SIR WILLIAM R. HAMILTON read a paper—

ON ANHARMONIC CO-ORDINATES.

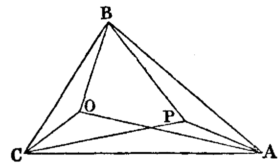

Fig. 1.

1. LET ABC be any given triangle; and let O, P be any two points in its plane, whereof O shall be supposed to be given or constant, but P variable. Then, by a well-known theorem, respecting the six segments into which the sides are cut by right lines drawn from the vertices of a triangle to any common point the three following *anharmonics of pencils* have a product equal to positive unity:—

$$(A.PCOB).(B.PAOC).(C.PBOA) = +1.$$

It is, therefore, allowed to establish the following system of three equations, of which any one is a consequence of the other two:—

$$\frac{y}{z} = (A.PCOB); \; \frac{z}{x} = (B.PAOC); \; \frac{x}{y} = (C.PBOA);$$

and, when this is done, I call the three quantities x, y, z, or any quantities proportional to them, the *Anharmonic Co-ordinates of the Point* P, with respect to the *given triangle* ABC, and to the *given point* O. And I *denote* that point P by the *symbol*,

$$P = (x, y, z); \text{ or, } P = (tx, ty, tz); \text{ \&c.}$$

2. When the variable point P takes the given position O, the three anharmonics of pencils above mentioned become each equal to unity; so that we may write then,

$$x = y = z = 1.$$

The given point O is therefore denoted by the symbol,

$$O = (1, 1, 1);$$

on which account I call it the *Unit-Point*.

3. When the variable point P comes to coincide with the given point A, so as to be at the vertex of the first pencil, but on the second ray of the second pencil, and on the fourth ray of the third, without being at the vertex of either of the two latter pencils, then the first anharmonic becomes indeterminate, but the second is equal to zero, and the third is infinite. We are, therefore, to consider y and z, but not x, as vanishing for this position of P; and consequently may write,

$$A = (1, 0, 0).$$

In like manner,

$$B = (0, 1, 0), \text{ and } C = (0, 0, 1);$$

and on account of these simple representations of its three corners, I call the given triangle ABC the *Unit-Triangle*.

4. Again, let the sides of this given triangle ABC be cut by a given transversal A′B′C′, and by a variable transversal LMN. Then, by another very well known theorem respecting segments, we shall have the relation,

$$(\mathrm{LBA'C}) \,.\, (\mathrm{MCB'A}) \,.\, (\mathrm{NAC'B}) = +1\,;$$

it is therefore permitted to establish the three equations,

$$\frac{m}{n} = (\mathrm{LBA'C}),\ \frac{n}{l} = (\mathrm{MCB'A}),\ \frac{l}{m} = (\mathrm{NAC'B});$$

where l, m, n, or any quantities proportional to them, are what I call the *Anharmonic Co-ordinates of the Line* LMN, with respect to the given triangle ABC, and the given transversal A′B′C′. And I *denote* the *line* LMN by the *symbol*,

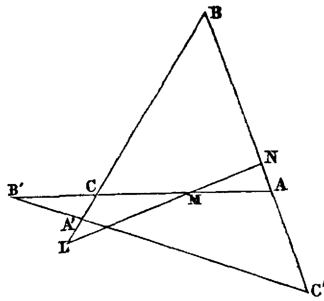

Fig. 2.

$$\overline{\mathrm{LMN}} = [l, m, n].$$

For example, if this variable line come to coincide with the given line A′B′C′, then

$$l = m = n\,;$$

so that this given line may be thus denoted,

$$\overline{\mathrm{A'B'C'}} = [1, 1, 1]\,;$$

on which account I call the *given transversal* A′B′C′ the *Unit-Line* of the Figure. The *sides*, BC, &c., of the given triangle ABC, take on this plan the symbols [1, 0, 0], [0, 1, 0], [0, 0, 1].

5. Suppose now that the *unit-point* and *unit-line* are related to each other, as being (in a known sense) *pole* and *polar*, with respect to the given or *unit-triangle;* or, in other words, let the lines OA, OB, OC be supposed to meet the sides BC, CA, AB of that given triangle, in points

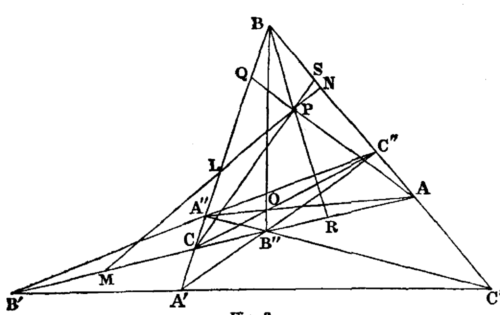

Fig. 3.

