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SOME PROPERTIES of the EIGENFUNCTIONS and EIGENVALUES INTRODUCED by BETHE for the LINEAR CHAIN of ATOMS

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U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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Some Properties of the Eigenfunctions and Eigenvalues Introduced by Bethe for the Linear Chain of Atoms.

Henry Unruh, Jr.*

The operations $S^2\psi$ and $S^{\pm}\psi$, in which ψ is an exact eigenfunction obtained by Bethe for the linear chain of localized spins, are discussed. The operations S^2 and $S^{\pm}$ are the usual square of the total angular momentum and the raising and lowering operators respectively. The results, valid for an arbitrary number of spin inversions, show, as expected, that all possible states of the system can be achieved by omitting all excitations associated with zero wave number but requiring each state to take all possible orientations in space.

Key Words: Spin waves, linear chain, ferromagnetism.

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1. Introduction

The exact energy eigenstates of a linear chain of N localized spins, each of spin 1/2, and subject to the isotropic Hamiltonian,

$$\mathcal{H}_I = -2J \sum_{j=1}^{N-1} (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z)$$

are well known[1*]. In $\mathcal{H}_I$ the subscript j locates the position of a spin in the lattice, and the spin operator for the spin in that location is

$$\underset{\sim}{S}_j = \underset{\sim}{i}\, S_j^x + \underset{\sim}{j}\, S_j^y + \underset{\sim}{k}\, S_j^z \, .$$

Introducing

$$S^2 = \left(\sum_{j=1}^{N} S_j^x \right)^2 + \left(\sum_{j=1}^{N} S_j^y \right)^2 + \left(\sum_{j=1}^{N} S_j^z \right)^2$$

and

$$S^{\pm} = \sum_{j=1}^{N} \left(S_j^x \pm i\, S_j^y \right) ,$$

* The notation we are using, and will use for the most part, is identical to that of Bethe (reference 1).

we have at once the following commutation relationships:

$$\left[S^2, S^{\pm}\right] = 0, \tag{1}$$

$$\left[\mathcal{H}_I, S^{\pm}\right] = 0, \tag{2}$$

$$\left[S^+, S^-\right] = 2S^z . \tag{3}$$

According to reference 1 the eigenstates of the system are given by

$$\psi(N, q) = \sum_{<m>} a(m_1, m_2, \ldots m_q)\, \alpha_1 \alpha_2 \cdots \beta_{m_1} \cdots \beta_{m_2} \cdots \cdots \beta_{m_q} \cdots \alpha_N , \tag{4}$$

with the restriction that

$$1 \leq m_1 < m_2 < \ldots < m_q \leq N ,$$

and

$$a(m_1, m_2, \ldots, m_q) = \sum_{P=1}^{q!} \exp\left[i\left(f_1^P m_1 + f_2^P m_2 + \ldots + f_q^P m_q + \tfrac{1}{2}\sum_{k<\ell} \phi_{k\ell}^P\right)\right]. \tag{5}$$

To obtain a numerical value for $a(m_1, \ldots, m_q)$ one first solves for the f's and ϕ's in the following equations:

$$
\left.
\begin{aligned}
Nf_1 &= 2\pi n_1 + \sum_{\ell=1}^{q} \phi_{1\ell} \; , \\
&\;\;\vdots \\
Nf_k &= 2\pi n_k + \sum_{\ell=1}^{q} \phi_{k\ell} \; , \\
\\
Nf_q &= 2\pi n_q + \sum_{\ell=1}^{q} \phi_{q\ell} \quad ,
\end{aligned}
\right\} \tag{6a}
$$

$$
\left.
\begin{aligned}
\tan\frac{\phi_{k\ell}}{2} &= \frac{\cos\frac{f_k+f_\ell}{2} - \cos\frac{f_k-f_\ell}{2}}{\sin\frac{f_k-f_\ell}{2}} \quad \text{for } k\neq\ell \quad , \\
\\
\phi_{kk} &= 0 \quad ,
\end{aligned}
\right\} \tag{6b}
$$

in which $n_k = 0, 1, 2. . . ,$ N-1, and $\phi_{\ell k}$. After the f's are obtained they may be written down (together with the associated ϕ's) in some arbitrary order. Such a sequence is one of the possible q ! sequences and is characterized by a particular value of the index P (which is written as a superscipt on the f's and ϕ's). Thus from equation (5) it is seen that $a(m_1, . . . , m_q)$ includes all possible permutations of the f's for an associated specification of an n_k set.

