c)}}.$$

Now if we make $x=0$, or $x=b$, or $x=-c$, either supposition will make $y=0$.

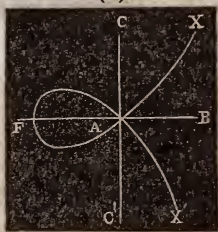
(b)



Hence we have three points in which the ordinate is zero, in this curve. At A, fig. (b), when $x=0$, at E when $x=b$, and at F when $x=-c$.

Every negative value of x less than c will give two equal values of y. Every such value of x greater than c will make y imaginary, and every positive value of x less than b will also make y imaginary,—hence figure (b) represents this curve.

(c)



When $c=0$, AF becomes a point, and the equation is represented by figure (a).

When $b=0$, and c retains its value, figure (c) represents the curve.

When $b=0$, and $c=0$ at the same time, the loop AF must be taken off.

Each of the values $x=0$, $x=b$, $x=-c$, reduces $\frac{dy}{dx}$ to infinity, hence at the three corresponding points, figure (b), at A, at E, and at F, the tangent is perpendicular to the axis of X.

Solving the equation $3x^2-2x(b-c)-bc=0$, will determine two real values for x, and thus define the points at which the tangent will be parallel to the axis of X.

We close this chapter with the following practical questions.

1. *Let the equation* xy^2+ax-b *represent a curve; has it any points of inflection?*

Ans. The points corresponding to $x=\frac{3b}{4a}$, and $y=\pm\sqrt{-\frac{1}{3}a}$ are points of inflection.

2. *The equation* $x^4-a^2x^2+a^3y=0$, *represents a curve; has that curve any points of inflection? If so, designate them.*

Ans. It has points of *inflection* corresponding to each of the points determined by making $x=\pm\frac{a}{\sqrt{6}}$, and therefore $y=\frac{5a}{36}$.

3. *Has the curve represented by the equation* $a^3y=x^4$ *any points of inflection?*

Ans. It has a double point of inflection at the origin of the co-ordinates.

4. *Determine whether the curve whose equation is*

$$y=3x+18x^2-2x^3$$

has a point of inflection?

Ans. At a point corresponding to $x=3$, and consequently $y=117$, is a point of inflection.

5. *Determine the point of inflection in the curve whose equation is*

$$y=\frac{ax^2}{a^2+x^2}.$$

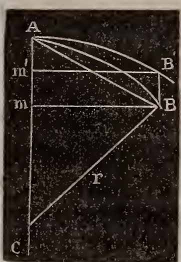
Ans. The point corresponds to $y=\frac{1}{4}a$.

CHAPTER III.

Osculating Curves.—Radius of Curvature.—Evolutes of Curves.

(Art. 36.) The *curvature* of a curve is its *deviation* from a tangent; and of two curves, that which departs most rapidly from its tangent, is said to have the greatest curvature.

From this definition it is obvious that the *greater* the radius, the *less* the curvature, and our object is now to find the relation existing between the radius and the curvature.



Let two circles touch each other internally at A. Conceive them to have a common tangent passing through A. Take any *very small* indefinite arc, as AB, and draw the chord AB, and the *equal* chord AB' to the other circle. The *curvature* of the inner circle is measured by Am, and of the outer circle by Am', because these are the *relative deviations* of these two curves from a tangent.

Let r be the radius of the inner circle, and R that of the exterior and larger circle. Also, let c represent the chord AB, it will therefore represent the equal chord AB'.

But the chord of a circle is a mean proportional between the diameter and the versed sine, therefore

$$c^2=2r(Am), \qquad \text{and} \qquad c^2=2R(Am').$$

Whence $$r(Am)=R(Am'),$$

which may be changed to the following form:

$$Am : Am' :: \frac{1}{r} : \frac{1}{R}$$

That is, *The curvature of two different circles varies inversely as their radii.*

(Art. 37.) A circle has the same degree or amount of curvature in every part; but other curves, the ellipse for example, has different degrees of curvature corresponding to different portions of its circumference, and each small portion of any ellipse or any other curve may be conceived to coincide with a small portion of some circle.

If a circle and a curve coincide at any particular point, it is an *axiomatic truth* that both the circle and the curve must have the same abscissa and ordinate corresponding with that point, and if the two curves coincide to any extent whatever, *the first and second differential coefficients of the circle will be equal to the first and second differential coefficients of the curve.*

The circle which thus changes its center and its radius to keep in coincidence with another curve, is called an *osculatory circle.*

The equation of a circle is of the second degree, therefore it can have but two differential coefficients, and if we are able to express the radius of a circle in terms of the first or second differential coefficients of the co-ordinates, or by any combination of them, that radius will correspond to the circle which will coincide with the curve having the same *variable co-ordinates.*

(Art. 38.) The object of this article is to express the radius of an osculatory circle in terms of the differentials of the co-ordinates.

In analytical geometry we found the general equation of the circle to be

$$(x-a)^2+(y-b)^2=R^2,$$

a and b being the co-ordinates of the center of the circle, and R the radius.

Differentiating and dividing by 2 produces

$$(x-a)dx+(y-b)dy=0. \qquad (1)$$

Differentiating again, regarding dx as constant, we obtain

$$dx^2+dy^2+(y-b)d^2y=0.$$

Whence $$(y-b)=-\frac{(dx^2+dy^2)}{d^2y}. \qquad (2)$$

This value of $(y-b)$ put in (1) transposed, &c. and

$$(x-a)=\frac{dy}{dx}\left(\frac{dx^2+dy^2}{d^2y}\right). \qquad (3)$$

Substituting the values of $(x-a)$ and $(y-b)$ as found in (2) and (3), in the equation of the circle, we shall have

$$R^2=\frac{dy^2}{dx^2}\left(\frac{dx^2+dy^2}{d^2y}\right)^2+\left(\frac{dx^2+dy^2}{d^2y}\right)^2.$$

Or $$R^2=\frac{(dx^2+dy^2)^3}{(dx\,d^2y)^2}.$$

Whence $$R=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y},$$

which is the general expression for the value of the radius of the osculatory circle.

(Art. 39.) To show the practical utility of the preceding formula, we will apply it to the general equation of the conic sections, which is

$$y^2=2px+qx^2. \quad \text{(Last eq. conic sections.)}$$

This equation, as we have before seen, will correspond to, or represent a circle, an ellipse, a parabola, or an hyperbola, according to the values assigned to $2p$ and q.

We can now obtain a general value for an *osculating radius*, which will apply to any of the conic curves whatever.

By differentiating the last equation, we have

$$dy=\frac{(p+qx)dx}{y}. \qquad (1)$$

Taking the differential again, regarding dx as constant, and we have

$$d^2y=\frac{qydx^2-(p+qx)dxdy}{y^2}=\frac{\{qy^2-(p+qx)^2\}dx^2}{y^3}. \qquad (2)$$

Whence $$R=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y}=\frac{\pm\{(p+qx)^2+y^2\}^{\frac{3}{2}}}{qy^2-(p+qx)^2}.$$

By substituting the value of y^2 in this last equation, we have

$$R=\frac{\pm\{(p+qx)^2+2px+qx^2\}^{\frac{3}{2}}}{2pqx+q^2x^2-(p+qx)^2}.$$

Or $$R=\frac{\pm\{(p+qx)^2+2px+qx^2\}^{\frac{3}{2}}}{-p^2}.* \qquad (3)$$

The signs should be so taken as to render R positive.

This last equation expresses the radius of curvature for each and all of the conic sections, *the origin being at the vertex of the major axis.* At that point we have $x=0$.

Whence $R=p$, for the radius of curvature at the vertex of the conic sections. For the parabola it is *half the parameter.* For the vertex of the origin of the ellipse, it is

$$R=p=\frac{B^2}{A},$$

which is half the parameter of the major axis.

The same value is found corresponding to the vertex of the hyperbola.

For the parabola $q=0$, and if we assume $p=1$, *what is the radius of curvature at the point corresponding to $x=10$?*

Ans. $90.\frac{44}{100}$.

---

* If we find the expression for the normal of the curve whose equation is $y^2=2px+qx^2$, and compare it with this equation, we shall perceive that the *radius of curvature* is equal to the cube of the normal divided by the square of half the parameter.

For another application, we require the radius of curvature for the ellipse at the vertex of the minor axis.

For this point $p=\frac{B^2}{A}$, $q=-\frac{B^2}{A^2}$, and $x=A$.

These values of p, q, and x, substituted in the equation, give $R=\frac{A^2}{B}$, *which is half the parameter of the minor axis.*

To come more directly to the utility of the theory, we now require the radius of curvature of the meridians on the equator at the poles and at the latitude of 42°, taking the diameter of the earth as given by John F. W. Herschel, and the length of a degree at each of these latitudes.

The equatorial radius is 3962.82 miles $=A$.

The polar radius is 3949.58 miles $=B$.

To find x corresponding to latitude 42°, we will make use of the mean radius 3956 miles, and subtract the cosine from the radius, and we obtain $x=1023$ miles nearly.

The radius of curvature at the equator is $\frac{B^2}{A}=3936.26$ miles.

The radius of curvature at the poles is $\frac{A^2}{B}=3976.2$ miles.

The radius of curvature in lat. 42° is found by the formula to be 3955.4 miles.*

Hence, the length of a degree on the meridian at the equator is $\frac{(3936.26)(3.1415962)}{180}=68.703$ miles.

In lat. 42° it is $\frac{(3955.4)(3.1415962)}{180}=69.031$ miles.

And in lat. 90° it is $\frac{(3976.2)(3.1415962)}{180}=69.377$ miles.

---

* To substitute for particular latitudes requires some care, as q is a fraction and negative. In this ellipse $q=-0.9933$ nearly, $p=3936.26$,

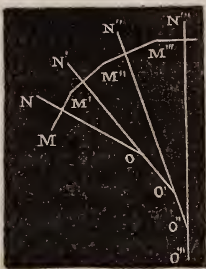
$(p+qx)^2=16579991.67$, $2px=8052967.5$, $qx^2=-1039507.13$.

Whence $R=\frac{(15540484.55)^{\frac{3}{2}}}{(3236.26)^2}=3955.4$.

(Art. 40.) An osculatory circle is one whose radius and position of the center are in a continual state of change.

Let M, M', M'', M''', &c. be points of a polygram inscribed in a curve. The perpendicular from O to MM' is the first radius of the osculatory circle, and a perpendicular from the point O' on to $M'M''$ is the second radius of the osculatory circle, and so on. The points O, O', O'', &c. if sufficiently near each other and properly connected, will form a *second curve*, which is called

THE EVOLUTE CURVE.



It is obvious from the figure, that the differences between two consecutive radii is OO', $O'O''$, &c. that is, *the difference between the radii of curvature at any two points of the involute is equal to the space between the points of the evolute intercepted between them.*

In the last article, we have seen that the osculatory circle must correspond to the following equations:

$$(x-a)dx+(y-b)dy=0. \qquad (1)$$

$$y-b=-\frac{(dx^2+dy^2)}{d^2y}. \qquad (2)$$

$$x-a=\frac{dy}{dx}\left(\frac{dx^2+dy^2}{d^2y}\right). \qquad (3)$$

Whence
$$b=y+\left(\frac{dx^2+dy^2}{d^2y}\right). \qquad (4)$$

And
$$a=x-\frac{dy}{dx}\left(\frac{dx^2+dy^2}{d^2y}\right). \qquad (5)$$

The values of a and b correspond to the points O, O', O'', and thus equations (4) and (5) will determine the evolute in any particular case.

It is obvious in the last figure that *the radius of curvature is normal to the involute and tangent to the evolute.*

As an example, let it be required to find the equation of the evolute of the common paraboloid; the equation of the involute is

$$y^2=2px.$$

This example requires the values of a and b deduced from equations (4) and (5), having

$$\frac{dy}{dx}=\frac{p}{y}, \quad dy^2=\frac{p^2dx^2}{y^2}, \text{ and } d^2y=-\frac{p^2dx^2}{y^3}.$$

Whence $$-\frac{dy^2+dx^2}{d^2y}=y+\frac{y^3}{p^2}, \quad \text{and} \quad b^2=\frac{y^6}{p^4}, \qquad (1)$$

And $$-\frac{dy}{dx}\left(\frac{dy^2+dx^2}{d^2y}\right)=+p+\frac{y^2}{p}, \quad a=x+p+\frac{y^2}{p}. \quad (2)$$

From the equation of the curve we have

$$\frac{y^6}{p^4}=\frac{8x^3}{p}, \quad \text{and} \quad \frac{y^2}{p}=2x.$$

These values substituted in (1) and (2), will give

$$b^2=\frac{8x^3}{p}. \quad (3) \qquad a=3x+p. \quad (4)$$

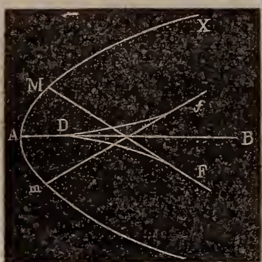
From (4) we obtain $x=\frac{a-p}{3}$, and $x^3=\frac{(a-p)^3}{27}$, which value put in (3), gives

$$b^2=\frac{8(a-p)^3}{27p},$$

showing the law of connection between (a) and (b), or it is the equation of the evolute curve of the common parabola.

Thus we might find the equation of the *evolute* of any other curve.

Corollary 1. If we make $b=0$, we shall have $a=p$, *showing that the evolute of the parabola meets the focus.*



Corollary 2. If we make a less than p, b will be *imaginary*, showing that the focus would then be a point of *inflection.*

Corollary 3. If we transfer the origin from A to D, we shall have

$$a'=a-p, \quad b'=b.$$

Then $$b'^2=\frac{8a'^3}{27p}.$$

Since every value of a' gives two equal values of b with contrary signs, the evolute is symmetrical in respect to the axis of X. The evolute DF corresponds to the involute AM, and the evolute Df to the involute Am.

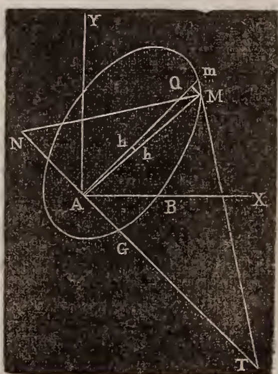
The evolute of one-fourth of the ellipse is the difference between MF and AD, that is, $\frac{A^2}{B}-\frac{B^2}{A}=$ the curve DF, along which the osculatory center moves.

CHAPTER IV.

On the differential expressions of Polar Curves.

(Art. 41.) Before discussing spirals, it is necessary to determine general expressions for the arc, the secant, the tangent, the sub-tangent, &c. of a polar curve.

The *sub-tangent in polar curves* is the part of the *perpendicular to the radius vector* of the point of contact intercepted between the pole and the point where the tangent meets this perpendicular.



Thus, Let A be the pole. Draw AM, Am, two consecutive radii, so near each other that Mm may be taken for the *differential of the arc.*

Let MQ be perpendicular to Am, then Qm is the *differential of the radius.*

Draw AT perpendicular to Am, or parallel to Qm, then, according to our *definition, AT is the sub-tangent* and *MT the tangent* to no particular arc, but corresponding to the *polar radius AM,* and the curve, whatever curve it may be.

In respect to the differential, it is immaterial whether we consider the angle MAB, or the angle MAG, as the integral angle, the differential hh is the same for either.

Let $AM=r$, and the angle $MAX=t$. Take $Ah=1$, and $hh=dt$. Then $QM=rdt$, $Qm=dr$, and if we put ds to represent Mm, the differential of the arc, the right angled triangle mQM will give us

$$ds=\sqrt{r^2dt^2+dr^2},$$

for the differential of an arc in respect to polar co-ordinates.

The differential sector AMm is measured by $\frac{1}{2}(Am.QM)$. That is, $\frac{1}{2}(r+dr)rdt$. But as dr is comparatively nothing in respect to r, the limit of this product is

$$\frac{r^2dt}{2},$$

which is the area of an elementary sector.

The similar triangles mQM and mAT give the proportion

$$mQ : QM :: mA : AT.$$

That is, $$dr : rdt :: r+dr : AT.$$

Passing to the limit, that is, taking r in the place of $r+dr$, we have

$$AT=\frac{r^2dt}{dr}, \quad \textit{the sub-tangent.}$$

The angle MAT being indefinitely less than mAT, we may regard MAT as a right angle, hence

$$MT=\sqrt{\overline{AM}^2+\overline{AT}^2}.$$

Or $$MT=r\sqrt{1+\frac{r^2dt^2}{dr^2}}, \quad \textit{the tangent.}$$

As MN is normal to the curve, the angle NMT is a right angle, and AM, or r, is a mean proportional between AT and AN, therefore

$$\frac{r^2dt}{dr} : r :: r : AN.$$

Or $$AN=\frac{dt}{dr}, \quad \textit{the sub-normal.}$$

In the right angled triangle ANM, we have

$$NM = \sqrt{\overline{AN}^2 + \overline{AM}^2}.$$

That is, $$NM = \sqrt{\frac{dt^2}{dr^2} + r^2}, \quad \textit{the normal.}$$

CHAPTER V.

On Transcendental Curves.

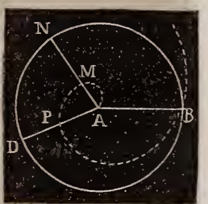
(Art. 42.) Curves are generally divided into two classes, algebraic and transcendental, according as their equations contain purely algebraic or transcendental quantities.

The curves hitherto examined, are algebraic; we now propose to illustrate and show some of the properties of some of the transcendental curves, beginning with

THE SPIRALS.

A spiral is a curve described by a point which moves along a right line in accordance with some fixed law, the line at the same time revolving uniformly about one extremity, *its pole.*

When the motion of the point commences at the pole and moves uniformly over the length of the line while the line makes one revolution, the spiral then described will be *the spiral of Archimedes.*



Thus, let AB be the line. A the pole. Let the point M commence at A, and when AB revolves to the position of AN, the point will be found at M, it having described the spiral curve AM in the same time.

When AB revolves to the position of AD, the point will be at P, it having described the curve MP while the line changed from AN to AD.

When the point arrives at B, the line is in the same position as at first, and the spiral $AMPB$ has been described.

Now let the line be indefinitely increased and the motion continued, and an infinite number of revolutions might be made.

To find the equation for this curve, let $AB=a$, the arc $BN=t$, and $AM=r$, *the radius vector* of the spiral at any point, as M.

Then by the definition

$$AM : AN = \text{arc } BN : \text{arc } BNDB.$$

That is, $$r : a = t : 2\pi a.$$

Whence $$r=\frac{t}{2\pi}, \textit{ the equation of the curve.}$$

The transcendental quantity in this equation is t, the arc of a circle; hence, this curve is a transcendental curve.

When t includes the entire circumference, it is equal to $2\pi a$, and then the equation becomes $r=a$. When $t=4\pi a$ then $r=2a$, and so on indefinitely.

THE HYPERBOLIC SPIRAL.

(Art. 43.) While the line AB revolves about the pole, let the generating point move along the line in such a manner, that the radius vectors shall be *inversely* proportional to the corresponding arcs, then the point will describe the *hyperbolic spiral.*

Let $AB=a$, $AM=1$, $AN=r$.

The arc $BN=t$.

Now from the definition we have

$$AN : AM :: \text{circ. } BNDGB : \text{arc } BN.$$

That is, $$r : 1 :: 2a\pi : t.$$

Whence $$r=\frac{2a\pi}{t}, \textit{ the equation of the curve.}$$

Let $AP=r$, then we must designate the arc BD by t, and the proportion will be the same as before, and so on for any point in the curve.

When AB has made one revolution, then $t=2a\pi$, and the equation becomes $r=1$, corresponding with the construction.

The equation shows that r cannot become zero until t becomes infinitely great; that is, the spiral will meet the pole after an infinite number of revolutions, and therefore the minor revolutions may be compared to the whirling of a top.

On the other hand, when t is very small, r will be correspondingly great; hence, the curve, after passing N, will run off and become nearly parallel to AB, and in that sense AB is an *asymptote* to the curve, and hence the name *hyperbolic spiral.*

The two preceding spirals, and indeed, all spirals that can be constructed or conceived of, are included in the general equation

$$r=at^n,$$

a representing a constant quantity, and n may be either positive or negative.

When n is positive, the spirals will pass through the pole, for if then we make $t=0$, we shall have $r=0$.

In the spiral of Archimedes $n=1$, and in the hyperbolic spiral $n=-1$, as we have just seen.

LOGARITHMIC SPIRAL.

While AB revolves uniformly about the pole, let the generating point move along the line AB in such a manner that the logarithms of the radius vectors may be proportional to the measuring arcs, it will describe the *logarithmic spiral.*

From this definition we have at once

$$t=\log. r,$$

for the equation of the logarithmic spiral, in which r represents the radius vector, and t the measuring arc.

(Art. 44.) We can now deduce some of the properties of the spirals by the application of the differential expressions for the polar curves, as determined in the preceding chapter.

Let mMh be a portion of a spiral curve, the radius vector $AM=r$, and the equation of the curve

$$r=at^n,$$

it is required to determine AT, MT, MN, and AN.

In (Chap. V,) we found

Sub-tangent $AT=\frac{r^2dt}{dr}$, and because $r=at^n$, we have

$$dr=nat^{n-1}dt.$$

Or $$\frac{dt}{dr}=\frac{1}{nat^{n-1}}.$$

Whence $$\frac{r^2dt}{dr}=\frac{a^2t^{2n}}{nat^{n-1}}=\frac{at^{n+1}}{n},$$ *the sub-tangent,* and when $n=1$, as it is in Archimedes' spiral, the value of the sub-tangent is at^2. But in that spiral $r=at$, and $a=\frac{1}{2\pi}$.

Whence $$at^2=\frac{r^2}{a}=2\pi r^2.$$

If $r=1$, the sub-tangent will be 2π, *the circumference of the measuring circle, and after two revolutions, it will be four times that length, and so on, as the squares of the number of revolutions.*

This property was discovered by Archimedes.

In the hyperbolic spiral $n=-1$, the corresponding sub-tangent is then $-a$, *a constant quantity.*

The tangent $MT=r\sqrt{1+\frac{r^2dt^2}{dr^2}}=r\sqrt{1+t^2}=at^n\sqrt{1+t^2}$.

The normal $MN=\sqrt{\frac{dt^2}{dr^2}+r^2}=\frac{1}{nat^{n-1}}\sqrt{1+n^2a^4t^{4n-2}}$.

The sub-normal $AN=\frac{dt}{dr}=\frac{1}{nat^{n-1}}$.

In the spiral of Archimedes the sides of the triangle MA, AT, and TM, are in the proportion of 1, t, and $\sqrt{1+t^2}$.

(Art. 45.) As the angle MAT is a right angle, we have

$$MA : AT :: \text{radius} : \tan.AMT.$$

That is, $$r : \frac{r^2dt}{dr} :: 1 : \tan.AMT.$$

Whence $$\tan.AMT=\frac{rdt}{dr}.$$

Let us apply this to the logarithmic spiral, the equation of which is

$$t = \log. r.$$

Whence $$dt = m\frac{dr}{r},$$

m being the modulus of the system.

Therefore $$\tan. AMT = \frac{rdt}{dr} = m.$$

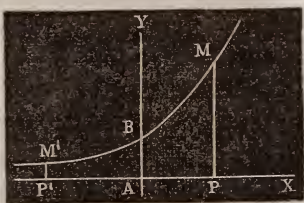
That is, the angle between the radius vector and the tangent to the spiral at the point of contact *is constant, and its trigonometrical tangent is equal to the modulus of the system.* If t is the Naperian log. of r, the angle will be 45°.

(Art. 46.) A *logarithmic curve* is not necessarily a spiral, for it is obvious that if we take rectangular co-ordinates and assume one ordinate to be a number, and the other a logarithm of that number, we shall thus have an equation which will produce a logarithmic curve.

The logarithmic equation is $y = a^x$, and taking x for the abscissa, and y the corresponding ordinate, the equation will mark out a curve, and a *particular curve* when a is given.

As is well known a is the base of the system, and x is the logarithm of the number y in that system.

We must also recollect that a cannot be 1, for every power of 1 is 1, and in that case the variations of x would produce no variations in y.



Let $M'BM$ be the logarithmic curve whose equation is $y = a^x$, and make $x = 0$, then we shall have

$$y = a^0 = 1 = AB,$$

and this will be the value of AB, whatever be the value of a, showing that all logarithmic curves will cut the axis of Y at the distance of unity, whatever be the system.

Let a be greater than 1, and x positive, then y will be positive and greater than 1, corresponding with the figure AP and PM.

When x is large, a small variation in x produces a much greater variation to y.

When x is negative, the equation becomes

$$y=a^{-x}=\frac{1}{a^x},$$

showing y to be a fraction, or less than unity, but y cannot become zero until x becomes infinite and negative, showing that the curve will meet the axis of Y at an infinite distance to the left of the origin. Hence AP' is an *asymptote* to the curve, and the curve itself can therefore be classed with the *hyperbolas* from whence comes the term *hyperbolic logarithms.*

The equation of the curve is $\quad y=a^x$.

Whence $\quad \log.y=x\log.a.$

$$\frac{dy}{dx}=y\log.a. \qquad (1)$$

But $\frac{dy}{dx}$ represents the tangent of the angle which the tangent line forms with the axis of X, hence that tangent will be parallel to the axis of X when $y=0$, and perpendicular to it when y is infinite.

But the most remarkable property of this curve is its *sub-tangent,* represented by the symbols $\left(y\frac{dx}{dy}\right)$, (Art. 24).

$$y\frac{dx}{dy}=\frac{1}{\log.a}.$$

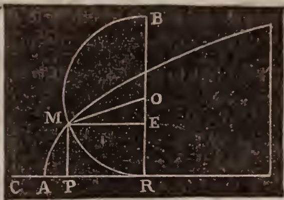
That is, the sub-tangent is *a constant quantity,* and equal to the *modulus of the system,* whichever system that may be.

(Art. 47.) Another important transcendental curve is

THE CYCLOID.

The cycloid is a curve described by the motion of a point in the circumference of a circle while the circle rolls along a right line, the point commencing to move from the line, and to make the curve complete, it must meet the line again, during which time the circle will make one revolution along the line.

Another revolution, and the point will describe another cycloid, and so on indefinitely.



Let M be a point in the circle BMR, and conceive it to roll along the line RC from R to C. The circular arc RM falls down upon and measures RA, and the point M moves over and describes the curve MA in the same time, and this curve MA is a portion of a cycloid.

To find the equation of this curve we must determine the relation between AP and PM.

Conceive A to be the origin of the co-ordinates, and put $AP=x$, $PM=y=RE$.

