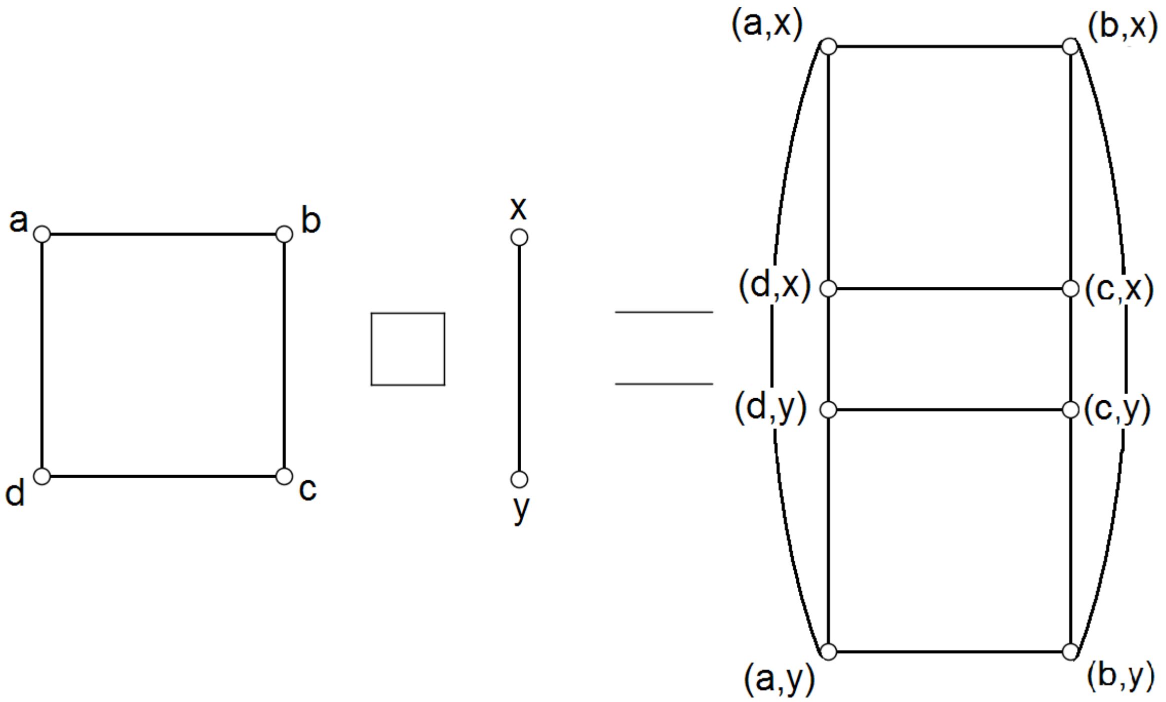
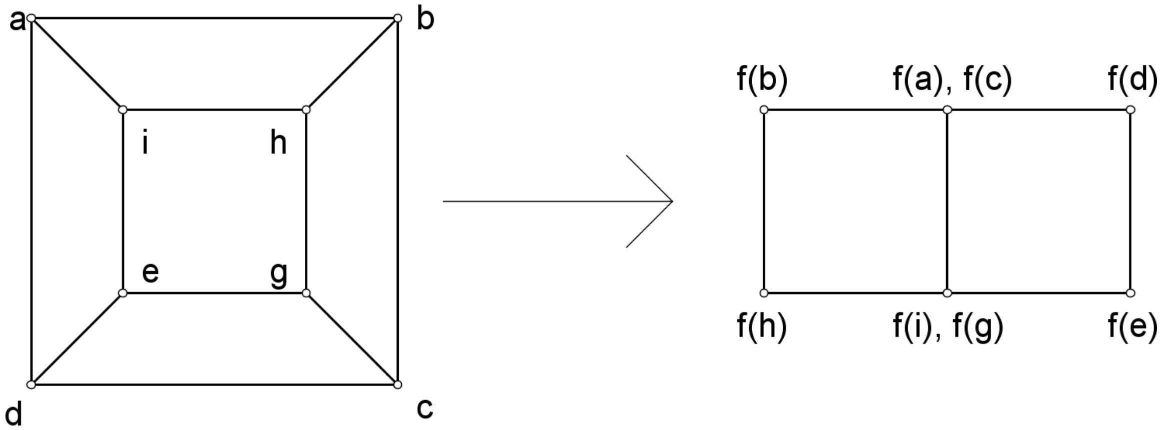
Definition. $W = X \square Y$ *is the graph where*
$V(X \square Y)$ *is all* (a, y) *such that* $a \in V(X)$, $y \in V(Y)$
$$E(X \square Y) = \{((a, y), (a, z)) \mid (y, z) \in E(Y)\} \cup \{((a, y), (b, y)) \mid (a, b) \in E(X)\}.$$

