ELEMENTS OF ALGEBRA

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PREFACE.

In the preparation of this work the authors have followed their usual plan of attempting to allow the light of modern mathematics to shine in upon the old, and to do this by means of a text-book which shall be usable in American high schools, academies, and normal schools.

In general, the beaten paths have been followed, experience having developed these and having shown their safety and value. But where there is an unquestionable gain in departing from these paths the step has been taken. For example, the subject of factoring has recently attracted the attention it deserves; in fact, several writers have carried it to an unjustifiable extreme; but there are few text-books that mention the subject after the chapter is closed; it is taught with no applications, and the student is usually left with the idea that it has none. The authors have departed from this plan, and have followed the chapter with certain elementary applications, using the method in solving easy quadratic and higher equations, making much use of it in fractions, and not ceasing to review it and its applications until it has come to be a familiar and indispensable tool. By following such a scheme the student knows much of quadratics before he reaches the chapter on the subject, and he enters upon it with increased intelligence and confidence.

The arrangement of chapters has been the subject of considerable experiment of late. But the plan adopted in this work is, in general, based upon the following:

1. The new should grow out of the old, as the expressions of algebra out of those of arithmetic, the negative number out of familiar concepts, factors out of elementary functions, quadratic and higher equations out of factoring, the theory of indices out of the three fundamental laws for positive integral indices, the complex number out of the surd, and so on.

2. The student's interest should be excited as early as possible, and it should be maintained by reviews and by applications to modern concrete problems. To this end the equation has been introduced in the first chapter, with simple applications, and general review exercises have been inserted at frequent intervals.

3. The new should be introduced where it is needed. To put the remainder theorem where it is usually placed, at the end of the work, is entirely unwarranted; it is needed just before factoring. To put complex numbers after quadratics is equally unscientific, for they are met on the very threshold of this subject.

Considerable attention has been given to the illustration of algebraic laws by graphic forms. The value of this plan is evident; the picture method, the coördination of the concrete and the abstract, the one-to-one correspondence between thought and thing — this has been recognized too long to require argument. This method of making algebraic abstractions seem real is followed in the presentation of certain fundamental laws (p. 37), in the study of certain common products (p. 51), but more especially in the treatment of the complex number (p. 236) — a subject usually passed with no understanding, — and (in the Appendix) in the study of equations.

Where the time and the maturity of the class allow, the Appendix may profitably be studied in connection with the several chapters to which it refers. This arrangement allows the teacher to cover the usual course, or to make it somewhat more elaborate if desired.

It need hardly be said that no class is expected to solve more than half of the exercises, the large number being inserted to allow of a change from year to year.

It is the hope of the authors that their efforts to prepare a text-book adapted to American schools of the twentieth century may meet the approval of teachers and students. It is believed that they have lessened the general average of difficulty of the old-style text-book, while greatly adding to the mathematical spirit.

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TABLE OF CONTENTS.

CHAPTER I.

INTRODUCTION TO ALGEBRA.

CHAPTER II.

ADDITION AND SUBTRACTION.

CHAPTER III.

MULTIPLICATION.

CHAPTER IX.

SIMPLE EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

CHAPTER X.

SIMPLE EQUATIONS INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

CHAPTER XI.

CHAPTER XII.

THE THEORY OF INDICES.

CHAPTER XIII.

COMPLEX NUMBERS.

CHAPTER XIV.

QUADRATIC EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

CHAPTER XV.

SIMULTANEOUS QUADRATIC EQUATIONS.

CHAPTER XVI.

CHAPTER XVII.

RATIO, VARIATION, PROPORTION.

CHAPTER XVIII.

SERIES.

CHAPTER XIX.

CHAPTER XX.

CHAPTER XXI.

APPENDIX.

TABLES.

ELEMENTS OF ALGEBRA.

CHAPTER I.

INTRODUCTION TO ALGEBRA.

I. ALGEBRAIC EXPRESSIONS.

1. There is no dividing line between the arithmetic with which the student is familiar and the algebra which he is about to study. Each employs the symbols of the other, each deals with numbers, each employs expressions of equality, and each uses letters to represent numbers.

In arithmetic the student has learned the meaning of 2^2; in algebra he will go farther and will learn the meaning of $2^{\frac{2}{3}}$. In arithmetic he has learned the meaning of $3 - 2$; in algebra he will go farther and will learn the meaning of $2 - 3$.

In arithmetic he has said,

If $2 \times$ some number equals 10,
the number must be $\frac{1}{2}$ of 10, or 5.

In algebra he will express this more briefly, thus:

If $\quad 2x = 10,$

then $\quad x = 5;$

indeed he may already have met this form in arithmetic.

By arithmetic he probably could not solve a problem of this nature: The square of a certain number, added to 5 times that number, equals 50; to find the number. But after studying algebra a short time, he will find the solution quite simple.

In arithmetic it is quite common to use a letter to represent a number, as r to represent the *rate* of interest, i to represent the *interest* itself, p the *principal*, etc. In algebra this is much more common. In arithmetic it is customary to denote multiplication by the symbol $\times$, the product of 5% and \$100 being written 5% $\times$ \$100, and the product of r and p by $r \times p$; but in algebra the latter product is represented by rp.

In expressing 5 times 2 we cannot write it 52, because that means $50 + 2$. But where only letters are used, or one numeral and one or more letters, *we may define the absence of a sign to mean multiplication.* Thus, ab means $a \times b$, that is, the product of the numbers represented by a and b; $5\,ab$ means 5 times this product.

EXERCISES. I.

If $a = 5$, $b = 7$, $c = 3$, $d = 1$, $e = 4$, find the value of each of the expressions in exs. 1–9.

1. $5\,abd$.
2. $\frac{2}{3}\,acde$.
3. $\dfrac{21\,ae}{4\,bcd}$.
4. $\sqrt{21\,bce}$.
5. $\dfrac{be - ad}{23\,c}$.
6. $\sqrt{\dfrac{35\,ab}{3\,cde}}$.
7. $\dfrac{2\,a + 4\,d}{b}$.
8. $\sqrt[3]{225\,ac}$.
9. $\dfrac{a + d - e}{cd}$.

If $a = 2$, $b = 3$, $c = 4$, $d = 5$, find the value of each of the expressions in exs. 10–17.

10. $\dfrac{abc}{bcd} + \dfrac{abc}{acd}$.
11. $\dfrac{a + d}{7\,b} - \dfrac{c - b}{3}$.
12. $\dfrac{a}{b} + \dfrac{b}{c} - \dfrac{d}{bc}$.
13. $\dfrac{4}{a} + \dfrac{6}{b} - \dfrac{8}{c} + \dfrac{10}{d}$.
14. $\dfrac{c}{a} + \dfrac{3\,d}{b} - \dfrac{2\,b}{3\,a}$.
15. $\dfrac{a}{4} - \dfrac{b}{6} + \dfrac{c}{8} - \dfrac{d}{10}$.
16. $\dfrac{a + c}{b} + \dfrac{d - b}{a} - 3$.
17. $\dfrac{a + b + c + d}{7} + \dfrac{5\,a - d}{5}$.

2. A collection of letters, or of letters and other number-symbols, connected by any of the signs of operation (+, –, ×, ÷, etc.) is called an *algebraic expression.*

E.g., $3x + 2a$ is an algebraic expression, but $3 + 2$ is an arithmetical expression. So $2a$ is an algebraic expression, 2 and a being connected by the (understood) sign of multiplication; also a, since that means $1a$.

3. An algebraic expression containing neither the + nor the – sign of operation is called a **term** or a **monomial.**

E.g., $\frac{4}{5}ab$, $5\sqrt{ax}$, $\frac{10abx}{23yz}$, are monomials. In the expression $2ax + 3by - 5y^2$, the expressions $2ax$, $3by$, and $5y^2$, are the terms, and each taken by itself is called a monomial. The broader use of the word *term* is given in § 46.

4. An algebraic expression made up of several terms or numbers connected by the sign + or – is called a **polynomial.**

The word means *many-termed.* On all such new words consult the **Table of Etymologies** in the Appendix.

5. A polynomial of two terms is called a **binomial**, one of three terms a **trinomial.** Special names are not given to polynomials of more than three terms.

E.g., $\frac{2}{3}a^2 - \frac{c}{d}$ is a binomial. $5\sqrt{a} - \frac{b}{c} + ab^2cd$ is a trinomial.

EXERCISES. II.

1. Select the algebraic expressions in the following list:

(a) $3a^2bc$. (b) $\frac{2}{3}a^2bcd$.

(c) $\frac{2a}{b} - c^3d^2$. (d) $x^2 + y^2 + z^2$.

(e) $2 - 3\sqrt{7} + 1$. (f) $2x^3 - 3x^2 - 9x + 1$.

2. Out of the algebraic expressions select the monomials.

3. Out of the polynomials select the binomials; trinomials.

6. In the operation of multiplication expressed by $a \times b \times c$, or abc, the a, b, and c are called the **factors** of the expression, and the expression is called a **multiple** of any of its factors.

Factors should be carefully distinguished from terms. The former are connected by signs of multiplication, expressed or understood; the latter by signs of addition or subtraction.

7. Any factor of an expression is called the **coefficient** of the rest of the product. The word, however, is usually applied only to some factor whose numerical value is expressed or known and which appears first in the product.

E.g., in the expression $3\,ax$, 3 is the coefficient of ax, and $3\,a$ is the coefficient of x.

Since $a = 1\,a$, the coefficient 1 may be understood before any letter.

8. As in arithmetic, the product of several equal factors is called a **power** of one of them.

E.g., $2 \times 2 \times 2$ is called the third power of 2 and is written 2^3; $aaaaa$ is called the fifth power of a and is written a^5.

9. The number-symbol which shows how many equal factors enter into a power is called an **exponent.**

E.g., in 2^3, 3 is the exponent of 2; in a^5, 5 is the exponent of a. The exponent affects only the letter or number adjacent to which it stands; thus, ab^3 means $abbb$.

The exponent should be carefully distinguished from the coefficient. In the expression $2\,ax^3$, 2 is the coefficient of ax^3, and $2\,a$ of x^3; 3 is the exponent of x.

Since x may be considered as taken once as a factor to make itself, x^1 is defined as meaning x. Hence, any letter may be considered as having an exponent 1.

There are other kinds of powers and exponents besides those which have just been defined, and these will be discussed later in the work.

10. The **degree of a monomial** is determined by the number of its literal factors.

E.g., a^5 is of the 5th degree, a^3b^4 of the 7th, $3abc$ of the 3d, and $5a$ of the 1st. A number, like 5, is spoken of as of *zero degree* because it has no literal factors.

11. The word *degree* is usually limited, however, by reference to some particular letter.

Thus, while $3a^2x^3$ is of the 5th degree, it is said to be of the 3d degree in x, or of the 2d degree in a, or of zero degree in other letters.

12. Terms of the same degree in any letter are called **like terms** in that letter.

Thus, $3ax^2$ and $5ax^2$ are like terms, being of the same degree in each letter. $3ax^2$ and $5bx^2$ are like terms in x.

13. The **degree of a polynomial** is the highest degree of any of its terms.

Thus, $ax^2 + bx + c$ is of the second degree in x.

14. As in arithmetic, one of the two equal factors of a second power is called the **square** (or *second*) **root** of that power, one of the three equal factors of a third power the **cube** (or *third*) **root**, one of the four equal factors of a fourth power is called the **fourth root**, etc.

The word *root* has also a broader meaning, as in "the square root of 2," an expression which is legitimate, although 2 is not a second power of any integral or fractional number. This meaning will be discussed later.

The square root of a is indicated either by $\sqrt{a}$ or by $a^{\frac{1}{2}}$, the cube root by $\sqrt[3]{a}$ or by $a^{\frac{1}{3}}$, the fourth root by $\sqrt[4]{a}$ or by $a^{\frac{1}{4}}$, etc. In $a^{\frac{1}{2}}$, the $\frac{1}{2}$ is called a fractional exponent, and the term is read "a, exponent $\frac{1}{2}$," or "the square root of a," or "a to the $\frac{1}{2}$ power," a reading which will be justified by the subsequent explanation of the word *power*.

From what has been stated it will be seen that one of the features of algebra is the representation of numbers by letters. The advantages of this plan will soon appear.

Thus, if a number is represented by n, 5 times the square of *that* number will be represented by $5\,n^2$. If two numbers are represented by a and b, 3 times the cube of the first, divided by 5 times the square root of the second, will be represented by $\frac{3\,a^3}{5\,b^{\frac{1}{2}}}$ or by $\frac{3\,a^3}{5\sqrt{b}}$.

15. Those terms of a polynomial which contain letters constitute the **literal part** of the expression.

E.g., the literal part of $x^2 + 2\,x + 1$ is $x^2 + 2\,x$.

The expression is also used with respect to factors. Thus, the literal part of $\frac{1}{2}\,a\sqrt{2}$ is a.

EXERCISES. III.

1. What is the numerical value of each *term* in the following expressions, if $a = 1$, $b = 2$, $c = 5$, $d = 3$?

(a) $ab^2c^3d^4$. (b) $c^2 + b^4 - 3\,a$.

(c) $2\,d^3 - 10\,c - 2\,b$. (d) $\frac{a}{b} + \frac{c}{10} + \frac{b}{d} + \frac{a}{3}$.

2. In ex. 1, what is the numerical value of each polynomial?

3. In $13\,a^2b^3x$, what is the coefficient of x? of b^3x? of a^2b^3x? What is the degree of the expression? What is its degree in x? What is the exponent of a? of b? of x?

4. In the following monomials name the coefficients of the various powers of x, and also the exponents of x:

(a) $\frac{x}{a}$. (b) x^5. (c) $\frac{a^{\frac{1}{2}}}{3}x^9$.

(d) $23\,a^2x^3$. (e) $4\,a^2b^3cx^9$. (f) $\frac{2}{3}\,a^3\sqrt{b}\,x$.

5. From ax^2, $3\,bx^3$, cx^2, a^3x, and $10\,abx^2$, select the like terms in x or any of its powers.

6. From $3\,ax^2$, $9\,mx$, $14\,ax^3$, ax^2, $9\,ax^2$, and $144\,x$, select the like terms.

7. Express algebraically that if $x^2+y^2+2\,xy$ be divided by $x+y$ the quotient is $x+y$. (Use fractional form.)

8. What is the degree of the polynomial ax^2+bx+c? What is its degree in x? What is its value if $a=b=c=1$, and $x=5$?

9. Express algebraically that if the sum of a^2, ab, and b^2 be divided by the square of the binomial $c-d$, the quotient is x.

10. What is the meaning of the expression

$$4\,a^2-3\,b^{\frac{1}{2}}+6\,c-d^{\frac{1}{4}}?$$

(That from 4 times the square of a certain number there has been subtracted, etc.)

11. Also of the following expressions:

(a) $a^2+2\,ab+b^2$. (b) a^2-b^2.

(c) $3\,a^3-4\,b^{\frac{1}{3}}+a^{\frac{1}{2}}$. (d) $a^3+3\,a^2b+3\,ab^2+b^3$.

12. Represent algebraically the sum of 3 times the square of a number, $\frac{2}{3}$ the cube root of a second number, and 5 times the 5th power of a third number. What is the value of the expression, if the three numbers are respectively 2, 8, 1?

13. Given $a=4$, $b=6$, $c=9$, $d=16$, $e=8$, find the value of each of the following, and designate the expression as a monomial, binomial, etc.:

(a) $2\,a^2bc^{\frac{1}{2}}$. (b) $d^{\frac{1}{2}}e^{\frac{1}{3}}-b$.

(c) $a^{\frac{1}{2}}+b+d+e^2$. (d) $\dfrac{25\,abcde}{72}$.

(e) $25\,d^{\frac{1}{4}}+a^2-\dfrac{b}{3}+5$. (f) $\frac{2}{3}\,b^3-c^{\frac{1}{2}}+e^{\frac{1}{3}}$.

II. THE EQUATION.

16. An equality which exists only for particular values of certain letters representing the **unknown quantities** is called an **equation**. These particular values are called the **roots** of the equation.

Thus, $x + 3 = 5$ is an equation because the equality is true only for a particular value of the unknown quantity x, that is, for $x = 2$. This equation contains only one unknown quantity.

$2 + 3 = 5$ expresses an equality, but it is not an equation as the word is used in algebra.

17. The discovery of the roots is called the **solution of the equation**, and these roots are said to *satisfy the equation.*

Thus, if $x + 5 = 9$, the equation is solved when it is seen that $x = 4$. This value of x satisfies the equation, for $4 + 5 = 9$.

18. If two algebraic expressions have the same value whatever numbers are substituted for the letters, they are said to be **identical**.

Thus, $a^2 + \frac{ab}{a}$ is identical to $a^2 + b$, and $a + b$ to $b + a$.

An identity is indicated by the symbol $\equiv$, as in $a^2 + b \equiv b + a^2$.

19. The part of an equation to the left of the sign of equality is called the **first member**, that to the right the **second member**, and similarly for an identity.

The two members are often spoken of as "the left side" and "the right side," respectively.

The extensive use of the equation is one of the characteristic features of algebra.

The importance and the treatment of the equation will best be understood by considering a few problems.

In each case we say, "Let $x =$ the number," meaning that x is to *represent* the unknown quantity.

1. *Find the number to twice which if* 3 *is added the result is* 11.

1. Let $x =$ the number.
2. Then $2x =$ twice the number.
3. Hence, $2x + 3 = 11.$ (Why?)
4. Subtracting 3 from these equals, the results must be equal, and $$2x = 11 - 3, \text{ or } 8.$$
5. Dividing these equals by 2, the results must be equal, and $$x = 4.$$

Check. To see if this value of x satisfies the equation, substitute it in step 3. Since $2 \times 4 + 3 = 11$, the result is correct. This is called checking or verifying the result.

20. **A check on an operation** is another operation whose result tends to verify the result of the first.

E.g., if $11 - 7 = 4$, then $4 + 7$ should equal 11; this second result, 11, verifies the first result, 4.

The secret of accurate work in algebra and in arithmetic lies largely in the continued use of proper checks.

21. **A check on a solution of an equation** is such a substitution of the root as shows that it satisfies the given equation.

This substitution must always be made in the original equation or in the statement of the problem. Thus, in the above solution it would not answer to substitute the root, 4, in step 4, because a mistake might have been made in getting step 4 from step 3.

2. *Two-thirds of a certain number, added to* 5, *equals* 17. *What is the number?*

1. Let $x =$ the number.
2. Then $\frac{2}{3}x + 5 = 17$, by the conditions of the problem.
3. Subtracting 5 from these equals, the results must be equal, and $$\frac{2}{3}x = 12.$$
4. Therefore, $x = 18.$

Check. $\frac{2}{3}$ of $18 = 12$, and $12 + 5 = 17$.

3. *72 divided by a certain number equals twice that number. What is the number?*

1. Let $x =$ the number.
2. Then $\frac{72}{x} =$ twice the number, by the conditions of the problem.
3. Therefore, $\frac{72}{x} = 2x$.
4. Multiplying these equals by x, the results must be equal, and
$$72 = 2x^2.$$
5. Dividing these equals by 2,
$$36 = x^2.$$
6. Extracting the square roots of these equals,
$$6 = x.$$

Check. $\frac{72}{6} = 12$, and $12 = 2 \times 6$.

4. *If from 35 a certain number is subtracted, the difference equals the sum of twice that number and 20. What is the number?*

1. Let $x =$ the number.
2. Then $35 - x = 2x + 20$. (Why?)
3. Then $35 = 3x + 20$, by adding x.
4. Then $15 = 3x$. (Why?)
5. Then $5 = x$. (Why?)

Check. (What should it be?)

From the preceding problems it will be seen that the two members of an equation are like the weights in two pans of a pair of scales which balance evenly; if a weight is taken from one pan, an equal weight must be taken from the other if the even balance is preserved; if a weight is added to one pan, an equal weight must be added to the other; and, in general, any change made in one side requires a like change in the other.

These facts are already known from arithmetic, where the equation is frequently met. Even in primary grades problems are given like $2 \times (?) = 12$, this being merely an equation with the symbol (?) in place of x.

22. The axioms. There are several general statements (of which a few have already been used) so obvious that their truth may be taken for granted. Such statements are called **axioms.**

The following are the axioms most frequently met in elementary algebra.

1. *Quantities which are equal to the same quantity, or to equal quantities, are equal to each other.*

That is, if $5 - x = 3$, and $1 + x = 3$, then $5 - x = 1 + x$.

2. *If equals are added to equals, the sums are equal.*

That is, if $x = y$, then $x + 2 = y + 2$.

3. *If equals are subtracted from equals, the remainders are equal.*

That is, if $x + 2 = 9$, then $x = 9 - 2$, or 7.

4. *If equals are added to unequals, the sums are unequal in the same sense.*

"In the same sense" means that if the first was greater than the second before the addition of the equals, it is after. Thus, if x is greater than 8, $x + 2$ is also greater than 10.

5. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*

That is, if x is less than 16, $x - 3$ is less than 13.

6. *If equals are multiplied by equal numbers, the products are equal.*

That is, if $\frac{x}{3} = 6$, $x = 3 \times 6$, or 18.

7. *If equals are divided by equals, the quotients are equal.*

That is, if $2x = 6$, $x = 6 \div 2$, or 3.

8. *Like powers of equal numbers are equal.*

That is, if $x = 5$, $x^2 = 25$. We here speak of x as a number because it represents one.

9. *Like roots of equal numbers are arithmetically equal.*

That is, if $x^2 = 36$, $x = 6$. The axiom says "arithmetically equal," because it will soon be found that there is an algebraic sense in which roots require special consideration.

These axioms should at once be learned by number.

23. Stating the equation. The greatest difficulty experienced by the student in the solution of problems is in the statement of the conditions in algebraic language. After the equation is formed the solution is usually simple.

While there is no method applicable to all cases, the following questions usually lead the student to the statement:

1. *What shall* x *represent?* In general, x represents the number in question.

E.g., in the problem, "Two-thirds of a certain number, plus 10, equals 30, what is the number?" x represents the *number*.

2. *For what number described in the problem may two expressions be found?*

Thus, in the above problem, 30 and "$\frac{2}{3}$ of a certain number, plus 10," are two expressions for the same number.

3. *How do you state the equality of these expressions in algebraic language?*

$$\tfrac{2}{3}x + 10 = 30.$$

EXERCISES. IV.

Form the equations for the following problems:

1. The difference of two numbers is 14 and the smaller is 3. What is the larger?

2. A's money is three times B's, and together they have $364. How much has B?

3. The sum of two numbers is 60 and the difference is 40. What is the smaller number?

Typical solutions. In the solution of problems involving equations, the axioms need not be stated in full except when this is required by the teacher. The check (which is a complete verification) should always be given in full, except when the teacher directs to the contrary. The following solutions may be taken as types:

1. *What is that number to whose square root if* 2 *is added the result is* 7 ?

1.	Let	$x =$ the number.	
2.	Then	$\sqrt{x} + 2 = 7$, by the conditions.	
3.	$\therefore$	$\sqrt{x} = 5.$	Ax. 3
4.	$\therefore$	$x = 25.$	Ax. 8

Check. $\sqrt{25} + 2 = 5 + 2 = 7.$

2. *What is that number from two-thirds of which if* 5 *is subtracted the result is* 10 ?

1.	Let	$x =$ the number.	
2.	Then	$\frac{2}{3}x - 5 = 10$, by the conditions.	
3.	$\therefore$	$\frac{2}{3}x - 5 + 5 = 15$, or $\frac{2}{3}x = 15.$	Ax. (?)
4.	$\therefore$	$x = 22\frac{1}{2}.$	Ax. (?)

Check. $\frac{2}{3}$ of $22\frac{1}{2} = 15$, and $15 - 5 = 10.$

3. *Find the value of* x *in the equation* $\sqrt{x} + 1 = \frac{1}{3} + 7.$

1.		$\sqrt{x} + 1 = \frac{1}{3} + 7.$	Given
2.	$\therefore$	$\sqrt{x} = 6\frac{1}{3}$, or $\frac{19}{3}.$	Ax. 3
3.	$\therefore$	$x = \frac{361}{9}$, or $40\frac{1}{9}.$	Ax. 8

Check. (Give it.)

4. *Find the value of* x *in the equation* $5x - 3 = x + 7.$

1.		$5x - 3 = x + 7.$	Given
2.	$\therefore$	$5x = x + 10.$	(Why ? See ex. 2, step 3)
3.	$\therefore$	$4x = 10$, for $5x - x$ means $5x - 1x.$	
4.	$\therefore$	$x = 2\frac{1}{2}.$	(Why ?)

Check. (Give it.)

EXERCISES. V.

1. Find the value of x in the equation $2x + 2 = 30 + x$.

2. Also in $\frac{20}{x} = 5$.

3. Also in $\frac{2}{x} = \frac{x}{2}$.

4. Also in $x^2 + 7 = 88$.

5. Also in $x^2 - 1 = 35$.

6. Also in $\frac{2}{3}x + 5 = \frac{1}{3}x + 20$.

7. Also in $22x + 30 = 17x + 70$.

8. Also in $250x - 20 = 20x + 440$.

9. Also in $12.75x + 6.25 = 7.25x + 17.25$.

10. What number is that which divided by 3 equals $\frac{4}{7}$?

11. What is the number whose half added to 16 equals 21?

12. What is the number whose twentieth part added to 10 equals 20?

$\frac{1}{20}x + 10 = 20$

13. What is that number to whose square if 5 is added the result is 41?

14. What is that number to whose square root if 5 is added the result is 41?

15. What is that number from one-third of which if 27 is subtracted the result is 5?

16. There is a number by which if 9 is divided the quotient is that number. Find it.

17. The sum of a certain number and 9 is equal to the sum of 1 and three times that number. Find the number.

18. The sum of a certain number, twice that number, and twice this second number, is 70. What is the first number?

19. The united ages of a father and son amount to 100 years, the father being 40 years older than the son. What is the age of the son?

Practical applications. The equation offers a valuable method for solving many practical problems, of which a few types will now be considered.

1. *What sum of money placed at interest for* 1 *year at* $4\frac{1}{2}\%$ *amounts to* \$836?

1. Let $x =$ the *number* of dollars.
2. Then $x + 0.04\frac{1}{2}x =$ the number of dollars in the principal + the interest.
3. But $836 =$ the number of dollars in the principal + the interest.
4. $\therefore$ $x + 0.04\frac{1}{2}x = 836.$
5. Or $1.04\frac{1}{2}x = 836.$
6. $\therefore$ $x = 800.$ Ax. 7
7. $\therefore$ the sum is \$800.

Check. $800 + 0.04\frac{1}{2}$ of $800 = 836.$

It should be noticed that since x stands for the *number* of dollars, when it is found that $x = 800$ it is known that the result is \$800.

In the applied problems of algebra, x *is always taken to represent an abstract number,* and the first step should always state definitely to what this abstract number is to refer.

2. *A commission merchant sold some produce on a commission of* 2%, *and paid* \$5 *for freight and cartage, remitting* \$117.50. *For how much did he sell the produce?*

1. Let $x =$ the *number* of dollars received.
2. Then $x - 0.02x =$ the number after deducting 2%.
3. And $x - 0.02x - 5 =$ the number after deducting for cartage also.
4. $\therefore$ $x - 0.02x - 5 = 117.50.$
5. $\therefore$ $0.98x = 122.50.$ (Why?)
6. $\therefore$ $x = 125.$ (Why?)

Check. $125 - 0.02$ of $125 - 5 = 117.50.$

3. *After deducting* $\frac{1}{10}$ *and then* $\frac{1}{6}$ *from a certain sum there remains* \$49.50. *Required the sum.*

1. Let $x =$ the *number* of dollars.
2. Then $x - \frac{1}{10}x = \frac{9}{10}x$, the number of dollars after deducting $\frac{1}{10}$.
3. From this $\frac{9}{10}x$ is to be taken $\frac{1}{6}$ of it,

$$\therefore \quad \frac{9}{10}x - \frac{1}{6} \text{ of } \frac{9}{10}x = \frac{18}{20}x - \frac{3}{20}x$$
$$= \frac{3}{4}x.$$

4. $\therefore \quad \frac{3}{4}x = 49.50.$
5. $\therefore \quad x = 49.50 \div \frac{3}{4}$
$$= 66.$$

$\therefore$ the sum is \$66.

Check. $66 - \frac{1}{10}$ of $66 = 59.40$. $\quad 59.40 - \frac{1}{6}$ of $59.40 = 49.50$.

EXERCISES. VI.

1. In how many years will \$100 double itself at 5% interest?

2. What sum of money put at interest for 2 years at 6% amounts to \$84?

3. In how many years will a sum of money double itself at 6% simple interest?

4. In how many years will \$80 amount to \$200, at 6% interest? ($80 + x \times 6\%$ of $80 = 200$.)

5. What is the rate per cent of premium for insuring a house for \$2000, when the premium is \$30?

6. Taking the number of units of area of a circle as being $3\frac{1}{7}$ times the square of the number of units of length in the radius, find the radius of the circle whose area contains $77\frac{7}{9}$ units.

7. After selling some goods on 5% commission, a merchant remits, as the net proceeds, \$79.80. How much is his commission? (Let $x =$ the *number* of dollars for which the goods were sold; after finding x take 5% of it.)

III. THE NEGATIVE NUMBER.

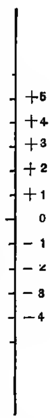
24. In remote times men could count only by what are often called *natural numbers,* that is, 1, 2, 3, 4, 5, $\cdots$. Such numbers suffice to solve an equation like $x - 3 = 0$, an equation in which x must evidently be 3.

Mankind then introduced the *unit fraction,* that is, a fraction with the numerator 1. Such numbers are necessary in solving an equation like $2x - 1 = 0$. (Solve it.)

Then came the *common fraction* with any numerator, as $\frac{2}{3}$, $\frac{5}{7}$, $\frac{10}{11}$, $\cdots$. Such numbers are necessary in solving an equation like $3x - 2 = 0$. (Solve it.)

The idea of number was then enlarged to cover the cases of $\sqrt{2}$, $\sqrt{7}$, $\sqrt[3]{5}$, $\cdots$, which are neither integers nor fractions with integral terms. Such numbers are necessary in solving an equation like $x^2 - 2 = 0$. (Solve it.)

25. Many centuries later the necessity was felt for further enlarging the idea of number in order to solve an equation like $x + 1 = 0$, or $x + a = 0$, a being one of the kinds of number above mentioned. This led to the consideration of *negative numbers,* -1, -2, -3, $\cdots$, and the meaning of these numbers will now be investigated.



26. If the mercury in a thermometer stands at 5° above a fixed point and then falls 1°, we say that it stands at 4° above that point. If it falls another degree, we say that it stands at 3° above that point, and the next time at 2°, and the next time at 1°.

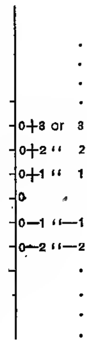
If the mercury then falls another degree, it becomes necessary to name the point at which it stands, and we call this point zero and designate it by the symbol 0.

If the mercury falls another degree, we must again name the point at which it stands, and instead of calling this point "1° below zero," we call it "minus 1°" or "negative 1°," and we designate it by the symbol $-1°$. Likewise, if the mercury falls 1° lower, we say that it stands at $-2°$, and so on.

27. Thus we find a new use for the word *minus* and the symbol −. Heretofore both the word and the sign have indicated an *operation*, subtraction; they now indicate the *quality* of a number, showing on which side of zero it stands, and thus they are *adjectives*.

In speaking of "west longitude," "west" is an adjective modifying "longitude"; in speaking of "minus latitude," "minus" is an adjective modifying "latitude"; so in "minus 2°," "minus" is an adjective.

28. It thus appears that our idea of number can be enlarged to include zero, and still further to include the series of natural numbers extended downward from zero.



If necessary to distinguish 1° above 0 from 1° below 0, the former is written $+1°$ and called either "plus 1°" or "positive 1°," and the latter is written $-1°$. But *unless the contrary is stated, a number with no sign before it is considered positive.*

29. It thus appears that *positive numbers may be represented as standing on one side of zero, and negative numbers on the other.*

Thus, if west longitude is called positive, east longitude is called negative, and *vice versa*; if north latitude is called positive, south latitude is called negative; if a man's capital is called positive, his debts are called negative, etc.

E.g., if the longitude of New York is 73° 58′ 25.5″ west and that of Berlin is 13° 23′ 43.5″ east, the former may be designated as $+73° 58' 25.5''$ and the latter as $-13° 23' 43.5''$, their difference being 87° 22′ 9″.

Similarly, if a man begins the year with $5000, and during the year oses his capital and gets $2000 in debt, he is $7000 worse off than at he beginning. It may then be said that he started with $5000 and nds with $-$ $2000, the difference being the $7000 which he lost.

30. Since two such expressions as $+ a$ and $- a$, or $+ 5°$ ind $- 5°$, represent different directions, but equal measures, hey are said to have the same **absolute value**.

The symbol $|-a|$ is read, "the absolute value of $-a$."

Hence, $|-5°| = |+5°|$, although $-5°$ does not equal $+5°$.

Since the difference between $-5°$ and $+5°$ on a thernometer is 10°, it appears that *we sometimes find the dif- ference between two numbers by adding absolute values.*

31. There are numerous signs used in algebra, as $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, exponents, etc. But *by the sign of a term is ilways meant the $+$ or $-$ sign, which indicates the quality f the term,* whether positive or negative.

Thus, in $a^2 + 7b$, the sign of $7b$ is plus (understood), while in $a^2/-7b$ t is minus.

32. Positive and negative numbers, together with zero, re often called **algebraic numbers**, positive numbers being alled **arithmetical.**

Zero is considered either as having no sign or as having both the lus and the minus signs.

EXERCISES. VII.

These are intended for oral drill and should be supplenented by many others of this type.

1. A ship in 8° west longitude (+ 8°) sails so as to lose ° in longitude. On what meridian is it then? Suppose t loses 7° more? 3° after that?

2. What is the difference in latitude between $+10°$ and $-20°$? between $+90°$ and $-90°$?

3. Show that $|5-7| = |-10+12| = |-22+20| = 2$.

4. What is meant by $|-4|$? by the absolute value of -8? of -3?

5. What is the absolute value of $10-17$? of $17-10$? of $-\frac{2}{3}$? of $+\frac{2}{3}$?

6. What other numbers have the same absolute value as $+3$, -5, $+10$, $-\sqrt{2}$, 0?

7. What is the difference in time between 50 years B.C. and 50 years A.D.? Indicate this by symbols.

8. Draw a line representing a thermometer scale; mark off $0°$, $30°$, $-25°$. What is the difference between $30°$ and $-25°$?

9. If the weight of a piece of iron is represented by $+10$ lbs., what will represent the weight of a toy balloon which pulls up with a force of 3 lbs.?

10. Suppose the piece of iron and the balloon mentioned in ex. 9 were fastened together. What would be their combined weight?

11. If the upward pull of a toy balloon is represented by $+3$ lbs., what will represent the *upward* pull of a piece of iron weighing 10 lbs.?

12. What is meant by saying that a person is worth $-\$1000$? Suppose \$2000 is added to his capital. How much is he then worth?

13. Draw a circumference and show that the difference between $50°$ and $-10°$ equals $|50°| + |-10°|$, or $60°$. Also that the difference between $10°$ and $-10°$ is $20°$.

14. If the weights of two pieces of iron are respectively 100 lbs. and 300 lbs., and to these are attached a balloon with an upward pull of 500 lbs., how shall the combined weight be represented?

IV. THE SYMBOLS OF ALGEBRA.

33. As already seen, algebra employs the symbols of rithmetic, often with a broader meaning, and introduces ew ones as occasion demands. The following classifica-on will enable the student to review the symbols thus ır familiar to him, and may add a few new ones to his st. Others will be considered from time to time as eeded.

1. Symbols of quantity.

a. *Arithmetical numbers, i.e.,* positive integers and frac-.ons.

b. *Algebraic numbers,* the above with the addition of neg-tive numbers and zero. Others will be considered later.

c. *Letters* denoting algebraic numbers; these are the ymbols of quantity chiefly used in algebra.

2. Symbols of quality.

a. *The symbols* $+$ *and* $-$ to indicate positive and nega-ive number, as in $+a$, $-b$, etc.

b. *The absolute value symbol,* as in $|-3|$, indicating hat the arithmetical quality of -3 is considered.

3. Symbols of operation.

a. *Addition,* $+$.

b. *Subtraction,* $-$.

c. *Multiplication,* $\times$, $\cdot$, and the absence of sign. Thus, $\times b$, $a \cdot b$, and ab, all indicate the product of a and b. It ; quite customary in algebra to say "a into b" for a times b."

d. *Division,* $\div$, $/$, $:$, and the fractional form. Thus, $\div b$, a/b, $a:b$, and $\frac{a}{b}$, all mean the quotient of a divided y b.

In arithmetic the symbol : is used only between numbers of the same denominations; but in algebra, where the letters represent abstract numbers, this distinction does not enter. For ease in typesetting the symbol / is often used in print; in writing, the fraction is usually employed.

e. *Involution and evolution* are indicated by exponents. Evolution is also indicated, as in arithmetic, by the symbol $\sqrt{}$, a contraction of r, the initial of *radix* (Latin, root). Thus,

a^3 means aaa,

$8^{\frac{1}{3}}$ means one of the three equal factors of 8, or 2.

4. Symbols of relation.

a. *Equality*, $=$.

b. *Identity*, $\equiv$; thus, $a \equiv a$, read "a is identical to a." Also read "stands for," as in $r \equiv$ rate, $P \equiv x^2 + 2xy$, etc.

c. *Inequality*: $>$ greater than, $<$ less than, $\neq$ not equal to, $\ngtr$ not greater than, $\nless$ not less than.

5. Symbols of aggregation.

The expression $m(a + b)$ means that $a + b$ is to be multiplied by m. The parenthesis about $a + b$ is called a *symbol of aggregation.*

The bar, brackets, and braces are also used, as in

$$m\{a - [b + x(a - \overline{b - c}) + xa] - d\}, \text{ and in}$$

$$\begin{array}{r|r|l} a & x^2 + 2a & x + c^2 \equiv (a + b)x^2 + (2a - b + c)x + c^2; \\ +b & -b & \\ & +c & \end{array}$$

but the term *parenthesis* is often employed to mean any symbol of aggregation. The subject is more fully discussed on p. 35.

6. Symbols of deduction.

since.

therefore.

7. **Symbol of continuation.**

$\cdots$, meaning "and so on," as in the sentence, "consider
e quantities a, a^2, a^3, $\cdots$."

34. Conventional order. Mathematicians have established
custom as to the order in which these signs shall be con-
dered when several are involved, as in an expression like

$$a + b \times c \div d \times ef^{\frac{1}{2}} - g + hk^2 - \frac{a}{b}.$$

In the above expression six operations are involved, as
llows:

	Direct.	Inverse.
Class I.	Addition.	Subtraction.
Class II.	Multiplication.	Division.
Class III.	Involution (Powers).	Evolution (Roots).

The mathematical custom is expressed in the following
nventions:

1. *If two or more operations of the same class come
gether (without symbols of aggregation), the operations
re to be performed in the order indicated.*

E.g., $2 + 3 - 4 + 1 = 2$, and $2 \times 8 \div 4 \times 2 = 8$.

2. *If two or more operations of different classes come
gether (without symbols of aggregation), the operations
the higher class are to be performed first.*

I.e., involution and evolution precede multiplication and division,
d these precede addition and subtraction.

E.g., $5 + 2 \times 8 \div 2^2 - \sqrt[3]{8} = 7$.

This conventional order can, of course, be varied by the
e of symbols of aggregation.

E.g., $2 + 3 \times 5 = 17$, but $(2 + 3) \times 5 = 25$.

There are also certain exceptions to this conventional order, but they are not of a nature to cause any confusion.

E.g., $ab \div cd$ means $(ab) \div (cd)$ and not $\frac{abd}{c}$, and similarly in other cases of the absence of sign where division is involved.

Similarly, when the sign of ratio (:) appears in a proportion it has not the same weight as the symbol $\div$. Thus, $2 + 3 : 12 - 2 = 1 : 2$ means $(2 + 3) : (12 - 2) = 1 : 2$.

EXERCISES. VIII.

1. If $a = 1$, $b = 2$, $c = 3$, $d = 4$, find the value of each of the following expressions:

(a) $(a + b^2)^2$. (b) $b(c + d)^2$.
(c) $5d/bc - a$. (d) $(9^{\frac{1}{2}} - d \div b)c^2$.
(e) $3a + b \times c - d$. (f) $(a + b)(c + d)$.
(g) $2 + a^2b^3d^{\frac{1}{2}} \div a + b$. (h) $2a \times b \div d \times c - a$.

2. Read the following expressions:

(a) $a + a^2 \equiv a + a^2$.
(b) $a/b \ngtr a$ if $b > 1$.
(c) $a^2 \equiv a + a^2 - a$, $\therefore a^2 < a + a^2$.
(d) $\because a = 2$, $\therefore a^2 = 4$, $a^3 = 8$, $a^4 = 16$, $\cdots$.
(e) $a^3 + a^2 \neq a^2$, and $a^2 + a^3 \nless a^2$, if a is positive.

3. Show that the following are equal when $a = 2$ and $b = 3$. That is, substitute 2 for a and 3 for b in each member.

(a) $(a + b)^2 = a^2 + 2ab + b^2$.
(b) $(b - a)^2 = b^2 - 2ba + a^2$.
(c) $(b^2 - a^2)/(b - a) = b + a$.
(d) $(a + b)(a^2 - ab + b^2) = a^3 + b^3$.
(e) $(b^3 - a^3)/(b - a) = b^2 + ba + a^2$.
(f) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

V. PROPOSITIONS OF ALGEBRA.

35. A **proposition** is a statement of either a truth to be monstrated or something to be done.

E.g., algebra investigates this proposition: The product of a^m and is a^{m+n}. It also considers such statements as this: Required the oduct of $a+b$ and $a-b$.

36. Propositions are divided into two classes, **theorems** d **problems.**

A **theorem** is a statement of a truth to be demonstrated.

E.g., The product of a^m and a^n is a^{m+n}.

A **problem** is a statement of something to be done.

E.g., Required the product of $a+b$ and $a-b$.

A **corollary** is a proposition so connected with another as t to require separate treatment.

The proof is usually substantially included in that of the proposition th which it is connected.

REVIEW EXERCISES. IX.

1. What is the degree of the expression $3ax^2y^3$? What its degree in x? in y? in x and y? in z?

2. Distinguish between coefficient and exponent. What the coefficient of x in the expression $\frac{x}{2}$? the exponent?

3. What is the meaning of the expression ab? of 26? $a\frac{x}{y}$? of $2\frac{3}{4}$? What is the value of ab if $a=2$, $b=6$? $a\frac{x}{y}$ if $a=2$, $x=3$, $y=4$?

4. What is meant by the *etymology* of a word? What the etymological meaning of *binomial?* of *trinomial? monomial?* of *aggregation?* of *theorem?* (See Table Etymologies.)

5. Show that if $a = 7$ and $b = 5$,

(a) $(a + b)(a - b) = a^2 - b^2$.
(b) $(a - b)^2 = a^2 + b^2 - 2ab$.
(c) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$.

6. Show that if $a = 3$, $b = 2$, $c = 1$,

(a) $(a + b)^2 - c^2 = (a + b + c)(a + b - c)$.
(b) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

7. What meaning has the number "minus 2" to you?

8. What is the value of $8^{\frac{1}{3}}$? of $9^{\frac{1}{2}}$? of $16^{\frac{1}{4}} \cdot 32^{\frac{1}{5}} \div 16^{\frac{1}{2}}$?

9. Show by substitution that 1 is a root of the equation in ex. 10.

10. How many terms in the equation $2x^2 + 3x - 4 = 1$? How many members?

11. Draw a diagram illustrating the fact that the absolute value of the difference between -5 and 10 is 15.

12. What is the degree of the polynomial $x^3 + 3x^2y^2 + 3xy^2 + 5y + 6$? What is the degree in x? in y? in z?

13. Write the following in algebraic language: The sum of the square of a number, 3 times the number, and 5, is equal to 9.

14. Represent algebraically the sum of the cube of a number, 5 times the square of the number, and 6, less half the number.

15. What is meant by solving an equation? by a root of an equation? by checking a solution? Illustrate with the equation $x - 2 = 0$.

16. What is the number from which if 5% be taken, and 10% from the remainder, and 20% from that remainder, the result is 41.04?

17. Write out three problems which you can now solve, but which you could not solve when you began to study algebra.

CHAPTER II.

ADDITION AND SUBTRACTION.

I. ADDITION.

37. In elementary arithmetic the word **number** includes only positive integers and fractions, or at most a few indicated roots like $\sqrt{2}$, $\sqrt[3]{5}$, $\cdots$. Hence, the word **sum**, as there used, applies only to the result of adding two positive numbers.

In algebra the word **sum** has a broader meaning, and includes the results of adding negative numbers and numbers some of which are positive and others negative.

E.g., consider the combined weight of these three articles: a 2-lb. weight, a 4-lb. weight, and a balloon which weighs -5 lbs. (*i.e.*, pulls upward with a force of 5 lbs.). Together they would evidently weigh 1 lb. Hence 1 lb. is said to be the sum of 2 lbs., 4 lbs., and -5 lbs.

So the result of adding a debt of \$100 to a capital of \$300 is a capital of \$200; hence, \$200 is said to be the sum of \$300 and $-$\$100.

38. In this broader view of addition two cases evidently arise:

1. Numbers with like signs.

$$2 \text{ lbs.} + 3 \text{ lbs.} = 5 \text{ lbs.}$$

A balloon pulling up 5 lbs. and one pulling up 8 lbs. together pull up 13 lbs., or $(-5 \text{ lbs.}) + (-8 \text{ lbs.}) = -13$ lbs.

2. Numbers with unlike signs.

A balloon pulling up 5 lbs. and a weight of 2 lbs. together pull up 3 lbs., or -5 lbs. $+ 2$ lbs. $= -3$ lbs.

39. From considerations like these we are led to define **the sum of two algebraic numbers** as follows:

1. *If two numbers have the same sign, their algebraic sum is the sum of their absolute values, preceded by their common sign.*

Thus, to add -3 and -2 means to add 3 and 2 and to place the sign $-$ before the result.

2. *If they have not the same sign, their algebraic sum is the difference of their absolute values, preceded by the sign of the one which has the greater absolute value.*

Thus, to add -3 and 2 means to find the difference between 3 and 2 and to place the sign $-$ before the result, since $|-3|>|2|$.

3. *In the special case where the two numbers have the same absolute value (i.e., where they are equal and of opposite signs), the sum is zero.*

E.g., $2+(-2)=0$.

4. *If one of two numbers is zero, their algebraic sum is the other number.*

Thus, $-3+0$ means -3.

40. *The algebraic sum of several numbers* is defined as the sum of the first two plus the third, that sum plus the fourth, $\cdots$.

Thus, $a+b+c+d$ means $a+b$ with c added, and that sum with d added. *I.e.*, $a+b+c+d$ means $[(a+b)+c]+d$.

EXERCISES. X.

1. Find the sum of $-20, +3, -47, +80$.
2. Also of $+2, -3, +5, -4, +9, -3, -6$.
3. Also of $2x^2, 5x^2, -6x^2, 8x^2$?
4. Also of $127\,mn, 62\,mn, -93\,mn, -17\,mn$?

5. $\$50 + \$7 + (-\$21) + (-\$30) = ?$

6. $5 + 219 + (-376) + (-40) + 10 + (-37) = ?$

7. $(-7) + 4 + (-2) + 18 + 13 + (-20) + (-6) = ?$

8. $3a + (-2a) + (-5a) + 8a + 6a + (-10a) = ?$

9. What is the sum of $3a$, $5a$, $-6a$, $8a$, $10a$, $-3a$, $17a$?

10. $12x^2y + 4x^2y + (-16x^2y) + (-3x^2y) + 10x^2y$ $(?)x^2y$?

11. 5 lbs. + 55 lbs. + (– 40 lbs.) + (– 27 lbs.) + 121 lbs. (– 19 lbs.) + (– 5 lbs.) = (?) lbs ?

12. What is the combined weight of two balloons weigh-;, respectively, – 10 lbs. and – 18 lbs., and three pieces iron weighing, respectively, 6 lbs., 12 lbs., and 14 lbs.?

13. On seven consecutive midnights in January, in Mon-al, the temperature was 30°, 18°, 10°, 4°, 0°, – 7°, – 20°. hat was the average midnight temperature for the week?

14. What is the combined weight, under water, of a piece cork weighing – 2 oz., a stone weighing 3 lbs., a piece of od weighing – 1 lb. 3 oz., and a piece of iron weighing bs.?

15. A merchant finds that he has cash in bank $575.50, ck worth $4875, due from customers $1121.50, that he es a note and interest amounting to $350.25 and bills ounting to $827, and that he owns a bond and mortgage $1000. Express his capital as the sum of these various ms with their proper signs.

16. A ship sailing up a river would go at the rate of 15 les an hour if it were not for the current; the current erages 5 miles an hour for the first 3 hours of the ship's gress, and 4 miles an hour for the next 2 hours. How has the ship gone at the end of 5 hours? Express this the sum of several algebraic numbers.

41. To **add** several literal expressions, called the **addends**, is to find a single expression called the **sum**, such that whatever values are substituted for the letters the value of the sum shall equal the sum of the values of the addends.

E.g., the sum of a, $2a$, $7a$, $-4a$, is $6a$; for suppose 1 is substituted for a, we have $1 + 2 + 7 - 4$, which is 6; and if 2 is substituted for a we have $2 + 4 + 14 - 8$, which is 12, and so for any other values.

Similarly, the sum of the addends in the annexed problem is $-4a + 4b - c$; for if $a = 2$, $b = \frac{1}{2}$, $c = 5$, we have $-10\frac{1}{2} + 7 - 7\frac{1}{2} = -11$, and similarly for any other values of a, b, c.

$$\begin{array}{rrr} 2a + & b - & 3c \\ & 4b + & c \\ -6a - & b + & c \\ \hline -4a + & 4b - & c \end{array}$$

$$\begin{array}{rrrcr} 4 + & \frac{1}{2} - & 15 & = & -10\frac{1}{2} \\ & 2 + & 5 & = & 7 \\ -12 - & \frac{1}{2} + & 5 & = & -7\frac{1}{2} \\ \hline -8 + & 2 - & 5 & = & -11 \end{array}$$

Since these values are entirely arbitrary, they are usually called *arbitrary values.*

42. Hence, it appears that *to add like terms is to add the coefficients, and to add polynomials is to add their like terms,* the literal parts being properly inserted in the sum.

The sum is supposed to be simplified as much as possible. Thus, the sum of $4a - b$ and $b + a$ is $5a$, not $4a + a$.

EXERCISES. XI.

1. Add $3x^2 + 2xy + 4y^2$, $4x^2 - 3xy - 2y^2$, $3x^2 + xy$.

2. Add $6\sqrt{m} + x$, $5\sqrt{m} - x - 3y$, $8\sqrt{m} - 2y$, and $3x$. Check the work by letting $m = 4$, $x = 1$, $y = 1$.

3. Add $2a + 3b - c$, $-4c$, $7a$, $-6b + 8c$, and $-a + b - c$. Check the work by letting $a = 1$, $b = 1$, $c = 1$.

4. Add $17x - 9y$, $3z + 14x$, $y - 3x$, $x - 17z$, and $x - 3y + 4z$. Check the work by letting $x = 1$, $y = 2$, $z = 3$.

5. Add $16m + 3n - p$, $p + 4q$, $-q + 7m - 3n$, $n - q$, and $3n + 2p$. Check the work by letting $m = 1$, $n = 1$, $p = 2$, $q = 4$, or by assigning any other arbitrary values.

II. SUBTRACTION.

ł3. **Subtraction** is the operation which has for its object, en the sum of two expressions and one of them, to find other.

Che given sum is called the **minuend**, the given addend is led the **subtrahend**, and the addend to be found is called **difference** or the **remainder.**

Chat is, the difference is the number which added to the trahend produces the minuend. In other words,

$$\textit{difference} + \textit{subtrahend} = \textit{minuend}.$$

E.g.,

$$\because 4 + 5 = 9, \quad \therefore \quad 4 = 9 - 5;$$
$$\because 4 + (-3) = 1, \quad \therefore \quad 4 = 1 - (-3);$$
$$\because 4 + (-5) = -1, \quad \therefore \quad 4 = -1 - (-5);$$
$$\because -4 + (-3) = -7, \quad \therefore -4 = -7 - (-3).$$

Chese results are illustrated as follows: the difference ween the temperature of 9° and that of 5° is 4°; that ween 1° and −3° (*i.e.*, 1° above 0 and 3° below 0) is 4°; t between −1° and −5° (*i.e.*, 1° below 0 and 5° below 0) ł°; that between −7° and −3° is −4°, that is, the mery must *fall* 4° from −3° to reach −7°.

We may, therefore, think of subtraction as the inverse of lition, or the process which undoes addition.

Example. What is the remainder after subtracting

$$3a^2 + 4ab - 5b^2 \text{ from } 4a^2 - 5ab + 2b^2?$$

$$\begin{array}{r} 4a^2 - 5ab + 2b^2 \\ 3a^2 + 4ab - 5b^2 \\ \hline a^2 - 9ab + 7b^2 \end{array}$$

What term added to $3a^2$ makes ? Evidently a^2.
What term added to $4ab$ makes ab? Evidently $-9ab$; for the ition of $-4ab$ makes 0, and the further addition of $-5ab$ makes ab.

Similarly, $7b^2$ is the term which added to $-5b^2$ makes $2b^2$.

$$\therefore 4a^2 - 5ab + 2b^2 - (3a^2 + 4ab - 5b^2) \equiv a^2 - 9ab + 7b^2.$$

Check. Let $a = 1$, $b = 2$. Then

$$\begin{array}{r} 4a^2 - 5ab + 2b^2 \\ 3a^2 + 4ab - 5b^2 \\ \hline a^2 - 9ab + 7b^2 \end{array} \qquad \begin{array}{r} 4 - 10 + 8 = 2 \\ 3 + 8 - 20 = -9 \\ \hline 1 - 18 + 28 = 11 \end{array}$$

Since this is an identity, it is true for any values of a and b. Hence, the work may be checked by letting $a = 1$, $b = 2$. The minuend then becomes 2, and the subtrahend -9, and the remainder 11, which is $2 - (-9)$.

44. Theorem. *The subtraction of a negative number should be interpreted as the addition of its absolute value.*

Given a and $-b$.

To prove that $a - (-b)$ equals a plus the absolute value of $-b$; that is,

that $a - (-b) = a + |-b|$ or $a + b$.

Proof. 1. $a - (-b)$ must be such a number that

$a - (-b) + (-b) \equiv a$. Def. of subt., § 43

2. Adding b to both members, and remembering that $(-b) + b \equiv 0$, § 39, 3

and that $a + 0 \equiv a$, § 39, 4

we have $a - (-b) \equiv a + b$,

which we were to prove.

COROLLARY. $\because a - (-b) = a + b$, $\therefore 0 - (-b) = b$. This is usually expressed by the phrase, *Minus a minus is plus.*

EXERCISES. XII.

1. From $3a - 4b + c$ subtract $2a - 5b - c$. Check the work by letting $a = 3$, $b = 1$, $c = 2$, or by assigning any other arbitrary values; also by adding the remainder and subtrahend.

2. From $3a - 5x + \frac{2}{3}m$ subtract $4a - m$. Check by letting $a = 5$, $x = 1$, $m = 3$, or by assigning any other

arbitrary values; also by adding the remainder and subtrahend.

3. From $13\,x + y - 3\,z$ subtract $5\,x - 7\,y + z$. Check as in exs. 1, 2.

4. From $7\,a^2 + 3\,ab - 6\,b^2$ subtract $a^2 + 3\,ab - 2\,b^2$. Check as in exs. 1, 2.

5. What expression added to $2\,x^2 - 3\,xy - 15\,y^2$ makes $-7\,x^2 - 3\,xy + z$? Check.

6. Perform the following subtractions, checking each as in exs. 1, 2.

(a)
$$\begin{array}{l} 3\,x^3 - 7\,x^2 + 2\,x - 13 \\ \underline{4\,x^3 - 2\,x^2 \qquad\quad + \;\; 1} \end{array}$$

(b)
$$\begin{array}{r} 1.5\,p^2 - 2\,pqr + 0.5\,r^2 \\ \underline{\qquad\quad - 3\,pqr - 1.3\,r^2} \end{array}$$

(c)
$$\begin{array}{l} 2\,a^{\frac{1}{2}} - 3\,ab + \quad b - \quad c \\ \underline{17\,a^{\frac{1}{2}} \qquad\quad - 13\,b + 12\,c} \end{array}$$

(d)
$$\begin{array}{l} 2\,a^5 - 3\,a^4b + 2\,a^3b^2 - 3\,a^2b^3 + 15\,ab^4 - b^5 \\ \underline{6\,a^5 \qquad\quad + 3\,a^3b^2 - \quad a^2b^3 \qquad\qquad + b^5} \end{array}$$

(e)
$$\begin{array}{l} \qquad 18\,a^2b^{\frac{1}{2}} + 4\,a \qquad\quad - 3\,b^3 \\ \underline{7\,a^3 \qquad\qquad - 2\,a + 4\,b^2 - \quad b^3} \end{array}$$

(f)
$$\begin{array}{l} 2\,xy - y^2 + 3\,x^2 \\ \underline{x^2 + y^2 - 3\,xy} \end{array}$$

7. What is the difference between the capital of a man who has a stock of goods worth \$5000, \$750 in the bank, and owes \$1000 on a mortgage, and that of one who has a stock of goods worth \$6000, has overdrawn his bank account \$275, and owns a \$500 mortgage?

8. If $P \equiv a^2 + 2\,ab + b^2$, $Q \equiv 2\,a^2 + ab + b^2$, and $R \equiv -4\,ab - 7\,b^2$, find the values of the following expressions. Check in each case by assigning arbitrary values to a and b.

(a) $P - Q$. (b) $P - R$. (c) $Q - R$.
(d) $Q - P$. (e) $P + Q - R$. (f) $P + R - Q$.
(g) $P - Q - R$. (h) $R - Q - P$. (i) $Q - P - R$.

45. Detached coefficients. Additions and subtractions may evidently be performed without the labor of writing down all of the letters. Since the coefficients of like terms are added, these coefficients may be detached and added separately, the coefficients of like terms being placed under one another. Missing terms are indicated by zeros.

Thus, the second of the following additions is the simpler:

(1)	(2)	Check.
$a^2 + 2ab + b^2$	$1 + 2 + 1$	$= 4$
$-3a^2 - ab + b^2$	$-3 - 1 + 1$	$= -3$
$4a^2 - 3ab - 3b^2$	$4 - 3 - 3$	$= -2$
$2a^2 - 2ab - b^2$	$2 - 2 - 1$	$= -1$

$$2a^2 - 2ab - b^2.$$

Since, if the arbitrary value 1 is assigned to each letter, the value of each term is its numerical coefficient, the check requires merely the addition of the coefficients.

EXERCISES. XIII.

Perform the following operations by using detached coefficients, checking the results by the above method.

1. Add $a^3b + a^2b^2 - 4ab^3$, $3a^3b - b^4$, $-a^2b^2 + b^4$, $4ab^3$.

2. Add $5x^4 - 2x^2y^2 + y^4$, $x^3y + xy^3$, $x^4 - xy^3$, $-x^3y + y^4$.

3. Add $x^3 - x^2y + xy^2 - y^3$, $2x^3 + 3x^2y - 4xy^2 + y^3$, $x^3 - y^3$.

4. Add $p^3 + 3p^2 + 4p - 6$, $-p^2 - 2p + 1$, $p^3 - 1$, $3p^3 + 2p + 3$.

5. From $a^2 + 2ab + b^2$ subtract $a^2 - 2ab + b^2$.

6. From $x^3 + x^2y + xy^2 + y^3$ subtract $x^3 - x^2y + xy^2 - y^3$.

7. Given $P \equiv x^3 + 3x^2y + 3xy^2 + y^3$, $Q \equiv -3x^2y + 3xy^2 - 3y^3$, $R \equiv x^3 - y^3$, find by using detached coefficients the values of the following, checking as above:

(a) $P - Q$. (b) $Q - R$. (c) $R - P$. (d) $Q - P$.
(e) $R - Q$. (f) $P - R$. (g) $P + Q + R$. (h) $P + Q - R$.

III. SYMBOLS OF AGGREGATION.

46. Symbols of aggregation, preceded by the symbols + and –, may be removed by considering the principles of addition and subtraction already learned.

Since $a + (b - c) \equiv a + b - c,$

and $a - (b - c) \equiv a - b + c,$

therefore, *a symbol of aggregation preceded by + may be neglected; if preceded by – it may be removed by changing the sign of each term within.*

E.g., $2a + (3b - c + a) \equiv 2a + 3b - c + a \equiv 3a + 3b - c.$

$2a - (3b - c + a) \equiv 2a - 3b + c - a \equiv a - 3b + c.$

For the same reasons, *any terms of a polynomial may be enclosed in a symbol of aggregation preceded by +; also in a symbol of aggregation preceded by – provided the sign of each term within is changed.*

E.g., $a + b - c + d \equiv a + (b - c + d) \equiv a + b - (c - d).$

The word **term** now takes on a broader meaning than that given in § 3. *E.g.*, in the expression $a - b(c - d)$, $b(c - d)$ is often considered as a *term*. So in general, where no confusion will arise, polynomials enclosed in symbols of aggregation, with or without coefficients, are often called *terms*.

E.g., $(a - b)x^2 + (a + b)x + (a^2 - b^2)$ may be considered as a trinomial.

EXERCISES. XIV.

Remove the symbols of aggregation in the following:

1. $p^2 + 2pq + q^2 - (q^2 - p^2).$
2. $a^2 - 3b^2 + (2a^2 + 7b^2 - c^2).$
3. $a^3 - (3a^2b + a^3 - b^3) - b^3 + 3a^2b.$
4. $2x^2 - 3xy + y^2 - (2x^2 + 3xy - y^2).$
5. $5m^3 - (3m^3 + 1) - (4m^4 + m^3 - 3) + (m^3 + 1).$

47. Several symbols of aggregation, one within another, may be removed by keeping in mind the principles already mentioned.

The order in which these symbols are removed cannot affect the result, but the simplest plan will be discovered by considering the following solution.

Simplify $a - [a + b - (c - \overline{d - e}) + c]$, (1) beginning with the inner symbol, (2) beginning with the outer symbol.

(1)	(2)
1. $a - [a + b - (c - \overline{d - e}) + c]$	1. $a - [a + b - (c - \overline{d - e}) + c]$
2. $\equiv a - [a + b - (c - d + e) + c]$	2. $\equiv a - a - b + (c - \overline{d - e}) - c$
3. $\equiv a - [a + b - c + d - e + c]$	3. $\equiv a - a - b + c - \overline{d - e} - c$
4. $\equiv a - a - b + c - d + e - c$	4. $\equiv a - a - b + c - d + e - c$
5. $\equiv -b - d + e$	5. $\equiv -b - d + e$

How many changes of signs were made throughout solution (1)? how many in solution (2)? Hence, which solution is the better?

From the second step of solution (2) could you have written down step 5 at once? Could you have done this from step 2 of solution (1)? On this account which is the better solution?

From the above solution it appears that *it is better to remove the outer parentheses first.* A little practice will enable the student to remove them all at sight if this plan is followed.

EXERCISES. XV.

Remove the symbols of aggregation in the following expressions, uniting like terms in each result.

1. $-[a^2 - (2\,ab - \overline{b^2 - a^2}) + b^2]$.
2. $4\,a^2 - \{5\,b^2 + a - [6\,a^2 - 3\,a - (b^2 - a)]\}$.
3. $a^2x - [ax^2 + a^2 - (a^2x - a^2) + x^2] - ax^2 + x^2$.
4. $10\,m^2 + 5\,mn - [6\,m^2 + n^2 - (2\,mn - \overline{m^2 + n^2})] - n^2$.
5. $-(-(-(\cdots-(-1)\cdots)))$, an even number of sets of parentheses; an odd number of sets.

IV. FUNDAMENTAL LAWS.

48. The following laws have thus far been assumed:

I. That $a + b \equiv b + a$, a and b being positive or negative integers, just as in arithmetic $3 + 4 = 4 + 3$. This is called the **Commutative Law of Addition**, because the order of the addends is changed (Latin *com*, intensive, + *mutare*, to change).

II. That $a + b + c \equiv a + (b + c)$, the letters representing positive or negative integers, or both, just as in arithmetic $3 + 4 + 5 = 3 + 9$. This is called the **Associative Law of Addition**, because b and c are associated in a group.

III. That $ab \equiv ba$, a and b being positive integers, just as in arithmetic $2 \cdot 3 = 3 \cdot 2$. This is called the **Commutative Law of Multiplication.**

That these laws are valid for the kinds of numbers indicated will now be proved, although the proof may be omitted by beginners if desired.

49. I. The Commutative Law of Addition.

1. If 3 marbles lie on a table, and 4 more are placed with them, the result is indicated by the symbols $3 + 4$.

2. If the original 3 marbles be removed, 4 will remain; and if the 3 be then replaced, the result will be indicated by the symbols $4 + 3$.

3. But the number of marbles has not been changed.

$$\therefore 3 + 4 = 4 + 3.$$

4. But this proof is independent of the particular numbers 3 and 4, and hence, a and b being any *positive integers*,

$$a + b \equiv b + a.$$

5. The proof is evidently substantially the same for several groups. Hence,

$$a + b + c + \cdots \equiv a + c + b + \cdots \equiv b + c + a + \cdots, \text{ etc.}$$

6. And since, if some of the terms are negative, we deal with their absolute values, adding or subtracting as indicated, and prefix the proper sign to the result, therefore the above proof is sufficiently general.

I.e., $a + b - c \equiv a - c + b$, because in any case we are to take the difference between the absolute values of $a + b$ and c, and prefix the proper sign.

50. II. THE ASSOCIATIVE LAW OF ADDITION.

To prove that $a + b + c \equiv a + (b + c)$, the letters representing positive or negative integers, or both.

1. $\because c + b + a \equiv (c + b) + a$. Def., § 40
2. $\equiv a + (c + b)$. Com. law, § 49
3. $\therefore a + b + c \equiv a + (b + c)$. Com. law, § 49

The proof is evidently similar, however many terms are involved or however the grouping is made.

51. III. THE COMMUTATIVE LAW OF MULTIPLICATION.

To prove that $ab \equiv ba$, the letters representing only positive integers.

```
*  *  *  *  *  .  .  .    a in a row
*  *  *  *  *  .  .  .
*  *  *  *  *  .  .  .
.  .  .  .  .  .  .  .
```

b rows.

1. Suppose a collection of objects arranged in b rows, a in a row, or, what is the same thing, in a columns, b in a column.

2. $\because$ there are b in one column, in a columns there are ab objects.

3. $\because$ there are a in one row, in b rows there are ba objects.

4. But the collection being the same, $ab \equiv ba$.

REVIEW EXERCISES. XVI.

1. Distinguish between an *equation* and an *identity*, illustrating each.

2. Show that $|2-3|=|3-2|$, and state a proposition covering such cases.

3. What is the etymological meaning of *coefficient?* of *subtraction?* of *literal?* of *minuend?*

4. Why is not the arithmetic definition of *sum* sufficient for algebra? What do you mean by *sum* in algebra?

5. What is the advantage in using detached coefficients in addition? Make up an example illustrating this.

6. What is the number which added to -5 equals 0? equals 2? Hence, what is the difference $0-(-5)$? $2-(-5)$?

7. Remove the symbols of aggregation in the following expressions. By beginning at the outside you can usually write the result at sight, except for simplifying.

(a) $[x+(x+y+\overline{x-y})-x]$.
(b) $a-\{a-[a-\overline{a-(b-a)}]\}$.
(c) $3a-[b+c-(a-b)+a]-b$.
(d) $x^2-[2x^2+y^2-(x^2-y^2-\overline{y^2-x^2})+y^2]-y^2$.

8. Enclose any two terms (after the first) in parentheses:

(a) $a^2-b^2-2c^2-3bc$.   (b) $3p^2-4pq-2q^2+r^2$.
(c) $4x^3-2x^2-7x+1$.   (d) $m^4-m^3+m^2-m+1$.

9. What is meant by the Commutative Law of Addition? Have you proved it for all kinds of numbers? If not, name a kind for which it has not yet been proved by you. Similarly for the Associative Law of Addition.

CHAPTER III.

MULTIPLICATION.

I. DEFINITIONS AND FUNDAMENTAL LAWS.

52. Multiplication originally had reference to positive integers and was a short form of addition. It was, for this case, defined as the operation of taking a number called the **multiplicand** as many times as an addend as there are units in an abstract number called the **multiplier**, the result being called the **product**.

E.g., in this limited sense, to multiply \$2 by 3 is to take \$2 3 times, thus, $3 \times \$2 = \$2 + \$2 + \$2 = \$6$.

But as mathematics progressed it became necessary to multiply by **simple fractions**, and hence to enlarge the definition to include this case.

By the primitive meaning of the word *times* it is impossible to take \$2 $\frac{2}{3}$ *of a time.* But the product of \$2 by $\frac{2}{3}$ may be *defined* as $\frac{2 \times \$2}{3}$.

So the product of c by $\frac{a}{b}$ may be *defined* as the product of a and c, divided by b, c being either integral or fractional.

As mathematics further progressed it became necessary to multiply by **negative numbers**, and hence to enlarge the definition to include this case. The natural definition will appear from a simple illustration.

Suppose 5 men move into a town, each paying \$1 a week in taxes. They are worth $5 \times \$1 = \$5$ a week to the town.

Suppose 5 such men move out. This may be represented by saying that the town gains -5 men, or, in money, $-\$5$.

Suppose 5 vagrants move in, each being a charge of \$1 a week. They are worth $5 \times (-\$1) = -\$5$ a week to the town.

Suppose 5 such vagrants move out. This may be represented by saying that the town gains -5 vagrants, or, in money, \$5.

Hence, it is *reasonable* to say that

\$1	multiplied by	$5 =$	\$5,	for the	first case;
\$1	" "	$-5 =$	$-\$5$,	"	second "
$-\$1$	" "	$5 =$	$-\$5$,	"	third "
$-\$1$	" "	$-5 =$	\$5,	"	fourth "

53. From such considerations **multiplication by a negative number** is defined as multiplication by the absolute value of the multiplier, the sign of the product being changed.

E.g., allowing the word *times* to indicate multiplication in general,
-2 times 3 means $-(|-2| \times 3)$, or $-(2 \times 3)$, or -6;
-2 " -3 " $-[|-2| \times (-3)]$ " $-[2 \times (-3)]$ " $-(-6)$, or $+6$.

54. General definition of multiplication. The above partial definitions may now be put into one general definition:

To multiply a number (the multiplicand) by an abstract number (the multiplier) is to do to the former what is done to unity to obtain the latter.

The result of multiplication is called the **product**, and the product of two abstract numbers is called a **multiple** of either.

E.g., consider the meaning of $3 \times \$2$. Since $3 = 1 + 1 + 1$, therefore, $3 \times \$2$ means $\$2 + \$2 + \$2 = \$6$.

Consider also $\frac{2}{3} \times \frac{5}{7}$. Since $\frac{2}{3} = (1 + 1) \div 3$, therefore, $\frac{2}{3} \times \frac{5}{7}$ means $(\frac{5}{7} + \frac{5}{7}) \div 3$, or $\frac{10}{7} \div 3$, or $\frac{10}{21}$.

Consider also $(-2) \times (-3)$. Since $-2 = -(1 + 1)$, therefore, $(-2) \times (-3)$ means $-[(-3) + (-3)]$, or $-(-6)$, or 6.

55. *The expression* $a \cdot 0$ *is defined to mean* 0.

This is the natural definition, because 2×0 must mean $0 + 0$.

And since it will be shown that the order of factors can generally be changed without altering the product, *the product* $0 \cdot a$ *is defined to be the same as* $a \cdot 0$, *or* 0.

56. The product of three abstract numbers is defined to be the product of the second and third multiplied by the first.

I.e., abc means $a(bc)$, the product of b and c multiplied by a.

The product of four or more abstract numbers may be understood from the above definition for three. *E.g.*, $abcd$ means cd multiplied by b, and that product by a.

57. Law of signs. From the definition it appears that *like signs produce plus, and unlike signs minus.*

I.e.,

$$+ \times + = +$$
$$+ \times - = -$$
$$- \times + = -$$
$$- \times - = +$$

58. Reading of products. As already stated, the original meaning of the word *times* referred to positive integers. The expressions $\frac{2}{3}$ *times*, $\frac{2}{3}$ *of a time*, and -2 *times* are meaningless in the original sense of the word. But with the extension of the definition of multiplication has come an extension of the meaning of the word *times*, so that it is now generally used for all products, as in § 53.

Thus, the expression $2\frac{1}{2}$ *times as much* is generally used, although it is impossible to pick up a book $2\frac{1}{2}$ times. So $(-2) \times (-3)$ is read, "minus 2 times minus 3," although we cannot look out of a window -2 times.

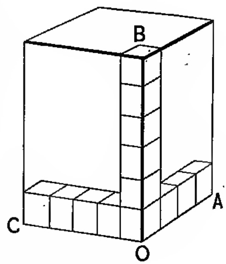
As already stated, the word *into* is sometimes used in algebra to indicate the product of two or more factors, $(-a)(-b)$ being read "$-a$ into $-b$."

The parentheses about negative factors are omitted when no misunderstanding is probable. Thus, $(-a) \cdot (-b)$ may be written $-a \times -b$, or even $-a \cdot -b$. But $-a^2$ and $(-a)^2$ are not the same, the former meaning $-aa$ and the latter $-a \cdot -a$, or $+a^2$.

59. The Associative and Commutative Laws of Multiplication. Before we are able to proceed with certainty in multiplication, it is necessary to show that we can change the order and grouping of the factors to suit our convenience.

For example, to prove that abc, which by definition means $a(bc)$, $\equiv (ab)c \equiv (ac)b \equiv b(ac) \cdots$.

Proof. 1. Suppose this solid to be composed of inch cubes, and to have the dimensions 4 in., 5 in., 6 in.



2. Then, since there are 4 cubes in the row OA, and there are 5 such rows in the layer CA, there are $(5 \cdot 4)$ cubes in that layer. And since there are 6 such layers, there are $6 \cdot (5 \cdot 4)$ cubes in all.

3. Similarly, since there are 6 in column OB, and there are 4 such columns in layer BA, there are $(4 \cdot 6)$ cubes in that layer. And since there are 5 such layers, there are $5 \cdot (4 \cdot 6)$ cubes in all.

4. Similarly, there are $4 \cdot (5 \cdot 6)$ cubes.

5. But the total number is the same,

$$\therefore 6 \cdot (5 \cdot 4) = 5 \cdot (4 \cdot 6) = 4 \cdot (5 \cdot 6).$$

6. And since the proof is independent of the numbers,

$$\therefore a \cdot (b \cdot c) \equiv b \cdot (c \cdot a) \equiv c \cdot (b \cdot a) \equiv (a \cdot b) \cdot c \equiv \cdots.$$

7. By taking d such solids it could be proved that

$$a \cdot (b \cdot c \cdot d) \equiv (a \cdot b) \cdot (c \cdot d) \equiv (a \cdot b \cdot c) \cdot d \equiv b \cdot a \cdot (d \cdot c) \equiv \cdots,$$

and similarly for any number of letters.

8. And since in multiplications involving negative numbers we proceed as if the numbers were positive, prefixing the proper sign, therefore the proof is general for all integers.

EXERCISES. XVII.

Perform the multiplications indicated:

1. $-2 \cdot -7$.
2. $-4 \cdot -\frac{3}{4}$.
3. $72 \cdot -\frac{1}{9} \cdot -\frac{3}{4}$.
4. $-\frac{1}{2} \cdot -\frac{1}{3} \cdot -\frac{1}{4}$.
5. $(-2)^2 \cdot (-3)^2$.
6. $(-1)^{17} \cdot (-2)^4$.
7. $4 \cdot 5 \cdot -3 \cdot 2 \cdot \frac{1}{3} \cdot \frac{1}{5}$.
8. $1 \cdot -2 \cdot 3 \cdot -4 \cdot 5 \cdot -6$.
9. $-1 \cdot -2 \cdot -3 \cdot -4$.
10. $5 \cdot 3 \cdot 1 \cdot -1 \cdot -3 \cdot -5$.
11. $1 \cdot (-2)^2 \cdot 3^3 \cdot (-4)^4$.
12. $(-1)^{100} \cdot (-1)^{99} \cdot (-2)^5$.
13. $4 \cdot 3 \cdot 2 \cdot 1 \cdot 0 \cdot -1 \cdot -2 \cdot -3 \cdot -4$.

60. The index law. Since $a^2 \equiv aa$, and $a^3 \equiv aaa$, therefore $a^2 \cdot a^3 \equiv aa \cdot aaa \equiv a^5$. Similarly, if m and n are positive integers,

$$a^m \equiv aaa \cdots \text{ to } m \text{ factors,}$$

and

$$a^n \equiv aaa \cdots \text{ to } n \text{ "}$$

$$\therefore \quad a^m \cdot a^n \equiv aaaa \cdots \text{ to } m + n \text{ "}$$

$$\therefore \quad a^m \cdot a^n \equiv a^{m+n}.$$

This is known as the **index law of multiplication.**

Hence, $2a^2b^3c^5 \cdot 5a^3b^2c^4 \equiv 10a^5b^5c^9$.

The cases in which m and n are negative, zero, or fractional are considered later.

EXERCISES. XVIII.

Perform the multiplications indicated:

1. $-a^2 \cdot (-a)^2$.
2. $25ab^2c^3d^4 \cdot 2a^4b^3c^2d$.
3. $-a \cdot -a^2 \cdot -a^3 \cdot -a^4 \cdot -a^5$.
4. $-a \cdot (-a)^2 \cdot (-a)^3 \cdot (-a)^4 \cdot (-a)^5$.
5. $x^x \cdot x^{x^2} \cdot x$.
6. $x^my^n \cdot x^ny^m \cdot x^2y^2$.
7. $x^2y^3z^4 \cdot xy^2z^3 \cdot x^6y^4z^2 \cdot xyz$.
8. $x^7 \cdot x^6 \cdot x^5 \cdot y^5 \cdot y^6 \cdot y^7 \cdot z^9 \cdot z^3 \cdot z$.

II. MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL.

61. I. *When the monomial is a positive integer*, as in the case of $a(b-c)$.

1. $\because$ $a = 1 + 1 + 1 + \cdots$ to a terms,
2. $\therefore a(b-c) \equiv (b-c)+(b-c)+(b-c)+\cdots$ to a terms, Def. mult. § 54
3. $\equiv b + b + b + \cdots$ to a terms, $- c - c - c - \cdots$ to a terms, § 49
4. $\equiv ab - ac.$ Def. mult. § 54

E.g., $2(x+y) \equiv (x+y)+(x+y) = 2x+2y.$

II. *When the monomial is a positive fraction*, as in the case of $\frac{x}{y}(b-c)$.

1. $\because$ $\frac{x}{y} = \frac{1+1+1+\cdots \text{ to } x \text{ terms}}{y},$
2. $\therefore \frac{x}{y}(b-c) \equiv \frac{(b-c)+(b-c)+\cdots \text{ to } x \text{ terms}}{y},$ Def. mult. § 54
3. $\equiv \frac{xb - xc}{y},$ as in I
4. $\equiv \frac{xb}{y} - \frac{xc}{y},$ because xb $y$ths minus xc $y$ths is the same as $(xb - xc)$ $y$ths.

III. *When the monomial is negative*, and either integral or fractional, as in the case of $(-m)(b-c)$.

1. $\because$ $-m = m \cdot 1$, preceded by the sign $-$,
2. $\therefore (-m)(b-c) \equiv m(b-c)$ preceded by the sign $-$, Def. mult. § 53
3. $\equiv (mb - mc)$ preceded by the sign $-$, I and II
4. $\equiv -mb + mc.$ § 46

62. From the results of these three cases it appears that:

To multiply a polynomial by a monomial is to multiply each term of the polynomial by the monomial and to add the products.

Since the multiplier is distributed among the terms of the multiplicand, this statement is known as the **distributive law of multiplication.**

E.g., $3a^2(a^4 - b) \equiv 3a^6 - 3a^2b$. This can be checked by letting $a = 1$, $b = 2$. Then $3a^2(a^4 - b) = 3 \cdot -1 = -3$, and $3a^6 - 3a^2b = 3 - 6 = -3$.

EXERCISES. XIX.

Perform the following multiplications, checking the results by assigning arbitrary values to the letters:

1. $a^2(a^2 + b^2 - c^2)$.
2. $5m^3xy(xz^2 - 3z^2x - 4)$.
3. $-7x^2y(-3xy^2 + 2xy)$.
4. $25ab^2c^3d^4(2a^4b^3 + 2c^2d)$.
5. $-5a[-3a + 2(a - 2)]$.
6. $-7m^2n^3(2m - 3n - 4mn + 6m^3n)$.

III. MULTIPLICATION OF A POLYNOMIAL BY A POLYNOMIAL.

63. Required the product of $(a + b)(c + d)$.

1. Let $m \equiv (a + b)$.
2. Then $m(c + d) \equiv mc + md$, § 61
3. $\equiv (a + b)c + (a + b)d$, $\because m \equiv (a + b)$
4. $\equiv ac + bc + ad + bd$. §§ 51, 61

From this it appears that *to multiply one polynomial by another is to multiply each term of the first by each term of the second and to add the products.*

This is the general form of the distributive law of multiplication.

The following example illustrates the process:

$$\begin{array}{ll} & x^2 + 2xy + y^2 \\ & \underline{x + y} \\ \text{Product by } x, & x^3 + 2x^2y + xy^2 \\ \text{Product by } y, & \underline{x^2y + 2xy^2 + y^3} \\ \text{Sum of products,} & x^3 + 3x^2y + 3xy^2 + y^3 \end{array}$$

$$\therefore (x + y)(x^2 + 2xy + y^2) \equiv x^3 + 3x^2y + 3xy^2 + y^3.$$

Check. Let $x = 1$, $y = 1$. Then

$$1 + 2 + 1 = 4$$
$$1 + 1 = 2$$
$$1 + 3 + 3 + 1 = 8, \text{ or } 2 \cdot 4.$$

Since the identity holds true for any values of x and y, it holds true if $x = y = 1$, as in the above check. It is evident, however, that the value 1 does not check the exponents. Where there is any doubt as to these, other values must be substituted.

EXERCISES. XX.

Perform the following multiplications, checking the results by assigning arbitrary values to the letters.

1. $(a + b)(x + y)$.
2. $(x + y)(x - y)$.
3. $(x^2 - y^2)(x + y)$.
4. $(p^2 + q^2)(x^2 - 3y^2)$.
5. $(4p^2 - 5q^2)(4p^2 + 5q^2)$.
6. $(a^2 + b^2 + c^2)(a + b + c)$.
7. $(a^3 + a^2 + a + 1)(a - 1)$.
8. $(5x^3 - x + 1)(3x^2 - x - 2)$.
9. $(2x + 3y - z)(2x - 3y + z)$.
10. $(x^4 + x^3 + x^2 + x + 1)(x - 1)$.
11. $(x + y)(x^3 + 3x^2y + 3xy^2 + y^3)$.
12. $(3a^2 - 2a)(5a^3 - 2a^2 - 3a + 4)$.
13. $(a - b)(a^7 + a^6b + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 + ab^6 + b^7)$.

64. A polynomial is said to be **arranged according to the powers of some letter** when the exponents of that letter in the successive terms either increase or decrease continually.

In the former case the polynomial is said to be *arranged according to ascending powers,* in the latter *according to descending powers* of the letter.

E.g., $x^5 + 3x^3 + x^2 + 1$ is arranged according to descending powers of x. If it is desired to have all of the powers represented, it is written $x^5 + 0x^4 + 3x^3 + x^2 + 0x + 1$.

The polynomial $x^3 - 3x^2y + 3xy^2 - y^3$ is arranged according to descending powers of x and ascending powers of y.

There is evidently an advantage in arranging both multiplicand and multiplier according to the powers of some letter, as shown by the following example:

NOT ARRANGED.

$$\begin{array}{l} y^2 + x^2 + 2xy \\ \quad x + y \\ \hline xy^2 + x^3 + 2x^2y \\ \qquad + y^3 + x^2y + 2xy^2 \\ \hline xy^2 + x^3 + 2x^2y + y^3 + x^2y + 2xy^2 \\ \equiv x^3 + 3x^2y + 3xy^2 + y^3 \end{array}$$

ARRANGED.

$$\begin{array}{l} x^2 + 2xy + y^2 \\ \quad x + y \\ \hline x^3 + 2x^2y + xy^2 \\ \quad x^2y + 2xy^2 + y^3 \\ \hline x^3 + 3x^2y + 3xy^2 + y^3 \end{array}$$

Check. Let $x = 1$, $y = 1$. Then $2 \cdot 4 = 8$.

The method at the right is evidently much simpler.

65. It is also evident that *the product of the terms of highest degree in any letter in the factors is the term of highest degree in that letter in the product.* Also that *the product of the terms of lowest degree in any letter in the factors is the term of lowest degree in that letter in the product.*

Hence, if the factors are both arranged according to the descending (or ascending) powers of some letter, the first term of the product will be the product of the first terms, and the last term will be the product of the last terms.

EXERCISES. XXI.

Perform the following multiplications, checking the results by assigning arbitrary values to the letters:

1. $x^3 - y^3$ by $x^3 + y^3$.
2. $a^2x + x^2a$ by $x^2a - a^2x$.
3. $x^2y^2 - x^3 - y^3$ by $y - x$.
4. $x + y + z$ by $x + y - z$.
5. $1 - a^2 + a^4 - a^6$ by $1 + a^2$.
6. $x^7 + x^5y + y^6$ by $x^2 - 3x + y$.
7. $\frac{1}{4}y^2 - \frac{1}{6}yz + \frac{1}{9}z^2$ by $\frac{1}{2}y - \frac{1}{3}z$.
8. $xyz - x^2 - y^2 - z^2$ by $x + y + z$.
9. $p^2 - 2pq + q^2$ by $p^2 + 2pq + q^2$.
10. $-a^2 + 3ab + b^2$ by $3ab - b^2 + a^2$.
11. $a^5 - a^4 + a^3 - a^2 + a - 1$ by $a + 1$.
12. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 - 2xy + y^2$.
13. $xy + 2xz - 3yz + x^2 + y^2 + 4z^2$ by $x - y - 2z$.

66. Detached coefficients may be employed in multiplication whenever it is apparent what the literal part of the product will be.

E.g., in multiplying $x^7 + pz + q$ by $x^6 - x + pq$ the coefficients cannot be detached to advantage.

But in multiplying $x^2 + 2xy + y^2$ by $x + 3y$, it is apparent that the exponents of x decrease by 1 while those of y increase by 1 in each factor, and that this law will also hold in the product. Hence, when the coefficients are known the product is known also, and the multiplication may be performed as follows:

$$\begin{array}{rl} & \textit{Check.} \\ 1+2+1 & = 4 \\ 1+3 & = \underline{4} \\ \hline 1+2+1 & \\ 3+6+3 & \\ \hline 1+5+7+3 & = 16 \end{array}$$

$$\therefore (x + 3y)(x^2 + 2xy + y^2) \equiv x^3 + 5x^2y + 7xy^2 + 3y^3.$$

67. If the coefficient of the first term of the multiplier is 1, as is frequently the case, the work can be materially simplified by the following arrangement:

The problem is the same as the preceding one.

$$\begin{array}{r|l} 1 & 1+2+1 \\ +3 & \underline{\quad\;\; 3+6+3} \\ & 1+5+7+3 \end{array}$$

Check. $4 \cdot 4 = 16$.

$x^3 + 5x^2y + 7xy^2 + 3y^3$.

68. In case any powers are lacking in the arrangement of the polynomial, zeros should be inserted to represent the coefficients of the missing terms.

E.g., to multiply $x^2 + xy + y^2$ by $x^2 + y^2$, either of the following arrangements may be used:

$$\begin{array}{r} 1+1+1 \\ \underline{1+0+1} \\ 1+1+1 \\ \underline{\qquad\qquad 1+1+1} \\ 1+1+2+1+1 \end{array}$$

$$\begin{array}{r|l} 1 & 1+1+1 \\ +0 & \\ +1 & \underline{\qquad\qquad 1+1+1} \\ & 1+1+2+1+1 \end{array}$$

Check. $2 \cdot 3 = 6$.

$x^4 + x^3y + 2x^2y^2 + xy^3 + y^4$

EXERCISES. XXII.

Perform the multiplications indicated in exs. 1–13, by detached coefficients, checking the results as usual.

$P \equiv x^3 - x^2y + xy^2 - y^3,\quad Q \equiv x - y,\quad R \equiv x^2 - xy + y^2.$

1. PQ.
2. PR.
3. QR.
4. P^2.
5. Q^2R.
6. R^2.
7. QR^2.
8. Q^2R^2.
9. $(x+y)^2$.
10. $(x-y)^2$.
11. $(x+y)^3$.
12. $(x-y)^3$.
13. $(x+y)(x-y)$.

14. Verify the following identities, (1) by substituting arbitrary values, (2) by expanding both sides of the identity, using detached coefficients or not as seems best:

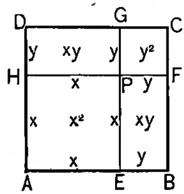
(a) $(x+y+z)^3 - (x^3+y^3+z^3) \equiv 3(x+y)(y+z)(z+x)$.

(b) $(x+y)^2 + (y+z)^2 + (z+x)^2 - (x^2+y^2+z^2) \equiv (x+y+z)^2$.

IV. SPECIAL PRODUCTS FREQUENTLY MET.

69. In exs. 9–13 on p. 50 five products were found which are so frequently used as to require memorizing. They are as follows:

1. $(x+y)^2 \equiv x^2 + 2xy + y^2$. Hence, *the square of the sum of two numbers equals the sum of their squares plus twice their product.*

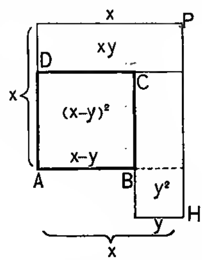


This theorem may be illustrated by a figure. Here the square $AC = (x+y)^2$, the square $AP = x^2$, the square $PC = y^2$, and there are two rectangles equal to $EF = xy$. And

$$\because AC = AP + 2\,EF + PC,$$

$$\therefore (x+y)^2 \equiv x^2 + 2\,xy + y^2.$$

2. $(x-y)^2 \equiv x^2 - 2xy + y^2$. Hence, *the square of the difference of two numbers equals the sum of their squares minus twice their product.*



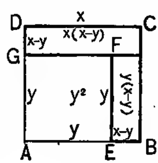
In the figure, $AP^2 = x^2$, $BH = y^2$, $AC = (x-y)^2$, and $DP = CH = xy$. And

$$\because AC = AP - 2\,DP + BH,$$

$$\therefore (x-y)^2 \equiv x^2 - 2\,xy + y^2.$$

Expressions of the form $x+y$, $x-y$, are called *conjugates* of each other.

3. $(x+y)(x-y) \equiv x^2 - y^2$. Hence, *the product of the two conjugate binomials equals the difference of their squares* (*i.e.*, the square of the minuend minus the square of the subtrahend).



In the figure, $AC = x^2$, $AF = y^2$, and $GC + FB = x(x-y) + y(x-y) = (x+y)(x-y)$. And

$$\because GC + FB = AC - AF,$$

$$\therefore (x+y)(x-y) \equiv x^2 - y^2.$$

4. $(x+y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$. Hence, *the cube of the sum of two numbers equals the sum of their cubes plus three times the square of the first into the second plus three times the square of the second into the first.*

5. $(x-y)^3 \equiv x^3 - 3x^2y + 3xy^2 - y^3$. (State the theorem.)

EXERCISES. XXIII.

By the help of the theorems of § 69 expand the expressions in exs. 1–18.

1. 42×38, *i.e.*, $(40+2)(40-2)$.

2. 23×17.

3. 95×85.

4. $(a^2+3)^2$.

5. $(a^7-2)^2$.

6. $(2p+1)^3$.

7. $(2x^2-1)^3$.

8. $(2x^2-y)^2$.

9. $[a-(b+c)]^2$.

10. $(2x^2+1)(2x^2-1)$.

11. $(a^2+3)(a^2-3)$.

12. $(a+b-ab)(a+b+ab)$.

13. $[(a+b)(a-b)]^2$.

14. $(a^2+2ab+b^2)(a^2-2ab+b^2)$.

15. $(x^4+y^4)(x^4-y^4)$.

16. 42^2.

17. $49 \cdot 51$.

18. 49^2.

19. Verify the following identities:

(a) $(a^2+b^2+c^2+d^2)(w^2+x^2+y^2+z^2) - (aw+bx+cy+dz)^2 \equiv (ax-bw)^2 + (cz-dy)^2 + (ay-cw)^2 + (dx-bz)^2 + (az-dw)^2 + (by-cx)^2$.

(b) $(x+y)^3 - x^3 - y^3 \equiv 3xy(x+y)(x^2+xy+y^2)^0$.

(c) $(x+y)^5 - x^5 - y^5 \equiv 5xy(x+y)(x^2+xy+y^2)^1$.

(d) $(x+y)^7 - x^7 - y^7 \equiv 7xy(x+y)(x^2+xy+y^2)^2$.

20. Expand the following expressions by the help of the theorems of § 69, checking by arbitrary values:

(a) $(x^2+y^2)^3$.

(b) $(x^3+y^3)^2$.

V. INVOLUTION.

70. The product of several equal factors is called a **power** of one of them (§ 8).

The broader meaning of the word *power* is discussed later (§ 130). At present the term will be restricted to positive integral power.

71. The operation of finding a power of a number or of an algebraic expression is called **involution.**

The student has already proved one important proposition in involution, *viz.*, that $a^m \cdot a^n \equiv a^{m+n}$, where the exponents are positive integers (§ 60).