Let the radius of the generating circle be r, and the arc MR, *the radius unity*, be z, then the value of the arc MR will be rz, which is equal to AR.

Now $$AP=AR-ME.$$

That is, $$x=rz-ME. \qquad (1)$$

But $$ME=\sqrt{BE.ER}=\sqrt{(2r-y)y}.$$

Whence $$x=\text{arc}(\sin.=\sqrt{2ry-y^2})-\sqrt{2ry-y^2}. \qquad (2)$$

If in this equation y be taken negative, the value of x will become imaginary, showing that M can never pass below the line AR. When $y=0$, x will equal an arc whose sine is 0, hence x will equal 0 also. When $y=2r$, x will equal the arc of 180° to the radius of r; y cannot be greater than $2r$, for then x would become imaginary, showing the absurdity of any such hopothesis.

(Art. 48.) But the properties of this curve are most easily deduced from its *differential equation.*

To find the differential equation of this curve we will differentiate (1), which is

$$x=rz-\sin.(rz).$$

$$dx=rdz-\cos.(rz)dz. \qquad (3)$$

But $$\sin.(rz)=\sqrt{2ry-y^2}.$$

Whence $$\cos.(rz)dz=\frac{(r-y)dy}{\sqrt{2ry-y^2}}. \qquad (4)$$

But cos.(rz) is OE, which is equal to $(r-y)$, therefore

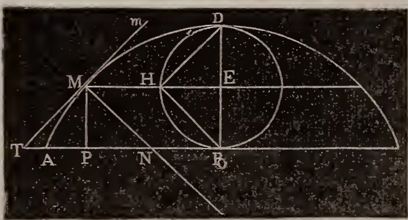
$$rdz = \frac{rdy}{\sqrt{2ry-y^2}}. \qquad (5)$$

Values taken from (4) and (5), and substituted in (3), will give

$$dx = \frac{rdy}{\sqrt{2ry-y^2}} - \frac{(r-y)dy}{\sqrt{2ry-y^2}}$$

Or $$dx = \frac{ydy}{\sqrt{2ry-y^2}},$$

which is the differential equation of the cycloid.



(Art. 49.) Now by the application of (Art. 24), we can readily find expressions for the tangent, sub-tangent, normal, and sub-normal of this curve

$$\text{The tan.} MT = y\sqrt{\frac{dx^2}{dy^2}+1} = y\sqrt{\frac{2r}{2r-y}}.$$

$$\text{Sub-tan. } TP = \frac{ydx}{dy} = \frac{y^2}{\sqrt{2ry-y^2}}.$$

$$\text{Normal } MN = \frac{y}{dx}\sqrt{dx^2+dy^2} = \sqrt{2ry}.$$

$$\text{Sub-normal } PN = \frac{ydy}{dx} = \sqrt{2ry-y^2}.$$

These values being determined, on the greatest ordinate BD, describe the generating circle. Take any point in the curve, as M, and draw ME parallel to AP. Join BH and HD, PM is parallel and equal to BE, each equal to y.

Now $BD=2r$, and by the property of the circle $HE = \sqrt{2ry-y^2}$.

Now PN and HE are equal, since each is equal to $\sqrt{2ry-y^2}$, and the two triangles MPN and HEB are equal, whence $MN = HB$ and $MNBH$ is a parallelogram.

Because MN is normal to the curve, and TMm a tangent at the point M, the angle NMm is a right angle equal to the angle BHD in the semi-circle, and as NM and BH are parallel, it follows that Mm, the tangent to the curve at M, is parallel to the corresponding chord of the generating circle described on the greatest ordinate.

(Art. 50.) Resuming the differential equation of the curve,

$$dx=\frac{ydy}{\sqrt{2ry-y^2}}.$$

Placing it in the form

$$\frac{dy}{dx}=\frac{\sqrt{2ry-y^2}}{y}=\sqrt{\frac{2r}{y}-1}.$$

Making $y=0$, we have $\frac{dy}{dx}=$infinity, and making $y=2r$, we have $\frac{dy}{dx}=0$, showing that at the point A, a tangent to the curve is perpendicular to the axis of X, and at the point D a tangent is parallel to the same axis.

(Art. 51.) To find the radius of curvature at any point, as M, we must apply the general equation, (Art. 38),

$$R=\pm\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxdy^2}. \qquad (1)$$

As the second differential of dx is not required by the formula, we may regard it as constant, therefore the differential of

$$dx=\frac{ydy}{\sqrt{2ry-y^2}}$$

is $$0=(yd^2y+dy^2)\sqrt{2ry-y^2}-\frac{ydy(rdy-ydy)}{\sqrt{2ry-y^2}} \; *$$

Reducing, and

$$0=(2ry-y^2)d^2y+rdy^2.$$

Whence $$d^2y=-\frac{rdy^2}{2ry-y^2}$$

* The denominator of this differential is omitted for obvious reasons.

Substituting the values of dx, dy, and d^2y, in (1), we have

$$R=\left(\frac{y^2dy^2}{2ry-y^2}+dy^2\right)^{\frac{3}{2}}\times\frac{(2ry-y^2)^{\frac{1}{2}}}{ydy}\times\frac{(2ry-y^2)}{rdy^2}$$

Or
$$R=\frac{(2ry)^{\frac{3}{2}}dy^3}{(2ry-y^2)^{\frac{3}{2}}}\times\frac{(2ry-y^2)^{\frac{3}{2}}}{rydy^3}.$$

Or
$$R=\frac{(2ry)^{\frac{3}{2}}}{ry}=2\sqrt{2ry}.$$

That is, the radius of curvature at any point, as M, is double of the corresponding normal MN. The radius of the curvature at A is therefore zero, and at D it is twice DB, or $4r$.

THE EVOLUTE.

(Art. 51.) In (Art. 40), we find the two following equations in which the quantities a and b represent the co-ordinates of the center of the osculatory circle; their relation, or one in terms of the other, will give the equation of the evolute.

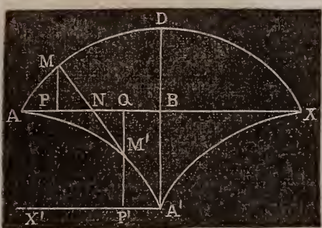
$$y-b=-\frac{(dx^2+dy^2)}{d^2y}.$$

$$x-a=\frac{dy}{dx}\left(\frac{dx^2+dy^2}{d^2y}\right)$$

The values of dx, dy, and d^2y, substituted in these equations and reduced, the results will be

$$y-b=2y, \quad \text{and} \quad x-a=-2\sqrt{2ry-y^2}.$$

Whence $\quad y=-b, \quad \text{and} \quad x=a-2\sqrt{2ry-y^2}.$



The first of these equations show that $QM'=PM$. The last article demonstrated that $MN=NM'$, therefore the two triangles PMN and NQM' are equal.

PN, the sub-normal, is equal to NQ. But this sub-normal is $\sqrt{2ry-y^2}$. Hence, PQ equals $2\sqrt{2ry-y^2}$, which subtracted from AQ, (a), gives x corresponding with the second equation.

If the value of x and y be taken from the last equation and substituted in the geometrical equation of the cycloid, we shall have, after a little reduction,

$$a = \text{arc}(\sin. = (\sqrt{-2rb - b^2}) - \sqrt{-2rb - b^2}, \qquad (1)$$

the equation of the evolute $AM'A'$.

To make this equation more clear, we will transpose the origin from A to A', BA' being equal to $2r$.

Take $A'P' = a'$, and $P'M' = b'$, then it is obvious that

$a = AB - A'P' = \pi r - a'$. And as $QM' = -b$, and $P'Q = 2r$, we shall have $\qquad -b = 2r - b'$.

Substituting these values in (1), we have

$$\pi r - a' = \text{arc}(\sin. = \sqrt{2rb' - b'^2}) + \sqrt{2rb' - b'^2}.$$

Whence $\quad a' = \pi r - \text{arc}(\sin. = \sqrt{2rb' - b^2}) - \sqrt{2rb' - b'^2}.$

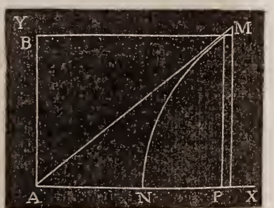
But $\quad \pi r - \text{arc}(\sin. = \sqrt{2rb' - b'^2}) = \text{arc}(\sin. = \sqrt{2rb' - b'^2}).$

Hence $\quad a' = \text{arc}(\sin. = \sqrt{2rb' - b'^2}) - \sqrt{2rb' - b'^2}.$

This equation has the same form and contains the same constant $(2r)$ as the equation of the cycloid, hence the curve $A'M'A$ is also a cycloid equal to the primitive one, — or the *involute* and *evolute* are equal.

OTHER GEOMETRICAL DIFFERENTIALS.

(Art. 53.) The differential of a plane geometrical surface is obvious, as the adjoining figure will illustrate.



Let A be the origin of co-ordinates, $AP = x$, and $PM = y$, and they may be regarded in reference to the rectangle PB, or to the triangle AMP, or to the curve NMP.

Let AP or x receive a small increment dx, then (ydx) is the *differential parallelogram*, which is the differential of the parallelogram BP, or of the triangle AMP, or of the *curve surface* MNP, according to the given relations between x and y.

(Art. 54.) If we conceive the surface $ABMP$ to revolve on AP as an axis, it will then describe a *cylinder*, and the *differential parallelogram* (ydx) will describe the differential of this cylinder, which will be measured by ($\pi y^2 dx$).

The revolution of the triangle AMP will describe a cone, the differential of which will also be represented or measured by ($\pi y^2 dx$). Also, the revolution of the curve surface NMP, on the same axis, will describe a segment of a circular solid, the differential of which is measured by the same expression ($\pi y^2 dx$).

Let the reader bear in mind that *different integrals may have the same differential*, as articles 53 and 54 illustrate.

(Art. 55.) Observe that $\sqrt{dx^2 + dy^2}$ is the differential of a line which is either a straight line or a curve, according to the relative values of x and y, and the revolution of this line on the axis of X will describe the differential of a surface, the surface of a cylinder, or of a cone, or of a curved surface, as the case may be.

THE INTEGRAL CALCULUS.

CHAPTER I.

(Art. 56.) The integral calculus is the converse of the differential, and all our rules of operation refer back to the same general principles.

Although the operations are the converse of those in the differential calculus, we must not infer that they are equally obvious, and one as easy as the other.

To cube a number, and to extract the cube root of a number, are converse operations, and the last can be deduced from the first, but it requires far more care and attention.

Without further remark we will proceed with the subject, commencing with the most simple case.

The differential of a simple quantity, as x, y, z, or any other single symbol, is expressed by writing the sign d before it, as dx, dy, &c. Hence, to pass from the differential quantity to its integral, we simply *remove that sign*, and for dx write x. To integrate $dx+dy-dz$, we simply write $x+y-z$, &c. &c.

The differential of x^3 is $3x^2dx$.

The differential of x^5 is $5x^4dx$.

The differential of y^m is $my^{m-1}dy$.

Now therefore, to integrate an exponential quantity consisting of a single term, we must frame a rule that will change $3x^2dx$ to x^3, $5x^4dx$ to x^5, and $my^{m-1}dy$ to y^m.

These operations can obviously be performed by the following

Rule (A). *Add one to the exponent, divide by the exponent so increased, and take away the differential factor.*

EXAMPLES.

1. *What is the integral of* $4x^5dx$? *Ans.* $\frac{2}{3}x^6$.

2. *What is the integral of* $7y^{\frac{1}{3}}dy$? *Ans.* $\frac{21}{4}y^{\frac{4}{3}}$.

3. *What is the integral of* $30P^{-\frac{1}{2}}dP$? *Ans.* $60\sqrt{P}$.

4. *What is the integral of* $\frac{2}{3}x^m dx$? *Ans.* $\frac{2}{3(m+1)}x^{m+1}$.

5. *Integrate the differential* $\frac{3adx}{x^4}$. *Ans.* $-\frac{a}{x^3}$.

6. *Integrate the differential* $-\frac{adx}{3x^{\frac{4}{3}}}$. *Ans.* $\frac{a}{\sqrt[3]{x}}$.

(Art. 57.) The differential of an equation like

$$y=ax+b, \qquad (1)$$

is $dy=adx$, and the attached constant b disappears.

Now take the reverse operation, and pass from the differential to the integral, and we have

$$y=ax, \qquad (2)$$

and the constant b is lost, and thus it might be in any other case — hence an integral obtained from a given differential may *require a correction*, which it is customary to denote by the symbol C.

This being the case, the *integral complete*, or equation (2), is

$$y=ax+C. \qquad (3)$$

To determine the value of C we must know the import of equation (1), or as most writers express it, we must *know the nature of the problem*, or the relation between the variables *at some particular point*.

Now from analytical geometry we know that equation (1) is the general equation of a straight line, and when $x=0$, y must equal b. Making this supposition in (3) we have

$$b=0+C.$$

This value of C put in (2), and we have $(y=ax+b)$, equation (1) completely restored.

Again, let us examine the first example under rule (A), $4x^5dx$ might have been the differential of $(\frac{2}{3}x^6+a)$, or of $(\frac{2}{3}x^6+m)$, or of any other constant attached to the variable part. Hence, it is very proper to designate $\frac{2}{3}x^6$ as the partial integral, and $(\frac{2}{3}x^6+C)$ as the complete or the corrected integral.

After we determine the value of C, the result is called a *particular* integral.

We cannot determine the value of C without some given or known condition, but with a condition it is very easy.

Thus, take the last integral $(\frac{2}{3}x^6+C)$, and suppose the value of the whole must be 4, when $x=1$, then $\frac{2}{3}+C=4$, and $C=3\frac{1}{3}$, and the particular integral is

$$\tfrac{2}{3}x^6+3\tfrac{1}{3}.$$

(Art. 58.) Rule (A) fails in one particular case, as in the following example :

Required the integral of $x^{-1}dx$.

By the rule $$\frac{x^{-1+1}}{-1+1}=\frac{1}{0}=\text{ infinity.}$$

But this is incorrect, for $x^{-1}dx$ is $\frac{dx}{x}$, which is the differential of the logarithm of x, (Art, 10), and therefore the integral is $(\log.x+C)$, and as the logarithm of x is *not an algebraic quantity, the rule failed.*

(Art. 59.) When we wish to indicate an integral we use the symbol $\int$, which is a prolongation of the letter S, the initial of the word sum, as the integral was conceived to be the sum of a great multitude of minute differentials.

Thus, if we wished to convey to the mind the integral of $mx^m dx$, we simply write $\int mx^m dx$, and so on for any other quantity.

Constant factors may be written without the sign. Thus, $\int a.x^m dx$ is the same as $a\int x^m dx$.

The more general index for this is $a\int Xdx$, X being a symbol indicating any algebraic function of x.

(Art. 60.) In general, the constant is arbitrary, since whatever value be assigned to it, it will disappear in taking the differential. This arbitrary nature of the constant fortunately enables us to cause the integral to fulfil any reasonable condition.

For example, *we require that the integral of* $5x^3dx$ *shall be* 100, *when* $x=a$.

The integral complete is

$$\int 5x^3dx=\frac{5x^4}{4}+C. \qquad (1)$$

Now by the condition, when we write a in the place of x, the second member of (1) is 100.

That is, $$\frac{5a^4}{4}+C=100.$$

Or $$C=100-\frac{5a^4}{4}.$$

This value of C put in (1) will give $\left(\frac{5x^4}{4}+100-\frac{5a^4}{4}\right)$, the integral required.

If in (1) we make $x=a$, and then $x=b$, the expressions may be generally expressed thus:

$$\int_{x=a} Xdx=A+C. \qquad \int_{x=b} Xdx=B+C.$$

Whence by subtraction,

$$\int_{x=b} Xdx-\int_{x=a} Xdx=B-A.$$

This indicates that the integral has been taken between *the limits a and b*, and it is usually written

$$\int_a^b Xdx=B-A.$$

the subtractive integral being placed below.

EXAMPLES.

1. *Find the integral of* $6x^3dx$ *between the limits of* $x=1$ and $x=3$. *Ans.* 120.

$$\int_1^3 6x^3dx=\tfrac{243}{2}-\tfrac{3}{2}=120.$$

2. *Find the integral of* $7x^2dx$, *between the limits of* $x=-1$ *and* $x=2$. *Ans.* 21.

$$\int_{-1}^2 7x^2dx=\tfrac{7}{3}x^3+C.$$

(Art. 61.) Many binomial differential expressions may be reduced to monomials by algebraical artifices, and then integrated by Rule (*A*) as illustrated by the following

EXAMPLES.

1. Let $$du=(a+bx^n)^m cx^{n-1}dx$$

be a differential equation, the integral of which is required.

Let the learner strictly *observe* its form. The exponent of the variable x within the parenthesis is n, without the parenthesis it is $n-1$, *one less.*

When this is the case, expressions in the above form are always integrable by the following process:

Place $$a+bx^n=z.$$

Then $$nbx^{n-1}dx=dz, \qquad x^{n-1}dx=\frac{dz}{nb}.$$

Substituting these expressions in the equation, and we have

$$du=\frac{c}{nb}z^m dz.$$

Integrating by the rule,

$$u=\frac{c.z^{m+1}}{nb(m+1)}+C.$$

And replacing the value of z, we finally have

$$u=\frac{c(a+bx^n)^{m+1}}{nb(m+1)}+C.$$

2. If $du=(1+3x^2)^3 6xdx$, $\qquad u=\dfrac{(1+3x^2)^4}{4}+C.$

3. If $du=-(3-2x^3)^{\frac{1}{3}}x^2dx$, $\qquad u=\dfrac{(3-2x^3)^{\frac{4}{3}}}{8}+C.$

4. If $du=(a+bx^2)^{\frac{3}{2}}xdx$, $\qquad u=\dfrac{(a+bx^2)^{\frac{5}{2}}}{5b}+C.$

5. If $du=\dfrac{xdx}{\sqrt{a^2+x^2}}$, $\qquad u=\sqrt{a^2+x^2}+C.$

6. If $du=\frac{xdx}{(1-x^2)^{\frac{3}{2}}}=(1-x)^{-\frac{3}{2}}xdx, \quad u=\frac{1}{\sqrt{1-x^2}}+C.$

7. If $du=\frac{x^2dx}{(1+x^3)^2}, \quad u=C-\frac{1}{3(1+x^3)}.$

(Art. 62.) Differential equations of the form represented in the last article, are always integrable when *m is a whole number, whatever may be the relation of the exponents within and without the parenthesis, or whatever may be the number of terms within the parenthesis.*

For example, if we require the integral of the differential equation

$$du=(a+bx+cx^2+\&c)^m exdx,$$

and if m is a whole number, we can expand the quantity in parenthesis, and then multiply each term by the part without the parenthesis, and we shall have a series of monomials, each one of which can be integrated by Rule (A).

This being understood, the integration of the following differentials can readily be obtained.

1. *Given* $du=(a+bx)^3xdx$ *to find* u.

By expanding $du=(a^3+3a^2bx+3ab^2x^2+b^3x^3)xdx$

$$=a^3xdx+3a^2bx^2dx+3ab^2x^3dx+b^3x^4dx.$$

Whence $$u=\frac{a^3x^2}{2}+\frac{3a^3bx^3}{3}+\frac{3ab^2x^4}{4}+\frac{b^3x^5}{5}+C.$$

N. B. We add but a single constant, for if we add to each term of the second member, the sum of them would be a constant quantity, which might be represented by C alone, hence, a single constant is all that is required.

We may also integrate the last example, and others similar to it, as follows:

Place $a+bx=z$. Then $dx=\frac{dz}{b}$, and $x=\frac{z-a}{b}$.

Whence $xdx=\frac{zdz-adz}{b^2}$, and $du=\frac{z^4dz}{b^2}-\frac{az^3dz}{b^2}$.

By investigation $u=\frac{z^5}{5b^2}-\frac{az^4}{4b^2}+C.$

Replacing the value of z, and

$$u=\frac{(a+bx)^5}{5b^2}-\frac{a(a+bx)^4}{4b^2}+C.$$

2. *Given* $du=(1-ax^n)^2bdx$ *to find* u.

$$Ans.\ u=bx-\frac{2abx^{n+1}}{n+1}+\frac{a^2bx^{2n+1}}{2n+1}+C.$$

3. *Given* $du=\left(\frac{1}{x}+x\right)^3x^2dx$ *to find* u.

$$Ans.\ u=\log.x+\frac{x^6}{6}+\frac{3x^4}{4}+\frac{3x^2}{2}+C.$$

(Art. 63.) Every equation in the form

$$du=Ax^m(a+bx)^n dx, \qquad (1)$$

can be integrated when either m or n is a whole positive number,

1st. Let m be a whole positive number, and n fractional or negative.

Place $a+bx=z,$ then $dx=\frac{dz}{b},$ $x^m=\frac{(z-a)^m}{b^m}.$

Whence $du=\frac{A}{b^{m+1}}(z-a)^m z^n dz. \qquad (2)$

Now as m is a whole number, this last equation can be integrated by (Art. 62.)

2d. Let n be a whole positive number, and m fractional or negative, and (1) corresponds at once to (Art. 62.)

EXAMPLES.

1. *Integrate the differential* $du=x^3(a+bx^2)^{\frac{1}{2}}dx.$

Place $a+bx^2=z.$ Then $xdx=\frac{dz}{2b},$ $x^2=\frac{z-a}{b}.$

$$2b^2du=z^{\frac{3}{2}}dz-az^{\frac{1}{2}}dz.$$

$$2b^2u=\frac{2z^{\frac{5}{2}}}{5}-\frac{2az^{\frac{3}{2}}}{3}=\frac{(6z-10a)z^{\frac{3}{2}}}{15}.$$

$$Ans.\ u=\frac{(3z-5a)}{15b^2}z^{\frac{3}{2}}=\frac{(3bx^2-2a)(a+bx^2)^{\frac{3}{2}}}{15b^2}.$$

2. *Integrate the differential* $du=2x(1-3x)^{-\frac{1}{2}}dx.$

$$Ans.\ u=-\frac{4}{9}(1-3x)^{\frac{1}{2}}+\frac{4}{27}(1-3x)^{\frac{3}{2}}+C.$$

3. *Integrate the differential* $du=\frac{x^2dx}{x-1}.$

$$Ans.\ u=\frac{x^2}{2}+x-\tfrac{3}{2}+\log.(x-1)+C.$$

4. *Integrate the equation* $dy=\frac{3(a+\sqrt{x})^2dx}{2\sqrt{x}}.$ (18th Ex. Art. 6.)

$$Ans.\ y=(a+\sqrt{x})^3+C.$$

5. *Integrate the differential*

$$du=\frac{(2a^2+4x^2)dx}{\sqrt{a^2+x^2}}. \quad \text{(16th Ex. Art. 6.)}$$

This example may be put in the following form :

$$\int du=\int\frac{2a^2dx}{\sqrt{a^2+x^2}}+\int\frac{4x^2dx}{\sqrt{a^2+x^2}}+C.$$

$$Ans.\ u=2x\sqrt{a^2+x^2}+C.$$

6. *Integrate the differential* $du=\frac{(1-3x)dx}{2\sqrt{1-x}}.$ (13th Ex. Art. 6.)

$$Ans.\ u=(1+x)\sqrt{1-x}+C.$$

(Art. 64.) We have seen in (Art. 10) of differential calculus that the differential of the logarithm of a number *is the differential of the number divided by the number,* multiplied by the *modulus* of the system. When the modulus is *one,* the system is the hyperbolic or Naperian, and the constant disappears, or it is not written. *A unit factor is not visible.*

Hence, whenever we observe differentials in the form

$$\frac{mdx}{x}, \text{ or } \frac{dx}{a+x}, \text{ or } \frac{(a+2x)dx}{ax+x^2}, \text{ or } \frac{(b+2cx)dx}{a+bx+cx^2},$$

we know that the integral must be the log. of the denominator, plus a constant.

Thus the integrals of the above expressions are

$$m\log.x+C, \quad \log.(a+x)+C, \quad \log.(a+bx+cx^2)+C, \text{ \&c.}$$

The logarithmic form of equations, or of differential expressions, is not always apparent in consequence of constant factors, but the form can be made apparent by a little algebraic artifice, as the following examples will illustrate.

EXAMPLES.

1. *Integrate the differential* $\dfrac{5x^3dx}{15x^4+21}$.

Put $15x^4+21=z$. Then $60x^3dx=dz$, or $5x^3dx=\dfrac{dz}{12}$.

Whence $\displaystyle\int\frac{5x^3dx}{15x^4+21}=\int\frac{1}{12}\cdot\frac{dz}{z}=\frac{1}{12}\int\frac{dz}{z}=\frac{1}{12}\log.(15x^4+21)+C$

2. *Integrate the differential equation* $du=\dfrac{(2+2y)dy}{2y+y^2}$.

Ans. $u=\log.(2y+y^2)+C$.

3. *Integrate* $du=\dfrac{2y^{\frac{1}{2}}dy}{1-y^{\frac{3}{2}}}$. *Ans.* $u=-\frac{4}{3}\log.(1-y^{\frac{3}{2}})+C$.

4. *Integrate* $du=\dfrac{(8-18y)ydy}{3+2y^2-3y^3}$.

Ans. $u=2\log.(3+2y^2-3y^3)+C$.

In the application of this branch of the science a sufficient number of examples will occur to exercise the student in logarithmic functions, and therefore we give no more at present.

(Art. 65.) We have seen that the differential of a product as xy is $(xdy+ydx)$.

Therefore, the integral of $(xdy+ydx)$ is xy, and this must serve as a fundamental rule for integration, and we now propose to show that this harmonizes with rule (A).

The differential expression obviously contains two variables, because we have dx and dy; hence, the integral will contain x and y, *but how connected,* or how related, *we are not supposed to know,* at the present moment.

But x must be *equal to,* or *greater,* or *less* than y. Let a be the difference between them, and that difference is constant.

That is, $$x=y\pm a. \qquad (1)$$

Whence $$ydx=ydy.$$

But $$xdy=(y\pm a)dy.$$

$$du=xdy+ydx=2ydy\pm ady.$$

Now by integration,

$$u=\int(xdy+ydx)=y^2\pm ay=(y\pm a)y=xy, \quad Ans.$$

Again, we can assume $x=ay$, in which a is greater, equal to, or less than one, as the case may demand.

$$dx=ady, \qquad ydx=aydy, \qquad xdy=aydy.$$

$$\int du=\int(xdy+ydx)=\int 2aydy=ay^2=ay.y=xy, \quad Ans.$$

Thus we perceive that either operation corresponds to rule (A) after the transformation is effected.