A″, B″, C″, which are, with respect to those sides, the harmonic conjugates of the points A′, B′, C′, in which the same sides are cut by the given transversal A′B′C′. Also, let the variable *point* P be situated upon the variable *line* LMN; and let Q, R, S be the intersections of AP, BP, CP with BC, CA, AB. Then, because

$$(BA'CA'') = (CB'AB'') = (AC'BC'') = -1,$$

we have

$$\begin{cases} -\frac{m}{n} = (LBA''C), -\frac{n}{l} = (MCB''A), -\frac{l}{m} = (NAC''B) \\ -\frac{n}{m} = (LCA''B), -\frac{l}{n} = (MAB''C), -\frac{m}{l} = (NBC''A); \end{cases}$$

as well as

$$\begin{cases} \frac{y}{z} = (QCA''B), \frac{z}{x} = (RAB''C), \frac{x}{y} = (SBC''A), \\ \frac{z}{y} = (QBA''C), \frac{x}{z} = (RCB''A), \frac{y}{x} = (SAC''B); \end{cases}$$

and therefore,

$$\frac{-lx}{nz} = (MARC); \ \frac{-my}{nz} = (LBQC).$$

But, by the pencil through P,

$$(MARC) = (LQBC);$$

and by the *definition* of the symbol (ABCD), for any four collinear points,

$$(ABCD) = \frac{AB}{BC} \cdot \frac{CD}{DA},$$

which is here throughout adopted, we have the *identity*,

$$(ABCD) + (ACBD) = 1;$$

therefore

$$(MARC) + (LBQC) = 1,$$

or,

$$lx + my + nz = 0.$$

6. We arrive then at the following *Theorem*, which is of fundamental importance in the present system of Anharmonic Co-ordinates:—

"If the unit-point O be the pole of the unit-line A′B′C′, with respect to the unit-triangle ABC, and if a variable point P, or (x, y, z), be situated anywhere on a variable right line LMN, or $[l, m, n]$, then the sum of the products of the corresponding co-ordinates of point and line is zero."

7. It may already be considered as an evident consequence of this Theorem, that any *homogeneous equation* of the p^{th} *dimension*,

$$f_p(x, y, z) = 0,$$

represents a *curve of the p^{th} order*, considered as the *locus* of the variable *point* P; and that any homogeneous equation of the q^{th} dimension, of the form

$$F_q(l, m, n) = 0,$$

may in like manner be considered as the *tangential equation* of a *curve of the q^{th} class*, which is the *envelope* of the variable *line* LMN. But any examples of such applications must be reserved for a future communication. Meantime, I may just mention that I have been, for some time back, in possession of an analogous method for treating Points, Lines, Planes, Curves, and Surfaces in Space, by a system of Anharmonic Co-ordinates.

8. As regards the *advantages* of the Method which has been thus briefly sketched, the *first* may be said to be its geometrical *interpretability*, in a manner *unaffected by perspective*. The *relations*, whether between *variables* or between *constants*, which enter into the formula of this method, are *all projective;* because they *all* depend upon, and are referred to, *anharmonic functions*, of groups or of pencils.

9. In the *second* place, we may remark that the great principle of *geometrical Duality* is recognised from the very outset. Confining ourselves, for the moment (as in the foregoing articles), to figures in a *given plane*, we have seen that the *anharmonic co-ordinates* of a *point*, and those of a *right line*, are deduced by processes absolutely *similar*, the one from a system of *four given points*, and the other from a system of *four given right lines*. And the *fundamental equation* ($lx + my + nz = 0$) which has been found to *connect* these *two systems* of co-ordinates, is evidently one of the most perfect *symmetry*, as regards *points* and *lines*. An analogous symmetry will show itself afterwards, in relation to points and planes.

10. The *third advantage* of the anharmonic method may be stated to consist in its possessing an *increased number of disposable constants*. Thus, within the plane, *trilinear* co-ordinates give us *only six* such constants, corresponding to the *three* disposable *positions* of the *sides* of that assumed *triangle*, to the perpendicular distances from which the co-ordinates are supposed to be proportional; but *anharmonics*, by admitting an *arbitrary unit-point*, enable us to treat *two other constants* as disposable, the *number* of such constants being thus raised from *six* to *eight*. Again, in *space*, whereas *quadriplanar* co-ordinates, considered as the *ratios of the distances* from *four assumed planes*, allow of only *twelve* disposable constants, corresponding to the possible selection of the *four planes of reference*, *anharmonic co-ordinates*, on the contrary, which admit either *five planes* or *five points* as *data*, and which might, therefore, be called *quinquiplanar* or *quinquipunctual*, permit us to dispose of no fewer than *fifteen constants as arbitrary*, in the general treatment of *surfaces*.

(*To be continued.*)