He has also learned how to raise the binomial $x \pm y$ to the second and third powers (§ 69).

It now becomes necessary to consider certain other theorems.

72. Notation. If m and n are positive integers,

$(a^m)^n$ means $a^m \cdot a^m \cdot a^m \cdots$ to n factors, each a^m;

a^{m^n} " $a \cdot a \cdot a \cdots$ to m^n " " a.

E.g., $(a^3)^2$ means $a^3 \cdot a^3 = a^{3+3} = a^6$;

a^{3^2} " $a \cdot a \cdot a \cdots$ to 3^2 factors, $\equiv a^9$;

a^{2^3} " $a \cdot a \cdot a \cdots$ to 2^3 " $\equiv a^8$.

73. a^1 has already been defined to equal a.

74. The expression a^0, a being either positive or negative, is defined to equal 1, for reasons hereafter set forth (§ 214).

75. Theorem. *The* n*th power of the* m*th power of an algebraic expression equals the* mn*th power of the expression.*

Given an algebraic expression a, and m and n positive integers.

To prove that $(a^m)^n \equiv a^{mn}$.

Proof. 1. $(a^m)^n$ means $a^m \cdot a^m \cdot a^m \cdots$ to n factors, each a^m,

2. $\equiv a^{m+m+m+\cdots \text{ to } n \text{ terms, each } m}$ § 60

3. $\equiv a^{mn}$.

76. Theorem. *The* m*th power of the product of several algebraic expressions equals the product of the* m*th powers of the expressions.*

Given the expressions $a, b, c, \cdots$, and m an integer.

To prove that $(abc\cdots)^m \equiv a^m b^m c^m \cdots$.

Proof. 1. $(abc\cdots)^m$ means $(abc\cdots)\cdot(abc\cdots)\cdot(abc\cdots)\cdots$, to m groups, each $(abc\cdots)$

2. $\equiv (aaa\cdots\text{to } m \text{ factors})\cdot(bbb\cdots\text{to } m \text{ factors})\cdot(ccc \cdots \text{to } m \text{ factors})\cdots$ § 59

3. $\equiv a^m b^m c^m$. Def. of power

77. Law of signs. Since

$$+a\cdot +a \equiv +a^2,$$

and

$$-a\cdot -a \equiv +a^2,$$

but

$$-a\cdot -a\cdot -a \equiv -a^3,$$

it is easily seen that

1. *Powers of positive expressions are positive;*
2. *Even powers of negative expressions are positive;*
3. *Odd powers of negative expressions are negative.*

EXERCISES. XXIV.

Express without parentheses exs. 1–12.

1. $(a^2x^m)^n$.
2. $(a^m b^m)^m$.
3. $(a^2b^3c^4d^5)^2$.
4. $(-a^2b^3c)^3$.
5. $(-ab^2c^3)^4$.
6. $-(a^2b^3c^4)^4$.
7. $(a^3)^4$, $(a^4)^3$.
8. $(a^2)^5$, $(a^5)^2$.
9. $(-a^m b^n)^{2mn}$.
10. $(-\frac{1}{2}a^2b^m)^{4m}$.
11. $-[-(a^2)^2]^2$.
12. $-(-a^m b^n c^p)^2$.
13. Prove that $(a^m)^n \equiv (a^n)^m$.
14. Is it true that $a^{m^n} \equiv a^{n^m}$? Proof.
15. Also that $(a^m b^n)^{mn} \equiv (a^n b^m)^{mn}$? Proof.

78. Powers of polynomials. A polynomial can be raised to any power by ordinary multiplication.

But in raising to the 4th power it is easier to square and then to square again, since $(a^2)^2 \equiv a^4$.

E.g., to expand $(x - 2y)^4$.

1. $(x - 2y)^2 \equiv x^2 - 4xy + 4y^2$. § 69
2. $(x^2 - 4xy + 4y^2)^2 \equiv [(x^2 - 4xy) + 4y^2]^2$ § 46
3. $\equiv (x^2 - 4xy)^2 + 2(x^2 - 4xy) \cdot 4y^2 + 16y^4$
4. $\equiv x^4 - 8x^3y + 16x^2y^2 + 8x^2y^2 - 32xy^3 + 16y^4$
5. $\equiv x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$.

Check. $(-1)^4 = 1 - 8 + 24 - 32 + 16 = 1$.

Similarly, to raise to the 6th power first cube and then square, since $(a^3)^2 \equiv a^6$.

But to raise to the 5th, 7th, 11th, or other powers of prime degree, multiply out by detached coefficients.

EXERCISES. XXV.

Expand the expressions in exs. 1–20.

1. $(20 + 1)^2$.
2. $(x^2 - 3y^3)^3$.
3. $(x + 3y)^4$.
4. $(2x - 7y)^4$.
5. $(x^m + y^n)^4$.
6. $(a + b + c)^3$.
7. $(\frac{1}{2}x - \frac{2}{3}y)^4$.
8. $(-x - 3y)^4$.
9. $(2x^3 - 3y^4)^5$.
10. $(a + 2b + c)^3$.
11. $(-a - b - c)^2$.
12. $(a^2 + 2ab + b^2)^2$.
13. $\left(\frac{x^2}{2} + \frac{x}{3} + \frac{1}{4}\right)^2$.
14. $\left(\frac{x^2}{4} - \frac{x}{3} - \frac{1}{2}\right)^2$.
15. $(a^{10} - b^8 + c^6)^2$.
16. $(x^4 + x^2y^2 + y^4)^2$.
17. $(\frac{1}{2}a + 2b + c)^2$.
18. $(31m^2 - 20n^2)^2$.
19. $(a - 2b + 3c)^2$.
20. $(a - b + c - d)^2$.

79. The Binomial Theorem. It frequently becomes necessary to raise binomials to various powers. There is a simple law for effecting this, known as the *Binomial Theorem.*

The student will discover most of this law in answering the following questions:

Expand $(a+b)^2$, $(a+b)^3$, $(a+b)^4$, $(a+b)^5$.

(a) How does the number of terms in each expansion compare with the degree of the binomial?

(b) How do the exponents of a change in the successive terms?

(c) How do the exponents of b change in the successive terms?

(d) In each case, what is the first coefficient? How does the second coefficient compare with the exponent of the binomial?

(e) In the case of the 4th power does the third coefficient equal $\frac{4 \cdot 3}{2}$? In the 5th power is it $\frac{5 \cdot 4}{2}$? What will it probably be in the 6th power? in the 7th? in the nth?

(f) In the case of the 4th power does the fourth coefficient equal $\frac{4 \cdot 3 \cdot 2}{2 \cdot 3}$? In the 5th power is it $\frac{5 \cdot 4 \cdot 3}{2 \cdot 3}$? What will it probably be in the 6th power? in the 7th? in the nth?

(g) In the case of the 4th power does the fifth coefficient equal $\frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 3 \cdot 4}$? In the 5th power is it $\frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4}$? What will it probably be in the 6th power? in the 7th? in the nth?

(h) In expanding $(a+b)^7$, what will be the coefficient of a^6b? of a^5b^2? (The student should now be able to answer without actual multiplication.)

80. Theorem. If the binomial $a + b$ is raised to the nth power, n integral and positive, the result is expressed by the formula

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \cdots$$

where:

1. *The number of terms in the second member is* n + 1;

2. *The exponents of* a *decrease from* n *to* 0, *while those of* b *increase from* 0 *to* n;

3. *The first coefficient is* 1, *the second is* n, *and any other is formed by multiplying the coefficient of the preceding term by the exponent of* a *in that term and dividing by* 1 *more than the exponent of* b.

The proof of this theorem, which has already been found inductively on p. 56, may be taken now or it may be postponed until later in the course. The proof is given in Appendix I.

81. Pascal's Triangle. The coefficients of the various powers of the binomial $f + n$ are easily found by a simple arrangement known as *Pascal's Triangle*, from the famous mathematician who made some study of its properties.

Coefficients for 1st power	1	1						
" " 2d "	1	2	1					
" " 3d "	1	3	3	1				
" " 4th "	1	4	6	4	1			
" " 5th "	1	5	10	10	5	1		
" " 6th "	1	6	15	20	15	6	1, and so on.	

Each number is easily seen to be the sum of the number above and the number to the left of the latter.

Write down the coefficients for the 7th, 8th, 9th, and 10th powers, thus enlarging Pascal's triangle.

For note on Pascal, see the *Table of Biography*.

82. Various powers of f + n. These are needed in the extraction of roots (§§ 128–133) and should be verified by the student.

$$(f+n)^2 \equiv f^2 + 2fn + n^2.$$
$$(f+n)^3 \equiv f^3 + 3f^2n + 3fn^2 + n^3.$$
$$(f+n)^4 \equiv f^4 + 4f^3n + 6f^2n^2 + 4fn^3 + n^4.$$
$(f+n)^5 \equiv$ (Expand it.)
$(f+n)^6 \equiv$ " "
$(f+n)^7 \equiv$ " "

Illustrative problems. 1. Expand $(2a - 3b^2)^3$.

1. $(2a-3b^2)^3 \equiv (2a)^3 + 3(2a)^2(-3b^2) + 3(2a)(-3b^2)^2 + (-3b^2)^3$
2. $\equiv 8a^3 - 36a^2b^2 + 54ab^4 - 27b^6.$

Check. $(-1)^3 = 8 - 36 + 54 - 27 = -1.$

In cases like this it is better to indicate the work in the first step and then simplify.

2. Expand $\left(\frac{x}{2} - y + z^2\right)^2$.

1. $\left(\frac{x}{2} - y + z^2\right)^2 \equiv \left[\left(\frac{x}{2} - y\right) + z^2\right]^2$
2. $\equiv \left(\frac{x}{2} - y\right)^2 + 2\left(\frac{x}{2} - y\right)z^2 + (z^2)^2$
3. $\equiv \frac{x^2}{4} - xy + y^2 + xz^2 - 2yz^2 + z^4.$

EXERCISES. XXVI.

Expand the following expressions:

1. $(x+y)^4$.
2. $(1-a)^8$.
3. $(x^2-y)^7$.
4. $(x-y)^{10}$.
5. $(a-2b)^2$.
6. $(2x+y^3)^2$.
7. $(x^2y-3y^2)^2$.
8. $(x+y-z)^2$.
9. $(\frac{1}{2}x - \frac{1}{3}y + \frac{1}{4}z)^2$.
10. $(\frac{1}{3}x^2y^3z^4 + \frac{1}{2}x^4y^3z^2)^2$.

11. $(a - b - c)^2$.

12. $(2a - 3b)^4$.

13. $(a - b + c)^3$.

14. $(3x + 2y^2)^5$.

15. $(\frac{1}{2}x^2 - \frac{1}{3}y^2)^3$.

16. $(3a^2 - 2ab + b^2)^3$.

17. $\left(x + \frac{1}{x}\right)^2$.

18. $\left(1 - \frac{1}{x}\right)^6$.

19. $\left(\frac{1}{2}x^3 + \frac{1}{y}\right)^2$.

20. $\left(\frac{1}{x^2} - 3x^3\right)^3$.

21. $\left(\frac{1}{x^2} - 2 + x^2\right)^3$.

22. $\left(\frac{4m^3}{3n^2} - \frac{2n}{3m^4}\right)^3$.

REVIEW EXERCISES. XXVII.

1. Solve the equation $184 - x^2 = 40$. Check.

2. What is the etymological meaning of *multiply?* of *abstract?* of *ascending?* of *descending?* of *commutative?*

3. Show that the arithmetic definition of multiplication is not broad enough for algebra. Explain the definition in § 54.

4. What is the broader meaning of the word *times* in algebra? Illustrate.

5. What is the Index Law of multiplication? Has it been proved by you for all kinds of exponents? If not, for what kind? Prove it.

6. What is meant by the Distributive Law of multiplication? Prove the law.

7. Make up an example illustrating the advantage of arranging the terms according to the powers of some letter in multiplication.

8. What are the advantages in using detached coefficients in multiplication? Illustrate by solving a problem.

CHAPTER IV.

DIVISION.

I. DEFINITIONS AND LAWS.

83. **Division** is the operation by which, having the product of two expressions and one of them (not zero) given, the other is found.

Thus, 6 is the product of 2 and 3; given 6 and 2, 3 can be found.

The given product is called the **dividend**, the given expression is called the **divisor**, and the required expression is called the **quotient**.

84. Since $0 = a \cdot 0$ (§ 55), it follows that $\frac{0}{a}$ should be defined to mean 0.

85. **Law of signs.** Since

$$+a \cdot +b \equiv +ab,$$
$$+a \cdot -b \equiv -ab,$$
$$-a \cdot +b \equiv -ab,$$

and

$$-a \cdot -b \equiv +ab,$$

it therefore follows, from the definition of division, that

$$+ab \div +a \equiv +b,$$
$$-ab \div +a \equiv -b,$$
$$-ab \div -a \equiv +b,$$

and

$$+ab \div -a \equiv -b.$$

That is, *like signs in dividend and divisor produce* +, *and unlike signs* −, *in the quotient.*

86. Index law. Since $a^{m-n} \cdot a^n \equiv a^m$, by the index law of multiplication (§ 60), therefore, $\frac{a^m}{a^n} \equiv a^{m-n}$, by the definition of division.

Hence, $10\,a^5b^5c^9 \div 5\,a^3b^2c^4 \equiv 2\,a^2b^3c^5$.

The above proof is based on the supposition that $m > n$, and that both are positive integers. The cases in which m and n are zero, negative, and fractional, and in which $m < n$, are considered later.

EXERCISES. XXVIII.

Perform the following divisions:

1. $-125 \div -25$.
2. $80 \div -16$.
3. $\frac{3\,a^2bc}{-bc}$.
4. $\frac{25\,a^7bc^6}{-5\,a^5bc}$.
5. $\frac{-10\,xy^2z^3}{-5\,y^2z^3}$.
6. $\frac{49\,x^{18}y^{20}z^3w}{-7\,wz^2}$.
7. $\frac{-70\,x^2y^3z^4}{7\,xy^2z^3}$.
8. $\frac{-35\,p^2q^{10}r^{12}}{-7\,p^2qr^{11}}$.
9. $\frac{-56\,a^{17}b^{14}c}{-8\,a^{16}bc}$.
10. $\frac{-27\,(a-b)}{a-b}$.

II. DIVISION OF A POLYNOMIAL BY A MONOMIAL.

87. 1. $\because ma + mb + mc \equiv m(a + b + c)$. § 61

2. $\therefore \frac{ma + mb + mc}{m} \equiv a + b + c$. Def. of division

3. Hence, *to divide a polynomial by a monomial is to divide each term of the polynomial by the monomial and to add the quotients.*

Thus, $\frac{17\,ab^3 - 34\,a^2b^2}{-17\,ab^2} \equiv 2\,a - b$. *Check.* Let $a = 2$, $b = 3$. Then $\frac{17 \cdot 2 \cdot 27 - 34 \cdot 4 \cdot 9}{-17 \cdot 2 \cdot 9} = 2 \cdot 2 - 3$, or $\frac{-306}{-306} = 1$.

EXERCISES. XXIX.

Perform the divisions indicated; check by assigning arbitrary values.

1. $\dfrac{27x^4y - 27xy^4}{-27xy}$.

2. $\dfrac{121m^2n^3 - 110m^3n^2}{-11m^2n^2}$.

3. $\dfrac{x^4 + 3x^3y + 3x^2y^2 + xy^3}{x}$.

4. $\dfrac{-3a^5b - 12a^4b^2 + 9a^3b^3}{-3a^3b}$.

5. $\dfrac{a^4 - 3a^3b + 3a^2b^2 - 7ab^3}{-a}$.

6. $\dfrac{34a^2b^3c - 17ab^2c^3 + 51a^3b^2c}{17ab^2c}$.

7. $\dfrac{200x^3y^5 - 75x^5y^3 + 125x^3y^3}{25x^3y^3}$.

8. $\dfrac{65x^{2m}y^2 - 52x^{3m}y^3 + 39x^{4m}y^4}{-13x^my^2}$.

9. $\dfrac{5p^4 - 15p^3q + 10p^2q^3 - 20p^5}{5p^2}$.

10. $\dfrac{2(a+b)^5 - 3(a+b)^3 + 2(a+b)^2}{(a+b)^2}$.

11. $\dfrac{48x^{17}y - 36x^{16}y + 72xy^{16} - 108xy}{12xy}$.

12. $\dfrac{(x^2 + 2xy + y^2)^3 + (x^2 + 2xy + y^2)^2}{-(x^2 + 2xy + y^2)}$.

13. $\dfrac{(2x-1)^7 + 5(2x-1)^5 - (2x-1)^3}{-(2x-1)^2}$.

14. $\dfrac{-52a^{20}b^6 - 78a^6b^{20} - 26a^{12}b^{12} - 130a^6b^8}{-26a^6b^6}$.

III. DIVISION OF A POLYNOMIAL BY A POLYNOMIAL.

88. As a preliminary to the explanation of this form of division it is necessary to observe the following important points:

1. In division, if dividend, divisor, and quotient are arranged according to the descending powers of some letter, then the first term of the quotient is the quotient of the first terms of dividend and divisor.

That is, in dividing $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$, the first term of the quotient is x^2. For it has been shown (§ 65) that the term of highest degree in any letter in the product (dividend) equals the product of the terms of highest degree in that letter in the multiplicand (divisor or quotient) and multiplier (quotient or divisor).

E.g., in dividing $x^7 + x^6y + 2x^5y^2 - 2x^2y^5 - xy^6 - y^7$ by $x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4$, the first term of the quotient is x^3. If the terms in each polynomial were written in reverse order, the first term of the quotient would evidently be $-y^3$.

2. If the product of the divisor and the first term of the quotient is subtracted from the dividend, a partial dividend is obtained which is the product of the divisor by the other terms of the quotient.

That is, in dividing $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$, we know (by 1) that x^2 is the first term of the quotient. Now if

$$\begin{array}{ll} \text{from} & x^3 + 3x^2y + 3xy^2 + y^3 \\ \text{we take } x^2(x+y) \text{ or} & \underline{x^3 + x^2y } \\ \text{the remainder} & 2x^2y + 3xy^2 + y^3 \end{array}$$

is a partial dividend and is the product of the divisor, $x + y$, by the other terms which follow in the quotient.

This is evident because the whole dividend is the product of $x + y$ by the quotient; hence, the $2x^2y + 3xy^2 + y^3$ is the product of $x + y$ by the other terms of the quotient.

It will be noticed that this is similar to the division with which the student has become familiar in arithmetic; each remainder is the product of the divisor and the rest of the quotient.

89. The operation of division can now be explained. Let it be required to divide $3x^2y + y^3 + x^3 + 3xy^2$ by $y + x$. It has been shown (§ 88) that, if the expressions are arranged according to the descending powers of x, the first term of the quotient is x^2.

		$x^2 + 2xy + y^2$	= quotient.
Divisor	$= x + y\ )$	$x^3 + 3x^2y + 3xy^2 + y^3$	= dividend.
If	$x^2(x + y)$ or	$x^3 + x^2y$	is subtracted,
the remainder		$2x^2y + 3xy^2 + y^3$	is a partial dividend, the product of $x + y$ by the rest of the quotient. ∴ the next term of the quotient is $2xy$.
Subtracting	$2xy(x + y)$ or	$2x^2y + 2xy^2$	
the remainder		$xy^2 + y^3$	is also a partial dividend, the product of $x + y$ by the rest of the quotient. ∴ the next term of the quotient is y^2.
Subtracting	$y^2(x + y)$ or	$xy^2 + y^3$	

there is no remainder, and the division is complete.

90. Exact division. If one of the partial dividends becomes identically 0, the division is said to be *exact*. If not, the degree of some partial dividend will be less than that of the divisor; such a partial dividend is called the **remainder.**

This subject will be further considered in the chapter on fractions.

If $D \equiv$ dividend, $d \equiv$ divisor, $q \equiv$ quotient, and $r \equiv$ remainder, then

$$D - r \equiv dq;$$

that is, if the remainder were subtracted from the dividend the result would be the product of the quotient and the divisor.

91. *Checks.* 1. Since the dividend is the product of the quotient and the divisor, one check is by multiplication.

$\because D - r \equiv dq$, any remainder should first be subtracted.

2. The work may be checked by arbitrary values.

92. Arrangement of work in division. The full form of the work is as follows:

$$
\begin{array}{rll}
 & x^2 + 2xy + 2y^2 & = \text{quotient.} \\
\text{Divisor} = x + y) & x^3 + 3x^2y + 4xy^2 + 5y^3 & = \text{dividend.} \\
 & \underline{x^3 + x^2y} & = x^2(x+y). \\
 & 2x^2y + 4xy^2 + 5y^3 & = \text{1st partial dividend.} \\
 & \underline{2x^2y + 2xy^2} & = 2xy(x+y). \\
 & 2xy^2 + 5y^3 & = \text{2d partial dividend.} \\
 & \underline{2xy^2 + 2y^3} & = 2y^2(x+y). \\
\text{(See check below.)} & 3y^3 & = \text{remainder.}
\end{array}
$$

It is better in practice to abridge this work as follows:

$$
\begin{array}{rl}
 & x^2 + 2xy + 2y^2 \\
x + y) & x^3 + 3x^2y + 4xy^2 + 5y^3 \\
 & \underline{x^3 + x^2y} \\
 & 2x^2y \\
 & \underline{2x^2y + 2xy^2} \\
 & 2xy^2 + 5y^3 \\
 & \underline{2xy^2 + 2y^3} \\
 & 3y^3
\end{array}
$$

It is still better to detach the coefficients if possible.

$$
\begin{array}{rl}
 & 1 + 2 + 2 \\
1 + 1) & 1 + 3 + 4 + 5 \\
 & \underline{1 + 1} \\
 & 2 \\
 & \underline{2 + 2} \\
 & 2 + 5 \\
 & \underline{2 + 2} \\
 & 3
\end{array}
$$

Check. Let $x = y = 1$.

$$(1 + 1)(1 + 2 + 2) = 1 + 3 + 4 + 5 - 3$$

or $2 \cdot 5 = 10$.

$x^2 + 2xy + 2y^2$, and $3y^3$ remainder.

Similarly, to divide $x^3 - 1$ by $x - 1$.

$$
\begin{array}{rl}
 & 1 + 1 + 1 \\
1 - 1) & 1 + 0 + 0 - 1 \\
 & \underline{1 - 1} \\
 & 1 \\
 & \underline{1 - 1} \\
 & 1 \\
 & \underline{1 - 1}
\end{array}
$$

Check. Let $x = 2$.

$$(2 - 1)(8 - 1) = 4 + 2 + 1$$

or $1 \cdot 7 = 7$.

$x^2 + x + 1$.

EXERCISES. XXX.

Perform the divisions indicated in exs. 1–14. Check the results by substituting such arbitrary values as shall not make the divisor zero.

1. $x^3 - y^3$ by $x - y$.
2. $x^{12} - a^{12}$ by $x^3 - a^3$.
3. $32a^5 - b^5$ by $2a - b$.
4. $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.
5. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
6. $a^3 + 3a^2 + 3a + 1$ by $a^2 + 2a + 1$.
7. $x^3 - 3x^2 + 3x + y^3 - 1$ by $x + y - 1$.
8. $x^4 - 2ax^3 + 2a^3x - a^4$ by $x^2 - a^2$.
9. 1 by $1 - x$, carrying the quotient to 6 terms.
10. $-a^5 - 2a^4 + 2a^3 + 6a^2 + a - 1$ by $-a^2 + a + 1$.
11. $-a^3 + 8a^5b - 14a^4b^2 + a^3b^3 + 6a^2b^4$ by $a^3 - 3a^2b + 2ab^2$.
12. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$ by $a^2 - 2ab + b^2$.
13. $x^3 + y^3 + z^3 - 3xyz$ by $x^2 + y^2 + z^2 - xy - yz - zx$.
14. $x^4 + x^2y^2 + y^4$ by $y^2 - xy + x^2$. (Rearrange the divisor.)

Perform the divisions indicated in exs. 15–31 by using detached coefficients, checking as above.

15. $x^5 - 5x^2 - 3000$ by $x - 5$.
16. $16x^4 - 81y^4$ by $2x + 3y$.
17. $3x^3 - 7x - 2 - 2x^2$ by $1 + x$.
18. $a^4 + 24a + 55$ by $a^2 - 4a + 11$.
19. $x^4 - 2a^2x^2 + a^4$ by $x^2 - 2ax + a^2$.
20. $x^8 - 3x^6 + 6x^4 - 7x^2 + 3$ by $x^4 - 2x^2 + 1$.
21. $p^5 + p^4 + 4p^2 - 9p + 3$ by $p^3 + p^2 - 3p + 1$.
22. $x^8 - x^5 + 2x^4 + 4x^3 - 7x^2 + 4x - 1$ by $x^2 + x - 1$.

23. $x^5 + 7x^3y^2 - 5x^4y - x^2y^3 + 2y^5 - 4xy^4$ by $(x - y)^3$.

24. $26a^2 + 4a^3 - 3a^4 + a^5 - 92a + 55$ by $a^2 - 3a + 11$.

25. $24m^4 - 14m^3 - 9m^2 - 84 + 43m$ by $7 - 3m + 4m^2$.

26. $x^8 - 3x^7 - 5x^5 + 2x^4 + 5x^3 + 4x^2 + 2$ by $x^3 + 2x - 1$.

27. $3m^6 + 7m^5 - 12m^4 + 2m^3 - 3m^2 + 13m - 6$ by $m^2 + 3m - 2$.

28. $x^7 - x^6 - 2x^5 + 5x^4 - 5x^3 + 8x^2 + 6x - 12$ by $x^3 - 2x^2 + 3$.

29. $x^8 + 2x^7 + 3x^6 + 4(x^5 + 1) + 5x^4 + 6x^3 + 7x^2 + 8x$ by $(x + 1)^2$.

30. $10m^6 - 11m^5 - 3m^4 + 20m^3 + 10m^2 + 2$ by $5m^3 - 3m^2 + 2m - 2$.

31. $x^7 + 2x^6 + 3x^5 + 3x^4 + 3x^3 + 3x^2 + 2x + 1$ by $x^5 + x^4 + x^3 + x^2 + x + 1$.

32. Divide the product of $(x - 1)(x - 2)(x - 3)(x - 4)$ by $2(4 - 3x) + x^2$.

33. Divide $q^5 + 1$ by $q + 1$, and hence tell the quotient of 100001 by 11 $(q = 10)$.

34. Divide $4t^4 + 2t^3 + 5t^2 + 8t + 1$ by $t + 1$, and hence tell the quotient of 42581 by 11.

35. Divide the sum of $\frac{3}{2}x^5 + 4x^4 + 7\frac{1}{2}x^3 + 11x^2 + 7x + 4$ and $\frac{3}{2}x^5 + 4x^4 + 6\frac{1}{2}x^3 + 9x^2 + 4x$ by their difference.

36. Divide $1 + x^2$ by $1 + x$ carrying the quotient to 5 terms. From the form of this quotient tell what the next 5 terms will be.

37. The product of two polynomials is $2m^4 - 13m^3n + 31m^2n^2 - 38mn^3 + 24n^4$. If one of them is $m^2 - 5mn + 6n^2$, what is the other?

Where the time allows, the work in Synthetic Division (Appendix II) should be taken at this point.

REVIEW EXERCISES. XXXI.

1. Solve the equation $2-(3-\overline{4-x})=3$.

2. Solve the equation $-2x+4=-12$. Check.

3. Solve the equation $\frac{2}{3}x+4=\frac{1}{3}x+4\frac{2}{3}$. Check.

4. What are the advantages in detaching the coefficients when practicable?

5. What is the etymological meaning of *quotient?* of *coefficient?* of *associative?*

6. The cube of a certain number, subtracted from 1, equals 9. Find the number.

7. What is the sign of the product of an odd number of negative numbers? Prove it.

8. If from twice a certain number we subtract 7 the result is 15. Find the number.

9. Three times a certain number, subtracted from 5, equals -10. Find the number.

10. Why do you arrange both dividend and divisor according to the powers of some letter?

11. Why do you avoid using such an arbitrary value in checking division as shall make the divisor zero?

12. If to three times a certain number we add 2 the result is five times the number. Find the number.

13. What is the value of

$$a\{a-b[a^2-2c(b^3-\overline{a-b}\div c)+b]-c\}$$

when $a=3$, $b=1$, $c=2$?

14. What is the Index Law of Division? Have you proved it for all values of the indices? If not, for what kinds of indices?

CHAPTER V.

ELEMENTARY ALGEBRAIC FUNCTIONS.

I. DEFINITIONS.

93. Every quantity which is regarded as depending upon another for its value is called a **function** of that other.

E.g., with a given principal and rate, the interest depends upon the time; hence in this case the interest is called a function of the time.

Similarly, the expression $x^2 - 3x + 21$ is a function of x, etc.

94. A function of x is usually indicated by some such symbol as $f(x)$, $F(x)$, $f_1(x)$, $P(x)$, $\cdots$.

Thus, if the expression $x^3 - x + 1$ is being considered, it may be designated by $f(x)$, read "function of x," or simply "function x."

If some other function of x, as $x^4 - x^3 + 2x^2 - x + 4$, is also being considered, it may be distinguished from the first one by designating it by $F(x)$, read "f major of x," or "f major function x."

$P(x)$, $f_1(x)$, $\cdots$ are read "P function x," "f-one function x," $\cdots$. The Greek letter ϕ (phi) is also very often used in this connection, $\phi(x)$ being read "phi function x."

95. If $f(x)$ is known in any discussion, $f(a)$ means that function with a put in place of x.

E.g., if $\quad f(x) \equiv x^2 + 2x + 1,$

then $\quad f(a) = a^2 + 2a + 1,$

$\quad f(2) = 2^2 + 2 \cdot 2 + 1 = 9,$

and $\quad f(0) = 0 + 0 + 1 = 1.$

96. A quantity whose value is not fixed is called a **variable**; if the value is fixed, it is called a **constant.**

E.g., in the expression $y^2 + 2y + 4$, y may have any value, and hence y is a variable. But when it is said that $y - 2 = 3$, the value of y is fixed, and hence y is a constant, 5.

97. Every algebraic expression which, in its simplest form, contains several variables is called a **function** of those variables.

E.g., $x^2 + 2xy + y^2$ is a function of x and y, and may be designated by $f(x, y)$, read "function of x and y," or simply "f of x and y."

But $x + y + a - y - x$ is not a function of x and y.

EXERCISES. XXXII.

1. If $f(x) \equiv x^4 - x^2 + x - 1$, what are $f(a)$, $f(a^2)$, $f(-2)$, $f(1)$, $f(0)$?

2. If $f(x) \equiv x^2 + x + 1$, and $F(x) \equiv x - 1$, find the value of $f(x) \cdot F(x)$. Check by letting $x = 2$, *i.e.*, by finding the value of $f(2) \cdot F(2)$.

3. If $f(x) \equiv x^3 + 3x^2 + 3x + 1$, and $\phi(x) \equiv x^2 + 2x + 1$, find the value of $f(x) \div \phi(x)$. Check by using $f(1) \div \phi(1)$.

4. If $f(x, y) \equiv x^3 - 3x^2y + 3xy^2 - y^3$, and $f_1(x, y) \equiv x - y$, find the value of $f(x, y) \cdot f_1(x, y)$; also of $f(x, y) \div f_1(x, y)$. Check by using $f(2, 1)$ and $f_1(2, 1)$.

5. If $F(x, y, z) \equiv x^3 + y^3 + z^3 - 3xyz$, and $f(x, y, z) \equiv x + y + z$, find the value of $F(x, y, z) \div f(x, y, z)$. Check by letting $x = y = z = 1$.

6. If $f_1(x) \equiv x^2 + 2x + 1$, $f_2(x) \equiv x^2 - 2x + 1$, and $f_3(x) \equiv x^2 - 1$, find the value of $f_1(x) \cdot f_2(x) \cdot f_3(x)$. Check by letting $x = 2$.

7. If $f(x) \equiv x^4 - 10x^3 + 35x^2 - 50x + 24$, find the values of $f(1)$, $f(2)$, $f(3)$, $f(4)$.

98. An algebraic expression is said to be **rational** with respect to any letter when it contains no indicated root of that letter. In the contrary case it is said to be **irrational** with respect to that letter.

E.g., $4a + \sqrt{2}$ is rational with respect to a,

but $2 + 4\sqrt{a}$ is irrational with respect to a.

So $x^2 - x\sqrt{a} + \sqrt[3]{a}$ is an irrational function of a, but it is a rational function of x.

99. A rational algebraic expression is said to be **integral** with respect to a letter when this letter does not appear in any denominator. In the contrary case it is said to be **fractional.**

E.g., $\frac{1}{2} - \frac{a}{3}$ is an integral *algebraic* expression, with respect to a,

but $2 - \frac{3}{a}$ is a fractional expression, with respect to a.

So $x^2 - \frac{x}{a} + \frac{1}{a^2}$ is an integral function of x,

but $x^2 - \sqrt{x}$ is not, because it is not rational,

and $\frac{1}{x} - \frac{2}{x^2}$ is not, because it has x in both denominators.

100. An algebraic expression is said to be **homogeneous** when all of its terms are of the same degree.

E.g., $7a^2x + 4a^3 + x^3$ is homogeneous, but $3a^2x + 4ax^3$ is not.

So $ax^2y + b^2xy^2 + c^3y^3$ is homogeneous *as to* x *and* y, but not as to x alone, nor as to y alone, nor as to a, x, and y.

101. An algebraic expression is said to be **symmetric** with respect to certain letters when those letters can be interchanged without changing the form of the expression.

E.g., $x^2 + 2xy + y^2$ is symmetric as to x and y, because if x and y are interchanged it becomes $y^2 + 2yx + x^2$ which is the same as the original expression. Similarly, $x^3 + y^3 + z^3 + axyz$ is symmetric as to x, y, and z, but not as to a, x, y, and z.

102. An algebraic expression is said to be **cyclic** with respect to certain letters in a given order when its value is not changed by substituting the second for the first, the third for the second, and so on to the first for the last.

E.g., $a(a-b)+b(b-c)+c(c-a)$ is cyclic as to a, b, and c; for if b is substituted for a, c for b, and a for c, it becomes $b(b-c)+c(c-a)+a(a-b)$, which is the same as the original expression.

It will be noticed that if an expression is symmetric it must be cyclic, for a cyclic change of letters is a special case of the general interchange of symmetry. But the converse is not true, for the special case does not include the general one.

E.g., $x^2+y^2+z^2-x(x+y^2)-y(y+z^2)-z(z+x^2)$ is cyclic but not symmetric; but $x^2+y^2+z^2-xy-yz-zx$ is symmetric and hence also cyclic.

The theory of cyclic functions is often called *cyclo-symmetry*, or, where no misunderstanding will result, simply symmetry.

EXERCISES. XXXIII.

Select from exs. 1–13 those expressions that are (1) homogeneous, (2) symmetric, (3) cyclic, as to any or all of the letters involved:

1. $a^2x-b^3x+c^4x$.
2. $x^{\frac{1}{2}}+2x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.
3. $x^2z-3xyz^2+y^2z^3$.
4. $ab+bc+ca+abc$.
5. $a^3+b^3+c^3-3abc$.
6. $abc-3ac^2+bc^2-c^3$.
7. $x^4-x^3y+x^2y^2-xy^3+y^4$.
8. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.
9. $a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2$.
10. $x^2+y^2+z^2+ax+by+cz+mxyz$.
11. $bc(b+c)+ca(c+a)+ab(a+b)+2abc$.

12. $a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2)$.

13. $(a + b)(a^2 + b^2 - c^2) + (b + c)(b^2 + c^2 - a^2)$
$+ (c + a)(c^2 + a^2 - b^2)$.

Select from exs. 14–20 those functions of x, of y, and of a that are (1) rational, (2) integral functions of those letters.

14. $\frac{2}{3}x - x\sqrt{a}$.

15. $x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$.

16. $x^2/a - x/a^2$.

17. $x^3 + x^2 + x + \sqrt{x}$.

18. $x^3 + 3x^2y + 3xy^2 + y^3$.

19. $x^m + x^{m-1} + x^{m-2} + \cdots + x^2 + x + 1$.

20. $x^m - x^{\frac{m}{2}}$, (1) when m is even, (2) when m is odd.

The applications of homogeneity and symmetry are numerous and valuable. If the time allows, they should be taken at this point. They are set forth at some length in Appendix III.

It should, however, be said that symmetry and homogeneity form two valuable checks, especially in multiplication. If two expressions are homogeneous their product is evidently homogeneous.

E.g., the product of $x^2 + 2xy - y^2$ and $x - y$ cannot be $x^3 + x^2y^2 - 3xy + y^3$, because the factors being homogeneous the product must be so.

Likewise, if two expressions are symmetric as to two or more letters, their product must be symmetric as to those letters.

E.g., the product of $x^2 - 2xy + y^2$ and $x + y$ cannot be $x^3 - x^2y + xy^2 + y^3$, because this is not symmetric as to x and y.

A knowledge of symmetry and homogeneity is of great value in factoring.

II. THE REMAINDER THEOREM.

103. If we consider the remainder arising from dividing a function of x, say $x^2 + px + q$, by $x - a$, we find an interesting law.

$$\begin{array}{r|l} & x + p + a = \text{quotient} \\ x - a \,) & x^2 + px + q \\ & \underline{x^2 - ax} \\ & (p + a)x + q \\ & \underline{(p + a)x - pa - a^2} \\ & a^2 + pa + q = \text{remainder.} \end{array}$$

That is, *the remainder is the same as the dividend with* a *substituted for* x.

Hence, if this law is general, we may find the remainder arising from dividing $x^2 + 2x - 3$ by $x - 2$ by simply substituting 2 for x in the dividend. This gives $2^2 + 2 \cdot 2 - 3 = 5$, the remainder.

Similarly, it is at once seen that, if this law is general, $x^{17} + 2x^9 - 3$ is exactly divisible by $x - 1$. (Why?)

That the law *is* general is proved on p. 75.

EXERCISES. XXXIV.

Assuming that the remainder can always be found as above stated, find the remainders arising from the following divisions:

1. $x^n - 1$ by $x - 1$.
2. $x^{75} - y^{75}$ by $x - y$.
3. $2x^5 - 64$ by $x - 2$.
4. $32a^5 - 1$ by $2a - 1$.
5. $(7x)^{10} + 1$ by $7x + 1$.
6. $x^4 - x^3 + x^2 - x + 1$ by $x - 3$.
7. $x^4 + x^3 - x^2 - x + 1$ by $x - 1$.
8. $3x^3 + 4x^2 - 2x - 36$ by $x - 2$.
9. $x^n + 1$ by $x + 1$, *i.e.*, by $x - (-1)$.

104. The Remainder Theorem. *If* $f(x)$ *is a rational integral algebraic function of* x, *then the remainder arising from dividing* $f(x)$ *by* $x - a$ *is* $f(a)$.

Proof. 1. Let q be the quotient and r the remainder.

2. Then $f(x) \equiv q(x-a) + r$. Def. of division

(*I.e.*, the dividend equals the product of the quotient and the divisor, plus the remainder, and this is true whatever the value of x.)

3. Step 2 is true if $x = a$, it being an identity.

4. But r does not contain x. (Why?)

5. $\therefore f(a) = q(a-a) + r = 0 + r = r$, from step 3.

6. *I.e.*, the remainder equals $f(a)$, or the dividend with a substituted for x.

105. Corollaries. 1. *If* $f(x)$ *is a rational integral algebraic function of* x, *then the remainder arising from dividing* $f(x)$ *by* $x + a$ *is* $f(-a)$.

For $x + a \equiv x - (-a)$; hence, $-a$ would merely replace a in the above proof.

2. *If* $f(a) = 0$, *then* $f(x)$ *is divisible by* $x - a$.

For the remainder equals $f(a)$, and this being 0 the division is exact.

3. *If* n *is a positive integer.*

(a) $x^n + y^n$ *is divisible by* $x + y$ *when* n *is odd.*

For, putting $-y$ for x, $x^n + y^n$ becomes $(-y)^n + y^n$, which equals 0 when n is odd, and not otherwise.

(b) $x^n + y^n$ *is never divisible by* $x - y$.

For, putting y for x, $x^n + y^n$ becomes $y^n + y^n$, which is not 0.

(c) $x^n - y^n$ *is divisible by* $x + y$ *when* n *is even.*

For, putting $-y$ for x, $x^n - y^n$ becomes $(-y)^n - y^n$, which equals 0 when n is even, and not otherwise.

(d) $x^n - y^n$ *is always divisible by* $x - y$.

For, putting y for x, $x^n - y^n$ becomes 0.

Illustrative problems. 1. Find the remainder arising from dividing $(x+1)^5 - x^5 - 1$ by $x+1$.

Substitute -1 for x, and $f(x)$ becomes $(-1+1)^5 - (-1)^5 - 1$, which equals $0 + 1 - 1$, or 0.

2. Also when $(x-m)^3 + (x-n)^3 + (m+n)^3$ is divided by $x+m$.

Substitute $-m$ for x, and $f(x)$ becomes $(-m-m)^3 + (-m-n)^3 + (m+n)^3$, which equals $-8m^3 - (m+n)^3 + (m+n)^3$, or $-8m^3$.

3. Also when $nx^{n+1} - (n+1)x^n + 1$ is divided by $x-1$.

Substitute 1 for x, and $n - (n+1) + 1 = 0$.

4. Find the remainder arising from dividing $x^5 + 5x^4 - 3x^3 - 2x + 7$ by $x+7$.

Here it is rather tedious to substitute -7 for x. If the student understands synthetic division (Appendix II) it is better to resort to it, as follows:

$$\begin{array}{r|llllll} & 1 & +5 & -3 & +0 & -2 & +7 \\ -7 & & -7 & 14 & -77 & 539 & -3759 \\ \hline & 1 & -2 & 11 & -77 & 537; & -3752 \text{ remainder.} \end{array}$$

Check. $[8 - (-3752)] \div 8 = 470.$

EXERCISES. XXXV.

Find the remainders in the following divisions:

1. $x^{2m} + y^{2m}$ by $x+y$.
2. $x^5 - 4x^2 + 3$ by $x+4$.
3. $x^{2m+1} + y^{2m+1}$ by $x+y$.
4. $32x^{10} - 33x^5 + 1$ by $x-1$.
5. $x^4 + 2x^2 - 3x - 7$ by $x-2$.
6. $x^{30} + x^{10} - 2$ by $x-1$; by $x+1$.
7. $x^3 + 3x^2 + 50$ by $x+5$; by $x-5$.
8. $x^6 + y^6$ by $x^2 + y^2$. (Substitute $-y^2$ for x^2.)
9. $x^{15} + y^{15}$ by $x^3 + y^3$.
10. $x^{20} + y^{20}$ by $x^4 + y^4$.

REVIEW EXERCISES. XXXVI.

1. Solve the equation $f(x) = f(2)$.

2. If $f(x) \equiv x - 1$, solve the equation $f(x) \cdot f(3) = 0$.

3. If $f(x) \equiv x - 1$, solve the equation $[f(x)]^2 = x^2 - 3$.

4. If $F(x) \equiv x^2 - 5x + 1$, solve the equation

$$F(x) = F(x) + 5x.$$

5. Is $ax^2 + bxy + ay^2$ symmetric as to x and y? as to a and b? as to a and x?

6. Is this a rational function of x:

$$\tfrac{1}{2}x^3 - x^2\sqrt{a} + 3x\sqrt[3]{a} - \sqrt[4]{a}\,?$$

Is it an integral function of x? Is it a rational function of a?

7. If $f(x, y)$ is symmetric as to x and y, is $[f(x, y)]^2$ also symmetric as to x and y? Illustrate by letting $f(x, y) \equiv x + y$.

8. May $f(x, y)$ be not symmetric as to x and y, and $[f(x, y)]^2$ be symmetric? Illustrate by letting $f(x, y) \equiv x - y$.

9. Do you see any advantage in having a function symbol, as $f(x)$, in the way of brevity?

10. Multiply $x^4 + 3x^3y + 4x^2y^2 + 3xy^3 + y^4$ by $x^2 - xy + y^2$, checking the result (1) by symmetry, (2) by homogeneity.

11. Multiply $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 + 2xy - y^2$ and check by symmetry or by homogeneity according to which one applies.

12. Divide $x^4 - y^4$ by $x - y$, checking the quotient by homogeneity.

13. Divide $x^5 + y^5$ by $x + y$, checking the quotient by symmetry.

CHAPTER VI.

FACTORS.

I. TYPES.

106. **The factors of a rational integral algebraic expression** are the rational integral algebraic expressions which multiplied together produce it.

In the expression $3x(x+1)(x^2+x+1)(x^3+2)$

3 is called a **numerical factor,**

x " " **monomial algebraic factor** of the first degree,

$x+1$ " " **linear binomial factor,**

x^2+x+1 " " **quadratic trinomial factor,** the term "quadratic" being applied to integral algebraic expressions of the second degree in some letter or letters.

x^3+2 is called a **cubic binomial factor,** the term "cubic" being applied to integral algebraic expressions of the third degree in some letter or letters.

E.g., in the expression $x^3(x+y+z)(x^2+y^2)$, x^3 is a monomial cubic factor, $x+y+z$ is a linear trinomial factor, and x^2+y^2 is a quadratic binomial factor.

107. Rational integral algebraic expressions which involve only rational numbers are said to exist in **the domain of rationality.**

E.g., $x^2+2x+\frac{1}{4}$, but not $x^2-\sqrt{2}$. The former has no *algebraic* fraction, and the latter involves an irrational number.

108. The product of two integral expressions in the domain of rationality is evidently another integral expression in that domain. We say that an expression is **reducible** in the domain of rationality if it is the product of several integral expressions in that domain, and **irreducible** in the contrary case.

E.g., $4x^2 - 9$ is reducible, because it equals $(2x + 3)(2x - 3)$, but $x^2 - 3$ is not reducible, the word "reducible" alone meaning "reducible in the domain of rationality."

109. A rational integral algebraic expression is said to be **factored** when its irreducible factors are discovered.

E.g., the factors of $x^4 - 1$ are $x^2 + 1$, $x + 1$, and $x - 1$. When $x^4 - 1$ is written in the form $(x^2 + 1)(x + 1)(x - 1)$, it is said to be factored, because $x^2 + 1$, $x + 1$, $x - 1$ are irreducible.

The expression $x - 1$ is irreducible, although it has the factors $\sqrt{x} + 1$ and $\sqrt{x} - 1$, because these are not rational.

The term "factorable" is applied only to rational integral expressions. *E.g.*, while $(\sqrt[4]{x} + 1)(\sqrt[4]{x} - 1) \equiv \sqrt{x} - 1$, expressions like $\sqrt{x} - 1$ are not spoken of as factorable.

110. **Factoring is the inverse of multiplication,** and like all inverse processes it depends on a knowledge of the direct process and of certain type forms already known.

E.g., because we know that

$$(x + y)^2 \equiv x^2 + 2xy + y^2,$$

therefore we know that the factors of

$x^2 + 2xy + y^2$ are $x + y$ and $x + y$,

and those of $m^2 + 2m + 1$ " $m + 1$ " $m + 1$.

111. Although all cases of factoring give rise to identities, the symbol $=$ is usually employed instead of $\equiv$ as being more easily written.

112. The type xy + xz, or the case of a monomial factor.

Since $x(y+z) \equiv xy + xz$, it follows that expressions in the form of $xy + xz$ can be factored.

E.g., $4x^2 + 2x = 2x(2x+1)$. *Check.* $6 = 2 \cdot 3$.

A polynomial may often be treated as a monomial, as in the second step of the following:

$$y^2 - my + ny - mn = y(y-m) + n(y-m)$$
$$= (y+n)(y-m).$$

Check. Let $y = 2$, $m = n = 1$. Then $3 = 3 \cdot 1$.

It must be remembered that an expression is not factored unless it is written as a single product, not as the sum of several products.

E.g., the preceding expression is not factored in the first step; only some of its *terms* are factored.

EXERCISES. XXXVII.

Factor the following expressions:

1. $x^7 + x^6y + x^4$.
2. $a^2 + 2ab + 3ac$.
3. $x^8 - x^4 - x^2 + x$.
4. $3x^9 - 4ax^6 + x^5$.
5. $9y^7 + 3by^5 + 6cy^4$.
6. $aby - ay + y^2 - by$.
7. $m^3 + 3m^2n + 3mn^2$.
8. $w^2 - wy + wx - wxy$.

113. The type $x^2 \pm 2xy + y^2$, or the square of a binomial.

Since $(x \pm y)^2 \equiv x^2 \pm 2xy + y^2$ (§ 69, 1, 2), it follows that expressions in the form of $x^2 \pm 2xy + y^2$ can be factored.

E.g., $x^2 + 4x + 4 = (x+2)^2$. *Check.* $9 = 3^2$.

$x^2 - 6xy + 9y^2 = (x - 3y)^2$. *Check.* $4 = (-2)^2$.

EXERCISES. XXXVIII.

Factor the following expressions:

1. $x^2 + 10x + 25$.
2. $4x^2 + 4xy + y^2$.
3. $25 + x^2 - 10x$.
4. $m^6 + 14m^3 + 49$.

5. $121x^2 - 22x + 1.$

6. $4x^2 + 4y(y - 2x).$

7. $9x^2 - 24xy + 16y^2.$

8. $81x^4 + 72x^2y^2 + 16y^4.$

9. $49z^2 + 81w^2 - 126zw.$

10. $(x + y)^2 + 2(x + y) + 1.$

11. $169a^2 + 169b^2 - 338ab.$

12. $a^2 + 4a + 4 + 2(a + 2) + 1.$

13. $a^2 + 2ab + b^2 + 2(a + b)y + y^2.$

14. $x^2 + 2xy + y^2 + 2xz + 2yz + z^2.$

15. $m^2 + n^2 + p^2 + 2mn - 2mp - 2np.$

114. The type $\mathbf{x^2 - y^2}$, or the difference of two squares.

Since $(x + y)(x - y) \equiv x^2 - y^2$ (§ 69, 3), it follows that expressions in the form of $x^2 - y^2$ can be factored.

E.g., $x^2 - 16 = (x + 4)(x - 4)$. *Check.* $-15 = 5 \cdot -3.$

$$x^4 - 16 = (x^2 + 4)(x^2 - 4)$$
$$= (x^2 + 4)(x + 2)(x - 2). \quad \textit{Check.} \quad -15 = 5 \cdot 3 \cdot -1.$$

$$x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2$$
$$= (x^2 + y^2)^2 - x^2y^2$$
$$= (x^2 + y^2 + xy)(x^2 + y^2 - xy). \quad \textit{Check.} \quad 3 = 3 \cdot 1.$$

EXERCISES. XXXIX.

Factor the following expressions:

1. $z^4 - 16y^2.$

2. $x^{16} - 1.$

3. $a^4 + a^2b^2 + b^4.$

4. $36x^2 - 9y^2.$

5. $16x^4 + 4x^2y^2 + y^4.$

6. $81x^4 + 9x^2 + 1.$

7. $x^2 + 2xy + y^2 - z^2.$

8. $(x + y)^2 - (x - y)^2.$

9. $a^2 + b^2 - x^2 - 1 + 2ab + 2x.$

10. $a^2 + 2ab + b^2 - (x^2 - 2xy + y^2).$

11. $4a^2 + 4a - 3 (= 4a^2 + 4a + 1 - 4).$

12. $x^2 - 6xy + 5y^2.$

13. $x^2 + 4xy + 3y^2.$

115. Forms of the factors. Although a rational integral algebraic expression admits of only one distinct set of irreducible factors, the forms of these factors may often appear to differ.

E.g., since $(x - 2y)(2x - y) = 2x^2 - 5xy + 2y^2$,
and $(2y - x)(y - 2x) = 2x^2 - 5xy + 2y^2$,
it might seem that $2x^2 - 5xy + 2y^2$ has two distinct pairs of factors.

This arises from the fact that the second pair is the same as the first, except that the signs are changed, each factor having been multiplied by -1. But this merely multiplies the whole expression by $-1 \cdot -1$, that is, by $+1$.

Hence, *the signs of any even number of factors may be changed without changing the product.*

E.g., $x^2 - 5x + 6 = (x - 2)(x - 3)$, or $(2 - x)(3 - x)$.

Check. $2 = -1 \cdot -2$, or $1 \cdot 2$.

$$\begin{aligned} x^4 - 1 &= (x^2 + 1)(x + 1)(x - 1) \\ &= (x^2 + 1)(-x - 1)(1 - x) \\ &= (-x^2 - 1)(x + 1)(1 - x). \end{aligned}$$

Check. Let $x = 2$. Then
$16 - 1 = 5 \cdot 3 \cdot 1 = 5 \cdot -3 \cdot -1 = -5 \cdot 3 \cdot -1$.

EXERCISES. XL.

Factor the following, giving the various forms of the results and checking each.

1. $1 - a^3$.
2. $x^3 - 1$.
3. $16 - x^4$.
4. $a^3 - b^6$.
5. $16x^4 - 81y^4$.
6. $2 + z^4 - 2z^2\sqrt{2}$.
7. $121 + x^2 - 22x$.
8. $z^6 + 2 - 2z^3\sqrt{2}$.
9. $x^{10} - 26x^5 + 168$.
10. $a^2 - c^2 + b^2 + 2ab$.
11. $16x^4 + 8x^2 + 1 - 25y^6$.
12. $-x^4 - 15x^3 + 23x^2 - 7x$.
13. $121x^2 + 121y^2 - 9 - 242xy$.
14. $4x^2 + 1 - y^2 - 2yz - z^2 + 4x$.

116. The type $x^3 \pm 3x^2y + 3xy^2 \pm y^3$, or the cube of a binomial.

Since $(x \pm y)^3 \equiv x^3 \pm 3x^2y + 3xy^2 \pm y^3$ (§ 69, 4, 5), it follows that expressions in the form of $x^3 \pm 3x^2y + 3xy^2 \pm y^3$ can be factored.

E.g., $8x^3 + 12x^2 + 6x + 1 = (2x)^3 + 3(2x)^2 + 3 \cdot 2x + 1$
$= (2x+1)^3$. *Check.* $27 = 3^3$.

$$27x^6 - 54x^4y + 36x^2y^2 - 8y^3 =$$
$$(3x^2)^3 - 3(3x^2)^2 \cdot 2y + 3 \cdot 3x^2(2y)^2 - (2y)^3$$
$$= (3x^2 - 2y)^3. \quad \textit{Check.} \quad 1 = 1^3.$$

$$\frac{x^3}{8} - \frac{x^2y^2}{4} + \frac{xy^4}{6} - \frac{y^6}{27} = \left(\frac{x}{2}\right)^3 - 3\left(\frac{x}{2}\right)^2\left(\frac{y^2}{3}\right) + 3\left(\frac{x}{2}\right)\left(\frac{y^2}{3}\right)^2 - \left(\frac{y^2}{3}\right)^3$$
$$= \left(\frac{x}{2} - \frac{y^2}{3}\right)^3.$$

Check. Let $x = 2$, $y = 3$. Then $1 - 9 + 27 - 27 = -8 = (1-3)^3$.

EXERCISES. XLI.

Factor the following expressions:

1. $1 - 3x + 3x^2 - x^3$.
2. $a^3 - 3a^2 + 3a - 1$.
3. $x^{12} - 3x^8 + 3x^4 - 1$.
4. $27x^3 - 27x^2 + 9x - 1$.
5. $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$.
6. $27a^9 - 27a^6 + 9a^3 - 1$.
7. $8x^3 - 12x^2y + 6xy^2 - y^3$.
8. $54x^2 - 27x + 8x^4 - 36x^3$.
9. $1.331x^3 - 7.26x^2 + 6.6x - 8$.
10. $64x^6y^6 - 48x^4y^4 + 12x^2y^2 - 1$.
11. $x^9y^6z^3 + 6x^6y^4z^2 + 12x^3y^2z + 8$.
12. $0.125x^6 - 0.75x^4 + 0.15x^2 - 1$.
13. $(a+b)^3 + 3(a+b)^2 + 3(a+b) + 1$.

117. The type $\mathbf{x^n \pm y^n}$. It has been shown (§ 105, Remainder Theorem, cor. 3) that

$x^n + y^n$ contains the factor $x + y$ when n is odd,
" " " " $x - y$ never,
$x^n - y^n$ " " " $x + y$ when n is even,
" " " " $x - y$ always.

Hence, it follows that expressions in the form of $x^n \pm y^n$ can often be factored.

E.g., $x^3 + y^3$ contains the factor $x + y$. The other factor can be determined by division. It may also be determined by noticing that $x^3 + y^3$ is symmetric and homogeneous, and that its factors must therefore be $x + y$ and $x^2 + kxy + y^2$, where k is to be determined. Letting $x = y = 1$,

$$x^3 + y^3 = (x + y)(x^2 + kxy + y^2)$$

becomes $$2 = 2(2 + k),$$

and therefore, $$k = -1,$$

whence $$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

This type occurs so often that the forms of the quotients should be memorized:

1. $\dfrac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \cdots$, the signs alternating.

2. $\dfrac{x^n - y^n}{x + y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \cdots$, the signs alternating.

3. $\dfrac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + x^{n-4}y^3 + \cdots$, the signs being all $+$.

We are thus able to write out the quotient of $(x^{15} + y^{15}) \div (x + y)$ at sight, and so for other similar cases.

The integral parts of the quotients in 1 and 2 are the same, but the remainders are different. *E.g.*, if n is odd there is no remainder in 1, but in 2 there is a remainder $-2y^n$.

When the exponent n exceeds 3 it is better to separate into two factors as nearly of the same degree as possible, and then to factor each separately.

E.g.,

$$\begin{aligned} x^8 - y^8 &= (x^4 + y^4)(x^4 - y^4) \\ &= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y), \end{aligned}$$

or the same with certain signs changed (§ 115).

This is better than to take out the linear binomial $x + y$ or $x - y$ first, which would give

$$x^8 - y^8 = (x + y)(x^7 - x^6y + x^5y^2 - x^4y^3 + x^3y^4 - x^2y^5 + xy^6 - y^7),$$

or

$$(x - y)(x^7 + x^6y + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^6 + xy^6 + y^7),$$

in which cases it would be difficult to discover the factors of the two expressions of the seventh degree.

So

$$x^{2n} - y^{2n} = (x^n + y^n)(x^n - y^n).$$

118. Binomials of the form $x^n \pm y^n$ which have not the factor $x \pm y$ may contain $x^m \pm y^m$.

E.g., $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$.

EXERCISES. XLII.

Factor the following expressions:

1. $x^7 + 1$.
2. $x^4 - 16$.
3. $x^6 - y^6$.
4. $1 - x^{16}$.
5. $x^3 + 8y^3$.
6. $x^5 + y^5$.
7. $32x^5 + 1$.
8. $x^{2m+1} + 1$.
9. $x^{12} + 4096$.
10. $a^2b^4 - b^2a^4$.
11. $729x^6 + y^6$.
12. $216a^6 - b^3$.
13. $(x + y)^3 + 1$.
14. $125a^3 + 27$.
15. $64x^6 - 729y^6$.
16. $27a^3 + 64b^3$.
17. $125x^7 - 27xy^3$.
18. $a^3 + a + b^3 + b$.
19. $(a - b)^3 - (a + b)^3$.
20. $m^2 - n^2 + 2n - 1$.

119. The type x^2+ax+b. Let $x^2+ax+b=(x+m)(x+n)$, in which m and n are to be determined. Then

$$x^2+ax+b \equiv x^2+(m+n)x+mn.$$

It therefore appears that if two numbers, m and n, can be found such that their sum, $m+n$, is a, and their product, mn, is b, the expression can be factored.

E.g., consider $x^2+10x+21$.

Here $10=3+7$,

and $21=3\cdot 7$,

$\therefore$ $x^2+10x+21=(x+3)(x+7)$. *Check.* $32=4\cdot 8$.

Consider also $x^2-3x-40$.

Here $-3=5-8$,

and $-40=5\cdot -8$,

$\therefore$ $x^2-3x-40=(x+5)(x-8)$. *Check.* $-42=6. -7$.

EXERCISES. XLIII.

Factor the following expressions:

1. x^2+3x+2.
2. x^2-x-2.
3. x^4+x^2-12.
4. x^2-5x+6.
5. $x^2-4x-165$.
6. $p^2-p-600$.
7. $a^2-3a-130$.
8. $x^2-4x-21$.
9. $a^2-11a-60$.
10. x^6-4x^3-45.
11. $4x^2+8x-45$.
12. $a^2+17a+66$.
13. $x^2+41x+420$.
14. x^4+16x^2+55.
15. $a^2-24a+135$.
16. $x^4y^2+4x^2y+3$.
17. x^4-15x^2-100.
18. $a^2-16a-225$.
19. $a^4x^4+5a^2x^2+6$.
20. $x^2+7xy+10y^2$.
21. $4a^2+2ab-2b^2$.
22. $a^6x^2-5a^3x-14$.
23. $m^2-38m+165$.
24. $x^2+11xy-26y^2$.
25. $m^2x^2-7mx-18$.
26. $m^2x^4+12mx^2+35$.

120. The type $ax^2 + bx + c$. Let

$$ax^2 + bx + c \equiv (mx + n)(px + q),$$

in which m, n, p, and q are to be determined. Then

$$ax^2 + bx + c \equiv mpx^2 + (mq + pn)x + qn.$$

It therefore appears that *the coefficient of* x, mq + pn, *is the sum of two numbers whose product,* mqpn, *is the product of the coefficient of* x², mp, *and the last term,* qn. Hence, if these numbers can be detected, the expression can be factored.

E.g., consider $6x^2 + 17x + 12$.

Here $17 = 9 + 8$,

and $6 \cdot 12 = 72 = 9 \cdot 8$.

$$\therefore \quad 6x^2 + 17x + 12 = 6x^2 + 9x + 8x + 12$$
$$= 3x(2x + 3) + 4(2x + 3)$$
$$= (3x + 4)(2x + 3). \quad \textit{Check.} \quad 35 = 7 \cdot 5.$$

Consider also $6x^2 + 7x - 3$.

Here $7 = 9 - 2$,

and $6 \cdot -3 = -18 = 9 \cdot -2$.

$$\therefore \quad 6x^2 + 7x - 3 = 6x^2 + 9x - 2x - 3$$
$$= 3x(2x + 3) - (2x + 3)$$
$$= (3x - 1)(2x + 3). \quad \textit{Check.} \quad 10 = 2 \cdot 5.$$

EXERCISES. XLIV.

Factor the following expressions:

1. $6x^2 + x - 12$.
2. $12p^2 - p - 1$.
3. $4x^2 - 4x - 3$.
4. $3a^2 + 8a + 4$.
5. $600a^2 - a - 1$.
6. $9x^2 - 17x - 2$.
7. $84x^2 - 5x - 1$.
8. $8a^2 + 22a + 12$.
9. $12p^2 - 7p + 1$.
10. $6p^2 + 25(p + 1)$.
11. $a^{12} - 7a^6b^6 - 8b^{12}$.
12. $16x^2 - 62x + 27$.
13. $16a^2 + 43ab + 27b^2$.
14. $40a^2 + 61ab - 84b^2$.
15. $16x^2y^2z^2 + 39xyz - 27$.
16. $30x^2 - 41xz - 15z^2$.

121. Application of the Remainder Theorem. The presence of a binomial factor is usually detected very readily by the use of this theorem (§ 104).

E.g., $x^3 - 4x + 3$ evidently contains the factor $(x - 1)$, and the other factor, $x^2 + x - 3$, can be found by division.

Similarly, consider $x^3 - 2x - 21$.

Trying $x - 1$ we have

$$f(1) = 1 - 2 - 21 \neq 0; \quad \therefore x - 1 \text{ is not a factor.}$$

Trying $x + 1$ we have

$$f(-1) = -1 + 2 - 21 \neq 0; \therefore x + 1 \text{ is not a factor.}$$

Trying $x - 3$ we have $f(3) = 0$; $\therefore x - 3$ is a factor.

If the student understands Synthetic Division (Appendix II), the test of divisibility is easily made by that process, thus:

$$\begin{array}{r|rrrr} & 1 & 0 & -2 & -21 \\ 3 & & 3 & 9 & 21 \\ \hline & 1 & 3 & 7; & 0 \end{array} \text{ remainder.}$$

Hence the factors are $x - 3$ and $x^2 + 3x + 7$.

Check. $-22 = -2 \cdot 11$.

Since the factors of -21 are ± 1 and ∓ 21, ± 3 and ∓ 7, the number of trials necessary is very limited.

EXERCISES. XLV.

Factor the following expressions:

1. $x^3 - 19x - 30$.
2. $x^3 - 3x - 2$.
3. $m^3 - 2mn^2 + n^3$.
4. $a^3 - a^2 - a - 2$.
5. $a^3 - a - 2 + 2a^2$.
6. $x^3 + 9x^2 + 20x + 12$.
7. $a^3 - 6a^2 + 11a - 6$.
8. $a^3 + 8a^2 - 112a + 256$.
9. $a^3 - a^2 - 15a + 12$.

For those who have studied symmetry as set forth in Appendix III, the cases of factoring given in Appendix IV are recommended at this point.

MISCELLANEOUS EXERCISES. XLVI.

122. *General directions.*

1. First remove all monomial factors.

2. Then see if the expression can be brought under some of the simple types given on pp. 81–87. This can probably always be done in cases of binomials and quadratic trinomials, and often in other cases.

3. If unsuccessful in this, the Remainder Theorem may be tried, especially with polynomials of the form

$$x^n + ax^{n-1}y + bx^{n-2}y^2 + \cdots.$$

4. Always check the results, and be sure that the factors are irreducible.

1. $x^4 + 4$.
2. $x^4 + 4y^4$.
3. $x^6 + y^6$.
4. $1 + x^2 + x^4$.
5. $x^8 - x^4y^4$.
6. $x^8 + y^8 + x^4y^4$.
7. $x^4 + x^2 + \frac{1}{4}$.
8. $x^4 - 2x^2y^2 + y^4$.
9. $a^8b^6c^4 - a^4b^6c^8$.
10. $x^2(x^2 + y^2) + y^4$.
11. $a^6 - a^3 - 110$.
12. $x^4 + x^2y + 2x^3y$.
13. $x^4 - 11x^2 + 1$.
14. $2x^2 + 11x + 12$.
15. $6x^2 - 23x + 20$.
16. $y^2 - z^2 + 2z - 1$.
17. $x^3y^3 + 2x^2y^2 + xy$.
18. $(x + y)^7 - x^7 - y^7$.
19. $a^4 - 15a^2b^2 + 9b^4$.
20. $ax^2 + (a + b)x + b$.
21. $ab + y^2 - ay - by$.
22. $12x^2y^2 - 17xy + 6$.
23. $x^4 - 8x^2y^2 + 16y^4$.
24. $(x + 1)^2 - 5x - 29$.
25. $(a + b)^3 + (a - b)^3$.
26. $16x^4 - 28x^2y^2 + y^4$.
27. $y^3 + 3y^2 + 6y + 18$.
28. $7x^3 + 96x^2 - 103x$.
29. $21a^2 + 26ab - 15b^2$.
30. $x^4 - (a^2 + b^2)x^2 + a^2b^2$.

31. $m^6n^6 + 1.$

32. $a^8 - 2a^4b^4 + b^8.$

33. $9x^2 - 16y^2.$

34. $x^{2n} - 11x^n + 28.$

35. $a^5 + a - 2a^3.$

36. $9a^{2m} - 5 - 4a^m.$

37. $10a^2 - 360b^4.$

38. $a^2(a^2 - 24) + 63.$

39. $x^{4m} + x^{2m} + 1.$

40. $a^2 - ac - bc - b^2.$

41. $x^4 + a^8 + x^2a^4.$

42. $x^2 + 12xy + 36y^2.$

43. $x^2 + 16x + 63.$

44. $m^2 - t^2 - w^2 + 2tw.$

45. $x^2 - 14x + 49.$

46. $(a^2 + 1)^3 - (b^2 + 1)^3.$

47. $a^3(a^3 - 1) - 56.$

48. $(x + y)^4 + 4(w + z)^4.$

49. $6 + 15a^2 - 19a.$

50. $5ab - bc + cd - 5ad.$

51. $8 - (x + y + z)^3.$

52. $x^3 + y^3 - 4x^2y - 4xy^2.$

53. $a^3b - ab^3 + a^2b + ab^2.$

54. $a^2(a + 1) - b^2(b + 1).$

55. $3xy(x + y) + x^3 + y^3.$

56. $4x^2y^2 - (x^2 + y^2 - z^2)^2.$

57. $2x + (x^2 - 4)y - 2xy^2.$

58. $121a^4 - 795a^2b^2 + 9b^4.$

59. $(a - 4)^2 - 4(a - 4) + 4.$

60. $(x - 5)^2 - 8(x - 5) + 12.$

61. $x^3(x - 2y) - y^3(y - 2x).$

62. $1 - (a - b) - 110(a - b)^2.$

63. $10 + 16(a + b) + 6(a + b)^2.$

64. $(m + n)^2 + 10(m + n) + 24.$

65. $2x^2 - x^2y + (y - 2)(xy - x)^2.$

66. $x^2 + y^2 - (w^2 + z^2) + 2(xy + wz).$

67. $(a + b)^5 - (a + b)^3 - (a + b)^2 + 1.$

II. APPLICATION OF FACTORING TO THE SOLUTION OF EQUATIONS.

123. To solve an equation is to find the value of the unknown quantity which shall make the first member equal to the second. Such a value is said to satisfy the equation (§ 17).

E.g., if $x^2 = 4$,

then $x^2 - 4 = 0$, or $(x+2)(x-2) = 0$;

$\therefore x = +2$ or -2. That is, either $+2$ or -2 will satisfy the equation; for if $x = +2$, then $(2+2)(2-2) = 0$; and if $x = -2$, then $(-2+2)(-2-2) = 0$.

If $x^2 + x = 6$,

then $x^2 + x - 6 = 0$,

whence $(x+3)(x-2) = 0$. This equation is evidently satisfied if either factor of the first member is 0. (Why?)

If $x + 3 = 0$, then $x = -3$, because $-3 + 3 = 0$;

and if $x - 2 = 0$, " $x = 2$, " $2 - 2 = 0$.

If $x^4 - 6x^3 + 11x^2 - 6x = 0$,

then $x(x-1)(x-2)(x-3) = 0$. This equation is evidently satisfied if any factor of the first member equals 0. (Why?)

Hence, x may equal 0, as one value;

or if $x - 1 = 0$, then $x = 1$, because $1 - 1 = 0$;

and if $x - 2 = 0$, " $x = 2$, " $2 - 2 = 0$;

and if $x - 3 = 0$, " $x = 3$, " $3 - 3 = 0$.

EXERCISES. XLVII.

Solve the following equations:

1. $x^2 - 1 = 0$.
2. $x^2 + 287 = 48x$.
3. $2x^2 + 2 = 5x$.
4. $6x^2 - 13x + 6 = 0$.
5. $x^2 = 2x + 143$.
6. $x^4 - 10x^2 + 21 = 0$.
7. $x^3 + 4x^2 + x = 6$.
8. $x^6 - 14x^4 + 49x^2 = 36$.
9. $x^4 - 13x^2 + 36 = 0$.
10. $2x^3 - 67x^2 + 371x = 0$.
11. $2x^3 - 7x^2 + 5x = 0$.
12. $x^4 - 15x^2 + 10x + 24 = 0$.

III. EVOLUTION.

124. If an algebraic expression is the product of two equal factors, one of those factors is called the **square root** of the expression. Similarly, one of three equal factors is called the **cube root**, one of four equal factors the **4th root**, ⋯ one of n equal factors the **nth root**.

The broader meaning of the word *root* is discussed later (§ 130).

The process of finding a root of an algebraic quantity is called **evolution**.

Evolution is, therefore, a particular case of factoring. It is evidently the inverse of Involution, as Root is the inverse of Power.

125. Symbolism. Square root is indicated either by the fractional exponent $\frac{1}{2}$ or by the old radical sign $\sqrt{}$, a form of the letter r, the initial of the Latin *radix* (root).

Similarly, $a^{\frac{1}{3}}$ or $\sqrt[3]{a}$ means the cube root of a,
$a^{\frac{1}{4}}$ " $\sqrt[4]{a}$ " " 4th " "
and, in general, $a^{\frac{1}{n}}$ " $\sqrt[n]{a}$ " " nth " "

For present purposes it is immaterial which set of symbols is used. The student should, however, accustom himself to the fractional exponent, which, while a little more difficult to write, has many advantages over the older radical sign as will be seen later.

126. Law of signs. Since any power of a positive quantity is positive, but even powers of a negative quantity are positive while odd powers are negative (§ 77), therefore,

1. *An even root of a positive quantity is either positive or negative.*

E.g., $4^{\frac{1}{2}} = \pm 2$, $81^{\frac{1}{4}} = \pm 3$.

2. *An odd root of any quantity has the same sign as the quantity itself.*

E.g., $8^{\frac{1}{3}} = 2$, $(-8)^{\frac{1}{3}} = -2$.

3. *An even root of a negative quantity is neither a positive nor a negative quantity.*

E.g., $\sqrt{-1}$ is neither $+1$ nor -1.

An even root of a negative quantity is said to be *imaginary*, and imaginary quantities are discussed later (Chap. XIII).

127. The root of a monomial power is easily found by inspection.

E.g., $\because \quad 4a^2b^4 \equiv 2 \cdot 2 \cdot a \cdot a \cdot b \cdot b \cdot b \cdot b,$

$$\therefore \quad \sqrt{4a^2b^4} \equiv \sqrt{(2 \cdot a \cdot b \cdot b) \cdot (2 \cdot a \cdot b \cdot b)} \equiv \pm 2 \cdot a \cdot b \cdot b$$
$$\equiv \pm 2ab^2.$$

Similarly, $\sqrt[3]{64x^3y^6} \equiv 4xy^2,$

$$\sqrt[5]{32x^{16}y^{30}} \equiv 2x^3y^6,$$
$$\sqrt[6]{64x^{12}} \equiv \pm 2x^2.$$

EXERCISES. XLVIII.

Simplify the following expressions:

1. $\sqrt{4\sqrt{a^8}}$.

2. $\sqrt[3]{-8a^6b^{12}}$.

3. $\sqrt[3]{3^3(a-2b)^6}$.

4. $\sqrt{16a^{4m}\sqrt[3]{a^{12n}b^{6p}}}$.

5. $\sqrt{2a^2\sqrt{2b^4\sqrt{4c^8}}}$.

6. $\sqrt{16x^{10}y^{20}z^{30}}$, $\sqrt[5]{32x^{10}y^{20}z^{30}}$.

7. $\sqrt[2x]{x^{2x}}$, $\sqrt[2x+1]{x^{2x+1}}$, $\sqrt[2x+1]{-x^{2x+1}}$.

8. $\sqrt{64x^{18}y^{12}}$, $\sqrt[3]{64x^{18}y^{12}}$, $\sqrt[6]{64x^{18}y^{12}}$.

9. $\sqrt{729a^{18}b^6}$, $\sqrt[3]{729a^{18}b^6}$, $\sqrt[6]{729a^{18}b^6}$.

10. $\sqrt[m]{a^{2m}b^{3m}}$, m being even; m being odd.

128. Roots extracted by inspection. The roots of the monomials given on p. 93 were extracted by inspection. Similarly, the square root of a square polynomial, the cube root of a cube polynomial, etc., can often be found by inspection.

Illustrative problems. 1. What is the square root of $x^4 + 4x^2y + 4y^2$.

1. $\because [\pm(f+n)]^2 \equiv f^2 + 2fn + n^2$, § 82
2. and $\because$ this polynomial can be arranged in a similar form, *viz.*,

$$(x^2)^2 + 2x^2(2y) + (2y)^2,$$

3. $\therefore$ it is evidently the square of $\pm(x^2 + 2y)$.

Check. $(\pm 3)^2 = 1 + 4 + 4 = 9$.

2. Find the cube root of $x^6 + 6x^4y + 12x^2y^2 + 8y^3$.

1. $\because (f+n)^3 \equiv f^3 + 3f^2n + 3fn^2 + n^3$, § 82
2. and $\because$ this polynomial can be arranged in a similar form, *viz.*,

$$(x^2)^3 + 3(x^2)^2 \cdot 2y + 3x^2(2y)^2 + (2y)^3,$$

3. $\therefore$ it is evidently the cube of $x^2 + 2y$.

Check. $3^3 = 1 + 6 + 12 + 8 = 27$.

3. Find the square root of

$$a^2 + 4b^2 + 9c^2 + 4ab - 6ac - 12bc.$$

1. $\because [\pm(x+y+z)]^2 \equiv x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$,
2. and $\because$ this polynomial can be arranged in a similar form, *viz.*,

$$a^2 + (2b)^2 + (-3c)^2 + 2a(2b) + 2a(-3c) + 2(2b)(-3c),$$

3. $\therefore$ it is evidently the square of $\pm(a + 2b - 3c)$.

Check. $0^2 = 1 + 4 + 9 + 4 - 6 - 12 = 0$.

4. Find the fifth root of

$$a^{10} - 5a^8b + 10a^6b^2 - 10a^4b^3 + 5a^2b^4 - b^5.$$

1. $\because$ there are 6 terms, and the polynomial is arranged according to the powers of a and b, it is the 5th power of a binomial (§ 82) whose first term is a^2 and whose second term is $-b$, if it is a 5th power.

2. But $(a^2 - b)^5$ equals the given polynomial. (Expand it.)

EXERCISES. XLIX.

Extract the square roots of exs. 1–6.

1. $\frac{4}{9}x^8 - \frac{4}{5}x^4 + \frac{9}{25}$.

2. $4a - 12\sqrt{ab} + 9b$.

3. $4m^2 - 12mx^2 + 9x^4$.

4. $9a^6b^8 - 30a^5b^4c^5 + 25a^4c^{10}$.

5. $4m^2 + 4mn + 12mp + n^2 + 6np + 9p^2$.

6. $4x^4 - 12x^3y + 16x^2y^3 + 9x^2y^2 - 24xy^4 + 16y^6$.

Extract the cube roots of exs. 7–12.

7. $\frac{64}{125}x^9 - \frac{16}{25}x^6 + \frac{4}{15}x^3 - \frac{1}{27}$.

8. $m^6 + 6m^4n + 12m^2n^2 + 8n^3$.

9. $8x^3 - 84x^2y + 294xy^2 - 343y^3$.

10. $8x^6 + 12x^5 + 18x^4 + 13x^3 + 9x^2 + 3x + 1$.

11. $m^6 - 3m^5 - 3m^4 + 11m^3 + 6m^2 - 12m - 8$.

12. $x^6 - 12x^5 + 54x^4 - 112x^3 + 108x^2 - 48x + 8$.

Extract the fourth roots of exs. 13, 14.

13. $\frac{81}{256}x^4 + \frac{9}{8}x^3y + \frac{3}{2}x^2y^2 + \frac{8}{9}xy^3 + \frac{16}{81}y^4$.

14. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$.

Extract the fifth roots of exs. 15, 16.

15. $80x^3 - 80x^4 + 32x^5 - 40x^2 + 10x - 1$.

16. $x^{10} - \frac{5}{2}x^8y + \frac{5}{2}x^6y^2 - \frac{5}{4}x^4y^3 + \frac{5}{16}x^2y^4 - \frac{1}{32}y^5$.

Extract the sixth roots of exs. 17, 18.

17. $a^6 - 12a^5 + 60a^4 - 160a^3 + 240a^2 - 192a + 64$.

18. $a^6 - 2a^5b + \frac{5}{3}a^4b^2 - \frac{20}{27}a^3b^3 + \frac{5}{27}a^2b^4 - \frac{2}{81}ab^5 + \frac{1}{729}b^6$.

129. Square root by the formula $f^2 + 2fn + n^2$. The subject is best understood by following the solution of a problem.

1. Required the square root of $4x^4 - 12x^2y + 9y^2$.

Let $f \equiv$ the found part of the root at any stage of the operation, and

$n \equiv$ the next term to be found.

Then $(f+n)^2 \equiv f^2 + 2fn + n^2$. § 82

The work may be arranged as follows:

Root	$= \pm(2x^2 - 3y)$		
Power $=$	$4x^4 - 12x^2y + 9y^2$	contains	$f^2 + 2fn + n^2$
$f^2 =$	$4x^4$		
$2f = 4x^2$	$-12x^2y + 9y^2$	"	$2fn + n^2$
$2f + n = 4x^2 - 3y$	$-12x^2y + 9y^2$	$=$	"

Explanation. 1. If a root is arranged according to the powers of some letter, the square obtained by ordinary multiplication will be so arranged (§ 65).

2. $\therefore$ the square is arranged according to the powers of x, so that the square root of the first term shall be the first term of the root.

3. $\because 4x^4 =$ the square of the first term, the first term is $2x^2$.

4. Subtracting f^2, the remainder, $-12x^2y + 9y^2$, contains $2fn + n^2$.

5. Dividing $2fn$ (*i.e.*, $-12x^2y$) by $2f$ (*i.e.*, $4x^2$), n is found to be $-3y$.

6. $\because f^2 = 4x^4$, and $2fn + n^2 = -12x^2y + 9y^4$, .. the sum of these is the square of $\pm(2x^2 - 3y)$.

Check. Let $x = y = 1$. Then $(-1)^2 = 4 - 12 + 9 = 1$.

We might, after a little practice, detach the coefficients. In the above example it would be necessary to remember that the powers of x decrease by two, while those of y increase by one.

E.g.,

	$4 - 12 + 9 \lfloor 2 - 3$	
	4	
4	$-12 + 9$	$\pm(2x^2 - 3y)$
$4 - 3$	$-12 + 9$	

2. Required the square root of

$$a^2 - 2\,ab^2 + b^4 + 4\,ac - 4\,b^2c + 4\,c^2.$$

$$\text{Root } = \pm(a - b^2 \quad + 2\,c)$$

$$\text{Power} = a^2 - 2\,ab^2 + b^4 + 4\,ac - 4\,b^2c + 4\,c^2 \text{ contains } f^2 + 2\,fn + n^2$$

$$f^2 = a^2$$

$2f \quad = 2\,a$	$-2\,ab^2 + b^4 + \cdots$	"	$2fn + n^2$
$2f + n = 2\,a - b^2$	$-2\,ab^2 + b^4$	$=$	"
$2f \quad = 2\,a - 2\,b^2$	$4\,ac - 4\,b^2c + 4\,c^2$ contains		$2fn + n^2$
$2f + n = 2\,a - 2\,b^2 + 2\,c$	$4\,ac - 4\,b^2c + 4\,c^2$	$=$	"

EXPLANATION. 1. See p. 96 for explanation down to $2f = 2\,a - 2\,b^2$.

2. $\because$

$$f^2 = a^2, \text{ and}$$

$$2fn + n^2 = -2\,ab^2 + b^4,$$

$$\therefore (f + n)^2 = a^2 - 2\,ab^2 + b^4, \text{ the square of } a - b^2.$$

3. $\because a - b^2$ has now been found, it may be designated by f.

4. $\therefore 4\,ac - 4\,b^2c + 4\,c^2$ contains $2\,fn + n^2$, the square of $a - b^2$ having been subtracted.

Check. Let $a = b = c = 1$. Then $2^2 = 1 - 2 + 1 + 4 - 4 + 4 = 4$.

Or let $a = 1$, $b = 2$, $c = 3$.

Then $$(1 - 4 + 6)^2 = 3^2 = 9$$

and $$1 - 8 + 16 + 12 - 48 + 36 = 9;$$

and so for any other arbitrary values.

130. Extension of the definition of root. If an algebraic expression is not the product of r equal factors, it is still said to have an rth root. In such a case **the rth root to n terms** is defined to be that polynomial of n terms found by proceeding as in the ordinary method of extracting the rth root of a perfect rth power.

E.g., the square root of $1 - x$ to 5 terms is

$$\pm(1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 - \cdots).$$

In the same way we may speak of the square root of numbers which are not perfect squares. Thus the square root of 2 to two decimal places is 1.41; to three decimal places, 1.414, and so on. We may also speak of the cube root of numbers which are not perfect cubes, and so on.

EXERCISES. L.

Extract the square roots of exs. 1–16.

1. $x^4 + 2x^3 - x + \frac{1}{4}$.

2. $1 + 8a + 22a^2 + 24a^3 + 9a^4$.

3. $9(a^2 - 1)^2 - 12(a^2 - 1)a + 4a^2$.

4. $x^6 - 6x^4 + 4x^3 + 9x^2 - 12x + 4$.

5. $x^6 - 2ax^5 + a^2x^4 - 2bx^3 + 2abx^2 + b^2$.

6. $25a^2 + 9b^2 + c^2 + 6bc - 10ca - 30ab$.

7. $10x^4 - 10x^3 - 12x^5 + 5x^2 + 9x^6 - 2x + 1$.

8. $9x^8 - 12ax^7 + 4a^2x^6 + 6a^3x^5 - 4a^4x^4 + a^6x^2$.

9. $16 - 8m - 23m^2 + 22m^3 + 5m^4 - 12m^5 + 4m^6$.

10. $9a^6b^4 - 12a^4b^5 + 4a^2b^6 + 24a^5b^3c^3 - 16a^3b^4c^3 + 16a^4b^2c^6$.

11. $9a^4 - 12a^2b^3 + 4b^6 + 24a^2c^4 - 16b^3c^4 + 16c^8 - 30a^2d + 20b^3d - 40c^4d + 25d^2$.

12. $4x^6y^2 - 12x^4y^3 + 9x^2y^4 + 4x^4y^2z - 6x^2y^3z + x^2y^2z^2 - 16x^3y^2z^3 + 24xy^3z^3 - 8xy^2z^4 + 16y^2z^6 + 4x^5yz - 6x^3y^2z + 2x^3yz^2 - 8x^2yz^4 + x^4z^2$.

13. $1 + x$ to 4 terms.

14. $1 - 2x$ to 4 terms.

15. $4 + 2x$ to 4 terms.

16. $9a^2 + 12ax$ to 2 terms.

17. Find x so that $a^4 + 6a^3 + 7a^2 - 6a + x$ shall be a perfect square.

18. Find m so that $4x^4 + 4x^3 + mx^2 + 4x + 4$ shall be a perfect square.

19. Find m so that $9a^4 + 12a^3 + 10a^2 + ma + 1$ shall be a perfect square.

20. Show that the square root of $2[(m+n)^4 + (m^4 + n^4)]$ is $2(m^2 + n^2 + mn)$.

131. The square roots of numbers are similarly found.

Required the square root of 547.56.

	Root $= 2\ \ 3.\ 4$			
	Power $=$ 5'47.56	contains $f^2 + 2fn + n^2$		
	$f_1^2 =$ 4 00.00			
$2f_1 = 40$	1 47.56	"	$2f_1n_1 + n_1^2$	$f_1 = 20$
$2f_1 + n_1 = 43$	1 29.00	=	"	$n_1 = 3$
$2f_2 = 46$	18.56	contains	$2f_2n_2 + n_2^2$	$f_2 = 23$
$2f_2 + n_2 = 46.4$	18.56	=	"	$n_2 = 0.4$

EXPLANATION. 1. ∵ the highest order of the power is 100's, the highest order of the root is 10's, and it is unnecessary to look below 100's for the square of 10's.

2. Similarly, it is unnecessary to look below 1's for the square of 1's, below 100ths for the square of 10ths, etc.

3. The greatest square in the 100's is 400, which is the square of 20, which may be called f_1 (read "f-one"), the first found part.

4. Subtracting, 147.56 contains $2fn + n^2$, because f^2 has been subtracted from $f^2 + 2fn + n^2$, where f stands always for the *found* part and n for the *next order* of the root.

5. $2fn + n^2$ is approximately the product of $2f$ and n, and hence, if divided by $2f$, the quotient is approximately n. $\therefore n = 3$.

6. $\therefore 2f + n = 43$, and this, multiplied by n, equals $2fn + n^2$.

7. ∵ f^2 has already been subtracted, after subtracting $2fn + n^2$ there has been subtracted $f^2 + 2fn + n^2$, or $(f + n)^2$, or 23^2.

8. Calling 23 the second found part, f_2, and noticing that $f_2 = f_1 + n_1$, it appears that 23^2, or f_2^2, has been subtracted.

9. ∴ the remainder 18.56 contains $2f_2n_2 + n_2^2$.

10. Dividing by $2f_2$ for the reason already given, $n_2 = 0.4$.

11. ∴ $2f_2 + n_2 = 46.4$, and $18.56 = 2f_2n_2 + n_2^2$, as before.

12. Similarly, the explanation repeats itself after each subtraction.

EXERCISES. LI.

Extract the square roots of exs. 1–6.

1. 958441.
2. 7779.24.
3. 32.6041.
4. 24.1081.
5. 0.900601.
6. 0.055696.

132. Cube root by the formula $f^3 + 3f^2n + 3fn^2 + n^3$.

Required the cube root of $8a^3 - 12a^2b + 6ab^2 - b^3$.

Let $f \equiv$ the found part of the root at any stage of the operation, and

$n \equiv$ the next term to be found.

Then $(f+n)^3 \equiv f^3 + 3f^2n + 3fn^2 + n^3$. § 82

The work may be arranged as follows:

			Root $= 2a - b$	
			Power $= 8a^3 - 12a^2b + 6ab^2 - b^3$ contains	
			$f^3 = 8a^3$	$f^3 + 3f^2n + 3fn^2 + n^3$
$3f^2$	$3fn$ $+n^2$	$3f^2 + 3fn$ $+n^2$	$-12a^2b + 6ab^2 - b^3$ contains	$3f^2n + 3fn^2 + n^3$
$12a^2$	$-6ab$ $+b^2$	$12a^2 - 6ab$ $+b^2$	$-12a^2b + 6ab^2 - b^3$ =	"

Explanation. 1. The cube is arranged according to the powers of a and b for a reason similar to that given in square root.

2. $\because 8a^3 =$ the cube of the first term, the first term is $2a$.

3. Subtracting f^3, the remainder, $-12a^2b + 6ab^2 - b^3$, contains $3f^2n + 3fn^2 + n^3$.

4. Dividing by $3f^2$ (*i.e.*, $12a^2$), n is found to be $-b$.

5. $\because f = 2a$, and $n = -b$, $\therefore 3f^2 + 3fn + n^2 = 12a^2 - 6ab + b^2$.

6. Multiplying by n, $-12a^2b + 6ab^2 - b^3$ must equal $3f^2n + 3fn^2 + n^3$. This together with f^3 completes the cube of $f + n$.

Check. Let $a = b = 1$. Then $1^3 = 8 - 12 + 6 - 1 = 1$.

EXERCISES. LII.

Extract the cube roots of exs. 1–6.

1. $8a^3 - 36a^2b + 54ab^2 - 27b^3$.
2. $a^3x^3 - 12a^2bx^3 + 48ab^2x^3 - 64b^3x^3$.
3. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.
4. $a^6 - 2a^5b + \frac{5}{3}a^4b^2 - \frac{20}{27}a^3b^3 + \frac{5}{27}a^2b^4 - \frac{2}{81}ab^5 + \frac{1}{729}b^6$.
5. $a^6 - 12a^5b + 54a^4b^2 - 112a^3b^3 + 108a^2b^4 - 48ab^5 + 8b^6$.
6. $x^3 + 3x^2y - 6x^2 + 3xy^2 - 12xy + 12x + y^3 - 6y^2 + 12y - 8$.

133. The cube roots of numbers are found by the same general method.

Required the cube root of 139,798,359.

Root = 5 1 9

Power = 139,798,359 cont's $f^3+3f^2n+3fn^2+n^3$

f^3 = 125,000,000

$3f^2$	$3fn + n^2$	$3f^2 + 3fn + n^2$	14,798,359 contains $3f^2n + 3fn^2 + n^3$ $f_1 = 500$
750,000	15,100	765,100	7,651,000 = $3f^2n + 3fn^2 + n^3$ $n_1 = 10$
780,300	13,851	794,151	7,147,359 contains $3f^2n + 3fn^2 + n^3$ $f_2 = 510$ 7,147,359 = $3f^2n + 3fn^2 + n^3$ $n_2 = 9$

Explanation. 1. ∵ the highest order of the power is hundred-millions, the highest order of the root is 100's (why ?), and it is unnecessary to look below millions for the cube of 100's. (Why ?)

2. Similarly, it is unnecessary to look below 1000's for the cube of 10's, below 1's for the cube of 1's, etc.

3. The greatest cube in the hundred-millions is 125,000,000, the cube of 500. ∴ 500 may be called f.

4. Subtracting, 14,798,359 contains $3f^2n + 3fn^2 + n^3$. (Why ?)

5. This is approximately the product of $3f^2$ and n, and hence if divided by $3f^2$ the quotient is approximately n. ∴ $n = 10$.

6. ∴ $3fn + n^2 = 15{,}100$, and $3f^2 + 3fn + n^2 = 765{,}100$, and this, multiplied by n, equals $3f^2n + 3fn^2 + n^3$.

7. ∵ f^3 has already been subtracted, after subtracting $3f^2n + 3fn^2 + n^3$ there has been subtracted $(f + n)^3$, or 510^3.

8. Calling 510 the second found part, f_2, it appears that $f_2{}^3$ has been subtracted. ∴ the remainder contains $3f^2n + 3fn^2 + n^3$.

9. The explanation now repeats itself as in square root.

EXERCISES. LIII.

Extract the cube roots of exs. 1–4.

1. (a) 10,077,696. (b) 31,855,013. (c) 125.751501.

2. (a) 367,061.696. (b) 997.002999.

3. (a) 551. (b) 975. Each to 0.001.

4. (a) 2. (b) 5. Each to 0.0001.

REVIEW EXERCISES. LIV.

Extract the cube roots of exs. 1–3.

1. $1 - x$ to 5 terms.

2. $64 - 48x + 9x^2$ to 3 terms.

3. $a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$.