(Art. 66.) We may take another view of this case. When we differentiate a product like xy, we conceive one letter, as x, constant, and the other variable, and thus we obtain xdy, *the partial differential.*

Then we conceive x to be variable and y constant, and under that supposition we obtain the *other partial differential* $(ydx.)$

Now if we take either of these *partial differentials* and integrate on the supposition that the letter having the sign d prefixed is the variable one, and the other constant, and we obtain xy for the integral, *and if we take each partial integral and integrate, we get* xy *twice, or* 2xy, *but we must take but one of them for the integral, for obvious reasons.*

The same principle holds good in relation to the three or more letters. The differential of xyz is

$$xydz+xzdy+yzdx.$$

Now if we integrate each of these expressions on the supposition xy is constant in the first term, xz constant in the second, and yz constant in the third, we shall have

$$xyz+xyz+xyz.$$

Here are three equal integrals, but we must take but one of these for the whole integral, because the differential was effected by *three distinct suppositions.*

This principle holds good in relation to a quotient, as $\frac{x}{y}$, the differential of which is

$$\frac{ydx-xdy}{y^2}, \text{ which may be written } \frac{dx}{y}-xy^{-2}dy.$$

Integrating each of these expressions on the supposition that y is constant in the first, and x constant in the second, we have

$$\frac{x}{y}+\frac{x}{y},$$

but we must only take one of these for the integral, for the same reason as before.

We may also change the form of this differential by substitution, so as to make rule (A) applicable to it.

Thus place $$du=\frac{ydx-xdy}{y^2}. \qquad (1)$$

Now put $y=tx$, t and x being variable, for if t were not variable, the fraction $\frac{x}{y}$, or its equal $\frac{x}{tx}=\frac{1}{t}$, would represent only a constant, which could have no differential, and therefore t as well as x must be variable.

If $y=tx$, $ydx=txdx$, and $xdy=txdx+x^2dt$.

Whence $$du=\frac{ydx-xdy}{y^2}=-\frac{x^2dt}{t^2x^2}=-t^{-2}dt.$$

Integrating by rule (A), and we have

$$u=\frac{1}{t}, \quad \text{but} \quad \frac{1}{t}=\frac{x}{y}.$$

EXAMPLES.

1. *Integrate* $(6xy-y^2)dx+(3x^2-2xy)dy$.

Ans. $3x^2y-y^2x$.

We integrate the first part on the supposition that y is constant, and the second on the supposition that x is constant, and we obtain

$$3x^2y-y^2x+3x^2y-y^2x,$$

and because we make two distinct suppositions, we divide by 2. Then *test* the result by taking the differential.

Again we may integrate the last example as follows:

Place $du=(6xy-y^2)dx+(3x^2-2xy)dy$, (1)

and assume $x=ay$, then $dx=ady$, and (1) becomes

$$du=(6a^2y^2-ay^2)dy+(3a^2y^2-2ay^2)dy=9a^2y^2dy-3ay^2dy.$$

Whence $u=3a^2y^3-ay^3=3a^2y^2.y-ay.y^2=3x^2y-xy^2$, *Ans.*

2. *Integrate the differential equation*

$$du=(2y^2x+3y^3)dx+(2x^2y+9xy^2+8y^3)dy.$$

Ans. $u=x^2y^2+3xy^3+2y^4+C$.

3. *Integrate* $du=\dfrac{(b^2+y^2)xdx+(a^2+x^2)ydy}{\sqrt{(b^2+y^2)(a^2+x^2)}}$.

Place $\sqrt{b^2+y^2}=P$, and $\sqrt{a^2+x^2}=Q$.

The several members will then reduce into the differential of a product. (See the author's Sequel, page 342.)

Ans. $u=\sqrt{b^2+y^2}.\sqrt{a^2+x^2}+C$.

4. *Integrate* $du=\dfrac{-3dy+3dz}{4(a-y+z)^{\frac{1}{4}}}$

Place $(a-y+z)^{\frac{1}{4}}=P$. *Ans.* $u=P^3+C$.

5. *Integrate* $du=6xdy+6ydx+3bdy+2cdx$.

Ans. $u=6xy+3by+2cx+C$.

Or $u=(2x+b)(3y+c)$.

6. *Integrate examples* 1, 2, 3, (Art. 3).

7. *Integrate* $du=\frac{3(ydy-xdx)}{(x^2-y^2)^{\frac{3}{2}}}$. (This is Ex. 15, Art. 6.)

Place $x=ay$. *Ans.* $u=\frac{3}{\sqrt{x^2-y^2}}+C$.

CHAPTER II.

On the Integration of Circular Differentials.

(Art. 67.) In (Art. 9), we have seen that if u designates an arc of a circle, and x its sign, we shall have, (*radius being unity*),

$$du=\frac{dx}{\sqrt{1-x^2}}.$$

Whence $$\int\frac{dx}{\sqrt{1-x^2}}=u+C. \qquad (1)$$

We know however, that when the sine of an arc is 0, the arc itself is 0, or 180°. Regarding it as 0, equation (1) becomes

$$0=0+C, \quad \text{or} \quad C=0.$$

Hence the whole integral *is the arc of a circle whose sine is* x, which is sometimes written

$$\int\frac{dx}{\sqrt{1-x^2}}=\text{arc}(\sin.=x). \qquad (2)$$

When we can integrate the first member of this equation in *numerical* or algebraic terms, we shall then have the numerical value of the arc of the circle, to compare with the numerical value of the sine.

(Art. 68.) When u is an arc and y its cosine, (radius unity,) we have

$$du=-\frac{dy}{\sqrt{1-y^2}}. \qquad \text{(Art. 9.)}$$

Whence $$\int -\frac{dy}{\sqrt{1-y^2}}=u+C. \qquad (3)$$

To determine the constant C we must take a particular case. Estimating the arc from the commencement of the first quotient the cosine of *zero arc is radius,* and from thence the cosine diminishes and becomes 0, when the arc becomes a quadrant. Hence when y is 0, the first member of equation (3) is $\frac{1}{2}\pi$, and u must also equal $\frac{1}{2}\pi$; therefore $C=0$, and the entire integral is

$$\int -\frac{dy}{\sqrt{1-y^2}}=u=\text{arc}(\cos.=y). \qquad (4)$$

Again, let u be the arc (always less than 90°, and estimated from the commencement of the first quadrant), and t the tangent of the same arc, then u and t will commence and vanish together, and integrals connecting them will require no correction.

Now from (Art. 9), we have at once

$$\int \frac{dt}{1+t^2}=u=\text{arc}(\tan.=t). \qquad (5)$$

When u is an arc estimated as above, and v its versed sine, we have

$$\int \frac{dv}{\sqrt{2v-v^2}}=u=\text{arc}(\text{vers. sin.}=v). \qquad (6)$$

(Art. 69.) It frequently happens that we have expressions to integrate in the form

$$\frac{dx}{\sqrt{a^2-x^2}}.$$

All such expressions indicate a circle, whose radius is a in place of unity, and x represents the sine of an arc if the expression is positive, and a cosine if it is negative.

In the above expression, if we suppose x represents the sine of an arc, and a the radius of the circle, and if we take z to represent the sine of the same arc in the circle of radius unity, we shall have

$$a : x :: 1 : z.$$

Whence $$z=\frac{x}{a}, \qquad dz=\frac{dx}{a}.$$

Therefore $$\frac{dx}{\sqrt{a^2-x^2}}=\frac{adz}{\sqrt{a^2-a^2z^2}}=\frac{dz}{\sqrt{1-z^2}}$$

And $$\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{dz}{\sqrt{1-z^2}}=\text{arc}(\sin.=z)=\text{arc}\left(\sin.=\frac{x}{a}\right)$$

the arc still taken in a circle whose radius is unity.

From this we may summarily conclude that the integral of $-\frac{dy}{\sqrt{a^2-y^2}}$ is an arc whose cosine is $\frac{y}{a}$.

(Art. 70.) We have just seen that the integral for a tangent to the circle of radius unity is

$$\int\frac{dt}{1+t^2}.$$

But suppose we have before us the expression $\frac{dt}{a^2+t^2}$, to be integrated, we would examine and see if it were not the differential of a tangent to a circle whose radius is a, and if so, determine its integral.

Let t be the tangent of an arc, and a the radius, and z be the tangent of the same arc to radius unity, then

$$a : t :: 1 : z.$$

$$t=az, \quad \text{and} \quad dt=adz.$$

Whence $$\frac{dt}{a^2+t^2}=\frac{adz}{a^2+a^2z^2}=\frac{1}{a}\left(\frac{dz}{1+z^2}\right).$$

$$\int\frac{dt}{a^2+t^2}=\frac{1}{a}\text{arc}(\tan.=z.)$$

Or $$\int\frac{dt}{a^2+t^2}=\frac{1}{a}\text{ arc}\left(\tan.=\frac{t}{a}\right),$$

the arc being estimated to the radius unity.

In view of the foregoing, and on inspecting equation (6) of (Art. 68), we will venture to conclude that

$$\int\frac{dv}{\sqrt{2av-v^2}}=\text{arc}\left(\text{versed sin.}=\frac{v}{a}\right).$$

CHAPTER III.

INTEGRATION BY SERIES.

Integration of Rational Fractions.

(Art. 71.) Expressions in the form Xdx, in which X is any algebraic function of x, and which cannot be integrated by any of the preceding artifices, the quantity X may be expanded into a series of simple terms, and then we can multiply each term by dx, and integrate. The sum of the integrals so found will be the approximate integral of Xdx, provided the series is converging, and is carried to a sufficient number of terms.

When the series is not converging, a little algebraic artifice can transform it into another which will converge.

In the preceding chapter we have integrated in terms of circular arcs. If we can also integrate the same expressions in algebraic terms, we shall have the numerical or algebraic measure of circular arcs. Thus, much useful truth is revealed by two methods of integrating the same quantity,—and this is one feature of the utility of the science.

For example, let us take the expression

$$\frac{dx}{\sqrt{1-x^2}},$$

from equation (2), (Art. 67), whose integral is the arc of a circle, the radius of which is unity, and sine x, and we shall have the numerical value of this arc. The above expression may be expanded as follows:

$$\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-\frac{1}{2}}=1+\tfrac{1}{2}x^2+\tfrac{1}{2}\cdot\tfrac{3}{4}x^4+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}x^6+\text{ \&c.}$$

Multiplying each term by dx, and integrating, we obtain

$$u=\sin.^{-1}x=x+\frac{x^3}{2.3}+\frac{3x^5}{2.4.5}+\frac{3.5x^7}{2.4.6.7}+\text{ \&c.}$$

This integral requires no correction, for if we make $x=0$, u will become 0 at the same time, as it should.

As x is the sine of an arc, it can never be taken greater than the *radius*, (*unity*,) and if we know the value of x for any particular arc, that value substituted in the second member will give the linear measure of the arc.

If we take $u=30°$, we know that the corresponding value of x is $\frac{1}{2}$, and taking ten terms of the series, we find

$$\text{Arc of } 30°=0.52359877.$$

Whence Arc of $180°=6(0.52359877)=3.14159262=\pi$.

N. B. This problem is the same as example 1, (Art. 20), in the differential calculus. We repeat it here to develop the method of integration, and to show the harmony and beauty of science.

For another example. One integration of the expression $\frac{dt}{1+t^2}$ is the arc of a circle whose tangent is t, and radius unity.

(See Eq. (5), (Art. 68.)

For another integration we expand $\frac{1}{1+t^2}$ by division, which produces

$$1-t^2+t^4-t^6+t^8-t^{10}+ \text{\&c.}$$

Multiplying each term by dt, and integrating, we have

$$\text{Arc}(\tan.=t)=t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\frac{t^9}{9}-\frac{t^{11}}{11}+ \text{\&c.}$$

This result is the same as equation (4), (Art. 20), differential calculus, and therefore we will not again carry out the numerical result.

For a third example. One integration of $\frac{dx}{x}$ is $\log.x$. We cannot obtain another integration of this differential, because we cannot expand it into a series.

But if we place $x=1+y$, then $dx=dy$.

And $\frac{dx}{x}=\frac{dy}{1+y}$. Whence $\int\frac{dx}{x}=\int\frac{dy}{1+y}=\log.(1+y)$.

For the second integral we can expand $\frac{1}{1+y}$ into the series

$$1-y+y^2-y^3+y^4-y^5+y^6-y^7+ \text{\&c.}$$

Multiplying each term by dy, and integrating, we obtain

$$\log.(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}-\frac{y^6}{6}+\text{ \&c. }+C. \quad (1)$$

To determine the value of C we make $y=0$, then $\log.1=C$. But $\log.1=0$, in all systems of logarithms; hence $C=0$ in this case.

Had we made $x=a+y$, the development of (1) would have been

$$\log.(a+y)=\frac{y}{a}-\frac{y^2}{2a^2}+\frac{y^3}{3a^3}-\frac{y^4}{4a^4}\text{ \&c. }+C. \quad (2)$$

Now to determine the value of C we make $y=0$, then $\log.a=C$, and the entire integral is

$$\log.(a+y)=\log.a+\frac{y}{a}-\frac{y^2}{2a^2}+\frac{y^3}{3a^3}-\frac{y^4}{4a^4}\text{ \&c.}$$

This result is the same in form as in (Art. 19), hence we omit carrying out the details, as it would be mere repetition.

For a fourth example,

Integrate $\frac{6xdx}{1+3x^2}$, or $\frac{2axdx}{1+ax^2}$.

We perceive at once that one integral is $\log.(1+3x^2)$. That is, the hyperbolic logarithm of $(1+3x^2)$ for the given differential, is plainly *the differential of the quantity divided by the quantity.*

We have got the *transcendental* integral, and now if we would obtain the algebraic or numerical integral, we must expand $(1+ax^2)^{-1}$, which is

$$1-ax^2+a^2x^4-a^3x^6+a^4x^8-a^5x^{10}, \text{ \&c.}$$

Multiplying each term by $2axdx$, and integrating, we have

$$\log.(1+ax^2)=ax^2-\frac{a^2x^4}{2}+\frac{a^3x^6}{3}-\frac{a^4x^8}{4}\text{ \&c. }+C.$$

which is the same as equation (1), example 3d, if we put y in place of ax^2.

(Art. 72.) All differentials in the form

$$x^{m-1}dx(a+bx^n)^{\frac{p}{q}}$$

can be integrated, term by term, after expanding the binomial and multiplying each term by the part without the parenthesis.

But the series of integrals thus obtained, may not converge, for convergency will depend on x being less or greater than unity, and also on the signs of m and n; hence the sum of the integrals, or the entire integral sought, may not be sufficiently near the truth to answer our purpose when obtained by such a process. Yet, when particular cases are given, algebraic artifices in the hands of a skilful operator, are equal to almost any emergency, and it is to such artifices we shall call the attention of the reader in some future chapter.

CHAPTER IV.

Integration of Rational Fractions.

(Art. 73.) A rational fraction, numerically considered, is one which is less than unity, algebraically considered. It may be written in the form

$$\frac{Px^3+Qx^2+Rx+S}{P'x^4+Q'x^3+R'x^2+S'x+T'},$$

the highest power of the variable *is greater by unity* in the denominator than in the numerator. If it were not, we would divide the numerator by the denominator, and thus obtain an integer term, and from the remainder and divisor, we would *then form our rational fraction.*

Such fractions can be separated into a series of *partial* fractions, whose denominators are binomials, provided the denominator is capable of being separated into binomial factors.

To separate a compound denominator into its simple factors, place the quantity equal to 0, and find the roots of the equation.

Let a denominator be $x^m+Px^{m-1}+Qx^{m-2}\ldots Gx+F$; place it equal to 0, and let the m roots of the equation be represented by a, b, c, &c., then by the theory of equations, the denominator will be the product of $(x-a)$, $(x-b)$, $(x-c)$, &c. to m factors.

These factors may be *real* or *imaginary, equal* or *unequal.* We shall commence with the most simple case, in which the factors of the denominator are real and unequal.

Let the denominator of a rational fraction consist of the three factors $(x+a)$, $(x+b)$, $(x+c)$; then the fraction will be equal to the sum of the three partial fractions,

$$\frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{x+c},$$

the numerators A, B, and C, are as yet undetermined constants.

EXAMPLES.

1. *Suppose it were required to integrate the rational fraction*

$$\frac{(2x^2-3)dx}{x^3-4x}.$$

The denominator is obviously the product of the factors x, $(x+2)$, $(x-2)$, therefore we may place the fraction equal to the three partial fractions, as follows:

$$\frac{(2x^2-3)dx}{x^3-4x}=\frac{Adx}{x}+\frac{Bdx}{x+2}+\frac{Cdx}{x-2}. \qquad (1)$$

The integral required will be equal to the sum of the integrals of the three partial fractions.

We can integrate the partial fractions after we determine the values of A, B, and C, and these values are determined in the following manner:

Divide (1) by dx, then we shall have

$$\frac{2x^2-3}{x^3-4x}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}.$$

an algebraic equation, and nothing more.

Reducing the second member to a common denominator, and

$$\frac{2x^2-3}{x^3-4x}=\frac{(x^2-4)A+(x^2-2x)B+(x^2+2x)C}{x^3-4x}$$

Omitting the common denominator, and transposing all to the first member, we have

$$(2-A-B-C)x^2+2(B-C)x+(4A-3)=0. \qquad (2)$$

As x represents a variable quantity, we are at liberty to make it equal zero. Or equation (2) will furnish the three equations

$$(2-A-B-C)x^2=0, \quad 2(B-C)x=0, \quad 4A-3=0.$$

In short, by the theory of indeterminate coefficients, we have $A=\frac{3}{4}$, $B=C$, and $2-A-B-C=0$. Whence $B=\frac{5}{8}$, and $C=\frac{5}{8}$. These values put in (1), and indicating the integration, we have

$$\int\left(\frac{2x^2-3}{x^3-4x}\right)dx=\tfrac{3}{4}\int\frac{dx}{x}+\tfrac{5}{8}\int\frac{dx}{x+2}+\tfrac{5}{8}\int\frac{dx}{x-2}+C.$$
$$=\tfrac{3}{4}\log.x+\tfrac{5}{8}[\log.(x+2)+\log.(x-2)]+C.$$
$$=\tfrac{3}{4}\log.x+\tfrac{5}{8}[\log.(x^2-4)]+C.$$

2. *Integrate the rational fraction* $\left(\frac{3x-5}{x^2-6x+8}\right)dx.$

Ans. $\frac{7}{2}\log.(x-4)-\frac{1}{2}\log.(x-2)+C.$

Place $x^2-6x+8=0$. Whence $x=2$, or 4.

Therefore put $\frac{3x-5}{x^2-6x+8}=\frac{A}{x-2}+\frac{B}{x-4}$, &c. &c.

(Art. 74.) The following example presents a case in which the denominator of the given fraction contains sets of equal factors.

3. *Integrate* $\frac{xdx}{(x-1)^2(x-2)^2}$.

Place $\frac{x}{(x-1)^2(x-2)^2}=\frac{A}{(x-1)^2}+\frac{A'}{x-1}+\frac{B}{(x-2)^2}+\frac{B'}{x-2}$. (1)

Clearing of denominators, and equating the coefficients of the like powers of x, we have

$$4A-4A'+B-2B'=0.$$
$$-4A+8A'-2B+5B'=1.$$
$$A-5A'+B-4B'=0.$$
$$A'+B'=0.$$

From these equations we obtain $A=1$, $A'=3$, $B=2$, and $B'=-3$. Whence

$$\int\frac{xdx}{(x-1)^2(x-2)^2}=\int\frac{dx}{(x-1)^2}+3\int\frac{dx}{x-1}+2\int\frac{dx}{(x-2)^2}-$$
$$3\int\frac{dx}{x-2}=-\frac{1}{x-1}+3\log.(x-1)-\frac{2}{x-2}-3\log.(x-2)+C.$$

We integrate the *first* and *third* of these partial fractions by (Art. 61.)

(Art. 75.) If the denominator of a rational fraction contain imaginary roots, it must contain a factor in the form

$$x^2+2ax+a^2+b^2.$$

since this expression placed equal to zero, will give two imaginary values to x, and since we know from the theory of algebra that imaginary roots necessarily exist in pairs, if there be m pairs of equal imaginary roots, there must be a factor in the denominator, in the form

$$(x^2+2ax+a^2+b^2)^m.$$

A rational fraction, as shown in the last article, can be separated into several partial fractions, and to the simple factor

$$x^2+2ax+a^2+b^2,$$

there will be a corresponding partial fraction

$$\frac{Mx+N}{(x+a)^2+b^2}dx,$$

which we propose to integrate.

Put $x+a=z$, then $dx=dz$. And put $N-aM=P$. Then the fraction becomes $\frac{Mz+P}{z^2+b^2}dz$, which is obviously the sum of two fractions.

Whence $$\int\frac{Mz+P}{z^2+b^2}dz=\int\frac{Mzdz}{z^2+b^2}+\int\frac{Pdz}{z^2+b^2}. \qquad (1)$$

The first term of the second member may be integrated thus:

$$\frac{M}{2}\int\frac{2zdz}{z^2+b^2}=\frac{M}{2}\log.(z^2+b^2). \qquad \text{(Art. 64.)}$$

The second term is integrated by (Art. 70.)

Whence $$\int\frac{Pdz}{z^2+b^2}=\frac{P}{b}\text{arc}\left(\tan.=\frac{z}{b}\right).$$

These values put in (1), give

$$\int\left(\frac{Mz+P}{z^2+b^2}\right)dz=\frac{M}{2}\log.(\overline{x+a}^2+b^2)+\frac{P}{b}\text{arc}\left(\tan.=\frac{x+a}{b}\right)$$

resuming the value of z in the second member.

It is proper to observe that an arc whose tangent is $\frac{x+b}{b}$, the sine of the same arc is $\frac{x+a}{\sqrt{(x+a)^2+b^2}}$, and the cosine is $\frac{b}{\sqrt{(x+a)^2+b^2}}$. These expressions afford the means of presenting the proposed integral under different forms, designating the arc by its sine and cosine, in place of its tangent.

EXAMPLES.

1. *Integrate* $\frac{(x^2-x+1)dx}{x^3+x^2+x-1}$.

The denominator is the product of the factors $(1+x)$ and $(1+x^2)$, therefore place

$$\frac{x^2-x+1}{(1+x)(1+x^2)}=\frac{A}{1+x}+\frac{Mx+P}{1+x^2}.$$

Clearing of denominators, &c. we find

$A=\frac{3}{2}$, $M=-\frac{1}{2}$, and $P=-\frac{1}{2}$. Whence

$$\int\frac{(x^2-x+1)dx}{(1+x)(1+x^2)}=\int\frac{3dx}{2(1+x)}-\int\frac{xdx}{2(1+x^2)}-\int\frac{dx}{2(1+x^2)}.$$

$$=\tfrac{3}{2}\log.(1+x)-\tfrac{1}{4}\log.(1+x^2)-\tfrac{1}{2}\tan.^{-1}x.$$

$$=\log.(1+x)^{\frac{3}{2}}-\log.(1+x^2)^{\frac{1}{4}}-\tfrac{1}{2}\tan.^{-1}x.$$

2. *Integrate* $\frac{(2x+3)dx}{x^3-x^2-2x}$.

Ans. $\frac{7}{6}\log.(x-2)+\frac{1}{3}\log.(x+1)-\frac{3}{2}\log.x.$

3. *Integrate* $\left(\frac{x^3-1}{x^2-4}\right)dx.$

Ans. $\frac{x^2}{2}+\frac{9}{4}\log.(x+2)+\frac{7}{4}\log.(x-2)+C.$

4. *Integrate* $\frac{(x-1)dx}{x(x^2+x+2)}$.

Ans. $-\frac{1}{2}\log.x+\frac{1}{2}\log.\sqrt{x^2+x+2}+\frac{5}{2\sqrt{7}}\tan.^{-1}\left(\frac{x+\frac{1}{2}}{\frac{1}{2}\sqrt{7}}\right)+C.$

By a review of the integration of rational fractions, we shall perceive that the partial fractions to be integrated will fall under some one or more of the following forms:

$$\int\frac{dx}{x+a},\quad \int x^m dx,\quad \int\frac{xdx}{(x^2+a^2)^m},\quad \int\frac{dx}{x^2+a^2}.$$

(Art. 76.) We will now investigate a formula for integrating differentials in the form $\frac{dx}{(x^2+a^2)^m}$.

In the first place we will assume the equation

$$\int\frac{dx}{(x^2+a^2)^m}=\frac{Kx}{(x^2+a^2)^{m-1}}+L\int\frac{dx}{(x^2+a^2)^{m-1}},\quad (1)$$

in which K and L are indeterminate coefficients, and (1) will become a practical formula, provided we can determine K and L in terms of a and m.

To test this we must differentiate equation (1), divide by dx, and clear the result of fractions, we shall then have

$$1=K(x^2+a^2)-2K(m-1)x^2+L(x^2+a^2).\quad (2)$$

This equation must be true for all values of x, it is true then when $x=0$, and this supposition gives

$$(K+L)a^2=1.\quad (3)$$

Equation (3) taken from (2), and the remainder divided by x^2, will produce

$$3K+L-2Km=0.\quad (4)$$

From (3) and (4) we obtain

$$K=\frac{1}{2(m-1)a^2},\text{ and } L=\frac{2m-3}{2(m-1)a^2}.$$

These values of K and L placed in (1), give $\int\frac{dx}{(x^2+a^2)^m}=$

$$\frac{x}{2(m-1)a^2(x^2+a^2)^{m-1}}+\frac{2m-3}{2(m-1)a^2}\int\frac{dx}{(x^2+a^2)^{m-1}}$$

for the formula required.

EXAMPLES.

1. *Integrate* $\frac{dx}{(x^2+1)^3}$.

