To Mr Bye

from his affectionate form, on account of his election to Tame Grammar School

FIRST THREE SECTIONS OF THE FIRST BOOK

OF

NEWTON'S PRINCIPIA.

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# NEWTON'S PRINCIPIA,

## FIRST BOOK, SECTIONS I., II., III.,

WITH

## NOTES AND ILLUSTRATIONS,

AND A

## COLLECTION OF PROBLEMS

PRINCIPALLY INTENDED AS EXAMPLES OF NEWTON'S METHODS.

BY

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*Principiis enim cognitis, multo facilius extrema intelligetis.*—CICERO.

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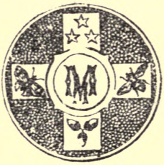

# PREFACE.

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In publishing the following work my principal intention is to explain difficulties which may be encountered by the student on first reading the *Principia*, and to illustrate the advantages of a careful study of the methods employed by Newton, by shewing the extent to which they may be applied in the solution of problems. I have also endeavoured to give assistance to the student who is engaged in the study of the higher branches of Mathematics, by representing in a geometrical form several of the processes employed in the Differential and Integral Calculus, and in the analytical investigations of Dynamics.

In my version of the first section and the beginning of the second I have adhered as closely as I could to the original form; and, in the cases in which sections have been interpolated, or the form of demonstration changed, I have indicated such changes and interpolations by brackets.

It is generally advisable not to deviate from Newton's words in the demonstrations of the Lemmas; but in many cases, I suppose purposely, he expressed himself very concisely, as in Lemmas IV. and X., and he was contented with simply giving the enunciation of Lemma V.; therefore in these cases interpolations have been made which, I believe, are in accordance with Newton's plan of demonstration.

Throughout the Problems and Theorems which depend upon the sixth proposition, the variations are replaced by equations. By this method of treating the subject I conceive that clearer ideas of the meaning of each step are obtained by the student.

In this edition I have introduced some notes on the geometrical solution of some problems relating to maxima and minima, and I have placed the investigations of the properties of the curves, which, after the conic sections, are the best examples for illustrating geometrical methods, in a more prominent position, at the end of the first section.

I have derived great assistance in the preparation of my notes from the study of Whewell's *Method of Limits*, and from several early editions of *Newton*, especially that of Carr.

With respect to the three Laws of Motion, I may remark that I have not commenced the work by enunciating and making observations upon them, partly because I should only have been repeating what has been said so well by Thompson, Tait, and Maxwell, whose works are in everybody's hands, and partly because in the course of reading recommended to students, for whose benefit my work was especially intended, those laws will have been already discussed in the elementary treatises on Dynamics.

The Problems are principally selected from the papers set in the Mathematical Tripos, and in the course of the College Examinations, and I have generally divided them into two portions, the first of which contains those problems which are capable of solution by more direct applications of the propositions which they illustrate, and are within the powers of a larger number of students In both portions I have been careful to introduce very few problems which are not capable of solution by methods given in the work.

At the end of the work I have given hints for the solution, and in many cases complete solutions, of the problems; and in doing so I am acting in direct opposition to my previously expressed opinion, but additional experience of fifteen years has shewn me that it a satisfaction to a student who has not been able to solve a problem to see a solution of it; and, even when he has been successful, to compare his solution with that of an older hand. The principal objection to the publication of solutions is that they are frequently referred to prematurely; but a wise student will treat them only as a *dernier ressort.*

In solving the Problems I have noticed two errors which should be corrected as follows:

XIII. 12. ....half the chord....is the harmonic mean, &c.

XXVIII. 6. ....velocity in a circle whose radius is the length of the unstretched string, &c.

Two sets of Problems have been numbered XXVII., the second is written XXVII. *bis* in the Solutions.

I take this opportunity to express my thanks to Mr. Stearn, of King's College, for his kindness in correcting the errors of the press and for many valuable suggestions.

PERCIVAL FROST.

CAMBRIDGE,
*February*, 1878.

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# NEWTON'S FIRST BOOK

## CONCERNING THE MOTION OF BODIES.

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### *SECTION I.*

### ON THE METHOD OF PRIME AND ULTIMATE RATIOS.

### LEMMA I.

*Quantities, and the ratio of quantities, which, in any finite time, tend constantly to equality, and which, before the end of that time, approach nearer to each other than by any assigned difference, become ultimately equal.*

If not, let them become ultimately unequal, and let their ultimate difference be $D$. Hence [since, throughout the time, they tend constantly to equality], they cannot approach nearer to each other than by the difference $D$, contrary to the hypothesis [that they approach nearer than by any assigned difference. Therefore, they do not become ultimately unequal, that is, they become ultimately equal].

*Variable Quantities.*

1. The *Quantities*, of which Newton treats in this Lemma, are variable magnitudes, described by a supposed law of construction, the variation of these magnitudes being due to the arbitrary progressive change of some element of the construction employed in the statement of the law.

When, in the progressive change of this element, it receives the last value which is assigned to it in any proposition, the hypothesis is said to arrive at its ultimate form, or to be indefinitely extended.



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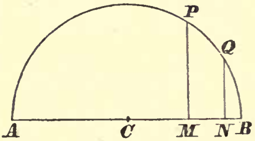
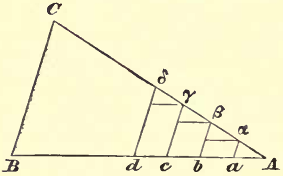
Thus, let $ABP$ be a semicircle, $ACB$ its diameter, $BP$ any arc, $PM$ the ordinate perpendicular to $ACB$, then, as the arc $BP$ gradually diminishes, $AM$ is a variable magnitude, continually increasing, and $BP$ is the element of the construction,




to the arbitrary change of which the variation of $AM$ is due; and if $BP$ may be made as small as we please, $AM$ may be made to approach to $AB$ nearer than by any difference that can be named, and the hypothesis approaches its ultimate form.

Again, if $ABC$ be a triangle, and $AB$ be divided into a number of equal portions, $Aa$, $ab$, $bc$, ..., and a series of parallelograms be inscribed upon those bases, whose sides $a\alpha$, $b\beta$, $c\gamma$, ... are parallel to $BC$ and terminated in $AC$, the sum of the areas of the parallelograms will be a variable magnitude, defined by that construction, and changing in a progressive manner, if the




number of parts into which $AB$ is divided be continually increased. In this case the number of parts is the variable element of the construction. In the ultimate form of the hypothesis, it will be shewn, Lemma II., that the sum of the parallelograms is the area of the triangle when the number is increased indefinitely.

2. The variation of a magnitude is *continuous*, when in the passage from *any* one value to *any* other, throughout its change,

it receives every intermediate value, without becoming infinite. When this is not the case, the variation is *discontinuous.*

According to the hypothesis in the last illustration, the number of parts into which $AB$ is divided being exact, the magnitude varies discontinuously, *i.e.* the sum of the areas does not pass through all the intermediate values between any two states of the progress.

If the hypothesis be changed, equal portions being set off commencing from $B$, and $Aa$ remaining over and above after $ba$, the last of the portions for which there is room, these equal portions could be made to diminish gradually, and the sum of the areas would in that case vary continuously.

*Tendency to Equality.*

3. Quantities are ultimately equal, when they are ultimately in a ratio of equality.

4. Quantities, which always remain *finite*, throughout the change of the hypothesis by which they are described, tend continually to equality, when their difference continually diminishes.

Thus, in fig. 1, page 2, let $BQ$ be an arc, always in a given ratio to $BP$, and let $QN$ be the corresponding ordinate; a $BP$ continually diminishes, $AM$ and $AN$ remain finite, and, since their difference continually diminishes, they tend continually to equality.

5. Quantities, which may become *indefinitely small*, or *indefinitely great*, as the hypothesis is indefinitely extended, tend continually to equality, when the ratio of their difference to either of them continually diminishes.

To illustrate this test of a tendency to equality, let us suppose, in fig. 1, page 2, that the arc $BP$ is double of the arc $BQ$; then, since $(\text{chd}\,BP)^2 = AB.BM$, and $(\text{chd}\,BQ)^2 = AB.BN$,

$$\therefore\ BM : BN :: (\text{chd}\,BP)^2 : (\text{chd}\,BQ)^2$$

$$:: (\text{arc}\,BP)^2 : (\text{arc}\,BQ)^2 :: 4 : 1 \text{ ultimately},$$

$$\therefore\ MN : BN :: 3 : 1 \text{ ultimately};$$

hence, we observe that $BM$ and $BN$ have a difference, which tends continually to become $3BN$, the ratio of which to either is finite, so that, although both tend to become indefinitely small as the hypothesis tends to its ultimate form, $BM$ and $BN$ do not satisfy the condition requisite for a tendency to equality.

*Observations on the Lemma.*

6. We will now proceed to examine the force of the other important terms employed in the statement of the first Lemma.

The expression "in any finite time" (tempore quovis finito), signifies what has been called the indefinite extension of the hypothesis from some definite state to its ultimate form.*

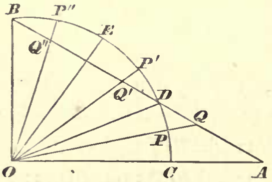
The law of the variation of the magnitudes under consideration is obtained by the examination of their construction while the element, to which the change is due, is at a finite distance from its final value, and the finite time is the supposed time occupied in the passage from this definite to the ultimate state.

In the first illustration, Art. 1, it denotes the progressive diminution of $BP$, from being a *finite* magnitude to the point of evanescence.

In the second, the progress from *any* finite number of equal portions to an indefinite number.

7. The expression "which *constantly* tend" (quæ constanter tendunt) signifies that, from the commencement of the *finite time* to the limit of the extension of the hypothesis, the differences continually diminish.

To illustrate this mode of expression, let $BC$ be a quadrant




* Whewell's *Doctrine of Limits.*

of a circle whose bounding radii are $OB$, $OC$, and let $BDA$ be a straight line cutting the arc $BDC$ and the radius $OC$ in $D$ and $A$, and let $OP$ be a radius revolving from $OC$ to $OB$, and cutting $BA$ in $Q$, $E$ the point of bisection of the arc $BD$.

$OP$ and $OQ$ twice tend *to* equality, viz. from $OC$ to $OD$ and from $OE$ to $OB$, and once *from* equality from $OD$ to $OE$; it is only from $OE$ to $OB$ that $OP''$ and $OQ''$ tend to equality constantly during the progress, and it is from some position between $OE$ and $OB$ that the finite time must be considered to commence.

8. "Before the end of that time" (ante finem temporis) implies that, however small the given difference may be, a less difference than that difference is arrived at, while the distance from the ultimate state is still finite, however near to the final state it may be necessary to proceed.

Thus, if, in the last figure, the angle $BOD$ be 60°, the radius one inch, and the given difference $\frac{36}{100000}$ or $\frac{18}{1000000}$ of an inch, the difference $PQ$ will be less than the given difference, if the revolving radius be $2'$ or $1'$, respectively, from the ultimate position; and so on, however small we choose the difference.

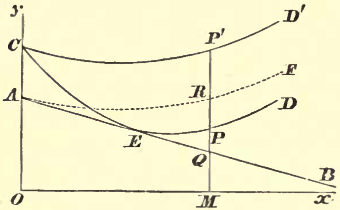
9. In the proof of the Lemma, if the ultimate difference be $D$, the quantities cannot approach nearer than by that given difference; otherwise, they would, in one part of the progression, have been tending *from* equality in order to arrive ultimately at that difference, contrary to the statement of the proposition in the words "ad æqualitatem constanter tendunt."

The nature of the proof, which is more difficult than may at first sight appear, can be illustrated as follows, by examining the effect of the omission of some of the points in the statement of the Lemma.

Draw $Oy$, $Ox$ at right angles, $AB$ any straight line meeting $Oy$ in $A$, $CED$ a curve touching $AB$ in $E$ and meeting $Oy$ in $C$, $CD'$ another touching a straight line parallel to $AB$ in $C$, $MQPP'$ a common ordinate.

As $OM$ diminishes until it becomes indefinitely small, $MQPP'$ moves up to $Oy$.

In both curves, the ordinates $MQ$ and $MP$ or $MP'$ have an ultimate difference $CA$, equal to $D$ suppose.




Omit the word "constanter," and the curve $CED$ is admissible in a representation of the approach of the quantities; because the ordinates approach, before the end of the time, nearer than by any assignable difference, as at $E$, although the condition of continual tendency to equality is not satisfied.

Omit the words "ante finem temporis," and $CD'$ will be sufficient; for, in this case, they tend continually to equality, but before the end of the time they do not approach nearer than by any assignable difference, and they are ultimately unequal.

In the case of the dotted line $ARF$ touching $AB$ at $A$, all the conditions are satisfied. $QM$ and $RM$ tend continually to equality, and their difference may be made less than any given difference before $OM$ vanishes.

### *Limit of a Variable Quantity.*

10. When a variable quantity tends continually to equality with a certain fixed quantity, and approaches nearer to this quantity than by any assignable difference, as the hypothesis determining its variation is approaching its ultimate form, this fixed quantity is called the *Limit of the variable quantity.*

The tests are: that there should be a tendency to equality; that this tendency should be continued from some finite condition; and that the approach should, during the progression to the ultimate form, be nearer than by any assignable difference.

Thus, as is mentioned in the Scholium at the end of the section, the variable quantity does not become equal to, or surpass the limit, before the arrival at the ultimate form.

*Limiting Ratio of Variable Quantities.*

11. If two quantities continually diminish or increase, and the ratio of these quantities tends continually to equality with a certain fixed ratio, and may be made to differ from that ratio by less than any assignable difference, as the hypothesis determining their variation is indefinitely extended, this fixed ratio is called the *limiting ratio* of the varying quantities.

*Ultimate Ratio of Vanishing Quantities.*

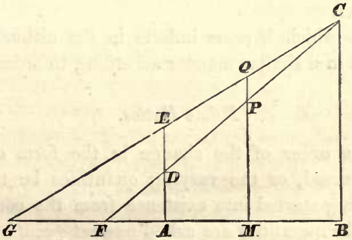
12. When the ultimate form of the hypothesis brings the quantities to a state of evanescence, they are called *vanishing quantities;* and the limiting ratio, or the limit of the ratio, is the *ultimate ratio of the vanishing quantities.*

The expression "vanishing quantities" does not imply that the quantities *are* indefinitely small while under examination, but only that they *will be* so in the ultimate form; which observation implies that *the ratio of the vanishing quantities* is not an equivalent expression with the *ultimate ratio of the vanishing quantities*, the former being taken "ante finem temporis."

"Ultimæ rationes illæ quibuscum quantitates evanescunt, revera non sunt rationes quantitatum ultimarum." See Scholium, at the end of the section.

Thus, let $GC$, $FC$ be two straight lines intersecting $AB$ in $G$, $F$, and draw $ADE$, $MPQ$, perpendicular to $AB$.

Let $\alpha$, $\beta$ be the areas $AMPD$, $AMQE$, then it is easily found



that $\alpha : \beta :: AD + MP : AE + MQ$; now, let $MPQ$ be supposed to move up to $ADE$, then, in the ultimate form of the hypothesis, $\alpha$ and $\beta$ vanish, and are called vanishing quantities from this circumstance.

Also, the *ultimate ratio* of the vanishing quantities is $AD : AE$.

In this case, since $MP : MQ$ is not equal to $AD : AE$, the ratio of the vanishing quantities, viz. $AD + MP : AE + MQ$, is different from $AD : AE$, the ultimate ratio.

### *Orders of Vanishing Quantities.*

13. When we have to consider various kinds of vanishing quantities, it is necessary to consider their relative magnitudes, and for this purpose if one of them be selected as a standard of small quantities, this quantity, and all the vanishing quantities of which the ultimate ratio to it is finite, are called vanishing quantities *of the first order.*

If $\alpha$, $\beta$ be any two vanishing quantities, and $\beta : \alpha$ vanish in the limit, $\beta$ is said to be a vanishing quantity of a higher order than $\alpha$.

If $\alpha$ be of the first order, and $\beta : \alpha^2$ be ultimately finite, $\beta$ is called a vanishing quantity *of the second order*, and so on for higher orders.

Trigonometrical functions give familiar illustrations of these orders; let $\theta$ be taken as the standard of vanishing quantities; $\sin\theta$ $\tan 2\theta$, $\sin\frac{1}{2}\theta$ are all of the first order, since their ratios to $\theta$ are ultimately 1, 2 and $\frac{1}{2}$; $\text{vers}\,\theta$, which is equal to $2\sin^2\frac{1}{2}\theta$ is of the second order, $\tan\theta - \theta$ and $\theta - \sin\theta$ are of the third order.

Quantities which become infinite in the ultimate state are also classified in a similar manner according to orders.

### *Prime Ratios.*

14. If the order of the change in the form of the hypothesis be reversed, or the varying quantities be tending from equality, having started into existence from the commencement of the time, the quantities are called *nascent quantities;* and the

ratio with which they commence existence is called the *prime ratio* of the nascent quantities.

*Application of Lemma I to the investigation of certain Limits.*

(1) *Limit of* $\frac{1+x}{2-x}$, *as* $x$ *gradually diminishes, and ultimately vanishes.*

Since the difference between $\frac{1+x}{2-x}$ and $\frac{1}{2}$ is $\frac{3x}{2(2-x)}$, this difference continually diminishes as $x$ gradually diminishes, and, by diminishing $x$ sufficiently, may be made less than any assignable difference.

Hence, $\frac{1+x}{2-x}$ will tend continually to equality with $\frac{1}{2}$, if we commence from some value of $x$ less than 2, and the difference may be made less than any assignable quantity *ante finem temporis*, therefore $\frac{1}{2}$ satisfies all the conditions of being the required limit.

(2) *Limit of* $\frac{2+x}{5+3x}$, *when* $x$ *increases indefinitely.*

Since the difference $\frac{2+x}{5+3x} - \frac{1}{3} = \frac{1}{3(5+3x)}$, which continually diminishes as $x$ increases, and may be made less than any assignable difference; therefore, as before, $\frac{1}{3}$ satisfies all the conditions of being a limit of $\frac{2+x}{5+3x}$.

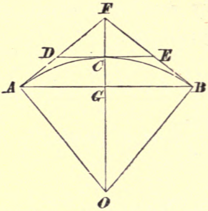
(3) *Tangents are drawn to a circular arc, at its middle point, and at its extremities. Shew that, when the arc diminishes, the area of the triangle formed by the chord of the arc, and the two tangents at the extremities, is ultimately four times that of the triangle formed by the three tangents.*

Let $C$ be the middle point of the arc, $AB$ the chord, $FA$, $FB$, $DCE$ the three tangents, and $O$ the centre of the circle,

$$\triangle FDE : \triangle FAB : FC^2 : FG^2.$$

$$\text{Now} \quad FC(FC+2CO) = FA^2 = FO.FG;$$

$$\therefore FC : FG :: FO : FC+2CO;$$




therefore, since $FC$ vanishes in the limit, $FC : FG :: CO : 2CO$ and $FG = 2FC$, ultimately;

$$\therefore\ \triangle FDE : \triangle FAB :: 1 : 4.$$

(4) *Limit of* $\dfrac{x^m - 1}{x - 1}$, *when* $x$ *differs from* 1 *by an indefinitely small quantity,* $m$ *being any number, integral or fractional, positive or negative.*

First, where $m$ is a positive whole number,

$$\frac{x^m - 1}{x - 1} = x^{m-1} + x^{m-2} + \ldots + x + 1,$$

which may be made to differ from $m$ by less than any assignable difference by taking $x$ sufficiently near to unity.

Next, let $m = \dfrac{p - q}{r}$, $p$, $q$, and $r$ being positive whole numbers, and let $x = y^r$;

$$\therefore\ \frac{x^m - 1}{x - 1} = \frac{y^{p-q} - 1}{y^r - 1} = \frac{1}{y^q} \cdot \frac{y^p - y^q}{y^r - 1} = \frac{1}{y^q} \cdot \frac{y^p - 1 - (y^q - 1)}{y^r - 1}$$

$$= \frac{1}{y^q} \cdot \frac{\dfrac{y^p - 1}{y - 1} - \dfrac{y^q - 1}{y - 1}}{\dfrac{y^r - 1}{y - 1}}.$$

This may be made to differ from $\dfrac{p - q}{r}$ or $m$ by a quantity less than any assignable quantity by taking $x$, and therefore $y$, sufficiently near to unity; hence, whether it be integral or fractional, positive or negative, $m$ is the limit required.

When we divide the numerator and denominator by $y - 1$, $y$ is not equal to 1, the time chosen being *ante finem temporis* while the difference is finite. See the direction in the Scholium referred to above: "Cave intelligas quantitates magnitudine determinatas, sed cogita semper diminuendas sine limite."

(5) *Limit of* $\dfrac{1^p + 2^p + 3^p + \ldots + n^p}{n^{p+1}}$, *when $n$ is indefinitely increased, $p$ being any positive number.*

Since this sum is the arithmetic mean of the $n$ fractions

$$\left(\frac{1}{n}\right)^p, \left(\frac{2}{n}\right)^p, \ldots \left(\frac{n}{n}\right)^p,$$

therefore, for all positive values of $p$, integral or fractional, it lies between $\left(\frac{1}{n}\right)^p$ and $\left(\frac{n}{n}\right)^p$ or 1, therefore its ultimate value lies between 0 and 1.

This being an important limit, we will investigate it first for the particular case in which $p$ is integral and positive, and then generally when $p$ is any positive quantity.

$$\text{Let } S_n = 1^p + 2^p + \ldots + n^p;$$

$$\text{then } S_{n+1} = 1^p + 2^p + \ldots + n^p + (n+1)^p;$$

$$\therefore S_{n+1} - S_n = (n+1)^p.$$

If therefore we assume that

$$S_n = An^{p+1} + Bn^p + \ldots + Ln + M,$$

$$\text{then } S_{n+1} = A(n+1)^{p+1} + B(n+1)^p + \ldots + L(n+1) + M;$$

$$\begin{aligned} \therefore (n+1)^p &= A\{(n+1)^{p+1} - n^{p+1}\} + B\{(n+1)^p - n^p\} + \ldots \\ &= A\{(p+1)n^p + \tfrac{1}{2}(p+1)pn^{p-1} + \ldots\} \\ &\quad + B\{pn^{p-1} + \tfrac{1}{2}p(p-1)n^{p-2} + \ldots) + \ldots, \end{aligned}$$

we obtain, by equating the coefficients, $p+1$ equations for determining the values of the $p+1$ constants $A, B, \ldots L$, which reduce the equation to an identity.

The first of these equations is $1 = (p+1)A$;

$$\therefore S_n = \frac{1}{p+1} . n^{p+1} + Bn^p + \ldots,$$

$$\text{and } \frac{S_n}{n^{p+1}} = \frac{1}{p+1} + \frac{B}{n} + \frac{C}{n^2} + \ldots + \frac{M}{n^{p+1}};$$

hence, if $n$ be increased, since the number of the terms following $\frac{1}{p+1}$ is finite, we may make the difference between $\frac{S_n}{n^{p+1}}$ and $\frac{1}{p+1}$ diminish until it becomes less than any assignable quantity;

therefore $\frac{1}{p+1}$ is the limit required.

Next, let $p$ be any positive quantity, and let $l$ be the limit of

$$\frac{1^p + 2^p + \ldots + n^p}{n^{p+1}};$$

$$\therefore \; 1^p + 2^p + \ldots + n^p = ln^{p+1} + Bn^\beta + Cn^\gamma + \ldots,$$

in which $p+1, \beta, \gamma \ldots$ are in descending order, and $\frac{Bn^\beta + Cn^\gamma + \ldots}{n^{p+1}}$ vanishes, when $n$ is made infinitely large;

$$\therefore \; 1^p + 2^p + \ldots + (n+1)^p = l\,(n+1)^{p+1} + B\,(n+1)^\beta + \ldots;$$

$$\therefore \; (n+1)^p = l\,\{(n+1)^{p+1} - n^{p+1}\} + B\,\{(n+1)^\beta - n^\beta\} + \ldots;$$

$$\therefore \; \left(1+\frac{1}{n}\right)^p = l.\,\frac{\left(1+\frac{1}{n}\right)^{p+1} - 1}{1+\frac{1}{n} - 1} + \frac{Bn^\beta}{n^{p+1}}.\,\frac{\left(1+\frac{1}{n}\right)^\beta - 1}{1+\frac{1}{n}-1} + \ldots;$$

therefore, observing that, when $n$ is increased indefinitely,

$$\frac{\left(1+\frac{1}{n}\right)^q - 1}{1+\frac{1}{n}-1} = q,$$

$$1 = (p+1)\,l + \text{limit of } \frac{\beta\,(1+\varepsilon)\,Bn^\beta + \gamma\,(1+\varepsilon')\,Cn^\gamma + \ldots}{n^{p+1}};$$

where $\epsilon$, $\varepsilon'$, ... vanish ultimately. Let $\varepsilon_1$ be the greatest of the quantities $\varepsilon$, $\varepsilon'$, ..., and let all the terms be positive, then

$$\beta\,(1+\varepsilon)\,Bn^\beta + \ldots \text{ is less than } (1+\varepsilon_1\,\beta\left(Bn^\beta + \frac{\gamma}{\beta}\,Cn^\gamma + \ldots\right),$$

and, since $\frac{\gamma}{\beta}$, $\frac{\delta}{\beta}$ ... are each less than 1,

$$\frac{\beta\,(1+\varepsilon)\,Bn^\beta + \ldots}{n^{p+1}} \text{ is less than } (1+\varepsilon_1)\,\beta \times \frac{Bn^\beta + Cn^\gamma + \ldots}{n^{p+1}},$$

which vanishes in the limit, hence $1=(p+1)\,l$ ultimately;

$$\text{therefore } \frac{1}{p+1} \text{ is the limit required.}$$

COR. $\dfrac{1}{p+1}$ is evidently also the limit of the sum

$$\frac{1^p+2^p+\ldots+(n-1)^p}{n^{p+1}}, \text{ since } \frac{n^p}{n^{p+1}} \text{ vanishes in the limit.}$$

(6) *If a straight line of constant length slide with its extremities in two straight lines, which intersect at a given angle $A$, and $BC$, $bc$ be two positions of the line intersecting in $P$, which become ultimately coincident, find the limits of the ratios $Cc : Bb$ and $PC : PB$.*

$$\text{By hypothesis, } BC^2=bc^2,$$

$$\text{but } BC^2=BA^2+CA^2-2BA.CA\cos A,$$

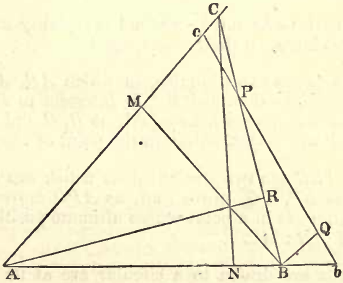
$$\text{and } bc^2=bA^2+cA^2-2bA.cA\cos A;$$

$$\therefore CA^2-cA^2=bA^2-BA^2+2\{BA(cA+Cc)-(BA+Bb)cA\}\cos A;$$

$$\therefore Cc(CA+cA)=Bb(BA+bA)+2(BA.Cc-cA.Bb)\cos A;$$

$$\therefore Cc : Bb :: BA+bA-2cA\cos A : CA+cA-2BA\cos A$$

$$:: BA-CA\cos A : CA-BA\cos A \text{ ultimately.}$$




Draw $CN$, $BM$ perpendicular to $AB$, $AC$, therefore the limit of the ratio $Cc : Bb$ is $BN : CM$.

Again, let $BQ$, drawn parallel to $AC$, meet $bc$ in $Q$,

$$\text{then } PC : PB :: Cc : BQ;$$

$$\text{also } Cc : Bb :: BN : CM \text{ ultimately,}$$

$$\text{and } Bb : BQ :: Ab : Ac;$$

$$\therefore\ Cc : BQ :: BN.AB : CM.AC \text{ ultimately.}$$

Draw $AR$ perpendicular to $BC$, then $BN.AB = BR.BC$ and $CM.AC = CR.BC$;

$$\therefore\ PC : PB :: BR : CR;$$

$$\therefore\ PC = BR \text{ and } PB = CR.$$

## I.

1. Are the limits of the ratios $y^2 : x$ equal in any of the three equations

$$(1)\ y^2 = ax^2,\quad (2)\ y^2 = ax - b^2,\quad (3)\ y^2 = ax - x^2,$$

when $x$ is indefinitely diminished?

2. Find the limit of $\dfrac{x+3}{1+3x}$,

   (1) when $x$ is indefinitely diminished,

   (2) when $x$ is indefinitely increased.

3. Find the ultimate ratio of the vanishing quantities $ax + bx^2$, $bx + ax^2$, when $x$ is made indefinitely small.

4. Prove that $a - bx$ and $b - ax$ tend to equality as $x$ diminishes to zero, and yet have not their limits equal.

5. $BAC$, $bAc$ are two triangles, in which $AB$, $Ab$ and $AC$, $Ac$ are coincident in direction, and $BC$, $bc$ intersect in $P$; prove that, if the areas of the triangles be equal, as $B$, $C$ and $b$, $c$ approach, each to each, $P$ will be ultimately in the point of bisection of $BC$.

6. $APQ$, $ABC$ are two straight lines which are intersected by two fixed lines $BP$, $CQ$, prove that, as $APQ$ moves up to $ABC$, $PC$ and $QB$ intersect in a point whose ultimate position divides $BC$ in the ratio of $AB : AC$.

7. Tangents are drawn to a circular arc at its middle point, and at its extremities, and the three chords are drawn. Prove that the triangle contained by the three tangents is ultimately one-half of that contained by the three chords, when the arc is indefinitely diminished.

8. $AP$ is a chord of a given circle, $AQ$ a chord near $AP$, find the position of the point of ultimate intersection of circles described on $AP$, $AQ$ as diameters, when $AQ$ approaches to and ultimately coincides with $AP$.

9. A circle passes through a fixed point, and cuts off from a fixed line a chord $PQ$ of constant length, prove that the chord of ultimate intersection of two consecutive circles bisects $PQ$.

10. $PN$ is an ordinate, and $PT$ a tangent to an ellipse, cutting the axis-major in $N$ and $T$ respectively; $A$ being the vertex, shew that as $P$ approaches $A$, $NT$ is ultimately bisected in $A$.

11. $APQ$ is a parabola, $PM$, $QN$ ordinates to the axis $AMN$, with centres $M$ and $N$ and radii $PM$, $QN$ two circles are drawn; prove that, when $N$ approaches indefinitely near to $M$, if the two circles intersect, the distance of their point of intersection from $PM$ is ultimately equal to the semi-latus rectum. What is the condition that the circles may intersect?

## II.

1. What is the test of tendency to equality? If two quantities diminish so that their difference diminishes, prove that they will tend to or from equality according as the ratio of their rates of decrease is greater or less than the ratio of the greater to the less.

2. $ABC$ is an isosceles triangle, base $BC$; $P$, $Q$ are points on the straight lines $CA$, $CB$ such that $AP$ is always twice $BQ$; prove that, if $PQ$ and $AB$ intersect in $R$, and $R'$ be the ultimate position of $R$, when $AP$ is indefinitely diminished,

$$R'B : AC :: AC : 2BC \sim AC.$$

3. $PMP'$ is a double ordinate of an ellipse, whose centre is $C$; $R$ is the point of ultimate intersection of the circles described on $PP'$ and the next consecutive double ordinate respectively, and $RT$ is the ordinate of $R$. Shew that $TM : CM :: BC^2 : AC^2$. What is the condition that these circles may intersect?

4. Two concentric and coaxial ellipses have the sum of the squares of their axes equal; if the curves approach to coincidence with each other, shew that the ratio of the distances of any one of their points of intersection from the axes will be ultimately equal to the inverse ratio of the squares of the axes.

5. If a triangle be inscribed in a given circle, prove that the algebraic sum of the small variations of its sides, each divided by the cosine of the angle opposite to it, will be equal to zero.

6. $ABC$, $APQ$ are drawn to cut a circle from an external point $A$; $BU$, $CT$ are tangents at $B$ and $C$ to the circle, meeting $APQ$ in $U$, $T$; shew that the ultimate ratio of $PU : QT$, when $APQ$ moves up to $ABC$, is $AB^2 : AC^2$.

7. $BCRA$ is a diameter of a circle whose centre is $C$, and $PRQ$ is a chord in it perpendicular to $BA$. $PR$ is bisected in $S$, and $CS$ meets the circle in $S'$. If tangents at $P$ and $S'$ meet $BA$ in $T$ and $T'$, shew that when $P$ moves up to $A$, $AT = 4AT'$ ultimately.

8. If the quadrilateral $ABCD$ be slightly displaced in its own plane, so as to occupy the position $abCD$, and $O$ be the point of intersection of $DA$, $CB$, prove that the point of ultimate intersection of $ab$ and $AB$ will be the foot of the perpendicular from $O$ upon $AB$.

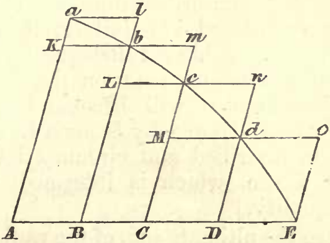
9. $PSp$, $QSq$ are focal chords of a parabola, prove that, ultimately, when $P$ moves up to $Q$,

$$PQ : pq :: SP^{\frac{3}{2}} : Sp^{\frac{3}{2}}.$$

10. The extremities of a straight line slide upon two given straight lines, so that the area of the triangle formed by the three straight lines is constant; find the limiting position of the chord of intersection of two consecutive positions of the circle described about that triangle.

## LEMMA II.

*If, in any figure AacE, bounded by the straight lines Aa, AE and the curve acE, any number of parallelograms Ab, Bc, Cd, &c. be inscribed upon equal bases AB, BC, CD, &c., and having sides Bb, Cc, Dd, &c. parallel to the side Aa of the figure; and the parallelograms aKbl, bLcm, cMdn, &c. be completed; then, if the breadth of these parallelograms be diminished, and the number increased indefinitely, the ultimate ratios which the inscribed figure AKbLcMdD, the circumscribed figure AalbmcndoE, and the curvilinear figure AabcdE have to one another, will be ratios of equality.*

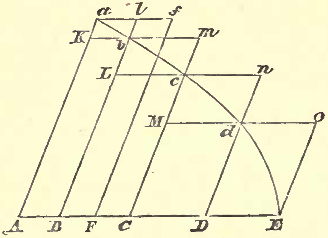



For the difference of the inscribed and circumscribed figures is the sum of the parallelograms *Kl*, *Lm*, *Mn*, *Do*, that is (since the bases of all are equal) a parallelogram whose base is *Kb*, that of one of them, and altitude the sum of their altitudes, that is, the parallelogram *ABla*. But this parallelogram, since its breadth is diminished indefinitely [as the number of parallelograms is increased indefinitely] becomes less than any assignable parallelogram; therefore, by Lemma I., the inscribed and circumscribed figures, and, *a fortiori*, the curvilinear figure, which is intermediate, become ultimately equal.

D

## LEMMA III.

*The same ultimate ratios are also ratios of equality, when the breadths of the parallelograms AB, BC, CD, ... are unequal, and all are diminished indefinitely.*




For, let $AF$ be equal to the greatest breadth, and the parallelogram $FAaf$ be completed. This parallelogram will be greater than the difference between the inscribed and circumscribed figures. But, when its breadth is diminished indefinitely, it will become less than any assignable parallelogram. [Therefore, *a fortiori*, the difference between the inscribed and circumscribed figures will become less than any assignable areas. Hence, by Lemma I., the ultimate ratios of the inscribed and circumscribed and the curvilinear figure, which is intermediate, will be ratios of equality.]

Cor. 1. Hence the ultimate sum of the vanishing parallelograms coincides [as to area] with the curvilinear figure.

Cor. 2. And, *a fortiori*, the rectilinear figure which is bounded by the chords of the vanishing arcs $ab$, $bc$, $cd$, &c., ultimately coincides with the curvilinear figure.

Cor. 3. As also the rectilinear circumscribed figure, which is bounded by the tangents at the extremities of the same arcs.

Cor. 4. And these ultimate figures, with respect to their perimeters $acE$, are not rectilinear figures, but curvilinear limits of rectilinear figures.

*Observations on the Lemmas II. and III.*

15. The statements of the propositions concerning limits of quantities and their ratios contain:

I. The hypothesis by which the quantities are defined.

II. The manner in which the hypothesis approaches its ultimate form.

III. The ultimate property when the hypothesis is thus indefinitely extended.

The strength of the proofs lies in the examination of the quantities while the hypothesis is in a finite state, before arrival at the ultimate form, and the deduction of properties by which the relations of the quantities can be pursued accurately to the ultimate state.

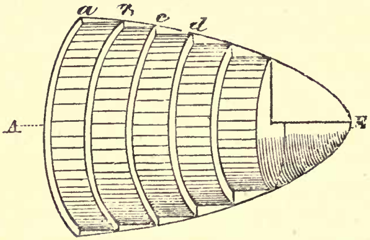
If in this manner we analyse the statement of Lemmas II. and III., the hypothetical constructions are given in the manner of describing the parallelograms; the extension of the hypothesis towards its ultimate form is the continual increase of the number of parallelograms *ad infinitum;* the ultimate property is the equality of the ratio of the sums of the parallelograms and the curvilinear area.

In the proof of the Lemmas, the continual decrease of the parallelograms $Al$ or $Af$ shews that the conditions of ultimate equality of two quantities are all satisfied, viz., that the sums of the two series of parallelograms, since they are finite, tend continually to equality, and that they approach nearer to each other than by any assignable difference "ante finem temporis," *i.e.*, while the number of the parallelograms still remains finite.

*Volumes of Revolution.*

16. In a manner exactly similar to Lemma II. it may be shewn that, if $Aa$ be perpendicular to $AE$, and the whole figure revolve round $AE$ as an axis, the ultimate ratios, which the sums of the volumes of the cylinders, generated respectively by the rectangles $Ab$, $Bc$, ... and $aB$, $bC$, ... and the volume of revolution generated by the curvilinear area $AEa$ will have to each other, will be ratios of equality.

The figure represents the cylinders generated by the inscribed rectangles.

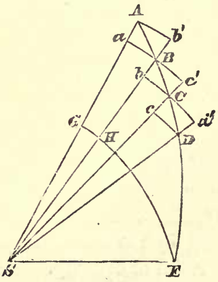



Thus the difference of the cylinders generated by *Ab* and *aB* is the annulus generated by the rectangle *ab*, and the difference of the two series of cylinders, which have all equal heights *AB*, *BC*, ..., is the sum of such annuli, and is easily seen to be the cylinder generated by *aB*, which, since the height continually diminishes, may be made less than any assignable volume, hence the conditions that the two series may have the same limit are satisfied, and hence also the volume of revolution, which is greater than one sum and less than the other, is ultimately in a ratio of equality to either sum.

The same argument applies when the revolution is only through a certain angle instead of being complete, in which case the cylinders are replaced by sectors of cylindrical volumes.

*Sectorial Areas.*

17. The Lemmas may be extended to sectorial areas.



Thus, let $SABCF$ be a sectorial area, and let the angle $ASF$ be divided into equal portions $ASB$, $BSC$, ... and the circular arcs $Ab'$, $aBc'$, $bCd'$, ... be drawn with centre $S$; then, since the difference of the two series of circular sectors is the sum of the areas $ab'$, $bc'$, ..., it is equal to the difference of the greatest and least of the sectors, viz. $AGHb'$; therefore the two areas $SAb'Bc'$... and $SaBbC$... tend continually to equality as the number of angles is increased and their magnitudes diminished, and the ratios which these areas have to each other and to the area $SABF$ are ultimately ratios of equality.

Similarly, as in Lemma III., if $ASB$, $BSC$, ... be unequal.

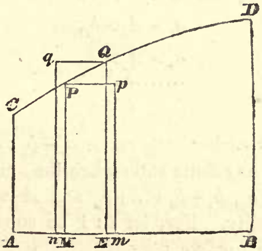
*Surfaces of Revolution.*

18. The following proposition is the extension of the principles of the Lemmas to the determination of a method for finding the area of a surface of a solid of revolution.

Let $CD$ be a plane curve which generates a surface of revolution by its revolution round $AB$, a line in its plane.

$CD$ is divided into portions, of which $PQ$ is one, $PM$, $QN$ are perpendicular to $AB$; $Pp$, $Qq$ are drawn parallel to $AB$, and each equal to $PQ$ in length; $pm$, $qn$ are perpendicular to $AB$. The surface generated by $CD$ shall be the limit of the sum of the cylindrical surfaces generated by such portions as $Pp$ or $Qq$.

For, the cylindrical surfaces generated by $Pp$ and $Qq$ are one less and the other greater than the surface generated by $PQ$,




since every portion of $Qq$ is at a greater, and every portion of $Pp$ at a less, distance from the axis than the corresponding portions of $PQ$.

But these surfaces are respectively $2\pi PM.Pp$ and $2\pi QN.Qq$, and their difference is $2\pi(QN-PM)PQ$, and the ratio of this difference to the surfaces themselves is $QN-PM : PM$ or $QN$, which ratio is ultimately less than any given ratio.

Hence the sums of the surfaces generated by the lines corresponding to $Pp$ and $Qq$ have the ratio of their difference to either sum less than the greatest value of the ratio $QN-PM : PM$, which may be made less than any finite ratio. Therefore the sums of the cylindrical surfaces and the curved surface, which is intermediate in magnitude to these sums, are ultimately in a ratio of equality.

### *Centre of Gravity.*

19. It is easily seen that the same methods are applicable to the determination of the position of the centre of gravity of any body, since it is known that, if a body be divided into any number of portions, the distance of the centre of gravity of the body from any plane is equal to the sum of the moments of all the portions divided by the sum of all the portions.

### *General Extension.*

20. The most general extension may be stated as follows: If any magnitude $A$ be divided into a series of magnitudes $A_1A_2...A_n$, each of which, when their number is increased indefinitely, becomes indefinitely small, and two series of quantities $a_1a_2...a_n$ and $b_1b_2...b_n$ can be found such that

$$a_1 > A_1 > b_1,$$
$$a_2 > A_2 > b_2,$$
$$\dots\dots\dots\dots\dots$$
$$a_n > A_n > b_n,$$

and also such that each of the ratios $a_1 - b_1 : a_1$, $a_2 - b_2 : a_2$, ... becomes less than any finite ratio when the number is increased; then $a_1 + a_2 + ... + a_n$, $b_1 + b_2 + ... + b_n$ and $A$ will be ultimately in a ratio of equality. For, let $l : 1$ be equal to the greatest of the ratios $a_1 - b_1 : a_1$, &c.;

$$\therefore\ a_1 - b_1 + a_2 - b_2 + ... : a_1 + a_2 + ...$$

is a ratio less than $l : 1$, and may therefore be made less than any assignable ratio by increasing the number. Therefore the two series $a_1+a_2+\ldots$ and $b_1+b_2+\ldots$ tend continually to equality, and the difference may be made, before the end of the time, less than any assignable magnitude; therefore the three magnitudes are ultimately in a ratio of equality.

21. COR. 1. "Omni ex parte" has not been adopted from the text of Newton, because it requires limitation, for the perimeters do not ultimately coincide with the perimeter of the curvilinear area.

In the figure for Lemma II. the perimeter of the inscribed series of parallelograms is

$$AK+Kb+bL+Lc+\ldots+DA=2AK+2AD,$$

and the limit of this perimeter is $2Aa+2AE$.

The perimeter of the other series of parallelograms, being $2Aa+2AE$ is constant throughout the change, and has properly no limit.

COR. 2. The perimeter of the figure bounded by the chords $ab$, $bc$, ... ultimately coincides with that of the curvilinear figure. This coincidence will be discussed under Lemma V.

COR. 3. The same is true for the figure formed by the tangents.

COR. 4. Instead of "propterea," as in Newton, it is advisable to *state*, as in Whewell's *Doctrine of Limits*, that, if a *finite* portion of a curve be taken, and many successive points in the curve be joined so as to form a polygon, the sides of which, taken in order, are chords of portions of the curves, when the number of those points is increased indefinitely, the curve will be the limit of the polygon.

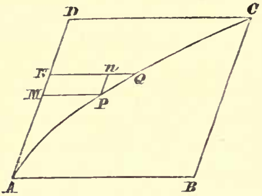
*Application to the Determination of certain Areas, Volumes, &c.*

(1) *Area of a parabola bounded by a diameter and an ordinate.*

Let $AB$, $BC$ be the bounding abscissa and ordinate. Complete the parallelogram $ABCD$.

Let $AD$ be divided into $n$ equal portions, of which suppose $AM$ to contain $r$, and $MN$ to be the $(r+1)^{\text{th}}$; draw $MP$, $NQ$

parallel to $AB$, meeting the curve in $P$, $Q$, and $Pn$ parallel to $MN$; the curvilinear area $ACD$ is the limit of the sum of the




series of parallelograms constructed, as $PN$, on the portions corresponding to $MN$.

But parallelogram $PN$ : parallelogram $ABCD$

$$:: PM.MN : CD.AD,$$

and, by the properties of the parabola,

$$PM : CD :: AM^2 : AD^2 :: r^2 : n^2,$$

$$\text{also } MN : AD :: 1 : n;$$

$$\therefore\ PM.MN : CD.AD :: r^2 : n^3;$$

therefore, parallelogram $PN = \frac{r^2}{n^3} \times$ parallelogram $ABCD$;

hence, the sum of the series of parallelograms

$$= \frac{1^2 + 2^2 + \ldots + (n-1)^2}{n^3} \times \text{parallelogram } ABCD,$$

and, when the number of parallelograms is increased indefinitely,

$$\frac{1^2 + 2^2 + \ldots + (n-1)^2}{n^3} = \frac{1}{3};$$

therefore, proceeding to the ultimate form of the hypothesis, the curvilinear area $ACD$ and the parabolic area $ABC$ will be, respectively, one-third and two-thirds of the parallelogram $ABCD$.

*Note* 1. If we had inscribed the series of parallelograms in $ABC$, $AB$ being divided into $n$ portions, we should have arrived at the result

$$\frac{1^{\frac{1}{2}} + 2^{\frac{1}{2}} + \ldots + (n-1)^{\frac{1}{2}}}{n^{\frac{3}{2}}}$$

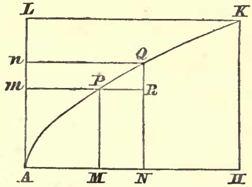
for the ratio of the series of parallelograms to the parallelogram $ABCD$, which might thus have been directly shewn to be ultimately $\frac{2}{3}$; but the former method is preferable, since the proof of the value of the limit depends upon simpler principles.

*Note* 2. If $BC$ had been divided into $n$ equal portions, the ratio of the parallelogram corresponding to $PN$ to the parallelogram $ABCD$ would have been $n^2 - r^2 : n^2$, and that of area $ABC$ to parallelogram $ABCD$ the limit of

$$\frac{n^2 - 1^2 + n^2 - 2^2 + \ldots + n^2 - (n-1)^2}{n^3} = 1 - \frac{1}{3} = \frac{2}{3}.$$

(2) *Volume of a paraboloid.*

Let $AKH$ be the area of a parabola, cut off by the axis $AH$




and an ordinate $HK$, which by its revolution round the axis generates a paraboloid.

Let $AH$ be divided into $n$ equal portions, and on $MN$ the $(r+1)^{\text{th}}$, as base, let the rectangle $PRNM$ be inscribed.

