$$\frac{d^2y}{dx^2} = \frac{ds}{dx} \times \frac{w}{T} = \sqrt{\left(1+\left|\frac{dy}{dx}\right|^2\right)} \times \frac{w}{T} \quad \ldots \text{(E)}$$

The equation (E) gives the general conditions of catenary curves, viz.:—the relations between the form of a curve, the horizontal strain, and the sectional weight, whether constant or variable.

The equations (A), (B), (C), and (D) of course hold good for all catenary curves. When the sectional weight of the chain is uniform (or w a constant) the curve of equilibrium will be that known as the Common Catenary.

In this case, the load l and the arc s being alike measured from the lowest point of the curve, $l = w \times s$ ∴ (from equation (B),

$$\frac{dy}{dx} = \left(\frac{l}{T} = \right)\frac{ws}{T} \quad \ldots \quad \ldots \quad \ldots \text{(1.)}$$

$$\text{But } \frac{ds}{dx} = \sqrt{\left(1+\left|\frac{dy}{dx}\right|^2\right)}$$

$$\therefore \frac{ds}{dx} = \sqrt{\left(1+\frac{w^2}{T^2}s^2\right)} \quad \ldots \quad \ldots \text{(2.),}$$

Let $\frac{T}{w} = a$ (which of course is a constant); then (2) gives, after substitution of $\frac{1}{a^2}$ for $\frac{w^2}{T^2}$,

$$\frac{ds}{dx} = \sqrt{1+\frac{s^2}{a^2}} \ldots \quad \ldots \quad \ldots \text{(3.)}$$

The definite solution of this differential equation (when the origin of x is taken at the lowest point of the curve) will be

$$s = \frac{a}{2}\left(E^{\frac{x}{a}} - E^{-\frac{x}{a}}\right)^{*} \quad \ldots \quad \ldots \text{(4.)}$$

$$\text{But, from (1), } \frac{dy}{dx} = \frac{s}{a}, \left(=\frac{1}{2}E^{\frac{x}{a}} - E^{-\frac{x}{a}}\right)$$

$$\therefore y = \frac{a}{2}\left(E^{\frac{x}{a}} + E^{-\frac{x}{a}} - 2\right) \quad \ldots \quad \ldots \text{(5.)}$$

(y being=0 when x=0.)

Let t = the amount of tension at the point (x, y) in the direction of the tangent of the curve. Then, by resolving t into the horizontal tension T and the load l, we find that

$$t^2 = T^2 + l^2 = w^2(a^2+s^2) \quad \ldots \quad \ldots \text{(6.)}$$

whence it is readily deduced that

$$t = w \times \frac{a}{2}\left(E^{\frac{x}{a}} + E^{-\frac{x}{a}}\right) \quad \ldots \quad \ldots \text{(7.)}$$

$$\text{Or, } t = w \times (y+a) \ldots \quad \ldots \quad \ldots \quad \ldots \text{(8.)}$$

The simple relations between the Tension, the Arc, and the Ordinate, expressed in equations (6) and (8) can be visibly displayed in a diagram. In Fig. 36, AOB is the Common Catenary. COM is the axis of y, and O is at once the lowest point of the curve and the origin.

The horizontal line DR is drawn at the distance CO=a below the axis of x, and is termed the Directrix. To find the tension (=t) at any point P, let fall PV perpendicular to the Directrix. Then, (from Equation (8)) $t = w \times$PV. From the centre C describe the circle of the radius CO=a. This might perhaps deserve the name of the Directing Circle, on account of properties to be now explained.

From P draw the horizontal line PN to meet the axis of y in N. From N draw the line NQ touching the circle in Q, and join the radius CQ. Then NQC is a right-angled triangle, of which we know the hypothenuse NC to be $=\frac{t}{w}$, and the side CQ to be $= a$. ∴ (from Equation (6), the other side NQ$= s$, =Arc OP.

It follows from this property of the *Directing Circle* that the line NQ is parallel to the tangent PT. For PT : TV : : Tangential Tension : Horizontal Tension. : : NC : CQ. And since NCQ and PTV are both right-angled triangles, and have the sides enclosing the angles at N and P respectively proportional, these angles must be equal, and consequently NQ and PT are parallel to one another.