At this time we should also note that (6b) can be written in this form

$$
\exp(i\phi_{k\ell}) = - \frac{\exp[i(f_k+f_\ell)] + 1 - 2\exp(if_k)}{\exp[i(f_k+f_\ell)] + 1 - 2\exp(if_\ell)} \quad . \tag{7}
$$

2. Calculations

We now evaluate $S^{+}\psi(N,q)$ for the special case when none of the f's are zero. At this point it is convenient to write $\psi(N,q)$ for this special case in the form $\psi_o(N,r)$ in which the r has replaced the q to make the notation consistent with that in reference 2. Now

$$S^{+}\psi_o(N,r) = \sum_{\langle m\rangle} a(m_1,\ldots m_r) S^{+} \alpha_1\alpha_2 \cdots \beta_{m_1} \cdots \beta_{m_i} \cdots \beta_{m_r} \cdots \alpha_N$$

$$= \sum_{\langle m\rangle} a(m_1,\ldots m_r) \sum_{i=1}^{r} \alpha_1\alpha_2 \cdots \beta_{m_1} \cdots \alpha_{m_i} \beta_{m_r} \cdots \alpha_N$$

$$= \sum_{\langle n\rangle} b(n_1,\ldots n_{r-1}) \alpha_1\alpha_2 \cdots \beta_{n_1} \cdots \cdots \beta_{n_{r-1}} \cdots \alpha_N, \qquad (8)$$

where

$$b(n_1,\ldots n_{r-1}) = \sum_{i=1}^{r} \sum_{m_i=n_{i-1}+1}^{n_i-1} a(n_1,n_2 \cdots n_{i-1}, m_i, n_i \cdots n_{r-1}), \qquad (9)$$

in which it is understood that when $i = 1$, m_1 runs from 1 to n_1-1, and when $i = r$, m_r runs from $n_{r-1}+1$ to N.

Combining (5) and (9), there results

$$b(n_1,\ldots n_{r-1}) = \sum_{P=1} \sum_{i=1} \sum_{m_i=n_{i-1}+1}$$

$$\times \exp\left[i\left(f_1^P n_1 + f_2^P n_2 + \ldots + f_{i-1}^P n_{i-1} + f_i^P m_i + f_{i+1}^P n_i + \ldots + f_r^P n_{r-1} + \tfrac{1}{2}\sum_{k<\ell} \phi_{k\ell}^P \right)\right].$$

The third sum can be evaluated, and if the second is written out explicitly we have

$$b(n_1 \ldots n_{r-1}) =$$

$$\sum_{P=1}^{r!} \Bigg\{ \left[\frac{\exp[if_1^P] - \exp[if_1^P n_1]}{1-\exp[if_1^P]} \right] \exp[i(f_2^P n_1 + \ldots + f_r^P n_{r-1} + \tfrac{1}{2}\sum_{k<\ell} \phi_{k\ell}^P)]$$

$$+\exp[i(f_1^P n_1)] \left[\frac{\exp[n_1+1] - \exp[if_2^P n_2]}{1-\exp[if_2^P]} \right] \exp[i(f_3^P n_2 + \ldots + f_r^P n_{r-1} + \tfrac{1}{2}\sum_{k<\ell} \phi_{k\ell}^P)]$$

$$+\ldots$$

$$+\exp[i(f_1^P n_1 + \ldots + f_{i-1}^P n_{i-1})] \left[\frac{\exp[if_i^P(n_{i-1}+1)] - \exp[if_i^P n_i]}{1-\exp[if_i^P]} \right] \exp[i(f_{i+1}^P n_i + \ldots + f_r^P n_{r-1} + \tfrac{1}{2}\sum_{k<\ell} \phi_{k\ell}^P)]$$

