PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

(A.)

FOR THE YEAR MDCCCXCII.

VOL. 183.

LONDON:
PRINTED BY HARRISON AND SONS, ST. MARTIN'S LANE, W.C.,
Printers in Ordinary to Her Majesty.
MDCCCXCIII.

WORKS PUBLISHED BY THE ROYAL SOCIETY.

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1893.

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St. Petersburg (continued).
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United States (continued).

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- p. United States Commission of Fish and Fisheries.
- AB. United States Geological Survey.
- AB. United States Naval Observatory.
- A. United States Weather Bureau, Department of Agriculture.

West Point (N.Y.)
- AB. United States Military Academy.

ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1892, by the PRESIDENT and COUNCIL.

The COPLEY MEDAL to RUDOLF VIRCHOW, For.Mem.R.S., for his Investigations in Pathology, Pathological Anatomy, and Prehistoric Archæology.

The RUMFORD MEDAL to NILS C. DUNÉR, for his Spectroscopic Researches on Stars.

A ROYAL MEDAL to JOHN NEWPORT LANGLEY, F.R.S., for his Work on Secreting Glands, and on the Nervous System.

A ROYAL MEDAL to the Rev. CHARLES PRITCHARD, D.D., F.R.S., for his Work on Photometry and Stellar Parallax.

The DAVY MEDAL to FRANÇOIS MARIE RAOULT, for his Researches on the Freezing Points of Solutions, and on the Vapour Pressures of Solutions.

The DARWIN MEDAL to Sir JOSEPH DALTON HOOKER, F.R.S., on account of his important Contributions to the progress of Systematic Botany, as evidenced by the "Genera Plantarum" and the "Flora Indica;" but more especially on account of his intimate association with Mr. DARWIN in the studies preliminary to the "Origin of Species."

The Bakerian Lecture, "On the Grand Currents of Atmospheric Circulation," was delivered for Professor JAMES THOMSON, F.R.S., by the President.

The Croonian Lecture, "Les Phénomènes psychiques et la Température du Cerveau," was delivered by Professor ANGELO MOSSO.

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(A)

VOL. 183.

LIST OF ILLUSTRATIONS.

PHILOSOPHICAL TRANSACTIONS.

I. *On the Physics of Media that are Composed of Free and Perfectly Elastic Molecules in a State of Motion.*

By J. J. WATERSTON.

Communicated by Captain BEAUFORT, *R.N.*, *F.R.S.*, *&c.*

Received December 11, 1845,—Read March 5, 1846.

[PLATES 1, 2.]

Introduction by Lord RAYLEIGH, *Sec.R.S.*

THE publication of this paper after nearly half a century demands a word of explanation; and the opportunity may be taken to point out in what respects the received theory of gases had been anticipated by WATERSTON, and to offer some suggestions as to the origin of certain errors and deficiencies in his views.

So far as I am aware, the paper, though always accessible in the Archives of the Royal Society, has remained absolutely unnoticed. Most unfortunately the abstract printed at the time ('Roy. Soc. Proc.,' 1846, vol. 5, p. 604; here reprinted as Appendix I.), gave no adequate idea of the scope of the memoir, and still less of the nature of the results arrived at. The deficiency was in some degree supplied by a short account in the 'Report of the British Association' for 1851 (here reprinted as Appendix II.), where is distinctly stated the law, which was afterwards to become so famous, of the equality of the kinetic energies of different molecules at the same temperature.

My own attention was attracted in the first instance to WATERSTON'S work upon the connection between molecular forces and the latent heat of evaporation, and thence to a paper in the 'Philosophical Magazine' for 1858, "On the Theory of Sound." He there alludes to the theory of gases under consideration as having been started by HERAPATH in 1821, and he proceeds :—

"Mr. HERAPATH unfortunately assumed heat or temperature to be represented by the simple ratio of the velocity instead of the square of the velocity—being in this apparently led astray by the definition of motion generally received—and thus was baffled in his attempts to reconcile his theory with observation. If we make this change in Mr. HERAPATH's definition of heat or temperature, viz., that it is proportional to the *vis viva*, or square velocity of the moving particle, not to the momentum, or simple ratio of the velocity, we can without much difficulty deduce, not only the primary laws of elastic fluids, but also the other physical properties of gases enumerated above in the third objection to NEWTON's hypothesis. In the Archives of the Royal Society for 1845–1846, there is a paper "On the Physics of Media that consists of perfectly Elastic Molecules in a State of Motion," which contains the synthetical reasoning upon which the demonstration of these matters rests. The velocity of sound is therein deduced to be equal to the velocity acquired in falling through three-fourths of a uniform atmosphere. This theory does not take account of the size of the molecules. It assumes that no time is lost at the impact, and that if the impacts produce rotatory motion, the *vis viva* thus invested bears a constant ratio to the rectilineal *vis viva*, so as not to require separate consideration. It also does not take account of the probable internal motion of composite molecules; yet the results so closely accord with observation in every part of the subject as to leave no doubt that Mr. HERAPATH's idea of the physical constitution of gases approximates closely to the truth. M. KRÖNIG appears to have entered upon the subject in an independent manner, and arrives at the same result; M. CLAUSIUS, too, as we learn from his paper "On the Nature of the Motion we call Heat" ('Phil. Mag.,' vol. 14, 1857, p. 108)."

Impressed with the above passage and with the general ingenuity and soundness of WATERSTON's views, I took the first opportunity of consulting the Archives, and saw at once that the memoir justified the large claims made for it, and that it marks an immense advance in the direction of the now generally received theory. The omission to publish it at the time was a misfortune, which probably retarded the development of the subject by ten or fifteen years. It is singular that WATERSTON appears to have advanced no claim for subsequent publication, whether in the Transactions of the Society, or through some other channel. At any time since 1860 reference would naturally have been made to MAXWELL, and it cannot be doubted that he would have at once recommended that everything possible should be done to atone for the original failure of appreciation.

It is difficult to put oneself in imagination into the position of the reader of 1845, and one can understand that the substance of the memoir should have appeared speculative and that its mathematical style should have failed to attract. But it is startling to find a referee expressing the opinion that "the paper is nothing but nonsense, unfit even for reading before the Society." Another remarks "that the whole investigation is confessedly founded on a principle entirely hypothetical, from which it is the object to deduce a mathematical representation of the phenomena of elastic media. It exhibits much skill and many remarkable accordances with the general facts, as well as numerical values furnished by observation. . . . The original principle itself involves an assumption which seems to me very difficult to admit, and by no means a satisfactory basis for a mathematical theory, viz., that the elasticity of a medium is to be measured by supposing its molecules in vertical motion, and making a succession of impacts against an elastic gravitating plane." These remarks are not here quoted with the idea of reflecting upon the judgment of the referee, who was one of the best qualified authorities of the day, and evidently devoted to a most difficult task his careful attention; but rather with the view of throwing light upon the attitude then assumed by men of science in regard to this question, and in order to point a moral. The history of this paper suggests that highly speculative investigations, especially by an unknown author, are best brought before the world through some other channel than a scientific society, which naturally hesitates to admit into its printed records matter of uncertain value. Perhaps one may go further and say that a young author who believes himself capable of great things would usually do well to secure the favourable recognition of the scientific world by work whose scope is limited, and whose value is easily judged, before embarking upon higher flights.

One circumstance which may have told unfavourably upon the reception of WATERSTON'S paper is that he mentions no predecessors. Had he put forward his investigation as a development of the theory of D. BERNOULLI, a referee might have hesitated to call it nonsense. It is probable, however, that WATERSTON was unacquainted with BERNOULLI'S work, and doubtful whether at that time he knew that HERAPATH had to some extent foreshadowed similar views.

At the present time the interest of WATERSTON'S paper can, of course, be little more than historical. What strikes one most is the marvellous courage with which he attacked questions, some of which even now present serious difficulties. To say that he was not always successful is only to deny his claim to rank among the very foremost theorists of all ages. The character of the advance to be dated from this paper will be at once understood when it is realised that WATERSTON was the first to introduce into the theory the conception that heat and temperature are to be measured by *vis viva*. This enabled him at a stroke to complete BERNOULLI'S explanation of pressure by showing the accordance of the hypothetical medium with the law of DALTON and GAY-LUSSAC. In the second section the great feature is the statement (VII.), that "in mixed media the mean square molecular velocity is inversely proportional to the specific weight of the molecules." The proof which WATERSTON gave is doubtless not satisfactory; but the same may be said of that advanced by MAXWELL fifteen years later. The law of AVOGADRO follows at once, as well as that of GRAHAM relative to diffusion. Since the law of equal energies was actually published in 1851, there can be no hesitation, I think, in attaching WATERSTON'S name to it. The attainment of correct results in the third section, dealing with adiabatic expansion, was only prevented by a slip of calculation.

In a few important respects Waterston stopped short. There is no indication, so far as I can see, that he recognised any other form of motion, or energy, than the translatory motion, though this is sometimes spoken of as vibratory. In this matter the priority in a wider view rests with Clausius. According to Waterston the ratio of specific heats should be (as for mercury vapour) 1·67 in all cases. Again, although he was well aware that the molecular velocity cannot be constant, there is no anticipation of the law of distribution of velocities established by Maxwell.

A large part of the paper deals with chemistry, and shows that his views upon that subject also were much in advance of those generally held at the time.

The following extract from a letter by Professor McLeod will put the reader into possession of the main facts of the case:—

"It seems a misfortune that the paper was not printed when it was written, for it shadows forth many of the ideas of modern chemistry which have been adopted since 1845, and it might have been the means of hastening their reception by chemists.

"The author compares the masses of equal volumes of gaseous and volatile elements and compounds, and taking the mass of a unit volume of hydrogen as unity, he regards the masses of the same volume of other volatile bodies as representing their molecular weight, and in the case of the elements he employs their symbols to indicate the molecules.

"In water he considers that the molecule of hydrogen is combined with half a molecule of oxygen, forming one of steam, and he therefore represents the compound as $HO_{\frac{1}{2}}$. He does not make use of the term "atom" (although he speaks of atomic weight on p. 18, but thinks it divisible), and if he had called the smallest proportion of an element which enters into combination an atom, he would probably have been led to believe that the molecules of some of the simple bodies contain two atoms, and he might have adopted two volumes to represent the molecule, as is done at the present time. The author calls one volume or molecule of chlorine Cl, one volume or molecule of hydrogen H, and one volume or molecule of hydrochloric acid $H_{\frac{1}{2}}Cl_{\frac{1}{2}}$. If he had regarded the molecules as containing two indivisible atoms, these bodies would have been represented, as now, by the formulæ Cl_2, H_2, and HCl respectively, all occupying two volumes. § 15 shows how near he was to this conception. Gerhardt in the Fourth Part of his 'Traité de Chimie Organique,' published in 1856, points out the uniformity introduced into chemical theory by the adoption of this system.

"For carbon he makes $C = 12$, as now accepted, although I do not find how he arrives at this number. He seems to have anticipated one of Ramsay's recent discoveries, that nitrous anhydride (hyponitrous acid, $ON_{\frac{2}{3}}$, No. 26 in the table) dissociates on evaporation into nitric oxide (binoxide of nitrogen, No. 23) and nitric peroxide (nitrous acid, No. 25).

"The values for the symbols for sulphur, phosphorus, and arsenic taken from the vapour densities (and which are multiples of what are believed to be the true atomic weights), cause some complexity in the formulæ of their compounds.

"There seem to be errors in the formulæ of alcohol and ether on p. 49, for they do not agree with those in the table. They ought probably to be written

$$2\,(HC_{\frac{1}{2}}) + O_{\frac{1}{2}}2H_{\frac{1}{2}} \text{ and } 4\,(HC_{\frac{1}{2}}) + O_{\frac{1}{2}}2H_{\frac{1}{2}}.$$

"Considering how nearly WATERSTON approached what is now believed to be the true theory, it is disappointing to read his controversy with ODLING in 1863 and 1864 ('Phil. Mag.,' vols. 26 and 27), where he seems to oppose the new formulæ then being introduced. He is very dogmatic about the constitution of hydrate of potash: he very properly insists that we can only obtain a knowledge of the molecular weight of bodies that can be volatilized, and of which the vapour densities can be determined, but he does not see the analogy between the hydrate and oxide of potassium with alcohol and ether, probably because he regards these latter bodies as combinations of water with different quantities of olefiant gas. He writes water $HO_{\frac{1}{2}} = 9$, alcohol $CH_2HO_{\frac{1}{2}} = 23$, and ether $C_2H_4 . HO_{\frac{1}{2}} = 37$, whilst he considers potassic hydrate $KO_{\frac{1}{2}} . HO_{\frac{1}{2}} = 56$, and oxide of potassium $KO_{\frac{1}{2}} = 47$, the hydrate having a higher molecular weight than the oxide. If we regard these compounds as derived from water by the replacement of hydrogen by ethyl and potassium respectively, the analogy between the two series is complete (ethyl was discovered in 1849 and is mentioned by WATERSTON).

$$\begin{array}{lll} H_2O & = 18 & \qquad H_2O = 18. \\ (C_2H_5)\,HO & = 46 & \qquad KHO = 56. \\ (C_2H_5)_2\,O & = 74 & \qquad K_2O = 94. \end{array}$$

"From a remark in the 'Phil. Mag.' (vol. 26, p. 520), I imagined that WATERSTON had arrived at the double atomic weights of many of the metals now adopted, for he gives that of iron as 56 and that of aluminium as 27 calculated from their specific heats, but there is an error in his arithmetic, for 3·3 divided by the specific heat of iron ·1138 gives 28·998, and 3·3 divided by the specific heat of aluminium ·2143 gives 15·399."

With the exception of some corrections relating merely to stops and spelling the paper is here reproduced exactly as it stands in the author's manuscript.—Dec. 1891.

[*Author's Introduction.*]

OF the physical theories of heat that have claimed attention since the time of BACON, that which ascribes its cause to the intense vibrations of the elementary parts of bodies has received a considerable accession of probability from the recent experiments of FORBES and MELLONI. It is admitted that these have been the means of demonstrating that the mode of its radiation is identical with that of light in the quantities of refraction and polarization. The evidence that has been accumulated in favour of the undulatory theory of light has thus been made to support with a great portion of its weight a like theory of the phenomena of heat; and we are, perhaps, justified in expecting that the complete development of this theory will have a much more important influence on the progress of science, because of its more obvious connection and intimate blending with almost every appearance of Nature. Heat is not only the subject of direct sensation and the vivifier of organic life, but it is manifested as the accompaniment of mechanical force. It is related to it both as cause and effect, and submits itself readily to measurement by means of the mechanical changes that are among the most prominent indications of its change of intensity. The undulatory theory at once leads us to the conclusion that, inasmuch as the temperature of a body is a persistent quality due to the motion of its molecules, its internal constitution must admit of it retaining a vast amount of living force. Indeed, it seems to be almost impossible now to escape from the inference that heat is essentially molecular *vis viva*. In solids, the molecular oscillations may be viewed as being restrained by the intense forces of aggregation. In vapours and gases these seem to be overcome; vibrations can no longer be produced by the inherent *vis insita* of the molecules struggling with attractive and repellant forces; the struggle is over and the molecules are free; but they, nevertheless, continue to maintain a certain temperature; they are capable of heating and being heated; they are endowed with the quality heat, which, being of itself motion, compels us to infer that a molecule in motion without any force to restrain or qualify it, is in every respect to be considered as a free projectile. Allow such free projectiles to be endowed with perfect elasticity, and likewise extend the same property to the elementary parts of all bodies that they strike against, and we immediately introduce the principle of the conservation of *vis viva* to regulate the general effects of their fortuitous encounters. Whether gases do consist of such minute elastic projectiles or not, it seems worth while to enquire into the physical attributes of media so constituted, and to see what analogy they bear to the elegant and symmetrical laws of aeriform bodies.

Some years ago I made an attempt to do so, proceeding synthetically from this fundamental hypothesis, and have lately obtained demonstration of one or two points where the proof was then deficient. The results have appeared so encouraging, although derived from very humble applications of mathematics, that I have been led to hope a popular account of the train of reasoning may not prove unacceptable to the Royal Society.—Sept. 1, 1845.

SECTION I.—OF A HOMOGENEOUS MEDIUM AND THE LAWS OF ITS ELASTICITY.

§ 1. The term medium is, perhaps, not quite appropriate to what is here intended to be signified. We speak of a resisting medium. of the medium of light, and in each expression something is referred to as intervening between bodies, and it is the quality of interposition that entitles it to the name. Here, for want of better, it is employed to denote a certain hypothetical condition of matter which it is the object of this Paper to show has physical properties that resemble those that have been found to belong to aeriform bodies. Inasmuch, therefore, as the word may be applied to a simple unmixed gas so as to speak of it as an oxygen medium or a hydrogen medium, &c., so far we may be allowed the use of it in treating of a hypothetical medium, which we have carefully to refrain from assimilating to any known form of matter until, by synthetical reasoning, circumstantial evidence has been accumulated sufficient to prove or render probable its identity.

To have a proper conception of what the medium is that forms the subject of speculation, we must imagine a vast multitude of small particles of matter, perfectly alike in every respect, perfectly elastic as glass or ivory—but of size, form and texture that requires not to be specified further than that they are not liable to change by mutual action—to be enclosed by elastic walls or surfaces in a space so much greater than their aggregate bulk as to allow them freely to move amongst each other in every direction. As all consideration of attractive forces is left out at present, it is obvious that each particle must proceed on a straight line until it strikes against another, or against the sides of the enclosure ; that it must then be reflected and driven into another line of motion, traversing backwards and forwards in every direction, so that the intestine condition of the multitude of these that form the medium may be likened to the familiar appearance of a swarm of gnats in a sunbeam.

The quality of perfect elasticity being common to all the particles, the original amount of *vis viva*, or living, acting force, of the whole multitude must for ever remain the same. If undisturbed by external action it cannot, of itself, diminish or increase, but must for ever remain as unchanged as the matter that is associated with it and that it endows with activity. Such is the case if we view the whole mass of moving particles as one object, but each individual of the multitude must at every encounter give or receive, according to the ever-changing angle and plane of impact, some portion of its force, so that, considered separately, they are for ever continually changing the velocity and direction of their individual motions ; striking against and rebounding from each other, they run rapidly in their zig-zag conflict through every possible mode of concurrence, and *at each point of the medium we may thus conceive that particles are moving in every possible direction and encountering each other in every possible manner during so small an elapsed interval of time that it may be viewed as infinitesimal in respect to any sensible period.* The medium must in this way become endowed with a permanent state of elastic energy or disposition to expand, uniformly sustained in every part and communicating to it the physical character of an elastic fluid.

The simplicity of this hypothesis facilitates the application of mathematics in ascertaining the nature and properties of such media, and the study acquires much interest from the analogies that it unfolds. For if the reasoning is correct, the physical laws common to all gases and vapours—those laws, namely, that concern heat and pressure—do actually belong to such media, and may be synthetically deduced from the constitution which has now been assigned to them.

The characteristic which renders a medium susceptible of mathematical treatment is that of its being composed of particles perfectly alike in every respect, but it is chiefly their identity in weight or mass that is the important point of distinction. A particle thus conforms to the definition that the eminent physicist AMPÈRE has given to the term molecule, which we may therefore adopt as a more significant name for the element of a medium.

The first department of the subject must naturally be devoted to the consideration of the circumstances that determine the equilibrium of such a homogeneous medium considered by itself. Its density, by which is to be understood not its specific gravity but the number of molecules in a constant volume,* may be supposed to vary without disturbing its homogeneity. The mean square velocity of the molecules (which in any infinitesimal portion of the medium may be assumed as uniform) we also have to consider as a variable quantity, and the physical qualities of a medium being dependent on these two elements of its constitution, it is necessary to determine clearly their mathematical relations.

§ 2. It is evident from the definition of the hypothesis, that the medium must exert some expansive force on the surface that encloses it; but the nature of the force is not strictly continuous, it is composed of a multitude of successive strokes. Nevertheless, their succession is certainly continuous, and it is not difficult to conceive how they may be sufficient to counterbalance and support a superincumbent weight. To obtain an exact idea of this, let us suppose that a small elastic plane whose weight is n times that of a molecule, is supported by a regular succession of such molecules striking its centre of gravity with a velocity v. We seek to know the condition of their mutual action when an equilibrium is maintained.

The following are the well-known equations that express the law of elastic collision. They are necessarily the foundation of all reasoning on the effects of the mutual action of elastic bodies by impact.

1. *The Meeting Impact.*

Two molecules, B and D, meet directly in an intermediate point and strike each other with the respective velocities β and δ. The velocities after impact are respectively :—

$$\beta_0 = -\beta + \frac{2(\delta+\beta)\mathrm{D}}{(\mathrm{B}+\mathrm{D})}; \quad \text{and} \quad \delta_0 = \delta - \frac{2(\delta+\beta)\mathrm{B}}{(\mathrm{B}+\mathrm{D})};$$

the direction of D's motion being reckoned positive.