Note this gives nm vertices in $V(X \square Y)$ and $V(X)$ disjoint copies of Y and $V(Y)$ disjoint copies of X.

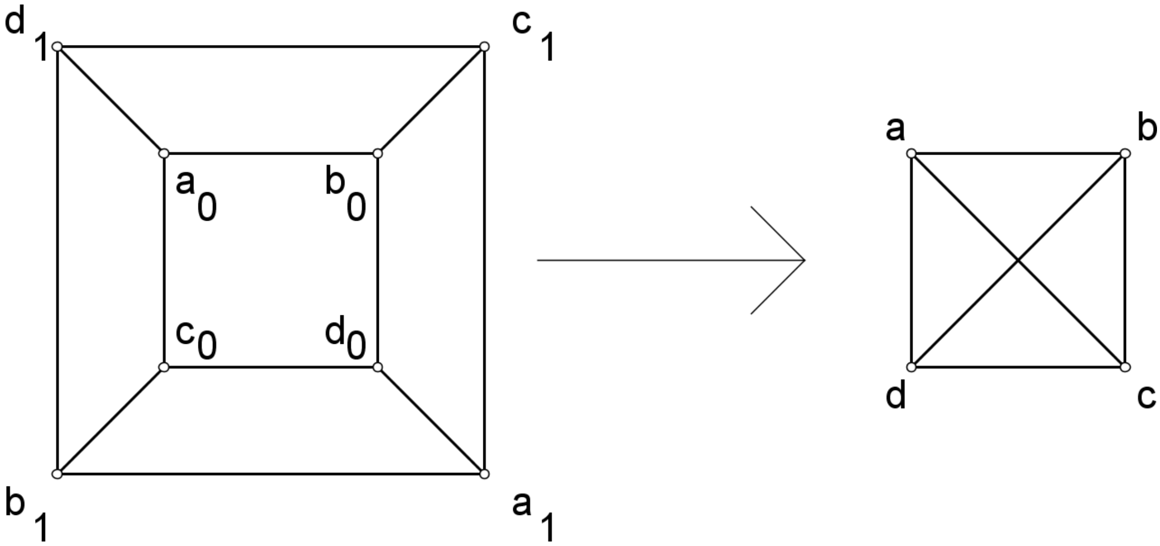


Definition. *$Y \xrightarrow{f} X$ is a graph homomorphism if for every edge $(a, b) \in E(Y)$ there is an edge $(f(a), f(b)) \in E(X)$.*



Definition. *A covering map is a homomorphism $Y \xrightarrow{f} X$ such that for every vertex $v \in V(Y)$ the edges $(v, v_i) \in E(Y)$ incident to v map bijectively to the edges $(f(v), f(v_i))$ incident to $f(v)$.*

A covering is n sheeted if for every $a \in X$ there are n vertices $a^s, \ s = 1, \ldots, n$ such that the $f^{-1}(a) = a^s$.



Theorem 1. *Given covering maps* $Y \xrightarrow{g} X$ *and* $Z \xrightarrow{h} W$ *then* $Y \square Z \xrightarrow{g \circ h} X \square W$ *is a covering map.*

Sketch of Proof

The $|V(Z)|$ disjoint $Y \subset Y \square Z$, cover the $X \subset X \square Z$. The edges $((k^s, z), (l^s, w)) \in E(Y \square Z)$ cover $(k, z), (l, w) \in E(X \square Z)$ where $f(k^s) = k \ \forall s$.