Substituting 1 for a, and 3 for m, the first application is as follows:

$$\int\frac{dx}{(x^2+1)^3}=\frac{x}{4(x^2+1)^2}+\tfrac{3}{4}\int\frac{dx}{(x^2+1)^2}. \qquad (1)$$

For a second application of the same formula, we write 1 for a, and 2 for m, then

$$\int\frac{dx}{(x^2+1)^2}=\frac{x}{2(x^2+1)}+\tfrac{1}{2}\int\frac{dx}{(x^2+1)}. \qquad (2)$$

But $$\int\frac{dx}{x^2+1}=\tan.^{-1}x. \qquad \text{(Art. 68.)} \qquad (3)$$

The result of (3) placed in (2), then that result placed in (1), and we have the final result as follows:

$$\int\frac{dx}{(x^2+1)^3}=\frac{x}{4(x^2+1)^2}+\frac{3x}{8(x^2+1)}+\tfrac{3}{8}\tan.^{-1}x+C.$$

2. *Integrate* $\dfrac{dx}{(x^2+6)^4}$.

$$\int\frac{dx}{(x^2+6)^4}=\frac{x}{36(x^2+6)^3}+\tfrac{5}{36}\int\frac{dx}{(x^2+6)^3}.$$

$$\int\frac{dx}{(x^2+6)^3}=\frac{x}{24(x^2+6)^2}+\tfrac{3}{24}\int\frac{dx}{(x^2+6)^2}.$$

$$\int\frac{dx}{(x^2+6)^2}=\frac{x}{12(x^2+6)}+\tfrac{1}{12}\int\frac{dx}{x^2+6}$$

$$\frac{dx}{x^2+6}=\frac{1}{\sqrt{6}}\tan.^{-1}\frac{x}{\sqrt{6}}.$$

Whence $$\int\frac{dx}{(x^2+6)^4}=\frac{x}{36(x^2+6)^3}+\frac{5x}{24.36(x^2+6)^2}+$$

$$\frac{5x}{36.96(x^2+6)}+\frac{5}{36.96\sqrt{6}}\tan.^{-1}\frac{x}{6}+C.$$

CHAPTER V.

Integration by Parts.

(Art. 77.) In the differential calculus we have found that the differential of a product, as (uv), gives the equation

$$d(uv)=udv+vdu. \quad \text{Whence} \quad uv=\int udv+\int vdu.$$

Therefore
$$\int udv=uv-\int vdu. \qquad (1)$$

From this we perceive that the integral of udv can be found whenever we are able to integrate vdu. This method of integrating udv is called *integration by parts.*

The utility of this method of integration principally consists in its application to binomial differentials in the form

$$x^{m-1}dx(a+bx^n)^p$$

which are not integrable by direct methods.

Many differentials in this form have already been integrated, but they were particular cases of this general form.

In the following general investigation we may regard m and n and p, fractional or negative.

In case p is a whole positive number, the binomial can be expanded, and each term can be integrated as before shown. When m and n are fractional, as they may be in particular examples, as

$$x^{\frac{1}{3}}dx(a+bx^{\frac{1}{2}})^p$$

place $x=z^6$, z being a new variable with an exponent equal to the *product of the two denominators.*

Whence
$$x^{\frac{1}{3}}dx(a+bx^{\frac{1}{2}})^p=6z^7dz(a+bz^3)^p$$

Hence, every binomial differential can be placed under the form

$$x^{m-1}dx(a+bx^n)^p$$

Place $x^{m-1}dx=dv,$ and $(a+bx^n)^p=u.$

Then $\dfrac{x^m}{m}=v,$ and $du=bpnx^{n-1}dx(a+bx^n)^{p-1}$

These values of u, du, v, and dv, substitued in (1) give

$$\int x^{m-1}dx(a+bx^n)^p = \frac{x^m(a+bx^n)^p}{m} - \frac{pnb}{m}\int x^{m+n-1}\,dx(a+bx^n)^{p-1} \quad (2)$$

Observe the identical equation

$$(a+bx^n)^p = (a+bx^n)^{p-1}(a+bx^n).$$

Multiply the second member as indicated, and then we shall have

$$(a+bx^n)^p = a(a+bx^n)^{p-1} + bx^n(a+bx^n)^{p-1}.$$

Multiply each term by $x^{m-1}dx$, and write the sign of integration, and it will stand thus:

$$\int x^{m-1}dx(a+bx^n)^p = a\int x^{m-1}dx(a+bx^n)^{p-1} + b\int x^{m+n-1}dx(a+bx^n)^{p-1}. \quad (3)$$

If we multiply (3) by $\frac{pn}{m}$, and add the product to (2), the last term will be eliminated, and after a little reduction, the result will be

Formula *A*.

$$\int x^{m-1}dx(X)^p = \frac{x^m(X)^p}{m+pn} + \frac{pna}{m+pn}\int x^{m-1}dx(X)^{p-1},$$

in which X represents the binomial $(a+bx^n)$.

If we multiply formula A by $(m+pn)$, change signs, transpose the first and last terms, and then divide by pna, we shall have

Formula *B*.

$$\int x^{m-1}dx(X)^{p-1} = -\frac{x^m(X)^p}{pna} + \frac{m+pn}{pna}\int x^{m-1}dx(X)^p.$$

Again, observing that the first members of (2) and (3) are identical, therefore the second members are equal. That is,

$$a\int x^{m-1}dx(X)^{p-1} + b\int x^{m+n-1}dx(X)^{p-1} = \frac{x^m(X)^p}{m} - \frac{pnb}{m}\int x^{m+n-1}dx(X)^{p-1}.$$

Transposing and reducing, we obtain

FORMULA *C*.

$$\int x^{m+n-1}dx(X)^{p-1}=\frac{x^m(X)^p}{b(m+pn)}-\frac{am}{b(m+pn)}\int x^{m-1}dx(X)^{p-1}.$$

Now if we transpose the first and third terms of this formula, and reduce the first member to unity, we shall have

FORMULA *D*.

$$\int x^{m-1}dx(X)^{p-1}=\frac{x^m(X)^p}{am}-\frac{b(m+pn)}{am}\int x^{m+n-1}dx(X)^{p-1}.$$

The formulas (*A*), (*B*), (*C*), (*D*), will apply to any possible binomial differential that can be presented.

When the exponent p is positive, and we wish to diminish it, we must use Formula *A*.

When p is negative, and we wish to increase it, that is, *diminish it numerically*, we must use Formula *B*.

When the exponent $(m-1)$ is positive, and we wish to diminish it, we use Formula *C*.

When that exponent is negative, and we wish to diminish it numerically, we use Formula *D*.

The formula in (Art. 76) is substantially the same as formula *B*.

(Art. 78.) It frequently happens that we are required to integrate binomial differentials in the form

$$\frac{x^{m+1}dx}{\sqrt{a^2-x^2}},$$

and for that purpose we can use formula *C*, and we now adjust that formula to this general case.

For this purpose we must write in formula *C*

a^2 for a, -1 for b, $-\frac{1}{2}$ for $(p-1)$.

$\frac{1}{2}$ for p, and 2 for n,

then formula *C* will become

FORMULA *c*.

$$\int\frac{x^{m+1}dx}{\sqrt{a^2-x^2}}=\frac{-x^m\sqrt{a^2-x^2}}{m+1}+\frac{a^2m}{m+1}\int\frac{x^{m-1}dx}{\sqrt{a^2-x^2}}.$$

This will apply when $(m+1)$ is positive, but when that exponent is negative, we require the converse of this formula, which we find by transposing the first and last terms, changing signs and dividing by $\frac{a^2m}{m+1}$.

This will give

FORMULA d.

$$\int\frac{x^{m-1}dx}{\sqrt{a^2-x^2}}=\frac{x^m\sqrt{a^2-x^2}}{a^2m}-\frac{m+1}{a^2m}\int\frac{x^{m+1}dx}{\sqrt{a^2-x^2}}.$$

We name this, formula d, because it can be drawn from the formula D, the same as c was drawn from C.

This formula must be applied when $(m-1)$ is negative.

A formula corresponding to the particular form

$$\frac{x^{m+1}dx}{\sqrt{a^2+x^2}},$$

can be deduced from C, by substituting in that formula

a^2 for a, 1 for b, $-\frac{1}{2}$ for $(p-1)$, $\frac{1}{2}$ for p, 2 for n, and we shall have

FORMULA C'.

$$\int\frac{x^{m+1}dx}{\sqrt{a^2+x^2}}=\frac{x^m\sqrt{a^2+x^2}}{m+1}-\frac{a^2m}{m+1}\int\frac{x^{m-1}dx}{\sqrt{a^2+x^2}}.$$

The converse of this is Formula d'.

$$\int\frac{x^{m-1}dx}{\sqrt{a^2+x^2}}=\frac{x^m\sqrt{a^2+x^2}}{a^2m}-\frac{m+1}{a^2m}\int\frac{x^{m+1}dx}{\sqrt{a^2+x^2}}.$$

(Art. 79.) It is desirable to have a formula applicable to the binomial differential, in the form

$$\frac{x^q dx}{\sqrt{2ax-x^2}},\quad\text{or}\quad x^{q-\frac{1}{2}}dx(2a-x)^{-\frac{1}{2}}.$$

To integrate this by formula C, we must place

$m+n-1=q-\frac{1}{2}$, $n=1$, $b=-1$, $p-1=-\frac{1}{2}$, or $p=\frac{1}{2}$.

Therefore $np=\frac{1}{2}$, and $m=q-\frac{1}{2}$, and for a in the formula we must write $2a$.

These substitutions will change formula C, into

$$\int \frac{x^{q-\frac{1}{2}}dx}{\sqrt{2a-x}} = \frac{-x^{q-\frac{1}{2}}\,d\,\sqrt{2a-x}}{q} + \frac{2aq-a}{q}\int \frac{x^{q-\frac{3}{2}}dx}{\sqrt{2a-x}}.$$

Observe that $x^{q-\frac{1}{2}} = \frac{x^q}{\sqrt{x}}$, and $x^{q-1}x^{\frac{1}{2}} = x^{q-\frac{1}{2}}$, and $\frac{x^{q-1}}{\sqrt{x}} = x^{q-\frac{3}{2}}$.

Therefore we can pass $x^{\frac{1}{2}}$ under the binomial radical in each term, and

$$\int \frac{x^q dx}{\sqrt{2ax-x^2}} = \frac{-x^{q-1}\,d\sqrt{2ax-x^2}}{q} + \frac{a(2q-1)}{q}\int \frac{x^{q-1}dx}{\sqrt{2ax-x^2}}.$$

To preserve uniformity of notation as much as possible, we will now write m in place of q, and we have

FORMULA d.

$$\int \frac{x^m dx}{\sqrt{2ax-x^2}} = -\frac{x^{m-1}\sqrt{2ax-x^2}}{m} + \frac{a(2m-1)}{m}\int \frac{x^{m-1}\,dx}{\sqrt{2ax-x^2}}.$$

This formula is to be used when m is positive. The converse of this is to be used when m is negative. To find the converse transpose the first and last terms, &c. and we have

FORMULA d'.

$$\int \frac{x^{m-1}dx}{\sqrt{2ax-x^2}} = \frac{x^{m-1}\sqrt{2ax-x^2}}{a(2m-1)} + \frac{m}{a(2m-1}\int \frac{x^m dx}{\sqrt{2ax-x^2}}.$$

Formulas d and d' diminish the numerical values of the exponent without the parenthesis, by unity.

When m is a whole positive number, the final differential in formula d will be of the form

$$\int \frac{dx}{\sqrt{2ax-x^2}} = \text{ver. sin.}^{-1}\frac{x}{a} + C. \qquad \text{(Art. 70.)}$$

As we have before observed, the formulas A, B, C, and D, are general, and some one of them will apply to any binomial differential that can be presented — but in consequence of the frequency of examples in which *the sign of the square root* appears over the binomial factor, it is expedient to adopt special formulas, as c, c', d, d', to meet such cases.

We now give a few practical examples, which, together with the formulas, will sufficiently illustrate the whole subject.

EXAMPLES.

1. *Integrate* $\frac{x^5\,dx}{\sqrt{1-x^2}}$. (Apply formula *c.*)

In the first operation $m+1=5$, and $a=1$. In the second $m+1=3$, and so on.

$$\int\frac{x^5\,dx}{\sqrt{1-x^2}}=-\tfrac{1}{5}x^4\sqrt{1-x^2}+\tfrac{4}{5}\int\frac{x^3\,dx}{\sqrt{1-x^2}}. \qquad (1)$$

$$\int\frac{x^3\,dx}{\sqrt{1-x^2}}=-\tfrac{1}{3}x^2\sqrt{1-x^2}+\tfrac{2}{3}\int\frac{x\,dx}{\sqrt{1-x^2}}. \qquad (2)$$

$$\int\frac{x\,dx}{\sqrt{1-x^2}}=-\sqrt{1-x^2}. \qquad (3)$$

To obtain the integral demanded, we must now take backward steps. That is, place the result obtained from (3) in (2), and then place that result in (1); and lastly add the arbitrary constant C, and we have

$$\int\frac{x^5\,dx}{\sqrt{1-x^2}}=-\left(\tfrac{1}{5}x^4+\tfrac{1}{3}\cdot\tfrac{4}{5}x^2+\frac{1.2.4}{1.3.5}\right)\sqrt{1-x^2}+C,$$

the integral sought.

2. *Integrate* $\frac{x^4\,dx}{\sqrt{1-x^2}}$. (Formula *c.*)

$$\int\frac{x^4\,dx}{\sqrt{1-x^2}}=-\frac{x^3\sqrt{1-x^2}}{4}+\tfrac{3}{4}\int\frac{x^2\,dx}{\sqrt{1-x^2}}. \qquad (1)$$

$$\int\frac{x^2\,dx}{\sqrt{1-x^2}}=-\frac{x\sqrt{1-x^2}}{2}+\tfrac{1}{2}\int\frac{dx}{\sqrt{1-x^2}}. \qquad (2)$$

$$\int\frac{dx}{\sqrt{1-x^2}}=\sin.^{-1}x. \quad \text{(Art. 67.)} \qquad (3)$$

Now the results obtained from (3) and (2) placed in (1) give

$$\int\frac{x^4\,dx}{\sqrt{1-x^2}}=-\left(\tfrac{1}{4}x^3+\frac{1.3}{2.4}x\right)\sqrt{1-x^2}+\frac{1}{2}\cdot\frac{3}{4}\sin.^{-1}+C.$$

N. B. When $(m+1)$ is *odd*, as it is in the first example, the final integral will be dependent on the integration of $\frac{x\,dx}{\sqrt{1-x^2}}$, or on $-\sqrt{1-x^2}$.

When $(m+1)$ is *even*, as it is in the second example, the final integral will be $\sin.^{-1}x$.

Hence, if $(m+1)$ be a whole number, whether odd or even, the complete integration is possible.

3. *Integrate* $\frac{dx}{x^3\sqrt{1-x^2}}$ (Formula *d.*)

In the first operation $m-1=-3$, $m=-2$.

In the second operation $m-1=-1$, $m=0$.

$$\int\frac{dx}{x^3\sqrt{1-x^2}}=-\frac{\sqrt{1-x^2}}{2x^2}+\tfrac{1}{2}\int\frac{dx}{x\sqrt{1-x^2}}. \qquad (1)$$

$$\int\frac{dx}{x\sqrt{1-x^2}}=\frac{\sqrt{1-x^2}}{0}-\frac{1}{0}\int\frac{x\,dx}{\sqrt{1-x^2}}. \quad \text{(Formula fails.)}$$

Here we perceive that the formula fails in the second operation, because $m=0$. Therefore we must find some other method of integrating

$$\frac{dx}{x\sqrt{1-x^2}}.$$

By an example in the differential calculus, (page 167), we learn that the differential of

$$*\log.\left(\frac{1+\sqrt{1-x^2}}{x}\right) \quad \text{is} \quad -\frac{dx}{x\sqrt{1-x^2}}.$$

Whence $$\int\frac{dx}{x\sqrt{1-x^2}}=-\log.\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad (2)$$

* This example was designed for page 167, but was omitted by mistake. We shall now place it among the miscellaneous examples.

And this value placed in (1) produces

$$\int \frac{dx}{x^3\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{2x^2} - \tfrac{1}{2}\log.\left(\frac{1+\sqrt{1-x^2}}{x}\right) + C.$$

(Art. 80.) The method of finding the integral of

$$\int \frac{dx}{x\sqrt{1-x^2}},$$

by mere reference to the differential calculus, is not satisfactory to a learner. It is therefore desirable to obtain the integral directly, as in other cases.

To this end assume $\sqrt{1-x^2} = z, \qquad xdx = -zdz.$

Whence $\int \frac{dx}{x\sqrt{1-x^2}} = \int -\frac{dz}{1-z^2} = \int \frac{dz}{z^2-1} = \frac{Adz}{z-1} + \frac{Bdz}{z+1}:$

$$A = -\tfrac{1}{2}. \qquad B = \tfrac{1}{2}.$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\tfrac{1}{2}\int \frac{dz}{z-1} + \tfrac{1}{2}\int \frac{dz}{z+1}.$$

$$= -\tfrac{1}{2}\log.(z-1) + \tfrac{1}{2}\log.(z+1).$$

$$= \tfrac{1}{2}\log.\frac{z+1}{z-1}$$

N. B. The product of two factors is the same when the signs of both factors are changed. Thus $+P$ multiplied into $-Q$ produces $-PQ$. Also, $-P$ into $+Q$, is $-PQ$.

Therefore we may change the signs of each factor in the second member of the equation above. Then we have

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\tfrac{1}{2}\log.\frac{1+z}{1-z} = -\tfrac{1}{2}\log.\frac{1+z}{1-z}.\frac{1+z}{1+z}.$$

$$= -\tfrac{1}{2}\log.\frac{\overline{1+z}^2}{1-z^2}$$

$$= -\tfrac{1}{2}\log.\frac{\overline{1+z}^2}{x^2}.$$

$$= -\log.\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

4. *Integrate* $\frac{x^2\,dx}{\sqrt{2ax-x^2}}$. (Formula *d.*)

Here $m=2$ in the first operation, and unity in the second operation.

$$\int\frac{x^2\,dx}{\sqrt{2ax-x^2}}=-\frac{x\sqrt{2ax-x^2}}{2}+\frac{3a}{2}\int\frac{xdx}{\sqrt{2ax-x^2}}. \quad (1)$$

$$\int\frac{xdx}{\sqrt{2ax-x^2}}=-\sqrt{2ax-x^2}+a\int\frac{dx}{\sqrt{2ax-x^2}}. \quad (2)$$

$$\int\frac{dx}{\sqrt{2ax-x^2}}=\text{ver. sin.}^{-1}\frac{x}{a}. \quad \text{(Art. 70.)} \quad (3)$$

Whence, by substitution, we have

$$\int\frac{x^2\,dx}{\sqrt{2ax-x^2}}=-\left(\frac{x}{2}+\frac{3a}{2}\right)\sqrt{2ax-x^2}+\frac{3a^2}{2}\text{ver. sin.}^{-1}\frac{x}{a}+C.$$

5. *Integrate* $\frac{x^{-3}\,dx}{a+bx}$. (Formula *D.*)

In which $p=0$, $m-1=-3$, $m=-2$, $n=1$, in the first operation.

Ans. $C-\frac{1}{2ax^2}+\frac{b}{a^2x}-\frac{b^2}{a^3}\log.\left(\frac{a+bx}{x}\right)$

6. *Integrate* $\frac{x^3\,dx}{\sqrt{a+bx}}$. (Formula *C′.*)

$m+1=2$, $a=a^2$, $b=1$.

$$\int\frac{x^3\,dx}{\sqrt{a+bx}}=\frac{x^2\sqrt{a+bx}}{3}-\frac{2a}{3}\int\frac{xdx}{\sqrt{a+bx}}. \quad (1)$$

$$\int\frac{xdx}{\sqrt{a+bx}}=\frac{a\sqrt{a+bx}}{b^2}-\frac{a^2\log.\sqrt{a+bx}}{b^2}. \quad \text{(Art. 62.)} \quad (2)$$

Place $a+bx=z$, and integrate by an independent process. Whence by substitution we shall have

$$\int\frac{x^3\,dx}{\sqrt{a+bx}}=\frac{x^2\sqrt{a+bx}}{b^2}-\frac{2a^2}{3b^2}\sqrt{a+bx}+\frac{2a^3}{3b^2}\log.\sqrt{a+bx}+C.$$

7. *Integrate* $\frac{x^3dx}{a+bx}$. (Formula *C.*)

$$\int\frac{x^3dx}{a+bx}=\frac{x^3}{3b}-\frac{ax^2}{2b^2}+\frac{a^2x}{b^3}-\frac{2a}{b^3}\log.(a+bx)+C.$$

8. *Integrate* $\frac{ax}{x^2(a+bx)^{\frac{7}{2}}}$. (Formula *D.*)

One operation gives

$$\int\frac{dx}{x^2(a+bx)^{\frac{7}{2}}}=-\frac{1}{ax(a+bx)^{\frac{5}{2}}}-\frac{7b}{2a}\int\frac{dx}{x(a+bx)^{\frac{7}{2}}}.$$

9. *Integrate* $dx\sqrt{a^2+x^2}$. (Formula *A.*)

Here $m-1=0$, $p=\frac{1}{2}$, $n=2$, $a=a^2$.

$$\int dx\sqrt{a^2+x^2}=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\int\frac{dx}{\sqrt{a^2+x^2}}.$$

In the differential calculus we are taught by an example, (page 167,) that the differential of $\log.(x+\sqrt{a^2+x^2})$ is

$$\frac{dx}{\sqrt{a^2+x^2}}.$$

Conversely then $\int\frac{dx}{\sqrt{a^2+x^2}}=\log.(x+\sqrt{a^2+x^2})$.

Therefore $\int dx\sqrt{a^2+x^2}=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\log.(x+\sqrt{a^2+x^2})$.

10. *Integrate* $dx\sqrt{x^2-a^2}$.

Ans. $\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log.(x+\sqrt{x^2-a^2})$.

(Art. 81.) The last two problems require us to integrate differentials in the form $\frac{dx}{\sqrt{x^2\pm a^2}}$, independently of the formulas in (Art. 77), and to infer the integral, as we have just done is not satisfactory, therefore we operate as follows:

The square root of $x^2\pm a^2$ obviously must contain $\pm x$, and some other quantity which we can represent by z. Therefore it is natural to place

$$\sqrt{a^2+x^2}=z-x. \qquad (1)$$

$$a^2=-2xz+z^2.$$

Whence $$(z-x)dz=zdx.$$

Or $$\frac{dz}{z}=\frac{dx}{z-x}=\frac{dx}{\sqrt{a^2+x^2}}.$$

Hence $$\int\frac{dx}{a^2+x^2}=\log.z=\log.(x+\sqrt{a^2+x^2}).$$

10. *Integrate* $\frac{nx^{n-1}dx}{(1+x)^{n+1}}$ (Formula B.)

N. B. This example, as well as several others, will be found in the differential calculus, in the first part of this volume.

Here, $m-1=n-1$, or $m=n$, but n in the formula referred to this example is 1, and $a=1$, $p-1=-n-1$, or $p=-m$.

$$\int x^{n-1}dx(1+x)^{-n-1}=\frac{x^n(1+x)^{-n}}{n}-\frac{(n-n)}{n}\int\frac{x^{n-1}dx}{(1+x)^n}.$$

But the last term of this equation is zero, because $(n-n)$ is zero, whence

$$n\int\frac{x^{n-1}dx}{(1+x)^{n+1}}=\frac{x^n}{(1+x)^n}, \text{ the integral sought.}$$

11. *Integrate* $-\frac{dx(1+\sqrt{1-x^2})}{x^2\sqrt{1-x^2}}$

This can be separated into two parts.

Thus $$\frac{-dx}{x^2\sqrt{1-x^2}}-\frac{dx}{x^2}.$$

The integral of the first part is $\frac{\sqrt{1-x^2}}{x}$, and of the second it is $\frac{1}{x^2}$, whence the whole integral is $\frac{1+\sqrt{1-x^2}}{x}$, which is equal to $\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$. The differential of this last quantity was demanded in the differential calculus.

CHAPTER VI.

Integration of Irrational Fractions.

In the last chapter it was found that differentials in the form $\frac{x^{m-1}dx}{\sqrt{a+bx^2}}$ cannot be integrated by parts, unless we can integrate the differential fraction in the form $\frac{dx}{\sqrt{a+bx^2}}$, which may be an irrational fraction.

(Art. 82.) The object of this chapter is to develop the general theory of integrating differentials in the form

$$\frac{dx}{\sqrt{A+Bx+Cx^2}}, \text{ and in the form } \frac{dx}{\sqrt{A+Bx-Cx^2}}$$

Our first object is to find equivalent expressions in which x^2 shall stand *without a coefficient,* and with the plus sign. In other words, the coefficient of x^2 must be $+1$. In the first case it is obvious that

$$\frac{dx}{\sqrt{A+Bx+Cx^2}}=\frac{dx}{\sqrt{C}\sqrt{\frac{A}{B}+\frac{B}{C}x+x^2}}=\frac{dx}{\sqrt{C}\sqrt{a+bx+x^2}}. \qquad (1)$$

If $\frac{A}{C}=a$, $\frac{B}{C}=b$. Or if $A=aC$, $B=bC$.

In the second case

$$\frac{dx}{\sqrt{A+Bx-Cx^2}}=\frac{dx}{\sqrt{-C}\sqrt{\frac{A}{-C}+\frac{B}{-C}x+x^2}}=\frac{dx}{\sqrt{-C}\sqrt{a+bx+x^2}} \qquad (2)$$

If $-\frac{A}{C}=a$, $-\frac{B}{C}=b$. Or if $A=-aC$, $B=-bC$.

By inspecting (1) and (2) we perceive that the integral in each case will depend on the integration of

$$\frac{dx}{\sqrt{a+bx+x^2}},$$

which integral must be multiplied by $\frac{1}{\sqrt{C}}$, for examples in the first case, and by $\frac{1}{\sqrt{-C}}$, for examples in the second case.

Hence our exclusive attention will be directed to the integration of

$$\frac{dx}{\sqrt{a+bx+x^2}},$$

the result of which we shall multiply by $\frac{1}{\sqrt{\pm C}}$ for a general formula.

Place $$\sqrt{a+bx+x^2}=z-x.\text{*} \qquad (1)$$

Squaring and reducing in part, we have

$$a+bx=z^2-2zx.$$

Whence $$(b+2z)dx=2(z-x)dz. \qquad (2)$$

Dividing (2) by (1), and

$$\frac{(b+2z)dx}{\sqrt{a+bx+x^2}}=2dz. \qquad (3)$$

Therefore

$$\int\frac{dx}{\sqrt{C}\sqrt{a+bx+x^2}}=\int\frac{2dz}{\sqrt{C}(b+2z)}=\frac{2}{\sqrt{C}}\int\frac{dz}{b+2z}. \qquad (4)$$

Again, place $b+2z=t$. Then $dz=\frac{dt}{2}$.

And $$\int\frac{dz}{b+2z}=\tfrac{1}{2}\int\frac{dt}{t}=\tfrac{1}{2}\log.t=\tfrac{1}{2}\log.(b+2z).$$

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* It is more natural to place the radical equal to $z+x$, but as both $-x$ and $+x$ will give x^2, we can take either, and the minus sign will give a more convenient result than the plus sign would do. But in numerical examples we may take either one.

But in (1) we find $z=x+\sqrt{a+bx+x^2}$.