2. *The Overtaking Impact.*

The two molecules, B and D, with the same velocities, β and δ, move in the same direction and D overtakes B; the velocities after impact are respectively :—

$$\beta_1 = \beta + \frac{2(\delta-\beta)\mathrm{D}}{(\mathrm{B}+\mathrm{D})}; \quad \text{and} \quad \delta_1 = \delta - \frac{2(\delta-\beta)\mathrm{B}}{(\mathrm{B}+\mathrm{D})};$$

the direction of D's motion being reckoned positive.

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* [Attention should be directed to this use of the word "density."—R.]

In the first of these let $\beta_0 = \beta$, $\delta = v$, $\mathrm{B} = n\mathrm{D}$; then shall

$$\beta = -\beta + \frac{2(v+\beta)}{n+1}; \quad \text{or} \quad \beta = \frac{v}{n},\text{*}$$

which evidently expresses the upward velocity given to the plane by the impulse of one molecule when the velocity of incidence and reflexion is the same. The plane ascends and descends the height due to this velocity, and then encounters the next in the succession of molecular impacts without any transference of force taking place between them; and n being taken an indefinitely great number, β is infinitesimal in respect to v, and the height through which the plane traverses is also infinitesimal, so that it is supported as if by a continuous force of upward pressure. The time between each impact is, according to the law of falling bodies, equal to the time taken by the force of gravity to destroy and reproduce the infinitesimal velocity v/n. This is $2v/gn$: the velocity which a free body gains or loses in a unit of time by the force of gravity being represented by g. The number of impacts in a unit of time is therefore $gn/2v = \mathrm{A}$. This, then, is the relation between the weight of the plane, in terms of that of the molecule unity, and the rapidity of the succession of impacts necessary to support it in a condition of statical equilibrium. Now, if the plane forms part of the surface that encloses the medium and that counterbalances by its weight the effect of the impacts of the confined molecules, such effect must correspond with the succession represented by A; and we deduce that *the elastic force of a medium, as represented by the weight or pressure required to confine it, is directly proportional to the number of molecular impacts that take place against a unit surface in a unit time with a constant velocity* (or $e \doteqdot \mathrm{A}$, if v is constant) . . . . I.

§ 3. Such being the nature of the elastic force, it will not be difficult to prove that it increases exactly as the density of the medium. The proposition stands thus: if the number of molecules in a volume of the medium be doubled, the number of impacts that take place on a constant surface in a constant time will also be doubled, the velocity being unchanged.

Suppose the number octupled, the mean distance is reduced to one-half. If they were equidistant and moving in one direction with the constant velocity, it is evident that eight times the previous number would pass the same imaginary plane in the same time, and if the plane were solid that eight times the previous number would impinge against it. Now, although all do not move in one direction, yet in both cases the same proportion of the whole must in each case do so. Whatever may be the density no preference can be assigned to one direction more than to another in the molecular movements; they must in every case be equally distributed in every

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* [The case is that where the particle (mass 1) and the plane (mass n) both reverse their velocities at impact. The conservation of *vis viva* is thereby secured, and the condition of momentum gives at once $n\beta = v$.—R.]

direction, and if the number is increased eight times in any one direction it must be so in every other.

This may be viewed in another light. Suppose in both cases, the density being 1 and 8 respectively, that the molecules are arrested in their motion. It is evident that opposite a unit of surface in density 8 there will, in the first row, be four times as many molecules as in density 1, and that the average distance between the rows is only one-half. Suppose the molecules to resume their motion, and compare density 8 with density 1, it is obvious that in half the time four times the number will impinge on the unit of surface, and in the same time eight times the number. Now it has been shown (§ 2) that the elastic force is proportional to the number of molecular impacts made with a constant velocity against a unit of surface in a unit of time, hence we deduce that *the elastic force* (e) *of a medium with a constant mean molecular velocity* (v) *is proportional to its density* (Δ^3) (or $e \doteqdot \Delta^3$, if v or v^2 is constant). . II.

§ 4. Hitherto the molecular velocity has been supposed constant. We have now to enquire how the elasticity of the medium is affected by a change in the velocity from v to mv. The intestine action of the medium may be viewed as the traversing of a certain mean distance, L, by the molecules in a given time, t; and in this time a certain mean number, A, of impacts take place against a unit of surface. If the velocity is increased m times, the distance L is traversed in $1/m$th the time t, or t/m, and in this reduced time the same number of impacts must take place as before took place in the time t; for there is nothing in the change of velocity simply that can alter the ratio that subsists between the mean distance traversed and the mean number of impacts, unless that ratio were subject to change without any change whatever in the medium, which is absurd; hence, in the original time, t, there is m times the original number of impacts, A.

It was shown in § 2 that if the weight of each of the molecules were represented by 1, their mean velocity by v, and weight of plane supported by their impinging action n, the number of impacts in a second or unit of time required to support the plane is $\frac{gn}{2v} = \mathrm{A}$, or $n = \frac{2}{g}\mathrm{A}v$, and this equation must evidently be maintained in altering the value of the terms. Now, it has been shown that in changing v to mv in a medium that does not alter its density we cause A to become $m\mathrm{A}$, and $\frac{2}{g}\mathrm{A}v$ becomes $\frac{2}{g}\mathrm{A}mvm = \frac{2}{g}\mathrm{A}vm^2 = nm^2$. Hence n, the weight of the plane, or measure of tension, must be increased m^2 times so that it may continue to equilibrate the impinging action. Thus, we deduce that while the molecular velocity increases from v to mv, the elasticity increases from n to m^2n or *the elasticity of a medium having a constant density is proportional to the mean square molecular velocity or* vis viva *of the medium* (or $e \doteqdot v^2$, when Δ^3 is constant)* . . . . III.

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* [II. and III. were given by D. BERNOULLI. See 'POGG. Ann.,' vol. 107, p. 490, 1859.—R.]

§ 5. Combining II. with III. it appears that when both the density and the *vis viva* are subject to change that the elasticity is equal to their product, or $e \doteqdot \Delta^3 v^2$, and this is the law that includes all the conditions of equilibrium of an enclosed homogeneous medium. One other condition only remains to be specified. *Under a constant pressure the density is inversely as the* vis viva *or mean square molecular velocity* ($\Delta^3 \doteqdot \frac{1}{v^2}$, if e is constant) . . . . IV.

§ 6. In concluding this part of the subject, we cannot fail of being sensible of the analogies that subsist between these synthetical deductions and the chief properties that distinguish aeriform fluids.

The first point that was inductively established is MARIOTTE'S law, viz.: *at the same temperature the density of air is as its compression.* This is analogous to the second deduction :—The square of the velocity being constant, the elastic force of a medium is proportional to its density. The accordance appears as complete as could be desired, and there is a residual evidence in favour of v^2 being identical with temperature, or being a quality that varies simultaneously with it.

The second point is DALTON and GAY-LUSSAC'S law of expansion. By experimenting upon the same weight of air at different temperatures under a constant pressure, these philosophers found that an increment of one degree caused always the same augmentation of bulk, and that this amounted to $\frac{1}{480}$th part of the space that it occupied at 32°. Thus, if the same law hold good at all temperatures, 480 cubic inches of air at this temperature should diminish one inch in bulk for every degree it was lowered in temperature, and would become zero in bulk at 480° below the freezing point of water, or $-448°$ on Fahrenheit scale.

Now in IV. we had $\Delta^3 \doteqdot \frac{1}{v^2}$, or $v^2 \doteqdot \frac{1}{\Delta^3}$, when e is constant; but $\frac{1}{\Delta^3}$ is the volume occupied by a constant number of molecules; hence with the same constant number of molecules the volume is as the mean square molecular velocity, and a constant increment of *vis viva* is followed by the same increment of volume under a constant pressure, and as the constant increment of volume (1 cubic inch) is to the constant increment of *vis viva* (1°) so is the volume (480 cubic inches) corresponding to a certain *vis viva* (32° Fahr.) to that *vis viva* (480°).

The analogy therefore still holds good, and the evidence continues in favour of the absolute temperature being represented by v^2.

When air is not allowed to expand and heat is applied, the elastic force increases with the temperature, and a rise of 1° causes an absolute increase in the elasticity, which is the same at all temperatures, and corresponds with the increase of bulk it would assume if allowed freely to expand. This is analogous to III., where, the density being constant or bulk unchanged, the elasticity is shown to be proportional to the mean square molecular velocity.

Thus, the laws of MARIOTTE and of DALTON and GAY-LUSSAC are represented by the formula $(448 + t)\,\Delta^3 = e$; in which $t =$ temperature, Fahrenheit scale; $\Delta^3 =$ density, and $e =$ elasticity.

The law of elasticity in the hypothetical medium is represented by the formula $v^2\Delta^3 = e$; in which v^2 is the mean square molecular velocity; $\Delta^3 =$ density, and $e =$ elasticity.

The *first* expresses physical laws that have been *found* to belong to a certain *existent* form of matter.

The *second* expresses physical laws that have been *proved* to belong to a certain *possible* form of matter.

The cause of the effect represented by $(448 + t)$ in the first is unknown, but has, at various times, by eminent authorities, been referred to *molecular motion.*

The corresponding term, v^2, of the second represents *molecular motion.*

SECTION II.—ON THE PHYSICAL RELATIONS OF MEDIA THAT DIFFER FROM EACH OTHER IN THE SPECIFIC WEIGHT OF THEIR MOLECULES.

§ 7. The synthetical deductions of last section apply to a homogeneous medium without respect to the absolute weight of its molecules, if the weight of each molecule is the same. This weight, common to all, may be viewed as the specific molecular weight of the medium, and distinguishes it from any other medium with a different specific molecular weight. We have now to enquire into the relations that subsist between the density and molecular velocity of two such media that have the same elasticity, or that are in equilibrium of pressure and also of *vis viva.*

We deduced from the law of impinging elastic bodies that if v represents the mean molecular velocity in feet per second, A the number of molecular impacts in a second upon a small elastic plane which is equal in weight to n molecules, then $n = \frac{2}{g}\mathrm{A}v$. Let ω represent the specific weight of the molecules, we have $\omega n = \frac{2\omega}{g}\mathrm{A}v = e =$ the elastic force exerted by the medium on a unit of surface;* and as this must in the present enquiry be assumed constant, we may easily remark how a change in ω affects v and A.

It is evident that since $\frac{2\omega}{g}\mathrm{A}v$ is a constant quantity and ω, A, and v variable, we have $\mathrm{A}v \doteqdot \frac{1}{\omega}$; but $e = \omega\Delta^3v^2$ (§ 5) $= \frac{2\omega}{g}\mathrm{A}v$, and, therefore, $\Delta^3 v = \frac{2}{g}\mathrm{A}$, or $\mathrm{A} = \frac{g}{2}\Delta^3 v$, and $\mathrm{A}v \doteqdot \frac{1}{\omega} \doteqdot \Delta^3 v^2$. Hence it is obvious that if Δ^3, the density or number of molecules in a constant volume, as well as e, the tension, are constant, while the molecular

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* e is the absolute weight of the small elastic plane that is supported by the succession of A number of molecular impacts per second, the weight of each of which is ω, and their common impinging velocity v feet per second.

velocity and specific weight are variable, these variables are bound by the relation expressed by $v^2 \doteqdot 1/\omega$, which signifies that if two media are in equilibrium of pressure and have in the same volume of each the same number of molecules, the squares of their molecular velocities must be inversely as their specific molecular weights. Hence, we deduce that *if any number of separate media have equal density and tension, the molecular velocity of each must be proportional to the inverse square root of their specific molecular weight, or to the inverse square root of the specific gravity of the media respectively** . . . . V.

§ 8. But media may be in equilibrium of pressure without being of equal density, for a deficiency of density may be compensated by an excess in the molecular velocity. It is plain that if e and v^2 are constant in any two media they may still be in equilibrium of pressure if ω is proportional to $1/\Delta^3$, or if the molecular weight of each medium is inversely as its density. If the specific molecular weights are in a constant ratio to each other and the tension and velocity also constant, the media must be kept in equilibrium of pressure if the density of each is reciprocally proportional to the specific molecular weight of the other.

We have supposed hitherto that the media are separate while their respective elasticities are compared. Let us now enquire into the effects of allowing them to have access to each other. The united media immediately obtain a heterogeneous character, for it requires no demonstration to convince us that the molecules of each will permeate through the volume occupied by the other, the vacuities in the space occupied by each presenting no more obstacle to the motion of one set of molecules than it does to the other; and as collision must take place amongst them in every possible manner and direction, the common space of the united media are free alike to each individual molecule of both to range through in its zig-zag course. Consequently, *media in contact with each other become gradually equally diffused through their common volume* . . . . VI.

The internal condition of the mixture must after a time become settled so that in any infinitesimal portion the same mean velocity will be found proper to the molecules of each medium respectively.

But as each of the two sets of molecules, although completely mixed together, preserve their specific weights, so must they have corresponding specific velocities that remain intact, notwithstanding that they as often impinge on molecules of the other set as on the molecules of their own kind. It is of consequence to settle what the ratio of these specific velocities is, for upon this point depends the nature of the *vis viva* equilibrium of different media, and we have to determine the relative condition of two media when they are in equilibrium both of pressure and of *vis viva*.

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* [The deduction of V. appears to be correct, though much embarrassed by the irrelevant g. In his first memoir on the Theory of Gases ('Pogg. Ann.,' vol. 100, 1857), Clausius arrives at the same conclusion. His assumption that the density (in Waterston's sense) of various gases is the same, appears to have been made upon chemical grounds.—R.]

§ 9. We must now refer back to the equations of impact (§ 2). It is apparent that the sum of the impinging *vis viva* of both molecules does not alter in either the meeting or overtaking impact; what is gained by one is lost by the other, or $\beta^2\text{B} + \delta^2\text{D} = \beta_0{}^2\text{B} + \delta_0{}^2\text{D} = \beta_1{}^2\text{B} + \delta_1{}^2\text{D}$. But in every case except one a transference of *vis viva* must take place from the one to the other.

The exception is found in the meeting impact when $\beta \doteqdot 1/\text{B}$ and $\delta \doteqdot 1/\text{D}$; then shall $\beta_0 = \beta$ and $\delta_0 = \delta$, but at the same time β_1 is not equal to β, or δ_1 to δ; in every other case β_0 is not equal to β nor to β_1, nor is δ_0 equal to δ or to δ_1.

It can seldom happen that the molecules strike each other directly. In taking account of the collective result of their fortuitous concourse we must view the position of the plane of concurrence and the respective inclinations of the line of motion of each molecule to it as three independent variables. The incident velocity of each is the absolute velocity resolved perpendicular to the plane, and the equations apply to this portion only of the *vis viva* of the molecules.

Although the variety in the mode of impact is infinite, it is certain that one direction of motion is as likely as any other, and hence, that the opposite of any direction is equally probable to the direction itself.

Let us confine our attention to any single case of impact and suppose that the directions of the motions of the two impinging molecules lie on one side of the plane of concurrence, then it appears that the nature of the impact must be *overtaking*. Again, let us suppose that they lie similarly disposed on the other side of the plane; the nature of the impact is again *overtaking*. Now, instead of having the opposite of both the original lines of motion, suppose the opposite of one only is taken; it is clear that the nature of the impact is in this case of the *meeting* kind; and the opposite of the other line of motion being taken while the first is in its original position, the impact is again of the meeting kind.

Each of these four cases are equally probable, and the resolved velocities, or the values of β and δ, are the same in all, but two are meeting impacts and two are overtaking, each couple having perfectly distinct numerical equations to define the relation between the incident and reflected *vis viva*.

We are thus obliged to infer that the intestine action of the medium must be viewed in this manner as divided into two kinds of impacts specifically distinct in the numerical relation that subsists between the velocity before and after concurrence, and when employing the equations for summing up the results of the whole indefinitely great multitude that take place in mixed media, the effect of any one *meeting* impact must be considered along with its counterpart overtaking impact with the same velocities.

§ 10. We have remarked that it is only the resolved portion of the whole *vis viva* of a molecule that is dealt with by the equations—that forms the force of impact—and it may be questioned whether the mean of these forces in each kind of molecules bears the same proportion to each other as the whole *vis viva* of each. That the ratio is the same is best seen by the *reductio ad absurdum* method of reasoning. If the ratio is different, the motions of the heavier molecules must be resolved in a different way from those of the lighter, or the plane of concurrence must incline to one set of molecules in a different manner to that of the other set. Now any such effect is quite inconsistent with the fundamental hypothesis, and would require us to admit the influence of a modifying power whose nature and mode of bringing about the effect in question is unknown.

The ratio ρ of the resolved to the absolute *vis viva* is actually one-third, and will become obvious in the next section, but it seems needless to require the demonstration in this place as all that we have to be assured of is the constancy of the ratio, whatever its actual value may be.

In seeking to demonstrate the nature of the *vis viva* equilibrium, the solitary condition that we have to reason from is that the mean value of $(\beta_0{}^2 + \beta_1{}^2)$ is equal to $2\beta^2$ (§ 2) and the mean value of $(\delta_0{}^2 + \delta_1{}^2) = 2\delta^2$. That this is a necessary condition is obvious, because if either were less there would be a continual transfer from the molecules B to the molecules D, or from D to B and *vice versâ.*

By squaring the equations in § 2 and adding, we have the following :—

$$\beta_0{}^2 + \beta_1{}^2 = 2\{\beta^2 - \frac{4\mathrm{D}}{\mathrm{B}+\mathrm{D}} \cdot \beta^2 + \frac{4\mathrm{D}^2}{(\mathrm{B}+\mathrm{D})^2} \cdot (\delta^2 + \beta^2)\}$$

$$\delta_0{}^2 + \delta_1{}^3 = 2\{\delta^2 - \frac{4\mathrm{D}}{\mathrm{B}+\mathrm{D}} \cdot \delta^2 + \frac{4\mathrm{B}^2}{(\mathrm{B}+\mathrm{D})^2} \cdot (\delta^2 + \beta^2)\}.$$

If in any case it happens that $\beta_0{}^2 + \beta_1{}^2 = \beta^2$, we shall have

$$\beta^2\left(\frac{4\mathrm{D}}{\mathrm{B}+\mathrm{D}}\right) = \frac{4\mathrm{D}^2}{(\mathrm{B}+\mathrm{D})^2} \cdot (\delta^2 + \beta^2),$$

or $\beta^2\mathrm{B} = \delta^2\mathrm{D}$, and $\delta_0{}^2 + \delta_1{}^2 = 2\delta^2$. Hence, if the squares of the impinging velocities happen to be in the inverse ratio of the molecular weights, then in either molecule the sum of the *vis viva* of the twofold encounter (one *meeting*, the other *overtaking* with the same impinging velocities) before impact, or $2\beta^2$, is equal to the sum after impact, or to $\beta_0{}^2 + \beta_1{}^2$.

But this is only one case out of an infinite number where the ratio is different. Generally, we may express the equation thus :

$$\beta_0{}^2 + \beta_1{}^2 = 2\beta^2 + p, \text{ and } \delta_0{}^2 + \delta_1{}^2 = 2\delta^2 + q.$$

Now, suppose that in an indefinitely great multitude of impacts the sum of all the individual values of $\beta_0{}^2 + \beta_1{}^2$ and $\delta_0{}^2 + \delta_1{}^2$ are taken, we shall have the mean of the values of the first equal to $\beta_m{}^2\rho$, in accordance with the necessary condition of permanence noticed above (by $\beta_m{}^2$ we mean to denote the mean molecular *vis viva* or mean square velocity of the B molecules, &c.). The mean value of $\beta^2 + p$ is, therefore, $\beta_m{}^2\rho$, but the mean value of β^2 is also evidently equal to $\beta_m{}^2\rho$, as above; hence, the mean value of p is 0, or the positive values of p balance the negative values. In the same way, it may be shown that the mean of the values of q is also 0. Hence, we deduce that $\beta_m{}^2\mathrm{B} = \delta_m{}^2\mathrm{D}$, or that *in mixed media the mean square molecular velocity is inversely proportional to the specific weight of the molecules* * . . . VII.

This is the law of the equilibrium of *vis viva.*

§ 11. Thus, it appears that the inverse ratio of the specific molecular weight is that which is naturally assumed by the mean square molecular velocity of media in contact, and according to the foregoing reasoning (§ 10), this is also the ratio that ensures an equilibrium of pressure between media of the same density, or which have the same number of molecules contained in the same volume. Thus, by combining V. with VII. we deduce that *media in equilibrium of pressure and* vis viva *are of equal density, or have specific gravities respectively proportional to their specific molecular weights* . . . . VIII.

§ 12. We may likewise remark that as the mean value of the product $\beta^2\mathrm{B}$ is equal to the mean $\delta^2\mathrm{D}$, or $\mathrm{B}_m{}^2 = \delta_m{}^2\mathrm{D}$, *there is the same amount of* vis viva *or mechanical force contained in equal volumes of all media that are in equilibrium of pressure and* vis viva . . . . IX.

§ 13. If different media are placed in contact they must diffuse themselves through their common volume with velocities proportional to their mean molecular velocity; but this velocity being in each inversely as the square root of its specific molecular weight, which is equal to the square root of its specific gravity, we may deduce, by combining VI. with VII., that *media in equilibrium of pressure and* vis viva *diffuse themselves through their common volume with velocities inversely proportional to the square root of their specific gravity* . . . . X.

