THE SOCIAL WELFARE UNDER STOCHASTIC DEMAND

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THESIS

THE SOCIAL WELFARE UNDER
STOCHASTIC DEMAND

by

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20. ABSTRACT (Continued)

variance of risk increases, using the least-square method we estimate the linear relation between the optimal output and mean or variance of risk.

In the second model we introduce the expected monopoly profit and observe how both the optimal price and output vary as the mean or variance of risk changes.

As the final step, we compare the results of two kinds of models, and find that which is the least affected by risk.

The Social Welfare Under
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In the second model we introduce the expected monopoly profit and observe how both the optimal price and output vary as the mean or variance of risk changes. As the final step, we compare the results of two kinds of models, and find that which is the least affected by risk.

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CHAPTER I

THEORY OF THE SOCIAL WELFARE FUNCTION

1.1 Introduction to the Models

When firms are competitive, the quantity of product is set at an equilibrium point where the price is equal to the marginal cost for each firm. But in the real world, there is sometimes only a single firm producing a product or demands of customers are not in steady state.

There is a respectably large body of economic literature that presumes to establish a relationship between the polar market structures of monopoly and pure competition and the consequent industry performance. Almost all of this literature considers only the case of a deterministic demand function.

This paper will consider what effect the introduction of a stochastic demand function will have upon the predicted market structure-performance correlation. Risk has been introduced into economic considerations in several ways. The most common device is to define the decision-makers' utility function in terms of an expected utility hypothesis.

An alternative approach is to ignore the expected utility hypothesis and treat the stochastic elements as strictly limited to optimization in terms of operationally defined economic variables. This latter approach will be followed.

Brown and Johnson* introduced models of a firm's decision-making process under uncertainty, and demand has generally been defined only as a single product. Brown and Johnson introduced two ways in which the risk enters the demand function: As an additive random variable; and as a multiplicative random variable. We shall choose additive random variable way since it facilitates analysis of the problem. We extend the model of Brown and Johnson by introducing risk when the mean is not equal to zero. Chapter I describes the models with normal distribution (μ,σ^2) of risk. Chapter II indicates the trend of the optimal output as the mean or variance of risk increases. Chapter III introduces another kind of model, maximization of the expected profit of a firm, and derives the trend of output and price when the mean or variance of risk increases. Chapter IV contains the conclusions.

We are interested in the social welfare function that we develop and extend in Chapters I and II. Following Brown and Johnson we define the social welfare function as the difference between expected willingness to pay and expected costs.

In the model of maximization of the social welfare function under linear demand, the optimal price will always

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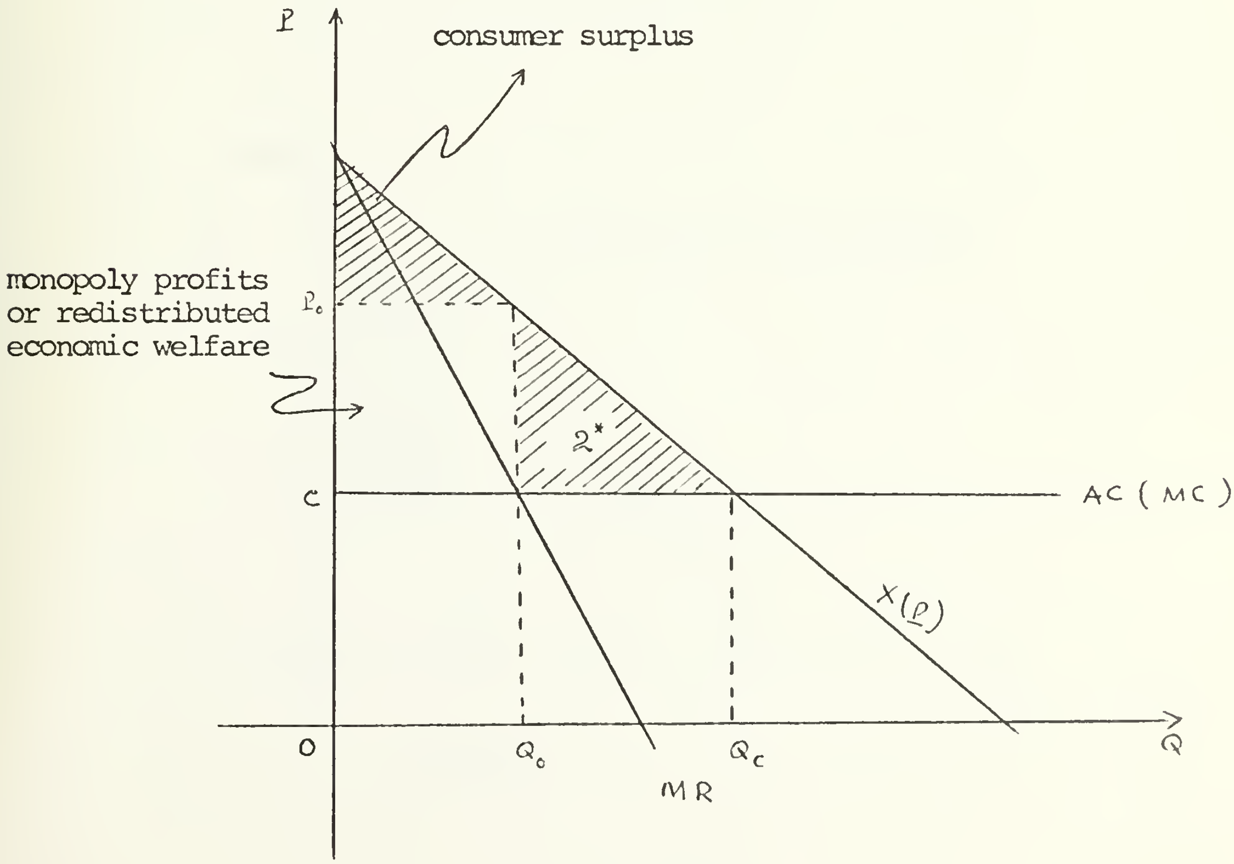
*"Public Utility Pricing and Output under Risk," by Gardner Brown, Jr. and M. Bruce Johnson, <u>The American Economic Review</u>, March 1969, page 119-128.

be constant and lower than, and the optimal output will generally be higher than, their counterparts in the riskless model, regardless of the manner in which the risk term enters the analysis.

In the model of Chapter III, the optimal price and output are lower or higher than the price and output in the riskless demand according to its mean and variance of risk.

1.2. Optimal Decisions under Deterministic Demand

Measure of welfare loss due to monopoly is consumer surplus (shaded area 2*). The monopoly profit would be redistributed from the monopolist-producer to consumers by a shift to competition.



MR: marginal revenue

MC: marginal cost

Q_o: optimal output under monopoly operation

P_o: optimal price under monopoly operation

Q_c: optimal output under competitive operation

Under deterministic demand, the aggregate social welfare is expressed as the shaded area in Figure 1 below and that is the sum of consumers' surplus and redistributed economic welfare. Assumed constant cost supply.

Q: quantity of output

P: unit price of output

C: unit cost of output

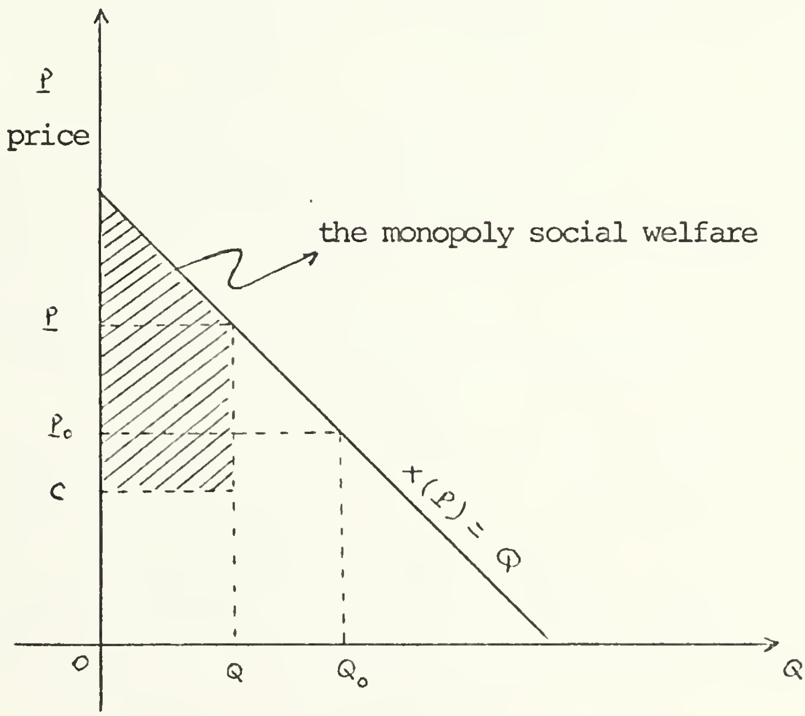


Figure 1

Assume the demand function is linear monotone decreasing as P increases, and differentiable, $Q = X(P)$. So the social welfare function W_o should be expressed as a function of quantity Q.

$$P = X^{-1}(Q)$$

$$W_o = \int_0^{Q_o} X^{-1}(Q)\, dQ - c\, Q_o$$

$$c = b + \beta$$

where b, β are constant and defined in Section 1.3

Now we find the optimal price and quantity of output such that the social welfare is maximized. The first conditions are:

$$\frac{\partial W_o}{\partial Q_o} = \frac{\partial}{\partial Q_o} \int_0^{Q_o} X^{-1}(Q)\, dQ - c = 0$$

$$= X^{-1}(Q_o) = c$$

$$\Longrightarrow \left[\begin{array}{l} P_o = c = b + \beta \\ \\ Q_o = X(b+\beta) \end{array} \right.$$

The second order conditions:

$$\frac{\partial^2 W_o}{\partial Q_o^2} = \frac{\partial}{\partial Q_o} X^{-1}(Q_o) < 0$$

Since the function Q is assumed to be linear and negatively sloping with respect to P, or, inversely, the function P is linear and downward sloping with Q:

$$\frac{\partial P}{\partial Q} < 0$$

That is,

$$\frac{\partial X^{-1}(Q)}{\partial Q} < 0$$

So that the welfare function is global maximum at $(P_o\ Q_o)$.