Cylinder generated by $PN$ : cylinder by $AHKL$

$$:: PM^2.MN : HK^2.AH.$$

$$\text{But } PM^2 : HK^2 :: AM : AH :: r : n,$$

$$\text{and } MN : AH :: 1 : n;$$

$$\therefore\ PM^2.MN : HK^2.AH :: r : n^2.$$

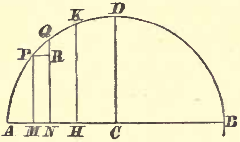
Hence cylinder generated by $PN = \frac{r}{n^2} \times$ cylinder by $AHKL$; therefore the sum of the cylinders inscribed is

$$\frac{1 + 2 + \ldots + (n-1)}{n^2} \times \text{circumscribed cylinder},$$

and the paraboloid is the limit of the series of inscribed cylinders; hence the volume of the paraboloid is half that of the cylinder on the same base and of the same altitude.

(3) *Volume of a spherical segment.*

Let $AHK$ generate, by its revolution round the diameter $AB$, the spherical segment whose height is $AH$.




Divide $AH$, as before, and make the same construction;

$$\text{then } PM^2 = AM.(AB - AM) = \frac{r}{n}AH.AB - \frac{r^2}{n^2}AH^2.$$

$$\text{Volume of cylinder generated by } PN = \pi PM^2.MN$$

$$= \pi PM^2.\frac{AH}{n} = \pi AH^2.\left(\frac{r}{n^2}AB - \frac{r^2}{n^3}AH\right),$$

whence, as before, the limit of the sum

$$= \pi AH^2(\tfrac{1}{2}AB - \tfrac{1}{3}AH),$$

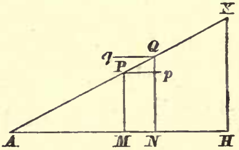
which is the volume proposed.

Cor. If $AH = \frac{1}{2}AB = AC$, the segment is a hemisphere whose volume is $\pi AC^2(AC - \frac{1}{3}AC) = \frac{2}{3}\pi AC^3$, which is two-thirds of the cylinder on the same base and of the same altitude.

(4) *Area of the surface of a right cone.*

As an illustration of the method of finding surfaces given above, suppose $AHK$ to be a right-angled triangle, which revolves round $AH$, a side containing the right angle, then the hypothenuse $AK$ generates a conical surface.

Let $MN$ be the $(r+1)^{\text{th}}$ portion of $AH$, after division into




$n$ equal portions; $MP$, $NQ$ ordinates parallel to $HK$; $Pp$, $Qq$ each equal to $PQ$ and parallel to $AH$.

The areas generated by $Pp$ and $Qq$ respectively are

$$2\pi PM.Pp \text{ and } 2\pi QN.Qq,$$

$$\text{and } PM : HK :: AM : AH :: r : n,$$
$$QN : HK :: AN : AH :: r+1 : n,$$
$$PQ : AK :: MN : AH :: 1 : n;$$

therefore the areas are $\frac{r}{n^2}.2\pi HK.AK$ and $\frac{r+1}{n^2}.2\pi HK.AK$ respectively; and the conical surface is intermediate in magnitude between

$$2\pi HK.AK \times \frac{1+2+\ldots+(n-1)}{n^2},$$

$$\text{and } 2\pi HK.AK \times \frac{1+2+\ldots+n}{n^2},$$

each of which has for its limit $\pi HK.AK$, which is therefore the area of the conical surface.

*Note.* The reader may notice the following method of obtaining the conical surface by development, although it is not related to the method of limits.

If a circular sector $KAK'$, traced on paper, be cut out, the bounding radii $AK$, $AK'$ can be placed in contact, so that the boundary $KLK'$ will form a circle.

The figure so formed will be conical, $AK$ will be the slant side, and $HK$ in the last figure will be the radius of the circular base, whose length will be the arc of the sector $KAK'$.

Hence, the area of the conical surface is equal to that of the sector $KAK' = \frac{1}{2}AK.2\pi HK = \pi HK.AK$.

(5) *Mass of a rod whose density varies as th $m^{\text{th}}$ power of the distance from one extremity.*

Let $AB$ be the rod, and let $MN$ be the $(r+1)^{\text{th}}$ portion, when its length has been divided into $n$ equal parts; and let $\rho.AM^m$ be the density at $M$, or the quantity of matter contained in an unit of length of the rod supposed of the same substance as the rod at the point $M$.

The quantity of matter in $MN$ is intermediate between

$$\rho.AM^m.MN \text{ and } \rho.AN^m.MN,$$

and the ratio of the difference of these to either of them is less than any assignable ratio when $n$ is indefinitely increased.

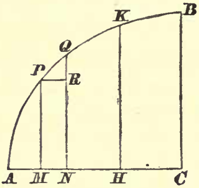
Therefore, since $AM = \frac{r}{n} AB$, and $MN = \frac{1}{n} AB$, the mass of the whole rod is the limit of

$$\rho . \frac{1^m + 2^m + \ldots + (n-1)^m}{n^{m+1}} AB^{m+1} = \frac{1}{m+1} \times \rho . AB^{m+1}$$

$= \left(\frac{1}{m+1}\right)^{\text{th}}$ of the mass of a rod of length $AB$ and of uniform density equal to that of the rod $AB$ at $B$.

(6) *Centre of gravity of the volume of a hemisphere.*

Let $CAB$ be a quadrant, which by its revolution round the radius $CA$ generates the hemisphere.




Let $MR$ be the rectangle which generates the $r^{\text{th}}$ inscribed cylinder, so that $CM = \frac{r}{n} \times CA$ and $MN = \frac{1}{n} \times CA$.

If the mass of a unit of volume be chosen as the unit of mass, the mass of the cylinder generated by $MR$ will be

$$\pi PM^2 . MN = \pi (CA^2 - CM^2) MN = \left(1 - \frac{r^2}{n^2}\right) \pi CA^2 . \frac{CA}{n};$$

hence, the mass of the series of inscribed cylinders will be

$$\pi CA^3 - \frac{1^2 + 2^2 + \ldots + n^2}{n^3} \pi CA^3,$$

and the mass of the hemisphere

$$= \pi CA^3 - \tfrac{1}{3} \pi CA^3 = \tfrac{2}{3} \pi CA^3.$$

Again, the moment of the mass of the cylinder generated by $MR$, with respect to the base of the hemisphere, will be

$$\pi PM^2 . MN . \tfrac{1}{2} (CM + CN),$$

which differs from $\pi PM^2.MN.CM$ by a quantity which vanishes compared with it, and is therefore ultimately $\left(\frac{r}{n^2}-\frac{r^3}{n^4}\right)\pi CA^4$; therefore the moment of the hemisphere, with respect to its base, is

$$(\tfrac{1}{2}-\tfrac{1}{4})\,\pi CA^4, \text{ or } \tfrac{1}{4}\pi CA^4;$$

hence the distance of the centre of gravity of the volume of the hemisphere from $C$, which is the moment with respect to the base divided by the mass, is $\frac{3}{8}.CA$.

## III.

1. Illustrate the terms "tempore quovis finito" and "constanter tendunt ad æqualitatem" employed in Lemma I. by taking the case of Lemma III. as an example.

2. Shew, from the course of the proof of Lemma II., that the ultimate ratio of vanishing quantities may be indefinitely small or great.

3. Shew that the ratio of the area of the parabolic curve, in which $PM^3 \propto AM$, to the area of the circumscribing parallelogram, of which one side is a tangent to the curve at $A$, is 3 : 4.

4. Shew that the volume of a right cone is one-third of the cylinder on the same base and of the same altitude.

5. $AHK$ is a parabolic area, $AH$ the axis, and $HK$ an ordinate perpendicular to the axis, $AHKL$ the circumscribing rectangle. Shew that the volumes generated by the revolution of $AHK$ round $AH$, $KL$, $AL$, and $HK$ are respectively $\frac{1}{2}$, $\frac{5}{6}$, $\frac{4}{5}$, and $\frac{8}{15}$ of the cylinder generated by the rectangle.

6. The volume of a spheroid is two-thirds of the circumscribing cylinder.

7. Find the centre of gravity of the volume of a right cone by the method of Lemma II.

8. Shew that the centre of gravity of a paraboloid of revolution is distant from the vertex two-thirds of the length of the axis.

9. Find the mass of a rod whose density varies as the distance from an extremity. Find also its centre of gravity, and shew that it is in one of the points of trisection of the rod.

10. The limiting ratio of an hyperboloid of revolution, whose axis is the transverse axis, to the circumscribing cylinder is 1 : 2 when the altitude is indefinitely diminished, and 1 : 3 when it is indefinitely increased.

## IV.

1. Prove that the areas of parabolic segments, cut off by focal chords, vary as the cubes of the greatest breadths of the segments.

2. Find the mass of a circle whose density varies as the $m^{\text{th}}$ power of the distance from the centre.

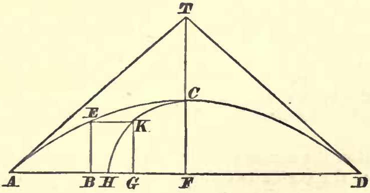
3. Shew that the abscissa and ordinate of the centre of gravity of a parabolic area, contained between a diameter $AB$ and ordinate $BC$, are $\frac{3}{5}AB$ and $\frac{3}{8}BC$ respectively.

4. A number of equal squares in one plane with their centres coincident are arranged consecutively, their sides making equal small angles, each with the adjacent ones; prove that the limit of the length of the serrated edge, when the number of squares is indefinitely increased, is equal to the circumference of a circle whose radius is a side of the square.

5. By supposing the axis of a parabola portioned off into successive lengths in the ratio 1 : 3 : 5, &c., apply Lemma III. to find the area contained by the curve and a double ordinate.

6. Find the volume generated by the revolution of an elliptic disc about an axis parallel to its major axis, and at such a given distance as not to intersect the disc.

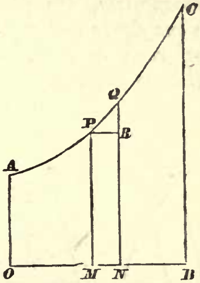
7. In the curve $ACD$, $BE$ is an ordinate perpendicular to $AD$, and $FC$ is the greatest value of $BE$, and $\dfrac{BE}{FC} = \sin\left(\dfrac{\pi AB}{AD}\right)$.




Shew that the area $ABE$ varies as $HG$, where $GK$ is the ordinate equal to $BE$ of the circle $CH$, whose centre is $F$ and radius $FC$.

8. In the curve of the last problem shew that the ratio of the area $ACD$ to the triangle whose sides are $AD$, and the tangents $AT$, $DT$ at the extremities, is $8 : \pi^2$.

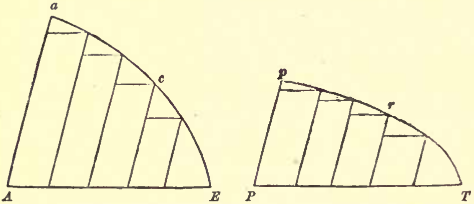
9. In the curve $APC$, in which the relation between any rectangular ordinate $PM$ and abscissa $OM$ is $\frac{OM}{OA} = \log \frac{PM}{OA}$,




prove that the area contained between the curve, the abscissa $OB$, and ordinate $BC$, is $OA(BC - AO)$.

## LEMMA IV.

*If in two figures AacE, PprT there be inscribed (as in Lemmas II., III.) two series of parallelograms, the number in each series being the same, and if, when the breadths are diminished indefinitely, the ultimate ratios of the parallelograms in one figure to the parallelograms in the other be the same, each to each, then the two figures AacE, PprT will be to one another in that same ratio.*




[Since the ratio, whose antecedent is the sum of the antecedents, and whose consequent is the sum of the consequents of any number of given ratios, is intermediate in magnitude between the greatest and least of the given ratios, it follows that the sum of the parallelograms described in *AacE* is to the sum in *PprT* in a ratio intermediate between the greatest and least of the ratios of the corresponding inscribed parallelograms; but the ratios of these parallelograms are ultimately the same, each to each, therefore the sums of all the parallelograms described in *AacE*, *PprT* are ultimately in the same ratio, and so the figures *AacE*, *PprT* are in that same ratio; for, by Lemma III., the former figure is to the former sum and the latter figure to the latter sum in a ratio of equality.] Q. E. D.

Cor. Hence, if two quantities of any kind whatever be divided into any, the same, number of parts, and those parts, when their number is increased and magnitude diminished indefinitely, assume the same given ratio each to each, viz. the first to the first, the second to the second, and so on in order, the whole quantities will be to one another in the same given ratio. For if, in the figures of this Lemma, the parallelograms be taken each to each in the same ratio as the parts, the sums of the parts will be always as the sums of the parallelograms; and, therefore, when the number of the parts and parallelograms is increased and their magnitude diminished indefinitely, the two quantities will be in the ultimate ratio of parallelogram to parallelogram, that is, (by hypothesis) in the ultimate ratio of part to part.

*Observations on the Lemma.*

22. The general proposition contained in the Corollary may be proved independently in the following manner:

Let $A$, $B$ be two quantities of any kind, which can be divided into the same number $n$ of parts, viz. $a_1$, $a_2$, $a_3 \ldots a_n$ and $b_1$, $b_2$, $b_3 \ldots b_n$ respectively, such that, when their number is increased and their magnitudes diminished indefinitely, they have a constant ratio $L : 1$ each to each, so that

$$a_1 : b_1 :: L(1+\alpha_1) : 1,$$
$$a_2 : b_2 :: L(1+\alpha_2) : 1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

where $\alpha_1$, $\alpha_2$, ... vanish when $n$ is increased indefinitely.

Then, $a_1 + a_2 + \ldots : b_1 + b_2 + \ldots$ being a ratio which is intermediate between the greatest and least of these ratios, each of which is ultimately $L : 1$, we have, proceeding to the limit,

$$A : B :: L : 1;$$

that is, $A$ and $B$ are in the ultimate ratio of the parts.

23. The proof given in the *Principia* is as follows: "For, as the parallelograms are each to each, so, componendo, is the sum of all to the sum of all, and so the figure $AacE$ to the figure $PprT$, for, by Lemma III., the former figure is to the former sum and the latter figure to the latter sum in a ratio of equality."

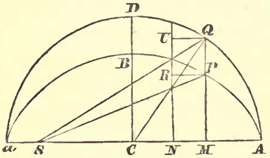
The proof given in the text is substituted for this, because the demonstration breaks down for any finite distance from the ultimate form of the hypothesis.

*Application to the determination of certain Areas, Volumes, &c.*

(1) *Area of an ellipse.*

Let $ACa$ be the major axis of an ellipse, $BC$ the semi-minor axis, $ADa$ the auxiliary circle, and let parallelograms be inscribed, whose sides are common ordinates to the two curves.

Let $PMNR$, $QMNU$ be any two corresponding parallelograms. The ratio of these parallelograms is $PM : QM$ or $BC : AC$.




Hence, area of ellipse : area of circle :: $BC : AC$, but area of circle $= \pi AC^2$; therefore area of ellipse $= \pi AC.BC$.

(2) *Area of a sector of an ellipse, pole in the focus.*

If $S$ be a focus of the ellipse, and $SP$, $SQ$ be joined,

$$\Delta SPM : \Delta SQM :: BC : AC,$$

and area $APM$ : area $AQM :: BC : AC$,

hence, area $ASP$ : area $ASQ :: BC : AC$,

but area $ASQ = \Delta SCQ +$ sector $ACQ$

$$= \tfrac{1}{2} SC.QM + \tfrac{1}{2} AC.\text{arc } AQ;$$

$\therefore$ area $ASP = \frac{1}{2}\{SC.PM + BC.\text{arc } AQ\}$.

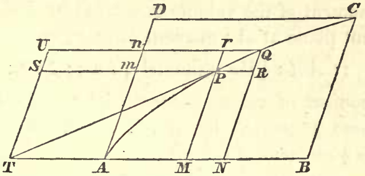
(3) *Area of a parabolic curve cut off by a diameter and an ordinate to the diameter.*

In the following investigation it is asserted that when a chord $PQ$ is drawn to a curve from a point $P$, as $Q$ moves up to $P$, $PQ$ assumes as its limiting position that of the tangent at $P$, which is deducible from the idea of a tangent being in the direction of the curve at the point of contact.

Let $AB$, $BC$ be the diameter and ordinate; $AD$ the tangent at $A$; $CD$ parallel to $AB$; $P$, $Q$ points near each other; $PM$, $QN$ and $Pm$, $Qn$ parallel respectively to $AD$ and $AB$.

Let $QP$ produced meet $BA$ in $T$, and complete the parallelograms $TAmS$, $TAnU$.

Then, since $QP$ is ultimately a tangent at $P$, $AT=AM$ ultimately, and the parallelogram $PU$ is ultimately double of




the parallelogram $Pn$, and the complements $PN$, $PU$ are equal; therefore the parallelograms $PN$, $Pn$ are ultimately in the ratio 2 : 1.

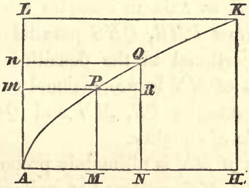
Hence, in the curvilinear areas $ABC$, $ACD$ two sets of parallelograms can be inscribed which are ultimately in the ratio 2 : 1, each to each; therefore area $ABC$ is ultimately double of area $ACD$, and is therefore two-thirds of $ABCD$.

(4) *Volume of a paraboloid of revolution.*

Let $AH$ be the axis of the parabola $APK$, $AHKL$ the circumscribing rectangle. Also let $PN$, $Pn$ be rectangles inscribed in the portions $AHK$, $AKL$.

$$\text{Volume generated by } PN = \pi PM^2 . MN = \pi . PM . PN.$$

$$\text{Volume generated by } Pn = \pi QN^2 . AM - \pi PM^2 . AM$$
$$= \pi AM . (QN + PM) . mn = \pi (QN + PM) . Pn;$$




$\therefore$ vol. by $PN$ : vol. by $Pn$ :: $PM.PN$ : $(QN+PM).Pn$, but $QN+PM=2PM$ and $PN=2Pn$, as in (3), and therefore vol. by $PN$=vol. by $Pn$ ultimately; hence, by Cor., Lemma IV., the volume of the paraboloid generated by $AHK$ is half the volume of the circumscribing cylinder generated by $AKL$.

(5) *Centre of gravity of a paraboloid of revolution.*

Since the volumes generated by $PN$ and $Pn$ are ultimately equal, the moment of the volume generated by $PN$ with respect to the tangent plane at $A$ : moment of that generated by $Pn$

$$:: AM : \tfrac{1}{2}Pm \text{ ultimately, } i.e. :: 2 : 1;$$

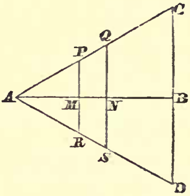
hence the moment of volume generated by $AHK$ is twice that of the volume generated by $AKL$, and the moment of the paraboloid $= \frac{2}{3}$ moment of the cylinder

$$= \tfrac{2}{3} \text{ volume of cylinder} \times \tfrac{1}{2}AH = \tfrac{2}{3} \text{ volume of paraboloid} \times AH;$$

hence the distance of the centre of gravity of the paraboloid from the vertex is two-thirds of the height of the paraboloid.

(6) *Centre of gravity and mass of a rod whose density varies as the distance from an extremity.*

Let $AB$ be the rod, $MN$ a small portion of it, then the density at $M \propto AM$.




Construct on $AB$ as axis an isosceles triangle $CAD$, whose base is $CD$, and draw $PMR$, $QNS$ parallel to $CD$; then $PR$, $QS$, $CD$ are proportional to the densities at $M$, $N$ and $B$; therefore the mass of $MN$ is proportional to a rectangle intermediate to the rectangles $PR$, $MN$ and $QS$, $MN$, which are ultimately in a ratio of equality.

Hence the mass of $MN$ is ultimately proportional to the mass of the rectangle $PR$, $MN$, supposed of uniform density, and the moment of $MN$, with respect to the line $CD$, is proportional to the moment of the same rectangle, since their distance is the same; hence, by the Lemma, the moment of the whole rod

: the moment of the triangle with respect to $CD$

:: the mass of the rod : the mass of the triangle;

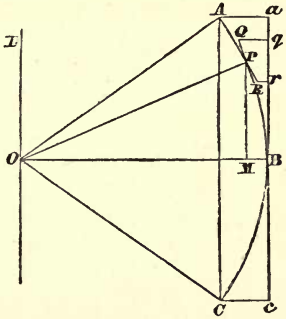
therefore, the distances of the centres of gravity of the rod and triangle from $CD$ being the same, the centre of gravity of the rod is at a distance $\frac{1}{3}AB$ from $B$.

Also, the mass of $MN$ being proportional to the area $PRN$, the mass of the rod is proportional to the area of the triangle $ACD$, and the mass of a rod of uniform density equal to that at $B$, and of length $AB$, being in the same proportion to the rectangle $AB$, $CD$, is therefore double of the mass of the rod.

(7) *Centre of gravity of a circular arc.*

Let $O$ be the centre of an uniform circular arc $ABC$, $OB$ the bisecting radius, $aBc$ a tangent at $B$, $OD$ parallel to $ac$, and $Aa$, $Cc$ parallel to $OB$.

Let $QR$ be the side of a regular polygon described about the




arc, $P$ the point of contact, $Qq$, $Rr$ perpendicular to $ac$, and $PM$ to $OB$. Then, since $OP$, $OB$ are perpendicular to $QR$, $qr$,

$$qr : QR :: OM : OP :: OM : OB;$$

but, since $OM$, $OB$ are the distances of the centres of gravity of $QR$ and $qr$ from $OD$, and $QR.OM = qr.OB$, the moments of $QR$ and $qr$ with respect to $OD$ are in a ratio of equality, and the same is true of every side of the circumscribing polygon; therefore, by Cor., Lemma IV., the moment of the arc, which is ultimately that of the polygon, is equal to the moment of $ac$

$$= ac.OB = \text{chord } AC.\text{radius } OB.$$

Hence, the distance of the centre of gravity of the arc from $O$

$$= \frac{\text{radius} \times \text{chord}}{\text{arc}}.$$

(8) *Surface of a segment of a sphere.*

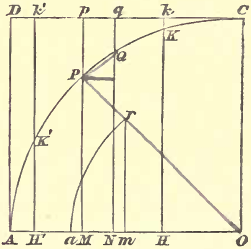
Let $AKH$ be the portion of a circle which generates by revolution round $AH$ the spherical segment, $O$ the centre of the circle, $PQ$ the chord of a small arc, $PM$, $QN$ perpendicular to $AH$.

Let $AOCD$ be the rectangle circumscribing the quadrant and generating the circumscribing cylinder.

Produce $MP$, $NQ$, $HK$ to meet $CD$ in $p$, $q$, $k$. Since $PQ$ is in its limiting position a tangent at $P$, $PQ$ is ultimately perpendicular to the radius $OP$, also $pq$ is perpendicular to $MP$;

$$\therefore\ PQ : pq :: OP : PM \text{ ultimately},$$

and the surface generated by $PQ$ is ultimately $2\pi PM.PQ$, Art. 18, $= 2\pi . OP.pq =$ the surface generated by $pq$.




The same is true for each side of the inscribed polygon when the number is indefinitely increased.

Hence the surface generated by $AK$, or the surface of the spherical segment, is equal to the surface of the circumscribed cylinder cut off by the plane of the base of the segment.

Cor. Hence, also, the surface of any belt of a sphere cut off by two parallel planes is equal to the corresponding belt of the cylindrical surface.

(9) *Centre of gravity of a belt of the surface of a sphere contained between parallel planes.*

The moment of the belt generated by $PQ$ with respect to the plane through $A$, perpendicular to $AH$, is evidently ultimately equal to that of the belt generated by $pq$; therefore the moment of any belt generated by $K'K$ is equal to that of the corresponding belt generated by $k'k$.

Hence, the centres of gravity of the two belts are coincident, viz. in the bisection of $HH'$, that is, the distance of the centre of gravity of a spherical belt, contained between parallel planes, is half-way between the two planes.

(10) *Volume of a spherical sector.*

Let the spherical sector be generated by the revolution of the sector $AOP$ about $AO$.

The volume of the spherical sector is equal to the limit of the sum of a series of pyramids whose vertices are in $O$, and the sum of whose bases is ultimately the area of the surface of the segment; also the volume of each pyramid is $\frac{1}{3}$ base $\times$ altitude.

Hence, the volume of the spherical sector is one-third of the area of the surface of the spherical segment $\times$ radius

$$=\tfrac{1}{3}.2\pi AD.Dp.AO=\tfrac{2}{3}\pi AM.AO^2=\tfrac{2}{3}\pi AO^3 \text{ vers} POA.$$

(11) *Centre of gravity of a spherical sector.*

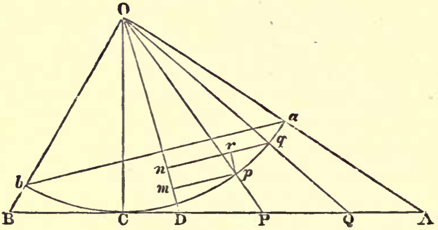
If we suppose each of the pyramids on equal bases, they may be supposed collected at their centres of gravity, whose distances are $\frac{3}{4}AO$ from $O$ ultimately, and they form a mass which may be distributed uniformly over the surface of a spherical segment whose radius is $\frac{3}{4}AO$, viz. that generated by $ar$, whose centre of gravity will be in the bisection of $am$, if $rm$ be perpendicular to $AH$.

Therefore the distance of the centre of gravity of the spherical sector from $O=\frac{1}{2}(Oa+Om)=\frac{3}{4}OA.\cos^2\frac{1}{2}POA$.

If the angle $POA$ become a right angle, the distance of the centre of gravity of the corresponding sector, which in this case will become the hemisphere, will be $\frac{3}{8}OA$, as in page 29.

(12) *To find the direction and magnitude of the resultant attraction of a uniform rod upon a particle, every particle of the*

*rod being supposed to attract with a force which varies inversely as the square of its distance from the attracted particle.*




Let $AB$ be the attracting rod, $O$ the particle attracted by the rod; draw $OC$ perpendicular to $AB$, join $OA$, $OB$, and let a circle be described with centre $O$ and radius $OC$ meeting $OA$, $OB$ in $a$, $b$. Let $OpP$, $OqQ$ be drawn cutting off the small portions $pq$, $PQ$ from the arc $aCb$ and the rod, respectively, and draw $PR$ perpendicular to $OQ$.

$$\text{Then } PR : PQ :: OC : OP \quad \text{ultimately},$$

$$\text{and } pq : PR :: Op : OP \quad \ldots\ldots\ldots\ldots;$$

$$\therefore\ pq : PQ :: Op^2 : OP^2 \quad \ldots\ldots\ldots\ldots,$$

and, if $aCb$ be of the same density as the rod and attract according to the same law,

attraction of $pq$ on $O$ : attraction of $PQ$ :: $\dfrac{pq}{Op^2} : \dfrac{PQ}{OP^2}$ ultimately.

Therefore the portions $PQ$, $pq$ of the rod and arc attract $O$ in the same direction with forces which are ultimately equal.

Hence, by Cor., Lemma IV., the resultant attraction of the rod is the same as that of the arc $aCb$, which, by symmetry, is in the direction $OD$, bisecting the angle $AOB$.

Again, draw $qn$ perpendicular to $OD$, $pr$ to $qn$; then, by similar triangles, $pqr$, $qOn$,

$$pq : qr :: Oq : On;$$

$$\therefore\ \frac{pq}{Oq^2} \cdot \frac{On}{Oq} = \frac{qr}{OC^2};$$

that is, the resultant attraction of $pq$ in the direction $OD$ is the same as that of $qr$ at the distance $OC$; hence the whole resultant attraction of $AB$ is

$$\frac{\mu . ab}{OC^2}, \text{ or } \frac{2\mu}{OC}\sin\tfrac{1}{2}AOB,$$

where $\mu$ is the attraction of a unit of mass at the unit distance.

## V.

1. Shew that the area of the sector of an ellipse contained between the curve and two central distances varies as the angle of the corresponding sector of the auxiliary circle.

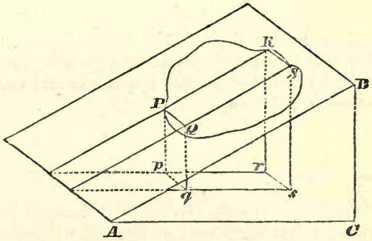
2. Prove that the volumes of two pyramids will be equal if they stand on the same base, and have their vertices in the same plane parallel to the base.

3. Find the volume of a paraboloid by comparison with the area of a triangle whose vertex and base are those of the generating parabola.

4. Find the centre of gravity of the paraboloid by reference to the same triangle.

5. Find the mass of a straight rod, whose density varies as the square of the distance from one extremity, by comparison with a cone whose axis is the rod.

6. Shew that the orthogonal projection of any plane area on another plane is the given area × the cosine of the inclination of the two planes.




As a first step, prove that, $pqsr$ being the projection of the inscribed parallelogram $PQSR$, $pqsr : PQSR :: \cos BAC : 1$.

7. Find the volume of a hemisphere by comparing the volumes generated by the quadrantal sector and the portion of the circumscribing square which is the difference between the square and the quadrantal sector.

## VI.

1. Find the volume of a paraboloid generated by the revolution of a semi-cubical parabola, in which $PM^2 \propto AM^3$, by means of a cone on the same axis.

2. Assuming that the area of a belt of a sphere cut off by two parallel planes varies as the perpendicular distance between them, find by the aid of Lemma IV. the area of any portion of the curve of sines.

3. Prove that, if $PQ$ be a small arc of an ellipse, and $CD$ be conjugate to $CP$, the limit of the sum of all the ratios $PQ : CD$, taken over the whole perimeter of the ellipse, will be $2\pi$.

4. $P$ is any point of a curve $OP$; $OX$, $OY$ any lines drawn at right angles through $O$, $PM$, $PN$ perpendicular to $OX$, $OY$ respectively. Prove that, if area $OPM$ : area $OPN$ : : $m$ : 1 always, and the whole system revolve about $OX$, volumes generated by $OPM$, $OPN$ will be as $m : 2$.

5. Prove that the surface generated by the revolution of a semi-circle round its bounding diameter is to the curved surface generated by the revolution of the same semi-circle round the tangent at the extremity of the diameter in the ratio of the length of the diameter to the length of the arc of the semi-circle.

6. Common ordinates $MPP'$, $NQQ'$ are drawn to two ellipses which have a common minor axis, and the outer of which touches the directrices of the inner; shew that the area of the surface generated by the revolution of $PQ$ about the major axis bears a constant ratio to the area $MP'Q'N$.

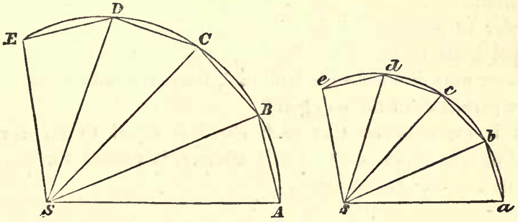
7. Prove that the area included between an hyperbola and the tangents at the vertices of the conjugate hyperbola is equal to the area included between the conjugate hyperbola and the tangents at the vertices of the hyperbola.

## LEMMA V.

*All the homologous sides of similar figures are proportional, whether curvilinear or rectilinear, and their areas are in the duplicate ratio of the homologous sides.*

[Similar curvilinear figures are figures whose curved boundaries are curvilinear limits of corresponding portions of similar polygons.

Let $SABCD...$, $sabcd...$ be two similar polygons, of which $SA$, $AB$, $BC$, ... are homologous to $sa$, $ab$, $bc$, ... respectively.




Then $AB : ab :: SA : sa$,

similarly, $BC : bc :: AB : ab :: SA : sa$,

$CD : cd :: BC : bc :: SA : sa$,

..................................

therefore, componendo,

$$AB + BC + CD + ... : ab + bc + cd + ... :: SA : sa.$$

Now this, being true for all similar polygons, will be true in the limit, when the number of the sides $AB$, $BC$, ... and $ab$, $bc$, ... is increased, and their lengths diminished indefinitely; if, therefore, $AE$, $ae$ be curves which pass through the angular points $A$, $B$, ... and $a$, $b$, ... of the polygons, these curves will be curvilinear limits of $AB + BC + ...$ and $ab + bc + ...$, and will be the boundaries of similar curvilinear figures; therefore the curved line $AE$ : the curved line $ae$

$$:: SA : sa :: SE : se.$$

Again, polygon $SABC...$ : polygon $sabc...$ :: $SA^2 : sa^2$, and this is true in the limit; hence, by Lemma III. Cor. 2, curvilinear area $SAE$ : curvilinear area $sae$

$$:: SA^2 : sa^2 :: AE^2 : ae^2 :: SE^2 : se^2.$$

Q. E. D.]

*Observations on the Lemma.*

24. In order to deduce the properties of similar curves, it is premised, as before mentioned under Cor. 4, Lemma III., that, if a *finite* portion of a curve be taken, and if a polygon be inscribed in the curve, the sides of which are chords taken in order of portions of the curve, and the number of sides of the polygons be increased indefinitely, and the magnitudes a the same time diminished indefinitely, the curve will be the limit of the perimeter of the polygon.*

It is *not* assumed that each chord is equal to the corresponding arc ultimately; this is afterwards proved for a continuous curve in Lemma VII.

*Criteria of Similarity.*

25. From the *definition* of similar curve lines, that they are curvilinear limits of homologous portions of similar polygons, the following criteria of similarity can be deduced, all of which are very convenient in practice; namely:

(1) One curve line is similar to another when, if *any* polygon be inscribed in one, a similar polygon can be inscribed in the other.

(2) If two curves be similar, and any point $S$ be taken in the plane of one curve, another point $s$ can be found in the plane of the other, such that, *any* radii $SP$, $SQ$ being drawn in the first, radii $sp$, $sq$ can be drawn in the second, inclined at

* Whewell's *Doctrine of Limits.*

the same angle as the former, and such that the following proportion will hold,

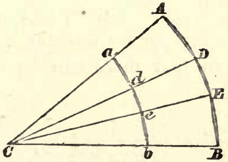
$$sp : sq :: SP : SQ.$$

(3) If two curves be similar, and in the plane of one curve *any* two lines $OX$, $OY$ be drawn, two other lines $ox$, $oy$ can be drawn in the plane of the other curve, inclined at the same angle, having the property that the abscissa and ordinate $OM$, $MP$ of any point $P$ in the first being taken, the abscissa and ordinate $om$, $mp$ of a corresponding point $p$ in the second will be proportional to the former, viz.,

$$om : mp :: OM : MP.$$

And the converse propositions can also be deduced, that if these proportions hold, the curves will be similar.

26. In order to illustrate test (1), let the arcs $AB$, $ab$ of two circles have the same centre $C$, and let the bounding radii be coincident in direction.




Let $ADEB$ be *any* polygon inscribed in $AB$, and let $CD$, $CE$ cut $ab$ in $d$, $e$; join $ad$, $de$, $eb$, these are parallel to $AD$, $DE$, $EB$ respectively, and $ad : de : eb :: AD : DE : EB$; hence $adeb$ is similar to $ADEB$; and therefore the arcs $ab$, $AB$ are similar.

27. Test (2) may be deduced as follows:

If $ABCD...$, $abcd...$, fig. p. 43, be corresponding portions of similar polygons, $AB$, $BC$, ... $ab$, $bc$, ... being homologous sides, and $AS$, $BS$, ... be drawn to any point $S$, construct the triangle $sab$ equiangular with $SAB$, and join $sc$, $sd$, ....

Then $sb : SB :: ab : AB :: bc : BC$, and $\angle SBC = \angle sbc$;

therefore $SBC$, $sbc$ are similar triangles;

hence $sc : SC :: sb : SB :: sa : SA$;

and similarly for $sd$, $se$, &c.

Hence, if two polygons be similar, and any point be taken in one, another point can be found in the other, such that the radii drawn to corresponding angular points will be proportional and include the same angles.

If we now increase the number of sides indefinitely and diminish their magnitude, the same property will hold with respect to the curvilinear limit of the polygon.

Test (3) can be deduced from test (1) in a similar manner.

*Centres of Similitude.*

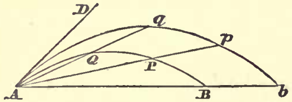
28. When two similar curves are so situated that a point can be found, such that the radii drawn from that point, either in the same or opposite directions, are in a constant ratio, such a point is called a *centre of similitude.*

If the radii be measured in the same direction, the point will be a centre of *direct* similitude, and of *inverse* similitude if they be measured in opposite directions.

It is easily shewn that there can be only one centre of similitude of one kind.

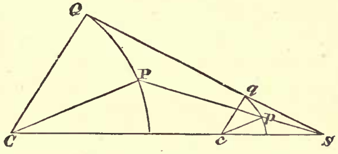
*Properties of similar curves and application of tests of Similarity.*

(1) *Similar conterminous arcs, which have their chords coincident, have a common tangent.*




Let $APB$, $Apb$ be similar conterminous arcs, $ABb$ the line of their chords, $AQq$, $APp$ any straight lines meeting the curves in $Q$, $q$ and $P$, $p$ respectively; then $A$ will evidently be a centre of direct similitude for the two curves; therefore $AQ : Aq :: AP : Ap$; hence $AP$, $Ap$ are similar portions of the curves, and arc$AP$ : arc$Ap$ :: $AP$ : $Ap$ :: $AB$ : $Ab$; therefore the arcs $AP$, $Ap$ vanish simultaneously, or, when $AP$ assumes its limiting position $AD$ for the curve $APB$, this is also the limiting position of $Ap$ for the curve $Apb$, that is, the curves have a common tangent.

(2) *To find the centres of direct and inverse similitude of any two circles.*




If one of the circles do not lie entirely within the other, let $S$ be the intersection of two common tangents to the circles which intersect in the *produced* line $Cc$ joining their centres, and let $CQ$, $cq$ be radii to the points of contact.

Draw $SpP$ through $S$ cutting the circles in $p$, $P$, then $cq$ is parallel to $CQ$, and $CP : cp :: CQ : cq :: CS : cS$;

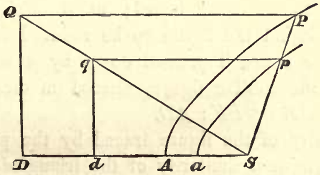
$$\therefore\ CS : CP :: cS : cp;$$

also $CPS$, $cpS$ are each greater or each less than a right angle, and $CSP$ is common to the triagles $CPS$, $cpS$; therefore the triangles are similar, Euclid VI. 7, and the sides about the angle $CSP$ are proportional, that is, $SP : Sp :: SC : Sc$; therefore $S$ is the centre of direct similitude.

Similarly, the intersection of two common tangents which cross between two circles is the centre of inverse similitude.

(3) *To find the condition of similarity of two conic sections.*

Let the conic sections be placed so that their directrices

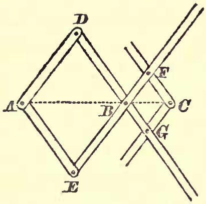


are parallel and foci coincident, and let $SpP$ be any line through the focus meeting them in $p$, $P$; draw $SaAD$ and $PQ$ perpendicular to the directrix $DQ$ of $AP$, and join $SQ$, and let $pq$, parallel to $PQ$, meet it in $q$, and draw $qd$ perpendicular to $SD$.

Then $Sd : SD :: Sq : SQ :: Sp : SP$; and, if the curves be similar, $Sp : SP$ will be a constant ratio; therefore $Sd : SD$ is a constant ratio, and $dq$ is a fixed straight line for all positions of $p$; also, since $pq : Sp :: PQ : SP$, $pq : Sp$ is a constant ratio; therefore $qd$ is the directrix of $ap$, and, the constant ratio being the same in both, the eccentricities are the same.

(4) Instruments, like the *Pantagraph* and the *Eidograph*, for copying plans on an enlarged or reduced scale are founded upon the properties of similar figures; as are also other methods of copying, such as by dividing plans or pictures into squares.

The Pantagraph is an instrument for drawing a figure similar to a given figure on a smaller or larger scale; one of its forms is as in the figure. $AD$, $EF$, $GC$ and $AE$, $DG$, $FC$ are two sets of parallel bars, joined at all the angles by




compass-joints; at $B$ is a point, which serves to fix the instrument to the drawing board; at $A$ is a point which is made to pass round the figure to be reduced or enlarged; at $C$ is a hole for a pencil pressed down by a weight, and the pencil traces the similar figure, altered in dimensions in the ratio of $BC : AB$ or $BF : AD$.

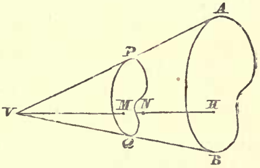
The similarity of the figure traced by the pencil is a consequence of continual similarity of the triangles $ABD$, $BFC$.

By changing the positions of the pegs at $F$ and $G$ the figure described by $C$ may be made of the required dimensions.

For a description of the Eidograph, invented by Professor Wallace, see the *Transactions of the Royal Society of Edinburgh*, vol. XIII.

(5) *Volume of a cone whose base is a plane closed figure of any form.*

Let $V$ be the vertex, $AB$ the base, $VH$ perpendicular to the base from $V$; let $VH$ be divided into $n$ equal portions, of




which $MN$ is the $(r+1)^{\text{th}}$; and let $PQ$ be the section through $M$ parallel to $AB$.

Take $VPA$ *any* generating line of the cone meeting the section $PQ$ and the base $AB$ in $PA$ respectively, then

$$PM : AH :: VM : VH;$$

therefore $PQ$ is similar to $AB$, $M$, $H$ being similarly situated points; and, by Lemma V.,

$$\text{area } PQ : \text{area } AB :: r^2 : n^2,$$

$$\text{also } MN : VH :: 1 : n;$$

therefore the volume of the cylinder whose base is $PQ$ and height $MN = \dfrac{r^2}{n^3} \times \text{area } AB.VH$, and the volume of the cone, by Lemma II., is one-third of the cylinder whose base is $AB$ and height $VH$.

## VII.

1. Apply a criterion of similarity to shew that segments of circles which contain equal angles are similar.

2. From the definition of an ellipse, as the locus of a point the sum of whose distances from two fixed points is constant, shew that ellipses are similar when the eccentricities are equal.

3. Prove that the centre of an ellipse is a centre of inverse similitude of two opposite equal portions of the circumference of the ellipse.

4. Employ the properties of similar figures to inscribe a square in a given semicircle.

5. Construct, by means of similar figures, two circles, each of which shall touch two given straight lines and pass through a given point.

6. Deduce the position of the centre of gravity of a circular sector from that of a circular arc; shew that the distance from the centre is $\frac{2}{3} \cdot \frac{\text{radius} \times \text{chord}}{\text{arc}}$.

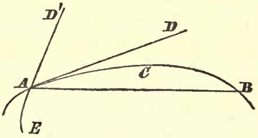
7. If $A$ be the vertex of a conical surface, $G$ the centre of gravity of the base, $H$ that of the volume of the conical figure, shew that $AH = \frac{3}{4}AG$.

8. Find the centre of gravity of the surface of a right cone on a circular base. Does the method apply to the surface of an oblique cone?

## LEMMA VI.

*If any arc ACB given in position be subtended by a chord AB, and if at any point A, in the middle of continuous curvature, it be touched by the straight line AD produced in both directions, then, if the points A, B approach one another and ultimately coincide, the angle BAD contained by the chord and tangent will diminish indefinitely and ultimately vanish.*

For, if that angle do not vanish, the arc $ACB$ will contain with the tangent $AD$ an angle equal to a




rectilineal angle, and therefore the curvature at the point $A$ will not be continuous, which is contrary to the hypothesis, that $A$ was in the middle of continuous curvature.

### *Definitions of a Tangent to a Curve.*

29. (1) If a straight line meet a curve in two points $A$, $B$, and if $B$ move up to $A$, and ultimately coincide with $A$, $AB$ in its limiting position will be a tangent to the curve at the point $A$.

If two portions of a curve $EA$ and $AB$ cut one another at a finite angle in $A$, there will be two tangents $AD$, $AD'$, which will be the limiting positions of straight lines $AB$ and $AE$, when $B$ and $E$ move up to $A$ along the different portions $EA$ and $BA$ of the curve respectively. And, similarly, if there be a multiple point in $A$, in which several branches of the curve cut one another at finite angles.

(2) The tangent is the direction of the side of the polygon, of which the curve is the curvilinear limit, when the number of sides are increased indefinitely.

This is founded on the same idea of a tangent as definition (1).

(3) The tangent to a curve at any point is the direction of the curve at that point.

In order to apply geometrical reasoning to the tangent by employing this definition, we are obliged to explain the notion of the direction of a curve, by taking two points very near to one another, and asserting that the direction of the curve is the limiting position of the line joining these points when the distance becomes indefinitely small, a statement which reduces this definition to the preceding.

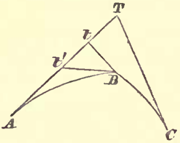
*Observations on the Lemma.*

30. "Curvatura Continua," if we consider curves as the curvilinear limits of polygons, requires the curves to be limits of polygons whose angles continually increase as the number of the sides increase, and may be made to differ from two right angles by less than any assignable angle before the assumption of the ultimate form of the hypothesis.

If, however, as we increase the number of sides and diminish their magnitude, one of the angles remains less than two right angles by any finite difference, the curvature of the curvilinear limit is discontinuous, and the form is that of a pointed arch, in which the two portions cut one another at a finite angle.

A curve may be of continued curvature for one portion between two points, while for another its curvature changes "per saltum."

Thus, if $ABC$ be a curve forming at $B$ a pointed arch, it




may be of continued curvature from $B$ to $A$ and from $C$ to $B$, though not from $C$ to $A$.

In this case the tangents in passing from $C$ to $A$ assume all positions intermediate to $CT$, $Bt$, and $Bt'$, $TA$, but at $B$ they pass from $Bt$ to $Bt'$ without assuming the intermediate positions.

31. "In medio curvaturæ continuæ," implies that the point $A$ in the enunciation of the Lemma is not such a point as $B$ in the last figure, but that, in passing from a point on one side of $A$ to another on the other side, the tangents pass through all the intermediate positions.

The curvature is supposed to be in the same direction in the figure of the Lemma, which in all curves of continuous curvature is possible, if $B$ be taken sufficiently near to $A$ at the commencement of the change in the construction.

If the point $A$ be not "in medio curvaturæ continuæ," two tangents $AD$, $AD'$ may be drawn at $A$ to the two parts of the curve, and the curve $BCA$ will make a finite angle with one of the tangents $AD'$.

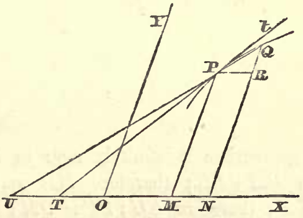
But, even in this case, the angle between the chord and that tangent which belongs to the portion of the curve considered continually diminishes and ultimately vanishes.

*The Subtangent.*

32. Def. The part of the line of abscissæ intercepted between the tangent at any point and the foot of the ordinate of that point is called the *subtangent*.

33. The subtangent may be employed as follows, to find a tangent at any point of a curve.

Let $OM$, $MP$ be the abscissa and ordinate of a point $P$ in




a curve, and let $Q$ be a point near $P$, $ON$, $NQ$ its abscissa and ordinate.

Let $QPU$ meet $OX$ the line of abscissæ in $U$; then, if $PR$ parallel to $OM$ meet $QN$ in $R$,

$$PM : MU :: QR : PR :: QN - PM : ON - OM.$$

Now as $Q$ approaches to $P$, the limiting position of $QPU$ is that of the tangent at $P$, viz. $tPT$, and $PM : MT$ is the limiting ratio of $QN - PM : ON - OM$.

*The Polar Subtangent and the Inclination of the Tangent to the Radius Vector, at any Point of a Spiral.*

34. DEF. Let $S$ be the pole, $PT$ the tangent to the curve at any point $P$, and let $ST$, perpendicular to $SP$, meet $PT$ in $T$; then $ST$ is called the *polar subtangent* at the point $P$.