$$+ \exp[i(f_i^P n_i + \ldots + f_i^P n_i)] \left[\frac{\exp[if_{i+1}^P(n_i+1)] - \exp[if_i^P n_{i+1}]}{1-\exp[if_{i+1}^P]} \right] \exp[i(f_{i+2}^P n_{i+1} + \ldots + f_r^P n_{r-1} + \tfrac{1}{2}\sum_{k<\ell} \phi_{k\ell}^P)]$$

$$+\ldots$$

$$+\exp[i(f_1^P n_1 + \ldots + f_{r-1}^P n_{r-1})] \left[\frac{\exp[if_r^P(n_{r-1}+1)] - \exp[if_r^P(N+1)]}{1-e[if_r^P]} \right] \exp[i(\tfrac{1}{2}\sum_{k<\ell} \phi_{k\ell}^P)] \Bigg\}$$

or $b(n_1 \ldots n_{r-1}) =$

$$\sum_{P=1}^{r!} \left\{ \left[\frac{\exp[if_1^P]}{1-\exp[if_1^P]} \right] \exp[i(f_2^P n_1 + \ldots + f_r^P n_{r-1} + \tfrac{1}{2} \sum_{k<\ell} \phi_{k\ell}^P)] \right.$$

$$- \exp[i(f_1^P n_1)] \left[\frac{1}{1-\exp[if_1^P]} - \frac{\exp[if_2^P]}{1-\exp[if_2^P]} \right] \exp[i(f_2^P n_1 + \ldots + f_r^P n_{r-1} + \tfrac{1}{2} \sum_{k<\ell} \phi_{k\ell}^P)]$$

$$- \ldots$$

$$- \exp[i(f_1^P n_1 + \ldots + f_i^P n_i)] \left[\frac{1}{1-\exp[if_i^P]} - \frac{\exp[if_{i+1}^P]}{1-\exp[if_{i+1}^P]} \right] \exp[i(f_{i+1}^P n_i + \ldots + f_r^P n_{r-1} + \tfrac{1}{2} \sum_{k<\ell} \phi_{k\ell}^P)]$$

$$- \ldots$$

$$\left. - \exp[i(f_1^P n_1 + \ldots + f_{r-1}^P n_{r-1})] \left[\frac{\exp[if_r^P (N+1)]}{1-\exp[if_r^P]} \right] \exp[i(\tfrac{1}{2} \sum_{k<\ell} \phi_{k\ell}^P)] \right\} . \quad (10)$$

Now consider one term of (10),

$$T_i^P = \exp[i(f_1^P n_1 + \ldots + f_i^P n_i)] \left[\frac{1}{1-\exp[if_i^P]} - \frac{\exp[if_{i+1}^P]}{1-\exp[if_{i+1}^P]} \right] \exp[i(f_{i+1}^P n_i + \ldots + f_r^P n_{r-1} + \tfrac{1}{2} \sum_{k<\ell} \phi_{k\ell}^P)] .$$

For any sequence of the f's that determines T_i^P one always has present another sequence, P', that does nothing more than interchange f_i^P and f_{i+1}^P. Therefore

$$\frac{T_i^P}{T_i^{P'}} = \frac{\left[\frac{1}{1-\exp[if_i^P]} - \frac{\exp[if_{i+1}^P]}{1-\exp[if_{i+1}^P]}\right]\exp[i\tfrac{1}{2}\sum_{k<\ell}\phi_{k\ell}^P]}{\left[\frac{1}{1-\exp[if_{i+1}^P]} - \frac{\exp[if_i^P]}{1-\exp[if_i^P]}\right]\exp[i\tfrac{1}{2}\sum_{k<\ell}\phi_{k\ell}^{P'}]} .$$

Now let

$$\tfrac{1}{2}\sum_{k<\ell}\phi_{k\ell}^P = \Phi + \tfrac{1}{2}\phi_{i,\,i+1}^P ,$$

so that

$$\tfrac{1}{2}\sum_{k<\ell}\phi_{k\ell}^{P'} = \Phi + \tfrac{1}{2}\phi_{i+1,\,i}^{P'} = \Phi - \tfrac{1}{2}\phi_{i,\,i+1}^P ,$$

and after some reduction,

$$\frac{T_i^P}{T_i^{P'}} = \left[\frac{\exp[if_i^P]\exp[if_{i+1}^P] + 1-2\exp[if_{i+1}^P]}{\exp[if_i^P]\exp[if_{i+1}^P] + 1-2\exp[if_i^P]}\right]\exp[i\phi_{i,\,i+1}^P] .$$