4. Factor $x^5 + x^2 - 4x^3 - 4$.

5. Show that $xyz(x^3 + y^3 + z^3) - (y^3z^3 + z^3x^3 + x^3y^3) \equiv (x^2 - yz)(y^2 - zx)(z^2 - xy)$.

6. Divide the product of $x^2 + x - 2$ and $x^2 + x - 12$ by the sum of $2x^2 + 6x + 1$ and $2 - x(10 + x)$.

7. Find the square root of

$$(x + 3)(x + 4)(x + 5)(x + 6) + 1.$$

8. Solve the equation

$$7 - 2\{6 - 3[5 - 2(4 - \overline{3 + 2x})]\} = 1.$$

9. Find the square root of

$$(2a - b)^2 - 2(2a^2 - 5ab + 2b^2) + (a - 2b)^2.$$

10. Find the three roots of the equation $x^3 - x^2 + 1 = x$.

11. Also of the equation $x^3 + 9x^2 + 8x - 60 = 0$.

12. If $a = -3$, $b = 0$, $c = 1$, $d = -2$, find the numerical value of $a - 2\{b + 3[c - 2a - (a - b)] + 2a - (b + 3c)\}$.

CHAPTER VII.

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE.

I. HIGHEST COMMON FACTOR.

134. The integral algebraic factor of highest degree common to two or more integral algebraic expressions is called their **highest common factor**.

E.g., a^2 is the highest common factor of a^3cd and $2a^2be^3$,

xy^2z " " " x^3y^2z " xy^2z^3,

$a-b$ " " " $(a-b)^2$ " a^2-b^2.

Consider, also, $2(a^3-b^3)$ and $4(b^2-a^2)$.

Here $2(a^3-b^3) = 2(a-b)(a^2+ab+b^2)$, or $-2(b-a)(a^2+ab+b^2)$,

and $4(b^2-a^2) = 4(b-a)(b+a)$, " $-4(a-b)(a+b)$.

Here either $a-b$ or $b-a$ is a common factor, and there being no other common *algebraic* factor, either is called the highest common factor. There is a common numerical factor, 2, but such factors have nothing to do with the algebraic divisibility of the expressions, and hence may be neglected.

In the last example, it is not usual to state both answers, $a-b$ and $b-a$, because $a-b = -1\cdot(b-a)$; that is, the two are the same except for a numerical factor, and numerical factors are not considered.

135. The arithmetical greatest common divisor must not be confounded with the algebraic highest common factor, although these are often called by the same name. The highest common factor has reference only to the degree of the expression.

E.g., consider the highest common factor of x^2-3x+2 and x^2-x-2.

Here $x^2-3x+2=(x-2)(x-1)$, or $(2-x)(1-x)$,

and $x^2-x-2=(x-2)(x+1)$ " $-(2-x)(x+1)$;

hence, the highest common factor is $x-2$, or $2-x$. Now if $x=5$, the expressions become 12 and 18, and the highest common factor becomes 3, or -3, although 6 is the greatest common divisor of 12 and 18.

The highest common factor is occasionally used in reducing fractions to their lowest terms.

136. Factoring method. The highest common factor of expressions which are easily factored is usually found by simple inspection.

E.g., to find the highest common factor of x^2-3x+2, x^3-x^2-2x, and $\frac{1}{2}x^2+\frac{1}{2}x-3$, we have:

1. $x^2-3x+2=(x-2)(x-1)$.
2. $x^3-x^2-2x=x(x-2)(x+1)$.
3. $\frac{1}{2}x^2+\frac{1}{2}x-3=\frac{1}{2}(x-2)(x+3)$.
4. $\therefore$ the highest common factor is $x-2$, or $2-x$.

EXERCISES. LV.

Find the highest common factor of each of the following sets of expressions:

1. $5a^2b^3c^4d^5$, $\frac{1}{2}a^5b^4c^3d^2$.
2. $15mnx^3$, $17mx^2yz$, $\frac{2}{3}abcx^{10}z$.
3. $10x^2yz$, $15ax^2yz^3$, $20amxz^{10}$.
4. x^2-y^2, y^3-x^3, $x^2-8xy+7y^2$.
5. x^3-y^3, y^2-x^2, $x^2-\frac{1}{2}xy-\frac{1}{2}y^2$.
6. x^2-4, x^2-x-6, $2-5x-3x^2$.
7. $2x^2-xy-y^2$, $4x^2+10xy+4y^2$.
8. $6a^2+19ab-7b^2$, $2a^2+ab-21b^2$.
9. $4a^2(a^3-b^3)$, $\frac{1}{2}ab^2(3a^2-5ab+2b^2)$.

137. If the factors of one of several algebraic expressions are known, but those of the others not, it is easy to ascertain, by division or by the Remainder Theorem, if the known factors of the one are factors of the other.

E.g., to find the highest common factor of $1 - x^2$ and $113x^7 - 4x^3 + 2x - 111$.

Here $\quad 1 - x^2 = (1 - x)(1 + x)$, or $-(x - 1)(x + 1)$.

But $x - 1$ is a factor of $113x^7 - 4x^3 + 2x - 111$, by the Remainder Theorem (§ 103), while $x + 1$ is not. $\therefore x - 1$ is the highest common factor.

EXERCISES. LVI.

Find the highest common factor of each of the following sets of expressions:

1. $x^5 - y^5,\ x^2 - y^2$.
2. $x^2 - 4,\ x^5 - 4x^2 - 16$.
3. $x^2 - 4,\ x^7 + 7x^2 + 100$.
4. $x^3 + 1,\ x^3 + ax^2 + ax + 1$.
5. $x^2 - 3x + 2,\ x^2 - 9x + 14$.
6. $x^2 - 9x + 14,\ 2x^3 - 5x^2 - 441$.
7. $x^3 + x^2 + x - 3,\ x^3 + 3x^2 + 5x + 3$.
8. $x^2 - 4,\ 5x^4 + 2x^3 - 23x^2 - 8x + 12$.
9. $2x^2 - 5xy + 3y^2,\ 6x^3 - 23x^2y + 25xy^2 - 6y^3$.
10. $a^3 - b^3,\ b^2 - a^2,\ 117a^3 - 117a^2b - 231ab + 231b^2$.
11. $x^3 - 1,\ x^2 - 1,\ 293x^5 - 200x^4 + 7x^3 - 50x^2 - 25x - 25$.
12. $1 - x^4,\ x^5 - 1,\ x^3 - 1 + 3x - 3x^2,\ 247x^2 - 240x - 7$.
13. $x^5 - 32,\ 16 - x^4,\ x^2 - 9x + 14,\ x^4 - 4x^2 + 6x - 12$.
14. $x^3 + 1,\ x^2 + 2x + 1,\ x^5 + 1,\ 324x^5 + 247x^4 + 100x^3 + 204x^2 - 27$.

138. Euclidean method. In case the highest common factor is not readily found by inspection of factors, a longer method, analogous to one suggested by Euclid (B.C. 300) for finding the greatest common divisor, may be employed.

139. This method depends upon two theorems:

1. *A factor of an algebraic expression is a factor of any multiple of that expression.*

Proof. 1. Let a, b, p, q be algebraic expressions, p and q being the factors of b.

2. Then $b = pq$.

3. $\therefore$ $ab = apq$. (Why?)

4. *I.e.*, if p is a factor of b, it is a factor of any multiple of b, as ab.

A similar proposition is readily seen to be true for numbers. *E.g.*, 5 is a factor of 35; and since multiplying 35 by any integral number does not take out this 5, therefore, 5 is a factor of any multiple of 35.

2. *A factor of each of two algebraic expressions is a factor of the sum and of the difference of any multiples of those expressions.*

Proof. 1. Let $b = pq$ and $b' = pq'$.

2. Then $ab = apq$ " $a'b' = a'pq'$. (Why?)

3. $\therefore$ $ab \pm a'b' = apq \pm a'pq' = p(aq \pm a'q')$. (Why?)

4. *I.e.*, if p is a factor of b and b', as in step 1, then it is also a factor of the sum and of the difference of any multiples of b and b', as ab and $a'b'$.

A similar proposition is true for numbers. *E.g.*, 5 is a factor of 60 and of 35, and also of the sum and of the difference of any multiples of these numbers.

140. The Euclidean method will best be understood by considering an example.

Required the highest common factor of

$$x^4 - x^3 + 2x^2 - x + 1 \text{ and } x^4 + x^3 + 2x^2 + x + 1.$$

$$\begin{array}{r|l}
x^4-x^3+2x^2-x+1 & x^4+x^3+2x^2+x+1 \,\underline{|1} \\
 & \underline{x^4-x^3+2x^2-x+1} \\
 & 2x \,|\, 2x^3 \qquad +2x \\
 & \overline{x^2 \qquad +1} \,|\, x^4-x^3+2x^2-x+1 \,\underline{|x^2-x+1} \\
 & \qquad \underline{x^4 \qquad + x^2} \\
 & \qquad -x^3+x^2-x+1 \\
 & \qquad \underline{-x^3 \qquad -x} \\
 & \qquad x^2 \qquad +1 \\
 & \qquad \underline{x^2 \qquad +1}
\end{array}$$

EXPLANATION. 1. The h.c.f. of the two expressions is also a factor of $2x^3 + 2x$, by th. 2 (§ 139).

2. It cannot contain $2x$, because that is not common to the two expressions.

3. $\therefore$ $2x$ may be rejected, and the h.c.f. must be a factor of $x^2 + 1$.

4. $x^2 + 1$ is a factor of $x^4 - x^3 + 2x^2 - x + 1$, by trial.

5. " " " $2x^3 + 2x$.

6. $\therefore$ " " " $x^4 + x^3 + 2x^2 + x + 1$. (Why ?)

7. $\therefore$ " is the h.c.f. (Why ?)

141. In order to avoid numerical fractions in the divisions, it is frequently necessary to introduce numerical factors. These evidently do not affect the *degree* of the highest common factor.

E.g., to find the highest common factor of $4x^3 - 12x^2 + 11x - 3$ and $6x^3 - 13x^2 + 9x - 2$.

$$\begin{array}{r|l}
 & 6x^3-13x^2+9x-2 \\
 & \qquad\qquad\qquad\quad 2 \\
4x^3-12x^2+11x-3 & 12x^3-26x^2+18x-4 \,\underline{|3} \\
 & \underline{12x^3-36x^2+33x-9} \\
 & 5 \,|\, 10x^2-15x+5 \\
 & 2x^2-3x+1 \,|\, 4x^3-12x^2+11x-3 \,\underline{|2x-3} \\
 & \qquad \underline{4x^3-6x^2+2x} \\
 & \qquad -6x^2+9x-3 \\
 & \qquad \underline{-6x^2+9x-3}
\end{array}$$

Here the introduction of the factor 2 and the suppression of 5 evidently do not affect the *degree* of the highest common factor.

142. In practice, detached coefficients should be used whenever the problem warrants.

E.g., to find the highest common factor of

$$3x^5y + 3x^4y + 2x^3y - x^2y - xy \text{ and } 2x^4 + 9x^3 + 9x^2 + 7x.$$

Here x is evidently a factor of the highest common factor. It may therefore be suppressed and introduced later, thus shortening the work.

But y is a factor of the first only, and hence may be rejected entirely.

The problem then reduces to finding the highest common factor of $3x^4 + 3x^3 + 2x^2 - x - 1$ and $2x^3 + 9x^2 + 9x + 7$.

$$\begin{array}{r|rrrrr|l}
 & 3\ + & 3\ + & 2\ - & 1\ - & 1 & \\
 & & & & & 2 & \\
2+9+9+7 & 6\ + & 6\ + & 4\ - & 2\ - & 2 & 3 \\
 & 6\ + & 27\ + & 27\ + & 21 & & \\
 & & -21\ - & 23\ - & 23\ - & 2 & \\
 & & & & - & 2 & \\
 & & 42\ + & 46\ + & 46\ + & 4 & 21 \\
 & & 42\ + & 189\ + & 189\ + & 147 & \\
 & -143 & -143\ - & 143\ - & 143 & & \\
 & & & 1\ + & 1\ + & 1 & 2+9+9+7 \mid 2+7 \\
 & & & & & & 2+2+2 \\
 & & & & & & \quad 7+7+7 \\
 & & & & & & \quad 7+7+7
\end{array}$$

$\therefore x(x^2 + x + 1)$ is the h.c.f.

143. The work can often be abridged by noticing the difference between the two polynomials.

E.g., in the case of $x^4 - 2x^3 + 3x^2 - 8x + 6$ and $x^4 - 4x^3 + 3x^2 - 6x + 6$. Here we have:

$$\begin{array}{r}
1 - 2 + 3 - 8 + 6 \\
1 - 4 + 3 - 6 + 6 \\
\hline
2 \mid 2 \qquad - 2 \\
1 \qquad - 1
\end{array}$$

$$x^2 - 1 = (x + 1)(x - 1).$$

By the Remainder Theorem $x - 1$ is a factor of each expression, and $x + 1$ is not; $\therefore x - 1$ is the highest common factor of the expressions.

144. The highest common factor of three expressions cannot be of higher degree than that of any two; hence, the highest common factor of this highest common factor and of the third expression is the highest common factor of all three. Similarly, for any number of expressions.

EXERCISES. LVII.

Find the highest common factor of each of the following sets of expressions:

1. $x^3 - 2x + 4, \; x^4 + x^3 + 4x.$
2. $2x^3 + 2x - 4, \; x^3 - 3x + 2.$
3. $x^4 + 4, \; x^4 - 2x^3 + x^2 + 2x - 2.$
4. $x^3 - 40x + 63, \; x^4 - 7x^3 + 63x - 81.$
5. $x^6 + y^6, \; x^4 - y^4, \; x^5 + x^3y^2 + x^2y^3 + y^5.$
6. $x^3(6x + 1) - x, \; 4x^3 - 2x(3x + 2) + 3.$
7. $x^4 - 15x^2 + 28x - 12, \; 2x^3 - 15x + 14.$
8. $7x^3 - 10x^2 - 7x + 10, \; 2x^3 - x^2 - 2x + 1.$
9. $x^2 - 4x - 117, \; x^4 - 13x^3 - x^2 + 14x - 13.$
10. $63a^4 - 17a^3 + 17a - 3, \; 98a^4 + 34a^2 + 18.$
11. $x^2 + 4x - 21, \; x^2 + 20x + 91, \; 2x^3 + 4x^2 - 70x.$
12. $8x^4 - 10x^3 + 7x^2 - 2x, \; 6x^5 - 11x^4 + 8x^3 - 2x^2.$
13. $9a^2 - 4b^2 + 4bc - c^2, \; 2b^2 + c^2 + 3ab - 3bc - 3ac.$
14. $(a - b)(a^2 - c^2) - (a - c)(a^2 - b^2), \; a^5 - b^5, \; ab - b^2 - ac + bc.$
15. $x^3 - 10(x^2 + 3) + 31x, \; x^2(x - 11) + 2(19x - 20), \; x^3 - 9x^2 + 26x - 24.$
16. $a^4b^2 + 4a^3b^3 + 3a^2b^4 - 4ab^2 - 4b^3, \; a^5b + 3a^4b^2 - a^3b^3 - 3a^2b^4 - 4a^2b + 4b^3.$
17. $3a^2 - 7ab + 2b^2 + 5ac - 5bc + 2c^2, \; 12a^2 - 19ab + 5b^2 + 11ac - 11bc + 2c^2.$

II. LOWEST COMMON MULTIPLE.

145. The integral algebraic multiple of lowest degree common to two or more algebraic expressions is called their **lowest common multiple.**

E.g., a^2b^3cd is the lowest common multiple of a^2bc and ab^3d.

Similarly, $\pm (a + b)^2 (a - b)$ is the lowest common multiple of $a^2 - b^2$, $b - a$, and $(a + b)^2$. For

1. $a^2 - b^2 = (a + b)(a - b)$.
2. $b - a = -(a - b)$.
3. $(a + b)^2 = (a + b)(a + b)$.
4. $\therefore$ either $(a + b)^2 (a - b)$ or $(a + b)^2 (b - a)$ contains the given expressions and is the common multiple of lowest degree.

The lowest common multiple of algebra must not be considered the same as the least common multiple when numerical values are assigned. *E.g.*, the lowest common multiple of $a + b$ and $a - b$ is $(a + b)(a - b)$; but if $a = 6$ and $b = 4$, the least common multiple of $6 + 4$ and $6 - 4$ is simply $6 + 4$.

146. So far as the **algebraic** multiple is concerned, numerical factors are not usually considered.

E.g., a^2b^3c is the lowest common multiple of $2ab^3c$, $\frac{1}{4}a^2b$, and $15ab$.

The lowest common multiple is used in reducing fractions to fractions having a lowest common denominator.

147. Factoring method. The lowest common multiple is usually found by the inspection of factors.

E.g., to find the lowest common multiple of $x^2 - 12x + 27$, $x^2 + x - 12$, and $15 - 2x - x^2$.

1. $x^2 - 12x + 27 = (x - 3)(x - 9)$.
2. $x^2 + x - 12 = (x - 3)(x + 4)$.
3. $15 - 2x - x^2 = -(x - 3)(x + 5)$.
4. $\therefore \pm (x - 3)(x + 4)(x + 5)(x - 9)$ is the lowest common multiple.

In practice, the result should be left in the factored form.

EXERCISES. LVIII.

Find the lowest common multiple of each of the following sets of expressions:

1. $-10a^2xyz,\ 5x^2yz^3,\ \frac{1}{3}a^3xy^3z.$
2. $x^3+y^3,\ x+y,\ xy-x^2-y^2.$
3. $a^2+b^2-2ab,\ b^2-a^2,\ a-b.$
4. $27-12x+x^2,\ x^2+2x-15.$
5. $x^4+4,\ 2-x^2,\ x^2+2,\ x-\sqrt{2}.$
6. $x^2+x-12,\ -36+13x-x^2,\ x^2-16.$
7. $x^3+y^3+3xy(x+y),\ x^3+y^3,\ x+y.$
8. $2xy-x^2-y^2,\ 2xy+x^2+y^2, x^2-y^2,\ x+y.$

148. Highest common factor method. Since the highest common factor contains all of the factors common to two expressions, it may be suppressed from either of them and the quotient multiplied by the other to obtain the l.c.m.

Proof. 1. Let $x = af,$
$y = bf,$

in which f is the highest common factor of x and y.

2. Then the lowest common multiple is evidently abf; *i.e.*, it is y multiplied by a.

E.g., to find the lowest common multiple of $2x^3+8x^2-3x-27$ and $2x^3+12x^2+x-45$.

$$\begin{array}{r}
2x^3+12x^2+x-45\\
2x^3+8x^2-3x-27\\
\hline
2\,\underline{|\,4x^2+4x-18}\\
2x^2+2x-9
\end{array}
\quad
\begin{array}{r|l}
2x^3+8x^2-3x-27 & x+3\\
2x^3+2x^2-9x & \\
\hline
6x^2+6x-27 & \\
6x^2+6x-27 & \\
\hline
\end{array}$$

$\therefore\ (2x^3+12x^2+x-45)(x+3)$ is the lowest common multiple.

EXERCISES. LIX.

Find the lowest common multiple of the sets of expressions in exs. 1–15.

1. $x^{12} + x^5,\ x^{17} + x^3.$
2. $3\,a^3 - 11\,a^2 + 4,\ 6\,a^2 - a - 2.$
3. $x^6 + 3\,x^5 + x + 3,\ x^3 - 8\,x + 3.$
4. $6\,x^2 + 13\,x + 6,\ 10\,x^2 - 3 + 13\,x.$
5. $x^2 + 2\,ax + a^2,\ x^2 + ab + (a + b)\,x.$
6. $6\,x^3 + 11\,x^2 - 9\,x + 1,\ 2\,x^2 + 3\,x - 2.$
7. $x^4 - x^3 + x^2 - x - 4,\ x^4 - x^2 + 2\,x - 8.$
8. $x^3 + 1 + 3\,(x^2 + x),\ x^4 + 1 + 4\,(x^3 + x) + 6\,x^2.$
9. $3\,x^3 - 15\,ax^2 + a^2x - 5\,a^3,\ 6\,x^4 - 25\,a^2x^2 - 9\,a^4.$
10. $x^2 + 20x + 91,\ 35 - 2x - x^2,\ x^4 + 6\,x^3 - 6\,x^2 + 6\,x - 7.$
11. $2\,x^4 - 2\,x^3 - x^2 - 4\,x - 7,\ 2\,x^4 + 6\,x^3 - 17\,x^2 + 8\,x - 35.$
12. $x^4 + x^3 + x + 1,\ 2\,x^5 - 3\,x^4 + 4\,x^3 + 2\,x^2 - 3\,x + 4.$
13. $x^7 + x^6 - x^4 - 6\,x^2 - 6x - 7,\ x^3 - x^6 - x^5 + x^4 - 6x^3 - x + 7.$
14. $x^7 + 2x^6 - 3\,x^5 + x^2 + 2\,x - 3,\ x^7 + 4\,x^6 - 7\,x^5 + x^2 + 4\,x - 7,\ x^5 + 1.$
15. $4\,a^2(3\,a + 2) - (27\,a + 18),\ 12\,a^3 - a\,(8\,a + 27) + 18,\ 6\,(3\,a - 2) + 27\,a^3 - 8.$
16. Find all of the algebraic expressions whose lowest common multiple is $x^3 - 4\,xy^2$.
17. Prove that the product of the lowest common multiple and the highest common factor of two expressions is the same as the product of the two expressions.
18. Investigate ex. 17 for the case of *three* expressions.
19. Find the lowest common multiple of $a^2 - 1$ and $a^2 - 4\,a + 3$. Can the result be checked by letting $a = 5$, 7, or any odd number above 3? Explain.

REVIEW EXERCISES. LX.

1. Factor $x(x-1)-a(a-1)$.

2. Solve the equation $4x^2+1=4x$.

3. Solve the equation $6x^2+11x-7=0$.

4. Extract the square root of x^2+1 to 3 terms.

5. Give a complete description of this expression as a function of x and y: $x^4+3x^3y+4x^2y^2+3xy^3+y^4$.

6. Show that the difference of the squares of any two consecutive numbers is equal to the sum of the numbers.

7. Find the lowest common multiple of $2x^5+x^4+4x^3+4x^2+2x+3$ and $6x^5-5x^4+12x^3-8x^2+5x-6$.

8. Find the lowest common multiple of x^4-x^2-2x-1, $2x^4-x^3-2x^2-2x-1$, and $3x^4-4x^3+6x^2-7x-8$.

9. Find the highest common factor of $x^4+2x^3-5x^2+15x+12$, $x^4+5x^3+5x^2+8x+16$, and $x^4+6x^3+10x^2+4x-16$.

10. In finding the highest common factor of two algebraic expressions, by what right may a factor be suppressed in one if it is not a factor of the other?

11. The highest common factor of two expressions is $4x^2-a^2$, and their lowest common multiple is $4x^4-5a^2x^2+a^4$. One of the expressions is $4x^3+4ax^2-a^2x-a^3$. Find the other.

12. Assign such values to a and b that the arithmetical least common multiple of a^3-b^3 and $a^3+b^3+2ab(a+b)$ shall not be the value of the algebraic lowest common multiple.

13. Prove that the difference between the cubes of the sum and difference of any two numbers is divisible by the sum of the square of the smaller number, and three times the square of the larger.

CHAPTER VIII.

FRACTIONS.

149. The symbol $\frac{a}{b}$, in which b is not zero, is defined to mean the division of a by b, and is called an **algebraic fraction.**

Hence, the algebraic fraction $\frac{a}{b}$ represents a quantity which, when multiplied by b, produces a.

The **terms of the fraction** $\frac{a}{b}$ are a and b, a being called the **numerator** and b the **denominator,** and either or both may be fractional, negative, etc.

The case in which b equals zero is discussed later.

There are two definitions of a fraction usually given in arithmetic: (1) The fraction $\frac{a}{b}$ is a of the b equal parts of unity; (2) The fraction $\frac{a}{b}$ is one bth of a.

Neither of these arithmetical definitions includes, for example, $\frac{2}{-3}$, $\frac{2}{\frac{2}{3}}$, $\frac{3}{\sqrt{2}}$, etc., for "2 of the -3 equal parts of unity" means nothing, and "one $\sqrt{2}$th of 3" is equally meaningless. Hence the broader algebraic definition.

In the first arithmetical definition above given, b names the part and hence is called the denominator (Latin, *namer*), and a numbers the parts and hence is the numerator (Latin, *numberer*). Hence the origin of these terms.

The fraction $\frac{a}{b}$ is, therefore, read "a divided by b," although the reading "a over b" is generally used in various languages, and is sanctioned by most teachers on the ground of brevity.

I. REDUCTION OF FRACTIONS.

150. Theorem of reduction. *The same factor may be introduced into or cancelled from both numerator and denominator of a fraction without altering the value of the fraction.*

Given the fraction $\frac{a}{b}$, and m any factor.

To prove that $\frac{a}{b} \equiv \frac{ma}{mb}$, that is, that the factor m may be introduced into both terms of $\frac{a}{b}$ or cancelled from both terms of $\frac{ma}{mb}$.

Proof. 1. $b \cdot \frac{a}{b} \equiv a.$ Def. of frac.

2. $\therefore$ $mb \cdot \frac{a}{b} \equiv ma.$ Ax. 6

3. $\therefore$ $\frac{a}{b} \equiv \frac{ma}{mb}.$ Ax. 7

An algebraic fraction is said to be **simplified** when all common algebraic factors, and hence the highest common factor, of both numerator and denominator have been suppressed, and there is no fraction or common numerical factor in either.

E.g., the fraction $\frac{a^2 + 2ab + b^2}{a^3 + b^3}$ is simplified when reduced to the form $\frac{a + b}{a^2 - ab + b^2}$ by cancelling the factor $a + b$.

But the fractions $\frac{\frac{a}{b} + b}{c}$ and $\frac{2a}{4b}$ are not simplified.

The student should notice that the theorem does not allow the cancellation of any *terms* of the numerator and denominator. No factor can be cancelled unless it is contained in every term of both numerator and denominator.

Usually the factors common to the two terms of the fraction can be found by inspection and cancelled; otherwise the highest common factor of both terms is found and then cancelled.

EXAMPLES. 1. Simplify the fraction $\frac{-a^3b^2cd^4}{a^2b^3cd^3}$.

1. Cancelling a^2, b^2, c, and d^3, the fraction reduces to $\frac{-ad}{b}$.

2. And since there are no other common factors, and the terms are integral, the fraction is simplified.

Check. Let $a = 3$, $b = d = 2$, $c = 1$. Then $\frac{-27 \cdot 4 \cdot 1 \cdot 16}{9 \cdot 8 \cdot 1 \cdot 8} = \frac{-6}{2}$.

2. Simplify $\frac{a^2 + 2ab + b^2}{a^2 - b^2}$.

1. This evidently equals $\frac{(a+b)^2}{(a+b)(a-b)}$.

2. Cancelling $a + b$, this reduces to $\frac{a+b}{a-b}$.

3. And since there are no other common factors, and the terms are integral, the fraction is simplified.

Check. Let $a = 2$, $b = 1$. Then $\frac{9}{3} = \frac{3}{1}$. (If a and b are given the same values, the denominator becomes zero, a case excluded, for the present, by the definition of fraction.)

3. Simplify $\frac{3x^2 + 26x - 77}{3x^2 - 10x + 7}$.

1. A factor of each term of the fraction is a factor of their difference, $36x - 84$ (§ 139, 2).

2. Hence of $3x - 7$, because the terms of the fractions do not contain 12.

3. Hence, if there is a common factor, it is $3x - 7$, because this is irreducible.

4. By substituting arbitrary values this is seen to be a probable factor, and the fraction reduces by division to $\frac{x+11}{x-1}$.

Check. Let $x = 2$. (Why not 1?) Then $\frac{-13}{-1} = \frac{13}{1}$.

4. Simplify $\frac{2x^3 + 9x^2 + 11x + 14}{3x^3 + 4x^2 + 7x + 2}$.

1. Here the simple factors are not as easily determined as the highest common factor, $x^2 + x + 2$.

2. Cancelling this, the fraction reduces to $\frac{2x + 7}{3x + 1}$.

3. ∴ the fraction is, by definition, simplified.

Check. Let $x = 1$. Then $\frac{36}{16} = \frac{9}{4}$.

If the student has not studied Appendix III, ex. 5 may be omitted.

5. Simplify $\frac{-a^3(b-c) - b^3(c-a) - c^3(a-b)}{(a-b)(b-c)(c-a)}$.

1. By the Remainder Theorem (§ 104) $a - b$ is a factor of both terms of the fraction. (We try $a - b$ because if there is any common factor it must be $a - b$, $b - c$, or $c - a$.)

2. Hence, because both terms are cyclic, $b - c$ and $c - a$ are factors.

3. And since the numerator is of the 4th degree, the other factor is a linear cyclic factor. Hence, it is $n(a + b + c)$.

4. Hence, the numerator is $n(a + b + c)(a - b)(b - c)(c - a)$. But by substituting the values $a = 2$, $b = 1$, $c = 0$, n is seen to be 1.

5. Hence, the fraction equals $a + b + c$.

Check. Let $a = 3$, $b = 2$, $c = 1$ (values different from those used for finding n). Then $\frac{-12}{-2} = 6$.

151. General directions for simplifying fractions. The preceding fractions were simplified in different ways. While there is no general method of attack, and the student must use his judgment as to the best plan to pursue, the following directions are of value:

1. *Cancel monomial factors first,* as in ex. 1.

2. *Then see if common polynomial factors can be readily discovered. Make free use of the Remainder Theorem.* Compare ex. 2.

3. *If common factors are not readily discovered, see if the difference between the numerator and denominator can be easily factored.* If so, try these factors, using arbitrary values or the Remainder Theorem, as in ex. 3.

4. *Never perform a multiplication until compelled to. Factor whenever possible. If the terms are cyclic and you have studied Appendix III, apply your knowledge of symmetry and homogeneity,* as in ex. 5.

5. *Let the method by finding the highest common factor be the final resort.* For one who is skillful in factoring, this tedious method ought rarely to be necessary. In ex. 5 students will probably use the Remainder Theorem instead of the method suggested.

6. *Always check the final result* by substituting arbitrary values or by some other simple device.

EXERCISES. LXI.

Simplify the following fractions and check each result:

1. $\dfrac{ab^2c^4}{bc^2\sqrt{a}}$.

2. $\dfrac{a^3-3a+2}{a^3+4a^2-5}$.

3. $\dfrac{a^2-b^2}{a^4-b^4}$.

4. $\dfrac{21x-10-9x^2}{3x^2-26x+35}$.

5. $\dfrac{x^8+y^8}{x^5+y^5}$.

6. $\dfrac{a^3+a^2+3a-5}{a^2-4a+3}$.

7. $\dfrac{x^4+x^2y}{x^4-y^2}$.

8. $\dfrac{6x^2+7xy-3y^2}{6x^2+11xy+3y^2}$.

9. $\dfrac{a^2bc^2d^3}{-ab^2cd^4}$.

10. $\dfrac{x^3-x^2-7x+3}{x^4+2x^3+2x-1}$.

11. $\dfrac{mx^2y-mxy^2}{nx^4y-nx^2y^3}$.

12. $\dfrac{x^2+y^2-z^2+2xy}{x^2-y^2-z^2+2yz}$.

1 13. $\dfrac{a^3 + a^2 - 2a}{a^3 - a^2 - 6a}$.

14. $\dfrac{x^3 + x^2 - 12x}{x^3 + 4x^2 + 5x + 20}$.

1 15. $\dfrac{3x^2y^2 + 4xy^3}{5x^2y^2 - 4x^3y}$.

16. $\dfrac{1 - a^2}{(1 + ax)^2 - (a + x)^2}$.

1 17. $\dfrac{a^2 + 3a - 10}{3a^2 + 2a - 16}$.

18. $\dfrac{x^3 - 5x^2 + 7x - 3}{2x^3 - 5x^2 + 4x - 1}$.

1 19. $\dfrac{x^3 + x^2y + xy^2}{x^5 + x^3y^2 + xy^4}$.

20. $\dfrac{a^6 - a^2x^4}{a^6 + a^5x - a^4x^2 - a^3x^3}$.

2 21. $\dfrac{m^3 - 39m + 70}{m^2 - 3m - 70}$.

22. $\dfrac{m^3 - 6m^2 + 11m - 6}{2m^3 - 14m + 12}$.

2 23. $\dfrac{x^2 - xy - 12y^2}{x^2 + 5xy + 6y^2}$.

24. $\dfrac{x^2 + (m - n)x - mn}{x(x + m) - n(x + m)}$.

25. $\dfrac{x^2 + (a + b)x + ab}{(x + a)(x + b)(x + c)}$.

26. $\dfrac{2a^2 - 10a - 28}{3a^3 - 27a^2 + 21a + 147}$.

27. $\dfrac{m^2x^2 - (m + y)mnx + mn^2y}{x^3 - (m + 1)nx^2 + mn^2x}$.

Omit the following unless Appendix III has been studied.

28. $\dfrac{a^2(b - c) + b^2(c - a) + c^2(a - b)}{abc(a - b)(b - c)(c - a)}$.

29. $\dfrac{(a - b)(b - c)(c - a)}{a^3(b - c) + b^3(c - a) + c^3(a - b)}$.

30. $\dfrac{ab(a - b) + bc(b - c) + ca(c - a)}{(a - b)(b - c)(c - a)}$.

31. $\dfrac{ab(a + b) + bc(b + c) + ca(c + a)}{(a + b)(b + c)(c + a)}$.

152. Reduction to integral or mixed expressions. Since the fraction $\frac{a}{b}$ indicates the division of a by b, it may be reduced to an integral form if the division is exact, and to a mixed form if the degree of the numerator equals or exceeds that of the denominator and the division is not exact.

E.g., $\frac{x^2 - y^2}{x + y} = x - y$, the division being exact.

$\frac{x^2 + y^2}{x + y} = x - y + \frac{2y^2}{x + y}$; that is, the division of the remainder by $x + y$ is indicated.

Check. On the last result. Let $x = y = 1$. Then $\frac{2}{2} = 1 - 1 + \frac{2}{2}$.

EXERCISES. LXII.

Reduce the fractions in exs. 1–10 to integral or mixed expressions, preferably by detaching the coefficients. Check each result.

1. $\frac{x^4 + y^4}{x + y}$.

2. $\frac{x^3 + y^3 + z^3 - 3xyz}{x + y + z}$.

3. $\frac{a^5 + 3a^2 - 1}{a^5 + 1}$.

4. $\frac{4x^3 + 4x^2 + 2x + 1}{2x^2 + x + 1}$.

5. $\frac{3x^2 + 2x - 1}{3x - 1}$.

6. $\frac{x^7 + x^4 + x^2 - 6}{x^6 + x^5 + x^3 + x - 6}$.

7. $\frac{2x^2 - 3xy + y^2}{x + y}$.

8. $\frac{a^3 - 3a^2b + 3ab^2 - b^3}{a^2 - 2ab + b^2}$.

9. $\frac{x^3 + 6x^2 + 12x + 8}{x + 2}$.

10. $\frac{x^3 + 4x^2y + 5xy^2 + 2y^3}{x^2 + 3xy + 2y^2}$.

11. Show that $\frac{1}{1 - a} = 1 + a + a^2 + a^3 + a^4 + \frac{a^5}{1 - a}$.

153. Reduction to equal fractions having a common denominator.

Theorem. *If* $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ *are any fractions whatever, and* m *is any common multiple whatever of* b, d, f, *it is possible to reduce the given fractions to equal fractions having the common denominator* m.

In arithmetic, for example, we can reduce the fractions $\frac{2}{3}, \frac{5}{8}, \frac{1}{12}$ to equal fractions having for their common denominators 24, 48, 96 ….

Proof. 1. ∵ m is a multiple of b, d, f, we may let

$$m = pb,$$

$$m = qd,$$

$$m = rf.$$

2. But $\frac{a}{b} = \frac{pa}{pb}, \quad \frac{c}{d} = \frac{qc}{qd}, \quad \text{and} \quad \frac{e}{f} = \frac{re}{rf}.$ § 150

3. ∴ $\frac{a}{b} = \frac{pa}{m}, \quad \frac{c}{d} = \frac{qc}{m}, \quad \text{and} \quad \frac{e}{f} = \frac{re}{m},$ by substituting the values of step 1.

In particular, if m is the lowest common multiple of the denominators, the fractions will be reduced to equal fractions having the lowest common denominator, a step of great importance in working with fractions.

E.g., to reduce the fractions $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$ to equal fractions having the lowest common denominator:

1. The l.c.m. of the denominators is $(x+y)(x-y)$.

2. $$\frac{x+y}{x-y} = \frac{(x+y)^2}{(x+y)(x-y)}.$$

3. $$\frac{x-y}{x+y} = \frac{(x-y)^2}{(x+y)(x-y)}.$$

EXERCISES. LXIII.

Reduce the following to equal fractions having the lowest common denominator:

1. $\frac{x}{yz}$, $\frac{y}{zx}$, $\frac{z}{xy}$.

2. $\frac{ab}{c^2d}$, $\frac{-b}{c^3d^2}$, $\frac{a^2b}{de^4}$.

3. $\frac{x}{y+z}$, $\frac{y}{z+x}$, $\frac{z}{x+y}$.

4. $\frac{xy}{x^2+y^2}$, $\frac{x^2y^2}{x^4-y^4}$, $\frac{x}{x-y}$.

5. $\frac{x+1}{x^2+4x+3}$, $\frac{x-1}{x^2-9}$, $\frac{x}{x-3}$.

6. $\frac{1}{m^2+6m+8}$, $\frac{2}{2m^2+7m+6}$.

7. $\frac{2m-2n}{m^2-mn+n^2}$, $\frac{4(m+n)}{5(m^2+mn+n^2)}$.

8. $\frac{a^2-b^2}{a^3-b^3}$, $\frac{(a+b)^2}{a^3+b^3}$, $\frac{a^2+b^2+2ab}{(a^2-b^2)^2}$.

9. $\frac{9x^2+12xy-5y^2}{3x^2-xy-10y^2}$, $\frac{6x^2-11xy+4y^2}{2x^2-5xy+2y^2}$.

10. $\frac{x-y}{x^2-y(x-y)}$, $\frac{x+y}{x^2+y(x+y)}$, $\frac{2x^3y^2}{x^6-y^6}$.

11. $\frac{x^3-2x^2+3x-4}{x^3+2x^2+3x+4}$, $\frac{x^3-2x^2-3x+4}{x^3-2x^2+3x+4}$.

12. $\frac{x+1}{x^2+5x+6}$, $\frac{x+2}{x^2+4x+3}$, $\frac{x+3}{x^2+3x+2}$.

13. $\frac{a-3}{a^2-9a+18}$, $\frac{2a+8}{a^2+a-12}$, $\frac{a+5}{a^2+8a+15}$.

14. $\frac{xy}{(y+z)(z+x)}$, $\frac{yz}{x^2+zy+zx+xy}$, $\frac{zx}{y^2+yz+xy+xz}$.

II. ADDITION AND SUBTRACTION.

154. Theorem. *Operations involving the addition and subtraction of fractions can be performed upon the numerators of equal fractions having a common denominator, the result being divided by this common denominator.*

Proof. 1. It has been proved in § 87 that

$$\frac{a}{k}+\frac{b}{k}+\frac{c}{k}\equiv\frac{a+b+c}{k}.$$

2. ∴ if the given fractions be reduced to equal fractions having the common denominator k, the operations can be performed as stated in the theorem.

For simplicity it is, of course, better to reduce to equal fractions having the lowest common denominator.

Thus, with numerical fractions,

$$\tfrac{2}{3}+\tfrac{5}{6}=\tfrac{4}{6}+\tfrac{5}{6}=\tfrac{9}{6}=\tfrac{3}{2}.$$

EXAMPLES. 1. Required the sum of $\frac{a}{b-c}$, $\frac{a}{b+c}$.

1. The l.c.m. of the denominators is $(b+c)(b-c)$.

2. $$\frac{a}{b-c}=\frac{(b+c)a}{(b+c)(b-c)}.$$ § 150

3. $$\frac{a}{b+c}=\frac{(b-c)a}{(b+c)(b-c)}.$$

4. $$\frac{a}{b-c}+\frac{a}{b+c}=\frac{(b+c)a+(b-c)a}{(b+c)(b-c)}$$ § 154

$$=\frac{2ab}{(b+c)(b-c)}.$$

Check. If $a=1$, $b=2$, $c=1$, then $\frac{1}{1}+\frac{1}{3}=\frac{4}{3}$. It is not permissible to let b and c have the same values, because that would make the common denominator zero, a case excluded for the present.

2. Simplify the polynomial $\frac{x}{x^2-1}+\frac{x+3}{x-1}-\frac{x-2}{x+1}$.

1. The l.c.m. of the denominators is x^2-1.

2. $$\frac{x+3}{x-1}=\frac{(x+1)(x+3)}{x^2-1}.$$

3. $$\frac{x-2}{x+1}=\frac{(x-1)(x-2)}{x^2-1}.$$

4. $$\therefore \frac{x}{x^2-1}+\frac{x+3}{x-1}-\frac{x-2}{x+1}=\frac{x+(x+1)(x+3)-(x-1)(x-2)}{x^2-1}$$
$$=\frac{x+x^2+4x+3-x^2+3x-2}{x^2-1}$$
$$=\frac{8x+1}{x^2-1}.$$

Check. Let $x=2$. Then $\frac{2}{3}+\frac{5}{1}-\frac{0}{3}=\frac{17}{3}$.

155. In a case like $\frac{x+y}{x^2+y^2}-\frac{x-y}{x^2+y^2}$, it must be remembered that the bar separating numerator and denominator is a sign of aggregation.

In this case the result is $\frac{x+y-(x-y)}{x^2+y^2}=\frac{x+y-x+y}{x^2+y^2}=\frac{2y}{x^2+y^2}$.

EXERCISES. LXIV.

Simplify the following expressions, checking each result by the substitution of such arbitrary values as do not make the denominators zero:

1. $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$.

2. $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

3. $\frac{2x}{3yz^2}+\frac{5z}{6y^2w}$.

4. $\frac{2-x}{1-x^2}+\frac{x-2}{1-x-2x^2}$.

5. $\frac{1}{x+y}+\frac{1}{x-y}$.

6. $\frac{x}{x+y}+\frac{x}{x-y}-\frac{2xy}{x^2-y^2}$.

7. $\frac{a+1}{a+2}-\frac{a+2}{a+3}+\frac{a-1}{a+2}.$

8. $\frac{(x+y)^2}{(x-y)^2}+\frac{x-y}{x+y}-\frac{x+y}{x-y}.$

9. $\frac{5x^4-7x^3-9x^2+11}{2x^4-3x^3+2x^2-1}-\frac{x-1}{x+3}.$

10. $\frac{a-4}{a^2-9a+20}+\frac{a-5}{a^2-11a}+\frac{a-3}{a^2-7a+12}.$

11. $\frac{1}{1+x}-\frac{2(1-x)}{(1+x)^2}+\frac{1+x^2}{(1+x)^3}-\frac{6x^2(1-x)}{(1+x)^4}.$

12. $\frac{a^2+ab+b^2}{a+b}-\frac{a^2-ab+b^2}{a-b}+\frac{2b^3-b^2+a^2}{a^2-b^2}.$

13. $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}.$

14. $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}.$

15. $\frac{xy}{(y-z)(z-x)}+\frac{yz}{(z-x)(x-y)}+\frac{zx}{(x-y)(y-z)}.$

16. $\frac{ax^2+byz}{(x-y)(x-z)}+\frac{ay^2+bzx}{(y-z)(y-x)}+\frac{az^2+bxy}{(z-x)(z-y)}.$

17. $\frac{x}{y}+\frac{2x^2+y^2}{xy}+\frac{3xy^2-3x^3-y^3}{x^2y}-\frac{4xy^3-2x^2y^2-y^4}{x^2y^2}.$

18. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-a)(b-c)}+\frac{1}{c(c-a)(c-b)}.$

19. $\frac{xy}{(y+z)(z+x)}+\frac{yz}{(z+x)(x+y)}+\frac{zx}{(x+y)(y+z)}$
$+\frac{2xyz}{(x+y)(y+z)(z+x)}.$

III. MULTIPLICATION.

156. Theorem. *The product of two fractions is a fraction whose numerator is the product of their numerators and whose denominator is the product of their denominators.*

Given the two fractions $\frac{a}{b}, \frac{c}{d}$.

To prove that $\frac{a}{b} \cdot \frac{c}{d} \equiv \frac{ac}{bd}$.

Proof. 1. Let $x = \frac{a}{b} \cdot \frac{c}{d}$.

2. Then $bdx = b \cdot \frac{a}{b} \cdot d \cdot \frac{c}{d}$ Ax. 6

3. $= ac$, for $b \cdot \frac{a}{b} = a$, by def. of division

4. $\therefore$ $x = \frac{ac}{bd}$. Ax. 7

5. $\therefore$ $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Ax. 1

COROLLARIES. 1. *Similarly for the product of any number of fractions.*

2. *The product* $a \cdot \frac{c}{d} = \frac{ac}{d}$, as defined in § 52.

For if $b = 1$, the identity $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ becomes $a \cdot \frac{c}{d} = \frac{ac}{d}$.

Illustrative problem. $\frac{x^2 - 8x + 15}{x^2 - 12x + 35} \cdot \frac{x^2 - 15x + 56}{x^2 - 17x + 72}$

$$\equiv \frac{(x-5)(x-3)(x-7)(x-8)}{(x-5)(x-7)(x-9)(x-8)} \equiv \frac{x-3}{x-9}.$$

Check. $\frac{8}{24} \cdot \frac{42}{56} = \frac{-2}{-8}$.

EXERCISES. LXV.

Perform the multiplications indicated, simplifying the results and checking as usual.

1. $\dfrac{7x^3y^2}{12x^2y^3}\cdot\dfrac{18x^4y^5}{28x^5y^4}$.

2. $\dfrac{x^3-y^3}{x^3+y^3}\cdot\dfrac{x^2-xy+y^2}{x^2+xy+y^2}$.

3. $\dfrac{27x}{8y+8x}\cdot\dfrac{x+y}{3}$.

4. $\dfrac{a^2+b^2+2ab}{a-b}\cdot\dfrac{1}{a^2-b^2}$.

5. $\dfrac{x^6-y^6}{x^3-y^3}\cdot\dfrac{x^2-y^2}{(x+y)^2}$.

6. $\dfrac{(a+b)(x+y)}{(a-b)(x-y)}\cdot\dfrac{a^2-b^2}{x+y}$.

7. $\dfrac{x^4-y^4}{(x-y)^2}\cdot\dfrac{x-y}{x^2+xy}\cdot\dfrac{x}{x^2+y^2}$.

8. $\dfrac{x^4+x^3y+xy^3+y^4}{x^2+2xy+y^2}\cdot\dfrac{x-y}{x^2+xy}$.

9. $\dfrac{x^2+x-12}{x^2-13x+40}\cdot\dfrac{x^2+2x-35}{x^2+9x+20}$.

10. $\dfrac{x^2+5x+6}{x^2+7x+12}\cdot\dfrac{x^2+9x+20}{x^2+11x+30}$.

11. $\dfrac{2a^2+5a+2}{6a^2+5a+1}\cdot\dfrac{9a^2+15a+4}{5a^2+12a+4}$.

Reduction of integral or mixed expressions to fractional form.

157. Theorem. *An integer can always be expressed as a fraction with any denominator.*

For since $$1\equiv\frac{b}{b},$$

$$\therefore a\equiv\frac{ab}{b},$$ by ax. 6 and § 156, cor. 2

158. Theorem. *A mixed expression can always be written in fractional form.*

For since $a + \frac{b}{c} \equiv \frac{ac}{c} + \frac{b}{c}$. § 157

$\therefore$ $a + \frac{b}{c} \equiv \frac{ac + b}{c}$. § 154

EXERCISES. LXVI.

Write the expressions in exs. 1–8 as fractions with the denominators indicated, as in § 157.

1. 5, denominator $25a$.
2. abc, " abc.
3. $x + y$, " $x - y$.
4. $x^3 + x^2 + x + 1$, " $x - 1$.
5. $x^4 - x^3 + x^2 - x + 1$, " $x + 1$.
6. $a^3 - b^3$, " $a^3 + b^3$.
7. $x^2 + xy + y^2$, " $x^2 - xy + y^2$.
8. $(a - b)(b - c)(c - a)$, " $(a + b)(b + c)(c + a)$.

Reduce the following to fractional forms, as in § 158, checking each result:

9. $4a - \frac{6ab - 2}{3b}$.
10. $a + b + \frac{b^2}{a - b}$.
11. $x^2 + x + 1 + \frac{2}{x - 1}$.
12. $x^3 - 3x - \frac{3x(3 - x)}{x - 2}$.
13. $1 + a + a^2 + a^3 + \frac{1}{a - 1}$.
14. $x^2 + 2xy + y^2 - \frac{(x^2 - y^2)^2}{x^2 - 2xy + y^2}$.

159. Theorem. *Any integral power of a fraction equals that power of the numerator divided by that power of the denominator.*

Given the fraction $\frac{a}{b}$, and the integer n.

To prove that $\left(\frac{a}{b}\right)^n \equiv \frac{a^n}{b^n}$.

Proof. 1. $\left(\frac{a}{b}\right)^n \equiv \frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\cdots$ to n factors Def. of power

2. $\equiv \frac{aaa\cdots \text{ to } n \text{ factors}}{bbb\cdots \text{ to } n \text{ factors}}$ § 156, cor. 1

3. $\equiv \frac{a^n}{b^n}$. Def. of power

EXERCISES. LXVII.

Express the quantities in exs. 1–6 without using the parentheses. Check each result.

1. $\left(\frac{x+y}{x-y}\right)^3$. 2. $\left(\frac{abc}{a-b}\right)^3$. 3. $\left(\frac{x+3}{x-3}\right)^2$.

4. $\left(\frac{a+b+c}{abc}\right)^2$. 5. $\frac{m^2+m+1}{(m+1)^2}$. 6. $\left(\frac{p+q+r}{p-q-r}\right)^2$.

Express the following quantities as powers of a fraction:

7. $\frac{a^2+10\,ab+25\,b^2}{1+4\,x+4\,x^2}$. 8. $\frac{a^4+9\,b^4+6\,a^2b^2}{81\,(a^2+b^2)+162\,ab}$.

9. $\frac{100\,x^4+20\,x^2+1}{x^4+20\,x^2+100}$. 10. $\frac{x^6+1+3\,x^2(x^2+1)}{x^6-1-3\,x^2(x^2-1)}$.

11. $\frac{4\,x^2+9\,y^2+12\,xy}{4\,x^2+9\,y^2-12\,xy}$. 12. $\frac{x^3-3\,x^2y+3\,xy^2-y^3}{x^3+3\,x^2+3\,x+1}$.

Illustrative problems in multiplication. 1. To find the product of $\frac{x-a}{x+a}$, $\frac{x-2a}{x+2a}$, and $\frac{x}{x-a}$.

1. By § 156, $\frac{x-a}{x+a}\cdot\frac{x-2a}{x+2a}\cdot\frac{x}{x-a}\equiv\frac{(x-a)(x-2a)x}{(x+a)(x+2a)(x-a)}$

2. $\equiv\frac{(x-2a)x}{(x+a)(x+2a)}$. § 150

Check. Let $x=3$, $a=1$. Then $\frac{2}{4}\cdot\frac{1}{5}\cdot\frac{3}{2}=\frac{3}{4\cdot 5}$.

2. To find the product of $\frac{a}{b}+\frac{b}{a}$ and $\frac{a}{b}-\frac{b}{a}$.

1. $\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{a}{b}-\frac{b}{a}\right)\equiv\left(\frac{a}{b}\right)^2-\left(\frac{b}{a}\right)^2$ § 69

2. $\equiv\frac{a^2}{b^2}-\frac{b^2}{a^2}$ § 159

3. $\equiv\frac{a^4-b^4}{a^2b^2}$. § 154

Check. Let $a=1$, $b=2$. Then

$$\left(\frac{1}{2}+2\right)\left(\frac{1}{2}-2\right)=\frac{1-16}{4},\text{ for }\frac{5}{2}\cdot\frac{-3}{2}=\frac{-15}{4}.$$

3. To find the product of $\frac{x^2+6x+5}{x^2+7x+12}\cdot\frac{x^2+8x+15}{x^2+5x+4}$.

1. $\frac{x^2+6x+5}{x^2+7x+12}\cdot\frac{x^2+8x+15}{x^2+5x+4}$

$\equiv\frac{(x+1)(x+5)}{(x+3)(x+4)}\cdot\frac{(x+3)(x+5)}{(x+1)(x+4)}$, by factoring

2. $\equiv\frac{(x+1)(x+5)(x+3)(x+5)}{(x+3)(x+4)(x+1)(x+4)}$ § 156

3. $\equiv\frac{(x+5)^2}{(x+4)^2}$. § 150

Check. $\frac{12}{20}\cdot\frac{24}{10}=\frac{36}{25}$.

EXERCISES. LXVIII.

Perform the multiplications indicated, simplify each result, and check.

1. $\left(\frac{2x}{3} - 1\right)^3.$

2. $\left(\frac{a}{b} + \frac{b}{a}\right)^2.$

3. $\left(1 + \frac{x}{y}\right)\left(1 - \frac{x}{y}\right).$

4. $\left(\frac{a}{b}\right)^{97} \cdot \left(\frac{b}{c}\right)^{99} \cdot \left(\frac{c}{a}\right)^{98}.$

5. $\frac{3}{xy} \cdot \left(x + \frac{xy}{3}\right) \cdot \left(1 - \frac{3}{3+y}\right).$

6. $\left[\frac{1}{3} + \frac{2x}{3(1-x)}\right] \cdot \left[\frac{3}{4} - \frac{3x}{2(1+x)}\right].$

7. $1 - \frac{a+b}{a-b}\left(\frac{a}{a+b} - \frac{a-b}{a} + \frac{a-b}{a+b}\right).$

8. $\left(\frac{p}{a} - \frac{q}{b}\right)\frac{r}{c} + \left(\frac{p}{a} - \frac{r}{c}\right)\frac{q}{b} + \left(\frac{q}{b} - \frac{r}{c}\right)\frac{p}{a}.$

9. $\left(\frac{1}{a} + \frac{1}{b} + \frac{a+b}{(a-b)^2}\right) \cdot \left(\frac{a^2 + ab}{a^2 - ab + b^2}\right).$

10. $\frac{1}{(a+b)^2} \cdot \left(\frac{1}{a^2} + \frac{1}{b^2}\right) + \frac{2}{(a+b)^3} \cdot \left(\frac{1}{a} + \frac{1}{b}\right) \equiv \frac{1}{a^2b^2}.$

11. $\left(\frac{a}{b} + \frac{b}{a}\right)^2 + \left(\frac{b}{c} + \frac{c}{b}\right)^2 + \left(\frac{c}{a} + \frac{a}{c}\right)^2$

$$\equiv 4 + \left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{b}{c} + \frac{c}{b}\right)\left(\frac{c}{a} + \frac{a}{c}\right).$$

12. $\frac{x^2 - (a+b+c)x + a(b+c)}{x^2 - 2ax + a^2} \cdot \frac{x^2 - b^2 - c^2 + 2bc}{x^2 - 2(b+c)x + (b+c)^2}.$

13. $\left(\frac{x^4}{16z^8} + \frac{x^3y}{12z^6} + \frac{x^2y^2}{9z^4} + \frac{4xy^3}{27z^2} + \frac{16y^4}{81}\right) \cdot \left(\frac{x}{2z^2} - \frac{2y}{3}\right).$

IV. DIVISION.

160. The fraction formed by interchanging the numerator and denominator of a fraction (of which neither term is zero) is called the **reciprocal** of that fraction.

E.g., 2 is the reciprocal of $\frac{1}{2}$, $\frac{2}{3}$ is the reciprocal of $\frac{3}{2}$, and $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$.

Evidently 1 and -1 are the only numbers which are their own reciprocals, respectively.

The term *reciprocal* is used only in relation to abstract numbers.

161. Theorem. *To divide any number by a fraction is equivalent to multiplying that number by the reciprocal of the fraction.*

Given the fraction $\frac{a}{b}$ and the number q.

To prove that $q \div \frac{a}{b} \equiv \frac{b}{a} \cdot q$.

Proof. 1. Let $x \equiv q \div \frac{a}{b}$.

2. $\therefore$ $\frac{a}{b} \cdot x \equiv q$, by def. of division, or Ax. 6

3. $\therefore$ $\frac{b}{a} \cdot \frac{a}{b} \cdot x \equiv \frac{b}{a} \cdot q$, by mult. by $\frac{b}{a}$. Ax. 6

4. $\therefore$ $x \equiv \frac{b}{a} \cdot q$, since $\frac{b}{a} \cdot \frac{a}{b} = 1$. §§156,150

5. $\therefore$ $q \div \frac{a}{b} \equiv \frac{b}{a} \cdot q$. Ax. 1

162. Corollaries. 1. *The reciprocal of a fraction equals* 1 *divided by the fraction.*

For $1 \div \frac{a}{b} \equiv \frac{b}{a} \cdot 1$, by the theorem.

2. $\frac{a+b+c}{m} = \frac{1}{m} \cdot a + \frac{1}{m} \cdot b + \frac{1}{m} \cdot c$.

For to divide by m is to multiply by its reciprocal.

Illustrative problems. 1. Perform the following division:

$$\frac{27}{8(x^2-y^2)} \div \frac{3x}{x-y}.$$

1. $\frac{27}{8(x^2-y^2)} \div \frac{3x}{x-y} \equiv \frac{x-y}{3x} \cdot \frac{27}{8(x+y)(x-y)}$ § 161

2. $\equiv \frac{(x-y)27}{3x \cdot 8(x+y)(x-y)}$ § 156

3. $\equiv \frac{9}{8x(x+y)}$, cancelling $3(x-y)$. § 150

Check. Let $x = 2, y = 1$. Then $\frac{27}{8 \cdot 3} \div \frac{6}{1} = \frac{9}{16 \cdot 3}$, for $\frac{9}{8} \div 6 = \frac{3}{16}$.

2. Perform the following division: $\frac{x^3 - a^3}{x + a} \div \frac{x - a}{x^3 + a^3}$.

1. $\frac{x^3-a^3}{x+a} \div \frac{x-a}{x^3+a^3} \equiv \frac{x^3+a^3}{x-a} \cdot \frac{x^3-a^3}{x+a}$ § 161

2. $\equiv \frac{(x^3+a^3)(x^3-a^3)}{(x-a)(x+a)}$ § 156

3. $\equiv (x^2 - xa + a^2)(x^2 + xa + a^2)$.

Check. Let $x = 2, a = 1$. Then

$$\frac{8-1}{2+1} \div \frac{2-1}{8+1} = (4-2+1)(4+2+1), \text{ for } \frac{7}{3} \div \frac{1}{9} = 21.$$

3. Perform the following division:

$$\left(\frac{a}{a-b} - \frac{b}{a+b}\right) \div \frac{a^2+b^2}{a^2-ab}.$$

1. $\frac{a}{a-b} - \frac{b}{a+b} \equiv \frac{a^2+b^2}{(a-b)(a+b)}$.

2. $\frac{a^2+b^2}{(a-b)(a+b)} \div \frac{a^2+b^2}{a^2-ab} \equiv \frac{a^2+b^2}{(a-b)(a+b)} \cdot \frac{a(a-b)}{a^2+b^2}$ § 161

$$\equiv \frac{a}{a+b}.$$

Check. Let $a = 2, b = 1$. Then $\frac{5}{3} \div \frac{5}{2} = \frac{2}{3}$.

EXERCISES. LXIX.

Perform the following divisions, simplifying each result, and checking.

1. $$\frac{a^2+2a-15}{a^2+8a-33} \div \frac{a^2+9a+20}{a^2+7a-44}.$$

2. $$\frac{x^2+x(a+b)+ab}{x^2+x(b+c)+bc} \div \left(\frac{x+a}{x+c}\right)^2.$$

3. $$\frac{a^2+b^2-c^2+2ab}{(a+b+c)^2} \div \frac{(a+b-c)^2}{abc}.$$

4. $$\left(\frac{6}{5-4x}-\frac{14}{2-x}\right) \div \frac{25x-29}{(5-4x)(2-x)}.$$

5. $$\left(\frac{a}{1+a}+\frac{a}{1-a}\right) \div \left(\frac{a}{1+a}-\frac{1-a}{a}\right).$$

6. $$\left(\frac{x}{x+1}-\frac{x-1}{x}\right) \div \left(\frac{x}{x+1}+\frac{x-1}{x}\right).$$

7. $$\left(\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}\right) \div \left(\frac{a+b}{a-b}-\frac{a-b}{a+b}\right).$$

8. $$\left(\frac{7a-13b}{a-3b}+\frac{2a-5b}{3b-a}-2\tfrac{2}{3}\right) \div \frac{1}{a-3b}.$$

9. $$\frac{x^2-6xy+9y^2}{x^2-4xy+4y^2} \div \left(\frac{x^2-9y^2}{x^2+4y^2} \div \frac{x^2+xy-6y^2}{x^2-xy-6y^2}\right).$$

10. $$\left(\frac{b}{3a-b}+\frac{3a}{3a-b}\right) \cdot \frac{3a-b}{9a^2+b^2} \div \left(\frac{1}{3a-b}-\frac{1}{3a+b}\right).$$

11. $$\frac{63x^2y^2}{m+n} \div \left\{\frac{14x(m-n)}{15(a+b)} \div \left[\frac{4(a-b)}{5x^2y} \div \frac{16(a^2-b^2)}{7(m^2-n^2)}\right]\right\}.$$

V. COMPLEX FRACTIONS.

163. A fraction whose numerator, denominator, or both, are fractional is called a **complex fraction**.

E.g., $\dfrac{\dfrac{a+b}{c}}{a^2+ab+b^2}$, $\dfrac{a}{\dfrac{b+c}{bc}}$, $\dfrac{\dfrac{a+b}{a}}{\dfrac{a-b}{a}}$ are complex fractions.

164. Complex fractions are simplified either by performing the division indicated, or by multiplying both terms by such a factor as shall render them integral.

E.g.,
$$\frac{\dfrac{a+b}{c}}{\dfrac{a^2-b^2}{c^2}} \equiv \frac{c^2}{a^2-b^2}\cdot\frac{a+b}{c} \qquad \S\ 150$$
$$\equiv \frac{c}{a-b}. \qquad \S\S\ 150,\ 156$$

Or,
$$\frac{\dfrac{a+b}{c}}{\dfrac{a^2-b^2}{c^2}} \equiv \frac{c\,(a+b)}{a^2-b^2}, \text{ by multiplying both terms by } c^2,$$
$$\equiv \frac{c}{a-b}, \text{ by cancelling } a+b.$$

Check. Let $a=2$, $b=1$, $c=1$. Then $\frac{3}{3}=1$.

It is obvious that the latter plan is the better when the multiplying factor is easily seen.

E.g., to simplify $\dfrac{\dfrac{x^2-y^2}{xy}}{\dfrac{x-y}{y^2}}$.

Multiplying both terms by xy^2, this equals
$$\frac{y\,(x^2-y^2)}{x\,(x-y)} \equiv \frac{y\,(x+y)}{x}.$$

Check. Let $x=2$, $y=1$. Then $\frac{3}{2}=\frac{3}{2}$.

EXERCISES. LXX.

1. Simplify $\dfrac{\dfrac{a+b}{a-b}}{\dfrac{a-b}{a+b}}$.

2. Simplify $\dfrac{\dfrac{x^2-y^2}{x}}{\dfrac{x+y}{y}}$.

3. Simplify $\dfrac{\dfrac{a-b}{2b+a}+\dfrac{1}{2}}{3\frac{1}{2}-\dfrac{5a+7b}{a+2b}}$.

4. Simplify $\dfrac{\dfrac{a+1}{a-1}+\dfrac{a-1}{a+1}}{\dfrac{a+1}{a-1}-\dfrac{a-1}{a+1}}$.

5. Simplify $\dfrac{\dfrac{1+a}{1+a^2}-\dfrac{1+a^2}{1+a^3}}{\dfrac{1+a^2}{1+a^3}-\dfrac{1+a^3}{1+a^4}}$.

6. Simplify $\dfrac{a(a-b)-b(a+b)}{\dfrac{a}{a+b}-\dfrac{b}{a-b}}$.

7. Simplify the reciprocal of $\dfrac{\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a-b}{a^6+b^6}}$.

8. Simplify the reciprocal of $\dfrac{a}{9b}-\dfrac{b}{15a}$.

Simplify the following expressions :

9. $$\frac{1+\dfrac{a}{1+a}}{a+\dfrac{1}{1+a}} \div \frac{(a+1)^2-a^2}{a^2+a+1}.$$

10. $$\frac{(a+b)^5-a^5-b^5}{(a+b)^3-a^3-b^3} \cdot \frac{\dfrac{1}{a}-\dfrac{1}{b}}{\dfrac{a^2}{b}-\dfrac{b^2}{a}}.$$

11. $$\frac{\dfrac{1}{a^4+a^2b^2+b^4}}{\dfrac{1}{a^3+b^3}\cdot(a+b)} \cdot \frac{a^3-b^3}{a-b}.$$

12. $$\frac{\left(1+\dfrac{a}{b}\right)\left(2+\dfrac{b}{a}\right)}{1+\dfrac{a}{b}+\dfrac{b}{a}} + \frac{\dfrac{3(a+b)}{b}}{\dfrac{a^3}{b^3}-1}.$$

13. $$\frac{1}{\left(1-\dfrac{y}{x}\right)\left(1-\dfrac{z}{x}\right)} + \frac{1}{\left(1-\dfrac{x}{y}\right)\left(1-\dfrac{z}{y}\right)} + \frac{1}{\left(1-\dfrac{x}{z}\right)\left(1-\dfrac{y}{z}\right)}.$$

14. $$\left\{\frac{\dfrac{1}{x^2}+\dfrac{1}{y^2}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}} - \frac{\dfrac{1}{x^2}-\dfrac{1}{y^2}}{\dfrac{1}{x^2}+\dfrac{1}{y^2}}\right\} \div \frac{8}{\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right)\left(\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}-2\right)}.$$

165. Continued fractions. Complex fractions of the form

$$\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\cdots}}}$$

are called **continued fractions.**

Such fractions are usually simplified to the best advantage by first multiplying the terms of the last fraction of the form $\cfrac{c}{d+\cfrac{e}{f}}$ by the last denominator, f, and so working up.

E.g., to simplify the fraction $\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}$.

Multiplying the terms of $\cfrac{1}{b+\cfrac{1}{c}}$ by c, the original fraction reduces to $\cfrac{1}{a+\cfrac{c}{bc+1}}$. Multiplying the terms of this fraction by $bc+1$, this reduces to $\dfrac{bc+1}{abc+a+c}$.

Check. Let $a=b=c=1$. Then $\dfrac{1}{1+\frac{1}{2}}=\dfrac{2}{3}$.

EXERCISES. LXXI.

1. Simplify $\cfrac{\cfrac{a}{b}}{1+\cfrac{a}{1-\cfrac{b^2}{a^2}}}$.

2. Simplify $\cfrac{1}{x-\cfrac{1+x^2}{x-\cfrac{1}{1-x}}}$.

3. Simplify $\dfrac{a}{b - \dfrac{c}{a - \dfrac{1}{b}}}$.

4. Simplify $\dfrac{1}{x - \dfrac{x^2 - 1}{x + \dfrac{1}{x - 1}}}$.

5. Simplify $\dfrac{1}{x^2 - \dfrac{x^3 - 1}{x + \dfrac{1}{x + 1}}}$.

6. Simplify $\dfrac{x + y}{x + y + \dfrac{1}{x - y + \dfrac{1}{x + y}}}$.

7. Simplify $\dfrac{a}{1 + \dfrac{a}{1 + a + \dfrac{a}{1 + a + a^2}}}$.

8. Simplify $a^3 + \dfrac{a^2}{a^2 + \dfrac{1}{a^3 - \dfrac{a^8 + a^8 - 1}{a^5}}}$.

9. Simplify $x + y + \dfrac{1}{x + y + \dfrac{1}{x + y + \dfrac{1}{x + y}}}$.

10. Simplify $(a + b)^2 + \dfrac{1}{(a - b)^2 + \dfrac{1}{(a + b)^2 + \dfrac{1}{(a - b)^2}}}$.

VI. FRACTIONS OF THE FORM $\frac{0}{0}$, $\frac{a}{0}$, AND $\frac{a}{\infty}$.

166. By the definition of fraction (§ 149) expressions of division in which the divisor (denominator) is zero were excluded. An interpretation of this exceptional case will now be considered.

When the absolute value of a variable quantity can exceed any given positive number, the quantity is said to increase *without limit*, or indefinitely.

E.g., in the series $\frac{1}{1}, \frac{1}{0.1}, \frac{1}{0.01}, \frac{1}{0.001}, \cdots$, the values of the successive terms are 1, 10, 100, ⋯ Hence, as the absolute values of the denominators are getting smaller, the absolute values of the fractions are getting larger and may be made to increase without limit.

The symbol for an infinitely great quantity is ∞, read "**infinity.**" This symbol must not, however, be understood to have a definite numerical meaning. It is merely an abbreviation for "a quantity whose absolute value has increased beyond any assignable limit."

Hence,
$$\infty + a = \infty,$$
$$a \cdot \infty = \infty,$$
and
$$\frac{\infty}{a} = \infty.$$

In fact, the symbol ∞ is not subject to any of the common laws of numbers.

167. If a is a constant finite quantity, the absolute value of $\frac{a}{x}$ can be made as small as we please by increasing x sufficiently. That is, $\frac{a}{x}$ can be brought as near 0 as we please. This is expressed by saying that the *limit* of $\frac{a}{x}$, as x increases indefinitely, is 0.

This is written, $\frac{a}{x} \doteq 0$ as x increases without limit, the symbol $\doteq$ being read "approaches as its limit."

168. The form $\frac{0}{0}$. The fraction $\frac{x^2 - a^2}{x - a}$ has a meaning for all values of x except $x = a$. But $\frac{x^2 - a^2}{x - a} \equiv \frac{x - a}{x - a} \cdot (x + a)$, and as $x \doteq a$ it is evident that $\frac{x - a}{x - a} \cdot (x + a) \doteq 1 \cdot (a + a) = 2a$.

Similarly, $\frac{x^2 - 1}{x - 1} \equiv \frac{x - 1}{x - 1} \cdot (x + 1)$, which $\doteq 2$ as $x \doteq 1$;

$\frac{x^2 - 4x + 4}{x - 2} \equiv \frac{x - 2}{x - 2} \cdot (x - 2)$, which $\doteq 0$ as $x \doteq 2$;

$\frac{x^3 - 1}{x - 1} \equiv \frac{x - 1}{x - 1} \cdot (x^2 + x + 1)$, which $\doteq 3$ as $x \doteq 1$.

But all these fractions approach the form $\frac{0}{0}$ as x approaches the limit assigned, and in the several cases the fractions approach different limits. And since the limits are undetermined at first sight, $\frac{0}{0}$ *is said to stand for an undetermined expression.*

This is commonly expressed by saying that $\frac{0}{0}$ is *indeterminate.* The limit, however, can often be determined by simple inspection.

169. The fact that the limit of $\frac{x^2 - 1}{x - 1}$ is 2 as $x \doteq 1$ is expressed in symbols thus:

$$\left.\frac{x^2 - 1}{x - 1}\right]_1 \doteq 2.$$

EXERCISES. LXXII.

Find the limit of each of the following expressions:

1. $\left.\frac{x^3 - 1}{x - 1}\right]_1$.

2. $\left.\frac{x^5 - 32}{x - 2}\right]_2$.

3. $\left.\frac{x^2 - a^2}{x - a}\right]_a$.

4. $\left.\frac{6x^2 - 13x + 6}{3x^2 + x - 2}\right]$

5. $\left.\dfrac{x^2+2x-8}{x-2}\right]_2$.

6. $\left.\dfrac{x^3-2x^2-5x+6}{x^2-4x+3}\right]_1$.

7. $\left.\dfrac{x^2+2x-8}{x+4}\right]_{-4}$.

8. $\left.\dfrac{x^3+2x^2+2x+1}{x+1}\right]_{-1}$.

170. The form $\frac{a}{0}$. This form should be interpreted to mean an expression whose absolute value is infinite.

For in the fraction $\frac{a}{x}$, as $x \doteq 0$ the absolute value of the fraction increases without limit.

Hence, the symbol $\frac{a}{0}$, while without meaning by the common notion of division, is interpreted to mean infinity.

171. The form $\frac{a}{\infty}$. This form should be interpreted to mean an expression whose absolute value is zero.

For as x increases without limit, $\frac{a}{x} \doteq 0$.

172. The form $\infty \cdot 0$. This form should be interpreted to mean an undetermined (indeterminate) expression. (Why?)

173. The relation of these forms to checks. The student has been cautioned against substituting any arbitrary values which make the denominator of a fraction zero. The reason is now apparent.

E.g., $\dfrac{1}{x-1}-\dfrac{2}{x^2-1}=\dfrac{1}{x+1}$. If $x=1$, this reduces to $\infty-\infty=\frac{1}{2}$, which checks nothing because ∞ has no exact numerical value.

Similarly in the case of

$$\frac{a^2-b}{a-b^2}+\frac{a-b^2}{a^2-b}=\frac{a^4+b^4+a^2+b^2-2ab(a+b)}{(a-b^2)(a^2-b)}.$$

If $a=b=1$, this reduces to $\frac{0}{0}+\frac{0}{0}=\frac{0}{0}$, which checks nothing because $\frac{0}{0}$ has no exact numerical value. But if $a=2$ and $b=1$, this reduces to $3+\frac{1}{3}=\frac{10}{3}$.

EXERCISES. LXXIII.

1. How should $\frac{\infty}{a}$ be interpreted? Why?

2. Also $\frac{\infty}{\infty}$.

3. Find the limit of $\left.\frac{x^3 - 2x^2 - 5x + 6}{x^2 - 4x + 3}\right]_3$.

4. Add $\frac{a}{(a-b)(a-c)}$, $\frac{b}{(b-c)(b-a)}$, $\frac{c}{(c-a)(c-b)}$.
What arbitrary values are excluded in the check? Why?

5. Similarly with $\frac{a-b}{b} + \frac{ab^2}{a^3 - b^3} + \frac{a^6}{a^3 - b^6}$. Why is it no check to let $a = b =$ any number?

6. Similarly with
$$\frac{1}{a-b} + \frac{1}{b-c} + \frac{1}{c-a} + \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2(a-b)(b-c)(c-a)}.$$

7. Similarly with
$$\left[\frac{a}{(a+b)^2} - \frac{b}{a^2 - b^2} + \frac{2b^2}{(a+b)^2(a-b)}\right] \cdot \frac{a+b}{a-b}.$$

8. Show that
$$\frac{1}{1-a} - \frac{1}{1+a} - \frac{2a}{1+a^2} + \frac{2a^2}{(1-a)(1+a^2)} - \frac{2a^2}{(1+a)(1+a^2)} = \frac{8a^3}{1-a^4}.$$

9. Verify the following identity, (1) by actually adding, (2) by the substitution of arbitrary values.
$$\frac{x^2y^2z^2}{b^2c^2} + \frac{(x^2-b^2)(y^2-b^2)(z^2-b^2)}{b^2(b^2-c^2)} + \frac{(x^2-c^2)(y^2-c^2)(z^2-c^2)}{c^2(c^2-b^2)} \equiv x^2 + y^2 + z^2 - b^2 - c^2.$$

10. Also $\frac{y^2z^2}{b^2c^2} + \frac{(y^2-b^2)(z^2-b^2)}{b^2(b^2-c^2)} + \frac{(y^2-c^2)(z^2-c^2)}{c^2(c^2-b^2)} \equiv 1.$

REVIEW EXERCISES. LXXIV.

1. What is the value of $\frac{x}{a}+\frac{x}{b-a}-\frac{a}{a+b}$, when $x=\frac{a^2(b-a)}{b(b+a)}$?

2. Show why the arithmetic definition of fraction is not sufficient for algebra.

3. Simplify the expression $\frac{a}{b}-\frac{a}{b}\left[\frac{b}{a}+\frac{a}{b}\left(\frac{b}{a}+1\right)\right]$.

4. Extract the square root of

(a) $9x^6/4-3x^4+11x^2/5-4/5+4/25x^2$.

(b) $1+4/x+20/x^3+25/x^4+10/x^2+24/x^5+16/x^6$.

(c) $178/7-20x/7y+9y^2/16x^2+4x^2/49y^2-15y/2x$.

5. Extract the cube root of

(a) $8x^3/a^3+48x^2/a^2+96x/a+64$.

(b) $\frac{1}{8}a^3-\frac{1}{4}a^2b-\frac{1}{27}b^3+\frac{3}{8}a^2c+\frac{1}{6}ab^2+\frac{1}{6}b^2c+\frac{1}{8}c^3+\frac{3}{8}ac^2-\frac{1}{4}bc^2-\frac{1}{2}abc$.

(c) $1-3x/2+3x^2/2-5x^3/4+3x^4/4-3x^5/8+5x^6/32-3x^7/64+3x^8/256-x^9/512$.

6. Prove that the sum of two quantities, divided by the sum of their reciprocals, equals the product of the quantities.

7. Show that by substituting $3(x+1)/(x-3)$ for x in either of the expressions $(3-4x+x^2)/(3+x^2)$, $2(3+x)/(3+x^2)$, it becomes identical with the other.

8. Raise $\frac{a}{b}-\frac{b}{a}$ to the fourth power. Check.

9. Raise $\frac{2a^2}{b}+\frac{b}{2a^2}$ to the sixth power. Check.

CHAPTER IX.

SIMPLE EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

I. GENERAL LAWS GOVERNING THE SOLUTION.

174. An **equation** has already been defined (§ 16) as an equality which exists only for particular values of certain letters, called the **unknown quantities.**

E.g., $x^2 = 4$ exists only for the two values $x = +2$ and $x = -2$.

175. An equation is said to be **rational, irrational, integral,** or **fractional,** according as the two members, when like terms are united, are composed of expressions which are rational, irrational (or partly so), integral, or fractional (or partly so), respectively, with respect to the unknown quantities.

E.g., $x + \sqrt{5} = 0$ is a rational integral equation;

$5 + \frac{1}{2}\sqrt{x} = 0$ is an irrational integral equation;

$\frac{1}{x} + 4 = x$ is a rational fractional equation;

$\frac{1}{(x+2)^{\frac{1}{2}}} = 4$ is an irrational fractional equation.

176. A rational integral equation which, when its like terms are united, contains no term of degree higher than the first with respect to the unknown quantities, is called a **simple** or a **linear equation.**

E.g., $x - 3 = 5$, $x^2 + x - 1 = x^2 + y$, are simple or linear equations. But $\sqrt{x} = 5$, $\frac{1}{x} = 2$, are not *as they now stand.*

177. Equations which are not simple are, however, often solved by the principles which govern the solution of simple equations.

E.g., $(x-1)(x-2)=0$ is an equation of the second degree. (Why?) But it is satisfied only if

$$x-1=0,$$

or if

$$x-2=0,$$

that is, if $x=1$, or if $x=2$. Hence, the solution of this equation of the second degree reduces to the solution of two linear equations.

EXERCISES. LXXV.

1. What is meant by the *roots* of an equation? (See § 16.) What are the two roots of the equation $x^2=25$?

2. What is meant by *solving* an equation? Solve the equations.

(a) $3x+5=0$. (b) $(x-2)(x-3)=0$.

(c) $(x+1)(x+2)=0$. (d) $(x+2)(x-3)=0$.

3. What is meant by an equation being *satisfied?* What values of x satisfy these equations?

(a) $(x+\frac{1}{2})(3x-2)=0$.

(b) $(2x-1)(2x+3)=0$.

(c) $x(x-1)(x-2)(x-3)=0$.

4. What is meant by the *members* of an equation? How do they differ from the terms?

5. Which of the following are simple equations with respect to x?

(a) $x^3+x^2+x-x^2-x^3=4$.

(b) $3x^2+x+7=2x^2+x(x+3)$.

(c) $\sqrt{x}+4=7$. (d) $\dfrac{1}{x}-\dfrac{x}{2}=3$.

(e) $x^2-x+1=0$. (f) $x(x+1)=x^2$.

178. Known and unknown quantities. It is the custom to represent the *unknown quantities* in an equation by the *last letters* of the alphabet, particularly by x, y, z.

This custom dates from Descartes, 1637.

179. Quantities whose values are supposed to be *known* are generally represented by the *first letters* of the alphabet, as by $a, b, c, \cdots$.

E.g., in the equation $ax + b = 0$, a and b are supposed to be known. Dividing both members by a, $x + b/a = 0$, which is satisfied if $x = -b/a$.

180. The *solution* of the simple equation has already been explained (§ 17). The general case, not involving fractional coefficients, will be understood from two illustrative problems and the series of questions in the following exercises.

1. Given the equation $5x - 2 = 3x + 8$, to find the value of x.

1.	$5x - 2 = 3x + 8$.	Given.	
2.	$5x = 3x + 10$.	Adding 2.	Ax. 2
3.	$2x = 10$.	Subtracting $3x$.	Ax. 3
4.	$x = 5$.	Dividing by 2.	Ax. 7

Check. Substitute 5 for x in the *original equation*, and

$$25 - 2 = 15 + 8.$$

2. Given the equation $2ax - a^2 = ax + 3a^2$, to find the value of x.

1.	$2ax - a^2 = ax + 3a^2$.	Given.	
2.	$2ax = ax + 4a^2$.	Adding a^2.	Ax. 2
3.	$ax = 4a^2$.	Subtracting ax.	Ax. 2
4.	$x = 4a$.	Dividing by a.	Ax. 7

Check. Substitute $4a$ for x in the original equation, and

$$8a^2 - a^2 = 4a^2 + 3a^2.$$

181. From these illustrative problems it will be observed that any term may be transferred from one member of an equation to the other if its sign is changed. This operation is called **transposition.**

E.g., if $x + 2 = 7$, transposing 2 we have

$$x = 7 - 2,$$

or $$x = 5.$$

It should be remembered, however, that the operation is really one of subtraction, 2 being taken from each member by ax. 3.

In general, transposition is always an operation of subtraction or addition.

EXERCISES. LXXVI.

The answers to the following questions will lead to the understanding of the steps to be taken in the solution of linear equations with one unknown quantity.

1. In which member do you seek to place the known quantities, and in which the unknown? Might this be changed about? What axioms are involved in this operation? (See ex. 1, steps 2 and 3, p. 147.)

2. Having done this, what is the next operation? What axiom is involved? (See ex. 1, step 4, p. 147.)

3. State, then, the two general steps to be followed in solving a linear equation with one unknown quantity.

4. How is the work checked?

Solve the following equations, checking the results.

5. $12x - 28 = 8 + 3x.$

6. $17 - x = 2x - 1.$

7. $27x - 127 = 11 - 19x.$

8. $2x + 3 = 4x + 5.$

9. $4x - 34 = 22 - 3x.$

10. $x + 2 + 3x + 4 = 5x + 6.$

11. $3x + 4x + 5x = 6x + 72.$

182. The axioms applied to the solution of equations. While it is true that the solutions of equations depend upon certain axioms (§ 22), it is necessary to consider the precise limitations of these axioms before proceeding further.

183. Two equations are said to be **equivalent** when all of the roots of either are roots of the other.

E.g., $x + 4 = 2x - 5$,

and $x + 5 = 2(x - 2)$, are equivalent equations,

for $x = 9$ is a root and the only root of each.

But $x = 2$ and $x^2 = 4$ are not equivalent equations, for -2 is a root of $x^2 = 4$, but not of $x = 2$.

The necessity for a consideration of the limitations of the use of the axioms is seen from the following:

Suppose 1. $x = 2$.

Then 2. $x^2 = 4$, by ax. 8.

But a root of equation 2 is not necessarily a root of equation 1, because *while equation* 2 *is true when equation* 1 *is true, it is not equivalent to equation* 1.

184. Axioms 6 and 7. *If equals are multiplied or divided by equals, the results are equal.*

This is true, but *it must not be interpreted to mean that if the two members of an equation are multiplied by equals, the resulting equation is equivalent to the given one.*

E.g., if the two members of the equation

$$x - 1 = 5$$

are multiplied by $x - 2$, we have

$$(x - 1)(x - 2) = 5(x - 2),$$

or $$x^2 - 8x + 12 = 0,$$

or $$(x - 6)(x - 2) = 0,$$

which has two roots, $x = 2$ and $x = 6$. Of these, only $x = 6$ satisfies the original equation. Hence, the resulting equation is not equivalent to the original one; there has been a new root introduced.

185. A new root which appears in performing the same operation upon both members of an equation is called an **extraneous root.**

EXERCISES. LXXVII.

What, if any, extraneous roots are introduced by multiplying both members of the following equations as indicated?

1.	$x + 2 = 5$	by	$x + 3$.
2.	$x - a = 0$	"	$x + a$.
3.	$x^2 - 1 = 0$	"	$x^2 - 5x + 6$.
4.	$x - 2 = 4x + 1$	"	3.
5.	$x - 5 = 5x - 21$	"	x.
6.	$x^2 + x = (1 - x)^2$	"	x.
7.	$3x - 4 = 4x - 3$	"	21.
8.	$(x + a)^2 = (x - a)^2$	"	$x + 1$.

186. Just as an extraneous root may be introduced by multiplying both members of an equation by equals, so a root may also be lost by the same process.

E.g., it is not permissible to multiply the two members of the equation

$$(x - 6)(x - 2) = 0$$

by $\frac{1}{x - 2}$, expecting thereby to obtain an equivalent equation, for we should have

$$x - 6 = 0,$$

which has only a single root, $x = 6$, whereas the original equation had two roots, $x = 6$ and $x = 2$. Hence, the resulting equation is not equivalent to the original one; a root has been lost by multiplying equals by equals.

In the same way, while if we multiply both members of the equation

$$x^3 - x = 0$$

by $\frac{1}{x}$, $\frac{1}{x + 1}$, $\frac{1}{x - 1}$, or $\frac{1}{x^2 - 1}$, the results will be equal, it is not true that we shall obtain equivalent equations.

187. Hence, it appears that *multiplying the two members of an equation* f(x) = 0 *by a function of* x *does not, in general, give an equivalent equation.* The operation may introduce an extraneous root, or it may suppress a root.

It should also be stated, in connection with extraneous roots, that no value is considered a root unless it makes the members identically equal. Hence, *a value that makes both members infinite is not a root,* for infinity is not identically equal to infinity (§ 166).

E.g., 1 is not a root of $\frac{2}{x-1} = \frac{x}{x-1}$, for it makes each member infinite.

EXERCISES. LXXVIII.

Would each resulting equation be equivalent to the given one, by multiplying both members as indicated below?

1. $\frac{1}{x} = 4$ by $\frac{x}{4}$.
2. $\frac{x^2-4}{x-2} = 0$ " $x-2$.
3. $\frac{x}{3} + \frac{3}{x} + \frac{2}{x} = 0$ " x.
4. $\frac{x^2}{x-1} = \frac{1}{x-1}$ " $x-1$.
5. $\frac{x^2-6x+5}{x^2-5x+4} = 5$ " x^2-5x+4.
6. $x^2-5x+6=0$ " $1/(x-2)$.
7. $2x^2-5=x^2-1$ " $1/(x+2)$.
8. $x^2-3x-28=0$ " $1/(x-7)$.
9. $\frac{x^2+9x+14}{x^2+14x+49} = \frac{1}{x+7}$ " $(x^2+14x+49)(x+7)$.
10. Also by $x^2+14x+49$; also by $x+7$.

188. Axioms 8 and 9. *Like powers or like roots of equals are equal.*

This is true, but *it must not be interpreted to mean that the equation formed by taking like powers or like roots of the members of a given equation is equivalent to that equation.*

E.g., if $x = 1$,

then $x^2 = 1$, or $x^2 - 1 = 0$, or $(x + 1)(x - 1) = 0$;

but this equation has two roots, $x = -1$, and $x = +1$, and of these, only $x = +1$ satisfies the original equation.

Similarly, if $x^2 = 4$,

it is true that $x = 2$;

but this equation is not equivalent to the original one. It should be written $x = +2$, and -2.

Students are liable to make a mistake by omitting the $\pm$ sign in extracting a square root, thus losing a root.

E.g., in the equation

$$x^2 + 2x + 1 = 4,$$

extracting the square root, $x + 1 = 2$,

$$\therefore x = 1.$$

It should be $x + 1 = +2$ and -2,

$$\therefore x = 1 \text{ and } -3.$$

EXERCISES. LXXIX.

What extraneous roots are introduced by squaring both members of the following equations?

1. $x = 0$.
2. $x + 3 = 3$.
3. $x - 2 = 2$.
4. $2x = 9$.
5. $x - 5 = 0$.
6. $4x = -28$.
7. $2x + 1 = 3$.
8. $\frac{x}{2} = 1$.
9. $\frac{x}{3} + 1 = 2$.

The discussion already given may be set forth in four theorems. These theorems, with strict proofs, may be found too abstract for most beginners, and hence they are given in Appendix V.

II. SIMPLE INTEGRAL EQUATIONS.

189. General directions for solution. From the suggestions in exs. 1–4, on p. 148, it appears that, to solve a simple integral equation, we

1. *Transpose the terms containing the unknown quantities to the first member, changing the signs* (axs. 2, 3, §§ 22, 181);
2. *Transpose the terms containing only known quantities to the second member, changing the signs;*
3. *Unite terms;*
4. *Divide by the coefficient of the unknown quantity;*
5. *Check the result by substituting in the original equation.*

EXERCISES. LXXX.

Solve the following equations:

1. $ax + b = bx + a$.
2. $(x - 1)(1 - x) = -x^2$.
3. $8x - (7 - x) = 29$.
4. $3x - 2(2 - x) = 21$.
5. $(2 - x)(5 - x) = x^2$.
6. $3(x - 1) = 4(x + 1)$.
7. $9x - 3(5x - 6) = -30$.
8. $2(x + 3) - 3(x + 2) = 0$.
9. $x(x^2 + 1) = x(x^2 - 1) + 9$.
10. $(x + 5)^2 = 21x + (4 - x)^2$.
11. $3x + 14 - 5(x - 3) = 4(x + 3)$.
12. $x(x - 1) - x(x - 2) = 2(x - 3)$.
13. $x(1 + x + x^2) = x^3 + x^2 + 3x - 17.5$.
14. $(x + 1)(x + 2) = (x + 3)(x + 4) - 50$.
15. $2(x + 1) + 3(x + 2) + 4(x + 3) = 101$.
16. $(x - 2)^2 - (x - 3)^2 = (x - 4)^2 - (x - 6)^2$.

III. SIMPLE FRACTIONAL EQUATIONS.

190. If the equation contains fractions, these may be removed by multiplying both members by the lowest common multiple of the denominators. This is called **clearing the equation of fractions.**

Unless the fractions are in their lowest terms before multiplying by the lowest common denominator, an extraneous root is liable to be introduced (§ 187).

It is not always advisable, however, to clear the equation of fractions at once, as is seen in the following illustrative problems.

Illustrative problems. 1. An equation which should be cleared of fractions at once: $\frac{x-3}{15}+\frac{x+7}{17}=8.$

1. The l.c.d. of the fractions is $15 \cdot 17$.
2. Multiplying both members by $15 \cdot 17$, by ax. 6,

$$17x - 51 + 15x + 105 = 15 \cdot 17 \cdot 8.$$

3. Subtracting $-51+105$, and uniting terms,

$$32x = 1986.$$

4. Dividing by 32,

$$x = 62\tfrac{1}{16}.$$

Check. $\frac{59\frac{1}{16}}{15}+\frac{69\frac{1}{16}}{17}=8$, or $3\frac{15}{16}+4\frac{1}{16}=8$.

2. An equation which need not be cleared of fractions at once: $x-\frac{x}{3}-\frac{3}{4}=\frac{1}{2}.$

1. Adding $\frac{3}{4}$ and uniting terms,

$$\tfrac{2}{3}x = \tfrac{5}{4}.$$

2. Multiplying both members by $\frac{3}{2}$ (or dividing both members by $\frac{2}{3}$),

$$x = \tfrac{15}{8}.$$

Check. $\frac{15}{8}-\frac{5}{8}-\frac{3}{4}=\frac{1}{2}.$

3. An equation which should be cleared of fractions part at a time: $\frac{3x+7}{15} - \frac{2x-4}{7x-12} = \frac{x+1}{5}$.

1. Multiplying both members by 15,

$$3x + 7 - \frac{15(2x-4)}{7x-12} = 3x + 3. \qquad \text{Ax. 6}$$

2. Transposing and uniting terms,

$$4 = \frac{15(2x-4)}{7x-12}. \qquad \text{Axs. 2, 3}$$

3. Dividing by 2 and multiplying by $7x - 12$,

$$14x - 24 = 15x - 30. \qquad \text{Axs. 6, 7}$$

4. Adding $24 - 15x$,

$$-x = -6. \qquad \text{Ax. 2}$$

5. Multiplying by -1,

$$x = 6. \qquad \text{Ax. 6}$$

Check. $\frac{25}{15} - \frac{8}{30} = \frac{7}{5}$.

4. An equation in which the fractions may be united to advantage before clearing: $\frac{x}{x-2} - \frac{x+1}{x-1} = \frac{x-8}{x-6} - \frac{x-9}{x-7}$.

1. Adding the fractions in each member separately,

$$\frac{x^2 - x - x^2 + x + 2}{(x-2)(x-1)} = \frac{x^2 - 15x + 56 - x^2 + 15x - 54}{(x-6)(x-7)}.$$

2. $\therefore$

$$\frac{2}{(x-2)(x-1)} = \frac{2}{(x-6)(x-7)}.$$

3. Dividing by 2 and clearing of fractions,

$$x^2 - 13x + 42 = x^2 - 3x + 2.$$

4. $\therefore$ $$-10x = -40. \qquad \text{(Why ?)}$$

5. $\therefore$ $$x = 4. \qquad \text{(Why ?)}$$

Check. $\frac{4}{2} - \frac{5}{3} = \frac{-4}{-2} - \frac{-5}{-3}$.

It will be noticed, in step 1, that *the bar of a fraction is a symbol of aggregation*, and in adding fractions or in clearing of fractions this must be taken into account.