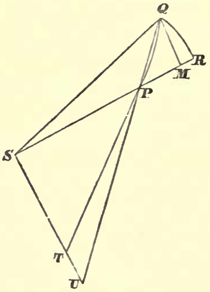
35. *To find the inclination of the tangent at any point of a curve to the radius vector.*

Let $Q$ be a point near $P$, $QM$ perpendicular to $SP$, produced if necessary, $QR$ the circular arc, centre $S$, meeting $SP$ in $R$.

Let $QP$ meet $ST$ in $U$, then

$$SU : SP :: QM : PM,$$

$$\text{and } MR : QM :: QM : SM + SR,$$




but, when $Q$ approaches indefinitely near to $P$, $QM$ vanishes compared with $SM + SR$; therefore $MR$ vanishes compared with $QM$ or $PM$; therefore $SU : SP :: QM : PR$, ultimately; therefore $ST : SP$ is the limiting ratio of $QR : PR$; or $QR : SQ \backsim SP$.

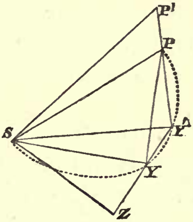
Hence $ST$, and also the trigonometrical tangent of the angle $SPT$ between the tangent and the radius vector can be found.

*Illustrations.*

(1) *If $SY$ be the perpendicular on the tangent $PY$ at $P$ in a curve, $Y$ will trace out a curve, called the pedal of the original curve; to shew that if $YZ$ be a tangent to the locus of $Y$, $SZ$ perpendicular to it, $SY^2 = SP.SZ$.*

Let $P'$ be a point near $P$, $SY'$ perpendicular on $P'P$, $SZ$ perpendicular on $Y'Y$.

Since angles $SYP$, $SY'P$ are right angles, a semicircle on $SP$ will pass through $Y$, $Y'$; therefore the angles $SY'Y$, $SPY$ in the same segment will be equal; the right angles $SZY'$, $SYP$ also are equal; therefore the triangles $SPY$, $SY'Z$ are similar, and $SZ : SY' :: SY : SP$; but, ultimately, as $P'$ moves




up to $P$, $P'PY'$ becomes the tangent at $P$, and $Y'YZ$ that at $Y$ to its locus, also $SY' = SY$;

$$\therefore\ SZ.SP = SY^2.$$

(2) *To find the subtangent in the semi-cubical parabola.*

In the semi-cubical parabola $PM^2 \propto OM^3$;

$$\therefore\ QN^2 - PM^2 : PM^2 :: ON^3 - OM^3 : OM^3,$$

$$\text{but } QN + PM = 2PM,$$

$$\text{and } ON^2 + ON.OM + OM^2 = 3\,OM^2, \text{ ultimately};$$

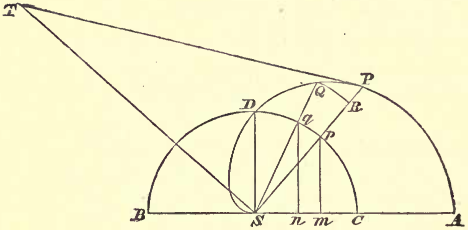
$$\therefore\ QN - PM : \tfrac{1}{2}PM :: ON - OM : \tfrac{1}{3}OM \text{ ultimately},$$

$$\text{and } QN - PM : PM :: ON - OM = MT;$$

therefore $MT$ is two-thirds of $OM$.

(3) *To find the inclination of the tangent at any point of a cardioid to the radius vector.*

DEF. If $BqpC$ be a circle, whose centre is $S$ and diameter $BC$, and $pm$ be drawn perpendicular to $BC$; then, if $Sp$ be produced to $P$, making $SP = Bm$, $P$ will trace out a *cardioid* $APS$.




Making the same construction as before, in Art. 35,

$$ST : SP :: QR : SP - SQ \text{ ultimately.}$$

Let $SQ$ meet the circle in $q$, and draw $qn$ perpendicular to $BC$,

$$\text{then } QR : pq :: SP : Sp \text{ ultimately,}$$

$$\text{also } pq : mn :: Sp : pm \text{ ..........,}$$

$$\therefore QR : mn :: SP : pm \text{ ..........;}$$

$$\text{but } mn = Bm - Bn = SP - SQ;$$

$$\therefore QR : SP - SQ :: SP : pm \text{ ultimately;}$$

$$\therefore ST : SP :: Bm : pm;$$

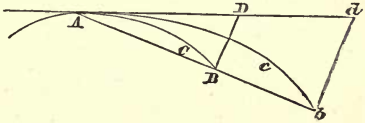
$$\text{hence } \angle PTS = \angle pBm = \tfrac{1}{2} \angle PSA;$$

and it follows that the cardioid cuts the axis $SCA$ at right angles, that it touches $SB$ at $S$, and that it cuts the circle $BDC$ at an angle equal to half a right angle.

## LEMMA VII.

*If any arc, given in position, be subtended by the chord AB, and at the point A, in the middle of continuous curvature, a tangent AD be drawn, and the subtense BD, then, when B approaches to A and ultimately coincides with it, the ulimate ratio of the arc, the chord, and the tangent to one another, is a ratio of equality.*

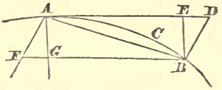
For whilst the point $B$ approaches to the point $A$, let $AB$, $AD$ be supposed always to be produced to points $b$ and $d$ at a finite distance, and $bd$ be drawn parallel to the subtense $BD$, and let the arc $Acb$ be always similar to the arc $ACB$, and have, therefore, $ADd$ for its tangent at $A$.




But, when the points $B$, $A$ coincide, the angle $bAd$, by the preceding Lemma, will vanish, and therefore the straight lines $Ab$, $Ad$, which are always finite, and the arc $Acb$, which lies between them [and is of continuous curvature in one direction, if the change commence when $B$ is near enough to $A$], will coincide ultimately, and therefore will be equal.

Hence, also, the straight lines $AB$, $AD$ and the intermediate arc $ACB$, which are always proportional to them, will vanish together, and have an ultimate ratio of equality to one another.

Cor. 1. Hence if $BF$ be drawn through $B$ parallel to the tangent, always cutting any straight line $AF$ passing through $A$ in $F$, then $BF$ will have ultimately to the vanishing arc $ACB$ a ratio of equality, since, if the parallelogram $AFBD$ be completed, it will always have a ratio of equality to $AD$.




Cor. 2. And if through $B$ and $A$ be drawn many straight lines $BE$, $BD$, $AF$, $AG$ cutting the tangent $AD$ and $BF$, parallel to it; the ultimate ratios of all the abscissæ $AD$, $AE$, $BF$, $BG$ and of the chord and arc $AB$ to one another will be ratios of equality.

Cor. 3. And, therefore, all these lines in every argument concerning ultimate ratios may be used indifferently one for the other.

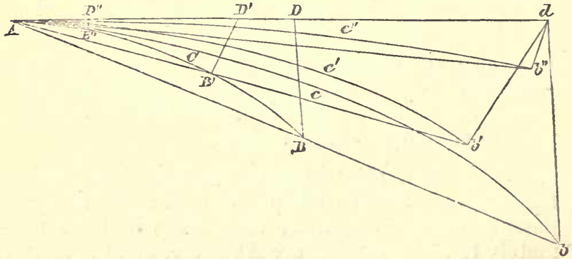
*Observations on the Lemma.*

36. Def. The *subtense* of the *angle of contact* of an arc is a straight line drawn from one extremity of the arc to meet, at a finite angle, the tangent to the arc at the other extremity.

This subtense is the secant which defines the limited line called, in the Lemma, "the tangent."

The chord is called by Newton the *subtense* of the *arc*, see Lemma XI.

37. In the construction for this Lemma, $BD$ must be a subtense, *i.e.* inclined throughout the change of position at a finite angle to the tangent, for, otherwise, the angles $BAD$



and $ADB$ being then both small, the ultimate ratio of the chord to the tangent might be any finite ratio instead of being one of equality.

This is the only limitation of the motion of $BD$; the figure representing changes which may take place in the approach towards the ultimate state of the hypothesis.

Here $b$, $d$ are the distant points, that is, points at a finite distance from $A$; $BD$, $B'D'$, $B''D''$ are consecutive positions of the subtense, when $B$ approaches towards $A$, and $db$, $db'$, $db''$ are parallel to these, $Ac'b'$, $Ac''b''$ are the forms of $Acb$ *changed* so as to be always similar to the corresponding portion of $ACB$ cut off by the chord.

It should be remarked that the curve $Acb$ is not intermediate in *magnitude* to the two lines $Ab$, $Ad$, but only in *position;* for example, $Ab$ may be equal to $Ad$, if $BD$ make equal angles with the two lines, and the curve line will then be greater than either $Ab$ or $Ad$; but it becomes in all cases less bent, until it is ultimately rectilinear; hence the three $Acb$, $Ab$, $Ad$ will be ultimately equal, the only alternative being that the curve may become doubled up, as in the figure,

which is precluded by the supposition that the curvature near $A$ is continued in the same direction throughout the passage from $B$ to $A$.

38. *The subtense ultimately vanishes compared with the arc.*

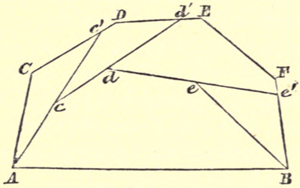
For $BD : ACB :: bd : Acb$, and, since $bd$ vanishes and $Acb$ remains finite in the limit, the ratio $BD : ACB$ ultimately vanishes. It will be afterwards seen that in curves of finite curvature $BD$ varies as the square of $ACB$ ultimately.

The ultimate equality of the lines $AD$, $AE$ with the chord or arc, whatever be the direction of the subtense, is due to the vanishing of $BD$, and therefore of $DE$ with respect to $AD$.

39. *If two curves of continuous curvature which do not intersect have a common chord, the length of the exterior curve will be*

*greater than that of the interior, provided that the curvature of the interior be always in the same direction.*

Let $AcdeB$, $ACDEFB$ any two polygons, having a common side $AB$, be such that the first lies entirely within the second




and that neither has internal angles, the perimeter of the first is less than that of the second.

For, produce $Ac$, $cd$, $de$ to meet the perimeter of the exterior in $c'$, $d'$, $e'$; then $AC + Cc' > Ac'$; $\therefore$ $ACDEFB > Ac'DEFB$;

$$\text{similarly } Ac'DEFB > Acd'EFB, \text{ and on on;}$$

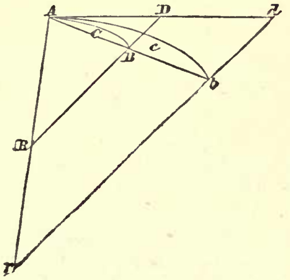
$$\therefore\ a\ fortiori,\ ACDEFB > AcdeB.$$

And, since the same is true in the limit, when the number of sides is increased indefinitely, the curvilinear limits of the polygons have the same property, and the proposition is proved.

## LEMMA VIII.

*If two straight lines $AR$, $BR$ make with the arc $ACB$, the chord $AB$, and the tangent $AD$, the three triangles $RACB$, $RAB$ and $RAD$, and the points $A$, $B$ approach one another; then the ultimate form of the vanishing triangles is one of similitude, and the ultimate ratio one of equality.*

For, whilst the point $B$ is approaching the point $A$, let $AB$, $AD$, $AR$ be always produced to points $b$, $d$, $r$ at a finite distance, and $rbd$ be always drawn parallel to $RD$, and let the arc $Acb$ be always similar to the arc $ACB$, and therefore have $Dd$ for the tangent at $A$.



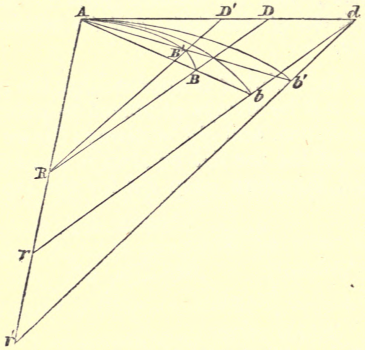

Then, when the points $B$, $A$ coincide, the angle $bAd$ will vanish, and therefore the three triangles $rAb$, $rAcb$, $rAd$ will coincide, and will therefore in that case be similar and equal. Hence also $RAB$, $RACB$, $RAD$, which are always similar and proportional to these, will be ultimately similar and equal to one another.

Cor. And hence, in every argument concerning ultimate ratios, these triangles can be used indifferently for one another.

*Observations on the Lemma.*

40. If $RB$ throughout the change in the hypothesis make a finite angle with $RA$, the three triangles $rAb$, $rAcb$, $rAd$ will

remain always finite, and will be ultimately identical and equal. But, if the angle $ARB$ be ultimately not finite, for example, if $RB$ revolve round a fixed point $R$, the three triangles $rAB$, ... will become infinite, since $r$ will move to $r'$ and so on to an infinite distance, and there will be the same kind of objection



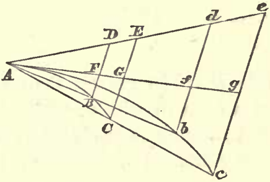

to dealing with these infinite triangles, as to reasoning immediately upon the relation of the triangles $RAB$, $RAD$ in the former case.

In this case we can at once deduce the equality of the triangles without producing $AD$ to a point $d$ at a finite distance. For, the ratio of the difference of $RAD$ and $RAB$ to $RAB$ is $BD : RB$, which vanishes ultimately, since $RB$ is finite in this case; hence $RAB$ and $RAD$ and also the curvilinear triangle, which is intermediate in magnitude to them, will be ultimately in a ratio of equality.

## LEMMA IX.

*If a straight line $AE$ and curve $ABC$, given in position, cut one another in a finite angle $A$, and ordinates $BD$, $CE$ bs drawn, inclined at another finite angle to that straight line, and meeting the curve in $B$, $C$; then, if the pointe $B$, $C$ move up together to the point $A$, the areas of the curvilinear triangles $ABD$, $ACE$ will be ultimately to one another in the duplicate ratio of the sides.*

For, as the points $B$, $C$ are approaching the point $A$, let $AD$, $AE$ be always produced to the points $d$, $e$ at a finite distance, such that $Ad : Ae :: AD : AE$; and




let the ordinates $db$, $ec$ be drawn parallel to $DB$, $EC$ meeting the chords $AB$, $AC$ produced in $b$, $c$.

Then [since $Ab : AB :: Ad : AD :: Ae : AE :: Ac : AC$, and therefore $Ab : Ac :: AB : AC$] a curve $Abc$ can be supposed to be drawn always similar to $ABC$, while $B$ and $C$ move up to $A$.

Let the straight line $Ag$ be drawn touching both curves at $A$, and cutting the ordinates $DB$, $EC$, $db$, $ec$ in $F$, $G$, $f$, $g$.

[Now areas $ABD$, $Abd$, by Lemma V., are always in the duplicate ratio of $AD$, $Ad$, and areas $ACE$, $Ace$ in the duplicate ratio of $AE$, $Ae$, and $AD : Ad :: AE : Ae$;

therefore $ABD : Abd :: ACE : Ace$,

and $ABD : ACE :: Abd : Ace$.]

If, then, the points $B$ and $C$ move up to $A$ and ultimately coincide with it, the angle $cAg$ will ultimately vanish, and the curvilinear areas *Abd*, *Ace* will coincide with the rectilinear triangles *Afd*, *Age*, and therefore will be ultimately in the duplicate ratio *Ad*, *Ae*.

But *ABD*, *ACE* are proportional to *Abd*, *Ace* always, also, *AD*, *AE* are proportional to *Ad*, *Ae*; therefore also areas *ABD*, *ACE* are ultimately in the duplicate ratio of *AD*, *AE*.

*Observations on the Lemma.*

41. By a finite angle is to be understood an angle less than two right angles, and neither indefinitely small nor indefinitely near to two right angles.

The angles between *AD* and the curve and between *AD* produced and *BD* are different finite angles, because otherwise *BD* would not meet the curve.

42. If the angle *DAF* be greater than a right angle, the figure may assume a form in which *AD* will lie below *ABC*; in this case *DB*, *EC*, ... must be produced to meet the tangent, and the argument may proceed in the same manner as before.

43. It is not necessary that *d* and *e* be fixed, but only that they remain at a finite distance from *A*, and that the proportion be retained; and the first part of this observation applies to *d* in the previous Lemmas.

The student, by reference to Arts. 37 and 40, will be able to exhibit the change in the figure which will correspond to a change of the position of *B* and *C* in the progress towards the ultimate position.

44. When the angle *CAG* vanishes, the curvilinear areas *Abd*, *Ace* coincide with the rectilinear triangles *Afd*, *Age*, and so are in the duplicate ratio of *Ad* : *Ae*. But if the angle *DAF* be not finite, those triangles will not themselves be finite, and the object aimed at by producing to a finite distance will not be attained.

The fact is, that the triangle *Adb* is made up of the triangle *Adf* and the curvilinear triangle *Afb*, of which the latter is indefinitely small ultimately, and the former is finite; therefore, in the Lemma, $Afb$ vanishes compared with $Adf$; but this will not be so if $Adf$ be indefinitely small, the ratio of the triangles $AFB$, $AGC$ must, therefore, be found by another process, and it will be found, by referring to Lemma XI., that the ratio will be ultimately that of the cubes of the arcs if the curvature of the curve at $A$ be finite.

## VIII.

1. $RQq$ is a common subtense to two curves $PQ$, $Pq$, which have a common tangent $PR$ at $P$. When $RQq$ approaches to $P$, $RQ$ and $Rq$ ultimately vanish; will the ratio $RQ : Rq$ be ultimately a ratio of equality?

2. If $PY$, a tangent to an ellipse at $P$, meet the auxiliary circle in $Y$, and $ST$ be perpendicular to the tangent at $Y$, $ST$ will vary inversely as $HP$.

3. If a subtense $BD$ be drawn to meet the tangent at $A$ at a finite angle $\alpha$, which remains constant as $B$ moves up to $A$, and $DB$ meet the normal at $A$ in $C$, shew that the ultimate ratio of $BC$ to $AB$ will be $\sec\alpha$.

4. In the curve in which the abscissa varies as the cube of the ordinate, shew that the subtangent is three times the abscissa.

5. Prove that the extremity of the polar subtangent from the focus of a conic section is always in a fixed straight line.

6. $AB$ is a diameter of a circle, $P$ a point contiguous to $A$, and the tangent at $P$ meets $BA$ produced in $T$; prove that ultimately the difference of $BA$, $BP$ will be equal to one-half of $TA$.

7. In any curve, if $Q$ be the intersection of perpendiculars to two consecutive radii vectores through their extremities, and $SY$ be the perpendicular from the pole $S$ on the tangent at $P$, prove that ultimately $SP^2 = SY.SQ$.

8. $PQ$, $pq$ are parallel chords of an ellipse whose centre is $C$; shew that, if $p$ move up to $P$, the areas $CPp$, $CQq$ will be ultimately equal.

9. From a point in the circumference of a vertical circle a chord and tangent are drawn, the one terminating at the lowest point, and the other in the vertical diameter produced; compare the velocities acquired by a heavy body in falling down the chord and tangent when they are indefinitely diminished.

10. A point moves so that the product of its distances from two fixed points is constant; shew that the normal to its path divides the angle between the two radii into two whose sines are proportional to the radii.

## IX.

1. On the radii vectores of a curve as diameters circles are described; find their envelope.

2. If the intercept $PQ$ between two curves of their common radius vector $OPQ$ be constant, and the normals at $P$ and $Q$ intersect in $N$, $ON$ will be at right angles to $OPQ$.

3. A right angle slides on any oval curve, so that the sides containing the right angle always touch the curve; shew that the angle one tangent makes with the tangent to the locus of the vertex is equal to that which the other tangent makes with the chord of contact.

Hence shew that, if the oval be an ellipse, the locus of the vertex will be a circle concentric with the ellipse.

4. A point moves so that the rectangle, whose sides are equal to the distances of the point from a given point and a given straight line, is equal to the square described on the perpendicular from the given point on the given line. Find the position of the point at which the tangent to the curve passes through the fixed point.

5. Two points $A$, $B$ describe two curves according to any finite and continuous law. If $A'$, $B'$ be the consecutive positions of $A$, $B$, and $ABC$, $A'B'C'$ be similar triangles, then the corresponding sides of the two triangles will ultimately intersect in the points $P$, $Q$, $R$, such that $\dfrac{AA'.BC}{QR} = \dfrac{BB'.CA}{RP} = \dfrac{CC'.AB}{PQ}$.

6. If $SP^2 = AB.PM$, where $PM$ is perpendicular to a fixed straight line, prove that the locus of the centre of the circle circumscribing the triangle formed by the tangent, the radius vector, and the polar subtangent, will be a straight line.

7. In the figure on page 30 let $FB'$ be taken equal to $AB$, and let the corresponding ordinate to the curve be $B'E'$; prove that the subtangent at $E'$ varies inversely as that at $E$.

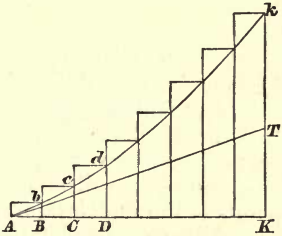
8. In the hyperbolic spiral, in which the radius vector varies inversely as the spiral angle, prove that the subtangent is constant.

9. In the spiral of Archimedes, in which the radius vector varies directly as the angle, prove that if a circle be described, of which a radius is the radius vector of the spiral, the polar subtangent will be equal to the arc of the circle subtended by the spiral angle.

## LEMMA X.

*The spaces which a body describes [from rest] under the action of any finite force, whether that force be constant or else continually increase or continually diminish, are in the very beginning of the motion in the duplicate ratio of the times.*

[Let the times be represented by lines measured from $A$, along $AK$, and the velocities generated at the end of those times by lines drawn perpendicular to $AK$. Suppose the time represented by $AK$ to be divided into a number of equal intervals, represented by $AB$,




$BC$, $CD$, ..., let $Bb$, $Cc$, $Dd$, ...$Kk$ represent the velocities generated in the times $AB$, $AC$, ...$AK$ respectively, and let $Abcd$ be the curve line which always passes through the extremities of these ordinates. Complete the parallelograms $Ab$, $Bc$, $Cd$, ....

In the interval of time denoted by $CD$, the velocity continually changes from that represented by $Cc$ to that represented by $Dd$, and therefore $CD$ being taken small enough, the space described in that time is intermediate between the spaces represented by the parallelograms $Dc$ and $Cd$; therefore the spaces described in the times $AD$, $AK$ are represented by areas which are intermediate between the sums of the parallelograms inscribed in, and circumscribed about, the curvilinear areas $ADd$, $AKk$ respectively.

Therefore, by Lemma II., the number of intervals being increased, and their magnitudes diminished indefinitely, the spaces described in the times $AD$, $AK$ are proportional to the curvilinear areas $ADk$, $AKk$.

Now the force being finite, the ratio of the velocity to the time is finite; therefore $Kk : AK$ is a finite ratio, however small the time is taken; hence, if $AT$ be the tangent to the curve line at $A$, meeting $Kk$ in $T$, $KT : AK$ will be a finite ratio; therefore the angle $TAK$ will be finite, or $AK$ will meet the curve at a finite angle.

Hence, by Lemma IX., if $AD$, $AK$ be indefinitely diminished, area $ADd$ : area $AKk$ :: $AD^2 : AK^2$; therefore, in the beginning of the motion, the spaces described are proportional to the squares of the times of describing them. Q. E. D.]

COR. 1. And hence it is easily deduced that the errors of bodies describing similar parts of similar figures in proportional times, which are generated by any equal forces acting similarly upon the bodies, and which are measured by the distances of the bodies from those points of the similar figures, to which the same bodies would have arrived in the same proportional times without the action of the disturbing forces, are approximately as the squares of the times in which they are generated.

COR. 2. But the errors which are generated by proportional forces, acting similarly at similar portions of similar figures, are approximately as the forces and the square of the times conjointly.

COR 3. The same is to be understood of the spaces which bodies describe under the action of different forces. These are, in the beginning of the motion, conjointly, as the forces and the squares of the times.

COR. 4. Consequently, in the beginning of the motion the forces are as the spaces described directly, and the squares of the times inversely.

Cor. 5. And the squares of the times are as the spaces described directly and the forces inversely.

The proof given in the original Latin is as follows:

Exponantur tempora per lineas $AD$, $AE$, et velocitates genitæ per ordinatas $DB$, $EC$; et spatia, his velocitatibus descripta, erunt ut areæ $ABD$, $ACE$ his ordinatis descriptæ, hoc est, ipso motus initio (per Lemma IX.) in duplicata ratione temporum $AD$, $AE$.

Q. E. D.

45. This proof has been amplified in order to exhibit in what manner the description of areas, by the flux of the ordinates, corresponds to that of spaces by the velocities represented by the ordinates; also to shew the propriety of the application of the ninth Lemma by reference to the definition of finite force which may be stated as follows: *A force is finite when the ratio of the velocity generated in any time to the time in which it is generated, is finite, however small the time be taken.*

*Observations on the Lemma.*

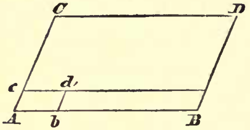
46. In the proof of this Lemma, time is represented by the length of a straight line, and a distance traversed by a body is represented by an area.

If the length of a straight line be always proportional to the period of time elapsed, the straight line will be a proper representation of the time. Thus a length of $n$ inches has the same ratio to one inch which an interval of $n$ seconds has to one second; and on this scale the length $n$ inches is a proper representation of $n$ seconds.

If an area be always in the same ratio to the unit of area that the length of a straight line is to the unit of length, the area will be a proper representation of the length of the straight line.

Thus, if $Ab$ be one foot, $AB$, $n$ feet, $Ac$ one inch, and $AC$, $t$ inches: complete the parallelograms $ABDC$, $Abdc$, and $Bc$, then $ABCD$ will contain $nt$ such areas as $Abdc$.

If now a particle move with a uniform velocity of $n$ feet a second, and $AC$ represent $t$ seconds, on the scale of one inch to a second, the parallelogram *Bc* will represent the space travelled over in the first second, since it contains *n* times the parallelo-



gram *Abdc*, and *ABDC* will represent the space travelled over in *t* seconds.

There will be no difficulty in the representation of a period of time by a line, or of a distance by an area, if the student bear in mind that periods of time and lengths of lines, although existing absolutely, are only estimated by their ratios to certain standard periods, and standard lengths, and they are therefore determined whenever these ratios are given, either directly in numbers or by the comparison of any magnitudes whatever *of the same kind.*

47. Cor. 1, 2. If bodies describe orbits under the action of certain forces, and small forces, extraneous to those under the action of which the orbits are described, be supposed to act upon the bodies, the orbits will be disturbed slightly, and the errors spoken of are the linear disturbances of the bodies, at any time, from the positions which they would have occupied at that time, if the extraneous forces had not acted.

Thus, in calculating the motion of the Moon considered as moving under the attraction of the Sun and Earth, it is convenient to estimate the motion which she would have, if subjected to the attraction of the Earth alone, and then to calculate what would be the disturbing effect of the Sun upon this orbit.

48. If *AB* be a portion of an orbit described by a body in any time, *AC* the portion of the orbit described when a disturbing force is introduced, *BC* is "quam proxime" the space which would have been described in the same time from rest by the action of the disturbing force alone. When the time is taken small, but not *indefinitely* small, the expression in the statement

of the corollaries, "approximately," is necessary for two reasons; for, in the first place, the position of the body in space is not the same at the end of any interval in the lapse of the time as if the body had moved from rest under the action of the disturbing force alone, and therefore the magnitude of the force is not generally the same either in direction or magnitude; and, in the second place, since the force is not generally uniform, the variation according to the duplicate ratio of the times is not exact, except in the limit.

But, when the times are taken very small, the variation of direction and magnitude of the force may be neglected, as an approximation to the true state of the case.

49. *Application of the method of Lemma X to determine the space described in a finite time from rest by a particle under the action of a* constant *force.*

Let $f$ be the measure of the acceleration caused by the constant force, so that at the time $t$ the velocity $V=ft$.

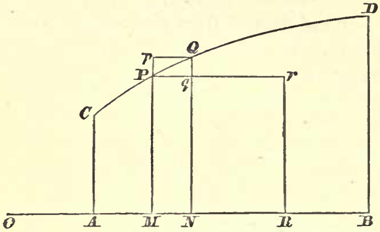
Since the velocity varies as the time, the curve $Ak$ in the figure of the Lemma is a straight line, $dD : AD$ being constant.

Therefore the space which is described in the time $t$, represented by $AK$, is represented by the area of the triangle $AKk$ or $\frac{1}{2}Kk.AK$. The space described in time $t$ from rest is therefore $\frac{1}{2}Vt=\frac{1}{2}ft^2$.

50. *General geometrical representation of the space described by a body when it moves with a variable velocity for a finite time.*

Prop. If a curve be found, such that the ordinate at each point represents the velocity corresponding to a time represented by the abscissa, then the space described by the body will be represented by the area bounded by the curve, the line of abscissæ, and the ordinates corresponding to the commencement and end of the time of motion.

Let $OA$, $OB$ represent the times at the commencement and end of the interval during which the motion of the body is to be examined. Let $OM$ be any other time, and let $AC$, $MP$, $BD$, perpendicular to $OAB$, represent the velocities at the ends of the times represented by $OA$, $OM$, $OB$; $CPD$ the curve which passes through the extremities of all such ordinates as $MP$.




Let $AB$ be divided into any number of small portions, such as $MN$; and let $NQ$ be the ordinate corresponding to $ON$. Complete the parallelograms $PMNq$, $QNMp$, and suppose corresponding parallelograms to be constructed on all the bases corresponding to $MN$.

The body during the time represented by $MN$ moves with a velocity, which, if $MN$ be taken small enough, will be intermediate in magnitude between the velocities represented by $PM$ and $QN$, and the space described during that time will be intermediate in magnitude between the spaces which would have been described with uniform velocity represented by $PM$ and $QN$, or between the spaces represented by the areas $PN$, $QM$.

Hence the whole space described in the interval of time represented by $AB$ is greater than that represented by the inscribed series and less than that by the circumscribed series of parallelograms, and each of these is, by Lemma II., ultimately equal to the area $ACDB$, when the number of portions into which $AB$ is divided is indefinitely increased, and their magnitudes diminished; therefore the proposition is proved.

51. Cor. 1. Since the area $PMNQ$ is ultimately equal to the rectangle $PM.MN$, it follows that *the measure of the velocity at any time is the limit of the quotient of the space described after that time by the time of describing it.*

52. Cor. 2. Let $MR$ represent the unit of time, and complete the parallelogram $PMRr$; then the area $PMRr$ represents

the space which would be described in an unit of time with a velocity represented by $PM$; whence it follows that *the velocity of a body at any instant may be measured by the space which it would describe if it moved with that velocity unchanged for an unit of time.*

*Measures of Variable Force, Kinetic Energy, Work of a Force.*

53. When a particle of mass $m$ is moving in a straight line under the action of an uniform force $F$, if $V$, $v$ be the velocities at the beginning and end of the interval of time $t$, and $s$ be the space described in that time, the following equations will hold: $m(v-V)=Ft$ and $\frac{1}{2}m(v^2-V^2)=Fs$.

These equations represent respectively that:

(1) The *increase of momentum* in a given *time* is equal to the *whole force* which has acted during that time.

(2) *Half the increase of vis viva*, or the *increase of the kinetic energy* in a given *space* is equal to the *work of the force* in that space.

If $F$ be a variable force, and $F_1$, $F_2$ be its least and greatest values during the time $t$, $m(v-V)$ will be greater than $F_1t$ and less than $F_2t$, each of which will become $Ft$ ultimately when $t$ is indefinitely diminished; and similarly for $\frac{1}{2}m(v^2-V^2)$.

Hence we obtain two measures of variable force in the form of the two limits:

(1) *The quotient of the increase of the momentum by the time, when the time is diminished indefinitely.*

(2) *The quotient of the increase of the kinetic energy by the space through which the force has acted, when that space is diminished indefinitely.*

54. In the velocity curve, Art. 50, the velocity $Qq$ is added in the time $MN$, the measure of the acceleration at the time $OM$ is therefore the limit of the ratio $Qq : Pq$, or the trigonometrical tangent of the angle which the tangent at $P$ to the velocity curve makes with the line of abscissæ.

55. *Geometrical representation of the momentum generated*

*by a finite and variable force acting for a finite time upon a particle moving in the direction of the action of the force.*

In the figure of p. 72, let $OA$, $OB$ represent the times at the commencement and end of the interval during which the action of the force is considered.

Let $AB$ be divided into any number of small portions, such as $MN$, and let $PM$, $QN$, perpendiculars to $AB$, represent the forces acting on the particle at the times $OM$, $ON$ respectively, and let parallelograms be constructed and the curve drawn as in Art. 50.

The momentum generated in the time $MN$, if $MN$ be taken small enough, will be intermediate between the momenta represented by the parallelograms $PN$ and $QM$; therefore, by Lemma II., the whole increase of momentum is represented by the area $ACDB$ bounded by the curve, the line of abscissæ, and the ordinates at the commencement and end of the finite interval of time represented by $AB$.

56. As in Arts. 51, 52, the measure of force given in (1) Art. 53 can be deduced; also that the force at any instant may be measured by the momentum which would be generated if the force were to continue unchanged for an unit of time.

57. *Geometrical representation of the kinetic energy generated by a force which acts upon a particle moving in the direction of the force's action through a finite space.*

Let $OAB$ be the line of motion of the particle, and when it arrives at $M$ let $PM$ perpendicular to $OAB$ represent the force, and let the construction be made as before.

The increase of kinetic energy in the passage from $M$ to $N$ is intermediate between the work done by the forces represented by $PM$ and $QN$, *i.e.* it is represented by an area which is intermediate between $PN$ and $QM$; therefore, by Lemma II, the increase of kinetic energy or the work of the force during the motion from $A$ to $B$ is represented by the area $ACDB$.

58. The measure of force given in (2), Art. 53, is deducible as before, since $PM.MN =$ area $PMNq$ ultimately.

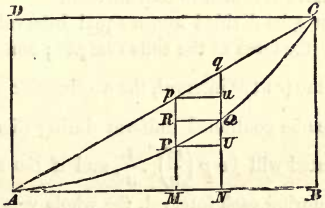
59. *In rectilinear motion of a particle under the action of any variable force, the sum of the kinetic and potential energies is constant.*

If the motion of the particle be considered only within the limits $A$, $B$, the area $PMBD$ represents the whole work which the force will be able to do as the particle moves from $M$ to the end of its path; this work is called the *Potential Energy*, and since the kinetic energy at $M$ is represented by the area $CAMP$, it follows that throughout the motion the sum of the kinetic and potential energies is constant.

*Application to the determination of the motion of a particle under various circumstances.*

(1) *To find the space travelled over in a given time by a body moving with a velocity which varies as the square of the time from the beginning of the motion.*

Let $AB$ represent the time, and let $BC$ perpendicular to $AB$ represent the velocity at the end of that time.




Let $AB$ be divided into any number of equal portions of which $MN$ is one, and let $MP$, $NQ$ represent the velocities at the ends of the times represented by $AM$, $AN$.

Then, since $MP : NQ : BC :: AM^2 : AN^2 : AB^2$, a parabola can be described touching $AB$ and passing through $P$, $Q$, $C$ and the extremities of all ordinates by which velocities are represented.

Hence the space described in the time represented by $AB$ is represented by the parabolic area $ABC$ or $\frac{1}{3}AB.BC$.

And if $p$ be the velocity at end of $1''$, $pt^2$ will be that at the end of $t''$; therefore $\frac{1}{2}pt^2.t = \frac{1}{2}pt^3$ will be the space described in the time $t$.

Note. The following method of representing the space serves to illustrate Art. 46.

Join $AC$, and let $pM$, $qN$ be the ordinates, and suppose the figure to revolve round $AB$; $pM$ generates a circle whose area $\propto pM^2 \propto AM^2$; therefore this circle may be taken to represent the velocity at the time corresponding to $AM$, and the solid generated by $pqNM$ represents the space described in time $MN$. The whole space is therefore represented by the cone generated by $ABC$, or $\frac{1}{3}AB.\pi BC^2$, which gives the same result as before.

(2) *To find the space described from rest at any time by a particle under the action of a force whose accelerating effect varies as the* $m^{\text{th}}$ *power of the time.*

This problem is more simply solved by applying directly the method of summation, since in order to find the area of the curve, constructed as in Lemma X., we should eventually be obliged to have recourse to that method.

Let the time $t$ be divided into $n$ equal intervals, and let the acceleration by the force at the time $t$ be $pt^m$; hence, at the commencement of the $(r+1)^{\text{th}}$ interval, the acceleration will be $p\left(\frac{rt}{n}\right)^m$, and, if the force be continued uniform during this interval, the velocity generated will be $p\left(\frac{rt}{n}\right)^m . \frac{t}{n}$, and if the same arrangement be made during each interval, the whole velocity generated will be $\frac{1^m + 2^m + \ldots + (n-1)^m}{n^{m+1}} pt^{m+1}$; hence, when the number of intervals is increased indefinitely, it follows, by the reasoning of Lemma II., that the velocity at the time $t = \frac{pt^{m+1}}{m+1}$.

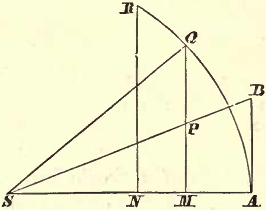
In the same manner, if the velocity at the commencement of each interval were continued uniform during the interval, the space described could be shewn to be

$$\frac{1^{m+1} + 2^{m+1} + \ldots + (n-1)^{m+1}}{n^{m+2}} \cdot \frac{pt^{m+2}}{m+1};$$

whence, proceeding to the limit, the space described in the time $t = \dfrac{pt^{m+2}}{(m+1)(m+2)}$.

(3) *To find the velocity acquired from rest, when a body is acted on by an attractive force whose accelerating effect varies as the distance from a fixed point.*

Let $S$ be the fixed point, $A$ the point from which the motion commences, and let $AB$, perpendicular to $SA$, represent the accelerating effect of the force at $A$. Join $SB$, and let $MP$, per-




pendicular to $SA$, meet $SB$ in $P$; then, since $PM : BA :: SM : SA$, $PM$ represents the accelerating effect of the force at $M$, and the square of the velocity acquired at $M$ is represented, Art. 57, by twice the area $BAPM$ or $SA.AB - SM.MP$.

With centre $S$ and radius $SA$ describe a circle $AQR$, and let $MPQ$, $NR$ be ordinates at $Q$, $R$; then, if $\mu D$ be the measure of the accelerating effect of the force at a distance $D$, (vel.)$^2$ at $M = \mu(SA^2 - SM^2)$; therefore the velocity at $M = \sqrt{(\mu)}\, QM$.

(4) *Time of describing a given space from rest under the action of a force varying as the distance from a fixed point.*

The time of describing $MN$ is ultimately, when $MN$ is indefinitely diminished, $\dfrac{MN}{\sqrt{(\mu)}\, QM} = \dfrac{QR}{\sqrt{(\mu)}\, SQ} = \dfrac{1}{\sqrt{(\mu)}} \times$ circular measure of $QSR$; therefore, if $t$ be the time from $A$ to $M$, $t\sqrt{(\mu)}$ will be the circular measure of $ASQ$.

Let $SA = a$, then the distance from $S$ at the time $t = a \cos\{t\sqrt{(\mu}\}$, and the velocity $= a\sqrt{(\mu)} \sin\{t\sqrt{(\mu)}\}$; hence, when $t\sqrt{(\mu)} = \pi$, the particle will come to rest at the point $A'$ on the opposite side of $S$, where $SA' = SA$, and, the time of oscillation from rest to rest, being $\frac{\pi}{\sqrt{(\mu)}}$, will be independent of the distance from which the motion commences.

(5) *Simple harmonic motion.*

DEF. The motion of a particle oscillating under the action of a force tending to a fixed point, and varying as the distance from it, is called *simple harmonic motion.*

From the preceding propositions the following construction for simple harmonic motion, which may also be taken as a definition, is obtained.

When a point $Q$ moves uniformly in a circle, and an ordinate $QM$ is drawn from its position at any instant to any diameter $AA'$, the motion of $M$, the foot of the ordinate, is *simple harmonic motion.**

DEF. The *amplitude* of a simple harmonic motion is the range $SA$ or $SA'$ on each side of the centre.

The *period* is the time which elapses from any instant until the moving point again moves in the same direction through the same position.

(6) *A particle is subject to the action of a force, whose accelerating effect varies as the distance from a fixed point, in the direction of which it acts, the particle is projected from a given point in a direction perpendicular to the direction of the force at that point, to find the path described by the particle.*

Let the force tend to $C$, and let $A$ be the point of projection, $P$ the position of the particle at any time.

Let $CB$, perpendicular to $CA$, be the distance in which a particle would be reduced to rest, if projected from $C$ with the velocity of projection; so that if $V$ be the velocity of projection, and $\mu CP$ be the accelerating effect of the force at $P$, $V^2 = \mu CB^2$ by (3).

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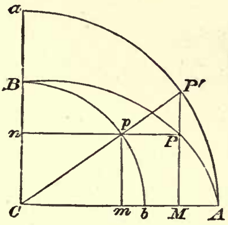
* Thomson's and Tait's *Natural Philosophy*, Art. 53.

Describe circles $Bb$, $Aa$ having the common centre $C$, and draw $CpP'$ cutting the circles in $p$ and $P'$, and draw $pn$ perpendicular to $CB$, and $pm$, $P'M$ to $CA$.




Referring to (4) supra, it will be seen that two particles starting respectively one from rest at $A$ and the other with the velocity of projection at $C$, under the action of the same force, would arrive simultaneously at $M$ and $n$, since the time in both cases is proportional to the angle $P'CA$.

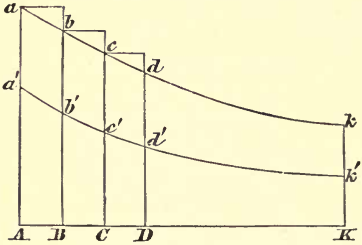
But the particle in the proposed problem is acted on at $P$ by a force which is represented by $PC$, whose accelerating effect parallel to $AC$ and $CB$ is represented by $MC$ and $PM$, therefore the acceleration in $AC$ is the same as that of the particle supposed to move in $AC$ from rest, and the retardation parallel to $BC$ the same as that of the particle in $CB$, projected from $C$, therefore $P$ is in the intersection of $np$ and $MP'$, and $PM : P'M :: pm : P'M :: Cp : CP' :: CB : CA$; therefore the required path of the particle is an ellipse whose semi-axes are $CA$ and $CB$.

Cor. 1. Area $ACP \propto$ area $ACP' \propto \angle ACP' \propto$ time from $A$ to $P$, hence the area swept out by the radius vector is proportional to the time.

Cor. 2. The square of the velocity at $P$ is the sum of the squares the velocities of the particles at $M$ and $n = \mu . P'M^2 + \mu . pn^2 = \mu . CD^2$, where $CD$ is the semi-diameter conjugate to $CP$.

(7) *The space described by a body moving in a medium, in which the resistance varies as the velocity, when no other force acts on the body, varies as the velocity destroyed.*

Let the time $AK$ be divided into equal intervals $AB$, $BC$, $CD$, ...; and let $Aa'$, $Bb'$, ... be the velocities at the beginning of the intervals, the space in time $AK$ is represented by the area $a'AKk'$.




Suppose the force of resistance to be constant throughout the intervals of time $AB$, $BC$, ..., and equal to the amount at the commencement of each, and let $Aa$, $Bb$, ... be the measures of the retarding effect of those forces, then the velocity destroyed s represented by the limit of the sum of the parallelograms $aB$, $bC$, ... or the area $aAKk$; hence the space described and the velocity destroyed vary respectively as the areas $a'AKk$ and $aAKk$; and, since the resistance varies as the velocity, the ratios $Aa' : Aa$, $Bb' : Bb$, &c., are all equal; therefore, by Lemma IV., the areas $a'AKk'$, $aAKk$ are in a constant ratio; hence the space described varies as the velocity destroyed.

## X.

1. If the square of the velocity of a body be proportional to the space described from rest, prove that the accelerating force is constant.

2. At what point of the proof of Lemma X. is it assumed that the body starts from rest?

3. State the proposition by which Lemma X. is replaced, when the body, instead of starting from rest, commences its motion with a given velocity.

4. If a body move from rest under the action of a force which varies as the square of the time from the beginning of the motion, shew that the velocity at any time will vary as the cube of the time, and the space described as the fourth power of the time.

5. If the velocity after a time $t$ from rest be equal to $a(2t+t^2)$, what will be the shape of the curve in the figure, and the space described in any time?

6. If the square of the velocity of a moving point vary as the time, find the space which will be described in a given time; and shew that the acceleration will vary inversely as the velocity.

7. If the curve employed in the proof of the Lemma be an arc of a parabola, the axis of which is perpendicular to the straight line on which the time is measured, prove that the accelerating effect of the force will vary as the distance from the axis of the parabola.

## XI.

1. If in the velocity curve of Lemma X. there should occur a point where the two parts of the curve cut one another at a finite angle, what would be the interpretation of this singularity? Explain also what a point of inflexion would imply.

2. A particle is placed in the line joining two centres of attracting force, the accelerating effect of each of which varies as the distance, find the time in which the particle oscillates.

3. When a body moves from rest at $A$ under the action of a force which varies as the square of the distance from $S$ ($=\mu . SM^2$ at $M$), the square of the velocity at $M = \frac{2}{3}\mu\,(SA^3 - SM^3)$.

4. If a body be acted on from rest by a repulsive force which varies as the distance from a fixed point, find the velocity when the body arrives at any position.

5. Two points move from rest in such a manner that the ratio of the times in which the same uniform acceleration would generate their respective velocities at those times is constant. Shew that their respective accelerations, at any time bearing that ratio, are equal.

6. Two forces reside at $S$, one attractive and whose accelerating effect on a particle varies as the distance from $S$, and the other constant and repulsive; prove that, if a particle be placed at $S$, it will move until it be brought to rest at a point which is double the distance from $S$ at which it would rest in equilibrium under the action of the forces.

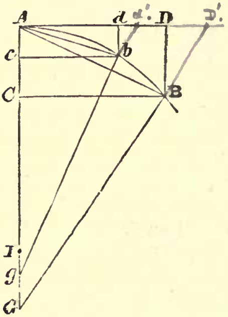
7. A particle moves from rest at $A$ under the action of a force tending to $S$, and varying as the distance from $S$, and in its path towards $S$ it strikes another particle of equal mass at rest at $B$; prove that, if the particles be perfectly elastic, they will meet again on the opposite side of $S$ at a distance equal to $SB$, and continue to impinge at $B$ and $B'$ for ever.

## LEMMA XI.

*The vanishing subtenses of the angle of contact, in all curves which have finite curvature at the point of contact, are ultimately in the duplicate ratio of the chords of the conterminous arcs.*

Case 1. Let $AB$ be the arc of a curve, $AD$ its tangent at $A$, $BD$ the subtense of the angle of contact, $BAD$, perpendicular to the tangent, $AB$ the chord of the arc.

Draw $AG$, $BG$ perpendicular to the tangent $AD$ and the chord $AB$ respectively, meeting in $G$; then let




the points $D$, $B$, $G$ move towards the points $d$, $b$, $g$, and let $I$ be the point of ultimate intersection of the lines $BG$, $AG$, when the points $B$, $D$ move up to $A$.

It is evident that the distance $GI$ may be made less than any assigned distance by diminishing $AB$.