For a known amount of Horizontal Tension (in terms of the sectional weight of chain) we can therefore very readily find the *direction* of the tangent at any given *elevation*. From these data it is possible to construct the curve without reference to tables of logarithms.

* E being the base of hyperbolic logarithms.

THE ELEVATION OF THE EXTERIOR RAIL ON RAILROAD CURVES, &c.

By Oliver Byrne, C.E.

Let w be the weight of the moving body or train, V its velocity in feet a second, R the radius of the curve, and g=the force of gravity at the surface of the earth; and let f represent the centrifugal force: then by a well-known dynamical expression

$$f = \frac{wV^2}{gR}$$

If R=one quarter of a mile=1320 feet, V=velocity=38·6 feet a second, and $g=32\frac{1}{6}$ feet,

$$\text{Then } f = \frac{w \times (38\cdot6)^2}{32\frac{1}{6} \times 1320} = \frac{193}{5500}w.$$

In this example the centrifugal force is about $\frac{1}{28}$ part of the weight.

If R=a mile=5280 feet, g as before=$32\frac{1}{6}$ feet, and V=60 miles an hour=88 feet a second, then

$$f = \frac{w \times 88^2}{32\frac{1}{6} \times 5280} = \frac{1}{22}w \text{ nearly.}$$

In this last case the centrifugal force, that urges the moving body to leave the curve, is equal $\frac{1}{22}$ part of the weight of the moving body. This force is in a great measure counteracted by the conical tread of the wheels, each pair of which is firmly fixed to an axle AB, Fig. 1. The rails C and D may be level, yet if the points of contact be at I and E, coned wheels will run round a circle whose centre is at Q. If the points of contact of the rails be I and F, then the centre will be at O; but if the wheels assume the position that the equal circles G and H come in contact with the rails C and D, the axle AB and KL would be parallel, and hence do not meet. The conical tread, the lateral play of the flanges, about half an inch on each side, and the centrifugal force of the weight moved on the curve, enlarge the diameter of the exterior wheel B, and diminish that of the interior A; hence there is a centripetal force directed towards the centres of the cones Q, O, &c., as the rails change their points of contact on the coned tires. Let d be the diameter of the wheels at the circles G and H, when they stand level, or rather when the line KL joining the points of contact is parallel to the axis AB. Let the outer diameter be increased by a variable small quantity z, as the rail touches nearer the flange; or let the outer diameter become $d+z$, it is evident that the inner diameter will become $z - z$. The value of z, generally varies from 0 to $\frac{3}{8}$ of an inch; so that the diameter of the wheels may be made to vary according to the circumstances of the motion from about 36 to 36·6 inches, sometimes more and sometimes less. Let r be the variable radius OT, answering to the increase z, and b= the gauge or breadth of the road CD; then $r+\frac{1}{2}b$ and $r-\frac{1}{2}b$ are the distances of the circles of contact from the centre O. By similar triangles

$$d+z : d-z :: r+\frac{b}{2} : r-\frac{b}{2}$$

$$r = \frac{bd}{2z}$$

If the breadth of the road=4·7 feet, the diameter of the wheel

$$\therefore d : z :: 2r : b$$

=3 feet, and z=one-tenth of an inch,

$$OT = r = \frac{12 \times 4\cdot7 \times 3 \times 12}{\cdot2} = 10152 \text{ inches} = 846 \text{ feet.}$$

Again, for the sake of example, let z equal $\frac{1}{40}$ of an inch, the breadth of the road=7 feet=84 inches, the diameter of the wheel at the level tread=30 inches. Required OT=r.

$$r = \frac{30 \times 84}{\cdot05} = 50400 \text{ inches} = 4200 \text{ feet.}$$

But the centripetal force corresponding to radius r is $p = \frac{wV^2}{gr}$ which acts in a contrary direction to the centrifugal force: they will hold each other in equilibrium when they become equal, and the train will have no inducement to fly off the track;

$$\therefore \frac{wV^2}{gr} = \frac{wV^2}{gR} \text{ or } r = R.$$

Consequently the vertex O of the imaginary cone must coincide with the centre of the curve of the railroad, to avoid slipping or dragging. But it was before shown that

$$r = \frac{bd}{2z} \quad \therefore R = \frac{bd}{2z} \quad \therefore z = \frac{bd}{2R}$$

which is the increment and decrement that the exterior and interior wheels respectively receive to produce equilibrium between the centripetal and centrifugal forces of the train.