1.3. <u>The Model with E(u) = 0</u>

Uncertainty may enter the demand function in this way:

$$D = X(P) + u \qquad \text{additive model}$$

P: price of production

X(P): function of P with certainty

u: continuous random variable

f(u): probability density fucntion of u

The following properties are attributed to $f(u)$:

$$\int_{-\infty}^{+\infty} u\, f(u)\, du = 0 \qquad ; \qquad F(\alpha) = \int_{-\infty}^{\alpha} f(u)\, du$$

$$\int_{-\infty}^{+\infty} u^2 f(u)\, du < \infty$$

Suppose the plant manager arbitrarily chooses values P and z before he knows the actual value of u. The actual sales of the product are:

$$S = \begin{cases} X(P) + u & \text{If} \quad z \geq X(P)+u \\ z & \text{If} \quad z \leq X(P)+u \end{cases}$$

where z: capacity of production

$X(P)+u$: actual demand

The linear cost function is:

$$c = b\, E(S) + \beta z$$

b: constant marginal cost of producing output

β: constant cost for one unit of z during the demand period

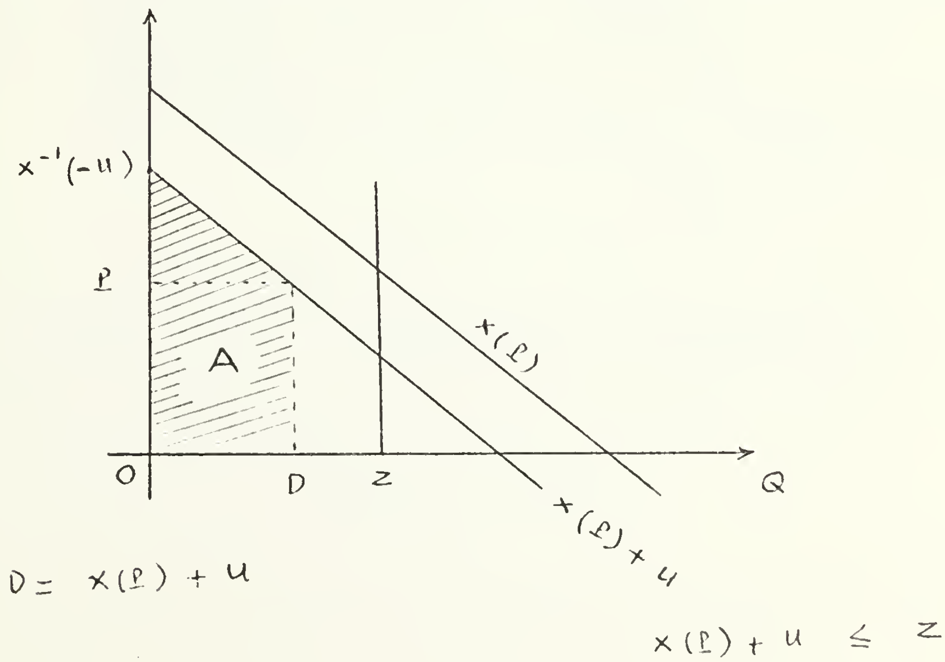
E(S): expected sale quantity

The welfare function is composed of the expected value of willingness to pay, the expected value of variable cost and the capacity cost known with certainty so the welfare function is chosen:

$$W_1 = E[\text{willingness to pay}] - E[\text{average variable cost}\cdot\text{sales}]$$
$$- E[\text{capacity cost}].$$

Now the problem is to find out the optimum point (P^*, z^*) of which the manager arbitrarily chooses before he knows an actual value of u in order to maximize the welfare function W. If the value of u is negative and large enough in absolute value, demand will be less than or equal to the capacity of the plant.

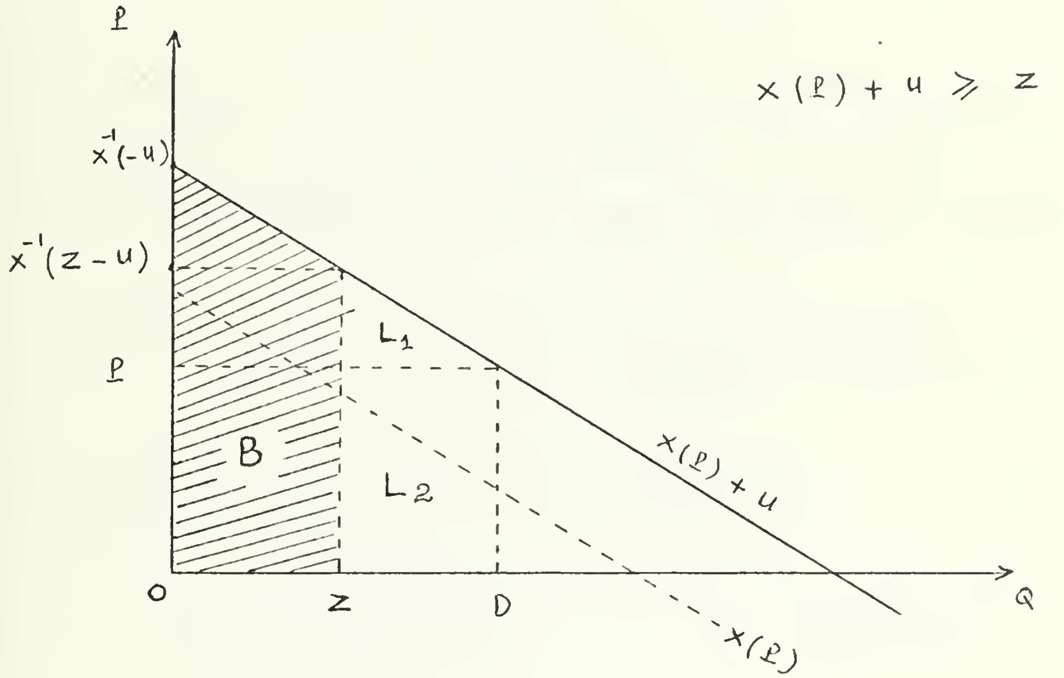
The area under the demand function is consumer's surplus. The area for consumers' surplus is equal to total revenue only in the case of "perfect," or "first degree" price discrimination.



The shaded area is:

$$A = \int_{-\infty}^{z-X(P)} f(u) \left[\int_{P}^{X^{-1}(-u)} [X(t)+u]dt + P\cdot[X(P)+u] \right] du$$

When u takes on large positive values, the actual demand will be greater than output z, the shaded areas are the consumers' surplus and total revenue.



$$B = \int_{z-X(P)}^{\infty} f(u) \left[\int_{P}^{X^{-1}(-u)} [X(t)+u]dt + P\cdot[X(P)+u] \right] du$$

$$- [E(L_1) + E(L_2)]$$

$$E(L_1) = \int_{z-X(P)}^{\infty} f(u) \int_{P}^{X^{-1}(z-u)} [X(t)+u-z]\, dt\, du$$

$$E(L_2) = \int_{z-X(P)}^{\infty} f(u)\ P\cdot[X(P)+u-z]\, du$$

for $u \geq z-X(P)$

L_1: the losses in surplus

L_2: the losses in revenue

The expected sales is equal to expected demand $X(P)+u$ over the entire range of random variable u such that $u \leq z-X(P)$ plus expected demand z over the range $u \geq z-X(P)$.

$$E[S] = \int_{-\infty}^{z-X(P)} [X(P)+u]\, f(u)\, du + \int_{z-X(P)}^{\infty} z\, f(u)\, du$$

Finally the welfare function is obtained

$$W_1 = A + B - C \qquad \text{(Appendix 1)}$$

Differentiate W_1 with respect to price and capacity, the first order conditions for maximization of the welfare function are:

$$\frac{\partial W_1}{\partial P} = PX'(P)\ F[z-X(P)] - bX'(P)\ F[z-X(P)] = 0$$

$$\frac{\partial W_1}{\partial z} = \int_{z-X(P)}^{\infty} f(u)\,[X^{-1}(z-u) - b]\ du - \beta = 0$$

(Appendix 1)

and the second order conditions:

$$\frac{\partial^2 W_1}{\partial P^2} < 0 \qquad \frac{\partial^2 W_1}{\partial z^2} < 0 \qquad \text{(Appendix 2)}$$

The simultaneous solutions of the two first-order conditions indicate that price equals short run marginal operating cost, and optimal capacity should be shosen such that marginal capacity cost is equal to the truncated mean of the difference between the willingness to pay and the actual price for the marginal disappointed purchaser of the commodity.

The second order conditions are satisfied so the welfare function is a global maximum at point (P_1* z_1*) the solutions of the first order conditions.

Appendix 1 $\quad u \approx N(0,\sigma^2)$

$$A = \int_{-\infty}^{z-X(P)} f(u) \int_{P}^{X^{-1}(-u)} [X(t)+u]\,dt\,du + \int_{-\infty}^{z-X(P)} P\cdot[x(P)+u]f(u)\,du$$

$$B = \int_{z-X(P)}^{\infty} f(u) \int_{P}^{X^{-1}(-u)} [X(t)+u]\,dt\,du + \int_{z-X(P)}^{\infty} P\cdot[X(P)+u]f(u)\,du$$

$$- E[L_1] - E[L_2]$$

$$A+B = \int_{-\infty}^{+\infty} f(u) \int_{P}^{X^{-1}(-u)} [X(t)+u]\,dt\,du$$

$$+ \int_{-\infty}^{+\infty} P[X(P)+u]f(u)\,du - E[L_1] - E[L_2]$$

$$A+B = \int_{-\infty}^{+\infty} f(u) \int_{P}^{X^{-1}(-u)} [X(t)+u]\,dt\,du + P\cdot X(P)$$

$$- E[L_1] - E[L_2]$$

$$W_1 = A+B-C$$

$$= A + B - b\,E(S) - \beta z$$

Defined: $$X'(P) = \frac{\partial X(P)}{\partial P} < 0$$

$$X''(P) = \frac{\partial^2 X(P)}{\partial P^2} = 0$$

Since X(P) is linear and downward sloping

$$(1) \quad \int_{-\infty}^{+\infty} f(u) \int_{P}^{X^{-1}(-u)} [X(t)+u]\, dtdu + P\, X(P)$$

$$(2) \quad - \int_{z-X(P)}^{\infty} f(u) \left[\int_{P}^{X^{-1}(z-u)} [X(t)+u-z]dt + P[X(P)+u-z] \right] du$$

$$(3) \quad -b \left[\int_{-\infty}^{z-X(P)} [X(P)+u]f(u)du + \int_{z-X(P)}^{\infty} z\, f(u)\, du \right] -\beta z$$

$$W_1 = (1) + (2) + (3)$$

Assume (1), (2), (3) converge to some limit as $u \to \infty$:

$$\frac{\partial(1)}{\partial P} = \int_{-\infty}^{\infty} f(u)[-X(P)-u]du + X(P) + PX'(P)$$

$$= -X(P) + X(P) + PX'(P) = P\, X'(P)$$

$$\frac{\partial(2)}{\partial P} = - \int_{z-X(P)}^{\infty} f(u) \left[-[X(P)+u-z] + X(P) + u - z + PX'(P) \right] du$$