Whence $\int\frac{dz}{b+2z}=\frac{1}{2}\log.(b+2x+2\sqrt{a+bx+x^2}.)$

Finally,

$$\int\frac{dx}{\sqrt{C}\cdot\sqrt{a+bx+x^2}}=\frac{1}{\sqrt{\pm C}}\log.(b+2x+2\sqrt{a+bx+x^2})+\text{ const.} \qquad (E)$$

EXAMPLES.

1. *Integrate* $\frac{dx}{\sqrt{1+x^2}}$. (Formula *E.*)

Here $C=1$, $B=0$, and $A=1$. Therefore $a=1$, $b=0$.

Whence $\int\frac{dx}{\sqrt{1+x^2}}=\log.(2x+2\sqrt{1+x^2})+c.$

But $\log.(2x+2\sqrt{1+x^2})=\log.(x+\sqrt{1+x^2})+\log.2$, and log. 2 may be united to c, and become part of the arbitrary constant.

Therefore $\int\frac{dx}{\sqrt{1+x^2}}=\log.(x+\sqrt{1+x^2})+c.$

N. B. In some of the following examples the results of the formulas may be reduced by expunging the factor (log.2) and conceiving it to be added to, and to become a part of the arbitrary constant.

2. $\int\frac{dx}{\sqrt{x^2-1}}=\log.(x+\sqrt{x^2-1})+c.$

3. $\int\frac{dx}{\sqrt{x+x^2}}=\log.(1+2x+2\sqrt{x+x^2})+c.$

4. $\int\frac{dx}{\sqrt{1+x+x^2}}=\log,(1+2x+2\sqrt{1+x+x^2})+c.$

5. $\frac{dx}{\sqrt{1-x^2}}=\frac{1}{\sqrt{-C}}\log.(x+\sqrt{x^2-1})+\log.2+c.$

N. B. Formulas in other works give

$$\frac{1}{\sqrt{-1}}\log.(x\sqrt{-1}+\sqrt{1-x^2})$$

for the integral of the second example ; both are correct ; indeed they are equal, as one can be reduced to the other as follows:

$$\log.\left(\frac{x+\sqrt{x^2-1}}{1}\right) \text{ is equal to } \log.(x+\sqrt{x^2-1}).$$

If we multiply numerator and denominator by $\sqrt{-1}$, we shall have

$$\log.\frac{(x+\sqrt{x^2-1})}{1}=\log.\left(\frac{x\sqrt{-1}+\sqrt{1-x^2}}{\sqrt{-1}}\right)=$$

$$\log.(x\sqrt{-1}+\sqrt{1-x^2})-\log.\sqrt{-1}.$$

Whence

$$\frac{1}{\sqrt{-1}}\log.(x\sqrt{x^2-1})+c=\frac{1}{\sqrt{-1}}\log.(x\sqrt{-1}+\sqrt{1-x^2})+c'.$$

The last expression is applicable to circular arcs, as will soon appear.

6. $$\int\frac{dx}{\sqrt{a+bx^2}}=\frac{1}{\sqrt{b}}\log.\left(x+\sqrt{\frac{a}{b}+x^2}\right)+c.$$

But $$\frac{1}{\sqrt{b}}\log.(x\sqrt{b}+\sqrt{a+bx^2})=\frac{1}{\sqrt{b}}\log.\left(+\sqrt{\frac{a}{b}+x^2}\right)+\frac{1}{\sqrt{b}}\log.b.$$

and either of these expressions differentiated, will produce the given differential.

(Art. 83.) In treating of circular arcs in the differential calculus, we have found that when the radius of a circle is unity, and the sine of the arc is x, the cosine of the same arc must be $\sqrt{1-x^2}$, and the differential of sine is

$$\frac{dx}{\sqrt{1-x^2}},$$

and this was the quantity to be integrated in example 5. Therefore another integral of that quantity is the arc of the circle corresponding to the sine x *plus*, an *arbitrary constant.*

Let z be that arc, then $x=\sin.\ z$, $\sqrt{1-x^2}=\cos.z$.

And $$\int \frac{dx}{\sqrt{1-x^2}}=z+C. \qquad (1)$$

But by example 5, we have

$$\int \frac{dx}{\sqrt{1-x^2}}=\frac{1}{\sqrt{-1}}\log.(x\sqrt{-1}+\sqrt{1-x^2})+c. \qquad (2)$$

Whence $$z+C=\frac{1}{\sqrt{-1}}\log.(x\sqrt{-1}+\sqrt{1-x^2})+c'.$$

To determine the relation between the constants, we will conceive the arc and its sign to commence at the same point and increase together, and suppose $x=0$, then will $z=0$, and the last equation will become $C=c'$, that is, the constants will be equal to each other, and therefore they may be omitted and the equation itself will become

$$z=\frac{1}{\sqrt{-1}}\log.(x\sqrt{-1}+\sqrt{1-x^2}).$$

Substituting the values of x and of $\sqrt{1-x^2}$ in this last equation, and we have

$$\log,(\sin.z\sqrt{-1}+\cos.z)=\sqrt{-1}.z.$$

Multiply each member by $\log.e$, observing that e is the base of the hyperbolic logarithms, and its log. is 1, and 1 as a factor *may be made visible or invisible.* We will make it visible in the second member, then

$$\log.(\sin.z\sqrt{-1}+\cos.z)=\sqrt{-1}.z\log.e.$$
$$=\log.(e^{\sqrt{-1}.z}).$$

We can now omit the sign (log.) in each member, which is in fact passing to the numbers ; then we shall have

$$\sin.z\sqrt{-1}+\cos.z=e^{\sqrt{-1}.z}. \qquad (1)$$

If we take z negative, we shall have

$$\sin.(-z)=-\sin.z, \qquad \cos.(-z)=\cos.z.$$

And the final result will be

$$-\sin.z\sqrt{-1}+\cos.z.=e^{-\sqrt{-1}.z}. \qquad (2)$$

By adding (1) and (2), and dividing by 2, we have

$$\cos.z=\frac{e^{\sqrt{-1}.z}+e^{-\sqrt{-1}.z}}{2}. \qquad (3)$$

Subtracting (2) from (1), and dividing, we obtain

$$\sin.z=\frac{e^{\sqrt{-1}.z}-e^{-\sqrt{-1}.z}}{2\sqrt{-1}}. \qquad (4)$$

By substituting nz for z in (1), we have

$$\sin.nz\sqrt{-1}+\cos.nz=e^{\sqrt{-1}.nz}. \qquad (5)$$

Going back to (1) and raising each member to the nth power, we have

$$(\sin.z\sqrt{-1}+\cos.z)^n=e^{\sqrt{-1}.nz}. \qquad (6)$$

The second members of (5) and (6) are identical, therefore

$$(\sin.z\sqrt{-1}+\cos.z)^n=\sin.nz\sqrt{-1}+\cos.nz. \qquad (7)$$

These expressions are purely algebraic symbols, expressing the relations between the arc and its sine and cosine, which, by proper artifices can be developed in numerical quantities.

Equation (7) is the same as appears in Robinson's Geometry, page 223, and its practical importance and utility is there shown.

(Art. 84.) In (Art. 71) the differential $\frac{dx}{\sqrt{1-x^2}}$ is integrated, and the result is a numerical series. But the integral found is a logarithmic expression. The two integrals deduced from the same differential must be equal to each other. That is,

$$\frac{1}{\sqrt{-1}}\log.(x+\sqrt{x^2-1})+C=x+\frac{x^3}{2.3}+\frac{3x^5}{2.4.5}+C'.$$

This is true for all values of x. Then by supposing $x=0$, we find $C=C'$, the two arbitrary constants equal to each other.

Hence, $$\frac{1}{\sqrt{-1}}\log.(x+\sqrt{x^2-1})=x+\frac{x^3}{2.3}+\frac{3x^5}{2.4.5}+\&c.$$

But this is an impractical equation on account of the presence of the imaginary factor.

Examples 2, 3, and 4, (Art. 82,) can be expanded into series and integrated term by term.

Then we can have the numerical values of $\log.(x+\sqrt{x^2-1})$, and of $\log.(1+2x+2\sqrt{x+x^2})$, and of $\log.(1+2x+2\sqrt{1+x+x^2})$, but it is not important to obtain them, because we have already found a simple and general logarithmic series in Chapter III, (Art. 71.)

(Art. 85.) We can find another integral to the differential $\frac{dx}{\sqrt{a+bx-x^2}}$ by another method of integration, as follows:

If we place $a+bx-x^2=0$, and resolve the quadratic, we shall find two real roots. Let them be represented by r and r'.

Then $$x^2-bx-a=(x-r)(x-r')$$

Changing signs

$$a+bx-x^2=-(x-r)(x-r')=(x-r)(r'-x).$$

This being understood, we can assume

$$\sqrt{a+bx-x^2}=\sqrt{(x-r)(r'-x)}=(x-r)t. \qquad (1)$$

By squaring, and afterwards dividing by $(x-r)$, we obtain

$$r'-x=(x-r')t^2. \qquad (2)$$

Taking the differential, and

$$-dx=t^2dx+2tdt(x-r).$$

Whence $$dx=-\frac{2dt.t(x-r)}{1+t^2}. \qquad (3)$$

Dividing (3) by (1) will give

$$\frac{dx}{\sqrt{a+bx-x^2}}=-\frac{2dt}{1+t^2}.$$

Whence $$\int\frac{dx}{\sqrt{a+bx-x^2}}=C-2\tan.^{-1}(t). \quad \text{(Art. 68.)}$$

$$=C-2\tan.^{-1}\sqrt{\frac{r'-x}{x-r}},$$

the value of t taken from (2).

CHAPTER VII.

Integration of Exponential Differentials.

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(Art. 86.) A differential in the form $Xa^x dx$ can easily be integrated, provided X be an algebraic function of x, and in such a form that successive differentials will terminate in a constant.

To establish a formula to integrate $Xa^x dx$, let us call to mind the well known equation

$$\int PdQ = PQ - \int QdP. \qquad (1)$$

Now let $\quad P = X, \quad$ and $\quad dQ = a^x dx. \qquad (2)$

To integrate (2) we will put $y = a^x$, whence $\log.y = x\log.a$, and $dy = \log.a.a^x dx$, or $\int a^x dx = \frac{y}{\log.a} = \frac{a^x}{\log.a}$

That is, $\quad Q = \frac{a^x}{\log.a}. \qquad (3)$

Again, assume

$$dP = dX = X'dx, \quad dX' = X''dx, \quad dX'' = X'''dx, \text{ \&c.} \qquad (4)$$

Here we perceive that X', X'', X''', &c. are the successive differential coefficients of X.

The values of P, Q, dP, dQ, taken from (2), (3), and (4), and substituted in (1), give

$$\int Xa^x dx = \frac{Xa^x}{\log.a} - \frac{1}{\log.a}\int X'a^x dx. \qquad (5)$$

Again $\quad \int X'a^x dx = \frac{X'a^x}{\log.a} - \frac{1}{\log.a}\int X''a^x dx. \qquad (6)$

And $\quad \int X''a^x dx = \frac{X''a^x}{\log.a} - \frac{1}{\log.a}\int X'''a^x dx. \qquad (7)$

&c. &c. &c.

If we substitute the values found in (7), in equation (6), and then that result in (5), equation (5) will become

$$\int Xa^x dx = \frac{Xa^x}{\log.a} - \frac{X'a^x}{(\log.a)^2} + \frac{X''a^x}{(\log.a)^3} - \frac{1}{(\log,a)^3}\int X'''a^x dx. \ (8)$$

If X'' is a constant quantity, $dX''=X'''dx=0$, and the series terminates with the third term, and in general the series will terminate with the term in which the last differential coefficient becomes constant.

If a becomes e, the base of the Naperian system, then log.a becomes log.$e=1$, and the formula preceding becomes

$$\int Xe^x dx=e^x(X-X'+X''-X'''+ \text{\&c. \&c.}) \qquad (9)$$

EXAMPLES.

1. *Integrate* $e^x x^3 dx$. *Ans.* $e^x(x^3-3x^2+6x-6)$.

In this example $X=x^3$. Hence $X'=3x^2$, $X''=6x$, $X'''=6$, and $X''''=0$.

2. *Integrate* $e^x x^2 dx$. *Ans.* $e^x(x^2-2x+2)$.

3. *Integrate* $e^x(x^2-\frac{3}{2})^2 dx$.
Ans. $e^x(x^4-4x^3+9x^2-18x+20\frac{1}{4}.)$

4. *Integrate* $e^x(x^3+3x^2-1)dx$. *Ans.* $e^x(x^3-1)$.

Remark.—We can extract the cube root of any number which is a perfect cube, with comparative ease, but when the root is a surd, we can only approximate to it by a series. So it is with a differential. When an exact integral exists, we can find it with comparative ease, but when no exact integral does exist, the approximate integral can be obtained only by a series.

In the Mathematical Operations, page 321, we required the differential of $\frac{e^x x}{1-x}$, and found it to be $\left(\frac{1+x-x^2}{(1-x)^2}\right)e^x dx$. Consequently the integral of this last expression is $\frac{e^x x}{1-x}$, and it is probable we can extract it from the differential as a particular case—but we could not be sure of integrating any other example of a similar form.

In this example $X=\frac{1+x-x^2}{(1-x)^2}$, hence X', X'', &c. do not converge toward a constant, and therefore it will be useless to apply the last formula.

The solution of such examples will depend much on the skill of the operator, guided by general principles.

$$\left(\frac{1+x-x^2}{(1-x)^2}\right)e^x\,dx=\frac{e^x\,dx}{(1-x)^2}+\frac{xe^x\,dx}{1-x}.$$

Whence $\int\frac{e^x dx}{(1-x)^2}+\int\frac{xe^x\,dx}{1-x}=$the required result. (1)

Let $P=\frac{x}{1-x}$, and $dQ=e^x dx$. Whence $Q=e^x$.

Substitute these values in equation (1), (Art. 86,) and we have

$$\int\frac{xe^x dx}{1-x}=\frac{e^x}{1-x}-\int\frac{e^x dx}{(1-x)^2}$$

Transpose the last term, and

$$\int\frac{e^x dx}{(1-x)^2}+\int\frac{xe^x dx}{1-x}=\frac{e^x}{1-x}. \qquad (2)$$

The first members of (1) and (2) are identical, therefore

The required result $=\frac{e^x}{1-x}$.

In the same manner integrate the following differential:

$$e^x dx\left(\frac{1}{x}+\log.x\right). \qquad \textit{Ans. } e^x\log.x.$$

(Art. 87.) When $X=\frac{1}{x^n}$, the successive differential coefficients of X will not approach a constant, and consequently formula (9) in such cases will be of no practical value, and we must return to first principles, and seek the integration of

$$\frac{a^x dx}{x^n}.$$

We will apply the principle of integrating by parts according to the fundamental formula,

$$\int PdQ=PQ-\int QdP. \qquad (1)$$

Here $P=\frac{1}{x^n}$, and $dQ=a^x dx$. Whence $Q=\frac{a^x}{\log.a}$.

Substituting the values of P, Q, dP, and dQ, in (1), we obtain

$$\int\frac{a^x dx}{x^n}=\frac{a^x}{x^n\log.a}+\frac{n}{\log.a}\int\frac{a^x dx}{x^{n+1}}.$$

Transposing the first and third terms, dividing by $\frac{n}{\log.a}$ and changing signs, we shall have

$$\int\frac{a^x dx}{x^{n+1}}=-\frac{a^x}{n.x^n}+\frac{\log.a}{n}\int\frac{a^x dx}{x^n}.$$

Now if we write n for $n+1$, we must write $n-1$ for n, then the preceding formula will become

$$\int\frac{a^x dx}{x^n}=-\frac{a^x}{(n-1)x^{n-1}}+\frac{\log.a}{n-1}\int\frac{a^x dx}{x^{n-1}}, \qquad (2)$$

a formula which produces a continual diminution of the exponent n. When n becomes 1, the formula fails, for then the factor $(n-1)$ in the second member, becomes 0.

Hence the differential $\frac{a^x dx}{x}$ must be integrated approximately by a series, no finite integral corresponding to it has been found, for the very probable reason that none exists.

By Maclaurin's theorem we expanded a^x (Art. 18,) into the series $1+\frac{cx}{1}+\frac{c^2x^2}{1.2}+\frac{c^3x^3}{1.2.3}$, &c. Multiplying each term by $\frac{dx}{x}$ and integrating, we shall have a converging series *when* x and c are each less than 1, or when the product cx is less than 1.

When the exponent n is a fraction, it will also be necessary to complete, or rather approximate to the integral by a series.

The following examples will illustrate and show the method of integrating exponential and logarithmic functions more clearly than anything else, for no general formulas can meet every case and condition.

EXAMPLES.

1. *Integrate* $\frac{a^x dx}{\sqrt{1+a^{nx}}}$.

Place $z=a^x$. Then $dx=\frac{dz}{z\log.a}$, and $z^n=a^{nx}$.

Whence $$\int\frac{a^x dx}{\sqrt{1+a^{nx}}}=\frac{1}{\log.a}\int\frac{dz}{\sqrt{1+z^n}},$$
which is easily expanded into a series, and then we can integrate term by term.

2. *Integrate* $x^m(\log.x)^n dx$.

Place $z=\log.x$, $x^m dx=dP$, and $P=(\log.x)^n=z^n$, and integrating by parts, we have
$$\int x^m(\log.x)^n dx=\frac{x^{m+1}(\log.x)^n}{m+1}-\frac{n}{m+1}\int x^m(\log.x)^{n-1}dx.$$

Substituting for n successively $x-1$, $n-2$, &c. we shall find
$$\int x^m(\log.x)^n dx=$$
$$\frac{x^{m+1}}{m+1}\left\{(\log.x)^n-\frac{n}{(m+1)}(\log.x)^{n-1}+\frac{n(n-1)}{(m+1)^2}(\log.x)^{n-2}-\&c\right\}$$

This series will terminate whenever n is a whole positive number.

This series fails when $m=-1$, *for then* $m+1=0$, *which would make the factor* $\frac{x^{m+1}}{m+1}=\frac{x^0}{0}$, *or infinite.*

But when $m=-1$ the differential becomes
$$\frac{(\log.x)^n dx}{x}.$$
and this is very easily integrated, for $\frac{dx}{x}$ is the differential of the $\log.x$. Therefore

Place $z=\log.x$. Then $dz=\frac{dx}{x}$, and the differential becomes $z^n dz$. Whence $\int z^n dz=\frac{z^{n+1}}{n+1}=\frac{(\log.x)^{n+1}}{n+1}$.

This is subject to the exception $n=-1$.

3. *Integrate* $\log.x\,dx$, *by parts.* *Ans.* $x(\log.x-1)$.

4. *Integrate* $\frac{dx}{x\log.^2x}$. *Ans.* $-\frac{1}{\log.x}$.

5. *Integrate* $\frac{dx}{(1-x)^2}\log.x.$ *Integral* $\frac{x\log.x}{1-x}+\log.(1-x)$

6. *Integrate* $\frac{dx}{x\sqrt{x}}\log.\left(\frac{1}{1-x}\right).$

Integral $2\log.\frac{1+\sqrt{x}}{1-\sqrt{x}}-\frac{2}{\sqrt{x}}\log.\left(\frac{1}{1-x}\right).$

Place $dQ=\frac{dx}{x\sqrt{x}}$, and $P=\log.\left(\frac{1}{1-x}\right).$

Integrating by parts will give us

$$-\frac{2}{\sqrt{x}}\log.\frac{1}{1-x}+2\int\frac{dx}{\sqrt{x}(1-x)}.$$

(Art. 88.) The method of integrating by parts produces

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This series is remarkable for its similarity to the series of Taylor, and it applies to the integration of quantities in the form Xdx, in which X is any function of x. The process is as follows:

$$\int Xdx=Xx-\int xdX. \qquad (1) \qquad \text{(By parts, Art. 86.)}$$

But $xdX=\frac{dX}{dx}xdx$. Integrating this last expression by parts, conceiving $\frac{dX}{dx}=P$, and $xdx=dQ$, we shall have

$$\int xdX=\int\frac{dX}{dx}xdx=\frac{dX}{dx}\cdot\frac{x^2}{2}-\int\frac{x^2}{2}\,\frac{d^2X}{dx}. \qquad (2)$$

Again,

$$\int\frac{x^2}{2}\cdot\frac{d^2X}{dx}=\int\frac{d^2X}{dx^2}\cdot\frac{x^2dx}{2}=\frac{x^3}{2.3}\,\frac{d^2X}{dx^2}-\int\frac{x^3}{2.3}\,\frac{d^3X}{dx^2}, \qquad (3)$$

Substituting in succession these values in (1), that equation will become

$$\int Xdx=Xx-\frac{dX}{dx}\cdot\frac{x^2}{1.2}+\frac{d^2X}{dx^2}\cdot\frac{x^3}{1.2.3}-\text{\&c.}+C,$$

the series in question.

To illustrate this series, let it be used to integrate $x^3 dx$, although in practice it never should be applied to such examples, because the integration is too simple to require it.

Hence $X = x^3$. Whence $\frac{dX}{dx} = 3x^2$. $\frac{d^2X}{dx^2} = 2.3.x$, and $\frac{d^3x}{dX^3} = 2.3$.

Therefore by the series

$$\int x^3 dx = x^3 x - \frac{3.x^2.x^2}{1.2} + \frac{2.3x.x^3}{1.2.3} - \frac{2.3.x^4}{1.2.3.4} + C.$$

The sum of this series is $\left(\frac{x^4}{4} + C\right)$, the true integral by the common method of integration.

The utility of this formula will be apparent in the following example, which is new to us, and it shows the beauty of analysis as clearly as any thing we ever met.

In the differential calculus, (Art. 18,) we find the following expressions:

$$d.\sin.x = \cos.x\, dx. \qquad (1)$$

$$d.\cos.x = -\sin.x\, dx. \qquad (2)$$

Whence $\sin.x = \int \cos.x\, dx + C$.

To apply the series of John Bernoulli, we must make $X = \cos.x$. Then, by successive differentiation, we have

$$\frac{dX}{dx} = -\sin.x, \quad \frac{d^2X}{dx^2} = -\cos.x, \quad \frac{d^3X}{dx^3} = \sin.x, \quad \frac{d^4X}{dx^4} = \cos.x, \quad \&c.$$

$$\sin.x = \int \cos.x\, dx = \cos.x.x + \frac{\sin.x.x^2}{1.2} - \frac{\cos.x.x^3}{1.2.3} - \frac{\sin.x.x^4}{1.2.3.4} + \frac{\cos.x.x^5}{1.2.3.4.5} + \frac{\sin.x.x^6}{1.2.3.4.5.6}, \ \&c.\ \&c.$$

It is not necessary to add the constant C, for if we make $x=0$, $\sin.x$ will equal 0, and each term of the second member will equal 0 at the same time, hence $C=0$.

Now divide every term of the last equation by cos.x, and for $\frac{\sin.x}{\cos.x}$ write its equal, tan.x, and we shall have

$$\tan.x=x+\tan.x.\frac{x^2}{1.2}-\frac{x^3}{1.2.3}-\tan.x\frac{x^4}{1.2.3.4}+\frac{x}{1.2.3.4.5}+$$
$$\tan.x\frac{x^6}{1.2.3.4.5.6}-\text{ \&c.}$$

Uniting the coefficients of the tan.x, and we have

$$\left(1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\text{ \&c.}\right)\tan.x$$
$$=\left(x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\frac{x^7}{1.2.3.4.5.6.7},\text{ \&c.}\right)$$

$$\text{Or }\tan.x=\frac{x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\frac{x^7}{1.2.3.4.5.6.7}+\text{ \&c.}}{1-\frac{x^2}{1.2}+\frac{x^2}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\text{ \&c.}}\quad(A)$$

But $$\tan.x=\frac{\sin.x}{\cos.x}.\quad(B)$$

Equations (A) and (B) are but different forms of expression for the tangent x, and as the fractions are irreducible, we may conclude at once that

$$\sin.x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\frac{x^7}{1.2.3.4.5.6.7}+\text{ \&c.}$$

$$\text{And }\cos.x=1-\frac{x^3}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\text{ \&c.}$$

These useful and beautiful formulas were found in our appendix to trigonometry, but the process there is much more complex than this one. They were also found in the differential calculus, (Art. 18.)

(Art. 89.) We can use this series to integrate a logarithmic differential.

For example, the differential of log.$(a+x)$ is $\frac{mdx}{a+x}$. Therefore $\log.(a+x)=\int\frac{mdx}{a+x}$, but this second member can be integrated by the series of Bernoulli. $\int\frac{mdx}{a+x}$.

Here $X=\frac{1}{a+x}$. Whence $\frac{dX}{dx}=-\frac{1}{(a+x)^2}$.

$$\frac{d^2X}{dx^2}=\frac{2}{(a+x)^3},\quad \frac{d^3X}{dx^3}=-\frac{2.3}{(a+x)^4},\quad \frac{d^4X}{dx^4}=\frac{2.3.4}{(a+x)^5},\ \&c.$$

$$\log.(a+x)=\int\frac{mdx}{a+x}=\frac{mx}{a+x}+\frac{mx^2}{2(a+x)^2}+\frac{mx^3}{3(a+x)^3}+\frac{mx^4}{4(a+x)^4}\ \&c.+C.$$

This is true for all values of x, it is true then when $x=0$, and making this supposition, we have $\log.a=C$. Whence

$$\log.(a+x)=\log.a+m\left(\frac{x}{a+x}+\frac{x^2}{2(a+x)^2}+\frac{x^3}{3(a+x)^3}+\frac{x^4}{4(a+x)^4}\ \&c.\right)\quad (1)$$

If we assume $m=1$, this equation will correspond to the *Naperian system* of logarithms, and if we assume $x=1$, the equation will become a very simple and practical formula for computing logarithms in that system.

It will then be the following:

$$\log.(a+1)=\log.a+\left(\frac{1}{a+1}+\frac{1}{2(a+1)^2}+\frac{1}{3(a+1)^3}+\frac{1}{4(a+1)^4}+\frac{1}{5(a+1)^5}\ \&c.\right)\quad (2)$$

In practice we may apply either (1) or (2), as we please; (1) has more scope than (2), because x can be any number, whole or fractional. By (2) we can find the log. of $(a+1)$ when the logarithm of a is known.

Either of these formulas can be used for computing common logarithms when m becomes known.

The value of m is discovered for the common system by comparing the log. of 10 in each system. (See Algebra, p. 241.)

To make a table of logarithms corresponding to any system, we are compelled to commence with the Naperian system. We must continue in that system until we obtain the Naperian log. of 10. Then we can find m, and then we can pass to the common system.