This example may also be treated like the following one if desired.

5. An equation in which the fractions should be reduced to mixed expressions before clearing:

$$\frac{5x-8}{x-2}+\frac{6x-44}{x-7}=\frac{x-8}{x-6}+\frac{10x-8}{x-1}.$$

1. Reducing to mixed expressions,

$$5+\frac{2}{x-2}+6-\frac{2}{x-7}=1-\frac{2}{x-6}+10+\frac{2}{x-1}.$$

2. Subtracting 11 and dividing by 2,

$$\frac{1}{x-2}-\frac{1}{x-7}=-\frac{1}{x-6}+\frac{1}{x-1}.$$

3. Adding the fractions in each member separately,

$$\frac{-5}{(x-2)(x-7)}=\frac{-5}{(x-6)(x-1)}.$$

4. Dividing by -5, and clearing,

$$x^2-7x+6=x^2-9x+14.$$

5. $\therefore$ $2x=8.$ (Why?)

6. $\therefore$ $x=4.$ (Why?)

Check. $\frac{12}{2}+\frac{-20}{-3}=\frac{-4}{-2}+\frac{32}{3}$, or $6+6\frac{2}{3}=2+10\frac{2}{3}$.

EXERCISES. LXXXI.

First determine which seems the best method of solving each of the following equations; then solve and (except as the teacher otherwise directs) check the solution by substituting the roots in the original equation.

1. $\frac{ab}{ax+bx}=1.$

2. $\frac{x-1}{x+1}=\frac{x-6}{x-3}.$

3. $\frac{50}{4x}+\frac{12}{x}=\frac{49}{10}.$

4. $\frac{2x^2}{x-3}=2x+15.$

5. $\frac{x}{2}+\frac{x}{3}=13-\frac{x}{4}.$

6. $0.5x+0.25x=1.5.$

7. $\frac{x}{a} = x - a + \frac{1}{a}.$

8. $\frac{x}{5} + \frac{x}{8} = 17 - \frac{x}{10}.$

9. $\frac{4x}{3} - \frac{5x}{7} = x - 8.$

10. $\frac{x-a}{x-b} = \frac{(2x-a)^2}{(2x-b)^2}.$

11. $\frac{1 + \frac{4bx}{a}}{bx} = \frac{\frac{4a}{bx} + 1}{a}.$

12. $\frac{x-a}{x+a} + \frac{3b-x}{2b+x} = 0.$

13. $\frac{6x+7}{12} = \frac{3x-4}{4x-3} + \frac{x}{2}.$

14. $\frac{x-2}{3} - \frac{x-4}{5} = \frac{x-6}{7}.$

15. $\frac{\frac{x}{5} + \frac{1}{2}}{3} - \frac{x - \frac{x}{2}}{2} + \frac{x}{5} = 0.$

16. $1 - \frac{a}{a + x + \frac{x^2}{a-x}} = \frac{b}{a}.$

17. $\frac{1}{x-3} + \frac{3}{x-9} = \frac{4}{x-6}.$

18. $\frac{b+x}{b+a} + \frac{b-x}{b-a} = \frac{b^2-ax}{a^2-b^2}.$

19. $\frac{5x+10.5}{x+0.5} + \frac{2x}{2x+1} = 9.$

20. $\frac{\frac{2}{3}}{\frac{2}{3}+x}-\frac{2}{3}=\frac{2}{3}-\frac{\frac{2}{3}x+\frac{2}{3}}{\frac{2}{3}+x}.$

21. $\frac{x}{2}-\frac{5x+4}{3}=7-\frac{8x-2}{3}.$

22. $\frac{8}{x+3}-\frac{9}{2x+6}=\frac{15}{7x+2}.$

23. $\frac{7x+5}{6}-\frac{5x-6}{4}=\frac{8-5x}{12}.$

24. $1=\frac{a}{b}\left(1-\frac{a}{x}\right)+\frac{b}{a}\left(1-\frac{b}{x}\right).$

25. $\frac{2x-5}{6}+\frac{6x+3}{4}=5x-\frac{35}{2}.$

26. $\frac{2x+3}{5}-\frac{6x+22}{15}=\frac{3x+17}{5(1-x)}.$

27. $\frac{1}{a-b}+\frac{a-b}{x}=\frac{1}{a+b}+\frac{a+b}{x}.$

28. $\frac{x+4a+b}{x+a+b}+\frac{4x+a+2b}{x+a-b}=5.$

29. $\frac{1}{x-1}-\frac{2}{x-2}=\frac{3}{x-3}-\frac{4}{x-4}.$

30. $\frac{x+3}{x+1}+\frac{x-6}{x-4}=\frac{x+4}{x+2}+\frac{x-5}{x-3}.$

31. $\frac{1}{(x+3)(x+5)}=\frac{1}{(x+9)(x-5)}.$

32. $\frac{1+3x}{5+7x}-\frac{9-11x}{5-7x}=\frac{14(2x-3)^2}{25-49x^2}.$

33. $\frac{2}{2x-1}-\frac{1}{x-3}-\frac{1}{x}+\frac{2}{2x-5}=0.$

34. $5x - \dfrac{2x-1}{3} + 1 = 3x + \dfrac{x+2}{2} + 7.$

35. $\frac{1}{3}\{\frac{1}{3}[\frac{1}{3}(\frac{1}{3}\,\overline{\tfrac{1}{3}x - 1}) - 1] - 1\} - 1 = 0.$

36. $\dfrac{5}{5x+8} - \dfrac{4}{2x+3} = \dfrac{1}{5}\left(\dfrac{3}{x+3} - \dfrac{8}{x+2}\right).$

37. $1 - \dfrac{x}{2}\left(1 - \dfrac{3}{4x}\right) = \dfrac{5x}{6}\left(7 - \dfrac{6}{7x}\right) - 35\frac{51}{56}.$

38. $\dfrac{3x^2 - 2x + 1}{5} = \dfrac{(7x-2)(3x-6)}{35} + \dfrac{9}{10}.$

39. $\dfrac{a(a+b)x}{a^2-b^2} - \dfrac{a^2-b^2}{a+b} - \dfrac{2bx}{b-a} = \dfrac{(5a+b)b}{a-b}.$

40. $\frac{1}{2}x - \frac{3}{4} + \frac{5}{6}x - \frac{7}{8} = \frac{9}{10} + \frac{11}{12}x - \frac{13}{14} - \frac{15}{16}x.$

41. $\dfrac{13x-10}{36} + \dfrac{4x+9}{18} - \dfrac{7(x-2)}{12} = \dfrac{13x-28}{17x-66}.$

42. $\dfrac{a}{x+1} - \dfrac{2(3a+5)}{x-1} + \dfrac{8a+15}{x-2} = \dfrac{3(a+2)}{x-3} - \dfrac{1}{x+2}.$

43. $\frac{1}{15}(2x-1) - \frac{1}{16}(3x-2) = \frac{1}{18}(x-12) - \frac{1}{24}(x+1).$

44. $\dfrac{(a+b)^2(x+1) + (a+b)(x+1) + (x+1)}{a+b+1} = (a+b)^2 + (a+b) + 1.$

45. $\dfrac{x+1}{3} - \dfrac{1}{2}\left(x - \dfrac{x+1}{2}\right) = \dfrac{x-2}{5} - \dfrac{1}{3}\left(x - \dfrac{x+3}{2}\right) + \dfrac{31}{60}.$

46. $\dfrac{3x^5 + 12x^4 + 44x^3 + 185x^2 + 8x + 98}{3x^4 + 18x^3 + 26x^2 + 15x + 14} = \dfrac{3x^3 + 44x + 2}{3x^2 + 6x + 2}.$

IV. IRRATIONAL EQUATIONS SOLVED LIKE SIMPLE EQUATIONS.

191. It often happens that irrational equations can be reduced to equivalent simple equations and thus solved.

E.g., $\sqrt{x} = 2$ can be reduced to the equivalent simple equation $x = 4$. In applying ax. 8 it is possible, however, that extraneous roots may be introduced (§ 185). That this is not the case in this instance is seen by substituting the value of x in the original equation.

192. A question at once arises, however, in dealing with equations like

$$\sqrt{x^2 + 2x + 1} + \sqrt{x^2 - 2x + 1} = 4.$$

Shall this be reduced to

$$\pm(x + 1) \pm (x - 1) = 4,$$

or shall only the positive roots be considered, as in

$$x + 1 + x - 1 = 4\,?$$

The former would give $x = \pm 2$, the latter only $x = 2$.

To answer this question, let $\sqrt{f(x)}$ and $\sqrt{F(x)}$, for brevity, represent the square roots of any two functions of x, like those already mentioned.

Then it is evident that an irrational equation of the form

$$\sqrt{f(x)} + \sqrt{F(x)} = a$$

involves four equations, *viz.*:

1. $+\sqrt{f(x)} + \sqrt{F(x)} = a;$
2. $+\sqrt{f(x)} - \sqrt{F(x)} = a;$
3. $-\sqrt{f(x)} + \sqrt{F(x)} = a;$
4. $-\sqrt{f(x)} - \sqrt{F(x)} = a,$

where $\sqrt{f(x)}$ and $\sqrt{F(x)}$ represent, in these four equations, only the positive square roots.

This is also seen in the case of $\sqrt{4} + \sqrt{9}$, which equals $(\pm 2) + (\pm 3)$.

193. Hence, any root which satisfies any one of the four equations is strictly a root of $\sqrt{f(x)} + \sqrt{F(x)} = a$.

By convention, however, only the roots which satisfy equation 1 *are usually considered.*

For example, consider the equation $\sqrt{x-2} + \sqrt{x-5} = 1$.

1. $\sqrt{x-5} = 1 - \sqrt{x-2}$. Ax. 3
2. $\therefore$ $x - 5 = 1 + x - 2 - 2\sqrt{x-2}$. Ax. 8
3. $\therefore$ $2\sqrt{x-2} = 4$. Ax. 3
4. $\therefore$ $x - 2 = 4$. Axs. 7, 8
5. $\therefore$ $x = 6$.

Substituting 6 for x in the given equation,

$$\sqrt{4} + \sqrt{1} = 1, \text{ or } (\pm 2) + (\pm 1) = 1.$$

While this is true in the form $(+2) + (-1) = 1$, the root 6, by the convention just given, is usually called extraneous.

194. Irrational equations can often be solved by isolating the radical and then applying ax. 8. For example, consider the equation $\sqrt{x-2} - \sqrt{x-5} = 1$.

1. We first isolate the radical $\sqrt{x-2}$, by adding $\sqrt{x-5}$ to each member.

2. $\therefore$ $\sqrt{x-2} = 1 + \sqrt{x-5}$.

3. Then, by squaring both members,

$$x - 2 = 1 + x - 5 + 2\sqrt{x-5}.$$

4. Then, isolating the radical $\sqrt{x-5}$, by subtracting $1 + x - 5$ and dividing by 2,

$$1 = \sqrt{x-5}.$$

5. $\therefore$ $1 = x - 5$, whence $x = 6$.

Check. $\sqrt{6-2} - \sqrt{6-5} = 1$.

195. If the equation contains several irrational expressions, there is no general rule for solution. The student must use his judgment as to which radical it is best to isolate first.

Illustrative problems. 1. Solve the equation

$$\sqrt{x+1} - 4\sqrt{x-4} + 5\sqrt{x-7} = 0.$$

1. Isolating the radical $4\sqrt{x-4}$ by adding it to both members, we have:

$$\sqrt{x+1} + 5\sqrt{x-7} = 4\sqrt{x-4}.$$

2. Squaring

$$x + 1 + 25x - 175 + 10\sqrt{x^2 - 6x - 7} = 16x - 64.$$

3. $\therefore$ $x - 11 = -\sqrt{x^2 - 6x - 7}.$ (Why?)

4. $\therefore$ $x^2 - 22x + 121 = x^2 - 6x - 7.$ (Why?)

5. $\therefore$ $x = 8.$ (Why?)

Check. $\sqrt{9} - 4\sqrt{4} + 5\sqrt{1} = 3 - 8 + 5 = 0.$

2. Solve the equation $\sqrt{x} = -2.$

Squaring, $x = 4$. But on substituting 4 for x, $\sqrt{4} = -2$. This satisfies the equation because $\sqrt{x}$ is both $+2$ and -2. But since the positive sign is usually taken with the radical (§ 193), 4 is usually called an extraneous root and the equation is said to be impossible. The equation $-\sqrt{x} = -2$ is not open to the same objection for it is satisfied by $x = 4$.

EXERCISES. LXXXII.

Solve the following equations, designating such roots as are usually called extraneous.

1. $\sqrt{x+2} - \sqrt{x+9} = 7.$

2. $\sqrt{x} + \sqrt{a+x} = a/\sqrt{x}.$

3. $\sqrt{x+19} + \sqrt{x+3} = 8.$

4. $2\sqrt{x-1} + \sqrt{4x+5} = 9.$

5. $\sqrt{8x+5} - 2\sqrt{2x-1} = 1.$

6. $4\sqrt{x+2} - \sqrt{x+7} = 5\sqrt{x-1}.$

V. APPLICATION OF SIMPLE EQUATIONS.

A. Problems Relating to Numbers.

Illustrative problems. 1. The sum of two numbers is 200, and their difference is 50. Find the numbers.

1. Let $x \equiv$ the lesser number.
2. Then $x + 50 =$ the greater number.
3. And $x + x + 50 =$ the sum.
4. But $200 =$ the sum.
5. $\therefore$ $x + x + 50 = 200$.
6. $\therefore$ $x = 75$,

and $x + 50 = 125$.

Check. The sum of 125 and 75 is 200, and their difference is 50.

Always check by substituting in the *problem* instead of the equation, because there may have been an error in forming the equation. The neglect to take this precaution often leads to wrong results.

2. What number must be added to the two terms of the fraction $\frac{7}{23}$ in order that the resulting fraction shall equal $\frac{59}{67}$?

1. Let $x \equiv$ the number to be added.
2. Then $\dfrac{7 + x}{23 + x} = \dfrac{59}{67}$.
3. $\therefore$ $67(7 + x) = 59(23 + x)$. (Why?)
4. $\therefore$ $469 + 67x = 1357 + 59x$.
5. $\therefore$ $8x = 888$. (Why?)
6. $\therefore$ $x = 111$. (Why?)

Check. $\dfrac{7 + 111}{23 + 111} = \dfrac{118}{134} = \dfrac{59}{67}$. That is, if 111 is added to both terms of the fraction $\frac{7}{23}$, the result equals $\frac{59}{67}$.

EXERCISES. LXXXIII.

1. What number is that which when subtracted from 28 gives the same result as when divided by 28?

2. Or, more generally, what number is that which when subtracted from n gives the same result as when divided by n? Check by supposing that $n = 3$, $n = 28$.

3. What number is that which when multiplied by 16 gives the same result as when added to 16?

4. Or, more generally, what number is that which when multiplied by n gives the same result as when added to n? Check by supposing that $n = 2$, $n = 16$.

5. What number is that which when divided by 12 gives the same result as when added to 12?

6. Generalize ex. 5 and check. (See exs. 2, 4.)

7. What number is that which when subtracted from 25 gives the same result as when multiplied by 25?

8. Generalize ex. 7 and check. (See exs. 2, 4, 6.)

9. What number must be added to 3 and 7 so that the first sum shall be $\frac{3}{4}$ of the second?

10. Or, more generally, what number must be added to a and b so that the first sum shall be $\frac{m}{n}$ of the second? Check by supposing that $a = 3$, $b = 7$, $m = 3$, $n = 4$.

11. Determine x, knowing that $a^4 - 5a^2 + 4a - x$ is algebraically divisible by $2a + 1$.

12. Divide the number 121 into two parts such that the greater exceeds the less by 73.

13. Or, more generally, divide the number n into two parts such that the greater exceeds the less by a.

14. Divide the number 121 into three parts such that the first exceeds the second by 85 and the second is four times the third.

15. Divide the number n into three parts such that the first exceeds the second by p and the third by q. Check by letting $n = 10$, $p = 1$, $q = 1$.

16. What is the value of n if $\frac{2a + n}{3n + 69a} = \frac{1}{33}$ when $a = \frac{1}{3}$?

17. If each of the two indicated factors of the two unequal products $52 \cdot 45$ and $66 \cdot 37$ is diminished by a certain number, the two products are equal. What is the number?

18. Divide the number 99 into four parts such that if 2 is added to the first, subtracted from the second, and multiplied by the third, and if the fourth is divided by 2, the results shall all be equal.

19. Or, more generally, divide the number n into four parts such that if a is added to the first, subtracted from the second, and multiplied by the third, and if the fourth is divided by a, the results shall all be equal. Check by letting $n = 10$, $a = 1$.

20. The square of a certain number is 1188 larger than that of 6 less than the number. What is the number?

21. The square of 13 times a certain number, less the square of 3 more than 12 times the number, equals the square of 9 less than 5 times the number. What is the number?

22. What number must be added to each term of the fraction $\frac{a}{b}$ that it may equal the fraction $\frac{c}{d}$? Check by letting $a = 3$, $b = 5$, $c = 9$, $d = 10$.

B. Problems Relating to Common Life.

Illustrative problems. 1. What sum gaining $6\frac{1}{4}\%$ of itself in a year amounts to \$157.50 in 2 yrs.?

1. Let $x \equiv$ the *number* of dollars.
2. Then $6\frac{1}{4}\% x =$ the *number* of dollars of interest for 1 yr.
3. $\therefore \quad x + 2 \cdot 6\frac{1}{4}\% x = 157.50.$ (Why?)
4. $\therefore \quad 1.12\frac{1}{2} x = 157.50.$
5. $\therefore \quad x = 140.$ (Why?)

Check. The interest on \$140 for 2 yrs. at $6\frac{1}{4}\%$ is \$17.50, and hence the amount is \$157.50.

2. The cost of an article is \$17.15, and this is 30% less than the marked price. What is the marked price?

1. Let $x \equiv$ the *number* of dollars of marked price.
2. Then $30\% x =$ the *number* of dollars of discount.
3. $\therefore \quad x - 30\% x = 17.15.$
4. $\therefore \quad 0.7 x = 17.15.$ (Why?)
5. $\therefore \quad x = 24.50.$ (Why?)

Check. \$24.50 less 30% of \$24.50 is \$17.15.

EXERCISES. LXXXIV.

23. What is that sum which diminished by $9\frac{1}{2}\%$ of itself equals \$1538.50?

24. How long will it take an investment of \$6024 to amount to \$7658.01 at $3\frac{1}{2}\%$ simple interest?

25. A man invests $\frac{3}{5}$ of his capital at 4% and the rest at $3\frac{1}{2}\%$, and thus receives an annual income of \$76. What is his capital?

26. A man invests one-fourth of his capital at 5%, one-fifth at 4%, and the rest at 3%, and thus secures an annual income of \$3700. What is his capital?

27. A train traveling 30 mi. per hour takes $2\frac{3}{8}$ hrs. longer to go from Detroit to Chicago than one which goes $\frac{1}{8}$ faster. What is the distance from Detroit to Chicago?

28. A loaned to B a certain sum at $4\frac{1}{2}\%$, and to C a sum \$200 greater at 5%; from the two together he received \$276 per annum interest. How much did he lend each?

29. The interest for 8 yrs. upon a certain principal is \$1914, the rate being $3\frac{1}{4}\%$ for the first year, $3\frac{1}{2}\%$ for the second, $3\frac{3}{4}\%$ for the third, and so on, increasing $\frac{1}{4}\%$ each year. What is the principal?

30. A bicyclist traveling a miles per hour is followed, after a start of m mi., by a second bicyclist traveling b mi. per hour, $b > a$. At these rates, in how many hours after the second starts will he overtake the first?

31. A capitalist has $\frac{2}{5}$ of his money invested in mining stocks which pay him 13%, $\frac{1}{3}$ in manufacturing which pays him 9%, and the balance in city bonds which pay him 3%. What is his capital, if his total income is \$26,640?

32. A man spends $\frac{1}{a}$th of his income for food, $\frac{1}{b}$th for rent, $\frac{1}{c}$th for clothing, $\frac{1}{d}$th for furniture, and saves e dollars. How much is his income?

33. Two trains start at the same time from Buffalo and New York, respectively, 450 mi. apart; the one from New York travels at the rate of 50 mi. per hour, and the other 0.8 as fast. How far from New York do they meet?

34. Two trains start at the same time from Syracuse, one going east at the rate of 35 mi. per hour and the other going west at a rate $\frac{1}{7}$ greater. How long after starting will they, at these rates, be exactly 100 mi. apart?

35. A train runs 75 mi. in a certain time. If it were to run $2\frac{1}{2}$ mi. an hour faster, it would run 5 mi. farther in the same time. What is the rate of the train?

36. A steamer can run 25 mi. an hour in still water. If it can run 90 mi. with the current in the same time that it can run 60 mi. against the current, what is the rate of the current?

37. The cost of publication of each copy of a certain illustrated magazine is $6\frac{1}{2}$ cts.; it sells to dealers for 6 cts., and the amount received for advertisements is 10% of the amount received for all magazines issued beyond 10,000. Find the least number of magazines which can be issued without loss.

38. A steamer and a sailboat go from M to N, the former at the rate of 35 mi. in 3 hrs. and the latter at the rate of 10 mi. in the same time. The sailboat has a start of $3\frac{1}{3}$ mi., but arrives at N 5 hrs. after the steamer. How long did it take the steamer to go from M to N, and what is the distance?

39. Two engines are used for pumping water from different shafts of a mine, their combined horse power being represented by 108. The first engine pumps 22 gals. every 10 secs. from a depth of 310 yds.; the second pumps 9 gals. more in the same length of time from a depth of 176 yds. Required the horse power of each.

40. There are two hoisting engines at a coal-pit mouth, the first capable of raising at the rate of 144 tons every 5 hrs. from a depth of 375 ft., and the second 80 tons every 3 hrs. from a depth of 540 ft. After the first had been running $1\frac{3}{4}$ hrs. the second began, and after 7 hrs. it had raised from the bottom of the mine $11\frac{1}{4}$ tons more than the first. Required the depth of the mine.

C. Problems Relating to Science.

Illustrative problems. 1. Alcohol is received in the laboratory 0.95 pure. How much water must be added to a gallon of this alcohol so that the mixture shall be 0.5 pure?

1. Let $x \equiv$ the *number* of gallons of water to be added.
2. Then $0.5(1+x)$ represents the alcohol in the mixture.
3. But 0.95 represents the alcohol in the original gallon.
4. $\therefore \quad 0.5(1+x) = 0.95.$
5. $\therefore \quad x = 0.9.$ (Why?)

Check. Adding 0.9 gal., there are 1.9 gals. of the mixture, 0.5 of which is the 0.95 gal. of alcohol.

2. Air is composed of 21 volumes of oxygen and 79 volumes of nitrogen. If the oxygen is 1.1026 times as heavy as air, the nitrogen is what part as heavy as air?

1. $21 \cdot 1.1026 + 79x = 100.$ (Why?)
2. $\therefore \quad x = 0.9727.$ (Why?)

EXERCISES. LXXXV.

41. How much water must be added to a 5% solution of a certain medicine to reduce it to a 1% solution?

42. How much pure alcohol must be added to a mixture of $\frac{4}{5}$ alcohol so that $\frac{9}{10}$ of the mixture shall be pure alcohol?

43. In midwinter in St. Petersburg the night is 13 hrs. longer than the day. How many hours of day? of night? At what time does the sun rise? set?

44. How many ounces of silver 700 fine (700 parts pure silver in 1000 parts of metal) and how many ounces 900 fine must be melted together to make 78 oz. 750 fine?

45. How many ounces of pure silver must be melted with 500 oz. of silver 750 fine to make a bar 900 fine?

46. How many pounds of pure water must be added to 32 lbs. of sea water containing 16% (by weight) of salt, in order that the mixture shall contain only 2% of salt?

47. In a certain composition of metal weighing 37.5 lbs., $18\frac{3}{4}$% is pure silver. How many pounds of copper must be melted in so that the composition shall be only 15.625% pure silver?

48. How many pounds of copper should be melted in with 94.5 lbs. of an alloy consisting of 3 lbs. of silver to 4 lbs. of copper so that the new alloy shall consist of 7 lbs. of copper to 2 lbs. of silver?

49. What per cent of the *water* must be evaporated from a 6% solution of salt (salt water which contains 6%, by weight, of salt) so that the remaining portion of the mixture may be a 12% solution?

50. The planet Venus passes about the sun 13 times to the earth's 8. How many months from the time when Venus is between the earth and the sun to the next time when it is in the same relative position?

51. Two bodies start at the same time from two points 243 in. apart, and move towards each other, one at the rate of 5 in. per second, and the other 2 in. per second faster. In how many seconds will they be 39 in. apart?

52. From two points d units apart two bodies move towards each other at the rate of a and b units a second, respectively. After how many seconds will they be c units apart for the first time ($c < d$)? together? c units apart for the second time?

53. These bodies (of ex. 52) move, from the two starting points, away from one another. How far are they apart after t secs.? When will they be e units apart ($e > d$)?

54. If sound travels 5450 ft. in 5 secs. when the temperature is 32°, and if the velocity increases 1 ft. per second for every degree that the temperature rises above 32°, how far does sound travel in 8 secs. when the temperature is 70°?

55. Seen from the earth, the moon completes the circuit of the heavens in 27 das. 7 hrs. 43 mins. 4.68 secs., and the sun in 365 das. 5 hrs. 48 mins. 47.8 secs., in the same direction. Required the time from one full moon to the next, the motions being supposed to be uniform. Answer correct to 0.0001 da.

56. In Spitzbergen (77° N. lat.) there is a certain part of the year in which the sun does not rise, remaining constantly below the horizon; there is also an equal length of time during which it does not set. The period in which it rises and sets is $1\frac{1}{2}$ months longer than the period of continued night. How many months in each of these three divisions of the year?

57. It is shown in physics that if $t \equiv$ the number of seconds which it takes a pendulum to swing from one state of rest to the next, through a small angle, then $t = \pi\sqrt{l/g}$, where $\pi \equiv 3\frac{1}{7}$, $g \equiv 32.2$, and $l \equiv$ the number of feet of length of the pendulum. Required the length of a 1-second pendulum; of a 2-seconds pendulum; of a pendulum which oscillates 56 times in 55 secs.

58. It is proved in physics that if $v \equiv$ the velocity of a body which started with an initial velocity of a ft. per second and has gained in velocity f ft. per second for t seconds, then $v = a + ft$. Suppose $v = 15$, $a = 0$, $t = 5$. Find f. (This is one of many exceptions to the custom of representing known quantities by the first and unknown quantities by the last letters of the alphabet.)

D. Problems Relating to Mensuration.

The following formulas are proved true in geometry and are probably already known to the student from his work in arithmetic. They are inserted for reference.

Symbols.

$$\pi \equiv 3.14159 \cdots, \text{ or nearly } 3\tfrac{1}{7}.$$

$r \equiv$ radius. $a \equiv$ area. $b \equiv$ base.

$c \equiv$ circumference. $h \equiv$ altitude (height).

Formulas.

Rectangle, $a = bh$. Triangle, $a = \frac{1}{2}bh$.

The square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides.

Circle, $c = 2\pi r$. $a = \pi r^2$.

Illustrative problem. What is the length of the radius of the circle whose circumference is 62.8318 units?

1. $\because$ $c = 2\pi r$,
2. $\therefore$ $62.8318 = 2 \cdot 3.14159 \cdot r$.
3. $\therefore$ $10 = r$.

EXERCISES. LXXXVI.

59. What is the altitude of a triangle whose area is 7 sq. in. and whose base is 2 in.?

60. What is the length of the base of a rectangle whose area is 18 sq. in. and whose altitude is $2\frac{1}{2}$ in.?

61. From the top of a flagstaff a line just reaches the ground; if a line a yard long is tied to this (no allowance being made for the knot), the whole line when tightly stretched touches the ground 20 ft. from the staff. Required the height of the staff.

62. What is the length of the radius of the circle whose area contains 25π sq. in.?

63. If the area of a triangle is $3\sqrt{3}$, and the base is $2\sqrt{3}$, required the altitude.

64. What is the length of the diameter of the circle whose circumference is 157.0795 in.?

65. The perimeter of a rectangle is 14 in., and the base is $33\frac{1}{3}\%$ longer than the altitude. Required the length of the diagonal.

66. Two rectangles of the same area have the following dimensions: the first, 15 ft. by 10 ft., and the second, 18 ft. by x ft. Required x.

67. What is the length of the radius of the circle the number of square units of whose area equals the number of linear units of circumference?

68. The perimeter of a triangle is 75 in.; the second side is $\frac{2}{3}$ of the first and the third is $\frac{5}{7}$ of the first. Required the length of each side.

69. The area of a triangle is 250 sq. ft., and the altitude is 25% more than the base. Required the length of the base. Is the resulting equation linear?

70. The perimeter of a triangle is 24 in., the first side is 2 in. longer than the second, and the second is 2 in. longer than the third. Required the length of each side.

71. A dock pile is $\frac{1}{4}$ above water and $\frac{1}{3}$ is driven into the soil; if the water at the dock is 7 ft. deep, what is the entire length of the pile and how many feet are above water?

E. Historical Problems.

Many problems which were of considerable difficulty prior to the introduction of our present algebraic symbols, about the opening of the seventeenth century, are now comparatively easy. They have considerable historical interest as showing the state of the science at various periods, and a few examples are here inserted.

EXERCISES. LXXXVII.

72. If 9 porters drink 12 casks of wine in 8 das., how many will last 24 porters 30 das.? (Tartaglia, a famous Italian algebraist, about 1550 A.D.)

73. Demochares lived $\frac{1}{4}$ of his life as a boy, $\frac{1}{5}$ as a young man, $\frac{1}{3}$ as a man, and 13 years as an old man. How old was he then? (Metrodorus, 325 A.D.)

74. Of 4 pipes, the first fills a cistern in 1 da., the second in 2 das., the third in 3 das., and the fourth in 4 das. How long will it take all running together to fill it?

75. In the center of a pond 10 ft. square grew a reed 1 ft. above the surface; but when the top was pulled to the bank it just reached the edge of the surface. How deep was the water? (From an old Chinese arithmetic, Kiu chang, about 2600 B.C.)

76. A horse and a donkey, laden with corn, were walking together. The horse said to the donkey: "If you gave me one measure of corn, I should carry twice as much as you; but if I gave you one we should carry equal burdens." Tell me their burdens, O most learned master of geometry. (Attributed to Euclid, the great writer on geometry at Alexandria, about 300 B.C.)

77. Heap, its whole, its seventh, it makes 19. (That is, what is the number which when increased by its seventh equals 19? From the mathematical work copied by the Egyptian Ahmes about 1700 B.C. from a papyrus written about a thousand years earlier.)

78. Find the number, $\frac{1}{3}$ of which and 1, multiplied by $\frac{1}{4}$ of which and 2, equals the number plus 13. (Mohammed ben Musa Al-Khowarazmi, the famous Persian mathematician, 800 A.D. From the title of his book comes the word Algebra, and from the latter part of his name — referring to his birthplace — comes our word Algorism.)

79. In a pond the top of a lotus bud reached $\frac{1}{2}$ ft. above the surface, but blown by the wind it just reached the surface at a point 2 ft. from its upright position. How deep was the water? (From a mathematical work by Bhaskara, a Hindu writer of about 1150 A.D. The work was named the Lilavati in honor of his daughter.)

80. Two anchorites lived at the top of a perpendicular cliff of height h, whose base was mh distant from a certain town. One descended the cliff and walked to the town; the other flew up a height, x, and then flew directly to the town. The distance traversed by each was the same. Find x. (Brahmagupta, a Hindu mathematician, about 640 A.D.)

81. An ancient problem relates that Titus and Caius sat down to eat, Caius furnishing 7 portions and Titus 8, all of equal value. Before they began Sempronius entered and they all ate equally and finished the food. Sempronius then laid down 30 denarii (pence) and said: "Divide these equitably between you in payment for my meal." How much should each receive?

F. Discussion of Problems.

196. Many problems can be suggested which admit of mathematical solution, but whose practical solutions are impossible by reason of the physical conditions imposed.

E.g., I can look out of the window 18 distinct times in 4 secs. What is the rate per second?

The answer, $4\frac{1}{2}$ times per second, while entirely correct from the mathematical standpoint, is physically impossible; for while I can look out 4 times, I cannot look out half of a time.

The problem might easily be changed, however, so as to demand the time required to look out once, the answer being $\frac{2}{9}$ of a second.

A similar absurdity appears in the result of the following problem: A father is 53 yrs. old and his son 28. After how many years will the father be twice as old as the son?

We have the equation

$$53 + x = 2(28 + x),$$

whence $$x = -3.$$

We are now met by the necessity of

(1) interpreting the meaning of the answer -3 *years after this time*, or

(2) changing the statement of the problem so as to avoid an answer which seems meaningless.

It is immaterial which course we take. We may say:

(1) -3 *years after this time* shall be understood to mean 3 *years before this time*, which is entirely in harmony with our interpretation of negative numbers (§ 29); or

(2) we may change the problem to read: "How many years ago was the father twice as old as the son?" For this latter question the solution would be

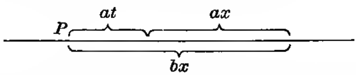
$$53 - x = 2(28 - x).$$

$$\therefore x = 3,$$

and the answer would be 3 years ago.

The discussion of results of this nature is well illustrated in an ancient problem known as the **Problem of the Couriers.**

A courier, A, travels at the uniform rate of a mi. per hour from P; after t hrs. a second one, B, starts in pursuit from P and travels at the uniform rate of b mi. per hour. After how many hours will B overtake A?



SOLUTION. 1. Let $x \equiv$ the number of hours required.

2. Then $\because a(t + x) = bx$, the distance B must travel,

$$\therefore x = \frac{at}{b - a}.$$

DISCUSSION. 1. If none of the quantities is zero, and $b > a$, the denominator is positive and $\therefore x$ is positive.

2. But if $b = a$, the denominator is zero and . . x is infinite (§ 170). *I.e.*, if they are traveling at the same rate B will never overtake A.

3. And if $b < a$, the denominator is negative and $\therefore x$ is negative. *I.e.*, if B is traveling slower than A he will never overtake him. But if the problem reads, "... after t hrs. B passes through P in pursuit," then the result would mean that they had been together $\left|\frac{at}{b - a}\right|$ hrs. *before* reaching P.

4. If either $t = 0$, or $a = 0$, the numerator is zero and $\therefore x = 0$, except when $b = a$, in which case x is undetermined (§ 168). This is evidently true, for if $t = 0$ and they are traveling at the same rate they will always be together. Or if $a = 0$ and $a = b$, then neither courier is traveling at all, and hence they are always together at P.

EXERCISES. LXXXVIII.

Solve the following and discuss the results according to the suggestions given above and in the problems.

1. A bicyclist starts out riding 10 mi. per hour, and is followed after 30 mins. by a second riding 8 mi. per hour. In what time will the second overtake the first?

2. A bicyclist starts from P, riding a mi. per hour; after t hrs. another follows and overtakes him in h hrs. At what rate did the second one ride? Discuss for $t = h = 0$.

3. A bicyclist starts from P, riding a mi. per hour; he is followed after t hrs. by a second rider traveling c times as fast. After how many hours will the second overtake the first? Discuss for (1) $c > 1,\ t \neq 0$, (2) $c = 1,\ t \neq 0$, (3) $c < 1,\ t \neq 0$, (4) $c = 1,\ t = 0$.

4. Two trains going from San Francisco to Chicago, on the same road, pass through Omaha, the first at 9.30 A.M., and the second at 10 A.M. The first train travels at the rate of 50 mi. per hour, and the second 10% slower. At what distance from Omaha are they together?

REVIEW EXERCISES. LXXXIX.

1. Are $x = 2$ and $x^4 = 16$ equivalent equations? Why?

2. Show that if x is a factor of every term of an equation, 0 is a root. *E.g.*, $x^2 + 2x = 5x$.

3. Solve the equation

$$3a - 2\{a + 3[a - 2(a - \overline{a - 2x})]\} = 11a.$$

4. Show that if both members of an equation have a common linear factor containing the unknown quantity, a root can be found by equating this factor to zero.

5. What is the fallacy in this argument?

1. Let $x = a$.
2. Then $x^2 = ax$, multiplying by x.
3. Then $x^2 - a^2 = ax - a^2$, subtracting a^2.
4. Then $(x + a)(x - a) = a(x - a)$, factoring.
5. ∴ $2a(x - a) = a(x - a)$, because $x + a = 2a$.
6. ∴ $2 = 1$, dividing by $a(x - a)$.

CHAPTER X.

SIMPLE EQUATIONS INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

197. A single linear equation containing two unknown quantities does not furnish determinate values of these quantities.

This means a single equation in which the similar terms have been united. *I.e.*, $x + y = x + 3$ is not included, because the x's have not been united.

E.g., $x - y = 1$ is satisfied if $x = 1$ and $y = 0$, or if $x = 2$ and $y = 1$, or if $x = 3$ and $y = 2$, etc.

198. But *two* linear equations containing *two* unknown quantities furnish, in general, determinate values. Similarly, as will be seen, a system of *three* linear equations containing *three* unknown quantities, a system of *four* linear equations containing *four* unknown quantities, $\cdots$ a system of n linear equations containing n unknown quantities, furnish, in general, determinate values of all of these quantities.

199. Equations all of which can be satisfied by the same values of the unknown quantities are said to be **simultaneous.**

E.g., $x + y = 7$, $x - y = 3$, are two equations which are satisfied if $x = 5$ and $y = 2$. Hence they are simultaneous.

But $x + y = 7$ and $x + y = 8$ cannot be satisfied by the same values of x and y, and hence they are not simultaneous.

The equations $x + 2y = 6$, $3x + 6y = 9$, are simultaneous; but each being derivable from the other they do not furnish determinate values.

I. ELIMINATION BY ADDITION OR SUBTRACTION.

200. The solution of two simultaneous equations involving two unknown quantities is made to depend upon the solution of a single equation involving but one of the unknown quantities. The usual process, by addition or subtraction, is seen in the following solutions:

1. Solve the system of equations

1. $4x + 3y = 41.$

2. $3x - 2y = 1.$

We first seek to give the y's coefficients having the same absolute values. This can be done by multiplying both members of the first by 2, and of the second by 3. Then

3. $8x + 6y = 82.$

4. $9x - 6y = 3.$

Add equations 3 and 4, member by member, and

5. $17x = 85.$

6. $\therefore$ $x = 5.$

Substitute this value in equation 1, and

7. $4 \cdot 5 + 3y = 41.$

8. $\therefore$ $3y = 21.$

9. $\therefore$ $y = 7.$

Check. Substitute these values in equation 2 (because y was obtained by substituting in equation 1), and $3 \cdot 5 - 2 \cdot 7 = 1.$

For brevity we shall hereafter use the expressions, in solutions, "Multiply 2 by 5," etc., meaning thereby, "Multiply *both members* of equation 2 by 5," etc.

201. When one of the unknown quantities has been made to disappear (as in passing from steps 3 and 4 to step 5 above) it is said to be *eliminated.*

In the above solution y was *eliminated by addition.* The quantity x may, however, be eliminated first, by subtraction, as in the following solution.

2. Solve the system of equations

$$1.\ 4x + 3y = 41.$$

$$2.\ 3x - 2y = 1.$$

3. $\because$ $12x + 9y = 123$, multiplying 1 by 3,

4. and $12x - 8y = 4$. (Why?)

5. $\therefore$ $17y = 119$, subtracting 4 from 3.

6. $\therefore$ $y = 7$. (Why?)

7. $\therefore$ $4x + 21 = 41$. (Why?)

8. $\therefore$ $4x = 20$. (Why?)

9. $\therefore$ $x = 5$. (Why?)

Check. In which equation should these values now be substituted? (Why?)

Other types are illustrated in the two problems following.

3. Solve the system of equations

$$1.\ \frac{x}{3} - \frac{y}{2} = 2.$$

$$2.\ \frac{x}{2} + \frac{y}{4} = 7.$$

It is not worth while here to clear of fractions. Simply multiply both members of the first by $\frac{1}{2}$, and

3. $\frac{x}{6} - \frac{y}{4} = 1.$

4. $\therefore$ $\frac{2x}{3} = 8$, adding 2 and 3.

5. $\therefore$ $x = 12$. (Why?)

It is now apparent that y can easily be found and the results checked in the usual way.

I.e., $y = 4$,

and $\frac{12}{3} - \frac{4}{2} = 2$, etc.

4. Solve the system of equations

$$1.\quad \frac{3}{x}+\frac{2}{y}=\frac{7}{4}.$$

$$2.\quad \frac{2}{x}+\frac{1}{y}=1.$$

These are not linear equations because, when cleared of algebraic fractions, they are of the second degree. But they can easily be solved by the methods of linear equations as here suggested.

3. $\frac{4}{x}+\frac{2}{y}=2.$ (Why ?)

4. $\therefore$ $\frac{1}{x}=\frac{1}{4}$, subtracting 1 from 3.

5. $\therefore$ $4=x$, multiplying by $4x$.

Hence, y is easily found to be 2, and the results check.

EXERCISES. XC.

Solve the following, checking each result by proper substitution:

1. $7x-3y=3.$
   $5x+7y=25.$

2. $3x+5y=5.$
   $4x-3y=26.$

3. $x+17y=53.$
   $8x+y=19.$

4. $5x+2y=1.$
   $13x+8y=11.$

5. $6x-5y=12.$
   $12x-11y=27.$

6. $1.7x+1.1y=13.$
   $1.3x-0.1y=1.$

7. $\frac{x}{3}-\frac{y}{7}=\frac{7}{2}.$
   $\frac{x}{2}+\frac{y}{5}=11.$

8. $\frac{x}{5}+\frac{y}{10}=3.$
   $\frac{x}{10}+\frac{y}{5}=3.$

9. $\frac{3}{x}-\frac{4}{y}=-\frac{15}{2}.$
   $\frac{2}{x}+\frac{5}{y}=\frac{31}{3}.$

10. $\frac{2x}{3}-\frac{y}{2}=5.$
    $\frac{x}{2}+\frac{2y}{3}=20.$

II. ELIMINATION BY SUBSTITUTION AND COMPARISON.

202. After finding the value of one unknown quantity by addition or subtraction the other is usually, but not necessarily, found by substitution. It is often more convenient to find each by substitution, especially when one of the coefficients is 1.

This method of **elimination by substitution** is illustrated in the following solution:

1. Given $x - \frac{2}{3}y = -5,$
2. and $3x + 2y = 45.$

From equation 1 we have:

3. $x = \frac{2}{3}y - 5.$ (Why?)

Substitute this value in equation 2, and

4. $2y - 15 + 2y = 45,$ from which
5. $4y = 60.$
6. $\therefore$ $y = 15.$

From this x is found, by substitution, to be 5, and the results check.

It is not necessary that the coefficient of x or y should be 1, although this is the case in which the method is most frequently employed. Consider, for example, the following solution:

1. Given $2x + 5y = 154,$
2. and $30x - 2y = 0.$
3. From equation 2, $x = \frac{1}{15}y.$
4. Substituting, $\frac{2}{15}y + 5y = 154,$

whence $y = 20.$

$\therefore$ $x = 2.$

203. A special form of substitution occurs when the value of one of the unknown quantities is found in each equation, and these values are compared. This is called **elimination by comparison.**

The method is illustrated in the following solution:

1. Given $x - \frac{2}{3}y = -5,$
2. and $3x + 2y = 45.$

Solving equations 1 and 2 for x, we have:

3. $x = \frac{2}{3}y - 5,$
4. and $x = 15 - \frac{2}{3}y.$

Substituting the value of x from step 3 in step 4, or, what is the same thing, comparing the values of x (by ax. 1), we have:

5. $\frac{2}{3}y - 5 = 15 - \frac{2}{3}y.$
6. $\therefore$ $\frac{4}{3}y = 20.$ (Why?)
7. $\therefore$ $y = 15.$ (Why?)
8. $\therefore$ $x = 5$, by substituting in step 3.

Check. Substituting in both of the original equations,

$$5 - \frac{2}{3} \cdot 15 = -5.$$
$$3 \cdot 5 + 2 \cdot 15 = 45.$$

EXERCISES. XCI.

Solve the following by substitution or comparison, checking the results as usual:

1. $x + y = 17.$ $3x + 2y = 44.$

2. $x + y = s.$ $x - y = d.$

3. $x = y.$ $3x + 5y = 120.$

4. $x + ay = b.$ $cx + y = d.$

5. $x - y - 1 = 0.$ $2x + y - 29 = 0.$

6. $ax + by = c.$ $a'x + b'y = c'.$

7. $x + y = 6912.$ $x = 4444 + y.$

8. $x + 2y = 30.$ $\frac{1}{2}x - \frac{1}{5}y = 3.$

9. $x + 17y = 300.$ $11x - y = 104.$

10. $x + 1\frac{2}{3}y = 26\frac{1}{12}.$ $4\frac{5}{6}y - x = 44\frac{7}{8}.$

11. $1.543689x - y = 1.543689.$ $x - 0.839286y = 0.839286.$

III. GENERAL DIRECTIONS.

204. The following general directions will be found of some value, although the student must use his judgment in each individual case.

1. *If the equations contain symbols of aggregation, decide whether it is better to remove them at once.*

It is usually best to remove them, as in a case like ex. 18, p. 188. But in a case like ex. 17, p. 188, it is evidently better to add at once.

2. *If the equations are in fractional form, decide whether it is better to eliminate without clearing of fractions.*

See pp. 181, 182, illustrative problems 3, 4. Much time is often wasted by clearing of fractions unnecessarily. This is also seen in the example on p. 193.

3. *If it seems advisable, clear of fractions and reduce each to the form* $ax + by = c$.

See illustrative problem 1, p. 186. The same course will naturally be followed with an example like ex. 8, p. 187.

4. *If the coefficient of either unknown quantity is* 1, *it is usually advisable to eliminate by substitution.*

See illustrative problem 1, p. 186, steps 4, 6, 7. This is, however, not often the case.

5. *Otherwise it is generally best to eliminate by addition or subtraction.*

This is the plan usually employed.

6. *If the unknown quantity is in an exponent, follow the plan suggested in* § 205.

It is here assumed that the root of the single equation derived from the two given equations satisfies those equations. For the proof of this fact see Appendix VI.

Illustrative problems. 1. Solve the system of equations

$$1.\quad \frac{1+x}{y}+3=5.$$

$$2.\quad \frac{2}{x}+5=\frac{y}{x}+2.$$

Here it is not best to attempt to eliminate without clearing of fractions. Multiplying both numbers of 1 by y,

3.	$1+x+3y=5y.$	Ax. 6
4. ∴	$x=2y-1.$	(Why?)
5.	$2+5x=y+2x$, from 2.	(Why?)
6. ∴	$3x-y=-2$, from 5.	(Why?)
7. ∴	$6y-3-y=-2$, substituting 4 in 6.	
8. ∴	$5y=1.$	(Why?)
9. ∴	$y=\frac{1}{5}$, and $x=\frac{-3}{5}.$	(Why?)

Check. Substituting in *both* given equations,

$2+3=5$, from 1,

and $-\frac{10}{3}+5=-\frac{1}{3}+2,$

or $\frac{5}{3}=\frac{5}{3}$, from 2.

205. Equations in which the unknown quantities appear as exponents are called **exponential equations.**

Exponential equations of the following type are easily solved by means of simple equations.

2. Solve the system of equations

$$a^{2x}\cdot a^{3y}=a^{32}.$$

$$a^{3x}\cdot a^{4y}=a^{44}.$$

1. ∵	$a^{2x}\cdot a^{3y}=a^{32},$	
2. ∴	$a^{2x+3y}=a^{32}.$	§ 60
3. ∴	$2x+3y=32.$	
4. Similarly,	$3x+4y=44.$	
Solving,	$x=4,\ y=8.$	

EXERCISES. XCII.

Solve the following, checking as usual:

1. $\frac{p^{2x}}{p^{5y}} = \frac{1}{p^{19}}.$
$\frac{p^{3x}}{p^{y}} = p^4.$

2. $\frac{x}{a} + \frac{y}{b} = \frac{1}{c}.$
$\frac{x}{m} - \frac{y}{n} = \frac{1}{p}.$

3. $\frac{a}{x} - \frac{b}{y} = c.$
$\frac{m}{x} - \frac{n}{y} = p.$

4. $\frac{3x}{5} + \frac{y}{4} = 13.$
$\frac{x}{3} - \frac{y}{8} = 3.$

5. $p^{2x} \cdot p^{5y} = p^{81}.$
$q^{3x} \cdot q^{y} = q^{14}.$

6. $4^x \cdot 16^y = 2^{80}.$
$16^x \cdot 2^{2y} = 4^9.$

7. $\frac{m^{4x}}{m^{3y}} = \left(\frac{1}{m}\right)^8.$
$\frac{m^{5x}}{m^{4y}} = \left(\frac{1}{m}\right)^6.$

8. $\frac{4x + 81}{10y - 17} = 6.$
$\frac{12x + 97}{15y - 17} = 4.$

9. $b^{11x} : b^{5y} = b^{20}.$
$b^{9x} : b^{7y} = b^{-4}.$

10. $7x - \frac{1}{5}y = 48.$
$5y + \frac{1}{7}x = 26.$

11. $x + \frac{1}{11}y = 71.$
$y - \frac{1}{13}x = 61.$

12. $17x - 13y = 144.$
$23x + 19y = 890.$

13. $\frac{x + y - 1}{x - y + 1} = a.$
$\frac{y - x + 1}{x - y + 1} = ab.$

14. $\frac{a}{x} - \frac{b}{y} + 1 = 0.$
$\frac{b^2}{x} + \frac{a^2}{y} = a - b.$

15. $\frac{x}{9} + \frac{y}{7} = 6.3.$
$\frac{x}{3} + \frac{53y}{56} = 39.2.$

16. $\frac{x + y}{x - y} = -\frac{15}{8}.$
$9x - \frac{3y + 44}{7} = 100.$

17. $a(x+y) - b(x-y) = 2a.$

$a(x-y) - b(x+y) = 2b.$

18. $10[x + 9(y - 8\overline{x+7})] = 6.$

$5[x + 4(y - 3\overline{x+2})] = 1.$

19. $\frac{a}{a+c}x - y = \frac{a-c}{b} - \frac{a}{a+c}y.$

$\frac{x}{c} + \frac{y}{a} = \frac{b}{ac}.$

20. $\frac{5y}{6} - \frac{4y-19}{3} = \frac{x}{6} + \frac{20-2y}{3}.$

$\frac{x+5y}{6} + 5 = \frac{2y+21}{3}.$

21. $\frac{2x}{y} = \frac{29}{14}.$

$y + 4x + 6 = \frac{4y^2 + 13xy - 12x^2}{4y - 3x - 1}.$

22. $\frac{13+x}{7} + \frac{3x-8y}{3} = x + y - 5\frac{1}{3}.$

$\frac{11-x}{2} + \frac{4x+8y-2}{9} = 8 - (y-x).$

23. $\frac{4x^2 + 2xy + 288 - 6y^2}{2x + 13 - 2y} = 2x + 3y - 131.$

$5x - 4y = 22.$

24. $\frac{7y + 13 - 5x}{4} + y = 2x - \frac{3y + 2x - 16}{3}.$

$x + \frac{5y+2x}{6} - \frac{3x - 12 + 8y}{5} = 4 - \frac{15 + 2y - 4x}{3}.$

IV. APPLICATIONS OF SIMULTANEOUS LINEAR EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

Illustrative problem. The sum of two numbers is 12 and 7 times the quotient of one divided by the other is 5. Required the numbers.

1.	Let	$x,\ y \equiv$ the numbers.	
2.	Then	$x + y = 12$, and	
3.		$7 \cdot \frac{x}{y} = 5$, by the conditions of the problem.	
4.	$\therefore$	$y = 12 - x$, from 2.	Ax. 3
5.	And	$7x = 5y$, from 3.	Ax. 6
6.	$\therefore$	$7x = 60 - 5x$.	(Why?)
7.	$\therefore$	$12x = 60$, and $x = 5$.	(Why?)
8.	$\therefore$	$y = 7$, from step 4.	

Check. The sum of 5 and 7 is 12, and 7 times $\frac{5}{7}$ is 5.

EXERCISES. XCIII.

1. The sum of two numbers is 30 and their difference is 17. Required the numbers.

2. What is that fraction which equals $\frac{1}{3}$ when 1 is added to the numerator, but equals $\frac{1}{4}$ when 1 is added to the denominator?

3. A number of two figures is 5 times the sum of its digits. If 9 is added to the number, the order of its digits is reversed. Required the number.

4. A man invests \$16,000 for 8 yrs. and \$11,000 for 6 yrs., and receives from the two \$8090 interest. Had the first been invested at the same rate as the second and the second at the same rate as the first, he would have received \$310 more interest in the same times. Required the rate at which each was invested.

5. The sum of two numbers is s and their difference is d. Required the numbers. From the result, deduce a rule for finding two numbers, given their sum and their difference.

6. The sum of two capitals, each invested at 5%, is $12,000, and the sum of 5 yrs. simple interest on the larger and 4 yrs. simple interest on the smaller is $2800. Required the capitals.

7. Divide the two numbers 80 and 90 each into two parts such that the sum of one part of the first and one part of the second shall equal 100, and the difference of the other two parts shall equal 30.

8. Two points move around a circle whose circumference is 100 ft.; when they move in the same direction they are together every 20 secs.; when in opposite directions they meet every 4 secs. Required their rates.

9. The boat A leaves the city C at 6 A.M.; an hour later the boat B leaves the city D, 80 mi. from C, and meets A at 11 A.M. They would also meet at 11 A.M. if B left at 6 A.M. and A 45 mins. later. Required their rates.

10. Of two bars of metal, the first contains 21.875% pure silver and the second 14.0675%. How much of each kind must be taken in order that when melted together the new bar shall weigh 60 oz., and 18.75% shall be pure silver?

11. A marksman fires at a target 500 yds. distant and hears the bullet strike $4\frac{1}{3}$ secs. after he fires; an observer standing 400 yds. from the target and 650 yds. from the marksman hears the bullet strike $2\frac{1}{8}$ secs. after he hears the report. Required the velocity of sound and the velocity of the bullet, each supposed to be uniform.

12. Find two numbers the sum of whose reciprocals is 5, and such that the sum of half of the first and one-third of the second equals twice the product of the two numbers.

13. Two bodies are 96 yds. apart. If they move towards each other with uniform (but unequal) rates, they will meet in 8 secs.; but if they move in the same direction the swifter overtakes the slower in 48 secs. Required the rate of each.

14. The sum of two numbers, one of one figure and the other of five figures, is 15,390. Writing the first number as the first digit to the left of the second number gives a number 4 times as large as that which is obtained by writing it as the last digit to the right. Required the numbers.

15. A reservoir has two contributing canals. If the first is open 10 mins. and the second 13 mins., 15 cu. yds. of water flow in; if the first is open 14 mins. and the second 5 mins., 2.4 cu. yds. more flow in. How many cubic yards of water per minute are admitted by each?

16. A silversmith has two silver ingots of different quality. He melts 13 oz. of the finer kind with 12 oz. of the other, the resulting ingot being 852 fine (see p. 169, ex. 44); but if he melts 1.5 oz. of the finer kind with 1 oz. of the other the resulting ingot is 860 fine. Required the fineness of each original ingot.

17. It is shown in physics that if a body starts with a velocity of u ft. per second, and if this velocity increases f ft. per second, then at the end of t secs. the body will have passed over $ut + \frac{1}{2}ft^2$. Suppose f is uniform and that in the 11th and 15th secs. the body passes through 24 ft. and 32 ft., respectively, find u and f.

V. SYSTEMS OF EQUATIONS WITH THREE OR MORE UNKNOWN QUANTITIES.

206. In general, three linear equations involving three unknown quantities admit of determinate values of these quantities. For one of the quantities can be eliminated from the first and second equations, and the same one from the first and third, thus leaving two linear equations involving only two unknown quantities. Similarly for a system of four linear equations containing four unknown quantities, and so on.

Illustrative problems. 1. Solve the following system of equations:

1. $5x - 3y + 4z = 17.$
2. $2x + 7y - 5z = 5.$
3. $9x - 2y - z = 8.$

We first proceed to eliminate z from 1 and 2.

4. $25x - 15y + 20z = 85$, multiplying 1 by 5.
5. $8x + 28y - 20z = 20$, multiplying 2 by 4.
6. $\therefore$ $33x + 13y = 105.$ (Why?)

We now proceed to eliminate z from 1 and 3.

7. $36x - 8y - 4z = 32$, multiplying 3 by 4.
8. $\therefore$ $41x - 11y = 49$, from 1 and 7.

We now proceed to eliminate y from 6 and 8.

9. $363x + 143y = 1155$, multiplying 6 by 11.
10. $533x - 143y = 637$, multiplying 8 by 13.
11. $\therefore$ $896x = 1792.$ (Why?)
12. $\therefore$ $x = 2.$ (Why?)
13. $\therefore$ $y = 3$, substituting in 6.
14. $\therefore$ $z = 4$, substituting in 1.

Check. Substitute in 2 and 3. (Why not in 1?)

$$4 + 21 - 20 = 5, \text{ and } 18 - 6 - 4 = 8.$$

2. Solve the following system of equations:

$$1.\quad \frac{1}{5x} + \frac{1}{7y} + \frac{1}{9z} = 1.$$

$$2.\quad \frac{2}{x} + \frac{3}{y} - \frac{4}{z} = 67.$$

$$3.\quad \frac{3}{x} - \frac{5}{y} + \frac{2}{z} = -38.$$

We first proceed to eliminate $\frac{1}{z}$ from 1 and 2.

4. $\frac{4}{5x} + \frac{4}{7y} + \frac{4}{9z} = 4$, from 1. (Why ?)

5. $\frac{2}{9x} + \frac{3}{9y} - \frac{4}{9z} = \frac{67}{9}$, from 2. (Why ?)

6. $\therefore$ $\frac{46}{5x} + \frac{57}{7y} = 103.$ (Why ?)

We then proceed to eliminate $\frac{1}{z}$ from 2 and 3.

7. $\frac{6}{x} - \frac{10}{y} + \frac{4}{z} = -76$, from 3. (Why ?)

8. $\therefore$ $\frac{8}{x} - \frac{7}{y} = -9$, from 2 and 7. (Why ?)

We then proceed to eliminate $\frac{1}{y}$ from 6 and 8.

9. $\frac{322}{5x} + \frac{399}{7y} = 721$, from 6. (Why ?)

10. $\frac{456}{7x} - \frac{399}{7y} = -\frac{513}{7}$, from 8. (Why ?)

11. $\frac{4534}{35x} = \frac{4534}{7}$, from 9 and 10. (Why ?)

12. $\therefore$ $\frac{1}{x} = 5$, and $x = \frac{1}{5}$. (Why ?)

13. $\therefore$ $\frac{1}{y} = 7$, and $y = \frac{1}{7}$. (Why ?)

14. $\therefore$ $\frac{1}{z} = -9$, and $z = -\frac{1}{9}$. (Why ?)

Check. Substitute in 1 and 2. (Why not in 3 ?)

$$1 + 1 - 1 = 1.$$
$$10 + 21 + 36 = 67.$$

The equations in ex. 2 are not linear in x, y, z (why ?), and it is unwise to clear of fractions (why ?). The equations are linear in $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, and it is better to solve as if these were the unknown quantities.

3. Solve the following system of equations:

1. $x + y - z = 6.$
2. $x + y + 2z = -3.$
3. $x - 2y - z = 0.$

Frequently systems of equations offer some special solution, as in this case.

Adding the equations, member by member,

4. $3x = 3.$

5. $\therefore x = 1.$

Subtracting 2 from 1, member by member,

6. $-3z = 9.$

7. $\therefore z = -3.$

8. $\therefore y = 2$, substituting in 1.

Check. Substitute in 2 and 3.

$$1 + 2 - 6 = -3.$$
$$1 - 4 + 3 = 0.$$

4. Solve the following system of equations:

1. $w + 2x + y - z = 4.$
2. $2w + x + y + z = 7.$
3. $3w - x + 2y - z = 1.$
4. $4w + 3x - y + 2z = 13.$

Eliminating z from 1 and 2,

5. $$3w + 3x + 2y = 11.$$

Also from 1 and 3,

6. $$2w - 3x + y = -3.$$

Also from 1 and 4,

7. $$6w + 7x + y = 21.$$

Eliminating y from 5 and 6,

8. $$w - 9x = -17.$$

9. Also from 6 and 7, $2w + 5x = 12.$

Eliminating w from 8 and 9,

10. $x = 2.$

11. $\therefore$ $w = 1$, substituting in 8.

12. $\therefore$ $y = 1$, substituting in 6.

13. $\therefore$ $z = 2$, substituting in 1.

Check. Substitute in 2, 3, and 4. (Why not in 1?)

$$2 + 2 + 1 + 2 = 7.$$
$$3 - 2 + 2 - 2 = 1.$$
$$4 + 6 - 1 + 4 = 13.$$

EXERCISES. XCIV.

Solve the following systems of equations:

1. $\dfrac{1}{x} + \dfrac{1}{y} = 1.$
$\dfrac{1}{x} + \dfrac{1}{z} = 2.$
$\dfrac{1}{y} + \dfrac{1}{z} = \dfrac{3}{2}.$

2. $\dfrac{x}{5} + \dfrac{y}{7} + \dfrac{z}{9} = 258.$
$\dfrac{x}{7} + \dfrac{y}{9} + \dfrac{z}{5} = 304.$
$\dfrac{x}{9} + \dfrac{y}{5} + \dfrac{z}{7} = 296.$

3. $7x - 3y = 1.$
$11z - 7u = 1.$
$4z - 7y = 1.$
$19x - 3u = 1.$

4. $5x - 6y + 4z = 15.$
$7x + 4y - 3z = 19.$
$x + y = 7.$
$x + 6z = 39.$

5. $x + y = 16.$
$z + x = 22.$
$y + z = 28.$

6. $x + y - z = 132.$
$x - y + z = 65.4.$
$-x + y + z = -1.2.$

7. $a^x \cdot a^{y+2} = a^{12}.$
$a^x \cdot a^{z+4} = a^{16}.$
$a^y \cdot a^{z+3} = a^{17}.$

8. $a_1x + b_1y + c_1z = d_1.$
$a_2x + b_2y + c_2z = d_2.$
$a_3x + b_3y + c_3z = d_3.$

9. $x = 21 - 4y.$
$z = 9 - \frac{2}{3}x.$
$y = 64 - 7\frac{1}{2}z.$

10. $x + y + z = 5.$
$3x - 5y + 7z = 75.$
$9x - 11z + 10 = 0.$

11. $\frac{6}{x} + \frac{12}{y} + \frac{18}{z} = 18.$
$\frac{12}{x} + \frac{18}{y} + \frac{6}{z} = 23.$
$\frac{18}{x} + \frac{6}{y} + \frac{12}{z} = 25.$

12. $7x - 2z + 3u = 17.$
$4y - 2z + v = 11.$
$5y - 3x - 2u = 8.$
$4y - 3u + 2v = 9.$
$3z + 8u = 33.$

13. $\frac{3y-1}{4} = \frac{6z}{5} - \frac{x}{2} + \frac{9}{5}.$
$\frac{5x}{4} + \frac{4z}{3} = y + \frac{5}{6}.$
$\frac{3x+1}{7} - \frac{z}{14} + \frac{1}{6} = \frac{2z}{21} + \frac{y}{3}.$

14. $a^{4x+3y+z} \cdot a^{3x+y+z} = a^{2z+15}.$
$a^{9x+y+3z} \cdot a^{6x-y+2z} = a^{5z+15}.$

15. $2w + x - 10y + 0.5z = 7.62.$
$3w - 2x + 2y + 3z = 8.26.$
$w + 3x + 5y - z = 8.61.$
$-6w - 2x + 3y + 10z = 25.51.$

VI. APPLICATIONS OF SIMULTANEOUS LINEAR EQUATIONS INVOLVING THREE UNKNOWN QUANTITIES.

Illustrative problem. A certain number of three figures is such that when 198 is added the order of the digits is reversed; the sum of the hundreds' digit and the tens' digit is the units' digit; and the number represented by the two left-hand digits is 4 times the units' digit. Required the number.

1. Let $a \equiv$ the hundreds' digit, b the tens', c the units'.
2. Then $100a + 10b + c \equiv$ the number.
3. Then, by the first condition,

$$100a + 10b + c + 198 = 100c + 10b + a.$$

4. By the second condition, $a + b = c$.
5. By the third condition, $10a + b = 4c$.
6. $\therefore$ the equations are

$$a - c = -2, \text{ from step 3.}$$
$$a + b - c = 0, \text{ from step 4.}$$
$$10a + b - 4c = 0, \text{ from step 5.}$$

7. Solving, $a = 1,\ b = 2,\ c = 3.$

$\therefore$ the number is 123.

Check by noting that 123 answers all of the conditions of the *original statement.*

EXERCISES. XCV.

1. What three numbers have the peculiarity that the sum of the reciprocals of the first and second is $\frac{1}{2}$, of the first and third $\frac{1}{3}$, and of the second and third $\frac{1}{4}$?

2. There is a certain number of six figures, the figure in units' place being 4; if this figure is carried over the other five to occupy the left-hand place, the resulting number is four times the original one. Required the original number.

3. Divide the number 96 into three parts such that the first divided by the second gives a quotient 2 and a remainder 3; and the second divided by the third gives a quotient 4 and a remainder 5.

4. The middle digit of a certain number of three figures is half the sum of the other two; the number is 48 times the sum of the digits. Subtracting 198 from the number, the order of the digits is reversed. Required the number.

5. Of 3 bars of metal, the first contains 750 oz. silver, $62\frac{1}{2}$ oz. copper, $187\frac{1}{2}$ oz. tin; the second, $62\frac{1}{2}$ oz. silver, 750 oz. copper, $187\frac{1}{2}$ oz. tin; and the third no silver, 875 oz. copper, 125 oz. tin. How many ounces from these bars must be melted together to form a bar which shall contain 250 oz. silver, $562\frac{1}{2}$ oz. copper, and $187\frac{1}{2}$ oz. tin?

6. Of three bars of metal, the first contains 750 oz. silver, 200 oz. copper, 50 oz. tin; the second, 800 oz. silver, 125 oz. copper, 75 oz. tin; and the third 700 oz. silver, 250 oz. copper, 50 oz. tin. How many ounces from these bars must be melted together to form a bar which shall contain 765 oz. silver, 175 oz. copper, and 60 oz. tin?

7. Two bodies, A and B, start at the same time from the points P and Q, respectively, and move at uniform rates towards one another, B faster than A; at the end of 18 secs., and again at the end of 30 secs., they are 48 ft. apart. Had they moved in the same direction, B following A, at the end of 40 secs. they would have been 48 ft. apart. Determine their rates and the distance PQ.

Solutions by determinants. The treatment of simultaneous linear equations by determinants is set forth in Appendix VII, and should be taken at this point if time allows.

REVIEW EXERCISES. XCVI.

1. Solve the equation $2.25x - 5 - 0.4x + 2.6 = 2x - 3$.

2. By the Remainder Theorem ascertain whether $10x^3 - 13x^2 - 5x + 3$ is exactly divisible by $2x - 3$.

3. Form an integral linear function of x which shall equal 37 when $x = 10$, and 4 when $x = -1$.

4. Form an integral linear function of x which shall vanish when $x = 2$, and which shall equal 4 when $x = 3$. (If $f(x) = mx + n$, then $2m + n = 0$ and $3m + n = 4$.)

5. Show that the following set of equations are not simultaneous and hence cannot be solved:

$$62x + 93y = 31.$$
$$2x + 3y = 4.$$

6. Simplify $\dfrac{x^2+x+1}{x-1+\dfrac{1}{x}} \cdot \dfrac{\dfrac{1}{x^2}-\dfrac{1}{x}+1}{x+1+\dfrac{1}{x}}$.

7. Write down by inspection the quotient of

$$\frac{1}{x^3}+\frac{7}{x^2}+\frac{13}{x}+4 \text{ by } \frac{1}{x^2}+\frac{3}{x}+1. \quad \text{Check.}$$

8. Multiply $\dfrac{1}{x^3}+\dfrac{3}{x^2}-\dfrac{2}{x}+7$ by $\dfrac{2}{x^2}-\dfrac{3}{x}+4$ by detached coefficients. Check.

9. Divide $\dfrac{1}{x^5}-\dfrac{3y}{x^4}+\dfrac{5y^2}{x^3}-\dfrac{5y^3}{x^2}+\dfrac{3y^4}{x}-y^5$ by $\dfrac{1}{x^2}-\dfrac{y}{x}+y^2$ by detached coefficients. Check.

10. Solve the equation

$$\frac{2x-1}{2x+1}+\frac{2x+5}{2x+7}-\frac{2x+1}{2x+3}-\frac{2x+3}{2x+5}=0.$$

11. Solve the system

$$0.2\,x + 0.3\,y + 0.4\,z = 25.$$
$$0.3\,x + 0.7\,y + 0.6\,z = 45.$$
$$0.4\,x + 0.8\,y + 0.9\,z = 58.$$

12. Solve the system

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x} + \frac{3}{y} - \frac{2}{z} = \frac{3}{x} - \frac{1}{y} = \frac{3}{4}.$$

13. Solve the system

$$3\,x - 5\,y + 4\,z = 0.5.$$
$$7\,x + 2\,y - 3\,z = 0.2.$$
$$4\,x + 3\,y - z = 0.7.$$

14. Solve the system

$$x + y + z = 3.824.$$
$$1.25\,x + 23.8\,y + 3.1\,z = 7.5276.$$
$$1.1\,x + 2\,y - 0.5\,z = 1.8505.$$

15. The sum of three capitals is \$111,000. The first is invested at 4%, the second at $4\frac{1}{2}$%, and the third at 5%, and the total annual interest is \$5120. If the first had been invested at $2\frac{1}{2}$%, the second at 3%, and the third at 4%, the total annual interest would have been \$3710. Required the capitals.

16. In each of three reservoirs is a certain quantity of water. If 20 gals. are drawn from the first into the second, the second will contain twice as much as the first; but if 30 gals. are drawn from the first into the third, the third will contain 20 gals. less than 4 times as much as the first; but if 25 gals. are drawn from the second into the third, the third will contain 50 gals. less than 3 times the second. How many gallons does each contain?

CHAPTER XI.

INDETERMINATE EQUATIONS.

207. A linear equation involving two unknown quantities can be satisfied by any number of values of those quantities.

E.g., in the equation $x + y = 5$

x can equal	$\cdots -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ 6, \cdots$
the corresponding values of y being	$7,\ \ 6,\ 5,\ 4,\ 3,\ 2,\ 1,\ 0,\ -1, \cdots.$

But of course this applies only to equations after like terms are united, and not to an equation like $x + y = x + 2$.

208. Equations like the above, which can be satisfied by an unlimited number of values of the unknown quantities are called **indeterminate equations.**

209. Since two equations containing three unknown quantities give rise, by eliminating one of these quantities, to a single equation containing only two, it follows that, in general, *Two equations, each containing three unknown quantities, are indeterminate as to all of these quantities.*

E.g., the two equations

$$2x + 3y + z = 10,$$

$$3x + 2y + z = 8,$$

give rise to the single equation

$$-x + y = 2,$$

or to $$5y + z = 14,$$

or to $$5x + z = 4,$$

all three of which are indeterminate.

210. Similarly, it is evident that, in general, n *linear equations, each containing* n + 1 *or more unknown quantities, are indeterminate.*

Roots of an indeterminate equation are often found by simple inspection.

E.g., to find the roots of $2x - 7y = 5$.

Let $x = 0, \quad 1, \quad 2, \quad 3, \; 4, \cdots$

then the corresponding values of y are $\frac{-5}{7}, \frac{-3}{7}, \frac{-1}{7}, \frac{1}{7}, \frac{3}{7}, \cdots$.

Similarly, find a set of roots of $x + 2y + 3z = 10$.

Let $z = 1;$

then $x + 2y = 7;$

and if $x = 0, \ 1, \ 2, \ 3, \cdots$

the corresponding values of y are $\frac{7}{2}$, 3, $\frac{5}{2}$, 2, $\cdots$.

That is, the equation is satisfied if

$$z = 1, \ x = 0, \ y = \tfrac{7}{2},$$

or if $z = 1, \ x = 1, \ y = 3$, etc.

Similarly, we may start with $z = 2$.

211. Sometimes it is desirable to find the various *positive integral roots* of an indeterminate equation. For practical purposes these may be found by simple inspection.

E.g., to find the positive integral roots of $5x + 3y = 19$. Here $x \ngtr 3$, because if $x > 3$, and integral, y is negative.

If $x = 3, \quad 2, \ 1$

then $y =$ a fraction, 3, a fraction.

$\therefore x = 2, \ y = 3$ are the only positive integral roots of the equation.

Graphs and discussion of equations. For those who have the time, the study of the graphic representation of linear equations, and the discussion of solutions (Appendix VIII) are strongly recommended at this point.

EXERCISES. XCVII.

1. Find three sets of roots of each of the following equations:

(a) $10x + 3y = -4.$

(b) $5x - 2y = 17.$

(c) $5x + 23y = 100.$

2. Find two entirely different sets of roots of each of the following equations:

(a) $x - 3y + 4z = 20.$

(b) $2x + 10y - z = 15.$

(c) $3x - 7y + 5z = 12.$

3. Find all of the positive integral roots of each of the following equations:

(a) $x + y = 5.$

(b) $2x + 10y = 30.$

(c) $3x + 5y = 20.$

4. Find all of the positive integral roots of

$$x + 2y + 3z = 14.$$

5. Find three sets of roots of the following system of equations:

$$x - 2y + 4z = 5.$$
$$2x - y + z = 1.$$

6. Find a set of roots of the following system of equations:

$$2w + 2x + 3y + z = 20.$$
$$3w + 3x + 2y + 2z = 25.$$
$$4w + 5x - y - z = 6.$$

CHAPTER XII.

THE THEORY OF INDICES.

I. THE THREE FUNDAMENTAL LAWS OF EXPONENTS.

212. It has already been proved that, when m and n are positive integers,

1. $a^m \cdot a^n \equiv a^{m+n}$. § 60

2. $a^m : a^n \equiv a^{m-n}$. § 86

3. $(a^m)^n \equiv a^{mn}$. § 75

It has also been stated (§ 125) that $a^{\frac{1}{2}}$ means the square root of a, $a^{\frac{1}{3}}$ means the cube root of a, and, in general, $a^{\frac{1}{n}}$ means the nth root of a, but the reason for this symbolism has not yet been given.

It is now proposed to investigate the meaning of the negative and the fractional exponents; that is, to find what meaning should be attached to symbols like 3^{-2}, $8^{\frac{2}{3}}$, $16^{-\frac{3}{4}}$, a^{-n}, $a^{-\frac{1}{n}}$, $\cdots$.

We shall then proceed to ascertain whether the three fundamental laws given above are true if m and n are fractional, negative, or both fractional and negative.

The necessity for this is apparent. We know that $a^m \cdot a^n \equiv a^{m+n}$, if m and n are positive integers, because a is taken first m times, and then n times, as a factor, and hence $m+n$ times in all. But we do not yet know that $a^{\frac{1}{n}} \cdot a^{\frac{1}{m}} = a^{\frac{1}{n}+\frac{1}{m}}$. Neither do we know that $a^{\frac{1}{m}} : a^{\frac{1}{n}} = a^{\frac{1}{m}-\frac{1}{n}}$, nor that $a^{-m} \cdot a^{-n} = a^{-m-n}$, nor that $a^{-m} \cdot a^{\frac{1}{n}} = a^{-m+\frac{1}{n}}$, etc.

II. THE MEANING OF THE NEGATIVE INTEGRAL EXPONENT.

213. The primitive idea of *power* (§ 8) was a product of equal factors. The primitive idea of *exponent* was the number which showed how many equal factors were taken.

According to this primitive idea the

3d power of a meant aaa, written a^3,
2d " " " aa, " a^2;

but there was no first power of a, because that is not the product of any number of a's, nor any zero power, fractional power, or negative power.

But since	a^3	means	aaa,	or $a^4 \div a$,
and	a^2	"	aa,	" $a^3 \div a$,
∴ it is reasonable to *define*	a^1	as	a,	" $a^2 \div a$,
and " " " "	a^0	"	1,	" $a \div a$,
" " " " "	a^{-1}	"	$\frac{1}{a}$,	" $1 \div a$,
" " " " "	a^{-2}	"	$\frac{1}{a^2}$,	" $\frac{1}{a} \div a$,
and, in general, to define	a^{-n}	"	$\frac{1}{a^n}$,	

n being a positive integer.

214. For this reason *we define*

a^1 to mean a,
a^0 " 1,
a^{-n} " $\frac{1}{a^n}$,

n being a positive integer.

But it is evident that $a \neq 0$.

Illustrative problems. 1. Express 2^{-3} as a decimal fraction.

1. 2^{-3}, by definition, means $\frac{1}{2^3}$.

2. $\frac{1}{2^3} = \frac{1}{8} = 0.125.$

2. Express $\frac{ab}{a^{-3}b^2}$ with positive exponents.

1. $\frac{ab}{a^{-3}b^2}$ means $\frac{ab}{\frac{1}{a^3}b^2}$. § 214

2. This equals $\frac{a^4}{b}$. § 161

3. Express $\frac{x^2y^3}{x^3y^2}$ in the integral form.

1. $\frac{x^2y^3}{x^3y^2} = \frac{y}{x}$ § 150

2. $= x^{-1}y.$ § 214

The expression $x^{-1}y$ is as much a fraction as is $\frac{y}{x}$, but it is not in the *form* of a common fraction.

4. Simplify $(2^{-2})^{-2}$.

1. 2^{-2} means $\frac{1}{2^2}$. § 214

2. $\left(\frac{1}{2^2}\right)^{-2}$ " $\left(\frac{1}{\frac{1}{2^2}}\right)^2$. § 214

3. This equals $(2^2)^2$ which equals 2^4. §§ 161, 75

5. Simplify $[(2^{-1})^{-1}]^{-1}$.

1. 2^{-1} means $\frac{1}{2}$.

2. $(\frac{1}{2})^{-1}$ " 2.

3. 2^{-1} " $\frac{1}{2}$.

4. $\therefore$ the expression reduces to $\frac{1}{2}$.

EXERCISES. XCVIII.

Express exs. 1–4 without exponents.

1. $\left[\left(\frac{2}{3}\right)^{-2}\right]^2$, $(2^{-2})^{-3}$.   2. $[(2^{-2})^{-2}]^{-2}$.

3. $\frac{1}{2^{-4}}$, $\frac{2}{4^{-2}}$, $\frac{3^{-2}}{2^{-3}}$, $\left(\frac{5}{7}\right)^{-2}$.   4. 2^{-3}, 3^{-2}, $\left(\frac{3}{2}\right)^{-1}$.

Express exs. 5–9 with positive exponents.

5. $\frac{a^{-d}b^{-c}c^{-a}}{a^{-2d}b^{-c}c^{-2a}}$.   6. $\frac{3x^{-5}y^{-6}z^{-7}}{4x^{-6}y^{-7}z^{-8}}$.

7. $\frac{x^m y^n}{x^{-m}y^{-n}}$, $\frac{x^{-m}y^{-n}}{x^m y^n}$.   8. $\frac{3^2a^3b^4c^5}{3^{-2}a^{-3}b^{-4}c^{-5}}$.

9. $2a^{-3}$, a^{-5}, $(-x)^{-x}$.

Express exs. 10–16 in the form of common fractions, with positive exponents for the factors.

10. $\left(\frac{x^3y^{-3}z}{x^{-3}y^3z}\right)^2$.   11. $\frac{a^{-x}b^{-y}}{y^{-a}x^{-b}}$, $\frac{2^{-3}3^{-2}}{3^{-3}2^{-2}}$.

12. $\frac{a^{-m-n}b^{-n-m}}{a^m b^n}$.   13. $\frac{3}{4}x^{-3}yz^{-4} \div \frac{4}{3}xy^2z^3$.

14. $[(x^{-a})^{-a}]^{-a}$.   15. $[(1-x)^{-2}(1-x^2)]^{-1}$.

16. $a^{-2}b^{-3}c^4d^5$, $a^{-m}b^{-n}c^{-p}$.

Simplify exs. 17–20.

17. $\frac{2^{-3}}{3^{-2}} \cdot \frac{4^{-5}}{5^{-4}} \cdot \frac{6^{-7}}{7^{-6}} \cdot \frac{6^3}{7^7}$.   18. $\frac{a^{-b}}{a^{-c}} \cdot \frac{c^{-a}}{c^{-b}} \cdot \frac{b^{-c}}{b^{-a}} \cdot a^b c^a b^c$.

19. $\frac{-a^{-2}}{(-a)^{-2}} \cdot \frac{1+a^{-3}}{3+a^{-1}} \cdot \frac{3a+1}{-a^{-2}}$.

20. $-a^{-2} \cdot (2-b)^2 \cdot 2^{-a} \cdot a^2 \cdot (b-2)^{-2}$.

III. THE MEANING OF THE FRACTIONAL EXPONENT.

215. We have now found the meaning of

1. The positive integral exponent greater than 1, the primitive meaning of exponent;
2. The unit exponent;
3. The zero exponent;
4. The negative integral exponent.

216. It remains to find the meaning which should attach to the **fractional exponent.**

The expression a^4 means $aaaa$,

and if the exponent is half as large,

a^2 or aa is the square root of a^4,

and if the exponent is half as large,

a^1 or a is the square root of a^2.

$\therefore$ if an exponent half as large indicates a square root,

$a^{\frac{1}{2}}$ should mean the square root of a.

Hence, $a^{\frac{1}{2}}$ is defined to mean the square root of a, and, in general, $a^{\frac{1}{n}}$ is defined to mean the nth root of a.

217. The reason for this is also seen from the fact that

$\because a^m \cdot a^m \cdots$ to n factors $\equiv a^{mn}$.

$\therefore a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdots$ " " should equal $a^{n\left(\frac{1}{n}\right)}$ or $a^{\frac{n}{n}}$ or a.

$\therefore a^{\frac{1}{n}}$ should be defined to mean the nth root of a.

218. And since $a^{mn} \equiv (a^m)^n$, so $a^{\frac{p}{q}}$ should be defined to be identical with $(a^{\frac{1}{q}})^p$.

Hence, *we define* $a^{\frac{p}{q}}$ *to mean the* p*th power of the* q*th root of* a, *and* $a^{-\frac{p}{q}}$ *to mean the reciprocal of* $a^{\frac{p}{q}}$.

219. The following identities involving fractional exponents are also true and will now be proved.

1. $a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots \equiv (abc\cdots)^{\frac{1}{n}}$. Proved in § 220
2. $(a^m)^{\frac{1}{n}} \equiv (a^{\frac{1}{n}})^m \equiv a^{\frac{m}{n}}$. " § 221
3. $a^{\frac{m}{n}} \equiv a^{\frac{pm}{pn}}$. " § 222
4. $(a^{\frac{1}{m}})^{\frac{1}{n}} \equiv a^{\frac{1}{mn}} \equiv (a^{\frac{1}{n}})^{\frac{1}{m}}$. " § 224

220. To prove that $a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots \equiv (abc\cdots)^{\frac{1}{n}}$.

1. Let $x = a^{\frac{1}{n}}b^{\frac{1}{n}}$.
2. ∴ $x^n = (a^{\frac{1}{n}}b^{\frac{1}{n}})^n$ Ax. 8
3. $= (a^{\frac{1}{n}})^n(b^{\frac{1}{n}})^n$ § 76
4. $= ab$. § 217
5. ∴ $x = (ab)^{\frac{1}{n}}$, or $a^{\frac{1}{n}}b^{\frac{1}{n}} \equiv (ab)^{\frac{1}{n}}$. Axs. 9, 1
6. ∴ $a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}} \equiv (ab)^{\frac{1}{n}}c^{\frac{1}{n}} \equiv (abc)^{\frac{1}{n}}$, and so on for any number of factors.

Similarly,
$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} \equiv \left(\frac{a}{b}\right)^{\frac{1}{n}}.$$

221. To prove that $(a^m)^{\frac{1}{n}} \equiv (a^{\frac{1}{n}})^m \equiv a^{\frac{m}{n}}$.

1. $(aaa\cdots \text{to } m \text{ factors})^{\frac{1}{n}} \equiv a^{\frac{1}{n}}a^{\frac{1}{n}}a^{\frac{1}{n}}\cdots$ to m factors. § 220
2. *I.e.*, $(a^m)^{\frac{1}{n}} \equiv (a^{\frac{1}{n}})^m$.
3. But $(a^{\frac{1}{n}})^m \equiv a^{\frac{m}{n}}$. Def. § 218

Hence, $a^{\frac{m}{n}}$ may be considered either as the mth power of the nth root of a (as defined in § 218) or as the nth root of the mth power of a.

But § 221 must be understood to apply only to the absolute values of the roots.

E.g., $(4^2)^{\frac{1}{2}} = 16^{\frac{1}{2}} = \pm 4,$

but $(4^{\frac{1}{2}})^2 = (\pm 2)^2 = +4.$

222. To prove that $a^{\frac{m}{n}} \equiv a^{\frac{pm}{pn}}$.

1. Let	$x = a^{\frac{m}{n}}$.	
2. $\therefore$	$x^n = a^m$.	Ax. 8 and § 221
3. $\therefore$	$x^{pn} = a^{pm}$.	Ax. 8 and § 75
4. $\therefore$	$x = a^{\frac{pm}{pn}}$.	Ax. 9 and § 218
5. $\therefore$	$a^{\frac{m}{n}} \equiv a^{\frac{pm}{pn}}$.	Ax. 1

Hence, both terms of a fractional exponent can be multiplied or divided by the same number without altering the value of the expression.

223. The student should understand clearly that § 222 is true *not because the exponent is a fraction.* The exponent is merely an expression in the *form* of a fraction, and hence a proof like that of § 150 has no application to this case. The laws of fractions apply to fractional exponents only as they are proved to do so.

224. To prove that $(a^{\frac{1}{m}})^{\frac{1}{n}} \equiv a^{\frac{1}{nm}} \equiv (a^{\frac{1}{n}})^{\frac{1}{m}}$.

1. Let	$x = (a^{\frac{1}{m}})^{\frac{1}{n}}$.	
2. $\therefore$	$x^n = a^{\frac{1}{m}}$.	Ax. 8
3. $\therefore$	$x^{mn} = a$.	Ax. 8
4. $\therefore$	$x = a^{\frac{1}{mn}}$.	Ax. 9
5. $\therefore$	$(a^{\frac{1}{m}})^{\frac{1}{n}} \equiv a^{\frac{1}{mn}}$, and similarly	
	$(a^{\frac{1}{n}})^{\frac{1}{m}} \equiv a^{\frac{1}{mn}}$.	

EXERCISES. XCIX.

Find the *absolute* value of each of the expressions in exs. 1–3.

1. $4^{\frac{1}{2}},\ 9^{\frac{1}{2}},\ 8^{\frac{1}{3}},\ 32^{\frac{1}{5}},\ 81^{\frac{1}{4}}.$

2. $25^{\frac{3}{2}},\ 125^{\frac{1}{3}},\ 32^{\frac{3}{5}},\ 64^{\frac{5}{6}},\ 625^{\frac{8}{4}}.$

3. $16^{-\frac{1}{2}},\ 36^{-\frac{3}{2}},\ 343^{-\frac{2}{3}},\ 1331^{-\frac{2}{3}},\ 14641^{-\frac{3}{4}}.$

Write in integral form, with negative or fractional exponents, the expressions in exs. 4–9.

4. $\dfrac{a}{\sqrt[4]{b}} + \dfrac{a+b}{\sqrt[3]{a-b}} - \dfrac{\frac{1}{a}}{\sqrt{a}+\sqrt[3]{a}} - \dfrac{1}{a^2}.$

5. $a\sqrt[3]{b} + b\sqrt[3]{a} + \sqrt[3]{a+b} - \sqrt[n]{a-b}.$

6. $\dfrac{1}{a\sqrt{a}} + \dfrac{1}{b\sqrt{b}} - \dfrac{\sqrt{a^3}+\sqrt{b^3}}{\sqrt{a^3b^3}} + \dfrac{1}{a^3} - \dfrac{1}{b^3}.$

7. $\sqrt{1 \div a^3},\ \sqrt[3]{1 \div a^2},\ \sqrt[5]{a^2 \div b^3},\ 1 \div \sqrt[7]{x^6y}.$

8. $a^2 \div (\sqrt{a} + \sqrt{1 \div a}) \div (a^3 + \sqrt[3]{a^2}) \div \sqrt{a^3}.$

9. $\sqrt[3]{a\sqrt{b}},\ \sqrt{\sqrt[3]{a \div b}},\ \sqrt{1 + b^3\sqrt{c}},\ \sqrt[3]{1 \div (a+b)^2}.$

Write the following without negative or fractional exponents, using the old form of radical sign ($\sqrt{\ }$) and the common fraction:

10. $a^{\frac{1}{2}}b^{\frac{1}{3}},\ a^{\frac{2}{3}}b^{\frac{1}{3}},\ x^{\frac{m}{n}},\ x^{\frac{m+1}{2}},\ x^{\frac{1}{2}}y^{\frac{1}{3}},\ z^{\frac{1}{6}}.$

11. $a^{-\frac{1}{5}},\ a^{-\frac{2}{3}}b^{-\frac{2}{3}},\ x^{\frac{7}{8}}y^{-\frac{7}{8}},\ x^{\frac{m+n}{m-n}}y^{\frac{m-n}{m+n}}.$

12. $\dfrac{a^{\frac{2}{3}}}{a^2 \div a^{-\frac{1}{3}}} - a^{\frac{1}{4}} \cdot a^{-\frac{2}{5}} \cdot a^{\frac{1}{3}} + (a - b^{-1})^{-2}.$

IV. THE THREE FUNDAMENTAL LAWS FOR FRACTIONAL AND NEGATIVE EXPONENTS.

225. Laws 1 and 2. *To prove that*

$$a^m \cdot a^n \equiv a^{m+n}$$

$$a^m : a^n \equiv a^{m-n}$$

if m *and* n *are fractional, negative, or both fractional and negative.*

a. *Let them be fractional and positive.* We have first to prove that $a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} \equiv a^{\frac{p}{q}+\frac{r}{s}}$.

1.	$a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} \equiv a^{\frac{ps}{qs}} \cdot a^{\frac{qr}{qs}}$	§ 222
2.	$\equiv (a^{ps})^{\frac{1}{qs}} \cdot (a^{qr})^{\frac{1}{qs}}$	§ 221
3.	$\equiv (a^{ps} \cdot a^{qr})^{\frac{1}{qs}}$	§ 220
4.	$\equiv (a^{ps+qr})^{\frac{1}{qs}}$	§ 60
5.	$\equiv a^{\frac{ps+qr}{qs}}$, or $a^{\frac{p}{q}+\frac{r}{s}}$.	§ 221

This shows that a case like $\sqrt[3]{a^2} \cdot \sqrt[5]{a^4}$ can be easily handled by fractional exponents, thus:

$$a^{\frac{2}{3}} \cdot a^{\frac{4}{5}} = a^{\frac{2}{3}+\frac{4}{5}} = a^{\frac{22}{15}}.$$

To see that $\sqrt[3]{a^2} \cdot \sqrt[5]{a^4}$ equals the 15th root of a^{22} is not so easy by the help of the old symbols alone.

We have also to prove that $a^{\frac{p}{q}} : a^{\frac{r}{s}} \equiv a^{\frac{p}{q}-\frac{r}{s}}$.

The proof is evidently identical with that just given, except that the sign of division replaces that of multiplication in the first member, and the sign of subtraction that of addition in the second member.

b. *Let one exponent be negative and either integral or fractional.* We have then to prove that

$$a^m \cdot a^{-n} \equiv a^{m+(-n)}, \text{ or } a^{m-n}.$$

1.	$a^m \cdot a^{-n} \equiv a^m \cdot \dfrac{1}{a^n}$	§§ 214, 218
2.	$\equiv \dfrac{a^m}{a^n}$	§ 156, cor. 2
3.	$\equiv a^{m-n}.$	§§ 86, 225, a

We have also to prove that $a^m : a^{-n} \equiv a^{m-(-n)} \equiv a^{m+n}$.

The proof is evidently identical with that just given, except that the sign of division replaces that of multiplication, and the sign of subtraction that of addition.

c. *Let both exponents be negative and either integral or fractional.* We have then to prove that

$$a^{-m} \cdot a^{-n} \equiv a^{-m+(-n)} \equiv a^{-m-n}.$$

1.	$a^{-m} \cdot a^{-n} \equiv \dfrac{1}{a^m} \cdot \dfrac{1}{a^n}$	§§ 214, 218
2.	$\equiv \dfrac{1}{a^m a^n}$	§ 156
3.	$\equiv \dfrac{1}{a^{m+n}}$	§§ 60, 225, a
4.	$\equiv a^{-m-n}.$	§§ 214, 218

As an illustration of the value of these laws, consider the case of $\dfrac{1}{\sqrt[5]{a^4}} : \dfrac{1}{\sqrt[4]{a^3}}$.

Here we have

$$a^{-\frac{4}{5}} : a^{-\frac{3}{4}} = a^{-\frac{16-15}{20}} = a^{-\frac{1}{20}},$$

or the 20th root of $\dfrac{1}{a}$, a result not so easily reached by the older notation.

226. Law 3. *To prove that* $(a^m)^n \equiv a^{mn}$, *if* m *and* n *are fractional, negative, or both fractional and negative.*

a. *Let* m *be fractional or negative or both,* n *being a positive integer.*

1. From §§ 60, 225, it follows that

$$a^p a^q a^r \cdots \equiv a^{p+q+r+\cdots}$$

if p, q, r, $\cdots$ are fractional, negative, or both fractional and negative.

2. And if $p = q = r = \cdots = m$, and there are n factors, then

$$(a^m)^n \equiv a^{mn},$$

whether m is positive or negative, integral or fractional, provided n is a positive integer.

b. *Let* m *and* n *be positive fractions. We then have to prove that* $(a^{\frac{p}{q}})^{\frac{r}{s}} \equiv a^{\frac{pr}{qs}}$.