$$+ f(z-X(P)) \left[\int_{P}^{X^{-1}(z-z+X(P))} [X(t)+z-X(P)]dt + P[X(P)+z-X(P)-z \right] \cdot$$

$$\cdot (-X'(P))$$

$$= - \int_{z-X(P)}^{\infty} PX'(P) \; f(u) \; du$$

$$\frac{\partial(3)}{\partial P} = -b \left[\int_{-\infty}^{z-X(P)} X'(P)f(u)du + [X(P)+z-X(P)] \right.$$

$$\left. \cdot f(z-X(P))(-X(P)) - zf(z-X(P))(-X'(P)) \right]$$

$$= -b \int_{-\infty}^{z-X(P)} X'(P) \; f(u) \; du$$

since:

$$\int_{-\infty}^{z-X(P)} [X(P)+u]f(u)du \quad \text{and} \quad \int_{z-X(P)}^{\infty} z \; f(u) \; du$$

are assumed to converge to some value as $u \to \infty$

$$\frac{\partial W_1}{\partial P} = \frac{\partial(1)}{\partial P} + \frac{\partial(2)}{\partial P} + \frac{\partial(3)}{\partial P}$$

$$= PX'(P) - \int_{z-X(P)}^{\infty} P\,X'(P)\,f(u)\,du - bX'(P)\int_{-\infty}^{z-X(P)} f(u)\,du$$

$$= P\,X'(P)\int_{-\infty}^{z-X(P)} f(u)\,du - b\,X'(P)\int_{-\infty}^{z-X(P)} f(u)\,du$$

$$\frac{\partial W_1}{\partial P} = (P-b)\,X'(P)\int_{-\infty}^{z-X(P)} f(u)\,du$$

$$\frac{\partial(1)}{\partial z} = 0$$

$$\frac{\partial(2)}{\partial z} = -\int_{z-X(P)}^{\infty} f(u)\left[\int_{P}^{X^{-1}(z-u)} (-1)\,dt + [X(X^{-1}(z-u))+u-z] \cdot \frac{\partial X^{-1}(z-u)}{z} - P\right] du + f(z-X(P)) \cdot$$

$$\cdot\left[\int_{P}^{X^{-1}(z-z+X(P))} [X(t)+z-X(P)-z]\,dt + P[X(P)+z-X(P)-z]\right]$$

$$= -\int_{z-X(P)}^{\infty} f(u)\,[-X^{-1}(z-u) + P - P]\,du$$

$$= \int_{z-X(P)}^{\infty} X^{-1}(z-u)\,f(u)\,du$$

$$\frac{\partial(3)}{\partial z} = -b \left[[X(P)+z-X(P)]\ f(z-X(P)) \right.$$

$$\left. + \int_{z-X(P)}^{\infty} f(u)\,du - z\ f(z-X(P)) \right] - \beta$$

$$= -\ b \int_{z-X(P)}^{\infty} f(u)\ du - \beta$$

$$\frac{\partial W_1}{\partial z} = \frac{\partial(1)}{\partial z} + \frac{\partial(2)}{\partial z} + \frac{\partial(3)}{\partial z}$$

$$= \int_{z-X(P)}^{\infty} X^{-1}(z-u)\ f(u)\ du \ - \ b \int_{z-X(P)}^{\infty} f(u)\ du - \beta$$

$$\frac{\partial\ W_1}{\partial\ z} = \int_{z-X(P)}^{\infty} [X^{-1}(z-u) - b]\ \ f(u)\ du - \beta$$

Appendix 2

$$\frac{\partial W_1}{\partial P} = \int_{-\infty}^{z-X(P)} [P - b]\ X'(P)\ f(u)\ du \qquad [2]$$

The function [2] is converged as $u \to \infty$ so we can take derivatives:

$$\frac{\partial^2 W_1}{\partial P^2} = \int_{-\infty}^{z-X(P)} X'(P)\ f(u)\ du$$

$$+ [P - b]\ X'(P)\ f(z-X(P))$$

$$X'(P) < 0 \quad ; \text{ IF } P \geq b \quad \text{then:}$$

$$\frac{\partial^2 W_1}{\partial P^2} < 0$$

$$\frac{\partial W_1}{\partial z} = \int_{z-X(P)}^{\infty} [X^{-1}(z-u) - b]\ f(u)\ du - \beta \qquad [3]$$

The function [3] is also converged when $u \to \infty$

$$\frac{\partial^2 W_1}{\partial z^2} = \int_{z-X(P)}^{\infty} \frac{\partial}{\partial z} [X^{-1}(z-u)] \; f(u) \; du$$

$$- [X^{-1}(z-z+X(P)-b] \; f(z-X(P))$$

$$= \int_{z-X(P)}^{\infty} \frac{\partial}{\partial z} [X^{-1}(z-u)] \; f(u) \; du$$

$$- [P-b] \; f(z-X(P))$$

$$\frac{\partial}{\partial z} [X^{-1}(z-u)] < 0$$

the same explanation in section 1-2 if $P \geqslant b$ then:

$$\frac{\partial^2 W_1}{\partial z^2} \leqslant 0$$

1.4. The Model with $E(u) = \mu$

In this section we introduce a random variable u with nonzero mean, probability density function f(u) and finite variance.

$$\int_{-\infty}^{+\infty} u\, f(u)\, du = \mu \quad , \quad F(\alpha) = \int_{-\infty}^{\alpha} f(u)\, du$$

$$\int_{-\infty}^{\infty} u^2 f(u)\, du < \infty \qquad\qquad \mu > 0$$

The sum of the areas A and B now becomes A'+B' and equals:

$$A' + B' = A + B + \quad P$$

and the expected sales:

$$E[S'] = \int_{-\infty}^{z-X(P)} [X(P)+u]\, f(u)\, du + \int_{z-X(P)}^{\infty} z\, f(u)\, du$$

The expression of the welfare function W_2 is almost the same as W_1 in the appendix 1

$$W_2 = A' + B' - bE[S] - \beta z$$

$$W_2 = \int_{-\infty}^{+\infty} f(u) \int_{P}^{X^{-1}(-u)} [X(t)+u]\, dtdu + P\, X(P) + \mu P$$

$$- \int_{z-X(P)}^{\infty} f(u) \left[\int_{P}^{X^{-1}(z-u)} [X(t)+u-z]dt + P[X(P)+u-z] \right] du$$

$$-b\left[\int_{-\infty}^{z-X(P)} [X(P)+u]f(u)du + \int_{z-X(P)}^{\infty} z\, f(u)du \right] -\beta z$$

Take the first derivatives and set equal to zero:

$$\frac{\partial W_2}{\partial P} = (P-b)\, X'(P) \int_{-\infty}^{z-X(P)} f(u)\, du = 0$$

$$\frac{\partial W_2}{\partial z} = \int_{z-X(P)}^{\infty} [X^{-1}(z-u) - b]\, f(u)\, du - \beta = 0 \qquad \text{(appendix 3)}$$

and the second order conditions are the same expression in appendix 2

$$\frac{\partial^2 W_2}{P^2} = \int_{-\infty}^{z-X(P)} X'(P)f(u)du + [P-b]X'(P)f(z-X(P))$$

$$\text{if} \quad P \geq b \quad \text{then} \quad \frac{\partial^2 W_2}{\partial P^2} < 0$$

$$\frac{\partial^2 W_2}{\partial z^2} = \int_{z-X(P)}^{\infty} \frac{\partial}{\partial z} [X^{-1}(z-u)]\, f(u)\, du - [P-b]f(z-X(P))$$

$$P \geq b \qquad \frac{\partial^2 W_2}{\partial z^2} < 0 \ .$$

The optimum price is independent with mean and variance of risk.

Appendix 3

$$(4) = \int_{-\infty}^{+\infty} f(u) \int_{P}^{X^{-1}(-u)} [X(t)+u]dtdu + P\,X(P) + \mu P$$

$$(5) = -\int_{z-X(P)}^{\infty} f(u) \left[\int_{P}^{X^{-1}(z-u)} [X(t)+u-z]dt + P[X(P)+u-z] \right] du$$

$$(6) = -b \left[\int_{-\infty}^{z-X(P)} [X(P)+u]f(u)du + \int_{z-X(P)}^{\infty} z\,f(u)\,du \right] - \beta z$$

$$W_2 = (4) + (5) + (6)$$

$$\frac{\partial(4)}{\partial P} = \int_{-\infty}^{+\infty} f(u)[-X(P)-u]du + PX'(P) + X(P) + \mu$$

$$= -X(P) - \mu + P\,X'(P) + X(P) + \mu$$

$$= P\,X'(P)$$

$\frac{\partial(5)}{\partial P}$ and $\frac{\partial(6)}{\partial P}$ are just the same expressions of $\frac{\partial(2)}{\partial P}$ and $\frac{\partial(3)}{\partial P}$ respectively in appendix 1.

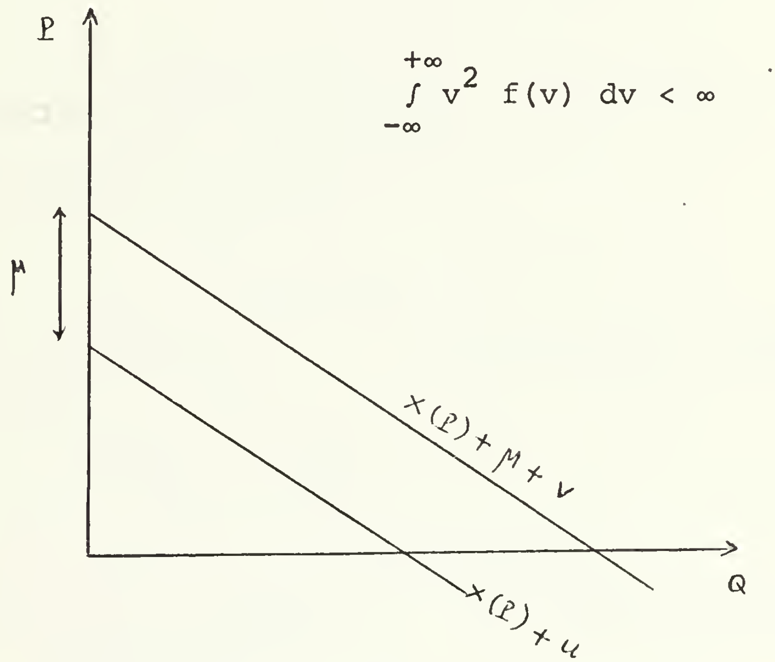
So that the first- and second-order conditions for maximization of welfare function are the same expressions in appendix 1 and appendix 2.