Substituting $\exp[i\phi_{i,\,i+1}^P]$ as given by (7), it is clear that $T_i^P/T_i^{P'} = -1$

or

$$T_i^P + T_i^{P'} = 0.$$

Therefore all the terms in (9) go to zero except possibly for the first and lost one, so that

$$b(n_1 \ldots n_{r-1}) = \sum_{P=1}^{r!} \left\{ \left[\frac{\exp[if_1^P]}{1-\exp[if_1^P]} \right] \exp[i(f_2^P n_1 + \ldots + f_r^P n_{r-1} + \tfrac{1}{2} \sum_{k<\ell} \phi_{k\ell}^P)] \right.$$

$$\left. - \exp[i(f_1^P n_1 + \ldots f_{r+1}^P n_{r-1})] \left[\frac{\exp[if_r^P(N+1)]}{1-\exp[if_r^P]} \right] \exp[1 \tfrac{1}{2} \sum_{k<\ell} \phi_{k\ell}^P] \right\}. \quad (11a)$$

Since all possible permutations of the f's appear in (11a), it is clear that for any sequence, P,

$$f_1^P, f_2^P, \ldots f_r^P$$

there is another sequence P′ such that

$$f_1^{P'} \equiv f_2^P, \; f_2^{P'} = f_3^P, \; \ldots \; f_r^{P'} = f_1^P .$$

Now adding the first term on the right-hand side of (11a) in the permutation P to the second term on the right-hand side of (11a) in the permutation P′, there results

$$\left[\frac{\exp[if_1^P]}{1-\exp[if_1^P]} \right] \exp[i(f_2^P n_1 + \ldots + f_r^P n_{r-1} + \tfrac{1}{2} \sum_{k<\ell} \phi_{k\ell}^P)]$$

$$- \exp[i(f_1^{P'} n_1 + \ldots + f_{r-1}^{P'} n_{r-1})] \left[\frac{\exp[if_r^{P'}(N+1)]}{1-\exp[if_r^{P'}]} \right] \exp[i \tfrac{1}{2} \sum_{k<\ell} \phi_{k\ell}^{P'}] =$$

$$\exp[i(f_2^P n_1 + \ldots + f_r^P n_{r-1})] \left[\frac{\exp[if_1^P]}{1-\exp[if_1^P]} \right]$$

$$\left[\exp[i \tfrac{1}{2} \sum_{k<\ell} \phi_{k\ell}^P] - \exp[if_1^P N] \exp[i \tfrac{1}{2} \sum_{k<\ell} \phi_{k\ell}^{P'}] \right]. \quad (11b)$$

Now

$$\sum_{k<\ell} \phi^{P}_{k\ell} = \phi_{12} + \phi_{13} + \phi_{14} + \ldots + \phi_{1,r-1} + \phi_{1r}$$

$$+ \phi_{23} + \phi_{24} + \ldots + \phi_{2,r-1} + \phi_{2r}$$

$$+ \phi_{34} + \ldots + \phi_{3,r-1} + \phi_{3r}$$

$$+ \vdots$$

$$+ \phi_{r-2,r-1} + \phi_{r-2,r}$$

$$+ \phi_{r-1,r} \quad ,$$

and

$$\sum_{k<\ell} \phi^{P'}_{k\ell} = \phi'_{12} + \phi'_{13} + \phi'_{14} + \ldots + \phi'_{1,r-1} + \phi'_{1r}$$