§ 14. Such are the principal points by which different media are related to each other. Their analogies to the properties of gases may be stated as follows:

(1.) The specific gravities of gases of the same *temperature* and pressure are respectively proportional to their atomic weight. [The combining equivalents or proportions may be viewed as simple multiples or divisors of the atomic weight or specific gravity.]

This is analogous to the VIII. deduction. Media in equilibrium of *vis viva* and pressure have specific gravities proportional to their molecular weight. It will be remarked that here again we have temperature represented by *vis viva.*

(2.) It is considered as almost proved that gases in equilibrium of pressure and temperature have, in equal volumes, the same absolute quantity of heat.

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* [This is the first statement of a very important theorem. (See also 'Brit. Assoc. Rep.,' 1851). The demonstration, however, of § 10 can hardly be defended. It bears some resemblance to an argument indicated and exposed by Professor TAIT ('Edinburgh Trans.,' vol. 33, p. 79, 1886). There is reason to think that this law is intimately connected with the Maxwellian distribution of velocities, of which WATERSTON had no knowledge.—R.]

We have deduced, in reference to media, that when they are in equilibrium of pressure and *vis viva* they have, in equal volumes, the same amount of *vis viva.*

(3.) It is considered probable that the quantity of heat contained in equal volumes of all gases in equilibrium of pressure and temperature is proportional to their absolute temperature; and, in general, that equal increments of heat cause equal increments of temperature or expansion.

In all media in equilibrium of pressure and *vis viva* the same increment of *vis viva* causes the same expansive effect. (See also § 6.)

(4.) Gases, however different in specific gravity, when placed in connection diffuse themselves equally through each other; and according to the elegant induction of Dr. GRAHAM ('Edin. Trans.,' 1831) the velocity of diffusion is inversely as the square root of their specific gravity.

This is very exactly responded to by the hypothesis. DALTON'S law of diffusion belongs to media as a necessary and the most obvious consequence of the constitution assigned to them (VI.) DR. GRAHAM'S law of diffusive velocity or volume applies also to media, because if placed in similar circumstances such an effect must depend on the molecular velocity, which, singularly enough, bears exactly the like ratio to their specific gravity (X.).

§ 15. In the first point of analogy it was stated the atomic weight of a gas corresponded with its specific gravity, but with the reservation that the combining proportions are simple multiples or quotients of the same. This appears to me to be the fair statement of the remarkable connection that is always found between the combining volume and combining weight. It seems impossible that the fact of a volume of every gas containing the same number of molecules can ever be inductively established, but all analogy leads us to this conclusion. One volume of oxygen combines with two volumes of hydrogen to form two volumes of aqueous vapour. If we inferred from this that one molecule of oxygen combined with two molecules of hydrogen to form one of steam, we must admit that the molecule of steam occupies double the volume of a molecule of hydrogen or oxygen. If it is admitted, on the other hand, that the oxygen molecule is capable of disintegration, and that half a molecule combines with one of hydrogen to form one of steam, the bulk of the three molecules are equal. This last is the view that is responded to by the hypothetical media. The objection to it is plausible from the natural repugnance to the idea of dividing what has been considered as an ultimate element into parts, and of supposing it possible that an element should have a strong affinity to itself—for this point is also involved. Half molecules of oxygen must have a powerful attraction to each other as they never appear separate. It is the same with all the other simple gases and vapours that combine in half or other fractional volumes. We have an analogous example among the hydrocarburets. Methyl is half a molecule of olefiant gas, and enters into distinct combinations similar in every respect to those of olefiant gas, but it never appears in a separate form. Olefiant gas enters into combination with itself in various proportions, forming etherine, ceten, elaen, &c., all of which are isomeric but have different specific gravities as vapours. These the chemist is unable to form directly; they are organic products and show a great molecular capability that is undoubtedly common to all bodies.

There has certainly as yet been no recognised example of a simple element appearing in the gaseous form with two specific gravities, but it would be rash to affirm that such was either impossible or absurd. Indeed, chlorine, bromine, iodine, and fluorine are so similar in their chemical qualities that it is not very unlikely they may be examples of isomeric combinations of one base in which condensation of atomic volume takes place simultaneously with increment of atomic mass. In the hydrocarburets, on the other hand, there is little, if any, condensation of atomic volume, their specific gravity in the liquid form being all nearly the same.

A strong argument in favour of the physical origin of the law of volumes is to be obtained from the combination of acids with ether. Some of these rise up into vapour in the same state of chemical union as when liquid, while others quite similar are decomposed in the act of rising and occupy double the volume of the others. Even those that are fully combined when they first rise, by an increase of temperature not very considerable, are decomposed and immediately assume a double volume. Oxalic and nitrous ethers are examples of the last kind. Sulphate and nitrate of pyroxilic ether are examples of disunion in the act of rising.

If the hypothetical law of volumes is true in all cases, we should have to distinguish between atomic weight and combining proportion. Thus, if the atomic weight of oxygen is 1 (or specific gravity of gas), its combining proportions are $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$ (?). If the atomic weight of hydrogen is 1, its ascertained combining proportions are $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, 3, 4, 5, 6, 8, 9, 16. If the atomic weight of nitrogen is 1, its combining proportions are $\frac{1}{2}$ and 1. The same for chlorine is $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$; for bromine they are $\frac{1}{2}$, 1, $1\frac{1}{2}$; for iodine, $\frac{1}{2}$, 1, $1\frac{1}{2}$; for arsenic, $\frac{1}{4}$, 1; for sulphur, $\frac{1}{6}$, $\frac{1}{3}$; for phosphorus, $\frac{1}{4}$, &c. These are necessarily derived from the specific gravity of the simple gases and of their compounds.

The labours of DUMAS, MITSCHERLICH, REGNAULT, and BINEAU, have extended the list of gases and vapours, whose specific gravities have been accurately measured, to nearly 150. Such determinations throw a light upon the atomic constitution of compounds which it is impossible to obtain from their mere chemical analysis, and form an important guide to theoretical discussion, when the arithmetic of volumes is properly applied.

SECTION III.—ON THE PHENOMENA THAT ATTEND THE CONDENSING AND DILATING OF MEDIA, AND THE MECHANICAL VALUE OF THEIR MOLECULAR VIS VIVA.

§ 16. In the first section a distinct idea of the elastic force of the medium was obtained by viewing it as a rapid succession of impacts on the lower surface of a gravitating elastic plane ; and the equilibrium as being maintained when the upward velocity given to the plane by the shock of one molecule was equal to the downward velocity given to it by gravity acting through half the infinitesimal portion of time that elapses between two successive impacts. During the first half of this time gravity acts in destroying the upward velocity : during the second half it acts and generates the same velocity downwards, and by applying the equation for the meeting impact (§ 2) we found the relation between the pressure and the number of impacts in a given time. This relation is expressed by $gn/2w = \mathrm{A}$, in which n is the number of molecules whose aggregate weight is the weight of the plane supported by A number of molecular impacts in a second of time : the impinging velocity being w, and g the accelerating force of gravity. It was also shown that the upward velocity given to the plane by one impact is w/n, and this is likewise the descending velocity with which it encounters the molecular shock.

We have now to examine the case where the encounter takes place with the plane at rest. Applying the equation for the meeting impact as in § 2, and putting $\epsilon = \beta = 0$; $n = \mathrm{B}$; $\mathrm{D} = 1$; $w = \delta$; we have

$$\epsilon_0 = \beta_0 = \frac{2w}{n+1} = \text{velocity upwards of the plane after the shock ;}$$

$$w_0 = \delta_0 = w - \frac{2wn}{n+1} = -w + \frac{2wn}{n+1} = w - \frac{w}{n+1}$$

$$= \text{velocity downwards of the molecules after the shock.*}$$

Thus, n being an indefinitely great number, we have the ascending velocity of the plane $\epsilon_0 = 2w/n$, being double what it was in the former case when the result of the impacts was statical equilibrium : and the decrement of molecular velocity $= w - w_0 = w/n$, which is a new and important feature. In the former case there was no decrement of molecular velocity : the molecule and plane continually meeting and retreating with velocity of impact and reflexion the same, and inversely proportional to their respective weight.

With its velocity $\frac{2w}{n}$ the gravitating plane ascends to the height $\left(\frac{2w}{gn}\right)^2 \frac{1}{2}g = \frac{2w^2}{gn^2}$, so that the weight of n molecules is raised through this height by the decrement w/n of the impinging velocity w of one molecule. Employing the differential notation,

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* [It is easy to see that in the case supposed $n\epsilon_0 = 2w$, when n is great, so that the velocity of the plane is $2w/n$; but in the next step there is an unfortunate error which runs through many of the subsequent deductions.

$$w_0 = \delta_0 = w - \frac{2wn}{n+1} = -\left(-w + \frac{2wn}{n+1}\right) = -\left(w - \frac{2w}{n+1}\right),$$

not $-\left(w - \frac{w}{n+1}\right)$. The *vis viva* expended in raising n to the height $2w^2/gn^2$ is thus $4w^2/n$, not $2w^2/n$.—R.]

$dw = w/n$; and $2dw\, w = 2w^2/n = dw^2$, the *vis viva* expended in raising n to the height $2w^2/gn^2$.

§ 17. If we recur to § 4 we may remark the necessity of considering molecular velocity in two points of view when applying the arguments of §§ 2 and 16 to an enclosed volume of a medium. The first point is that upon the molecular velocity of impact depends the intensity of the shock on the plane, the ascending velocity given to it, and therefore also the time between the impacts, if the weight of the plane is considered constant. Thus as any one velocity is to the time of ascent and descent of the plane caused by that velocity, so is the mean of the impinging velocities to the mean of the time intervals, or inversely as the number of impacts in a unit of time; and the equilibrium does not require that the succession of impacts should be regular; the rapidity of the succession may fluctuate, but the average time and velocity must be constant. The second point is that any augmentation of velocity causes an increase in the frequency of the encounters (§ 4). In the equation $ng/2w = \mathrm{A}$, if A were not a function of w it would remain unchanged, if n and w increased or diminished in the same proportion; but it was shown in § 4 that it was proportional to w, when Δ^3, the density, is constant, and to Δ^3 or n or c when w is constant; hence $\mathrm{A} = wc\,\Delta^3$, in which c is a constant factor that has to be determined. We have also to determine the ratio between w^2, the mean square impinging velocity, and v^2, the mean square absolute molecular velocity, in the equation $ng/2w = \mathrm{A} = wc\,\Delta^3$, or $n = \frac{2}{g}\, wc\,\Delta^3$.

Suppose the unit of volume in which the medium is confined to be a cube, the upper side of which is the plane n, and let v^2 be the mean square velocity of the molecules, so that if the squares of the respective velocities of all the molecules be added together, the sum will at all times be equal to $\Delta^3 v^2$. Resolve the motion of each molecule at any instant into the six rectangular directions parallel to the side of the cube and add up the squares of the resolved velocities that are perpendicular to one side; it is evident that the sum must be $\frac{1}{6}\,\Delta^3 v^2$, as the force is equally distributed in every direction, and in the stratum of the medium next the plane n one-sixth of the force of the molecules that happen to be in the stratum at any given instant is directed perpendicularly upon the plane. Suppose the breadth of the stratum is $1/\Delta$, the number of molecules that at all times are to be found moving in it is Δ^2, and half of these are diminishing their distance from the plane, and half increasing their distance with the mean square velocity $\frac{1}{3}\,v^2$.

The molecules moving equally in every direction must necessarily impinge equally in every possible direction on the plane, so that if their lines of motion were brought from every point of the surface of the plane where they impinge and made to issue from one central point, they would radiate equally to every part of the hemisphere; and as soon as those belonging to any one direction have impinged and thus withdrawn from forming part of the constant aggregate force $\frac{1}{6}v^2\,\Delta^2$, their place is immediately supplied by others of the same entering the stratum. The time taken by the set whose velocity is u and inclination to the plane θ to traverse the breadth of the stratum is evidently as $\frac{1}{u \sin \theta}$, and in a unit of time the number of the impacts in the succession of these belonging to the set is proportional to $u \sin \theta$. But this is the value of the resolved velocity of the set. Referring back to the reasoning in §§ 2, 16, the supporting effect of each impact on the heavy plane n was shown to be proportional also to the velocity of impact or molecular velocity resolved perpendicular to the plane. The supporting effect of each set in a unit of time is therefore as the square of the resolved or impinging velocity of the set. But the mean of all the square impinging velocities is $\frac{1}{3}v^2$, and half the molecules in the stratum are continually approaching the plane; the supporting effect of their continuous action is therefore the same as would be derived from the medium reduced to half density advancing against the plane with the uniform velocity $\sqrt{\frac{1}{3}v^2}$. Now as Δ^3 represents the number of molecules in the cubical unit of volume, the side of the cube being the unit of length, and $\sqrt{\frac{1}{3}v^2}$ the number of such units traversed in a unit of time, the supporting effect of the medium on the heavy plane n in the unit of time, is the same as that derived from $\frac{1}{2}\Delta^3\sqrt{\frac{1}{3}v^2}$ molecules impinging with the velocity $\sqrt{\frac{1}{3}v^2}$. Hence it is obvious that $A = \frac{1}{2}\Delta^3\sqrt{\frac{1}{3}v^2}$, and $w = \sqrt{\frac{1}{3}v^2}$, or $c = \frac{1}{2}$, and $3w^2 = v^2$. Also $n = \frac{2}{g} w^2 c \, \Delta^3 = \frac{v^2 \Delta^3}{3g}$; or $v^2 \Delta^3 = 3gn$; or $v = \sqrt{3g \frac{n}{\Delta^3}}$.

Thus we obtain an expression for the square root of the mean square molecul velocity in terms of the height of a uniform atmosphere $\frac{n}{\Delta^3}$, or what is the same, in terms of the ratio of the number of molecules in a column of the medium of the height of a uniform atmosphere to the number in the column of a unit height: and since $\sqrt{2gh}$ expresses the velocity acquired by a body in falling through the height h, we arrive at the following deduction. *The mean square molecular velocity of a medium is equal to the square of the velocity that a body acquires in falling through one-and-a-half times the height of a uniform atmosphere;* if the pressure of the medium is estimated from the effects of the molecular impacts on a perfectly rigid and elastic surface. If it were estimated by the effect on molecular elastic surfaces, there is reason to believe that the mean square velocity is double the amount specified (see Sec. 4) . . . . XI.*

§ 18. Suppose that the cubical volume of the medium receives such an increment of *vis viva* that under the constant pressure n the volume from 1 becomes $1 + \frac{2w^2}{gn^2}$; it has been shown in § 5 that the molecular *vis viva* must also from 1 have become

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* [The author here arrives at the correct conclusion,

$$v^2 = 2g \, . \, \tfrac{3}{2}h. \text{ —R.]}$$

$1 + \frac{2w^2}{gn^2}$ that it may sustain the same pressure with the reduced density. But the molecular *vis viva* in the cubical volume is $\Delta^3 v^2$: hence we have the proportion

$$1 : 1 + \frac{2w^2}{gn^2} :: \Delta^3 v^2 : \Delta^3 v^2 \left(1 + \frac{2w^2}{gn^2}\right);$$

the increment of *vis viva* in the contained medium is thus $\Delta^3 v^2 \frac{2w^2}{gn^2}$. But it has been shown in § 17 that $w^2 = \frac{1}{3}v^2$, and $gn = \frac{1}{3}v^2 \Delta^3$, hence $\Delta^3 v^2 \frac{2w^2}{gn^2} = \frac{2v^2}{n}$.

Now in § 16 it was shown that to raise the weight of n molecules to the height $2w^2/gn^2$ the *vis viva* expended was $2w^2/n = \frac{2}{3} v^2/n$; comparing this, the *vis viva* expended in the act of increasing the volume, with $2v^2/n$, the increment of *vis viva* required to support the increased volume, it is evident that the ratio is exactly one-third. Thus we deduce that *to effect one increment of expansion in the volume of a medium subsisting under a constant pressure, four-thirds of an increment of vis viva are required: one of which thirds is expended in the act and does not appear in the medium: the remaining three-thirds, or one increment, appears in the medium and assists in sustaining its augmented volume** . . . . XII.

§ 19. This result is a necessary consequence of the perfect conservation of *vis viva* in the impinging action of perfectly elastic bodies. To enlarge a volume that is pressed upon is to raise a weight; is to expend mechanical force; is to expend molecular *vis viva*, and the last train of reasoning has led us to the relation between the molecular force expended and the work performed by it.

The mechanical value of the whole of the *vis viva* of the medium may be ascertained by the following proportion

$$\frac{2v^2}{3n} : \frac{2v^2}{3n^2} :: \Delta^3 v^2 : \frac{\Delta^3 v^2}{gn}. \quad \text{But } gn = \tfrac{1}{3}\Delta^3 v^2, \text{ therefore } \frac{\Delta^3 v^2}{gn} = 3.$$

Thus the *vis viva* expended in raising n, or the constant pressure, through the increment of the unit of volume, is to that increment as the whole *vis viva* of the medium is to three units of volume. Hence we deduce that if a medium is supposed to expand to four times its original volume with its original tension undiminished, it will in doing so expend as much as the whole of its original molecular force. This may be more concisely expressed by: *The molecular vis viva of a medium is equal to its tension acting through three times its volume*† . . . . XIII.

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* [The corrected version of XII. will be "To effect one increment of expansion in the volume of a medium subsisting under a constant pressure, *five*-thirds of an increment of *vis viva* are required; *two* of which thirds is expended in the act, and does not appear in the medium; the remaining three-thirds, or one increment, appears in the medium, and assists in sustaining its augmented volume."—R.]

† [This is the virial equation applicable under the supposed conditions.—R.]

§ 20. If a medium is not allowed to increase in volume while its *vis viva* is increasing, no force will of course be expended, and each increment of *vis viva* engenders a like increment of tension. Thus *if we compare the amount of vis viva required to produce an increment of molecular vis viva in the medium, in the two cases of constant pressure and constant volume; it is manifest that the ratio is 4 to 3, or 4/3*[*] . . . . XIV.

§ 21. If we suppose the heavy plane n instead of being raised by the medium to descend upon it through the same differential height $2w^2/gn^2$, it is obvious that the same differential *vis viva* that was formerly abstracted is now communicated to the molecules of the unit volume. Force is exerted by the descending weight upon the medium and is transferred to its molecules.[†] Thus it is evident that the conversion of mechanical force into molecular *vis viva* is subject to the same law as the conversion of molecular *vis viva* into mechanical force. This law is expressed in XII. and XIV. The following is another form of annunciation which refers to an experimental method of ascertaining it if such media were actual existents. *The ratio of the increment of vis viva evolved by a small condensation of a medium to the diminution of molecular vis viva required to maintain the same condensation under a constant pressure is* $\frac{1}{3}$[‡] . . . . XV.

§ 22. If a medium is compressed or dilated and the molecular *vis viva* evolved in it or given out from it by the act of condensation and dilatation be retained, let us enquire into the ratio of the density to the pressure. The preceding reasoning has shown that the increment or decrement of *vis viva* is equal to one-third of the increment or decrement of density, or $\frac{dv^2}{v^2} = \frac{d\Delta^3}{3\Delta^3}$; hence $\frac{2dv}{v} = \frac{d\Delta}{\Delta}$, which being integrated gives $v^2 \doteq \Delta$. But $v^2\,\Delta^3 = e$, therefore $\Delta^4 \doteq e$, and $v^2 \doteq \sqrt[4]{e}.$ Thus we deduce that *if a medium is compressed or dilated from a given condition of density and vis viva, the mean square molecular velocity varies as the fourth root of the tension or as the cube root of the density*[§] . . . . XVI.[‖]

§ 23. The tendency of media to have their *vis viva* augmented when being forced into smaller volume is very similar to the rise of temperature that appears in air when being condensed. Thus tinder may be inflamed by the sudden compression of a small quantity of air, and on charging an air gun the condenser and force pump become so hot as to be painful to touch. Again, mercury may be frozen if exposed to a jet of air escaping from a state of high compression and expanding against the

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[*] [The ratio of specific heats, commonly called γ, should be 5 : 3, not 4 : 3.—R.]

[†] Note A (motion indestructible as matter).

[‡] [This result also requires correction.—R.]

[§] Note B (vapours).

[‖] [The corrected argument is:—Since $dv^2/v^2 = \frac{2}{3}\,d\Delta^3/\Delta^3$, we get $dv/v = d\Delta/\Delta$, or $v \doteq \Delta$. Accordingly $v^2 \doteq \Delta^2 \doteq (\Delta^3)^{\frac{2}{3}}$. But $v^2\Delta^3 = e$; therefore, $\Delta^5 \doteq e$, and $v^2 \doteq e^{\frac{2}{5}}$. Also $e \doteq (\Delta^3)^{\frac{5}{3}}$, or $p \doteq \rho^\gamma$, where $\gamma = \frac{5}{3}$.—R.]

atmospheric pressure. Media also lose their *vis viva* if allowed to expand against pressure.