But, since the angles $ABD$ and $GAB$ are equal, and also the right angles $BDA$, $ABG$, the triangles $ABD$, $GAB$ are similar; therefore $BD : AB :: AB : AG$, or $BD.AG = AB^2$, and, similarly, $bd.Ag = Ab^2$;

$$\therefore\ AB^2 : Ab^2 = BD.AG : bd.Ag;$$

therefore the ratio $AB^2 : Ab^2$ is a ratio compounded of the ratios of $BD : bd$ and $AG : Ag$.

But, since $GI$ may be made less than any assigned length, the ratio $AG : Ag$ may be made to differ from a ratio of equality less than by any assigned difference; therefore the ratio $AB^2 : Ab^2$ may be made to differ from the ratio $BD : bd$ less than by any assigned difference.

Hence, by Lemma I., the ultimate ratio $AB^2 : Ab^2$ is the same as the ultimate ratio of $BD : bd$. Q.E.D.

Case 2. Let now the subtenses $BD'$, $bd'$ be inclined at any given angle to the tangent; then, by similar triangles $D'BD$, $d'bd'$, $BD' : bd' :: BD : bd$, but ultimately $BD : bd :: AB^2 : Ab^2$; therefore ultimately $BD' : bd' :: AB^2 : Ab^2$. Q.E.D.

Case 3. And, although the angle $D'$ be not a given angle, if $BD'$ converge to a given point, or be drawn according to any other [fixed] law [by which the angle $D'$ remains finite, since $BD'$ is a subtense], still the angles $D'$, $d'$, constructed by this law common to both, will continually approach to equality and become nearer than by any assigned difference, and will be therefore ultimately equal, by Lemma I., and hence $BD'$, $bd'$ will be ultimately in the same ratio as before. Q.E.D.

Cor. 1. Hence, since the tangents $AD$, $Ad$, the arcs $AB$, $Ab$ and their sines $BC$, $bc$ become ultimately equal to the chords $AB$, $Ab$, their squares also will be ultimately as the subtenses $BD$, $bd$.

Cor. 2. The squares of the same lines also will be ultimately as the sagittæ of the arcs, which bisect the chords, and converge to a given point; for those sagittæ are as the subtenses $BD$, $bd$.

Cor. 3. And therefore the sagittæ will be ultimately in the duplicate ratio of the times in which a body describes the arcs with a given velocity.

Cor. 4. The rectilinear triangles $ADB$, $Adb$ are ultimately in the triplicate ratio of the sides $AD$, $Ad$, and in the sesquiplicate ratio of the sides $DB$, $db$; since these triangles are in the ratio compounded of $AD : Ad$ and $BD : bd$. So also the triangles $ABC$, $Abc$ will be ultimately in the triplicate ratio of the sides $BC$, $bc$. The sesquiplicate ratio may be regarded as the subduplicate of the triplicate, or as compounded of the simple and the subduplicate ratios.

Cor. 5. And, since $DB$, $db$ are ultimately parallel and in the duplicate ratio of $AD$, $Ad$ [therefore, this being a property of a parabola,] at every point at which a curve has finite curvature an arc of a parabola can be drawn which will ultimately coincide with the curve; and the curvilinear areas $ADB$, $Adb$ will be ultimately two-thirds of the rectilinear triangles $ADB$, $Adb$; and the segments $AB$, $Ab$ the third parts of the same triangles. And hence these areas and these segments will be in the triplicate ratio as well of the tangents $AD$, $Ad$ as of the chords and arcs $AB$, $Ab$.

## SCHOLIUM.

But, in all these propositions, we suppose the angle of contact to be neither infinitely greater nor infinitely less than the angles of contact which circles have with their tangents; that is, that the curvature at the point $A$ is neither infinitely great nor infinitely small; in other words, that the distance $AI$ is of finite magnitude.

For $DB$ might be taken proportional to $AD^3$, in which case no circle could be drawn through the point $A$ between the tangent $AD$ and the curve $AB$, and the angle of contact would be infinitely less than that of any circle.

And, similarly, if different curves be drawn in which $DB$ varies successively as $AD^4$, $AD^5$, $AD^6$, &c., a series of angles of contact will be presented which may be continued to an infinite number, of which each will be infinitely less than the preceding. And if curves be drawn in which $DB$ varies as $AD^2$, $AD^{\frac{3}{2}}$, $AD$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, &c., another infinite series of angles of contact will be obtained, of which the first will be of the same kind as in the circle, the second infinitely greater, and each infinitely greater than the preceding. But, moreover, between any two of these angles an infinite series of other angles of contact can be inserted, of which each may be infinitely greater or infinitely less than any preceding; for example, if between the limits $AD^2$ and $AD^3$ there be inserted $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$, $AD^{\frac{17}{6}}$, &c. And, again, between any two angles of this series there can be inserted a new series of intermediate angles differing from one another by infinite intervals. Nor does the nature of the case admit any limit.

The propositions which have been demonstrated concerning curved lines and the included areas are easily applied to curved surfaces and solid contents.

These Lemmas have been premised for the sake of escaping from the tedious demonstrations by the method of *reductio ad absurdum*, employed by the old geometers. The demonstrations are certainly rendered more concise by the method of indivisibles; but, as there is a harshness in the hypothesis of indivisibles, and on that account it is considered to be an imperfect geometrical method, it has been preferred to make the demonstrations of the following propositions depend on the ultimate sums and ratios of vanishing quantities and on the prime sums and ratios of nascent quantities, *i.e.* on the limits of sums and ratios; and therefore to premise demonstrations of those limits as concise as possible. By these demonstrations the same results are deducible as by the method of indivisibles; and we may employ the principles which have been established with greater safety. Consequently, if, in what follows, quantities should be treated of as if they consisted of particles [indefinitely small parts], or small curve lines should be employed as straight lines, it would not be intended to convey the idea of indivisible, but of vanishing divisible quantities, not that of sums and ratios of determinate parts, but of the limits of sums and ratios; and it must be remembered that the force of such demonstrations rests on the method exhibited in the preceding Lemmas.

An objection is made, that there can be no ultimate proportion of vanishing quantities; inasmuch as before they have vanished the proportion is not ultimate, and when they have vanished it does not exist. But by the same argument it could be maintained that there could be no ultimate velocity of a body arriving at a certain position at which its motion ceases; for that this velocity, before the body arrives at that position, is not the ultimate velocity; and that, when it arrives there, there is no velocity. And the answer is easy: that, by the ultimate velocity is to be understood that, when the body is moving, neither before it reaches the last position and the motion ceases nor after it has reached it, but at the instant at which it arrives; *i.e.* the very velocity *with which* it arrives at the last position and *with which* the motion ceases.

And, similarly, by the ultimate ratio of vanishing quantities is to be understood the ratio of the quantities, not before they vanish nor after, but *with which* they vanish. Likewise, also, the prime ratio of nascent quantities is the ratio *with which* they begin to exist. And a prime or ultimate sum is that *with which* it begins to be increased or ceases to be diminished.

There is a limit which the velocity can attain at the end of the motion, but cannot surpass. This is the ultimate velocity. And the like can be stated of the limit of all quantities and proportions commencing or ceasing to exist. And, since this limit is certain and definite, to determine it is strictly a geometrical problem. And all geometrical propositions may be legitimately employed in determining and demonstrating other propositions which are themselves geometrical.

It may also be argued that, if the ultimate ratios of vanishing quantities be given, the ultimate magnitudes will also be given, and thus every quantity will consist of indivisibles, contrary to what Euclid has demonstrated of incommensurable quantities, in his tenth book of the *Elements*.

But this objection rests on a false hypothesis. Those ultimate ratios with which quantities vanish are not actually ratios of ultimate quantities, but limits to which the ratios of quantities decreasing without limit are continually approaching; and which they can approach nearer than by *any* given difference, but which they can never surpass, nor reach before the quantities are indefinitely diminished

The argument will be understood more clearly in the case of infinitely great quantities. If two quantities, of which the difference is given, be increased infinitely, their ultimate ratio will be given, namely, a ratio of equality, yet in this case the ultimate or greatest quantities of which that is the ratio will not be given.

In what follows, therefore, if at any time, for the sake of facility of conception, the expressions *indefinitely small*, or *vanishing*, or *ultimate* be used concerning quantities, care must be taken not to understand thereby quantities determinate in magnitude, but to conceive them in all cases quantities to be diminished without limit.

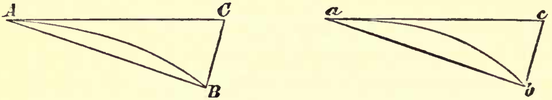
### *Curvature of Curves.*

60. The curvature of a curve at any point is greater or less as the amount of deflection from the tangent at that point, in the immediate neighbourhood of the point, is greater or less.

Two curves will have the same curvature at two points, taken one in each, if at equal distances from the points of contact, in the immediate neighbourhood of the points, they have the same deflection from the tangents at those points.

61. An exact geometrical test of equality of curvature may be obtained as follows:

If $AB$, $ab$ be two curves which have the same curvature at $A$, $a$ respectively, draw the tangents $AC$, $ac$ and take $AC = ac$.




Draw subtenses $BC$, $bc$ inclined at equal angles to the tangents.

If $BC$ and $bc$ were equal, for all equal values of $AC$, $ac$, the curves would be equal and similar. If $BC : bc$ be ultimately a ratio of equality, when $AC$, $ac$ are taken indefinitely small, the curves will have the same deflection from the tangents in the immediate neighbourhood of $A$, $a$, or the curves will have the same curvature at those points.

If the chords $AB$, $ab$ be drawn, it will be an immediate consequence that the ultimate ratio of the angles $BAC$, $bac$ will be a ratio of equality. These angles are called *the angles of contact.*

Hence, curves will have the same curvature at two points, one in each, if, equal tangents being drawn at those points, and subtenses inclined at any equal angles to the tangents, the limiting ratio of the subtenses be a ratio of equality, or if the limiting ratio of the angles of contact be a ratio of equality.

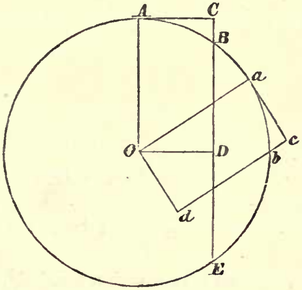
62. The curvature of one curve will be infinitely greater or infinitely less than that of another if the limiting ratio of the subtense of the first to that of the second be infinitely great or infinitely small.

63. The ratio of the curvature of one curve to that of another at two points, or of the curvature of the same curve at two different points, is the limiting ratio of the subtenses drawn from the extremities of equal tangents and inclined at equal angles to the tangents.

64. The curvature of a curve is said to be finite, at any point, when the ratio of the curvature at that point to that of any circle whose radius is finite, is a finite ratio.

65. *The curvature of a circle is the same at every point.*

Let $A$, $a$ be any two points on a circle, $AC$, $ac$ equal tangents at $A$, $a$, $Cb$, $cb$ subtenses perpendicular to the tangents, $OD$, $Od$ perpendicular to the subtenses produced; therefore $CD=cd$, each being equal to the radius, and $BD=bd$; hence $BC=bc$ always, and therefore ultimately, when the arcs are indefinitely diminished, $BC:bc$ is a ratio of equality; therefore




the circle has the same curvature at any two points.

66. *In different circles the curvatures vary inversely as the radii.*

In the last figure, produce $CB$ to the circumference in $E$. Then, $AC^2=CB.CE$; also, if $A'C'$ be a tangent to another circle, and $A'C'$ be taken equal to $AC$, and the same construction be made, $A'C'^2=C'B'.C'E'$; therefore $CB.CE=C'B'.C'E'$, and $CB:C'B'::C'E':CE$; and when $AC$, $A'C'$ are indefinitely diminished, $CE=2AO$; therefore $CB:C'B'::A'O':AO$, ultimately, or the curvatures are inversely proportional to the radii.

*Measure of Curvature.*

67. The curvature of a circle is the same at every point; the curvatures of different circles vary inversely as the diameters

of the circles; and a circle can be constructed of any degree of finite curvature by varying the magnitude of the diameter.

Hence, a circle can always be found whose curvature at any point is equal to that of a curve at a fixed point.

The curvature of a curve at any point is therefore completely determined when the diameter of the circle is found, which has the same curvature as the curve at the given point.

The diameter of the circle, which has the same curvature as the curve at a given point, is called *the diameter of curvature of the curve at that point.*

The chord of the circle, drawn in any direction, is called the *chord of curvature in that direction.*

The circle itself is called *the circle of curvature*, and is the circle which has the same tangent as the curve at any point, and also the same curvature.

68. Any other curve might have been chosen to establish a standard measure of finite curvature; but, since no curve but the circle has the same curvature at every point, it would then have been necessary, after selecting the curve, to specify the point, the curvature at which might be made the measure of curvature.

Thus, if the standard curve were a parabola, we must choose the curvature of the parabola at the vertex or at the extremity of the latus rectum or at some determinate point, by which to obtain the measure.

The inconvenience is obvious.

*General Properties of the Circle of Curvature.*

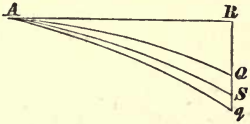
69. If a circle be drawn touching a curve at a given point, and cutting it at a second point, as the second point approaches indefinitely near the point of contact, the circle will assume a limiting magnitude, and will evidently satisfy the condition of having the same curvature as the curve at that point.

70. Since a tangent at any point is the limiting position of a side, terminated in that point, of a polygon inscribed in the curve, when the number of sides is increased indefinitely,

so the circle of curvature at any point is the limiting circle which passes through three consecutive angular points of the polygon, one of which coincides with the point.

71. *No circle can be drawn whose circumference lies between a curve and its circle of curvature, in the neighbourhood of the point at which the circle of curvature is drawn.*

For, let $AQ$ be the arc of the curve, $Aq$ of the circle of curvature; and let, if possible, another circle be drawn, of which the arc $AS$ lies between the curve and circle, and having therefore the same tangent $AR$ at $A$; and let $RQ$, the subtense perpendicular to the tangent, cut the circles in $S$, $q$.




Then $SR : qR$ will be ultimately in the inverse ratio of the diameters of the circles; therefore $SR$ will be ultimately unequal to $qR$; but, since $qR$ and $QR$ are ultimately in a ratio of equality, $SR$, which is intermediate in magnitude, will be ultimately equal to either, which is absurd; therefore no circle, &c.

This proposition corresponds to *Euclid* III., Prop. XVI.

72. *The circle of curvature generally cuts the curve.*

For the curvature of the curve at different points taken along the curve continually increases or continually diminishes, until it arrives at a maximum or minimum value.

If therefore the circle of curvature be drawn at any point, on the side on which the curvature is increasing, as we proceed from the point, the curve will lie within the circle, and on the other side, on which the curvature is diminishing, the curve will lie without the circle; which proves the proposition for the general position of the point.

For the particular case, in which the point is at a position of maximum or minimum curvature, as at the extremities of the axes of an ellipse, if the curvature be a maximum the curvature at adjacent points on either side will be less than that of the circle of curvature at the point under consideration; therefore the circle will lie entirely within the curve on both sides near the point of maximum curvature; and, similarly, it will lie without the curve at points of minimum curvature.

We can illustrate this by reference to the polygon inscribed in the curve; see the figure in the following page.

If, in the curve, equal chords $AB$, $BC$, $CD$, $DE$,... be placed in order, generally the angles $ABC$, $BCD$, $CDE$,... will increase or decrease, commencing from any point, which property of the polygon will have in the curvilinear limit, when the chords are diminished indefinitely, the corresponding property, that the curvature decreases or increases continually.

Suppose the angles are increasing from $B$; make the angles $CBA'$, $CDE'$ equal to the angle $BCD$, and $BA'$, $DE'$ equal to $BC$, $CD$...; then a circle through $B$, $C$, $D$ will pass also through $A'$ and $E'$, and these points will be on opposite sides of the perimeter of the polygon, whence, if we proceed to the limit, the circle of curvature at a point in the middle of increasing curvature will cut the curve.

If the angles $ABC$ and $DEF$ be each less than the angles $BCD$, $CDE$, supposed equal, the curvature will decrease and then increase, and the circle about $BCD$ will pass through $E$, and $BA$, $EF$ will lie within the circle, and, proceeding to the limit, the circle of curvature will lie without the curve, near the point of minimum curvature.

### *Evolute of a Curve.*

73. Def. If the circles of curvature be drawn at every point of a curve, the centres of those circles will lie in a curve which is called the *evolute* of the proposed curve.

### *Properties of the Evolute.*

74. *The extremity of a string unwrapped from the evolute of a curve traces out the curve.*

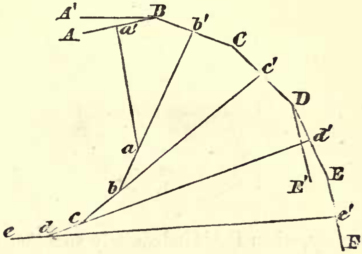
Let $ABCDE$ be any equilateral polygon, and let $a'a$, $b'b$, $c'c$ $d'd$ be drawn perpendicular to the sides from the middle points

$a'$, $b'$, &c., these intersect in the angular points $abcd$... of another polygon.

If a string were wrapped round $a'abcd$... the extremity $a'$ would as the string was unwrapped pass through the points $a'b'c'd'$.

Let now the number of sides of the polygon be increased and the magnitude diminished indefinitely.

The points $a'b'c'$... will be ultimately in the curve which is the limit of the polygon, and since $a$, $b$, $c$... are the centres of the circles described about $ABC$, $BCD$,..., $a$, $b$, $c$,... will be ultimately the centres of the circles of curvature at $a'b'c'$..., and




the curve, which is the limit of the polygon $abcd$..., will be the evolute of the curve $a'b'c'$..., and the property proved for the polygons will be true for the limits of the polygons, therefore the extremity of the string unwrapped from the evolute will trace the curve of which it is the evolute. This property gives rise to the name of evolute.

DEF. The curves formed by the unwrapping of a string from a curve are called *involutes*.

75. *The tangent to the evolute of a curve is a normal to the curve.*

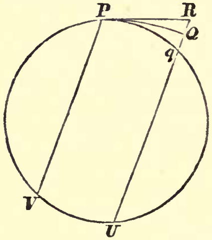
Since $b'b$ is ultimately the tangent to the evolute and is perpendicular to $BC$, which is ultimately the tangent to the curve $a'b'c'$..., therefore the tangent to the evolute is a normal to the curve.

*Propositions on Diameters and Chords of Curvature.*

76. *If a subtense be drawn from the extremity of an arc of finite curvature, in any direction, the chord of curvature parallel to that direction will be the limit of the third proportional to the subtense and the arc.*

Let $PQ$, $Pq$ be arcs of a curve and its circle of curvature at $P$, let $PR$ be the common tangent, and $RQq$ the direction of a common subtense, meeting the circle in $U$.

Draw the chord $PV$ parallel to $RQ$. Then, since $Rq.RU=PR^2$, $RU$ is the third proportional to $Rq$ and $PR$.




But, ultimately, when $PQ$ is indefinitely diminished, $RU=PV$, and $PR=PQ$, by Lemma VII. also, $Rq=RQ$ by the property of the circle of curvature.

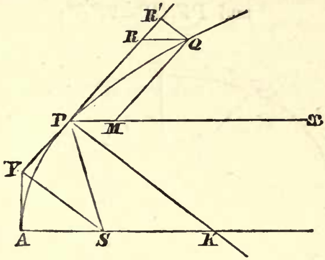
Therefore $PV$ is the limit of the third proportional to $RQ$ and $PQ$.

Cor. *The diameter of curvature is the limit of the third proportional to the subtense perpendicular to the tangent and the arc.*

77. *The two chords of curvature at any point of a parabola drawn through the focus, and in the direction of the diameter, are each equal to four times the focal distance of that point.*

Let $AP$ be a parabola, $P$ any point, $RQ$ a subtense parallel to the diameter $PMx$, $QM$ the ordinate at $Q$, $S$ the focus. Then, by a property of the parabola, $QM^2=4SP.PM$; therefore $4SP$ is a third proportional to $PM$ and $QM$, *i.e.* to $RQ$ and $PR$.

Hence, $4SP$ is the limit of the third proportional to the subtense $QR$ and the arc $PQ$, and is therefore equal to the chord of curvature at $P$ in direction of the diameter.




And, since $PS$, $PM$ are equally inclined to the tangent at $P$, the chords in those directions are equal; therefore the chord of curvature through $S$ is four times the focal distance $SP$.

78. *One-fourth of the diameter of curvature at any point of a parabola is a third proportional to the perpendicular from the focus on the tangent at that point, and the focal distance of that point.*

For, draw $SY$, $QR'$ perpendicular to $PR$, and let $PI$ be the diameter of curvature at $P$.

$$\text{Then } PI.\,QR' = PQ^2 = PR^2 \text{ ultimately}, = 4SP.\,QR;$$

$$\therefore\ PI : 4SP :: QR : QR' :: SP : SY;$$

since the triangles $SYP$, $QR'R$ are similar; therefore $\frac{1}{4}PI$ is a third proportional to $SY$ and $SP$.

79. *The chord of curvature at any point of an ellipse drawn through the centre of the ellipse is a third proportional to the diameter through that point and the diameter conjugate to it.*

Let $P$ be any point in an ellipse, $PCG$ the diameter, $DCD'$ conjugate to it, $Q$ any point near $P$, $QR$ a subtense parallel to $CP$, $QM$ an ordinate parallel to $DC$, $PV$ the chord of curvature drawn through $C$.

$$\text{Then } PV.\,QR = PQ^2 = QM^2 \text{ ultimately},$$

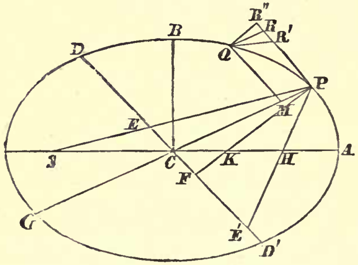
$$\text{and } QM^2 : PM.MG :: CD^2 : CP^2;$$

$$\therefore\ PV.\,QR : QR.MG :: CD^2 : CP^2 \text{ ultimately};$$

$$\therefore\ PV : 2CP :: CD^2 : CP^2 \text{ ultimately};$$

$$\therefore\ PV.CP : CP^2 :: 2CD^2 : CP^2,$$

$$\text{and } PV.CP = 2CD^2;$$




or $PV$ is a third proportional to $PG$ and $DCD'$.

80. *The chord of curvature at any point through the focus is a third proportional to the major axis, and the diameter parallel to the tangent at that point.*

Draw the focal distance $SP$ cutting the diameter $DCD'$ in $E$, let $PV'$ be the chord of curvature through $S$, and draw the subtense $QR'$ parallel to $SP$.

$$\text{Then } PV' : PV :: QR : QR' \text{ ultimately}$$

$$:: CP : PE \text{ by similar triangles};$$

$$\therefore\ PV'.PE = PV.CP = 2CD^2;$$

$$\therefore\ PV' \text{ is a third proportional to } 2PE \text{ and } DCD',$$

and $2PE$ is equal to the major axis.

Similarly for the other focus $H$.

81. *The diameter of curvature at any point is a third proportional to twice the perpendicular from the point on the diameter parallel to the tangent and that diameter.*

Draw $QR''$ perpendicular to the tangent, and $PF$ perpendicular to $DCD'$, and let $PI$ be the diameter of curvature.

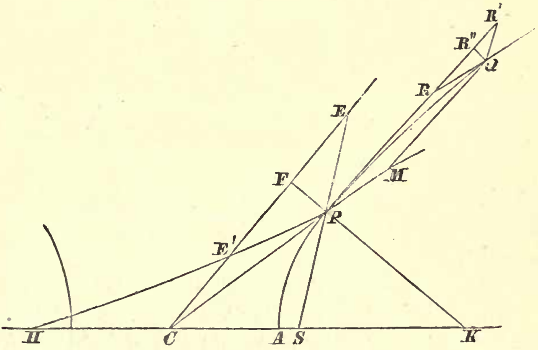
$$PI : PV :: QR : QR'' :: CP : PF;$$

$$\therefore\ PI.PF = PV.CP = 2CD^2;$$

$$\therefore\ PI \text{ is a third proportional to } 2PF \text{ and } DCD'.$$

82. Since the chord of curvature in any direction varies inversely as the subtense $QR$ drawn in that direction, it is easily seen that, if $PL$ be the portion of the chord intercepted between $P$ and $DCD'$, the chord of curvature at $P$ in the direction $PL$ will be the third proportional to $2PL$ and $DCD'$.

83. The propositions concerning the chords and diameter of curvature of an ellipse may be proved in the same words for the hyperbola, employing the following figure.




84. *The radius of curvature at any point of a conic section is to the normal in the duplicate ratio of the normal to the semi-latus rectum.*

Let $PK$ be the normal, $PO$ the radius of curvature at $P$, $L$ the semi-latus rectum.

(i) For the parabola,

$$PO : 2SP :: SP : SY :: SY : SA;$$

$$\therefore\ PO : 2SY :: SP : SA :: 4SP.SA : L^2;$$

but $PK = 2SY$; $\therefore\ PK^2 = 4SP.SA$; $\therefore\ PO : PK :: PK^2 : L^2$.

(ii) For the ellipse or hyperbola,

$$PO.PF = CD^2 \text{ and } PK.PF = BC^2;$$

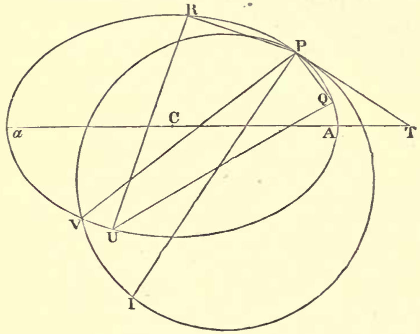
$$\therefore\ PO : PK :: CD^2 : BC^2 :: AC^2 : PF^2;$$

but $PF.PK = BC^2 = L.AC$; $\therefore\ AC : PF :: PK : L$;

$$\therefore\ PO : PK :: PK^2 : L^2.$$

85. *To find the chord common to a conic section and the circle of curvature at any point.*

If a circle intersect a conic section in four points, as $PQUR$, and these points be joined in pairs by two lines, these lines will




be equally inclined to the axis of the conic section. Thus, in the conic section, $PQ$, $RU$ are equally inclined to the axis.

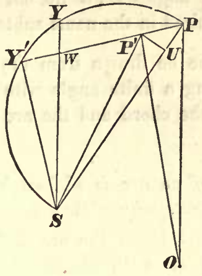
For, if $UR$, $QP$ intersect in $O$, $OR.OU = OP.OQ$, hence the diameters of the ellipse parallel to $UR$, $QP$ will be equal, and therefore equally inclined to the axis.

Let $Q$ and $R$ move up to and ultimately coincide with $P$, then the intersecting circle becomes the circle of curvature at $P$, and $PQ$ is in the direction $PT$ of the tangent, ultimately, and $RU$ assumes the position of the chord common to the conic section and the circle of curvature at $P$. Hence, if $PV$ be drawn at an equal inclination with $PT$ to the axis, $PV$ will be the common chord required.

And, if $VI$ be drawn perpendicular to $PV$, meeting the normal at $P$ in $I$, $PI$ will be the diameter of curvature at $P$.

86. *To find the radius of curvature of a curve defined by the relation between the radius vector and the perpendicular from the pole on the tangent.*

Let $PY$, $PP'Y'$ be the directions of consecutive sides of a polygon inscribed in a curve, $SY$, $SY'$ perpendiculars on these sides; draw $PO$, $P'O$ perpendicular to the same sides, inter-




secting in $O$, and $P'U$ perpendicular to $SP$, and let $SY$, $PY'$ intersect in $W$.

A semicircle on $SP$ as diameter passes through $Y$ and $Y'$;

$\therefore \angle YPW = \angle YSY' = \angle POP'$, and $\angle WYP = \angle OP'P$;

therefore the triangles $POP'$, $WPY$ are similar;

$$\therefore PO : PP' :: PW : YW,$$

$$\text{also } PP' : SP :: PU : PY',$$

by similar triangles $P'UP$, $SY'P$, and $PW = PY'$ ultimately;

$$\therefore PO : SP :: PU : YW :: SP \sim SP' : SY \sim SY' \text{ ultimately.}$$

Also, if $PV$ be the chord of curvature through $S$,

$$PV : 2PO :: SY : SP;$$

$$\therefore PV : 2SY :: SP \sim SP' : SY \sim SY' \text{ ultimately.}$$

*Observations on the Lemma.*

87. In the proof of Lemma XI., $AI$ is the limit of the third proportional to $BD$ and $AB$, hence it is the diameter of curvature of the curve at $A$.

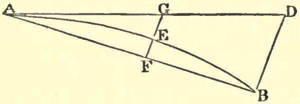
88. For an example of a law according to which, in Case 3, the directions of the subtenses may be determined, we may suppose that they always pass through a point given in position at a finite distance from $A$, or that they always touch a given curve; but it must be observed that the case in which they

touch a curve which has the same tangent $AD$ at $A$ is excluded, since in this case the angles $D'$, $d'$ do not in the limit remain finite, a property required in the name subtense.

89. DEF. If a line be drawn from the middle point of an arc of a curve, making a finite angle with the chord, the part intercepted between the chord and the arc is called the *sagitta* of the arc.

90. *The sagitta of an arc is ultimately one quarter of the subtense drawn at the extremity of the arc parallel to the sagitta.*

Let the sagitta $FE$ bisect the arc $AB$ in $E$, and be produced to the tangent at $A$ in $G$, and let $BD$ be a subtense parallel to $FE$.




$$\text{Then } EG : BD :: AE^2 : AB^2 \text{ ultimately}; \quad \therefore BD = 4EG,$$

$$\text{also } BD : FG :: AD : AG :: AB : AE \text{ ultimately};$$

$$\therefore BD = 2FG = 4EG; \text{ hence } FE = EG = \tfrac{1}{4}BD \text{ ultimately}.$$

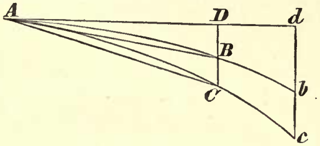
91. COR. 5. The parabola mentioned in this corollary is a parabola of curvature at that point; for, since $DB$ is taken in any given direction, the proportion $BD : bd :: AD^2 : Ad^2$ proves that the curve is ultimately in the form of a parabola, and that, therefore, the line through $A$ drawn in the given direction is the corresponding diameter of the parabola of curvature.

Hence the axis of the parabola may be taken in any assigned direction.

If the subtenses be perpendicular to the tangent, the parabola of curvature will be the parabola whose curvature at the vertex will determine the curvature of the curve, since the axis will be perpendicular to the tangent, and if $4AU$, in the figure page 104, be the third proportional to the subtense and arc, the limiting position of $U$ will be the focus of the parabola.

By means of this corollary the proposition alluded to under Lemma IX., Art. 44, is established; viz. that the ratio of the areas which takes place of the duplicate ratio, obtained in that Lemma, is the triplicate ratio of the same lines, when the line, $AE$, instead of cutting the tangent at a finite angle, coincides with the tangent.

92. Scholium. Let $AB$, $AC$ be two curves, having a common tangent $AD$ at $A$, and let subtenses $DB$, $DBC$ of the




angles of contact be drawn from $D$ at any point in the tangent in the same direction, and let $BD \propto AD^m$, $CD \propto AD^n$ in the curves $AB$, $AC$ respectively. Draw $dbc$ a common ordinate from a fixed point $d$, parallel to $DBC$. Then

$$AD^m : Ad^m :: BD : bd,$$

$$\text{and } AD^n : Ad^n :: CD : cd,$$

and if $m$ be greater than $n$, $= n + r$ suppose,

$$AD^n . AD^r : Ad^n . Ad^r :: BD : bd;$$

$$\therefore \; CD . AD^r : cd . Ad^r :: BD : bd$$

$$:: BD . AD^r : bd . AD^r;$$

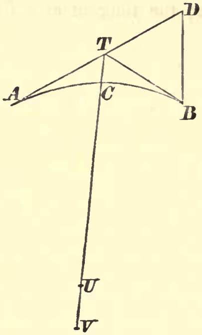
$$\therefore \; CD : BD :: cd . Ad^r :: bd . AD^r,$$

and since $b$, $c$, $d$ are fixed, and $AD$ vanishes in the limit, therefore $CD$ is indefinitely greater than $BD$; also, since the angles of contact $BAD$, $CAD$ are ultimately proportional to $BD$, $CD$, it follows that, if in two curves the subtenses vary according to different powers of the arcs or tangents, the angle of contact of that curve in which the index of the power is the least will be infinitely greater than the angle of contact of the other.

*Illustrations.*

(1) *Two tangents $AT$, $BT$ are drawn at the extremities of an arc $AB$, to prove that $AT$ is ultimately equal to $BT$, when $AB$ is indefinitely diminished.*

Draw $TCUV$ in any direction making a finite angle with the tangents, and meeting the circles of curvature at $A$ and $B$ in $UV$.




Then since the circle of curvature at $A$ is the limit of the circle which passes through $C$ and has the tangent $AT$ at $A$, and similarly for that at $B$, we have ultimately

$$TA^2 : TB^2 :: TC.TU : TC.TV,$$

and $TU = TV$ ultimately; $\therefore$ $TA = TB$ ultimately.

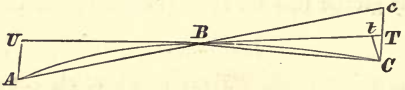
Cor. If $BD$ be any subtense of the arc $AB$,

$$AT + TB = AB = AD \text{ ultimately};$$

therefore $AD$ will be ultimately bisected by the tangent $BT$.

(2) *If $BT$ be a tangent at $B$, $AB$, $BC$ equal chords of a curve of finite curvature, drawn from $B$, and $AB$ be produced to $c$, making $Bc = AB$, and $Cc$ be joined meeting $BT$ in $T$, $cT$ will ultimately be equal to $CT$, when the arcs $AB$, $CB$ are diminished indefinitely.*

Let $AU$ be drawn parallel to $CT$, meeting the tangent at $B$ in $U$, and let two circles touch $UBT$ at $B$ and pass one through




$A$ and the other through $C$, and let $BV$, $BV'$ be chords of these circles drawn parallel to $AU$ or $CT$, then $AU.BV = AB^2$, and

$CT.BV' = BC^2$; but $BV = BV'$ ultimately, since the two circles are each ultimately the circle of curvature at $B$ and $AB = BC$, therefore $AU = CT$ ultimately.

Through $B$ draw $RBR'$ parallel to $AC$, meeting $AU$ in $R'$ and $Cc$ in $R$, then $R'U = RT$, therefore $2RT$ is the difference between $AU$ and $CT$, hence $RT$ ultimately vanishes compared with $CT$, and since $CR = Rc$, therefore $CT = Tc$ ultimately.

(3) *If, from the point of contact of a curve with its tangent, equal distances be measured along the curve and tangent, the line joining their extremities will ultimately be parallel to the normal at the point of contact.*

In the last figure, let, $BC$, $BT$ be equal distances, measured along the arc and the tangent; join $CT$, let the tangent at $C$ meet $BT$ in $D$, produce $BT$ to $F$ making $DF = DC$, take $BE =$ the chord $BC$, and join $EC$, $TC$, and $FC$.

Since the arc $BC$ is intermediate in magnitude between $BD + DC$ and $BC$, therefore, $BT$ being equal to arc $BC$, the point $T$ lies always between $E$ and $F$. But the triangles $BCE$, $DCF$ being both isosceles, each of the angles $BEC$, $BFC$ will ultimately be a right angle, therefore the angle $BTC$, which is less than $BEC$ and greater than $BFC$, will also ultimately be a right angle.

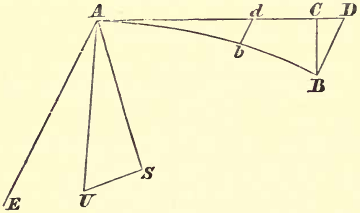
Hence $CT$ will ultimately be parallel to the normal at $B$.

NOTE. In order to shew the danger of falling into an error by a careless employment of the propositions proved in the first section, the following fallacious proof may be noticed of the above proposition.

In the figure page 102, join $BC$, then $BT : CB$ will be ultimately a ratio of equality, by Lemma VII; therefore $CBT$ being an isosceles triangle ultimately, $CT$ will be perpendicular to the line bisecting the angle $CBT$, and therefore to the tangent $BT$, since $BT$ and $BC$ will ultimately coincide with the bisecting line.

The fact is that Lemma VII. only allows us to assert that $BT$ and the chord $BC$ differ by a quantity $Tt$, which vanishes compared with either of them, and therefore $Tt$ may $\propto BC^2$; but, by Lemma XI, $CT \propto BC^2$; hence $Tt : CT$ may possibly be a finite ratio, or $CT$ may be ultimately inclined at any finite angle to $BT$, at least as far as the reasoning given in the above proof is concerned.

(4) *To construct for the focus of the parabola of curvature whose axis is in a given direction.*




Let $AB$ be a curve of finite curvature, $BD$, $bd$ subtenses parallel to $AE$ the given direction. Draw $AU$ perpendicular to $AD$, and $AS$ making angle $UAS = UAE$; then since $AE$ is a diameter of the parabola by Art. 91, $AS$ is in the direction of the focus.

Also, if $4AS$ be taken a third proportional to $BD$ and $AD$, the limiting position of $S$ will be the focus of the parabola.

(5) *To find the locus of the focus of the parabola of curvature, when its axis changes its direction.*

Let $BC$ be perpendicular to $AD$, and $AU$ be chosen so that $4AU.BC = AC^2$, then the limiting position of $U$ is the focus of the parabola whose curvature at the vertex is the same as that of the curve at $A$; also, if $S$ be the focus of the parabola whose axis is parallel to $DB$, $4AS.DB = AD^2 = AC^2$, ultimately; therefore $AU : AS :: BD : BC$, and $\angle\, SAU = \angle\, DBC$; hence if we join $SU$, the triangles $SAU$, $CBD$ will be similar, and $\angle\, ASU = \angle\, BCD =$ a right angle; therefore the locus of $S$ is a circle on $AU$ as diameter.

(6) *$ABC$ is an arc of finite curvature, and is divided so that $AB : BC :: m : n$, a constant ratio; join $AB$, $AC$, $BC$, and shew that, ultimately, $\Delta ABC$ : segment $ABC :: 3mn : (m + n)^2$.*

For, by Cor. 5, Lemma XI.

$$\text{seg } AB : \text{seg } ABC :: AB^3 : ABC^3 :: m^3 : (m+n)^3$$

$$\text{seg } BC : \text{seg } ABC :: n^3 : (m+n)^3;$$

$$\therefore \text{seg } AB + \text{seg } BC : \text{seg } ABC :: m^3 + n^3 : (m+n)^3,$$

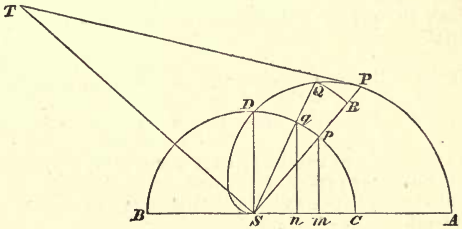
$$\text{and } \Delta\, ABC = \text{seg } ABC - \text{seg } AB - \text{seg } BC;$$

$$\therefore\ \Delta\, ABC : \text{seg } ABC :: 3\,(m^2 n + mn^2) : (m+n)^3$$

$$:: 3mn : (m+n)^2.$$

(7) *To find the chord of curvature, at any point of the cardioid, through the focus.*

It is easily seen from p. 56 (3), that $SY$ being perpendicular to $PT$, the triangles $PSY$, $pBm$, and $CBp$ are similar;




$$\therefore SY : SP :: Bm : Bp :: Bp : BC;$$

$$\therefore SY^2 : SP^2 :: SP : BC, \text{ since } Bm = SP,$$

$$\therefore SY^2 BC = SP^3, \text{ and } (SY^2 - SY'^2)\, BC = SP^3 - SP'^3;$$

$$\therefore SP \sim SP' : SY \sim SY' :: 2SY.BC : 3SP^2 \text{ ultimately};$$

$$\therefore \text{by Art. 86, chord of curvature} : 2SY :: 2SP : 3SY;$$

therefore the chord of curvature through $S = \frac{4}{3}SP$.

## XII.

1. Prove that the focal distance of the point in the parabola at which the curvature is one-eighth of that at the vertex is equal to the latus rectum.

2. Prove that the diameter of curvature at the vertex of the major axis of an ellipse is equal to the latus rectum: and shew that the ratio of the curvature at the extremities of the axes is that of the cubes of the axes.

3. Shew that at no point of an ellipse will the circle of curvature pass through the centre, if the eccentricity be less than $\sqrt{\frac{1}{2}}$.

4. Find for what point of an ellipse the circle of curvature passes through the other extremity of the diameter at that point, shew that the distance of this point from the centre is the side of the square of which $AB$ is the diagonal.

5. In a rectangular hyperbola, the diameter of curvature at any point, and the chords of curvature through the focus and centre are in geometrical progression.

6. Prove that at a point $P$ in an ellipse for which the minor axis is a mean proportional between the radius of curvature and the normal, $PC = AC - BC$. Shew that this is impossible unless $AC \geqq 2BC$.

7. If the radius of curvature for an ellipse at $P$ be twice the normal, prove that $CP = CS$.

If moreover $AC = 2BC$, prove that $CP = 3PM$.

8. If the circle of curvature at a point $P$ of a parabola pass through the other extremity of the focal chord through $P$, and the tangent at $P$ meet the axis in $T$, prove that the triangle $PST$ will be equilateral.

9. Prove that the distance of the centre of curvature, at any point of a parabola, from the directrix is three times that of the point.

10. If the circle of curvature at a point on a parabola touch the directrix, the focal distance of the point will be $\frac{9}{16}$ of the latus rectum.

11. $PQ$ is a normal at a point $P$ of a rectangular hyperbola, meeting the curve again in $Q$, prove that $PQ$ is equal to the diameter of curvature at $P$.

12. Prove that the portion of the normal intercepted between the line joining the extremities of the two chords of curvature through the foci of an ellipse, and the point of contact $P$, is $\dfrac{2BC^2}{PF}$.

13. A fixed hyperbola is touched by a concentric ellipse. If the curvatures at the point of contact be equal, the area of the ellipse will be constant.

14. Shew that the directrices of all parabolas touching a curve of finite curvature at any given point, and having the same curvature at that point as the curve, pass through a fixed point.

## XIII.

1. Prove that the chord of curvature through the vertex $A$ of a parabola : $2PY :: 2PY : AP$, $Y$ being the intersection of the tangents at $P$ and $A$.

2. Apply the property that the radius of curvature at any point of an ellipse is to the normal in the duplicate ratio of the normal to the semi-latus rectum, to shew that the radius of curvature at the extremity of the major axis is equal to the semi-latus-rectum.

3. Assuming only that a curve has a subnormal of constant length, prove geometrically that its radius of curvature varies as the cube of its normal.

4. If $Pp$ be any chord of an ellipse, $PT$, $pT$ tangents at $P$ and $p$, shew that the curvatures at $P$ and $p$ are as the cubes of $pT$ and $PT$.

5. Shew that the sum of the chords of curvature through a focus of an ellipse at the extremities of conjugate diameters is constant. Also, if $\rho$, $\sigma$ be the radii of curvature at those points, prove that $\rho^{\frac{2}{3}} + \sigma^{\frac{2}{3}}$ is constant.

6. Prove that the chords of curvature through any two points on an ellipse in the direction of the line joining them are in the same ratio as the squares on the diameters parallel to the tangents at the points.

7. Prove that the distances of the centre of curvature at any point of an ellipse and of that point from the minor axis are in the duplicate ratio of the distances of the point and the directrix from the same axis.

8. An hyperbola touches an ellipse, having a pair of conjugate diameters of the ellipse for its asymptotes. Prove that the curves have the same curvature at the point of contact.

9. Shew that, if $D$ be the diameter of an ellipse parallel to the tangent at a point $P$, whose eccentric angle is $\phi$, the length of the chord common to the ellipse and circle of curvature at $P$ will be $D \sin 2\phi$

10. Determine a parabola of curvature in magnitude and position for any point in a circle, when the subtenses are inclined at 45° to the tangent.

11. If $x$, $y$ be the coordinates of a point $P$ of a curve $OP$, passing through the origin $O$, the diameter of curvature at $O$ will be $\dfrac{x^2+y^2}{x\sin\alpha \sim y\cos\alpha}$ ultimately, $\alpha$ being the inclination of the tangent at $O$ to the line of abscissæ. Hence shew that, if the equation of a curve, referred to rectangular areas, be $y^2+2ay-2ax=0$, the radius of curvature at the origin will be $2\sqrt{2}.a$.

12. A circle is a circle of curvature, at a fixed point in the circumference, to an ellipse, one focus of which lies on the circle, shew that the locus of the other focus is also a circle.

13. Prove that the chord of curvature at any point $P$ of an ellipse in any direction $PQ$ is half the harmonic mean between the two tangents drawn from $P$ to the confocal conic that touches $PQ$, the tangents being reckoned positive when drawn towards the interior of the ellipse.

## XIV.

1. If $AEB$ be the chord, $AD$ the tangent, and $BD$ the subtense, for an arc $ACB$ of finite curvature at $A$, find the limit of the ratio area $ACBE$ : area $ACBD$, as $B$ approaches $A$.

2. An arc of continuous curvature $PQR$ is bisected in $Q$, $PT$ is the tangent at $P$; prove that, ultimately, as $R$ approaches $P$, the angle $RPT$ is bisected by $PQ$.

3. If $AB$ be an arc of finite curvature bisected in $C$, and $T$ be a point in the tangent at $A$, at a finite distance from $A$, prove that the angle $BTC$ will be ultimately three times the angle $CTA$, when $B$ moves up to $A$.

4. Two curves touch one another, and both are on the same side of the common tangent. If in the plane of the curves this tangent revolve about the point of contact, or if it move parallel to itself, the prime ratio of the nascent chords in the former case will be the duplicate of their prime ratio in the latter case.

5. $ACB$ is a small arc of finite curvature; prove that the mean of the distances of every point of the arc from the chord $AB$ is equal to $\frac{2}{3}$ of the distance of the middle point of the arc from the chord, and that the mean of the distances of every point of the arc from the tangent at either extremity of the arc is equal to $\frac{1}{3}$ of the distance of the middle point of the arc from the same tangent.

6. When on an arc of continuous curvature there is no point where the curvature is a maximum or minimum, the circles of curvature at the extremities of the arc cannot intersect.

7. If $S$ be any point in the plane of a curve, $P$ any point on the curve, $Y$ the corresponding point on the pedal for which $S$ is the pole, $V$ the point where $PS$ cuts the circle of curvature at $P$, $V'$ the corresponding point for the pedal, then $4SP.SV' = PV.YV'$.

8. The radius of curvature in a curve increases uniformly with its inclination to a fixed radius. Prove that the area between the curve, its evolute, and the two radii of curvature of lengths $a$, $b$, which contain an angle $\phi$, is $\frac{1}{6}(a^2 + ab + b^2)\phi$.

9. A curve is such that the radius vector makes half the angle with the normal that it does with a fixed line; find the chord of curvature through the pole.

10. In a segment of an arc of finite curvature a pentagon is inscribed, one side of which is the chord of the arc, and the remaining sides are equal. Shew that the limiting ratio of the areas of the pentagon and segment, when the chord moves up towards the tangent at one extremity, is 15 : 16.

11. $APQ$ is a curve of continued and finite curvature, $P$ and $Q$ are two points in it, whose abscissæ along the normal at $A$ are always in the ratio $m : 1$, and from $B$, $C$, two points in the normal, straight lines $BPb$, $CPc$, $BQb'$, $CQc'$ are drawn to meet the tangent at $A$. Shew that, when $P$ and $Q$ move up to $A$, the areas or the triangles $bPc$, $b'Qc'$ are ultimately in the ratio $m^{\frac{3}{2}} : 1$.