1.5. Extension of the Model in Section 1.4

In the previous section we introduced u as a continuous random variable with nonzero mean. Now we would like to move the demand function upward a distance μ, and random variable u becomes random variable v with the probability density function f(v) but the mean of v is equal to zero, and the same variance with u.

$$\int_{-\infty}^{+\infty} v\, f(v)\, dv = 0$$

$$\int_{-\infty}^{+\infty} v^2\, f(v)\, dv < \infty$$



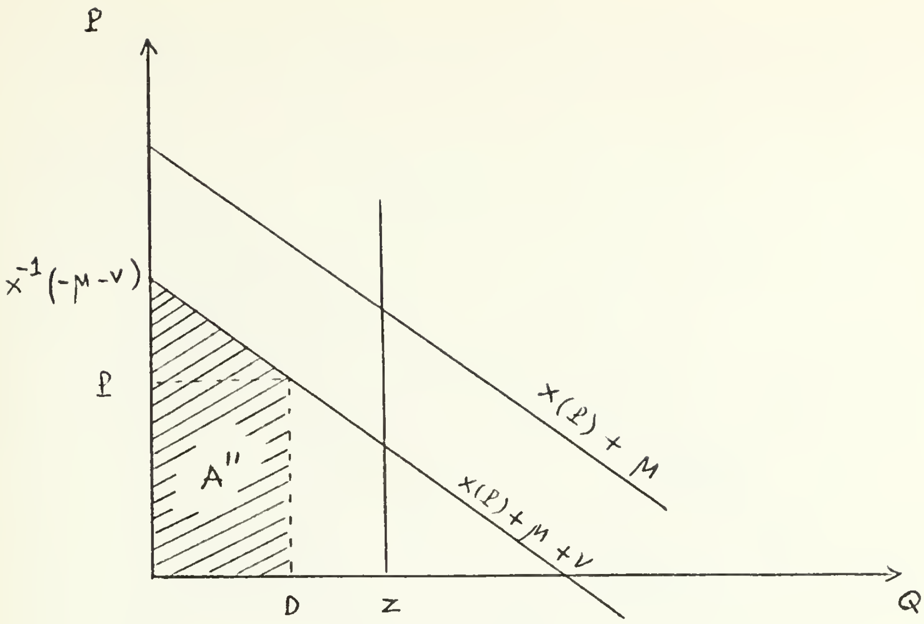
If the value of v is negative, and large enough to make demand smaller than the capacity z.

The area A" now equals:

$$A'' = \int_{-\infty}^{z-X(P)-\mu} f(v) \left[\int_{P}^{X^{-1}(-\mu-v)} [X(t)+v+\mu]dt + P\cdot[X(P)+\mu+v] \right] dv$$

$$X(P)+\mu+v \leq z$$

$$v \leq z-X(P)-\mu$$



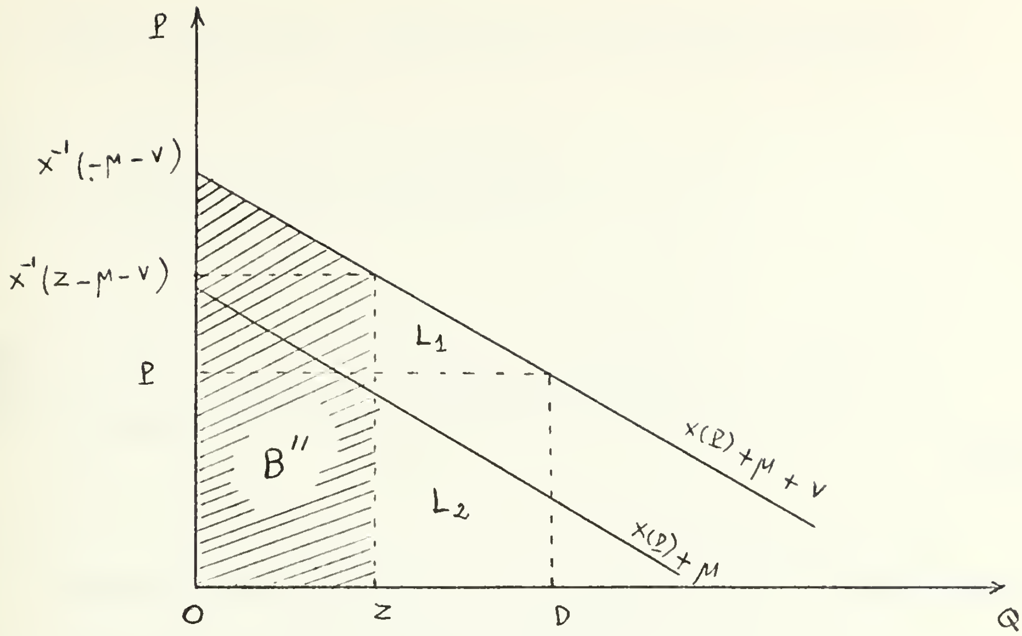
If v is positive and large enough, demand will be greater than z and area B" equals:

$$B'' = \int_{z-X(P)-\mu}^{\infty} f(v) \left[\int_{P}^{X^{-1}(-\mu-v)} [X(t)+\mu+v]dt + P[X(P)+\mu+v] \right] dv$$

$$- E[L_1] - E[L_2]$$

$$E[L_1] = \int_{z-X(P)-\mu}^{\infty} f(v) \int_{P}^{X^{-1}(z-\mu-v)} [X(t)+\mu+v-z]dtdv$$

$$E[L_2] = \int_{z-X(P)-\mu}^{\infty} P[X(P)+\mu+v-z]\ f(v)\ dv$$



$$A''+B'' = \int_{-\infty}^{+\infty} f(v) \int_{P}^{X^{-1}(-\mu-v)} [X(t)+\mu+v]\, dt dv$$

$$+ P[X(P)+\mu] - E[L_1] - E[L_2]$$

Expected sales equals:

$$E[S''] = \int_{-\infty}^{z-X(P)-\mu} [X(P)+\mu+v] f(v)\, dv + \int_{z-X(P)-\mu}^{\infty} z\, f(v)\, dv$$

The welfare function W_3 is:

$$W_3 = A'' + B'' - b\, E[S''] - \beta z$$

The first-order conditions for maximization:

$$\frac{\partial W_3}{\partial P} = \int_{z-X(P)-\mu}^{\infty} [P-b]\, X'(P)\, f(v)\, dv = 0$$

$$\frac{\partial W_3}{\partial z} = \int_{z-X(P)-\mu}^{\infty} f(v)\,[X^{-1}(z-\mu-v) - b]\, dv - \beta = 0$$

Appendix 4

The optimum price is still equal to b, independent of any constant value that is added to X(P).

The second-order conditions are negative if $P \geq b$

$$\frac{\partial^2 W_3}{\partial z^2} < 0 \quad , \quad \frac{\partial^2 W_3}{\partial P^2} < 0$$

Appendix 5

Appendix 4

$$W_3 = A'' + B'' - b\,E[S] - \beta z$$

$$(7) = \int_{-\infty}^{\infty} f(v) \int_{P}^{X^{-1}(-\mu-v)} [X(t)+\mu+v]\,dt\,dv + P[X(P)+\mu]$$

$$(8) = -\int_{z-X(P)-\mu}^{\infty} f(v) \left[\int_{P}^{X^{-1}(z-\mu-v)} [X(t)+\mu+v-z]\,dt + P[X(P)+\mu+v-z] \right] dv$$

$$(9) = -b \left[\int_{-\infty}^{z-X(P)-\mu} [X(P)+\mu+v]\,f(v)\,dv + \int_{z-X(P)-\mu}^{\infty} z\,f(v)\,dv \right] - \beta z$$

$$W_3 = (7) + (8) + (9)$$

Assume that all integrals involved in taking derivatives exist when v increases toward infinity.

$$\frac{\partial(7)}{\partial P} = \int_{-\infty}^{+\infty} f(v)[-X(P)-\mu-v]dv + X(P) + \mu + P\,X'(P)$$

$$= P\,X'(P)$$

$$\frac{\partial(8)}{\partial P} = -\int_{z-X(P)-\mu}^{\infty} f(v)\left[-[X(P)+\mu+v-z] + X(P) + \mu + v - z + P\,X'(P)\right]dv +$$

$$+ f(z-X(P)-\mu)\left[\int_{P}^{X^{-1}(z-\mu-z+X(P)+\mu)=P} [X(t)+\mu+z-X(P)-\mu-z]dt + P[X(P)+\mu+z-X(P)-\mu-z](-X'(P))\right]$$

$$= -\int_{z-x(P)-\mu}^{\infty} P\,X'(P)\,f(v)\,dv$$

$$\frac{\partial(9)}{\partial P} = -b\left[\int_{-\infty}^{z-X(P)-\mu} X'(P)f(v)dv + [X(P)+\mu+z-X(P)-\mu]\cdot f(z-X(P)-\mu)(-X'(P)) - zf(z-X(P)-\mu)\cdot(-X'(P))\right]$$

$$= -b\int_{-\infty}^{z-X(P)-\mu} X'(P)\,f(v)\,dv$$

$$\frac{\partial W_3}{\partial P} = \frac{\partial(7)}{\partial P} + \frac{\partial(8)}{\partial P} + \frac{\partial(9)}{\partial P}$$

$$= PX'(P) - \int_{z-X(P)-\mu}^{\infty} PX'(P)\ f(v)\ dv - b \int_{-\infty}^{z-X(P)-\mu} X'(P)\ f(v)\ dv$$

$$\frac{\partial W_3}{\partial P} = \int_{-\infty}^{z-X(P)-\mu} [P-b]\ X'(P)\ f(v)\ dv = 0$$

$$\frac{\partial(7)}{\partial z} = 0$$

$$\frac{\partial(8)}{\partial z} = - \int_{z-X(P)-\mu}^{\infty} f(v) \left[\int_{P}^{X^{-1}(z-\mu-v)} -1\ dt \right.$$

$$\left. + [X(X^{-1}(z-\mu-v)) + \mu + v - z] - P \right] dv$$

$$+ f(z-X(P)-\mu) \cdot$$

$$\cdot \left[\int_{P}^{X^{-1}(z-\mu-z+X(P)+\mu)=P} [X(t)+\mu+z-X(P)-\mu-z]dt \right.$$

$$\left. + P[X(P)+\mu+z-X(P)-\mu-z] \right]$$

$$= \int_{z-X(P)-\mu}^{\infty} X^{-1}(z-\mu-v)\ f(v)\ dv$$

$$\frac{\partial(9)}{\partial z} = -b \left[[X(P)+\mu+z-X(P)-\mu] \; f(z-X(P)-\mu) \right.$$

$$\left. + \int_{z-X(P)-\mu}^{\infty} f(v) \; dv - z \; f(z-X(P)-\mu) \right] - \beta$$

$$= -b \int_{z-X(P)-\mu}^{\infty} f(v) \; dv - \beta$$

$$\frac{\partial W_3}{\partial z} = \frac{\partial(7)}{\partial z} + \frac{\partial(8)}{\partial z} + \frac{\partial(9)}{\partial z}$$

$$= \int_{z-X(P)-\mu}^{\infty} X^{-1}(z-\mu-v) f(v) dv - b \int_{z-X(P)-\mu}^{\infty} f(v) dv - \beta$$

$$\frac{\partial W_3}{\partial z} = \int_{z-X(P)-\mu}^{\infty} [X^{-1}(z-\mu-v) - b] \; f(v) \; dv - \beta = 0$$