Thus by XVI., if at 60° Fahr. a vessel containing air of double density is allowed to empty itself into the atmosphere, the decrement of temperature in the air that remains ought to be $-48°$ by the XVI. deduction. If on the contrary its density is suddenly increased from 1 to 2, its temperature ought to rise to 196°.*

Thus the analogy between media and gases and between *vis viva* and temperature is still maintained, and the phenomena of latent heat in gases appear also in media as the transference of force during a change of volume : *out* of the medium when it expands, and exerts, or gives out, mechanical force ; *into* the medium when it is compressed and acted upon, or receives mechanical force.

The phenomena of latent heat thus appear to be the conversion of mechanical force into molecular *vis viva;* the visible into the invisible, as in condensation : and molecular *vis viva* into mechanical force; the invisible into the visible, as in expansion.

It is a necessary consequence of the conservation of *vis viva* or indestructibity of force among perfectly elastic bodies.

§ 24. Several experiments have been made on the *ratio* of the increment of temperature evolved by a small condensation of a volume of air to the diminution of temperature required to produce the same condensation under a constant pressure, and found it to be $\frac{3}{8}$. Mr. IVORY ('Phil. Mag.,' 1827) has proved that this ratio is constant under every change of temperature or density so long as DALTON and GAY-LUSSAC'S law is maintained, or the air thermometer is an exact measure of heat. MM. GAY-LUSSAC and WELTER have also proved this experimentally for a considerable range. Mr. IVORY has also expressed his opinion that the nascent value of this ratio will be found to be $\frac{1}{3}$, and that the cause is probably connected with the proportion that subsists between the linear and the solid increments of expansion.

The value of this ratio in all media, whatever may be their condition of density or *vis viva*, is $\frac{1}{3}$ (§ 21) ; and the synthetical demonstration rests on the same fundamental principle that determines the proportion of linear to solid increments of expansion.

This ratio $\frac{1}{3}$ applies only to infinitesimal changes of volume, and it slowly increases with the amount of condensation. When the medium is compressed from 1 to 1·20 the ratio becomes $\frac{3}{8}$.†

In the experiments of MM. GAY-LUSSAC and WELTER referred to in the 'Mécanique Céleste,' the condensation did not exceed $\frac{1}{40}$th part of an atmosphere. The discrepancy may be exactly ascertained by performing the same experiments as it were on the medium by computation. The absolute temperature is denoted by v^2

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* Note C (temperature of condensed air).

† Note D (to find the compression that corresponds with a given ratio between latent and sensible heat.)

and the absolute zero corresponds to $-461°$ of Fahr. scale (taking RUDBERG's constant of expansion, see § 6). By employing the equation $e = v^2 \Delta^3$ and $\left(\frac{e_0}{e_1}\right)^{\frac{1}{4}} = \frac{{v_0}^2}{{v_1}^2}$; (XVI), it is easy, by substituting the barometric height for e, to compute the result of any given experiment as if it had been performed on the hypothetical medium. This I have done in the case of MM. CLÉMENT and DÉSORMES' experiments. Referring to the account given of them in the 'Mécanique Céleste,' the minus interior pressure of the medium would have been 3·42 millims. ; the mean result is given as 3·61, the difference being only about $\frac{1}{130}$th part of an inch of mercury. If the experiment gave the nascent ratio, the minus pressure at the end would have been one-fourth of the minus pressure at the beginning, or 3·45 millims. In MM. GAY-LUSSAC and WELTER's experiments, the difference of pressure at the end was 4·44 millims. ; the nascent ratio in a medium would in a like experiment be 4·09 millims., the difference being about $\frac{1}{70}$th of an inch of mercury.

The evidence afforded by these experiments may be summed up as follows :—

The initial ratio of the increment of *vis viva* under a constant volume to the increment required to effect the same change of *vis viva* under a constant pressure in the medium is . . . . $\frac{1}{1·33}$.

In air the same ratio of the increments of heat under the same circumstances by MM. CLÉMENT and DÉSORMES' experiments, is . . . . $\frac{1}{1·35}$.

In air the same ratio, by MM. GAY-LUSSAC and WELTER's experiments, is . $\frac{1}{1·37}$.

A difference in the reading of the height of the mercury in the manometer of $\frac{1}{130}$th and $\frac{1}{70}$th part of an inch would bring the respective experiments to coincide with the theory.*

§ 25. In the more recent of Mr. JOULE's physical researches that gentleman has applied mechanical force to the compressing of air surrounded with water, to collect the heat evolved, and has found that about 800 lbs. descending through the height of one foot increases the temperature of a pound of water one degree. The same result nearly was obtained by forcing water through narrow tubes. Mechanical force was expended, and the same proportionate amount of heat was produced in the water. It is remarkable that the same mechanical value was found for the heat generated by the magneto-electric machine. Such accordance in the results, as Mr. JOULE remarks, seems strongly to favour the *vis viva* or vibratory theory of heat.

If air is similar in its constitution to a medium we may employ the deductions of this section to determine the mechanical value of any quantity of heat applied to it,

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* [The fair agreement of the erroneously deduced value of γ, viz., $\frac{4}{3}$, with observation, was doubtless the reason of the author not discovering his mistake of calculation. We know that upon his principles the calculated value should be $\frac{5}{3}$, which accords much less well with the results observed for ordinary gases than does $\frac{4}{3}$. It should be borne in mind that the observed value, $\gamma = 1·405$, has not, even at the present time, been reconciled with theory, although reasons may be given for a departure from $\gamma = \frac{5}{3}$. —R.]

and assuming the specific heat of air to be 0·238 that of water,* we may ascertain the mechanical value of 1° applied to 1 lb. of water, which is equal to $\frac{1}{0\cdot238}$ degree applied to 1 lb. of air. Since 820 cubic feet of air at 60° and 30 in. tension weigh as much as one cubic foot of water, we have $\frac{820}{0\cdot238} = 3444$ cubic feet of air which, heated one degree without being allowed to change its volume, requires as much heat as one cubic foot of water to raise it one degree. The absolute temperature at 60° is 520°, and one degree added augments by $\frac{1}{521}$ part the absolute heat or molecular *vis viva* of the air. But the whole *vis viva* in 3444 cubic feet of air at the temperature 60° and pressure 30 in., is equal to the whole pressure of the atmosphere on a square foot, acting through three times 3444 feet in height, or 10,332 feet (XIII.). The pressure of a column of 30 inches of mercury on a base of 1 square foot or 144 square inches is equal to 14·722 lbs. × 144 = 2120 lbs. This weight raised through 10,332 ft., corresponds to 21,904,000 lbs. raised one foot high, and $\frac{1}{521}$ part of this, or 42,043 lbs., raised one foot high represents the absolute mechanical effect of 1° of heat applied to one cubic foot of water. Dividing this by 62½, the number of lbs. in a cubic foot of water, we get 673 lbs. raised one foot high equal to the mechanical effect corresponding to 1° of heat applied to 1 lb. of water. This compared with Mr. JOULE's result is not unsatisfactory considering the difficulties that attend the experiments that afford the data.†‡

SECTION IV.—ON THE RESISTANCE OF MEDIA TO A MOVING SURFACE.§

26. The simplest case of resistance is that attending the motion of a rigid and perfectly elastic plane moving in the direction of its perpendicular.

Let the velocity of its motion be z, which we must at first assume to be indefinitely smaller than v, the square root of the mean square molecular velocity. Let a molecule with velocity u, impinge on the front surface of the moving plane at an angle θ; the impinging velocity is $u \sin \theta$; and applying the formula for the meeting impact (§ 2) the velocity of reflexion is $u \sin \theta + 2z$, and the square of this is $u^2 \sin^2 \theta + u \sin \theta\, 4z + 4z^2$. The increment of molecular *vis viva* received from the

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* Note E (specific heat of air).

† Note F (M. CLAPEYRON's view of the motive power of heat examined).

‡ [This is an independent calculation of the mechanical equivalent of heat, quite distinct from that of MAYER.—R.]

§ [The weak point in the argument of this section appears to be the neglect of the effect of the altered velocities of the reflected molecules in disturbing the condition of those about to impinge. The results can only apply when the dimensions of the obstacle are small in comparison with the free path of the molecules.

The non-agreement of his theory with observations upon the resistance experienced by obstacles which do not comply with the above condition, unfortunately led the author to take the step in the wrong direction explained in §§ 27, 28. But it is proper to note that the author speaks with hesitation (§ 29). —R.]

plane is therefore $u \sin \theta \, 4z$. To find the increment given to the medium in a unit of time we have first to consider the number of impacts in the unit of time if the plane were at rest, and then the additional number owing to its motion. Referring back to the reasoning in § 17, it has been shown that the impinging velocity $u \sin \theta$ is repeated $u \sin \theta$ times in a unit of time ; hence the increment of *vis viva* in the unit is $u^2 \sin^2 \cdot \theta \, 4z$, and as the mean of all the values of $u^2 \sin^2 \theta$ is $w^2 = \frac{1}{3} v^2$, we have—following the reasoning in § 17, and supposing the surface of the plane to be equal to the side of the cube that contains Δ^3 molecules—the increment of *vis viva given* to the medium by the front surface in a unit of time is $\frac{1}{6} \Delta^3 v^2 \, 4z$. By applying the same reasoning to the back surface of the plane the same amount of *vis viva* is found to be taken from the medium on that side.

We have now to consider the additional number of impacts due to the motion of the plane. Let us first suppose that no change of density is caused by the motion. The action of a medium on a surface at rest is the same as that of a uniform current of molecules whose mean distance is $\sqrt[3]{\frac{1}{2} \Delta^3}$ and velocity w (see § 17), and in the same manner the surface meeting this current with the velocity z, the effect is the same as if the velocity of the current were increased to $w + z$. The additional number of impacts due to the motion of the plane is therefore $\frac{1}{2} \Delta^3 z$, and the mean increment of *vis viva* to each being $4z$, the whole increment in the unit of time is $2 \Delta^3 z^2$. The same reasoning applied to the action on the back surface shows that the diminution in the number of impacts is also $\frac{1}{2} \Delta^3 z$, and the mean decrement of *vis viva* caused by each impact being also $4z$, we have the decrement of *vis viva* in a unit of time also $2 \Delta^3 z^3$. The sum of these $4 \Delta^3 z^2$ is the force required to move the plane with the velocity z. The weight n, whose pressure is equal to this force, is found as in the last equation of § 17 : there we had $w^2 \Delta^3 = ng$, and in the same way here we have $4z^2 \Delta^3 = n_0 g$, or $n_0 = \frac{4}{g} z^2 \Delta^3$.

This result differs very much from the actual resistance of a body moving in air as observed by ROBINS and HUTTON. In HUTTON'S Dictionary it is mentioned that the resistance to a surface of one square foot, moving 20 feet per second, was found to be 12 ozs., and that it increased as the square of the velocity. Now if we compute the resistance by the formula $\frac{1}{g} z^2 \Delta^3 = n_0$, which is the common theory at low velocities, we shall find n_0 to be 15 ozs. nearly, when z is 20 feet per second, the weight of a cubic foot of air being represented by $\frac{1000}{820}$ oz.

Here then is a notable discrepancy ; the resistance of the medium that represents air in specific gravity and tension appears to be four times greater than it ought to be.

§ 27. We have all along assumed, for the sake of simplicity and to avoid any addition to the fundamental hypothesis, that the surfaces upon which the medium acts are perfectly rigid as well as perfectly elastic, although no such surface, so far as we know, exists in nature. If molecular *vis viva* is heat, the molecules of solids must be in motion as well as those of gases; and the nature of the motion must be such as to permit the equilibrium of *vis viva* to be established between them respectively. They must also be perfectly elastic; but a surface composed of them cannot be assumed as perfectly rigid.

The original hypothesis in respect to gases involves the necessity of making certain assumptions respecting the physical condition of the surfaces upon which media are supposed to act.

(1.) That they are composed of molecules in a state of vibratory motion which results from the struggle that their *vis insita* makes with the attractive and repulsive forces of aggregation.

(2.) The nature of these vibrations and forces may remain undefined, further than that the *vis insita* proper to a molecule is alternately destroyed at the extremities, and reproduced in the middle of each vibration when it and the surrounding molecules are in equilibrium of *vis viva*.

(3.) The impact of the molecules of a medium on the molecules of the solid surface is that of perfectly elastic bodies, and enables the equilibrium of *vis viva* to be established between them.

This equilibrium must be effected by a continual interchange of *vis viva*, the molecules of the solid giving to the molecules of the medium and *vice versa*.

In the case of a heavy molecular plane supported by the elasticity of the medium, as detailed in § 2 and § 16, the impacts that take place on the lower surface, establish both the molecular *vis viva* equilibrium, and also the statical equilibrium of the heavy plane.

Does this new condition of surface upon which the medium acts, make any change in the relation $ng/2v = \mathrm{A}$, that was shown to subsist between the impinging velocity of succession, and weight supported when the surface was assumed as perfectly rigid? Then, the molecules of the medium encountered at each impact the whole *vis insita* of the heavy plane and communicated directly to its centre of gravity a certain infinitesimal velocity. Now, it strikes a vibrating molecule which *afterwards* communicates a certain effect or infinitesimal velocity on the centre of gravity of the molecular plane. Let us endeavour to gain a clear idea of the numerical relation between the effect and the cause that produces it, viz., the impinging force of the free molecule of the medium.

The centre of gravity of the whole molecular plane being at rest while the centre of gravity of each of its molecules is in a state of intense vibration, it is evident that the track or orbit described by the centre of gravity of one of its molecules must be exactly imitated but on an infinitely reduced scale and in a reverse direction by the motion of the centre of gravity of the remaining molecules of the plane. The action and reaction of the molecular forces are equal. At any point of the orbit of one of the lower molecules of the plane let a free molecule of the medium impinge. If they are equal in mass they will exchange impinging velocities ; this interchange modifies the vibration and disturbs the harmony between the simultaneous motions of the centres of gravity above referred to ; the motion of the centre of gravity of the molecule changes suddenly, not so with the centre of gravity of the remainder of the plane. If the connection between the plane and the molecules were broken at the instant of impact, it is clear that the centre of gravity of the remaining molecules of the plane must continue to move in the direction and with the velocity it had at the instant. Now as the concurrence may take place at any part of the vibration, either going or returning, it is plain that the mean motion of the centre of gravity of the remainder of the plane caused by the transference of *vis viva* from the molecule of the plane to that of the medium is zero. But the centre of gravity of the remainder of the plane reciprocates the active effect of the molecular force on the new velocity until it is destroyed at the end of the first vibration ; the molecular force acting as much on the remainder of the plane as upon the molecule. The destruction of this by the mutual binding force destroys in the opposite direction the same amount of *vis insita* in the plane, or generates it in the same direction, and as we have to attend only to the effect upon the centre of gravity of the plane made by the motion transferred from the medium to the molecule of the plane, the mean effect must be equal to the mean incident *vis insita* of the molecules of the medium; in short the same as if the striking molecule cohered to the plane after impact. This is the case if the plane is at rest when struck, but a condition of statical equilibrium requires that the infinitesimal descending motion by gravity should be equal to the ascending infinitesimal motion given by the impetus of the striking molecules. The upward velocity therefore given to the plane by this impetus is only one half what it would be if the plane were at rest when struck (see § 2). Thus the expression $\frac{2v}{n+1}$ (see § 2) becomes $\frac{v}{n+1}$, and v/n becomes $v/2n$, and $gn/2v = \mathrm{A}$ becomes $gn/v = \mathrm{A}$.

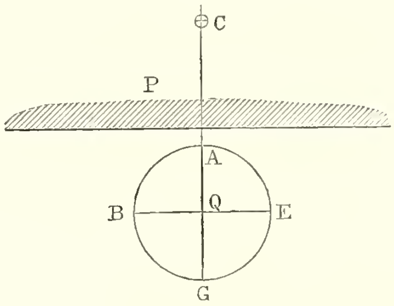
These alterations make no difference in the subsequent reasoning until we come to § 17 where the equation for A is employed, and in consequence of its change of value the terminal equation $\Delta^3 v^2 = 3gn$ is changed to $\Delta^3 v^2 = 6gn$.

This alters the value of v from $\sqrt{3g\frac{n}{\Delta^3}}$, the velocity acquired in falling through one and a half uniform atmospheres, to $\sqrt{6g\frac{n}{\Delta^3}}$, the velocity acquired in falling through three uniform atmospheres, and the numerical value of v in the medium that corresponds with air at the temperature of melting ice is 2244 feet per second.

§ 28. As this change in the value of v reconciles the discrepancy in the theory of resistance, and in the subjects of the two concluding sections, it may be proper to illustrate by diagram the general principle that the mean impinging effect of free molecules on a cluster of cohering molecules is the same as if the striking molecules cohered at the instant of impact.

Let P be the molecular plane consisting of $(n - 1)$ molecules, and Q one of its lower molecules at rest between attractive and repellent forces.

Fig. 1.



(1.) Let Q receive an impulse in the direction GQ, so that its initial velocity may be $QE = v$; it will proceed along the line QA until its motion is subdued at some point A by the molecular repulsive force; at the instant when the centre of gravity of the molecule Q, and centre of gravity of P, are at their minimum distance both move together with the common velocity v/n, and this is the velocity communicated to their common centre of gravity, which is not disturbed further by their mutual action during the vibratory motion that ensues. In this case there is both molecular *vis viva* communicated to Q, and also the velocity v/n to the common centre of gravity, and a free molecule of the medium has lost the velocity v.

(2.) Suppose, in the next instance, that the centre of gravity of $P + Q$ or n to be stationary, while Q is continuing its vibrations, and let it be struck when at the centre of its descending vibration by a free molecule having the same velocity QE. They will be reflected from each other without gain or loss of motion, and Q will move back towards A instead of forward to G. Here there is a loss of the down motion and the gain of an up motion. By the first (1.) the centre of gravity of P continues to ascend with the velocity $\frac{v}{n - 1}$ that it has at the instant of impact; and as the downward force of Q that subdued this is gone, the centre of gravity of the whole $P + Q$ or n molecules acquires the permanent upward velocity v/n. By the second (2.) the same velocity v/n is given to the common centre of gravity by the gain of the up motion as in the first case of impact. The result of the second mode of impact is thus to communicate the velocity $2v/n$ to the common centre of gravity.

(3.) If the same kind of impact takes place with Q in the middle of the ascending vibration, it is evident that the force of impact is zero at that point, and the upward velocity to the centre of gravity zero. Now $\left(\frac{2v}{n} + 0\right) \div 2$ (impacts) $= \frac{v}{n}$. This result is more obvious if we assume the velocity of the free molecules $v + \lambda$ and that of $Q = v$. The result of the first of these is $\frac{2v + \lambda}{n}$, and of the second $\frac{\lambda}{n}$. Half the sum of these is $\frac{v + \lambda}{n}$, and thus the mean result of these two equally probable impacts is the transmission of the *vis insita* of the impinging molecule to the plane as if it cohered to it after impact.

(4.) At either of the extremities of the vibration the same law obviously applies, but it is the two impacts where *vis viva* is taken from and again returned to Q whose effects ought to be viewed together. In the meeting impact let the velocity of Q from v change to u with a different direction; then according to the second case of impact the upward effect on the plane is to give it the ascending velocity $\frac{v + \overline{v - u}}{n}$, and the loss of velocity is $\overline{v - u}$. Let this loss be returned in an overtaking impact so that $\overline{v - u}$ shall become v; then according to the first case of impact the upward effect on the plane is u/n. In these two impacts Q returns to its original condition of motion, and the mean effect is v/n. A continual and equal interchange of *vis viva* being necessary to the persistent molecular condition of the plane and of the medium, the same is effected by means of impacts which take place equally in the ascending and descending vibration. This equality seems to be a necessary condition because the motions that are taken account of are the velocities of impact resolved in a vertical direction only, and the plane of impact cannot now be assumed always to be horizontal as in the case of the rigid plane; hence the absolute velocity in the vibration and the resolved impinging velocity are independent variables.

§ 29. Such is the view of the phenomena which seems to authorise the change that has been imposed on the value of the mean square molecular velocity. It has no pretension to be considered as a demonstration, and we are therefore not permitted to make use of it as a synthetical deduction from the hypothesis.

Nevertheless, if it is admitted as being probable, the probability is increased if it reconciles at once all the discrepancies that have been met with, and at the same time neither affects any one of the preceding deductions where the analogy to the properties of gases is perfect nor introduces any other point of discordance.

If we now revise the mode of estimating the law of resistance in § 26 it is obvious that the mean increment of velocity communicated by the plane now considered as molecular to the free molecules of the medium is not $2z$, but z, and hence the mean increment of *vis viva* in each incident molecule is not $4z$, but $2z$, and the increment in a unit of time not $2\Delta^3 z^2$, but $\Delta^3 z^2$. The sum of the front increment and back decrement is not $4\Delta^3 z^2$, but $2\Delta^3 z^2$; and as $w^2 \Delta^3$ is no longer equal to ng, but to $2ng$, we have $2\Delta^3 z^2 = 2ng$, or $n = \frac{1}{g}, \Delta^3 z^2$, which is the equation derived from the common theory of the resistance of the atmosphere at low velocities.