12. $AB$ is an arc of finite curvature at $A$, and a point $P$ is taken such that $AP : PB$ is in the constant ratio of $m : n$. Tangents at $A$ and $B$ intersect the tangent at $P$ in $T$ and $R$, and $AB$ is joined. Prove that the ultimate ratio of the area $ATRB$ to the segment $APB$, as $B$ moves up to $A$, is $3(m^2 + mn + n^2) : 2(m+n)^2$.

13. The tangent to a curve at a point $B$ meets the normal at a point $A$ in $T$, $C$ is the centre of curvature at $A$, and $O$ a point on $AC$; prove that, in the limit, when $B$ moves up to $A$, the difference of $OA$ and $OB$ bears to $AT$ the ratio $OC : OA$.

14. $O$ is a point within a closed oval curve, $P$ any point on the curve, $QPQ'$ a straight line drawn in a given direction, such that $QP = PQ' = PO$; prove that, as $P$ moves round the curve, $Q$, $Q'$ trace out two closed loops, the sum of whose areas is twice the area of the original curve.

## NOTE ON MAXIMA AND MINIMA.

93. When a variable magnitude changes its value in consequence of the change of some element of its construction, the law of its variation can be graphically represented by the form of a curve in which the ordinate and abscissa of every point represent respectively the corresponding values of the variable magnitude and of the element on which it depends.

Examples of this mode of representation have been given in Arts. 55 and 57, in which the time or space is the element upon which depends the velocity or kinetic energy, which are the variable magnitudes respectively considered.

94. This graphic representation enables us to obtain a property of any maximum or minimum value of a variable magnitude which is applicable to the solution of a variety of problems.

For, let $Ox$ be the line of abscissæ and $B$ a point in the auxiliary curve at which the tangent $RBS$ to the curve is parallel to $Ox$, and let the abscissa $OA$ represent the corresponding value of the element, then the ordinate $AB$ is a maximum or minimum according as the portion of the curve $PBQ$ in the neighbourhood of $B$ is concave or convex to the line $Ox$.

Let a chord $PQ$ be drawn parallel to the tangent $RBS$, the two points $P$ and $Q$ one on each side of $B$ have equal ordinates $MP$, $NQ$, which, as $PQ$ moves up to and continues parallel to the tangent, become nearer and nearer and are ultimately equal to the maximum or minimum value, while the difference between the corresponding abscissæ ultimately vanishes.

Hence is derived the following theorem:

*If a variable magnitude have a maximum or minimum value there will be two values of the element of construction, one greater and the other less than the critical value, which will give equal values of the variable magnitude.*

95. *Stationary value of a magnitude.*

Let the equal ordinates $MP$, $NQ$ be produced to meet the tangent in $R$ and $S$, then by Lemma XI., $PR$ and $QS$ vanish compared with $AM$ or $AN$, and the ratio of the rates of increase of the ordinate to that of the abscissa, which is generally finite, vanishes for the critical case of a maximum or minimum; on this account the magnitude is said to have a *stationary* value.

One or two examples are sufficient to shew the application of this method.

96. *To find at what point on the bank of an oval pond a person must land in order to pass from a given point on the pond to a given point on the bank in the shortest possible time, having given the ratio of his rates by land and by water.*

Let $A$, $B$ be the two given points, $P$ the point at which he must land, and let $nv$, $v$ be the velocities by water and along the bank. On opposite sides of $P$ there are two points $Q$, $R$ at which if he land the time to $B$ will be the same, in $AR$ take $AM = AQ$, then $MR$ in water and $QR$ on land are described in the same time, therefore $n.QR = MR$, which is true, however near $Q$ and $R$ may be to $P$; therefore $\cos\phi = n$, where $\phi$ is the angle between $AP$ and the tangent at $P$; whence, when the exact form of the oval is given, the position of $P$ can be found.

97. *To find the chord of an oval, which, drawn through a given point, cuts off a maximum or minimum segment.*

Through the fixed point $A$ it is possible to draw two chords $PAQ$ and $pAq$, one on each side of the required chord, for which the areas cut off are exactly equal; take away the common part, and the remainders $PAp$, $QAq$ are equal; therefore, ultimately, when the angle between them vanishes, $PA.pA = QA.qA$, and the chord which cuts off a maximum or minimum area must be bisected by the fixed point.

98. *If a triangle of constant shape be described about a given triangle, prove that when the area is a maximum the normals to*

*the sides of the circumscribed triangle at the angular points of the given triangle will meet in a point.*

Let $ABC$ be the given triangle, $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ two positions of the circumscribing triangle whose areas are equal, the triangle of maximum area being intermediate in position.

Since the angles at $\alpha$, $\alpha'$ are equal, the points $\alpha$, $\alpha'$ lie in the same segment of a circle whose base in $BC$, and the angles $\alpha C\alpha'$, $\alpha B\alpha'$ are equal. Hence the triangles $\alpha C\alpha'$, $\beta C\beta'$, $\beta A\beta'$, $\gamma A\gamma'$, &c., are ultimately proportional to $C\alpha^2$, $C\beta^2$, ....

But the sum of the areas $\alpha C\alpha'$, $\beta A\beta'$, $\gamma B\gamma'$ are ultimately equal to the sum of $\beta C\beta'$, $\gamma A\gamma'$, $\alpha B\alpha'$,

$$\therefore\ \alpha C^2 - \beta C^2 + \beta A^2 - \gamma A^2 + \gamma B^2 - \alpha B^2 = 0.$$

Let the normals at $A$ and $C$ meet in $N$;

$$\therefore\ \alpha C^2 - \beta C^2 = \alpha N^2 - \beta N^2,$$

$$\beta A^2 - \gamma A^2 = \beta N^2 - \gamma N^2;$$

$$\therefore\ \alpha B^2 - \gamma\beta^2 = \alpha N^2 - \gamma N^2 = \alpha D^2 - \gamma D^2,$$

if $ND$ be perpendicular to $\alpha\gamma$;

$$\therefore\ \alpha B - \gamma B = \alpha D - \gamma D;\ \therefore\ BD = 0,$$

which proves the proposition.

## XV.

1. In an arc $AB$ of continuous curvature $n$ points $P_1$, $P_2$, .. are taken so that the polygon $AP_1P_2..B$ has a maximum area; prove that, when the arc $AB$ is indefinitely diminished, the arcs $AP_1$, $P_1P_2$, .. are all equal.

2. Find the greatest rectangle which can be inscribed in a triangle, one side of which is on a side of the triangle.

3. Prove that the diagonals of the greatest rectangle which can be inscribed in an ellipse, having its sides parallel to the axes, are the equi-conjugate diameters.

4. Prove that the parallelograms of smallest area which can be described about a given ellipse are those which have their sides parallel to conjugate diameters.

5. A point $O$ is taken on the major axis $AA'$ of an ellipse produced, and a line is drawn through $O$ cutting the ellipse in the points $P$ and $P'$. Prove that when the area of the quadrilateral $APP'A'$ is a maximum the projection of $PP'$ upon $AA'$ is equal to the semi-axis-major.

6. Prove that the quadrilateral of maximum area that can be formed with four straight lines $AB$, $BC$, $CD$, $DA$, of given lengths is such that a circle can be described about it. Hence prove that the curve of given length which on a given chord encloses a maximum area is an arc of a circle.

7. From a point $T$ on the exterior of two oval curves tangents $TP$, $TQ$ are drawn to the inner; shew that, when the arc $PQ$ is a minimum or maximum, the radii of curvature at $P$ and $Q$ are in the ratio $TP \sec\alpha : TQ \sec\beta$, where $\alpha$, $\beta$ are the angles which $TP$, $TQ$ respectively make with the normal at $T$.

8. Find the ultimate intersection of the chords common to an ellipse and two consecutive circles of curvature, and shew that when the common chord attains its maximum length for a given ellipse, it cuts the ellipse at angles whose tangents are as $1 : 3$.

9. A triangle inscribed in a closed oval curve moves so that two of its sides cut off constant areas. Prove that when the area cut off by the third side is stationary the three lines formed by joining each angular point of the triangle to the intersection of tangents at the other two points are concurrent.

10. Any two normal chords of an ellipse at right angles to each other cut off equal areas from the curve. Hence find the position of the normal chord which cuts off the minimum area.

11. An endless string just reaches round the circumference of an oval, and when it is cut at any point it is unwrapped until it becomes a tangent at the point of section; shew that the involute so formed will have a maximum or minimum length if the point of section be chosen so that the length of the oval shall be equal to the circumference of the circle of curvature at that point.

## DIGRESSION

## ON THE PROPERTIES OF CERTAIN CURVES.

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### THE CYCLOID.

99. Def. If, in one plane, a circle roll along a straight line, any point on its circumference will describe a curve called a *Cycloid*.

Let $C$, $D$ be the points where the tracing point $P$ meets the straight line, on which it rolls; $A$ the point where it is furthest from $CD$, $AB$ being the corresponding diameter of the circle.

The rolling circle is called the *generating circle*, $AB$ is called the *axis*, $A$ the *vertex*, $CD$ the *base*, and $C$, $D$ the *cusps*.

100. Let $RPS$ be the generating circle in any position, then, since the points of the base and circle come successively in contact without slipping, $CS = \text{arc } PS$, $CB$ and $BD$ are each half of the circumference of the circle, and $BS = \text{arc } RP$.

101. *To draw a tangent to a cycloid.*

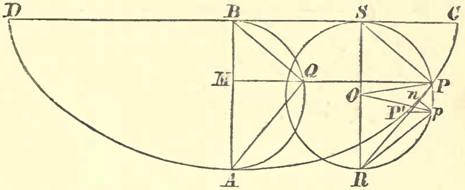
Let the generating circle be in the position $RPS$, then, considering a circle as the limit of a regular polygon of a large number of sides, it will roll by turning about the point of contact, which will be at rest for an instant, being an angular point of the polygon; therefore for an instant $P$ will move perpendicular to $SP$, or in the direction $PR$ of the supplemental chord, which will therefore be the tangent to the cycloid at $P$.

If $AQB$ be the circle on $AB$ as diameter, $PQM$ an ordinate perpendicular to $AB$, the tangent at $P$ will be parallel to the chord $QA$.

102. *To find the length of the arc of a cycloid.*

Let $RPS$ be the position of the generating circle corresponding to the point $P$ in the cycloid, $PR$ being the tangent at $P$.

When the circle has turned through any angle $POp$ the centre $O$ will have moved through a distance equal to $Pp$, and the motion of the generating point will be the resultant




of $Pp$ due to the rotation, and $pP' = Pp$ parallel to the base due to the translation of the centre of the circle; and $PP'$ will ultimately coincide with $PR$. Draw $pn$ perpendicular to $PR$, then, since $Pp = P'p$, $PP' = 2Pn = 2\,(RP - Rp)$ ultimately. Hence the arc of the cycloid measured from the vertex increases twice as fast as the chord of the generating circle, which is a tangent to the cycloid, and they vanish simultaneously, therefore the arc of the cycloid is double of the chord of the generating circle, or referring to the circle on the axis $AB$ as diameter, the arc $AP$ is double of the corresponding chord $AQ$.

103. *To find the relation between the arc and abscissa.*

Let $AM$ be the abscissa of the point $P$, then

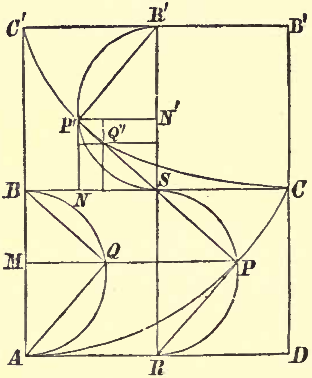
$$AM : AQ :: AQ : AB;$$

$$\therefore\ AP^2 = 4AQ^2 = 4AB \, . \, AM.$$

104. *To shew that the evolute of a given cycloid is an equal cycloid, and that the radius of curvature of a cycloid is twice the normal.*

Let $APC$ be half the given cycloid, $AB$ the axis, $A$ the vertex, and $BC$ the base. Produce $AB$ to $C'$, making $BC'$ equal to $AB$, and complete the rectangle $BCB'C'$, and let the semicycloid $C'P'C$ be generated by a circle, whose diameter is equal to that of the generating circle of the given cycloid, rolling on $C'B'$; $C$ being the vertex, $CB'$ the axis of this cycloid.

Let $SPR$, $SP'R'$ be two positions of the respective generating circles, having their diameters $RS$, $SR'$ in the same straight line, $P$, $P'$ being the corresponding points of the cycloids; join $SP$, $PR$ and $SP'$, $P'R'$.




By the mode of generation, arc $SP = SC$, and arc $SPR = BC$;

$$\therefore \text{ arc} PR = BS = C'R' = \text{arc} P'R';$$

$$\therefore \angle PSR = \angle P'SR'; \text{ and } PSP' \text{ is a straight line.}$$

Also, $\text{arc} P'S = \text{arc} PS$; $\therefore$ chd. $P'S =$ chd. $PS$;

$$\therefore P'SP = 2P'S = P'C \text{ the cycloidal arc;}$$

also $P'SP$ touches the cycloid $C'P'C$ at $P'$;

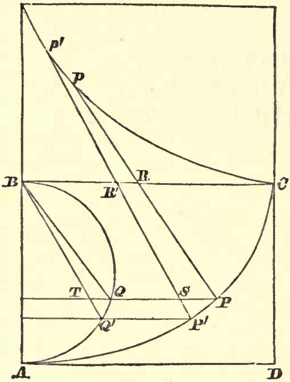
therefore, a string fixed at $C'$, and wrapped over the arc of the semicycloid, will, when unwrapped, have its extremity in the arc of the given cycloid; hence, the evolute of a semicycloid is an equal semicycloid, and the radius of curvature at $P$ is $2PS$ or twice the normal. If another equal semicycloid be described by the circle rolling on $B'C'$ produced, the extremity of the string wrapped on this curve will trace out the remainder of the given cycloid.

Thus a pendulum may be made to oscillate in a given cycloid.

105. *To find the area of the cycloid.*

Let $P$, $P'$ be two points very near each other in a cycloid,

$Q$, $Q'$ corresponding points in the generating circle, $p$, $p'$ in the evolute, $R$, $R'$ the intersections of the base with normals $Pp$,




$P'p'$, $T$, $S$ the intersections of $BQ'$ and $P'p'$ with $PQ$. Then $pR = PR = BQ$, and $\Delta p'PS = 4\Delta p'RR'$ ultimately $= 4\Delta\, BQT$; therefore trapezium $PRR'S = 3\Delta BQT$ ultimately, and the same being true for all the inscribed triangles and trapeziums, whose sums are ultimately the areas of the semicircle and semicycloid, therefore, by Cor., Lemma IV., the area of the cycloid is three times that of the generating circle.

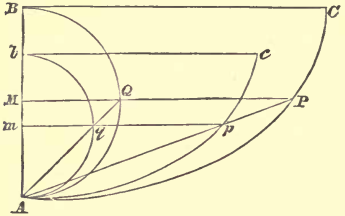
106. The following method of finding the area of a cycloid is independent of the properties of the evolute.

In the figure of Art. 104 let $P'$ be any point in the cycloid $CP'C'$, $P'S$ the chord of the generating circle which touches the cycloid, and let $Q'$ be a point in the cycloid near $P'$, then the arc $P'Q'$ ultimately coincides with $P'S$. Let $Q'N'$, $Q'N$ be the complements of the parallelogram whose diagonal is $P'S$, and sides parallel and perpendicular to the base, these are equal ultimately; therefore, by Lemma IV., the cycloidal area $CNP' =$ circular segment $SP'N'$.

The exterior portion $CB'C'$ is equal to the area of the semicircle, and the whole parallelogram $BCB'C'$ is the rectangle under the diameter and semi-circumference of the generating circle, and is equal to four times the area of the semicircle; therefore the cycloidal area $CC'B'$ is three times the area of the semicircle.

107. *All cycloids are similar.*

Let two cycloids $APC$, $Apc$ be placed so that their vertices are the same, and their axes coincident in direction, and describe




circles on the axes $AB$, $Ab$ as diameters. Draw $AqQ$ cutting the circles in $q$, $Q$. Then, since the segments $Aq$, $AQ$ are similar, arc $Aq$ : arc $AQ$ :: $Aq : AQ$; and, if $mqp$, $MQP$ be ordinates to the cycloids, arcs $Aq$, $AQ = qp$, $QP$ respectively; therefore $qp : QP :: Aq : AQ$, and $ApP$ is a straight line. Also $Ap : AP :: Aq : AQ :: Ab : AB$, a constant ratio; hence the cycloids satisfy the condition of similarity, and in this position of the cycloids the point $A$ is a centre of direct similitude.

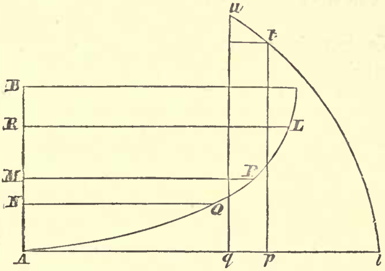
108. *To construct a cycloid which shall have its vertex at a given point, its base parallel to a given straight line, and which shall pass through a given point.*

Let $A$ be the given vertex, $AB$ perpendicular to the given line, $P$ the given point. In $AB$ take any point $b$, and with the generating circle, whose diameter is $Ab$, describe a cycloid $Apc$, join $AP$ intersecting this cycloid in $p$.

Take $AB$ a fourth proportional to $Ap$, $AP$, and $Ab$; then $AB$ will be the diameter of the generating circle of the required cycloid; for, since $Ap : AP :: Ab : AB$, and all cycloids are similar, $P$ is a point in the cycloid whose axis is $AB$.

109. *A particle slides down the smooth arc of a cycloid, whose axis is vertical, and vertex downwards, to find the time of an oscillation.*

Let $AB$ be the vertical axis of the cycloidal arc $APL$, $L$ the point from which the particle begins to move, $PQ$ a small arc of




its path, $LR$, $PM$, $QN$ perpendicular to $AB$; and take $Al$, $Ap$, $Aq$ on the tangent at $A$ respectively equal to $AL$, $AP$, $AQ$.

Suppose a point to move from $l$ to $A$ in the same time as the particle moves on the cycloid from $L$ to $A$, their velocities being always equal at equal distances from $A$.

Let $v$ be the velocity at $P$ or $p$, and $T$ the time of falling from $B$ to $A$, so that $v^2 = 2gRM$ and $2AB = gT^2$; therefore $v^2T^2 = 4AB.RM = 4AB.AR - 4AB.AM = AL^2 - AP^2$, Art. 103, $= Al^2 - Ap^2$.

Describe a circle with centre $A$ and radius $Al$, and draw the ordinates $pt$, $qu$, then $Al^2 - Ap^2 = pt^2$, and $pt = vT$; and if $\tau$ be the time from $P$ to $Q$, $PQ = pq = v\tau$ ultimately, hence

$$tu : Al :: pq : pt :: \tau : T;$$

therefore, if a point move in the circle from $l$ with uniform velocity $\frac{Al}{T}$, the point moving in $lA$ will always be in the foot of the ordinate and the motion in $lA$ or $LA$ will therefore be a simple harmonic motion, by (5) page 78.

The time from $L$ to $A$ is the time of describing the quadrant $\frac{1}{2}\pi Al$ with velocity $\frac{Al}{T}$, $= \frac{1}{2}\pi T = \frac{1}{2}\pi\sqrt{\frac{2AB}{g}}$.

The length of the string which, by the contrivance of Art. 104, makes a particle oscillate in this cycloid is $2AB = l$ suppose; therefore the time of the oscillation of a cycloidal pendulum of length $l$ from rest to rest $= \pi\sqrt{\frac{l}{g}}$.

NOTE. The time from $L$ to $P$ is $\sqrt{\frac{2AB}{g}} \times \cos^{-1}\frac{AP}{AL}$.

110. We can shew that the motion on the cycloid is a simple harmonic motion by the first definition, (5) page 78; for, referring to the figure, page 115, since the tangent at $P$ is parallel to $AQ$, the acceleration along the curve at $P$ is $g \cdot \frac{AQ}{AB} = g \cdot \frac{AP}{2AB}$, which varies as $AP$, and, by (4) page 77, the time from $L$ to $A$ is obtained.

111. *To find the time of a very small oscillation of a simple pendulum suspended from a point.*

A *simple pendulum* is an imaginary pendulum consisting of a heavy particle called the *bob*, suspended from a *point* by means of a rod or string without weight.

In this case the pendulum describes the small arc of a circle which may be considered the same as a cycloidal arc, the axis of which is half the length $l$ of the pendulum, therefore the time of oscillation from rest to rest is $\pi\sqrt{\frac{l}{g}}$.

112. *To count the number of oscillations made by a given pendulum in any long time.*

In consequence of the liability to error in counting a very great number of oscillations, since in the case of a seconds pendulum there would be 3600 oscillations for each hour, it becomes necessary to adopt some contrivance for diminishing the labour. For this purpose the pendulum is made to oscillate nearly in the same time as that of a clock; it is then placed in front of that of the clock, so that when they are simultaneously near their lowest positions the bob of the pendulum and a cross marked on the pendulum of the clock may be in the field of view of a fixed telescope.

Suppose that after $n$ oscillations of the given pendulum they are again in coincidence close to the same position; if there be $m$ such coincidences in the whole time of observation, the number of oscillations in that time will be $mn$; thus the only labour has been to count the $n$ oscillations, and to estimate the number of the coincidences before the last one observed.

113. *To measure the accelerating effect of gravity by means of a pendulum.*

Let $g$ be the measure of this effect or the velocity generated by the force of gravity in a second.

Let $l$ be the length of a simple pendulum which makes $n$ oscillations in $m$ hours, then $\frac{3600m}{n}$ = number of seconds in one oscillation $= \pi\sqrt{\frac{l}{g}}$; therefore $g = \frac{\pi^2 l n^2}{(3600)^2 m^2}$, in whatever unit of length $l$ is estimated.

This would be a very exact method of determining $g$, if we could form a simple pendulum; but it is impossible to do this, and it is only by calculations of a nature too difficult to be explained here that it can be shewn how to deduce the length of the simple pendulum, which would oscillate in the same time as a pendulum of a more complicated structure.

114. The seconds pendulum at any place is the simple pendulum which at the mean level of the sea at that place would oscillate in one second.

If $L$ be the length of the seconds pendulum, $l$ the length of a pendulum making $n$ oscillations in $m$ hours,

$$\therefore \pi\sqrt{\frac{l}{g}} = \frac{3600m}{n}, \text{ and } \pi\sqrt{\frac{L}{g}} = 1, \therefore L = \frac{n^2 l}{(60)^4 m^2}.$$

115. *To determine the height of a mountain by means of a seconds pendulum, the force of gravity at any point being supposed to vary inversely as the square of the distance from the centre of the earth.*

Let $L$ be the length of a seconds pendulum, $x$ the height of the mountain above the mean level of the sea, $a$ the radius of the earth, all expressed in feet; and let $n$ be the number of seconds lost in 24 hours by the pendulum at the top of the mountain.

If $g$ be the measure of the accelerating effect of gravity at the mean level of the sea, then $\frac{ga^2}{(a+x)^2}$ will be its value at the top of the mountain, and the time of oscillation at the top will be $\pi\sqrt{\left\{\frac{L}{g}\left(\frac{a+x}{a}\right)^2\right\}}$, or $\frac{a+x}{a}$ seconds, since $\pi\sqrt{\frac{L}{g}}=1$; hence, writing $N$ for $24\times60\times60$, $(N-n)\frac{a+x}{a}=N$, and $1+\frac{x}{a}=\frac{N}{N-n}=1+\frac{n}{N}+\frac{n^2}{N^2}\dots$; $\therefore\ x=\frac{a}{N}n+\frac{a}{N^2}n^2$, nearly, but $a=4000\times1760\times3$ and $N=24\times60\times60$, therefore the height of the mountain will be $244{\cdot}4n+{\cdot}0027n^2$; thus, if $n=10$, the height will be $2444{\cdot}7$ feet.

NOTE. The attraction of the mountain would make a sensible variation from the law of the inverse square, this law being true only if the earth consisted of homogeneous spherical strata.

116. *To find the number of seconds lost in a day, in consequence of a slight error in the length of the seconds pendulum; and conversely.*

Let $N$ be the number of seconds in a day, $L$ the length of the seconds pendulum, $L+\lambda$ that of the incorrect pendulum, $N-n$ the number of its oscillations in a day;

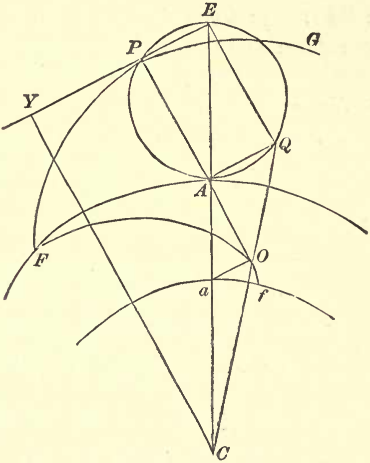
$$\therefore\ (N-n)\,\pi\sqrt{\frac{L+\lambda}{g}}=N.\pi\sqrt{\frac{L}{g}};\ \therefore\ \frac{\lambda}{L}=\frac{2n}{N}\text{ nearly.}$$

## THE EPICYCLOID AND HYPOCYCLOID.

117. DEF. The curve traced out by a point on the circumference of a circle, which rolls upon that of a fixed circle, is called an *Epicycloid* if the concavities of the two circles be in opposite directions, a *Hypocycloid* if the concavities be in the same direction.

118. *To shew that the evolute of an epicycloid is a similar epicycloid.*

Let $FA$ be the fixed circle, $APE$ the rolling circle in any position, $P$ the generating point, $CAE$ a line drawn through




the point of contact, meeting the rolling circle in $A$, $E$; and let $GPF$ be the epicycloid, of which $PA$ and $PE$ will be a normal and tangent.

Draw the chord $EQ$ parallel to $PA$ and join $CQ$ meeting $PA$ produced in $O$. Since $EQ$ is parallel to $AO$,

$$CO : CQ :: CA : CE;$$

therefore $O$ and $Q$ describe similar figures. But $Q$, being the other extremity of the diameter through $P$, will describe an epicycloid similar and equal to $GPF$, being at its cusp when $P$ is at $G$ the greatest distance from $C$.

Draw $Oa$ parallel to $QA$ and therefore perpendicular to $PO$, meeting $CA$ in $a$, then $O$ generates an epicycloid $fF$ by the rolling of a circle $AOa$, whose diameter is $Aa$, on a fixed circle of radius $Ca$.

Also $PO$ the normal to $GF$ is perpendicular to $aO$ and is therefore a tangent to $fF$, hence $fF$ is the evolute of the given epicycloid and is a similar epicycloid.

Let $a$, $b$ be the radii of the fixed and rolling circles for the given epicycloid, then

$$Aa : CA :: OQ : CQ :: AE : CE :: 2b : a+2b;$$

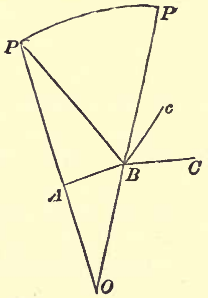
therefore $Aa : AE :: a : a+2b$, and if $a=\infty$, $Aa=AE$, and $AF$, $af$ become straight lines, whence the evolute of a cycloid is an equal cycloid.

119. Since $AO : PA :: AO : EQ :: CA : CE$, therefore $PO : PA :: 2(a+b) : a+2b$, which gives $PO$ the radius of curvature at $P$ of the given epicycloid; this will be found independently of the evolute in Art. 121 below.

120. *To find the length of any arc of the epicycloid.*

By the properties of the evolute, see the last figure, the arc $OF$ of the evolute $=OP=2AP.\dfrac{a+b}{a+2b}$, and the arc of the epicycloid generated by $Q$, measured from $Q$ to the highest point, $=OF\dfrac{a+2b}{a}=2AP.\dfrac{a+b}{a}$; therefore the arc $GP$ from the highest point $G$ of the epicycloid $GPF=2EP.\dfrac{a+b}{a}$.

121. *To find the radius of curvature at any point of an epicycloid.*



Let $AB$, $BC$ be consecutive sides of a fixed regular polygon of $m$ sides, $AB$, $Bc$ sides of another regular polygon of $n$ sides equal to those of the former, on the outside of which it rolls, in a position in which two sides are coincident.

Let $P$ be any angular point of the rolling polygon; $P$ will generate a figure composed of a series of circular arcs such as $PP'$, $P'$ being the position of $P$ when $Bc$ coincides with $BC$. Produce $PA$, $P'B$ to meet in $O$.

Then $\angle APB = \dfrac{\pi}{n}$, and $\angle PBP' = \angle cBC = \dfrac{2\pi}{m} + \dfrac{2\pi}{n}$;

$$\therefore \angle POB = 2\pi\left(\frac{1}{m} + \frac{1}{n}\right) - \frac{\pi}{n} = \pi\left(\frac{2}{m} + \frac{1}{n}\right);$$

$$\therefore PO : PB :: \sin 2\pi\left(\frac{1}{m} + \frac{1}{n}\right) : \sin \pi\left(\frac{2}{m} + \frac{1}{n}\right).$$

When the number of sides is indefinitely increased, the polygons ultimately become circles, the curve traced out by $P$ becomes an epicycloid, and $PO$ the radius of curvature at $P$.

If $a$, $b$ be the radii of the fixed and rolling circles $m.AB = 2\pi a$ and $n.AB = 2\pi b$, ultimately; therefore $m : n :: a : b$;

$$\therefore PO : PA :: 2\left(\frac{1}{m} + \frac{1}{n}\right) : \frac{2}{m} + \frac{1}{n} :: 2(a+b) : a + 2b;$$

therefore the radius of curvature is $2PA \, . \, \dfrac{a+b}{a+2b}$, where $PA$ is the part of the normal intercepted between the generating point and the point of contact.

If $a = \infty$, or the fixed circle become a straight line, the epicycloid will become a cycloid, and the radius of curvature will be twice the normal, as in Art. 104.

122. *To find the area of an epicycloid.*

In the last figure, area $APP'B = \triangle PAB +$ sector $PBP'$; now

sector $PBP' = \frac{1}{2}PB^2 . 2\pi\left(\dfrac{1}{m} + \dfrac{1}{n}\right)$ and $\triangle PAB = \frac{1}{2}PB^2 \sin\dfrac{\pi}{n}$;

$$\therefore \text{area } APP'B = \triangle PAB\left\{1 + \frac{2(m+n)}{m}\right\} \text{ ultimately;}$$

hence, by Lemma IV. Cor., the area of the segment of the epicycloid included between two normals and the fixed circle is $\left(3+\frac{2b}{a}\right)\times$ the corresponding segment of the rolling circle. Compare Art. 105.

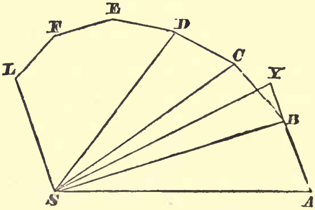
123. The corresponding properties of the hypocycloid may be proved in a similar manner; and the results obtained will be the same as for the epicycloid, if in the latter the sign of $b$ be changed.

Thus, if the diameter of the fixed be double that of the rolling circle, the hypocycloid will become a straight line, which agrees with the result of Art. 121, since $a+2b=0$, and therefore the radius of curvature at every point will be infinite.

## THE EQUIANGULAR SPIRAL.

124. Def. The *equiangular spiral* is a curve which cuts all the radii drawn from a fixed point at a constant angle.

125. If a series of radii $SA$, $SB$, $SC$, ... be drawn inclined at equal angles, and $AB$, $BC$, $CD$, ... make equal angles $SAB$, $SBC$, ... with these radii respectively, the curvilinear limit




of the polygon $ABCD$ ..., when the equal angles $ASB$, $BSC$, ... are indefinitely diminished, will be an equiangular spiral.

126. *To find the length of an arc of an equiangular spiral contained between two radii.*

Let $\alpha$ be the constant angle $SAB$, and let $SL$ be the $n^{\text{th}}$

radius from $SA$; then, since the triangles $ASB$, $BSC$, ... are similar, $SA : SB :: SB : SC$ ....

Let $SB=\lambda . SA$, then $BC=\lambda . AB$, $CD=\lambda^2 . AB ... FL=\lambda^{n-1} . AB$;

$$\therefore\ AB+BC+...+FL : AB :: 1+\lambda+...+\lambda^{n-1} : 1 :: 1-\lambda^n : 1-\lambda$$
$$:: SA-\lambda^n . SA : SA-SB :: SA-SL : SA-SB,$$

but $AB \cos\alpha = SA - SB \cos ASB = SA - SB$ ultimately, and $AB+BC+...$ is ultimately the arc of the spiral; therefore arc $AL = (SA - SL) \sec\alpha$.

127. *To find the area of an equiangular spiral bounded by two radii.*

Employing the same construction as above,

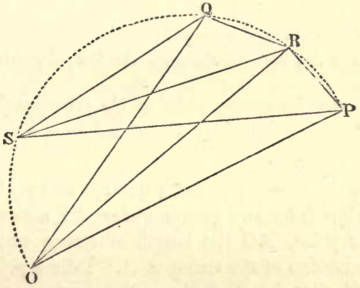
$$\triangle ASB + \triangle BSC + \triangle CSD + ... : \triangle ASB :: 1+\lambda^2 ...+\lambda^{2n-2} : 1$$
$$:: 1-\lambda^{2n} : 1-\lambda^2 :: SA^2 - SL^2 : SA^2 - SB^2,$$

but $SB^2 = SA^2 - 2SA . AB \cos\alpha + AB^2$ and $\triangle ASB = \frac{1}{2} SA . AB \sin\alpha$;

$$\therefore\ SA^2 - SB^2 = 4 \triangle ASB \times \cot\alpha, \text{ ultimately};$$
$$\therefore \text{ area } ASL = \tfrac{1}{4}(SA^2 - SL^2)\tan\alpha.$$

128. *To find the radius and chord of curvature through the pole at any point of an equiangular spiral.*

Let $SP$, $SQ$ be radii drawn to two points $P$ and $Q$, near to




one another, let $PR$, $QR$, tangents to the spiral at $P$ and $Q$, intersect in $R$, and let the normals $PO$, $QO$ intersect in $O$; join $OR$, $SR$.

Then, since angles $SQR$, $SPR$ are equal to two right angles, and each of the angles $OQR$, $OPR$ is a right angle, the circle which passes through $P$, $R$, and $Q$ will also pass through $S$ and $O$, and $OR$ will be its diameter; therefore $\angle OSR$ is a right angle. Hence, proceeding to the limit, $O$ is the centre of the circle of curvature at $P$, and $OSP$ is a right angle. Therefore if $\alpha$ be the angle of the spiral, $OP = SP \operatorname{cosec} \alpha$ will be the radius of curvature, and $2SP$ the chord of curvature through the pole.

129. The following is an illustration of Art. 86.

If $PV$ be the chord of curvature through $S$,

$$SY' - SY : SP' - SP :: 2SY : PV;$$

but in the equiangular spiral $SY : SY' :: SP : SP'$;

$$\therefore\ SY' - SY : SP' - SP :: SY : SP; \text{ whence } PV = 2SP.$$

## THE CATENARY.

130. Def. The *Catenary* is the curve in which a uniform and perfectly flexible string, of which the extremities are suspended at two points, would hang under the action of gravity, supposed to be a constant force acting in parallel lines.

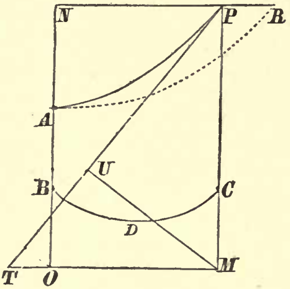
The *directrix* is a horizontal straight line whose depth below the lowest point is equal to the length of string whose weight is equal to the tension at the lowest point.

The *axis* is the vertical through the lowest point.

131. *The tension at any point of the catenary is equal to the weight of the string which, if suspended from that point, would extend to the directrix.*

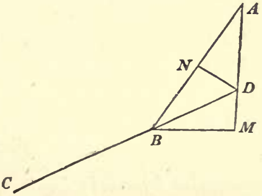
Let $A$ be the lowest point of a uniform and perfectly flexible string hanging from two points under the action of gravity, $P$ any other point, $AO$ the length of string whose weight is equal to the tension of the string at $A$. Take a point $B$ in $OA$, and let $OM$, $BC$ drawn horizontally meet a vertical $PM$ in $M$ and $C$.

If a string pass round smooth pegs at $APCB$, it is evident that there will be a position of equilibrium whatever be the length of the string, or the position of $BC$, and for some length and some position of $BC$ the tangent at $A$ will be horizontal.




Also, since $BDC$ will hang symmetrically, the tensions of the string at $B$ and $C$ will be equal, and $BDC$ may be removed and replaced by equal lengths $BO$, $CM$ of the string, without disturbing the equilibrium of $AP$, therefore the tension of the catenary at $P$ is equal to the weight of a string of length $PM$.

132. The proposition of the preceding article may be proved by considering the catenary as the limit of the polygon formed by a series of equal rods of the same substance jointed freely at the extremities and suspended from two fixed points, when the length of the rods is indefinitely diminished.




The equilibrium will be undisturbed if each rod be replaced by two weights at the extremities, each equal to half that of the rod, connected by a rod without weight.

Let $AB$, $BC$ be two consecutive positions of the rods, weights equal to those of the rods being placed at $A$, $B$, $C$; let $AM$ be vertical and $BM$ horizontal, and produce $CB$ to meet $AM$ in $D$; draw $DN$ perpendicular to $AB$.

The forces which keep $B$ in equilibrium act in the directions of the sides of the triangle $ABD$, and are proportional to them.

Therefore the difference of the tensions of $AB$ and $BC$ is to the weight of the rod $AB$ as $AB - BD : AD$, that is, ultimately, as $AN : AD$ or $AM : AB$; hence the difference of the tensions is the weight of a rod of length $AM$.

Therefore, proceeding to the limit, the difference of tensions at any two points of the catenary is equal to the weight of string, which is equal in length to the vertical depth of one point below the other, whence the truth of the proposition.

133. *$P$ is a point in a catenary, $PM$ perpendicular to the directrix, $PT$ a tangent at $P$, $MU$ perpendicular to $PT$; to shew that $PU$ is equal to the arc measured from the lowest point, and that $MU$ is constant.*

Let $PT$, fig. for Art. 131, meet the direction $OM$ in $T$, and let $AO$ be the axis, then since the arc $AP$ supposed to become rigid is in equilibrium under the action of the tensions at $A$ and $P$ and the weight, and these forces are in the directions of the sides of the triangle $TPM$,

$$\therefore AP : AO : PM :: PM : MT : TP :: PU : MU : PM,$$

by similar triangles $TPM$, $MPU$;

$$\therefore PU = AP \text{ and } MU = AO.$$

134. *To draw a tangent to a catenary at any point.*

With centre $M$ and radius equal to $AO$ describe a circle, and draw $PU$ touching this circle in $U$; then, since $MU$, which is perpendicular to $PU$, is equal to $AO$, $PU$ will be the tangent at $P$.

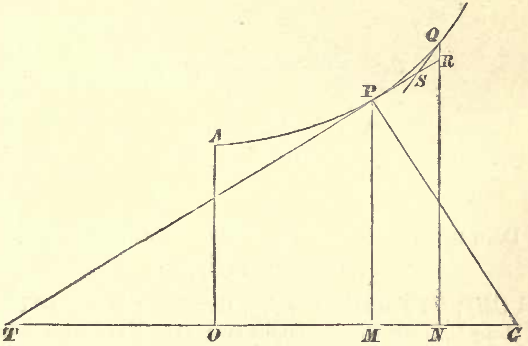
135. *If a rectangular hyperbola be described, having centre $O$ and semi-transverse axis $OA$, the ordinate of the hyperbola will be equal to the arc of the catenary.*

For, let $AR$ be the hyperbola, therefore

$$RN^2 = ON^2 - OA^2 = PM^2 - UM^2 = PU^2; \ \therefore RN = PU = AP.$$

136. *To find the radius and vertical chord of curvature of a catenary.*

Let $PQ$ be a small arc of a catenary, $RSPT$, $QS$ tangents at $P$ and $Q$, $PM$, $QN$ ordinates, $TOM$ the directrix.




Since $QRS$ is a triangle of the forces acting upon $PQ$,

$$\text{tension at } P : \text{weight of } PQ :: RS : QR,$$

$$\therefore PM : PQ :: RS : QR :: \tfrac{1}{2}PQ : QR, \text{ ultimately};$$

therefore $2PM$ is the vertical chord of curvature, and $PG$, the part of the normal intercepted between the point $P$ and the directrix is equal to the radius of curvature at $P$.

Also $PG : PM :: PT : TM ::$ tension at $P$ : tension at $A$ $:: PM : AO$, therefore the radius of curvature is a third proportional to $AO$ and $PM$.

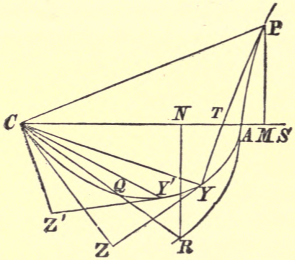
## THE LEMNISCATE.

137. DEF. The *Lemniscate* is the locus of the feet of the perpendiculars drawn from the centre of a rectangular hyperbola upon the tangent.

138. *To find the inclination to the tangent at any point of the radius from the centre of the lemniscate.*

Let $CY$ be perpendicular on $PT$ the tangent at the point $P$

in the hyperbola, then $CY.CP = PF.CD = AC^2$, since $AC = BC$ and $CP = CD$ in the rectangular hyperbola.




Draw the ordinate $PM$, then $CT.CM = AC^2 = CY.CP$;

$$\therefore\ CY : CT :: CM : CP;$$

and $CMP$, $CYT$ are right angles; therefore $\angle PCM = \angle TCY$.

Draw $CZ$ perpendicular on the tangent at $Y$ to the lemniscate; then $ZCY$ and $YCP$ are similar triangles, see page 55;

$$\therefore\ \angle ZYC = \angle CPY = \text{complement of twice } \angle YCA.$$

139. *To find the perpendicular on the tangent at any point of the lemniscate.*

$$CZ.CP = CY^2, \text{ and } CY.CP = AC^2;$$

$$\therefore\ CZ : CY :: CY^2 : AC^2;$$

$$\therefore\ CZ.AC^2 = CY^3.$$

140. *To find the chord of curvature through the centre, and the radius of curvature at any point of the lemniscate.*

Let $YV$ be the chord of curvature;

$$\therefore\ YV : 2CZ :: CY - CY' : CZ - CZ', \text{ ultimately, Art. 86,}$$

$$\text{and } (CZ - CZ')\, AC^2 = CY^3 - CY'^3,$$

$$\therefore\ CY - CY' : CZ - CZ' :: AC^2 : 3CY^2 :: CY : 3CZ;$$

$\therefore\ YV = \frac{2}{3} CY$, or the chord of curvature through the centre is two-thirds of the radius vector.

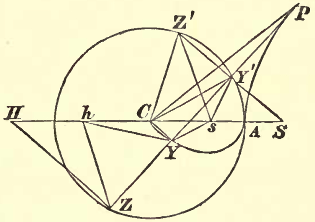
Also, the radius of curvature : $\frac{1}{2}YV$

$$:: CY : CZ :: AC^2 : CY^2 :: CP : CY,$$

hence the radius of curvature is $\frac{1}{3}CP$, or $\frac{1}{3}$ of the radius at the corresponding point of the hyperbola.

141. *Poles of the lemniscate.*

Let $S$, $H$ be the foci of the hyperbola, $s$, $h$ the middle points of $CS$ and $CH$; $s$, $h$ are called the *poles* of the lemniscate.




Draw $SY'$, $HZ$ perpendicular to the tangent to the hyperbola at $P$, and let $SY'$ meet the auxiliary circle again in $Z'$, and join $sY'$, $sZ'$, $sY$, $hY$, and $hZ$.

Since $Cs = sS$, the perpendicular from $s$ on $YY'$ bisects it; therefore $sY' = sY$, similarly $hY = hZ = sZ'$.

$$\text{Now} \quad SC.Ss = \tfrac{1}{2}SC^2 = AC^2 = SY'.SZ';$$

therefore a circle can be drawn circumscribing $CsY'Z'$; therefore $\angle Y'sZ' = \angle Y'CZ'$; also $\triangle Y'sZ' = \frac{1}{2}\triangle Y'CZ'$, since the altitude of $Y'CZ'$ is double of that of $Y'sZ'$;

$$\therefore sY'.sZ' = \tfrac{1}{2}CY'.CZ' = \tfrac{1}{2}CA^2;$$

therefore $sY.hY = \frac{1}{2}CA^2$, which is the property of the poles of the lemniscate.

For this proof I am indebted to Prof. Tait.

## XVI.

1. If a line move parallel to the base of a cycloid, find the limit of the ratio of the segment of the cycloid to the corresponding segment of the generating circle, when the line becomes indefinitely near to the tangent at the vertex.

2. A balloon was found to be sailing steadily before the wind at an invariable elevation above the earth. A seconds pendulum suspended to the car was observed to make 2997 oscillations in 50 minutes; shew that the height of the balloon was 4 miles and 7 yards nearly, the radius of the earth being 4000 miles.

3. If a particle be made to oscillate in a cycloid on a smooth inclined plane, whose inclination to the horizon is 30°, and the base of the cycloid be horizontal, find the radius of the generating circle in order that the particle may perform a complete oscillation in $n$ seconds.

4. If $P$ be a point in a cycloid, and $O$ the corresponding position of the centre of the generating circle, shew that $PO$ will touch another cycloid of half the dimensions.

5. Shew that the limit of the whole length of an epicycloid or hypocycloid, corresponding to a complete revolution of the generating round the fixed circle, is eight times the radius of the latter, when that of the former is indefinitely diminished.

6. Prove that the epicycloid of one cusp is the pedal of a circle referred to a point in its circumference.

7. Shew that the evolute of an equiangular spiral is a similar spiral, and that the extremities of the diameters of curvature lie in a similar spiral.

8. An equiangular spiral rolls along a straight line, shew that its pole describes a straight line.

9. Prove that, if a catenary roll on a fixed straight line, its directrix will always pass through a fixed point.

10. If $SY$ be drawn perpendicular to the tangent to a lemniscate at a point $P$, and $SA$ be the greatest value of $SP$, prove that $SP^3 = SA^2 . SY$; $S$ being the centre.

## XVII.

1. From the consideration that the diameter of curvature is the limit of the third proportional to the subtense perpendicular to the tangent and the arc, prove that the radius of curvature of a cycloid at any point is twice the normal cut off by the base.

2. On the normal to a cycloid a constant length is measured both inwards and outwards; find the area included between the loci of the points so obtained.

3. $P$, $Q$ are consecutive points on an epicycloid of two cusps; from $p$, $q$, the corresponding points of contact of the rolling with the fixed circle, $pm$, $qn$ are drawn perpendicular to the cusp-line; prove that the elementary area $PQpq$ is twice the elementary area $pmnq$. Hence find the area of the epicycloid and of its evolute.

4. Prove that the diameter through the point of a rolling circle which generates an epicycloid always touches another epicycloid generated by a circle of half the dimensions.

5. A hypocycloid of $n$ cusps has at any point a tangent drawn, prove that the length of the tangent, intercepted between the generating circle and the point of contact, is to the arc measured from the point to the vertex of the branch in which the point is taken, as $n : 2(n-1)$.

6. A bead slides on a hypocycloid being acted on by a force which varies as the distance from the centre of the hypocycloid and tending to it; prove that the time of oscillation will be independent of the arc of oscillation.