1. Let	$x = (a^{\frac{p}{q}})^{\frac{r}{s}}$.	
2. Then	$x^s = (a^{\frac{p}{q}})^r$	Ax. 8 and § 221
3.	$= a^{\frac{pr}{q}}$.	§ 226, a
4. ∴	$x^{qs} = a^{pr}$.	Ax. 8 and § 221
5. ∴	$x = a^{\frac{pr}{qs}}$.	Ax. 9
6. ∴	$(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}$.	

c. *Let* n *be negative and either integral or fractional,* m *being positive. We have then to prove that* $(a^m)^{-n} \equiv a^{-mn}$.

1.	$(a^m)^{-n} \equiv \dfrac{1}{(a^m)^n}$	§ 214
2.	$\equiv \dfrac{1}{a^{mn}}$	§ 75
3.	$\equiv a^{-mn}$.	§ 214

d. *Let* m *be negative and either integral or fractional,* n *being positive. We have then to prove that* $(a^{-m})^n \equiv a^{-mn}$.

1.	$(a^{-m})^n \equiv \left(\frac{1}{a^m}\right)^n$	§ 214
2.	$\equiv \frac{1}{a^{mn}}$	§ 75
3.	$\equiv a^{-mn}$.	§ 214

e. *Let* m *and* n *be negative and either integral or fractional. We have then to prove that* $(a^{-m})^{-n} \equiv a^{mn}$.

1.	$(a^{-m})^{-n} \equiv \left(\frac{1}{a^m}\right)^{-n}$	§ 214
2.	$\equiv \left(1 \div \frac{1}{a^m}\right)^n \equiv (a^m)^n$	§ 214
3.	$\equiv a^{mn}$.	§ 75

The value of this law may be seen by the solution of a few problems. Consider for example the case of

$$\sqrt[3]{\left(\frac{1}{1 \div \sqrt[4]{a^3}}\right)^4}.$$

This expression, thus written in the older style, does not strike the eye as simple; but since $1 \div \sqrt[4]{a^3}$ may be written $a^{-\frac{3}{4}}$, the expression reduces to $(a^{\frac{3}{4}})^{\frac{4}{3}}$, which equals a.

Consider also the more complicated expression

$$x \cdot \sqrt[r^2-q^2]{\left(\frac{1}{\sqrt[q]{x^r}} \div \frac{1}{\sqrt[r]{x^q}}\right)^{qr}}.$$

Writing this with fractional and negative exponents, we have

$$x \cdot (x^{-\frac{r}{q}+\frac{q}{r}})^{\frac{qr}{r^2-q^2}} = x \cdot x^{-\frac{r^2-q^2}{qr} \cdot \frac{qr}{r^2-q^2}} = x \cdot x^{-1} = x^0 = 1.$$

To simplify this without the assistance of negative and fractional exponents would be more difficult.

V. PROBLEMS INVOLVING FRACTIONAL AND NEGATIVE EXPONENTS.

227. It has now been proved that we can operate with expressions involving negative or fractional exponents just as if these exponents were positive integers. Exercises involving such exponents will now be given.

The student should see the distinct advantage in using the fractional exponent instead of the old form of radical sign, except in cases like the expression of a single root, *and in using the negative exponent,* except in cases like the expression of a simple fraction. This has been shown on p. 215, but it is worth while to consider the matter further, that the student may become entirely familiar with the use of the modern symbols.

E.g., while it is easier to write $\sqrt{a}$ than $a^{\frac{1}{2}}$, and $\frac{1}{a}$ than a^{-1}, because we are more accustomed to the forms $\sqrt{a}$ and $\frac{1}{a}$, it is much easier to see that

$$(x^{-\frac{2}{3}})^{-\frac{3}{4}} = x^{\frac{1}{2}},$$

than to see that the equivalent expression

$$\frac{1}{\sqrt[4]{(1 \div \sqrt[3]{x^2})^3}} = \sqrt{x}.$$

Similarly, it is easier to recognize in

$$x^{\frac{8}{28}} + 2x^{\frac{3}{21}} + 1 = 0$$

the quadratic form

$$x^{\frac{2}{7}} + 2x^{\frac{1}{7}} + 1 = 0,$$

than to recognize it in

$$\sqrt[28]{x^8} + 2\sqrt[21]{x^3} + 1 = 0.$$

It is doubtful if students would readily grasp the significance of the form $a^3 + 2a^2\sqrt[6]{a} + a\sqrt[3]{a}$; but when written $a^{\frac{6}{2}} + 2a^{\frac{13}{6}} + a^{\frac{4}{3}}$ it is seen to be the square of $a^{\frac{3}{2}} + a^{\frac{2}{3}}$.

Illustrative problems. 1. Remove the parentheses from $(x^{-1} \div y^{-1})^{-2}$, expressing the result with positive exponents.

$$(x^{-1} \div y^{-1})^{-2} = x^2 \div y^2. \qquad \S\ 226$$

2. Multiply $x^{-2} + x^{-1} + 1$ by $x^{-2} - x^{-1} + 1$.

Since we can multiply as if the exponents were positive, we have the following:

$$\begin{array}{lllll|c}
 & & & & & \textit{Check.} \\
x^{-2} & + x^{-1} & + 1 & & & 3 \\
x^{-2} & - x^{-1} & + 1 & & & \underline{1} \\
\hline
x^{-4} & + x^{-3} & + x^{-2} & & & \\
 & - x^{-3} & - x^{-2} & - x^{-1} & & \\
 & & x^{-2} & + x^{-1} & + 1 & \\
\hline
x^{-4} & & + x^{-2} & & + 1 & 3
\end{array}$$

Detached coefficients should be used in practice.

3. Divide $x^{-3} + 3x^{-2} + 3x^{-1} + 1$ by $x^{-1} + 1$.

Since we can divide as if the exponents were positive, we have the following:

$$\begin{array}{l}
\text{Quotient} = x^{-2} + 2x^{-1} + 1 \\
x^{-1} + 1 \,|\, x^{-3} + 3x^{-2} + 3x^{-1} + 1 \\
\qquad\qquad\quad \underline{x^{-3} + x^{-2}} \\
\qquad\qquad\qquad 2x^{-2} + 3x^{-1} \\
\qquad\qquad\qquad \underline{2x^{-2} + 2x^{-1}} \\
\qquad\qquad\qquad\qquad x^{-1} + 1 \\
\qquad\qquad\qquad\qquad \underline{x^{-1} + 1}
\end{array}$$

Check. $8 \div 2 = 4.$

Detached coefficients should be used in practice.

4. Solve the equation $x^{-\frac{2}{3}} - 3x^{-\frac{1}{3}} + 2 = 0$.

1. $\because \quad x^2 - 3x + 2 \equiv (x-2)(x-1),$
2. $\therefore \quad x^{-\frac{2}{3}} - 3x^{-\frac{1}{3}} + 2 \equiv (x^{-\frac{1}{3}} - 2)(x^{-\frac{1}{3}} - 1).$
3. $\therefore \quad (x^{-\frac{1}{3}} - 2)(x^{-\frac{1}{3}} - 1) = 0.$
4. $\therefore \quad x^{-\frac{1}{3}} = 2,$ or $x^{-\frac{1}{3}} = 1.$
5. $\therefore \quad x^{-1} = 2^3 = 8,$ or $x^{-1} = 1^3 = 1.$
6. $\therefore \quad x = 8^{-1} = \frac{1}{8},$ or $x = 1,$ and these roots check.

EXERCISES. C.

Remove the parentheses and simplify in exs. 1–8.

1. $[(-a^3)^2]^5$.
2. $(x^{-1}+y^{-1})^3$.
3. $\{[(x^2-y^{-2})^{-1}]^3\}^{-2}$.
4. $[(a^{m+n})^{m-n}\cdot(a^{n^3})^{\frac{1}{n}}]^{\frac{1}{m^2}}$.
5. $[(a^{-p})^q]^{-\frac{1}{p}}:[(a^q)^{-r}]^{-\frac{1}{r}}$.
6. $\{[(x^{-2})^{-2}]^{-2}\}^{-2}$, $(a^{\frac{3}{4}}b^{\frac{5}{6}})^{\frac{2}{3}}$.
7. $(2x^{-2}\div y^{-2})^{-3}$, $\sqrt[3]{64[(x-y)^{-6}]^{\frac{1}{2}}}$.
8. $(x^{m-n})^{m+n}\cdot x^{n^2}:x^{m^2}$, $[(x^{\frac{2}{3}}y^{-\frac{5}{6}}z^{\frac{4}{5}})^{-\frac{7}{8}}]^{\frac{1}{4}}$.

Express with positive integral or fractional exponents, in simplest form, exs. 9–14.

9. $\sqrt[p]{a^{-q}b^{2p}}$.
10. $\sqrt[3]{x^7y^6z^5}$.
11. $\sqrt[m]{a^{2m}b^{3m^2}}$.
12. $\sqrt[m+n]{a^{m^2-n^2}}$.
13. $a^{-\frac{p}{q}}b^{-x}\sqrt[3]{cd^2}$.
14. $\sqrt[m]{a^{-m}b^{-2m}c^{-m^2}}$.

Perform the multiplications indicated in exs. 15–19.

15. $3a^{-\frac{2}{3}}\cdot 4a^{-\frac{5}{6}}\cdot 2a^{\frac{3}{4}}$; $ax^{-m}y^{-n}\cdot bx^ny^m$.
16. $a^{\frac{3}{2}}\cdot a^{\frac{2}{3}}$; $3a^{-2}b^{-3}c^4\cdot 4a^{-3}b^5c^{-4}$; $x^{-x}\cdot(-x^x)$.
17. $5\sqrt[3]{x^2y}\cdot 2x^{\frac{3}{4}}y^2$; $-a^{-4}b^4c^5d^{-5}\cdot -a^5b^{-5}c^4d^{-4}$.
18. $(x^2+2xy+y^2)\cdot(x^{-2}-2x^{-1}y^{-1}+y^{-2})$.
19. $(x^{-3}+3x^{-2}y+3x^{-1}y^2+y^3)\cdot(x^{-2}+2x^{-1}y+y^2)$.

Perform the divisions indicated in exs. 20–30.

20. $\dfrac{a^{\frac{4}{5}}b^{\frac{3}{4}}}{c^{\frac{2}{3}}} : \dfrac{a^{\frac{3}{4}}b^{\frac{2}{3}}}{c^{\frac{5}{6}}}.$

21. $4\,ab^{\frac{2}{3}}c^{\frac{1}{4}} : 2\,b^{\frac{4}{5}}c^{\frac{7}{8}}.$

22. $(a^{\frac{7}{8}}b^{\frac{5}{6}} - a^{\frac{3}{4}}b^{\frac{5}{7}} + 4\,a^{\frac{3}{8}}b^{\frac{2}{5}}) : a^{\frac{2}{3}}b^{\frac{4}{5}}.$

23. $4\,x^{-4} + 11\,x^{-2} - 45$ by $2\,x^{-1} - 3.$

24. $a^{-10} - a^{-5} + 1$ by $a^{-2} - a^{-1} + 1.$

25. $(4\,x^{\frac{4}{3}}y^{\frac{5}{3}} - 9\,x^{\frac{8}{5}}y^{\frac{7}{4}}) : (2\,x^{\frac{2}{3}}y^{\frac{5}{6}} + 3\,x^{\frac{4}{5}}y^{\frac{7}{8}}).$

26. $x^{-3} + 2\,x^{-2}y^{-1} - 3\,y^{-3}$ by $x^{-1} - y^{-1}.$

27. $3\,a^{-\frac{3}{4}} : 5\,a^{-\frac{5}{6}},\ x^2 : \{[x^{-\frac{1}{2}}y^{-\frac{1}{3}}(x^2y^2)^{\frac{2}{3}}]^{-\frac{1}{2}}\}^{-6}.$

28. $16\,x^{-3} + 6\,x^{-2} + 5\,x^{-1} - 6$ by $2\,x^{-1} - 1.$

29. $\sqrt[8]{4\,x^{-1}y^2z^{\frac{1}{2}}} : [(1 : \sqrt[4]{12\,x^3y^{-\frac{2}{3}}z^2}) \cdot \sqrt[12]{108\,x^{-3}y^2z^{-4}}].$

30. $x^{-5} - 2\,x^{-4} - 4\,x^{-3} + 19\,x^{-2} - 31\,x^{-1} + 15$ by $x^{-3} - 7\,x^{-1} + 5.$

31. Find the remainder when $x^{-5} - 11\,x^{-3} + 10$ is divided by $x^{-1} - 1.$

32. Also when $x^{-3} + (a - 3)\,x^{-2} + (b - 3\,a)\,x^{-1} - 3\,b$ is divided by $x^{-1} - 3.$

33. Factor $2\,x^{-3} - 9\,x^{-2} - 8\,x^{-1} + 15$, negative exponents being allowed in the factors.

34. Also $6\,x^{-3} + x^{-2} - 5\,x^{-1} - 2.$

35. Also $6\,x^{-3} + 17\,x^{-2} - 18\,x^{-1} - 45.$

36. Also $1 - 9\,x^{\frac{1}{5}} - 486\,x^{\frac{2}{5}}$, fractional exponents being allowed in the factors.

VI. IRRATIONAL NUMBERS. SURDS.

228. Rational and irrational algebraic expressions have already been defined (§ 98). But in algebra it is often necessary to use numbers which are irrational.

229. A **rational number** is a number expressible as the quotient of two integers.

E.g., $3 = \frac{3}{1}$, $0.666 \cdots = \frac{2}{3}$, $\frac{5}{7}$.

230. An **irrational number** is a number which is not rational.

E.g., $2^{\frac{1}{2}}$ or $\sqrt{2}$, $(1 + 2^{\frac{1}{2}})^{\frac{1}{3}}$ or $\sqrt[3]{1 + \sqrt{2}}$, $\sqrt{-1}$.

231. Irrational numbers which are not even roots of negative numbers are often called *surds*, but in elementary works the term is still further limited to irrational roots of rational numbers, or to such roots combined with rational numbers.

E.g., $\sqrt{2}$ and $3 + \sqrt[3]{5}$ are the types here treated, but not $\sqrt{2 + \sqrt[3]{3}}$ and $\sqrt{-5}$.

232. Surds are classified as follows:

1. According to the root index, as

quadratic, or of the second order,	as	$\sqrt{5}$,
cubic, " " third "	"	$\sqrt[3]{7}$,
quartic, or biquadratic,	"	$\sqrt[4]{x}$,
quintic,	"	$\sqrt[5]{5}$,
sextic	"	$\sqrt[6]{a}$,

and in general as

n-tic, n being a positive integer, " $\sqrt[n]{a}$.

2. **Similar or dissimilar** (if they have a single term), according as the surd factors are or are not the same.

E.g., $2\sqrt{3}$, $4\sqrt{3}$, $-7\sqrt{3}$ are similar surds.

$2\sqrt{3}$, $3\sqrt{2}$ are dissimilar surds.

$\sqrt[3]{2}\cdot\sqrt{3}$, $5\sqrt{3}$ are similar as to $\sqrt{3}$ but dissimilar as to $\sqrt[3]{2}$.

3. **Pure or mixed** (if they have a single term), according as they do not or do contain either real factors or dissimilar surd factors.

E.g., $\sqrt{3}$ is a pure surd, but $2\sqrt{3}$ and $\sqrt{5}\cdot\sqrt[3]{3}$ are mixed surds.

4. According to the number of terms in the expression when simplified, as

monomial surds, as $\sqrt{2}$, $3\sqrt[3]{2}$,

binomial " " $\sqrt{2}+\sqrt[3]{5}$, $5+\sqrt{2}$,

trinomial " " $2+\sqrt{3}+\sqrt[4]{7}$,

and, in general, **polynomial** surds.

5. According to simplicity. A surd is said to be in its *simplest form* when all the factors that are perfect roots are expressed without the root sign, when the index is as small as possible, and there are no fractions under the radical sign.

E.g., $\sqrt{9}$, $\sqrt[4]{4}$, $\sqrt{\frac{1}{2}}$, $\sqrt[3]{a^3x}$, are not in the simplest form. For

$$\sqrt{9}=3,$$

$$\sqrt[4]{4}=\sqrt{\sqrt{4}}=\sqrt{2},$$

$$\sqrt{\tfrac{1}{2}}=\sqrt{\tfrac{2}{4}}=\sqrt{\tfrac{1}{4}\cdot 2}=\tfrac{1}{2}\sqrt{2}.$$

$$\sqrt[3]{a^3x}=a\sqrt[3]{x}.$$

The fractional exponent is, in general, more convenient in all operations involving surds. The two forms of the radical symbol are used here in order that both may be familiar.

233. Convention as to signs. When we consider an expression like $\sqrt{4}+\sqrt{9}$ we see that it reduces to $(\pm 2)+(\pm 3)$, and hence to

$$+2+3=5,$$
$$+2-3=-1,$$
$$-2+3=1,$$
$$-2-3=-5.$$

But for simplicity it is agreed among mathematicians that in expressions of this kind *only the absolute values of the roots shall be considered* unless the contrary is stated.

Hence, $\sqrt{4}+\sqrt{9}=2+3=5$, but $\pm\sqrt{4}\pm\sqrt{9}=5,\ -1,\ 1$, or -5. (Compare § 192.)

EXERCISES. CI.

1. Classify according to the index of the root:

(a) $\sqrt[4]{5}$. (b) $\sqrt[5]{7}$. (c) $a^{\frac{1}{3}}$. (d) $x^{\frac{1}{2}}$.

2. Classify as similar or dissimilar:

(a) $2\sqrt{2},\ 5\sqrt{2},\ 8\cdot 2^{\frac{1}{2}}$. (b) $2\sqrt[3]{5},\ -\sqrt[3]{5},\ \frac{1}{2}\sqrt[3]{5}$.

3. Select the surds from the following:

(a) $\sqrt{2}$. (b) $4^{\frac{1}{2}}$. (c) $\sqrt{\sqrt{2}+\sqrt{3}}$.

4. Classify as pure or mixed:

(a) $\sqrt{47}$. (b) $3\sqrt[3]{5}$. (c) $ab^{\frac{1}{3}}$. (d) $\sqrt{2}\cdot\sqrt[3]{3}$.

5. Classify according to the number of terms:

(a) $a^{\frac{1}{2}}b^{\frac{1}{2}}$. (b) $\sqrt{2}+\sqrt[3]{5}$. (c) $2+\sqrt{3}+\sqrt[3]{4}$.

6. Find the value of each of these expressions:

(a) $\sqrt{4}+\sqrt{9}+\sqrt{16}$. (b) $\sqrt[3]{8}+\sqrt{25}+\sqrt[4]{16}+\sqrt[5]{32}$.

(c) $\sqrt[3]{1728}-\sqrt{144}+\sqrt{169}-13$.

234. Reduction of surds. It has been shown (§ 217) that $a \equiv (a^n)^{\frac{1}{n}}$. Hence, it follows that *a number can be reduced to the form of a surd of any order.*

E.g., $2 = \sqrt[3]{8}$, the *form* of a surd of the 3d order.

Similarly, $\sqrt{2}$ can be reduced to the *form* of a surd of the 5th order, for $2^{\frac{1}{2}} = (2^{\frac{5}{2}})^{\frac{1}{5}}$, or $\sqrt[5]{2^{\frac{5}{2}}}$, or $\sqrt[5]{32^{\frac{1}{2}}}$.

Similarly, $\sqrt[5]{4} = \sqrt[10]{4^2} = \sqrt[10]{16}$, a surd of the 10th order.

Hence, *mixed surds can always be reduced to pure surds.*

E.g., $\because \quad a\sqrt[n]{b} \equiv \sqrt[n]{a^n b},$

$\therefore \quad 3\sqrt[3]{5} = \sqrt[3]{3^3 \cdot 5} = \sqrt[3]{135}.$

235. Since it is desirable to have the number under the radical sign as small an integer as possible, it is often necessary *to reduce surds to their simplest forms* (§ 232, 5).

E.g.,

$$\sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{2}{4}} = \sqrt{\tfrac{1}{4} \cdot 2} = \tfrac{1}{2}\sqrt{2}.$$

$$\sqrt[3]{135} = \sqrt[3]{3^3 \cdot 5} = 3\sqrt[3]{5}.$$

$$\sqrt{\frac{5}{18}} = \sqrt{\frac{5}{3^2 \cdot 2}} = \sqrt{\frac{10}{3^2 \cdot 2^2}} = \frac{1}{6}\sqrt{10}.$$

Hence, in the case of fractions under the radical sign *we multiply both terms by the smallest number which will make the denominator the required power, then extract the indicated root of the denominator, and reduce the remaining surd as much as possible.*

E.g.,

$$\sqrt{\frac{12}{13}} = \sqrt{\frac{4}{13^2} \cdot 39} = \frac{2}{13}\sqrt{39}.$$

236. Since in multiplying surds it is desirable to have them of the same order, it is often necessary *to reduce several surds to equivalent surds of the same order,* the order always being as low as possible.

E.g., $\sqrt{2} \cdot \sqrt[3]{3} = 2^{\frac{1}{2}} \cdot 3^{\frac{1}{3}} = 2^{\frac{3}{6}} \cdot 3^{\frac{2}{6}} = (2^3 \cdot 3^2)^{\frac{1}{6}} = \sqrt[6]{8 \cdot 9} = \sqrt[6]{72}.$

EXERCISES. CII.

1. Reduce the following numbers to the *forms* of surds of the orders indicated:

(a) 5, 3d order. (b) 2, 6th order.
(c) $\frac{1}{8}$, 4th " (d) 10, 5th "
(e) 11, 2d " (f) 12, 3d "
(g) -2, 2d " (h) -5, 3d "
(i) 3, 5th " (j) -2, 6th "

2. Reduce the following to pure surds:

(a) $2\sqrt{3}$. (b) $3\sqrt{2}$. (c) $2\sqrt[5]{2}$.
(d) $5\cdot 2^{\frac{1}{2}}$. (e) $a^{\frac{1}{2}}b^{\frac{1}{2}}c$. (f) $a\sqrt{2a^3}$.
(g) $3\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{5}$. (h) $ab^{\frac{1}{2}}c$.

3. Reduce the following numbers to the *forms* of surds of the orders indicated:

(a) $\sqrt[3]{abc^2}$, 9th order. (b) $\sqrt[7]{a^6}$, 14th order.
(c) $\sqrt[5]{5}$, 30th " (d) $3^{\frac{1}{3}}$, 15th "
(e) $5^{\frac{1}{4}}$, 20th " (f) $10^{\frac{1}{3}}$, 15th "
(g) $\sqrt[4]{4}$, 8th " (h) $\sqrt[15]{5}$, 60th "

4. Reduce the following to equivalent surds of the same order, the order being as low as possible in each case:

(a) $\sqrt{a}$, $\sqrt[3]{b}$. (b) $\sqrt{3}$, $\sqrt[3]{3}$, $\sqrt[7]{2}$.
(c) $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $4^{\frac{1}{4}}$. (d) $\sqrt[8]{4}$, $\sqrt[4]{3}$, $\sqrt[12]{5}$.
(e) $a^{\frac{1}{2}}b^{\frac{2}{3}}$, $a^{\frac{2}{3}}b^{\frac{1}{2}}$. (f) $\sqrt[5]{2}$, $\sqrt[2]{5}$, $\sqrt[20]{3}$.
(g) $7^{\frac{1}{2}}$, $9^{\frac{1}{6}}$, $11^{\frac{1}{9}}$. (h) 2, $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{4}$, $\sqrt[5]{5}$.

237. Addition and subtraction of surds. Irrational expressions may evidently be added and subtracted the same as rational expressions, by taking advantage of some convenient unit.

E.g.,

		Check.
$a\sqrt{x}\ +$	$b\sqrt[3]{x} - c\sqrt[n]{z}$	1
$-$	$c\sqrt[3]{x} + c\sqrt[n]{z}$	0
$a\sqrt{x}\ +$	$b\sqrt[3]{x} + c\sqrt[n]{z}$	3
$2a\sqrt{x} +$	$(2b - c)\sqrt[3]{x} + c\sqrt[n]{z}$	4

Similarly, required the sum of $\sqrt{24}$, $\sqrt{54}$, and $-\sqrt{96}$. Here we have, each surd being reduced to its simplest form,

$$\begin{aligned}\sqrt{24} &= \sqrt{4\cdot 6} = 2\sqrt{6}\\ \sqrt{54} &= \sqrt{9\cdot 6} = 3\sqrt{6}\\ -\sqrt{96} &= -\sqrt{16\cdot 6} = -4\sqrt{6}\end{aligned}$$

Hence, the sum is $\sqrt{6}$

Similarly, required the sum of $\sqrt{8}$, $\sqrt{27}$, $-2\sqrt{2}$, and $\sqrt{48}$. Here we have $2\sqrt{2} + 3\sqrt{3} - 2\sqrt{2} + 4\sqrt{3} = 7\sqrt{3}$.

In general, however, the sums of surds can only be indicated as $\sqrt{3} + \sqrt[3]{7}$, $-\sqrt[n]{a} + \sqrt[m]{b}$.

EXERCISES. CIII.

Simplify the following:

1. $\sqrt{72} + \sqrt{108} - \sqrt{32} - \sqrt{243}$.
2. $\sqrt[3]{24} + \sqrt[3]{375} - \sqrt[3]{648} + 10\sqrt[3]{3}$.
3. $\sqrt{a^4b} + \sqrt[4]{a^{12}b^2} - \sqrt[5]{a^{10}b^5}\cdot\sqrt{b}$.
4. $(a^6b)^{\frac{1}{3}} - a^2\sqrt[3]{b} + a^2b^3c$.
5. $\sqrt{147} + \sqrt{243} - \sqrt{363} + \sqrt{432} - \sqrt{507}$.
6. $\sqrt[3]{1715} + \sqrt[3]{3645} + \sqrt[3]{6655} + \sqrt[3]{8640} - 39\sqrt[3]{5}$.
7. $\sqrt[3]{x^4 + 5x^3 + 6x^2 - 4x - 8} + \sqrt[3]{x^4 - 4x^3 + 6x^2 - 4x + 1}$.

238. Multiplication of surds. In general, products involving irrational numbers must be indicated, as $3\sqrt{2}$, or expressed approximately, as $3\sqrt{2} = 3 \cdot 1.414 \cdots = 4.24 \cdots$.

E.g.,
$$3\sqrt{2} \cdot 2\sqrt[3]{3} = 3\sqrt[6]{8} \cdot 2\sqrt[6]{9} \qquad \S\ 236$$
$$= 6\sqrt[6]{72}. \qquad \S\ 220$$

This result, while it still leaves a root to be extracted and a multiplication to be performed, is more compact than the indicated product $3\sqrt{2} \cdot 2\sqrt[3]{3}$.

Similarly, to square $3\sqrt{2} + 2\sqrt[3]{3}$.
$$(3\sqrt{2} + 2\sqrt[3]{3})^2 = (3\sqrt{2})^2 + 2(3\sqrt{2})(2\sqrt[3]{3}) + (2\sqrt[3]{3})^2 \qquad \S\ 69,\ 1$$
$$= 18 + 12\sqrt[6]{72} + 4\sqrt[3]{9}.$$

It is understood that no results are to be expressed approximately, in decimal form, unless so stated.

EXERCISES. CIV.

Perform the following multiplications:

1. $3\sqrt{\frac{3}{4}} \cdot 2\sqrt[3]{\frac{4}{5}}$.
2. $\sqrt{2} \cdot \sqrt[3]{3} \cdot \sqrt[3]{2} \cdot \sqrt{3}$.
3. $(3 - 5\sqrt{3})^2$.
4. $\sqrt{a} \cdot \sqrt[3]{a} \cdot \sqrt[4]{a} \cdot \sqrt[5]{a}$.
5. $\sqrt{7} \cdot \sqrt[3]{7} \cdot \sqrt[6]{7}$.
6. $(\sqrt{a - b} + \sqrt{a + b})^2$.
7. $2\sqrt{2} \cdot 3\sqrt{3} \cdot 5\sqrt{6}$.
8. $\sqrt[9]{121} \cdot \sqrt[3]{11} \cdot \sqrt[9]{14641}$.
9. $(2 + 8\sqrt{3})(4 - 5\sqrt{3})$.
10. $3\sqrt{2} \cdot 2\sqrt[3]{3} \cdot 4\sqrt[4]{5} \cdot 5\sqrt[6]{6}$.
11. $\sqrt{\sqrt{a}} + \sqrt{b} \cdot \sqrt[3]{\sqrt{a}} - \sqrt{b}$.
12. $(\sqrt{2} + \sqrt{3})(2\sqrt{2} - 5\sqrt{3})$.
13. $5\sqrt[3]{(a + 2b)^2} \cdot 3\sqrt[3]{(a + 2b)^2}$.

239. Division of surds. To divide an irrational number *by a rational number* is equivalent to multiplying by the reciprocal of the rational number, and hence it may be considered as a case of multiplication.

E.g., $\frac{a+\sqrt{b}}{c}$ is merely $\frac{1}{c}\cdot(a+\sqrt{b})$, or $\frac{a}{c}+\frac{1}{c}\sqrt{b}$.

240. Division by a surd usually reduces, without much difficulty, to division by a rational number, as shown in the following example:

To divide $\sqrt{2}+\sqrt{3}$ by $\sqrt{5}$, we have:

$$\frac{\sqrt{2}+\sqrt{3}}{\sqrt{5}}=\frac{\sqrt{5}(\sqrt{2}+\sqrt{3})}{\sqrt{5}\cdot\sqrt{5}},$$

assuming that we can multiply both terms of the fraction by $\sqrt{5}$ without changing the value, as we can in the case of rational multipliers (§ 150). This equals

$$\frac{\sqrt{10}+\sqrt{15}}{5}, \text{ or } \tfrac{1}{5}(\sqrt{10}+\sqrt{15}).$$

241. In the preceding example we have reduced the fraction to an equivalent fraction with a rational denominator. The process of rendering a quantity rational is called **rationalization.**

The advantage of rationalizing the denominator is seen by considering the computation necessary to find the approximate value of $\frac{\sqrt{2}+\sqrt{3}}{\sqrt{5}}$. Here there are three square roots to be extracted, followed by one addition and by one division with a long divisor.

But in the case of $\frac{1}{5}(\sqrt{10}+\sqrt{15})$ there are only two square roots to be extracted, followed by one addition and by one division with a short divisor.

242. The factor by which an expression is multiplied to produce a rational expression is called a **rationalizing factor.**

E.g., $\sqrt{8}$ can be rationalized by multiplying it by $\sqrt{2}$.

243. Since *the problem of division by surds reduces to that of the rationalization of the divisor,* exercises in rationalization will first be considered.

Illustrative problems. 1. By what expression may $a^{\frac{2}{3}}b^{\frac{1}{4}}$ be multiplied in order that the product shall be rational.

1. $\because \qquad x^{\frac{1}{n}} \cdot x^{1-\frac{1}{n}} = x,$

2. $\therefore \qquad a^{\frac{2}{3}}b^{\frac{1}{4}} \cdot a^{1-\frac{2}{3}}b^{1-\frac{1}{4}} = ab.$

3. $\therefore$ $a^{1-\frac{2}{3}}b^{1-\frac{1}{4}}$, or $a^{\frac{1}{3}}b^{\frac{3}{4}}$, is a rationalizing factor. There are evidently any number of rationalizing factors, since we may multiply this one by any rational expression. This is, however, the simplest one.

2. By what expression may $\sqrt[5]{a^4} \cdot \sqrt[4]{b^5}$ be multiplied in order that the product shall be rational?

1. $\sqrt[5]{a^4} \cdot \sqrt[4]{b^5} \equiv a^{\frac{4}{5}}b^{\frac{5}{4}} \equiv a^{\frac{4}{5}}b^{\frac{1}{4}}b.$

2. Evidently $a^{\frac{4}{5}}b^{\frac{1}{4}}b \cdot a^{1-\frac{4}{5}}b^{1-\frac{1}{4}}$ will equal ab^2, a rational expression.

3. $\therefore$ $a^{\frac{1}{5}}b^{\frac{3}{4}}$ is a rationalizing factor.

3. By what expression may $a + \sqrt{b}$ be multiplied in order that the product shall be rational?

1. $\because \qquad (x - y)(x + y) \equiv x^2 - y^2,$ § 69

2. $\therefore \qquad (a - \sqrt{b})(a + \sqrt{b}) \equiv a^2 - b$, a rational expression.

3. $\therefore \qquad a - \sqrt{b}$ is a rationalizing factor.

244. And, in general, *the conjugate of a binomial quadratic surd* (§ 69, 3) *is a rationalizing factor of that surd.*

4. Find a rationalizing factor for $\sqrt{a} \pm \sqrt{b} \pm \sqrt{c}.$

1. $\because (x + y + z)(-x + y + z)(x - y + z)(x + y - z)$

$$\equiv 2x^2y^2 + 2y^2z^2 + 2z^2x^2 - x^4 - y^4 - z^4,$$

2. $\therefore$ any trinomial quadratic surd of the form $\sqrt{a} \pm \sqrt{b} \pm \sqrt{c}$ can be rationalized by multiplying it by the product of the other three trinomials. *E.g.*, the rationalizing factor for $\sqrt{2} - \sqrt{3} + \sqrt{5}$ is

$$(\sqrt{2} + \sqrt{3} + \sqrt{5})(-\sqrt{2} + \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5}).$$

EXERCISES. CV.

Find the simplest rationalizing factor for each of the following expressions:

1. $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$.
2. $\sqrt{7}-\sqrt{5}$.
3. $a^{\frac{7}{8}}b^{\frac{5}{6}}c^{\frac{3}{2}}$.
4. $\sqrt{a}-\sqrt{b}$.
5. $2+\sqrt{3}$.
6. $\sqrt{a+b}+c$.
7. $3-\sqrt{2}$.
8. $\sqrt{5}-\sqrt{2}-\sqrt{3}$.
9. $\sqrt{5}-1$.
10. $\sqrt{5}+\sqrt{7}+\sqrt{11}$.
11. $a^{\frac{1}{16}}b^{\frac{7}{8}}c^{\frac{n-1}{n}}$.
12. $\sqrt{2}+\sqrt{7}-\sqrt{11}$.
13. $\sqrt{7}+\sqrt{5}$.
14. $\sqrt{a+b}+\sqrt{a-b}$.

Illustrative problems in division. 1. Divide $\sqrt[3]{12}$ by $\sqrt[3]{3}$.

$$\frac{\sqrt[3]{12}}{\sqrt[3]{3}}=\sqrt[3]{\frac{12}{3}}=\sqrt[3]{4}. \qquad \text{§ 220}$$

2. Divide $\sqrt{5}$ by $\sqrt[3]{2}$.

1. $$\frac{\sqrt{5}}{\sqrt[3]{2}}=\frac{\sqrt[6]{5^3}}{\sqrt[6]{2^2}}=\sqrt[6]{\frac{5^3}{2^2}} \qquad \text{§ 220}$$

2. $$=\sqrt[6]{\frac{2^4\cdot 5^3}{2^6}}=\tfrac{1}{2}\sqrt[6]{2000}.$$

3. Divide $\sqrt{2}+\sqrt{3}$ by $\sqrt{2}-\sqrt{3}$. That is, rationalize the denominator of the fraction $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$.

1. The rationalizing factor for the denominator is evidently
$$\sqrt{2}+\sqrt{3}.$$

2. $$\frac{(\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}=\frac{2+2\sqrt{6}+3}{2-3}=-(5+2\sqrt{6}).$$

EXERCISES. CVI.

Perform the divisions indicated in exs. 1–16.

1. $6 : 4\sqrt[4]{24}$.

2. $24 : (2\sqrt{7} - 6)$.

3. $3a^2b^3 : 2\sqrt[3]{a^2b}$.

4. $15\sqrt{24} : 3\sqrt[3]{24}$.

5. $58 : (8 + \sqrt{35})$.

6. $12\sqrt[3]{192} : 4\sqrt[5]{729}$.

7. $16\sqrt[3]{a^2bc^2} : 8\sqrt[4]{a^3b^2c}$.

8. $90 : (5\sqrt{3} - \sqrt{30})$.

9. $10\sqrt{12} : 2\sqrt{18} : 4\sqrt{8}$.

10. $\sqrt[4]{(a^3 - 2b)^3} : \sqrt{(a^3 - 2b)}$.

11. $(\sqrt{12} - \sqrt{18} + \sqrt{6}) : \sqrt{2}$.

12. $(3\sqrt{5} - 8\sqrt{2}) : (3\sqrt{3} - 4\sqrt{5})$.

13. $(18 - 16\sqrt{5}) : (4 - \sqrt{5} - 2\sqrt{3})$.

14. $(7\sqrt{12} - 4\sqrt{27}) : (8\sqrt{3} + 2\sqrt{2})$.

15. $(15\sqrt{8} + 10\sqrt{7} - 8\sqrt{2} + 5) : -4\sqrt{5}$.

16. $(3\sqrt{3} - 2\sqrt{2}) : (5\sqrt{8} - 3\sqrt{2} - 2\sqrt{3})$.

Rationalize the denominators of the fractions in exs. 17–23.

17. $\dfrac{11^{\frac{1}{2}} + 5^{\frac{1}{2}}}{11^{\frac{1}{2}} - 5^{\frac{1}{2}}}$.

18. $\dfrac{30}{2 - 3^{\frac{1}{2}} + 5^{\frac{1}{2}}}$.

19. $\dfrac{7 + 3\sqrt{7}}{12 - 6\sqrt{11}}$.

20. $\dfrac{3 - 5^{\frac{1}{2}} - 2^{\frac{1}{2}}}{3 + 5^{\frac{1}{2}} + 2^{\frac{1}{2}}}$.

21. $\dfrac{2}{(a^2 + b)^{\frac{1}{2}} + (a^2 - b)^{\frac{1}{2}}}$.

22. $\dfrac{2m}{(a + m)^{\frac{1}{2}} + (a - m)^{\frac{1}{2}}}$.

23. $\dfrac{1}{x(1 - a^2)^{\frac{1}{2}} - y(1 + a^2)^{\frac{1}{2}}}$.

245. Roots of surds. The roots of perfect powers of surd expressions can often be found by inspection or extracted in the ordinary way.

1. To find the square root of $a + 4\sqrt{ab} + 4b$.

1. $\because$ $\sqrt{f^2 + 2fn + n^2} \equiv \pm(f + n)$,

2. $\therefore$ $\sqrt{a + 4\sqrt{ab} + 4b} = \pm(\sqrt{a} + 2\sqrt{b})$.

Check. $\sqrt{9} = \pm 3$.

2. To find the fifth root of the perfect fifth power

$$a^{\frac{5}{2}} - 5a^2b^{\frac{1}{3}} + 10a^{\frac{3}{2}}b^{\frac{2}{3}} - 10ab + 5a^{\frac{1}{2}}b^{\frac{4}{3}} - b^{\frac{5}{3}}.$$

This is readily seen to be $a^{\frac{1}{2}} - b^{\frac{1}{3}}$. § 82

To check, let $a = b = 1$. Then $0^5 = 0$. If, however, we wish to check the exponents, let a equal any square and b equal any cube. *E.g.*, let $a = 9$, $b = 8$. Then

$$(3 - 2)^5 = 243 - 810 + 1080 - 720 + 240 - 32.$$

3. To find the square root of $7 + 4\sqrt{3}$.

1. If this can be brought into the form $f^2 + 2fn + n^2$, the root will be in the form $\pm(f + n)$. § 69

2. We first make the coefficient of the second term 2, because of the $2fn$, and have $7 + 2\sqrt{12}$.

3. And $\because$ 12 is the product of 3 and 4, and 7 is the sum of 3 and 4, we have

$$\sqrt{7 + 4\sqrt{3}} = \sqrt{4 + 2\sqrt{3 \cdot 4} + 3} = \pm(\sqrt{4} + \sqrt{3}) = \pm(2 + \sqrt{3}).$$

Check. Square $2 + \sqrt{3}$.

4. To find the square root of $8 - 2\sqrt{15}$.

1. As in ex. 3 we attempt to bring this into the form $f^2 + 2fn + n^2$.

2. $\because$ 15 is the product of 5 and 3, and 8 is their sum, we have

$$\sqrt{8 - 2\sqrt{15}} = \sqrt{5 - 2\sqrt{15} + 3} = \pm(\sqrt{5} - \sqrt{3}).$$

Of these results, only the positive one is usually considered in practice.

Check. Square $\sqrt{5} - \sqrt{3}$.

EXERCISES. CVII.

1. Extract the square roots of

(a) $a - 2\sqrt{2ab} + 2b$. (b) $a - 2a^2 + a^3$.

(c) $3a - 8\sqrt{3a} + 16$. (d) $a^{\frac{1}{2}} - 2a^{\frac{1}{4}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$.

(e) $2a - \sqrt{200a} + 25$. (f) $x^4 + 2x^2\sqrt[3]{y} + \sqrt[3]{y^2}$.

2. Extract the cube roots of

(a) $8 - 12\sqrt{a} + 6a - a\sqrt{a}$.

(b) $a - 3\sqrt[3]{a^2b^2} + 3b\sqrt[3]{ab} - b^2$.

(c) $x^3 - 3x^2\sqrt[4]{y} + 3x\sqrt{y} - \sqrt[4]{y^3}$.

(d) $x^4\sqrt{x} - 3x^3\sqrt[3]{y} + 3x\sqrt[6]{x^3y^4} - y$.

3. Extract the fifth roots of

(a) $1 - 5y^{\frac{2}{5}} + 10y^{\frac{4}{5}} - 10y^{\frac{6}{5}} + 5y^{\frac{8}{5}} - y^2$.

(b) $32 - 80\sqrt[5]{y^2} + 80\sqrt[5]{y^4} - 40y\sqrt[5]{y} + 10y\sqrt[5]{y^3} - y^2$.

4. Extract the square roots of

(a) $8 - 2\sqrt{7}$. (b) $\frac{3}{2} + \sqrt{2}$.

(c) $8 + \sqrt{60}$. (d) $9 - 4\sqrt{2}$.

(e) $10 - \sqrt{96}$. (f) $\frac{5}{6} + \frac{1}{3}\sqrt{6}$.

(g) $10\sqrt{7} + 32$. (h) $112 + 40\sqrt{3}$.

5. Extract the square roots of

(a) $2x + 2\sqrt{x^2 - y^2}$. (b) $2x + 2\sqrt{x^2 - 1}$.

(c) $ab - 2a\sqrt{ab - a^2}$.

(d) $x^2 + x + y + 2x\sqrt{x + y}$.

VII. THE BINOMIAL THEOREM.

246. It has been shown (§ 80, the proof being given in Appendix I) that if n is a positive integer

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \cdots.$$

It was proved by Sir Isaac Newton that this is true even if n is negative or fractional. The proof is, however, too difficult for the student at this time.

Assuming that the binomial theorem is true whether n is positive or negative, integral or fractional, it offers a valuable exercise in the use of negative and fractional exponents.

E.g., $\because\ (a+b)^n$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \cdots,$$

$$\therefore\ \sqrt{a+b} = (a+b)^{\frac{1}{2}}$$

$$= a^{\frac{1}{2}} + \tfrac{1}{2}a^{\frac{1}{2}-1}b + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}a^{\frac{1}{2}-2}b^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{2\cdot 3}a^{\frac{1}{2}-3}b^3 + \cdots$$

$$= a^{\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{1}{2}}b - \tfrac{1}{8}a^{-\frac{3}{2}}b^2 + \tfrac{1}{16}a^{-\frac{5}{2}}b^3 - \cdots.$$

$$\therefore\ \sqrt{5} = (4+1)^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}} + \tfrac{1}{2}\cdot 4^{-\frac{1}{2}} - \tfrac{1}{8}\cdot 4^{-\frac{3}{2}} + \tfrac{1}{16}\cdot 4^{-\frac{5}{2}} - \cdots$$

$$= 2 + \tfrac{1}{4} - \tfrac{1}{64} + \tfrac{1}{512} - \cdots.$$

$$\frac{1}{(1+x)^3} = (1+x)^{-3}$$

$$= 1 + (-3)x + \frac{-3(-3-1)}{2}x^2 + \frac{-3(-3-1)(-3-2)}{2\cdot 3}x^3 + \cdots$$

$$= 1 - 3x + 6x^2 - 10x^3 + \cdots.$$

EXERCISES. CVIII.

1. Expand to four terms $(1 + x)^{-1}$.

2. Also $1 / \sqrt{1 - x}$.

3. Also $\sqrt{14} = \sqrt{16 - 2} = 4\,(1 - \frac{1}{8})^{\frac{1}{2}}$.

4. Find the 5th term in the expansion of $(1 - x)^{-2}$.

5. Also in the expansion of $(1 + x)^{\frac{2}{3}}$.

6. Also in the expansion of $(1 - x)^{\frac{3}{4}}$.

7. Find $\sqrt{10}$ by expanding $(9 + 1)^{\frac{1}{2}}$ to four terms, reducing these to decimal fractions and adding.

8. Similarly for $\sqrt{82} = (81 + 1)^{\frac{1}{2}}$.

9. Similarly for $\sqrt[3]{28} = (27 + 1)^{\frac{1}{3}}$.

10. Similarly for $\sqrt{37} = (36 + 1)^{\frac{1}{2}}$.

REVIEW EXERCISES. CIX.

1. Divide $x^{\frac{1}{2}} - 4\,x^{\frac{3}{8}}a^{\frac{1}{8}} + 6\,x^{\frac{1}{4}}a^{\frac{1}{4}} - 4\,x^{\frac{1}{8}}a^{\frac{3}{8}} + a^{\frac{1}{2}}$ by $x^{\frac{1}{4}} - 2\,x^{\frac{1}{8}}a^{\frac{1}{8}} + a^{\frac{1}{4}}$.

2. Simplify $3\,(a^{\frac{1}{2}} + x^{\frac{1}{2}})^2 - 4\,(a^{\frac{1}{2}} + x^{\frac{1}{2}})\,(a^{\frac{1}{2}} - x^{\frac{1}{2}}) + (a^{\frac{1}{2}} - 2\,x^{\frac{1}{2}})^2$.

3. Simplify $\dfrac{a^m + b^n}{a^{-m} + b^{-n}} \cdot \dfrac{a^n - b^m}{a^{-n} - b^{-m}}$.

4. By inspection find the square root of

   (a) $4\,a^{-\frac{1}{2}} + 4 + a^{\frac{1}{2}}$.

   (b) $a^{\frac{4}{5}} - 2\,a^{\frac{3}{5}} + 5\,a^{\frac{2}{5}} - 4\,a^{\frac{1}{5}} + 4$.

   (c) $x + y + z + 2\,x^{\frac{1}{2}}y^{\frac{1}{2}} - 2\,x^{\frac{1}{2}}z^{\frac{1}{2}} - 2\,y^{\frac{1}{2}}z^{\frac{1}{2}}$.

   (d) $a^2 + 4\,a^{\frac{3}{2}}y^{\frac{1}{3}} + 10\,ay^{\frac{2}{3}} + 12\,a^{\frac{1}{2}}y + 9\,y^{\frac{4}{3}}$.

5. Simplify $(3^{\frac{3}{4}} + 3^{\frac{1}{2}} + 3^{\frac{1}{4}} + 1)(3^{\frac{1}{4}} - 1)$.

6. Factor $36\,x^{\frac{2}{7}} - 65\,x^{\frac{1}{7}} - 36$, fractional exponents being allowed in the factors.

7. Also $4\,x^{\frac{4}{9}} - 4\,x^{\frac{2}{9}}y^{\frac{2}{9}} + 9\,y^{\frac{4}{9}}$.

8. Solve the equation $x^{\frac{2}{3}} + 3\,x^{\frac{1}{3}} + 2 = 0$.

9. Also $4\,x^{\frac{2}{3}} - 15\,x^{\frac{1}{3}} + 14 = 0$.

10. Also $x^{\frac{2}{5}} - 5\,x^{\frac{1}{5}} + 6 = 0$.

11. Extract the square root of

$$a^2b^{-1} + \tfrac{1}{4}\,a^{-1}b^2 + 2\,(a^{\frac{3}{2}} - b^{\frac{3}{2}})\,(ab)^{-\frac{1}{4}}.$$

12. Also of $25\,x^{-4} - 30\,x^{-3}y + 49\,x^{-2}y^2 - 24\,x^{-1}y^3 + 16\,y^4$.

13. Extract the cube root of

$$x^{-6} - 9\,x^{-5} + 33\,x^{-4} - 63\,x^{-3} + 66\,x^{-2} - 36\,x^{-1} + 8.$$

14. Also of

$$8\,x^2 + 48\,x^{\frac{5}{3}} + 60\,x^{\frac{4}{3}} - 80\,x - 90\,x^{\frac{2}{3}} + 108\,x^{\frac{1}{3}} - 27.$$

15. Also of

$$8\,a^{\frac{3}{4}} + 48\,a^{\frac{5}{8}}b + 60\,a^{\frac{1}{2}}b^2 - 80\,a^{\frac{3}{8}}b^3 - 90\,a^{\frac{1}{4}}b^4 + 108\,a^{\frac{1}{8}}b^5 - 27\,b^6.$$

16. Also of

$$\tfrac{1}{64}\,x^{-\frac{3}{5}} + \tfrac{3}{32}\,x^{-\frac{1}{2}} + \tfrac{3}{8}\,x^{-\frac{2}{5}} + \tfrac{7}{8}\,x^{-\frac{3}{10}} + \tfrac{3}{2}\,x^{-\frac{1}{5}} + \tfrac{3}{2}\,x^{-\frac{1}{10}} + 1.$$

17. If $a^b = b^a$, show that $\left(\dfrac{a}{b}\right)^{\frac{a}{b}} = a^{\frac{a}{b}-1}$.

18. Simplify $\left(\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right)^2 + \left(\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)^2$.

19. Simplify

$$\sqrt[3]{a-1}\cdot\sqrt[3]{a+1}\cdot\sqrt[3]{a^2+a+1}\cdot\sqrt[3]{a^2-a+1}\cdot\sqrt[3]{(a^6-1)^2}.$$

CHAPTER XIII.

COMPLEX NUMBERS.

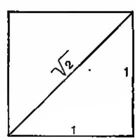
I. DEFINITIONS.

247. Certain steps in the growth of the number system have already been set forth in § 24, but are here repeated for reasons which will be obvious.

1. **The positive integer** suffices for the solution of the equation $x - 3 = 0$, since $x = 3$ satisfies the equation. We can represent such a number by a line three units long, as in the annexed figure, the unit being of any convenient length.

2. **The positive fraction.** If, however, we attempt to solve the equation $3x - 2 = 0$, either we must say that the solution is impossible or we must extend the idea of number to include the positive fraction. Then $x = \frac{2}{3}$ satisfies the equation. We can represent such a number by dividing a line one unit long into three parts and taking two of them.

3. **The surd.** If we attempt to solve the equation $x^2 - 2 = 0$, either we must say that the solution is impossible or we must extend the idea of number to include the surd. Then $\sqrt{2}$ satisfies the equation. We can represent $\sqrt{2}$ by the diagonal of a square whose side is one unit long. This is evident because the square on the hypotenuse equals the sum of the squares on the two sides of the right-angled triangle.



4. **The negative number.** If we attempt to solve the equation $x + 2 = 0$, either we must say that the solution is impossible or we must extend the idea of number to include the negative number. Then $x = -2$ satisfies the equation. We can represent such a number by supposing the negative sign to denote direction, a direction opposite to that which we assume for positive numbers.

248. The numbers thus far described in this chapter are called **real numbers.**

249. The imaginary number. If we attempt to solve the equation $x^2 + 1 = 0$, either we must say that the solution is impossible or we must extend the idea of number still further.

The equation $x^2 + 1 = 0$

leads to $x^2 = -1$,

which leads to $x = \pm\sqrt{-1}$,

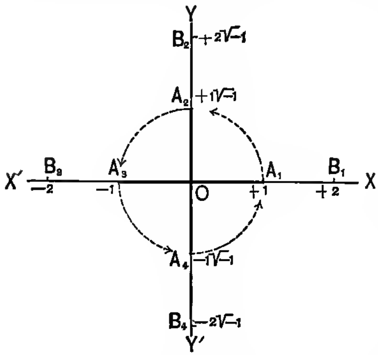
which cannot be a positive or a negative integer, fraction, or surd (§ 126).

250. We call an even root of a negative number an *imaginary number.*

The term "imaginary" is unfortunate, since these numbers are no more imaginary than are fractions or negative numbers. We cannot imagine looking out of a window -2 times or $\frac{1}{2}$ of a time any more than $\sqrt{-1}$ times. The "imaginary" is merely another step in the number system. The name is, however, so generally used that it should continue to designate this new form of number.

To the ancients, negative numbers were as "imaginary" as $\sqrt{-1}$ is to us. It was only when some one drew a picture of $\sqrt{2}$ (see § 247, 3), of -1, and later of $\sqrt{-1}$, that these were understood.

251. As with fractions, surds, and negative numbers, it is necessary to represent the imaginary graphically by a line, or in some other concrete way, in order to make its nature clear to the beginner.



In this figure the multiplication of $+1$ by -1 swings the line OA_1 through 180° to the position OA_3.

As a matter of custom this line is supposed to swing as indicated by the arrows, opposite to the movement of clock-hands, *counter-clockwise.*

252. That is, since $(\sqrt{-1})^2$ means $\sqrt{-1}\cdot\sqrt{-1}$ or -1, the multiplication of $+1$ by $\sqrt{-1}\cdot\sqrt{-1}$ swings $+1$ through 180°; therefore the multiplication of $+1$ by $\sqrt{-1}$ should be regarded as swinging it through half of this angle, or 90°, to the position OA_2.

Or we may say that since multiplication by $\sqrt{-1}$ twice, carries OA through 180°, therefore multiplication by $\sqrt{-1}$ once should carry it through 90°.

Similarly, -1 multiplied by $\sqrt{-1}\cdot\sqrt{-1}$, or -1 multiplied by -1, swings OA_3 the rest of the way around to OA_1; hence, -1 multiplied by $\sqrt{-1}$ should be looked upon as swinging it to the position OA_4.

253. Hence, *we represent* $+1\sqrt{-1}$ (*or* $+\sqrt{-1}$), $+2\sqrt{-1}$, $+3\sqrt{-1}$, $\cdots$, *by integers on the perpendicular* OY, upward from O, and $-1\sqrt{-1}$ (*or* $-\sqrt{-1}$), $-2\sqrt{-1}$, $-3\sqrt{-1}$, $\cdots$, *by integers on the negative side of this line, i.e.*, on OY', downward from O.

254. Hence, it appears that *the symbols* $+\sqrt{-1}$ *and* $-\sqrt{-1}$ *are, like* $+$ *and* $-$, *symbols of quality* and may be looked upon as indicating direction.

E.g.,	$+3$	indicates 3 units to the right,
	-3	" " " left,
	$+3\sqrt{-1}$	" " up,
	$-3\sqrt{-1}$	" " down.

255. Since $\sqrt{ab} \equiv \sqrt{a}\cdot\sqrt{b}$, we say that $\sqrt{-3}$ shall equal $\sqrt{3\cdot -1} = \sqrt{3}\cdot\sqrt{-1}$. Hence,

Every imaginary number can be written in the form a $\sqrt{-1}$, *where* a *is real, though possibly a surd or a fraction, and* $\sqrt{-1}$ *is the imaginary unit.*

E.g., to represent $3\sqrt{-1}$, we measure 3 units upward from the O point on the line $X'X$; to represent $-\sqrt{-2}$, we reduce this to the form $-\sqrt{2}\cdot\sqrt{-1}$, then construct a line equal to $\sqrt{2}$, as in § 247, 3, and lay this off on OY'.

EXERCISES. CX.

Solve the following equations, expressing the results in the form $a\sqrt{-1}$.

1. $x^2 = -9$.
2. $3x^2 + 2 = 0$.
3. $5x^2 = -5$.
4. $x^2\sqrt{2} = -3$.
5. $x^2 + 5 = 0$.
6. $5x^2 = -125$.
7. $x^2 + 4 = 0$.
8. $x^2 + 20 = -5$.

Represent graphically the following imaginary numbers:

9. $\sqrt{-4}$.
10. $\sqrt{-5}$.
11. $-5\sqrt{-1}$.
12. $\sqrt{-32}$.
13. $3\sqrt{-1}$.
14. $\sqrt{2}\cdot\sqrt{-2}$.
15. $-\sqrt{-16}$.
16. $2\sqrt{-9}$.
17. $-\frac{1}{2}\sqrt{-12}$.

256. The complex number. If we attempt to solve the equation $x^2 - 4x + 5 = 0$ by factoring, we may write it in the form

$$x^2 - 4x + 4 - (-1) = 0,$$

or
$$(x-2)^2 - (-1) = 0,$$

or
$$(x - 2 + \sqrt{-1})(x - 2 - \sqrt{-1}) = 0,$$

whence
$$x = 2 - \sqrt{-1},$$

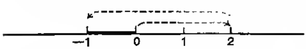
or
$$x = 2 + \sqrt{-1}.$$

Hence, it appears that each root is the algebraic sum of a real number and an imaginary.

Such a number is said to be **complex**.

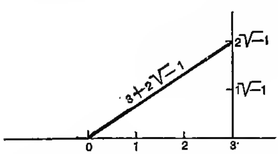
257. As with positive and negative integers, fractions, surds, and imaginaries, we proceed to make the nature of the complex number more clear by resorting to a graphic representation.

If we wish to represent the sum of 2 and -3, we pass from zero 2 units to the right and then 3 units to the left, and we say that *the sum is the distance from* 0 *to the point where we stop.*



The fact that the absolute value of the sum is less than the sum of the absolute values of the addends is no longer strange to us, because we have become accustomed to this in dealing with negative numbers.

258. Similarly, to represent the sum of 3 and $2\sqrt{-1}$ we pass from zero 3 units to the right and then 2 units upward (for $2\sqrt{-1}$) and we say, as before, that *the sum is the distance from* 0 *to the point where we stop.*



The fact that the absolute value of the sum is less than the sum of the absolute values of the addends is no more strange than it is in the case of $2 + (-3)$.

EXERCISES. CXI.

Represent graphically the following complex numbers:

1. $4+\sqrt{-4}$.
2. $5-2\sqrt{-1}$.
3. $5+2\sqrt{-1}$.
4. $-\frac{1}{3}-\sqrt{-\frac{1}{3}}$.
5. $-5-2\sqrt{-1}$.
6. $-3-3\sqrt{-1}$.
7. $-\frac{1}{2}+\frac{1}{2}\sqrt{3}\cdot\sqrt{-1}$.
8. $-\frac{1}{2}-\frac{1}{2}\sqrt{3}\cdot\sqrt{-1}$.

259. Symbolism of complex numbers. Instead of writing the symbol $\sqrt{-1}$, the letter i is usually employed.

This letter, standing for *imaginary*, seems to have been first used in this sense by Euler in 1777.

Then
$$\sqrt{-4}=2\sqrt{-1}=2i,$$
$$\sqrt{-3}=i\sqrt{3}, \text{ etc.}$$

Also,
$$i^2=-1,$$
$$i^3=-1\cdot i=-i,$$
$$i^4=(i^2)^2=(-1)^2=1,$$
$$i^5=1\cdot i=i,$$
$$i^6=i\cdot i=i^2=-1,$$
$$i^7=-1\cdot i=-i,$$
$$i^8=-i\cdot i=-(i)^2=-(-1)=1;$$

and, in general,
$$i^{4n}=1,$$
$$i^{4n+1}=i,$$
$$i^{4n+2}=-1,$$
$$i^{4n+3}=-i.$$

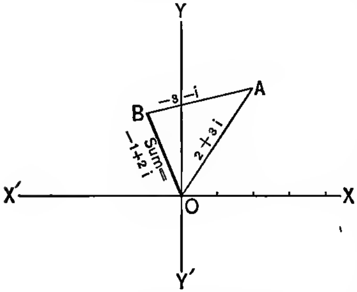
EXERCISES. CXII.

Represent graphically the following complex numbers:

1. $2+3i$.
2. $4+2i$.
3. $i^2+\frac{1}{2}i$.
4. i^3+i^2.
5. i^4+i^3.
6. i^5+2i^4.

II. OPERATIONS WITH COMPLEX NUMBERS.

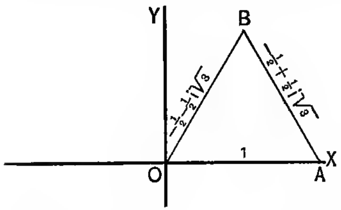
260. Complex numbers are subject to all of the laws of rational numbers and the operations do not materially differ from those already familiar to the student.



Illustrative problems. 1. Represent graphically the sum of $2+3i$ and $-3-i$.

Starting from O we lay off $+2$ (to the right), then $3i$ (upward), OA being $2+3i$. From A we then lay off -3 (to the left), then $-i$ (one unit downward), reaching B. Then the sum is OB, *the distance from O to the point where we stop.*

2. Add 1, $-\frac{1}{2}+\frac{1}{2}i\sqrt{3}$, and $-\frac{1}{2}-\frac{1}{2}i\sqrt{3}$; then represent the sum graphically.



$$\begin{array}{rl} & 1 \\ & -\frac{1}{2}+\frac{1}{2}i\sqrt{3} \\ & -\frac{1}{2}-\frac{1}{2}i\sqrt{3} \\ \hline \text{Sum} = & 0 \end{array}$$

Graphically, we lay off 1 from O to A. From A we lay off $-\frac{1}{2}$, then, $\frac{1}{2}i\sqrt{3}$ (*i.e.*, $\frac{1}{2}\cdot i\cdot 1.73\cdots$, or $0.87\,i$), reaching B. From B we lay off $-\frac{1}{2}$, then $-\frac{1}{2}i\sqrt{3}$, reaching O. Hence, the sum is zero.

3. Multiply $2+3i$ by $3-2i$.

$$\begin{array}{lll} 2+3i & & \\ 3-2i & & \\ \hline 6+9i & = 6+9i & = \ \ 6+9i \\ \quad -4i-6i^2 = & -4i-6(-1) = & \ \ 6-4i \\ \hline & & 12+5i \end{array}$$

Simply *multiply by* i *as if it were any other letter, but in finally simplifying remember that* $i^2=-1$.

4. Divide $12 + 5i$ by $3 - 2i$.

Multiply both terms of the fraction

$$\frac{12 + 5i}{3 - 2i}$$

by the conjugate of the denominator. Then

$$\frac{(3 + 2i)(12 + 5i)}{(3 + 2i)(3 - 2i)} = \frac{26 + 39i}{9 - 4(-1)} = \frac{26 + 39i}{13} = 2 + 3i.$$

5. Cube $-\frac{1}{2} + \frac{1}{2}i\sqrt{3}$.

$$\because (f + n)^3 \equiv f^3 + 3f^2n + 3fn^2 + n^3,$$

$$\therefore (-\tfrac{1}{2} + \tfrac{1}{2}i\sqrt{3})^3$$
$$= -\tfrac{1}{8} + 3 \cdot \tfrac{1}{4} \cdot \tfrac{1}{2}i\sqrt{3} + 3 \cdot (-\tfrac{1}{2}) \cdot \tfrac{3}{4} \cdot (-1) + \tfrac{3}{4}(-1) \cdot \tfrac{1}{2}i\sqrt{3}$$
$$= -\tfrac{1}{8} + \tfrac{3}{8}i\sqrt{3} + \tfrac{9}{8} - \tfrac{3}{8}i\sqrt{3}$$
$$= 1.$$

Hence, $-\frac{1}{2} + \frac{1}{2}i\sqrt{3}$ is a cube root of 1.

6. Extract the square root of $-16 + 30i$.

$$\because a + 2\sqrt{ab} + b \equiv [\pm(\sqrt{a} + \sqrt{b})]^2, \qquad \S\ 245$$

and $\because -16 + 30i$ can be written $9 + 2\sqrt{-9 \cdot 25} + (-25)$,

$$\therefore -16 + 30i = 9 + 2\sqrt{-9 \cdot 25} + (-25)$$
$$= [\pm(3 + \sqrt{-25})]^2$$
$$= [\pm(3 + 5i)]^2.$$

$\therefore \pm(3 + 5i)$ is the required square root.

The solution is seen to consist simply of making the coefficient of the square root 2, and then separating -16 into two parts whose product is -225. (See § 245, 3.)

The addition (including subtraction) of complex numbers has been represented graphically. It is also possible to represent the other operations graphically, but the explanation is too difficult for an elementary text-book.

7. Extract the square root of $a^2 + 2abi - b^2$.

This is evidently the same as $a^2 + 2abi + (bi)^2$.

Hence, the square root is $\pm(a + bi)$.

EXERCISES. CXIII.

1. Find the following sums and represent each solution graphically.

(a) $5-7i$ and $5+7i$. (b) $-2-3i$ and $2+3i$.

(c) 1, -1, i, and $-i$. (d) $-6+2i$ and $6+2i$.

(e) 1, $\frac{1}{2}+\frac{1}{2}i\sqrt{3}$, $-\frac{1}{2}+\frac{1}{2}i\sqrt{3}$, -1, $-\frac{1}{2}-\frac{1}{2}i\sqrt{3}$, and $\frac{1}{2}-\frac{1}{2}i\sqrt{3}$.

2. Multiply

(a) $3-4i$ by $5+2i$. (b) $-\frac{1}{2}+\frac{1}{2}i$ by $\frac{1}{2}+\frac{1}{2}i$.

(c) $2+9i$ by $9+2i$. (d) $-4+2i$ by $-4-2i$.

(e) $-\frac{1}{2}+\frac{1}{2}i\sqrt{3}$ by $-\frac{1}{2}-\frac{1}{2}i\sqrt{3}$.

3. Divide

(a) 10 by $3-i$. (b) $4+22i$ by $7+i$.

(c) $1+8i$ by $2+i$. (d) $1+8i$ by $2+3i$.

(e) $7+61i$ by $4+7i$. (f) $3+6i$ by $3-6i$.

4. Raise the following to the powers indicated:

(a) i^{17}. (b) $(2+3i)^2$. (c) $(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^2$.

(d) $(2+i)^4$. (e) $(3-5i)^3$. (f) $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})^3$.

(g) $(a+bi)^2$. (h) $(2-7i)^3$. (i) $(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^3$.

5. Extract the square root of

(a) $3+4i$. (b) $5+12i$. (c) $-5-12i$.

(d) $-45-28i$. (e) $24-10i$. (f) $15-8i$.

(g) $\frac{13}{4}+21i$. (h) $-\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$. (i) $-\frac{3}{4}+i$.

REVIEW EXERCISES. CXIV.

1. Simplify the expression

$$-i/[10+2\sqrt{5}+(\sqrt{5}+1)i].$$

2. Also the expression $(\sqrt{3}+i)^3/i(-1+\sqrt{-3})^5$.

3. Also the expression

$$i\left(\frac{-1+\sqrt{-3}}{2}\right)^3\left(\frac{1+i}{\sqrt{2}}\right)^2\left(\frac{-1-\sqrt{-3}}{2}\right)^3.$$

4. By factoring, solve the equation $8x^2-35x+12=0$.

5. By the Remainder Theorem determine whether $x-i$ is a factor of x^4+5x^2+4.

6. Find the times between 4 and 5 o'clock at which the hands of a watch are at right angles.

7. By factoring, find four different roots of the equation $x^4-1=0$. (Two are imaginary.) Check.

8. By substituting the three numbers

$$1,\ -\tfrac{1}{2}+\tfrac{1}{2}i\sqrt{3},\ -\tfrac{1}{2}-\tfrac{1}{2}i\sqrt{3},$$

for x, show that they are the roots of the equation $x^3-1=0$.

9. Find to two decimal places the values of x and y in the following:

$$\frac{x}{3.579}+\frac{y}{5.793}=7.935.$$

$$\frac{x}{9.753}+\frac{y}{7.539}=5.397.$$

10. The sum of two numbers is 16, and the sum of their reciprocals is double the difference of their reciprocals. What are the numbers?

CHAPTER XIV.

QUADRATIC EQUATIONS INVOLVING ONE UNKNOWN QUANTITY.

I. METHODS OF SOLVING.

261. A **quadratic equation** (or equation of the second degree) involving one unknown quantity is an equation which can be reduced to the form $ax^2 + bx + c = 0$, a, b, c being known quantities and a not being zero.

E.g.,
$$3x^2 + 2x + 3 = 0,$$
$$x^2 + 1 = 0,$$
$$\tfrac{1}{2}x^2 + x\sqrt{2} = 0,$$

are quadratic equations involving one unknown quantity.

So is the equation

$$2x^3 + 3x^2 - 5x + 7 = (2x^2 + 1)(x - 1),$$

because it can be reduced to the form $ax^2 + bx + c = 0$.

Similarly for
$$\frac{2}{x} + \frac{x}{2} = 0,$$

although, in general, multiplication by any $f(x)$ is liable to introduce an extraneous root (§ 185).

But
$$0 \cdot x^2 + 4x - 5 = 0$$

is not a quadratic equation; neither is

$$2x^3 + x + 1 = x^3 + x^2 + 3x,$$

nor
$$x^2 + x + 1 = (x + 1)(x - 1).$$

The equation $$x^6 + x^3 + 4 = 0$$
is not a quadratic equation in x, but it is one in x^3, for it is the same as
$$(x^3)^2 + (x^3) + 4 = 0.$$
So $$\frac{1}{x^2} + \frac{1}{x} + 2 = 0$$
is, without reduction, a quadratic equation in $\frac{1}{x}$, or x^{-1}, and
$$(a + x^2)^2 + 2(a + x^2) + 3 = 0$$
is a quadratic equation in $a + x^2$, and
$$x^2 + x + 3\sqrt{x^2 + x} = 4$$
is a quadratic equation in $\sqrt{x^2 + x}$.

262. The quadratic equation $ax^2 + bx + c = 0$ is said to be **complete** when neither b nor c is zero; otherwise to be **incomplete.**

The coefficient a cannot be zero, because the equation is to be a quadratic (§ 261).

E.g., $x^2 + 2x - 3 = 0$ is a complete quadratic equation,
but $x^2 - 3 = 0$
and $x^2 + 2x = 0$ are incomplete.

Older English works speak of an equation of the form
$ax^2 + c = 0$ as a **pure quadratic,**
and $ax^2 + bx + c = 0$ as an **affected quadratic.**

The following are further examples of complete (affected) quadratic equations:

$$(x - 1)^{\frac{1}{2}} + (x - 1)^{\frac{1}{4}} + 5 = 0, \text{ in } (x - 1)^{\frac{1}{4}};$$
$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt[4]{x}} + 7 = 0, \text{ in } \sqrt[4]{x};$$
$$x^{\frac{1}{5}} + 2x^{\frac{1}{10}} + 1 = 0, \text{ in } x^{\frac{1}{10}}.$$

263. Solution by factoring. (a) **The type**

$$(ax + b)(cx + d) = 0.$$

One of the best methods of solving the ordinary quadratic equation is by factoring, as already shown in § 123.

Illustrative problems. 1. Solve the equation

$$x^2 + 16x + 63 = 0.$$

1. This reduces to $(x + 9)(x + 7) = 0$. § 119

2. This is satisfied if either factor is zero, the other remaining finite (§ 123). Hence, either

$$x + 9 = 0, \text{ or } x + 7 = 0.$$

3. $\therefore x = -9$, or $x = -7$.

Check. Substituting these values in the *original* equation (§ 189),

$$81 - 144 + 63 = 0,$$
$$49 - 112 + 63 = 0.$$

2. Solve the equation $2x^2 = 7$.

1. This reduces to $x^2 = \frac{7}{2}$. Ax. 6

2. $\therefore$ $x = \pm\sqrt{\frac{7}{2}} = \pm\frac{1}{2}\sqrt{14}$. Ax. 9, § 235

That is, it is not worth while to factor as in ex. 1. But the problem can be so solved; for

$$x^2 - \tfrac{7}{2} = 0.$$
$$\therefore (x - \sqrt{\tfrac{7}{2}})(x + \sqrt{\tfrac{7}{2}}) = 0.$$
$$\therefore x = \pm\sqrt{\tfrac{7}{2}} = \pm\tfrac{1}{2}\sqrt{14}.$$

Check. Substituting in the original equation,

$$2 \cdot \tfrac{1}{4} \cdot 14 = 7.$$

3. Solve the equation $6x^2 - 7x + 2 = 0$.

1. This reduces to $(2x - 1)(3x - 2) = 0$. § 120

2. $\therefore$ $2x - 1 = 0$, or $3x - 2 = 0$. § 123

3. $\therefore$ $2x = 1$, or $3x = 2$,

and $x = \frac{1}{2}$, or $x = \frac{2}{3}$.

Check. $\frac{3}{2} - \frac{7}{2} + 2 = 0$, $\frac{8}{3} - \frac{14}{3} + 2 = 0$.

EXERCISES. CXV.

Solve the equations:

1. $x^2 = x$.
2. $x^2 = 7 - 6x$.
3. $\frac{1}{x^2} - \frac{1}{x} = 6$.
4. $x + \frac{1}{2} = \frac{1}{2x}$.
5. $9x^2 - 1 = 0$.
6. $x^2 = 2(12 - 5x)$.
7. $x^2 + 17x = 0$.
8. $8x - x^2 - 12 = 0$.
9. $x^2 - 2x - 15 = 0$.
10. $x(10 + x) = -21$.
11. $x^2 + 5x - 14 = 0$.
12. $6x^2 + 7x + 2 = 0$.
13. $x^2 + 19x + 18 = 0$.
14. $x^2 + 26x = -120$.
15. $x^2 - 12x - 85 = 0$.
16. $x(4 - x) + 77 = 0$.
17. $x^2 - 22x + 121 = 0$.
18. $3x^2 - 10x + 3 = 0$.
19. $x^2 - 24x + 143 = 0$.
20. $10x^2 + 29x = -10$.

264. (b) **The type $(x + a)(x - a) = 0$.**

It frequently happens that it is easier to arrange the first member as the difference of two squares than to factor in the form suggested on p. 248, especially when the numbers are such that the linear factors involve surds.

E.g., to solve the equation $x^2 + 4x + 1 = 0$. Here $x^2 + 4x$ are the first two terms of a square, $x^2 + 4x + 4$. The equation may be written

$$x^2 + 4x + 4 - 3 = 0,$$

or

$$(x + 2)^2 - 3 = 0,$$

or

$$(x + 2 + \sqrt{3})(x + 2 - \sqrt{3}) = 0,$$

since we are not confined to the domain of rationality (§ 107) in our solutions.

$$\therefore x + 2 + \sqrt{3} = 0, \text{ or } x + 2 - \sqrt{3} = 0,$$

and

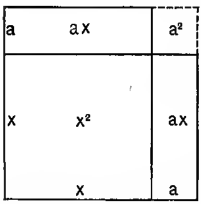
$$x = -2 - \sqrt{3}, \text{ or } x = -2 + \sqrt{3}.$$

Check. $4 \pm 4\sqrt{3} + 3 - 8 \mp 4\sqrt{3} + 1 = 0$.

265. The addition of an absolute term to two terms so that the trinomial shall be a square is called **completing the square.**

E.g., to complete the square of $x^2 + 2x$ we must add 1; to complete the square of $x^2 + x$ we must add $\frac{1}{4}$.

266. Since $(x + a)^2 \equiv x^2 + 2ax + a^2$, it is seen that *the quantity which must be added to* $x^2 + 2ax$ *to complete the square is the square of half the coefficient of* x.



E.g., to complete the square for $x^2 + 8x$, add 16, $x^2 + 8x + 16$ being $(x + 4)^2$. To complete the square for $x + 6\sqrt{x}$ with respect to $\sqrt{x}$, add 9, $x + 6\sqrt{x} + 9$ being $(\sqrt{x} + 3)^2$.

From the annexed figure it is readily seen that if we have $x^2 + ax + ax$, or $x^2 + 2ax$, the square on $x + a$ will be completed by adding a^2 in the corner.

EXERCISES. CXVI.

Complete the squares in exs. 1–16.

1. $\frac{1}{x^2} + \frac{2}{x}$.	2. $\frac{x^2}{a^2} + 2\frac{x}{a}$.
3. $x - \sqrt{x}$.	4. $x^2 - 6x$.
5. $x^2 + \frac{1}{2}x$.	6. $x^2 + 30x$.
7. $x^2 - \frac{2}{3}x$.	8. $x^2 - 34x$.
9. $4x^2 + 8x$.	10. $x^2 + 10x$.
11. $x^2 - 100x$.	12. $x^2 - 2x$; $x^2 + 2x$.
13. $9x^2 + 36x$.	14. $(x-1)^2 + 4(x-1)$.
15. $100x^2 + 20x$.	16. $(x+a)^2 + 2(x+a)$.

17. In general, to complete the square for $x^2 + px$ what must be added?

Illustrative problems. 1. Solve the equation

$$x^2 + 3x + 2 = 0.$$

1. Completing the square for $x^2 + 3x$, the equation may be written

$$x^2 + 3x + \tfrac{9}{4} - \tfrac{1}{4} = 0.$$

2. $\therefore \quad (x + \tfrac{3}{2})^2 - \tfrac{1}{4} = 0.$

3. $\therefore \quad (x + \tfrac{3}{2} + \tfrac{1}{2})(x + \tfrac{3}{2} - \tfrac{1}{2}) = 0,$

or $\quad (x + 2)(x + 1) = 0.$

4. $\therefore \quad x = -2, \text{ or } -1.$

Check. $4 - 6 + 2 = 0,\ 1 - 3 + 2 = 0.$

2. Solve the equation $x - \sqrt{x+1} = 0.$

1. $x = \sqrt{x+1}.$ Ax. 3

2. $\therefore \quad x^2 = x + 1, \text{ or } x^2 - x - 1 = 0.$ Axs. 8, 3

3. $\therefore \quad x^2 - x + \tfrac{1}{4} - \tfrac{5}{4} = 0.$

4. $\therefore \quad (x - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})(x - \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}) = 0.$

5. $\therefore \quad x = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}.$

Check. $\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5} - \sqrt{\tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{5}}$

$= \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5} - \tfrac{1}{2}\sqrt{6 \pm 2\sqrt{5}}$

$= \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5} - \tfrac{1}{2}\sqrt{1 \pm 2\sqrt{5} + 5}$ § 245

$= \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5} - \tfrac{1}{2}(1 \pm \sqrt{5}) = 0.$

EXERCISES. CXVII.

Solve the equations:

1. $\dfrac{x}{4} + \dfrac{25}{x} = 3.$

2. $1 \Big/ \left(x + \dfrac{1}{x}\right) = 1.$

3. $x^2 - 2x = -2.$

4. $x^2 + 6x + 2 = 0.$

5. $x^2 - 9x - \tfrac{1}{5} = 0.$

6. $x^2 - 6x + 2 = 0.$

7. $x^2 - 7x + 5 = 0.$

8. $x^2 + 10x + 5 = 0.$

9. $x^2 + 10x + 25 = 0.$

267. Solution by making the first member a square. The method of § 264 may be modified by making the first member the square of a binomial of the form $x + a$.

E.g., to solve the equation $x^2 + 4x + 1 = 0$.

The first member would be a square if the 1 were 4, *i.e.*, if 3 were added. Hence, adding 3 to both members,

1. $x^2 + 4x + 4 = 3.$ Ax. 2

2. ∴ $(x + 2)^2 = 3,$

3. ∴ $x + 2 = \pm\sqrt{3},$ Ax. 9

4. ∴ $x = -2 \pm \sqrt{3}.$

Check. $(-2 \pm \sqrt{3})^2 + 4(-2 \pm \sqrt{3}) + 1$

$$= 4 \mp 4\sqrt{3} + 3 - 8 \pm 4\sqrt{3} + 1 = 0.$$

268. It therefore appears that *the equation* $x^2 + px + q = 0$ *can be solved by*

1. *Subtracting* q *from each member;* then

2. *Completing the square,* by adding the square of half the coefficient of x (§ 266) to each member; and then

3. *Extracting the square root* of each member and solving the simple equations which are thus obtained.

The ± sign in step 3 of the above solution is placed only in the second member, because no new values of x would result if it were placed in both members.

Suppose it were placed in both members. Then

$$\pm(x + 2) = \pm\sqrt{3}; \text{ that is}$$

(1) $+(x + 2) = +\sqrt{3},$ whence $x = -2 + \sqrt{3},$

(2) $+(x + 2) = -\sqrt{3},$ " $x = -2 - \sqrt{3},$

(3) $-(x + 2) = +\sqrt{3},$ " $-x = 2 + \sqrt{3}$ and $\therefore x = -2 - \sqrt{3},$

(4) $-(x + 2) = -\sqrt{3},$ " $-x = 2 - \sqrt{3}$ " $\therefore x = -2 + \sqrt{3}.$

That is, $x = -2 \pm \sqrt{3}$, as in step 4 of the solution.

Illustrative problems. 1. Solve the equation

$$x^2 + x + 1 = 0.$$

1. $x^2 + x = -1.$ Ax. 3
2. $x^2 + x + \frac{1}{4} = -1 + \frac{1}{4} = -\frac{3}{4}.$ Ax. 3
3. $x + \frac{1}{2} = \pm \frac{1}{2} i \sqrt{3}.$ Ax. 9
4. $\therefore$ $x = -\frac{1}{2} \pm \frac{1}{2} i \sqrt{3}.$ Ax. 3

Check. $(-\frac{1}{2} \mp \frac{1}{2} i \sqrt{3}) + (-\frac{1}{2} \pm \frac{1}{2} i \sqrt{3}) = -1.$

2. Solve the equation $x^2 + 3x + \sqrt{x^2 + 3x + 7} - 23 = 0.$

1. This may be written in quadratic form, thus,

$$x^2 + 3x + 7 + \sqrt{x^2 + 3x + 7} - 30 = 0,$$

a quadratic in $\sqrt{x^2 + 3x + 7}$. This quantity may now be represented by y, for simplicity, and

2. $y^2 + y - 30 = 0.$
3. $\therefore$ $y^2 + y + \frac{1}{4} = \frac{121}{4}.$
4. $\therefore$ $y + \frac{1}{2} = \pm \frac{11}{2}.$
5. $\therefore$ $y = -\frac{1}{2} \pm \frac{11}{2} = 5$, or -6.
6. $\therefore$ $\sqrt{x^2 + 3x + 7} = 5$, or -6.

This evidently gives rise to two quadratic equations in x. First consider the case of $y = 5$.

7. Then $x^2 + 3x + 7 = 25.$
8. $\therefore$ $x^2 + 3x - 18 = 0.$
9. $\therefore$ $(x + 6)(x - 3) = 0$, and $x = -6$, or 3,

results which easily check.

If $y = -6$, we have

10. $x^2 + 3x + 7 = 36,$
11. whence $x^2 + 3x + \frac{9}{4} = \frac{125}{4}.$
12. $\therefore$ $x + \frac{3}{2} = \pm \frac{5}{2} \sqrt{5}$, and $x = -\frac{3}{2} \pm \frac{5}{2} \sqrt{5}.$

This pair of results checks, provided we remember that

$$\sqrt{x^2 + 3x + 7} = 5 \text{ or } -6.$$

For, substituting 5 and -6 for $\sqrt{x^2 + 3x + 7}$, we have

$$18 + 5 - 23 = 0,$$

$$29 - 6 - 23 = 0.$$

3. Solve the equation $2x^2 - 2x = 5$.

1.	$x^2 - x = \frac{5}{2}$.	Ax. 7
2.	$x^2 - x + \frac{1}{4} = \frac{11}{4}$.	Ax. 2
3.	$x - \frac{1}{2} = \pm \frac{1}{2}\sqrt{11}$.	Ax. 9
4.	$x = \frac{1}{2}(1 \pm \sqrt{11})$.	Ax. 2

Check. $(6 \pm \sqrt{11}) - (1 \pm \sqrt{11}) = 5$.

It is often possible, in cases of this kind, to avoid fractions by the exercise of a little forethought. This equation may be written

1′. $4x^2 - 4x = 10$.

2′. ∴ $(2x)^2 - 2(2x) + 1 = 11$, a quadratic in $2x$.

3′. ∴ $2x - 1 = \pm\sqrt{11}$.

4′. ∴ $2x = 1 \pm \sqrt{11}$.

5′. ∴ $x = \frac{1}{2}(1 \pm \sqrt{11})$.

EXERCISES. CXVIII.

Solve the equations:

1. $x^2 - \frac{5}{6}x = 1$.

2. $6x + 40 - x^2 = 0$.

3. $x^2 + 8x = 65$.

4. $x^2 + \frac{5}{3}x - \frac{28}{3} = 0$.

5. $x^2 + 0.9x = 8.5$.

6. $2.5x^2 - 4\frac{2}{3}x = 304$.

7. $3\frac{1}{3}x^2 - 4x = 96$.

8. $x^2 + 132x = -1331$.

9. $x^2 + 6x + 25 = 0$.

10. $7x^2 - 5x - 150 = 0$.

11. $4x^2 - 5x + 62 = 0$.

12. $4.05x^2 - 7.2x = 1476$.

13. $(x + a)^2 + 2(x + a) + 1 = 0$.

14. $\left(x + \frac{1}{x}\right)^2 - 3\left(x + \frac{1}{x}\right) + 2 = 0$.

15. $(x^2 + 2x)^2 - 3(x^2 + 2x) + 2 = 0$.

16. $(x^2 + x - 1)^2 + 4(x^2 + x - 1) + 4 = 0$.

17. $(x + 4)(12x - 5) + 4\frac{1}{2} = (7x^2 - 10)8 - 12.75x$.

269. Solution by formula. Every quadratic equation can be reduced to the form $ax^2 + bx + c = 0$ (§ 261).

This equation can be solved by any of the methods already suggested and it will be found that

$$x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}.$$

Hence, the roots of any quadratic equation which has been reduced to the form $ax^2 + bx + c = 0$ can be written down at sight.

E.g., the roots of

$$6x^2 - 13x + 6 = 0 \text{ are } -\frac{-13}{2\cdot 6} \pm \frac{1}{2\cdot 6}\sqrt{(-13)^2 - 4\cdot 6\cdot 6}$$
$$= \tfrac{13}{12} \pm \tfrac{1}{12}\sqrt{169 - 144}$$
$$= \tfrac{13}{12} \pm \tfrac{5}{12} = \tfrac{3}{2} \text{ or } \tfrac{2}{3}.$$

Similarly, the roots of

$$\frac{2}{x^2} - \frac{3}{x} + 1 = 0 \text{ are } \frac{1}{x} = -\frac{-3}{2\cdot 2} \pm \frac{1}{2\cdot 2}\sqrt{(-3)^2 - 4\cdot 2\cdot 1}$$
$$= \tfrac{3}{4} \pm \tfrac{1}{4}\sqrt{9 - 8}$$
$$= \tfrac{3}{4} \pm \tfrac{1}{4} = 1 \text{ or } \tfrac{1}{2}.$$
$$\therefore x = 1 \text{ or } 2.$$

270. In particular, the roots of

$$x^2 + px + q = 0 \text{ are } x = -\frac{p}{2} \pm \tfrac{1}{2}\sqrt{p^2 - 4q}.$$

E.g., the roots of $x^2 + x + 1 = 0$ are $-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 - 4}$
$$= -\tfrac{1}{2} \pm \tfrac{1}{2}i\sqrt{3}.$$

271. *The formulas*

$$x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac},$$

$$x = -\frac{p}{2} \pm \tfrac{1}{2}\sqrt{p^2 - 4q},$$

are so important that they should be memorized and freely used in the solution of such quadratic equations as are not readily solved by factoring.

EXERCISES. CXIX.

Write out, at sight, the roots of equations 1–30, and then simplify the results.

1. $x^2 - 3x + 1 = 0.$
2. $x^2 + 6x + 2 = 0.$
3. $x^2 + 4x - 4 = 0.$
4. $x^2 - 5x + 1 = 0.$
5. $x^2 + 2x + 2 = 0.$
6. $x^2 + 2x - 24 = 0.$
7. $x^2 - 2x + 3 = 0.$
8. $x^2 - 5x - 36 = 0.$
9. $x^2 + 2x - 3 = 0.$
10. $x^2 + 7x - 44 = 0.$
11. $x^2 - 5x - 36 = 0.$
12. $x^2 + 10x + 5 = 0.$
13. $x^2 + 7x + 10 = 0.$
14. $x^2 - 4x - 12 = 0.$
15. $12x^2 + x - 6 = 0.$
16. $x^2 + 4x - 45 = 0.$
17. $x^2 - 7x + 12 = 0.$
18. $x^2 - 3x - 28 = 0.$
19. $3x^2 - 2x + 1 = 0.$
20. $x^2 - 16x + 60 = 0.$
21. $4x^2 + 5x + 6 = 0.$
22. $x^2 + 10x + 21 = 0.$
23. $2x^2 + 3x + 1 = 0.$
24. $6x^2 - 37x + 6 = 0.$
25. $x^2 - 2.1x - 1 = 0.$
26. $6x^2 + 5x - 56 = 0.$
27. $x^2 - 11x - 60 = 0.$
28. $x^2 + 0.6x + 0.3 = 0.$
29. $x^2 - 10x + 16 = 0.$
30. $x^2 + 0.7x + 0.1 = 0.$

31. What are the roots of the equation $ax^2 + bx + c = 0$, if $b^2 = 4ac$?

32. Show that if $b^2 - 4ac$ is negative the two roots are complex.

33. Show that if $b^2 - 4ac$ is positive the two roots are real.

34. Show that if $b^2 - 4ac$ is a perfect square the two roots are rational.

272. Summary of methods of solving a quadratic equation. From the preceding discussion it appears that *a quadratic equation is solved by forming from it two simple equations whose roots are those of the quadratic.*

E.g., to solve the quadratic equation

$$x^2 + 7x + 12 = 0,$$

we may write it in the form

$$(x + 3)(x + 4) = 0,$$

whence $x + 3 = 0$, or $x + 4 = 0$, *two simple equations whose roots,* -3, -4, *are those of the quadratic.*

Or we may write it in the form

$$x^2 + 7x + (\tfrac{7}{2})^2 - (\tfrac{1}{2})^2 = 0,$$

whence $$[(x + \tfrac{7}{2}) + \tfrac{1}{2}][(x + \tfrac{7}{2}) - \tfrac{1}{2}] = 0,$$

and therefore $$x + \tfrac{7}{2} + \tfrac{1}{2} = 0,$$

or $$x + \tfrac{7}{2} - \tfrac{1}{2} = 0,$$

two simple equations whose roots, -3, -4, *are those of the quadratic.*

Or we may write it in the form

$$x^2 + 7x + (\tfrac{7}{2})^2 = (\tfrac{1}{2})^2,$$

whence $$x + \tfrac{7}{2} = \tfrac{1}{2},$$

or $$x + \tfrac{7}{2} = -\tfrac{1}{2},$$

two simple equations whose roots, -3, -4, *are those of the quadratic.*

Or we may simply write out the results from a formula obtained by one of the above methods.

For expressions easily factored the first method is the best; otherwise it is usually better to use the formula at once.

Illustrative problems. 1. Solve the equation

$$\frac{x+3}{x+5}-\frac{x+1}{x+3}=\frac{3x-5}{3x-7}-\frac{3x-3}{3x-5}.$$

The denominators are such as to suggest adding the fractions in each member separately before clearing of fractions. Then

1. $$\frac{4}{(x+3)(x+5)}=\frac{4}{(3x-5)(3x-7)}.$$

2. Multiplying by $\frac{1}{4}(x+3)(x+5)(3x-5)(3x-7)$,

$$(3x-5)(3x-7)=(x+3)(x+5). \qquad \text{Ax. 6}$$

3. $\therefore$ $$8x^2-44x+20=0, \qquad \text{(Why ?)}$$

or $$2x^2-11x+5=0.$$

4. This is easily factored (§ 263), and

$$(x-5)(2x-1)=0.$$

5. $\therefore$ $$x=5 \text{ or } \tfrac{1}{2}.$$

Check. For $x=5$, $\frac{4}{80}=\frac{4}{80}$; for $x=\frac{1}{2}$, $\frac{16}{77}=\frac{16}{77}$.

2. Solve the equation $\dfrac{1}{x-1}+\dfrac{1}{x-2}+\dfrac{1}{x-3}=0.$

Multiplying by $(x-1)(x-2)(x-3)$ we have

1. $$3x^2-12x+11=0.$$

2. This is not so easily factored as in the first problem; hence, applying the formula (§ 271), we have

$$x=-\frac{-12}{2\cdot 3}\pm\frac{1}{2\cdot 3}\sqrt{(-12)^2-4\cdot 3\cdot 11}$$
$$=2\pm\tfrac{1}{3}\sqrt{3}.$$

Check. $$\frac{1}{1\pm\frac{1}{3}\sqrt{3}}+\frac{1}{\pm\frac{1}{3}\sqrt{3}}+\frac{1}{-1\pm\frac{1}{3}\sqrt{3}}$$
$$=\frac{1\mp\frac{1}{3}\sqrt{3}}{1-\frac{1}{3}}\pm\sqrt{3}+\frac{-1\mp\frac{1}{3}\sqrt{3}}{1-\frac{1}{3}}$$
$$=\tfrac{3}{2}\mp\tfrac{1}{2}\sqrt{3}\pm\sqrt{3}-\tfrac{3}{2}\mp\tfrac{1}{2}\sqrt{3}=0.$$

3. Solve the equation $x^2+2x=0$.

This factors into $x(x+2)=0$, whence $x=0$ or -2.

And, in general, if x is a factor of every term of an equation, $x=0$ is one root.

EXERCISES. CXX.