It will be remarked that the resistance is as much derived from the minus pressure behind as from the resistance in front, whereas the common theory only takes account of the inertia of the front which is assumed at low velocities as constituting the whole of the resistance.*

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* Note G (objection to undulatory theory of heat).

SECTION V.—ON THE VERTICAL EQUILIBRIUM OF A MEDIUM, SUPPOSING IT TO FORM THE ATMOSPHERE OF A PLANET.*

§ 30. Suppose the height of the atmosphere AB to form the axis of a parabola of which the vertex A is at the summit. If a body begins to fall from A it is evident from the law of falling bodies that its acquired velocity at any point is porportional to the ordinate of the parabola at that point.

Divide AB into an infinite number of parts so that the length of each shall be proportional to the ordinate of the parabola at that part. Suppose that in each of these parts one molecule is vibrating upwards and downwards, striking against the upper and lower molecules of the adjacent parts with a velocity proportional to the ordinate of the parabola and equal therefore to what a body would acquire in falling from the vertex. It is evident that each of the parts or infinitesimal divisions will be traversed in the same time dt by its molecule, and that the impinging velocities of each pair are equal, so that there is a perfect equilibrium and constancy of phenomena; but the upper impact of a molecule against the one above it is made with less velocity than the lower impact against the one below it, because the accelerating force of gravity increases the velocity during the interval of descent, and the acceleration is represented by the increment of the parabola's ordinate in that interval. If g be the accelerating force of gravity, or velocity bestowed on a falling body every unit of time, the acceleration in each interval of descent, or infinitesimal division of the height AB, is evidently gdt. If this constant increment of velocity should by any cause be reduced in any given proportion, the aggregate effect must evidently be the same as if the force of gravity g were reduced in the same proportion.

In such a vertical column of single molecules it is apparent that the equilibrium acquires† a continually increasing velocity in the molecular motion from the summit to the base; and since the *vis viva* of a molecule is measured by the square of its velocity, it is also obvious that the molecular *vis viva* increases in the simple proportion of the distance from the summit. And knowing v^2 the amount of *vis viva* in the molecules at the base, we also know the height of the column $v^2/2g$, which is simply the height due to the molecular velocity.

§ 31. In a medium the nature of the action that sustains the upper molecules must be the same. The mean of the upper molecular impacts of a stratum must have less force

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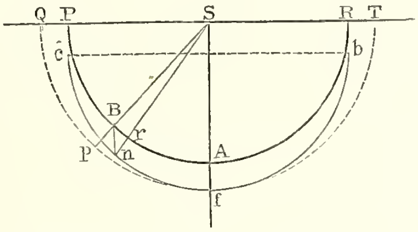
* [This section attempts to deal with one of the most difficult points in the theory. That the loss of velocity suffered by every ascending molecule will lead to a smaller mean velocity above than below seems, at first sight, inevitable. This consideration was urged by GUTHRIE ('Nature,' vol. 8, p. 67, 1873); and, in his reply (p. 85), MAXWELL narrates that a similar argument, which occurred to him in 1866, nearly upset his belief in calculation. WATERSTON'S result really depends upon an assumption that, at a given height, the molecular velocities are all the same; whereas, according to the true Maxwellian law, all velocities are to be found at all heights. The force of this consideration will be appreciated when it is remembered that those molecules which at any time move at a low level with low velocities, would not of themselves reach a high level at all.—R.]

† [? requires.—R.]

than the mean of the lower, the difference being the accelerating effect of gravity in the breadth of the stratum. We have also to remember that the nature of the equilibrium of a medium requires that if the velocities of all the molecules that pass a horizontal or other plane in a given time are resolved perpendicular to that plane, the sum of the squares of these resolved velocities are equal in opposite directions. In a constant infinitesimal time the absolute acceleration of velocity is evidently greater with the vertically moving molecules than with those moving obliquely to the vertical; thus the aggregate effect of the accelerating force of gravity in increasing the molecular velocity must be less than if it acted upon them directly as in the vertical column (§ 30). The question is, how much less? for in such proportion must we consider the force of gravity to be reduced, supposing it to act uniformly on all the molecules of the atmosphere. The retardation of the ascending molecules of a stratum is equal to the acceleration of the descending molecules. Let us consider the latter.

We assume from the original hypothesis that in any infinitesimal area the lines of molecular motion lie equally in every direction, so that if supposed to issue from one point S, they would be directed equally to every point of the surface of a sphere of which S is the centre. Let SB represent one of these velocities and B*n* the vertical acceleration by gravity in the infinitesimal time *dt*. With the radius SB describe a hemisphere having its base on the horizontal plane PS. It is evident that the locus of the point *n* is the surface of another hemisphere with its base at the distance of B*n* below the plane PS, and that the area of the space between the bases is equal to the area between the surfaces of the hemispheres. Now, if gravity acted on each line of molecular motion, instead of acting only in the vertical, the common increment of velocity that would affect all is B*p* = B*n* = *gdt*; and this, when B*n*/BS is infinitesimal, is equal to the quotient of the area between the concentric hemispheres *qpf*T, PBAR, by the surface of the inner hemisphere. But the actual increase of BS is *nr* = *n*S − BS, and the mean increment is found by adding up all the particular values of *nr*, and dividing the sum by the number of values, or, what amounts to the same thing, taking the quotient of the area between the equal hemispheres PBAR, *cnfb* by the surface of one of them. Now the first is equal to the area between the bases, which is equal to the product of a great circle by B*n*, and the latter is well known to be equal to two great circles; therefore the quotient is $\frac{1}{2}$B*n*.

Fig. 2.



Thus, it seems to be clear that the molecules of a medium are collectively only half as much affected by gravity as if they all moved in vertical lines; but it has been shown (§ 30) that if they moved in vertical lines the height of the atmosphere would be the height due to the molecular velocity; but as the increments of their velocity by gravity is only one-half what they would receive if their motion were vertical, that height must be computed as if the force of gravity were only one-half the actual amount. Thus, if v^2 be the mean square velocity at a depth H below the summit of the atmosphere, the height due to this with the full effect of gravity is $v^2/2g$, and with half effect it is $v^2/g = \mathrm{H}$. Thus we arrive at the following deduction. *The molecular* vis viva *increases simply as the depth below the summit of the atmosphere, and the height of the summit above any stratum is equal to the quotient of the mean square molecular velocity at that point by the accelerating force of gravity, or to double the height that a free and unresisted projectile would ascend if projected vertically with an initial velocity equal to the square root of the mean square molecular velocity in the stratum*. . . . . XVII.

§ 32. To ascertain the law of density we have the equation in § 17, modified as in § 27 to $\frac{1}{6}\Delta^3 v^2 = gn$, for the reasons given in last section. By this we have $v^2/g = 6n/\Delta^3 = \mathrm{H}$, which applies to any part of the atmosphere at all heights.

Differentiating the equation $\frac{6n}{\Delta^3} = \mathrm{H}$, we have $\frac{6dn}{\Delta^3} - \frac{6n3d\Delta}{\Delta^4} = d\mathrm{H}$. But $\frac{dn}{\Delta^3}$ is evidently equal to $d\mathrm{H}$, and $\frac{6n3d\Delta}{\Delta^4} = \frac{6n}{\Delta^3} \times \frac{3d\Delta}{\Delta} = \mathrm{H}\,\frac{3d\Delta}{\Delta}$; therefore $\frac{6dn}{\Delta^3} - \frac{6n3d\Delta}{\Delta^4} = d\mathrm{H} = 6d\mathrm{H} - \mathrm{H}\,\frac{3d\Delta}{\Delta}$; or $\frac{3d\Delta}{\Delta} = \frac{5d\mathrm{H}}{\mathrm{H}}$; and by integration we have $\Delta^3 \doteqdot \mathrm{H}^5$. Thus we deduce that *the density of the medium at any depth below the summit of an atmosphere is proportional to the fifth power of that depth* . . . . XVIII.

§ 33. As we had $v^2 \doteqdot \mathrm{H}$ we may further deduce that $v^2\Delta^3 = \mathrm{H}^6$, or that *the elastic force of the atmospheric medium at any point is proportional to the sixth power of the depth of that point below the summit and to the sixth power of the mean square molecular velocity* . . . . XIX.

These deductions are all embraced by the equations $v^2/g = \mathrm{H} \doteqdot \Delta^{\frac{3}{5}}$, and $v^2\Delta^3 \doteqdot \mathrm{H}^6 \doteqdot \overline{v^2}|^6$.

§ 34. To compare these results with what is known of the physical condition of our atmosphere, we have first the obvious correspondence between the diminution of molecular *vis viva* and of temperature in ascending. No sufficient explanation of this has, I believe, been yet offered, for it is needless to attempt to do so by supposing the specific heat of air to increase as its density diminishes, as no difference of specific heat disturbs the equilibrium of the temperature of bodies placed in horizontal contact. The very fact of a gaseous atmosphere presenting a constant inequality of temperature at different elevations seems to prove that the law of the vertical equilibrium of temperature is essentially different from the law of horizontal equilibrium.

The actual rate of diminution is very fluctuating and uncertain, varying from 200 to 500 feet for each degree. The formula of our hypothesis applies only to the condition of an atmosphere resting on a horizontal base ; such, indeed, as may be found only during a balloon ascent.

M. Gay-Lussac, in his celebrated ascent from the neighbourhood of Paris, found the depression amount to 72½° Fahr. in 7634 yards. This corresponds to 316 feet of elevation to 1°, if the rate is uniform. We have already determined (§ 27) the value of v in air at the temperature of melting ice to be 2244 feet per second; hence $v^2/g = 156{,}593$ feet $=$ H, the height of the atmosphere at this temperature (being nearly 30 miles). Now, taking Rudberg's expansion of dry air, the value of v^2 in degrees of Fahr. is 493° at this temperature, and $\frac{156593}{493} = 317{\cdot}6$, which is the elevation that ought to correspond to 1° by the hypothesis, in which also the rate is uniform. If Dalton and Gay-Lussac's constant of expansion is preferred, the elevation for 1° is 328 feet.

The hypothesis requires that the diminution of temperature should be uniform, and the best authorities agree that it approximates to uniformity at considerable elevations. In M. Gay-Lussac's table of observations taken during his ascent, the indications of the thermometer are somewhat irregular, as might be expected from the manner of making the observations, and the formula (Laplace's) employed to compute the elevations may not, perhaps, answer so well for balloon ascents as it has been found to do in mountainous elevations. We have also to keep in view that the atmosphere absorbs a large proportion of the Sun's rays in their passage through, besides being supplied with heat from the ground irregularly according to the varying characteristics of its surface. Taking all these circumstances into account, the accordance between theory and M. Gay-Lussac's extreme observations is nearer than might be expected, and probably will not be found so exact at lesser elevations.

But the hypothesis admits of being tested without employing any empirical barometric formula, because, if it is correct, the tension as shown by the column of mercury ought to vary as the sixth power of the absolute temperature (from zero at $-$ 461° Fahr.) (XIX). But the observations must be taken at stationary points during the ascent, so that time may be allowed for the thermometer to acquire the temperature of the stratum of air in which the balloon rests. Let z° be the absolute temperature of Fahr. zero; then for any two observations we ought to have $\left(\frac{z+t_1}{z+t_2}\right)^6 = \frac{e_1}{e_2}$; in which t_1, e_1, are the temperature and tension at any one altitude and t_2, e_2, the same at any other. The value of z eliminated from this equation ought to be 448 or 461. I have tried this with M. Gay-Lussac's sixteenth and last observations, which appear to be the most favourable for accuracy, and z comes out equal to 467. The rate of diminution also in the interval of 1800 metres between these two observations agrees well with theory, being 310 feet for each degree Fahr.*

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* Note H (Formula for measuring heights by thermometer).

§ 35. We must now endeavour to show that an atmosphere of mixed media follows exactly the same law of equilibrium.

Atmospheres of different homogeneous media supposed separate from each other must evidently have the same height if the mean square molecular velocity at the base of each of them is the same, and, consequently, the molecular *vis viva* in each atmosphere will, at the same height, be proportional to the specific molecular weight of the medium. If, on the contrary, the molecular *vis viva* at the base of each is the same, then will the height which is proportional to the mean square molecular velocity follow the inverse ratio of the specific molecular weight, which is also equal to the direct ratio of the mean square molecular velocity (VII.). Thus, a hydrogen atmosphere ought to be four times the height of an oxygen atmosphere, &c. When mixed, the molecules of each of the media at the same height are necessarily in equilibrium of *vis viva*, and the mean space occupied by each molecule is therefore the same (§ VIII.). To see distinctly that the condition of the mixture is exactly that of a homogeneous medium of equal specific gravity, or, what is the same, whose specific molecular weight is equal to the sum of the products of the specific molecular weight of each by its proportionate volume—and thus having in equal volumes the same amount of *vis viva* as the mixture—we have only to recollect that a constant increment of descent in the atmosphere corresponds in all parts of it to the same constant increment of mean square molecular velocity, whatever the molecular velocity may be, or whatever the weight of the molecules with which it is associated, and the increment of *vis viva* in each medium for the same constant increment of mean square molecular velocity is as the product of its specific weight by its constituent volume. But the increment of mean square molecular velocity in the homogeneous being the same as in each of the constituents of the heterogeneous medium, and the product of its specific weight by its constituent volume being equal to the sum of the products of the specific weight and constituent volume of each, it is obvious that for the same increment of descent through the atmosphere the increment of *vis viva* in the homogeneous is the same as in the mixed medium, and that generally the physical condition of an atmosphere consisting of various media mixed together is exactly the same as if it were composed of one homogeneous medium whose specific gravity is equal to that of the mixture.

If each constituent of an atmosphere were supposed to form an atmosphere by itself, and ranged by each other side to side, and having all the same height, the ratio of their densities or proportional number of molecules in a constant volume would be the same at all heights, but the molecular *vis viva* would be respectively as the specific weight of each. If in this condition they were brought together so that all might occupy the space of one, an immediate change in the molecular *vis viva* of each medium would ensue, the heavy molecules losing and the light molecules gaining *vis viva* until the *vis viva* equilibrium is established; and this, as well as their united density, corresponds with the same qualities of the homogeneous medium at the same height. All this is an obvious corollary of what precedes. *The vertical condition of equilibrium of an atmosphere is the same whether that atmosphere consists of one homogeneous medium or of a mixture of different media having the same specific gravity* . . . . XX.

§ 36. The relation between the total height of an atmosphere of the medium to the height due to the molecular square velocity (XVII.) enables us easily to estimate the effect of diminished gravity, and assign the limits of temperature at the base, beyond which an atmosphere cannot be retained.

If the force of gravity is constant, it has been shown that the height of the atmosphere is equal to the height a body would ascend with the molecular velocity, and supposed to be acted upon by half the force of gravity; but as it really diminishes as we ascend, according to the ratio $\left(\frac{R}{R+H}\right)^2$ (in which R is the radius of the planet and H the height above its surface), the true height must be computed with the variable half-force $\frac{g}{2}\left(\frac{R}{R+H}\right)^2$, instead of with the constant half-force $\frac{g}{2}$.

Let v be the initial velocity of the vertical projectile at the surface of the planet, w its velocity at the height h; then shall $\frac{dh}{w} = dt$, and $dt\ \frac{g}{2}\left(\frac{R}{R+H}\right)^2 = -\,dw =$ the retardation in the differential time. Substituting the value of dt, the differential of the time in this equation, we have $-\,dw = \frac{dh}{w}\cdot\frac{g}{2}\cdot\left(\frac{R}{R+h}\right)^2$, and multiplying by $2w$ we have $-\,2w\,dw = -\,dw^2 = dh\,g\ .\left(\frac{R}{R+h}\right)^2$. Integrating this expression gives $v^2 - w^2 = Rg\left(1 - \frac{R}{R+h}\right)$, so that when $w^2 = 0$, we have by eliminating h (which then represents the total height of the atmosphere) $h = \frac{Rv^2}{Rg - v^2}$; being its value in terms of the radius of the planet, of the mean square molecular velocity at its surface, and of the force of gravity at its surface. In the former expression for H, where the force of gravity was supposed constant, we had $H = v^2/g$, or $v^2 = gH$. Substituting this value of v^2 in the equation for h, we have $h = \frac{H}{1 - \frac{H}{R}} = H\left(1 + \frac{H}{R}\right)$, if H^2/R is infinitesimal in respect to unity.

Thus the correction to be applied for the diminishing power of gravity in ascending increases as the square of the height, and employing the preceding data, the total height of the earth's equilibrated dry atmosphere, considered as a medium at the temperature of melting ice, is by this theory 157,776 feet, being 1183 feet more than the last determination, with constant force of gravity; and the correction to be added to the height computed with constant force of gravity is in feet $1{\cdot}2 \times H^2$, the square of the height in miles.

§ 37. We may express the last equation in a more general form with the molecular *vis viva* as the constant instead of the mean square molecular velocity. If the specific weight of the medium that corresponds with air is taken as unity, and $1 \times v^2$ is the value of the molecular *vis viva* at the surface of the planet: with any other medium whose specific weight is s, its mean square molecular velocity with the same *vis viva* is v^2/s, (VII); and the absolute height of its atmosphere is $h = \frac{\mathrm{R}v^2}{\mathrm{R}gs - v^2}$. This equation gives the absolute height of an atmosphere under every variety of condition, and determines the limit of *vis viva* at the surface of a planet beyond which the medium cannot be retained, for when h is infinite we have $\mathrm{R}gs - v^2 = 0$ or $v^2/s = \mathrm{R}g$. With regard to the medium that represents the atmosphere of our planet, we have already determined the value of v^2 to be $(2244)^2$, when the absolute temperature is 493° and $s = 1$; consequently, when $v^2/s = \mathrm{R}g$, the surface temperature must be 65,760° Fahr. for air, and 4556° for a hydrogen atmosphere whose specific weight s is $\frac{1}{14 \cdot 432}$. At these surface temperatures such atmospheres would slowly evaporate into space.*

At the surface of the Moon the limit of temperature for an atmosphere of air is 3008° absolute or 2505° on Fahr. scale. For a hydrogen atmosphere it is 208° absolute or − 253° Fahr. But the proximity of the Earth reduces these limits respectively 100° and 7°, so that if the Moon's surface had even a higher temperature than 2405°, the Earth, according to this theory, would then gradually withdraw the whole of any atmosphere of air that it might then have possessed.

By employing the same equation, it appears that the temperature at the surface of a body like the Sun in magnitude and mass requires to be 13,400° to sustain an atmosphere identical in constitution and height to that of the Earth.†

§ 38. It will not fail to be remarked that the positive evidence in favour of the reasoning of this section turns on one point. Does the law of vertical equilibrium of temperature correspond with the law of vertical equilibrium of *vis viva?* We have seen that the correspondence is more exact than might be expected, although it is extremely difficult to put the question to the test of direct experiment. In a column of air 318 feet high the temperature at the bottom ought to be 1° higher than at the top in any state of the atmosphere. If air is made to circulate quickly in two tubes of this height lined with non-conducting material, the difference of temperature ought to be very distinctly shown by thermometers at the top and bottom. The quick motion of the air downwards and upwards may be expected to compensate for the disturbing effect of the sides of the tube, and even to cause their internal surface to assume the proper atmospherical gradient of temperature.

The accuracy of the formula for measuring heights by the barometer that may be derived from the deductions of this section depends on the integrity of the law of diminution of temperature, but as this varies from local causes, the theoretical rule does not seem to apply so well as those in common use, which are partly empirical.

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* Note K (central heat).

† Note L (nebular hypothesis).

But a strictly theoretical formula may be deduced if it is permitted to assume that the change of temperature between two stations is uniform, whatever that difference may amount to, and likewise that the change of density is conformable.*

SECTION VI.—ON THE VELOCITY WITH WHICH IMPULSES ARE TRANSMITTED THROUGH A MEDIUM.†

§ 39. The reasoning on the subject of this section is founded on the principle that the velocity must correspond with the average velocity resolved through the medium in any one direction.

We have seen in § 17 that the mean square velocity resolved in one direction is equal to one-third of that mean square velocity, and it is easy to prove, if all the velocities of the molecules are equal, that the average resolved velocity in one direction is equal to one-half the common velocity.

As the equal lines representing the molecular velocities on one side of a plane may be assumed to radiate equally in every direction from one point, they will spread to every point of the hemisphere, resting on the plane; let perpendiculars be dropped from these points upon the plane. *The quotient of the sum of these divided by their number is equal to half the common length of the equal lines.* The proof of this is derived from the integration of simple circular functions that give the quotient of the sum of the sines of a hemisphere divided by their number, or by the surface of the hemisphere, equal to half the radius. Thus, if θ be the inclination of the radiating lines to the plane, $v\,d\theta \sin\theta \cos\theta\, 2\pi$ represents the aggregate of the perpendiculars upon the base of the hemisphere, and $d\theta \cos\theta\, 2\pi$ represents their aggregate number. Collecting the quotients of the first by the second for every value of θ from 0° to 90°, or what is the same, integrating these functions, and dividing the first by the second, we have the quotient equal to $\frac{1}{2}v$, which is the mean velocity resolved perpendicular to the stratum when the molecular velocity v is constant.