Appendix 5

$$\frac{\partial W_3}{\partial P} = \int_{-\infty}^{z-X(P)-\mu} [P-b]\ X'(P)\ f(v)\ dv$$

This integral exists when v increases toward infinity so we can take the derivative

$$\frac{\partial^2 W_3}{\partial P^2} = \int_{-\infty}^{z-X(P)-\mu} X'(P) f(v) dv + [P-b]\ X'(P)\ f(z-X(P)-\mu) \cdot (-X'(P))$$

$$= \int_{-\infty}^{z-X(P)-\mu} X'(P) f(v) dv - [P-b](X'(P))^2\ f(z-X(P)-\mu)$$

$$X'(P) < 0 \quad \text{if} \quad P \geq b \quad \text{then:}$$

$$\frac{\partial^2 W_3}{\partial P^2} < 0$$

$$\frac{\partial W_3}{\partial z} = \int_{z-X(P)-\mu}^{\infty} [X^{-1}(z-\mu-v) - b]\ f(v)\ dv \quad - \beta$$

Assume this integral exists when $v \to \infty$ then:

$$\frac{\partial^2 W_3}{\partial z^2} = \int_{z-X(P)-\mu}^{\infty} \frac{\partial}{\partial z} [X^{-1}(z-\mu-v) - b]\ f(v)\ dv$$

$$- [X^{-1}(z-\mu-z+X(P)+\mu) - b]\ f(z-X(P)-\mu)$$

$$= \int_{z-X(P)-\mu}^{\infty} \frac{\partial}{\partial z} [X^{-1}(z-\mu-v) - b]\ f(v)\ dv$$

$$- [P-b]\ f(z-X(P)-\mu)$$

$$\frac{\partial^2 W_3}{\partial z^2} < 0 \qquad \text{if} \quad P \geq b \ .$$

CHAPTER II

Numerical Computations and Graphs

2.1. The Trend of Output when Variance of Risk Increases

In the general case, demand function has a linear form, downward sloping.

$$X(P) = a + cP$$

- a: a constant (> 0)
- c: a constant (< 0)
- P: unit price of output
- b, β: are positive constants and are defined in the previous section

Under the stochastic demand, we see that the optimum price always equals b, the short run marginal operating cost, and is independent of the mean and variance of risk. But the optimum output z is dependent on the mean and variance of risk. Now we study the optimal output z when variance of risk increases.

It is easier when we use the model in Section 1.3,

$$E[u] = 0$$

The optimum point:

$$P_1{}^* = b$$

$z_1{}^*$ such that:

$$\int_{z_1^*-X(b)}^{\infty} [X^{-1}(z_1^*-u) - b]\, f(u)\, du - \beta = 0 \qquad (2')$$

$$u \simeq N(0,\sigma^2)$$

$$f(u) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{u^2}{\sigma^2}}$$

$$X(P) = a + cP$$

and $\qquad X^{-1}(z-u) = \dfrac{z-u-a}{c}$

Substituting in $(2')$:

$$\int_{z-a-cb}^{\infty} \left[\frac{z-u-a}{c} - b\right] f(u)\, du - \beta = 0$$

$$= D \int_{D}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} du - \sigma \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{D^2}{\sigma^2}} - c\beta = 0 \qquad (3')$$

where: $D = z-a-cb$

The function $(3')$ is an implicit function of z and σ.

$$\frac{\partial z}{\partial \sigma} = -\frac{f\sigma}{fz}$$

$f\sigma$: derivative of $(3')$ with respect to

fz: derivative of $(3')$ with respect to z

$$\frac{\partial z}{\partial \sigma} = \frac{\frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}\frac{D^2}{\sigma^2}}}{\int_D^\infty \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} du} > 0 \qquad \text{appendix 6}$$

$$u \geq z-a-cb$$

$\frac{\partial z}{\partial \sigma} > 0$ for any original point (z,σ),

that means that when σ increases then z increases, or σ and z vary in the same direction.

Appendix 6

Call:

$$f(\sigma z) = D \int_D^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} du - \int_D^{\infty} \frac{1}{\sqrt{2\pi}} \frac{u}{\sigma} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} du - c\beta = 0$$

$$\int_D^{\infty} \frac{1}{\sqrt{2\pi}} \frac{u}{\sigma} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} du = \sigma \int_{D/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} Xe^{-\frac{1}{2}X^2} dX$$

$$= \sigma \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{D^2}{\sigma^2}}$$

$$f(\sigma,z) = D \int_D^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} du - \sigma \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{D^2}{\sigma^2}} - c\beta = 0$$

$$D = z-a-cb$$

$$fz = \frac{\partial f(\sigma,z)}{\partial z}$$

$$= \int_D^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} du - D \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\frac{D^2}{\sigma^2}} - \frac{\sigma}{\sqrt{2\pi}} \left(-\frac{1}{2\sigma^2}(2D)\right) e^{-\frac{1}{2}\frac{D^2}{\sigma^2}}$$

$$fz = \int_D^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} du$$

$fz > 0$ for any D .

$$f\sigma = \frac{\partial f(\sigma, z)}{\partial \sigma}$$

$$= D \int_D^{\infty} \frac{1}{\sqrt{2\pi}} [- \frac{1}{\sigma^2} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} + \frac{u^2}{\sigma^4} e^{-\frac{1}{4}\frac{u^2}{\sigma^2}}] \quad du$$

$$- \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{D^2}{\sigma^2}} - \frac{1}{\sqrt{2\pi}} \frac{D^2}{\sigma^2} e^{-\frac{1}{2}\frac{D^2}{\sigma^2}}$$

$$-f\sigma = \frac{D}{\sigma^2} \int_D^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} du - \frac{D}{\sigma} \int_{D/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} X^2} dX$$

$$+ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{D^2}{\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{D^2}{\sigma^2}} > 0$$

$$\frac{\partial z}{\partial \sigma} = \frac{-f\sigma}{fz} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{D^2}{\sigma^2}}}{\int_D^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} du} > 0$$

$$\frac{\partial z}{\partial \sigma} > 0$$

2.2. The Trend of Output When Mean of Risk Distribution Increases

When the mean of risk distribution is not equal to zero, the optimum decisions are the solutions of the first-order conditions in section 1.4.

$$P_2^* = b$$

z_2^* such that:

$$f(\mu,z) = \int_{z_2^*-X(b)}^{\infty} [X^{-1}(Z_2^*-u) - b]\ f(u)\ du - \beta = 0$$

$$u \simeq N(\mu,\sigma^2) \qquad \sigma\text{:constant}$$

$$f(u) = \frac{1}{\sqrt{2\pi}}\ \frac{1}{\sigma}\ e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2}$$

with the same demand function in section 2.1. The implicit function $f(\mu,z)$ becomes:

$$f(\mu,z) = D \int_{D}^{\infty} f(u)\ du - \int_{D}^{\infty} u\ f(u)\ du - c\beta = 0$$

$$f(\mu,z) = \frac{D-\mu}{\sigma} \int_{D}^{\infty} \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du - \frac{\sigma}{\sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{D-\mu}{\sigma}\right)^2} - c\beta = 0$$

$$D = z-a-cb$$

$$\frac{\partial z}{\partial \mu} = \frac{-f\mu}{fz}$$

$$f\mu = \frac{\partial f(\mu, z)}{\partial \mu}$$

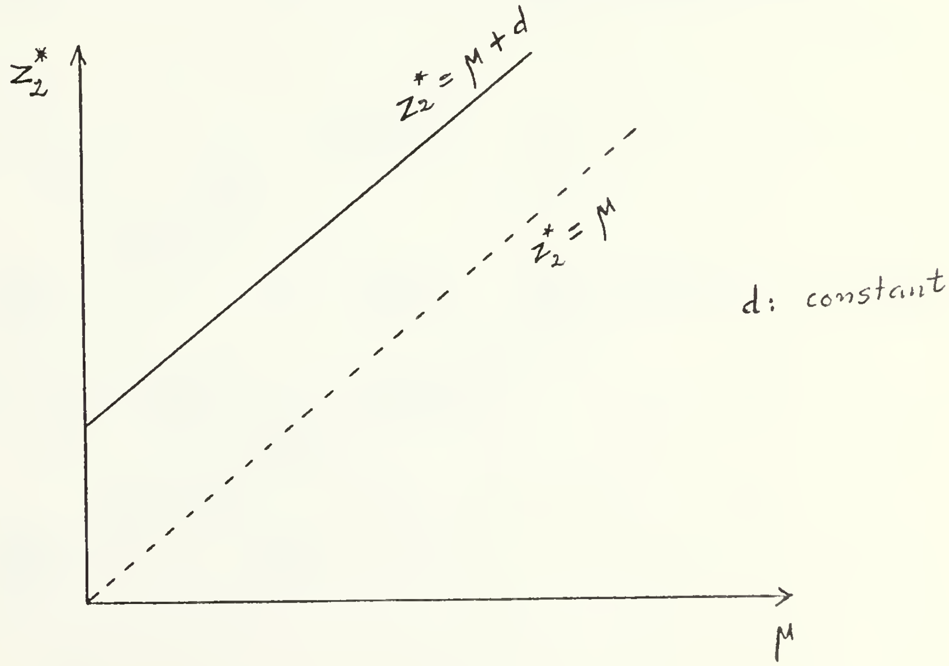
where

$$fz = \frac{\partial f(\mu, z)}{\partial z}$$

And, as appendix 7 shows:

$$\frac{\partial z}{\partial \mu} = 1 > 0$$

So that the optimum output increases the same amount as the mean of risk distribution increases, or decreases the same amount of the mean of risk distirbution decreasing.