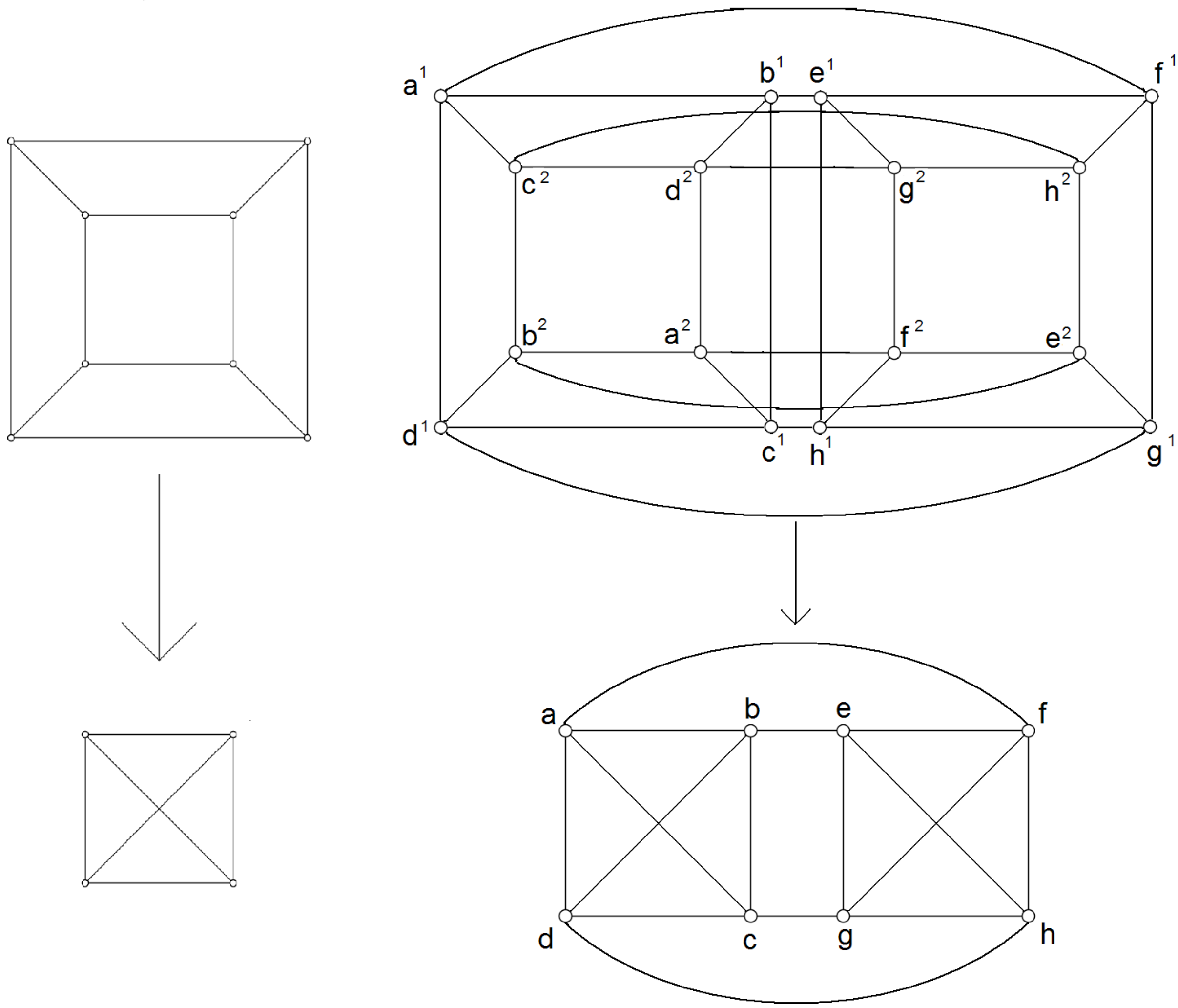


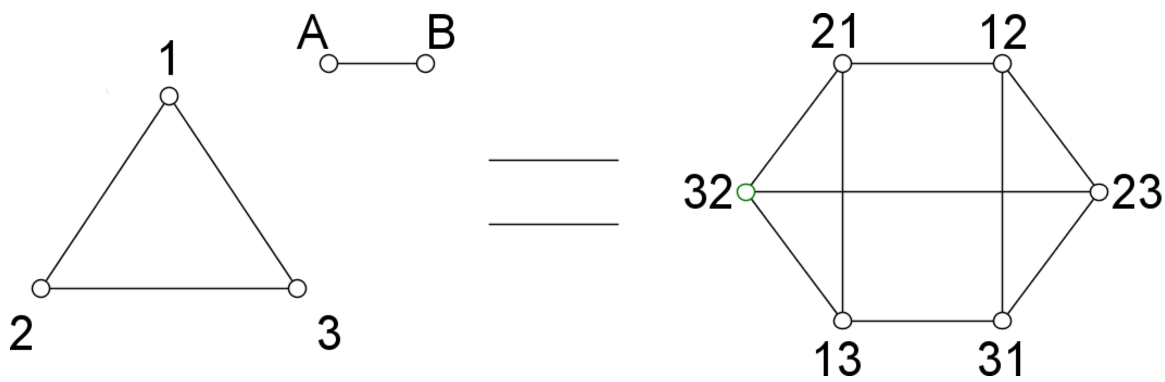
Figure 1: $Q_3 \to K_4$ is a covering map, so $Q_3 \square K_2 \to K_4 \square K_2$ is also.

Hence $Y \square Z \xrightarrow{f} X \square Z$

Then the composition $Y \square Z \xrightarrow{f} X \square Z \xrightarrow{g} X \square W$ gives that $Y \square Z \xrightarrow{f \circ g} X \square W$ is a covering map.