$$+ \phi'_{23} + \phi'_{24} + \ldots + \phi'_{2,r-1} + \phi'_{2r}$$

$$+ \phi'_{34} + \ldots + \phi'_{3,r-1} + \phi'_{3r}$$

$$+ \vdots$$

$$+ \phi'_{r-2,r-1} + \phi'_{r-2,r}$$

$$+ \phi'_{r-1,r}$$

$$
\begin{aligned}
&= \phi_{23} + \phi_{24} + \phi_{25} + \dots + \phi_{2r} + \phi_{21} \\
&\qquad + \phi_{34} + \phi_{35} + \dots + \phi_{3r} + \phi_{31} \\
&\qquad\qquad + \phi_{45} + \dots + \phi_{4r} + \phi_{41} \\
&\qquad\qquad\quad + \\
&\qquad\qquad\quad \vdots \\
&\qquad\qquad\qquad + \phi_{r-1,r} + \phi_{r-1,1} \\
&\qquad\qquad\qquad\qquad + \phi_{r1} \\
&= - \phi_{12} - \phi_{13} - \phi_{14} - \phi_{15} - \dots - \phi_{1,r-1} - \phi_{1r} \\
&\quad + \phi_{12} + \phi_{13} + \phi_{14} + \phi_{15} + \dots + \phi_{1,r-1} + \phi_{1r} \\
&\qquad + \phi_{23} + \phi_{24} + \phi_{25} + \dots + \phi_{2,r-1} + \phi_{2r} - \phi_{12} \\
&\qquad\quad + \phi_{34} + \phi_{35} + \dots + \phi_{3,r-1} + \phi_{3r} - \phi_{13} \\
&\qquad\qquad + \\
&\qquad\qquad \vdots \\
&\qquad\qquad\qquad + \phi_{r-1,r} - \phi_{1,r-1} \\
&\qquad\qquad\qquad\qquad - \phi_{1r},
\end{aligned}
$$

and finally

$$\sum_{k<\ell} \phi'_{k\ell} = \sum_{k<\ell} \phi_{k\ell} - 2 \sum_{j=2}^{r} \phi_{1j}.$$

Therefore the last bracketed term in (11a) is

$$\left[\exp[\, i \tfrac{1}{2} \sum_{k<\ell} \phi^{P}_{k\ell}] - \exp[\, iNf_1^{P}] \exp[\, i\tfrac{1}{2} \sum_{k<\ell} \phi^{P'}_{k\ell}] \right] =$$

$$\exp[\, i \tfrac{1}{2} \sum_{k<\ell} \phi^{P}_{k\ell}] - \exp[\, iNf_1^{P}] \exp[\, i\tfrac{1}{2} \sum_{k<\ell} \phi^{P}_{k\ell}] \exp[\, -i \sum_{k=2} \phi 1j] =$$

$$\exp[\, \tfrac{1}{2} \sum_{k<\ell} \phi^{P}_{k\ell}] \left[1 - \exp[\, i (Nf_1^{P} - \sum_{j=2} \phi_{1j})] \right].$$

But from the first of equation (6a) it is clear that the result is zero. (It is understood that terms like $\phi_{kk} = 0$ in (6b)). Therefore the terms on the right hand side of (11a) drop out in pairs, leaving us with the result that

$$b(n_1 \ldots n_{r-1}) = 0.$$

When this result is substituted into (8) we have what we set out to accomplish,

$$S^{+} \psi_{o}(N, r) = 0 , \qquad r < \frac{N}{2} \; . \tag{12}$$

It should be emphasized that (12) has been proved only for the case when all r of the f's are non zero, since otherwise zero denominators would have occurred, thus requiring a more careful analysis than was done. It should also be emphasized that (12) has been proved only for the case $r<\frac{N}{2}$. This is because we have made use of the f's supplied by Bethe's theory, and he has supplied them only for the case $r<\frac{N}{2}$, which, however, is all that is necessary. Indeed it can be shown that

$$S^{+}\psi_{o}(N, r) \neq 0 \text{ if } r>\frac{N}{2}. \tag{13}$$

This may be done as follows: We realize first of all that the eigenfunctions have always been written so that a(m's) coefficients contain indices that locate the position of the inverted spins but are not dependent on the location of the individual up spins. Mathematically, there is no justification for this; one could just as well make the a(m's) dependent on the location of the up spins rather than the inverted ones and there would be a one to one correspondence of all eigenstates obtained under these conditions with the states expressed by (4) which have been used exclusively so far. Specifically, if we consider $\psi_{o}(N, r)$ with $r<\frac{N}{2}$ we may write a Φ_{o} as just described which has the a(m's) depending on the location of the α's and these are N-r in number. We write the subscript zero on Φ because each distribution of the β's determines uniquely a distribution of the α's, which is to say that the a(m's) in Φ are dependent on all the indices locating the α's. Now there is nothing in the system to distinguish up from down. Such a space inversion is equivalent to exchanging S^{+} and S^{-} and exchanging the α's and β's. Therefore when we have