Solve the following:

1. $\frac{1}{x+1} - \frac{2}{1-x} = \frac{13}{4x-1}.$

2. $\frac{3}{3-x} - \frac{2}{2-x} = \frac{1}{1-3x}.$

3. $\frac{2x+1}{x+1} - \frac{x+1}{x+2} = \frac{x-6}{x-7}.$

4. $\frac{4x}{2x-1} - \frac{x+1}{x} = \frac{x+5}{x+4}.$

5. $\frac{1}{a-x} - \frac{1}{a-2x} = \frac{1}{a-5x}.$

6. $\frac{3}{x^2-1} - \frac{6}{x-1} + \frac{1}{x+1} - 2 = 0.$

7. $\frac{2}{3}x^2 - \frac{x}{2} - \frac{16}{15} = \frac{34}{69}x^2 - \frac{76}{115}x + \frac{5}{6}.$

8. $\frac{x-2a}{2a} + \frac{x-3b}{3b} - \frac{x^2-6ab}{6ab} = 0.$

9. $\sqrt{2-x} + \sqrt{3+x} - \sqrt{11+x} = 0.$

10. $(1+2x)^{\frac{1}{2}} - (3+x)^{\frac{1}{2}} + (2-x)^{\frac{1}{2}} = 0.$

11. $\frac{6+5x}{4(5-x)} - \frac{3x-4}{5(5+x)} + \frac{5-7x}{25-x^2} - \frac{89}{105} = 0.$

12. $\frac{4(2-\sqrt{x})}{\sqrt{x}+x} = \frac{\sqrt{x}-x}{2+\sqrt{x}} + \frac{3x^2}{4(\sqrt{x}+x)(2+\sqrt{x})}.$

13. $\frac{4(2+\sqrt{x})}{\sqrt{x}-x} = \frac{\sqrt{x}+x}{2-\sqrt{x}} + \frac{3x^2}{4(\sqrt{x}-x)(2-\sqrt{x})}.$

II. DISCUSSION OF THE ROOTS.

273. The number of roots. The roots of the equation $ax^2 + bx + c = 0$ have been shown to be

$$-\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}.$$

This shows that every quadratic equation has two roots.

It is also true that no quadratic equation has more than two different roots.

For, suppose the equation $x^2 + px + q = 0$ has three different roots, r_1, r_2, r_3. Then by substituting these for x we have

1. $r_1^2 + pr_1 + q = 0,$
2. $r_2^2 + pr_2 + q = 0,$
3. $r_3^2 + pr_3 + q = 0,$ whence
4. $r_1^2 - r_2^2 + p(r_1 - r_2) = 0.$

Dividing by $r_1 - r_2$, which by hypothesis $\neq 0$,

5. $r_1 + r_2 + p = 0.$

Similarly, taking equations 2 and 3,

6. $r_2 + r_3 + p = 0,$
7. $\therefore r_1 - r_3 = 0,$ by subtracting. Ax. 3

But this is impossible because, by hypothesis, $r_1 \neq r_3$. Hence, it is impossible that the equation shall have three different roots, and so for any greater number.

It must be observed, however, that a quadratic equation need not have two *different* roots. For example, the equation

$$x^2 - 4x + 4 = 0$$

reduces to $(x-2)(x-2) = 0,$

and the roots are 2 and 2; that is, the equation has two roots, but they are equal.

274. The nature of the roots. The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation

$$ax^2 + bx + c = 0.$$

In this discussion a, b, c are supposed to be *real.*

If the discriminant is positive, the two roots are real and unequal.

For then $-\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$ can involve no imaginary.

In particular, *if the discriminant is a perfect square, the two roots are rational.*

For then $\sqrt{b^2 - 4ac}$ is rational.

If the discriminant is zero, the two roots are equal.

For then $-\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} = -\frac{b}{2a} \pm 0.$

In this case, $-\frac{b}{2a}$ is called a **double root.**

If the discriminant is negative, the two roots are complex.

For then $-\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}$ contains the imaginary $\sqrt{b^2 - 4ac}$.

Since the two complex roots enter together the instant that b^2 becomes less than $4ac$, we see that *complex roots enter in pairs.*

For example, in the equation

$$x^2 + 3x - 7 = 0$$

the roots are real, since $3^2 - 4(-7)$ is positive.

In $$2x^2 + x - 3 = 0$$

the roots are rational, since $1 - (-24)$ is a perfect square.

In $$3x^2 + 2x + 1 = 0$$

the roots are complex, since $4 - 12$ is negative.

275. Since the equation $ax^2 + bx + c = 0$ has for its roots $-\frac{b}{2a} + \frac{1}{2a}\sqrt{b^2 - 4ac}$ and $-\frac{b}{2a} - \frac{1}{2a}\sqrt{b^2 - 4ac}$, it follows that

$$\left[x - \left(-\frac{b}{2a} + \frac{1}{2a}\sqrt{b^2 - 4ac}\right)\right] \cdot \left[x - \left(-\frac{b}{2a} - \frac{1}{2a}\sqrt{b^2 - 4ac}\right)\right] = 0.$$

Hence, *any quadratic function of* x *can be factored*

1. In the domain of *rationality*,
   if the discriminant is *square;*
2. In the domain of *reality*,
   if the discriminant is *positive;*
3. In the domain of *complex numbers*,
   if the discriminant is *negative;*
4. Into two *equal factors*,
   if the discriminant is *zero*.

Illustrative problems. 1. What is the nature of the roots of the equation $x^2 + x + 1 = 0$?

$\because b^2 - 4ac = 1 - 4 = -3$, the two roots are complex.

2. What is the nature of the roots of the equation

$$x^2 + 6x + 9 = 0?$$

$\because b^2 - 4ac = 36 - 36 = 0$, the two roots are equal.

3. What is the nature of the roots of the equation

$$4x^2 + 8x + 3 = 0?$$

$\because b^2 - 4ac = 64 - 48 = 16$, the roots are real, unequal, and rational.

4. Can $f(x) = 5x^2 + 3x - 7$ be factored?

$\because b^2 - 4ac = 9 + 140 = 149$, which is not a square, $f(x)$ cannot be factored in the domain of rationality.

EXERCISES. CXXI.

What is the nature of the roots of equations 1–10?

1. $5x^2 + 1 = 0.$
2. $a^2x^2 + \frac{1}{4} - ax = 0.$
3. $x^2 - x + 1 = 0.$
4. $2x^2 - x - 20 = 0.$
5. $3x^2 + x + 7 = 0.$
6. $3x^2 + 4x + 5 = 0.$
7. $7x^2 - x - 3 = 0.$
8. $x^2 + 50x + 625 = 0.$
9. $\frac{1}{4}x^2 + x + 1 = 0.$
10. $12x^2 - 12x + 3 = 0.$

Of the following functions of x select those which can be factored in the domain of rationality and factor them.

11. $3x^2 - 7.$
12. $2x^2 + 7x + 3.$
13. $6x^2 + x - 1.$
14. $2x^2 - 5x + 3.$
15. $7x^2 + 2x - 6.$
16. $55x^2 - 27x + 2.$
17. $6x^2 + 7x - 3.$
18. $11x^2 - 23x + 2.$
19. $2x^2 + 3x - 4.$
20. $132a^2 + 51a - 21.$
21. $40x^2 + 34x + 6.$
22. $121x^2 + 11x + 12.$
23. $80x^2 + 70x + 60.$
24. $56x^2 + 113x + 56.$
25. $65x^2 - 263x - 42.$
26. $105x^2 - 246x + 33.$

Reduce the following to the form $ax^2 + bx + c = 0$, and state the nature of the roots:

27. $\dfrac{x + \sqrt{x}}{x - \sqrt{x}} = 4.$
28. $\dfrac{(x + a)^2}{(a - b)^2} - \dfrac{x + b}{a - b} = 3.$
29. $\dfrac{1}{x} - x - \dfrac{8}{3} = 0.$
30. $\dfrac{2x + b}{a} - \dfrac{4x - a}{2x - b} = 0.$
31. $2 - \dfrac{1}{x - 1} = \dfrac{1}{(x - 1)^2}.$
32. $\dfrac{\sqrt{x}}{\sqrt{x} - 5} + \dfrac{20 - \sqrt{x}}{\sqrt{x}} = 3.$

276. Relation between roots and coefficients. The roots of the equation $x^2 + px + q = 0$ are

$$x_1 = -\frac{p}{2} + \tfrac{1}{2}\sqrt{p^2 - 4q},$$

$$x_2 = -\frac{p}{2} - \tfrac{1}{2}\sqrt{p^2 - 4q}.$$

Their sum is $x_1 + x_2 = -p$,

and their product $$x_1x_2 = \left(-\frac{p}{2}\right)^2 - \left(\tfrac{1}{2}\sqrt{p^2 - 4q}\right)^2$$

$$= \frac{p^2}{4} - \frac{p^2 - 4q}{4}$$

$$= q.$$

That is, *in an equation of the type* $\mathrm{x}^2 + \mathrm{px} + \mathrm{q} = 0$,

1. *The sum of the roots is the coefficient of* x *with the sign changed;*

2. *The product of the roots is the absolute term.*

These relations evidently give a valuable check upon our solutions. Any solution which contradicts these laws is incorrect.

E.g., if the student finds the roots of the equation $x^2 - x - 30 = 0$ to be -6 and 5, there is an error somewhere in the solution, because their sum is not the coefficient of x with its sign changed.

EXERCISES. CXXII.

Solve the following, checking by the above laws.

1. $x^2 + 1 = 0$.
2. $x^2 - 1 = 0$.
3. $x^2 + x = 0$.
4. $x^2 - x - 1 = 0$.
5. $x^2 - 6x + 8 = 0$.
6. $x^2 - x - 2 = 0$.
7. $x^2 - 5x + 4 = 0$.
8. $x^2 - 17x + 16 = 0$.
9. $x^2 - 12x + 27 = 0$.
10. $x^2 + 24x + 144 = 0$.

277. Formation of equations with given roots. Since if

$$x = r_1 \text{ and } \quad x = r_2,$$

then $$x - r_1 = 0, \quad \text{"} \; x - r_2 = 0,$$

and hence $(x - r_1)(x - r_2) = 0$, a quadratic equation; therefore it is easy to form a quadratic equation with any given roots.

E.g., to form the quadratic equation whose roots are 2 and -3.

1. $\because \quad x = 2, \quad \therefore x - 2 = 0.$
2. $\because \quad x = -3, \quad \therefore x + 3 = 0.$
3. $\therefore \quad (x - 2)(x + 3) = 0,$ or $x^2 + x - 6 = 0.$

Similarly, to form the equation whose roots are $\frac{1}{2} \pm \frac{1}{4} i$.

1. $\because \quad x = \frac{1}{2} + \frac{1}{4} i, \quad \therefore x - \frac{1}{2} - \frac{1}{4} i = 0.$
2. $\because \quad x = \frac{1}{2} - \frac{1}{4} i, \quad \therefore x - \frac{1}{2} + \frac{1}{4} i = 0.$
3. $\therefore (x - \frac{1}{2} - \frac{1}{4} i)(x - \frac{1}{2} + \frac{1}{4} i) = 0$, and this may, if desired, be written in the form

$$x^2 - x + \tfrac{5}{16} = 0,$$

or $$16 x^2 - 16 x + 5 = 0.$$

EXERCISES. CXXIII.

Form the equations whose roots are given below.

1. $\frac{2}{3}, \frac{3}{2}.$
2. $\sqrt{2}, \sqrt{3}.$
3. $i, -i.$
4. $\sqrt{2}, -3.$
5. $3, -11.$
6. $-7, -8.$
7. $-\dfrac{a}{2}, -\dfrac{a}{2}.$
8. $\dfrac{1}{a}\sqrt{b}, \dfrac{1}{b}\sqrt{a}.$
9. $-a \pm 2bi.$
10. $3 + 2i, 3 - 2i.$
11. $-\frac{1}{2} \pm \frac{1}{2} i \sqrt{3}.$
12. $5 + 3i, 5 - 3i.$
13. $\dfrac{a}{b}\sqrt{-1}, -\dfrac{a}{b}\sqrt{-1}.$
14. $a + 2\sqrt{-1}, a - 2\sqrt{-1}.$

III. EQUATIONS REDUCIBLE TO QUADRATICS.

278. Thus far the student has learned how to solve any equation of the first or second degree involving one unknown quantity, and simultaneous equations of the first degree involving several unknown quantities.

It is not within the limits of this work to consider general equations of degree higher than the second. It often happens, however, that special equations of higher degree can be solved by factoring, as already explained, or by reducing to quadratic form.

A few of the more common cases will now be considered, some having already been suggested in the exercises.

279. The type $ax^{2n} + bx^n + c = 0$. This is a quadratic in x^n, and (§ 269)

$$x^n = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac},$$

whence
$$x = \sqrt[n]{-\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}}.$$

Illustrative problems. 1. Solve the equation

$$x^6 + 10x^3 + 16 = 0.$$

This is a quadratic in x^3 and is easily solved by factoring.

$$\because (x^3 + 8)(x^3 + 2) = 0,$$

$$\therefore x^3 = -8, \text{ or } -2.$$

$$\therefore x = -2, \text{ or } -\sqrt[3]{2}.$$

Check for $x = -\sqrt[3]{2}$. $4 - 20 + 16 = 0$.

We might also solve by the above formula, thus:

$$x = \sqrt[3]{-5 \pm \tfrac{1}{2}\sqrt{100 - 64}}$$

$$= \sqrt[3]{-2}, \text{ or } \sqrt[3]{-8} = -2.$$

2. Solve the equation $x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0.$

This may be arranged

$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 1 = 0, \text{ a quadratic in } x + \frac{1}{x}.$$

Solving (§ 270), $\qquad x + \frac{1}{x} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}.$

$$\therefore x^2 - (-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5})x + 1 = 0,$$

and (§ 270) $\qquad x = \tfrac{1}{4}(\sqrt{5} - 1 \pm i\sqrt{10 + 2\sqrt{5}}),$

or $\qquad \tfrac{1}{4}(-\sqrt{5} - 1 \pm i\sqrt{10 - 2\sqrt{5}}).$

3. Solve the equation $x^{-\frac{1}{2}} + x^{-\frac{1}{4}} - 2 = 0.$

This is a quadratic in $x^{-\frac{1}{4}}$. Solving by factoring,

$$x^{-\frac{1}{4}} = 1, \text{ or } -2.$$

$$\therefore x = 1, \text{ or } \frac{1}{(-2)^4}.$$

Check for $x = \frac{1}{(-2)^4}$. $\quad (-2)^2 + (-2) - 2 = 0.$

If $(-2)^4$ had been written 16, there would appear to be an extraneous root, but by writing it $(-2)^4$ we know that the 4th root is -2.

4. Solve the equation $x^2 = 21 + \sqrt{x^2 - 9}.$

This may be arranged

$$(x^2 - 9) - (x^2 - 9)^{\frac{1}{2}} - 12 = 0.$$

The solution often seems easier if y is put for the unknown expression in the quadratic. Here, let $y = (x^2 - 9)^{\frac{1}{2}}$. Then

$$y^2 - y - 12 = 0,$$

or $\qquad (y - 4)(y + 3) = 0,$

whence $\qquad y = 4, \text{ or } -3.$

$\therefore \qquad x^2 - 9 = 16, \text{ or } (-3)^2,$

and $\qquad x^2 = 25, \text{ or } 9 + (-3)^2,$

and $\qquad x = \pm 5, \text{ or } \pm\sqrt{9 + (-3)^2}.$

Check for $x = \pm\sqrt{9 + (-3)^2}$. $\quad 9 + (-3)^2 = 21 + \sqrt{9 + (-3)^2 - 9}$, or $18 = 21 - 3$, because $\sqrt{(-3)^2} = -3$. If the $(-3)^2$ were written 9, there would appear to be an extraneous root.

5. Solve the equation $(x^2 + x + 3)(x^2 + x + 5) = 35.$

In equations of this kind there is often an advantage in letting y equal some function of x. Here, let $y = x^2 + x + 3$. Then

$$y(y + 2) = 35,$$

or
$$y^2 + 2y - 35 = 0,$$

or
$$(y + 7)(y - 5) = 0,$$

whence
$$y = -7, \text{ or } 5.$$

Hence,
$$x^2 + x + 3 = -7, \text{ or } 5,$$

and each of these equations can be solved for x.

It would answer just as well to let $y = x^2 + x + 5$, in which case we should have $(y - 2)y = 35$.

EXERCISES. CXXIV.

Solve the following:

1. $\sqrt{x - 1} = x - 1.$
2. $x - x^{\frac{1}{2}} - 20 = 0.$
3. $7x - 4x^{\frac{1}{2}} - 20 = 0.$
4. $x^{\frac{1}{4}} + x^{\frac{1}{2}} - 20 = 0.$
5. $x^6 - 28x^3 + 27 = 0.$
6. $7x^{\frac{2}{3}} + x^{\frac{1}{3}} - 350 = 0.$
7. $x^{\frac{3}{2}} + x - 6x^{\frac{1}{2}} = 0.$
8. $x^2 + 5x - 1 = \dfrac{35}{x^2 + 5x + 1}.$
9. $(x^2 + 3)^2 + (x^2 + 3) - 42 = 0.$
10. $x - (a + b)x^{\frac{1}{2}} - 2a(a - b) = 0.$
11. $(x^2 + 2x + 3)(x^2 + 2x + 6) = -2.$
12. $(x^2 + 3x - 4)(x^2 + 3x + 2) + 8 = 0.$
13. $\sqrt{x}/(21 - \sqrt{x}) + (21 - \sqrt{x})/\sqrt{x} = 2.5.$
14. $\dfrac{1}{x^2 + x + 2} + \dfrac{1}{x^2 + x + 4} - \dfrac{6}{x^2 + x + 8} = 0.$

280. Radical equations have already been discussed (§ 191) in the special case in which they lead to simple equations, and several problems have been given in connection with the study of quadratics.

Whenever they lead to quadratic equations their solution is possible, and a few cases somewhat more elaborate than those already given will now be considered.

Illustrative problems. 1. Solve the equation

$$2x^2 + 3x - 3\sqrt{2x^2 + 3x - 4} - 2 = 0.$$

1. This may be arranged

$$2x^2 + 3x - 4 - 3\sqrt{2x^2 + 3x - 4} + 2 = 0.$$

2. Let $y = \sqrt{2x^2 + 3x - 4}.$

3. Then $y^2 - 3y + 2 = 0.$

4. $\therefore$ $(y-2)(y-1) = 0.$

5. $\therefore$ $y = 2$, or 1.

6. $\therefore$ $2x^2 + 3x - 4 = 2$, or 1,

two quadratic equations in x, which give

$$x = -\tfrac{3}{4} \pm \tfrac{1}{4}\sqrt{57},\ 1, \text{ or } -\tfrac{5}{2}.$$

Check for $x = -\frac{5}{2}$. $\frac{25}{2} - \frac{15}{2} - 3 - 2 = 0.$

2. Solve the equation $x - 1 = 2 + 2x^{-\frac{1}{2}}$.

1. This may be written

$$(\sqrt{x} - 1)(\sqrt{x} + 1) - \frac{2(\sqrt{x} + 1)}{\sqrt{x}} = 0.$$

2. Or $(\sqrt{x} + 1)\left(\sqrt{x} - 1 - \dfrac{2}{\sqrt{x}}\right) = 0.$

3. $\therefore$ $\sqrt{x} + 1 = 0$, and $\sqrt{x} = -1$, and $x = (-1)^2$, or $\sqrt{x} - 1 - \dfrac{2}{\sqrt{x}} = 0$, and $x - \sqrt{x} - 2 = 0$, a quadratic in $\sqrt{x}$.

4. $\therefore$ $(\sqrt{x} - 2)(\sqrt{x} + 1) = 0.$

5. $\therefore$ $\sqrt{x} = 2$, and $x = 4$, or $\sqrt{x} = -1$, and $x = (-1)^2$.

$\therefore$ there are three roots, two being alike, 4, $(-1)^2$, $(-1)^2$. All three are easily seen to check. The reason for writing $(-1)^2$ instead of $+1$ is explained on p. 267, exs. 3 and 4.

3. Solve the equation $\sqrt{x+3} - \sqrt{x+8} = 5\sqrt{x}$.

1.	$2x + 11 - 2\sqrt{(x+3)(x+8)} = 25x$.	Ax. 8
2. ∴	$-2\sqrt{x^2 + 11x + 24} = 23x - 11$.	Ax. 3
3. ∴	$21x^2 - 22x + 1 = 0$, squaring, etc.	
4. ∴	$(21x - 1)(x - 1) = 0$, and $x = \frac{1}{21}$, or 1.	

In checking, each root is found to be extraneous. This might have been anticipated because in squaring the first member of step 2 the $(-2)^2$ was called 4, and hence, when the result was placed under the radical sign for checking, and the root taken as positive, a failure to check was natural.

Had the original equation been $\sqrt{x+3} + \sqrt{x+8} = 5\sqrt{x}$, the root 1 would have checked; had it been $-\sqrt{x+3} + \sqrt{x+8} = 5\sqrt{x}$, the root $\frac{1}{21}$ would have checked.

EXERCISES. CXXV.

Solve the following:

1. $\sqrt[3]{x+3} - \sqrt[3]{x-4} - 1 = 0$.
2. $x^2 + x = 4 + \sqrt{10 - x^2 - x}$.
3. $\sqrt{1+4x} - \sqrt{1-4x} = 4\sqrt{x}$.
4. $\sqrt{x^2 - 8x + 31} + (x-4)^2 = 5$.
5. $x^2 + 5x - 10 = \sqrt{x^2 + 5x + 2}$.
6. $\sqrt{x-2} + \sqrt{3+x} - \sqrt{19+x} = 0$.
7. $x^{\frac{1}{2}}(x^2-1)^{\frac{1}{4}} - 2x(x^2-1)^{\frac{1}{2}} - \frac{1}{4} = 0$.
8. $\sqrt{1+2x} - \sqrt{4+x} + \sqrt{3-x} = 0$.
9. $\sqrt[3]{x+8} + \sqrt[3]{x-6} - \sqrt[3]{8x-10} = 0$.
10. $\sqrt{4x-2} + 2\sqrt{2-x} - \sqrt{14-4x} = 0$.
11. $3\sqrt{x^2 - 7x + 12} = \sqrt{7}\ \ \sqrt{x^2 - 7x + 12}$.
12. $\sqrt{(x-1)(x-2)} + \sqrt{(x-3)(x-4)} = \sqrt{2}$.

281. Reciprocal and binomial equations. A **reciprocal equation** is an equation in which the coefficients of the terms equidistant from those of highest and lowest degree, respectively, have the same absolute value and have the same signs throughout or opposite signs throughout.

E.g., the following:

$$x^2 - 1 = 0,$$

$$ax^3 + bx^2 - bx - a = 0,$$

$$ax^4 - bx^3 + cx^2 - bx + a = 0,$$

$$x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$$

They are called *reciprocal*, because they are unaltered when for the unknown quantity is written its reciprocal.

E.g., when $\frac{1}{x}$ is written for x in the equation

$$ax^2 + bx + a = 0,$$

it becomes

$$\frac{a}{x^2} + \frac{b}{x} + a = 0,$$

which, by multiplying both members by x^2, reduces to

$$a + bx + ax^2 = 0,$$

the original equation.

282. Since x can be replaced by $\frac{1}{x}$, *the roots of reciprocal equations enter in pairs,* each root being the reciprocal of the other root of that pair, excepting the two roots $+1$ and -1, each of which is its own reciprocal.

E.g., $x^2 + x + 1 = 0$ has for its roots

$$x_1 = -\tfrac{1}{2} + \tfrac{1}{2} i \sqrt{3},$$

$$x_2 = -\tfrac{1}{2} - \tfrac{1}{2} i \sqrt{3},$$

and each is the reciprocal of the other, because their product is 1 (§ 162).

So $x^2 + 1 = 0$ has for its roots the reciprocals i and $-i$.

Similarly in the case of $x^3 - 2x^2 - 2x + 1 = 0$. Here

$$x^3 + 1 - 2x(x+1) = 0,$$

whence $(x+1)(x^2 - x + 1 - 2x) = 0,$

and therefore $x + 1 = 0$, and $x = -1$,

or $x^2 - 3x + 1 = 0$, and $x = \frac{3}{2} \pm \frac{1}{2}\sqrt{5}$.

In this case, $\frac{3}{2} + \frac{1}{2}\sqrt{5}$ and $\frac{3}{2} - \frac{1}{2}\sqrt{5}$ are reciprocals, because their product is 1 (§ 162), and the other root, -1, is its own reciprocal. And in general, in the case of reciprocal equations of odd degree, one root is always its own reciprocal.

This is seen in the case of $x^3 - 1 = 0$.

283. Reciprocal equations can often be reduced to equations of lower degree by the factoring method set forth in the preceding example, or by dividing by some power of the unknown quantity, as in the following case:

Solve $x^4 + x^3 + x^2 + x + 1 = 0$.

Divide by x^2, and
$$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0,$$
an equation already considered (§ 279).

It reduces to
$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 1 = 0,$$
a quadratic in $x + \frac{1}{x}$. Solving for $x + \frac{1}{x}$, we have

$$x + \frac{1}{x} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}. \qquad \text{§ 270}$$

$$\therefore x^2 + (\tfrac{1}{2} \mp \tfrac{1}{2}\sqrt{5})x + 1 = 0,$$

two quadratics in x.

These equations may now be solved for x, each giving two values. The final roots are four in number, as would be expected. They are given on p. 267, and in more complete form on p. 273.

284. Equations of the form $x^n + p = 0$ are called **binomial equations.** In this case, no restriction is placed on p; it may be positive, negative, integral, fractional, real, imaginary, etc.

The solution of binomial equations in which $p = \pm 1$ evidently depends upon the solution of a reciprocal equation.

E.g., $$x^5 - 1 = 0$$
reduces to $$(x - 1)(x^4 + x^3 + x^2 + x + 1) = 0,$$
whence $$x - 1 = 0, \text{ and } x = 1,$$
or $$x^4 + x^3 + x^2 + x + 1 = 0,$$
a reciprocal equation, the one just considered in § 283, with four roots.

Since if $x^5 - 1 = 0$, or $x^5 = 1$, x is the fifth root of 1, and since $x^5 - 1 = 0$ has 5 roots (§§ 279, 283), *viz.*:

$$x_1 = 1,$$
$$x_2 = \tfrac{1}{4}(\sqrt{5} - 1 + i\sqrt{10 + 2\sqrt{5}}),$$
$$x_3 = \tfrac{1}{4}(\sqrt{5} - 1 - i\sqrt{10 + 2\sqrt{5}}),$$
$$x_4 = \tfrac{1}{4}(-\sqrt{5} - 1 + i\sqrt{10 - 2\sqrt{5}}),$$
$$x_5 = \tfrac{1}{4}(-\sqrt{5} - 1 - i\sqrt{10 - 2\sqrt{5}}),$$

therefore, there are 5 fifth roots of 1.

Similarly, there are 2 square roots of any number, 3 cube roots, $\cdots$ n nth roots.

Thus the two square roots of 1 are evidently $+1, -1$, which may be obtained by extracting the square root directly or by solving the equation $x^2 - 1 = 0$.

The three cube roots are readily found by solving the equation $x^3 - 1 = 0$.

Here $$x^3 - 1 = 0$$
leads to $$(x - 1)(x^2 + x + 1) = 0,$$
whence $$x - 1 = 0, \text{ and } x = 1,$$
or $$x^2 + x + 1 = 0, \text{ solved in § 282.}$$

EXERCISES. CXXVI.

Solve the following:

1. $2x^2 + 5x + 2 = 0.$

2. $x^3 + x^2 + x + 1 = 0.$

3. $10x^2 - 29x + 10 = 0.$

4. $2x^3 - 3x^2 - 3x + 2 = 0.$

5. $x^4 + x^3 - 4x^2 + x + 1 = 0.$

6. $x^4 - x^3 - 4x^2 - x + 1 = 0.$

7. $x^4 + 4x^3 + 2x^2 + 4x + 1 = 0.$

8. $x^4 - 5x^3 + \frac{83}{16}x^2 - 5x + 1 = 0.$

9. $x^4 - \frac{5}{6}x^3 - \frac{19}{3}x^2 - \frac{5}{6}x + 1 = 0.$

10. $2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0.$

11. $12x^4 + 4x^3 - 41x^2 + 4x + 12 = 0.$

12. $x^3 + 1 = 0.$ 13. $x^5 + 1 = 0.$ 14. $x^6 - 1 = 0.$

15. What are the 2 square roots of 1? the 3 cube roots? the 4 fourth roots?

16. What are the 3 cube roots of 8?

17. What are the 6 sixth roots of 1?

18. Show that the product of any two of the fourth roots of 1 equals one of the four roots, and that the cube of either imaginary root equals the other.

19. Show that the product of any two of the cube roots of 1 equals one of the three roots, and that the square of either complex cube root equals the other.

20. Show that the sum of the 2 square roots of 1, the sum of the 3 cube roots, the sum of the 4 fourth roots, the sum of the 5 fifth roots, are all equal to zero.

285. Exponential equations have already been considered in § 205. Only in certain cases can they be solved by linear or quadratic methods.

E.g., $2^{x^2} : 8^x = 16 : 1.$

This may be written

$$2^{x^2} : 2^{3x} = 2^4,$$

or $$2^{x^2-3x} = 2^4,$$

whence $$x^2 - 3x = 4,$$

giving $x = 4$, or -1. Each result checks.

The equation $2^{x+1} + 4^x = 8$ may be written

$$2 \cdot 2^x + 2^{2x} = 8,$$

or $(2^x)^2 + 2(2^x) - 8 = 0$, a quadratic in 2^x.

Hence, solving, $2^x = 2$ or, -4.

If $2^x = 2$, $x = 1$, a result which checks.

If $2^x = -4$, we cannot find x.

EXERCISES. CXXVII.

Solve the following:

1. $64^{x^2} : 2^x = 4.$
2. $3^{x^2} : 81^x = (3^{11})^7.$
3. $2^x \cdot 2^{x^2+1} = 2.$
4. $3^{2x} \cdot 9^x = 27^{x^2} \cdot 3.$
5. $2^{x^2} \cdot 16^x = \frac{1}{16}.$
6. $a^{x^2} : (a^x)^2 = (a^8)^5.$
7. $2 \cdot 4^{3\sqrt{x}} = 2^{3x-8}.$
8. $(3^x)^x \cdot 3^x = 27^{14}.$
9. $(4^{\frac{3x}{2}}/8)^x = 2^{3x+9}.$
10. $2 \cdot 6^{2x+4} = 3^{3x} \cdot 2^{x+9}.$
11. $\dfrac{a^{88x^2} \cdot a^{18x}}{a^8} = 1.$
12. $9^{x^2} \cdot 9^{7x} = \dfrac{1}{81^5}.$
13. $\dfrac{m^{18x^2}}{m^{77x}} : m^{18} = 1.$
14. $a^{85x^2} \cdot (a^2)^{87x} = \dfrac{1}{a^{85}}.$
15. $2 \cdot 5^{2-x} \cdot 25 = 2/25^{x-1}.$

IV. PROBLEMS INVOLVING QUADRATICS.

Illustrative problems. 1. What number is 0.45 less than its reciprocal?

1. Let $x =$ the number.

2. Then $x = \frac{1}{x} - 0.45.$

3. ∴ $x^2 + 0.45x - 1 = 0.$

4. ∴ $x = -0.225x \pm 0.5\sqrt{0.2025 + 4}$
$= 0.8, \text{ or } -1.25.$

Check. $0.8 = 1.25 - 0.45.$ $-1.25 = -0.8 - 0.45.$

Hence, either result satisfies the condition. But if the problem should impose the restriction "in the domain of positive numbers," -1.25 would be excluded; if "in the domain of negative numbers," 0.8 would be excluded; if "in the domain of integers," both results would be excluded and no solution would be possible.

2. A reservoir is supplied with water by two pipes, A, B. If both pipes are open, $\frac{11}{12}$ of the reservoir will be filled in 2 mins.; the pipe A alone can fill it in 5 mins. less time than B requires. Find the number of minutes in which the reservoir can be filled by A alone.

1. Let $x =$ the *number* of minutes required by A.

2. Then $x + 5 =$ " " " B.

3. Then $\frac{1}{x} =$ part filled by A in 1 min.,

and $\frac{1}{x+5} =$ " " B " 1 min.

4. ∴ $\frac{2}{x} + \frac{2}{x+5} =$ " " both in 2 mins. $= \frac{11}{12}.$

5. ∴ $11x^2 + 7x - 120 = 0$, or $(x - 3)(11x + 40) = 0.$

6. ∴ $x = 3, \text{ or } -\frac{40}{11}.$

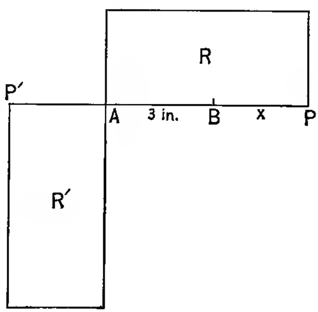
Here each root satisfies the *equation;* but the conditions of the *problem* are such as to limit the result to the domain of positive real numbers. Hence, $-\frac{40}{11}$, being meaningless in this connection, is rejected.

3. The number of students in this class is such as to satisfy the equation $2x^2 - 33x = 140$. How many are there?

1. $2x^2 - 33x - 140 = 0$.
2. $\therefore (x - 20)(2x + 7) = 0$.
3. $\therefore x = 20$, or $-\frac{7}{2}$.

Here, too, the conditions of the problem are such as to limit the result, this time to the domain of positive integers. Hence, "$-\frac{7}{2}$ of a student," being meaningless, is rejected.

4. A line, AB, 3 in. long, is produced to P so that the rectangle constructed with the base AP and the altitude BP has an area 14.56 sq. in. Find the length of BP.



1. Let x = the *number* of inches in BP.
2. Then the area $R = (3 + x)x = 14.56$.
3. $\therefore x^2 + 3x - 14.56 = 0$.
4. $\therefore x = 2.6$, or -5.6.

Here we are evidently not limited as in probs. 2 and 3. The negative root may be interpreted to mean that AB is produced to the left. BP' is -5.6 in., *i.e.*, 5.6 in. to the left, and the rectangle becomes R', which is, however, identically equal to R.

EXERCISES. CXXVIII.

In each exercise discuss the admissibility of both roots.

A. Relating to Numbers.

1. What number is $\frac{7}{12}$ of its reciprocal?

2. What number is $\frac{7}{12}$ greater than its reciprocal?

3. What is the number which multiplied by $\frac{3}{5}$ of itself equals 1215?

4. Separate the number 480 into two factors, of which the first is $\frac{5}{6}$ of the second.

5. The sum of a certain number and its square root is 42. Required the number.

6. Find a number of which the fourth and the seventh multiplied together give for a product 112.

7. One-fourth of the product of $\frac{3}{7}$ of a certain number and $\frac{5}{6}$ of the same number is 630. Find the number.

8. The square of 5 more than a certain number is 511,250 more than 10 times the number. Required the number.

9. The product of the numbers $2x3$ and $4x6$, written in the decimal system, is 115,368. What figure does x represent?

10. Separate the number 3696 into two factors such that if the smaller is diminished by 4 and the larger increased by 7 their product will be the same as before.

11. Of three certain numbers, the second is $\frac{4}{5}$ of the first, and the third is $\frac{5}{6}$ of the second; the sum of the squares of the numbers is 469. What are the numbers?

B. Relating to Mensuration.

For formulas see p. 172.

12. How many sides has a polygon which has 54 diagonals?

13. The area of a rectangle is 120 sq. in., and its diagonal is 17 in. Required its length and breadth.

14. The base of a triangle of area 16.45 sq. in. is 2.3 in. more than the altitude. Required the base.

15. The length of a rectangle of area 70 sq. in. is 3 in. more than the breadth. Required the dimensions.

16. Divide a line 16 in. long into two parts which shall form the base and altitude of a rectangle of 63.96 sq. in.

17. The hypotenuse of a right-angled triangle is 10 in., and one of the sides is 2 in. longer than the other. Required the lengths of the sides.

18. In a right-angled triangle one of the sides forming the right angle is 6 in., and the hypotenuse is double the other side. Find the length of the other side.

19. A square and a rectangle have together the area 220 sq. in. The breadth of the rectangle is 9 in., and the length of the rectangle equals the side of the square. Required the area of the square.

20. From the vertex of a right angle two bodies move on the arms of the angle, one at the rate of 1.5 ft., and the other 2 ft., per second. After how many seconds are they 50 ft. apart?

21. What is the result if, in the preceding example, 1.5, 2, and 50 are replaced by m, n, d?

22. A square is 78 sq. in. greater than a rectangle. The breadth of the rectangle is 7 in., and the length is equal to the side of the square. Required the side of the square.

23. If the sides of a certain equilateral triangle are shortened by 8 in., 7 in., and 6 in., respectively, a right-angled triangle is formed. Required the length of the side of the equilateral triangle.

24. If two sides of a certain equilateral triangle are shortened by 22 in. and 5 in., respectively, and the third is lengthened by 3 in., a right-angled triangle is formed. Required the length of a side of the equilateral triangle.

25. On an indefinite straight line given two points, A and B, d units apart, to find on this line a point, P, such that $AP^2 = BP \cdot AB$. Draw the figure showing the positions of the two points. (This is the celebrated geometric problem of "The Golden Section.")

26. Four places, A, B, C, D, are represented by the corners of a quadrilateral whose perimeter is 85 mi. The distance BC is 24 mi., and CD is 14 mi. The distance from A to D by the way of B and C is $\frac{4}{15}$ as great as the square of the distance from A direct to D. How far is it from A to B? also from A to D?

27. About the point of intersection of the diagonals of a square as a center, a circle is described; the circumference passes through the mid-points of the semi-diagonals; the area between the circumference and the sides of the square is 971.68 sq. in. Required the length of the side of the square. (Take $\pi = 3.1416$.)

28. A mirror 56 in. high by 60 in. wide has a frame of uniform width and such that its area equals that of the mirror. What is the width of the frame?

C. Relating to Physics.

29. If a bullet is fired upward with a velocity of 640 ft. per sec., the number of seconds elapsing before it strikes the earth is represented by t in the equation $0 = 320t - \frac{1}{2}gt^2$, in which $g = 32$ ft. Find t.

30. Two points, A and B, start at the same time from a fixed point and move about the circumference of a circle in opposite directions, each at a uniform rate, and meet after 6 secs. The point A passes over the entire circumference in 9 secs. less time than B. Required the time taken by A, and also by B, in passing over the whole circumference.

31. It is shown in physics that if two forces are pulling from a point, P, and are represented in direction and intensity by the lines PA, PB, the resultant force is represented by PC, the diagonal of their parallelogram. Two forces, of which the first is 23 lbs. greater than the second, act at right angles from a point. Their resultant is 37 lbs. Required the intensity of each force.

32. Two forces, of which the first is 47 lbs. less than the second, act at right angles from a point. Their resultant is 65 lbs. Required the intensity of each force.

33. It is proved in physics that if a body starts with a velocity ("initial velocity") of u ft. per sec., and if this increases a ft. per sec. (the "acceleration"), then in t secs. the space s described is $s = ut + \frac{1}{2}at^2$. Suppose the initial velocity is 40 ft. per sec., and the body moves with an acceleration of -2 ft. per sec., find when it will be 400 ft. from the starting point.

34. Suppose a body starts from a state of rest, and the acceleration is 18 ft. per sec., find the time required to pass over the first foot; the second; the third. (See ex. 33.)

35. Two points, A and B, start at the same time from a fixed point and move about the circumference of a circle in the same direction, each at a uniform rate, and are next together after 8 secs. The point A passes over the entire circumference in 18 secs. less time than B. Required the time taken by A in passing over the whole circumference.

36. It is shown in physics that if $h \equiv$ the number of feet to which a body rises in t secs. when projected upward with a velocity of u ft. per sec., then $h = ut - \frac{1}{2}gt^2$, where $g = 32$. Find the time that elapses before a body which starts with a velocity of 64 ft. per sec. is at a height of 28 ft.

37. A body is projected vertically upward with a velocity of 80 ft. per sec. When will it be at a height of 64 ft.? (See ex. 36.)

D. Miscellaneous.

38. A reservoir can be filled by two pipes, A and B, in 9 mins. when both are open, and the pipe A alone can fill it in 24 mins. less time than B can. Required the number of minutes that it will take A alone to fill it.

39. A reservoir has a supply pipe, A, and an exhaust pipe, B. A can fill the reservoir in 8 mins. less time than B can empty it. If both pipes are open, the reservoir is filled in 6 mins. Required the number of minutes which it will take to fill it if A is open and B is closed.

40. Two travelers, A and B, set out at the same time from two places, P and Q, respectively, and travel so as to meet. When they meet it is found that A has traveled 30 mi. more than B, and that A will reach Q in 4 das., and B will reach P in 9 das. after they meet. Find the distance between P and Q.

REVIEW EXERCISES. CXXIX.

1. Solve $\dfrac{x^2+m^2}{m^2+n^2}=\dfrac{x}{n}$.

2. Solve $\dfrac{x}{2}+\dfrac{2}{x}=\dfrac{x}{3}+\dfrac{8}{x}$.

3. Factor $(x^2+x-13)^2-49$.

4. Factor $x^3-6x^2-37x+210$.

5. Solve $\dfrac{2x}{x-4}+\dfrac{4x-3}{x+1}-9=0$.

6. Solve $\left(\dfrac{x-1}{x+1}\right)^2-\left(\dfrac{x-2}{x+2}\right)^2=0$.

7. Solve $\dfrac{x-a}{x-b}-\dfrac{x-b}{x-a}=\dfrac{a^2-b^2}{a^2-ax}$.

8. Simplify $\dfrac{2x^2-x+2}{4x^3+3x+2}\cdot\dfrac{4x^2-1}{2x-1}$.

9. Solve $\left(\dfrac{x^2-11x+19}{x^2+x-11}\right)^2=\dfrac{3(2-x)}{2+x}$.

10. Solve $18(x+1)^2(x+2)^2=8(x-3)^2(x+1)^2$.

11. Solve $(x-3)^3-3(x-2)^3+3(x-1)^3-x^3=9-x$.

12. Find the square root of

$$x^2y^{-2}+\tfrac{1}{4}y^2x^{-2}-xy^{-1}+\tfrac{1}{2}yx^{-1}-\tfrac{3}{4}.$$

13. If $x^2+xy+z=0$, and $w^2+wy+z=0$, where $x\neq w$, prove that $w+x+y=0$. (Subtract and factor.)

14. Find the lowest common multiple of

$$(b^2+c^2-a^2+2bc)(c+a-b)$$

and $$(a^2-b^2-c^2+2bc)(a+b+c).$$

CHAPTER XV.

SIMULTANEOUS QUADRATIC EQUATIONS.

I. TWO EQUATIONS WITH TWO UNKNOWN QUANTITIES.

286. 1. **When one equation is linear.** While this is not a case in simultaneous quadratics, since one equation is linear, it forms a good introduction to the general subject.

In this case, one of the unknown quantities can be found in terms of the other in the linear equation, and the value substituted in the quadratic. The problem then becomes that of solving a quadratic equation.

E.g., to solve the system $x - 2y = 3.$

$$x^2 + y^2 = 26.$$

Here we have

1. $x = 3 + 2y$, from the first equation.
2. $\therefore \quad (3 + 2y)^2 + y^2 = 26.$ (Why?)
3. $\therefore \quad 5y^2 + 12y - 17 = 0.$
4. $\therefore \quad (5y + 17)(y - 1) = 0.$
5. $\therefore \quad y = -\frac{17}{5}$, or 1.
6. $\therefore \quad x = 3 + 2y = -\frac{19}{5}$, or 5.

Check, for $x = -\frac{19}{5}$, $y = -\frac{17}{5}$.

$$-\tfrac{19}{5} + \tfrac{34}{5} = \tfrac{15}{5} = 3.$$

$$\tfrac{361}{25} + \tfrac{289}{25} = \tfrac{650}{25} = 26.$$

In checking, the roots must be properly arranged in pairs.

E.g., in the preceding example

$x = -\frac{19}{5}$ when and only when $y = -\frac{17}{5}$,

and $x = 2$ " " " " $y = 5$.

EXERCISES. CXXX.

Solve the following systems of equations:

1. $x/y = 2.$
$xy = 8.$

2. $x + y = 100.$
$xy = 2400.$

3. $x + y = 9.$
$xy = 45.$

4. $x - y = 11.$
$6/x = y/10.$

5. $x - y = 24.$
$xy = 4212.$

6. $\frac{1}{16}x^2 + y^2 = 122.$
$\frac{1}{2}x - y = 13.$

7. $x + y = 1.25.$
$xy = 0.375.$

8. $2x^2 + y^2 - 100 = 0.$
$7x - y - 50 = 0.$

9. $x^2 + y^2 = 1274.$
$x = 5y.$

10. $3x + 4y - 8 = 10.$
$x^2 - y^2 = -5.$

11. $5(x + y) = xy.$
$xy = 180.$

12. $\frac{1}{6}x^2 + \frac{1}{4}y^2 - 60 = 0.$
$\frac{1}{6}x + \frac{1}{4}y - 5 = 0.$

13. $x + y = -6.$
$xy = -2592.$

14. $14x^2 - 122y^2 = 100.$
$x = 3y.$

15. $\frac{1}{6}x^2 + \frac{1}{5}y^2 = 11.$
$\frac{2}{3}x + \frac{1}{5}y = 5.$

16. $27x + 33y - 60 = 0.$
$8x^2 + 10y^2 - 18 = 0.$

17. $x^2 + xy + y^2 = 63.$
$x - y = -3.$

18. $0.01x^2 + 0.5y - 2 = 0.$
$0.1x - 0.25y - 3 = 0.$

19. $(7 + x)(6 + y) = 80.$
$x + y = 5.$

20. $0.01x^2 + 400y - 25 = 0.$
$0.5x + y - 10 = 0.$

21. $x^2 + y^2 = 500.$
$\frac{x + y}{x - y} = 3.$

22. $x + y - 4 = 0.$
$\frac{1}{x} + \frac{1}{y} - 1 = 0.$

287. 2. **When both equations are quadratic.** In this case, x can be found in terms of y in either equation, but, *in general*, the value will involve y^2. In this case, the value of x substituted in the other equation will involve y^4, and hence *the result will be an equation of the fourth degree.*

E.g., given the system $x^2 - y^2 = -3$.

$$2x^2 + 3x + y = 7.$$

From the first equation

$$x = \pm\sqrt{y^2 - 3}.$$

Substituting in the second,

$$2(y^2 - 3) \pm 3\sqrt{y^2 - 3} + y = 7.$$

Isolating the radical, squaring, and reducing, we have

$$2y^4 + 2y^3 - 30y^2 - 13y + 98 = 0,$$

an equation of the fourth degree.

288. Hence, *in general, two simultaneous quadratic equations involving two unknown quantities cannot be solved by means of quadratics.*

It is only in special cases that such systems admit of solution by quadratics, and four pairs of roots should always be expected.

A few of the more common of these special cases will now be considered.

EXERCISES. CXXXI.

To what single equations of the fourth degree do the following systems reduce?

1. $x^2 + y = 7.$
   $x + y^2 = 11.$

2. $y^2 + 2x - xy = 5.$
   $x^2 + x + y = 4.$

3. $2x^2 + 3x - y^2 = 0.$
   $x^2 - 3y^2 + y = 0.$

4. $x^2 + xy + y^2 + x - 5 = 0.$
   $2x^2 + y^2 - x + y - 3 = 0.$

289. When one equation is homogeneous. In this case a solution is always possible. For if $ax^2 + bxy + cy^2 = 0$ is the homogeneous equation we can divide by y^2 and have $a \cdot \frac{x^2}{y^2} + b \cdot \frac{x}{y} + c = 0$, a quadratic in $\frac{x}{y}$. Hence, $\frac{x}{y}$ can be found and x will then be known as a multiple of y, and this value can then be substituted in the other equation.

E.g., to solve the system

1. $$x^2 - \tfrac{5}{2}xy + y^2 = 0.$$
2. $$x^2 + 3x - 4y + 4 = 0.$$
3. From 1 $$\left(\frac{x}{y}\right)^2 - \frac{5}{2}\left(\frac{x}{y}\right) + 1 = 0.$$
4. $\therefore$ $$2\left(\frac{x}{y}\right)^2 - 5\left(\frac{x}{y}\right) + 2 = 0,$$

or $$\left(2 \cdot \frac{x}{y} - 1\right)\left(\frac{x}{y} - 2\right) = 0.$$

5. $\therefore$ $$\frac{x}{y} = \frac{1}{2}, \text{ or } 2, \text{ and } x = \frac{y}{2}, \text{ or } 2y.$$

Substituting $x = \frac{y}{2}$ in equation 2, we have

6. $$\frac{y^2}{4} + \frac{3y}{2} - 4y + 4 = 0.$$
7. $\therefore$ $$y^2 - 10y + 16 = 0.$$
8. $\therefore$ $$y = 2, \text{ or } 8, \text{ and } \therefore x = \frac{y}{2} = 1, \text{ or } 4.$$

Substituting $x = 2y$ in equation 2 and reducing, we have

9. $$y^2 + \tfrac{1}{2}y + 1 = 0.$$
10. $\therefore$ $$y = -\tfrac{1}{4} \pm \tfrac{1}{4}i\sqrt{15}.$$
11. $\therefore$ $$x = 2y = -\tfrac{1}{2} \pm \tfrac{1}{2}i\sqrt{15}.$$
12. $\therefore$ $$x = 1, 4, -\tfrac{1}{2} + \tfrac{1}{2}i\sqrt{15}, -\tfrac{1}{2} - \tfrac{1}{2}i\sqrt{15},$$

and $$y = 2, 8, -\tfrac{1}{4} + \tfrac{1}{4}i\sqrt{15}, -\tfrac{1}{4} - \tfrac{1}{4}i\sqrt{15},$$

these roots being taken in pairs in the order indicated.

Check. All of these roots check. While the substitution of the complex roots takes time and patience, it is the only method of determining the correctness of the solution.

EXERCISES. CXXXII.

Solve the following systems of equations:

1. $x^2 + y^2 - bxy = 0.$
$$x + y = a.$$

2. $3x^2 + 3xy - y^2 = 5.$
$$x^2 - 2xy + y^2 = 0.$$

3. $5x^2 + 4xy - y^2 = 0.$
$$x^2 + x + y = 5.$$

4. $x^2 + xy + x - y = -2.$
$$2x^2 - xy - y^2 = 0.$$

5. $x^2 + 3xy + 3x - y = 2.$
$$x^2 + 2xy - 3y^2 = 0.$$

6. $x^2 - y^2 + x + y = \frac{13}{36}.$
$$36(x^2 + y^2) = 97xy.$$

7. $2x^2 + 3xy + 4y = 18.$
$$x^2 + 4xy = 12y^2.$$

8. $3x^2 + 4xy + 3x - y = 3.$
$$x^2 + xy = 0.$$

9. $x^2 + 4x + 3y + y^2 = -2.$
$$x(x + 2y) - 15y^2 = 0.$$

10. $x(x + y) + y(y + x) = 4xy.$
$$x(x + y) + y + x = 24.$$

11. $x^2 - 3x + 4y + 2xy = 24.$
$$x^2 + 3xy = 4y^2.$$

12. $147x^2 + 196xy + 57y^2 = 0.$
$$x^2 + 2xy + 33 = 0.$$

290. When both equations are homogeneous except for the absolute terms. In this case a solution is always possible by quadratics. For if

$$a_1x^2 + b_1xy + c_1y^2 = d_1,$$

and
$$a_2x^2 + b_2xy + c_2y^2 = d_2,$$

we can multiply both members of the first by d_2, and of the second by d_1, and subtract, and

$$(a_1d_2 - a_2d_1)x^2 + (b_1d_2 - b_2d_1)xy + (c_1d_2 - c_2d_1)y^2 = 0.$$

This may now be treated as in § 289.

E.g., to solve the system

1. $$x^2 + 3xy - 2y^2 = 2.$$

2. $$2x^2 - 5xy + 6y^2 = 3.$$

Multiplying both members of equation 1 by 3, and of equation 2 by 2, and subtracting, we have:

3. $$x^2 - 19xy + 18y^2 = 0.$$

This equation is easily reducible. If it were not, we should divide by y^2 and proceed as in § 289.

4. $\therefore$ $$(x - 18y)(x - y) = 0.$$

5. $\therefore$ $$x = 18y, \text{ or } y.$$

Substituting $18y$ for x in 1, we have

6. $$324y^2 + 54y^2 - 2y^2 = 2.$$

7. $\therefore$ $$y = \pm \tfrac{1}{2}\sqrt{\tfrac{1}{47}} = \pm \tfrac{1}{94}\sqrt{47},$$

and
$$x = 18y \quad = \pm \tfrac{9}{47}\sqrt{47}.$$

Substituting y for x in 1, we have

8. $$y^2 + 3y^2 - 2y = 2.$$

9. $\therefore$ $$y = \pm 1, \text{ whence } x = \pm 1.$$

Check. All of these results check.

E.g., try $x = \pm \frac{9}{47}\sqrt{47}$, $y = \pm \frac{1}{94}\sqrt{47}$.

Substituting these values in equation 1,

$$\tfrac{81}{47} + \tfrac{27}{94} - \tfrac{2}{188} = 2.$$

Substituting in equation 2,

$$\tfrac{162}{47} - \tfrac{45}{94} + \tfrac{3}{94} = 3.$$

291. Since §§ 289 and 290 depend upon finding the value of $\frac{x}{y}$, or of $\frac{y}{x}$, we can also solve by letting $\frac{y}{x}=v$, or $y=vx$, then finding v.

E.g., in the preceding example we had the system

1. $$x^2 + 3xy - 2y^2 = 2.$$
2. $$2x^2 - 5xy + 6y^2 = 3.$$

Let $\frac{y}{x} = v$, or $y = vx$. Then, from 1, we have

3. $$x^2 + 3vx^2 - 2v^2x^2 = 2.$$
4. ∴ $$x^2 = \frac{2}{1 + 3v - 2v^2}.$$

Similarly, from 2, we have

5. $$2x^2 - 5vx^2 + 6v^2x^2 = 3.$$
6. ∴ $$x^2 = \frac{3}{2 - 5v + 6v^2}.$$

Equating the values of x^2,

7. $$\frac{2}{1 + 3v - 2v^2} = \frac{3}{2 - 5v + 6v^2}.$$

Reducing,

8. $$18v^2 - 19v + 1 = 0,$$

or $$(18v - 1)(v - 1) = 0.$$

9. ∴ $$v = \tfrac{1}{18}, \text{ or } 1.$$
10. ∴ $$y = vx = \tfrac{1}{18}x, \text{ or } x.$$

This is substantially the same as step 5 of the preceding solution (p. 289), and the rest of the work is as given there.

In the same way we may let $\frac{x}{y} = v$, or $x = vy$. We should then have, from equation 1,

$$v^2y^2 + 3vy^2 - 2y^2 = 2.$$

∴ $$y^2 = \frac{2}{v^2 + 3v - 2}.$$

Similarly, from 2, $$y^2 = \frac{3}{2v^2 - 5v + 6}.$$

Equating these values of y^2, v can be found as above.

EXERCISES. CXXXIII.

Solve the following systems of equations:

1. $x^2 + 2xy = 39.$ $xy + 2y^2 = 65.$

2. $x^2 + 3xy = 2.$ $3y^2 + xy = 1.$

3. $x^2 + 3xy = 54.$ $xy + 4y^2 = 115.$

4. $2x^2 + 3xy = 27.$ $xy + y^2 = 4.$

5. $m^2x^2 + n^2y^2 = q^2.$ $\frac{x^2}{a^2} = \frac{y^2}{b^2}.$

6. $7x^2 - 5xy = 18.$ $\frac{x^2}{y^2} + 3 = \frac{7}{y^2}.$

7. $3xy + y^2 - 18 = 0.$ $4x^2 + xy - 7 = 0.$

8. $x^2 - xy + y^2 = 21.$ $y^2 - 2xy = -15.$

9. $x^2 + xy + y^2 = 139.$ $5y^2 - 4xy = -75.$

10. $ax^2 + b(x^2 + y^2) = m.$ $cy^2 + d(x^2 + y^2) = n.$

11. $x^2 - 2xy + y^2 = 57.$ $169x^2 + 2y^2 = 177.$

12. $3x^2 - 5xy + 2y^2 = 14.$ $2x^2 - 5xy + 3y^2 = 6.$

13. $2x^2 + 2xy + y^2 = 73.$ $x^2 + xy + 2y^2 = 74.$

14. $32y^2 - 2xy - 11 = 0.$ $x^2 + 4y^2 = 10.$

15. $3x^2 + 13xy + 8y^2 = 162.$ $x^2 - xy + y^2 = 7.$

16. $(3x + y)(3y + x) = 384.$ $(x - y)(x + y) = 40.$

17. $3x^2 + 4xy + 5y^2 - 48 = 0.$ $4x^2 + 5xy - 36 = 0.$

18. $2x^2 + 3xy - 3y^2 + 124 = 0.$ $7x^2 - xy - y^2 + 49 = 0.$

292. When the equations are symmetric with respect to the two unknown quantities. In this case a solution is always possible by quadratics. The solution is accomplished by letting $x = u + v$, and $y = u - v$, and first solving for u and v.

E.g., given the system

1. $$x^2 + 3xy + y^2 = 41.$$

2. $$x^2 + y^2 + x + y = 32.$$

Let $x = u + v$ and $y = u - v$. Then, by substituting in 1, we have

3. $$5u^2 - v^2 = 41, \text{ or } v^2 = 5u^2 - 41.$$

Substituting in 2,

4. $$u^2 + v^2 + u = 16.$$

Substituting here the value of v^2 from 3,

5. $$6u^2 + u - 57 = 0,$$

or $$(6u + 19)(u - 3) = 0.$$

6. $\therefore$ $$u = -\tfrac{19}{6}, \text{ or } 3.$$

Substituting this value of u in 3,

7. $$v = \pm \tfrac{1}{6}\sqrt{329}, \text{ or } \pm 2.$$

8. $\therefore x = u + v = \dfrac{-19 \pm \sqrt{329}}{6}$, 5, or 1, four values as we should expect (§ 287).

9. Since the equations are symmetric with respect to x and y, y must have the same values, always arranged so that $x + y$ shall equal $2u$. (Why?)

10. $\therefore$ for $$x = \frac{-19 + \sqrt{329}}{6}, \frac{-19 - \sqrt{329}}{6}, 5, 1,$$

we have $$y = \frac{-19 - \sqrt{329}}{6}, \frac{-19 + \sqrt{329}}{6}, 1, 5$$

All of the results check.

It should be noticed that a set of equations like

$$x - y = 1,\ x^2 + y^2 = 25,$$

is symmetric with respect to x and $-y$. Hence, if

$$x = 3, \text{ or } -2,\ y = 2, \text{ or } -3.$$

EXERCISES. CXXXIV.

Solve the following systems of equations:

1. $x^2 + y^2 = 41.$
   $x - y = 1.$

2. $x^2 + xy + y^2 = 19.$
   $x + y = 5.$

3. $x^2 - xy + y^2 = 3.$
   $x^2 + xy + y^2 = 7.$

4. $x^2 + y^2 + 3(x + y) = 4.$
   $3x^2 + 4xy + 3y^2 = 3.$

5. $x + \sqrt{xy} + y = 14.$
   $x^2 + xy + y^2 = 84.$

6. $x^2 - 2.5xy + y^2 = 0.$
   $2(x + y)^2 = 3.6(x^2 + y^2).$

7. $\dfrac{1}{x+2} + \dfrac{1}{y+2} = \dfrac{55}{63}.$
   $\dfrac{1}{x} + \dfrac{1}{y} = 7.$

8. $x - y = 4.$
   $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{26}{5}.$

9. $x(x + y) - 40 = 0.$
   $y(y + x) - 60 = 0.$

10. $2x^2 + xy + 2y^2 = 79.58.$
    $x^2 - 2xy + y^2 = 21.29.$

293. 3. **When equations above the second degree are involved.** *In general, such systems cannot be solved by quadratics,* although they can be solved in special cases.

E.g.,
$$x^3 + x^2y + y^3 = 11.$$
$$x - y = -1.$$

Here $x = y - 1$; hence,
$$(y-1)^3 + (y-1)^2y + y^3 = 11,$$
or
$$3y^3 - 5y^2 + 4y - 12 = 0, \text{ a cubic equation.}$$

Now a cubic equation may sometimes be solved by factoring, as here, for this reduces (§ 104) to
$$(y-2)(3y^2 + y + 6) = 0,$$
whence
$$y = 2, \text{ or } \tfrac{1}{6}(-1 \pm i\sqrt{71}),$$
whence
$$x = 1, \text{ " } \tfrac{1}{6}(-7 \pm i\sqrt{71}).$$

294. If the equations are symmetric with respect to the unknown quantities, they often yield to the method given in § 292.

E.g., to solve the system

1. $$x^3 + y^3 = 91.$$
2. $$x + y = 7.$$

Let $x = u + v$, $y = u - v$. Then

3. $$2u^3 + 6uv^2 = 91, \text{ from 1.}$$
4. $$u = \tfrac{7}{2}, \quad \text{" } 2.$$
5. $\therefore$ $$\tfrac{343}{4} + 21v^2 = 91, \text{ and } v = \pm \tfrac{1}{2}.$$
6. $\therefore x = u + v = 4$, or 3, and $\therefore y = 3$, or 4, by symmetry.

This system is easily solved in other ways, as by dividing the members of 1 by the members of 2, etc.

EXERCISES. CXXXV.

Solve the following systems of equations:

1. $x^3 + y^3 = 72.$
   $x + y = 6.$

2. $x^4 + y^4 = 97.$
   $x + y = 1.$

3. $x^4 + y^4 = 337.$
   $x - y = 1.$

4. $x^3 - y^3 = 279.$
   $x - y = 3.$

5. $x^5 + y^5 = 4149.$
   $x + y = 9.$

6. $x^2 + y^2 + xy(x + y) = 154.$
   $x^3 + y^3 - 3(x^2 + y^2) = 50.$

7. $\dfrac{1}{x^3} - \dfrac{1}{y^3} = 19.$
   $\dfrac{1}{x} - \dfrac{1}{y} = 1.$

8. $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = \dfrac{5}{2}.$
   $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{10}{xy}.$

9. $\sqrt{x + y} + \dfrac{4}{\sqrt{x + y}} - 4 = 0.$
   $\dfrac{x^2 + y^2}{xy} = \dfrac{34}{15}.$

295. Special devices will frequently suggest themselves, but it is not worth while to attempt to classify them. A few are given in the following illustrative problems.

1. Solve the system

$$1.\quad x^2y^2 + xy - 6 = 0.$$

$$2.\quad x^2 + y^2 = 5.$$

From 1 we have

3. $(xy - 2)(xy + 3) = 0$, whence $xy = 2$, or -3.

4. Adding $2xy = 4$ or -6 to, and subtracting it from, the respective members of 2, we have

5. $$x^2 + 2xy + y^2 = 9, \text{ or } -1.$$
$$x^2 - 2xy + y^2 = 1, \text{ " } \ 11.$$

6. $\therefore$ $$x + y = \pm 3, \text{ or } \pm i,$$
$$x - y = \pm 1, \text{ " } \pm\sqrt{11}.$$

Adding, and dividing by 2,

7. $x = \dfrac{\pm 3 \pm 1}{2}$, or $\dfrac{\pm i \pm \sqrt{11}}{2}$

$$= 2,\ 1,\ -1,\ -2,\ \frac{i + \sqrt{11}}{2},\ \frac{i - \sqrt{11}}{2},\ \frac{-i + \sqrt{11}}{2},\ \frac{-i - \sqrt{11}}{2}.$$

On account of symmetry, y must have the same values, arranged so as to satisfy step 6.

$$\therefore y = 1,\ 2,\ -2,\ -1,\ \frac{i - \sqrt{11}}{2},\ \frac{i + \sqrt{11}}{2},\ \frac{-i - \sqrt{11}}{2},\ \frac{-i + \sqrt{11}}{2}.$$

All of the results check.

E.g., consider the last ones,

$$x = \frac{-i - \sqrt{11}}{2}, \quad y = \frac{-i + \sqrt{11}}{2}.$$

Substituting in equation 1,

$$\left(\frac{-i - \sqrt{11}}{2} \cdot \frac{-i + \sqrt{11}}{2}\right)^2 + \frac{-i - \sqrt{11}}{2} \cdot \frac{-i + \sqrt{11}}{2} - 6$$
$$= (-3)^2 + (-3) - 6 = 9 - 3 - 6 = 0.$$

Substituting in equation 2,

$$\left(\frac{-i - \sqrt{11}}{2}\right)^2 + \left(\frac{-i + \sqrt{11}}{2}\right)^2 = \frac{10 + 2i\sqrt{11}}{4} + \frac{10 - 2i\sqrt{11}}{4} = 5.$$

2. Solve the system

$$1. \quad x = a\sqrt{x+y}.$$

$$2. \quad y = b\sqrt{x+y}.$$

Adding,

3. $$x + y = (a+b)\sqrt{x+y}, \text{ or}$$

$$x + y - (a+b)\sqrt{x+y} = 0, \text{ or}$$

4. $$\sqrt{x+y}\,(\sqrt{x+y} - \overline{a+b}) = 0.$$

5. $\therefore$ $$\sqrt{x+y} = 0, \text{ or } a+b.$$

Substituting in 1 and 2,

$$x = 0, \text{ or } a(a+b).$$

$$y = 0, \text{ `` } b(a+b).$$

The results check.

3. Solve the system

$$1. \quad x^4 + x^2y^2 + y^4 = 481.$$

$$2. \quad x^2 + xy + y^2 = 37.$$

Factoring 1, by § 114,

3. $$(x^2 + xy + y^2)(x^2 - xy + y^2) = 481.$$

4. $\therefore$ $$37(x^2 - xy + y^2) = 481, \text{ or}$$

$$x^2 - xy + y^2 = 13.$$

Subtracting from 2,

5. $$2xy = 24, \text{ whence } xy = 12.$$

Adding to 2, and subtracting from 4,

6. $$x^2 + 2xy + y^2 = 49.$$

$$x^2 - 2xy + y^2 = 1.$$

7. $\therefore$ $$x + y = \pm 7.$$

$$x - y = \pm 1.$$

8. $\therefore x = 4, \ -4, \ 3, \ -3, \quad y = 3, \ -3, \ 4, \ -4.$

Graphs. For the graphic representation of quadratic equations, and for the discussion of the number of roots of simultaneous quadratic equations with two unknown quantities, see Appendix IX. If Appendix VIII has been studied, this may be taken at this point.

MISCELLANEOUS EXERCISES. CXXXVI.

Solve the following systems of equations:

1. $x^3 + y^3 = b.$
   $x + y = a.$

2. $x^2 - xy + y^2 = 124.$
   $x^2 - y^2 = 44.$

3. $x^{\frac{3}{2}} + y^{\frac{2}{3}} = 3x.$
   $x^{\frac{1}{2}} + y^{\frac{1}{3}} = x.$

4. $y\sqrt{y} = 17\sqrt{y} + 4x.$
   $x^3 = 4\sqrt{y} + 17x.$

5. $x^2 + y^2 = 25x^2y^2.$
   $12xy = 1.$

6. $x^2 + y^2 - x - y = \frac{7}{4}.$
   $xy = 1.$

7. $\sqrt{x} + \sqrt{y} = 12.$
   $x^2 + y^2 = 3026.$

8. $3(x^2 + y^2) = 10(x + y).$
   $9(x^4 + y^4) = 34(x^3 + y^3).$

9. $(x^2 + xy + y^2)\sqrt{x^2 + y^2} = 185.$
   $(x^2 - xy + y^2)\sqrt{x^2 + y^2} = 65.$

10. $\frac{y}{x} - \frac{9\sqrt{x}}{y} - \frac{81}{xy} = (2y + 9)\frac{\sqrt{x}}{y}.$
    $\frac{\sqrt{y}}{x} + 3\sqrt{\frac{x}{y}} = \frac{9}{x\sqrt{y}} + \sqrt{x}.$

11. $\sqrt{x^2 + 144} + \sqrt{y^2 + 144} = 35.$
    $xy = 144.$

12. $\sqrt{x^2 - y^2} - \sqrt{x^2 + y^2} + 2 = 0.$
    $\sqrt{x + y} - \sqrt{x - y} = 1.5.$

13. $x + y - 2\sqrt{xy} - \sqrt{x} + \sqrt{y} = 2.$
    $\sqrt{x} + \sqrt{y} = 7.$

14. $\sqrt{x} + \sqrt{y} = x - y = x - \sqrt{xy} + y.$

15. $x^2 - 6xy + 9y^2 - 4x + 12y = -4.$
    $x^2 - 2xy + 3y^2 - 4x + 5y = 53.$

II. THREE OR MORE UNKNOWN QUANTITIES.

In general, three simultaneous quadratic equations involving three unknown quantities cannot be solved by quadratics.

Many special cases, however, admit of such solution.

The same is true if one equation is linear and the other two are quadratic, or if one is of a degree higher than 2.

If, however, two are linear and the other quadratic, a solution is possible by quadratics, as in illustrative problem 2 on p. 299.

Illustrative problems. 1. Solve the system

1. $3xy = 2x + 2y.$

2. $2yz = 3y + 2z.$

3. $4zx = 5z - 3x.$

Dividing both members of 1, 2, 3, by xy, yz, zx, respectively, we have

4. $$3 = \frac{2}{x} + \frac{2}{y}.$$

5. $$2 = \frac{2}{y} + \frac{3}{z}.$$

6. $$4 = \frac{5}{x} - \frac{3}{z}.$$

Adding 5 and 6,

7. $$6 = \frac{5}{x} + \frac{2}{y}.$$

Eliminating y, with 4 and 7,

8. $$3 = \frac{3}{x}, \text{ whence } x = 1.$$

$$\therefore y = 2,\ z = 3.$$

Check.

$$6 = 2 + 4,$$

$$12 = 6 + 6,$$

$$12 = 15 - 3.$$

2. Solve the system

$$1.\quad x + y - 2z = -9.$$
$$2.\quad 3x + 2y + z = 9.$$
$$3.\quad x^2 + y^2 + z^2 = 30.$$

Eliminating z from 1 and 2,

4. $$y = \frac{9 - 7x}{5}.$$

Eliminating y from 1 and 2,

5. $$z = \frac{27 - x}{5}.$$

Substituting 4 and 5 in 3, and reducing,

6. $$5x^2 - 12x + 4 = 0, \text{ or}$$
$$(x - 2)(5x - 2) = 0.$$

7. $\therefore$ $$x = 2, \text{ or } \tfrac{2}{5}.$$
$\therefore$ $$y = -1, \text{ or } \tfrac{31}{25}.$$
$\therefore$ $$z = 5, \text{ or } 5\tfrac{8}{25}.$$

Check for the second set of values.

$$\tfrac{2}{5} + \tfrac{31}{25} - 10\tfrac{16}{25} = -9.$$
$$\tfrac{6}{5} + \tfrac{62}{25} + 5\tfrac{8}{25} = 9.$$
$$\tfrac{4}{25} + \tfrac{961}{625} + \tfrac{17689}{625} = \tfrac{18750}{625} = 30.$$

EXERCISES. CXXXVII.

Solve the following systems of equations:

1. $4y^3 = 9xz.$
$x^3 = 36yz.$
$9z^3 = 4xy.$

2. $x^2 + y^2 + xy = 19.$
$y^2 + z^2 + yz = 37.$
$z^2 + x^2 + zx = 28.$

3. $\dfrac{x^2 + y^2}{xyz} = \dfrac{5}{6}.$
$\dfrac{z^2 + x^2}{xyz} = \dfrac{5}{3}.$
$\dfrac{y^2 + z^2}{xyz} = \dfrac{13}{6}.$

4. $\dfrac{xyz}{x + y} = \dfrac{9}{2}.$
$\dfrac{xyz}{y + z} = 2.$
$\dfrac{xyz}{z + x} = \dfrac{18}{7}.$

III. PROBLEMS INVOLVING QUADRATICS.

EXERCISES. CXXXVIII.

1. The difference of two numbers is 11, the sum of their squares 901. What are the numbers?

2. The sum of two numbers is 30, the sum of their squares 458. What are the numbers?

3. Find two numbers whose sum, whose product, and the difference of whose squares are all equal.

4. The sum of the squares of two numbers is 421, the difference of the squares 29. What are the numbers?

5. A certain fraction equals 0.625, and the product of the numerator and denominator is 14,440. Required the fraction.

6. The sum of the areas of two circles is 24,640 sq. in., and the sum of their radii is 112 in. Required the lengths of their radii.

7. The product of the numbers $2x3$ and $4y6$, in which x and y stand for the tens' digit, x being twice y, is 103,518. What are the tens' digits?

8. If a certain two-figure number, the sum of whose digits is 11, is multiplied by the units' digit, the product is 296. Required the number.

9. Three successive integers are so related that the square of the greatest equals the sum of the squares of the other two. Required the numbers.

10. Separate the number 102 into three parts such that the product of the first and third shall be 102 times the second, and the third shall be $\frac{3}{2}$ of the first.

11. Two cubes have together the volume 407 cu. in., and the sum of one edge of the one and one of the other is 11 in. Required the volume of each.

12. If the product of two numbers is increased by their sum, the result is 89; if the product is diminished by their sum, the result is 51. Required the numbers.

13. One of the sides forming the right angle of a right-angled triangle is $\frac{3}{7}$ the other, and the area of the triangle is 5082 sq. in. Required the lengths of the sides.

14. There are two numbers such that the product of the first and 1 more than the second is 660, and the product of the second and 1 less than the first is 609. What are the numbers?

15. A sum of money at interest for 5 yrs. amounts to $4600. Had the rate been increased 1% it would have amounted to $40 more than this in 4 yrs. Required the capital and the rate.

16. The product of the numbers $x17$ and $2y2$, in which x stands for the hundreds' digit of the first and y for the tens' of the second, and in which $y = x + 3$, is 83,054. Required the values of x and y.

17. Find a two-figure number such that the product of the two digits is half the number, and such that the difference between the number and the number with the digits interchanged is $\frac{3}{2}$ of the product of the two digits.

18. In going 1732.5 yds. the front wheel of a wagon makes 165 revolutions more than the rear wheel; but if the circumference of each wheel were 27 in. more, the front wheel would, in going the same distance, make only 112 revolutions more than the rear one. Required the circumference of each wheel.

19. The floor of a certain room has 210 sq. ft., each of the two side walls 135 sq. ft., and each of the two end walls 126 sq. ft. Required the dimensions of the room.

20. A certain cloth loses $\frac{1}{8}$ in length and $\frac{1}{16}$ in width by shrinking. Required the length and width of a piece which loses 3.68 sq. yds., and which has its perimeter decreased 3.4 yds. by shrinking.

21. A rectangular field is 119 yds. long and 19 yds. wide. How much must the width be decreased and the length increased in order that the area shall remain the same while the perimeter is decreased 24 yds.?

22. Two points move, each at a uniform rate, on the arms of a right angle toward the vertex, from two points 50 in. and 136.5 in., respectively, from the vertex. After 7 secs. the points are 85 in. apart, and after 9 secs. they are 68 in. apart. Required the rate of each.

23. There are two lines such that if they are made the sides of a right-angled triangle the hypotenuse is 17 in.; but if one be made the hypotenuse and the other a side, the remaining side is such that the square constructed upon it contains 161 sq. in. How long are the two lines?

24. There is a fraction whose numerator being increased by 2 and denominator diminished by 2, the result is the reciprocal of the fraction; but if the denominator is increased by 2 and the numerator diminished by 2, the result is $1\frac{1}{15}$ less than the reciprocal. Required the fraction.

25. If the numerator of a fraction is decreased by 2, and the new fraction added to the original one, the sum is $1\frac{2}{3}$; if the denominator is decreased by 2, and the new fraction added to the original one, the sum is $2\frac{1}{60}$. Required the fraction.

REVIEW EXERCISES. CXXXIX.

1. Solve $a^{x^2-1}/a^{y^2+1}=a^9,\ b^{x-1}/b^y=1.$

2. Form the equation whose roots are 0, i, i.

3. Solve $a^x/a^y=a^2,\ b^y\cdot b^z=(b^3)^3,\ c^x/c^z=1/c^{-3}.$

4. Solve $x+y=a+b,\ x/a-y/b=a/b-b/a.$

5. Solve $4y+5z=11,\ 3z+6x=9,\ 8x-3y=4.$

6. Construct an integral quadratic function of x such that $f(2)=0$ and $f(3)=0$.

7. Simplify

$$\{(x^{a-b}\cdot x^{b-c})^a\cdot(x^a\div x^c)^a\}/\{(x^a x^c)^a\div(x^{a+c})^c\}.$$

Solve the following:

8. $$x^4+x^2y^2+y^4=61.$$
$$x^2-xy+y^2=\frac{42}{xy}.$$

9. $$x+y=2xy=x^2-y^2.$$

10. $$x+y+(x+y)^{\frac{1}{2}}-12=0.$$
$$x^2+y^2-45=0.$$

11. $$\frac{1}{x+y}+\frac{1}{x-y}=\frac{3}{4}.$$
$$2x^3+6xy^2=\frac{9}{64}(x^2-y^2)^3.$$

12. $$(3x+4y)(7x-2y)+3x+4y=44.$$
$$(3x+4y)(7x-2y)-7x+2y=30.$$

13. $$17(x+y)^{-\frac{1}{2}}-7(x+y)^{\frac{1}{2}}x^{-1}=10x(x+y)^{-\frac{3}{2}}.$$
$$(x-y)^{\frac{1}{2}}=y-1.$$

CHAPTER XVI.

INEQUALITIES.

MAXIMA AND MINIMA.

296. Having given two real and unequal numbers, a and b, $a - b$ cannot be zero. If $a - b$ is positive, a is said to be *greater than* b; if negative, a is said to be *less than* b.

E.g., $3 > 2$ because $3 - 2$ is positive,
$-2 > -3$ " $-2 - (-3)$ is positive,
$-3 < -2$ " $-3 - (-2)$ is negative.

If $a > 0$, then a is positive, and if
$a < 0$, " " " negative.

297. The inequalities $a > b$, $c > d$ are called **inequalities in the same sense**, and similarly for $a < b$, $c < d$. But $a > b$, $c < d$ are called **inequalities in the opposite sense**, and similarly for $a < b$, $c > d$.

298. In this chapter the letters used to represent numbers will be understood to represent *positive and real finite numbers*, except as the minus sign indicates a negative number.

299. Just as we distinguish two classes of equalities, (1) equations and (2) identities, so in inequalities we have two classes, (1) those which are true only for particular values of a quantity called the unknown quantity, and (2) those which are true for all values of the letters.

E.g., $x + 2 > 3$ is true only when $x > 1$, but $a + b > b$ is always true.

300. If a variable quantity, x, cannot be greater than a constant, m, but can equal it or approach indefinitely near it in value, then m is called the **maximum** value of x.

Similarly, if $x \not< m$ but can equal it or approach indefinitely near it in value, then m is called the **minimum** value of x.

E.g., $(x-1)^2 \not< 0$, because it is the square of a real quantity and hence cannot be negative. But $(x-1)^2$ can equal 0 by letting $x = 1$. Hence, 0 is the minimum value of $(x-1)^2$.

Since we shall need the subject of inequalities in only a few cases in our subsequent work, we shall present but a few of the fundamental theorems. It is evident, however, that the subject is an extensive one, covering simple inequalities, quadratic inequalities, etc., together with simultaneous inequalities corresponding to simultaneous equations.

301. The axioms of inequalities. The following axioms have already been assumed and used:

Ax. 4. *If equals are added to unequals, the sums are unequal in the same sense.*

Ax. 5. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*

These are easily demonstrated, thus:

1. If $a > b$, then $a - b$ is positive. § 296

2. Then $$a - b \equiv a + k - k - b$$ $$\equiv (a + k) - (k + b)$$

and this expression is positive.

3. $\therefore$ $$a + k > b + k.$$ § 296

Similarly for ax. 5.

Theorems. Three important theorems of inequalities will now be proved, the first two corresponding to axs. 6 and 8.

302. Theorem. *If unequals are multiplied by equals, the products are unequal in the same or in the opposite sense, according as the multiplier is positive or negative.*

Proof. 1. If $a > b$, then $a - b$ is positive. § 296

2. Then $k(a - b)$ is positive,
and $-k(a - b)$ is negative. § 296

3. ∴ $ka - kb$ is positive,
and $-ka - (-kb)$ is negative.

4. ∴ $ka > kb$,
and $-ka < -kb$. § 296

In this discussion the multiplier is supposed to be neither zero nor infinite.

303. Theorem. *If* $a > b$, *then* $a^m > b^m$.

Proof. 1. $a - b$ is positive. § 296

2. ∴ $(a^{m-1} + a^{m-2}b + \cdots + ab^{m-2} + b^{m-1})(a - b)$ is positive, because the multiplier is evidently a positive quantity.

3. ∴ $a^m - b^m$ is positive, because this is the product of the expressions.

4. ∴ $a^m > b^m$. § 296

304. Theorem. *If* $a \neq b$, $a^2 + b^2 > 2ab$.

Proof. 1. $(a - b)^2 > 0$, because $(a - b)^2$ is positive, being the square of a real number. It is not 0, for $a \neq b$.

2. ∴ $a^2 - 2ab + b^2 > 0$.

3. ∴ $a^2 + b^2 > 2ab$.

Evidently $a^2 + b^2 = 2ab$, if $a = b$.

Illustrative problems. 1. Prove that $x^2 > 2x - 1$, if $x \neq 1$.

We have $x^2 + 1 > 2x$, by § 304.

2. $x^{p+q} + y^{p+q} > x^p y^q + x^q y^p$, if $x \neq y$.

1. This is true if $x^{p+q} - x^p y^q + y^{p+q} - x^q y^p$ is positive.
2. Or if $x^p(x^q - y^q) - y^p(x^q - y^q)$ is positive.
3. Or if $(x^p - y^p)(x^q - y^q)$ is positive.
4. But both factors are positive if $x > y$, and both factors are negative if $x < y$, and in either case their product is positive.

3. Which is greater, $2 + \sqrt{3}$, or $2.5 + \sqrt{2}$?

1.	$2 + \sqrt{3} \gtreqless 2.5 + \sqrt{2}$, according as	
2.	$7 + 4\sqrt{3} \gtreqless 8\frac{1}{4} + 5\sqrt{2}$, squaring.	§ 303
3. Or as	$-1\frac{1}{4} + 4\sqrt{3} \gtreqless 5\sqrt{2}$.	Ax. 5
4. Or as	$49\frac{9}{16} - 10\sqrt{3} \gtreqless 50$.	§ 303
5. Or as	$-10\sqrt{3} \gtreqless \frac{7}{16}$.	Ax. 5

6. But a negative number is less than a positive one.

$$\therefore 2 + \sqrt{3} < 2.5 + \sqrt{2}.$$

4. Solve the inequality $2x - \dfrac{x}{3} + \dfrac{1}{2} > 3x - \dfrac{1}{3} + \dfrac{x}{6}$.

1.	$12x - 2x + 3 > 18x - 2 + x$.	§ 302
2. $\therefore$	$-9x > -5$.	Ax. 5
3. $\therefore$	$x < \frac{5}{9}$, and $\frac{5}{9}$ is the maximum value.	§ 300

Check. If $x = \frac{5}{9}$, the inequality becomes an equation. If $x > \frac{5}{9}$, the sense of the inequality is reversed.

5. Solve the inequality $x^2 - 5x + 6 < 0$.

1. $(x-2)(x-3) < 0$, and hence is negative.
2. The smaller factor, $x - 3$, is negative, and the other positive.
3. $\therefore x > 2$ and $x < 3$, or $2 < x < 3$.

6. Show that the minimum value of $x^2 - 8x + 22$ is 6.

1. Let $x^2 - 8x + 22 = y$, in which we have to find the minimum value of y.

2. Then $x^2 - 8x + 22 - y = 0$.

3. $\therefore x = 4 \pm \sqrt{y - 6}$, and y cannot be less than 6 without making x complex.

7. Divide the number 6 into two parts such that their product shall be the maximum.

1. Let x and $6 - x$ be the parts.

2. Then $x(6 - x) = y$, in which we have to find the maximum value of y.

3. Solving for x, $x = 3 \pm \sqrt{9 - y}$, and y cannot be greater than 9 without making x complex.

4. When $y = 9$, $x = 3$; $\therefore$ the parts are 3 and 3.

Check. $3 \cdot 3 = 9$; but $2(6 - 2) = 2 \cdot 4 = 8$, a smaller number.

EXERCISES. CXL.

1. What is the nature of the inequality resulting from subtracting unequals from equals? Prove it.

2. Investigate the addition of unequals to unequals.

3. Also the subtraction of uuequals from unequals.

4. Show that the maximum value of $4x - x^2$ is 4, and that 2 is the value of x which makes this $f(x)$ a maximum.

5. If $f(x) \equiv x^2 + x + 1$, show that $x = -0.5$ renders $f(x)$ a minimum, and find the minimum.

6. Prove that, in general, $x^3 + 1 > x^2 + x$. What is the exception?

7. Also that $(x + y)^2 > 4xy$.

8. Solve the inequality $x^2 + 5x > -6$.

9. Prove that $(a+b)(b+c)(c+a) > 8\,abc$.

10. Prove that the minimum value of $x^2-10\,x+35$ is 10.

11. Solve the inequality $5\,x+2 > 3\,x+\frac{x}{2}-7$. Check the result.

12. Solve the inequality $\frac{x-3}{x-4} > 0$. Check.

13. Required the length of the sides of the maximum rectangle of perimeter 16.

14. Prove that if the sum of two factors is k, a constant, the maximum value of their product is $k^2/4$.

15. Show that if a square is inscribed in a square whose area is 16, its corners lying on the sides of the larger square, its area $\nless 8$.

16. If a, b, c are three numbers such that any two are together greater than the third, then

$$a^2+b^2+c^2 < 2\,ab+2\,bc+2\,ca.$$

17. Solve the inequality $x^2-3\,x < 10$.

18. Solve the inequality $x(x-10) < 11$.

19. Find the maximum value of $8\,x-x^2$, and also the value of x that renders this $f(x)$ a maximum.

20. Find the minimum value of $x(x+10)$, and also the value of x that renders this $f(x)$ a minimum.

21. Required the area of the largest rectangle having the perimeter 20 inches. How do the sides compare in length?

22. Required the area of the largest rectangle having the perimeter p inches. How do the sides compare in length?

CHAPTER XVII.

RATIO, VARIATION, PROPORTION.

I. RATIO.

305. The **ratio** of one number, a, to another number, b, of the same kind, is the quotient $\frac{a}{b}$.

Thus, the ratio of $2 to $5 is $\frac{\$2}{\$5}$, or $\frac{2}{5}$, or 0.4, but there is no ratio of $2 to 5 ft., or $10 to 2. Here, as elsewhere in algebra, however, the letters are understood to represent pure (abstract) number.

A ratio may be expressed by any symbol of division, *e.g.*, by the fractional form, by $\div$, by /, or by : ; but the symbols generally used are the fraction and the colon, as $\frac{a}{b}$, or $a:b$.

306. In the ratio $a:b$, a is called the **antecedent** and b the **consequent**.

307. The ratio $b:a$ is called the **inverse** of the ratio $a:b$.

308. If two variable quantities, x, y, have a constant ratio, r, one is said to **vary** as the other.

E.g., the ratio of any circumference to its diameter is $\pi = 3.14159$; hence, a circumference is said to vary as its diameter.

If $\frac{x}{y} = r$, then $x = ry$. The expression "x varies as y" is sometimes written $x \propto y$, meaning that $x = ry$.

If $x = r \cdot \frac{1}{y}$, x is said to vary inversely as y.

309. If two variable quantities, x, y, have the same ratio as two other variable quantities, x', y', then x and y are said to *vary as* x' and y'. And if any two values of one variable quantity have the same ratio as the corresponding values of another variable quantity which depends on the first, then one of these quantities is said to vary as the other.

E.g., the circumference c and diameter d of one circle have the same ratio as the circumference c' and diameter d' of any other circle; hence, c aud d are said to vary as c' and d'.

If two rectangles have the same altitude, their areas depend on their bases; and since any two values of their bases have the same ratio as the corresponding values of their areas, their areas are said to vary as their bases.

310. Applications in geometry. Similar figures may be described as figures having the same shape, such as lines, squares, triangles whose angles are respectively equal, circles, cubes, or spheres. It is proved in geometry that in two similar figures

1. *Any two corresponding lines vary as any other two corresponding lines.*

2. *Corresponding areas vary as the squares of any two corresponding lines.*

3. *Corresponding volumes vary as the cubes of any two corresponding lines.*

E.g., in the case of two spheres, the circumferences vary as the radii, the surfaces vary as the squares of the radii, the volumes vary as the cubes of the radii.

These facts are easily proved. Let s, s' stand for the surfaces of two spheres of radii r, r', respectively. Then we know from mensuration that

$$s = 4\pi r^2, \text{ and } s' = 4\pi r'^2,$$

$$\therefore \frac{s}{s'} = \frac{4\pi r^2}{4\pi r'^2} = \frac{r^2}{r'^2}.$$

Hence, the surfaces vary as the squares of the radii. In like manner the volumes might be considered.

Illustrative problems. 1. If the ratio of x^2 to 3 is 27, find the value of x.

$\because \dfrac{x^2}{3} = 27$, $\therefore x^2 = 3 \cdot 27 = 81$, $\therefore x = \pm 9$, and each value checks.

2. If a sphere of iron weighs 20 lbs., find the weight of a sphere of iron of twice the surface.

1. Let r_1, r_2 be the respective radii.

2. Then $4\pi r_1^2 = \frac{1}{2} \cdot 4\pi r_2^2$, because the surface of a sphere $= 4\pi r^2$. (p. 311.)

3. $\therefore \dfrac{r_2}{r_1} = \sqrt{2}$.

4. And $\because$ the volumes (aud hence the weights) vary as the cubes of the radii (§ 310), and $\because \dfrac{r_2^3}{r_1^3} = (\sqrt{2})^3 = 2\sqrt{2}$.

5. $\therefore$ the second sphere weighs $2\sqrt{2}$ times as much as the first.

$$2\sqrt{2} \cdot 20 \text{ lbs.} = 56.57 \text{ lbs., nearly.}$$

EXERCISES. CXLI.

1. The ratio of 625 to x^3 is 5. Find x.

2. Find x in the following ratios:

(a) $4 : x^2 = 9$. (b) $x^2 : 27 = 300$. (c) $x = \frac{1}{144} : x$.

(d) $\dfrac{x^2}{63} = 7$. (e) $\dfrac{36}{x} = x$.

3. Find x in the following ratios:

(a) $\dfrac{x^2}{15} = 2.4$. (b) $\dfrac{3}{x^3} = \dfrac{81}{8}$. (c) $\dfrac{49x^4}{432} = \dfrac{3}{49}$.

(d) $7 : x = 4.9$. (e) $x^3 : 5 = \frac{25}{8}$.

4. One cube is 1.2 times as high as another. Find the ratio of (1) their surfaces, (2) their volumes.

5. The surfaces of a certain sphere and a certain cube have the same area. Find, to 0.01, the ratio of their volumes.

311. Applications in business. Of the numerous applications of ratio in business, only a few can be mentioned, and not all of these commonly make use of the word "ratio."

In computing interest, the simple interest varies as the time, if the rate is constant; as the rate, if the time is constant; as the product of the rate and the number representing the time in years (if the rate is by the year), if neither is constant.

I.e., for twice the rate, the interest is twice as much, if the time is constant; for twice the time, the interest is twice as much, if the rate is constant; but for twice the time and 1.5 times the rate, the interest is $2 \cdot 1.5$ times as much.

The common expressions "2 out of 3," "2 to 5," "6 per cent" (merely 6 out of 100) are only other methods of stating the following ratios of a part to a whole, $\frac{2}{3}$, $\frac{2}{7}$, $\frac{6}{100}$, or the following ratios of the two parts, $\frac{2}{1}$, $\frac{2}{5}$, $\frac{6}{94}$.

E.g., to divide $100 between A and B so that A shall receive $2 out of every $3, is to divide it into two parts

(1) having the ratio 2 : 1, or
(2) so that A's share shall have to the whole the ratio 2 : 3, or
(3) so that B's share shall have to the whole the ratio 1 : 3.

EXERCISES. CXLII.

1. Divide $1000 so that A shall have $7 out of every $8.

2. Divide $500 between A and B so that A shall have $0.25 as often as B has $1.25.

3. The area of the United States is 3,501,000 sq. mi., and the area of Russia is 8,644,100 sq. mi. Express the ratio of the former to the latter, correct to 0.01.

4. The white population of the United States in 1780 was 2,383,000; in 1790, 3,177,257; in 1880, 43,402,970; in 1890, 54,983,890. What is the ratio of the population in 1790 to that in 1780? in 1890 to that in 1880?

312. Applications in physics. (a) *Specific gravity.* The specific gravity of any substance is the ratio of the weight of that substance to the weight of an equal volume of some other substance taken as a standard.

In the case of solids and liquids, distilled water is usually taken as the standard. Thus, the specific gravity of mercury, of which 1 l weighs 13.596 kg, is 13.596, because a liter of water weighs 1 kg, and

$$13.596 \text{ kg} : 1 \text{ kg} = 13.596.$$

In the case of gases either hydrogen or air is usually the standard.

The following table will be needed for reference:

SPECIFIC GRAVITIES.

Mercury,	13.596.	Silver,	10.5.
Nickel,	8.9.	Gold,	19.3.

WEIGHTS OF CERTAIN SUBSTANCES.

1 l of water, 1 kg. 1 cm^3 of water, 1 g.

1 cu. ft. of water, about 62.5 lbs.

EXAMPLE. What is the weight of 1 cu. in. of copper?

1. 1 cu. ft. of water weighs 62.5 lbs.
2. $\therefore$ 1 cu. in. of water weighs 62.5 lbs $\div$ 1728.
3. $\therefore$ 1 cu. in. of copper weighs $8.9 \cdot 62.5$ lbs. $\div$ 1728, or 5.15 oz.

EXERCISES. CXLIII.

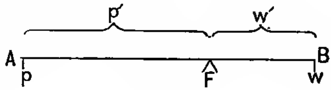
1. What is the weight of a cubic foot of gold?