7. If, along the several normals to an epicycloid, a system of particles move from the curve under the action of a force, tending to the centre of the fixed circle, and varying as the distance, prove that they will all arrive at the fixed circle at the same instant.

8. A plane curve rolls along a straight line, shew that the radius of curvature of the path of any point, fixed with respect to the curve, is $\frac{r^2}{r - \rho \sin\phi}$, $r$ being the distance of the fixed point from the point of contact, $\phi$ the angle between this line and the fixed line, and $\rho$ the radius of curvature of the curve at the point of contact.

9. In an equiangular spiral, which is its own evolute, the area included between the curve and $PQ$, the radius of curvature at $P$ touching the evolute in $Q$, is $\frac{1}{4}PQ^2 \tan\alpha$, where $\alpha$ is the angle of the spiral, and $PQ$ is supposed not to cut the curve between $P$ and $Q$.

10. Prove, by the method of Lemma IV., that the area included between a catenary, the axis, the directrix, and the ordinate at any point $P$ is twice the area of the triangle formed by the axis, the tangent at the vertex, and the straight line drawn perpendicular to the tangent at $P$ from the point of intersection of the axis and directrix.

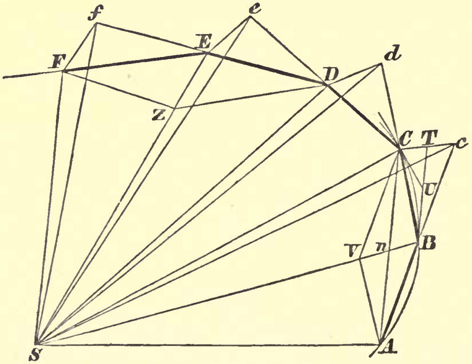
## *SECTION II.*

### CENTRIPETAL FORCES.

### PROP. I. THEOREM I.

*When a body revolves in an orbit, subject to the action of forces tending to a fixed point, the areas which it describes by radii drawn to the fixed centre of force, are in one fixed plane, and are proportional to the times of describing them.*

Let the time be divided into equal parts, and in the first interval let the body describe the straight line




$AB$ with uniform velocity, being acted on by no force. In the second interval it would, if no force acted, proceed to $c$ in $AB$ produced, describing $Bc$ equal to $AB$; so that the equal areas $ASB$, $BSc$ described by radii $AS$, $BS$, $cS$ drawn to the centre $S$, would be completed in equal intervals.

But, when the body arrives at $B$, let a centripetal force tending to $S$ act upon it by a single instantaneous impulse, and cause the body to deviate from the direction $Bc$, and to proceed in the direction $BC$.

Let $cC$ be drawn parallel to $BS$, meeting $BC$ in $C$, then, at the end of the second interval, the body will be found at $C$, in the same plane with the triangle $ASB$, in which $Bc$ and $cC$ are drawn. Join $SC$; and the triangle $SBC$, between parallels $SB$, $Cc$, will be equal to the triangle $SBc$, and therefore also to the triangle $SAB$.

In like manner, if the centripetal force act upon the body successively at $C$, $D$, $E$, &c., causing the body to describe in the successive intervals of time the straight lines $CD$, $DE$, $EF$, &c., these will all lie in the same plane; and the triangle $SCD$ will be equal to the triangle $SBC$, and $SDE$ to $SCD$, and $SEF$ to $SDE$.

Therefore equal areas are described in the same fixed plane in equal intervals; and, componendo, the sums of any number of areas $SADS$, $SAFS$, are to each other as the times of describing them.

Let now the number of these triangles be increased, and their breadth diminished indefinitely; then their perimeter $ADF$ will be ultimately a curved line; and the instantaneous forces will become ultimately a centripetal force, by the action of which the body is continually deflected from the tangent to this curve, and which will act continuously; and the areas $SADS$, $SAFS$, being always proportional to the times of describing them, will be so in this case. Q.E.D.

COR. 1. The velocity of a body attracted towards a fixed centre in a non-resisting medium is reciprocally proportional to the perpendicular dropped from that centre upon the tangent to the orbit.

For the velocities at the points $A$, $B$, $C$, $D$, $E$ are as the bases $AB$, $BC$, $CD$, $DE$, $EF$ of equal triangles, and, since the triangles are equal, these bases are reciprocally proportional to the perpendiculars from

$S$ let fall upon them. [And the same is true in the limit, in which case the bases are in the direction of tangents to the curvilinear limit, therefore the velocity, &c.]

Cor. 2. If on chords $AB$, $BC$ of two arcs described in equal successive times in a non-resisting medium by the same body the parallelogram $ABCV$ be completed, and the diagonal $BV$ of this parallelogram be produced in both directions in that position which it assumes ultimately when those arcs are diminished indefinitely, it will pass through the centre of force.

Cor. 3. If, on $AB$, $BC$ and on $DE$, $EF$ chords of arcs described in a non-resisting medium in equal times, the parallelograms $ABCV$, $DEFZ$ be completed, the forces at $B$ and $E$ will be to one another in the ultimate ratio of the diagonals $BV$, $EZ$, when the arcs are indefinitely diminished.

For the velocities of the body represented by $BC$, $EF$ in the polygon are compounded of the velocities represented by $Bc$, $BV$ and $Ef$, $EZ$; and those represented by $BV$, $EZ$, which are equal to $cC$, $fF$, in the demonstration of the proposition were generated by the impulses of the centripetal force at $B$ and $E$, and are thus proportional to those impulses. [And the same is true in the limit, in which case the ultimate ratio of the impulses at any two points is the ratio of the continuous forces at those points].

Cor. 4. The forces by which any bodies moving in non-resisting media are deflected from rectilinear motion into curved orbits, are to one another as those sagittæ of arcs described in equal times, which converge to the centre of force and bisect the chords, when those arcs are indefinitely diminished.

For the diagonals of the parallelograms $ABCV$, $DEFZ$ bisect each other, and these sagittæ are halves of the diagonals $BV$, $EZ$ when the arcs are indefinitely diminished. [And the same will be true whether $ABC$ and $DEF$ be parts of the same or of different orbits described by bodies of equal mass, if the arcs be described in equal times].

Cor. 5. And therefore the accelerating effects of the same forces are to that of the force of gravity as those sagittæ are to vertical sagittæ of the parabolic arcs which projectiles describe in the same time.

Cor. 6. All the same conclusions are true by the Second Law of Motion, when the planes, in which the bodies move together with the centres of force which are situated in those planes, are not at rest, but are moving uniformly and parallel to themselves.

The statement of the proposition in the original Latin is,

"Areas, quas corpora in gyros acta radiis ad immobile centrum virium ductis describunt, et in planis immobilibus consistere, et esse temporibus proportionales."

*Observations on the Proposition.*

142. In all cases of motion of bodies it is of great importance for the student to distinguish between the forces themselves under the action of which the bodies may be moving, and the effects which these forces produce.

It is only by an examination of the motion of a body that we are able to infer that it is, or is not, acted on by any force; if we find that the body is moving with uniform velocity in a straight line, we infer that it is, during such motion, acted upon by no force, or that the forces which are acting upon it are in equilibrium; if we find that there is any change of direction or velocity, gradual or abrupt, we infer that the body is moving under the action of some force or forces; if the change be gradual, we infer that such forces are *finite*, by which we mean that the forces require a finite time to produce a finite change whether of direction or velocity; if, on the contrary, the change be abrupt, we infer that the forces are what are called *impulsive*, that is, such as produce a finite change in an instant.

Since then, in order to make any inference with respect to the forces supposed to act, a clear conception of the motion of a body must be first attained, it becomes necessary for the student to be able to describe the motion of a particle of matter as he would that of a point, independently of the causes of such motion.

In doing this he must give a geometrical description of the line traced by the point either in a plane or in space, and then he must describe the rate, uniform or variable, with which this line is traversed.

He may then proceed to attribute any change of direction or velocity to the action of forces upon the particle whose motion he has been examining.

143. In accordance with this method of separating the geometry of the motion from the causes of the deviations, the first proposition would be stated in such a manner as the following:

"When a point moves in a curve, in such a manner that the accelerations at every point are in the direction of a fixed point, the areas, which it describes by radii drawn to the fixed point to which the accelerations tend, are in one fixed plane, and are proportional to the times of describing them."

And, generally, if the words *force* and *body*, employed by Newton, be replaced by *acceleration* and *point*, the resulting statements will be in accordance with this geometrical method of description. It will then be easy to use such terms in the proofs as will not imply, in the manner of expression, the action of force; thus, instead of saying "let a centripetal force tending to $S$ act upon the body by a single instantaneous impulse," we may use the words "let a finite velocity be communicated to the point in the direction of $S$."

144. It should be carefully observed that, before proceeding to the limit, it is proved that *any* polygonal areas $SADS$, $SAFS$, are proportional to the times of description of their perimeters; so that ultimately these areas become *finite* curvilinear areas, described in *finite* times.

145. In proceeding to the ultimate state of the hypothesis, it is concluded readily from Lemmas II. and III. that the curvilinear areas are the limits of the polygons; but a greater

difficulty arises in the transition from the *discontinuous* motion under the action of instantaneous impulsive forces to the *continuous* motion under the action of a continuous force tending to $S$. For, in the curvilinear path of the body which is the limit of the perimeter of the polygon, the direction of the motion at the angular points of the polygon is different, and also the deflection from the direction of motion is twice as great in the polygon as it is in the curve.

Now, although we may assume that the curvilinear limit of the perimeter of the polygon may be described under the action of some force, is that force the same which is the limit of the series of impulses?

The centripetal force supposed to act with a simple instantaneous impulse, "impulsu unico et magno," is supposed to generate a finite velocity at once, which effect a finite force cannot produce.

If, instead of this imaginary impulse, we suppose a force finite, but very great, and acting for a very short time, the effect upon the figure would be to round off the angular points of the polygon.

The transition from the impulses to the continuous force, in the ultimate form of the hypothesis, must be considered as axiomatic, like the ultimate equality of the ratio of the finite arc to the perimeter of the inscribed polygon.

146. We can, however, shew that if the curvilinear limit of the polygon be described under the action of some continuous force tending to $S$, the effect of this force, estimated by the quantity of motion generated in the interval between the impulses, will be ultimately the same as that generated by the impulse.

Consider first the geometrical properties of the limit of the polygonal perimeter. Let $BT$, $CU$ be tangents at $B$, $C$ to the curvilinear limit, and let $Cc$ intersect $BT$ in $T$, fig. page 136.

Now, since $Cc$ ultimately vanishes compared with $Bc$, $BC$ and $Bc$ or $AB$ and $BC$ are ultimately in a ratio of equality, and $Cc$ is ultimately bisected by $BT$, by (2) page 102; also, $CU = BU = UT$ ultimately, by (1) page 102.

Consider next the effects produced by the different kinds of force which act in the two cases.

In the polygonal path, the impulsive force at $B$ generates a velocity with which the body describes $Cc$ in the time $t$ in which $AB$ or $BC$ is described, the measure of the effect of the impulse is therefore the velocity $\frac{Cc}{t}$.

In the curvilinear path, the deflection from the direction $BT$ at $B$, in the same time $t$, is $TC$, by means of the continuous action of finite forces, and if we suppose the force ultimately uniform in magnitude and direction, the measure of the accelerating effect of the force will be $\frac{2TC}{t^2}$, and the velocity generated in that time will be $\frac{2TC}{t^2}.t=\frac{Cc}{t}$.

Hence the effects of the finite and impulsive forces, measured by the quantity of motion produced, are the same.

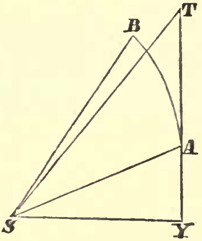
147. We can also shew that a continuous force, which generates the same quantity of motion as the impulse at $B$ in the time from $B$ to $C$, would cause the body on arriving at $C$ to move in the direction of the tangent to the curvilinear limit of the perimeter.

For the velocity due to the action of the finite force at the end of time $t$ being ultimately $\frac{2TC}{t}$ in the direction $TC$, and that in the direction $BT$ being $\frac{BT}{t}=\frac{2TU}{t}$; therefore $TC$, $UT$ represent the velocities in those directions; therefore $UC$ is the direction of motion at $C$, that is, the body moves in the direction of the tangent at $C$.

148. Cor. 1. The corollary may be proved directly from the proposition, for the proportionality of the areas to the times of describing them will be true if the force suddenly cease to act, in which case the body will proceed in the direction of the tangent.

Let $V$ be the velocity at the point $A$, $ASB$ the curvilinear area described in any time $T$, $AT=V.T$ the space described if the force cease to act. Join $ST$ and draw $SY$ perpendicular to $AT$, then area $ASB$ = triangle $SAT = \frac{1}{2}V.T \times SY$, also area $ASB \propto T$; therefore $V$ varies inversely as $SY$.

Again, if $h$ be twice the area described in the unit of time




employed in estimating the accelerating effect of the force tending to $S$ and the velocity of the body,

$$2.\text{area } SAB = hT; \quad \therefore \; h = V.SY.$$

By the use of this area the proportions employed in subsequent propositions by Newton may be converted into equations, for the convenience of calculation.

If bodies move in curves for which the areas, described in the same time, are not equal, $V \propto \dfrac{h}{SY}$.

149. Cor. 4. The statement in this corollary requires modification, for, unless the forces be considered only with reference to their accelerating effects, or unless the bodies be supposed of equal mass, the forces will not be proportional to the sagittæ.

150. Cor. 5. The object of this corollary is to determine the numerical measure of the central force which governs the motion of a body, when the circumstances of the motion are known; for it supplies us with the ratio of this force to the force of gravity on the same body at any place, the measure of which can be determined by experiment.

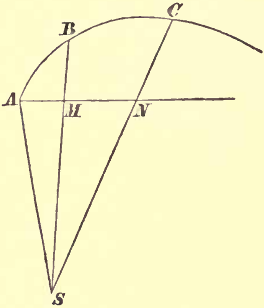
*Applications of the Proposition.*

151. Prop. *When the force, instead of tending to a fixed point, acts in parallel lines, the property of the motion enunciated*

*in teh proposition may be replaced by the property that the resolved part of the space described perpendicular to the direction of the force is proportional to the times.*

This is immediately deducible from the second law of motion, since there is no force in the direction perpendicular to that of he forces, and the velocity in that direction is uniform.

That this is the result of the properties in the proposition may be shewn by removing the centre of force to an infinite distance.




Let $S$ be the centre of force, $AMN$ perpendicular to $SB$, the area $ABCS$ is proportional to the time of describing $AC$, and the areas $AMNS$ and $ABCS$ are ultimately equal when $S$ is removed to an infinite distance in $BMS$, hence the triangle $ASN$ is proportional to the time, and therefore the base $AN$, which varies as the triangle $ASN$, is also proportional to the time, and therefore, since $CN$ is ultimately perpendicular to $AN$, the proposition is proved.

152. PROP. *If a body describe a curvilinear orbit about a force tending constantly to a fixed point, the area described in a given time will be unaltered, if the force be suddenly increased ar diminished, of if the body be acted on at any moment by an impulsive force tending to that point.*

For, if in the polygon the impulse at any point $B$ be increased or diminished by any force tending to or from $S$, the only effect will be to remove the vertex $C$ of the triangle $SBC$ to some other point in the line $cC$ parallel to $BS$, hence the area will be unaltered, and the argument which establishes the equality of polygonal areas in a given time will proceed as before. Hence in the limit the curvilinear areas described in a given time will be unaltered.

If the new force introduced at $B$ be impulsive, the angle $ABC$ will remain less than two right angles when we proceed to the limit, and the two parts of the curve will cut one another at a finite angle.

Hence, in any calculation made upon supposition of such changes of force, the value of $h$, Art. 148, will be the same before and after the change of the force.

## *Apses.*

153. DEF. In any orbit described under the action of a force tending to a fixed centre, a point at which the direction of the motion is perpendicular to the central distance is called an *apse*, the distance from the centre is called an *apsidal distance*, and the angle between consecutive apsidal distances is called an *apsidal angle*.

Thus, in the ellipse about the centre, the four extremities of the axes are apses; there are two different apsidal distances, and every apsidal angle is a right angle.

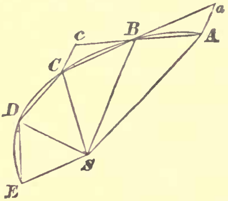
In the ellipse about a focus, the apses are at the greatest and least distances, and the apsidal angle is two right angles.

154. *In a central orbit described under the action of forces tending to a fixed point, each apsidal distance will divide the orbit symmetrically, if the forces be always equal at equal distances.*

It is easily shewn that, in any orbit described by a body under the action of forces tending to a fixed point, the forces depending only upon the distance, if a second body be projected at any point with the velocity of the first in the opposite direction, it will proceed to describe the same orbit in the reverse direction, under the action of the same forces.

For, let $ABC$ be a portion of the polygon whose limit is the curvilinear path of the body, and produce $AB$ to $c$, and $CB$ to $a$, making $Bc = AB$, and $Ba = CB$.

The impulse at $B$ is measured by $cC$ when the body describes $ABC$, and if the motion be reversed, the same impulse at $B$ would cause the body to move in $BA$, with the velocity which it had in $AB$, since $aA = cC$. And the same is true




throughout the polygonal path, hence the assertion is true for the whole path, described under the action of impulses which are always the same at the same points, and therefore, proceeding to the limit, the statement made for any orbit is proved.

Hence, since the forces are equal at equal distances on both sides of the apse, the path of the body from an apse being similar and equal to the path which would be described if the velocity were reversed at the apse, is similar to the path described in approaching the apse; whence the proposition is established.

155. *There are only two different apsidal distances, and all apsidal angles are equal.*

For, after passing a second apse, the curve being symmetrical on both sides, a third apse will be in such a position that the apsidal distance is the same as for the first apse, and all the apsidal angles are shewn similarly to be equal.

156. Cor. Hence a central orbit can never re-enter itself unless the ratio of the apsidal angle to a right angle be commensurable, and if it be so, the curve will always re-enter.

*Illustrations.*

(1) *If a body describe an ellipse under the action of a force tending to one of the foci, the square of the velocity varies inversely*

*as the distance from that focus, and directly as the distance from the other.*

$$\text{For } BC^2 : SY^2 :: HZ : SY :: HP : SP;$$

$$\therefore \text{(vel.)}^2 \propto \frac{1}{SY^2} \propto \frac{HP}{SP}.$$

(2) *The velocity is greatest when the body is at the extremity of the major axis which is nearer to the focus to which the force tends, and least at the other extremity.*

For $SY$ is the least in the first and greatest in the second position.

(3) *The velocity at an extremity of the minor axis is a geometric mean between the greatest and least velocities.*

For at this point $HZ = BC$, and at the extremities of the major axis the values of $HZ$ are $Sa$ and $SA$, and $BC^2 = SA \cdot Sa$.

(4) *In the equiangular spiral described under the action of a force tending to the focus, the velocity* $\propto \dfrac{1}{SP}$.

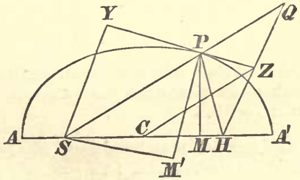
For, $SY \propto SP$.

(5) *If the force tend to the centre of the elliptic orbit described by a body, the time between the extremities of conjugate diameters will be constant.*

For the area $PCD$ is constant.

(6) *The velocity at any point of an ellipse about a force tending to a focus is compounded of two uniform velocities, one perpendicular to the radius vector, and the other perpendicular to the major axis.*

Let $S$ be the centre of force, $SY$, $HZ$ perpendiculars on the tangent at $P$, join $SP$, $CZ$. Then $HZ$, $ZC$ parallel to $PS$, and



$CH$ are perpendicular to the three directions; therefore the velocity represented by $HZ$ in magnitude is the resultant of the two represented by $CZ$ and $HC$; but the velocity perpendicular to $HZ = \frac{h}{SY} = \frac{h}{b^2} . HZ$; therefore the velocities perpendicular to $HC$ and $CZ$ are $\frac{h}{b^2} ae$ and $\frac{h}{b^2} a$.

## XVIII.

1. If different bodies be projected with the same velocity from a given point, all being attracted by forces tending to one fixed point, shew that the areas described by the lines drawn from the fixed point to the bodies will be proportional to the sines of the angles of projection.

2. When a body describes a curvilinear orbit under the action of a force tending to a fixed point, will the direction of motion or the curvature of the orbit at any point be changed, if the force at the point receive a finite change?

3. A body moves in a parabola about a centre of force in the vertex, shew that the time of moving from any point to the vertex varies as the cube of the distance of the point from the axis of the parabola.

4. In a parabolic orbit described round a force tending to the focus, shew that the velocity varies inversely as the normal at any point. Shew also that the sum of the squares of the velocities ta the extremities of a focal chord is constant.

5. If the velocity at any point of an ellipse described about the centre can be equal to the difference of the greatest and least velocities, the major axis cannot be less than double of the minor.

6. If an ellipse be described under the action of a force tending to the centre, shew that the velocity will vary directly as the diameter conjugate to that which passes through the body; also that the sum of the squares of the velocities at the extremities of conjugate diameters will be constant.

7. In an ellipse described round a force tending to the focus, compare the intervals of time between the extremities of the same latus rectum, when $AC = 2CS$.

8. In the ellipse described about the focus $S$, $ASHA'$ being the major axis, time in $AB$ : time in $BA'$ : : $\pi - 2e : \pi + 2e$.

9. If the velocities at three points in an ellipse described by a particle, the acceleration of which tends to either of the foci, be in arithmetical progression, prove that the velocities at the opposite extremities of the diameters passing through these points will be in harmonical progression.

10. If $v_1$, $v_2$ be the velocities at the extremities of a diameter of an ellipse described about the focus, and $u$ the velocity at either of those points when it is described about the centre, prove that $u(v_1 + v_2)$ will be constant.

11. In a central orbit, the velocity of the foot of the perpendicular from the centre of force on the tangent varies inversely as the length of the chord of curvature through the centre of force.

12. A particle is describing a parabola about its focus $S$; if $P$ and $Q$ be two points of its path, shew that its velocity at $Q$ will be compounded of the velocity at $P$ and a velocity which will be constant if the angle $PSQ$ be constant.

## XIX.

1. A body describes a parabola about a centre of force in the focus; shew that its velocity at any point may be resolved into two equal constant velocities, respectively perpendicular to the axis and to the focal distance of the point.

2. A body describes an ellipse under the action of a central force tending to one of the foci; shew that the sum of the velocities at the extremities of any chord parallel to the major axis varies inversely as the diameter parallel to the direction of motion at those points.

3. A body moves in an ellipse under the action of a force tending to the centre; shew that the component of the velocity at any point perpendicular to either focal distance is constant; and that the sum of the squares of the velocities, at the extremities of any pair of semi-conjugate diameters, resolved in any given direction is constant.

4. In an ellipse described about a focus, the time of moving from the greatest focal distance to the extremity of the minor axis is $m$ times that from the extremity of the minor axis to the least focal distance; find the eccentricity, and shew that, if there be a small error in $m$, the corresponding error in the eccentricity will vary inversely as $(1+m)^2$.

5. If the velocity of a body in a given elliptic orbit be the same at a certain point, whether it describe the orbit in a time $t$ about

one focus, or in a time $t'$ about the other, prove that, $2a$ being the major axis, the focal distances will be $\frac{2at'}{t+t'}$ and $\frac{2at}{t+t'}$.

6. A body describes a parabola about the focus; if the segments $PS$, $Sp$ of the focal chord $PSp$ be in the ratio $n:1$, prove that the time in $pA$ : time in $AP$ :: $3n+1 : n^2(n+3)$.

7. If $SY$ be perpendicular to the tangent to a curve at $P$, and $P$ and $Y$ both move as if under the action of a central force tending to $S$, prove that the radius of curvature at $P$ will vary as $SY$.

8. If $P$, $Q$ be any two points in an ellipse described by a particle under the action of a force tending to the centre, prove that the velocity acquired in passing from $P$ to $Q$ will be in the direction $QP'$, where $P'$ is the other extremity of the diameter through $P$.

9. Two points $P$, $P'$ are moving in the same ellipse, in the same directions, with accelerations tending to the centre $C$; shew that the relative velocity of one with regard to the other is parallel and proportional to $CT$, where $T$ is the point of intersection of the tangents at $P$ and $P'$. If the points move in opposite directions, what will be their relative velocity?

10. Two particles revolve in the same direction in an oval orbit round a centre of force $S$, which divides the axis unequally, starting simultaneously from the extremities of a chord $PQ$, drawn through $S$. Prove that, when they first arrive in positions $R$, $T$ respectively, such that the angle $RST$ is a minimum, the time from $R$ to the next apse will be an arithmetic mean between the times from $P$ to the next apse and to $Q$ from the last apse.

11. Two equal particles are attached to the extremities of a string of length $2l$, and lie in a smooth horizontal plane with the string stretched; if the middle point of the string be drawn with uniform velocity $v$ in a direction perpendicular to the nitial direction of the string, shew that the path of each particle will be a cycloid, and that the particles will meet after a time $\frac{l\pi}{2v}$.

12. If the velocity in a central orbit can be resolved into two constant components, one perpendicular to the radius vector, and the other to a fixed straight line, shew that the curve must be a conic.

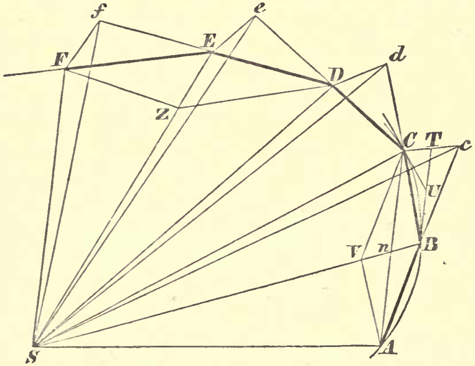
13. The velocity in a cardioid described about a force tending to the pole varies in the inverse sesquiplicate ratio of the distance.

14. The velocity in the lemniscate varies inversely as the cube of the central distance, when a particle moves in the curve round a force tending to the centre.

## PROP. II. THEOREM II.

*Every body, which moves in any curve line described in a plane, and describes areas proportional to the times of describing them about a point either fixed or moving uniformly in a straight line, by radii drawn to that point, is acted on by a centripetal force tending to the same point.*

Case 1. Let the time be divided into equal intervals, and in the first interval let the body describe $AB$ with uniform velocity, being acted on by no force; in the second interval it would, if no force acted, proceed to $c$ in $AB$ produced, describing $Bc$ equal to $AB$; and the triangles $ASB$, $BSc$ would be equal. But




when the body arrives at $B$, let a force, acting upon it by a single impulse, cause the body to describe $BC$ in the second interval of time, so that the triangle $BSC$ is equal to the triangle $ASB$, and therefore also to the triangle $BSc$; therefore $BSC$ and $BSc$ are between the same parallels, hence $BS$ is parallel to $cC$, and therefore $BS$ was the direction of the impulse at $B$.

Similarly, if at $C$, $D$, ... the body be acted on by impulses causing it to move in the sides $CD$, $DE$, ... of a polygon, in the successive intervals, making the triangles $CSD$, $DSE$, ... equal to $ASB$ and $BSC$, the impulses can be shewn to have been in the directions $CS$, $DS$, .... Hence, if *any* polygonal areas be described proportional to the times of describing them, the impulses at the angular points will all tend to $S$.

The same will be true if the number of intervals be increased and their length diminished indefinitely, in which case the series of impulses will approximate to a continuous force tending to $S$, and the polygons to curvilinear areas, as their limits. Hence the proposition is true for a *fixed* centre.

Case 2. The proposition will also be true if $S$ be a point which moves uniformly in a straight line, for, by the second law of motion, the relative motion will be the same, whether we suppose the plane to be at rest, or that it moves together with the body which revolves and the point $S$, uniformly in one direction.

Cor. 1. In non-resisting media, if the areas be not proportional to the times, the forces will not tend to the point to which the radii are drawn, but will deviate *in consequentiâ*, *i.e.* in that direction towards which the motion takes place, if the description of areas be accelerated; but if it be retarded, the deviation will be *in antecedentiâ*.

Cor. 2. And also in resisting media, if the description of areas be accelerated, the directions of the forces will deviate from the point to which the radii are drawn in that direction towards which the motion takes place.

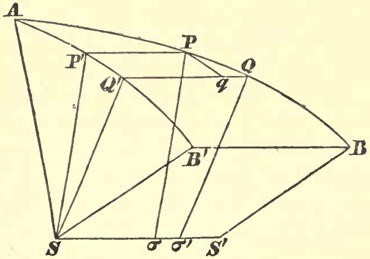
## SCHOLIUM.

A body may be acted on by a centripetal force compounded of several forces. In this case, the meaning of the proposition is, that that force, which is the resultant of all, tends to $S$. Moreover, if any force act continually in a line perpendicular to the plane of the areas described, this force will cause the body to deviate from the plane of its motion, but will neither increase nor diminish the amount of area described, and therefore must be neglected in the composition of the forces.

*Observations on the Proposition.*

157. The description of an area round a point in motion may be explained by the following construction for the relative orbit, in the case of motion about a point which is itself moving uniformly in a straight line.

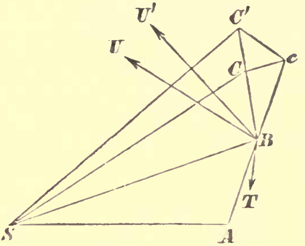
Let $SS'$ be the line in which $S$ moves uniformly, and let the body move from $A$ to $B$ in the same time that $S$ moves from $S$ to $S'$, and let $P$, $\sigma$ be simultaneous positions of the body and of $S$.




If $PP'$ be drawn equal and parallel to $\sigma S$, and the same construction be made for every point in the path of the body, the curve $AP'B'$, which is the locus of $P'$, will be the orbit which the body would appear to describe to an observer at $S$, who referred all the motion to the body; for $SP'$ will be equal and parallel to $\sigma P$, and therefore the distance of the body, and the direction in which it is seen, will be the same in the two cases.

If $Q$, $Q'$ be corresponding points near $P$ and $P'$, and the force at $\sigma$ be supposed to act impulsively, the relative motion round $\sigma$ will be unaltered if we apply to both $P$ and $\sigma$ velocities equal to that of $\sigma$ and in a contrary direction, but in this case $\sigma$ will be reduced to rest and the velocity of $P$ will be the velocity relative to $\sigma$. Take $PQ$ and $\sigma\sigma'$, which are described in the same time, to represent the velocities of $P$ and $\sigma$, and let $Qq$ be equal and parallel to $\sigma'\sigma$, then $Pq$ represents the velocity of $P$ relative to $\sigma$; and, since $Q'q = S\sigma - \sigma'\sigma = P'P$, $P'Q'$ is equal and parallel to $Pq$, and therefore the velocity in the orbit $AB'$ about $S$ at rest is equal to the relative velocity about $S$ in motion.

158. Cor. 1. Reverting to the polygonal area, if the tri-



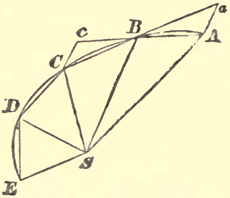

angle $SBC'$ be greater than the triangle $SAB$, the impulse at $B$ will not be in the direction $BS$, but $BU$, parallel to $cC'$, that is, if the areas be not proportional to the times but be in an increasing ratio, the direction of the force will deviate towards the direction in which the description of areas is accelerated; and *vice versâ*, when the description is retarded.

159. Cor. 2. The effect of a resisting medium is to retard the motion, or, supposing it the limit of a series of impulses, we must conceive an impulse at $B$, in the case of the polygon, in the direction $BA$; if therefore the description of areas be accelerated, the impulse applied at $B$ in the direction $BU'$ must act still further *in consequentiâ* than that in $BU$, in order that, with the impulse corresponding to the resistance of the medium, it may produce a resultant impulse in the direction of $BU$. The effect of the resistance alone is to retard the description of areas.

If the force act *in consequentiâ*, the resultant of this force and the resistance of the medium may act in the direction $BS$, and the proportionality of the areas to the times be preserved.

160. PROP. *Let $ABCDE$ be any plane curve, $S$ any point in the plane, to shew that, generally, the curve can be described under the action of a force tending to or from $S$, with finite velocity, the velocity at any given point being any given velocity.*

For arcs $AB$, $BC$, ... can be measured from any point $A$, along the curve, such that the areas $SAB$, $SBC$,... are all equal,




and of any magnitude. Also a body can be made, by some force to move along the curve with finite velocity, so as to describe the arcs $AB$, $BC$, ... in equal times, unless the tangent to one of the arcs, as $DE$, pass through $S$, in which case, if the arcs be indefinitely diminished, $DE$, $AB$ will not be finite ultimately.

Hence by Prop. II. a body *can* move with finite velocity under the action of some force tending to or from $S$, *generally*.

161. NOTE 1. Since in making the motion of the body such that it shall describe equal areas in equal times we are only concerned with the ratio of the velocities, the velocity at any point $A$ may be any given velocity.

162. NOTE 2. Or if we please we may suppose the force at any point any given force; for, in the case of the polygon, the velocity generated by the impulse at $B$ is to the velocity in $AB$ as $cC$ to $Bc$, hence the impulse at $B$ may be of any magnitude if we choose the velocity in $AB$ properly.

163. NOTE 3. The ratio of the velocities will be the same at two given points, for all forces tending to a given centre, under the action of which the curve can be described.

164. NOTE 4. Hence a body can move throughout any ellipse under the action of a centripetal force tending to the centre or focus, the force depending only on the distance, since in these cases the curve is symmetrical on opposite sides of any apse; or about any point within the ellipse, if the forces do not depend only on the distance, since no point within an ellipse lies on any tangent.

165. NOTE 5. In the case of an oval, $S$ being an external point, a body can move with finite velocity under the action of a force tending to the point $S$, in the portion which is concave to $S$, and from $S$, in that which is convex to $S$, but not from one portion to the other.

## XX.

1. If an ellipse be described so that the sum of the areas swept out by radii drawn to the vertices is proportional to the times of describing them, prove that the resultant acceleration will tend to the centre.

2. A body is moving in a parabola, and the time from the vertex to any point varies as the cube of the ordinate; shew that this motion could be caused by the action of a central force.

3. A body was moving in a circle, and it was observed that the time of describing any arc from a fixed point varied as the sum of the arc and the perpendicular distance from one extremity of the arc on the diameter through the other; shew that the body was acted on by a central force.

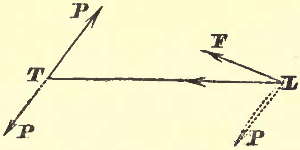
4. A heavy particle falls from the cusp to the vertex of a cycloid, whose axis is vertical; shew that a particle could describe the cycloid in the same manner under the action of a constant force directed to a certain moving point.

5. From the centre of a planet a perpendicular is let fall upon the plane of the ecliptic; prove that the foot of this perpendicular will move as if it were a particle acted on by a force tending to the sun's centre.

## PROP. III. THEOREM III.

*Every body, which describes areas proportional to the times of describing them by radii drawn to the centre of another body which is moving in any manner whatever, is acted on by a force compounded of a centripetal force tending to that other body, and of the whole accelerating force which acts upon that other body.*

Let the first body be $L$, the second $T$, $T$ moves under the action of some force $P$, $L$ under the action of




another force $F$. At every instant apply to both bodies the force $P$ in the contrary direction to that in which it acts, as represented by the dotted arrows.

$L$ will continue to describe about $T$, as before, areas proportional to the times of describing them, and since there is now no force acting on $T$, $T$ is at rest or moves uniformly in a straight line.

Therefore, by Theorem II., the resultant of the force $F$ and the force $P$ applied to $L$ tends to $T$.

Hence $F$ is compounded of a centripetal force tending to $T$, and of a force equal to that which acts on $T$. Q.E.D.

Cor. 1. Hence, if a body $L$ describe areas proportional to the times of describing them by radii drawn to another body $T$; and from the whole force which acts upon $L$, whether a single force or compounded of several forces, be taken away the whole accelerating force which acts upon the other body $T$; the whole remaining force, which acts upon $L$, will tend to the other body $T$ as a centre.

Cor. 2. And if these areas be very nearly proportional to the times of describing them, the remaining force will tend to the other body very nearly.

Cor. 3. And, conversely, if the remaining force tend very nearly to the other body $T$, the areas will be very nearly proportional to the times.

Cor. 4. If the body $L$ describe areas which are very far from being proportional to the times of describing them, by radii drawn to another body $T$, and that other body $T$ be at rest, or move uniformly in a straight line, then either there will be no centripetal force tending to that other body $T$, or such centripetal force will be compounded with the action of other very powerful forces, and the whole force compounded of all the forces, if there be many, may be directed towards some other centre fixed or moving.

The same will hold, when the other body moves in any manner whatever, if the centripetal force spoken of be understood to be that which remains after taking away the whole force acting upon the other body $T$.

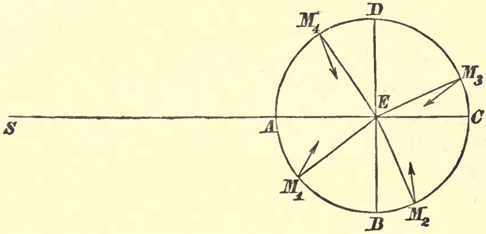
## SCHOLIUM.

Since the equable description of areas is a guide to the centre to which that force tends, by which a body is principally acted on, and by which it is deflected from rectilinear motion, and retained in its orbit, we may, in what follows, employ the equable description of areas as a guide to the centre, about which all curvilinear motion in free space takes place.

### *Illustration.*

166. As an illustration of the last propositions and their corollaries, we may state some of the observed facts in the motion of the Moon, Earth, and Sun, and make the deductions corresponding to them.

Suppose the Moon's orbit relative to the Earth to be nearly circular, and let $ABCD$ be this orbit, $E$ the Earth.




(1) The areas described by the radii drawn from the Moon to the Earth are nearly proportional to the times of describing; hence the resultant force on the Moon tends nearly to $E$.

(2) If $ES$ the line joining the centres of the Earth and Sun meet the Moon's relative orbit about the Earth in $A$, $C$, and $DEB$ be perpendicular to $DS$, the description of areas will be accelerated as the Moon moves from $D$ to $A$ and from $B$ to $C$, and retarded from $A$ to $B$ and from $C$ to $D$; hence the direction of the resultant force on the Moon in the positions $M_1$, $M_2$, $M_3$, $M_4$, will be in the directions of the arrows slightly inclined to the radii drawn to $E$.

From these observed facts, we see that when the force, under the action of which $E$ moves, is applied to the Moon in the contrary direction, the remaining force tends in the directions of the arrows.

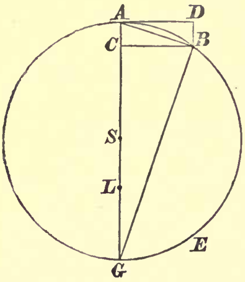
By the supposition that the Earth and Moon are acted on by forces tending to the sun, whose distance compared with $EM$ is very great, and that the differences of the forces on these bodies are not very great, the accelerating effect of the force on the Moon in $DAB$ being greater than that on the Earth, and in $BCD$ less, the circumstances of the description of areas in the motion of the Moon are accounted for.

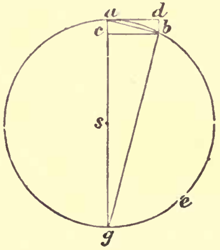
## PROP. IV. THEOREM IV.

*The centripetal forces on equal bodies, which describe different circles with uniform velocity, tend to the centres of the circles, and are to each other as the squares of arcs described in the same time, divided by the radii of the circles.*

The bodies move uniformly, therefore the arcs described are proportional to the times of describing them; and the sectors of circles are proportional to the arcs on which they stand, therefore the areas described by radii drawn to the centres are proportional to the times of describing them; hence, by Prop. II., the forces tend to the centres of the circles.

Again, let $AB$, $ab$ be small arcs described in equal times,




$AD$, $ad$ tangents at $A$, $a$; $ACSG$, $acsg$ diameters through $A$, $a$. Join $AB$, $ab$, and draw $BC$, $bc$ perpendicular to $AG$, $ag$.

When the arcs $AB$, $ab$ are indefinitely diminished, since $AC$, $ac$ are sagittæ of the double of arcs $AB$, $ab$ described in equal times, they are ultimately, by Prop. I., Cor. 4, as the forces at $A$ and $a$.

But $AC.AG = (\text{chd}\, AB)^2$ and $ac.ag = (\text{chd}\, ab)^2$;

$\therefore$ force at $A$ : force at $a$ :: $AC : ac$ ultimately,

$$:: \frac{(\text{chd}\,AB)^2}{AG} : \frac{(\text{chd}\,ab)^2}{ag} :: \frac{(\text{arc}\,AB)^2}{AG} : \frac{(\text{arc}\,ab)^2}{ag}, \text{ Lem. VII.}$$

Take $AE$, $ae$ two arcs described in *any* equal finite times, then $AE : ae :: AB : ab$, since the bodies move uniformly, and this is also true in the limit;

$$\therefore \text{ force at } A : \text{ force at } a :: \frac{AE^2}{AS} : \frac{ae^2}{as}. \quad \text{Q.E.D.}$$

Cor. 1. Since these arcs are proportional to the velocities of the bodies, the centripetal forces will be in the ratio compounded of the duplicate ratio of the velocities directly, and the simple ratio of the radii inversely.

That is, if $V$, $v$ be the velocities in the two circles, $R$, $r$ the radii, $F$, $f$ the centripetal forces, $AE : ae :: V : v$;

$$\therefore F : f :: \frac{V^2}{R} : \frac{v^2}{r}.$$

Cor. 2. And since the circumferences of the circles are described in their periodic times, the velocities are in the ratio compounded of the ratio of the radii directly and the ratio of the periodic times inversely; hence the centripetal forces are in the ratio compounded of the ratio of the radii directly, and of the ratio of the squares of the periodic times inversely.

If $P$, $p$ be the periodic times in the two circles respectively,

$$V : v :: \frac{2\pi R}{P} : \frac{2\pi r}{p} :: \frac{R}{P} : \frac{r}{p};$$

$$\therefore F : f :: \frac{V^2}{R} : \frac{v^2}{r} :: \frac{R}{P^2} : \frac{r}{p^2}.$$

Cor. 3. Hence, if the periodic times be equal, and therefore the velocities proportional to the radii, the centripetal forces will be as the radii; and conversely.

If $P=p$, then $V:v::R:r$;

$$\therefore\ F:f::\frac{V^2}{R}:\frac{v^2}{r}::R:r.$$

Cor. 4. Also if the periodic times be in the subduplicate ratio of the radii, the centripetal forces will be equal.
That is, if $P^2:p^2::R:r$, then $F=f$, by Cor. 2.

Cor. 5. If the periodic times be as the radii, and therefore the velocities equal, the centripetal forces will be reciprocally as the radii; and conversely.

Cor. 6. If the periodic times be in the sesquiplicate ratio of the radii, and therefore the velocities reciprocally in the subduplicate ratio of the radii, the centripetal forces will be reciprocally as the squares of the radii; and conversely.
That is, if $P^2:p^2::R^3:r^3$,

$$\text{then } V^2:v^2::\frac{R^2}{P^2}:\frac{r^2}{p^2}::\frac{1}{R}:\frac{1}{r};$$

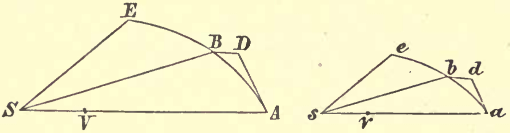
$$\therefore\ F:f::\frac{V^2}{R}:\frac{v^2}{r}::\frac{1}{R^2}:\frac{1}{r^2}.$$

Cor. 7. And, generally, if the periodic times vary as any power $R^n$ of the radius $R$, and, therefore, the velocity vary inversely as the power $R^{n-1}$, the centripetal force will vary inversely as $R^{2n-1}$; and conversely.

Cor. 8. All the same proportions can be proved concerning the times, velocities, and forces, by which bodies describe similar parts of any figures whatever, which are similar and have centres of force similarly situated, if the demonstrations be applied to those cases, *uniform description of areas* being substituted for *uniform velocity*, and *distances of the bodies from the centres of force* for *radii of the circles*.

Let $AE$, $ae$ be similar arcs of similar curves described by bodies about forces tending to similarly situated points $S$, $s$; and let $AB$, $ab$ be small arcs described in equal times; $BD$, $bd$ subtenses parallel to $SA$, $sa$; $AV$, $av$ chords of curvature at $A$, $a$, so that

$$AV : av :: AS : as.$$




Then, force at $A$ : force at $a$ :: $DB : db$, ultimately,

$$:: \frac{AB^2}{AV} : \frac{ab^2}{av} :: \frac{AB^2}{SA} : \frac{ab^2}{sa}, \text{ ultimately;}$$

and if $V$, $v$ be the velocities at $A$, $a$ since $AB$, $ab$ are described in equal times, $AB : ab :: V : v$, ultimately;

$$\therefore \text{ force at } A : \text{force at } a :: \frac{V^2}{SA} : \frac{v^2}{sa}, \text{ as Cor. 1.}$$

Again, if $AB$, $ab$ be small similar arcs described in times $T$, $t$, instead of being arcs described in equal times, and $P$, $p$ be the times of describing similar finite arcs $AE$, $ae$,

$$T : P :: \text{area } ASB : \text{area } ASE :: \text{area } asb : \text{area } ase :: t : p;$$

therefore, when $AB$, $ab$ are indefinitely diminished, $T : t :: P : p$.

$$\text{Hence, } F : f :: \frac{V^2}{SA} : \frac{v^2}{sa} :: \frac{AB^2}{T^2.SA} : \frac{ab^2}{t^2.sa} \text{ ultimately,}$$

$$:: \frac{SA}{T^2} : \frac{sa}{t^2} :: \frac{SA}{P^2} : \frac{sa}{p^2}, \text{ as Cor. 2.}$$

Cor. 9. It follows also from the same proposition, that the arc which a body, moving with uniform velocity in a circle under the action of a given centripetal force, describes in any time, is a mean proportional between the diameter of the circle, and the space through which the body would fall from rest under the action of the same force and in the same time.

For, let $AL$ be the space described from rest in the same time as the arc $AE$, then since, if $BD$ be perpendicular to the tangent at $A$, $BD$ will be ultimately the space described by the body, under the action of the force at $A$, in the time in which the body describes the arc $AB$, and the times are proportional to the arcs;

$$\therefore\ AL : BD :: AE^2 : AB^2;$$

$$\therefore\ AL.AG : BD.AG :: AE^2 : AB^2;$$

$$\text{and } BD.AG = (\text{chd}\,AB)^2 = (\text{arc}\,AB)^2, \text{ ultimately};$$

$$\text{therefore } AL.AG = AE^2, \text{ or } AL : AE :: AE : AG.$$

Q. E. D.

## SCHOLIUM.

The case of the sixth Corollary holds for the heavenly bodies, and on that account the motion of bodies acted upon by a centripetal force, which decreases in the duplicate ratio of the distance from the centre of force, is treated of more fully in the following section.

Moreover, by the aid of the preceding proposition and its corollaries, the proportion of a centripetal force to any known force, such as gravity, can be obtained. For, if a body revolve in a circle concentric with the earth by the action of its own gravity, this gravity is its centripetal force.

But, from the falling of heavy bodies, by Cor. 9, both the time of one revolution and the arcs described in any given time are determined.

And by propositions of this kind Huygens, in his excellent tract, *De Horologio Oscillatorio*, compared the force of gravity with the centrifugal force of revolving bodies.