We have explained this whole subject several times before — but its great utility and beautiful philosophy is a sufficient excuse for a repetition in connection with this new formula.

This new series does not converge as rapidly as some others, but it is more symmetrical, and was obtained by fewer steps than any other.

To find the Naperian logarithm of 2, we must make $a=1$ in equation (2), then $\log.a=0$, and $\frac{1}{a+1}=\frac{1}{2}$. Hence we may write the series

$$\frac{1}{2}+\frac{1}{2(2)^2}+\frac{1}{3(2)^3}+\frac{1}{4(2)^4}+\frac{1}{5(2)^5}, \text{ \&c. \&c.}$$

Now if we take $\frac{1}{2}$, or .5, and divide it by 2 continuously, we shall have

$$\frac{1}{2}+\frac{1}{(2)^2}+\frac{1}{(2)^3}+\frac{1}{(2)^4}, \text{ \&c.}$$

The first term of this series divided by 1, the second by 2, the third by 3, &c. &c., and the sum of these will be the logarithm sought. The work would stand thus :

1).5	.5000000
2).25	.1250000
3).125	.0416666
4).0625	.0156250
5).03125	.0062500
6).015625	.0026041
7).0078125	.0011161
8).00390625	.0004882
9).001953125	.0002170
10).0009765625	.0000976
&c.	&c.
	.6930636

This operation continued and extended to a greater number of decimals, would give the true hyperbolic log. of 2. The foregoing is only designed to show the practical form of making the computation.

If we multiply the log. of 2 by 3, we shall have the log. of 8 at once. Then make $a=8$, and again apply the formula, and we shall find the log. of 9, which divided by 2 will give the log. of 3, &c. &c.

(Art. 90.) If the object were simply to obtain the best practical formula for computing logarithms, we would integrate the differential $\frac{adx}{a^2-x^2}$ by two different methods. First, by rational fractions, (Art. 73); second, by expanding $\frac{a}{a^2-x^2}$ into a series by division, and multiplying each term by dx, and integrating. Then we should have

$$\int\frac{adx}{a^2-x^2}=\tfrac{1}{2}\log.(a+x)-\tfrac{1}{2}\log.(a-x)=\tfrac{1}{2}\log.\left(\frac{a+x}{a-x}\right)+C.$$

$$\int\frac{adx}{a^2-x^2}=\frac{x}{a}+\frac{x^3}{3a^3}+\frac{x^5}{5a^5}+\frac{x^7}{7a^7}+\&c.+C'.$$

Whence $\log.\left(\frac{a+x}{a-x}\right)=2\left(\frac{x}{a}+\frac{x^3}{3a^3}+\frac{x^5}{5a^5}+\frac{x^7}{7a^7}+\&c.\right)+$ (1)

Equating the second members, and making $x=0$, there will result $C=C'$, and no constant appears in (1).

If we make $\frac{a+x}{a-x}=1+\frac{1}{z}$, then will $\frac{x}{a}=\frac{1}{2z+1}$, and (1) will become

$$\log.\left(1+\frac{1}{z}\right)=\log.(z+1)-\log.z=$$

$$2\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\&c.\right)\qquad(2)$$

the most approved formula yet found.

If we make $a=2$, and $x=1$, in formula (1) we shall have the Naperian log. of 3 at once.

The operation is as follows:

	.5	.500000000
3)	.125	.041666666
5)	.03125	625
7)	78125	1116071
9)	1953125	217013
11)	048828125	44390
13)	1220703125	9390
15)	30517578125	2033
17)	7629394	439
		.549306002
		2
		1.098612004

We commenced the operation with .5, divided this by 4, producing the next line below .125. This we again divided by 4, thus finding the next line below, and so on. These sums we again divided by 1, 3, 5, 7, and so on, producing the next column. The result is true as far as six places of decimals — the seventh decimal should be 3 in place of 0.

The double of this logarithm will be the hyperbolic log. of 9. Then we might make $z=9$, and formula (2) would give the hyperbolic log. of 10.

CHAPTER VIII.

The Integration of Circular Differentials of Multiple Arcs.

(Art. 91.) The object of this chapter is to investigate and show the method of integrating differentials in the form

$$\sin.^2 x\,dx, \quad \sin.^3 x\,dx, \quad \text{or in general,} \quad \sin.^n x\,dx.$$

For a clear comprehension of this, we must re-examine the method of taking the differentials of functions in the form

$$\sin.3x, \quad \text{or} \quad \sin.nx.$$

Let $z=nx$. Then $dz=ndx$, $\sin.z=\sin.nx$, $d.\sin.z=\cos.z\,dz$.

That is, $d.\sin.nx=n\cos.nx\,dx$, and $d.\cos.nx=-n\sin.nx\,dx$.

Whence $\int\cos.nx\,dx=\frac{\sin.nx}{n}$, and $\int\sin.nx\,dx=-\frac{\cos.nx}{n}$. (A)

These primitive formulas will serve as our general rules of integration in this chapter.

As a preliminary step, we require formulas for $\sin.^2x$, $\sin.^3x$, $\sin.^4x$, or in short, $\sin.^n x$ expressed in a series of the simple dimensions of the sines or cosines of multiples of that arc.

Let y and x be any two arcs, then by trigonometry we have

$$\sin.(y+x)=\sin.y\cos.x+\cos.y\sin.x.$$

$$\sin.(y-x)=\sin.y\cos.x-\cos.y\sin.x.$$

By adding these two equations and transposing $\sin.(y-x)$ we obtain

$$\sin.(y+x)=2\sin.y\cos.x-\sin.(y-x). \qquad (1)$$

Now suppose $y=nx$. Then (1) becomes

$$\sin.(n+1)x=2\sin.nx\cos.x-\sin.(n-1)x. \qquad (2)$$

Making $n=1, 2, 3, 4, 5$, &c. in succession, we form the following table:

$$\sin.2x=2\sin.x\cos.x-0. \qquad (3)$$

$$\sin.3x=3\sin.x-4\sin.^3x. \qquad (4)$$

$$\sin.4x=(4\sin.x-8\sin.^3x)\cos.x. \qquad (5)$$

$$\sin.5x=5\sin.x-20\sin.^3x+16\sin.^5x. \qquad (6)$$

&c. &c.

Again, if we take the trigonometrical equations

$$\cos.(y+x)=\cos.y\cos.x-\sin.y\sin.x.$$

$$\cos.(y-x)=\cos.y\cos.x+\sin.y\sin.x.$$

Add them and transpose as before, we shall have

$$\cos.(y+x)=2\cos.y\cos.x-\cos.(y-x). \qquad (7)$$

And as before suppose $y=nx$, (7) will become

$$\cos.(n+1)x=2\cos.nx\cos.x-\cos.(n-1)x.$$

Making $n=1, 2, 3, 4$, &c. in succession, we shall find

$$\cos.2x=2\cos.^2x-1. \qquad (8)$$

$$\cos.3x=4\cos.^3x-3\cos.x. \qquad (9)$$

$$\cos.4x=8\cos.^4x-8\cos.^2x+1. \qquad (10)$$

$$\cos.5x=16\cos.^5x-20\cos.^3x+5\cos.x. \qquad (11)$$

&c. &c.

By the aid of the well known equation, $\sin.^2x+\cos.^2x=1$, combined with equations (3), (4), &c. to (11), as circumstances may require, we obtain

$$\sin.x=\sin.x. \qquad (12)$$

(8) substituted, $\sin.^2x=1-\cos.^2x=\frac{1}{2}(1-\cos.2x.)$ (13)

(4) transposed, $\sin.^3x=\frac{1}{4}(-\sin.3x+3\sin.x.)$ (14)

$(13)^2$, (10) & (8) sub. $\sin.^4x=\frac{1}{8}(\cos.4x-4\cos.2x+3.)$ (15)

(6) reduced by (4), $\sin.^5x=\frac{1}{16}(\sin.5x-5\sin.3x+10\sin.x.)$ (16)

If we multiply equations (12), (13), (14), (15), and (16), by dx, and integrate, we shall have the following series of equations:

$$\int\sin.x\,dx=-\cos.x. \qquad (17)$$

$$\int\sin.^2x\,dx=-\tfrac{1}{4}\sin.2x+\tfrac{1}{2}x. \qquad (18)$$

$$\int\sin.^3x\,dx=\tfrac{1}{4}\cdot\frac{\cos.3x}{3}-\tfrac{3}{4}\cos.x. \qquad (19)$$

$$\int\sin.^4x\,dx=\tfrac{1}{8}\cdot\frac{\sin.4x}{4}-\tfrac{1}{4}\sin.2x+\tfrac{3}{8}x. \qquad (20)$$

$$\int\sin.^5x\,dx=-\tfrac{1}{16}\cdot\frac{\cos.5x}{5}+\tfrac{5}{16}\cdot\frac{\cos.3x}{3}-\tfrac{10}{16}\cos.x. \qquad (21)$$

&c. &c.

(Art. 92.) If we turn back to equations (8), (9), (10), &c. in the last article, we can find a series of equations expressing the powers of the cosine of x, as follows:

$$\cos.x=\cos.x. \qquad (22)$$

$$2\cos.^2x=\cos.2x+1. \qquad (23)$$

$$4\cos.^3x=\cos.3x+3\cos.x. \qquad (24)$$

$$8\cos.^4x=\cos.4x+4\cos.2x+3. \qquad (25)$$

&c. &c.

Multiplying by dx and integrating, we have

$$\int \cos. x\, dx = \sin. x. \qquad (26)$$

$$\int \cos.^2 x\, dx = \tfrac{1}{4}\sin. 2x + \tfrac{1}{2}x. \qquad (27)$$

$$\int \cos.^3 x\, dx = \tfrac{1}{12}\sin. 3x + \tfrac{3}{4}\sin. x. \qquad (28)$$

$$\int \cos.^4 x\, dx = \tfrac{1}{32}\sin. 4x + \tfrac{1}{4}\sin. 2x + \tfrac{3}{8}x. \qquad (29)$$

&c. &c.

(Art. 93.) We may also integrate circular functions in the following manner. We give but one example, which is intended as a general illustration.

Integrate $\cos.^3 x\, dx$. Place $\cos. x = z$.

Then $dx = -\dfrac{dz}{\sin. x} = -\dfrac{dz}{\sqrt{1-z^2}}$. But $\cos.^3 x = z^3$.

Whence $\int \cos.^3 x\, dx = -\int \dfrac{z^3 dz}{1-z^2}$, and the integral of this last expression is to be found in the second and third equations of the first example in (Art. 79), which is

$$\tfrac{1}{3}z^2\sqrt{1-z^2} + \tfrac{2}{3}\sqrt{1-z^2}.$$

Replacing $\cos. x$ for z, and we have

$$\int \cos.^3 x\, dx = \tfrac{1}{3}\cos.^2 x\sqrt{1-\cos.^2 x} + \tfrac{2}{3}\sqrt{1-\cos.^2 x}.$$

The integral of this same quantity is found in (28) of the last article. Therefore

$$\tfrac{1}{3}\cos.^2 x\sqrt{1-\cos.^2 x} + \tfrac{2}{3}\sqrt{1-\cos.^2 x} = \tfrac{1}{12}\sin. 3x + \tfrac{3}{4}\sin. x,$$

but this result reveals nothing new.

In a similar manner we might integrate many of the differentials in the preceding article, and thus find many other equations between sines and cosines.

(Art. 94.) We may integrate differentials in the form $\sin. x \cos.^3 x\, dx$ by the same general principles.

Assume $\cos. x = z$. Then, as in (Art. 93),

$$\cos.^3 x\, dx = -\frac{z^3 dz}{\sqrt{1-z^2}}.$$

But $\sin.x=\sqrt{1-z^2}$; therefore $\sin.x\cos.^3x\,dx=-z^3dz$.

And $$\int\sin.x.\cos.^3x\,dx=-\frac{z^4}{4}+C.$$

In general terms $$\int\sin.x\cos.^n dx=-\frac{z^{n+1}}{n+1}=-\frac{\cos.x^{n+1}}{n+1}+C.$$

The same general principle will enable us to integrate a differential in the form

$$\frac{dx}{\sin.^3x}.$$

Place $\sin.x=z$. Then $dx=\frac{dz}{\cos.x}$, and $\frac{dx}{\sin.^3x}=\frac{dz}{\cos.x.z^3}$.

Whence $$\int\frac{dx}{\sin.^3x}=\int\frac{dz}{z^3\sqrt{1-z^2}}.$$

This last differential has already been integrated by (Formula *d*), example 3, (Art. 79.)

CHAPTER IX.

Successive Integrations.

(Art. 95.) In the differential calculus we perceive that every equation of the first degree between two variables is susceptible of being differentiated but once. An equation of the second degree can be differentiated twice. An equation of the third degree three times, and so on.

For instance, if $y=ax^3+bx^2+cx+d$. (1)

We shall have by successive differention

$$\frac{dy}{dx}=3ax^2+2bx+c. \quad (2)$$

$$\frac{d^2y}{dx^2}=6ax+2b. \quad (3)$$

$$\frac{d^3y}{dx^3}=6a \quad (4)$$

Another differentiation and division by dx, would give

$$\frac{d^4y}{dx^4}=0.$$

Hence, if n be the degree of an equation, the $(1+n)$ differential coefficient will be zero, unless the independent variable is a reciprocal quantity, as in the following example:

Let $y=\frac{1}{x}$, then $\frac{dy}{dx}=-\frac{1}{x^2}$, $\frac{d^2y}{dx^2}=\frac{2}{x^3}$, $\frac{d^3y}{dx^3}=-\frac{2.3}{x^4}$, &c.

Now by inspecting the preceding examples it is obvious that the nth differential of y divided by the nth power of the differential dx, *must be equal to* X, C, *or* 0, *that is, some function of* x, *represented by* X, *or to a constant quantity* C, *or to* 0.

Successive integration is the converse of successive differentials, and to illustrate the operation, we will take equation (4) in the first example, and integrate it.

$$\frac{d^3y}{dx^3}=6a.$$

Multiplying each side by dx, and we shall have

$$\frac{d^3y}{dx^2}=6adx.$$

Now, regarding all powers of dx greater than the first, as constant, and integrating, the first integral will be

$$\frac{d^2y}{dx^2}=6ax+C.$$

Multiplying again by dx, and integrating as before, the second integral will be

$$\frac{dy}{dx}=3ax^2+cx+C'.$$

And again, and the final integral will be

$$y=ax^3+\frac{c}{2}x^2+c'x+C''.$$

This is the same in form as equation (1), and would be identical if we could restore the identical constants b, c, and d in place of c, c', and C''.

In *abstract* examples, particular constants cannot be determined.

If we have $\frac{d^3y}{dx^3}=0$, the first integral will be

$$\frac{d^2y}{dx^2}=C,$$

That is, *some constant;* and the final integral will be

$$y=\tfrac{1}{2}Cx^2+C'x+C''.$$

If we have $\frac{d^3y}{dx^3}=X$, X being any function of x, the first integral will be

$$\frac{d^2y}{dx^2}=\int Xdx+C.$$

The second, $\frac{dy}{dx}=\int dx\int Xdx+Cx+C'.$

The last, $y=\int dx\int dx\int Xdx+\tfrac{1}{2}Cx^2+C'x+C''.$ (A)

This last equation may be indicated thus:

$$y=\int^3 Xdx^3+\tfrac{1}{2}Cx^2+c'x+c''. \qquad (B)$$

The sign $\int^3$ indicates three successive integrals.

(Art. 96.) A differential in the form

$$\frac{d^2y}{dx^2}=a+y^2, \quad \text{or } =y^2, \quad \text{or } =3y^3,$$

Or in general equal Y, Y being any function of y, can be integrated as follows. Multiply each member by $2dy$.

Then $$\frac{2dy \,.\, d^2y}{dx \,.\, dx}=Ydy.$$

The first member, we perceive, is the differential of $\frac{dy^2}{dx^2}$, on the supposition that dx is constant and dy variable, (and dy is always variable, or we could have no second differential of y), therefore, by integration, we have

$$\frac{dy^2}{dx^2}=2\int Ydy+C.$$

Or $$\frac{dy}{\sqrt{C+2\int Ydy}}=dx.$$

Whence $$x=\int -\frac{dy}{\sqrt{C+2\int Ydy}}+C'.$$

For a particular example, we require the integral of

$$a^2 d^2y+ydx^2=0.$$

$$\frac{d^2y}{dx^2}=-\frac{y}{a^2}. \qquad \frac{2dy}{dx}\cdot\frac{d^2y}{dx}=-\frac{2ydy}{a^2}.$$

$$\frac{dy^2}{dx^2}=-\frac{y^2}{a^2}+C. \qquad \frac{dy}{dx}=\sqrt{C-\frac{y^2}{a^2}}.$$

$$x=\int\frac{dy}{\sqrt{C-\frac{y^2}{a^2}}}+C'.$$

$$\int\frac{dy}{\sqrt{C-\frac{y^2}{a^2}}}=\int\frac{ady}{\sqrt{a^2C-y^2}}=a\int\frac{dy}{\sqrt{a^2C-y^2}}.$$

If we make $Ca^2=1$, then $C=\frac{1}{a^2}$, and the last integral becomes

$$a\sin.^{-1}y+C'. \qquad \text{Or} \quad x=a.\sin.^{-1}y+C'.$$

For a second example, we give the following:

$$\frac{d^2y}{dx^2}=-\frac{Py}{b}. \qquad y=C\sin.\left(x\sqrt{\frac{P}{b}}+f\right)$$

This is solved on pages 350, 351, of the author's Operations.

3. *Integrate the differential* $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\,d^2y}=a.$

In the differential calculus we put $\frac{dy}{dx}=p.$

Whence $$\frac{d^2y}{dx^2}=\frac{dp}{dx}.$$

These values substituted in the given differential, will reduce it to

$$\frac{dx(1+p^2)^{\frac{3}{2}}}{dp}=a$$

Whence $dx = \frac{adp}{(1+p^2)^{\frac{3}{2}}}$, and $pdx = dy = \frac{apdp}{(1+p^2)^{\frac{3}{2}}}$.

Integrating the last two expressions, we obtain

$$x = C + \frac{ap}{\sqrt{1+p^2}}, \qquad y = C' - \frac{a}{\sqrt{1+p^2}},$$

and eliminating p, we find

$$(x-C)^2 + (y-C')^2 = a^2,$$

the general equation of the circle, C and C' being the co-ordinates of the center.

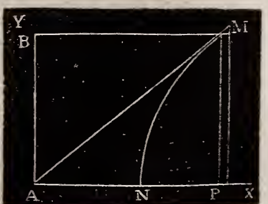
This result was expected, because the given differential corresponded to a constant radius of curvature. See (Art. 51.)

CHAPTER X.

Geometrical Integrals.

(Art. 97.) In this chapter we propose to show the application of the integral calculus to geometry; an operation of the greatest utility in finding the measure of surfaces and solids.

In (Articles 53, 54, and 55), we have shown geometrical differentials, and by the integration of these, we shall have the corresponding integrals, which will be the measure of surfaces or solids, as the case may be.



To commence with the most simple case, *we require the area of the space* NPM, *one side of which is bounded by the curve* NM, *on the supposition that the curve is a portion of a parabola, and the point* N *its vertex.*

Let N be the zero point, $NP = x$, $PM = y$, and it is obvious that ydx is the differential of the area required.

By the equation of the curve we have $y^2 = 2px$.

Whence $ydy=pdx, \qquad ydx=\frac{y^2dy}{p}.$

$$\int ydx=\frac{y^3}{3p}+C.$$

When $x=0$, $y=0$, and therefore $C=0$, and the whole integral

$$NPM=\frac{y^3}{3p}=\frac{y.2px}{3p}=\tfrac{2}{3}xy.$$

That is, the area of any portion of the parabola bounded by the axis and ordinate is measured *by two-thirds of the rectangle made by the abscissa and ordinate.*

The same was shown in analytical geometry.

(Art. 98.) *Now let the area of the same space be required on the supposition that the curve is a circle and the radius unity.*

The same notation as before. The equation of the curve is

$$(1-x)^2+y^2=1. \qquad (1)$$

Whence $\int ydx=\int\frac{y^2dy}{1-x}=\int\frac{y^2dy}{\sqrt{1-y^2}}.$

The part $\frac{1}{\sqrt{1-y^2}}$ expanded into a series by the binomial, and afterwards multiplied by y^2, produces

$$y^2+\frac{y^4}{2}+\frac{3y^6}{2.4}+\frac{3.5y^8}{2.4.6}+\frac{3.5.7y^{10}}{2.4.6.8}+\text{\&c.}$$

Multiply each term by dy, and integration will give

$$\frac{y^3}{3}+\frac{y^5}{2.5}+\frac{3y^7}{2.4.7}+\frac{3.5y^9}{2.4.6.9}+\frac{3.5.7y^{11}}{2.4.6.8.11}+\text{\&c.} \qquad (2)$$

This integral requires no correction, or rather, the value of the correction is 0, because x and y vanish together.

Here y is the sine of the *arc NM*, and it may be of any value from 0 to 1.

When $y=1$, the space *NPM* will represent a quadrant, and its numerical value will be the sum of the series

$$\frac{1}{3}+\frac{1}{2.5}+\frac{3}{2.4.7}+\frac{3.5}{2.4.6.9}+\frac{3.5.7}{2.4.6.8.11}+\text{\&c.}$$

Four times this series is the value of the whole circle.

When we make $y=1$, series (2) does not converge with sufficient rapidity to meet with practical favor, therefore we assume $y=\frac{1}{2}$, knowing that $y=\frac{1}{2}$ when the arc is 30°.

Substituting $y=\frac{1}{2}$ in series (2), we have for the area of the *half segment NPM* of 30°, the following series

$$\frac{1}{3.2^3}+\frac{1}{2.5.2^5}+\frac{3}{2.4.7.2^7}+\frac{3.5}{2.4.6.9.2^9}+\frac{3.5.7}{2.4.6.8.11.2^{11}} \&c.$$

The sum of this series carried sufficiently far, will be found to be .04529302, the double of this, or .09058604, is the area of a segment containing 60° of arc in a circle whose radius is unity.

In any other circle whose radius is r, the area of the segment containing the same number of degrees is $(.090586\, r^2.)$

(Art. 99.) To find the area of the whole circle by the aid of this semi-segment .04529302, we will turn back to the figure and conceive a line drawn from M to the center of the circle. The hypotenuse of the triangle so formed is 1, and the perpendicular $y=\frac{1}{2}$, therefore the base is $\sqrt{3}$, and the area of the triangle is .21650637.

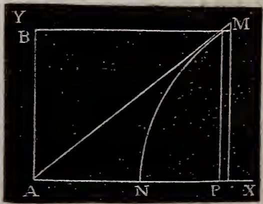
Whence to *NPM*..............	.04529302
Add the triangle $\frac{1}{4}.\frac{1}{2}\sqrt{3}$.......	.21650637
Sum is sector of 30°...........	.26179939
Multiply by 12,..........	12
Area of the whole circle,.....	$3.14159268=\pi$

To find the circumference of this circle we will for the moment represent it by x, and the radius by r. But the area of any circle is the product of the circumference into half the radius, therefore $\frac{r.x}{2}=\pi$, and $x=2\pi$ on the supposion that $r=1$.

Hence, when the radius is 1, the length or measure of 180° of the circumference is 3.14159268, the most important number in mathematical science.

(Art. 100.) Take the same figure as before, and conceive

NM to be a portion of an ellipse, N being at one extremity of the greater axis.



As before, let N be the origin of co-ordinates, $NP=x$, $PM=y$, and from N to the center of the circle, or to the center of the ellipse, we will designate by A. Then, *if the curve is a circle*, $y=\sqrt{2Ax-x^2}$. *If the curve is an ellipse*, $y=\frac{B}{A}\sqrt{2Ax-x^2}$, B being the semi-minor axis of the ellipse, according to customary notation.

Now we have just seen that the area of a circular segment is represented by

$$\int ydx=\int dx\sqrt{2Ax-x^2}. \qquad (a)$$

The area of an elliptic segment on the same abscissa x, and corresponding on the same major axis $2A$, must therefore be represented by

$$\frac{B}{A}\int dx\sqrt{2Ax-x^2}. \qquad (b)$$

These segments are any like portion of the circle, and the ellipse; when $x=A$ the segments will be quadrants. But like portions of any two magnitudes are to each other as the wholes are. Therefore

area circle : area ellipse : : $\int dx\sqrt{2Ax-x^2} : \frac{B}{A}\int dx\sqrt{2Ax-x^2}$

Or area circle : area ellipse : : $1 : \frac{B}{A}$.

But the area of the circle in question is $=\pi A^2$.

Whence πA^2 : area of ellipse : : $1 : \frac{B}{A}$.

Or Area of ellipse$=\pi AB$.

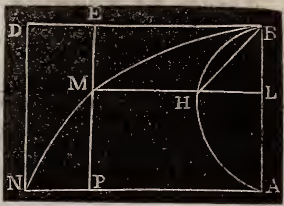
But πAB is obviously the area of a circle whose radius is $\sqrt{AB}$, a mean proportional between A and B.

N. B. This article is the same as propositions 10, 11, and 12, of the ellipse in Robinson's Geometry.

(Art. 101.) In this article we propose to find the area enclosed by the cycloid.

Let r be the radius of the generating circle, NA the axis of x, and N the zero point.

According to custom, place $NP=x$, and $PM=y$. The object is to find the double area NAB.



The differential of the segment NPM is ydx, hence the integral of this, is the segment itself: and if we suppose x to be equal to NA, the segment will be the area of NAB.

The differential equation of the cycloid is

$$dx=\frac{ydy}{\sqrt{2ry-y^2}}. \quad \text{(Art. 48.)} \quad (1)$$

Whence $$\int ydx=\int\frac{y^2\,dy}{\sqrt{2ry-y^2}}=NPM+C.$$

This differential has already been integrated in (Art. 80), it is the fourth example. All we have to do is to change x to y, and a to r, in that example, and the result is

$$-\left(\frac{y}{2}+\frac{3r}{2}\right)\sqrt{2ry-y^2}+\frac{3r^2}{2}\text{ vers. sin.}^{-1}\frac{y}{r}+C.$$

But when we make $NPM=0$, $y=0$, and therefore $C=0$.

When we take $y=2r$, the above result becomes

$$\frac{3r^2}{2}\text{ vers. sin.}^{-1}\frac{2r}{r}=\frac{3\pi r^2}{2}.$$

The double of this, $(3\pi r^2)$, *is three times the area of the generating circle,* the area required.

We can, however, obtain this result by a more simple process, if we take into consideration the external space DBN.

The rectangle $ABDN$ is obviously $(NA).(AB)$, which is $(\pi r)(2r)$, or $2\pi r^2$, and now if we can obtain the value of NDB, and subtract it, the remainder will be the area NAB.