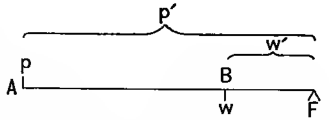
2. What is the weight of 1 cm^3 of nickel? of silver?

3. The specific gravity of ice is 0.92, of sea-water 1.025. To what depth will a cubic foot of ice sink in sea-water?

4. From ex. 3, how much of an iceberg 500 ft. high would show above water, the cross-section being supposed to have a constant area?

313. (b) *Law of levers.* If a bar, AB, rests on a fulcrum, F, and has a weight, w, at B, then by exerting enough pressure, p, at A the weight can be raised. In the first figure the pressure is downward (positive pressure); in the second it is upward (negative pressure).





There is a law in physics that, if p', w' represent the number of units of distance AF, FB, respectively, and p, w the number of units of pressure and weight, respectively, then

$$\frac{pp'}{ww'} = 1.$$

In the first figure p, w, p', w' are all considered as positive; in the second figure p is considered as negative because the pressure is upward, and w' is considered as negative because it extends the other way from F. Hence, the ratio $pp' : ww' = 1$ in both cases.

Example. Suppose $AF = 25$ in., $FB = 14$ in., in the first figure. What pressure must be applied at A to raise a weight of 30 lbs. at B?

1. By the law of levers $\frac{25\,p}{14 \cdot 30} = 1.$

2. $\therefore p = \frac{14 \cdot 30}{25} = 16.8$, and $\therefore$ the pressure must be 16.8 lbs.

EXERCISES. CXLIV.

1. Two bodies weighing 20 lbs. and 4 lbs. balance at the ends of a lever 2 ft. long. Find the position of the fulcrum.

2. The radii of a wheel and axle are respectively 4 ft. and 6 in. What force will just raise a mass of 56 lbs., friction not considered?

REVIEW EXERCISES. CXLV.

1. In each figure on p. 315, what must be the distance AF in order that a pressure of 1 kg may raise a weight of 100 kg 3 dm from F?

2. If a sphere of lead weighs 4 lbs., find the weight of a sphere of lead of (1) twice the volume, (2) twice the surface, (3) twice the radius.

3. A nugget of gold mixed with quartz weighs 0.5 kg; the specific gravity of the nugget is 6.5, and of quartz 2.15. How many grams of gold in the nugget?

4. A vessel containing 1 l and weighing 0.5 kg is filled with mercury and water; it then weighs, with its contents, 3 kg. How many cm^3 of each in the vessel?

5. What pressure must be exerted at the edge of a door to counteract an opposite pressure of 100 lbs. halfway from the hinge to the edge? one-third of the way from the hinge to the edge?

6. Explain Newton's definition of number: Number is the abstract ratio of one quantity to another of the same kind. What kinds of numbers are represented in the following cases: 5 ft. : 1 ft., 1 ft. : 5 ft., the diagonal to the side of a square, the circumference to the diameter of a circle?

7. The depths of three artesian wells are as follows: A 220 m, B 395 m, C 543 m; the temperatures of the water from these depths are: A 19.75° C., B 25.33° C., C 30.50° C. From these observations, is it correct to say that the increase of temperature is proportional to the increase of depth? If not, what should be the temperature at C to have this law hold?

THE THEORY OF RATIO.

314. A ratio is called a **ratio of greater inequality**, of **equality**, or of **less inequality**, according as the antecedent is greater than, equal to, or less than the consequent.

315. Theorem. *A ratio of greater inequality is diminished, a ratio of equality is unchanged in value, and a ratio of less inequality is increased by adding any positive quantity to both terms.*

Given the ratio $a : b$, and p any positive quantity.

To prove that $\frac{a+p}{b+p} \lesseqgtr \frac{a}{b}$ according as $a \gtreqless b$.

Proof. 1. $\frac{a+p}{b+p} \lesseqgtr \frac{a}{b}$ according as

$ab + pb \lesseqgtr ab + ap$. § 302, ax. 6

2. Or as $pb \lesseqgtr ap$, or as $b \lesseqgtr a$. § 301

3. *I.e.*, as $a \gtreqless b$.

316. Theorem. *If* $\frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{c}}{\mathrm{d}} = \frac{\mathrm{e}}{\mathrm{f}} = \cdots$, *then each of these ratios equals* $\frac{\mathrm{a}+\mathrm{c}+\mathrm{e}+\cdots}{\mathrm{b}+\mathrm{d}+\mathrm{f}+\cdots}$.

Proof. 1. Let $\frac{a}{b} = k$. Then $k = \frac{c}{d} = \frac{e}{f} = \cdots$.

2. $\therefore$ $a = kb$,

$c = kd$,

$e = kf, \cdots$.

3. $\therefore a + c + e + \cdots = k(b + d + f + \cdots)$. Ax. 2

4. $\therefore \frac{a+c+e+\cdots}{b+d+f+\cdots} = k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots$. Ax. 7

EXERCISES. CXLVI.

1. Prove that the product of two ratios of greater inequality is greater than either.

2. Consider ex. 1 for two ratios of equality; of less inequality. Then state the general theorem and prove it.

3. Find the value of x, knowing that if x is subtracted from both terms of the ratio $\frac{1}{2}$ the ratio is squared.

4. Is the value of a ratio changed by raising both terms to the same power? State the general theorem and prove it.

5. Prove (or show that it has been proved) that the value of a ratio is not changed by multiplying both terms by the same number.

6. As in § 315, consider the effect of subtracting from both terms of a ratio any positive number not greater than the less term. State the theorem and prove it.

7. Which is the greater ratio, $\frac{a+5b}{a+6b}$ or $\frac{a+6b}{a+7b}$?

8. Which is the greater ratio, $\frac{x-2y}{y-2x}$, or $\frac{x-3y}{3y-2x}$?

9. Which is the greater ratio, $\frac{a+b+c}{a-b-c}$, or $\frac{a-b+c}{a+b-c}$?

10. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $\frac{a^2+b^2+c^2}{ab+bc+cd}=\frac{ab+bc+cd}{b^2+c^2+d^2}$.

11. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$, prove that $k=\frac{3a+5c-6e}{3b+5d-6f}$.

12. If $\frac{a}{b}>\frac{c}{d}$, the letters standing for positive numbers, prove that $\frac{a}{b}>\sqrt{\frac{a^2+c^2}{b^2+d^2}}>\frac{c}{d}$.

II. VARIATION.

317. It has already been stated (§ 308) that the expressions "x varies as y," "x varies inversely as y," simply mean that the ratios $x:y$, $x:\frac{1}{y}$, are respectively equal to some constant. These are merely special cases of $x=f(y)$; for $x:y=k$ reduces to $x=ky$, whence x is a function of y; similarly $x:\frac{1}{y}=k$ reduces to $x=\frac{k}{y}$, whence x is a function of y.

Although there is nothing in the theory of variation which is not substantially included in the theory of ratio, the phraseology and notation of the subject are so often used in physics as to require some further attention.

Two illustrations from physics will be given in this connection, the one relating to the pressure of gases and the other to electricity. While neither requires much algebra for its consideration, each offers an excellent illustration of the use of variation in physics. No preliminary knowledge of physics is necessary, however, to the work here given.

318. Boyle's law for the pressure of gases. It is proved in physics that if p is the number of units of pressure of a given quantity of gas, and v is the number of units of volume, then p varies inversely as v when the temperature remains constant.

This law was discovered in the seventeenth century by Robert Boyle.

E.g., if the volume of a gas is 10 dm³ under the ordinary pressure of the atmosphere ("under a pressure of one atmosphere"), it is

$\frac{1}{2}$ as much when the pressure is 2 times as great,

$\frac{1}{n}$ " " " " " " n "

n times " " " " " " $\frac{1}{n}$ "

the temperature always being considered constant.

EXAMPLE. A toy balloon contains 3 l of gas when exposed to a pressure of 1 atmosphere. What is its volume when the pressure is increased to 4 atmospheres? decreased to $\frac{1}{8}$ of an atmosphere?

1. ∵ the volume varies inversely as the pressure, it is $\frac{1}{4}$ as much when the pressure is 4 times as great.

2. Similarly, it is 8 times as much when the pressure is $\frac{1}{8}$ as great.

3. ∴ the volumes are 0.75 l and 24 l.

EXERCISES. CXLVII.

1. If a cylinder of gas under a certain pressure has its volume increased from 20 l to 25 l, what is the ratio of the pressures?

2. A certain gas has a volume of 1200 cm^3 under a pressure of 1033 g to 1 cm^2. Find the volume when the pressure is 1250 g.

3. A cubic foot of air weighs 570 gr. at a pressure of 15 lbs. to the square inch. What will a cubic foot weigh at a pressure of 10 lbs.?

4. Equal quantities of air are on opposite sides of a piston in a cylinder that is 12 in. long; if the piston moves 3 in. from the center, find the ratio of the pressures. Draw the figure.

5. A liter of air under ordinary pressure weighs 1.293 g when the barometer stands at 76 cm. Find the weight when the barometer stands at 82 cm, the weight varying as the height of the barometer.

6. If the volume of a gas varies inversely as the height of the mercury in a barometer, and if a certain mass occupies 23 cu. in. when the barometer indicates 29.3 in., what will it occupy when the barometer indicates 30.7 in.?

319. Problems in electricity. The great advance in electricity in recent years renders necessary a knowledge of such technical terms as are in everyday use.

When water flows through a pipe some *resistance* is offered due to friction or other impediment to the flow of the water.

When electricity flows through a wire some *resistance* is offered. This *resistance is measured in ohms.* An *ohm* is the resistance offered by a column of mercury 1 mm^2 in cross-section, 106 cm long, at 0° C.

A certain *quantity* of water flows through the pipe in a second, and this may be stated in gallons or cubic inches, etc.

A certain *quantity* of electricity flows through the wire. This *quantity is measured in amperes.* An *ampere* is the current necessary to deposit 0.001118 g of silver a second in passing through a certain solution of nitrate of silver.

A certain *pressure* is necessary to force the water through the pipe. This pressure may be measured in pounds per sq. in., kilograms per cm^2, etc.

A certain *pressure* is necessary to force the electricity through the wire. This *pressure is measured in volts.* A *volt* is the pressure necessary to force 1 ampere through 1 ohm of resistance.

Hence, in considering the water necessary to do a certain amount of work (as to turn a water-wheel) it is necessary to consider not merely the *pressure*, for a little water may come from a great height, nor merely the *volume*, nor merely the *resistance* of the pipe; all three must be considered.

Hence, in considering the electricity necessary to do certain work it is necessary to consider not merely the *voltage*, for a little electricity may come with a high pressure, nor merely the *amperage*, nor merely the *number of ohms of resistance;* all three must be considered.

The names of the electrical units mentioned come from the names of three eminent electricians, Ohm, Ampère, and Volta.

320. It is proved in physics that the resistance of a wire varies directly as its length and inversely as the area of its cross-section.

That is, if a mile of a certain wire has a resistance of 3.58 ohms, 2 mi. of that wire will have a resistance of $2 \cdot 3.58$ ohms, or 7.16 ohms. Also, 1 mi. of wire of the same material but of twice the sectional area will have a resistance of $\frac{1}{2}$ of 3.58 ohms, or 1.79 ohms.

From these laws and definitions, the most common problems and statements concerning electrical measurements will be understood.

EXERCISES. CXLVIII.

1. If the resistance of 700 yds. of a certain cable is 0.91 ohm, what is the resistance of 1 mi. of that cable?

2. The resistance of a certain electric lamp is 3.8 ohms when a current of 10 amperes is flowing through it. What is the voltage?

3. If the resistance of 130 yds. of copper wire $\frac{1}{16}$ in. in diameter is 1 ohm, what is the resistance of 100 yds. of $\frac{1}{32}$ in. copper wire?

4. The resistance of a certain wire is 9.1 ohms, and the resistance of 1 mi. of this wire is known to be 1.3 ohms. Required the length.

5. Three arc lamps on a circuit have a resistance of 3.12 ohms each; the resistance of the wires is 1.1 ohms, and that of the dynamo is 2.8 ohms. Find the voltage for a current of 14.8 amperes.

6. The resistance of a dynamo being 1.6 ohms, and the resistance of the rest of the circuit being 25.4 ohms, and the electromotive force being 206 volts, find how many amperes flow through the circuit.

THEORY OF VARIATION.

321. Theorem. *If* $x \propto y$ *and* $y \propto z$, *then* $x \propto z$.

Proof. 1. If $x \propto y$, then $x = ky$. § 308

2. If $y \propto z$, then $y = k'z$. § 308

3. ∴ $x = ky = kk'z$. Substn.

4. ∴ $x \propto z$. § 308

Note that in step 2 we cannot use the same constant as in step 1.

E.g., if the edge of a cube varies as the diagonal of a face, and the diagonal of a face varies as the diagonal of the cube, then the edge must vary as the diagonal of the cube.

322. Theorem. *If* $x \propto yz$, *then* $y \propto x / z$.

Proof. 1. $x = kyz$. (Why ?)

2. ∴ $y = \frac{1}{k} x / z$. Ax. 7

3. ∴ $y \propto x / z$. § 308

E.g., if the area of a rectangle varies as the product of the (numbers representing the) base and altitude, then the base varies as the quotient of the (number representing the) area divided by the (number representing the) altitude.

323. Theorem. *If* $w \propto x$ *and* $y \propto z$, *then* $wy \propto xz$.

Proof. 1. $w = kx$ and $y = k'z$. (Why ?)

2. ∴ $wy = kk'xz$. (Why ?)

3. ∴ $wy \propto xz$. (Why ?)

E.g., if the surface of a sphere varies as the square of the diameter, and $\frac{1}{3}$ of the radius varies as the radius, then the product of the surface and $\frac{1}{3}$ of the radius varies as the product of the radius and the square of the diameter.

324. Theorem. *If* x ∝ y *when* z *is constant, and if* x ∝ z *when* y *is constant, then* x ∝ yz *when both* y *and* z *vary.*

To understand this statement consider a simple illustration: The area of a triangle (p. 172) varies as the altitude when the base is constant, and as the base when the altitude is constant; but it varies as the product of their numerical values when both base and altitude vary.

Proof. 1. Let the variations of y and z take place separately.

2. Let x change to x' when y changes to y', z remaining unchanged. Then

$$\because x \propto y, \therefore \frac{x}{x'} = \frac{y}{y'}.$$

3. Let x' change to x'' when y' remains unchanged and z changes to z'. Then

$$\because x \propto z, \therefore \frac{x'}{x''} = \frac{z}{z'}.$$

4. $\therefore \frac{x}{x'} \cdot \frac{x'}{x''}$, or $\frac{x}{x''}$, equals $\frac{y}{y'} \cdot \frac{z}{z'}$, or $\frac{yz}{y'z'}$.

5. *I.e.*, x changes to x'' as yz changes to $y'z'$, or $x \propto yz$.

Illustrative problems. 1. If $x \propto y$, and if $x = 2$ when $y = 5$, find x when $y = 11$.

$\because x \propto y$ means that $x = ky$, $\therefore 2 = k \cdot 5$, and $k = \frac{2}{5}$. $\therefore x = \frac{2}{5}y$. When $y = 11$, $x = \frac{2}{5} \cdot 11 = 4.4$.

2. The volumes of spheres vary as the cubes of their radii. Two spheres of metal are melted into a single sphere. Required its radius.

1. $v = kr^3$ and $v' = kr'^3$. § 308
2. $\therefore$ the volume of the single sphere is $k(r^3 + r'^3)$.
3. Call v'' this volume, and r'' the radius; then

$$v'' = k(r^3 + r'^3) = kr''^3.$$

4. $\therefore r''^3 = r^3 + r'^3$, and $\therefore r'' = (r^3 + r'^3)^{\frac{1}{3}}$.

EXERCISES. CXLIX.

1. If $x \propto z$ and $y \propto z$, prove that $xy \propto z^2$.

2. If $x \propto z$ and $y \propto z$, prove that $x + y \propto z$.

3. If $x + y \propto x - y$, prove that $x^2 + y^2 \propto xy$.

4. If $w \propto x$ and $y \propto z$, prove that $w/y \propto x/z$.

5. If $10x + 3y = 7x - 4y$, show that $x \propto y$.

6. If $a^x \propto b^y$, and if $x = 3$ when $y = 5$, prove that $a^{x-3} = b^{y-5}$.

7. If $x \propto y$, and if $x = a$ when $y = b$, find the value of x when $y = c$.

8. If $x \propto y$, and if $x = 7$ when $y = 11$, find the value of x when $y = 7$.

9. If $x \propto y$, prove that $px \propto py$, p being either a constant or a variable.

10. What is the radius of the circle which is equal to the sum of two circles whose radii are 3 and 4, respectively?

11. Prove that the volume of the sphere whose radius is 6 is equal to the sum of the volumes of three spheres whose radii are 3, 4, and 5, respectively.

12. The illumination from a given source of light varies inversely as the square of the distance. How much farther from an electric light 20 ft. away must a sheet of paper be removed in order to receive half as much light?

13. Kepler showed that the squares of the numbers representing the times of revolution of the planets about the sun vary as the cubes of the numbers representing their distances from the sun. Mars being 1.52369 as far as the earth from the sun, and the time of revolution of the earth being 365.256 das., find the time of revolution of Mars.

III. PROPORTION.

325. The equality of two ratios forms a **proportion.**

Thus, $\frac{2}{3}=\frac{4}{6}$, $a:b=c:d$, $x/y=m/n$, are examples of proportion. The symbol :: was formerly much used for =.

326. There may be an equality of several ratios, as $1:2=4:8=9:18$, the term **continued proportion** being applied to such an expression.

Three quantities, a, b, c, are said to be in continued proportion when $a:b=b:c$.

327. There may also be an equality between the products of ratios, as $\frac{2}{3}\cdot\frac{5}{7}=\frac{1}{7}\cdot\frac{10}{3}$, such an expression being called a **compound proportion.**

328. In the proportion $a:b=c:d$, a, b, c, d are called the **terms,** a and d being called the **extremes** and b and c the **means.** The term d is called the **fourth proportional** to a, b, c.

329. In the proportion $a:b=b:c$, b is called the **mean proportional** between a and c, and c is called the **third proportional** to a and b.

330. If one quantity varies directly as another, the two are said to be **directly proportional,** or simply **proportional.**

E.g., at retail the cost of a given quality of sugar varies directly as the weight; the cost is then proportional to the weight. Thus, at 4 cts. a pound 12 lbs. cost 48 cts., and 4 cts. : 48 cts. = 1 lb. : 12 lbs.

331. If one quantity varies inversely as another, the two are said to be **inversely proportional.**

E.g., in general, the temperature being constant, the volume of a gas varies inversely as the pressure, and the volume is therefore said to be inversely proportional to the pressure.

Illustrative problems. 1. What are the mean proportionals between 5 and 125?

1. $$\frac{5}{x} = \frac{x}{125}.$$
2. $\therefore$ $625 = x^2$.
3. $\therefore$ $\pm 25 = x$, and both results check.

2. What is the fourth proportional to 1, 5, 9?

1. $$\frac{1}{5} = \frac{9}{x}.$$
2. $\therefore$ $x = 5 \cdot 9 = 45$, and the result checks.

3. What number must be added to the numbers 1, 6, 7, 18 so that the sums shall form a proportion?

1. $$\frac{1+x}{6+x} = \frac{7+x}{18+x}.$$
2. $\therefore$ $18 + 19x + x^2 = 42 + 13x + x^2$.
3. $\therefore$ $x = 4$.

Check. $\frac{5}{10} = \frac{11}{22}$.

EXERCISES. CL.

1. State which of the following, other things being equal, are directly and which are inversely proportional:
   (a) Volume of gas, pressure.
   (b) Price of bread, price of wheat.
   (c) Distance from fulcrum, weight.
   (d) Amount of work done, number of workers.
2. Given $1.43 : x = 4.01 : 2$, find the value of x.
3. Also in $27 : x = x : 48$.
4. What are the mean proportionals between 1 and 1?
5. Also between $1 + i$ and $2(1 - i)$, where $i \equiv \sqrt{-1}$?
6. What is the third proportional to $1 + i$ and -2?

332. The **applications of proportion** are found chiefly in geometry and physics. Other methods are now generally employed for business problems.

In the two illustrative examples below, the first three steps are explanatory of the statement of the proportion and may be omitted in practice. In the first problem the ratios are written in the fractional form in order that the reasons involved may appear more readily.

Illustrative problems. 1. The time of oscillation of a pendulum is proportional to the square root of the number representing its length; the length of a 1-sec. pendulum being 39.2 in., what is the length of a 2-sec. pendulum?

1. Let $x =$ the *number* of inches of length.

2. Then $\frac{x}{39.2} =$ the ratio of the lengths.

3. And $\frac{2}{1} =$ the ratio of the corresponding times of oscillations.

4. $\because$ the time is proportional to the square root of the number representing the length,

$$\therefore \frac{\sqrt{x}}{\sqrt{39.2}} = \frac{2}{1}.$$

5. $\therefore \frac{x}{39.2} = \frac{4}{1}$, whence $x = 39.2 \cdot 4 = 156.8$. Axs. 8, 6

6. $\because x =$ the number of inches, $\therefore$ the pendulum is 156.8 in. long.

2. A mass of air fills 10 dm^3 under a pressure of 3 kg to 1 cm^2. What is the space occupied under a pressure of 5 kg to 1 cm^2, the temperature remaining constant?

1. Let $x =$ the *number* of dm^3 under a pressure of 5 kg to 1 cm^2.

2. Then $x : 10 =$ the ratio of the volumes.

3. And $5 : 3 =$ the ratio of the corresponding pressures.

4. $\because$ the volume is inversely proportional to the pressure,

$$\therefore x : 10 = 3 : 5.$$

5. $\therefore$ $x = 10 \cdot 3 : 5 = 6.$ Ax. 6

6. $\because x =$ the number of dm^3, $\therefore$ the space is 6 dm^3.

EXERCISES. CLI.

1. How long is a pendulum which oscillates 56 times a minute?

2. A cube of water 1.8 dm on an edge weighs how many kg?

3. If a pipe 1.5 cm in diameter fills a reservoir in 3.25 mins., how long will it take a pipe 3 cm in diameter to fill it?

4. If a projectile 8.1 in. in length weighs 108 lbs., what is the weight of a similar projectile 9.37 in. long?

5. If a metal sphere 10 in. in diameter weighs 327.5 lbs., what is the weight of a sphere of the same substance 14 in. in diameter?

6. Of two bottles of similar shape one is twice as high as the other. The smaller holds 0.5 pt. How much does the larger hold?

7. If a sphere whose surface is 16π cm^2 weighs 5 kg, what is the weight of a sphere of the same substance whose surface is 32π cm^2?

8. If the length of a 1-sec. pendulum be considered as 1 m, what is the time of oscillation of a pendulum 6.4 m long? 62.5 m long?

9. A body weighs 25 lbs. 5000 mi. from the earth's center. How much will it weigh 4000 mi. from the center? (Weight varies inversely as the square of the distance from the earth's center.)

10. The distance through which a body falls from a state of rest is proportional to the square of the number representing the time of fall. If a body falls 176.5 m in 6 secs., how far does it fall in 3.25 secs.? in 1 sec.? in 2 secs.?

THEORY OF PROPORTION.

333. Theorem. *In any proportion in which the numbers are all abstract, the product of the means equals the product of the extremes.*

Proof. 1. If $\frac{a}{b}=\frac{c}{d}$, then, by multiplying by bd,

2. $ad = bc$. Ax. 6

334. Theorem. *If the product of two abstract numbers equals the product of two others, either two may be made the means and the other two the extremes of a proportion.*

Proof. 1. If $ad = bc$, then, by dividing by bd,

2. $\frac{a}{b}=\frac{c}{d}$. Ax. 7

Similarly, $\frac{b}{a}=\frac{d}{c}$, etc.

335. Theorem. *If* $\mathrm{a : b = c : d}$, *then* $\mathrm{a : c = b : d}$.

The proof is left for the student.

The old mathematical term for the interchange of the means is "alternation." The first proportion is "taken by alternation" to get the second. The term, while of little value, is still used.

336. Theorem. *If* $\mathrm{a : b = c : d}$, *then* $\mathrm{b : a = d : c}$.

The proof is left for the student.

The old mathematical term for this change is "inversion."

337. Theorem. *If* $\mathrm{a : b = c : d}$, *then* $\mathrm{a + b : b = c + d : d}$.

The proof is left for the student.

The old mathematical term for this change is "composition."

338. Theorem. *If* $\mathrm{a : b = c : d}$, *then* $\mathrm{a - b : b = c - d : d}$.

The proof is left for the student.

The old mathematical term for this change is "division."

339. Theorem. *If*

$$a : b = c : d, \textit{ then } a + b : a - b = c + d : c - d.$$

Proof. 1. $\frac{a+b}{b} = \frac{c+d}{d}$. § 337

2. $\frac{a-b}{b} = \frac{c-d}{d}$. § 338

3. $\therefore \frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d}$. Ax. 7

4. $\therefore$ $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. § 161

The old mathematical term for this change is "composition and division."

There is sometimes an advantage in applying this principle in solving fractional equations. *E.g.*, given the equation

$$\frac{x^2 + 3x - 1}{x^2 - 3x + 1} = \frac{x^2 - 4x + 2}{x^2 + 4x - 2}.$$

$$\frac{2x^2}{6x - 2} = \frac{2x^2}{-8x + 4}.$$

$$\therefore x = 0, \text{ or } \tfrac{3}{7}.$$

340. Theorem. *The mean proportionals between two numbers are the two square roots of their product.*

Proof. 1. $\frac{a}{x} = \frac{x}{b}$.

2. $\therefore$ $x^2 = ab$. § 333, or ax. 6

3. $\therefore$ $x = \pm\sqrt{ab}$. Ax. 9

Illustrative problems. 1. If $a : b = c : d$, prove that

$$a + b + c + d : b + d = c + d : d.$$

1. This is true if $ad + bd + cd + d^2 = bc + bd + cd + d^2$. § 334
2. Or if $ad = bc$. Ax. 3
3. But $ad = bc$. § 333
4. $\therefore$ reverse the process, deriving step 1 from step 3, and the original proportion from step 1.

2. Solve the equation $\dfrac{\sqrt{x+2}+\sqrt{x-3}}{\sqrt{x+2}-\sqrt{x-3}}=1\frac{1}{2}$.

We may clear of fractions at once, isolate the two radicals, and square; but in this and similar cases § 339 can be used to advantage. Writing the second member $\frac{3}{2}$ and applying § 339, we have

1. $$\frac{2\sqrt{x+2}}{2\sqrt{x-3}}=\frac{5}{1}.$$

2. ∴ $$\frac{x+2}{x-3}=25.$$

3. ∴ $$x+2=25x-75.$$

4. ∴ $$x=\tfrac{77}{24}.$$

Check. Substitute $\frac{77}{24}$ for x in the original equation, and reduce; then

$$\frac{3\sqrt{\frac{5}{6}}}{2\sqrt{\frac{5}{6}}}=1\tfrac{1}{2}.$$

3. Find a mean proportional between $1+i$ and $-2-14i$.

1. By § 340 this equals $\pm\sqrt{(1+i)(-2-14i)}$

2. $=\pm\sqrt{12-16i}$

3. $=\pm 2\sqrt{3-4i}$

4. $=\pm 2\sqrt{4-2\sqrt{-4}-1}$

5. $=\pm 2(2-i)$. § 245

EXERCISES. CLII.

1. Find the value of x in $2:3+i=x:5$.

2. Find the third proportional to $1-\sqrt{2}$ and $1-3\sqrt{2}$.

3. If $a:b=c:d$, prove that $a^3+b^3:c^3+d^3=\dfrac{a^4}{a+b}:\dfrac{c^4}{c+d}$.

4. Also that $\dfrac{a+c}{b+d}=\dfrac{c}{d}$.

5. Also that $bc+cd:c-a=2bcd+cd^2-da^2:cd-bc$.

6. Also that $\sqrt{a-b}:\sqrt{c-d}=\sqrt{a}-\sqrt{b}:\sqrt{c}-\sqrt{d}$.

7. If $a : b = b : c$, prove that $a + c > 2b$.

8. If $a : b = b : c$, prove that $(a + c)b$ is a mean proportional between $a^2 + b^2$ and $b^2 + c^2$.

9. Find the two mean proportionals between

(a) 2 and 98. (b) 50 and -2.

(c) 3 and 432. (d) -7 and -847.

10. Given $16 - 6x : 3 = 2 + x : x$, to find x.

11. Given $\sqrt{x + 7} + \sqrt{x - 7} : \sqrt{x + 7} - \sqrt{x - 7} = 6 : 1$, to find x.

12. Given $1 + x : 13 - x = x - 2 : x^2 - 21 = x + 4 : 37 - x^2$, to find x.

13. Given $a - b : \dfrac{(a + b)^2}{2ab} - 1 = x : a + b + \dfrac{2b^2}{a - b}$, to find x.

14. Given $3a^2 + 2ab - 8b^2 : 5a^2 + 4ab - 12b^2 = x : 5a - 6b$, to find x.

15. Given $x : y = a + b - \dfrac{ab}{a + b} : a - b + \dfrac{ab}{a - b}$, and $x + y : a^3 = 2 : 1$, to find x and y.

16. Given $\sqrt{x - 5} : \sqrt{7 + x} = 1 : 2$, to find x.

17. Find the value of x in

$$3 + 4x - x^2 : 3 - 4x + x^2 = 2 + x : 2 - x.$$

18. If $\dfrac{ax + cy}{by + dz} = \dfrac{ay + cz}{bz + dx} = \dfrac{az + cx}{bx + dy}$, prove that each of these ratios equals $\dfrac{a + c}{b + d}$.

19. If $\dfrac{a - b}{ay + bx} = \dfrac{b - c}{bz + cx} = \dfrac{c - a}{cy + az} = \dfrac{a + b + c}{ax + by + cz}$, prove that each of these ratios equals $\dfrac{1}{x + y + z}$.

CHAPTER XVIII.

SERIES.

341. A **series** is a succession of terms formed according to some common law.

E.g., in the following, each term is formed from the preceding as indicated:

$1,\ 3,\ 5,\ 7,\ \cdots,$ by adding 2;

$7,\ 3,\ -1,\ -5,\ \cdots,$ by subtracting 4, or by adding -4;

$3,\ 9,\ 27,\ 81,\ \cdots,$ by multiplying by 3, or by dividing by $\frac{1}{3}$;

$2,\ 2,\ 2,\ 2,\ \cdots,$ by adding 0, or by multiplying by 1.

In the series $0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \cdots$, each term after the first two is found by adding the two preceding terms.

342. An **arithmetic series** (also called an arithmetic progression) is a series in which each term after the first is found by adding a constant to the preceding term.

E.g., $-7,\ -1,\ 5,\ 11,\ \cdots,$ the constant being 6,

$2,\ 2,\ 2,\ 2,\ \cdots,$ " " " 0,

$98,\ 66,\ 34,\ 2,\ \cdots,$ " " " -32.

343. A **geometric series** (also called a geometric progression) is a series in which each term after the first is formed by multiplying the preceding term by a constant.

E.g., $3,\ -6,\ +12,\ -24,\ \cdots,$ the constant being -2,

$10,\ 5,\ 2\frac{1}{2},\ 1\frac{1}{4},\ \cdots,$ " " " $\frac{1}{2}$,

$2,\ 2,\ 2,\ 2,\ \cdots,$ " " " 1.

344. The terms between the first and last are called the **means** of the series.

I. ARITHMETIC SERIES.

345. Symbols. The following are in common use:

n, the number of terms of the series.

s, " sum " " " "

$t_1, t_2, t_3, \cdots t_n$, the terms of the series.

In particular, a, or t_1, the 1st term, and l, or t_n, the nth or last term.

d, the constant which added to any term gives the next; d is usually called the *difference*.

346. Formulas. There are two formulas in arithmetic series of such importance as to be designated as fundamental.

I. t_n, or $l = a + (n - 1)d$.

Proof. 1. $t_2 = a + d$, by definition.

$t_3 = t_2 + d = a + 2d.$

$t_4 = t_3 + d = a + 3d.$

$\vdots \quad \vdots \quad \vdots$

2. $\therefore$ $t_n = t_{n-1} + d = a + (n - 1)d.$

3. Or $l = a + (n - 1)d.$

E.g., the 50th term in the series 2, 7, 12, 17, $\cdots$ is $2 + 49 . 5 = 247$.

II. $s = \frac{n(a + l)}{2}$, or $\frac{n(t_1 + t_n)}{2}$.

Proof. 1. $s = a + (a + d) + (a + 2d) + \cdots (l - d) + l.$

2. Hence, $s = l + (l - d) + (l - 2d) + \cdots (a + d) + a$, by reversing the order.

3. $\therefore 2s = (a + l) + (a + l) + \cdots (a + l).$ Ax. 2

4. $\therefore 2s = n(a + l)$, $\because$ there is an $(a + l)$ in step 3 for each of the n terms in step 1.

5. $\therefore$ $s = \frac{n(a + l)}{2}.$

E.g., the sum of the first 50 terms of the series 2, 7, 12, 17, $\cdots$, of which l has just been found, is

$$\frac{50(2 + 247)}{2} = 6225.$$

347. It is evident that from formulas I and II various others can be deduced.

E.g., given d, l, s, to find n. The problem merely reduces to that of eliminating a from I and II, and solving for n.

1. From I, $a = l - (n-1)d.$

2. Substituting in II, $s = \dfrac{n[2l-(n-1)d]}{2}.$

3. $\therefore$ $$n^2 - \frac{2l+d}{d}\cdot n + \frac{2s}{d} = 0.$$

4. $\therefore$ $$n = \frac{2l+d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$$ § 269

Illustrative problems. 1. Which term of the series 25, 22, 19, $\cdots$ is -125?

1. Given $a = 25$, $d = -3$, $l = -125$, to find n.
2. $\because$ $l = a + (n-1)d$, $-125 = 25 + (n-1)(-3)$.
3. Solving, $n = 51$.

2. Insert arithmetic means between 5 and 41 so that the 4th of these means shall have to the next to the last, less 1, the ratio 1 : 2.

1. The means are $5+d$, $5+2d$, $\cdots$ $41-2d$, $41-d$.

2. $\therefore$ $$\frac{5+4d}{41-2d-1} = \frac{1}{2}.$$

3. $\therefore d = 3$, and the means are 8, 11, 14, 17, $\cdots$ 35, 38.

3. The sum of three numbers of an arithmetic series is 12 and the sum of their squares is 56. Find the numbers.

In this and similar cases it is advisable to take $x-y$, x, $x+y$, y being the common difference. In the case of four numbers it is advisable to take $x-3y$, $x-y$, $x+y$, $x+3y$, $2y$ being the difference.

1. $(x-y) + x + (x+y) = 12$, $\therefore x = 4$.
2. $(x-y)^2 + x^2 + (x+y)^2 = 56$, $\therefore 3x^2 + 2y^2 = 56$.
3. $\therefore$ $y = \pm 2$.
4. $\therefore$ the numbers are 4 ∓ 2, 4, 4 ± 2; that is, 2, 4, 6, or 6, 4, 2.

348. The following table gives the various formulas of arithmetic series, and these should be worked out from formulas I and II by the student.

	Given.	To find.	Result.
1	$a\ d\ n$	l	$l = a + (n-1)d.$
2	$a\ d\ s$		$l = -\frac{1}{2}d \pm \sqrt{(a-\frac{1}{2}d)^2 + 2ds}.$
3	$a\ n\ s$		$l = 2s/n - a.$
4	$d\ n\ s$		$l = s/n + (n-1)d/2.$
5	$a\ d\ n$	s	$s = \frac{1}{2}n[2a + (n-1)d].$
6	$a\ d\ l$		$s = \frac{1}{2}(l+a) + (l^2 - a^2)/2d.$
7	$a\ n\ l$		$s = \frac{1}{2}n(a+l).$
8	$d\ n\ l$		$s = \frac{1}{2}n[2l - (n-1)d].$
9	$d\ n\ l$	a	$a = l - (n-1)d.$
10	$d\ n\ s$		$a = s/n - \frac{1}{2}(n-1)d.$
11	$d\ l\ s$		$a = \frac{1}{2}d \pm \sqrt{(l+\frac{1}{2}d)^2 - 2ds}.$
12	$n\ l\ s$		$a = 2s/n - l.$
13	$a\ n\ l$	d	$d = (l-a)/(n-1).$
14	$a\ n\ s$		$d = 2(s - an)/(n^2 - n).$
15	$a\ l\ s$		$d = (l^2 - a^2)/(2s - l - a).$
16	$n\ l\ s$		$d = 2(nl - s)/(n^2 - n).$
17	$a\ d\ l$	n	$n = (l - a + d)/d.$
18	$a\ d\ s$		$n = [d - 2a \pm \sqrt{(2a-d)^2 + 8ds}]/2d.$
19	$a\ l\ s$		$n = 2s/(a+l).$
20	$d\ l\ s$		$n = [d + 2l \pm \sqrt{(2l+d)^2 - 8ds}]/2d.$

Illustrative problem. Find the number of terms in the arithmetic series whose first term is 25, difference -5, and sum 45.

We may substitute in formula 18, but it is quite as easy to use the two fundamental formulas which the student will carry in his mind.

1. From I, $l = 25 + (n - 1)(-5) = 30 - 5n.$

2. " II, $45 = \frac{25 + 30 - 5n}{2}n.$

3. $\therefore$ $n^2 - 11n + 18 = 0.$

4. $\therefore$ $(n-2)(n-9) = 0$, and $n = 2$, or 9.

The explanation of the two results appears by writing out the series.

$$25,\ 20,\ (15,\ 10,\ 5,\ 0,\ -5,\ -10,\ -15).$$

The part enclosed in parentheses has 0 for its sum.
Hence, the sum of 2 terms is the same as the sum of 9 terms.

EXERCISES. CLIII.

1. Find t_{200} in the series 1, 3, 5, $\cdots$.

2. Find s, given $a = 40$, $n = 101$, $d = 5$.

3. Find s, given $a = 1$, $l = 200$, $n = 200$.

4. Given $t_1 = -1\frac{1}{3}$ and $t_{15} = 59\frac{1}{3}$, find d.

5. Find t_{20} in the series 540, 480, 420, $\cdots$.

6. Find n, given $s = 29{,}000$, $a = 40$, $l = 540$.

7. Insert 7 arithmetic means between -5 and 11.

8. Insert 12 arithmetic means between -18 and 125.

9. Find s, given $a = 14$, $n = 8$, $d = -4$. Write out the series.

10. How many multiples of 17 are there between 350 and 1210?

11. What is the sum of the first 200 numbers divisible by 5? by 7?

12. Show that the sum of any $2n+1$ consecutive integers is divisible by $2n+1$.

13. What is the sum of the first 50 odd numbers? the first 100? the first n?

14. What is the sum of the first 50 even numbers? the first 100? the first n?

15. Given $l = 11$, $d = 2$, $s = 32$, to find n. Check the result by writing out the series.

16. How long has a body been falling when it passes through 53.9 m during the last second?

17. Suppose every term of an arithmetic series to be multiplied by k; is the result an arithmetic series?

18. The sum of four numbers of an arithmetic series is 0 and the sum of their squares is 20. Find the numbers.

19. The sum of four numbers of an arithmetic series is 12 and the sum of their squares is 116. Find the numbers.

20. The sum of three numbers of an arithmetic series is 21 and the sum of their squares is 179. Find the numbers.

21. Find five numbers of an arithmetic series such that the sum of the first and fifth is 46, and that the ratio of the fourth to the second is 1.3.

22. $100 is placed at interest annually on the first of each January for 10 yrs., at 6%. Find the total amount of principals and interest at the end of 10 yrs.

23. Find the nth term and the sum of the first n terms:

(a) $1 + \frac{3}{4} + \frac{1}{2} + \cdots$. (b) $11 + 9 + 7 + \cdots$.

II. GEOMETRIC SERIES.

349. Symbols. The following are in common use:

n, s, a, l and t_1, t_2, $\cdots t_n$, as in arithmetic series;
r, the constant by which any term may be multiplied to produce the next; r is usually called the *rate* or *ratio*.

350. Formulas. There are two formulas in geometric series of such importance as to be designated as fundamental.

I. t_n, or $l = ar^{n-1}$.

Proof. 1. $t_2 = ar$, by definition.

$$t_3 = t_2 r \quad = ar^2.$$
$$t_4 = t_3 r \quad = ar^3.$$
$$\vdots \quad \vdots \quad \vdots$$

2. $\therefore$ $t_n = t_{n-1}r = ar^{n-1}$.

3. Or $l = ar^{n-1}$.

E.g., the 7th term of the series 16, 8, 4, $\cdots$ is

$$l = 16 \cdot (\tfrac{1}{2})^{7-1} = 16 \cdot \tfrac{1}{64} = \tfrac{1}{4}.$$

II. $s = \dfrac{ar^n - a}{r-1} = \dfrac{lr-a}{r-1}$.

Proof. 1. $s = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$.

2. $\therefore$ $rs = \quad ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n$, by multiplying by r.

3. $\therefore$ $rs - s = ar^n - a$, by subtracting, (2) − (1).

4. $\therefore$ $(r-1)s = ar^n - a$, and $s = \dfrac{ar^n - a}{r-1}$, by dividing by $(r-1)$.

5. And $\because ar^n = ar^{n-1} \cdot r = lr$, $\therefore s = \dfrac{lr-a}{r-1}$.

E.g., the sum of the first 7 terms of the series 16, 8, 4, $\cdots$, of which l has just been found, is

$$\frac{\frac{1}{4} \cdot \frac{1}{2} - 16}{\frac{1}{2} - 1} = 31\tfrac{3}{4}.$$

351. It is evident that from formulas I and II various others can be deduced.

E.g., given l, a, n, to find r. $\because l = ar^{n-1}$, $\therefore r = (l/a)^{\frac{1}{n-1}}$.

Given n, l, s, to find a. The problem reduces to that of eliminating r from I and II and solving, if possible, for a.

1. From II, $$r = \frac{s-a}{s-l}.$$

2. Substitute this in I, and $$l = a\left(\frac{s-a}{s-l}\right)^{n-1},$$

or $$l(s-l)^{n-1} - a(s-a)^{n-1} = 0.$$

Here it is impossible to isolate a. When the numerical values of l, s, n are given, a can frequently be determined by inspection.

For example, given $n = 4$, $l = 8$, $s = 15$, to find a. Here
$$8 \cdot 7^3 = a(15-a)^3,$$
and a evidently equals 8, or 1. Either value checks, for the series may be 8, 4, 2, 1, or 1, 2, 4, 8.

Illustrative problems. 1. Find the sum of five consecutive powers of 3, beginning with the first.

1. Here $a = 3$, $r = 3$, $n = 5$.
2. $s = (ar^n - a)/(r-1) = (3 \cdot 3^5 - 3)/2 = 363$.

2. Of three numbers of a geometric series, the sum of the first and second exceeds the third by 3, and the sum of the first and third exceeds the second by 21. Find the numbers.

1. Let x, xy, xy^2 be the numbers.
2. Then $x + xy = xy^2 + 3$, or $x + xy - 3 = xy^2$.
3. And $x + xy^2 = xy + 21$, or $-x + xy + 21 = xy^2$.
4. $\therefore$ $x + xy - 3 = -x + xy + 21$, or $x = 12$.
5. $\therefore$ $4y^2 - 4y - 3 = 0$, by substituting in 2.
6. $\therefore$ $(2y+1)(2y-3) = 0$, and $y = -\frac{1}{2}$, or $\frac{3}{2}$.
7. $\therefore$ the numbers are 12, -6, 3, or 12, 18, 27. Each set checks.

352. The following table gives the various formulas of geometric series. They should be worked out from formulas I and II by the student, excepting those for n. The formulas for n require logarithms and may be taken after Chap. XIX.

	Given.	To find.	Results.
1	$a\ r\ n$		$l = ar^{n-1}$.
2	$a\ r\ s$		$l = [a + (r-1)s]/r$.
3	$a\ n\ s$	l	$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$.
4	$r\ n\ s$		$l = (r-1)sr^{n-1}/(r^n - 1)$.
5	$a\ r\ n$		$s = a(r^n - 1)/(r-1)$.
6	$a\ r\ l$		$s = (rl - a)/(r-1)$.
7	$a\ n\ l$	s	$s = (l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}})/(l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}})$.
8	$r\ n\ l$		$s = l(r^n - 1)/(r^n - r^{n-1})$.
9	$r\ n\ l$		$a = l/r^{n-1}$.
10	$r\ n\ s$		$a = s(r-1)/(r^n - 1)$.
11	$r\ l\ s$	a	$a = rl - (r-1)s$.
12	$n\ l\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$.
13	$a\ n\ l$		$r = (l/a)^{\frac{1}{n-1}}$.
14	$a\ n\ s$		$r^n - sr/a + (s-a)/a = 0$.
15	$a\ l\ s$	r	$r = (s-a)/(s-l)$.
16	$n\ l\ s$		$r^n - sr^{n-1}/(s-l) + l/(s-l) = 0$.
17	$a\ r\ l$		$n = (\log l - \log a)/\log r + 1$.
18	$a\ r\ s$		$n = \{\log[a + (r-1)s] - \log a\}/\log r$.
19	$a\ l\ s$	n	$n = (\log l - \log a)/[\log(s-a) - \log(s-l)] + 1$.
20	$r\ l\ s$		$n = \{\log l - \log[lr - (r-1)s]\}/\log r + 1$.

EXERCISES. CLIV.

1. The sum of how many terms of the series 4, 12, 36, ⋯ is 118,096?

2. Find the sum of the first ten terms of the series $3^{\frac{1}{2}}, -2^{\frac{1}{2}}, \frac{2}{3}\cdot 3^{\frac{1}{2}}, \cdots$.

3. Find the geometric mean between

 (a) 1 and 4. (b) -2 and -8.

4. Find the sum of five numbers of a geometric series, the second term being 5 and the fifth 625.

5. What is the fourth term of the geometric series whose first term is 1 and third term $\frac{1}{25}$?

6. The arithmetic mean between two numbers is 39 and the geometric mean 15. Find the numbers.

7. Prove that the geometric mean between two numbers is the square root of their product (§ 343).

8. Prove that the arithmetic mean between two unequal positive numbers is greater than the geometric mean.

9. To what sum will \$1 amount at 4% compound interest in 5 yrs.? (Here $a = \$1$, $r = 1.04$, $n = 6$.)

10. In ex. 9, suppose the rate were 4% a year, but the interest compounded semiannually?

11. The sum of the first eight terms of a certain geometric series is 17 times the sum of the first four terms. What is the rate?

12. Find the 10th term and the sum of the first ten terms of the series:

 (a) $1, \frac{1}{2}, \frac{1}{4}, \cdots$. (b) $1, -2, 4, -8, \cdots$.

 (c) $1, 2, 4, \cdots$. (d) $32, -16, 8, -4, \cdots$.

353. Infinite geometric series. If the number of terms is infinite and $r<1$, then s approaches as its limit $\frac{a}{1-r}$ (§ 167).

This is indicated by the symbols $s \doteq \frac{a}{1-r}$, n being infinite.

The symbol $\doteq$ is read "approaches as its limit" (p. 140).

Proof. 1. $\because r<1$, the terms are becoming smaller, each being multiplied by a fraction to obtain the next.

2. $\therefore l \doteq 0$, and $\therefore lr \doteq 0$, although they never reach that limit.

3. $\therefore s \doteq \frac{0-a}{r-1}$, by formula II.

4. $\therefore s \doteq \frac{a}{1-r}$, by multiplying each term of the fraction by -1.

E.g., consider the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\cdots$, where n is infinite. Here $s \doteq \frac{a}{1-r}$, or $\frac{1}{1-\frac{1}{2}}$, or 2. That is, the greater the number of terms, the nearer the sum approaches 2, although it never reaches it for finite values of n.

EXERCISES. CLV.

1. Given $s \doteq 8$, $a=4$. Find r.
2. Given $s \doteq 10\frac{2}{3}$, $r=\frac{1}{4}$. Find a.
3. Given $s \doteq 1$, $r=\frac{2999}{3000}$. Find a.
4. Given $s=155$, $r=2$, $n=5$. Find a.
5. Given $s=124.4$, $r=3$, $n=4$. Find a.
6. Find the limits of the following sums, n being infinite:

(a) $20+10+5+2\frac{1}{2}+\cdots$. (b) $\frac{7}{2}+\frac{7}{8}+\frac{7}{32}+\frac{7}{128}+\cdots$.

(c) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$.

(d) $10+1+0.1+0.01+\cdots$.

354. Circulating decimals. If the fraction $\frac{3}{11}$ is reduced to the decimal form, the result is $0.272727\cdots$, and similarly the fraction $\frac{11}{72} = 0.152777\cdots$. The former constantly repeats 27, and the latter constantly repeats 7 after 0.152.

When, beginning with a certain order of a decimal fraction, the figures constantly repeat in the same order, the number is called a **circulating decimal**, and the part which repeats is called a **circulate**.

A circulate is represented by a dot over its first and last figures.

$0.272727\cdots$ is represented by $0.\dot{2}\dot{7}$;
$0.152777\cdots$ " " " $0.152\dot{7}$.

A circulating decimal may be reduced to a common fraction by means of the formula $s \doteq \frac{a}{1-r}$, as follows:

1. To what common fraction is $0.\dot{2}\dot{7}$ equal?

1. $0.\dot{2}\dot{7} = 0.27 + 0.0027 + 0.000027 + \cdots$.
2. This is a geometric series with $a = 0.27$, $r = 0.01$, n infinite.
3. $\therefore s \doteq \frac{0.27}{1 - 0.01} = \frac{27}{99} = \frac{3}{11}$.

2. To what common fraction is $0.152\dot{7}$ equal?

1. $0.152\dot{7} = 0.152 + 0.0007 + 0.00007 + \cdots = 0.152 +$ a geometric series with $a = 0.0007$, $r = 0.1$, n infinite.
2. $\therefore s \doteq \frac{0.0007}{1 - 0.1} = \frac{7}{9000}$.
3. To this must be added 0.152, giving $0.152\frac{7}{9}$, or $\frac{1375}{9000}$, or $\frac{11}{72}$.

EXERCISES. CLVI.

Express as common fractions:

1. $0.\dot{3}$.	2. $0.0\dot{4}\dot{5}$.	3. $0.\dot{0}00\dot{1}$.
4. $0.\dot{1}4\dot{7}$.	5. $1.2\dot{3}7\dot{5}$.	6. $5.\dot{0}50\dot{4}$.
7. $0.\dot{0}4\dot{5}$.	8. $2.0\dot{0}347\dot{1}$.	9. $0.2345\dot{6}$.

III. MISCELLANEOUS TYPES.

355. Of the other types of series, some can be treated by the methods which have just been considered.

Illustrative problems. 1. Defining a harmonic series as one the reciprocals of whose terms form an arithmetic series, insert three harmonic means between 2 and 4.

This reduces to the insertion of three arithmetic means between $\frac{1}{2}$ and $\frac{1}{4}$.

1. $\because$ $a = \frac{1}{2}$, $n = 5$, and $l = \frac{1}{4}$,

2. $\therefore$ $\frac{1}{4} = \frac{1}{2} + 4d$, and $d = -\frac{1}{16}$.

3. $\therefore$ the arithmetic series is $\frac{1}{2}, \frac{7}{16}, \frac{3}{8}, \frac{5}{16}, \frac{1}{4}$,

and " harmonic " $2, 2\frac{2}{7}, 2\frac{2}{3}, 3\frac{1}{5}, 4$.

2. Sum to 20 terms the series $1, -3, 5, -7, 9, -11, \cdots$.

Here the odd numbers of the terms form an arithmetic series with $d = 4$, and the even ones form an arithmetic series with $d = -4$. There are ten terms in each set. Summing separately, we have

$$190 - 210 = -20.$$

3. What is the harmonic mean between a and b?

1. If h is the harmonic mean, $\frac{1}{a}, \frac{1}{h}, \frac{1}{b}$ must form an arithmetic series (ex. 1).

2. $\therefore$ $$\frac{1}{h} - \frac{1}{a} = \frac{1}{b} - \frac{1}{h}.$$

3. $\therefore$ $$h = \frac{2ab}{a+b}.$$

E.g., the harmonic mean between 3 and 4 is $\frac{24}{7}$. For, taking the reciprocals of 3, $\frac{24}{7}$, and 4, we have $\frac{1}{3}, \frac{7}{24}, \frac{1}{4}$, or $\frac{8}{24}, \frac{7}{24}$, and $\frac{6}{24}$, which form an arithmetic series.

4. Find the sum of n terms of the series $1, 2x, 3x^2, 4x^3, \cdots$.

Here the coefficients form an arithmetic series and the x's a geometric. Such a series is called *arithmetico-geometric.*

Let $s = 1 + 2x + 3x^2 + \cdots + (n-1)x^{n-2} + nx^{n-1}$;

then $xs = \qquad x + 2x^2 + \cdots + (n-2)x^{n-2} + (n-1)x^{n-1} + nx^n$.

Subtracting,

$$(1-x)s = 1 + x + x^2 + \cdots + x^{n-2} + x^{n-1} - nx^n.$$

$$\therefore s = \frac{1-x^n}{(1-x)^2} - n\frac{x^n}{(1-x)}.$$

EXERCISES. CLVII.

1. Sum the series $3, 6, \cdots 3(n-1), 3n$.

2. Sum to $2n$ terms the series $1, -2, +3, -4, \cdots$.

3. Sum the series $1, 4x, 7x^2, 10x^3, \cdots$, to n terms.

4. Sum the series $1, -3, +5, -7, + \cdots$ to $2n$ terms.

5. Insert a harmonic mean between 2 and 2; between -2 and -2.

6. Prove that no two unequal numbers can have their arithmetic, geometric, and harmonic means equal, or any two of these equal.

7. Show that the sum of the first n terms of the series $1, -2, +4, -8, +16, \cdots$ is $\frac{1}{3}(1 \pm 2^n)$, the sign depending on whether n is odd or even.

8. Find the sum of $1 + 2x + 3x^2 + 4x^3 + \cdots$ to n terms by writing the series $(1 + x + x^2 + \cdots) + (x + x^2 + x^3 + \cdots) + (x^2 + x^3 + \cdots) + (x^3 + \cdots)$, etc., summing each group separately, and adding the sums.

9. The number of balls in a triangular pile is evidently $1 + (1+2) + (1+2+3) + \cdots$, depending on the number of layers. How many balls in such a pile of 10 layers?

CHAPTER XIX.

LOGARITHMS.

356. About the year 1614 a Scotchman, John Napier, invented a scheme by which multiplication can be performed by addition, division by subtraction, involution by a single multiplication, and evolution by a single division.

357. In considering the annexed series of numbers it is apparent that

$2^0 = 1$	$2^6 = 64$
$2^1 = 2$	$2^7 = 128$
$2^2 = 4$	$2^8 = 256$
$2^3 = 8$	$2^9 = 512$
$2^4 = 16$	$2^{10} = 1024$
$2^5 = 32$	$2^{11} = 2048$

1. $\because \quad 2^3 \cdot 2^5 = 2^8,$

$\therefore \quad 8 \cdot 32 = 2^8 = 256.$

$\therefore$ the product can be found by adding the exponents $(3 + 5 = 8)$ and then finding what 2^8 equals.

2. $\because \quad 2^9 : 2^3 = 2^6,$

$\therefore \quad 512 : 8 = 64.$

$\therefore$ this quotient can be found from the table by a single subtraction of exponents.

3. $\because \quad (2^5)^2 = 2^5 \cdot 2^5 = 2^{10},$

$\therefore \quad 32^2 = 1024.$

4. $\because \quad \sqrt{2^{10}} = \sqrt{2^5 \cdot 2^5} = 2^5,$

$\therefore \quad \sqrt{1024} = 32.$

5. The exponents of 2 form an arithmetic series, while the powers form a geometric series.

In like manner a table of the powers of any number may be made and the four operations, multiplication, division, involution, evolution, reduced to the operations of addition, subtraction, multiplication, and division of exponents.

358. For practical purposes, *the exponents of the powers to which 10, the base of our system of counting, must be raised to produce various numbers* are put in a table, and these exponents are called the **logarithms** of those numbers.

In this connection the word *power* is used in its broadest sense, 10^n being considered as a power, whether n is positive, negative, integral, or fractional. The logarithm of 100 is written "log 100."

E.g., $10^3 = 1000$, $\therefore \log 1000 = 3$. $10^2 = 100$, $\therefore \log 100 = 2$.

$10^0 = 1$, $\therefore \log 1 = 0$. $10^1 = 10$, $\therefore \log 10 = 1$.

$10^{-1} = \frac{1}{10}$, $\therefore \log 0.1 = -1$. $10^{-2} = \frac{1}{10^2}$, $\therefore \log 0.01 = -2$.

$10^{\frac{301}{1000}}$, that is, the thousandth root of 10^{301}, is nearly 2,

$$\therefore \log 2 = 0.301, \text{ nearly.}$$

Although log 2 cannot be expressed exactly as a decimal fraction, it can be found to any required degree of accuracy.

EXERCISES. CLVIII.

1. What is the logarithm of 10^{-3}? of 1000^3? of 10^9?

2. What is the logarithm of $10^4 \cdot 10^3$? of $10^7 : 10^3$?

3. What is the logarithm of $\sqrt[3]{10^4} \cdot 10^6 \cdot 10^8$? of $\sqrt[10]{10}$?

4. What is the logarithm of $10^3 \cdot 10^8 \cdot 10^5$? of 0.001 of $10^2 \cdot 10^4$? of $10^3 \cdot 10^5 \cdot 10^9$?

5. Between what two consecutive integers does log 800 lie, and why? also log 3578? log 27?

6. Between what two consecutive negative integers does log 0.02 lie, and why? also log 0.009? log 0.0008?

7. If the logarithm of 2 is 0.301, what is the logarithm of 2^{1000}? ($2 = 10^{\frac{301}{1000}}$, $\therefore 2^{1000} = ?$ $\therefore$ the logarithm of $2^{1000} = ?$)

359. Since 2473 lies between 1000 and 10,000, its logarithm lies between 3 and 4. It has been computed to be 3.3932. The integral part 3 is called the *characteristic* of the logarithm, and the fractional part 0.3932 the *mantissa*.

That is, $10^{\frac{33932}{10000}}$, or $10^{3.3932} = 2473$, $\therefore \log 2473 = 3.3932$.

$\because 10^{3.3932} : 10^1 = 10^{2.3932}$, $\therefore 10^{2.3932} = 247.3$, $\therefore \log 247.3 = 2.3932$.

Similarly, $10^{1.3932} = 24.73$, $\therefore \log 24.73 = 1.3932$.

" $10^{0.3932} = 2.473$, $\therefore \log 2.473 = 0.3932$.

" $10^{0.3932-1} = 0.2473$, $\therefore \log 0.2473 = 0.3932 - 1$.

360. It is thus seen that

1. *The characteristic can always be found by inspection.*

Thus, because 438 lies between 100 and 1000, hence log 438 lies between 2 and 3, and $\log 438 = 2 +$ some mantissa.

Similarly, 0.0073 lies between 0.001 and 0.01, hence log 0.0073 lies between -3 and -2, and $\log 0.0073 = -3 +$ some mantissa.

Since 5 lies between 1 and 10, log 5 lies between 0 and 1, and equals $0 +$ some mantissa.

2. *The mantissa is the same for any given succession of digits, wherever the decimal point may be.*

Thus, $\log 2473 = 3.3932$, and $\log 0.2473 = 0.3932 - 1$.

3. *Therefore, only the mantissas need be put in a table.*

Instead of writing the negative characteristic after the mantissa, it is often written before it, but with a minus sign above; thus, $\log 0.2473 = 0.3932 - 1 = \bar{1}.3932$, this meaning that only the characteristic is negative, the mantissa remaining positive.

Negative numbers are not considered as having logarithms, but operations involving negative numbers are easily performed. *E.g.*, the multiplication expressed by $1.478 \cdot (-0.007283)$ is performed as if the numbers were positive, and the proper sign is prefixed.

EXERCISES. CLIX.

1. What is the characteristic of the logarithm of a number of three integral places? of 6? of 20? of n?

2. What is the characteristic of the logarithm of 0.3? of any decimal fraction whose first significant figure is in the first decimal place? the second decimal place? the 20th? the nth?

3. From exs. 1, 2 formulate a rule for determining the characteristic of the logarithm of any positive number.

4. If $\log 39{,}703 = 4.5988$, what are the logarithms of

(a) 39,703,000? (b) 397.03? (c) 3.9703?
(d) 0.00039703? (e) 0.39703? (f) 3970.3?

361. The fundamental theorems of logarithms.

I. *The logarithm of the product of two numbers equals the sum of their logarithms.*

1. Let $a = 10^m$, then $\log a = m$.
2. Let $b = 10^n$, " $\log b = n$.
3. $\therefore ab = 10^{m+n}$, and $\log ab = m + n = \log a + \log b$.

Thus, $\log(5 \times 6) = \log 5 + \log 6$.

II. *The logarithm of the quotient of two numbers equals the logarithm of the dividend minus the logarithm of the divisor.*

1. Let $a = 10^m$, then $\log a = m$.
2. Let $b = 10^n$, " $\log b = n$.
3. $\therefore \frac{a}{b} = \frac{10^m}{10^n} = 10^{m-n}$, and $\log \frac{a}{b} = m - n$.

Thus, $\log(40 \div 5) = \log 40 - \log 5$.

III. *The logarithm of the* n*th power of a number equals* n *times the logarithm of the number.*

1. Let $a = 10^m$, then $\log\ a = m$.
2. $\therefore\ a^n = 10^{mn}$, and $\log a^n = nm = n \log a$.

IV. *The logarithm of the* n*th root of a number equals* $\frac{1}{n}$*th of the logarithm of the number.*

1. Let $a = 10^m$, then $\log\ a = m$.
2. $\therefore\ a^{\frac{1}{n}} = 10^{\frac{m}{n}}$, and $\log a^{\frac{1}{n}} = \frac{m}{n} = \frac{1}{n} \cdot \log a$.

Th. III might have been stated more generally, so as to include Th. IV, thus: $\log a^{\frac{x}{y}} = \frac{x}{y} \cdot \log a$. The proof would be substantially the same as in ths. III and IV.

EXERCISES. CLX.

Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, $\log 7 = 0.8451$, and $\log 514 = 2.7110$, find the following:

1. $\log 60$.
2. $\log 24$.
3. $\log 7^8$.
4. $\log \sqrt[3]{2}$.
5. $\log 625$.
6. $\log 7^{\frac{1}{2}}$.
7. $\log \sqrt[5]{3^4}$.
8. $\log \sqrt[3]{21}$.
9. $\log 35$.
10. $\log 514^3$.
11. $\log 1.05$.
12. $\log 257$.
13. $\log 1050$.
14. $\log 154{,}200$.
15. $\log \sqrt[3]{514}$.
16. $\log 10.28$.
17. $\log 154.2$.
18. $\log 3.598$.
19. $\log 0.3084$.
20. $\log 30.84$.
21. $\log 15.42^{\frac{1}{2}}$.
22. $\log 1799 [= \log (\frac{1}{2} \cdot 514 \cdot 7)]$.
23. Show how to find $\log 5$, given $\log 2$.

362. Explanation of table. *Given a number to find its logarithm.* In the table on pp. 354 and 355 only the mantissas are given. For example, in the row opposite 71, and under 0, 1, 2, $\cdots$ will be found:

N	0	1	2	3	4	5	6	7	8	9
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567

This means that the mantissa of log 710 is 0.8513, of log 711 it is 0.8519, and so on to log 719. Hence,

$$\log 715 = 2.8543, \qquad \log 7.18 = 0.8561,$$
$$\log 71{,}600 = 4.8549, \qquad \log 0.0719 = \bar{2}.8567.$$

And $\because$ 7154 is $\frac{4}{10}$ of the way from 7150 to 7160, $\therefore$ log 7154 is about $\frac{4}{10}$ of the way from log 7150 to log 7160.

$\therefore \log 7154 = \log 7150 + \frac{4}{10}$ of the difference between log 7150 and log 7160

$$= 3.8543 + \tfrac{4}{10} \text{ of } 0.0006$$
$$= 3.8543 + 0.0002 = 3.8545.$$

Similarly, $\log 7.154 = 0.8545,$

and $\log 0.07154 = \bar{2}.8545.$

The above process of finding the logarithm of a number of four significant figures is called **interpolation**. It is merely an approximation available within small limits, since numbers do not vary as their logarithms, the numbers forming a geometric series while the logarithms form an arithmetic series. It should be mentioned again that the mantissas given in the table are only approximate, being correct to 0.0001. This is far enough to give a result which is correct to three figures in general, and usually to four, an approximation sufficiently exact for many practical computations.

N	0	1	2	3	4	5	6	7	8	9
0	0000	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	0294	9345	9395	9445	9494
9	9542	9590	9638	0685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2089
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3027	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4820	4843	4857	4871	4886	4900
31	4914	4928	4042	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5509	5611	5623	5635	5647	5658	5670
37	5682	5694	5765	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5055	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6787	6776	6785	6794	6803
48	6812	6821	6630	6839	6848	6857	6866	6875	6884	6893
49	6002	6911	6920	6928	6937	6946	6955	6964	6972	6981
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

In all work with logarithms *the characteristic should be written before the table is consulted*, even if it is 0. Otherwise it is liable to be forgotten, in which case the computation will be valueless.

Illustrative problems. 1. Find from the table log 4260.

The characteristic is 3.

The mantissa is found to the right of 42 and under 6; it is 0.6294.

$\therefore \log 4260 = 3.6294.$

2. Find from the table log 42.67.

The characteristic is 1.

$$\begin{aligned}\log 42.7 &= 1.6304\\ \log 42.6 &= \underline{1.6294}\\ \text{difference} &= 0.0010\\ \tfrac{7}{10} \text{ of } 0.0010 &= 0.0007\\ \therefore \log 42.67 &= 1.6294 + 0.0007\\ &= 1.6301.\end{aligned}$$

EXERCISES. CLXI.

From the table find the following:

1. log 28.	2. log 443.	3. log 9.823.
4. log 2.34.	5. log 6.81.	6. log 700.3.
7. log 8940.	8. log 43.41.	9. log $\sqrt[4]{125}$.
10. log 3855.	11. log 2.005.	12. log 9.821^5.
13. log 1003.	14. log 3.142.	15. log 24,000.
16. log 23.42.	17. log $\sqrt{4.28}$.	18. log 0.2346.
19. log $75.55^{\frac{1}{3}}$.	20. log 0.0007.	21. log 0.00323.
22. log 0.2969.	23. log 0.0129^3.	24. log 0.000082.

363. *Given a logarithm to find the corresponding number.* The number to which a logarithm corresponds is called its **antilogarithm.**

E.g., $\because \log 2 = 0.3010$, $\therefore$ antilog $0.3010 = 2$.

The method of finding antilogarithms will be seen from a few illustrations. Referring again to the row after 71 on p. 355, we have:

N	0	1	2	3	4	5	6	7	8	9
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567

Hence, we see that

antilog $0.8513 = 7.1$, antilog $5.8531 = 713{,}000$,

antilog $\bar{2}.8567 = 0.0719$, antilog $\bar{1}.8555 = 0.717$.

Furthermore, $\because$ 8540 is halfway from 8537 to 8543,

$\therefore$ antilog 2.8540 is about halfway from antilog 2.8537 to antilog 2.8543.

$\therefore$ antilog 2.8540 is about halfway from 714 to 715.

$\therefore$ antilog $2.8540 = 714.5$.

Similarly, to find antilog $\bar{1}.8563$.

$$\begin{array}{ll} \text{antilog } \bar{1}.8567 = 0.719 & \quad \bar{1}.8563 \\ \text{antilog } \bar{1}.8561 = 0.718 & \quad \bar{1}.8561 \\ \qquad\qquad\quad \overline{6} & \qquad\quad \overline{2} \end{array}$$

$\therefore$ antilog $\bar{1}.8563 = 0.718\frac{2}{6} = 0.7183$.

The interpolation here explained is, as stated on p. 353, merely a close approximation; it cannot be depended upon to give a result beyond four significant figures except when larger tables are employed.

This is sufficient in many numerical computations. *E.g.*, we speak of the distance to the sun as 93,000,000 mi., using only two significant figures.

EXERCISES. CLXII.

From the table find the following:

1. antilog 0.3234.
2. antilog 2.4271.
3. antilog $\bar{2}.9193$.
4. antilog 5.2183.
5. antilog 3.9286.
6. antilog $\bar{1}.7929$.
7. antilog 0.8996.
8. antilog 4.7834.
9. antilog 3.9320.
10. antilog 2.0000.
11. antilog 1.9850.
12. antilog 0.7076.
13. antilog 10.5445.
14. antilog 3.6987.
15. antilog 0.9485 − 4.
16. antilog 0.6585 − 6.
17. antilog 0.6120 − 2.
18. antilog 0.9290 − 3.

364. Cologarithms. In cases of division by a number n it is often more convenient to add the logarithm of $\frac{1}{n}$ than to subtract the logarithm of n. The logarithm of $\frac{1}{n}$ is called the **cologarithm** of n.

$$\because \log \frac{1}{n} = \log 1 - \log n = 0 - \log n,$$

$$\therefore \operatorname{colog} n = -\log n.$$

Also, $\operatorname{colog} n = 10 - \log n - 10$, often a more convenient form to use.

E.g., $\therefore$ $\log 6 = 0.7782.$

$\therefore$ $\operatorname{colog} 6 = -0.7782.$

This may also be written $10 - 0.7782 - 10$, or $9.2218 - 10$.

The object of this is seen when we consider the addition of several logarithms and cologarithms; it is easier to add if all the mantissas are positive, subtracting the 10's afterwards.

In general, $\operatorname{colog} n = 10p - \log n - 10p$; that is, we may use 10, 20, or any multiple of 10, as may be most convenient.

The cologarithm can evidently be found by mentally subtracting each digit from 9, excepting the right-hand significant one (which must be subtracted from 10) and the zeros following, and then subtracting 10.

E.g., to find colog 6178.

$$\begin{array}{rcccccl} & 9. & 9 & 9 & 9 & 10 & \\ \log 6178 = & 3. & 7 & 9 & 0 & 9 & \\ \hline \operatorname{colog} 6178 = & 6. & 2 & 0 & 9 & 1 & -\ 10. \end{array}$$

To find colog 41.5.

$$\begin{array}{rcccccl} & 9. & 9 & 9 & 10 & 0 & \\ \log 41.5 = & 1. & 6 & 1 & 8 & 0 & \\ \hline \operatorname{colog} 41.5 = & 8. & 3 & 8 & 2 & 0 & -\ 10. \end{array}$$

To find colog 0.013.

$$\begin{array}{rcccccl} & 9. & 9 & 9 & 9 & 10 & \\ \log 0.013 = & \bar{2}. & 1 & 1 & 3 & 9 & \\ \hline \operatorname{colog} 0.013 = & 11. & 8 & 8 & 6 & 1 & -\ 10 = 1.8861. \end{array}$$

In case the characteristic exceeds 10 but is less than 20, colog n may be written $20 - \log n - 20$, and so for other cases; but these cases are so rare that they may be neglected at this time.

The advantage of using cologarithms will be apparent from a single example:

To find the value of $\dfrac{317 \cdot 92}{6178 \cdot 0.13}$.

USING COLOGARITHMS.

$$\begin{array}{rl} \log 317 = & 2.5011 \\ \log 92 = & 1.9638 \\ \operatorname{colog} 6178 = & 6.2091 - 10 \\ \operatorname{colog} 0.13 = & 10.8861 - 10 \\ \hline \log 36.32 = & 1.5601 \end{array}$$

$$\therefore \frac{317 \cdot 92}{6178 \cdot 0.13} = 36.32.$$

NOT USING COLOGARITHMS.

$$\begin{array}{rl} \log 317 = & 2.5011 \\ \log 92 = & 1.9638 \\ \hline \log (317 \cdot 92) = & 4.4649 \\ \hline \log 6178 = & 3.7909 \\ \log 0.13 = & \bar{1}.1139 \\ \hline \log (6178 \cdot 0.13) = & 2.9048 \\ \hline \log (317 \cdot 92) = & 4.4649 \\ \log (6178 \cdot 0.13) = & 2.9048 \\ \hline \log 36.32 = & 1.5601 \end{array}$$

365. Various bases. Thus far we have considered logarithms as exponents of powers of 10. But it is evident that any other *base* might be taken. Logarithms *to the base 10,* such as we have thus far considered, are sometimes called *common* or *Briggs logarithms,* the latter designation being in honor of Henry Briggs, who is said to have suggested this base to Napier.

If 2 were the base, log 8 would be 3, because $2^3 = 8$. Similarly, log 16 would be 4, and so on.

Where a different base than 10 is used (which is not the case in practical calculations), or where more than one base is used in the same discussion, the base is indicated by a subscript; thus, $\log_2 32 = 5$, because $2^5 = 32$.

366. Computations by logarithms. A few illustrative problems will now be given covering the types which the student will most frequently meet. It is urged that all work be neatly arranged, since as many errors arise from failure in this respect as from any other single cause.

Since π enters so frequently into computations, the following logarithms will be found useful:

$$\log \pi = 0.4971, \qquad \log \frac{1}{\pi} = \bar{1}.5029.$$

1. Find the value of $\dfrac{0.007^3}{0.03625}$.

$$\begin{aligned} \log 0.007 &= 0.8451 - 3 \\ 3 \cdot \log 0.007 &= 2.5353 - 9 \\ \text{colog } 0.03625 &= 11.4407 - 10 \\ & 13.9760 - 19 \\ &= 0.9760 - 6 = \log 0.000009462. \end{aligned}$$

$\therefore 9.462 \cdot 10^{-6} = Ans.$

It will be noticed that the negative characteristic is less confusing if written by itself at the right.

2. Find the value of $0.09515^{\frac{1}{3}}$.

$$\log 0.09515 = 0.9784 - 2.$$

$\because$ the characteristic (-2) is not divisible by 3, this may be written

$$\log 0.09515 = 1.9784 - 3.$$

Then $\frac{1}{3} \log 0.09515 = 0.6595 - 1 = \log 0.4566.$

$\therefore 0.4566 = Ans.$

3. Given a, r, l, in a geometric series, to find n. Compute the value if $l = 256$, $a = 1$, $r = 2$.

1. From § 350, $l = ar^{n-1}$.
2. $\therefore$ $\log l = \log a + (n-1)\log r.$ § 361
3. $\therefore$ $\dfrac{\log l - \log a}{\log r} + 1 = n.$

$$\log 256 = 2.4082$$
$$\log 1 = 0,\ \log 2 = 0.3010;$$
$$2.4082 \div 0.3010 = 8.$$

4. $\therefore$ $n = 8 + 1 = 9.$

4. Find the value of $\dfrac{2.706 \cdot 0.3 \cdot 0.001279}{86{,}090}$.

This may at once be written $\dfrac{2{,}706 \cdot 3 \cdot 1.279}{8.609} \cdot 10^{-8}$, thus simplifying the characteristics. Then

$$\begin{aligned} \log 2.706 &= 0.4324 \\ \log 3 &= 0.4771 \\ \log 1.279 &= 0.1069 \\ \text{colog } 8.609 &= \underline{9.0650 - 10} \\ \log 1.206 &= 0.0814 \end{aligned}$$

$\therefore 1.206 \cdot 10^{-8} = Ans.$

5. Given $2^x = 7$, find x, the result to be correct to 0.01.

$$x \log 2 = \log 7.$$

$$\therefore x = \frac{\log 7}{\log 2} = \frac{0.8451}{0.3010} = 2.81.$$

This division might be performed by finding the antilogarithm of $(\log 0.8451 - \log 0.3010)$, a plan not expeditious in this case.

6. The weight of an iron sphere, specific gravity 7.8, is 14.3 kg. Find the radius.

$$v = \tfrac{4}{3}\pi r^3 \cdot 1 \text{ cm}^3 = \text{volume in cm}^3.$$

$$\therefore \text{ weight} = \tfrac{4}{3}\pi r^3 \cdot 7.8 \cdot 1 \text{ g} = 14{,}300 \text{ g}.$$

$$\therefore \quad r = \left(\frac{3 \cdot 14{,}300}{4\pi \cdot 7.8}\right)^{\frac{1}{3}}, \text{ the number of centimeters of radius.}$$

$$\begin{aligned} \log 3 &= 0.4771 \\ \log 14{,}300 &= 4.1553 \\ \text{colog } 4 &= 9.3979 - 10 \\ \text{colog } \pi &= 9.5029 - 10 \\ \text{colog } 7.8 &= 9.1079 - 10 \\ 3 &\underline{|2.6411} \\ \log 7.593 &= 0.8804 \end{aligned}$$

$\therefore$ radius $= 7.593$ cm.

EXERCISES. CLXIII.

In the following exercises give the result to four significant figures.

1. Find the value of $37^{\frac{1}{37}}$.

2. Given $x^4 = x^7 : 15$. Find x.

3. Find the value of $(32/29)^{\frac{4}{5}}$.

4. Find the value of $\sqrt{\pi \cdot 5.927}$.

5. Find the value of $(5.376/\pi)^{\frac{1}{3}}$.

6. Find the value of $(37/2939)^{1\frac{2}{3}}$.

7. Given $227{,}600 = 7^{n-1}$. Find n.

8. Find the value of $\sqrt[10]{2\sqrt[10]{2} : \sqrt{10}}$.

9. Find the value of $(3.64/7.985)^6$.

10. Find the value of $\sqrt[11]{4.257^9 \sqrt[3]{0.8}}$.

11. Find the value of $(1402/3999)^{-3.8}$.

12. Find the value of $\sqrt[100]{100}$.

13. Find the value of $(22.8 \div 0.09235)^{\frac{1}{2}}$.

14. Find the value of $(24.73^5 \div 31.97^4)^{\frac{1}{9}}$.

15. Find the value of $(44 \cdot 8.37)^{\frac{1}{3}} \div 0.227^{\frac{1}{2}}$.

16. Find the value of $4\,\pi r^2$, when $r = 2.0\dot{6}$.

17. Also of $\frac{4}{3}\,\pi r^3$.

18. Given $x : 5.127 = 0.325 : 2936$. Find x.

19. Find the value of $\frac{4}{3}\,a^2 b\pi$, when $a = 19.63$, $b = 19.57$.

20. Given a, r, s, in a geometric series, show that

$$n = \frac{\log[a + (r-1)s] - \log a}{\log r},$$

and compute the value of n when $a = 1$, $r = 2$, $s = 511$.

21. Also, given r, l, s, show that

$$n = \frac{\log l - \log[lr - (r-1)s]}{\log r} + 1.$$

Compute the value of n when $r = 3$, $l = 729$, $s = 1092$.

22. Also, given a, l, s, show that

$$n = \frac{\log l - \log a}{\log(s-a) - \log(s-l)} + 1.$$

Compute the value of n when $a = 3$, $l = 729$, $s = 1092$.

23. Find the values of $\sqrt{2}$, $\sqrt[2.6]{2.6}$, $\sqrt[2.7]{2.7}$, $\sqrt[3]{3}$, each to 3 decimal places. Which of these is greatest? From this it may be inferred that the value of n that makes $\sqrt[n]{n}$ greatest is about what?

24. Solve the equation $5^x = 6$. (First take the logarithm of each member.)

25. Also the equation $\sqrt[x]{5} = 10$.

CHAPTER XX.

PERMUTATIONS AND COMBINATIONS.

367. The different groups of 2 things that can be selected from a collection of 3 different things, without reference to their arrangement, are called the **combinations** of 3 things taken 2 at a time.

E.g., representing the 3 things by the letters a, b, c, we can select 2 things in 3 ways, ab, ac, bc.

In general, the different groups of r things which can be selected from a collection of n different things, without reference to their arrangement, are called the **combinations of n things taken r at a time.**

So the combinations of the 4 letters a, b, c, d, taken 3 at a time, are abc, abd, acd, bcd; taken 2 at a time, ab, ac, ad, bc, bd, cd.

EXERCISES. CLXIV.

1. What is the number of combinations of 5 things taken 2 at a time? Represent them by letters.

2. What is the number of combinations of 5 things taken 3 at a time? Represent them by letters.

3. Write out the combinations of the letters w, x, y, z, taken 4 at a time; 3 at a time; 2 at a time; 1 at a time.

4. How does the number of combinations of 6 things taken 2 at a time compare with the number taken 4 at a time?

368. The different groups of 2 things which can be selected from 3 things, varying the arrangements in every possible manner, are called the **permutations** of 3 things taken 2 at a time.

E.g., the permutations of the letters a, b, c, taken 2 at a time, are ab, ba, ac, ca, bc, cb.

In general, the different groups of r things which can be selected from n different things, varying the arrangement in every possible manner, are called the **permutations of n things taken r at a time.**

In all this work the things are supposed to be different, and not to be repeated, unless the contrary is stated.

369. The number of combinations of n things taken r at a time is indicated by the symbol C_r^n. The number of permutations of n things taken r at a time is indicated by the symbol P_r^n.

EXERCISES. CLXV.

1. Show that $P_2^4 = 12$.

2. Show that $P_3^4 = 2 \cdot P_2^4$.

3. Show that $P_2^4 = 2 \cdot C_2^4$.

4. Find the value of P_2^5; of P_3^5.

5. Show that $C_1^n = n$, and $C_n^n = 1$.

6. Show that $P_1^3 = 3$, and in general that $P_1^n = n$.

7. Using the letters a, b, c, show that $C_2^3 = 3$.

8. Write out the permutations of the letters of the word *time*, taken all together.

9. Write out the permutations of the letters a, b, c, d taken 2 at a time; 3 at a time.

370. Theorem. *The number of permutations of* n *different things taken* r *at a time is* $n(n-1)(n-2)\cdots(n-r+1)$.

Proof. 1. Since we are to take r things we may suppose there are r places to be filled.

The first place may be filled in any one of n ways.

Thus, with a, b, c, d, we may fill the first place with a, b, c, or d.

2. For every way of filling the first place there are $n-1$ ways of filling the first and second.

Thus, if the first place be filled with a, we may fill the first and second with ab, ac, ad.

3. ∴ for n ways of filling the first place there are $n(n-1)$ ways of filling the first two.

E.g.,

ab,	*ac*,	*ad*,
ba,	*bc*,	*bd*,
ca,	*cb*,	*cd*,
da,	*db*,	*dc*,

giving $4 \cdot 3 = 12$ ways in all.

4. For every way of filling the first two places there are $n-2$ ways of filling the first, second, and third.

Thus, if the first 2 places be filled with ab, the first 3 can be filled with abc, abd, *i.e.*, in $4-2$ ways.

5. ∴ for $n(n-1)$ ways of filling the first two places there are $n(n-1)(n-2)$ ways of filling the first three.

E.g.,

abc,	*abd*,	*adc*,	*adb*,
acb,	*acd*,	*bca*,	*bcd*,
bda,	*bdc*,	*cda*,	*cdb*,

and the same with the first two letters interchanged in each.

6. Similarly, the number taken 4 at a time is $n(n-1)(n-2)(n-3)$, and the same reasoning evidently shows that the number of permutations of n things r at a time is

$$n(n-1)(n-2)\cdots(n-\overline{r-1})$$

or $$n(n-1)(n-2)\cdots(n-r+1).$$

COROLLARY. *If* $\mathrm{n}=\mathrm{r}$, $\mathrm{P}^{\mathrm{n}}_{\mathrm{n}}=\mathrm{n}(\mathrm{n}-1)\cdots3\cdot2\cdot1$. *Hence, the number of permutations of* n *things taken all together is* $\mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)\cdots3\cdot2\cdot1$.

EXERCISES. CLXVI.

1. Find the value of P^{100}_2.

2. Find the value of P^{20}_4.

3. Prove that $P^{n-1}_{n-1}=\frac{1}{n}P^n_n$.

4. Prove that $P^n_n=P^n_r\cdot P^{n-r}_{n-r}$.

5. Find the value of P^5_2; of P^6_3. Prove this by writing out the permutations of the letters a, b, c, $\cdots$.

6. Show from the theorem (§ 370) that P^n_r is greater as r is greater.

7. Show from the corollary that P^n_n is the product of all integers from 1 to n inclusive.

8. Find the number of permutations of the letters of the word *number* taken all together.

9. Find the number of permutations of the letters of the word *courage* taken 3 at a time; taken all together.

10. By writing out the permutations and the combinations of the letters a, b, c, d, e, taken 2 at a time, ascertain how P^5_2 compares with C^5_2.

371. Factorials. The product

$$n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1,$$

that is of all integers from 1 to n inclusive, is called **factorial n.**

Thus, factorial $3 = 1\cdot 2\cdot 3 = 6$,

" $4 = 1\cdot 2\cdot 3\cdot 4 = 24$, etc.

Factorial n is represented by several symbols. In writing it is customary to use $\underline{|n}$, this being a symbol easily made. In print, on account of the difficulty of setting the $\underline{|n}$, it is customary to use the symbol $n!$ or (especially in Germany) Πn.

Π is a Greek letter corresponding to P, and may be thought of as standing for *product.*

We shall use in print only the symbol $n!$

372. It therefore appears that

(1) $P_n^n = n!$

(2) $P_r^n = \dfrac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1} = \dfrac{n!}{(n-r)!}$

EXERCISES. CLXVII.

1. Show that $P_4^{10} = \dfrac{10!}{6!}$.

2. Show that $5! = 120$.

3. Find the value of $\dfrac{10!}{5!}$.

4. Also of $\dfrac{12!}{10!}\cdot\dfrac{8!}{6!}\cdot\dfrac{4!}{2!}$.

5. Prove that $n! = n(n-1)(n-2)\cdot(n-3)!$

6. Prove that $(n!)^2 = n^2(n-1)^2(n-2)^2\cdots 3^2\cdot 2^2\cdot 1$.

7. In how many ways can 10 persons be placed in a row?

373. Theorem. *The number of permutations of* n *different things taken* r *at a time, when each of the* n *things may be repeated, is* n^r.

Proof. After the first place has been filled, the second can be filled in n ways, since repetition is allowed. So for the subsequent places.

Hence, instead of having

$$P_r^n = n(n-1)(n-2)\cdots(n-r+1),$$

we have $$n \cdot n \cdot n \cdots n = n^r.$$

EXERCISES. CLXVIII.

1. Find the value of P_4^6, repetitions being allowed.

2. Find the value of P_5^5, repetitions being allowed.

3. How many numbers are there containing 4 digits?

4. How many ways are there of selecting 3 numbers from 50 on a combination lock, repetitions being allowed?

5. How many ways are there of selecting 3 numbers from 10 on a combination lock, repetitions being allowed?

6. Show that P_n^n, repetitions being allowed, is n^n. From this tell how many 9-figure numbers are possible, all zeros being excluded.

7. From ex. 6, how many 10-figure numbers are possible, zeros being admitted except in the highest order.

8. How many possible integral numbers can be formed from the digits 1, 2, 3, 4, or any of them, repetitions of the digits being allowed?

9. The chance of guessing correctly, the first time, the three numbers on which a combination lock of 100 numbers is set, is 1 out of how many?

374. Theorem. *The number of combinations of* n *different things taken* r *at a time is*

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

Proof. 1. For each combination of r things there are $r!$ permutations.

2. $\therefore$ for C^n_r combinations there are $C^n_r \times r!$ permutations.

3. But it has been shown that this number of permutations is

$$n(n-1)(n-2)\cdots(n-r+1). \qquad \S\, 370$$

4. $\therefore C^n_r \times r! = n(n-1)(n-2)\cdots(n-r+1)$,

and $$C^n_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

COROLLARIES. 1. $C^n_r = P^n_r / r!$

2. $C^n_r = \dfrac{n!}{r!(n-r)!}$.

For we may multiply both terms of the fraction

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$$

by $(n-r)!$, giving

$$\frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{r!(n-r)!}$$

which equals $$\frac{n!}{r!(n-r)!}$$

This is a more convenient formula to write and to carry in mind. Practically, of course, it gives the same result as the other. *E.g.*,

By the theorem, $$C^5_3 = \frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1};$$

by the corollary, $$C^5_3 = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 2\cdot 1}.$$

EXERCISES. CLXIX.

1. If $P_n^n = 3{,}628{,}800$, find n.

2. Find the values of P_3^7; of P_5^{10}; of C_3^8.

3. If $P_2^n = 56$, find n, and explain why there should be two results.

4. In how many ways can 3 persons be selected from a class of 20?

5. In how many ways can the letters of the word *cat* be arranged?

6. Prove that $C_r^n = C_{n-r}^n$, by substituting in the formula of § 374, cor. 2.

7. What is the number of combinations of 20 things taken 5 at a time?

8. In how many ways can the letters of the word *number* be arranged?

9. How many numbers can be formed by taking 4 out of the 5 digits 1, 2, 3, 4, 5?

10. How many triangles are formed from 4 lines, each of which intersects the other 3?

11. How many changes can be rung with a peal of 7 bells, a particular one always being last?

12. In how many ways may the letters of the word *united* be arranged, taken all at a time?

13. How many changes can be rung with a peal of 5 bells, using each bell once in each change?

14. In how many ways can a consonant and a vowel be chosen out of the letters of the word *numbers?*

15. How many numbers between 2000 and 5000 have the hundreds figure 7 and are divisible by 2?

16. In how many ways may the letters of the word *rate* be arranged, taken any number at a time?

17. In how many ways can 5 persons be seated about a circular table, one of them always occupying the same place?

18. How many different arrangements (permutations) can be made by taking 5 of the letters of the word *triangle?*

19. On an examination 15 questions are given, of which the student has a choice of 10. In how many ways may he make his selection?

20. How many different arrangements can be made of the letters of the word *algebra*, it being noted that two of the letters are alike?

21. There are four points in a plane, no three being in the same straight line. How many straight lines can be drawn connecting two points?

22. How many different signals can be made with 5 different flags, displayed on a staff 3 at a time? 4 at a time? 2 at a time? altogether? any number at a time?

23. Suppose a telegraphic system consists of two signs, a dot and a dash; how many letters can be represented by these signs taken 1 at a time? 2 at a time? 3 at a time? 4 at a time?

24. Prove that the number of permutations of n different things taken r at a time is $n-r+1$ times the number of permutations of the n things taken $r-1$ at a time.

CHAPTER XXI.

THE BINOMIAL THEOREM.

375. The binomial theorem is stated in § 80, and a proof, which may be used in connection with that section, is given in Appendix I.

It is now proposed to consider this theorem in the light of Chapter XX.

376. Theorem. *If the binomial* a + b *is raised to the* n*th power,* n *integral and positive, the result is expressed by the formula*

$$(\mathrm{x}+\mathrm{a})^{\mathrm{n}} \equiv \mathrm{x}^{\mathrm{n}} + \mathrm{C}_1^{\mathrm{n}}\,\mathrm{x}^{\mathrm{n}-1}\mathrm{a} + \mathrm{C}_2^{\mathrm{n}}\,\mathrm{x}^{\mathrm{n}-2}\mathrm{a}^2$$
$$+ \mathrm{C}_3^{\mathrm{n}}\,\mathrm{x}^{\mathrm{n}-3}\mathrm{a}^3 + \cdots \mathrm{C}_{\mathrm{n}-1}^{\mathrm{n}}\,\mathrm{x}\mathrm{a}^{\mathrm{n}-1} + \mathrm{a}^{\mathrm{n}}.$$

Proof. 1. By multiplication we know that

$$(x+a)(x+b)$$
$$\equiv x^2 + (a+b)x + ab,$$
$$(x+a)(x+b)(x+c)$$
$$\equiv x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc,$$
$$(x+a)(x+b)(x+c)(x+d)$$
$$\equiv x^4 + (a+b+c+d)x^3$$
$$+ (ab+ac+ad+bc+bd+cd)x^2$$
$$+ (abc+abd+acd+bcd)x + abcd.$$

There is evidently a law running through all these expansions, relating to the exponents and the coefficients of x.

2. We might infer from step 1 that if there were n factors, the product would have for the coefficient

of x^n, 1;

of x^{n-1}, $a + b + c \cdots n$;

of x^{n-2}, the combinations of the letters $a, b, \cdots n$, taken 2 at a time;

of x^{n-3}, the combinations of these letters taken 3 at a time;

⋮

of x, the combinations of these letters taken $n-1$ at a time.

3. This inference is correct; for the term containing x^n can be formed only by taking the product of the x's in all the factors, and hence its coefficient is 1.
The terms containing x^{n-1} can be formed only by multiplying the x's in all but one factor by the other letter in that factor; hence the x^{n-1} term will have for its coefficient $(a + b + \cdots n)$.
The terms containing x^{n-2} can be formed only by multiplying the x's in all but 2 factors by the other letters in those factors, *i.e.*, by a and b, a and c, a and d, etc.; hence the x^{n-2} term will have for its coefficient $(ab + ac + ad + \cdots)$.
The reasoning is evidently general for the rest of the coefficients.

4. If, now, we let $a = b = c = \cdots = n$, we have

$$(x + a)^n \equiv x^n + C_1^n x^{n-1}a + C_2^n x^{n-2}a^2 + C_3^n x^{n-3}a^3 + \cdots + C_{n-1}^n x a^{n-1} + a^n.$$

As stated in § 246, the binomial theorem is true whether n is positive or negative, integral or fractional. While the proof of this fact cannot satisfactorily be presented without the differential calculus, the fact itself should be recognized.

The following exercises will serve to recall the application of the theorem, although they do not differ materially from those already met by the student in the exercises following §§ 80, 246.

Illustrative problems. 1. Required the square root of $1 + x$ to 3 terms.