Appendix 7

$$f(\mu,z) = \frac{D-\mu}{\sigma}\int_D^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du - \frac{\sigma}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{D-\mu}{\sigma}\right)^2} - c\beta = 0$$

$$fz = \frac{1}{\sigma}\int_D^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du + \frac{D-\mu}{\sigma}\left(-\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{D-\mu}{\sigma}\right)^2}\right)$$

$$- \frac{\sigma}{\sqrt{2\pi}}\left(-\frac{1}{2}\quad 2\left(\frac{D-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right)\, e^{-\frac{1}{2}\left(\frac{D-\mu}{\sigma}\right)^2}\right.$$

$$fz = \frac{1}{\sigma}\int_D^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du > 0$$

$$fz > 0$$

$$f\mu = -\frac{1}{\sigma}\int_D^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du + \frac{D-\mu}{\sigma}\int_D^\infty \frac{1}{\sqrt{2\pi}}\left(\frac{u-\mu}{\sigma}\right) \cdot$$

$$\cdot e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du - \sigma\,\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{D-\mu}{\sigma}\right)^2} \cdot \frac{D-\mu}{\sigma^2}$$

$$= -\frac{1}{\sigma}\int_D^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du + \frac{D-\mu}{\sigma^2}\int_D^\infty \frac{1}{\sqrt{2\pi}}\left(\frac{u-\mu}{\sigma}\right) \cdot$$

$$\cdot e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du - \frac{D-\mu}{\sigma}\,\frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{D-\mu}{\sigma}\right)^2}$$

$$= -\frac{1}{\sigma}\int_D^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du$$

$$\frac{\partial z}{\partial \mu} = \frac{-f\mu}{fz} = 1 > 0$$

2.3. Consequence of Section 1.5

If u is normally distributed with mean μ and variance σ^2, and v is also normally distributed with mean zero and variance σ^2, the same variance as u.

The demand function is just the same as in the previous section,

$$X(P) = a + cP$$

so the solutions of section 1.4 become:

$$P_2^* = b$$

z_2^* is the solution of this equation

$$f(\mu,z) = (D-\mu) \int_D^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} du - \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{D-\mu}{\sigma}\right)^2} - c\beta = 0$$

$$D = z-a-cb \qquad \text{(appendix 7)}$$

$$f(\mu,z) = (D-\mu)\left[1 - \Phi\left(\frac{D-\mu}{\sigma}\right)\right] - \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{D-\mu}{\sigma}\right)^2} - c\beta = 0$$

and the solutions of section 1.5 are the same solutions of section 1.4 above (Appendix 8)

Thus the demand under normal distribution of risk with mean μ and variance σ^2 is the same as adding a positive constant μ to the demand and the risk becomes normally distributed with mean zero and the same variance; or the second model in sections 1.4 and 1.5 are just the same under those conditions above.

Appendix 8

The first-order conditions of section 1.5 are:

$$P_3{}^* = b$$

$z_3{}^*$ is the solution of equation $f^*(\mu z)$

$$f^*(\mu z) = \int_{z-X(P)-\mu}^{\infty} [X^{-1}(z-\mu-v) - b]\ f(v)\ dv - \beta = 0$$

$$f^*(\mu,z) = \int_{D-\mu}^{\infty} (D-\mu)\ f(v)\ dv - \int_{D-\mu}^{\infty} v\ f(v)\ dv - c\beta = 0$$

$$= (D-\mu)\left[1 - \Phi\left(\frac{D-\mu}{\sigma}\right)\right] - \frac{\sigma}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{D-\mu}{\sigma}\right)^2} - c\beta = 0$$

$$\Phi\left(\frac{D-\mu}{\sigma}\right) = \int_{-\infty}^{D} \frac{1}{\sqrt{2\pi}}\, \frac{1}{\sigma}\, e^{-\frac{1}{2}\frac{v^2}{\sigma^2}}\, dv \qquad D = z_3^* - a - cb$$

These solutions are the same as the solutions in section 1.4.

2.4. Numerical Computations and Graphs

$a = 10$ units $\qquad b = \$2$

$c = -1 \qquad \beta = \$1$

$X(P) = 10 - P$

Use the model in section 1.3 with mean of risk distribution equal to zero; the optimum output z is the solution of this equation:

$$D \int_{D}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}\frac{u^2}{\sigma^2}} du - \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{D^2}{\sigma^2}} - c\beta = 0$$

There are many optimum output z when variance σ^2 has some special values, keeping the mean constant and equal to zero.

z*	σ
7	.5
7.1	1
7.32	1.5
7.615	2
8	2.5
8.42	3
8.88	3.5
9.375	4
9.9	4.5

The relationship between z* and σ is expressed by the implicit function above. Now we try to express z as a linear function of σ.

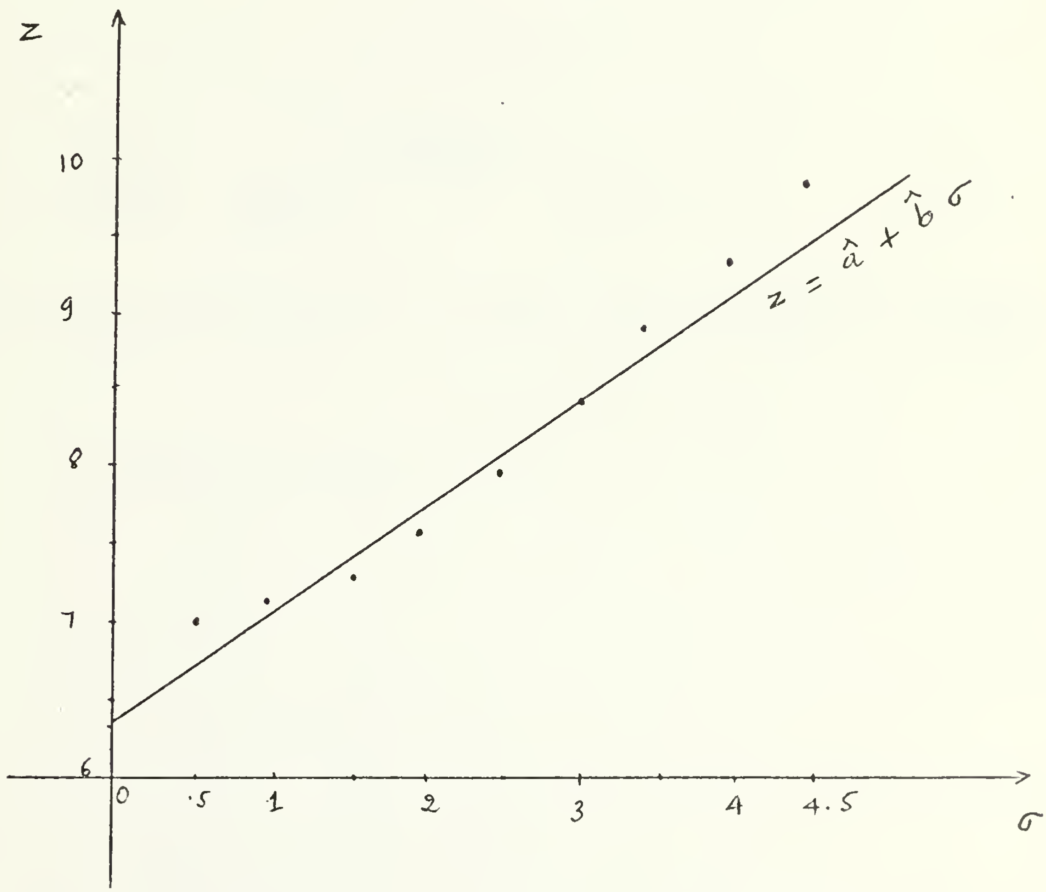
$$z = \hat{a} + \hat{b}\sigma$$

Using the least squares method, we can obtain the value of $\hat{a}$ and $\hat{b}$:

$$\hat{a} = 6.3163$$
$$\hat{b} = .745$$

$$z = 6.3163 + .745\sigma$$

This function is upward sloping with σ, so z increases as the value of σ increases:



Now we hold the variance σ^2 constant and equal to 1. Let the mean of risk distribution vary and observe the optimum output z*. Under these conditions the optimum output z* is the solution of the equation $f(\mu,z) = 0$ in section 2.2. We rewrite that equation as the following:

$$f(\mu,z) = (D-\mu) \int_D^\infty \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2}(\frac{u-\mu}{\sigma})^2} du - \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{D-\mu}{\sigma})^2} - c\beta = 0$$

or:

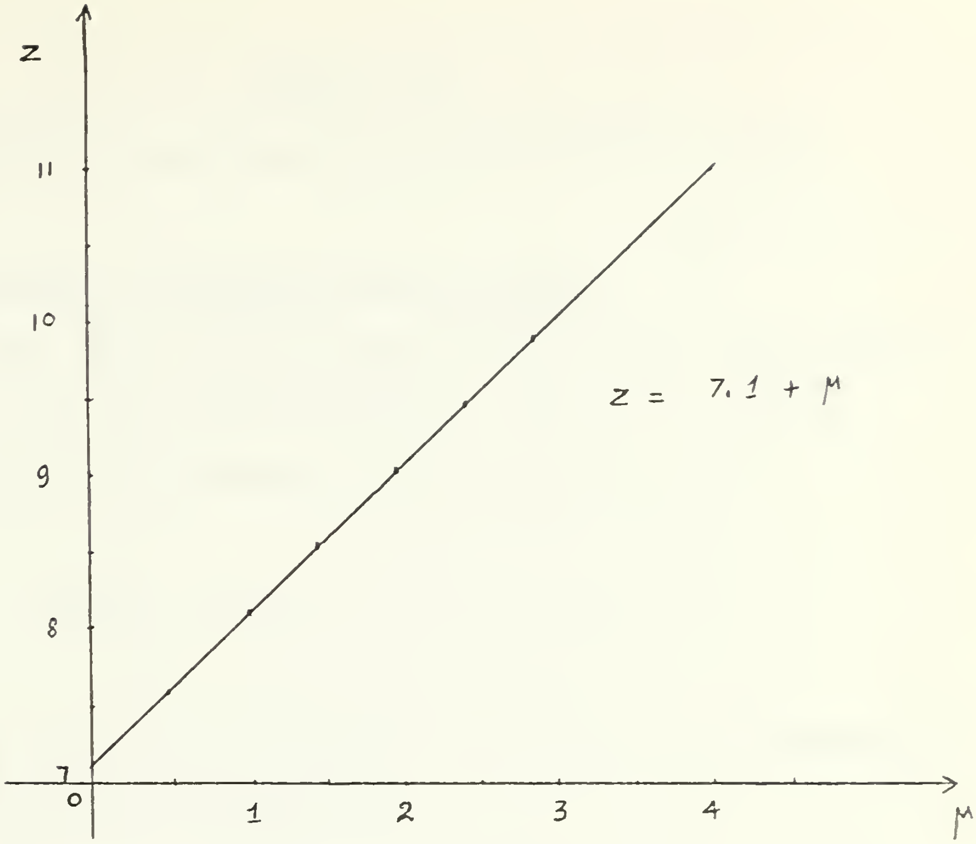
$$(D-\mu)\left[1 - \Phi\left(\frac{D-\mu}{\sigma}\right)\right] - \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{D-\mu}{\sigma})^2} - c\beta = 0$$

These are the solutions of z in the case variance σ is equal to 1.

z*	μ
7.1	0
7.6	.5
8.1	1
8.6	1.5
9.1	2
9.6	2.5
10.1	3
10.6	3.5
11.1	4

We can express z as a linear function of μ, this function is upward sloping with μ.

$$z = 7.1 + \mu \ .$$


Z
11
10
9
8
7
0
1
2
3
4
μ
Z = 7.1 + μ

CHAPTER III

THEORY OF MAXIMIZATION OF THE EXPECTED PROFIT

3.1. The Model of Maximization of Profit

In this model, we also set the quantity z and price P before we know the actual demands. Demand is a random variable, and it is a function of P and u:

$$\text{Demand} = X(P) + u$$

where u is normally distributed with mean μ and variance σ^2.