The differential of the space $NDEM$ is $(EM)dx$. But $EM=2r-y$.

Hence $d.NDEM=(2r-y)dx$.

If in this we substitute the value of dx from (1), we shall have

$$d.NDEM=\frac{(2r-y)ydy}{\sqrt{2ry-y^2}}=dy\sqrt{2ry-y^2}.$$

Whence, by integration, $NDEM=\int dy\sqrt{2ry-y^2}$.

By comparing this with expression (a) in (Art. 100), we perceive that the second member *is the area of the segment* of a circle whose radius is r and abscissa y.

That is, $NMED=ALH$.

Hence $NBD=$ the semi-circle AHB.

But the area of the semi-circle is $\frac{\pi r^2}{2}$.

Whence $ANB=2\pi r^2-\frac{\pi r^2}{2}$, and the double of this is $3\pi r$, *the area sought.*

That is, *the area of the cycloidical space is three times the area of the generating circle.*

(Art. 102.) While on the cycloid, let us require the value of any portion of its arc, as BM.

Let s be the length of an arc, whatever be the curve; then we have seen in (Art. 55), that

$$ds=\sqrt{dx^2+dy^2}. \qquad (1)$$

In short, this equation is obvious in a primary point of view.

But $dx=\frac{ydy}{\sqrt{2ry-y^2}}$, and this value put in (1), produces

$$ds=\sqrt{\frac{y^2dy^2}{2ry-y^2}+dy^2}=dy\sqrt{\frac{2r}{2r-y}}. \qquad (2)$$

By integration, $s=-2\sqrt{2r}.\sqrt{2r-y}+C. \qquad (3)$

If we estimate the arc from B, making $z=0$, then $y=2r$, and $0=0+C$, which shows that there is no constant to add, and

$$s=-2\sqrt{2r}.\sqrt{2r-y}, \qquad (4)$$

which is the general expression of any arc estimated from B.

If $BM=s$, the corresponding value of y is AL, and $2r-y=BL$. Hence,

$$BM=-2\sqrt{AB}\cdot\sqrt{BL}=-2\sqrt{AB.BL}.$$

But $\sqrt{AB.BL}=\pm BH$. Therefore $BM=2BH$.

That is, *The arc of a cycloid estimated from the vertex is twice the corresponding chord of the generating circle: And the arc BMN is twice the diameter of the generating circle, and the entire cycloidical arc is four times the diameter of the same circle.*

It is necessary to have a formula for the circumference of the ellipse,—we will therefore substitute the values of dx and dy, drawn from the equation of the ellipse in the general formula for the arc of a curve, that is, in $\sqrt{dx^2+dy^2}$, and integrate.

Let the center be the origin of co-ordinates, and the equation of the ellipse is

$$A^2y^2+B^2x^2=A^2B^2.$$

For convenience we shall use the eccentricity of the ellipse, which is the distance from either focus to the center, when $A=1$, or $e^2=\frac{A^2-B^2}{A^2}$, whatever be the value of A. Then $1-e^2=\frac{B^2}{A^2}$.

Whence $$y^2=(1-e^2)(A^2-x^2).$$

$$\frac{dy}{dx}=-\frac{(1-e^2)x}{y}=-\frac{x\sqrt{1-e^2}}{\sqrt{A^2-x^2}}$$

In short, $$ds=\sqrt{dx^2+dy^2}=\frac{Adx\sqrt{1-\frac{e^2x^2}{A^2}}}{\sqrt{A^2-x^2}}.$$

Expanding $\sqrt{1-\frac{e^2x^2}{A^2}}$ into a series, we obtain

$$ds=\frac{Adx}{\sqrt{A^2-x^2}}\left(1-\frac{e^2x^2}{2A^2}-\frac{e^4x^4}{2.4A^4}-\frac{3e^6x^6}{2.4.6A^6}-\&c.\right)$$

Multiplying each term of the series by the factor without the parenthesis, we shall have a series of differentials which can be integrated separately, one by one, which together, will show the approximate value of any arc corresponding to any assumed value of x less than A. When we make $x=A$, after integration we have one-fourth of the circumference, which multiplied by 4, gives the whole circumference, which is

$$2\pi A\left(1-\frac{e^2}{2.2}-\frac{3e^4}{2.2.4.4}-\frac{3.3.5e^6}{2.2.4.4.6.6}-\text{\&c.}\right)$$

(Art. 103.) The curvature of the circle is uniform, and as we have found the value of the whole circumference (2π), to radius unity, therefore we shall have, by simple division, the value of any required portion of it. But it is not so with *other curves.*

To find the lengths of other curves between any proposed limits, we must integrate the expression $\sqrt{dx^2+dy^2}$, taking x or y between the proposed limits, (the equation of the curve given,) and the relation between x and y in each particular case. It is not necessary that we should give examples in every curve, we will therefore select the most interesting, as one of the *spirals.*

The general equation for the spirals is $r=at^n$, (Art. 43,) in which r is the varying radius, t the measuring arc, and a a constant quantity.

The differential equation of *an arc* in respect to polar co-ordinates is

$$ds=\sqrt{r^2dt^2+dr^2}, \qquad \text{(Art. 41.)}$$

therefore the integral of this expression is s, or the length of a spiral curve between proposed limits.

From the equation of the curve, $dr^2=n^2a^2t^{2n-2}dt^2$.

Whence $\int\sqrt{r^2dt^2+dr^2}=\int dt(t^2+n^2)^{\frac{1}{2}}at^{n-1}$.

When $n=1$, as is the case in the spiral of Archimedes, the differential becomes

$$a\sqrt{1+t^2}.dt.$$

By the ninth example (Art. 80,) we find that the integral of this is

$$\frac{at\sqrt{1+t^2}}{2}+\frac{a}{2}\log.(t+\sqrt{1+t^2})+C.$$

N. B. We shall find an expression of the same form for an arc of the common *parabola.*

If we take the logarithmic spiral whose equation is $t=\log.r$, we shall find

$$\sqrt{r^2 dt^2+dr^2}=dr\sqrt{2}.$$

Whence $r\sqrt{2}+C$, (or simply $r\sqrt{2}$, commencing at the origin of the radius vectors,) expresses the arc of this curve, "and we see that though there is between this origin and any point of the curve at an infinite distance from it, an infinite number of revolutions, yet they include an arc of finite length, which is equal to the diagonal of the square described on the radius vector."

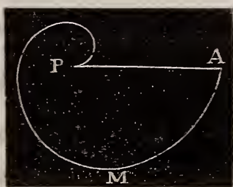
AREA OF SPIRALS.

(Art. 104.) In (Art. 41) will be found the differential of a polar sector, or the differential area of a polar curve, equal to

$$\frac{r^2 dt}{2}, \qquad (1)$$

in which $r=\frac{t}{2\pi}$ when we apply it to the spiral of Archimedes.

Whence $$\int\frac{r^2 dt}{2}=\int\pi r^2 dr=\frac{\pi r^3}{3}=\frac{t^3}{24\pi^2}.$$



If we assume $t=2\pi$ one revolution, this last expression for the *area corresponding* will be $\frac{\pi}{3}$, showing that the space *PMA* included in the first revolution from the pole is equal to one-third the area of the circle, whose radius is equal to the radius vector at the end of the first revolution.

If we make $t=2\pi$, the area described in two revolutions is $\frac{8\pi}{3}$, but this includes the first revolution *described the second time*, hence the area actually enclosed after two revolutions, will be

$$\frac{8\pi}{3}-\frac{\pi}{3}=\frac{7\pi}{3}.$$

Again, *if we take the logarithmic spiral* whose equation is $t=\log.r$ and apply it in (1), we shall have

$$\int\frac{r^2\,dt}{2}=\int\frac{rdr}{2}=\frac{r^2}{4}+C.$$

If we estimate the area from the pole where $r=0$, which makes $C=0$, and the whole area is $\frac{r^2}{4}$, that is, *the area of the Naperian logarithmic spiral is equal to one-fourth of the square described on the radius vector.*

Again, *if we take the hyperbolic spiral,* then $r=-1$, and the general equation of the spirals becomes $r=\frac{a}{t}$.

Whence $\int\frac{r^2\,dt}{2}=-\int\frac{adr}{2}=-\frac{ar}{2}$. But $-\frac{ar}{2}=-\frac{a^2}{2t}$.

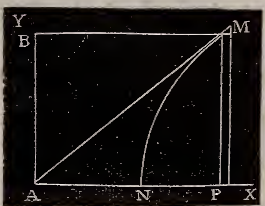
This area is infinite when $t=0$, but we can find the area included between any two radius vectors, b and c, by integrating between the limits $t=b$ and $t=c$, which will give

$$\frac{a^2}{2}\left(\frac{1}{b}-\frac{1}{c}\right).$$

CHAPTER XI.

Geometrical Integrals, continued.

THE AREAS OF CURVED SURFACES, OR SURFACES OF REVOLUTION, DETERMINED BY INTEGRATION.



(Art. 105.) If any curve, as NM, revolve on an axis, as NP, the axis of X, it will describe a curved surface.

If the curve is a circle, the surface so described will be the surface of a sphere. If the curve NM be a portion of a parabola, the surface it will describe will be a parabolic surface, &c. &c.

If NM be a straight line, the surface described by revolving on the axis of X will be the surface of a cone, or if a portion of the line only revolve, the surface so described will be the surface of a conic frustrum.

It is obvious that we must obtain the general differential expression for these surfaces, and it is obvious that this difference is measured by the revolution of a small portion of the arc at M. Or, by the revolution of the differential of the arc, which is $=\sqrt{dx^2+dy^2}$.

And this line revolves at the extremity of the radius y.

Therefore $2\pi y\sqrt{dx^2+dy^2}$ is the differential in question. In words, *The differential of a surface of revolution is equal to the circumference of a circle perpendicular to the axis, into the differential of the arc of the meridian curve.*

Our first application of this formula will be to the circle, because most persons are more familiar with that curve than with any other;—therefore

1. *Required the surface of a segment of a sphere corresponding to the co-ordinates* x, y, *the origin being at the circumference.*

Let R be the radius: Then $(R-x)^2+y^2=R^2$. (1)

From which we find $\sqrt{dx^2+dy^2}=\dfrac{Rdy}{\sqrt{R^2-y^2}}$.

Whence $2\pi y\sqrt{dx^2+dy^2}=\dfrac{2\pi R.ydy}{\sqrt{R^2-y^2}}$.

$$\int 2\pi y\sqrt{dx^2+dy^2}=2\pi R.\int\frac{ydy}{\sqrt{R^2-y^2}}=-2\pi R\sqrt{R^2-y^2}+C$$

We perceive by the figure, that commencing at N, the zero point, making $x=0$, $y=0$, and the area in question equals 0. That is, the equation above becomes

$$0=-2\pi R\sqrt{R^2}+C.$$

Or $$C=2\pi R^2.$$

Substituting this value of C and the general integral becomes

$$-2\pi R\sqrt{R^2-y^2}+2\pi R^2$$

When we take $y=R$, the surface corresponds with a hemisphere, and the factor $\sqrt{R^2-y^2}$, then becomes 0, showing that the surface of a hemisphere whose radius is R, is $2\pi R^2$. Therefore the surface of the whole sphere is $4\pi R^2$.

Again, when we take $x=2R$, y will again be 0, and the first term of the expression for the general integral becomes

$$-2\pi R\sqrt{R^2}=\pm 2\pi R^2,$$

showing that the area of the surface of revolution is

$$\pm 2\pi R^2+2\pi R^2=0, \quad \text{or} \quad 4\pi R^2,$$

corresponding to $y=0$, the latter value is the surface of the whole sphere.

(Art. 106.) Conceive NM to be a straight line, then the area of the surface of revolution will be the surface of a cone. And in that cone we shall have

$$x : y=a : b. \qquad \text{Or } x=\frac{ay}{b}.$$

$$\sqrt{dx^2+dy^2}=\sqrt{\frac{a^2}{b^2}dy^2+dy^2}=\frac{dy}{b}\sqrt{a^2+b^2}.$$

$$\int 2\pi y\sqrt{dx^2+dy^2}=\frac{2\pi}{b}\int ydy(\sqrt{a^2+b^2})=\frac{2\pi y^2}{2b}(a^2+b^2)^{\frac{1}{2}}+C.$$

If we conceive the area to commence at the point N where $x=0$, we shall have the area equal 0, and $y=0$, which will give $C=0$, and the whole integral will be

$$\frac{2\pi y^2}{b}\times\frac{\sqrt{a^2+b^2}}{2}.$$

But $\sqrt{a^2+b^2}=NM$, and if we make $y=b$, the surface will be

$$2\pi y\left(\frac{NM}{2}\right)$$

an expression which is obviously *the circumference of the base of a cone multiplied by the half of its slant hight.* The same rule as was found in geometry.

When the curve NM is a parabola, the surface of revolution is called a *paraboloid.*

In that case we have $\quad y^2=2px, \qquad ydy=pdx.$

Whence $\sqrt{dx^2+dy^2}=\sqrt{\frac{y^2}{p^2}dy^2+dy^2}=\frac{dy}{p}\sqrt{y^2+p^2}.$

Or $\int 2\pi y\sqrt{dx^2+dy^2}=\frac{2\pi}{p}\int ydy\sqrt{y^2+p^2}=\frac{2\pi}{3p}(y^2+p^2)^{\frac{3}{2}}+C.$

When $y=0$, the surface of the revolution is 0. Therefore

$$0=\frac{2\pi}{3p}(p^2)^{\frac{3}{2}}+C.$$

Or $$C=-\frac{2\pi p^3}{3}.$$

Whence the entire integral between the limits $y=0$ and $y=b$, may be written thus:

$$\frac{2\pi}{3p}\left((b^2+p^2)^{\frac{3}{2}}-p^2)\right).$$

When the curve is an ellipse, the result comes out in a series more tedious than interesting.

When the curve NM is a portion of the cycloid, we have

$$dx=\frac{ydy}{\sqrt{2ry-y^2}}.$$

Whence $$dx^2+dy^2=\frac{y^2dy^2}{2ry-y^2}+dy^2=\frac{2rdy^2}{2r-y}.$$

And $$2\pi y\sqrt{dx^2+dy^2}=2\pi\sqrt{2r}\left(\frac{ydy}{\sqrt{2r-y}}\right). \qquad (1)$$

Whence $$\int 2\pi y\sqrt{dx^2+dy^2}=2\pi\sqrt{2r}\int\frac{ydy}{\sqrt{2r-y}}.$$

To integrate this last expression, place $\sqrt{2r-y}=z$. (2)

Then $y=2r-z^2$, and $ydy=-4rzdz+2z^3dz$.

$$\int\frac{ydy}{\sqrt{2r-y}}=-4rz+\tfrac{2}{3}z^3.$$

Substituting this value in (1), restoring the value of (z) at the same time, and we shall have

$$2\pi\sqrt{2r}\left(-4r\sqrt{2r-y}+\tfrac{2}{3}(2r-y)^{\frac{3}{2}}\right)+C.$$

To find the value of C we must consider that when $y=0$, the surface sought must be 0. Hence, the following equation must be true :

$$0=2\pi\sqrt{2r}\left(-4r\sqrt{2r}+\tfrac{2}{3}(2r)^{\frac{3}{2}}\right)+C.$$

$$0=2\pi\left(-8r^2+\frac{8r^2}{3}\right)+C.$$

Whence $$C=\frac{32\pi r^2}{3}.$$

If we make $y=2r$, the value of half the surface sought is numerically the same as C, because $\sqrt{2r}$ can be taken with the minus sign. Hence, the whole surface equals $\frac{64}{3}\pi r^2$, *which is sixty-four thirds the area of the generating circle.*

The preceding examples are sufficient to illustrate the theory of finding the area of surfaces by integration.

CHAPTER XII.

Geometrical Integrals, continued.

THE VOLUME, OR CUBATURE OF SOLIDS OF REVOLUTION.

(Art. 107.) The motion of a line is conceived to form a surface, and the motion of a plane, or the revolution of a plane on an axis, may be conceived to form a solid or a geometrical volume.

But it is not necessary to conceive a revolution of a plane to obtain a solid; we can take a solid like a parallelopipedon, a cone, or a pyramid, and conceive it to *increase* or *decrease* by the motion of one of its surfaces, and thus we have its differential.

The integral of this differential, corrected if necessary, will give the volume to that differential. We shall give both methods, calling particular attention to the first problem, on account of its simplicity and its elementary character. It is this :

1st. *Find the volume of a pyramid, by integration.*

Let G be the vertex of a pyramid, and assume $GF=1$, and designate the corresponding base $FOHE$ by b. Let $GA=x$, and $AIQM$, the corresponding base.

But these bases are in proportion to the squares of the distances from the vertex.

Therefore $1 : x^2 :: b : bx^2=AIQM$.

The *differential of this pyramid is obviously* $bx^2\,dx$.

Hence the pyramid itself is $\int bx^2\,dx$.

Bu $$\int bx^2\,dx=\frac{bx^3}{3}+C.$$

This is true for all values of x, it is true then when $x=0$, and making this supposition, the last equation becomes $0=0+C$, or $C=0$. Therefore $\frac{bx^3}{3}$ is the whole integral, or the solidity of the pyramid.

But $\frac{bx^3}{3}=(bx^2)\frac{x}{3}$. That is, *the base multiplied by one-third of the altitude* gives the cubical contents of a pyramid.

N. B. When the base is a circle the pyramid becomes a cone, to which the same rule applies, namely,

The area of the base multiplied by one-third of the altitude.

Scholium. A sphere may be conceived to be composed of a *great multitude* of pyramids, the base of each one being a very small portion of the surface of the sphere, and the altitude of each one the radius of the sphere. Therefore, *the volume of a sphere is equal to its surface multiplied into one-third of its radius.*

Again, we may conceive the triangle GAI to revolve on the axis GA, thus forming a cone. The radius of the base of that cone will be AI, which we will designate by y, then πy^2 will be the area of the base, and $\pi y^2\,dx$ will be the *differential of the cone.*

Hence the cone itself will be $\int \pi y^2\,dx$.

We can integrate this, provided we can find the relation between x and y.

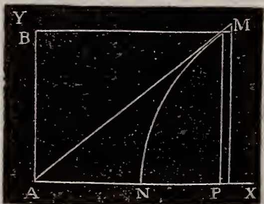
Let $GF=1$, and $FE=a$, GA being x.

Then by proportional triangles, $1 : a=x : y$. $y=ax$.

Whence $\int \pi y^2 dx=\int \pi a^2 x^2 dx=\frac{\pi a^2 x^3}{3}=\frac{\pi a^2 x^2}{3}.x=\frac{\pi}{3}x.y^2$

Which is $\left(\pi y^2 . \frac{x}{3}\right)$, *the area of the base multiplied by one-third of the altitude.*

(Art. 108.) *Required the volume of any solid of revolution, as the segment of a circle, a segment of a paraboloid, the segment of an ellipsoid, &c.*



Let N be the origin of co-ordinates, R the radius of a circle, $NP=x$, $PM=y$ as before.

Now it is obvious that the revolution of the segment NPM, on the axis NP, will produce a solid, and it is also obvious that the revolution of PM, (ydx), on the center P, is $\pi y^2 dx$ *the differential of the volume.* Hence, the volume of revolution between the limits $x=a$ and $x=b$, is found by

$$\int_a^b \pi y^2 dx.$$

For the segment of a sphere we have the equation

$$(R-x)^2+y^2=R^2. \qquad (1)$$

From which $\quad y^2=2Rx-x^2.$

Whence

$$\int \pi y^2 dx=\int (2\pi Rxdx-\pi x^2 dx)=\pi Rx^2-\frac{\pi x^3}{3}+C.$$

When $x=0$, the area is 0, therefore $C=0$, and the integral corresponding to any segment x, is $\left(\pi Rx^2-\frac{\pi x^3}{3}\right)$, *the solidity of the segment.*

When $x=R$, the segment is a hemisphere, and its solidity is $\frac{2}{3}\pi R^3$, and when $x=2R$, or when the segment contains the *whole sphere,* its volume is $\frac{4}{3}\pi R^3$, the same result as was found in elementary geometry.

This result also corresponds to the scholium in (Art. 107), for in (Art. 105) we found the surface of a sphere to be $4\pi R^2$, which multiplied by $\frac{R}{3}$ produces $\frac{4\pi R^3}{3}$, the solidity of the sphere as before.

(Art. 109.) We may change the origin of co-ordinates from the surface to the center of volume at pleasure, or we may in fact change it to any other known point.

For example, we will recompute the last problem, taking the center of the sphere for the *zero point.*

In that case $x^2+y^2=R^2$, or $y^2=R^2-x^2$.

Whence $\int \pi y^2 dx=\int (\pi R dx-\pi x^2 dx)=\pi R^2 x-\frac{\pi x^3}{3}+C.$

When $x=0$, the volume $=0$, and therefore $C=0$.

When $x=R$, the volume corresponds to a hemisphere and the expression to $\pi R^3-\frac{\pi R^3}{3}=\frac{2\pi R^3}{3}$, the same as before.

N. B. In the expression $\pi R^2 x-\frac{\pi x^3}{3}$, x cannot be taken greater than R. In case it be so taken, the numerical value of the whole would be *minus*, but magnitudes cannot be essentially negative.

Let us now require the volume of an ellipsoid, the ellipse revolving on its major axis.

The equation for the ellipse is

$$A^2y^2+B^2x^2=A^2B^2.$$

Whence $$y^2=B^2-\frac{B^2}{A^2}x^2.$$

$$\int \pi y^2 dx=\int \left(\pi B^2 dx-\pi\frac{B^2}{A^2}x^2 dx\right)=\pi B^2 x-\pi\frac{B^2}{3A^2}x^3+C.$$

The origin being the center when $x=0$, the volume corresponding equals 0, and consequently $C=0$.

Hence, the value of any segment of an ellipsoid must be

$$\pi B^2 x\left(1-\frac{x^2}{3A^2}\right).$$

If we make $x=A$, the expression will correspond to a semi-ellipsoid, and it will reduce to

$$\tfrac{2}{3}\pi B^2 A, \quad \text{or} \quad \pi B^2 . \tfrac{2}{3}A.$$

That is, *two-thirds of the circumscribing cylinder.*

If we suppose $A=B$, the ellipse will become a circle, and the semi-ellipsoid will become an hemisphere, and the expression above will become $\frac{2}{3}\pi R^2$, as it ought to do.

(Art. 110.) If an ellipse revolve on its minor axis, it will describe an *oblate* spheroid, and the differential of the volume will be

$$\pi x^2 dy.$$

But $x^2=A^2-\frac{A^2}{B^2}y^2$. Whence $\pi x^2 dy=\pi\left(A^2-\frac{A^2}{B^2}y^2\right)dy.$

$$\int \pi x^2 dy=\pi A^2 y-\frac{\pi A^2 y^3}{3B^2}.$$

If we make $y=B$, this solid will be expressed by

$$\left(\pi A^2 B-\frac{\pi A^2 B}{3}\right), \quad \text{or} \quad \tfrac{2}{3}B . \pi A^2.$$

This is also *two-thirds of the circumscribing cylinder.*

Comparing the two solids generated by the revolution on each axis, we find

$$\text{oblate solid} : \text{prolate solid} :: BA^2 : AB^2$$
$$:: A : B.$$

To find the volume of a paraboloid, we have the equation of the parabola $y^2=2px.$

This value of y^2 placed in the general expression $\int \pi y^2 dx$ will give us $2\pi pxdx$ for the differential of this solid. Hence the solid itself is πpx^2, which requires no correction.

But $\pi px^2 = \frac{2\pi px}{2} . x = \pi y^2 . \frac{x}{2}$, and this we perceive is *one-half of the circumscribing cylinder.*

Scholium. Let y be the radius of a circle, then its area will be πy^2. Let h be the altitude of a cylinder.

Now conceive a cylinder, a cone, a paraboloid, and a sphere, to equal circumferences, and each the same altitude h. By a short retrospect we find

The volume of a cylinder $= \pi y^2 . h.$

The volume of a cone $= \pi y^2 . \frac{h}{3}.$

The volume of a paraboloid $= \pi y^2 . \frac{h}{2}.$

The volume of a sphere $= \pi y^2 . \frac{2h}{3}.$

Calling the *cylinder* 1, we have for the *cone* $\frac{1}{3}$, for the paraboloid $\frac{1}{2}$, and for the *sphere* $\frac{2}{3}$.

The proportion in whole numbers is, cylinder 6, cone 2, paraboloid 3, sphere 4.

These proportions were discovered by Archimedes, and it is said that he requested them to be engraved on his tomb.

For another example, we require the volume generated by the revolution of a cycloid on its base.

The differential of any revoloid is $\pi y^2 dx$.

For the cycloid we have $dx = \frac{ydy}{\sqrt{2ry - y^2}}$. (Art. 48.)

Whence $\int \pi y^2 dx = \int \frac{\pi y^3 dy}{\sqrt{2ry - y^2}} = \pi \int \frac{y^3 dy}{\sqrt{2ry - y^2}}$.

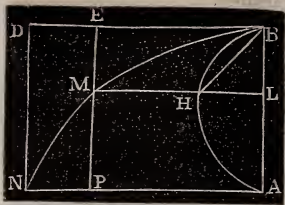
This is integrated by formula d, (Art. 78,) as follows:

$$\int \frac{y^3 dy}{\sqrt{2ry - y^2}} = -\frac{y^2}{3}\sqrt{2ry - y^2} + \frac{5r}{3}\int \frac{y^2 dy}{\sqrt{2ry - y^2}}. \quad (1)$$

$$\int \frac{y^2 dy}{\sqrt{2ry - y^2}} = -\frac{y}{2}\sqrt{2ry - y^2} + \frac{3r}{2}\int \frac{ydy}{\sqrt{2ry - y^2}}. \quad (2)$$

$$\int \frac{ydy}{\sqrt{2ry-y^2}} = -\sqrt{2ry-y^2} + r\int \frac{dy}{\sqrt{2ry-y^2}}. \qquad (3)$$

$$\int \frac{dy}{\sqrt{2ry-y^2}} = \text{arc}\left(\text{ver. sin.} = \frac{y}{r}\right). \qquad (4) \quad (\text{Art. } 79.)$$



These integrals require no constant, for when we make $y=0$ the volume will be 0, as it ought to be.

If we make $y=2r$, the corresponding volume will be half the volume sought, and (4) will become π.