1. $\because \quad (a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots,$

2. $\therefore \quad (1+x)^{\frac{1}{2}} = 1^{\frac{1}{2}} + \frac{1}{2}\cdot 1^{-\frac{1}{2}}\cdot x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}\cdot 1^{-\frac{3}{2}}\cdot x^2 + \cdots$

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \cdots.$$

2. Expand to 4 terms $(a - 2b)^{-3}$.

1. $\because (x+y)^n \equiv x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{2\cdot 3}x^{n-3}y^3 + \cdots,$

2. $\therefore (a-2b)^{-3} = a^{-3} + (-3)a^{-4}(-2b) + \frac{-3\cdot -4}{2}a^{-5}(-2b)^2 + \frac{-3\cdot -4\cdot -5}{2\cdot 3}a^{-6}(-2b)^3 + \cdots$

$$= a^{-3} + 6a^{-4}b + 24a^{-5}b^2 + 80a^{-6}b^3 + \cdots.$$

3. Expand to 3 terms $(1+x)^{-\frac{1}{4}}$.

As above, $(1+x)^{-\frac{1}{4}} = 1 + (-\frac{1}{4})x + \frac{(-\frac{1}{4})(-\frac{1}{4}-1)}{2}x^2 + \cdots$

$$= 1 - \tfrac{1}{4}x + \tfrac{5}{32}x^2 \cdots.$$

EXERCISES. CLXX.

Expand the following binomials:

1. $(x+5)^7$.

2. $(x^2-2a)^5$.

3. $\left(1-\dfrac{x}{2}\right)^{10}$.

4. $\left(\dfrac{a}{b}+\dfrac{b}{a}\right)^6$.

5. $(40+1)^4$.

6. $(3a-\frac{2}{3}b^2)^6$.

7. $(1+x)^{\frac{1}{4}}$, to 4 terms.

8. $(a+b)^{\frac{3}{4}}$, to 4 terms.

9. $\sqrt{a^2-x^2}$, to 3 terms.

10. $(1+x)^{-\frac{2}{3}}$, to 4 terms.

11. $(1-2a)^{\frac{3}{4}}$, to 4 terms.

12. $(3x-2y)^{\frac{2}{3}}$, to 4 terms.

13. $\sqrt[5]{31}=(32-1)^{\frac{1}{5}}$, to 3 terms.

14. $\dfrac{1}{\sqrt{1+x}}=(1+x)^{-\frac{1}{2}}$, to 4 terms.

15. $(1-x)^{-1}$, to 5 terms, checking by performing the division $\dfrac{1}{1-x}$.

16. $(1-x)^{-2}$, to 5 terms, checking by performing the division $\dfrac{1}{1-2x+x^2}$.

17. $(1+x)^{-2}$, to 5 terms, checking by performing the division $\dfrac{1}{1+2x+x^2}$.

APPENDIX.

I. PROOF OF THE BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS (p. 57).

If n is a positive integer

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \cdots.$$

Proof. 1. The law is evidently true for the 2d power, for $(a+b)^2 \equiv a^2 + 2\,ab + b^2$, or, as the theorem says,

$$\equiv a^2b^0 + 2\,a^1b^1 + a^0b^2. \qquad \S\ 69$$

2. It is also true for the 3d power, for $(a+b)^3 \equiv a^3 + 3\,a^2b + 3\,ab^2 + b^3$, or, as the theorem says,

$$\equiv a^3b^0 + 3\,a^2b^1 + 3\,a^1b^2 + a^0b^3. \qquad \S\ 69$$

3. Now if it were true for the kth power we should have $(a+b)^k \equiv a^k + ka^{k-1}b + \frac{k(k-1)}{2}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{2\cdot 3}a^{k-3}b^3 + \cdots,$ and if this were multiplied by $a+b$ we should have

4. $(a+b)^{k+1}$

$$\equiv a^{k+1} + \left.\begin{matrix} k \\ +1 \end{matrix}\right| a^kb \quad \left.\begin{matrix} +\dfrac{k(k-1)}{2} \\ +\quad k \end{matrix}\right| a^{k-1}b^2 + \left.\begin{matrix} \dfrac{k(k-1)(k-2)}{2\cdot 3} \\ +\dfrac{k(k-1)}{2} \end{matrix}\right| a^{k-2}b^3 + \cdots$$

$$\equiv a^{k+1} + (k+1)\,a^kb + \frac{(k+1)k}{2}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{2\cdot 3}a^{k-2}b^3 + \cdots.$$

5. But here we see that *if the theorem were true for any power*, as the kth, it would be true for the next higher power, as the $(k+1)$th.

6. But the theorem *is* true for the 3d power (step 2), and

$\therefore$ it is true for the $(3+1)$th or 4th power, by step 5;

$\therefore$ " " " $(4+1)$th " 5th " " "

and so on for all integral powers.

II. SYNTHETIC DIVISION (p. 67).

If the divisor is a binomial of the first degree, there is often a considerable gain by resorting to a form of division known as **synthetic**.

The process is best understood by following the solution of a problem.

Required the quotient of $x^3 - 3x^2 + 3x + 4$ by $x - 1$. The ordinary long form would be as follows, the heavy numerals being the ones reserved in the synthetic form given below:

$$
\begin{array}{r|l}
 & x^2 - 2x + 1 \\
\hline
x - 1 & x^3 - 3x^2 + 3x + 4 \\
 & x^3 - \mathbf{1}x^2 \\
\cline{2-2}
 & \quad - 2x^2 \\
 & \quad - 2x^2 + \mathbf{2}x \\
\cline{2-2}
 & \qquad\qquad\quad x + 4 \\
 & \qquad\qquad\quad x - \mathbf{1} \\
\cline{2-2}
 & \qquad\qquad\qquad 5 \text{ rem.}
\end{array}
$$

This may be abridged by writing the quotient below, as follows:

$$
\begin{array}{r|l}
x & x^3 - 3x^2 + 3x + 4 \\
-1 & \quad - \mathbf{1}x^2, + \mathbf{2}x, - \mathbf{1} \\
\hline
 & x^2 - \ 2x + 1; \qquad 5 \text{ rem.}
\end{array}
$$

Here the first term of the quotient, x^2, is multiplied by -1, this product subtracted from $-3x^2$ and the remainder immediately divided by x to get the next term, $-2x$, and so on.

Since it is easier to add than to subtract, it is usual to change the sign of the second term of the divisor and add. Doing this, and detaching the coefficients, we have *the common form for synthetic division,* as follows:

$$\begin{array}{r|l} 1 & 1 - 3 + 3 + 4 \\ +1 & \quad\ \ 1 - 2 \quad 1 \\ \hline & 1 - 2 + 1; \quad 5 \text{ rem.} \end{array}$$

Check. Let $x = 2$. Then $(8 - 12 + 6 + 4 - 5) \div 1 = 4 - 4 + 1$.

In case any powers of a letter are wanting in arranging according to descending powers of that letter, zero coefficients should be introduced as usual.

EXERCISES. CLXXI.

Perform the following divisions by the synthetic process, detaching the coefficients, and checking in the usual way.

1. $a^3 + b^3$ by $a + b$.
2. $x^6 - y^6$ by $x - y$.
3. $a^3 - 4a + 3$ by $a - 1$.
4. $x^4 - 2x^2 + 1$ by $x + 1$.
5. $1 + x + x^2 + x^3$ by $1 + x$.
6. $x^2 - 29x + 190$ by $x - 10$.
7. $x^3 + 3x^2a - 4xa^2$ by $x - a$.
8. $x^3 + 2x^2 - 4x + 1$ by $x - 1$.
9. $x^4 - 3x^2 + 2x + 6$ by $x + 1$.
10. $x^3 + 3x^2 + 3x + 28$ by $x + 4$.
11. $5x^4 + 4x^3 + 3x^2 + 2x + 1$ by $x + 1$.
12. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$ by $a - b$.
13. $x^4 - 10x^2 + 9$ by $x + 3$; also by $x - 3$.
14. $3x^3 - 2x^2 - 7x - 2$ by $x + 1$; also by $x - 2$.
15. $2x^2 + 3xy - 2y^2$ by $x + 2y$; also by $y - 2x$.

III. THE APPLICATIONS OF HOMOGENEITY, SYMMETRY, AND CYCLO-SYMMETRY (p. 73).

The applications of homogeueity, symmetry, and cyclo-symmetry are very extensive and they materially simplify the study of algebra. The principle which lies at the foundation of these applications is as follows:

If two algebraic expressions are homogeneous, symmetric, or cyclic, their sum, difference, product, or quotient is also homogeneous, symmetric, or cyclic, respectively.

The truth of this principle follows from the definitions and from previous proofs. *E.g.*, by the law of the formation of the product of two polynomials it appears that each term of one factor is multiplied by each term of the other; hence, if one factor is homogeneous and of the third degree and the other is homogeneous and of the second degree, then the product must be homogeneous and of the fifth degree.

The converse is not necessarily true. *E.g.*, the sum of two non-symmetric expressions may be symmetric, as the sum $a^2 + b^2 + c$ and $c(c - 1)$.

These considerations suggest some valuable checks on the four fundamental operations. Since algebraic expressions are often homogeneous, symmetric, or cyclic, these checks will be of service throughout the study of the subject.

E.g., the product of $x^2 + y^2$ and $x + y$ is $x^3 + xy^2 + x^2y + y^3$. This may be checked by arbitrary values, or by noticing that the product must be homogeneous, of the third degree, and symmetric as to x and y.

In the same way the square root of $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$ must be symmetric, the product of $(a - b)(b - c)(c - a)$ must be cyclic and the quotient of $27a^3b^3 + c^3$ by $3ab + c$ must be symmetric as to a and b; otherwise there must be an error in the operation.

It so happens that many of the expressions dealt with in higher algebra are, or can be made, symmetric or homogeneous, or both, and hence the value of these checks becomes the more apparent as the student progresses in mathematics.

EXERCISES. CLXXII.

Perform the operations here indicated, checking each by substituting arbitrary values and also by (1) homogeneity, (2) symmetry, or (3) cyclo-symmetry, as seems best.

1. $(x^4 + y^4)(x^8 - x^4y^4 + y^8)$.

2. $(2x + y - z)(2x - y + z)$.

3. $(81a^4b^4 - 256c^4) \div (3ab + 4c)$.

4. $(a + b + c)(bc + ca + ab) - abc$.

5. $(x^4 + x^2y^2 + y^4) \div (x^2 + y^2 + xy)$.

6. $-(a - b)(b - c)(c - a)(a + b + c)$.

7. $a^3(b - c) + b^3(c - a) + c^3(a - b)$.

8. $a^4(b - c) + b^4(c - a) + c^4(a - b)$.

9. $-(a - b)(b - c)(c - a)$.

10. $a^2(b - c) + b^2(c - a) + c^2(a - b)$.

11. $(a^3 + b^3 + 1 - 3ab) \div (a + b + 1)$.

12. $(x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$.

13. $(x^4 - 18x^2y^2 + y^4) \div (x^2 - y^2 + 4xy)$.

14. $(x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z)$.

15. $(x + y - 2z)^2 + (y + z - 2x)^2 + (z + x - 2y)^2$.

16. $(k - 2l - 3m)^2 + (l - 2m - 3k)^2 + (m - 2k - 3l)^2$.

17. $(a^2 - b^2 - c^2 + d^2 + 2bc - 2ad) \div (a + b - c - d)$.

18. $(p + q + r)^3 + (p^3 - q - r)^3 + (q - r - p)^3$
$+ (r - p - q)$.

19. $(x + y + z)^3 - (y + z - x)^3 - (z + x - y)^3$
$-(x + y - z)^3$.

Symbolism of symmetric expressions. Since the terms of a symmetric expression are so closely related in form, it is often necessary to write only the *types* of these terms.

E.g., if a trinomial is symmetric as to a, b, and c, and if one term is ab, the others are at once known to be bc and ca. The term ab is therefore called a *type term.*

The Greek letter Σ (*sigma*, our S) is used to mean "*the sum of all expressions of the type* $\cdots$."

E.g., in $f(x, y, z)$, Σx^2y means "the sum of all expressions of the form x^2y" which can be made from the three given letters.

That is, $\Sigma x^2y \equiv x^2y + x^2z + y^2x + y^2z + z^2x + z^2y$. This polynomial is called the *expansion* of Σx^2y.

If these same three letters are under discussion,

$$\Sigma x^2 \equiv x^2 + y^2 + z^2, \text{ but } (\Sigma x)^2 \equiv (x + y + z)^2.$$

In case of any doubt, the letters under discussion are written below the Σ, thus:

$$\sum_{abc} (a + b) \equiv (a + b) + (b + c) + (c + a)$$

$$\sum_{xy} (x^2 + y) \equiv (x^2 + y) + (y^2 + x).$$

If an expression is known to be cyclic, Σ has a slightly different meaning. It then stands for "*the sum of expressions of this type, which can be formed by a cyclic interchange of the letters.*"

E.g., if only cyclic expressions involving three letters are under discussion,

$$\Sigma(a - b) \equiv (a - b) + (b - c) + (c - a),$$

instead of

$$(a - b) + (b - a) + (b - c) + (c - b) + (c - a) + (a - c);$$

and

$$\Sigma a(b + c - 2a)^2 \equiv a(b + c - 2a)^2 + b(c + a - 2b)^2 + c(a + b - 2c)^2.$$

EXERCISES. CLXXIII.

Expand the expressions in exs. 1–8.

1. Σxy, where only x, y, z are involved.

2. Σa^2b, " a, b "

3. $\Sigma (a+b)^2$ " a, b, c "

4. Σab^2 " " "

5. $\sum_{xyz} x^3y^2z$.

6. $\sum_{abc} a^3 - 3\,abc$.

7. $\sum_{a \cdots e} a^2 + 2\,\Sigma ab$.

8. $\sum_{abc} a^3 + 3\,\Sigma a^2b + 6\,abc$.

In *cyclic* functions involving only a, b, c, what is the expansion of the expressions in exs. 9–14?

9. $\Sigma a(b+c)$.

10. $\Sigma a^2(b-c)$.

11. $\Sigma a^2(b^3-c^3)$.

12. $\Sigma a^2b^2(a-b)$.

13. $\Sigma a^2(a-b+c)$.

14. $\Sigma (b-c)^3(b+c-2\,a)$.

Show that the following identities are true, by expanding both members. Those involving negative signs are cyclic. Except as stated to the contrary, only a, b, c are involved.

15. $\Sigma a(b-c) \equiv 0$.

16. $(\Sigma a)^2 - \Sigma a^2 \equiv 2\,\Sigma ab$.

17. $(\sum_{a \cdots d} a)^2 \equiv \Sigma a^2 + \Sigma\, 2\,ab$.

18. $\frac{1}{2}[(\sum_{a \cdots e} a)^2 - \Sigma a^2] \equiv \Sigma ab$.

19. $\Sigma (a-b)(a+b-c) \equiv 0$.

20. $\Sigma (a-b)^3 \equiv 3\,(a-b)(b-c)(c-a)$.

21. $\Sigma a^2(b-c) \equiv -(a-b)(b-c)(c-a)$.

22. $\Sigma ab(a-b) \equiv -(a-b)(b-c)(c-a)$.

23. $(\Sigma a)(\Sigma ab) - abc \equiv (a+b)(b+c)(c+a)$.

24. $(\Sigma a)(\Sigma a^2) + 2\,abc \equiv (a+b)(b+c)(c+a) + \Sigma a^3$.

Illustrative problems. The preceding principles render it easy to simplify certain expansions which would otherwise require considerable labor. The process will be understood from a few problems.

1. Expand $(a+b+c)^2$.

1. The expression is symmetric and homogeneous.

2. $\therefore$ the expanded form contains only the types a^2, ab, with numerical coefficients.

3. $\therefore$ it is of the form $m\Sigma a^2 + n\Sigma ab$, where we have to determine m and n.

4. Considering the expression as a binomial, $(\overline{a+b}+c)^2$, we shall evidently have $a^2 + 2\,ab + b^2 +$ some terms which do not contain a^2 or ab.

5. $\therefore$ the coefficients of the type a^2 are all 1, and those of the type ab are all 2. $\therefore m = 1$, $n = 2$.

6. $\therefore$ the result is $\Sigma a^2 + 2\,\Sigma ab$, or $a^2 + b^2 + c^2 + 2\,(ab + ca + bc)$.

Check. Let $a = b = c = 1$. Then

$$3^2 = 1^2 + 1^2 + 1^2 + 2\,(1+1+1) = 9.$$

2. Simplify

$$(a+b+c)^2 + (a+b-c)^2 + (b+c-a)^2 + (c+a-b)^2.$$

1. As in problem 1, the types are a^2, ab, and the expanded form is $m\Sigma a^2 + n\Sigma ab$, where we have to determine m and n.

2. In the four trinomials we have a^2, a^2, $(-a)^2$, $(-a)^2$, or $4\,a^2$, as shown in problem 1. $\therefore m = 4$.

3. Also $2\,ab$, $2\,ab$, $-2\,ab$, $-2\,ab$, or $0 \cdot ab$. $\therefore n = 0$.

4. $\therefore$ the result is $4\,\Sigma a^2$, or $4\,(a^2 + b^2 + c^2)$.

Check. Let $a = b = c = 1$. Then

$$3^2 + 1^2 + 1^2 + 1^2 = 4\,(1^2 + 1^2 + 1^2) = 12.$$

This particular problem is so simple that there is no great gain by using the Σ symbolism.

3. Expand $(\Sigma a)^2$, where $\Sigma a \equiv a + b + c + d + e + \cdots$

1. What can be said of $(\Sigma a)^2$ as to symmetry? homogeneity?

2. $\therefore$ the expanded form contains only what types?

3. $\therefore$ it is of what form, and what coefficients are to be determined? (See problem 1.)

4. What are these coefficients in the expansion of $(a + b)^2$?

5. Will the addition of other letters, as $c + d + e + \cdots$, affect these coefficients of a^2 and ab?

6. $\therefore$ what values have the coefficients m and n, and what is the result?

4. Expand $(\Sigma a)^3$, where $\Sigma a \equiv a + b + c + d + e + \cdots$.

1. The types are evidently of the third degree, and therefore must be a^3, a^2b, abc. (Why?)

2. In expanding $(\overline{a + b} + c)^3$, we have (§ 69)

$$\overline{a + b}^3 + 3\,\overline{a + b}^2 \cdot c + 3\,\overline{a + b} \cdot c^2 + c^3,$$

in which the coefficient of a^3 is evidently 1, of a^2b is 3, and of abc (found only in $3\,\overline{a + b}^2 \cdot c$) is 6.

3. The addition of other letters, $d + e + \cdots$, will not affect the coefficients of a^3, a^2b, or abc.

4. $\therefore (\Sigma a)^3 \equiv \Sigma a^3 + 3\,\Sigma a^2b + 6\,\Sigma abc$.

5. Expand $(x + y + z)^3 - (y + z - x)^3 - (z + x - y)^3 - (x + y - z)^3$

1. What are the types?

2. $\because$ we have x^3, $-(-x)^3$, $-x^3$, $-x^3$, what is the coefficient of Σx^3?

3. $\because$ we have $3\,x^2y$, $-3\,x^2y$, $-(-3\,x^2y)$, $-3\,x^2y$, what is the coefficient of Σx^2y?

4. $\because$ we have $6\,xyz$ (as in problem 4), $-(-6\,xyz)$, $-(-6\,xyz)$, $-(-6\,xyz)$, what is the coefficient of Σxyz?

5. $\therefore$ the result is $24\,xyz$.

Check. Let $x = y = z = 1$. Then, etc.

EXERCISES. CLXXIV.

Σ is limited to three letters in each of the following exercises, except as otherwise indicated.

1. Expand $(\Sigma a)^4$.

2. Expand $(\Sigma_{a \cdots d} a)^4$.

3. Show that, if $\Sigma a = 0$, $(\Sigma a^2)^2 \equiv 4(\Sigma ab)^2$.

4. Show that $\Sigma a \cdot (\Sigma a^2 - \Sigma ab) \equiv \Sigma a^3 - 3abc$.

5. Show that, if $\Sigma a = 0$, $\Sigma(a+b)^3 + \Sigma a^3 = 0$.

6. Show that $(a+b)(b+c)(c+a) \equiv \Sigma a^2 b + 2abc$.

7. Simplify $(a-b-c)^2 + (b-a-c)^2 + (c-a-b)^2$.

8. Show that $\Sigma x \cdot (\Sigma x - 2x) \cdot (\Sigma x - 2y) \cdot (\Sigma x - 2z)$
$\equiv 2\Sigma x^2 y^2 - \Sigma x^4$.

9. Simplify $(a-2b-3c)^2 + (b-2c-3a)^2$
$+ (c-2a-3b)^2$.

10. Show that $(-a+b+c)(a-b+c)(a+b-c)$
$\equiv \Sigma a^2(b+c) - \Sigma a^3 - 2abc$.

11. Show that $\Sigma(a-b) \equiv 0$.

12. Show that $(a+b+c)(-a+b+c)(a-b+c)$
$(a+b-c) \equiv \Sigma 2a^2b^2 - \Sigma a^4$.

13. Show that $(a+b)(b+c)(c+a) \equiv \Sigma ab^2 + 2abc$.

14. Show that $\Sigma a \cdot \Sigma a^2 \equiv ab(a+b) + bc(b+c)$
$+ ca(c+a) + \Sigma a^3$.

15. Show that $\Sigma a \cdot \Sigma ab \equiv a^2(b+c) + b^2(c+a)$
$+ c^2(a+b) + 3abc$.

16. Show that $(\Sigma a - 2a)(\Sigma a - 2b)(\Sigma a - 2c)$
$\equiv a^2(b+c) + b^2(c+a) + c^2(a+b) - \Sigma a^3 - 2abc$.

IV. APPLICATION OF THE LAWS OF SYMMETRY AND HOMOGENEITY TO FACTORING (p. 88).

Since many of the expressions in mathematics are symmetric or homogeneous or both, the application of the laws of symmetry and homogeneity is of great importance.

E.g., to factor $ac^2 + ba^2 + cb^2 - ab^2 - bc^2 - ca^2$, it should be noticed that

1. It is homogeneous, of the third degree, and cyclic.

2. $\therefore$ either it has 3 linear factors, $a - b$ being one (why ?), or else it has 1 linear factor, $a + b + c$ (why ?) and 1 quadratic factor. (Why ?)

3. And $\because$ it vanishes for $a = b$, $\therefore$ $a - b$ is a factor, and $\therefore$ $b - c$ and $c - a$. (Why ?)

4. There are no more literal factors (why ?), but there may be a numerical factor n.

5. Then $ac^2 + ba^2 + cb^2 - ab^2 - bc^2 - ca^2 \equiv n(a - b)(b - c)(c - a)$, and if $a = 2$, $b = 1$, $c = 0$, this reduces to

$$2 = -2 \cdot n,$$

whence $$n = -1.$$

6. $\therefore$ the expression equals $-(a - b)(b - c)(c - a)$.

Check by letting $a = 3$, $b = 2$, $c = 1$, or other values.

EXERCISES. CLXXV.

Factor the following:

1. $\Sigma x^4(y - z)$.
2. $\Sigma a^4(b^2 - c^2)$.
3. $\Sigma x^3(y - z)$.
4. $\Sigma a^4(b^3 - c^3)$.
5. $(\Sigma a)^3 - \Sigma a^3$.
6. $\Sigma a^4 - 2\,\Sigma a^2b^2$.
7. $(\Sigma a)(\Sigma ab) - abc$.
8. $\Sigma ab(a + b) + 2\,abc$.
9. $\Sigma a(b^2 + c^2) + 2\,abc$.
10. $\Sigma a(b - c)^2 + 8\,abc$.
11. $\Sigma a(b + c)^2 - 4\,abc$.
12. $a^3 - b^3 + c^3 + 3\,abc$.
13. $\Sigma(a - b)(a + b - c)^2$.
14. $\Sigma(a - b)(a + b - 2\,c)^2$.

15. $\Sigma a^3(b+c)+abc\Sigma a$. 16. $4a^2b^2-(a^2+b^2-c^2)^2$.

17. $(\Sigma x)^3+\Sigma x^3-\Sigma(x+y)^3$.

18. $(\Sigma x)^4+\Sigma x^4-\Sigma(x+y)^4$.

19. $a^3+b^3+c^3-3abc$. One factor must be $a\pm b$ or Σa. (Why?)

20. $\Sigma a^3+3(a+b)(b+c)(c+a)$.

21. $\Sigma(a-b)^3$, Σ referring to a, b, c.

22. $\Sigma(a^2-b^2)^3$, Σ referring to a, b, c.

23. $(\Sigma a)(\Sigma ab)-(a+b)(b+c)(c+a)$.

24. $x(y^3-z^3)+y(z^3-x^3)+z(x^3-y^3)$.

25. $(s-a)^2+(s-b)^2-(s-c)^2$, where $s=a+b+c$.

26. $\Sigma a^2(b-c)$, *i.e.*, $a^2(b-c)+b^2(c-a)+c^2(a-b)$.

27. $(x-a)^2(b-c)+(x-b)^2(c-a)+(x-c)^2(a-b)$.

28. $(a+b)(a-b)^3+(b+c)(b-c)^3+(c+a)(c-a)^3$.

29. $\Sigma(a-b)(a^2+b^2)$, *i.e.*, $(a-b)(a^2+b^2)$
$+(b-c)(b^2+c^2)+(c-a)(c^2+a^2)$.

30. $(x+y+z)^3-(x+y-z)^3-(y+z-x)^3$
$-(z+x-y)^3$.

31. $(x-a)(x-b)(a-b)+(x-b)(x-c)(b-c)$
$+(x-c)(x-a)(c-a)$.

32. $a(b+c)(b^2+c^2-a^2)+b(c+a)(c^2+a^2-b^2)$
$+c(a+b)(a^2+b^2-c^2)$.

33. Find three factors, only, of $(x-y)^{2n+1}+(y-z)^{2n+1}$
$+(z-x)^{2n+1}$.

34. Also of $(\Sigma x)^{2n+1}-\Sigma x^{2n+1}$, Σ referring to x, y, z.

The type $\Sigma x^2 + \Sigma\, 2\, xy$, the square of a polynomial.

Since $(\Sigma x)^2 \equiv \Sigma x^2 + \Sigma\, 2\, xy$ (p. 385), it follows that expressions in the form of $\Sigma x^2 + \Sigma\, 2\, xy$ can be factored.

E.g., $x^2 + y^2 + z^2 + 2\,xy + 2\,yz + 2\,zx = (x + y + z)^2$.

Check. $9 = 3^2$.

Similarly, $4\,a^2 + 9\,b^2 + c^2 - 12\,ab - 6\,bc + 4\,ca = (2\,a - 3\,b + c)^2$.

Check. Let $a = b = 1$, $c = 2$. Then $1 = 1^2$.

EXERCISES. CLXXVI.

Factor the following:

1. $4\,x^2 + 9\,y^2 + 1 + 12\,xy + 6\,y + 4\,x$.

2. $1 + 4\,a^2 + 9\,b^4 + 4\,a + 6\,b^2 + 12\,ab^2$.

3. $4 + 16\,a^2 + 25\,b^4 + 16\,a + 20\,b^2 + 40\,ab^2$.

4. $5\,x^2 + y^4 + 9 + 2\,xy^2\sqrt{5} + 6\,x\sqrt{5} + 6\,y^2$.

5. $a^2 + b^4 + 9\,c^6 + d^2 + 2\,ad + 6\,ac^3 + 2\,ab^2 + 6\,b^2c^3$
   $+ 2\,b^2d + 6\,c^3d$.

The following miscellaneous exercises review some of the elementary cases of factoring.

6. $4\,x^4 + 8\,x^2y^2 + 9\,y^4$.
7. $4\,x^4 - 4\,x^2y^2 + 9\,y^4$.
8. $5\,x^2 + 5 + 3\,x + 3\,x^3$.
9. $x^2y^2 - 4\,x^2 + 4 - y^2$.
10. $a^4 + b^4 - 16 + 2\,a^2b^2$.
11. $1 + 4\,xy - 4\,y^2 - x^2$.
12. $x^2a + ay^2 + by^2 + bx^2$.
13. $x^6 + 144 - 16\,x^2 - 9\,x^4$.
14. $by + 2\,bx + 2\,ay + 4\,ax$.
15. $x^2 + wxy - 4\,wy^2 - 4\,xy$.
16. $x^2 + 10\,x + y^2 + 10\,y + 25 + 2\,xy$.
17. $x^2 - 12\,x + 4\,y^2 + 36 - 24\,y + 4\,xy$.

V. GENERAL LAWS GOVERNING THE SOLUTION OF EQUATIONS (p. 152).

Theorem. *If the same quantity is added to or subtracted from the two members of an equation, the result is an equivalent equation.*

Given $A = B$, an equation, and C any quantity.

To prove that $A \pm C = B \pm C$ is an equivalent equation.

Proof. 1. If for certain values of the unknown quantities, A and B take numerically equal values, it is evident that $A \pm C$ and $B \pm C$ must also take equal values.

2. $\therefore$ any root of $A = B$ is also a root of $A \pm C = B \pm C$.

3. If for certain values of the unknown quantities $A \pm C$ and $B \pm C$ take numerically equal values, it is evident that A and B must also take equal values, because we obtain their values by subtracting from the equal values of $A \pm C$ and $B \pm C$ the same number.

4. $\therefore$ any root of $A \pm C = B \pm C$ is also a root of $A = B$.

5. Since any root of $A = B$ is also a root of $A \pm C = B \pm C$, and any root of $A \pm C = B \pm C$ is also a root of $A = B$, it follows that the two equations are equivalent.

COROLLARY. *Every equation can be put into the form* $A = 0$.

For in subtracting from the two members of an equation a quantity equal to its second member, an equivalent equation is obtained of which the second member is 0.

Theorem. *If the two members of an equation are multiplied or divided by the same quantity, which is neither zero nor capable of becoming zero or infinitely great, the result is an equivalent equation.*

Given the equation $A = B$, and the factor C, which by the conditions cannot be 0 or infinitely great.

To prove that $AC = BC$ is an equation equivalent to $A = B$.

Proof. 1. $\because A = B, \quad \therefore A - B = 0.$

2. $\therefore C(A - B) = 0.$ Ax. 7

3. Every root of 1, making $A - B = 0$, must also make $C(A - B) = 0$, because C is not infinitely large.

If $C = \infty$, then $C(A - B)$ would be undetermined, by § 172.

4. $\therefore$ every root of 1 is a root of 2.

5. Conversely, every root of 2, making $C(A - B) = 0$, must also make $A - B = 0$, because C is not zero.

If $C = 0$, then $A - B$ would equal $\frac{0}{0}$, an undetermined quantity by § 168.

6. $\therefore$ every root of 2 is a root of 1.

7. From 4 and 6, the equations are equivalent.

The necessity for the limitations on the value of the multiplier is evident from a simple example. In the equation

$$x^2 + x = 0$$

we cannot expect to get an equivalent equation by dividing by x or multiplying by $\frac{1}{x}$, for $x = 0$ and $\frac{1}{x} = \infty$, and the simple equation

$$x + 1 = 0$$

is evidently not equivalent to the quadratic equation $x^2 + x = 0$.

Theorem. *If the two members of a rational fractional equation are multiplied by the lowest common denominator of the fractions, the result is, in general, an equivalent equation.*

Proof. 1. The equation can be transformed so that the second member is 0.

2. The first member, being a rational fractional expression, can then be reduced to the form $\frac{A}{B}$, in which B is the lowest common denominator of the fractions, after they are added and reduced, and hence is prime to A.

3. $\therefore$ the equation can be reduced to $\frac{A}{B} = 0$, the members of which it is proposed to multiply by B.

4. There can be no values of the unknown quantity which make A and B zero at the same time, since B is prime to A.

5. $\therefore$ in order that $\frac{A}{B} = 0$ it is necessary and sufficient that $A = 0$. $\therefore$ the equation $A = 0$ is equivalent to the equation $\frac{A}{B} = 0$.

To illustrate the theorem, consider the following cases:

In the equation $\frac{4}{x} = x$, it is legitimate to multiply by x, giving $4 = x^2$, whence $x = +2$, or -2, either root satisfying the original equation.

But in the equation $\frac{x^2}{x} = 1$, it is not legitimate to multiply by x, for then $x^2 = x$, and $x^2 - x = 0$, whence $x(x-1) = 0$, $x = 0$, or 1. But $x = 0$ does not satisfy the original equation, because $\frac{0}{0}$ does not necessarily equal 1.

Similarly we cannot solve $\frac{x^2 + 5x + 6}{x + 2} = 3$ by multiplying by $x + 2$.

Theorem. *If both members of an equation are raised to any integral power, the resulting equation contains all of the roots of the given equation, but in general is not equivalent to it.*

Given the equation $A = B$.

To prove that the equation $A^m = B^m$ contains all of the roots of the equation $A = B$, but in general is not equivalent to it.

Proof. 1. From $A = B$ it follows that $A - B = 0$.

2. From $A^m = B^m$ it follows that $A^m - B^m = 0$.

3. But whether m is odd or even, $A^m - B^m$ contains the factor $A - B$.

4. ∴ equation 2 becomes

$$(A - B)(A^{m-1} + A^{m-2}B + \cdots) = 0,$$

and is satisfied by $A = B$.

5. ∴ equation 2 contains the roots of equation 1.

6. But from equation 4,

$$A^{m-1} + A^{m-2}B + \cdots = 0,$$

and hence $A^m = B^m$ contains other roots than $A = B$, and hence is not equivalent to it.

To illustrate let $x = 2$;

squaring, $x^2 = 4$,

an equation containing the root

$$x = 2,$$

but also the extraneous root

$$x = -2.$$

If we cube, $x^3 = 8$,

and this again contains the root

$$x = 2,$$

but it also contains the extraneous roots

$$x = -1 \pm \sqrt{-3}.$$

VI. EQUIVALENT SYSTEMS OF EQUATIONS (p. 185).

It has been shown that the solution of a system of equations is made to depend upon the solution of a second system derived from the first. But it has not yet been shown that extraneous roots are not introduced by this operation.

Two systems of equations, each having the same roots as the other, are called **equivalent systems**.

Theorem. *Given a system of two equations*

$$(1)\quad \mathrm{f}(\mathrm{x}, \mathrm{y}) = 0, \quad \mathrm{F}(\mathrm{x}, \mathrm{y}) = 0,$$

and a, b *two numbers* ($\mathrm{b} \neq 0$), *then*

$$(2)\quad \mathrm{a} \cdot \mathrm{f}(\mathrm{x}, \mathrm{y}) + \mathrm{b} \cdot \mathrm{F}(\mathrm{x}, \mathrm{y}) = 0, \quad \mathrm{f}(\mathrm{x}, \mathrm{y}) = 0$$

is an equivalent system.

Proof. 1. ∵ a solution of system (1) makes both $f(x, y)$ and $F(x, y)$ equal zero, it makes both $a \cdot f(x, y)$ and $b \cdot F(x, y)$ equal zero, and hence satisfies system (2).

2. ∵ a solution of system (2) makes

$$f(x, y) = 0$$

and $$a \cdot f(x, y) + b \cdot F(x, y) = 0,$$

it must therefore make $a \cdot f(x, y) = 0$,

and hence, $$b \cdot F(x, y) = 0,$$

and hence, $$F(x, y) = 0, \because b \neq 0.$$

Hence, it is a solution of system (1).

This theorem justifies the solution of two simultaneous linear equations by addition, subtraction, and substitution. For it shows that we may multiply the members by any numbers (a, b), add or subtract (since b may be negative) the equations member for member, and combine this result with the equation $f(x, y) = 0$.

VII. DETERMINANTS (p. 198).

The practical solution of simultaneous linear equations, while possible by the methods already given, is frequently tedious. For this reason mathematicians often resort to a simpler method, that of **determinants.**

The theory of determinants is comparatively modern, and although it is not practicable to enter into the subject at any length at this time, the elementary notions are so simple and so helpful, and the applications so common, that a brief presentation of the subject will be of value.

The symbol $\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}$ is merely another way of writing $a_1b_2 - a_2b_1$. The symbol is called a **determinant**, and the letters a_1, a_2, b_1, b_2 are called its *elements*.

This is a determinant of the *second order*; *i.e.*, there are *two* elements on each side of the square. It will be noticed that the expanded form is simply the difference of the diagonals.

In a determinant, the horizontal lines of elements are called *rows*, the vertical ones *columns*.

In the above determinant the rows are a_1, b_1 and a_2, b_2; the columns are $\begin{matrix} a_1 \\ a_2 \end{matrix}$ and $\begin{matrix} b_1 \\ b_2 \end{matrix}$.

When the determinant $\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}$ is written in the form $a_1b_2 - a_2b_1$ it is said to be *expanded*.

It is understood that the expanded form is to be simplified in all cases. *E.g.*; while $\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} = 2 \cdot 7 - 5 \cdot 3$, the result should be stated as -1.

EXERCISES. CLXXVII.

1. Expand the following determinants:

$$\begin{vmatrix} a & b \\ y & x \end{vmatrix}, \quad \begin{vmatrix} x & y \\ b & a \end{vmatrix}, \quad \begin{vmatrix} a & y \\ b & x \end{vmatrix}, \quad \begin{vmatrix} x & b \\ y & a \end{vmatrix}.$$

2. Also the following:

$$\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix}, \quad \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}, \quad \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}.$$

3. Also the following:

$$\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}, \quad \begin{vmatrix} 5 & 0 \\ 7 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 0 \\ 18 & 30 \end{vmatrix}, \quad \begin{vmatrix} 0 & 19 \\ 0 & 27 \end{vmatrix}, \quad \begin{vmatrix} 0 & a \\ 0 & b \end{vmatrix}, \quad \begin{vmatrix} 0 & 0 \\ a & b \end{vmatrix}.$$

4. From exs. 1 and 2, state what changes can be made in a determinant of the second order without changing its value.

5. From ex. 3, what is the value of a determinant if either a row or a column is made up of zeros?

6. Expand the following determinants:

$$\begin{vmatrix} 12 & 6 \\ 5 & 7 \end{vmatrix}, \quad \begin{vmatrix} 8 & 1 \\ 2 & 3 \end{vmatrix}, \quad \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix}.$$

7. Expand the following determinants:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

8. From ex. 7, state the effect on the value of a determinant of the second order of changing the rows into columns and the columns into rows.

9. Expand the following determinants:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 + b_1 & b_1 \\ a_2 + b_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 - b_1 & b_1 \\ a_2 - b_2 & b_2 \end{vmatrix}.$$

10. From ex. 9, state the effect on the value of a determinant of the second order, of increasing the elements of one column by the corresponding elements of another, or of diminishing the elements of one column by the corresponding elements of another.

11. Expand the following determinants:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \qquad \begin{vmatrix} 2\,a_1 & b_1 \\ 2\,a_2 & b_2 \end{vmatrix}, \qquad \begin{vmatrix} ma_1 & b_1 \\ ma_2 & b_2 \end{vmatrix}.$$

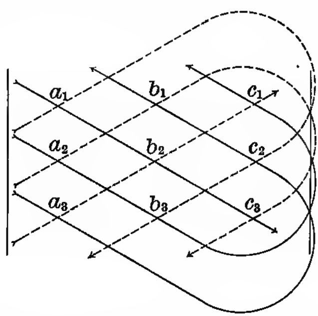
12. From ex. 11, state the effect on the value of a determinant of the second order, of multiplying the elements of a column by any number.

A determinant of the second order is no more easily written than is its expanded form. But one of the third order (one with three elements on the side of the square) is materially more condensed than is its expanded form.

The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is the general form of a determinant of the third order, and it stands for

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

The expansion of a *third* order determinant is easily written by following the arrows in this arrangement. *This method of expansion holds only for determinants of the second and third orders,* all that we shall treat in this work.



The fact that the student is rarely called upon, in elementary algebra, to solve a system of more than three simultaneous linear equations makes it undesirable to enter, at this time, upon the theory of determinants of an order higher than the third.

EXERCISES. CLXXVIII.

1. Expand the following determinants, the rows of the first being the columns of the second:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}.$$

2. Also the following:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

3. From exs. 1 and 2, state the effect on the value of a determinant of the third order, of changing the rows into columns and the columns into rows.

4. Expand the following determinants:

$$\begin{vmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 7 & 8 & 9 \end{vmatrix}, \qquad \begin{vmatrix} 2+4 & 4 & 6 \\ 1+3 & 3 & 5 \\ 7+8 & 8 & 9 \end{vmatrix}, \qquad \begin{vmatrix} 6 & 4 & 6 \\ 4 & 3 & 5 \\ 15 & 8 & 9 \end{vmatrix}.$$

5. Also the following:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad \begin{vmatrix} a_1+b_1 & b_1 & c_1 \\ a_2+b_2 & b_2 & c_2 \\ a_3+b_3 & b_3 & c_3 \end{vmatrix}.$$

6. From exs. 4, 5, state the effect on the value of a determinant of the third order, of increasing the elements of one column by the corresponding elements of another.

7. Expand the following determinants:

$$\begin{vmatrix} 0 & a & b \\ 0 & c & d \\ 0 & e & f \end{vmatrix}, \qquad \begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix}.$$

8. From ex. 7, what is the value of a determinant of the third order if either a row or a column is made up entirely of zeros?

9. Expand the following determinants:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad \begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix}.$$

10. From ex. 9, state the effect on the value of a determinant of the third order of multiplying the elements of a column by any number.

In the preceding exercises certain general theorems have been proved by the student for determinants of the second and third orders. These will now be presented formally, the proof, however, referring only to determinants of these orders.

Theorem. *The value of a determinant is unchanged if the rows are changed to columns and* vice versa.

Given the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

To prove that it equals the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Proof. Each expands into

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

The proof for the determinant of the second order is left for the student. Take the determinants

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \text{ and expand.}$$

Theorem. *If each element of a column (or row) of a determinant is multiplied by any factor, the determinant is multiplied by that factor.*

Proof. Consider the determinants $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

By the law of expansion every term of the expanded form contains one a (*i.e.*, a_1, a_2, or a_3) and only one; hence, if every a is multiplied by m, the m will appear once and only once as a factor of every term of the expanded form. Similarly, for any column or row.

For example, consider the determinant $\begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix}$. If we multiply either column or either row by 2, the determinant is multiplied by 2. This is seen by expanding

$$\begin{vmatrix} 6 & 2 \\ 10 & 7 \end{vmatrix}, \quad \begin{vmatrix} 3 & 4 \\ 5 & 14 \end{vmatrix}, \quad \begin{vmatrix} 6 & 4 \\ 5 & 7 \end{vmatrix}, \quad \begin{vmatrix} 3 & 2 \\ 10 & 14 \end{vmatrix},$$

the results being $42 - 20 = 22$ in each case, while the original determinant equals $21 - 10 = 11$.

Theorem. *If a column (or row) is made up entirely of zeros, the determinant equals zero.*

Proof. As in the preceding theorem, every term of the expanded form contains an a; hence, if every a is zero, the expanded form vanishes. Similarly, for any other row or column.

This is seen, for a special case, in the determinant $\begin{vmatrix} 1 & 2 & 0 \\ 4 & 7 & 0 \\ 3 & 6 & 0 \end{vmatrix}$ which expands into $1 \cdot 7 \cdot 0 + 4 \cdot 6 \cdot 0 + 3 \cdot 0 \cdot 2 - 1 \cdot 6 \cdot 0 - 4 \cdot 2 \cdot 0 - 3 \cdot 7 \cdot 0 = 0$.

The same may be seen in the case of determinants like

$$\begin{vmatrix} 2 & 4 \\ 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 3 \\ 0 & 2 \end{vmatrix}, \text{ etc.}$$

Theorem. *If each element of a column is multiplied by any number, and added to the corresponding element of any other column, the value of the determinant is not changed.*

Given the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

To prove that it equals the determinant $\begin{vmatrix} a_1 + mb_1 & b_1 & c_1 \\ a_2 + mb_2 & b_2 & c_2 \\ a_3 + mb_3 & b_3 & c_3 \end{vmatrix}$.

Proof. Expanding the second determinant, it equals

$$(a_1 + mb_1)\,b_2c_3 + (a_2 + mb_2)\,b_3c_1 + (a_3 + mb_3)\,b_1c_2$$
$$-(a_3 + mb_3)\,b_2c_1 - (a_2 + mb_2)\,b_1c_3 - (a_1 + mb_1)\,b_3c_2$$

which equals

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$$

the other terms all cancelling out.

That is, the two determinants are equal.

The proof is the same whatever columns (or rows) are taken, and for the second order as well as for the third.

COROLLARIES. 1. *The elements of any column (or row) may be added to or subtracted from the corresponding elements of any other column (or row) without changing the value of the determinant.*

For m may equal 1 or -1.

2. *If two columns (or rows) are identical, the determinant equals zero.*

For, if the elements of one are subtracted from the corresponding elements of the other, a column (or row) will be composed of zeros.

3. *If the elements of one column are the same multiples of the corresponding elements of another, the determinant equals zero.* (*Why?*)

Illustrative problems. 1. Expand the determinant

$$\begin{vmatrix} 27 & 25 \\ 42 & 41 \end{vmatrix}.$$

Subtracting the second column from the first, the determinant equals $\begin{vmatrix} 2 & 25 \\ 1 & 41 \end{vmatrix} = 82 - 25 = 57.$

This is much easier than finding the value of $27 \cdot 41 - 42 \cdot 25$.

2. Expand $\begin{vmatrix} 8 & 21 \\ 6 & 15 \end{vmatrix}$.

Factoring the second column by 3, and then subtracting it from the first, we have

$$3\begin{vmatrix} 8 & 7 \\ 6 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & 7 \\ 1 & 5 \end{vmatrix} = 3(5 - 7) = -6.$$

3. Expand $\begin{vmatrix} 10 & 17 & 3 \\ 20 & 16 & 4 \\ 30 & 15 & 5 \end{vmatrix}$.

Subtracting the first row from the second and that from the third,

$$\begin{vmatrix} 10 & 17 & 3 \\ 10 & -1 & 1 \\ 10 & -1 & 1 \end{vmatrix} = 0. \qquad \text{(Why ?)}$$

General directions for expanding determinants.

1. *Remove factors from columns or rows.*

2. *Endeavor to make the absolute values of the elements as small as possible by subtracting corresponding elements of rows or columns, or multiples of those elements.*

3. *Endeavor to bring in as many zeros as possible.*

4. *Endeavor to make the elements of two columns (or rows) identical, so that the determinant may be seen to be zero (if that is its value) without expanding.*

5. *After thus simplifying as much as possible, expand.*

EXERCISES. CLXXIX.

Expand the determinants or prove the identities as indicated.

1. $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix}$.　　2. $\begin{vmatrix} 97 & 96 \\ 63 & 62 \end{vmatrix}$.　　3. $\begin{vmatrix} 3 & 9 \\ 13 & 39 \end{vmatrix}$.

4. $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{vmatrix}$.　　5. $\begin{vmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 97 & 5 & 17 \end{vmatrix}$.　　6. $\begin{vmatrix} a & b & a \\ a^2 & b^2 & ab \\ a^3 & b^3 & ab^2 \end{vmatrix}$.

7. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \equiv (a-b)(b-c)(c-a)$.

8. $\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix} \equiv 2\,abc$.　　9. $\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} \equiv 4\,abc$.

10. $\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} \equiv 4\,a^2b^2c^2$.

Application of determinants to the solution of a system of two linear equations.

On solving the system

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

the roots are found to be

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1},$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

It is at once seen that

1. *Each denominator is the determinant* $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, *made up of the coefficients of* x *and* y.

2. *The numerator for* x *is the same determinant with* c *put for* a (*the coefficient of* x).

3. *The numerator for* y *is also the same determinant, with* c *put for* b (*the coefficient of* y).

Illustrative problem. Solve the system

$$3x + 11y = 64,$$
$$17x - 7y = 16.$$

Here $x = \dfrac{\begin{vmatrix} 64 & 11 \\ 16 & -7 \end{vmatrix}}{\begin{vmatrix} 3 & 11 \\ 17 & -7 \end{vmatrix}} = \dfrac{16\begin{vmatrix} 4 & 11 \\ 1 & -7 \end{vmatrix}}{\begin{vmatrix} 14 & 11 \\ 10 & -7 \end{vmatrix}} = \dfrac{16(-28-11)}{2(-49-55)} = \dfrac{8 \cdot -39}{-104} = 3.$

$\therefore y = 5$, by substitution.

EXERCISES. CLXXX.

Solve by determinants, checking in the usual way.

1. $\dfrac{24}{x} + \dfrac{93}{y} = 4\frac{1}{7}.$
   $\dfrac{42}{x} - \dfrac{31}{y} = 1.$

2. $\dfrac{49}{x} - \dfrac{56}{y} = 7.$
   $\dfrac{56}{x} - \dfrac{49}{y} = 21\frac{1}{8}.$

3. $23x - 30y = 2.$
   $10x + 7y = 61.$

4. $23x + 10y = 252.$
   $19x + 17y = 154.7.$

5. $41x - 37y = 4.$
   $43x + 39y = 82.$

6. $235x - 234y = 236.$
   $411x + 410y = 412.$

7. $52x - 39y = 13.$
   $18x + 18y = 15.$

8. $0.5x - 0.3y = 0.021.$
   $0.6x + 2y = 0.332.$

Three linear equations with three unknown quantities. On solving the system

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3,$$

the roots are found to be

$$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2},$$

$$y = \frac{a_1d_2c_3 + a_2d_3c_1 + a_3d_1c_2 - a_3d_2c_1 - a_2d_1c_3 - a_1d_3c_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2},$$

$$z = \frac{a_1b_2d_3 + a_2b_3d_1 + a_3b_1d_2 - a_3b_2d_1 - a_2b_1d_3 - a_1b_3d_2}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}.$$

It is at once seen that the same law already set forth holds here, and that the roots may be expressed thus:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

It is thus seen that the roots of three linear simultaneous equations can be written down, in the determinant form, at sight. It then becomes merely a matter of simplifying. Whether it is easier to solve by determinants, or to solve by elimination through addition and subtraction, depends largely on the size of the coefficients. If the coefficients are small, there is usually no advantage in using determinants; if they are large there is often a great gain. In the problems on the next two pages the coefficients are not in general large enough to make it worth while to use determinants except for practice.

Illustrative problem. Solve the system

$$11x + 9y + 33z = 52,$$
$$13x + 11y + 71z = 0,$$
$$17x + 15y + 80z = 30.$$

The common denominator for x, y, z is

$$\begin{vmatrix} 11 & 9 & 33 \\ 13 & 11 & 71 \\ 17 & 15 & 80 \end{vmatrix} = \begin{vmatrix} 2 & 9 & 33 \\ 2 & 11 & 71 \\ 2 & 15 & 80 \end{vmatrix} = \begin{vmatrix} 2 & 9 & 33 \\ 0 & 2 & 38 \\ 0 & 4 & 9 \end{vmatrix} = 2 \cdot 2 \begin{vmatrix} 1 & 9 & 33 \\ 0 & 1 & 19 \\ 0 & 4 & 9 \end{vmatrix}$$
$$= 2 \cdot 2 (9 - 4 \cdot 19) = -4 \cdot 67.$$

(How is the second determinant obtained from the first? the third from the second? and so on.)

The numerator for x is

$$\begin{vmatrix} 52 & 9 & 33 \\ 0 & 11 & 71 \\ 20 & 15 & 80 \end{vmatrix} = 2 \cdot 5 \begin{vmatrix} 26 & 9 & 33 \\ 0 & 11 & 71 \\ 3 & 3 & 16 \end{vmatrix} = 10 \begin{vmatrix} 26 & -17 & 7 \\ 0 & 11 & 71 \\ 3 & 0 & 13 \end{vmatrix}$$
$$= 10 (26 \cdot 11 \cdot 13 - 3 \cdot 17 \cdot 71 - 3 \cdot 11 \cdot 7)$$
$$= -10 \cdot 134.$$
$$\therefore x = \frac{-10 \cdot 134}{-4 \cdot 67} = 5.$$

The numerator for y is

$$\begin{vmatrix} 11 & 52 & 33 \\ 13 & 0 & 71 \\ 17 & 30 & 80 \end{vmatrix} = 2 \begin{vmatrix} 11 & 26 & 0 \\ 13 & 0 & 32 \\ 17 & 15 & -29 \end{vmatrix} = 2(17 \cdot 32 \cdot 26 - 29 \cdot 13 \cdot 26 - 11 \cdot 15 \cdot 32)$$
$$= -2 \quad 938.$$
$$\therefore y = \frac{-2 \cdot 938}{-4 \cdot 67} = 7.$$

We may now find z by substitution. Or the numerator for z is

$$\begin{vmatrix} 11 & 9 & 52 \\ 13 & 11 & 0 \\ 17 & 15 & 30 \end{vmatrix} = 2 \begin{vmatrix} 2 & 9 & 26 \\ 2 & 11 & 0 \\ 2 & 15 & 15 \end{vmatrix} = 2 \cdot 2 \begin{vmatrix} 0 & -2 & 26 \\ 1 & 11 & 0 \\ 0 & 4 & 15 \end{vmatrix},$$

by factoring by 2 and subtracting the second row from each of the others. This equals $4 \cdot 134$.

$$\therefore z = \frac{4 \cdot 134}{-4 \cdot 67} = -2.$$

EXERCISES. CLXXXI.

Solve by determinants, checking all numerical results in the usual way.

1. $x + y = 10.$
   $y + z = 10.$
   $x + z = 6.$

2. $5x - 3z = 3.$
   $2x + y = 5.$
   $3y + z = 7.5.$

3. $\frac{x}{a} + \frac{y}{b} = 3.$
   $\frac{y}{b} + \frac{c}{z} = 5.$
   $\frac{x}{a} + \frac{z}{c} = 4.$

4. $\frac{x}{4} - y + \frac{z}{3} = 3.25.$
   $\frac{x}{3} + \frac{y}{2} - z = 7.5.$
   $\frac{x}{2} - \frac{y}{3} + \frac{z}{4} = 4.$

5. $12x + 7y = 109.$
   $5y - 2z = 11.$
   $4x + 3z = 26.$

6. $7x - 3y - 2z = 16.$
   $2x - 5y + 3z = 39.$
   $5x + y + 5z = 31.$

7. $4x + 9y + z = 16.$
   $2x + 3y + z = 4.$
   $x + y + z = 1.$

8. $3x + 3y + 3z = 144.$
   $7x + 7y + 5z = 306.$
   $9x + y - z = 154.$

9. $p^2x + q^2y + r^2z = s^2.$
   $p^3x + q^3y + r^3z = s^3.$
   $px + qy + rz = s.$

10. $a^2x + b^2y + c^2z = a + b + c.$
    $ax + by + cz = 1.$
    $x + y + z = 0.$

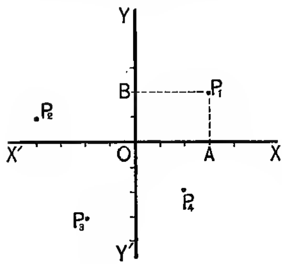
11. $3x + 4y + 2z = 47.$
    $5x - 3y + 7z = 41.$
    $7x - 2y - 5z = 24.$

12. $2x - 3y + 4z = -18.$
    $3x + 4y - 5z = 34.$
    $x + y + z = 0.$

13. $123x + 17y - 139z = 1.$
    $51x + 37y - 97z = -9.$
    $5x + 31y - 35z = 1.$

VIII. GRAPHIC REPRESENTATION OF LINEAR EQUATIONS (p. 202).

In the annexed figure the two lines XX' and YY', intersecting at right angles at O, are called *rectangular axes.*



A segment, OA, on OX is called the *abscissa* of any point, as P_1, on a perpendicular to XX' at A.

A segment, OB, on OY is called the *ordinate* of any point, as P_1, on a perpendicular to YY' at B.

The abscissa and ordinate together are called the *coördinates* of the point, the abscissa always being named first.

Abscissas to the right of O are called positive, those to the left negative. Ordinates above O are called positive, those below negative.

E.g., in the figure the coördinates of P_1 are 3, 2; those of P_2 are -4, 1; those of P_3 are -2, -3; those of P_4 are 2, -2; those of A are 3, 0; those of O are 0, 0. The ordinate of any point on XX' is evidently 0, and the abscissa of any point on YY' is 0 also.

Hence, when the axes are given a point in their plane is fixed when its coördinates are known. Conversely, when a point is fixed its coördinates with respect to any given axes are evidently fixed also.

A point, as P_1, is designated by its coördinates.

Thus, P_1 is designated by (3, 2), P_2 by $(-4, 1)$, P_3 by $(-2, -3)$, and P_4 by $(2, -2)$.

If the coördinates are unknown, they are designated by x and y, the point being designated by the symbol (x, y). That is, if it is desired to speak of two general points, as we speak of two unknown quantities in algebra, they may be designated either as P_1, P_2, or as (x_1, y_1), (x_2, y_2).

EXERCISES. CLXXXII.

In each exercise draw a pair of rectangular axes and take $\frac{1}{8}$ inch as the unit of measure for laying off the coördinates.

1. Represent the points (2, 5), (– 4, – 7).

2. Also (5, 0), (0, 5).

3. Also (0, 0), (2, 2), (– 4, – 4). Join these. Do they, or do they not, lie in the same straight line?

4. Similarly for the points (– 3, 0), (0, 3), (3, 0), (0, – 3).

5. What kind of a figure is formed by joining, in order, the points (2, 4), (– 2, 4), (– 2, – 4), (2, – 4)?

6. Also the points (6, 3), (3, 3), (3, – 5)?

The graph of an equation. The equation $y = x - 1$ is satisfied if

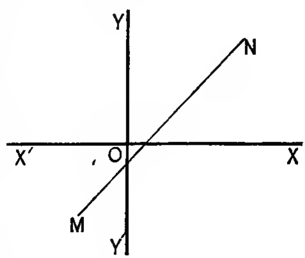
$$x = \cdots - 1, \quad 0, \ 1, \ 2, \cdots$$

while

$$y = \cdots - 2, \ -1, \ 0, \ 1, \cdots.$$

The points (– 1, – 2), (0, – 1), (1, 0), ··· may, therefore, be thought of as lying on a line representing this equation.

Hence, in the figure the line *MN* is considered the graphic representation of the equation $y = x - 1$. Such a graphic representation is called the **graph of the equation.**

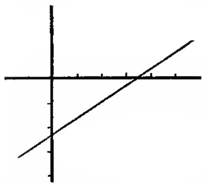


The word *locus* is sometimes used instead of *graph*. *Locus* is a Latin word meaning *place*, and the line is the place where the points are found. Strictly, therefore, it is the *graph of the equation* and the *locus of the points* which we have.

In a case like that of $y = x - 1$, y is a function of x. Hence, *the abscissas represent the variable* x, *and the ordinates represent the function.*

A simple equation containing two unknown quantities can always be represented graphically by a straight line.

This is the reason why it is called a *linear* equation, a term which has, however, been extended to include all equations of the first degree.



Hence, it is necessary to locate only two points to determine the graph of a simple equation.

The easiest plan usually consists in letting $x = 0$ and finding the corresponding value of y; then letting $y = 0$ and finding the corresponding value of x.

E.g., to draw the graph of the equation $2x - 3y = 7$. If $x = 0$, then $y = -\frac{7}{3}$; if $y = 0$, then $x = \frac{7}{2}$. Hence, draw a line through $(0, -\frac{7}{3})$ and $(\frac{7}{2}, 0)$, as in the figure.

Since the line represents the equation it is evident that the coördinates of any point on the graph satisfy the equation. That is,

a single *linear equation* involving two unknown quantities has an infinite number of *roots*.	a single *straight line* has an infinite number of *points*.

EXERCISES. CLXXXIII.

Draw the graphs of the following equations:

1. $x - y = 0$.
2. $x + y = 0$.
3. $2x - y = 8$.
4. $7x - 4y = 10$.
5. $-2x + 3y = 5$.
6. $16x + 2y = 7$.

It is also apparent that although a single

linear equation has an infinite number of *roots*, *two linear equations involving two unknown quantities* have in general but one common pair of *roots*.

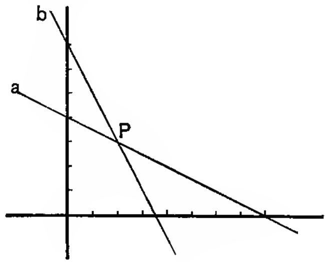
straight line has an infinite number of points, *two straight lines* have in general but one common *point*.

For example, the two equations

$$(a)\ x + 2y = 8$$

$$(b)\ 2x + y = 7$$

have the graphs *a* and *b*.



The two *equations* have the common pair of *roots* whose *values* are

$$x = 2,\ y = 3.$$

The two *graphs* have the common *point* whose *coördinates* are

$$x = 2,\ y = 3.$$

Hence, two linear equations involving two unknown quantities can be solved by means of graphs, although this is not advisable in practice.

EXERCISES. CLXXXIV.

Draw the graphs of the following pairs of equations and show that the intersections represent the solutions.

1. $x + y = 0.$
$x - y = 0.$

2. $5x + 2y = 16.$
$3x - y = 3.$

3. $3x + 5y = 12.$
$x + y = 2.$

4. $5x + 7y = 11.$
$7x + 5y = 1.$

Discussion of Solutions.

While in general

two *linear equations* involving *two unknown quantities* have a single common pair of *roots*, they may not, for they may be *inconsistent* or they may be *indeterminate*.

two *straight lines* in a *plane* have a single common *point*, they may not, for they may be *parallel* or they may *coincide*.

E.g., the *equations*

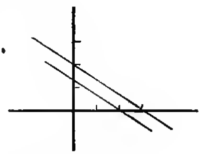
$$2x + 3y = 6$$
$$2x + 3y = 4$$

have evidently no common pair of roots, since that would make $6 = 4$. Hence, they are called *inconsistent*.

E.g., the *graphs* of

$$2x + 3y = 6$$
$$2x + 3y = 4$$

are *parallel*.



Also the *equations*

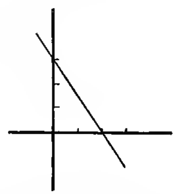
$$\frac{x}{2} + \frac{y}{3} = 1$$
$$3x + 2y = 6$$

have no determinate solution, for the members of the second are merely six times those of the first. They are, therefore, *equivalent* equations, and reduce to a single *indeterminate equation*.

Also the *graphs* of

$$\frac{x}{2} + \frac{y}{3} = 1$$
$$3x + 2y = 6$$

coincide.



It is a mistake quite often made by students to think that it is possible to solve *any* two equations like

$$8x + 4y = 5.$$
$$10x + 5y = 8.$$

They may not be simultaneous, as in this case.

EXERCISES. CLXXXV.

Discuss the following systems of equations, solving if possible and drawing graphs in all cases.

1. $x + y = 1.$
   $x - y = 1.$

2. $x + y = 4.$
   $x - y = 6.$

3. $y + 3x = 6.$
   $y = 4.$

4. $x + 3y = 6.$
   $x = 4.$

5. $2x + 3y = 8.$
   $\frac{x}{2} + \frac{y}{3} = 1.$

6. $6x + 10y = 10.$
   $\frac{x}{5} + \frac{y}{3} = 2.$

7. $x - 1.5y = 10.$
   $2x - 3y = 5.$

8. $7x + 35y = 15.$
   $3x + 15y = 8.$

9. $10x + 6y = 5.$
   $\frac{x}{3} + \frac{y}{5} = \frac{1}{6}.$

10. $8x - 12y = 2.$
    $\frac{x}{3} - \frac{y}{2} = \frac{1}{12}.$

11. $6x + 0.8y = 10.$
    $3x + 0.4y = 6.$

12. $1.02x - 0.01y = 20.1.$
    $0.2x - 0.1y = 1.$

In the same way it may happen that three equations involving three unknown quantities may be inconsistent or indeterminate.

Illustrative problems. 1. Solve the following system:

$$\begin{aligned} 1.\quad 9x + 6y + 3z &= 30. \\ 2.\quad 6x + 4y + 2z &= 20. \\ 3.\quad x + 2y + 3z &= 14. \end{aligned}$$

Equations 1 and 2 are easily seen to be equivalent. Hence, there are only two independent equations, involving three unknown quantities, and they are indeterminate.

2. Solve the following system:

$$1.\quad 6x + 1\tfrac{1}{3}y + 2z = 14\tfrac{2}{3}.$$
$$2.\quad 9x + 2y + 3z = 22.$$
$$3.\quad x + 2y + 3z = 14.$$

Equations 1 and 2 are easily seen to be equivalent. Hence, there are only two independent equations, involving three unknown quantities. But these two are determinate as to x, for subtracting 3 from 2 we have $8x = 8$, $\therefore x = 1$. But y and z are indeterminate. That is, these two equations are inconsistent except for $x = 1$.

3. Solve the following system:

$$1.\quad 9x + 6y + 3z = 30.$$
$$2.\quad 6x + 4y + 2z = 30.$$
$$3.\quad x + 2y + 3z = 14.$$

Equations 1 and 2 are easily seen to be inconsistent; for if the members of 1 are multiplied by $\frac{2}{3}$, $6x + 4y + 2z = 20$ instead of 30.

EXERCISES. CLXXXVI.

Discuss the following systems of equations with respect to their being indeterminate or inconsistent.

1. $x + 2y - 2z = 0.$
   $2x - y + z = 10.$
   $3x + y - z = 10.$

2. $5x + 3y - z = -17.$
   $6x + 4y = -14.$
   $x + y + z = 3.$

3. $6x + 9y + 12z = 5.$
   $2x + 3y + 4z = 32.$
   $3x + 2y - z = 8.$

4. $7x + 11y + 4z = 22.$
   $2x + 3y + 4z = 9.$
   $5x + 2y - z = 6.$

5. $10x + 5y - 15z = 5.$
   $2x + y - 3z = 0.$
   $3x + 2y + z = 6.$

6. $2x - 3y - 4z = -5.$
   $x + 6y + 3z = 10.$
   $x + 5y + 3z = 9.$

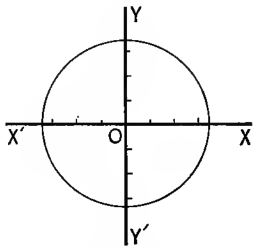
IX. GRAPHS OF QUADRATIC EQUATIONS (p. 296).

The student has already learned in Appendix VIII how graphically to represent a simple equation. Furthermore he has learned that to every point on the graph corresponds one root and only one of the equation, and *vice versa,* a "one-to-one correspondence" between points and roots.

He has also learned that

as, in general, two *straight lines* in a plane have one common *point* and only one,	so, in general, two *linear equations* have one common *root* and only one.

We shall now consider the graphs of equations of degrees above the first.



Illustrative problems. 1. Required the graph of the equation

$$x^2 + y^2 = 10.$$

$\because y = \pm\sqrt{10 - x^2}$, $\therefore$ by giving x various values (noticing that $x^2 \ngtr 10$ for real values of y) we have corresponding values of y as follows:

$$x = \pm\sqrt{10}, \pm 3, \pm\sqrt{8}, \pm\sqrt{7}, \pm\sqrt{6}, \pm\sqrt{5}, \quad \pm 2, \pm\sqrt{2}, \pm 1, \quad 0.$$
$$y = \quad 0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, \quad \pm 2, \pm\sqrt{5}, \pm\sqrt{6}, \pm\sqrt{8}, \pm 3, \pm\sqrt{10}.$$

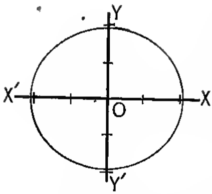
Taking the approximate square roots, and laying off the abscissas and ordinates as indicated, and then connecting the successive points, the graph is the circumference of a circle.

So, in general, *the graph of every equation of the form* $x^2 + y^2 = k^2$ *is the circumference of a circle.*

2. Required the graph of the equation $2x^2 + 3y^2 = 10$.

$\because y = \pm\frac{1}{3}\sqrt{6(5 - x^2)}$, $\therefore$ by giving x various values (noticing that $x^2 \ngtr 5$ for real values of y) we have corresponding values of y as follows:

$$x = \pm\sqrt{5}, \quad \pm 2, \quad \pm\sqrt{3}, \pm\sqrt{2}, \quad \pm 1, \quad \pm 0.$$
$$y = \quad 0, \pm\tfrac{1}{3}\sqrt{6}, \pm\tfrac{2}{3}\sqrt{3}, \pm\sqrt{2}, \pm\tfrac{2}{3}\sqrt{6}, \pm\tfrac{1}{3}\sqrt{30}.$$



Taking the approximate square roots, and laying off the abscissas and ordinates as indicated, and then connecting the successive points, the graph is a curve known as an *ellipse*.

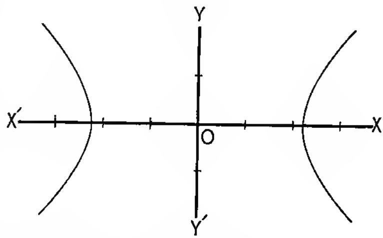
So, in general, *the graph of every equation of the form* $ax^2 + by^2 = c$, *where* a, b, c *are positive, is an ellipse.*

3. Required the graph of the equation $2x^2 - 3y^2 = 10$.

$\because y = \pm\frac{1}{3}\sqrt{6(x^2-5)}$, $\therefore$ by giving x various values (noticing that $x^2 \nless 5$ for real values of y) we have corresponding values of y as follows:

$$x = \pm\sqrt{5},\quad \pm\sqrt{6},\quad \pm\sqrt{7},\quad \pm\sqrt{8},\quad \pm 3,\quad \pm\sqrt{10},\ \cdots.$$
$$y = \quad 0,\quad \pm\tfrac{1}{3}\sqrt{6},\quad \pm\tfrac{2}{3}\sqrt{3},\quad \pm\sqrt{2},\quad \pm\tfrac{2}{3}\sqrt{6},\quad \pm\tfrac{1}{3}\sqrt{30},\ \cdots.$$

Taking the approximate square roots, and laying off the abscissas and ordinates as indicated, and then connecting the successive points the graph is the curve known as the *hyperbola*.

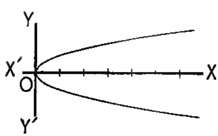


So, in general, *the graph of every equation of the form* $ax^2 - by^2 = c$, *where* a, b, c, *are positive, is an hyperbola.*

4. Required the graph of the equation $y^2 = 8x$.

$\because y = \pm\frac{1}{2}\sqrt{2x}$, $\therefore$ by giving x various values (noticing that $x \nless 0$ for real values of y) we have corresponding values for y as follows:

$$x = 0,\quad 1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 6,\ \cdots.$$
$$y = 0,\quad \pm\tfrac{1}{2}\sqrt{2},\quad \pm 1,\quad \pm\tfrac{1}{2}\sqrt{6},\quad \pm\sqrt{2},\quad \pm\tfrac{1}{2}\sqrt{10},\quad \pm\sqrt{3},\ \cdots.$$



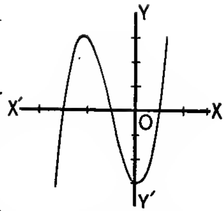
Taking the approximate square roots, and laying off the abscissas and ordinates as indicated, and connecting the successive points, the graph is a curve known as the *parabola*.

So, in general, *the graph of every equation of the form* $y^2 = ax$ *is a parabola.*

The ellipse, hyperbola, and parabola are curves formed by cutting into a right circular cone, and hence are called *conic sections.*

5. Required the graph of the equation

$$x^3 + 3x^2 - x - 3 = y.$$



Giving x various values, we have corresponding values of y as follows:

$x = \cdots -4, -3, -2, -1, \quad 0, 1, \ 2, \ 3, \quad 4, \cdots.$

$y = \cdots -15, \ 0, \ 3, \ 0, -3, 0, 15, 48, 105, \cdots.$

The curve is seen to be one which can be cut by a straight line in three points, and this is the general characteristic of graphs of cubic equations.

EXERCISES. CLXXXVII.

1. Required the graph of the equation $x^2 + y^2 = 25$. In how many points at the most could a straight line cut this curve?

2. Similarly for $y^2 = 18x$.

3. Similarly for $2x^2 + 5y^2 = 10$.

4. Similarly for $2x^2 - 5y^2 = 10$.

5. In how many points, at the most, can a straight line cut the graph of a quadratic equation, judging by the results of exs. 1–4?

6. Required the graphs of the equations

$$x^2 + y^2 = 13,$$
$$x - y = 5,$$

drawn with respect to the same axes. What are the abscissas and ordinates of the points of intersection of the two graphs? How do these compare with the common roots of the two equations?

7. Similarly for the equations

$$2x^2 + 3y^2 = 35,$$
$$3x^2 + 2y^2 = 30.$$

8. Similarly for the equations

$$y^2 = 10x,$$
$$y^2 - 2x^2 = 80.$$

9. Required the graph of the equation $y = x^3 - 9x$. In how many points does this curve cut the X axis?

10. Required the graph of the equation

$$y = x^4 - 5x^3 + 9x^2 - 5x.$$

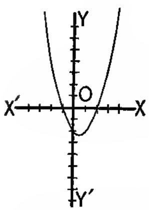
In how many points could a straight line cut this curve?

Graphic representation of the roots. In the equation

$$y = x^2 - x - 2$$

we have the following corresponding values:

$$x = \cdots -3, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad \cdots.$$
$$y = \cdots \quad 10, \quad 4, \quad 0, \quad -2, \quad -2, \quad 0, \quad 4, \quad 10, \quad \cdots.$$



The graph is shown in the annexed figure. When $y = 0$, $x = -1$, or 2. That is, *the values of* x, *in the equation* $x^2 - x - 2 = 0$, *are the abscissas* of the intersection of the graph with the X axis.

Similarly, any equation $f(x) = 0$ can be solved by writing it $f(x) = y$ and plotting it. *The abscissas of the intersections of the graph with the* X *axis will then be the roots of the equation.*

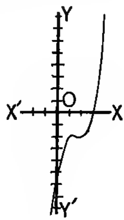
Imaginary roots show themselves by a curve which does not reach the X axis.

E.g., in studying the equation $x^3 - 5x^2 + 8x - 6 = 0$, let $y = f(x)$. Then we have the following corresponding values:

$$x = \cdots \quad -1, \quad 0, \quad 1, \quad 1.5, \quad 2, \quad 3, \quad 4.$$
$$y = \cdots -20, \quad -6, \quad -2, \quad -1.875, \quad -2, \quad 0, \quad 10.$$

The curve does not reach the X axis between 1 and 2. In solving the equation the roots are found to be $1+i$, $1-i$, and 3.

The fact that *complex roots enter in pairs* is readily understood by a study of the graph.

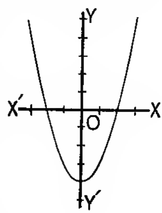


E.g., consider the equation $x^2-4=0$.

Let $f(x)=y$. We then have the following corresponding values:

$$x = \quad 0, \ \pm 1, \ \pm 2, \ \pm 3, \ \cdots.$$
$$y = -4, \ -3, \ \ 0, \ \ 5, \ \cdots.$$

If, now, we make each y 3 units greater, *i.e.*, if we lift the curve 3 units (or, what is the same thing, lower the X axis 3 units) x will equal ± 1 when $y=0$. *I.e.*, the roots will approach each other. This can be done by making the equation $x^2-1=y$.



If we lift the curve another unit (or lower the X axis 4 units), making the equation $x^2=y$, x will have only the double root 0 when $y=0$; *i.e.*, the two roots are now equal.

If we lift the curve another unit (or lower the X axis 5 units), making the equation $x^2+1=y$, x will have the imaginary value $\pm i$ when $y=0$, the two imaginaries entering together. In other words, *complex roots* (of which pure imaginaries form a special class) *enter in pairs.*

Roots of simultaneous equations. We have seen that two linear equations, involving but two unknown quantities, can be solved by finding the point of intersection of their graphs.

Similarly, if we have two equations like

I. $$x^2+3y^2=28,$$

II. $$2x^2-y^2=-7,$$

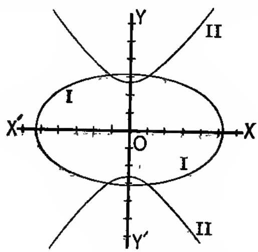
the coördinates of the common points of their graphs represent the common roots of the equations.

From I, we have $y = \pm \frac{1}{3}\sqrt{3(28 - x^2)}$.

Hence, if

$x = \quad 0, \qquad \pm 1, \pm 2, \qquad \pm 3, \qquad \pm 4, \pm 5, \pm\sqrt{28},$

then $\quad y = \pm \frac{2}{3}\sqrt{21}, \pm 3, \pm 2\sqrt{2}, \pm\frac{1}{3}\sqrt{57}, \pm 2, \pm 1, \qquad 0.$



The graph is marked I in the annexed figure. It is an ellipse.

From II, we have $y = \pm\sqrt{2x^2 + 7}$.

Here if $x = \quad 0, \qquad \pm 1, \pm 2, \qquad \pm 3, \pm 4,$

then $\qquad y = \pm\sqrt{7}, \pm 3, \pm\sqrt{15}, \pm 5, \pm\sqrt{39},$

The graph is marked II in the annexed figure. It is an hyperbola, a two-branched figure.

The common roots are

$$x = 1, \quad 1, \quad -1, \quad -1.$$
$$y = 3, \quad -3, \quad 3, \quad -3.$$

From the preceding figure we see confirmed the fact already mentioned, that two simultaneous quadratic equations involving two unknown quantities have, in general, four roots, the two curves intersecting in four points.

Two of the points may coincide, as in Fig. 1, giving a double root, or there may be two double roots, as in Fig. 2, or two of the roots may be imaginary, as in Fig. 3, or both pairs of roots may be imaginary.

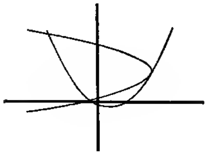


FIG. 1.

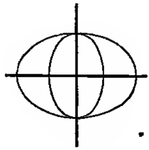


FIG. 2.

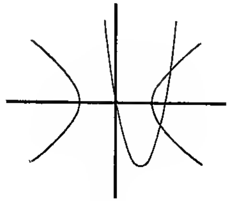


FIG. 3.

From similar considerations it may be inferred that there are 6 roots common to two simultaneous equations of which one is a quadratic and the other a cubic. In general, if one equation is of the *m*th degree and the other of the *n*th, there are *mn* roots.

EXERCISES. CLXXXVIII.

Represent graphically the following sets of simultaneous equations, and find at least one value of x and one of y common to the two.

1. $x^2 + y^2 = 8.$
   $y^2 = 2x.$

2. $x^2 + y = 7.$
   $x + y^2 = 11.$

3. $x + 3y = 10.$
   $x^2 + 5 = 2y.$

4. $2x^2 + y^2 = 19.$
   $x^2 - y^2 = 8.$

5. $2x^2 + y^2 = 3.$
   $x^2 - 3y^2 = -2.$

6. $x^3 + x^2 - x - 3 = y.$
   $x + y = 5.$

7. $x^2 + y^2 = 5.$
   $y = x^3 + 3x^2 - 4x + 2.$

TABLE OF BIOGRAPHIES.

THE simple equation was known to the Egyptians, its solution appearing in the oldest deciphered mathematical work extant, the papyrus of Ahmes. The quadratic equation was solved by the Greeks, and indeterminate equations formed a considerable portion of the works of Diophantus. The Hindus, Persians, and Arabs next took up the science and made considerable progress in the study of equations and series. The Arabs gave to algebra its name.

The sixteenth century saw a great revival of learning in general and of algebra in particular. The cubic and quartic equations were now solved.

The seventeenth century saw modern symbolism established, thus forming elementary algebra as it is known to-day.

The following table contains the names mentioned in this work, together with a few others prominent in the history of algebra. The notes are chiefly from those prepared by the authors for their translation of Fink's "History of Mathematics" (Chicago, The Open Court Publishing Co., 1900), to which reference is made for a more complete account of the development of the science.

Abel, Niels Henrik. Born at Findöe, Norway, August 5, 1802; died April 6, 1829. Proved the impossibility of the algebraic solution of the quintic equation.

Ahmes. An Egyptian scribe. Lived about −1700. Wrote the earliest deciphered mathematical manuscript extant, on arithmetic, algebra, and mensuration.

Al Khowarazmi, Abu Jafar Mohammed ibn Musa. First part of ninth century. Native of Khwarazm (Khiva). Arab mathematician and astronomer. The title of his work gave the name to algebra.

Ampère, André-Marie. Born at Lyons, France, in 1775; died at Marseilles in 1836. Founder of the science of electro-dynamics.

Apollonius of Perga, in Pamphylia. Taught at Alexandria between − 250 and − 200, in the reign of Ptolemy Philopator. Solved the general quadratic with the help of conics.

Argand, Jean Robert. Born at Geneva, 1768; died *c.* 1825. Private life unknown. One of the inventors of the present method of geometrically representing complex numbers (1806).

Aristotle. Born at Stageira, Macedonia, − 384; died at Chalcis, Eubœa, − 322. Great philosopher. Represented unknown quantities by letters; suggested the theory of combinations.

Aryabhatta. Born at Pataliputra on the Upper Ganges, 476. Hindu mathematician. Wrote on algebra, including quadratic equations, permutations, indeterminate equations, and magic squares.

Bézout, Étienne. Born at Nemours in 1730; died at Paris in 1783. Prominent in the study of symmetric functions and determinants.

Bhaskara Acharya. Born in 1114. Hindu mathematician and astronomer. Author of the "Lilavati" and the "Vijaganita," containing the elements of arithmetic and algebra. One of the most prominent mathematicians of his time.

Bombelli, Rafaele. Italian. Born *c.* 1530. His algebra (1572) summarized all then known on the subject, especially the cubic.

Boyle, Robert. Born in Ireland, Jan. 25, 1627; died Dec. 30, 1691. Celebrated physicist.

Brahmagupta. Born in 598. Hindu mathematician. Contributed to geometry, trigonometry, and algebra.

Briggs, Henry. Born at Warley Wood, near Halifax, Yorkshire, February, 1560–1; died at Oxford Jan. 26, 1630–1. Savilian professor of geometry at Oxford. Among the first to recognize the value of logarithms; those with decimal base bear his name.

Burgi, Joost (Jobst). Born at Lichtensteig, St. Gall, Switzerland, 1552; died at Cassel in 1632. One of the first to suggest a system of logarithms. The first to recognize the value of making the second member of an equation zero.

Cardan, Jerome (Hieronymus, Girolamo). Born at Pavia, 1501; died at Rome, 1576. Professor of mathematics at Bologna and Padua. Mathematician, physician, astrologer. First to publish (1545) the solution of the cubic equation.

Cataldi, Pietro Antonio. Italian mathematician, born 1548; died at Bologna, 1626. Professor of mathematics at Florence, Perugia, and Bologna. Pioneer in the use of continued fractions.

Cauchy, Augustin Louis. Born at Paris, 1789; died at Sceaux, 1857. Professor of mathematics at Paris. One of the most prominent mathematicians of his time. Contributed to the theory of determinants, series, and algebra in general.

Cramer, Gabriel. Born at Geneva, 1704; died at Bagnols, 1752. Added to the theory of equations and revived the study of determinants (begun by Leibnitz).

D'Alembert, Jean le Rond. Born at Paris, 1717; died there, 1783. Physicist, mathematician, astronomer. Contributed to the theory of equations.

De Moivre, Abraham. Born at Vitry, Champagne, 1667; died at London, 1754. Contributed to the theory of complex numbers and of probabilities.

Descartes, René, du Perron. Born at La Haye, Touraine, 1596; died at Stockholm, 1650. Discoverer of analytic geometry. Contributed extensively to algebra.

Diophantus of Alexandria. Lived about 275. Most prominent of Greek algebraists, contributing to indeterminate equations.

Euclid. Lived about -300. Taught at Alexandria in the reign of Ptolemy Soter. The author or compiler of the most famous text-book of geometry ever written, the "Elements," in thirteen books.

Euler, Leonhard. Born at Basel, 1707; died at St. Petersburg, 1783. One of the greatest physicists, astronomers, and mathematicians of the eighteenth century.

Ferrari, Ludovico. Born at Bologna, 1522; died in 1562. Solved the biquadratic.

Ferro, Scipione del. Born at Bologna, *c.* 1465; died between Oct. 29 and Nov. 16, 1526. Professor of mathematics at Bologna. Investigated the geometry based on a single setting of the compasses, and was the first to solve the special cubic $x^3 + px = q$.

Gauss, Karl Friedrich. Born at Brunswick, 1777; died at Göttingen, 1855. The greatest mathematician of modern times. Prominent as a physicist and astronomer. The theories of numbers, of functions, of equations, of determinants, of complex numbers, of hyperbolic geometry, are all largely indebted to his great genius.

Harriot, Thomas. Born at Oxford, 1560; died at Sion House, near Isleworth, July 2, 1621. The most celebrated English algebraist of his time.

Horner, William George. Born in 1786; died at Bath, Sept. 22, 1837. Chiefly known for his method of approximating the real roots of a numerical equation (1819).

Lagrange, Joseph Louis, Comte. Born at Turin, Jan. 25, 1736; died at Paris, April 10, 1813. One of the foremost mathematicians of his time. Contributed extensively to the calculus of variations, theory of numbers, determinants, and theory of equations.

Maclaurin, Colin. Born at Kilmodan, Argyllshire, 1698; died at York, June 14, 1746. Professor of mathematics at Edinburgh. Contributed to the study of conics and series.

Metrodorus. Lived about 325. Author of many algebraic problems.

Napier, John. Born at Merchiston, then a suburb of Edinburgh, 1550; died there in 1617. Inventor of logarithms.

Newton, Sir Isaac. Born at Woolsthorpe, Lincolnshire, Dec. 25, 1642, (O. S.); died at Kensington, March 20, 1727. Lucasian professor of mathematics at Cambridge (1669). The world's greatest mathematical physicist. Invented fluxional calculus (*c.* 1666). Contributed extensively to the theories of series, equations, curves, etc.

Ohm, Georg Simon. Born at Erlangen, Germany, in 1781; died July 6, 1854. Celebrated physicist.

Pascal, Blaise. Born at Clermont, 1623; died at Paris, 1662. Physicist, philosopher, mathematician. Contributed to the theory of numbers, probabilities, and geometry.

Recorde, Robert. Born at Tenby, Wales, 1510; died in prison, at London, 1558. Professor of mathematics and rhetoric at Oxford. Introduced the sign = for equality.

Tartaglia, Nicolo. (Nicholas the stammerer. Real name, Nicolo Fontana.) Born at Brescia, *c.* 1500; died at Venice, *c.* 1557. Physicist and arithmetician; known for his work on cubic equations.

Taylor, Brook. Born at Edmonton, 1685; died at London, 1731. Physicist and mathematician. Known for his work in series.

Viète (Vieta), François. Born at Fontenay-le-Comte, 1540; died at Paris, 1603. The foremost algebraist of his time.

Volta, Alessandro. Born at Como, Italy, Feb. 18, 1745; died March 5, 1827. Celebrated physicist.

Wallis, John. Born at Ashford, 1616; died at Oxford, 1703. Savilian professor of geometry at Oxford. Suggested (1685) the modern graphic interpretation of the imaginary.

TABLE OF ETYMOLOGIES.

THE following table will serve to make more clear to students the meaning of many words used and defined in elementary algebra.

L., Latin. *G.*, Greek. *dim.*, diminutive.

Abscissa. L. cut off.

Absolute. L. *absolutus*, *ab*, from, + *solvere*, loosen. That is, complete, unrestricted.

Abstract. L. *abs*, away, + *trahere*, draw.

Add. L. *ad*, to, + *-dere*, for *dare*, put, place.

Affected. L. *ad*, to, + *facere*, do, make; *i.e.*, to act upon, influence. Hence compounded; an equation of several degrees.

Aggregation. L. *ad*, to, + *gregare*, collect into a flock, from *grex*, flock.

Algebra. Arabic, *al*, the, + *jabr*, redintegration, consolidation. The title of Al Khowarazmi's work (see Table of Biographies) was *'ilm al-jabr wa' l muqābalah*, the science of redintegration and equation; of this long title only *al-jabr* survives.

Alternation. L. *alter*, other.

Antecedent. L. *ante*, before, + *cedere*, go.

Antilogarithm. L. and G. *anti-*, against, opposite to, + logarithm. See Logarithm. The number standing opposite to the logarithm.

Arithmetic. G. *arithmos*, number.

Ascend. L. *ad-*, to, + *scandere*, climb.

Associative. L. *ad-*, to, + *sociare*, join.

Axiom. G. *axioma*, that which is thought fit, a requisite.

Binomial. L. *bi-*, two-, + *nomen*, name.

Characteristic. G. *charakterizein*, designate; from G. *character*, an instrument for graving, from *charassein*, to scratch.

Circulate. From circle. L. dim. of *circus*, a ring, G. *kirkos* or *krikos*, a circle, ring.

Commutative. L. *com-*, intensive, + *mutare*, change.

Comparison. L. *com-*, together with, + *par*, equal.

Complement. L. *complementum*, that which fills; from *com-*, intensive, + *plere*, fill.

Complete. See Complement.
Complex. L. *com.*, together, + *plectere*, weave.
Composition. L. *com-*, together, + *ponere*, place.
Compound. Same etymology as Composition.
Consequent. L. *con-*, together, + *sequi*, follow.
Constant. L. *con-*, together, + *stare*, stand.
Continued. L. *con-*, together, + *tenere*, hold.
Corollary. L. *corollarium*, a gift, money paid for a garland of flowers, from *corolla*, dim. of *corona*, a crown.
Cube. G. *cubos*, a die, a cube.

Decimal. L. *decem*, ten.
Deduce. L. *de*, down, away, + *ducere*, lead.
Define. L. *de-*, + *finire*, limit, settle, define.
Degree. L. *de*, down, + *gradus*, step.
Denominator. L. namer, from *de*, + *nominare*, name, from *nomen*, name.
Descend. L. *de*, down, + *scandere*, climb.
Detach. Ital. *des-*, privative, + *-tacher*, fasten.
Determinant. L. *de-*, + *terminare*, bound, limit.
Determine. See Determinant.
Discriminant. L. *dis-*, apart, + *cernere* = G. *krinein*, separate.
Distribute. L. *dis-*, apart, + *tribuere*, give.
Divide. L. *di-*, for *dis-*, apart, + *videre*, see.
Domain. L. *dominium*, dominion, from *dominus*, lord.

Eliminate. L. *e*, out, + *limen*, a threshold. To turn out of doors.
Equal. L. *aequalis*, equal, from *aequus*, plain.
Equation. See Equal.
Evolution. L. *e*, out, + *volvere*, roll. To unfold the root.
Exponent. L. *ex*, out, + *ponere*, put; *i.e.*, to set forth, indicate.
Extraneous. L. *extra*, outside.
Extreme. L. *extremus*, superlative of *exter*, outer.

Factor. L. a doer, from *facere*, do.
Fraction. L. *fractus*, broken, from *frangere*, break.
Function. L. *functus*, performed, from *fungi*, perform.

Graph. G. *graphein*, write.

Homogeneous. G. *homos*, the same, + *genos*, race.

Identical. L. *idem*, the same.
Imaginary. L. *imago*, an image.
Indeterminate. L. *in-*, privative, + *determinate*. See Determinants.
Index. L. *indicare*, point out, show.
Infinite. L. *in-*, not, + *finitus*, bounded.
Inspection. L. *in*, on, in, at, + *specere*, look.

Integer. L. *in*, privative, + *tangere*, touch; *i.e.*, untouched, whole, sound.
Inverse. L. *in*, on, toward, + *vertere*, turn.
Involution. L. *in*, in, + *volvere*, roll. To roll the root into a power.

Limit. L. *limes* (*limit-*), a cross-path, boundary.
Linear. L. *linea*, line.
Literal. L. *littera*, *litera*, a letter.
Logarithm. G. *logos*, proportion, ratio, + *arithmos*, number.

Mantissa. L. an addition.
Maximum. L. greatest, superlative of *magnus*, great.
Mean. L. *medius*, middle.
Minimum. L. least.
Minuend. L. *minuere*, lessen.
Monomial. G. *monos*, single, + L. *nomen*, name.
Multiple. L. *multus*, many, + *-plus*, like English -fold, from *plicare*, fold.

Negative. L. *ne*, not, + *que*, a generalizing suffix.
Notation. L. *notatio*, a marking, from *nota*, a mark, a sign.
Number. L. *numerus*, number.
Numerator. L. numberer.

Operation. L. *opus*, work.
Ordinate. L. *ordo* (*ordin-*), a row.

π. Initial of G. *periphereia*, periphery, circumference.

Polynomial. G. *polus*, many, + L. *nomen*, name.
Positive. L. *positivus*, settled, from *ponere*, put.
Power. L. *posse*, to be able.
Problem. G. *problema*, a question proposed for solution; from *pro*, before, + *ballein*, throw.
Product. L. *pro-*, forth, + *ducere*, lead.
Proportion. L. *pro*, for, before, + *portio*, a share.
Proposition. L. *pro*, before, + *ponere*, place.
Pure. L. *purus*, clean.

Quadratic. L. *quadratus*, a square, from *quattuor*, four.
Quantity. L. *quantus*, how much, from *quam*, how.
Quartic. L. *quattuor*, four.
Quotient. L. *quot*, how many.

Radical. L. *radix*, root.
Ratio. L. a reckoning, calculation, from *reri*, think, estimate.
Rational. L. *ratio*. See Ratio.
Real. L. *realis*, belonging to the thing itself, from *res*, thing.
Reciprocal. L. *re-*, back, + adjective formative *-cus*.
Reduce. L. *re-*, back, + *ducere*, lead.
Remainder. L. *re-*, behind, back, + *manere*, remain.
Root. L. and G. *radix*, a root.

Series. L. a row.
Similar. L. *similis*, like.
Simplify. L. *simplex*, simple.

Simultaneous. L. *simultim*, at the same time, from *simul*, together.

Solution. L. *solvere*, loose.

Square. L. *quadra*, a square, from *quattuor*, four.

Substitute. L. *sub*, under, + *statuere*, set up.

Subtract. L. *sub*, + *trahere*, draw.

Subtrahend. See Subtract.

Sum. L. *summa*, highest part.

Surd. L. *surdus*, deaf. A mistranslation of the G. *alogos*, which does not mean stupid (hence deaf), but inexpressible.

Symbol. G. *symbolos*, a mark, from *syn*, together, + *ballein*, put.

Symmetry. G. *syn*, together, + *metron*, measure.

Theorem. G. *theorema*, a sight, a principle contemplated.

Transpose. L. *trans*, over, + *ponere*, place.

Trinomial. L. *tres* (*tri-*), three, + *nomen*, name.

Vary, variation. L. *varius*, different.