Let the gauge of the road $= b = 64$ inches, the diameter of the wheels at the points where they rest level on the rails=36 inches; R, the radius $= 600$ feet, $2\,R \times 12 = 14{,}400$ inches; hence in this case

$$x = \frac{64 \times 36}{14400} = \cdot 16 \text{ of an inch.}$$

The coning of the wheels will not compensate for much more than this, as there is only a play of about half an inch on each side between the flanges and the rails.

Let Fig. 2 represent part of the tire of a railway car wheel; the rise from A to C half an inch in $3\frac{1}{2}$ inches, that is,

$$AB : BC :: 7 : 1.$$

But in order that the statement may be general and applicable to all cases let $AB : BC :: n : 1.$

When the wheels stand level on level rails, let E be one of the points of contact, DB the space given for the play of the wheel between the flange and the rail; this for the sake of example I put $= \frac{1}{2}$ an inch $= v$. The rail and wheel may touch at any other point between C and A, yet the space DB is unaltered, and the elevation will be as n to 1. The diameter through E is put equal d.

$$n : 1 :: v : \frac{v}{n} \quad \therefore \frac{2v}{n} = x;$$

giving to v what I have just supposed to be its greatest value, half an inch, $x = \frac{1}{7}$, and

$$r = \frac{ndb}{4v} = \frac{bd}{2x}$$

It is clear that, as x increases, r decreases. Let $d = 3$ feet, $b = 4{\cdot}7$ feet, $x = \frac{1}{7}$ of an inch$=\frac{1}{84}$ part of a foot;

$$\therefore\ r = 3 \times 4{\cdot}7 \times \frac{84}{2} = 592{\cdot}2 \text{ feet}$$

the least possible radius of curvature on the suppositions made, in which the two forces balance each other, supposing the two rails to be exactly level. I will assume another case.

Let $v = \frac{1}{4}$ of an inch, and $\frac{1}{n} = \frac{1}{7}$;

$$x = \frac{2v}{n} = \frac{1}{14} \text{ of an inch}$$

$$r = \frac{bd}{2x} = 3 \times 4{\cdot}7 \times \frac{14}{2} \times 12 = 1184{\cdot}4 \text{ feet.}$$

Again, let $v = \frac{1}{8}$ of an inch, $z = \frac{2v}{n} = \frac{1}{28}$

$$\therefore\ 2z = \tfrac{1}{168} \text{ of a foot}$$

$$\therefore\ r = 3 \times 4{\cdot}7 \times \tfrac{168}{2} = 2368{\cdot}8 \text{ feet.}$$

Hence it may be inferred that the coning of the wheels, without a rise in the outer rail, cannot be depended upon in curves of less than 2000 feet radius.

What I have said on this subject up to this is mere preliminary matter. I will now find the elevation of the exterior rail for any radius R of a railroad curve. Let x=the required elevation; b, as before, the gauge of the road; $\frac{wx}{b}$ will be the force drawing the train to the interior rail on account of the elevation x. This force must hold the centrifugal force in equilibrium, hence