This value put in (3) will give

$$\int \frac{ydy}{\sqrt{2ry-y^2}} = \pi r.$$

And this substituted in (2) will give

$$\int \frac{y^2 dy}{\sqrt{2ry-y^2}} = \frac{3\pi r^2}{2}.$$

And this placed in (1) and multiplied by π produces

$$\pi\int \frac{y^3 dy}{\sqrt{2ry-y^2}} = \frac{5\pi^2 r^3}{2}$$

This being half the volume sought, the whole must be

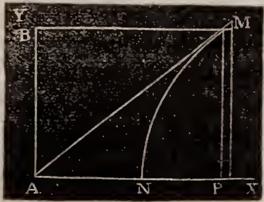
$$5\pi^2 r^3.$$

But $\pi(2r)^2$ represents the base of the circumscribing cylinder. And $2\pi r$ represents its altitude.

Therefore $8\pi^2 r^3$ is its solidity.

Hence, *the solid required is five-eighths of the circumscribing cylinder.*

(Art. 110.) Now conceive the curve to revolve on the axis of Y, N the center of revolution, and NP the radius. On the supposition that NPM is a portion of a parabola, we require the volume generated by the revolution of the curve on the axis of Y, the origin being at N, or the axis being changed from A to N.



To solve this problem we must obtain a new and corresponding expression for the differential of the volume.

It is obvious that ydx is the differential of the revolving surface. This revolving at the extremity of the radius x, will revolve through a space equal to $2\pi x$. Hence, the differential of the volume of revolution will be expressed by

$$2\pi xydx.$$

This may be applied to any curve (revolving on the axis of Y and center N,) as well as to the parabola. We take the parabola because the integral comes out in a definite form.

The equation of the parabola is $y^2=2px.$

Whence $x=\frac{y^2}{2p}.$ And $2\pi xydx=\frac{\pi y^4dy}{p^2}$

$$\int 2\pi xydx=\pi\int\frac{y^4dy}{p^2}=\frac{\pi y^5}{5p^2}+C.$$

But this volume requires no correction, for when $x=0$, $y=0$, therefore $C=0$, and the volume sought is

$$\frac{\pi y^5}{5p^2}. \quad \text{But} \quad y^4=4p^2x^2.$$

Whence $\frac{\pi y^5}{5p^2}=\frac{4\pi x^2y}{5}=V$, the volume sought.

Let us observe that πx^2 is the base of the circumscribing cylinder, and y being its altitude, πx^2y is the volume of the cylinder.

Now by proportion,

$$\text{cylinder} : V :: \pi x^2y : \frac{4\pi x^2y}{5}.$$

$$:: 1 : \frac{4}{5}.$$

Whence $V=\frac{4}{5}$ *of its circumscribing cylinder.*

Scholium. Hence the volume around the axis of Y bounded by a portion of a parabola, is one-fifth of its circumscribing cylinder.

CHAPTER XIII.

On the Integration of Homogeneous and Linear Differentials.

(Art. 111.) An equation is said to be homogeneous when the sum of the exponents of the variables is the same in every term.

Differentials of this form can always be integrated. In such cases we place one of the variables equal to the other multiplied by an assumed variable factor, but we shall illustrate by

EXAMPLES.

1. *Integrate the differential*

$$x^2dy=y^2dx+xydx.$$

This equation is homogeneous. Therefore place $x=vy$. Then the equation becomes

$$v^2y^2dy=y^2dx+vy^2dx,$$

which is divisible by y^2, and

$$v^2dy=dx+vdx=(1+v)dx. \qquad (1$$

But $x=vy$. Therefore $dx=vdy+ydv$. (2

The value of (dx) substituted in (1), it becomes

$$v^2dy=vdy+ydv+v^2dy+vydv.$$

Or $$0=vdy+ydv+vydv. \qquad (3)$$

Dividing each term by (vy), we have

$$\frac{dy}{y}+\frac{dv}{v}+dv=0.$$

By integrating each term, we obtain

$$\log.y+\log.v+v=C.$$

Or $$\log.(vy)+v=C.$$

That is, $\log.x+\frac{x}{y}=C$, the result sought.

2. *Integrate the differential*

$$(x^2+xy)dy=(x-y)ydx.$$

As before, let $x=vy$, then

$$(v^2y^2+vy^2)dy=(vy-y)ydx.$$

And $$(v^2+v)dy=(v-1)dx. \qquad (1)$$

Because $x=vy$, $dx=vdy+ydv$, and this value of dx substituted in (1), produces

$$(v^2+v)dy=v^2dy+vydv-vdy-ydv.$$

By reducing, $$2vdy=vydv-ydv.$$

Dividing by vy, and we obtain

$$\frac{2dy}{y}=dv-\frac{dv}{v}.$$

By integrating $$2.\log.y=v-\log.v+C.$$

That is $$\log.y+(\log.y+\log.v)=v+C.$$

Or $$\log.y+\log.x=\frac{x}{y}+C,$$ the integral sought

3. *Integrate the differential*

$$xdy-ydx=dx\sqrt{x^2+y^2}$$

If we place $y=vx$, then

$$xdy-vxdx=dx\sqrt{x^2+v^2x^2}.$$

Dividing by x, $$dy-vdx=dx\sqrt{1+v^2}. \qquad (1)$$

Because $y=vx$, $dy=vdx+xdv$, and $dy-vdx=xdv$. (2)

Equating (1) and (2), we have

$$xdv=dx\sqrt{1+v^2}.$$

$$\frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x}$$

By integrating, we find

$$\log.(v+\sqrt{1+v^2})=\log.x+\log.C=\log.Cx.$$ (See Art. 81.)

Passing to numbers, $$v+\sqrt{1+v^2}=Cx$$

Restoring the value of v $$\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}=Cx.$$

Multiplying by x, $y+\sqrt{x^2+y^2}=Cx^2$.

4. *Integrate the differential equation*

$$du=\frac{ydx-xdy}{x^2+y^2}. \qquad (1)$$

Let $x=vy$. Then $dx=vdy+ydv$.

The values of x and dx substituted in (1), and reduced, will give

$$du=-\frac{dv}{1+v^2}.$$

By integrating, we find $u=-\text{arc}(\tan.=v.)$ (Art. 68.)

Restoring the value of v, and $u=-\text{arc}\left(\tan.=\frac{x}{y}\right)+C$.

5. *Integrate the differential*

$$du=\frac{ax^2dy-axydx}{(x^2+y^2)^{\frac{3}{2}}}.$$

Place $x=vy$. Then substituting the values of x and dx, we shall find after reduction

$$du=-\frac{avdv}{(1+v^2)^{\frac{3}{2}}}.$$

Whence $$u=-a\int\frac{vdv}{(1+v^2)^{\frac{3}{2}}}.$$

Integrating the second member by formula B, (Art. 77,) we find $u=\frac{a}{\sqrt{1+v^2}}$. Restoring the value of v, $\frac{x}{y}$, we obtain $u=\frac{ay}{\sqrt{x^2+y^2}}+C$, for the result.

(Art. 112.) Differential equations may sometimes appear in the form

$$dy+Pydx=Qdx, \qquad (1)$$

in which P and Q are functions of x.

The object of this article is to show the integration of such differentials. Equations of this kind being of the first degree, in respect to y and dy are sometimes called *linear equations.*

Place $$y=zX, \qquad (2)$$

X being some function of x, to be determined by circumstances, as we are about to explain.

The differential of (2) is

$$dy=zdX+Xdz.$$

This value of dy substituted in (1), produces

$$zdX+X(dz+Pzdx)=Qdx. \qquad (3)$$

Now X being arbitrary, we can so assume it that

$$zdX=Qdx. \qquad (4)$$

Then $$X(dz+Pzdx)=0.$$

Whence $$X=0, \quad \text{or} \quad dz+Pzdx=0.$$

From the last $$\frac{dz}{z}=-Pdx.$$

By integration $$\log.z=-\int Pdx. \qquad (5)$$

But $$1=\log.e. \qquad (6)$$

By the multiplication of (5) and (6), we have

$$\log.z=-\int Pdx.\log.e=\log.e^{-\int Pdx}.$$

Passing to numbers, $$z=e^{-\int Pdx}. \qquad (7)$$

From (4) $$dX=\frac{Qdx}{z}=Q(e^{\int Pdx})dx \qquad (8)$$

By integration $$X=\int Q(e^{\int Pdx})dx. \qquad (9)$$

The values of z and X, (7), (8), substituted in (2), give

$$y=e^{-\int Pdx}\int Q(e^{\int Pdx})dx, \qquad (10)$$

the formula for the integral value of y.

EXAMPLES.

1. *Integrate the differential*

$$dy-\frac{xydx}{1+x^2}=\frac{adx}{1+x^2}.$$

Here $$P=-\frac{x}{1+x^2}. \qquad Q=\frac{a}{1+x^2}.$$

$$\int Pdx = -\int \frac{xdx}{1+x^2} = -\log.\sqrt{1+x^2}.$$

Comparing this with (5), $\log z = \log.\sqrt{1+x^2}$.

Or $z = -\sqrt{1+x^2}$. (1)

From (8) $dX = \frac{Qdx}{z} = -\frac{adx}{(1+x^2)\sqrt{1+x^2}} = -\frac{adx}{(1+x^2)^{\frac{3}{2}}}$.

By integration $X = -\frac{ax}{\sqrt{1+x^2}} + C$. (Formula B, Art. 77.)

But $y = zX = ax - C\sqrt{1+x^2}$, the integral sought.

2. *Integrate the differential*

$$dy - \frac{aydx}{1-x} = \frac{bdx}{1-x}.$$

Here $P = -\frac{a}{1-x}$. $Q = \frac{b}{1-x}$. $Pdx = -\frac{adx}{1-x}$.

$$\int Pdx = a\log.(1-x).$$

Hence $\log.z = a\log.(1-x) = \log.(1-x)^a$.

Whence $z = (1-x)^a$. (1)

From (8) we have $dX = \frac{bdx}{(1-x)(1-x)^a} = \frac{bdx}{(1-x)^{a+1}}$.

By integration $X = -\frac{b}{a(1-x)^a} + C$. (2)

The product of (1) and (2) will give y for the first member, whence

$$y = -\frac{b}{a} + C(1-x)^a. \qquad (3)$$

N. B. If this is truly the integral sought, its differential will produce the example. We will thus verify it.

$$dy = aC(1-x)^{a-1}dx.$$

Multiply both members of this equation by $(1-x)$, and

$$(1-x)dy = aC(1-x)^a dx.$$

$$\frac{(1-x)dy}{dx} = aC(1-x)^a.$$

From (3) we have

$$ay+b=aC(1-x)^{a}.$$

Whence $$(1-x)dy=(ay+b)dx.$$

Or $$dy-\frac{aydx}{1-x}=\frac{bdx}{1-x},$$ the given differential.

Thus we might verify the first example.

3. *Integrate the differential*

$$\frac{dy}{dQ}=\frac{2g}{a}\cos.Q+2am.y.$$

$$Ans.\ y=Ce^{2amQ}+\frac{2g\sin.Q}{a(1+4a^2m^2)}-\frac{4gm\cos.Q}{1+4a^2m^2}.$$

This example is solved in the author's Operations. It is the last problem in that work.

4. *Integrate the differential*

$$dy+2axydx=bx^2dx.$$

Here $P=2ax$, and $Q=bx^2$. $\int Pdx=ax^2$.

Whence $z=e^{-ax^2}$, and $dX=\frac{Qdx}{z}=\frac{bx^2dx}{e^{-ax^2}}=bx^2e^{ax^2}dx.$

$$X=b\int x^2e^{ax^2}dx.$$

The integral of this last expression depends on a series, and therefore it can only be found approximately, and as the differential applies to no particular problem or question in philosophy, we leave it thus:

$$y=be^{-ax^2}\int x^2e^{ax^2}dx+C.$$

Miscellaneous Examples.

1. Draw the line indicated by the equation

$$y^2=4x^2-12x+9.$$

2. Draw the line indicated by the equation

$$2y+3x-6=0.$$

3. Draw the lines indicated by the equations

$$-y+x=0, \quad \text{and} \quad y+x=0.$$

4. Determine the angle formed by the intersection of the two lines indicated by the equations

$$2y+4x+1=0, \qquad y-10x+3=0.$$

Ans. The obtuse angle is 147° 43′ 27″6.
The acute angle is 32° 16′ 34″4.

5. Determine the angle formed by the intersection of the two lines whose equations are

$$y=3x+7, \qquad y=-7x+10,$$

and find the co-ordinates of the point of intersection.

Ans. The acute angle is 26° 34′ 8″, the obtuse angle is therefore 153° 33′ 52″, and if we represent the required co-ordinates by x', y', we shall find $x'=0.31$, $y'=7.9293$.

6. Describe the circle whose equation is

$$x^2+y^2+8x-6y=11.$$

That is, find the radius and the co-ordinates of the center.

Ans. Let x', y', represent the co-ordinates of the center, and R the radius, we shall find $R=6$, $x'=3$, $y'=-4$.

7. Describe the circle whose equation is

$$x^2+y^2-4x-4y=8.$$

8. Describe the curve whose equation is

$$y=2x^3-5x^2+2.$$

9. The hypotenuse of a right angled triangle is constant, but the perpendicular varies: what will be the corresponding va-

riation of the other side, and what effect will be produced on the acute angles?

10. What is the differential of $u=3x+5x^3+b$?

Ans. $du=(3+15x^2)dx$.

11. What is the differential of $u=(a+\sqrt{x})^n$?

$$Ans.\ du=\frac{n(a+\sqrt{x})^{n-1}dx}{2\sqrt{x}}.$$

12. What is the differential coefficient of $u=\dfrac{1+x}{\sqrt{1-x}}$.

$$Ans.\ \frac{du}{dx}=\frac{3-x}{2(1-x)^{\frac{3}{2}}}$$

13. What is the differential coefficient of $u=a+\dfrac{4\sqrt{x}}{3+x^2}$.

$$Ans.\ \frac{du}{dx}=\frac{6(1-x^2)}{(3+x^2)^2x^{\frac{1}{2}}}.$$

14. What is the first derived polynomial of the algebraic equation

$$x^3-17x^2+54x-350=0?$$

Ans. $3x^2-34x+54=0$.

15. What is the differential of $u=a^x+by$?

$$Ans.\ du=\left(\log.a\,dx+\frac{dy}{y}\right)(a^x+by).$$

16. What is the differential of $u=x\log.x$?

Ans. $du=(1+\log.x)dx$.

17. Differentiate $u=\log.\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$.

$$Ans.\ du=-\frac{dx}{x\sqrt{1-x^2}}.$$

18. Find the arc whose logarithmic tangent varies three times as rapidly as the logarithmic cosine.

Let the arc be represented by x. Then the problem requires that $3d.(\log.\cos.x)=d.(\log.\tan.x)$.

Ans. $x=35°\ 16'\ 9''$.

19. Find the arc whose log. tangent varies five times as rapidly as the log. sine of the same arc.

Here $$5d.(\log \sin.x)=d.(\log. \tan.x).$$

Ans. $x=63° \ 25' \ 52''$.

20. Find the values of x which will render the function

$$y=ax^3-b^2x^2+C,$$

a maximum or minimum.

Ans. y is a minimum when $x=0$, and a maximum when $x=\frac{2b^2}{3a}$.

21. Divide the number 60 into two such parts that the square of one part diminished by 3 times the rectangle of the two parts shall be the greatest possible.

Ans. The parts are $22\frac{1}{2}$ and $37\frac{1}{2}$.

22. Find the greatest value of y corresponding to the equation $$y^2=\frac{x^3}{a-x}.$$

Ans. When y is greatest, $x=\frac{3a}{2}$, when least, $x=0$.

23. Required the sub-tangent of the curve whose equation is

$$xy^2=a^2(a-x).$$

Ans. $-\frac{2(ax-x^2)}{a}$.

24. The sub-tangent to a curve is $-\frac{2ax-x^2}{a}$, find the equation to that curve.

Ans. $xy^2=a^2(a-x.)$

N. B. To resolve this, we place the general expression for a sub-tangent $\left(y\frac{dx}{dy}\right)$ equal to $-\frac{2(ax-x^2)}{a}$, and separate the variables and integrate.

25. What is the length of the longest straight inflexible pole that can be put up a chimney, when the hight from the floor to the mantel is $=a$, and the depth from the front to the back $=b$?

Ans. $a\sqrt{1+\left(\frac{b}{a}\right)^{\frac{2}{3}}}+b\sqrt{1+\left(\frac{a}{b}\right)^{\frac{2}{3}}}$.

26. Find the equation of the curve whose sub-normal is $4x^{\frac{1}{3}}y$?

Ans. $y=3x^{\frac{4}{3}}+b.$

27. The tangent of a certain curve is represented by $y\sqrt{\frac{x+1}{x}}$, what is the equation of that curve, and what is the expression for its sub-normal?

Ans. The equation for the curve is $y=2\sqrt{x}+b.$

The sub-normal is represented by $\frac{y}{\sqrt{x}}.$

28. Required the area of a curve whose equation is $xy^2=a.$

Ans. The area is $=2xy.$

29. Find the equation of a curve whose area is expressed by twice the ordinate.

Ans. $x=2\log.y+b.$

An equation in which x is the abscissa and y the ordinate.

30. The sub-tangent of a curve is expressed by twice the rectangle of its co-ordinates. Find the equation of that curve.

Ans. $y=\frac{1}{2}\log.x+b.$

31. The expression for a tangent to a curve is $\frac{Ry}{x}$. Find the equation to that curve.

Ans. We place $y\sqrt{1+\frac{dx^2}{dy^2}}$, the general expression for a tangent equal to the given expression $\frac{Ry}{x}$; and by reduction and integration we find the curve to be a circle.

32. The sub-normal of a curve is $\frac{9}{2}x^2+3x+\frac{1}{2}$, find the equation of the curve.

Ans. $y^2=3x^3+3x^2+x+C.$

33. Find the equation of the curve whose area is expressed by two-thirds of the product of its co-ordinates.

Ans. $y^2=Cx$, but we may assume $C=2p$, then we have $y^2=2px$, the common parabola.

34. Find the equation of the curve whose sub-tangent is equal to the rectangle of its ordinate and sub-normal, x being the abscissa, and y the ordinate, the curve commencing at the origin of the co-ordinates.

Ans. $y^3=\frac{9}{4}x^2$, a cubical parabola.

35. In latitude 40° north when the sun's declination is 10° north, what time in the day will the variation of the sun's altitude be the greatest possible?

Ans. When the sun is due east or west.

N. B. In spherical trigonometry, we learn that

$$\cos.P=\frac{\sin.A-\sin.L\cos.D}{\cos.L\sin.D},$$

an equation in which $A=$ the sun's altitude, $L=$ the latitude, D the sun's polar distance, and P the angular distance of the sun from the meridian.

This problem requires us to find when dA shall be the greatest possible, L and D being constant quantities ; P will vary in consequence of the variation of A.

$$-\sin.PdP=\frac{\cos.A\,dA}{\cos.L\sin.D}. \qquad (1)$$

But $\cos.A : \sin.P :: \sin.D : \sin.Z$,

Z being the sun's azumuth. Whence $\cos.A=\frac{\sin.P\,\sin.D}{\sin.Z}$.

This placed in (1), and reduced, we find

$$dA=-dP.\cos.L\sin.Z. \qquad (2)$$

That is, dA is the variation of altitude for any small interval of time corresponding to dP, (the variation of the angle P being uniform,) therefore, as $-dP$, and $\cos.L$ are constant quantities, dA is greatest when $(\sin.Z)$ is greatest, or when the center of the sun is due east or west.

By means of the right angled spherical triangle we find in Lat. 40° north, when the sun's declination is 10° north, the sun must be due east 5h. 11m. 28s. before it comes to the meridian, and the same interval after meridian would bring it due west, provided the declination did not change during the interval.

36. In the last example we required the time of day when the variation of the sun's altitude is zero.

Ans. This is answered by placing $dA=0$ in (2) of last example. Then the second member of that equation is 0, which makes $Z=0$ or $dP=0$. Then the sun is on the meridian, or it is apparent noon.

37. The area of a curve is represented by x^2y, what is the sub-tangent to that curve?

Ans. $\frac{x^2}{1-2x}$.

38. The sub-normal of a curve is $\frac{a^2}{x}$, what is the equation of the curve?

Ans. $y^2=2a^2 \log. x$.

39. A curve is expressed by $\int\frac{ydy}{r-x}=\int dx$, what curve is it, or what is the equation of the curve?

Ans. $y=\sqrt{2rx-x^2}$, showing that it is the equation of the circle, the origin being on the curve.

40. The base of a right angled triangle is a, and the perpendicular x, and hypotenuse y; x and y are variable: what relation must exist between x, y, and a, when the variation of x is n times that of y?

Ans. $y=nx$, and $x=\frac{a}{\sqrt{n^2-1}}$.

41. Taking a triangle as designated in the preceding proposition, a variation of the perpendicular and hypotenuse will necessarily involve a variation in the acute angles. Determine that variation.

Ans. The acute angles will vary by a quantity whose sine or tangent is measured by $\frac{adx}{y^2}$, in words,

The sine or tangent is equal to the base multiplied by the variation of the perpendicular, and that product divided by the square of the hypotenuse.

Ex.—The base of a right angled triangle is 80, the perpendicular 60, and the hypotenuse 100 feet.

If the perpendicular be increased or diminished $\frac{1}{10}$ of a foot, what will be the corresponding variations of the acute angles?

Ans. The nat. sine or tan. is $\frac{80 \cdot \frac{1}{10}}{10000} = 0008$, $2' \, 45''$

log. .0008	—4.903090	
Add	10	
log. sine	6.903090	(See Robinson's Geom.
sub. log. of 1″	4.685575	page 161.)
log. of 165″	2.217515	

42. Integrate the equation $\frac{dy}{dx} = B - A \cot.x$.

Ans. $y = C + Bx - A \log.\sin.x$.

43. The hypotenuse of a right angled triangle is given. Required its dimensions when the perpendicular added to twice the base is a maximum.

Ans. If h represent the hypotenuse, $h\sqrt{\frac{1}{5}}$ is the perpendicular, $2h\sqrt{\frac{1}{5}}$ is the base.

44. The area of a curve is represented by $\frac{2}{5}y^{\frac{3}{2}}$, x and y being the co-ordinates; the curve commencing at the origin. What is its sub-normal?

Ans. The equation of the curve is $y = \frac{25x^2}{36}$, and the value of its sub-normal is $\frac{25}{18}xy$.

45. What is the sun's longitude when its variation in longitude is 10 times its variation in declination?

N. B. Let D represent the sun's declination, L its longitude, E the obliquity of the ecliptic. Then the fundamental equation is $\sin.D = \sin.E \sin.L$. (1) (Radius unity.)

By differentiation $\cos.DdD = \sin.E \cos.LdL$. (2)

The condition requires $dL = 10dD$.

This substituted in (2) and reduced, produces

$$\cos.D=10\sin.E\cos.L. \qquad (3)$$

By squaring (1) and (3), and adding them, observing that $\sin.^2D+\cos.^2D=1$, and still further reducing, we shall have

$$\cos.^2L=\frac{\cot.^2E}{99}, \quad \text{or} \quad \cos.L=\frac{\cot.E}{\sqrt{n^2-1}},$$

when n represents the ratio of the variation expressed generally. And this is the answer in *general terms*.

Ans. The sun's longitude is 76° 37′ 12″ from the equinoxes, that is to say, lon. 76° 37′ 12″, 103° 22′ 48″, 256° 37′ 12″, and longitude 283° 22′ 48″.

46. In latitude 42° north, when the sun's declination was 12° north, the shadow of a perpendicular post, 10 feet high, extended 22 feet horizontally, it being in the forenoon. What was the time of day, and what time must elapse for the shadow to contract $\frac{3}{10}$ of a foot? The semi-diameter of the sun being 15′ 54″.

Let $A=$ the altitude of the sun at the time the shadow extended 22 feet. Then the tangent of the apparent altitude of the upper limb is found by the following proportion:

R : tan.A : : 22 : 10 tan.A=log.11.000000—log.1.342423=
log. tan.A=9.657577=24° 26′ 39″
When the shadow was 21.7 feet, the alt. was =24° 44′ 30″

The difference of these altitudes is 17′ 51″=1071″, which we take for the *differential* of the first altitude.

The altitudes computed from a shadow correspond to the upper limb of the sun,—therefore to obtain the true altitude of the sun's center at the same time, we must subtract the sun's semidiameter and the refraction.

In this case the sun's semi-diameter=15′ 54″, and the refraction 2′ 8″, both subtractive. Hence, from 24° 26′ 39″ we take 18′ 2″, and we have 24° 8′ 37″ for the sun's altitude when the shadow of 10 feet perpendicular extended 22 feet horizontally.

Let $L=$ the latitude of the observer, and $D=$ the sun's polar distance, then with the true altitude of the sun's center, we find its meridian distance P=68° 12′ 40″, and the time from *apparent noon* is 4h 32m 51s, or it is 7h 27m 9s apparent time A. M.

(See Robinson's Geometry, page 211.)

Solar time is deduced from the spherical equation

$$\cos.P=\frac{\sin.A-\sin.L\cos.D}{\cos.L\sin.D}, \quad \text{(See Robinson's Geom. p. 209.)}$$

in which L and D are constant quantities, and P varies in consequence of the variation of A, therefore by taking the differential, we shall have

$$-\sin P\,dP=\frac{\cos.A.dA}{\cos.L\sin.D}.$$

Or $$dP=-\frac{\cos.A\,dA}{\sin.P\cos.L\sin.D}, \text{ (radius unity.)}$$

The minus sign indicates that P decreases while A increases, which is true whatever be the time of day.

To find the value of dP, we have $A=24°\ 8'\ 38''$, $dA=1071''$, $L=42°$, $D=78°$, and $P=68°\ 12'\ 40''$.

$\cos.A$ 24° 8′ 38″ (radius 1)............		-1.960243
dA 1071″............................		3.029789
		2.990032
sin. 68° 12′ 40″............	-1.967810	
cos. 42°..................	-1.871073	
sin. 78°..................	-1.990404	
		-1.829287
$dP=1448''$........................		3.160745

Fifteen seconds of arc correspond to *one second* of time, therefore 1448″ corresponds to 1 minute $36\frac{1}{2}$ seconds, and in this interval the shadow will contract three-tenths of a foot.

If to 7h 27m 9s we add 1m 36s, we shall have 7h 28m 45s for the mean time when the shadow extended 21.7 feet.

If the second altitude be corrected, and the time corresponding be computed, the result will be 7h 28m 48s, a result within *three seconds* of the differential method, but the differential method is the most accurate for small differences.

47. The logarithmic differential of the sine of an arc is six times the logarithmic differential of the cosine of the same arc? What is the arc?

Ans. 22° 12′ nearly.