§ 40. But the hypothesis does not admit of the molecules having all the same

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* Note M (barometric formula).

† [The idea of the direct connection between the velocity of sound and that of the molecules is of great interest, and leads at once to the conclusion that the velocity of sound is independent of density, but proportional to absolute temperature. The next person to raise the question was STEFAN ('POGG. Ann.', vol. 118, 1863, p. 494), but his calculation is as defective as that of the author. On WATERSTON'S principles, the ratio of the velocity of sound to the molecular velocity of mean square should be $\sqrt{5/3}$, as was shown by MAXWELL (PRESTON, 'Phil. Mag.,' vol. 3, 1877, p. 453). In the 'Philosophical Magazine' for 1858 (vol. 16, p. 481) WATERSTON returned to the subject. It is curious that he regarded the ordinary hydrodynamical investigation, not merely as needlessly indirect, but as inconsistent with the molecular theory. A result in harmony with experiment cannot be obtained on the basis of a hypothetical medium constituted of elastic spheres, for such a medium would have a ratio of specific heats different from that observed in gases.—R.]

velocities; we have therefore to enquire into the effects of this diversity upon the velocity of transmission.

The molecules in a small sphere of the medium at any given instant may be classed in respect to their velocity into sets, and of each set we are allowed, by the hypothesis which we are following, to assume that there is an equal number moving in every direction, and that since continuous uniformity in the density requires that the number contained in the spherical space should be always the same, the exit of one of a set may be conceived to be immediately followed by the entrance of another of the same moving in the same direction and with the same velocity. One-half the number in a set is increasing their distance from a given plane, and the other half diminishing their distance. Let the motions of one of these halves be resolved in the direction perpendicular to this plane, and let us add together such resolved spaces as are described by all the molecules of the set that happen to be in the sphere during a constant time for so long as they remain in it, and divide by the constant number of molecules of the set in the sphere at all times; the quotient must evidently be the mean velocity in that direction and set, and must be the uniform rate with which an impulse is conveyed in one direction by means of an infinite series of impacts, the space between two impacts measured in the constant direction being the step forward made by the infinitesimal portion of the impulse contained in the traversing motion of the molecule from one of the impacts to the other.

The number moving in any one direction with the velocity u is equal to the number moving in any other direction with the same velocity, and each of these numbers takes the same time to traverse the sphere. If we compare this time with that taken by the molecules of another velocity or set, it is obvious that these times must be inversely as the velocities, and *the number that continuously pass through the sphere or any other constant space in a constant time must be as the velocity:* for this may be estimated as if there were continuous currents of molecules moving in every possible direction with the respective velocities; the encounters that may be imagined to interrupt this continuity being infinite in number do not alter the general average of velocity or direction or proportionate number, and therefore each velocity and direction may be taken as constant. Now, for any one velocity u viewed thus as constant, the mean resolved velocity in one direction of all the molecules that happen to move with this velocity at any instant is, as above demonstrated, equal to $\frac{1}{2}u$; *but if we add up the resolved spaces traversed by all the molecules that have been in the sphere with this velocity during a constant time, and divide by the constant number that are in the sphere at any instant,* we require to multiply $\frac{1}{2}u$ by a factor that is proportional to u, so that $\frac{1}{2}u^2$, the resulting product, is proportional to the mean distance traversed in a constant time by the molecules that have appeared in a constant space to move with this velocity during the constant time.

Now, suppose an impulse to be given to the medium at any point, and an indefinitely long cylinder of the medium to extend from this point; the impulse given to the molecules at its extremity can be conveyed through the medium only by means of the molecular encounters, each of which contributes to this effect with different velocities resolved in the direction of the cylinder. It has been shown that the mean traversed distance in a given space during a given time due to any one molecular velocity is as the square of that velocity multiplied by the proportion of the number of molecules associated with it. Therefore, the united effect of all velocities must be equal to the sum of these products, and this sum is unity, or the whole medium multiplied by the mean square molecular velocity. But it was shown that if the molecular velocity were constant and equal to v, the absolute value of the mean resolved velocity, or traversed distance, during a constant time in one direction, would be $\frac{1}{2}v$. Now, as the traversed distance or transmitting effect of this velocity in comparison to the other velocities u, &c., as they actually exist in the medium, is as v^2 to u^2, &c., we arrive at the conclusion that $\frac{1}{2}v$ is actually the uniform velocity with which an impulse is transmitted through a medium.

§ 41. According to the reasoning in §§ 27 and 28, the value of v is the velocity acquired in falling through three uniform atmospheres. This gives $\frac{1}{2}v = 1122$ feet per second at the temperature of melting ice, and 1176 at the temperature of 80°, being an increase of $\frac{9}{8}$th of a foot for each degree. In the article, "Acoustics," 'Penny Cycl.,' it is stated that 1125 feet per second at 62° accords nearly with the mean of the best experiments, and that the difference for 1° is $\frac{8}{7}$ths of a foot. The velocity by the hypothesis is therefore about $\frac{1}{40}$ part greater than is found by observation, which, in a distance of 8 miles, amounts to a difference of one second in the time of travelling that distance. This is probably a greater difference than can be allowed between observation and a correct theory. M. MOLL's observations, which seem to be standard authority, were taken with such precautions that an error of 1 in 40 seconds can hardly be admitted. They likewise agree remarkably well with Mr. GOLDINGHAM's observations at Madras.

In taking astronomical observations of the same kind with the same instrument, it has lately been discovered that two individuals differ sensibly from each other. This has led to the suspicion that in all observations there is a personal error due to some obscure physiological cause that allows a small interval of time between sensation and perception, or volition. If this were the same for the sense of hearing as for vision, it could not affect the results of experiments on sound; but it is quite possible that it may be different, and the very fact that such personal errors do exist, may justify a suspicion that such an effect might interfere and prevent a perfect degree of accuracy from being obtained.

We have made no hypothesis of the nature of the impinging surfaces of the molecules. May the discrepancy arise from something omitted in this?

In other media, according to this theory, the velocity varies as the inverse square root of their specific gravity, and at different temperatures as the square root of the absolute temperatures. It depends wholly on the thermometer, and is quite independent of the indications of the barometer. This accords with the generally received theory. The molecular velocity of watery vapour being to that of air as $\sqrt{14{\cdot}4}$ to $\sqrt{9}$, the moisture in the atmosphere ought to have the effect of accelerating the velocity about 4 feet per second in temperate latitudes and 10 feet in the tropics at a maximum.

Notes.

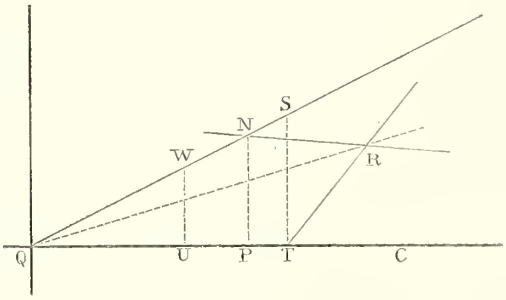
Note A.—Motion Indestructible as Matter.

The force of the descending weight is apparently *expended*, but it is only *transferred* to the medium. If the elements of matter are perfectly elastic, this kind of transference must be of general occurrence wherever force is exerted, because the exertion of force is then but its transference either from its invisible constant condition, as in the medium, to its visible transitory condition, as in the ascent of the weight, or *vice versâ*. In the former case force exists in the matter of the medium without change ; in the latter it is being transferred to the agent of gravitation, so to speak, and apparently released and disconnected from matter. The force of a medium, when it equilibrates a force of gravity, is similar to the force of a wind or a current of water on a stationary surface. Is it not possible to view all forces as inseparable from some form of matter, and all the phenomena of nature, as not consisting of the creation and annihilation of force, but in its transference from one form of matter to the other ?

Note B.—Vapours.

This enables us conveniently to represent the relation between the density and the square root of the mean square molecular velocity of a medium while it is being dilated or compressed.

Fig. 3.



Take Q as the origin of co-ordinates, and let QP represent v and PN the $\sqrt{\frac{1}{\Delta}}$. Join NQ. If the medium is compressed so that $\sqrt{\frac{1}{\Delta}}$ becomes TS, then shall v become QT, and if it dilates so that $\sqrt{\frac{1}{\Delta}}$ becomes UW, then shall v become QU.

Each point in the positive quarter of the co-ordinate axes represents a medium of a given density and temperature or *vis viva;* the sixth power of the ordinate x represents its density, and the square of its abscissa y its *vis viva.* Thus, $x^6y^2 = c$ is the expression for the tension or elastic force of a medium whose point on this chart (as it may be called) is defined in position by the co-ordinates x, y.

If c is supposed constant, x, y to vary, their *locus* traces out a hyperbolic curve (such as NR) whose equation is $x^6y^2 = c$. It is a curve of constant pressure, of which kind is STPC for one atmosphere drawn on the accompanying large chart of vapours (Plate 1). The sine of the inclination of its tangent to the axis of y is $\frac{1}{3}\,x/y$. Any gas expanding or contracting under a constant pressure traces out a curve of this kind with its varying density and *vis viva.*

It is remarkable that if the points corresponding to the density and *vis viva* of a vapour in contact with its generating liquid are laid down on this chart (fig. 3), they range themselves in a straight line, such as TR, that issues from some point advanced on the axis QC. As this fact applies to all vapours that have been experimented upon, it seems to point to the true physical law of their equilibrium with the liquid. On the accompanying chart of vapours I have projected the points of several sets of experiments. It may be viewed as a portion of the fig. 3 enlarged, the point Q, or origin of co-ordinates, being about 40 inches to the left of the outer margin.

The following details will be sufficient, with the chart, to enable any one to satisfy himself of the truth, and, if he pleases, to construct the formula of any new vapour by means of *two* simple experiments on its tension.

In vapours, as well as gases, the pressure or tension being equal to the product of the absolute temperature or $(t + 461)$ by the density, to find the latter we have only to divide the tabular tension (in inches of mercury) opposite t° Fahr. scale by the former. The sixth root of the quotient is the value of x, and the square root of $(t + 461)$ is the corresponding value of y. In the accompanying chart I have projected several sets of tables of pressures in this way. The unit value of x is $\frac{1}{3}\,10\sqrt{10}$ inches long, and the unit value of y, or square root of absolute temperature, is $\frac{10}{6}$ths of an inch in length.

It will be remarked how nearly the experiments of SOUTHERN and the French Academy on steam range themselves in one line. To observe this more distinctly I have drawn the straight line SF through SOUTHERN's pressure at 212°, and the French Academy's observation at 429·4. The divergence at the four lowermost experiments of SOUTHERN is more apparent than real, the greatest difference being equivalent to only $\frac{3}{100}$ths of an inch of mercury.

The general equation for a straight line TR (fig. 3) is $x = (y - G) \tan H$, in which $G = QT$ and $H = RTC$. Each vapour being represented by such a line with two constants G and H, to find these constants, which may be done by two experiments on any one vapour, let e_0 be the tension at t_0 temperature Fahr., and e_1 the tension at t_1 temperature; then since $y_0^2 = 461 + t_0$ and $y_0^2x_0^6 = e_0$, we have

$x_0 = \left(\frac{e_0}{y_0{}^2}\right)^{\frac{1}{6}} = \overline{y_0 - \mathrm{G}} \tan \mathrm{H}$, or $\tan \mathrm{H} = \sqrt[6]{\frac{e_0}{461 + t_0}} \times \frac{1}{\sqrt{461 + t_0} - \mathrm{G}}$; and by 2nd experiment $\tan \mathrm{H} = \sqrt[6]{\frac{e_1}{461 + t_1}} \times \frac{1}{\sqrt{461 + t_1} - \mathrm{G}}$. From these we eliminate*

$$\mathrm{G} = \frac{\sqrt{461 + t^0}\sqrt[6]{\frac{461 + t_0}{461 + t_1} \times \frac{e_1}{e_0}} - \sqrt{461 + t_1}}{\sqrt{\frac{461 + t_0}{461 + t_1} \times \frac{e_1}{e_0}} - 1}. \dagger$$

From the two observations corresponding to the points S and F, I have computed the value of G = 19·4923 and tan H = 0·092308, which define the equation for the steam line that best answers to the experiments of SOUTHERN and the French Academy combined. This equation is $\tan^6 \mathrm{H} \left(\sqrt{461 + t} - \mathrm{G}\right)^6 (461 + t) = e$, or by putting cot H as the common denominator we have the following formula for the tension of steam at all temperatures :—

$$\left\{\frac{\sqrt{461 + t} - 19{\cdot}4923}{10{\cdot}883}\right\}^6 (461 + t) = e \text{ (in inches of mercury).}$$

It is singular that the points of projection belonging to each of the many sets of experiments on steam range themselves in a line, but these lines do not coincide, except in the case of SOUTHERN and the French Academy. Each set is thus consistent with the general law, and on this account their want of accordance with each other is difficult to explain. It might be caused by an error of the standard scales, but this is hardly possible; or impurity of the water, which is not very likely where every precaution has been so carefully attended to. Dr. URE's line of observations is more inclined to the axis than SF, the cot H in the formula being 10·3 and G = 19·8. The line of the American Institute's experiments has yet a greater elevation, the cot H being about 9·8.

The other lines of vapour on the chart explain themselves. It will be remarked that the deflections from the straight are all of a zig-zag character, there is no general bending to one side or another, and this seems to show that there is some physical law upon which the equilibrium of vapours with their liquids is arranged that is represented by the above function of the temperature. As it seems to apply to all vapours, it is probably not beyond the grasp of physical research, if the *vis viva* theory is admitted, for it evidently does not depend on the chemical qualities of the body any more than does the law of volumes in gases.

It must be confessed that as yet we have but few materials wherewith to found a process of investigation. The fundamental point is the specific heat of steam.

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* [? determine.—R.]

† If DALTON and GAY-LUSSAC's constant of expansion is preferred, 448 is to be substituted in the place of 461.

If it is more than about one-third that of water, if for equal volumes it is not the same as that of air and the other gases, then is the theory incomplete and altogether defective.

It is stated to be about 0·84, but this requires confirmation. The specific heat of the vapour of alcohol ought to be only $\frac{1}{5}$th that of the liquid. The vapour of ether only $\frac{1}{7}$th that of the liquid. Vapour of etherine $\frac{1}{4}$th. Vapour of sulphuret of carbon $\frac{1}{4}$th. Vapour of oil of turpentine $\frac{1}{10}$th. Vapour of bromine $\frac{1}{4}$th, &c.

These are from MM. DE LA RIVE and MARCET'S experiments.

Is there any direct evidence in favour of or against this view? So far as I can discover, little, if any, on either side. It is quite undetermined as yet by experiment, and is so surrounded by practical difficulties that it will probably long remain so.

But analogy favours it in so many ways as to make it highly probable. Liquid etherine has four times the specific heat that its vapour ought to have, judging from its specific gravity. Now, olefiant gas is isomeric with etherine, and its specific gravity is one-half that of etherine vapour. It can hardly be doubted that the specific heat of liquefied olefiant gas for equal weights is the same as that of etherine, which corresponds very nearly with naphtha, turpentine, and the other hydrocarburets. If it is so it will be double what it ought to be, if for equal volumes it is the same as that of air and the other gases. Now, the eminent French chemists who have at different times made experiments on the specific heat of this gas agree that it is nearly 1·5 times that of air, while Mr. HAYCRAFT ('Edin. Trans.'), on the other hand, with his simple and apparently most efficient apparatus, found it to be the same as that of air, and accounts for the higher number of the French chemists by the great difficulty there is in freeing it from ethereal vapours.

It is the same with carbonic acid gas (the only other exception to the law of equal specific heat for equal volumes). The French chemists agree that its specific heat lies between 1·175 and 1·258; Mr. HAYCRAFT, by many experiments, that if carefully dried, it offers no exception to the general law. Judging from the analogy of other similar binary compounds, there cannot be a doubt that the specific heat of liquefied carbonic acid is double, if not three times, that of the gas.

It would be a most valuable addition to our knowledge if this great change in the specific heat of a body when it becomes vapour could be thoroughly established, and it is in vain to proceed with the subject of vapours until it is so.

Sulphuric ether is probably better adapted for the experiment than any other body. Its boiling temperature is very low, and there is a vast disproportion between the specific heat of the liquid and what may be expected in the vapour. Suppose a volume of it and of air are maintained in equilibrium of pressure and temperature by means of a bent tube with mercury. If a sudden small and equal dilatation is made in both at the same instant, the difference of pressure that will then become apparent will indicate the value of the latent heat of the vapour in terms of its specific heat, and, as we know the value of the same in terms of the specific heat of water, we shall have the ratio of the specific heat of the vapour to that of water. The air follows the line RQ (fig. 3) in its expansion, while the vapour is obliged to keep to its line RT.

The latent heat of vapours is another subject where there is room for much additional research. Dr. Ure, in his 'Dictionary of Chemistry,' has given a table of eight vapours, but none of the liquids appear to have been pure. The ether boiled at 112°, and the specific gravity of the alcohol was 0·825. If a correction is made for this want of purity, it is singular that the latent heat of each is almost exactly in the inverse ratio of the specific weight of its vapour. This may indicate that the heat required to vaporize a molecule of each of these bodies is the same, and amounts to 3000°, referred to the constant specific heat of a gaseous molecule, which is the proper theoretical standard.

Is this the measure of the force of liquid cohesion? On the *vis viva* theory of heat this, for 1 lb. of water, amounts to the force required to raise 1 lb. to the height of about 680,000 feet. According to Mitscherlich, the *vis viva* generated by the union of the constituent elements of the same quantity of water amounts to ten times this force.

Considerable attention has lately been given to thermo-chemistry; but it is to be regretted that no notice has been taken of the permanent change of specific gravity that is usually found to ensue in chemical mixtures that evolve heat. It would be interesting to ascertain if there is any harmonious connection between the quantity of heat evolved and the change of atomic volume. Dr. Ure has given a curious example of hydrated nitric acid, where the permanent change of volume appears to be the same as would be caused by a permanent change of temperature equal to the heat evolved.

Since only two experiments are required to fix a line of vapour on the chart, it would not be a very arduous undertaking to accomplish this for all bodies that throw off vapours at accessible temperatures. We might then have the means of answering the various questions that cannot fail to suggest themselves on looking at the chart; and, first of all, do the vapours of arsenic, iodine, camphor, salts of ammonia, and the other solids that rise into vapour before becoming liquid, follow the general law? This question has yet to be determined.

How are the lines of vapour of the simple bodies related to each other? We have only one example as yet, viz., the vapour of mercury by M. Avogadro. It will be remarked that the line drawn through MR, the third and the second last observation, agrees very well with the position of the other points. This line produced meets the axis at 50°, which is certainly lower than the temperature assigned by Dr. Faraday's delicate experiment as the point of no vapour. In judging of this discrepancy, however, we must recollect that the density is represented not by the ordinate to the line of vapour, but to its sixth power; hence, at the temperature of 75°, the density indicated by the line on the chart is only $\frac{1}{120,000,000}$th of the density of an atmosphere. The formula that corresponds to this line has the constant $G = 22{\cdot}6065$ and $\cot H = 20{\cdot}0023$.

It will be remarked that the line for DALTON'S ether (which, from its low boiling point, must have been nearly pure), THOMPSON'S pyroxilic spirit, and the well determined line for steam, are nearly parallel; is this parallelism perfect? It is also remarkable that the projection of two experiments by DALTON on aqueous ammonia is exactly parallel with the steam line, and further, that the same parallelism is maintained by the vapours of liquefied ammoniacal gas and carbonic acid (by THILORIER). It would be extremely desirable if Dr. FARADAY'S experiments on chlorine and the other more condensible gases could be repeated on a large scale so as to determine their position on the chart, and by two or three observations on each to eliminate the constants G and H. It is by such experiments and those of M. CAGNIARD DE LA TOUR, made at the other extremity of the scale of heat and pressure, and likewise by Mr. PERKINS, all of which may be classed under the head of chemical physics, that we may expect to extort from nature some of her most hidden secrets, to come in sight of new continents in the world of natural science, not dreamt of in our philosophy, because removed beyond the bounds of suggestive analogy. Such pressures appear to us great, and are certainly dangerous to operate with; but in respect to those which exist in nature, and that everywhere surround us, restrained by internal forces, they can only be considered as infinitesimal.

Note C.—Temperature of Compressed Air.

These changes of temperature are certainly much greater than are said to have been observed by DARWIN, DALTON, and others. Not having access to the original account of these experiments, I am unable to ascertain how far they accord with the theory; but the specific heat of air is so small in comparison to that of the materials of which thermometers are composed that the actual difference of temperature in a single condensation or dilatation must be much greater than what is indicated by any thermometric apparatus.

A more effectual way of ascertaining this seems to be by continually and quickly repeating the same condensation with different portions of air, so that after some time, by proper care, the condensing syringe ought to exhibit the temperature of the air at its maximum tension.