The expression of profit is:

$$\Pi = PQ_s - c'z$$

where: Q_s = the actual sales

$Q_s = \min[X(P) + u, z]$

c' = cost per unit in production

Therefore:

$$\Pi = P[X(P) + u] - c'z \qquad \text{if} \quad X(P) + u < z$$

$$\Pi_2 = Pz - c'z \qquad \text{if} \quad X(P) + u \geq z$$

$$E[\Pi] = \int_{-\infty}^{z-X(P)} \Pi \, f(u) \, du + \int_{z-X(P)}^{\infty} \Pi_2 \, f(u) \, du$$

Our problem is to try to maximize the expected profit . The first-order conditions for maximization are:

$$\frac{\partial E(\Pi)}{\partial P} = [X(P) + P\,X'(P) - z + \mu]\,\Phi(Y) + z - \frac{\sigma}{\sqrt{2\pi}}\,e^{-\frac{1}{2}Y^2} = 0$$

$$Y = \frac{z-X(P)-\mu}{\sigma}$$

$$\frac{\partial E(\Pi)}{\partial z} = P[1 - \Phi(Y)] - c' = 0$$

Appendix 9

The second-order conditions for maximization are:

$$\frac{\partial^2 E(\Pi)}{\partial P^2} = 2X'(P)\,\Phi(Y) + [X(P) + P\,X'(P) - z + \mu]\,f(z-X(P))\cdot$$

$$\cdot(-X'(P)) - X'(P)\,Y\cdot\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}Y^2}$$

$$\frac{\partial^2 E(\Pi)}{\partial z^2} = -P\,f(z-X(P)) < 0$$

Appendix 9

where: $\Phi(Y) = \int_{-\infty}^{z-X(P)} f(u)\,du$

In our model with the linear demand $X(P) = 10 - P$, u is normally distributed (μ,σ^2). Substituting in the expressions above:

$$\frac{\partial^2 E(\Pi)}{\partial P^2} = -2\ \Phi(Y) - P\ f(Y) < 0$$

So that the solutions of the first-order conditions are the global maximum solutions of the problem.

Appendix 9

$$E[\Pi] = \int_{-\infty}^{z-X(P)} \Big[P[X(P) + u] - c'z \Big] f(u)\, du$$

$$+ \int_{z-X(P)}^{\infty} [P-c']z\, f(u)\, du$$

$$\frac{\partial E(\Pi)}{\partial P} = \int_{-\infty}^{z-X(P)} \Big[X(P) + u + P\, X'(P) \Big] f(u)\, du$$

$$+ \Big[P[X(P)+z-X(P)] - c'z \Big] \quad f(z-X(P))(-X'(P))$$

$$+ \int_{z-X(P)}^{\infty} z\, f(u)\, du - [P - c']z\, f(z-X(P))(-X'(P))$$

$$= [X(P) + P\, X'(P)]\ \Phi\left(\frac{z-X(P)-\mu}{\sigma}\right) + \int_{-\infty}^{z-X(P)} u\, f(u)\, du$$

$$+ z\left[1 - \Phi\left(\frac{z-X(P)-\mu}{\sigma}\right)\right]$$

$$\frac{\partial E(\Pi)}{\partial P} = [X(P) + P\, X'(P) + \mu - z]\ \Phi(Y) + z - \frac{\sigma}{\sqrt{2\pi}}\, e^{-\frac{1}{2} Y^2} = 0$$

$$\frac{\partial E(\Pi)}{\partial z} = \int_{-\infty}^{z-X(P)} -c' \; f(u)\; du + \left[P(X(P) + z - X(P)) - c'z \right] \cdot$$

$$\cdot f(z-X(P)) + \int_{z-X(P)}^{\infty} [P-c'] \; f(u) \; du - [P-c'] \; z \; f(z-X(P))$$

$$\frac{\partial E(\Pi)}{\partial z} = P[1 - \Phi(Y)] - c' = 0$$

$$\frac{\partial^2 E(\Pi)}{\partial P^2} = [X'(P) + X'(P)] \int_{-\infty}^{z-X(P)} f(u)\; du + [X(P) + PX'(P) - z + \mu] \cdot$$

$$\cdot f(z-X(P))(-X'(P)) - X'(P) \left(\frac{z-X(P)-\mu}{\sigma}\right) \frac{1}{\sqrt{2\pi}} \; e^{-\frac{1}{2}\left(\frac{z-X(P)-\mu}{\sigma}\right)^2}$$

$$= 2X'(P) \; \Phi(Y) + [X(P) + PX'(P) + \mu - z] \; f(z-X(P))$$

$$- X'(P) \; Y \frac{1}{\sqrt{2\pi}} \; e^{-\frac{1}{2} Y^2}$$

$$\frac{\partial^2 E(\Pi)}{\partial P^2} = 2X'(P) \; \Phi(Y) + [X(P) + PX'(P) + \mu - z] \; f(z-X(P))(-X'(P))$$

$$- X'(P) \; Y \frac{1}{\sqrt{2\pi}} \; e^{-\frac{1}{2} Y^2}$$

$$\frac{\partial^2 E(\Pi)}{\partial z^2} = -P \; f(z-X(P)) < 0$$

3.2. The Optimal Point (P*,z*) under a Change in Mean and Variance of the Distribution u

We introduce the same demand function, and the same value of a, c, b, β .

$$X(P) = a + cP$$

where: $a = 10$ $\quad b = 2$

$c = -1$ $\quad \beta = 1$

$c' = b+\beta = 3$

The first-order conditions now become:

$$(5) \quad [a + 2cP + \mu - z]\, \Phi(Y) + z - \frac{\sigma}{\sqrt{2\pi}}\, e^{-\frac{1}{2} Y^2} = 0$$

$$(6) \quad P[1 - \Phi(Y)] - c' = 0$$

Now we keep the mean constant and equal to zero. The following results are the solutions of equations (5) and (6) according to each value of variance σ^2.

σ	P*	z*	$\mu = 0$
.5	6.41	3.63	
1	6.32	3.745	
1.5	6.22	3.847	
2	6.11	3.935	
2.5	6	4	
3	5.88	4.03	
3.5	5.75	4.0575	
4	5.61	4.04	

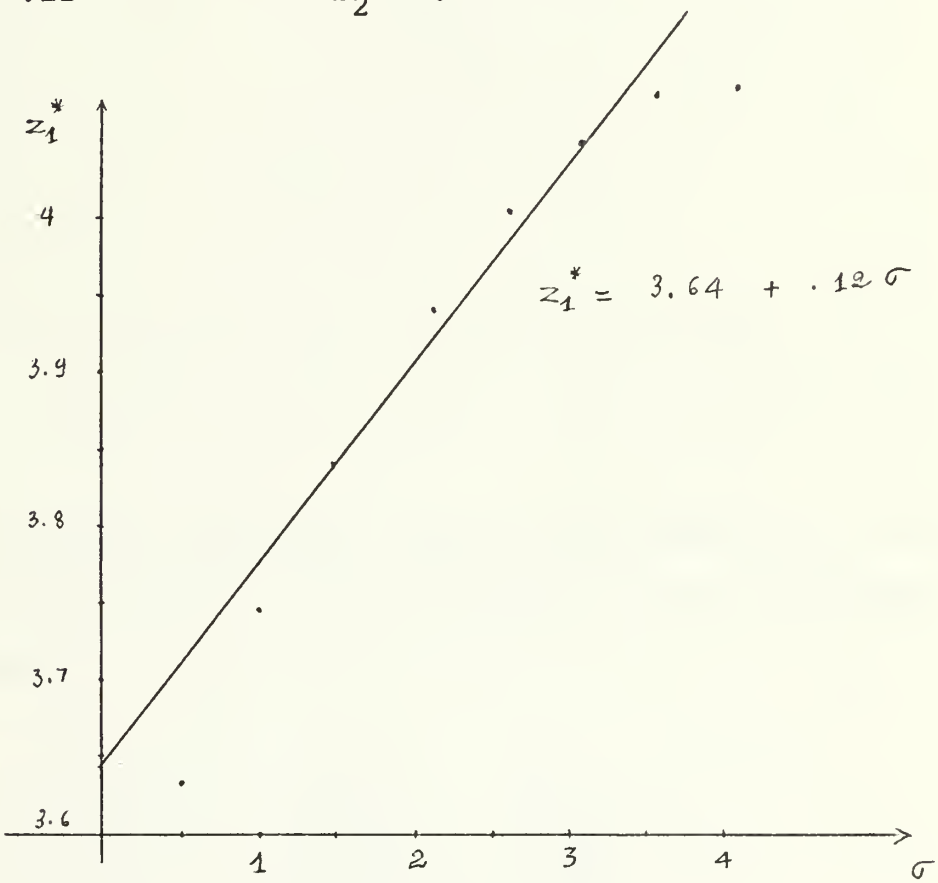
As variance increases, the price slowly decreases, and the optimum output increases a little.