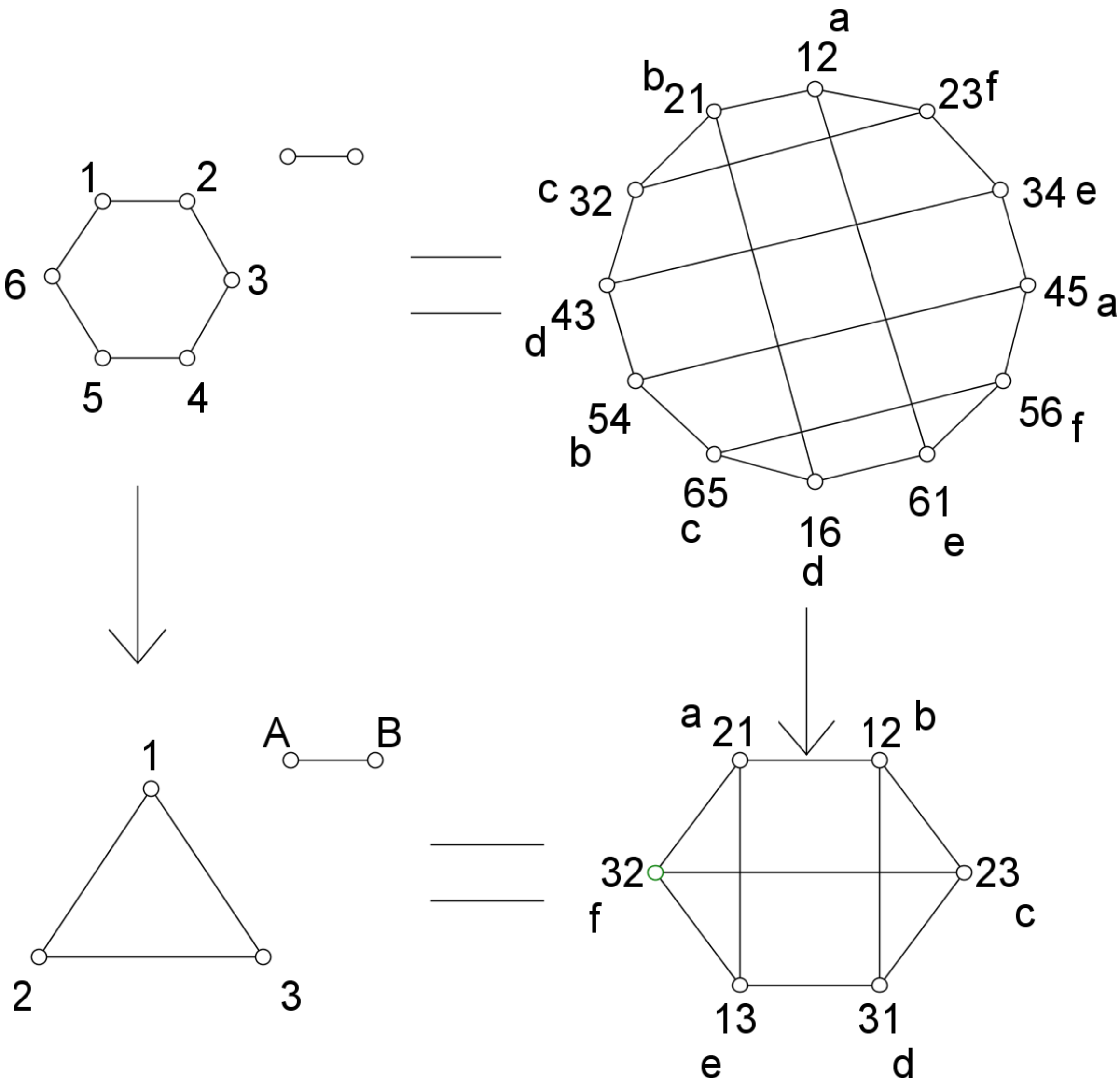
Definition. *$W = [Z, X]$ is the graph where $V(W)$ is the set of homomorphisms $Z \xrightarrow{f} X$ and $E(W)$ is the set of homomorphisms $Z \Box K_2 \xrightarrow{f} X$ connecting the two vertices correponding to each homomorphism, $Z_i \xrightarrow{f} X$ where $Z_i \subset Z \Box K_2$.*

Due to the similarity of $[Z, X]$ to the set theoretic exponent, we refer to $[Z, X]$ as the graph exponential.