$r > \frac{N}{2}$ to start with, we wind up with an eigenfunction with $r' > \frac{N}{2}$ so that

$$S^+\psi_o(N, r) = 0$$

is equivalent to

$$S^-\psi_o(N, N-r) = 0$$

or

$$S^-\psi_o(N, r') = 0, \quad r' > \frac{N}{2}. \tag{14}$$

At this point it is convenient to introduce

$$S^2 = S^-S^+ + S^z + (S^z)^2 \tag{15}$$

$$= S^+S^- - S^z + (S^z)^2 \tag{16}$$

Thus, using (12), (15), and (16)

$$S^2\psi_o(N, r) = \tfrac{1}{2}(N-2r)[\tfrac{1}{2}(N-2r) + 1]\psi_o(N, r)$$

$$= S^+S^-\psi_o(N, r) + \tfrac{1}{2}(N-2r)[\tfrac{1}{2}(N-2r)-1]\psi_o(N, r) \tag{17}$$

for $r < \frac{N}{2}$. Similarly we find, using (14), (15) and (16),

$$S^2\psi_o(N, r) = \tfrac{1}{2}(N-2r)[\tfrac{1}{2}(N-2r)-1]\psi_o(N, r), \tag{17b}$$

for $r > \frac{N}{2}$. Thus $\psi_o(N, r)$ is an eigenstate of S^2, which is expected since $[S^2, \mathcal{H}_1] = 0$. But we now see from (17) that for $r < \frac{N}{2}$

$$S^+S^-\psi_o(N, r) = (N-2r)\psi_o(N, r), \tag{18}$$

that is,

$$S^-\psi_o(N, r) \neq 0 \qquad r < \frac{N}{2}. \tag{19}$$

The corresponding statement when a space inversion is made is (13), so its correctness is established.

Now since $[S^-, \mathcal{H}_I] = 0$, $S^-\psi_o(N, r)$ must be an eigenfunction of $\mathcal{H}_I$, that is,

$$S^-\psi_o(N, r) = K\psi_1(N, r),$$

or

$$= K'\psi_o(N, r+1)$$

since the total number of inversions certainly increases by one. The subscript on ψ represents the number of inversions having associated f values of zero. We may exclude $K'\psi_o(N, r+1)$ as a possibility since then $S^+\psi_o(N, r+1)$ would be zero [according to (12)] so we would have a contradiction in (18). Therefore

$$S^-\psi_o(N, r) = K\psi_1(N, r).$$

If we assume all the state functions normalized, the constant may be evaluated as follows:

$$(S^- \psi_o(N, r),\ S^- \psi_o(N, r)\ ) = K^* K,$$

$$(\psi_o(N, r),\ S^+ S^- \psi_o(N, r)\ ) = K^* K,$$

so that, using (18),

$$K^* K = (N-2r),$$

and finally,

$$S^- \psi_o(N, r) = (N-2r)^{1/2} \psi_1(N, r)\ . \qquad (20)$$

Now (18) becomes

$$S^+ \psi_1(N, r) = (N-2r)^{1/2} \psi_o(N, r)\ . \qquad (21)$$

We now assume for some particular h, where $h<N-2r$, it is true that

$$S^- \psi_h(N, r) = \{(h+1)(N-2r-h)\}^{1/2} \psi_{h+1}(N, r)\ , \qquad (22)$$

and

$$S^+ \psi_{h+1}(N, r) = \{(h+1)(N-2r-h)\}^{1/2} \psi_h(N, r)\ , \qquad (23)$$

where all the state functions are orthonormal. We now establish the correctness of (22) and (23) by induction. Operating on (23) with S^-, and with the help of (4), there results