If air is a medium we have in XVI. the means of computing the temperature that ought to be shown by a thermometer placed at the bottom of the syringe.

Thus, t_0, e_0 being the temperature and tension of the air outside, e_1 the tension corresponding to the load on the eduction valve of the syringe; then $\frac{e_1}{e_0} = \left(\frac{t_0 + 461}{t_1 + 461}\right)^4$, and $\left(t_0 + 461\right)\sqrt[4]{\frac{e_1}{e_0}} - 461 = t_1$, the temperature of the air when condensed.

The air engines of the atmospheric railway may perhaps afford the means of testing this equation.

Note D.—To find the Compression that Corresponds to a Given Ratio of Latent to Sensible Heat.

This may be computed by means of IV. and XVI.

Suppose the medium is compressed so that the mean molecular distance changes from 1 to $\frac{x}{x+1}$, th *vis viva* increases from 1 to $\frac{x+1}{x}$ (XVI.), the increment being $1/x = \text{L}$.

By withdrawing molecular *vis viva* under a constant pressure, let the mean distance change from 1 to $\frac{x}{x+1}$; the molecular *vis viva* must be diminished from 1 to $\left(\frac{x}{x+1}\right)^3$ (IV.), the decrement being $\frac{(x+1)^3 - x^3}{(x+1)^3} = \text{K}$. The ratio K/L is given to find $\left(\frac{x+1}{x}\right)^3$, which is the amount of compression from unity that makes the ratio between the sensible and latent, or evolved *vis viva*, equal to K/L.

If we put $x/(x+1) = y$, the equation resolves itself into $y^3 + y^2 + y = \text{K/L} = \frac{8}{3}$ in the case given. This equation may easily be solved by inspection of a table of square and cube numbers. $1/y^3$ is the compression from unity to give the ratio K/L of the sensible to the latent heat.

Note E.—Specific Heat of Air.

It is probable that the specific heat of mercury and water are better determined than that of any other bodies. Assuming that the specific heat of liquid mercury (0·033) is the same as that of its vapour, and that all gases and vapours have for the same volume the same specific heat when *in equilibrio* of pressure and temperature, the specific heat of air in terms of that of water is 0·238. The mean experimental value is, according to the French chemists, 0·267. Mercury is thus the only liquid, so far as is known, whose specific heat is the same as what it ought to be in the state of vapour, if it conforms to the general law. Water is nearly three times greater than steam ought to be. Alcohol five times that of its vapour. Ether seven times, &c.; see Note B.

This accordance of the specific heat of mercury with its vapour seems to prove that there is little or no part of the heat required to raise the temperature of the liquid absorbed in a latent form. This is an important point in the *vis viva* theory.

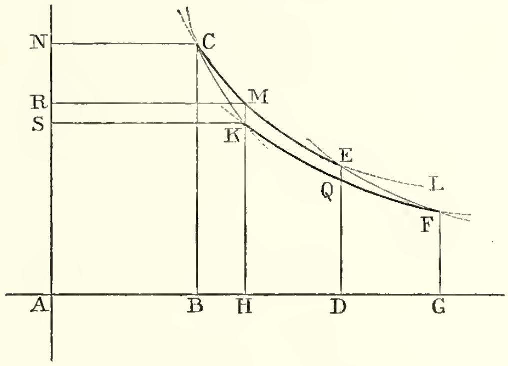
It is remarkable that there is no instance, so far as yet known, of mercury combining in fractional parts of a volume. Arsenic, sulphur, phosphorus, and some others combine in fractional volumes, and their specific heat in the form of vapour by the theory of gases is the same fractional part of the specific heat of the solid. This is somewhat confirmatory of the theoretical interpretation of the law of volumes (§ 15). The vaporous molecule of mercury is indivisible. The vaporous molecule of arsenic is divisible into four parts: of sulphur into six parts; of phosphorus into four parts; of hydrogen into two parts; of oxygen into two parts; of water ($O_{\frac{1}{2}}2H_{\frac{1}{2}}$) into three parts; of alcohol ($2(HC_{\frac{1}{2}}) + 2H_{\frac{1}{2}}O_{\frac{1}{2}}$) into five parts; of sulphuric ether ($4(HC_{\frac{1}{2}}) + 2H_{\frac{1}{2}}O_{\frac{1}{2}}$) into seven parts; sulphuric acid, anhydrous ($3O_{\frac{1}{2}} + S_{\frac{1}{6}}$) into four parts; etherine $4(HC_{\frac{1}{2}})$ into four parts, &c. We may thus predict the specific heat of pyroxilic spirit to be 0·83, and of pyroxilic ether 0·72.

Note F.—M. Clapeyron's *Fundamental Position.*

The density and tension of a medium expanding according to Marriotte's law, are represented linearly by the co-ordinates to the common hyperbola CMEL referred to its asymptote AG; the abscissa AB, &c., representing the volume, and the ordinates BC, &c., the tension.

We have shown that each incremental expansion is made at the expense of the molecular *vis viva* of the medium, so that to maintain the expansion, according to Marriotte's law, the loss of *vis viva* must be continually made up; and the amount required to be supplied for any given expansion, as from B to D, is to the constant original amount as the area CEDB is to 3 CB.AB, or to the original tension CB acting through three times the original volume AB (XIII.).

Fig. 4.



Suppose the medium to expand from B to H (against pressure) without having its loss of *vis viva* supplied; then, according to § 22, its tension from CB, or unity, becomes $KH = \left(\frac{AB}{AH}\right)^{\frac{4}{3}} CB$. Assuming the original volume and tension as unity and $BH = y$, we have $KH = x = \left(\frac{1}{1+y}\right)^{\frac{4}{3}}$ and $x\,dy = dy\left(\frac{1}{1+y}\right)^{\frac{4}{3}}$ = the differential of the asymptotal area CKHB.

To integrate this put $\frac{1}{\Delta^3} = 1 + y$, and differentiating gives $dy = -\frac{3d\Delta}{\Delta^4}$; and since

$x = \left(\frac{1}{1+y}\right)^{\frac{4}{3}} = \Delta^4$, we have $\int x\, dy = 3 - 3\Delta = 3 - \frac{3}{(1+y)^{\frac{1}{3}}}$ = area CKHB in terms of ABCN unity. In this, y being made infinite, we have the area of the whole asymptotal space = 3ABCN, which accords with § 19, as the asymptotal area represents the collective force of expansion from the original volume to infinity.

If, during the expansion from B to H, *vis viva* were supplied to the medium so as to maintain the original quantity unimpaired, the point K in the hyperbola $x = \left(\frac{1}{1+y}\right)^{\frac{4}{3}}$ would coincide with M, the point in the common hyperbola $x = \frac{1}{1+y}$, or MH $= \frac{\text{AB}}{\text{AH}}.\text{CB}$. Now, it is evident that MH : KH : : $\frac{1}{1+y} : \left(\frac{1}{1+y}\right)^{\frac{4}{3}}$: : $3 : \frac{3}{(1+y)^{\frac{1}{3}}}$; but the preceding integral gave $\frac{3}{(1+y)^{\frac{1}{3}}}$ = 3ABCN − CKHB, therefore MH : KH : : 3AB.CB : 3AB.CB − CKHB = ratio of original *vis viva* of the medium to the force remaining after expansion from B to H. Thus, MH : MK : : original *vis viva* : decrement of *vis viva* owing to expansion; and three times the area KNAS is equal to the asymptotal area on the other side of KH. These relations evidently hold good in whatever part of the conic hyperbola the point C may be taken.

Suppose the medium is maintained at its original *vis viva* while it expands from C to E, it will exert the mechanical force CEDB = p, and absorb the *vis viva* CEDB, the original quantity in the medium being 3AB.CB. From E let it expand to F without being supplied with *vis viva;* then, as before, FG $= \left(\frac{\text{AD}}{\text{AG}}\right)^{\frac{4}{3}}$.ED and $\frac{\text{LF}}{\text{FG}}$ is the proportion of original *vis viva* expended represented by EFGD = (m), its equivalent mechanical force exerted. Let it now be compressed from F to K, the *vis viva* communicated to the medium being continually withdrawn. The amount withdrawn and the force exerted is represented by the area FGHK = q. From K let the medium be compressed without withdrawing the *vis viva* generated until the original tension CB and density AB are regained. The force of compression and *vis viva* communicated to the medium in the last operation is represented by the area CKHB = n. For shortness put the area KQDH = s, CEFK = δ, and EQF = ϵ. It is evident since the molecular *vis viva* through CEL is constant and through KQF constant, that $\frac{\text{LF}}{\text{LG}} = \frac{\text{MK}}{\text{MH}}$ and $\frac{\text{MH}}{\text{LG}} = \frac{\text{MK}}{\text{LF}}$. Also, $\frac{\text{MH}}{\text{LG}} = \frac{\text{AG}}{\text{AH}} = \frac{\text{MK}}{\text{LF}}$, or AG.LF = AH.MK = SK.MK = $n = m$. But $q + e = s + m = s + n$, and $s + n + \delta - \epsilon = p$. In this equation substitute for $s + n$ its equal, $q + e$, and we have $q + \epsilon + \delta - \epsilon = q + \delta = p$. Thus, the curvilinear area δ, or CEFK, is the excess of the force exerted by the medium expanding from C to E at the higher constant temperature, over the force exerted upon the same, compressing it from F to K at the lower constant temperature. It is also the excess of the *vis viva* absorbed in the first part of the process over the *vis viva* given out in the last part.

This is the interpretation of M. CLAPEYRON'S fundamental position* applied to the hypothetical medium. M. CLAPEYRON assumes that the quantity of heat taken from the body A (p. 349, 'TAYLOR'S Memoirs') during the expansion from C to E is necessarily equal to the quantity given to the body B during compression from F to K, and that from E to F and K to C *no* heat is taken from or given to the volume of gas, a proportion only of the sensible heat being reduced to a latent form and again thrown out. Thus, there appears the anomaly of the mechanical force δ being exerted or generated without any loss of heat. M. CLAPEYRON'S view is here so much at variance with the *vis viva* theory of heat that it seemed proper to enter into a full explanation of the same condition in the hypothetical medium. The case is very instructive, and throws light on the *vis viva* theory, which is at the same time the means of clearing up the anomaly in causation of mechanical force seeming to be generated without expenditure of heat. M. CLAPEYRON'S conclusions, so far as gases are concerned, are quite independent of any hypothesis, and seem to be strictly deduced from the laws of MARIOTTE and DALTON and GAY-LUSSAC combined with the relation that has been found to subsist between simultaneous increments of sensible and of latent heat. It is satisfactory to observe that they agree, so far as they go, with the physical properties of the medium.

Note G.—Objection to Theory.

The only difficulty I can discover in the *vis viva* theory of heat applies in some measure to the undulatory theory of light. The ethereal medium that transmits the undulations is affected by vibrations of the elements of matter, but there is not the least symptom of it affecting by its resistance the planetary motions, and yet theory shows that it must permeate through the very substance and heart of all bodies with such quick and subtle power as not in any sensible manner to be affected in its equilibrium by any part of their motion except what is vibratory. Now, the *vis viva* theory of heat shows that the greatest ordinary velocities of this vibratory motion do not much exceed 2000 feet per second, but the velocity of the earth in its orbit is upwards of 50 times this amount. In the first case, if a hot body were isolated from all surrounding matter, the whole of its motion would be withdrawn from it in a very short time by the ethereal medium. This we can affirm inductively from the laws of the radiation of heat. In the second case not the slightest resistance is made manifest. The ether only affects and is affected by vibratory motion. Any other kind, however great, it neither affects nor is affected by. What are we to infer from this incongruity ? If vibratory motion differed in no other point from the rectilinear and rotatory motion of masses of matter, than in the sudden change of direction, it is plain that if there was no resistance in the one case there could be no resistance in the other, and *vice versâ*. We are, therefore, compelled to infer that the disturbance

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* Essay on the motive power of heat, 'Journ. Polyt.,' translated in 'TAYLOR'S Memoirs,' vol. 1, p. 349.

of the ethereal medium is not caused directly by the motion of material particles as with the grosser media of gases and liquids when affected by the motion of bodies passing through them. Something takes place in the vibratory motion of the elements of matter that does not take place in their collective motion.

Molecular vibration consists in the struggle, as it were, between the *vis insita* of molecules and the forces that bind them together. It seems, therefore, as if the disturbance of the medium which answers to radiation of heat and light were derived from the disturbance of the molecular forces, and that when there was no such disturbance there was no resistance.

But the difficulty yet remains of a body moving in a material medium without resistance. It is almost inconceivable in the present state of our knowledge of molecular physics. If it is at all possible (for there is almost an absurdity involved in asserting that it is possible) it can only be by means of a persistent or continuous relation of intense mutual organized activity between the ether and the perfectly elastic elements of matter, of which we can as yet have very little conception.

The disturbance of molecular forces that allows the medium to absorb the *vis insita* of vibrating molecules may be likened to the ascent and descent of planetary bodies to and from the sun while revolving in elliptic orbits. There is a disturbance when the centripetal force on a body increases or diminishes, so that if a body revolved in a circle there is no disturbance, no resistance, no absorption of its *vis insita* by the active medium, and if it revolves in an eccentric ellipse there is disturbance, resistance, and absorption. The number of luminous vibrations in a second is, by the undulatory theory, from 458 million million to 727 million million, and the vibrations that cause heat are probably not very different in the velocity of their succession. If a set of molecules thus vibrating in a perfectly cold region took one second to dissipate their molecular *vis viva* by radiation, this is sufficient time for so vast a multitude of revolutions in the molecular orbits that the loss of *vis insita* in our vibration is probably as infinitesimal as that of the planets during one revolution, and we might conjecture that this infinitesimal ratio was a function of the ratio of the elasticity of the ether to the molecular velocity of the vibrations, the elasticity of the ether being apparent only as an active centripetal force. Such a retardation would certainly follow if the velocity of the action of gravitation were not infinite, but there still remains to be suggested the physical condition of a medium that offers no resistance. Is such an entity possible?

Note H.—Formula for Measuring Heights by the Thermometer.

The tension of the atmospheric medium varies, as we have seen, in the proportion of the sixth power of the depth below the summit, and the elasticity of steam varies as the sixth power of the ordinate to the line SF on the chart multiplied by the absolute temperature or square of the corresponding abscissa (see Note B). This suggests a method of measuring heights by the thermometer that requires little or no computation.

Let ST on the chart of vapours, which there represents the sixth root of the density of steam at 212°, be also taken to represent the whole height of the atmosphere above a point where the water boils at the temperature of 212°. Then from ϖ, the ordinate to the steam line at 200°, draw ϖq parallel to QS, the line of one atmosphere pressure, or rather, it ought to be converging to the point where the cord QS produced meets the axis. It is evident that SQ represents very nearly the height where water boils at 200°. Now, if such lines are drawn at each degree between 200° and 212°, they will divide Sq into parts that are sensibly equal.

The value of each of these parts depends on the value given to ST, which, according to our theory, is in feet 318 times the absolute temperature of the air at the station where water boils at the temperature corresponding to the square of the abscissa of the ordinate ST.

The following is the accurate formula by the theory for any vapour of which the constant G (see Note B) is known :—

$$317{\cdot}6\left\{1-\left(\frac{\sqrt{\tau+461}-\mathrm{G}}{\sqrt{t+461}-\mathrm{G}}\right)\left(\frac{\tau+461}{t+461}\right)^{\frac{1}{6}}\right\}(\mathrm{T}+461)=h,$$

in which T is the temperature of the air at the lower station, $t =$ temperature at which the liquid boils at the lower station, $\tau =$ temperature at which the liquid boils at the upper station (all expressed in degrees on Fahr. scale), G the first constant of the vapour (see Note B), and h the difference of height between the stations in feet. By boiling temperature is meant the temperature at which the tension of the vapour is the same as that of the external atmosphere.

Let us apply the formula to the vapour of water, in which G is 19·4923, and let us take an example where T is 60°, $t = 212°$, and $\tau = 211°$; the value of h is 528·6 feet. It will be found, by taking other values for T and τ, that this elevation for 1° difference in the boiling point increases about $\frac{7}{6}$ths of a foot for each degree that τ diminishes, and increases exactly 1 foot for every degree that T increases, and *vice versâ.*

Professor FORBES, who has discussed this subject fully in a recent paper, finds that his observations indicate a number between 540 and 550 feet. The accordance is thus satisfactory; by the formula, 550 feet per 1° is the mean value for $\mathrm{T} = 75°$, $t = 212°$, $\tau = 202°$, which corresponds with Professor FORBES' mean; and what it wants of uniformity is too small to be discovered in practice.

Suppose that we wished to employ the vapour of sulphuric ether for this purpose. It boils at 96° under a pressure of 30 in. DALTON'S observation on this ether gives a line of vapour which has $\mathrm{G} = 16{\cdot}86$. Compute the above equation with this value of G, and $t = 96°$, $\tau = 95°$, $\mathrm{T} = 60°$. The result is $h = 568$, being $\frac{1}{13}$th greater than in the former case of steam.

The less this ratio, the more exact, of course, is this method of estimating heights, and it depends greatly on the value of the constant H (see Note B), for the less the inclination of the line of vapour is to the axis the smaller is the height that corresponds to a difference of 1° in the boiling point. For mercury this is only 248 feet, but the temperature is inconveniently high.

Pure sulphuric ether appears to be the best adapted, as its boiling temperature is low, and the disadvantage of the great amount of the difference of elevation for 1° might, perhaps, be compensated for, by employing a very delicate alcohol thermometer, with the divisions of the scale large and extending only from 60° to 100°. The ether need not have access to the air; the equilibrium of pressure may be indicated by the rising of a thin metallic capsule air-tight and elastic.

Such an instrument would require little more than the heat of the hand to bring it into action, and perhaps might be made sufficiently delicate to be used as a machine for taking levels and making sections of a country.

Note K.—Central Heat.

It is not difficult to compute the temperature that the atmosphere would have if it were supposed to be continued through a shaft to the centre of the earth. Suppose the force of gravity to diminish with the central radius, which is the case in a homogeneous sphere, the temperature of the air at the centre would be about 29,000°, and the density much greater than that of any known solid.

Heat, if it is motion, cannot be propagated upwards without loss or conducted downwards without gain, any more than can a body when projected upwards retain its original velocity without diminution, or when projected downwards without increase.

The temperature in mines (according to M. CORDIER) increases in descending at the rate of 1° to 50 or 60 feet. Is this the natural condition of vertical equilibrium of molecular *vis viva* within the crust of the globe? If such were the case some regular difference might be found between the top and bottom temperatures in pillars or lofty buildings if precautions could be taken to prevent horizontal conduction.

Note L.—Nebular Hypothesis.

The *vis viva* theory appears to harmonize well with the Nebular Hypothesis of LAPLACE. The intense activity of the molecules of the Sun's mass may be viewed as the result of, or to have been originally produced by, its centripetal force while condensing. The motion generated is not lost, as it is in appearance when inelastic bodies meet each other with equal momenta. The clashing together of the descending elastic matter is followed by equal recoil in the opposite direction, and molecular *vis viva* is generated. We see this take place on a minute scale when metals are hammered, or compressed, or rubbed. Friction and every other expenditure of mechanical force gives birth to heat or molecular *vis viva*, which is dissipated by radiation and conduction.

The observations of Sir JOHN HERSCHEL and Professor FORBES with the actinometer ('Phil. Trans.,' 1842) have recently supplied a knowledge of the absolute value of the solar radiation before it passes through the atmosphere. They have found it to amount to 388·4 actines, each actine being one-millionth of a metre in thickness of ice melted per minute. This is equal to 1·835 ft. of ice melted in a mean solar day.

Now, since it is known that ice requires 140° of heat to melt it, and the mechanical value of 1° in water is equal to the weight of the water raised through 673 feet against the force of gravity on the Earth's surface (§ 25), we have the means of computing exactly the absolute mechanical power of the solar radiation—the absolute force thrown out by the Sun in a given time.

There are various ways of reckoning this and obtaining a clear conception of it with reference to different standards. The fundamental principles are contained in Section 3, and an example of their application is given in detail in § 25. The results of four computations are as follows:—

1. During one year the solar force upon a square foot at the Earth's mean distance from the Sun is equal to 20 tons raised 20 miles, or to about one ton raised one mile per day, which is equivalent to $\frac{1}{4}$th of a horse-power, according to engineers' mode of reckoning.

2. At the Sun its amount in one year is equivalent to the descent of a stratum of the Sun's surface (and of its mean density) $3\frac{1}{3}$ miles thick through its own breadth.

3. If the Sun is supposed to contract uniformly throughout its mass so that its radius becomes $3\frac{1}{3}$ miles less in consequence of the general increase of density, the force generated is sufficient to supply the solar radiation for about 9000 years.

4. If a mass equal to the Earth descended to the Sun's surface from its mean distance, it would acquire a velocity of 390 miles per second, and the *vis viva* generated when it strikes the Sun would amount to the force thrown out by the Sun in 45 years.