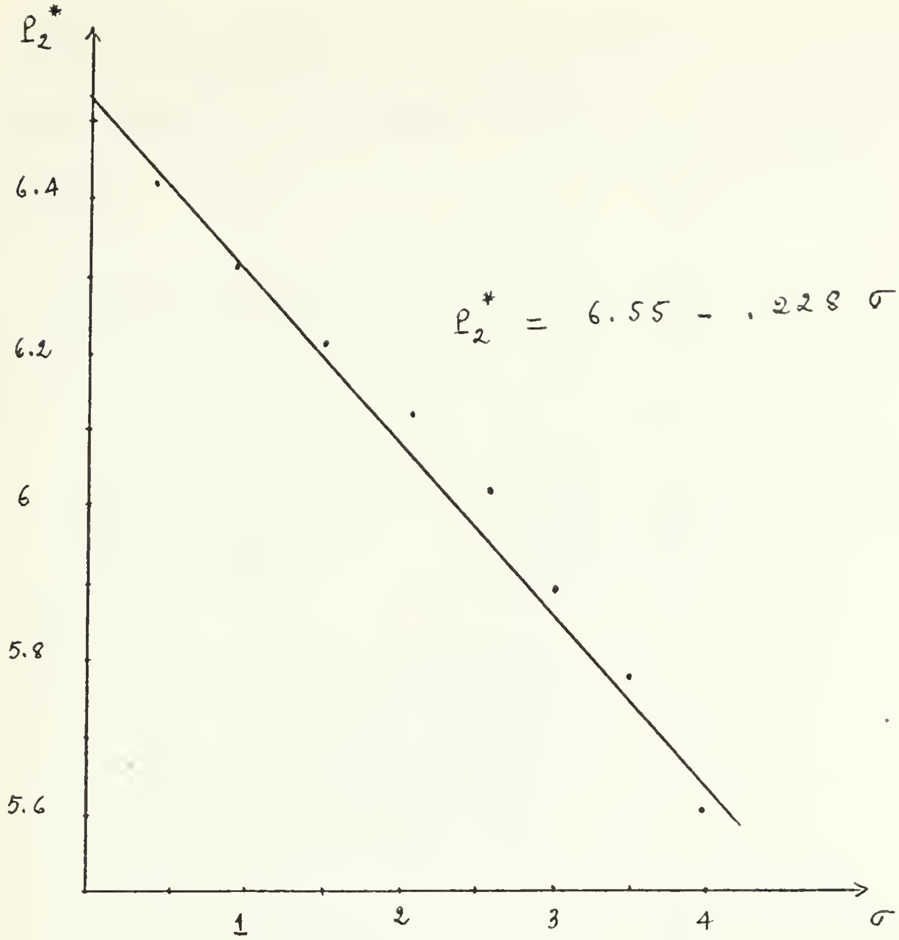
Using least squares, we can find the function $z^*(\sigma)$, and $P^*(\sigma)$ as the linear function of σ:

$z_1^* = \hat{a}_1 + \hat{b}_1\sigma$; $P_2^* = \hat{a}_2 + \hat{b}_2\sigma$

$\hat{a}_1 = 3.64$ $\hat{a}_2 = 6.55$

$\hat{b}_1 = .12$ $\hat{b}_2 = .228$





Now keep the variance σ constant and equal to 1. The solutions of equation (5) and equation (6) are tabulated as follows:

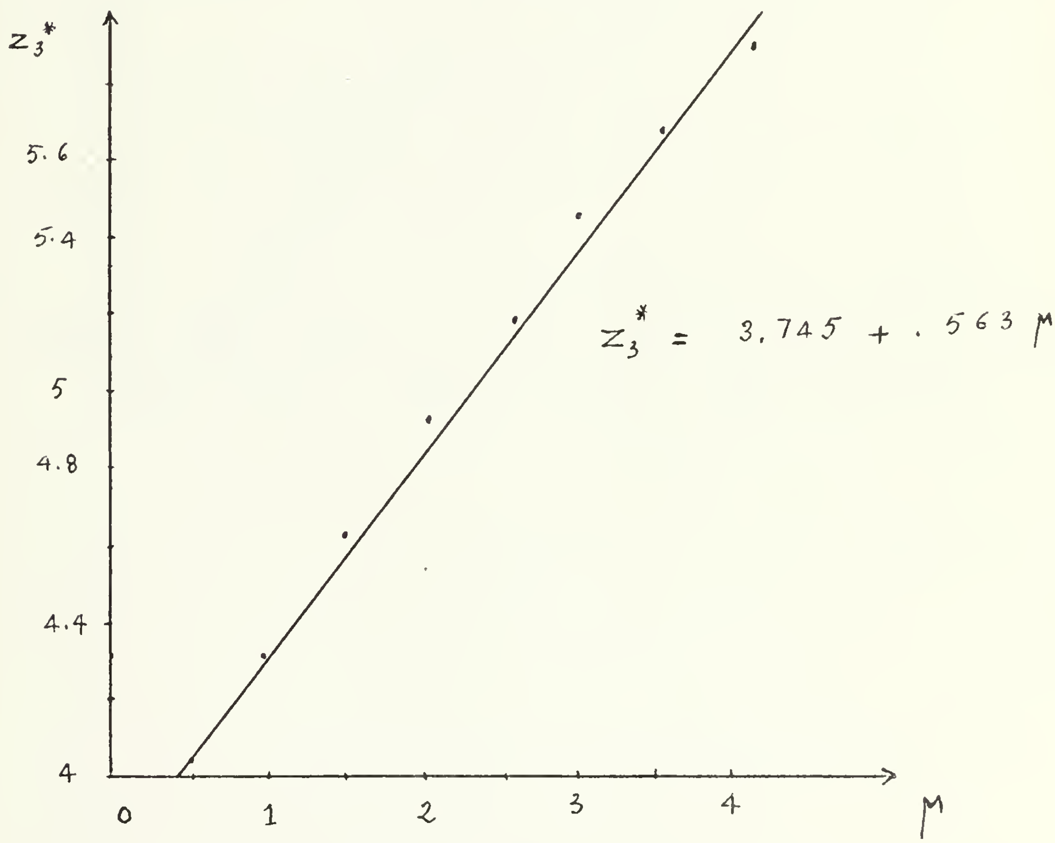
μ	P*	z*	σ = 1
.5	6.6	4.015	
1	6.84	4.315	
1.5	7.1	4.595	
2	7.35	4.88	
2.5	7.61	5.15	
3	7.87	5.43	
3.5	8.12	5.71	
4	8.38	6.	

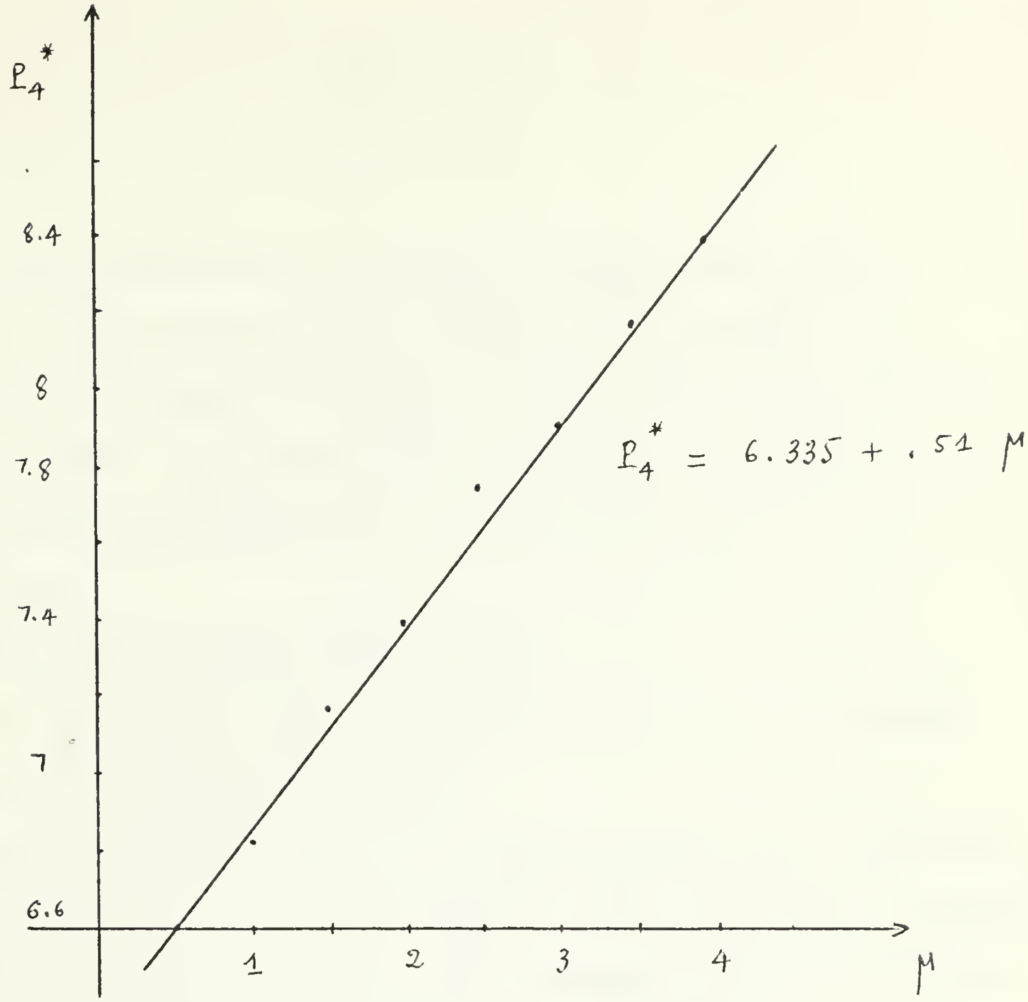
When μ increases, both price and optimum output increase. We can estimate the linear relationship between z* and μ, P* and μ.

$$z_3{}^* = \hat{a}_3 + \hat{b}_3 \mu \quad ; \quad P_3{}^* = \hat{a}_4 + \hat{b}_4 \mu$$

$\hat{a}_3 = 3.745$ $\qquad$ $\hat{a}_4 = 6.335$

$\hat{b}_3 = .563$ $\qquad$ $\hat{b}_4 = .51$




P_4^*
8.4
8
7.8
7.4
7
6.6
1
2
3
4
μ
$P_4^* = 6.335 + .51\ \mu$

CHAPTER IV

CONCLUSION

We have developed two kinds of models, the first is to maximize the expected social welfare, the second is to maximize the expected monopoly profit; both models involve random demand. There are three essential factors in determining the optimal price and production level decisions for a firm.

a) the shape of the demand curve

b) the manner in which risk enters the demand function

c) the distribution of risk u.

Under riskless demand, the optimum price and output for maximization of the welfare function are determined. Adding risk to the demand function X(P) reduces the optimum price.

If the mean and variance of risk u are allowed to vary, the optimal price P* remains constant and equals the short run marginal operating cost, but the optimal output varies directly with mean or variance of risk u. Also, the optimal output under risk is greater than the equilibrium output in the deterministic model.

When the variance of risk stays constant and equals one, the optimal output varies directly and the same scale with the mean of risk u.

The optimal solutions are the same when the random linear demand curve X(P) + u with normal distribution (μ,σ^2) of

risk u is shifted upward a distance μ with normal distribution $(0,\sigma^2)$ of risk u.

It is easy to understand this effect because of the symmetry property of normal distributions. In Chapter III the demand function and the risk u for the expected profit model are the same as in Chapters I and II. With the mean of risk set equal to zero and allowing the variance to increase, the optimum price decreases slowly and the optimum output increases slowly.

Allowing the mean to vary while the variance is set to one causes optimal price and output to vary directly but very little. Therefore, this model is less sensitive to risk than the models of Chapters I and II. If the demand curve is not linear downward sloping and additively separable in P and u or risk has a complicated distribution, then it is difficult to draw conclusions for operating the firms.

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