$$S^-S^+\psi_{h+1}(N, r) = \{(h+1)(N, 2r-h)\}^{1/2} S^-\psi_h(N, r) ,$$

$$(S^+S^- - 2S^z)\psi_{h+1}(N, r) = (h+1)(N-2r-h)\psi_{h+1}(N, r) ,$$

$$S^+S^-\psi_{h+1}(N, r) = (h+2)(n-2r-h-1)\psi_{h+1}(N, r) . \tag{24}$$

Now

$$S^-\psi_{h+1}(N, r) = K\psi_{h+2}(N, r)$$

or

$$= K'\psi_{h+1}(N, r+1).$$

Choosing the latter, we may substitute back into (24), giving

$$K'S^+\psi_{h+1}(N, r+1) = (h+2)(N-2r-h-1)\psi_{h+1}(N, r)$$

and, using (23),

$$K'\{(h+1)(N-2r)\}^{1/2}\psi_h(N, r+1) = (h+2)(N-2r-h-1)\psi_{h+1}(N, r),$$

which is impossible since $\psi_h(N, r+1)$ is orthogonal to $\psi_{h+1}(N, r)$. If we choose the former, we may determine K according to

$$(S^- \psi_{h+1}(N, r),\ S^- \psi_{h+1}(N, r)\ ) = K^* K$$

or

$$K^* K = (\psi_{h+1}(N, r),\ S^+ S^- \psi_{h+1}(N, r)\ )$$

$$= (h+2)(H-2r-h-1)\ ,$$

when using (24). Then

$$S^- \psi_{h+1}(N, r) = \{(h+2)(N-2r-h-1)\}^{1/2} \psi_{h+2}(N, r)$$

and

$$S^+ \psi_{h+2}(N, r) = \{(h+2)(N-2r-h-1)\}^{1/2} \psi_{h+1}(N, r);$$

i.e., if (22) and (23) are true for a particular h, then they must also be true for h+1. Since (22) and (23) are true for h = 0, we immediately have

$$(S^-)^h \psi_o(N, r) = A\psi_h(N, r), \qquad h < N-2r$$

$$= 0, \qquad h \geq N-2r, \tag{25}$$

where

$$A^2 = \prod_{j=0}^{h} (j+1)(N-2r-j).$$

So far we have proved that each of the h inversions introduced by the S^- operator has an f value of zero. Further, it is apparent by looking at the explicit specification of ψ following (4) that the S^- operation introduces inversions whose location indices do not appear in the a(m) coefficients. Now since there is only one state of the system, $\psi_h(N, r)$, for each specified set of f values and since such a state can be generated from $\psi_o(N, r)$ for the same r specified f values, it follows that the states, $\psi_o(N, r)$, and those generated from them be means of S^- will include all possible states of the system. We therefore have established, that only the <u>h</u> inversions introduced by the S^- operator have an <u>f</u> value of zero. Further, the inversions introduced by the S^- operator, and only those, have location indices that do not appear in the a(m) coefficients. Therefore, we have proved the following statement: Those inversions, and only those, introduced by the S^- operator, i.e., all those inversions whose location indices do not appear in the a(m) coefficients, individually have an associated f value of zero. This statement, together with the result (25), is most important for use in reference (2). It should be noted that in reference (2) the f's are replaced by k's and non-normalized state functions are used.

3. Summary

To summarize what has been accomplished we note that we have used the exact eigenstates of a linear chain that were derived by Bethe. These states are complete and are characterized by collections of f's, which can be real or complex as well as zero or non-zero and which become the usual wave numbers in the simple spin wave theory. We then focused our attention on all the f's collectively that are zero and found that they may be given a physical interpretation that is distinct from the non-zero f's. This interpretation is that excitation of the $f = 0$ modes simply corresponds to tipping the total angular momentum vector with respect to the vertical and does not lead to a diminution of the magnitude of the vector, while excitation of $f \neq 0$ modes corresponds to a diminution of the vector but does not change its orientation. We may make this interpretation since from (1), (17b) and (25) it can be seen that the eigenvalue of S^2 is independent of h as long as $h < N-2r$. These results are exact and permit an arbitrary number of inversions in the system and so are not limited to the usual spin wave theory in which the temperature must be low compared to the Curie temperature to insure that the relative number of inversions be small.

4. References

1. H. A. Bethe, Z. Physik, 71, 205, (1931).

2. H. Unruh, Jr., NBS Tech. Note 327.

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