The preceding results may be proved in this manner. In any circle let a regular polygon be supposed to be described of any number of sides. And if a body moving with a given velocity along the sides of the polygon be reflected by the circle at each of its angular points, the force with which it impinges on the circle at each of the reflections will be proportional to the velocity; and therefore the sum of the forces, in a given time, will vary as the velocity and the number of the reflections conjointly. But if the number of sides of the polygon be given, the velocity will vary as the space described in a given time, and the number of reflections in a given time will vary, in different circles, inversely as the radii of the circles, and, in the same circle, directly as the velocity. Hence, the sum of the forces exerted in a given time varies as the space described in that time increased or diminished in the ratio of that space to the radius of the circle; that is, as the square of that space divided by the radius, and therefore, if the number of sides be diminished indefinitely so that the polygon coincides with the circle, the sum of the forces varies as the square of the arc described in the given time divided by the radius.

This is the *centrifugal force* by which the body presses against the circle, and to this the opposite force is equal, by which the circle continually repels the body towards the centre.

### *Symbolical representation of Areas, Lines, &c.*

167. In the statement of the proposition the words "arcuum quadrata applicata ad radios," in the text of Newton, is rendered the squares of arcs divided by the radii. Such expressions as $\frac{AB^2}{AG}$ may be regarded as representations of lines (*e.g.* this expression denotes $AC$) whose lengths are determined by such constructions as the following:

To $AG$ apply a rectangle whose area is that of the square on $AB$, and let $AC$ be the side adjacent to $AG$; $AC$ is thus obtained by applying the square on $AB$ to $AG$. The propriety of the symbol $\frac{AB^2}{AG}$ employed to represent a line $AC$, assumed from algebra, is obvious, since the number of units of area in the square on $AB$ and in the rectangle whose sides are $AG$,

$AC$ are the same; hence, if $m$, $n$, $r$ be the number of units of length in these lines, $m^2 = n \times r$ and $r = \frac{m^2}{n}$.

168. If symbols of this kind, viz. $\frac{AB^2}{AG}$, be used in the same manner as a fraction, we may either treat them numerically, considering $AB^2$ to represent the number of units of area contained in the square on $AB$, and $AG$ as the number of units of length in $AG$, and thus apply the rules of Arithmetical Algebra; or we may look upon $AB^2$ as the absolute representation of an area, and $AG$ as that of a line, in which case $\frac{AB^2}{AG}$ would have no meaning except by interpretation. In this interpretation we are guided by the principles upon which Symbolical Algebra is applied to any science, the laws of operation by symbols being the same in Arithmetical and Symbolical Algebra, and the symbols being interpreted so that these laws are not contradicted. Thus if, in the application to Geometry, the symbol $A$ be supposed to denote an area equal to that of a rectangle whose sides are represented by $a$ and $b$, the assumption that $A = ab$ or $ba$ will imply that $ab = ba$, hence the laws remain the same as in Arithmetical Algebra, and $\frac{A}{a} = b$; so that the interpretation is legitimate, that, if a rectangle be applied to $a$, whose area is $A$, $\frac{A}{a}$ will denote the other side of the rectangle.

*Observations on the Proposition.*

169. In the statement of the proposition the word 'equal' has been inserted before 'bodies' in order to make the theorem correct, whether we suppose the centripetal force to be estimated with reference to the momentum or the velocity generated.

It would, perhaps, be better to state the proposition as follows: "The resultant of the forces, under the action of which bodies describe different circles with uniform velocity, are centripetal and tend to the centres of the circles, and their accelerating effect are to each other, &c.," for it is not known, prior to the proof, that the forces are centripetal.

170. CORS. 1 and 9. The first corollary asserts that the centripetal forces on bodies moving in different circles vary as $\frac{V^2}{R}$, but the ninth shews that the accelerating effects of the centripetal forces are in each circle equal to $\frac{V^2}{R}$.

For, if $V$ be the velocity, $F$ the accelerating effect of the force in any circle, $T$ the time of describing any arc, $VT$ will be the length of the arc, $\frac{1}{2}FT^2$ will be the space through which the body would move under the action of the same force continued constant, in the same time in which the arc is described,

$$\therefore\ \tfrac{1}{2}FT^2 : VT :: VT : 2R;\ \therefore\ V^2 = FR.$$

171. Scholium. In uniform circular motion the centripetal force is employed in counteracting the tendency of the body to move in a straight line, which it would do, according to the first law of motion, with the uniform velocity which it has at any point of the circle, if the centripetal force were suddenly to cease to act. This tendency to recede is called a centrifugal force improperly; for the effect of a force being to accelerate or retard the motion of a body, or to alter its direction, if the tendency could properly be termed a force and the centripetal force which counteracts it were removed, it would accelerate or retard the motion of the body, or alter its direction, which it does not.

The only sense in which the term centrifugal force can be used with propriety as a force may be obtained by the consideration of relative equilibrium, in which case, if the same centripetal force acted on the body, the centrifugal force would keep it in equilibrium, supposing the body were at rest as it would appear to be to an observer moving with it.

Thus, if a body be supported on the surface of the earth, since the body describes a circle about the axis of the earth with uniform velocity, the pressure of the support and the attraction of the earth must have a resultant, whose direction will pass through the centre of this circle, and whose magnitude will be such as would cause the body to describe it; this resultant and the centrifugal force will be in statical equilibrium.

172. In this case of circular motion the force is exerted not in accelerating or retarding the motion, but in changing its direction.

Thus, referring to the figure of Prop. I., if the direction of the impulse at $B$ bisect the angle $ABC$, the triangle $CBc$ will be isosceles, and $BC = Bc = AB$; therefore the velocities in $BC$ and $AB$ will be equal, and the effect of the impulse has been to change the direction without altering the velocity of the body.

Hence, the regular polygon inscribed in a circle, centre $S$, can be described with uniform velocity under the action of impulses tending to the centre; and, by similar triangles $SBC$, $CBc$,

$$Cc : BC :: BC : BS.$$

And if $V$ be the uniform velocity in the polygon, $T$ the time in a side $BC$, $BC = V.T$; therefore $Cc = \frac{V^2 T^2}{BS}$.

If now the number of sides be indefinitely increased, $Cc$ will be ultimately twice the space through which the body will be drawn from the tangent by the continuous force, see Art. 146; therefore $\frac{Cc}{T^2} = \frac{V^2}{BS}$ will be the measure of the accelerating effect of the centripetal force tending to the centre of the circle.

*Illustrations of Circular Motion.*

(1) *A small body is attached by an inelastic string to a point on a smooth horizontal table, to determine the tension of the string when the body describes a circle.*

If the body be set in motion by a blow perpendicular to the string, the string will remain constantly stretched, and the only force which acts on the body in the horizontal plane being in the direction of the fixed point, the areas described round this point will be proportional to the time, and the body will move in a circle with uniform velocity.

Let $v$ be the velocity of projection, and $l$ the length of the string, then the accelerating effect of the tension of the string is $\frac{v^2}{l}$; that is, $\frac{v^2}{l}$ is the velocity which would be generated in an unit of time from rest by the action of this tension continued constant, therefore the tension of the string : the weight of the body :: $\frac{v^2}{l} : g$.

Ex. If a velocity of two feet a second be communicated perpendicular to a string whose length is a yard,

$$v^2 : lg :: 4 : 3 \times 32 :: 1 : 24,$$

hence the tension is $\frac{1}{24}$th of the weight, and the time of revolution is evidently $\frac{2\pi l}{v}$ seconds $= \frac{6\pi''}{2} = 9{\cdot}4''$, nearly.

(2) *If a particle be attached by a string of given length to a point in a rough horizontal plane, and a given velocity be communicated to it, perpendicular to the string supposed tight, find the tension of the string at any time, the time in which it will be reduced to rest, and the whole arc described.*

Let $V$ be the velocity of projection, $l$ the length of the string in feet, $v$ the velocity at any time $t$. Since the particle describes a small arc ultimately with uniform velocity the accelerating effect of the tension at the time $t$ is $\frac{v^2}{l}$. Again, if $\mu$ be the coefficient of friction, the retarding effect of friction is $\mu g$, which is constant, hence the velocity destroyed in the time $t$ since friction is the only force acting in the direction of the tangent is $\mu gt$, and $v = V - \mu gt$.

Therefore the particle comes to rest in $\frac{V}{\mu g}$ seconds after describing the arc $\frac{V^2}{2\mu g}$ feet.

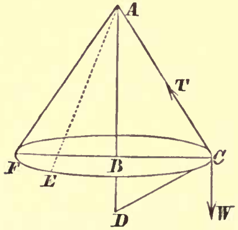
The tension of the string at the time $t$: the weight of the particle :: $\frac{v^2}{l} : g :: \frac{(V-\mu gt)^2}{l} : g$; therefore the tension $\propto \left(\frac{V}{\mu g} - t\right)^2$ $\propto$ the square of the time which will elapse before the particle comes to rest.

(3) *Supposing that the Moon describes a circle with uniform velocity about the centre of the Earth as its centre, to find the ratio of the centripetal acceleration of the Moon's motion to gravity at the Earth's surface.*

Let $n=$ number of seconds in the Moon's periodic time, $R=$ the radius of the Moon's orbit in feet; therefore the velocity of the Moon is $\frac{2\pi R}{n}$ and $\frac{1}{R}\cdot\left(\frac{2\pi R}{n}\right)^2$ is the measure of the accelerating effect of the force exerted on the Moon, and the measure of the same for gravity at the Earth's surface $=32.2$; hence, the ratio required is $4\pi^2 R : 32.2n^2$.

(4) *A body is suspended by a string from a fixed point, and being drawn out of the vertical is projected horizontally so as to describe a horizontal circle with uniform velocity. Find the velocity and the tension of the string.*

Let $A$ be the point of suspension, $BC$ the radius of the circle described; therefore, the circle being described uniformly, the resultant force on the body tends to the centre $B$, and the measure of the accelerating effect of this resultant force is $\frac{V^2}{BC}$, in the direction $CB$. Let $T$, $W$ be the tension of the string and the weight of the body, acting in $CA$ and parallel to $AB$ respectively, therefore $T : W :: CA : AB$;




also, $\frac{V^2}{BC} : g :: CB : AB$, Art. 171, $\therefore V^2 = \frac{g.BC^2}{AB}$,

and, if $CD$ be perpendicular to $AC$, $BC^2 = AB.BD$; and the velocity will be that due to falling through the space $\frac{1}{2}BD$.

## XXI.

1. If the cube of the velocity, in circles uniformly described, be inversely proportional to the periodic time, shew that the law of force will vary inversely as the square of the radii.

2. Compare the areas described in the same time by the planets, supposed to move in circular orbits about the Sun in the centre exerting a force which varies inversely as the square of the distance.

3. If the forces by which particles describe circles with uniform velocity vary as the distance, shew that the times of revolution will be the same for all.

4. If the velocity of the Earth's motion were so altered that bodies would have no weight at the equator, find approximately the alteration in the length of a day, assuming that, before the alteration, the centrifugal force on a body at the equator was to its weight :: 1 : 288.

5. A particle moves uniformly on a smooth horizontal table, being attached to a fixed point by a string, one yard long, and it makes three revolutions in a second. Compare the tension of the string with the weight of the particle.

6. A body moves in a circular groove under the action of a force to the centre, and the pressure on the groove is double the given force on the body to the centre, find the velocity of the body.

7. If a locomotive be passing a curve at the rate of twenty-four miles an hour, and the radius of the curve be $\frac{1}{15}$ of a mile, prove that the resultant of the forces which retain it on the line, viz. of the action of the rails on the flanges of the wheels, and the horizontal part of the forces which act perpendicular to the inclined road-way, will be $\frac{1}{100}$ of the weight of the locomotive, nearly.

8. If a body be attached by an extensible string to a fixed point in a smooth horizontal table, find the velocity with which the body must move in order to keep the string constantly stretched to double its length.

If $W$ be the weight of the body, and $nW$ be the weight which if suspended at the extremity of the string would just double its length, $l$ the length of the string, shew that the square of the required velocity $= 2nlg$.

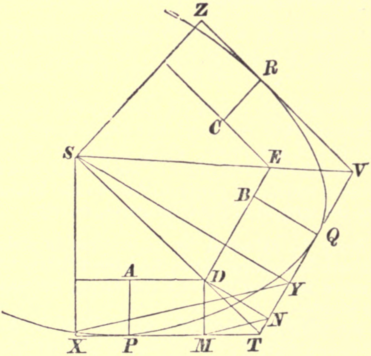
9. A man stands at the North Pole and whirls 24 lbs. troy weight on a smooth horizontal plane by a string a yard long at the rate of 100 turns a minute; he finds that the difference of the forces which he has to exert according as he whirls it one way or the opposite is roughly 39 grains; find the period of the rotation of the earth.

10. Two equal bodies lie on a rough horizontal table, and are connected by a string which passes through a small ring on the table; if the string be stretched, find the greatest velocity with which one of the bodies can be projected in a direction perpendicular to its portion of the string without moving the other body.

## PROP. V. PROBLEM I.

*Having given the velocity with which a body is moving at any three points of a given orbit, described by it under the action of forces tending to a common centre, to find that centre.*

Let the three straight lines $PT$, $TQV$, $VR$, touch the given orbit in the points $P$, $Q$, $R$ respectively, and let them meet in $T$ and $V$.




Draw $PA$, $QB$, $RC$ perpendicular to the tangents, and inversely proportional to the velocities of the body at the points $P$, $Q$, $R$. Through $A$, $B$, $C$ draw $AD$, $DBE$, $CE$ at right angles to $PA$, $QB$, $RC$ meeting in $D$ and $E$. Join $TD$, $VE$; $TD$ and $VE$ produced, if necessary, shall meet in $S$ the required centre of force.

For, the perpendiculars $SX$, $SY$, let fall from $S$ on the tangents $PT$, $TQV$, are inversely proportional to the velocities at $P$, $Q$ (Prop. I. Cor. 1), and are therefore directly as the perpendiculars $AP$, $BQ$, or as the perpendiculars $DM$, $DN$ on the tangents. Join $XY$, $MN$, then, since $SX : SY :: DM : DN$ and the angles $XSY$, $MDN$ are equal, therefore the triangles $SXY$, $MDN$ are similar; therefore $SX : DM :: XY : MN :: XT : MT$, and the angles $SXT$, $DMT$ are right angles; therefore, $S$, $D$, $T$ are in the same straight line. Similarly $S$, $E$, $V$ are in the same straight line, and therefore the centre $S$ is the point of intersection of $TD$, $VE$. Q. E. D.

## XXII.

1. If $AB$, $BC$, $CD$, the three sides of a rectangle, be the directions of the motion of a body at three points of a central orbit, and the velocities be proportional to these sides respectively, prove that the centre of force will be in the intersection of the diagonals of the rectangle.

2. If the velocities at three points of a central orbit be respectively proportional to the opposite sides of the triangle formed by joining the points, and have their directions parallel to the same sides, prove that the centre of force will be the centre of gravity of the triangle.

3. Three tangents are drawn to a given orbit, described by a particle under the action of a central force, one of them being parallel to the external bisector of the angle between the other two. If the velocity at the point of contact of this tangent be a mean proportional between those at the points of contact of the other two, prove that the centre of the force will lie on the circumference of a certain circle.

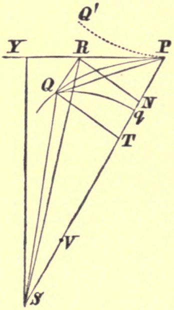
4. If the velocities be inversely proportional to the sides of the triangle formed by the tangents at the three points, the centre of force will be the point of concourse of the straight lines joining each an angular point of this triangle to the intersection of the tangents to its circumscribing circle at the ends of the opposite side.

5. If the velocity of a particle describing an ellipse under the action of a centre of force vary as the diameter parallel to the direction of its motion directly, and as its distance from one of the axes inversely, prove that the centre of force will be at an infinite distance.

## PROP. VI. THEOREM V.

*If a body revolve about a fixed centre of force, in any orbit whatever, in a non-resisting medium, and if, at the extremity of a very small arc, commencing from any point in the orbit, a subtense of the angle of contact at that point be drawn parallel to the radius from that point to the centre of force, then the force at that point tending to the centre will be ultimately as the subtense directly and the square of the time of describing the arc inversely.*

Let $PQ$ be the small arc, $PS$ the radius drawn from $P$ to $S$, the centre of force. $RQ$ the subtense of the




angle of contact at $P$, parallel to $PS$. $T$ the time of describing $PQ$. $F$ the accelerating effect of the force at $P$.

Then, when the body leaves $P$, it would, if not acted on by the central force, move in the direction $PR$, and if the force $F$ continued constant in magnitude and direction throughout the time $T$, $QR$ would be the space through which it would have been drawn by $F$ in that time; therefore ultimately, $F = \frac{2QR}{T^2} \propto \frac{QR}{T^2}$.

Cor. 1. Draw $QT$ perpendicular to $SP$, and let $h$ = twice the area described in an unit of time. Then area $PSQ = \frac{1}{2}ht$, Prop. I., also, since triangle $PSQ = \frac{1}{2}SP.QT$, and area $PSQ$ = triangle $PSQ$, ultimately, Lemma VIII., therefore $hT = SP.QT$, ultimately; hence, ultimately, $F = 2\,\frac{QR}{T^2} = \frac{2h^2}{SP^2}\cdot\frac{QR}{QT^2}$.

COR. 2. Draw $SY$ perpendicular on $PR$. Then, $\triangle\, PSQ = \triangle\, PSR = \frac{1}{2}SY.PR$;

$$\therefore\ hT = SY.PR = SY.PQ, \text{ ultimately;}$$

$$\text{hence, ultimately, } F = 2\,\frac{QR}{T^2} = \frac{2h^2}{SY^2}\cdot\frac{QR}{PQ^2}.$$

COR. 3. If the orbit have finite curvature at $P$, and $PV$ be the chord of the circle of curvature whose direction passes through $S$, $PV.QR = PQ^2$, ultimately;

$$\therefore\ F = \frac{2h^2}{SY^2.PV}.$$

COR. 4. If $V$ be the velocity at $P$, then $V = \frac{PQ}{T}$, and

$$F = \frac{2QR}{T^2} = \frac{2QR}{PQ^2}\cdot\left(\frac{PQ}{T}\right)^2, \text{ ultimately;}$$

$$\therefore\ F = \frac{2V^2}{PV}, \text{ or } V^2 = 2F.\frac{PV}{4};$$

that is, the velocity at any point of a central orbit at which the curvature is finite is that which would be acquired by a body moving from rest under the action of the central force at that point continued constant, after passing through a space equal to a quarter of the chord of curvature at that point drawn in the direction of the centre of force.

COR. 5. Hence, if the form of any curve be given, and the position of any point $S$, towards which a centripetal force is continually directed, the law of the centripetal force can be found, by which a body will be deflected from its direction of motion, so as to remain in the curve. Examples of this investigation will be given in the following problems.

*Observations on the Proposition.*

173. In Newton's enunciation of the proposition, the sagitta of the arc, which bisects the chord and is drawn in the direction of the centre of force, is employed instead of the subtense used in the text, but these are ultimately proportional by Art. 90.

The variations by which Newton expresses the results of the first three corollaries are replaced by equations, in order to facilitate the comparison of the motion of bodies in different orbits and the forces acting upon them.

174. The figure employed in proof of the proposition is drawn upon supposition that the force is attractive, the orbit being concave to the centre of force; the same proof will apply also to the case of a repulsive force, if the curve be drawn in the direction of the dotted line $PQ'$ and the same construction be made.

The exception, however, should be made, that the method fails in the particular positions in which the body is at the points of contact of tangents drawn from the centre of force to the curve; in such cases $QR$ does not ultimately meet the tangent at a finite angle or is not a subtense; the result of the proposition is therefore not demonstrated for these particular positions. A further discussion of the case is given on the next proposition.

175. In the proof it is assumed that the body moves ultimately in the same manner as if the force $P$ remained constant in magnitude and direction, in which case the body would describe a parabola, whose axis is parallel to $PS$, and which is evidently the parabola which has at $P$ the same curvature as the curve. By this consideration the proposition contained in Cor. 4 can be readily proved. For, since the body moves in a parabola under the action of a constant force in parallel lines, the velocity at $P$ is that acquired by falling from the directrix under the action of the force at $P$, continued constant, *i.e.* through a space equal to the distance of the focus of the parabola, which is equal to a quarter of the chord of curvature at $P$, drawn through $S$.

176. The supposition that the force at $P$ continued constant in magnitude and direction, causes the body to move in a curve which is ultimately coincident with the path of the body, may be justified by considering that if $PQ'$ be the arc of the parabola described on this supposition in the same time as the arc $PQ$ actually described, the error $Q'Q$ is due to the change in the magnitude of the forces and the direction of their action in the two cases; now, the greatest difference of magnitude varies as the difference of $SP$ and $SQ$ ultimately, and the ratio of the error from this cause to $Q'R$ vanishes ultimately; also, since $\angle PSQ$ vanishes ultimately, the ratio of the error, arising from the change of direction, to $Q'R$ vanishes; therefore, $Q'Q : Q'R$ vanishes, and the curves may be considered ultimately coincident.

177. It is evident that the results of the Proposition and of the fourth corollary are true of the resultant of any forces, under the action of which any plane orbit is described, for this resultant may be supposed ultimately constant in direction and magnitude, in which case the curve described is a parabola. Hence, as in Art. 175, if $F$ be the accelerating effect of the resultant of the forces, $QR$ the subtense parallel to the direction of the resultant,

$$V^2 = 2F.\frac{PV}{4}, \text{ and } F = 2 \text{ limit } \frac{QR}{T^2}.$$

*Homogeneity.*

178. Cor. 1, 2. In the expressions for $F$ obtained in these corollaries, it is of great importance to observe the dimensions of the symbols. Thus $hT$ represents an area and $h$ is of two dimensions in linear space and of $-1$ in time; therefore $h^2.QR$ is of five in space, and of $-2$ in time, and $SP^2.QT^2$ of four dimensions in space; hence, $\dfrac{2h^2.QR}{SP^2.QT^2}$ is of one dimension in space and of $-2$ in time, and represents either twice the space through which a force would draw a body in an unit of time, or the velocity generated by the force in an unit of time, either of which may be taken as the measure of the accelerating effect of the force; moreover, this unit is the same by which the magnitude of $h$ is determined.

Hence, if the actual areas, lines, &c., be represented by the symbols, and not the *number* of units, as mentioned in Art. 168, every term of an equation or of a sum or difference must be homogeneous, or of the same number of dimensions, both in space and time; for example, $PQ + V.T$ representing a line, $V$ must be of $-1$ dimensions in time.

## *Tangential and Normal Forces.*

179. *To find the accelerating effect of the components of the forces, under the action of which a body describes any plane curve, taken in the directions of the normal and tangent at any point.*

Let $PQ$ be a small arc of the curve described under the action of any forces, $T$, $N$ the measures of the accelerating effect of these forces, in the direction of the tangent and perpendicular to it. Then, if $V$ be the velocity at $P$, $t$ the time of describing $PQ$, the forces may be supposed ultimately to remain constant; therefore, if $QR$ be perpendicular to $PR$, we shall have ultimately $QR = \frac{1}{2}N.t^2$, and $PR = V.t + \frac{1}{2}T.t^2 = V.t$ since the ratio of $T.t^2 : Vt$ vanishes ultimately; hence, if $\rho$ be the radius of curvature at $P$, $2\rho = \dfrac{PR^2}{QR} = \dfrac{2V^2}{N}$ ultimately; therefore $\dfrac{V^2}{\rho}$ will be the measure of the normal acceleration estimated towards the centre of curvature.

Again, if $V'$ be the velocity at $Q$, $V'$ will be ultimately the component of the velocity in the direction $PR$; therefore, by Art. 53, we obtain two measures of the tangential acceleration, the limits of $\dfrac{V'^2 - V^2}{2PQ}$ and $\dfrac{V' - V}{t}$.

180. *To find the velocity at any point of an orbit described under the action of any forces in one plane.*

Let $AB$ be any arc of an orbit, $V$, $v$ the velocities at $A$ and $B$, and suppose the arc $AB$ divided into a large number of small portions, of which $PQ$ is one, $v_r$, $v_{r+1}$ velocities at $P$ and $Q$, $T$ the accelerating effect of the tangential component of the forces at $P$,

$$v_{r+1}^2 - v_r^2 = 2T.PQ \text{ ultimately},$$

and $v^2 - V^2$ is obtained by taking the limit of the sum of the magnitudes $2T.PQ$ corresponding to the different arcs when their number is indefinitely increased.

That this is rigidly correct may be shewn by considering that $v_{r+1}^2 - v_r^2 : 2T.PQ$ is ultimately a ratio of equality; therefore, by Cor., Lemma IV., or Art. 22, the limiting ratio of the sums is also a ratio of equality.

In the case of a central force, whose accelerating effect is $F$, $T = F\cos RPS$;

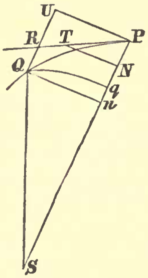
$\therefore\ v_{r+1}^2 - v_r^2 = 2F.PQ\cos RPS = 2F(SP - SQ)$ ultimately,

whence $v^2 - V^2$, if $F$ depend only on the distance.

*Radial and Transversal Forces.*

181. *To find the accelerating effect of the components of force, under the action of which a body describes any plane curve, taken in the direction of a radius vector drawn from a fixed point, and perpendicular to it.*

Let $PQ$ be a small arc described in the time $T$; $QRU$, $PU$ parallel and perpendicular to $SP$; $P$, $Q$ the measures of the accelerating effects of the components in $PS$ and $PU$; $PR$ a




tangent at $P$. If $V$ be the velocity at $P$, make $PT = V.T$, draw $TN$ perpendicular to $SP$, and let $Qq$ be the arc of a circle, centre $S$.

Since the forces may be considered ultimately constant in magnitude and direction, $\frac{1}{2}P.T^2 = Nn = Nq + \dfrac{Qn^2}{2Sq}$ ultimately.

Let $h$ be twice the area which would be described in an unit of time by radii from $S$, if the transverse force $Q$ ceased to act, then $Qn.SP = TN.SP = h.T$; therefore $\frac{Qn^2}{2Sq} = \frac{h^2T^2}{2SP^3}$ ultimately; and if $P'$ be the measure of the accelerating effect of a force, under the action of which the body would move in $PS$, so that its distance from $S$ would be always equal to that of the body in $PQ$ at the same time, $\frac{1}{2}P'.T^2 = Nq$ ultimately; therefore $P = P' + \frac{h^2}{SP^3}$.

Again, if at $Q$ $h'$ correspond to $h$, $h' - h$, the increase of $h$, will be due to the increase of velocity in direction $PU$, which is equal to $Q.T$ ultimately; therefore $(h' - h)\,T = Q.T^2.SP$ ultimately; hence $Q = \frac{h' - h}{SP.T}$ ultimately.

### *Angular Velocity.*

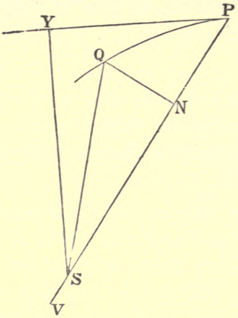
182. Def. *Angular velocity* of a point moving about a fixed point is the rate at which angles are described by radii drawn to the fixed point.

*Uniform angular velocity* is measured by the angle described in an unit of time.

*Variable angular velocity* is measured by the angle which would be described by a radius in an unit of time, if moving with uniform angular velocity equal to the angular velocity at the time under consideration; this is the limit of the angle, described in a time $T$, divided by $T$, when $T$ is indefinitely diminished.

183. *To find the angular velocity in a central orbit.*

Let $PQ$ be a small arc described in the time $T$, draw $QN$ perpendicular to $SP$, then $h.T =$ twice the area $PSQ = QN.SP$ ultimately; and, if the angles be supposed estimated in circular measure, $\angle PSQ = \frac{QN}{SQ} = \frac{QN}{SP}$ ultimately; therefore the angular velocity, which is $\angle \frac{PSQ}{T}$ ultimately, $= \frac{h}{SP^2}$.




184. *To find the angular velocity of the perpendicular on the tangent from the centre of force.*

Draw $SY$ perpendicular on the tangent $PY$, and let $PV$ be the chord of curvature through $S$.

The angle described by $SY$ in the time $T$ is equal to the angle between the tangents at $P$ and $Q$, or to twice the angle $PVQ$; therefore angular velocity of $SY$ : angular velocity of $SP :: 2 \angle PVQ : \angle PSQ :: 2SQ : QV$ ultimately; hence the angular velocity of $SY = \dfrac{2h}{PV.SP}$.

*Illustrations.*

(1) *To find the tension of a string by which a body is attached to the centre of a vertical circle in which it revolves.*

Let $P$ be the position of the body at any time, $CP$, $CA$ radii drawn to $P$ and the lowest point, and let $v$, $u$ be the velocities at $P$ and $A$. Draw $PM$ perpendicular to $CA$. Then $u^2 - v^2 = 2g.AM$ and $\dfrac{v^2}{CA}$ is the accelerating effect of the forces in the direction $PC$, viz. the tension of the string and the component of the weight of the body. Let $T$ be the tension of the string and $m$ the mass of the body;

$$\therefore \frac{v^2}{CA} = \frac{T}{m} - g.\frac{CM}{CP} \text{ and } T : mg :: v^2 + g.CM : g.CA;$$

therefore the tension of the string : the weight of the body

$$:: u^2 - 2g.CA + 3g.CM : g.CA.$$

NOTE 1. In order that the complete circle may be described, since the string must be stretched at the highest point where $-CA$ must be written for $CM$, $u^2 =$ or $> 5g.CA$, and if the circle be just described, the tension at the lowest point will be six times the weight.

NOTE 2. If the body oscillate, the extent of the oscillation will be given by the consideration that at the extremity $P'$ of the arc of oscillation there will be no velocity, therefore $u^2 = 2g.AM'$, and $AM'$ is less than $AC$, otherwise the string would not be stretched, so that the tension at $A$ : the weight $:: 2AM' + AC : AC$.

(2) *Find the force under the action of which a body may describe the equiangular spiral uniformly.*

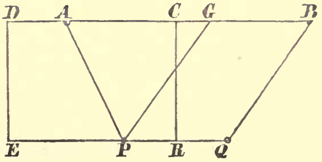
The velocity being constant, there is only a normal force measured by $(\text{vel.})^2 \div$ radius of curvature $= \dfrac{V^2 \sin\alpha}{SP}$, Art. 128.

(3) *Find the force tending to the pole of the cardioid, under the action of which the curve is described.*

Since $PV = \frac{4}{3}SP$, and $(\text{vel.})^2 = \dfrac{h^2}{SY^2} = \dfrac{h^2.BC}{SP^3}$, see page 105,

therefore the accelerating effect of the force is $\dfrac{3h^2.BC}{2SP^4} \propto \dfrac{1}{SP^4}$.

(4) *Two equal rings $P$, $Q$ slide on a string which passes round two fixed pegs $A$, $B$ in a smooth horizontal plane; the rings are brought together, and then projected with equal velocities, so as to keep the string stretched symmetrically. Shew that the tension of he string varies inversely as the distance $AP$.*



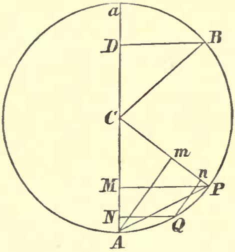
The figure represents the position of the system at any time. Let $CR$ bisect $AB$ and $PQ$, and let $DE$ be drawn parallel to $CR$, so that $EP = PA$, then $EPR = AP + PR$ is constant; therefore $DE$ is fixed, and $P$ moves in a parabola whose focus is $A$ and directrix $DE$.

Also, the tensions of the string in $PA$, $PQ$ being equal and equally inclined to the tangent to $P$'s path, the resultant of these tensions, which are the only forces acting in the plane of the curve, acts in the normal, hence the rings move with uniform velocity equal to the velocity of projection $V$, and if $T$ be the measure of the accelerating effect of the tension, $PG$ the normal, $\rho$ the radius of curvature, $2T \cos APG = \dfrac{V^2}{\rho}$, and $2\rho \cos APG$ $=$ chord of curvature through $A = 4PA$; therefore

$$T = \frac{V^2}{4PA} \propto \frac{1}{PA}.$$

(5) *A body revolves in a smooth circular tube under the action of a force tending to any point in the circumference, and varying as the distance from that point. Find the pressure on the tube, and the point where there is no pressure, the motion commencing from a given point.*

Take $A$ the centre of force, $C$ that of the circle; let $B$ be the point of starting, $PQ$ a small arc, $BD$, $PM$, $QN$ ordinates to the




diameter through the centre of force, $Am$, $Qn$ perpendicular on $CP$; let $\mu . PA$ be the measure of the accelerating effect of the force at $P$; therefore $\mu . mA$, $\mu . Pm$ are those of the tangential and normal forces, $= \mu . PM$ and $\mu . AM$ respectively.

(vel.)$^2$ at $Q$ − (vel.)$^2$ at $P = 2\mu . PM . PQ = 2\mu . CP . MN$ ultimately, see Art. 179, whence, taking the limit of the summation for all the small arcs in $BP$, (vel.)$^2$ at $P = 2\mu . CP . DM$.

Also, $\dfrac{(\text{vel.})^2 \text{ at } P}{CP} = \mu . AM \mp$ the accelerating effect of the pressure of the tube, the upper or lower sign being taken according as the pressure is from or towards $C$; therefore the pressure on the tube has for the measure of its accelerating effect

$$\pm \mu (AM - 2DM) = \pm (3AM - 2AD);$$

hence the pressure is outwards from $B$ until $AM = \frac{2}{3}AD$, at which point there is no pressure, and inwards from that point to the corresponding one on the opposite side, having its greatest value at $A$, and the outward pressure at $B$ is half the inward pressure at $A$.

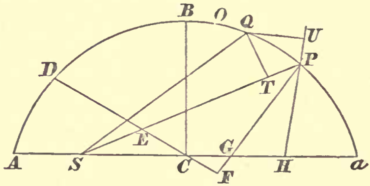
(6) *If in a smooth elliptic tube a particle be placed at any point, and be acted on by two forces which tend to the foci and vary inversely as the square of the distances from those points, shew that the pressure at any point will vary as the curvature.*

Let $O$ be the point of starting, $PQ$ a small arc described by the body, $QT$, $QU$ perpendiculars on $SP$, $HP$.

Take $\dfrac{\mu}{SP^2}$, $\dfrac{\mu'}{HP^2}$, $R$, as the measures of the accelerating effects of the forces, and of the pressure of tube outwards.

Then, employing the usual letters for the lines of the figure, the accelerating effect of the tangential component of force to $S$ is

$$\frac{\mu}{SP^2} \cdot \frac{PT}{PQ} = \frac{\mu(SP - SQ)}{SP . SQ . PQ} = \frac{\mu}{PQ . SQ} - \frac{\mu}{PQ . SP} \text{ ultimately;}$$




and similarly for the force tending to $H$;

$$\therefore (\text{vel.})^2 \text{ at } P - (\text{vel.})^2 \text{ at } Q = \left(\frac{2\mu'}{HP} - \frac{2\mu'}{HQ}\right) - \left(\frac{2\mu}{SQ} - \frac{2\mu}{SP}\right);$$

$$\therefore (\text{vel.})^2 \text{ at } P = \frac{2\mu'}{HP} - \frac{2\mu'}{HO} - \frac{2\mu}{SO} + \frac{2\mu}{SP}.$$

Also, $\dfrac{(\text{vel.})^2 \text{ at } P}{\rho} = \left(\dfrac{\mu'}{HP^2} + \dfrac{\mu}{SP^2}\right)\dfrac{PF}{PE} - R$, if $\rho$ be the radius of curvature at $P$, and $2\rho . \dfrac{PF}{PE} = PV = \dfrac{2CD^2}{AC} = \dfrac{2SP.HP}{AC}$;

$$\therefore R.\rho = \frac{\mu'.SP}{AC.HP} + \frac{\mu.HP}{AC.SP} - \frac{2\mu'}{HP} - \frac{2\mu}{SP} + \frac{2\mu'}{HO} + \frac{2\mu}{SO}$$

$$= \frac{2\mu'}{HO} + \frac{2\mu}{SO} - \frac{\mu'}{AC} - \frac{\mu}{AC} = \left(\frac{\mu' SO}{HO} + \frac{\mu HO}{SO}\right)\frac{1}{AC},$$

which is constant; therefore $R$ varies as the curvature.

## XXIII.

1. A body is attached to a point by a thread, and is projected so as to describe a vertical circle, prove that, if $T_1$, $T_2$ be the tensions of the string at the extremities of any diameter, the arithmetic mean between $T_1$, $T_2$ is independent of the position of the diameter, and that $T_2 \sim T_1$ is six times the component of the weight in the direction of the diameter.

2. A string of given length $l$ is capable of sustaining a weight $W$. One end is fixed, and a given weight $P$ less than $W$, attached to the other end, oscillates in a vertical plane, find the greatest arc through which the weight can oscillate without breaking the string.

3. A ring slides on a string hanging over two pegs in the same horizontal line, find the tension of the string at the lowest point, if the ring begin to fall from the point in the horizontal line through the pegs, the string being stretched.

4. $AB$ is the vertical axis of a cycloid, $A$ the highest point, $AM$, $AN$ are the abcissæ of points at which a body begins to slide down the arc of the cycloid, and at which it leaves the curve; prove that $N$ is the middle point of $MB$.

5. If in a central orbit the direction of motion change uniformly, prove that the normal force will vary as the radius of curvature.

6. Given the Sun's motion in longitude at apogee and perigee to be 57′ 10″ and 61′ 10″; find the eccentricity of the Earth's orbit, supposed to be an ellipse about the Sun in one of the foci.

7. Prove that the angular velocity of a projectile about the focus of its path varies inversely as its distance from the focus.

8. A particle, constrained to move on an equiangular spiral, is attracted to the pole by a force proportional to the distance, prove that, at whatever point the particle be placed at rest, the times of describing a given angle about the centre of force will be the same.

9. A body slides down a smooth cycloidal arc, whose axis is vertical and vertex downwards, find the pressure at any point of the cycloid, and shew that, if it fall from the highest point, the pressure at the lowest point will be twice the weight of the body.

10. Find the law of force, tending to the centre, under the action of which a lemniscate can be described.

## XXIV.

1. Two straight lines $AB$ and $BC$ are united at $B$; $AB$ revolves about $A$, and $BC$ about $B$ with the same uniform angular velocity; shew that the acceleration on $C$ tends to $A$ and varies as $CA$.

2. A particle describes an ellipse, the centre of force being situated at any point within the figure. Shew that at the point where the *true* angular velocity is equal to the *mean* angular velocity, the radius vector is a mean proportional between the semiaxes.

3. A particle begins to move from any point of a smooth parabolic tube, being attracted to the focus by a force which varies inversely as the square of the distance; find the greatest pressure.

4. If $SY$ be the perpendicular on the tangent at a point $P$ of an orbit, described about a centre of force $S$, prove that the acceleration at $P$ will be equal to the product of the velocities of $P$ and $Y$ divided by $SY$.

5. A smooth cone is placed with its axis vertical and vertex upwards, shew that there is a certain portion of the surface upon which a particle can describe a circle, if properly projected and acted on by gravity and by a force tending to the vertex and varying as the distance.

6. Shew that the force required for the description of an ellipse about the vertex $A$ varies as $\frac{AP}{AN^3}$, where $PN$ is the perpendicular on the axis.

7. If a particle describe an ellipse under the action of a force tending to any fixed point $O$, the force will vary as $\frac{DD'^6}{OP^2 . PP'^3}$, where $P$ is the position of the particle, $PP'$ the chord through $O$, and $DD'$ the diameter parallel to this chord.

8. Shew that in the elliptic orbit described under the action of a force tending to a focus, the angular velocity round the other focus varies inversely as the square of the diameter parallel to the direction of motion.

9. A particle moves in a circular tube, under the action of a force which tends to a point in the tube, and whose accelerating effect varies as the distance, shew that, if the particle begin to move from a point at a distance from the centre of force equal to the radius, there will be no pressure on the tube at an angular distance from the centre of force equal to $\cos^{-1}\frac{2}{3}$.

10. A particle moves in a smooth elliptic groove, under the action of two forces tending to the foci and varying inversely as the squares of the distances, the forces being equal at equal distances. Prove that, if the velocity at the extremity of the axis major be to that at the extremity of the axis minor as $AC$ to $BC$, then the velocity at any point will vary inversely as the normal; find the pressure on the tube.

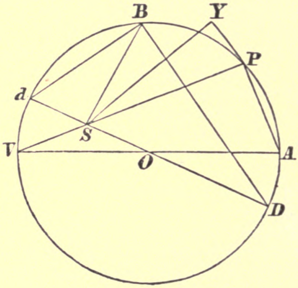
11. Determine the relation between $\mu$ and $\lambda$ and the velocity of projection, in order that an ellipse may be described under the action of forces $\frac{\mu}{SP^2}$, $\frac{\mu}{HP^2}$ to the foci and $\lambda . CP$ to the centre, acting simultaneously.

12. A particle is attached to a point $C$ by a string, and is attracted by a force which tends to a point $S$, and varies inversely as the square of the distance from $S$. Find the least velocity with which the particle can be projected from a point in $CS$, or $CS$ produced, so as to describe a complete circle. If $CS$ be less than the length of the string, prove that the tension will be a maximum at a point $D$, where $SD$ is perpendicular to $CS$, and that if $CS$ be half the length of the string, the two minimum and the maximum tensions will be as 0, 4 and $3\sqrt{3}$.

## PROP. VII. PROBLEM II.

*A body moves in the circumference of a circle, to find the law of the centripetal force, tending to any given point in the plane of the circle.*

Let $APV$ be the circumference of the circle, $S$ the given point to which the centripetal force tends, $PV$ the




chord of the circle drawn through $S$ from $P$, the position of the body at any time, and $VOA$ the diameter through $V$. Join $PA$, and draw $SY$ perpendicular to $PY$, the tangent to the curve at $P$.

By Prop. VI. Cor 3, if $F$ be the measure of the accelerating effect of the centripetal force, $F = \dfrac{2h^2}{SY^2.PV}$, and, since the angles $SPY$, $VAP$ are equal, and also the right angles $PYS$, $APV$, the triangles $SPY$, $VAP$ are similar, and $SY : SP :: PV : VA$;

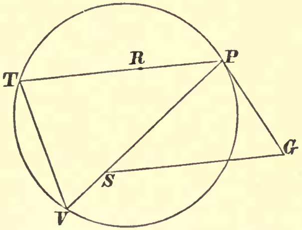
$$\therefore F = \frac{2h^2.VA^2}{SP^2.PV^3};$$

therefore, since $h$ and $VA$ are given, $F$ varies inversely as $SP^2.PV^3$.

Cor. 1. Hence, if the given point $S$ to which the centripetal force tends, be situated on the circumference of the circle, $V$ will coincide with $S$, and $F$ vary inversely as $SP^5$.

Cor. 2. The force, under the action of which a body $P$ revolves in a circle $APTV$, is to the force, under the action of which the same body $P$ can revolve in the same circle in the same periodic time about any other centre of force $R$, as $RP^2.SP$ to $SG^3$, $SG$ being a straight line drawn from the first centre $S$, parallel to the distance $RP$ of the body from the second centre of force $R$, to meet $PG$, a tangent to the circle.

For, by the construction of this proposition, since the periodic times are the same, the areas described in




a given time are the same; therefore, $h$ is the same for both centres, hence, if $PRT$ be the chord through $R$, the force to $S$ : the force to $R$ :: $RP^2.PT^3$ $SP^2.PV^3$; but, by similar triangles $TPV$, $GSP$, $PT : PV :: SP : SG$; therefore force to $S$ : force to $R$ :: $RP^2.SP^3 : SP^2.SG^3 :: RP^2SP : SG^3$.

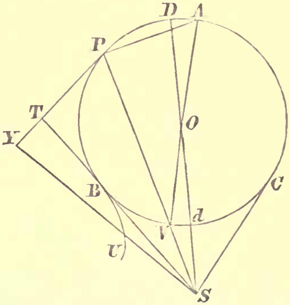
Cor. 3. The force, under the action of which a body $P$ revolves in any orbit about a centre of force $S$, is to the force, under the action of which the same body $P$ can revolve in the same orbit in the same periodic time about any other centre of force $R$, as

$RP^2.SP$ to $SG^3$, $SG$ being the straight line drawn from the first centre of force $S$, parallel to $RP$ the distance of $P$ from the second centre of force $R$, to meet $PG$ the tangent to the orbit.

For, in each case, the body may be supposed for a short time to be moving in the circle of curvature, and the forces are the same as those which would retain the body in the circular orbit; therefore, since the areas described in a given time are equal, the ratio of the forces is $RP^2.SP : SG^3$.

*Observations on the Proposition.*

185. In the figure employed in the proposition, the force is supposed to be attractive, but the investigation of the law of force applies also to the case in which the centre of force



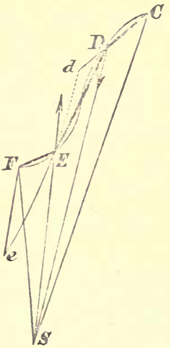

$S$ is exterior to the circle, in which case the force is repulsive through the arc $BC$, which is convex to the centre of force, and contained between the tangents drawn from $S$ to the circle.

It is important, however, to observe that this problem is to find what would be the law of force tending to $S$, under the action of which a body would be moving, supposing that it could move in the circle, or any portion of the circle, under the action of such a force, but it does not assert the possibility of such a motion, which is considered in Art. 165.

In fact, the *complete* description of a circle $ABC$, under the sole action of a central force tending to an external point $S$, is impossible, because, as the body approaches the point $B$, the component of the velocity perpendicular to $SB$ remains finite however near the body approaches $B$, and since there is no force to generate a velocity in the opposite direction, the body must proceed to describe an arc $BU$ on the opposite side. $SB$ would be a tangent to both curves, because the velocity in direction $BS$ becomes larger than any finite quantity, as the body approaches $B$, and therefore the angle between $BS$ and the direction of motion is indefinitely small at $B$.

That a finite velocity in the direction perpendicular to $SB$ could remain up to $B$, may be shewn by producing $SB$ to $T$ in the tangent $PY$ at $P$; then the component of the velocity at $P$ perpendicular to $SB$ is $\frac{h}{SY} \cdot \frac{SY}{ST} = \frac{h}{ST} = \frac{h}{SB}$, when the body arrives at a point very near to $B$.

186. The force at a point indefinitely near to $B$ cannot be properly determined by the method of Prop. VI., because the lines parallel to the direction of the force from which the mea-




sures of the force are obtained are not subtenses, or sagittæ, since they are in this case not inclined at a finite angle to the tangent.

But it can be seen in another manner from the polygon of Prop. I, that the force is infinitely great, when the distance from $B$ becomes infinitely small.

Thus, if $CDEF$ be a portion of the polygon whose limit touches the radius from $S$ between $D$ and $E$, the angle between $DE$ and $DS$ or $ES$ may be made as small as we please compared with the angle between $CD$ and $DE$, hence the velocity generated by the impulse in the directions $DS$ and $SE$ will become infinitely great compared with the velocities in $CD$ and $EF$. In the figure, the impulses at $D$ and $E$, whose directions are denoted by the arrows, have corresponding to them in the limit the forces on opposite sides of the tangent, which are attractive and repulsive respectively.

187. COR. 1. For the reasons given above, a limitation should be made, viz., when $P$ is at a finite distance from $S$. In this case $PV = SP$ and $F = \frac{8h^2R^2}{SP^5}$, $R$ being the radius of the circle.

We may also observe here that the *possibility* of a description of a circle is not asserted, but only the law of force required *in case of* description of any portion of the circle. The complete description of the single circle is, in fact, impossible, for, under the action of the force obtained, the body would pass to the other side of the tangent on arriving at $S$, then proceed to describe another equal circle, and, on arriving again at $S$, return into the original circle.