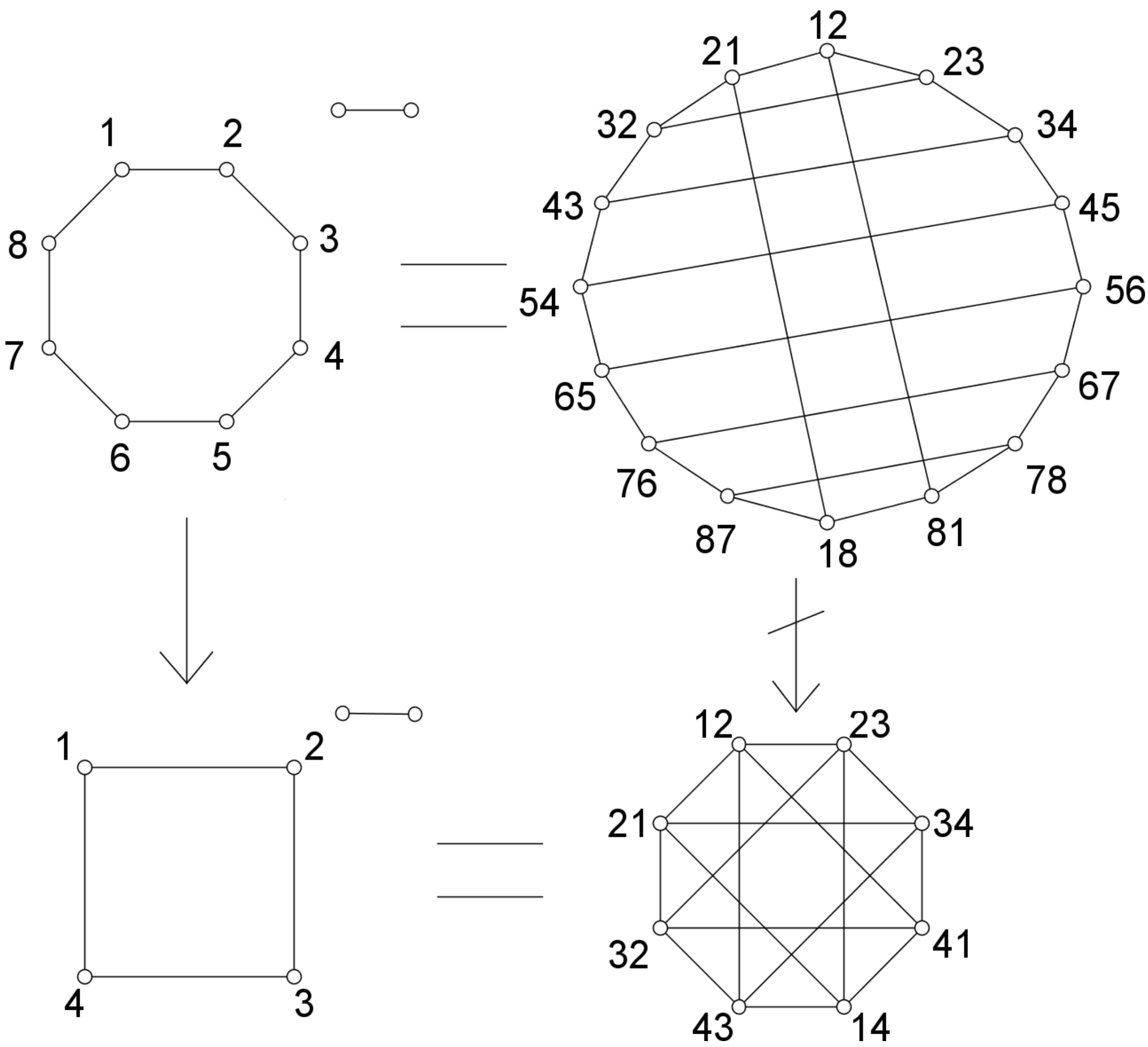


If $Y \xrightarrow{f} X$ is a covering map, then is there a covering map $[Z, Y] \xrightarrow{f'} [Z, X]$?

There is a covering map $[K_2, C_6] \to [K_2, C_3]$



There is no covering map $[K_2, C_8] \to [K_2, C_4]$.



Definition. *Let $C_4(X)$ denote the set of injective maps of C_4 into X. That is, all maps of C_4 into X such that all the vertices of C_4 map to distinct vertices of X.*

Theorem 2. *Let $Y \xrightarrow{f} X$ be an n sheeted covering map. Then $[K_2, Y] \xrightarrow{f} [K_2, X]$ is a covering map if and only if $C_4(Y) \to C_4(X)$ is n to 1.*

Sketch of Proof

A vertex $v \in V([K_2, Y])$ is of the form $K_2 \to Y$.
An edge $e \in E([K_2, Y])$ is of the form $K_2 \square K_2 \to Y$.
The homomorphism $K_2 \square K_2 \to Y$ produces as its image C_4 or P_3 or P_2.

Since $Y \xrightarrow{f} X$ is a covering map, there is a bijection between the edges incident to $v \in V([Z, Y])$ and incident to $f(v) \in V([Z, X])$ formed by the homomorphisms $K_2 \square K_2 \to P_2$ or P_3 in Y and X.

Since $\exists W \neq C_4$ such that $W \to C_4$ then $[Z, Y]$ covers $[Z, X]$ if and only if $f^{-1}(C_4(X)) = C_4(Y)$, that is, when there is an n to 1 function between $C_4(Y)$ and $C_4(X)$.