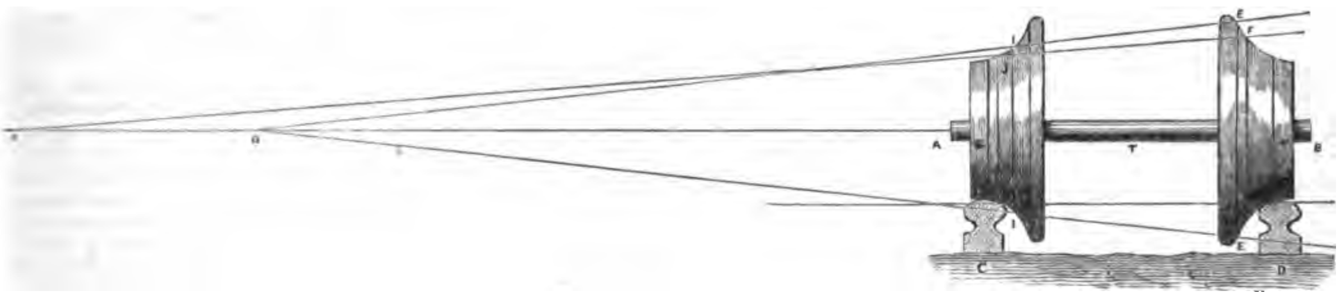
$$\frac{wx}{b} = \frac{wV^2}{gR}\ ;\ \text{V being the velocity.} \quad \therefore\ x = \frac{bV^2}{gR}$$

Example.—Let R=1910 feet, $g=32{\cdot}2$, $b=4{\cdot}7$ feet, V=44 feet a second=30 miles an hour;

$$\therefore\ x = \frac{4{\cdot}7 \times 44^2}{32{\cdot}2 \times 1910} = \cdot 224.$$

This value of x involves the elevation given by *coning* the wheels; the proper coning for a radius of 1910 feet=R, as before established, is

Fig. 1.



$$z = \frac{bd}{2R} = \frac{36 \times 56\frac{1}{2}}{2 \times 12 \times 1910} = \cdot 044$$

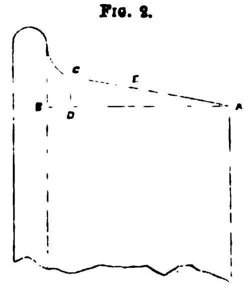
on the supposition that b=36, and d=4 ft. $8\frac{1}{2}$ in.

·224
·044

·180 inches rise of outer rail.

When the coning of the wheels is so defective that the proper circles on the wheels cannot be employed, then the outer rail is pressed and broken by the flanges. It is in this particular that engineers have erred. Wheel-makers have no system of coning, so that the wheels they make may run on a curve with a given velocity without injuring the flange and outer rail. In my Pocket-Book for Railroad and Civil Engineers, for the first time I laid down the means of effecting this object; it is simple, when the rise (x) of the outer rail is formed for any radius (R) and velocity (V); if the coning (z) cannot be established more or less without constraint, the wheels will not run on the curve with freedom. In fact, at all the speeds on every curve, the apices O, Q, &c. (Fig. 1) must coincide with the centre of the railroad curve, or damage will be done to the running gear.

Fig. 2.



Want of skill in this particular is, in America, made up by a range given to the kingbolt, a lateral-motion beam, and other contrivances.

Example.—Let the diameter of the wheels $= d = 33\frac{1}{2}$ inches, at the points where they touch on the level rails; say the coning allows the diameter of the outer wheel to be increased, while the diameter of the inner wheels are diminished ·13 inch; what is the least radius with this play, that these wheels will accommodate themselves to? and what is the elevation of the outer rail for a velocity of 50 miles an hour on a track of 819 feet radius, the gauge = 4 feet $8\frac{1}{2}$ inches?

$$R = \frac{33{\cdot}5 \times 56{\cdot}5}{2 \times \cdot 13} = \frac{b'd}{2z} = 7280 \text{ inches} = 606 \text{ ft. } 8 \text{ in.}$$

the least radius that these wheels will accurately accommodate.

50 miles = 264,000 feet; velocity $V = 73\frac{1}{3}$ feet a second.

$$x = \frac{bV^2}{gR} = \frac{4{\cdot}7 \times (73\frac{1}{3})^2}{32{\cdot}2 \times 819} = \cdot 958,$$

for a radius of 819 feet,

$$z = \frac{bd}{2R} = \frac{2{\cdot}8 \times 4{\cdot}7}{2 \times 819} = \cdot 008$$

$$\therefore\ \cdot 958 - \cdot 008 = \cdot 95, \text{ elevation of the outer rail.}$$