188. COR. 2. The orbit being the same, and also the periodic times about $S$ and $R$ being equal, the value of $h$, in the two cases, is the same; also, the force tending to $S$ for the orbit being of the same magnitude at $P$ as that under the action of which the circle of curvature would be described, and $SY$, $PV$ being the same in the orbit and the circle, $h$ is also the same, Prop. VI. Cor. 3; and, similarly, $h$ is the same in the circle and orbit described about $R$; therefore it is the same in the circle described about $S$ and $R$ as centres of force, and hence Cor. 2 applies.

*Absolute Force.*

189. If the force upon a body placed at any distance from the point $S$ vary inversely as the $n$th power of that distance, the magnitude of the force, or its ratio to any given force, as that of gravity, will be determined when the distance $SP$ is given. The measure of the accelerating effect of the force is written $\frac{\mu}{SP^n}$, where $\mu$ the constant part of this measure is an algebraical symbol of $n+1$ dimensions in linear space. If the unit of space $= a$, $\frac{\mu}{a^n}$ is the measure of the accelerating effect of the force on a body at an unit of distance, and $\mu$ is called the *Absolute Force*, being the measure of the accelerating effect of the force at an unit of distance $\times$ the $n$th power of that unit. The absolute force is not the measure of the accelerating effect of any force, unless the symbols be treated numerically, in which case $\mu$ is twice the number of units of space through which a constant force, equal to the force at an unit of distance, would draw a body from rest in an unit of time.

*Law of Force in a Circular Orbit.*

190. The law of force may be expressed in terms of the distance $SP$, for $SD$, $Sd$ being the greatest and least distances of the body from $S$, $SD.Sd = SP.SV$; see figure, page 188.

$$\therefore SP.PV = SP^2 \pm SD.Sd,$$

$+$ or $-$ according as $S$ is within or without the circle;

$$\therefore F = \frac{2h^2.AV^2.SP}{(SP^2 \pm SD.Sd)^3}.$$

If $S$ be on the circumference $Sd = 0$, therefore $F = \frac{2h^2.AS^2}{SP^5}$.

If $S$ be exterior to the circle, $SD.Sd = SB^2$, and the lower sign must be taken; therefore $F = \frac{2h^2AV^2.SP}{(SP^2 - SB^2)^3}$.

*Velocity in the Circular Orbit.*

191. *To find the velocity in the circular orbit described under the action of a force tending to any point in the plane of the orbit.*

$$\text{The velocity at } P = \frac{h}{SY} = \frac{h}{SP} \cdot \frac{SP}{SY} = \frac{h}{SP} \cdot \frac{VA}{PV} \propto \frac{1}{SP.PV}$$

COR. If $S$ be in the circumference of the circle, and $\frac{\mu}{SP}$ be the accelerating effect of the force, $\mu = 2h^2 SA^2$;

$$\text{hence the velocity at } P = \frac{h.VA}{SP^2} = \left(\frac{\mu}{2}\right)^{\frac{1}{2}} \cdot \frac{1}{SP^2}.$$

Or, we may employ the result of Prop. VI., Cor. 4,

$$V^2 = F \cdot \frac{PV}{2} = \frac{\mu}{SP^5} \cdot \frac{SP}{2}; \quad \therefore V = \left(\frac{\mu}{2}\right)^{\frac{1}{2}} \cdot \frac{1}{SP^2} \propto \frac{1}{SP^2}.$$

*Periodic Time.*

192. *To find the periodic time in a circular orbit described under the action of a force tending to a point in the circumference.*

Let $P$ be the periodic time, $R$ the radius of the circle, and let $\frac{\mu}{SP^5}$ be the measure of the accelerating effect of the force at $P$, then $h.P$ = twice the area of the circle $= 2\pi R^2$,

$$\text{and } \mu = 2h^2 AS^2 = 8h^2 R^2; \; \therefore P = \frac{4\sqrt{2}\pi R^3}{\mu^{\frac{1}{2}}}.$$

193. *To compare the periodic times in the same circle when described under the action of a force tending to a point in the circumference, and a force tending to the centre, of the same magnitude as that of the first force at a distance equal to the radius of the circle.*

Let $P'$ be the periodic time, and $V$ the uniform velocity in the circle in the second case, $V^2 = \frac{\mu}{R^5} \cdot R$; $\therefore V = \frac{\mu^{\frac{1}{2}}}{R^2}$,

$$\text{and } P' \cdot V = 2\pi R; \; \therefore P' = \frac{2\pi R^3}{\mu^{\frac{1}{2}}} = \frac{P}{2\sqrt{2}}.$$

*Illustrations.*

(1) *When the force in a circular orbit tends to a point within the circle, to find the point at which the true angular velocity is equal to the mean angular velocity.*

The true angular velocity $= \dfrac{h}{SP^2}$, the mean $= \dfrac{2\pi}{P} = 2\pi \cdot \dfrac{h}{2\pi R^2}$; therefore at the required point $SP = R$, or the perpendicular from the required point upon the line joining $S$ to $O$ the centre of the circle bisects $OS$.

(2) *A body describes a circle under the action of a force tending to a point within it, the measure of whose accelerating effect at the greatest and least distances SD and Sd are the radius and twice the diameter respectively, the unit of time being a second; find the number of seconds in passing from D to d.*

$$\text{Since } \frac{8h^2R^2}{SD^2 . Dd^3} = R, \ \frac{8h^2R^2}{Sd^2 . Dd^3} = 4R;$$

$$\therefore SD = 2Sd, \text{ and } 3Sd = Dd = 2R; \ \therefore h = 2R . Sd = \tfrac{4}{3}R^2;$$

and the number of seconds from $D$ to $d = \dfrac{\pi R^2}{h} = \dfrac{3\pi}{4}$.

## XXV.

1. If $\mu$ be the absolute force in a circular orbit described under the action of a force tending to a point in the circumference, prove that the time in a quadrant commencing from the extremity of the diameter through the centre of force will be $(\pi + 2) R^3 (\frac{1}{2}\mu)^{-\frac{1}{2}}$.
In what unit of time is the result expressed?

2. A point describes a circle, with an acceleration tending to any point within the circle. Prove that, if three points be taken at which its velocities are in harmonical progression, the velocities at the other extremities of the diameters, passing through those points, will also be in harmonical progression.

3. In the case of a centre of force $S$ within a circle, if two points $L$, $M$ be taken, such that $LS$, $MS$ make equal angles with the diameter through $S$, and on the same side of it, then the forces at $L$ and $M$ will be to each other in the inverse ratio of the squares on $OL$ and $OM$.

4. The sum of the reciprocals of the velocities at the extremities of any diameter is independent of the position of the centre of force, and varies as the periodic time.

5. Prove that, when a circular orbit is described about an internal point, the sum of the square roots of the accelerations at the extremities of any chord passing through that point varies inversely as the square root of the length of the chord.

6. **Prove that, if the law of force tending to $S$, a point without a circle, be the law of force under which part of the circle can be described, the body will move near $B$ as if acted on by a force tending to $B$ and varying inversely as the cube of the distance from $B$.**

7. $OE$ is a radius perpendicular to the diameter through $S$ in a circular orbit about a central force tending to a point $S$ within the circle, $SB$ an ordinate, perpendicular to $OS$, shew that, if the force at $B$ be an arithmetic mean between the forces at the greatest and least distances, $OE^3 = SB \cdot SE^2$.

8. Prove that, if a circle be described about a force tending to a point in the circumference, and $PQ$ be a chord parallel to the diameter through that point, the times of describing equal small arcs near $P$ and $Q$ will differ by a quantity which varies as $PQ$.

9. When a particle is describing a circle under the action of a central force, shew that at every instant the angular velocities about all points in the circumference are the same.

10. The period in an orbit described under the action of a central force, whose accelerating effect is $\mu r^n$ is given to be $\lambda a^m \div \mu^{\frac{1}{2}}$, $a$ being a line and $\lambda$ a number, find $n$.

11. Apply the proposition contained in Cor. 3, to prove that if in an elliptic orbit described under the action of a force tending to the centre, the force vary as the distance from the centre, then the force tending to the focus will vary inversely as the square of the focal distance.

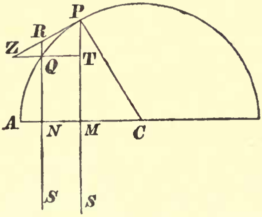
12. Deduce, by Cor. 3, the law of force, when a parabola is described under the action of a force tending to the focus, from the constant force parallel to the axis, under the action of which the same parabola may be described.

13. Shew, by the method of projections, that the centripetal force at any point $P$ tending to a fixed point $O$ in the axis major of an ellipse under which the ellipse can be described, varies as $\left(\frac{OQ}{PQ}\right)^3 \cdot OP$, $POQ$ being the chord of the ellipse through $O$.

## PROP. VIII. PROBLEM III.

*A body moves in a semicircle $PQA$ under the action of a force tending to a point $S$ so distant that the lines $PS$, $QS$ drawn from the body to that point may be considered parallel; to find the law of force.*

Let $CA$ be a semidiameter of the semicircle drawn from the centre perpendicular to the direction in which the force acts, cutting $PS$, $QS$ in $M$ and $N$, and join $CP$.




Let $PRZ$ be the tangent at $P$, $ZQT$ perpendicular to $PMS$, meeting $PRZ$ in $Z$, and let $SNQ$ meet $PRZ$ in $R$.

Then the force at $P = \frac{2h^2 QR}{SP^2 . QT^2}$ ultimately, if the arc $PQ$ be indefinitely diminished, and $SP$ may be considered constant; also, by Euclid III. 36,

$$QR . (RN + QN) = RP^2,$$

and, since $RQ$ is parallel to $PT$, and the triangles $PZT$, $CPM$ are similar,

$$RP : QT :: ZP : ZT :: CP : PM;$$

$$\therefore \frac{QT^2}{QR} = \frac{QT^2}{RP^2} . \frac{RP^2}{QR} = \frac{PM^2}{CP^2} . (RN + QN)$$

$$= \frac{2PM^3}{CP^2} \text{ ultimately};$$

hence force at $P = \frac{h^2 CP^2}{SP^2 . PM^3} \propto \frac{1}{PM^3}$.

*Aliter.*

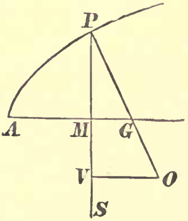
In fig. page 190 draw $OE$ a semidiameter perpendicular to $SD$, and let the distance $SP$ cut the circle in $V$, and $OE$ in $M$, then, by the preceding proposition, $F=\frac{8h^2R^4}{SP^2.PV^3}$, and, if $S$ be very distant, the ratio $PM : SM$ or $SO$ will vanish; therefore, $SP=SO$ ultimately, and $PV$ is ultimately perpendicular to $OE$ and equal to $2PM$;

$$\therefore F=\frac{h^2R^4}{SO^2.PM^3}\propto\frac{1}{PM^3}.$$

## SCHOLIUM.

A body moves in an ellipse, hyperbola or parabola, under the action of a force tending to a point so situated and so distant that the lines drawn from the body to that point may be considered parallel, and perpendicular to the major axis of the ellipse, the axis of the parabola or the transverse axis of the hyperbola. To shew that the force varies inversely as the cube of the ordinates.

Let $AMG$ be the axis to which the direction of the forces may be considered perpendicular, $PM$, $PG$




the ordinate and normal, $PO$ the diameter of curvature, and $PV$ the chord of curvature in direction $PS$.

$$\text{Then } F = \frac{2h^2}{SY^2.PV} = \frac{2h^2}{SP^2.PV} \cdot \frac{PG^2}{PM^2}, \because \frac{SP}{SY} = \frac{PG}{PM},$$

$$\therefore F \propto \frac{PG^2}{PM^2.PV} \propto \frac{PG^3}{PM^3.PO} \propto \frac{1}{PM^3};$$

since $PO \propto PG^3$, Art. 84.

*Observations on the Proposition.*

194. It has been shewn in Art. 151, that the equable description of areas may, in the case of forces acting in parallel lines, be replaced by the uniformity of the resolved part of the velocity in the direction perpendicular to that of the forces. In the proof given in the text, when $S$ is removed to an infinite distance, $h$ and $SP$ are both infinite magnitudes, but the expression $\frac{h}{SP}$ is finite, for area $SPQ$ described in the time $T$ is ultimately equal to area $SMN$, whose base is equal to $uT$, $u$ being the component of the velocity perpendicular to the direction of the forces; therefore $hT = uT.SP$ ultimately, and $\frac{h^2}{SP^2} = u^2$, hence the acceleration due to the force, when a body describes the semicircle, is $\frac{u^2R^2}{PM^3}$.

195. The accelerating effect of the force, acting in parallel lines, may be obtained directly from the proposition of Art. 151, as follows.

Let $u$ be the constant component of the velocity $V$, perpendicular to the direction of the force, and let $F$ be the accelerating effect of the force, therefore $F = \frac{2V^2}{PV} = \frac{V^2}{PM}$;

$$\text{also } V \,.\, u :: CP : PM; \;\therefore\; F = \frac{u^2.CP^2}{PM^3}.$$

*Extension of Scholium.*

196. *When a body describes any curve under the action of a force tending to a point S, so distant that the lines drawn from S*

*to the body may be considered parallel; to find the law of force and the velocity at any point.*

Let $AP$ be any curve, $AMG$ the line to which the forces are perpendicular, $PM$, $PG$ the ordinate and normal at the point $P$, $PV$ the chord of curvature in the direction of the force, $PO$ the diameter of curvature.

Let $F$ be the accelerating effect of the force at $P$, $u$ the component of the velocity $V$ in the direction $AMG$;

$$\therefore\ V : u :: PG : PM,$$

$$\text{also } PV : PO :: PM : PG;$$

$$\therefore\ F = \frac{2V^2}{PV} = \frac{2u^2.PG^2}{PM^2.PO} \cdot \frac{PO}{PV} = \frac{2u^2.PG^3}{PO.PM^3},$$

$$\text{and the velocity} = u.\frac{PG}{PM}.$$

*Illustrations.*

(1) *A cycloid is described by a particle, under the action of a force acting in a direction parallel to the axis; find the acceleration and the velocity at any point.*

In the cycloid $PO = 4PG$, and $PM.AB = PG^2$, $AB$ being the length of the axis;

$$\therefore\ F = \frac{2u^2.PG^2}{PM^3} \cdot \frac{PG}{PO} = \frac{u^2.AB}{2PM^2} \propto \frac{1}{PO^4},$$

$$\text{and the velocity at } P = u.\frac{PG}{PM} = u.\frac{AB}{PG} \propto \frac{1}{PO}.$$

(2) *A particle moves in a catenary under the action of forces acting in vertical lines; find the accelerating effect of the force and the velocity at any point.*

Let $AM$ be the directrix, $AB$ the ordinate at the lowest point.

$$\text{Then } PG : PM :: PM : AB \text{ and } PO = 2PG;$$

$$\therefore\ F = \frac{2u^2.PG^3}{PO.PM^3} = \frac{u^2.PM}{AB^2} \propto PM \propto PO^{\frac{1}{2}},$$

$$\text{and the velocity at } P = u.\frac{PG}{PM} = u.\frac{PM}{AB} \propto PM.$$

## XXVI.

1. A body is moving in a semicircle under the action of a force tending to a point, so distant that the lines drawn from the body to that point may be considered parallel; if the centre of force be transferred to the centre of the circle, when the direction of the body's motion is perpendicular to that of the force, its magnitude at that point being unaltered, prove that the body will continue to move in the circle.

2. If a cycloid be described under the action of forces in the direction of the base, the force at any point will vary inversely as $AM . MQ$; $AM$, $MQ$ being the abscissa and ordinate of the corresponding point of the generating circle.

3. A catenary is described under the action of a horizontal force, prove that the force varies as the distance from the directrix directly, and the cube of the arc from the lowest point inversely.

4. If the same parabola be described by particles when the force tends to the focus, and when it is parallel to the axis, the velocities will be equal at the points at which the forces are equal.

5. A parabola having its vertex at $A$ and its axis coincident with $AB$ the diameter of a semicircle, is described so as to cut the semicircle in $P$; prove that, if a body move in the semicircle under the action of a force perpendicular to $AB$, the time of moving from $A$ to $P$ will vary as the difference between $AB$ and the latus rectum. Prove also, that if a second body move from $A$ to $P$ in the parabola in the same time under the action of a force perpendicular to its axis, and the velocities in the two curves at $P$ be equal, the latus rectum of the parabola will be $\frac{1}{3}AB$.

## PROP. IX. PROBLEM IV.

*If a body revolve in an equiangular spiral, required the law of centripetal force tending to the pole of the spiral.*

Draw $SY$ from $S$, the pole of the spiral, perpendicular to the tangent $PY$, and let $PV$ be the chord of curvature at $P$, whose direction passes through $S$; then $F'$, the measure of the accelerating effect of the force tending to the pole, is $\dfrac{2h^2}{SY^2.PV}$; but, if $\alpha$ be the angle of the spiral, $SY=SP\sin\alpha$ and $PV=2SP$, Art. 128;

$$\therefore\ F=\frac{h^2}{\sin^2\alpha\, SP^3}=\frac{\mu}{SP^3}\propto\frac{1}{SP^3}.$$

197. *To find the velocity of a body describing an equiangular spiral under the action of a force tending to the pole.*

If $\dfrac{\mu}{SP^3}$ be the accelerating effect of the force tending to $S$,

$$V^2=F.\tfrac{1}{2}PV=\frac{\mu}{SP^3}.SP;\ \therefore\ V=\frac{\mu^{\frac{1}{2}}}{SP}.$$

198. *To find the time of describing any arc of the equiangular spiral.*

Let $AL$ be any arc, $SA$, $SL$ bounding radii, $P$ the time of describing the arc. Then area $SAL=\frac{1}{4}(SA^2\sim SL^2)\tan\alpha$, Art. 127;

$$\therefore\ P=\frac{2\times\text{area}\,SAL}{h}=\frac{SA^2\sim SL^2}{2h}\tan\alpha=\frac{SA^2\sim SL^2}{2\mu^{\frac{1}{2}}\cos\alpha}.$$

199. *In any orbit, described under the action of a force tending to any point $S$, when the angle between the tangent $PY$ and the radius $SP$ is a maximum or minimum, the velocity is equal to the velocity in a circle at the same distance about the same force in the centre.*

For, the curve, near this point, may be considered an equiangular spiral ultimately, since the angle is constant for a short time; therefore the chord of curvature is $=2SP$, and $V^2=F.SP$

## XXVII.

1. In different equiangular spirals, described under the action of forces tending to the poles which are equal at equal distances, shew that the angular velocity varies at any point as the force and the perpendicular on the tangent conjointly.

2. The angular velocity of the perpendicular on the tangent is equal to that of the radius.

3. The velocity of approach towards the focus, called the paracentric velocity, varies inversely as the distance.

4. A body is describing a circle, whose radius is $a$, with uniform velocity, under the action of a force, whose accelerating effect at any distance $r$ is $\frac{\mu}{r^3}$. Prove that, if the direction of its motion be deflected inwards through any angle $\beta$ without altering the velocity, the body will arrive at the centre of force after a time $\frac{a^2}{2\mu^{\frac{1}{2}} \sin\beta}$.

5. Deduce from the time in an equiangular spiral the time of passing from one point to another, when a body moves along a straight line with a velocity which varies inversely as the distance from a fixed point in that line.

6. A body describes an equiangular spiral in a resisting medium with uniform angular velocity under the action of a force tending to the pole; prove that the force to the pole varies as the distance and the resistance as the velocity.

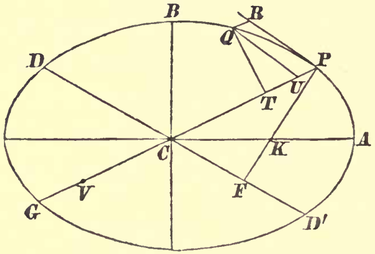
7. Two particles of equal mass $m$, and at a distance $2a$ apart, are projected simultaneously with velocity $V$ in the same direction perpendicular to the line joining them, the only force acting is a mutual force of attraction varying inversely as the cube of the distance between the particles, and equal at the distance $2a$ to $mf$. Prove that, if after a time $\sqrt{\left(\frac{a}{f} \cdot \frac{V^2 - 2af}{V^2 - af}\right)}$ one of the particles be stopped and kept at rest, the other will proceed to describe an equiangular spiral about it as pole.

8. Three particles $A$, $B$, $C$ start from rest and move with uniform velocities, $A$ always directing its course towards $B$, $B$ towards $C$, and $C$ towards $A$. Prove that if their velocities be proportional to $b^2c$, $c^2a$, $a^2b$, where $a$, $b$, $c$ are the initial distances of $B$ from $C$, $C$ from $A$, and $A$ from $B$ respectively, they will describe similar equiangular spirals with a common pole.

## PROP. X. PROBLEM V.

*If a body be revolving in an ellipse, to find the law of centripetal force tending to the centre of the ellipse.*

Let $CA$, $CB$ be the semiaxes of the ellipse, $P$ the position of the body at any time, $PCG$, $DCD$ conjugate diameters, $Q$ a point near $P$, $QT$, $PF$ perpendiculars from $Q$ and $P$ on $PC$, $DD'$; draw $QU$ an ordinate to $PCG$, $QR$ a subtense parallel to $CP$.




$$\text{Then } F = \frac{2h^2}{CP^2} \cdot \frac{QR}{QT^2} \text{ ultimately.}$$

But, by similar triangles $QTU$, $PFC$,

$$\frac{QT^2}{QU^2} = \frac{PF^2}{CP^2}, \text{ and } \frac{QU^2}{PU.UG} = \frac{CD^2}{CP^2};$$

$$\therefore \frac{QT^2}{PU.UG} = \frac{PF^2.CD^2}{CP^4} = \frac{AC^2.BC^2}{CP^4},$$

$$UG = 2CP \text{ ultimately, and } PU = QR;$$

$$\therefore \frac{QT^2}{2QR} = \frac{AC^2.BC^2}{CP^3} \text{ ultimately;}$$

$$\therefore F = \text{limit of } \frac{2h^2.QR}{CP^2.QT^2} = \frac{h^2.CP}{AC^2.BC^2} \propto CP;$$

therefore the force is proportional to the distance from the centre.

*Aliter.*

Let $CY$ be perpendicular on the tangent at $P$, and $PV$ be the chord of curvature at $P$ which passes through the centre $=\frac{2CD^2}{CP}$, Art. 79.

Then $F=\frac{2h^2}{CY^2.PV}=\frac{h^2.CP}{PF^2.CD^2}=\frac{h^2}{AC^2.BC^2}.CP=\mu.CP.$

COR. 1. And conversely, if the force be as the distance, a body will revolve in an ellipse having its centre in the centre of force, or in a circle, which is a particular kind of ellipse.

COR. 2. And the periodic times will be the same in all ellipses described by bodies about the same centre of force.

For the periodic time in any ellipse

$$=\frac{2 \times \text{area of ellipse}}{h}=\frac{2\pi AC.BC}{h},$$

and the forces, at different distances in the same or different ellipses, vary as the distance; therefore $\frac{h^2}{AC^2.BC^2}=\mu$ is the same in different ellipses, therefore the periodic times in different ellipses is the same, and $=\frac{2\pi}{\sqrt{\mu}}$.

## SCHOLIUM.

If the centre of an ellipse be supposed at an infinite distance, the ellipse will become a parabola, and the body will move in this parabola; and the force, now tending to a centre at an infinite distance, will be constant and act in parallel lines. This theorem is due to Galileo. And, if the parabola be changed into an hyperbola, by the change of inclination of the plane cutting the cone, the body will move in this hyperbola under the action of a repulsive force tending from the centre.

200. *To find the velocity in the elliptic orbit under the action of a force tending to the centre, the measure of whose accelerating effect is $\mu \times$ distance.*

The velocity at $P = \frac{h}{CY} = \frac{h.CD}{PF.CD} = \frac{h.CD}{AC.BC} = \sqrt{\mu}.CD.$

*Aliter.*

$$(\text{Vel.})^2 \text{ at } P = F.\frac{PV}{2} = \mu.CP.\frac{CD^2}{CP}; \quad \therefore \text{ vel. at } P = \sqrt{\mu}.CD.$$

201. *If a hyperbolic orbit be described under the action of a repulsive force tending from the centre, the force will vary as the distance, and the velocity at any point as the diameter of the conjugate hyperbola parallel to the tangent at the point.*

This may be proved exactly as in the case of the ellipse, employing the proper figure.

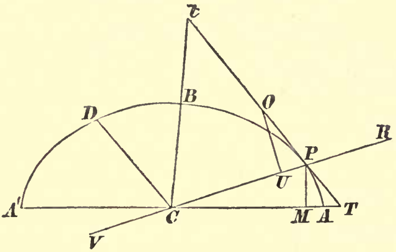
202. *To find the time in any arc of an elliptic orbit about a force tending to the centre.*

If $P$ be any point of the orbit, $Q$ the corresponding point in the auxiliary circle, time in $AP \propto$ area $ACP \propto$ area $ACQ \propto \angle ACQ$; therefore time in $AP$ : periodic time :: $\phi : 2\pi$, if $\phi$ be the circular measure of $\angle ACQ$, and periodic time $= \frac{2\pi}{\sqrt{\mu}}$; therefore time in $AP = \frac{\phi}{\sqrt{\mu}}$.

203. *If, at a given point, the velocity of a body be known, and the direction of its motion; to determine the curve which the body will describe under the action of a given centripetal force, which varies as the distance from the point to which it tends.*

Let $Pt$ be the direction of motion at $P$, $V$ the velocity at $P$, $\mu.CP$ the measure of the accelerating effect of the force tending to $C$. On $PC$ produced, if necessary, take $PV$ equal to four times the space through which a body must move from rest, under the action of the force at $P$ continued constant, in order to acquire the given velocity $V$; so that $V^2 = 2\mu CP.\frac{1}{4}PV$.

Draw $CD$ parallel to $Pt$, a mean proportional to $CP$ and $\frac{1}{2}PV$, and let an ellipse be constructed with $CP$, $CD$ as semi-conjugate diameters, then $PV$ is the chord of curvature at $P$ through $C$.




In this ellipse let a body revolve under the action of a force tending to $C$, whose magnitude at $P$ is that of the given force, see Arts. 160, 162, then, when it arrives at the point $P$, it will be moving in the direction $Pt$, also the square of the velocity at $P = \mu . CD^2 = \mu . CP . \frac{1}{2}PV = V^2$, or the velocity at $P$, in the constructed ellipse, is $V$. Hence the body revolving in this ellipse is under the same circumstances as the proposed body, in all respects which can influence the motion of a body; therefore the proposed body will describe the ellipse constructed as above.

A direct solution of the problem, which is solved synthetically in this Article, is given in pages 78 and 79.

204. *Geometrical construction for the position and magnitude of the axes of the elliptic orbit, described by a body about the centre, when the velocity at a given point is known, and also the direction of motion.*

Produce $CP$ to $R$, making $PR$ a third proportional to $CP$ and $CD$; bisect $CR$ in $U$, and draw $UC$ perpendicular to $CR$, meeting the tangent at $P$ in $O$, and with centre $O$ describe a circle passing through $C$, $R$, and cutting the tangent in $T$ and $t$;

$$\therefore\ PT . Pt = CP . PR = CD^2;$$

Let $TC$ intersect the ellipse in $A$, $A'$, and draw $PM$ parallel to the diameter conjugate to $ACA'$;

$$\text{then } PT^2 : CD^2 :: TA.TA' : CA^2$$
$$:: CT^2 - CA^2 : CA^2;$$
$$\therefore\ PT^2 : PT.Pt :: CT^2 - CT.CM : CT.CM;$$
$$\therefore\ PT : Pt :: MT : CM;$$

hence $CT$ is parallel to $PM$, and $CT$, $Ct$ are in the directions of conjugate diameters; but $TCt$ is a right angle, therefore $CT$, $Ct$ being in the direction of perpendicular conjugate diameters, are the directions of the axes of the ellipse, and if $PM$, $Pm$ be perpendiculars from $P$ upon these directions, the semiaxes are mean proportionals between $CM$, $CT$, and $Cm$, $Ct$. Q.E.F.

205. *Equations for determining the position and dimensions of the orbit.*

Let $\mu.R$ be the measure of the accelerating effect of the force at the distance $CP = R$, $V$ the velocity, $\alpha$ the angle between $CP$ and the direction of motion at the given point $P$. Let $a$, $b$ be the semiaxes of the ellipse, $\varpi$ the angle which $CP$ makes with the major axis.

$$\text{Then } V^2 = \mu.CD^2 \text{ and } CD^2 + CP^2 = a^2 + b^2;$$

$$\therefore\ a^2 + b^2 = \frac{V^2}{\mu} + R^2 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\text{Also } V.R\sin\alpha = h = \sqrt{\mu}.ab;$$

$$\therefore\ ab = \frac{V.R\sin\alpha}{\sqrt{\mu}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and, by the properties of the ellipse,

$$\frac{R^2}{a^2}\cos^2\varpi + \frac{R^2}{b^2}\sin^2\varpi = 1 \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The equations (1), (2), and (3) determine $a$, $b$, and $\varpi$, whence the magnitude and position of the ellipse is determined.

We can obtain an equation for $\varpi$, immediately in terms of the data, as follows:

$$\left(\frac{R^2}{b^2} - 1\right)\sin^2\varpi = \left(1 - \frac{R^2}{a^2}\right)\cos^2\varpi, \text{ by (3)},$$

$$\frac{R^2}{a^2} + \frac{R^2}{b^2} = \operatorname{cosec}^2\alpha\left(1 + \frac{\mu R^2}{V^2}\right), \text{ by (1) and (2)},$$

$$\frac{R^4}{a^2b^2} = \operatorname{cosec}^2\alpha . \frac{\mu R^2}{V^2}, \text{ by (2)},$$

$$\therefore \left(\frac{R^2}{b^2} - 1\right)\left(1 - \frac{R^2}{a^2}\right) = \cot^2\alpha;$$

$$\therefore \frac{\cos^2\varpi}{\dfrac{R^2}{b^2} - 1} = \frac{\sin^2\varpi}{1 - \dfrac{R^2}{a^2}} = \frac{\sin\varpi \cos\varpi}{\cot\alpha}$$

$$= \frac{\cos^2\varpi - \sin^2\varpi}{\operatorname{cosec}^2\alpha\left(1 + \dfrac{\mu R^2}{V^2}\right) - 2};$$

$$\therefore \cot 2\varpi = \frac{1}{2}\tan\alpha\left(\cot^2\alpha - 1 + \operatorname{cosec}^2\alpha . \frac{\mu R^2}{V^2}\right)$$

$$= \cot 2\alpha + \operatorname{cosec} 2\alpha . \frac{\mu R^2}{V^2} \quad \text{..................(4)};$$

whence $\varpi$ is known immediately from the initial circumstances of the motion.

206. If the force be repulsive, the equations for determining $a$, $b$, $\varpi$ will be

$$a^2 - b^2 = R^2 - \frac{V^2}{\mu} \quad \text{......................(1)},$$

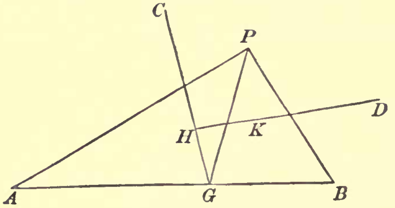
$$ab = \frac{VR\sin\alpha}{\sqrt{\mu}} \quad \text{........................(2)},$$

$$\text{and } \frac{R^2}{a^2}\cos^2\varpi - \frac{R^2}{b^2}\sin^2\varpi = 1 \quad \text{................(3)}.$$

The direction and magnitude of the axes of the hyperbola may be determined geometrically, by observing that the asymptotes are the diagonals of the parallelograms of which the conjugate semi-diameters are sides, and that the axes bisect the angles between the asymptotes.

207. *When a particle is acted on by any number of forces, which tend to different centres, and vary as the distances from those centres, to find the resultant attraction.*

Let $\mu . R$, $\mu' . R$ be the magnitudes of two of the forces at the distance $R$, $A$, $B$ the centres to which they tend, $P$ the position of a particle acted on by the forces.




Let $G$ be the centre of gravity of two particles at $A$ and $B$ whose masses are in the ratio of $\mu$ to $\mu'$, join $PA$, $PB$, $PG$.

The components of the force $\mu . PA$, in the directions $PG$, $GA$, are $\mu . PG$ and $\mu . GA$, and those of the force $\mu' . PB$, in the directions $PG$, $GB$, are $\mu' . PG$, and $\mu' . GB$, but $\mu . GA = \mu' . GB$, therefore the resultant of the forces tending to $A$ and $B$ is $(\mu + \mu')\, PG$, which is a single force of magnitude $(\mu + \mu')\, R$, at the distance $R$, tending to the centre of gravity of masses $\mu$, $\mu'$ placed at $A$ and $B$.

Let $\mu'' R$ be the magnitude of a force at the distance $R$, tending to $C$, the resultant attraction is that of a force tending to the centre of gravity $H$ of particles at $C$ and $G$, whose masses are in the ratio $\mu'' : \mu + \mu'$, which varies as the distance from $H$, and whose magnitude at the distance $R$ is $(\mu + \mu' + \mu'')\, R$.

And generally, the resultant of any number of forces is a single force, tending to the centre of gravity of a system of particles, placed at the different centres, whose masses are proportional to the magnitudes of the forces at the unit distance, and whose magnitude at any distance is the sum of those of the forces at the same distance.

208. Cor. 1. If every particle of a solid of any form attract with a force which varies as the mass of the particle and the distance conjointly, the resultant attraction of the solid upon any body will be the same as that of the whole mass of the solid

collected into its centre of gravity and attracting according to the same law.

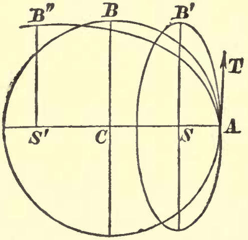
209. Cor. 2. If any of the forces be repulsive, as that whose centre is $B$, $G$ will lie in $AB$ or $BA$ produced, according as $\mu'$ is greater or less than $\mu$, and the resultant of the forces, tending to $A$ and from $B$, will be $(\mu' - \mu)$ $PG$ from $G$, or $(\mu - \mu')$ $PG$ towards $G$.

*Illustrations.*

(1) *A body revolves in a circular orbit about a force which varies as the distance, and tends to the centre of the circle, and the centre of force is suddenly transferred to a point in the radius which at the moment of change passes through the body; to find the subsequent motion of the body.*

Since the force varies as the distance, and is attractive, the orbit will be an ellipse. And, since the force is a finite force, the body will move in the same direction as before, at the moment of the change. Also, the velocity will, for the same reason, be unaltered at that moment.

Let $CA$ be the radius passing through the body at the moment of change, $CB$ perpendicular to $CA$, $\mu.CA$ the force at distance $CA$, $V$ the velocity in the circle.




Then $V^2 = \mu.CA.CA = \mu.CA^2$; and if $S$, the new point to which the force tends, be in $CA$, let $AB'$ be the ellipse described, $SA$ will be one of the semi-axes of the ellipse, since $A$ is an apse, and, $SB'$ being the other, if a body revolved in this ellipse round $S$, $\mu . SB'^2$ would be the square of the velocity at $A$, that is, $\mu . SB'^2 = \mu . CA^2$, and therefore $SB' = CA = CB$; hence the magnitude and position of the two semi-axes $SA$ and $SB'$ are known, and the ellipse is completely determined.

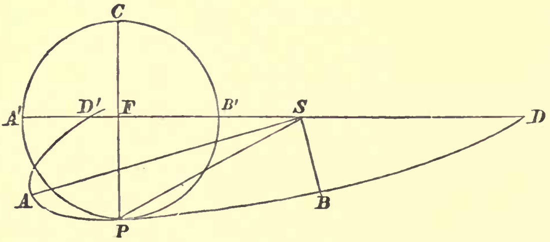
The ellipse lies without the circle at $A$, because, the velocity being unaltered, the force has been diminished in the ratio of $SA : CA$, and therefore the curvature diminished in that ratio.

If $S$ had been in $AC$ produced, as at $S'$, the force would have been increased, and the orbit $AB''$ would be within the circle near $A$.

The greatest distance from $CA$ which the body reaches is in all cases the same for this law of force, because the component of the force perpendicular to $CA$ is the same at the same distance from $CA$ in whatever curve the body moves; therefore, in each orbit, the velocity being the same at $A$, the velocity perpendicular to $AC$ is destroyed by the force at the same distance from $AC$.

(2) *A body is describing a circle about a force which varies as the distance and tends to the centre; if the centre to which the force tends be suddenly transferred to a point in the circumference, at an angular distance of* 60° *from the position of the particle at any time, to determine the orbit described.*

The orbit is an ellipse, since the force is attractive.




Let $P$ be the position of the body at the instant the centre of force is transferred from $C$, the centre of the circle, to $S$, where $SCP$ is an equilateral triangle.

The velocity at $P$ is $\sqrt{\mu}.CP=\sqrt{\mu}.SP$; and, since it is unaltered by the change of the centre of force, the semi-diameter conjugate to $SP$ is equal to $SP$.

Draw $DSD'$ perpendicular to $CP$, meeting it in $F$, and take $SD=SD'=SP$. Construct an ellipse having $SP$, $SD$ as equal conjugate semi-diameters; $SA$, $SB$ the semi-axes bisect the angles $PSD$, $PSD'$. The ellipse so described will be the orbit required.

Prove the following construction:

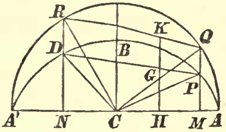
On $CP$ as diameter describe a circle cutting $SD'$ in $B'$, $A'$ $SA'$, $SB'$ are the lengths of the semi-axes.

Explain why the orbit is exterior to the circle.

(3) *Two bodies whose masses are $m$, $m'$ revolve in an ellipse under the action of a force tending to the centre; shew that, if they be at one time at the extremities of two conjugate diameters they will always be so, and in this case find the locus of their centre of gravity.*

Let $P$, $D$ be their positions at any time, $CP$, $CD$ being semi-conjugate diameters. Let the ordinates $MP$, $ND$, meet the auxiliary circle in $Q$ and $R$.

Since the angles $ACQ$, $ACR$ are always proportional to the times, $RCQ$ will always be a right angle; therefore the bodies will always be at the extremities of conjugate diameters.




Let $GH$ be the ordinate of their centre of gravity.

Join $RQ$ and produce $HG$ to meet $RQ$ in $K$;

$$\therefore KH : GH = QM : PM, \text{ a constant ratio,}$$

$$\text{also, } RK : KQ = DG : GP, \ \ldots\ldots\ldots\ldots\ldots\ldots;$$

hence $CK$ is constant, or the locus of $K$ is a circle, and the locus of $G$ is an ellipse, whose axes are proportional to those of $APD$.

Shew that the semi-major axis : $CA$ :: $(m^2+m'^2)^{\frac{1}{2}} : m+m'$.

(4) *A body is composed of matter which attracts with a force varying as the distance; shew that, however a particle be projected, unless it strike the body, it will describe its orbit in the same periodic time.*

This is obvious immediately from Art. 208, relating to the resultant of attracting forces.

(5) *A body moves in an ellipse under the action of a force varying as the distance; if the velocity at any point be slightly increased in the ratio* $1+n:1$, *find the consequent changes in the axes of the ellipse.*

*If, when the change takes place, the body be at the end of one of the equal conjugate diameters, shew that the eccentricity will be unaltered, and that the apse line will regrede through a small angle, whose circular measure is* $\dfrac{nab}{a^2-b^2}$.

When $V$ is changed to $(1+n)\,V$, $CD$ is changed to $(1+n)\,CD$; let the corresponding changes of $a$, $b$ and $\varpi$ be $\alpha a$, $\beta b$ and $\gamma$; $\alpha$, $\beta$, $\gamma$, and $n$ being so small that we may neglect their squares. Then by the equations of Art. 205,

$$(1+\alpha)^2a^2+(1+\beta)^2b^2=(1+n)^2CD^2+R^2=a^2+b^2+2n\,CD^2;$$

$$\therefore\ \alpha a^2+\beta b^2=n\,.\,CD^2.$$

$$\text{Again } (1+\alpha)a\,.\,(1+\beta)b=(1+n)\,CD\,.\,R\sin\alpha=(1+n)ab;$$

$$\therefore\ \alpha+\beta=n, \text{ and } \alpha(a^2-CD^2)=\beta\,(CD^2-b^2),$$

$$\therefore\ \frac{\alpha}{a^2-R^2}=\frac{\beta}{R^2-b^2}=\frac{n}{a^2-b^2}.$$

In the particular case $2R^2=a^2+b^2$, $\therefore\ \alpha=\beta=\frac{1}{2}n$, hence, $a$ and $b$ being altered in the same proportion, the eccentricity will be unaltered.

$$\text{Also, } \frac{R^2}{a^2}\cos^2(\varpi+\gamma)+\frac{R^2}{b^2}\sin^2(\varpi+\gamma)=1+n$$

$$\text{and } \frac{R^2}{a^2}\cos^2\varpi+\frac{R^2}{b^2}\sin^2\varpi=1;$$

$$\therefore\ \left(\frac{R^2}{b^2}-\frac{R^2}{a^2}\right)\{\sin^2(\varpi+\gamma)-\sin^2\varpi\}=n;$$

$$\therefore\ \left(\frac{R^2}{b^2}-\frac{R^2}{a^2}\right)\sin(2\varpi+\gamma)\sin\gamma=n;$$

and, since the axes bisect the angles between equal conjugate diameters, $ab = R^2 \sin 2\varpi$, therefore $\gamma$, being expressed in circular measure, $= \dfrac{nab}{a^2 - b^2}$.

(6) *In any position of a particle describing an ellipse, under the action of a force tending to the centre, the centre of force is suddenly transferred to the focus. Find the axes of the new orbit and shew that its major-axis bisects the angle between the focal distance and the major-axis of the given ellipse.*

Employing the equations of Art. 205, if $\alpha$, $\beta$ be the semi-axes of the new orbit, $P$ the position of particle when the centre s transferred to $S$, since the semi-diameter conjugate to $SP$ in the new orbit will be equal to $CD$,

$$\alpha^2 + \beta^2 = CD^2 + SP^2 = SP \,.\, HP + SP^2 = 2a \,.\, SP,$$

$$\text{and } SY^2 : BC^2 :: SP : HP :: SP^2 : CD^2;$$

$$\therefore \alpha\beta = CD \,.\, SY = b \,.\, SP;$$

$$\therefore (\alpha^2 - \beta^2)^2 = 4(a^2 - b^2)\, SP^2, \text{ and } \alpha^2 - \beta^2 = 2aeSP,$$

$$\therefore \alpha^2 = a(1+e)\, SP, \text{ and } \beta^2 = a(1-e)\, SP.$$

$$\text{Also } \frac{SP^2}{\alpha^2}\cos^2\varpi + \frac{SP^2}{\beta^2}\sin^2\varpi = 1,$$

$$\therefore \frac{a(1-e^2)}{SP} = (1-e)\cos^2\varpi + (1+e)\sin^2\varpi = 1 - e\cos 2\varpi;$$

therefore $2\varpi = \angle PSA$, or the major-axis of the new orbit bisects the angle between $PS$ and the major-axis of the original orbit.

NOTE. By the construction of Art. 204, since $PR$ is a third proportional to $SP$ and $CD$, and therefore is equal to $HP$, the circle which determines $T$ and $t$ passes through $H$, and the arcs $HT$, $TR$ are equal, that is, $ST$ bisects the angle $PSA$.

## XXVII.

1. Shew that the velocity in an ellipse about the centre is the same at the points whose conjugate diameters are equal as that in a circle at the same distance.

2. A body is revolving in a circle under the action of a force tending to the centre, the law of force at different distances being that the force varies as the distance; find the orbits described when the circumstances are changed at any point as follows:

i. The force is increased in the ratio of $1 : n$.

ii. The velocity is increased in the ratio $1 : n$.

iii. The force becomes repulsive, remaining of the same magnitude.

iv. The direction is changed by an impulse in the direction of the centre, measured by the velocity equal to that in the circle.

3. If a body be projected from an apse, with a velocity double of that in a circle at the same distance, find the position and magnitude of the axes of its orbit.

4. A particle is revolving in a circle acted on by a force which varies as the distance; the centre of force is suddenly transferred to the opposite extremity of the diameter through the particle and becomes repulsive; shew that the eccentricity of the hyperbolic orbit $= \frac{1}{2}\sqrt{5}$.

5. A body is moving under the action of a force tending to a fixed centre, and varying as the distance. The force suddenly ceases, and after an interval commences to act again. Prove that the radii of curvature of the orbit at the points where the body ceases and recommences to be attracted are equal.

6. A body moves in an ellipse about a centre of force in the centre, and its velocity is observed when it arrives at its greatest distance, and again after a lapse of one-third of its periodic time. If these velocities be in the ratio of $2 : 3$, prove that the eccentricity of the ellipse will be $\sqrt{\frac{5}{8}}$.

7. The particles of which a rectangular parallelepiped is composed attract with a force which varies as the distance, and a body is projected so as to describe a curve on one of the faces supposed smooth; find the periodic time.

8. An elastic ball, moving in an ellipse about the centre, on arriving at the extremity of the minor axis strikes directly another ball at rest; find the orbits described by both bodies.

9. A body is projected in a direction making an angle $\cos^{-1}\frac{1}{\sqrt{3}}$ with the distance from a point to which a force tends, varying as the distance from it, and the velocity $= \sqrt{\frac{2}{3}} \times$ velocity in the circle at the same distance; prove that one axis is double of the other and that the inclination of the major axis to the distance is $\frac{1}{2}\cos^{-1}\frac{1}{3}$.

10. From points in a line $CA$ between $C$ and $A$ particles are projected at right angles to $CA$ with velocities proportional to their distances from $A$, $C$ being a centre to which the force tends, and the force varying as the distance; find the ellipse of greatest area which is described.

11. Two particles are projected in parallel directions from two points in a straight line passing through a centre of force, the acceleration towards which varies as the distance, with velocities proportional to their distances from that centre. Prove that all tangents to the path of the inner cut off, from that of the outer, arcs described in equal times.

12. An hyperbola and its conjugate are described by particles round a force in the centre. They are at an apse at the same instant; shew that they will always be at the extremities of conjugate diameters. Also if $v$, $v'$ be their velocities, $v^2 - v'^2 = \mu (a^2 - b^2)$.

13. An ellipse and an hyperbola have the same centre and foci. They are described by particles, under the action of forces in the centre of equal intensity. If $a$, $a'$ be their semi-transverse axes, the square of the velocity of each body at a point where the curves cut will be $\mu\, a^2 - a'^2)$.

14. If any number of particles be moving in an ellipse about a force in the centre, and the force suddenly cease to act, shew that, after the lapse of $\frac{1}{2\pi}$ of the period of a complete revolution, all the particles will be in a similar, concentric, and similarly situated ellipse.

15. A particle is describing an ellipse under the action of a force tending to the centre. Prove that its angular velocity about a focus is inversely proportional to its distance from that focus.

## XXVIII.

1. $CX$, $CY$ are straight lines inclined at any angle, and a force tends to $C$, and varies as the distance from $C$. If from various points in $CY$ different particles are projected parallel to $CX$ at the same moment, and with the same velocity, they will all arrive at $CX$ at the same time and place; and they will also do so, if the force cease to act for any interval of time.

2. A number of particles move in hyperbolas, under the action of the same repulsive force from their common centre. Shew that, if the transverse axes coincide, and the particles start from the vertex at the same instant, they will always lie in a straight line