The density of the Sun being little more than that of water, it is possible that the mere gradual contraction of its bulk, or natural subsidence of the mass, may generate sufficient force to supply the amount of radiation without any diminution of temperature, and it would appear from the third computation that the decrement of the apparent diameter of the Sun owing to such condensation may not amount to more than $\frac{1}{50}$th of a second in 9000 years.

Note M.—Barometric Formula.

This may include the effect of aqueous vapour by the formula in Note B. Let τ = mean dew point, t = mean temperature of the air at the two stations. Find

$$P = \left\{ \frac{\sqrt{\tau + 461} - 19{\cdot}736}{10{\cdot}422} \right\}^6 (t + 461),$$

then

$$H = (\log B - \log b) \, . \, \frac{1250801}{1 - 0{\cdot}376 \left(\frac{2P}{B + b} \right)} \cdot \left(\frac{461 + t}{9958 + t} \right)$$

is the correct theoretical formula, in which

H = height in feet between the two stations.

B = height of barometer at lower station.

b = height of barometer at higher station brought to the same temperature as B.

Addition to Notes B. *and* H. *of the Paper, "On the Physics of Certain Media," recently submitted to the Royal Society.*

Received January 27, 1846.

In Note B it was shown that the formula of a vapour might be obtained from two experiments on its tension, and in Note H, that the function which defines the law of density in vapours is analogous to what defines the law of tension in ascending the atmosphere, thereby enabling us to construct a rule for measuring heights by the thermometer. It may, perhaps, be useful to add what relates to the law of the tension of mixtures of air and vapour.

In some cases it seems impossible to clear vapours entirely of permanently elastic matter, and it will be allowed to be very desirable, in a practical point of view, that we should be able to deduce the necessary constants from experiments made upon them in their usual state of commixture. It will be found, I believe, that this may be accomplished by means of the data afforded by not less than three experiments if the volume occupied by the gas and vapour remains constant, or if the proportionate changes in it are capable of being accurately determined. We do not require to know anything of the quantity of air enclosed with the vapour: this forms one of the three unknown quantities involved in the three equations afforded by the experiments; the other two being the constants G and H that develop the law of density of the pure vapour.

In the accompanying chart (Plate 2), which is drawn on the same scale as the general chart of vapours given in Note B, it may be remarked how the straight lines of vapour are transformed into a high order of hyperbolas when any permanently elastic matter is allowed to contribute its effect of tension. The mode of laying off the points is simply as follows. Suppose we wish to know the effect that air of $\frac{6}{100}$ths of an inch of tension at 51° has upon the chart line of aqueous vapour; we have $F^6 t = F^6 (51 + 461) = 0{\cdot}06$. From this we obtain the value of F^6, which we employ in the general equation for such mixtures, viz., $F^6 t + t \left(\frac{\sqrt{t} - G}{H} \right)^6 = e$, or $\sqrt[6]{F^6 + \left(\frac{\sqrt{t} - G}{H} \right)^6} = \sqrt[6]{\frac{e}{t}}$ = the ordinate on the chart to the abscissa $\sqrt{t}$. The curve traced out by these coordinates is one leg of a species of hyperbola. The apex of this hyperbola has its ordinate F′F′ (see chart) equal to the element F in the equation, and the corresponding abscissa is G, which is equal to $\sqrt{t}$ at this point which is the zero of the vapour. It makes $\frac{\sqrt{t} - G}{H} = 0$, and $\sqrt[6]{\frac{c}{t}} = F$.

As the tension of a vapour is excessively small for a considerable range of temperature above its zero point, the curve, which begins at F′, does not sensibly leave the tangent at its vertex, F′E,—which is also parallel to the axis at the distance from it, F,—for some distance beyond the point of contact. It then takes a sudden bend, having the greatest curvature at the point where the tension of the vapour is nearly one half the tension of the air, and ascends along the line of vapour converging towards it as an asymptote. This curve answers very well to the general run of the experiments on aqueous vapour at low temperatures, and those of Professor MAGNUS that have recently appeared in the 14th number of the 'Scientific Memoirs' correspond with it almost exactly.

Are we then to infer from this coincidence that the general divergence from the straight at low temperatures is the effect of a minute portion of air that clings to the water, in spite of all the precautions taken to prevent it, and that it only becomes sensible when the tension of the vapour, *per se*, has descended to the same attenuated proportion; or is the law that is represented by the general equation of Note B, defective to this trifling extent?

Although no attempt has yet been successful to give a physical interpretation of the function of the temperature that represents the density of a vapour, yet it must be considered as a circumstance favourable to the possibility of doing so on the *vis viva* theory, that it corresponds so far with several of the laws of gases or media as like them to involve the sixth power of an element of the temperature. Thus in XVI. (§ 22) it was shown that when a medium was compressed the *vis viva* increased as the mean molecular distance diminished, or, what is the same, that the sixth power of the molecular velocity increased in the same ratio as the density. This actually enables the condition of a gas in respect to density and temperature, while dilating or being compressed, to be represented on the chart of vapours, and has already been referred to in Note B. The physical demonstration of this peculiarity of function depends ultimately (as shown in Section III.) on the six rectangular directions of space. It seems highly probable, therefore, that the same primary cause shapes the function in the case of vapours, and we may thus be led to hope that in the liquid condition of bodies their molecules are arranged upon a plan more simple and less interwoven with the essential nature of the molecular forces than might otherwise have been anticipated.

In the upper curve, FCS, the ordinates represent the sixth root of the respective densities when the air and vapour are in equal proportion at the ordinary pressure of the atmosphere. The vapour is that of the sulphuret of carbon employed by M. MARX* ('Scient. Mem.,' Part 14), which has 117°·12 as its boiling temperature. With half its volume filled with air it assumes the ordinary pressure of the atmosphere (30 inches) at 80°·55. At this point it crosses the line of the constant pressure of one atmosphere (see Note B) as may be seen on the accompanying chart. It may also deserve to be remarked that the curve crosses this line in a much more sloping direction than the straight line of the pure vapour, and that, consequently, according to what is stated in Note H, such a mixture is so much better adapted to the measurement of heights by the thermometer, inasmuch as 1° represents a much less difference of height in the atmosphere when applied to mixtures of air and a vapour, than with the same vapour in its pure state. We may thus, perhaps, with such mixtures, be enabled to construct an instrument for measuring heights by means of the thermometer which will have all the advantages that are anticipated in Note H from employing the pure vapour of a volatile liquid with a chart line of density having the smallest possible inclination to the axis.

As an example of this let us take the mixture represented by the upper line of density, FCS.

At the temperature 80°·55 and the mercury in the barometer standing at 30 inches, a small quantity of air saturated with sulphuret of carbon is enclosed and at the lower temperature, 70°, the tension of the mixture of air and vapour reduced to 26·76; it is required from these data to determine the height corresponding to a lowering of 1 degree in that temperature which brings the tension to an equilibrium with the external atmosphere.

We have first to compute G by the formula given in Note B as follows:—

$$G = \frac{\sqrt{t_0}\sqrt[6]{\frac{t_0 e_1}{t_1 e_0}} - \sqrt{t_1}}{\sqrt[6]{\frac{t_0 e_1}{t_1 e_0}} - 1};$$

in which $t_0 = 461 + 70$, $t_1 = 461 + 80·55$, $e_0 = 26·76$, $e_1 = 30·00$.

By this we obtain G = 8·742. It is making use of the small arc, CS, as if it were a straight line.

We have next to employ the value of G in the formula of Note H, viz.:—

$$317·6 \left\{1 - \left(\frac{\sqrt{\tau} - G}{\sqrt{t} - G}\right)\left(\frac{\tau}{t}\right)^{\frac{1}{6}}\right\} \cdot T = h.$$

In this the absolute temperature at the lower of two stations is denoted by T, and t is the temperature at which the tension of the enclosed air and vapour equilibrates the atmospheric pressure at the lower station, the same at the upper station being τ.

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* [? Magnus.—R.]

Suppose

$$T = (461 + 60), \quad t = (461 + 80{\cdot}55), \quad \text{and} \quad \tau = (461 + 70)\,;$$

the formula computed gives $h = 3127$, which, divided by $t - \tau = 10{\cdot}55$, gives $296\frac{1}{2}$ feet as the value of 1° in such an instrument. This amount varies but little, through a considerable range of temperature and pressure.

This value may be obtained by observing the temperature at the bottom and at the summit of a known height, and dividing the elevation in feet between the two stations by the difference. Neither the law of the vapour nor the amount of air enclosed with it is required.

What if we dismiss the vapour altogether and enclose dry air only? It is evident that the line CS will then become parallel to the axis and distant from it F, the sixth root of F^6, the density, which is constant. The element G becomes infinitely negative, and $\frac{\sqrt{\tau} - G}{\sqrt{t} - G} = 1$, thus simplifying considerably the expression for h, which is now converted into $h = 317{\cdot}6 \left\{1 - \left(\frac{\tau}{t}\right)^{\frac{1}{6}}\right\} T$.

Let $t = b^6$, and $\tau = (b - \beta)^6 = b^6 - 6b^5\beta$, when β/b is a small fraction. By division we have on this hypothesis $\tau/t = 1 - 6\,\beta/b$, and $(\tau/t)^{\frac{1}{6}} = 1 - \beta/b$, which converts the equation for h into $h = 317{\cdot}6\,\frac{\beta}{b}\,T$. To express $\frac{\beta}{b}$ in terms of t and τ, we have $\overline{t - \tau} = 6b^5\beta$, and $\frac{t - \tau}{6t} = \frac{\beta}{b}\cdot$ Hence, so long as this fraction is small in comparison to unity, we have the following simple expression for the height in terms of the indications of the thermometer:—

$$h = 317{\cdot}6 \left(\frac{T}{t} \cdot \frac{t - \tau}{6}\right)\cdot$$

This gives the nearly constant value, 53 feet, for each degree of Fahr. thermometer, at moderate elevations and ordinary temperatures.

This is the lowest possible value for difference of temperature that can be obtained. In ascending through an increment of the height of the atmosphere, we experience one decrement of temperature, and five decrements of density, which, together, make six decrements of tension. These six decrements of tension must be effected in the enclosed air of the instrument before an equilibrium is established, and as the density is a constant quantity they must be produced by means of a lowering of temperature to the amount of six decrements. Thus, six decrements of temperature in the instrument correspond to the same differential height as one decrement of temperature in the atmosphere, or six degrees correspond to 317·6 feet, the difference of height that causes a difference of 1° while in its natural condition of vertical equilibrium.

It appears, therefore, that dry air is in every respect the best in theory for measuring heights with the thermometer by means of such an instrument as is referred to in Note H. The theory upon which its theory rests has been shown to agree with

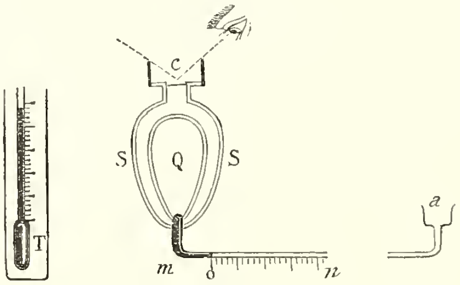
M. Gay-Lussac's observations during his balloon ascent, § 34, and also with Professor Forbes's original determination of the nearly constant difference of level in the atmosphere that corresponds with the same constant difference in the boiling point of water.

Suppose, then, we take a small glass vessel, in which there is fixed a delicate thermometric apparatus with a large scale. We heat it to about 100°, and the inside being perfectly dry we close it and make it perfectly air-tight by means of a thin, finely polished silver capsule. From 100° down to 60° corresponds to an altitude of only about 2100 feet, so that for greater elevations or a greater range of the barometer we would require to seal it at a higher temperature, or what amounts to the same thing, partially exhaust the air while fixing the capsule. In afterwards employing this instrument the polished capsule will be a concave mirror so long as the pressure of the atmosphere exceeds the tension of the enclosed air. It will become a plane reflector when they are in equilibrium, and convex when the tension exceeds the atmospheric pressure.

Now, the image of an object is so different in these three kinds of reflectors, that I conceive it will be possible to recognise the point of equilibrium with very considerable accuracy, or, perhaps, better by means of an eye-piece adjusted to a certain angle of reflection.

The principal difficulty in such an instrument would probably be in getting the temperature of the air and of the thermometer to be perfectly the same at the instant of equilibrium. M. Brequet's metallic spiral thermometer is, perhaps, the best adapted, and would make the apparatus very portable. But it is the practical artist only who can judge if such an instrument can be made effective.

Fig. 5.



The annexed sketch is another form of the apparatus, to be used with a delicate mercurial thermometer that may show the temperature of the atmosphere at the station. Q is an air thermometer, with a bead of mercury as index, which, before observing with it, must be blown into the bulb by putting the finger upon the open end, *a*.

The air in Q, having now the same temperature and density as the atmosphere outside the bead of mercury, is allowed to fall into the stem of the instrument, and the reading of the scale marked off. Heat is then applied to the shut glass bottle, SS, that encloses Q, until the capsule, C, shows the equilibrium; the air in Q, of course, has the same temperature, and has expanded accordingly against the constant pressure of the external atmosphere; the reading of the mercury in the scale, n, will now show the amount of expansion, and, therefore, the ratio of the constant density of the air enclosed in SS to the density of the atmosphere at the station. Thus, we ascertain the absolute temperature and absolute density of the atmosphere at every station from scales with as large a reading as we please.

To obtain the law of tension of a vapour by three experiments made upon it at different temperatures when mixed with an unknown quantity of air, let t_0, t, t_1 be the three absolute temperatures found by adding 461 to the reading of Fahr. scale, and e_0, e, e_1 the corresponding tensions. Also, let G and H represent the two unknown constants of the vapour, and F^6 the constant that represents the quantity of the enclosed air, or number of gaseous molecules, which is the same at all temperatures, while the number of vaporous molecules $\left(\frac{\sqrt{t}-G}{H}\right)^6$ changes with the temperature t. These expressions mean the number of molecules in a constant volume, so that the experiments require to be made with the enclosed volume over the liquid constant. The general expression for the observed tension is

$$e = t\,F^6 + t\left(\frac{\sqrt{t}-G}{H}\right)^6 \quad \ldots\ldots\ldots\ldots \quad (1).$$

By eliminating F^6 from each of the three experiments, we have

$$F^6 = \frac{e_0}{t_0} - \left(\frac{\sqrt{t_0}-G}{H}\right)^6 = \frac{e}{t} - \left(\frac{\sqrt{t}-G}{H}\right)^6 = \frac{e_1}{t_1} - \left(\frac{\sqrt{t_1}-G}{H}\right)^6 \quad \ldots \quad (2).$$

From the first and second of these, we have

$$H^6\left(\frac{e_1}{t_1} - \frac{e}{t}\right) = (\sqrt{t_1}-G)^6 - (\sqrt{t}-G)^6 \quad \ldots\ldots\ldots \quad (3).$$

From the second and third, we have

$$H^6\left(\frac{e}{t} - \frac{e_0}{t_0}\right) = (\sqrt{t}-G)^6 - (\sqrt{t_0}-G)^6 \quad \ldots\ldots\ldots \quad (4).$$

Dividing (3) by (4), we obtain a known ratio R,

$$R = \frac{\frac{e_1}{t_1} - \frac{e}{t}}{\frac{e}{t} - \frac{e_0}{t}} = \frac{(\sqrt{t_1}-G)^6 - (\sqrt{t}-G)^6}{(\sqrt{t}-G)^6 - (\sqrt{t_0}-G)^6} \quad \ldots\ldots\ldots \quad (5).$$

From this we may eliminate G by trial and error. The shortest way of making this computation is, perhaps, the following :—

Assume any value A for G, and substitute it in (5), and compute the corresponding value of R, which let us denote by N. If we make the proportion, as the differential of N is to the differential of A, so is the difference between N and R to δ, the difference between A and G, we have

$$\tfrac{1}{6} \times \frac{\{(\surd t_1 - \mathrm{A})^6 - (\surd t - \mathrm{A})^6\} - \{(\surd t - \mathrm{A})^6 - (\surd t_0 - \mathrm{A})^6\}\,\mathrm{R}}{\{(\surd t - \mathrm{A})^5 - (\surd t_0 - \mathrm{A})^5\}\,\mathrm{N} - \{(\surd t_1 - \mathrm{A})^5 - (\surd t - \mathrm{A})^5\}} = \delta, \text{ and } \mathrm{A} - \delta = \mathrm{A}_1,$$

which approximates nearer to G. Substituting this value in the place of A in the above equation, we obtain the next value of δ, which call δ_1, and $\mathrm{A}_1 - \delta_1 = \mathrm{A}_2$, which approximates still nearer to G.

We arrive more quickly at the exact value of G by making $\mathrm{A} - \frac{\mathrm{N}}{\mathrm{R}}\,\delta = \mathrm{A}_1$, and $\mathrm{A}_1 - \frac{\mathrm{N}_1}{\mathrm{R}}\,\delta_1 = \mathrm{A}_2$. Having thus found A_1, A_2, N, N_1, N_2, we may lay off A as the ordinate to N, A_1 to N_1, and A_2 to N_2, then, drawing a curve through these points, the ordinate to it opposite R is G, which, in this way, may be obtained very exactly.

As an example the following three observations are taken from Professor MAGNUS's experiments on the elastic force of steam, that have recently appeared in the 14th number of the 'Scientific Memoirs.'

$$\begin{array}{llll} e_0 = 0{\cdot}178 & t_0 = 493 & = (461 + 32) \\ e\ = 3{\cdot}793 & t\ = 585{\cdot}1 & = (461 + 124{\cdot}1) \\ e_1 = 29{\cdot}920 & t_1 = 673 & = (461 + 212) \end{array}$$

Computing the preceding formulæ with these data, a few trials give G = 19·625. Then, by (3) or (4), we get H = 10·62, and by (2), from the first experiment at the lowest temperature, we obtain the value of F^6, and thence $\mathrm{F}^6\,(51 + 461) = 0{\cdot}08$, or $\frac{1}{375}$th part of an atmosphere of permanently elastic matter at 51° Fahr.

The line on the chart which answers to the experiments of SOUTHERN and the French Academy has G = 19·492, and H = 10·83.

It is obvious that one of the experiments ought to be taken at as low a temperature as possible, and that F^6 should be computed from its data.

The general formula for vapour, $t\left(\frac{\surd t - \mathrm{G}}{\mathrm{H}}\right)^6 = e$, is easy to compute when the tension for a given temperature is required; but when the temperature that corresponds to a given tension is sought, the equation does not admit of direct solution. The following is, perhaps, the simplest method of overcoming the difficulty. It is founded on the property of the tangent to the curve of constant pressure, alluded to in Note B.

The length of that portion of it intercepted between the point of contact and the axis of the curve, is equal to three times the length of the abscissa.

The tension given being ϵ, and the corresponding absolute temperature τ being required, assume any absolute temperature as near to τ as may be guessed roughly; then, by the formula for the given vapour, $t\left(\frac{\sqrt{t}-\mathrm{G}}{\mathrm{H}}\right)^6 = e$, compute e. The value of τ may then be found directly, with all necessary precision, by the following equation:

$$\tau = \left\{\frac{4\mathrm{H}\sqrt[6]{\epsilon} + 3\mathrm{G}\sqrt[6]{t}}{\frac{\mathrm{H}}{\sqrt{t}}\sqrt[6]{\epsilon} + 3\sqrt[6]{t}}\right\}^2.$$

From this value of τ subtract 461, and we have the temperature required on Fahr. scale that corresponds to the elastic force ϵ.

J. J. WATERSTON.

December 15, 1845.

EXPLANATION OF TABLE OF GASES AND VAPOURS.

Received February 19, 1846.

Having found the following Table useful to refer to while studying the subject of gases and vapours, I have been led to hope that it might be made available, so far as it goes, in shortening the labour of drawing up a complete view of their physical constants. That such a condensed view of their physical character and constitution is a desideratum will probably be generally admitted, and principally with reference to theoretical chemistry does it seem to be of importance to have their molecular characteristics placed before the eye in a clear and concise manner.

The tables of this description that are usually inserted in chemical treatises do not, perhaps, give to the arithmetic of volumes all the clearness that it is susceptible of. This is a consequence of employing the chemical equivalent or lowest combining proportion as the unit, whether or not it happens to correspond with the specific gravity of the gas, and it is generally either half this ratio or, as in the case of sulphur, of phosphorus, and of arsenic, even a smaller fraction of it.

Thus we have $\mathrm{H}+\mathrm{O}$ the symbol for water. In Dr. TURNER'S 'Chemistry' its constitution is thus defined: 1, or one equivalent of hydrogen, + 8, or one equivalent of oxygen, = 9 the equivalent of water; and, by volume, 100 of hydrogen combines with 50 of oxygen to form 100 of steam. If we take 16 as the equivalent of oxygen, which corresponds with its specific gravity, then $\mathrm{HO}_{\frac{1}{2}}$ expresses distinctly the constitution of steam both by weight and volume. Another objectionable point may sometimes be remarked though it has now almost disappeared, the combining ratio by volume is inserted before any determination of the fact has been made: thus

Dr. TURNER defines the peroxide of hydrogen, 100 volumes, to consist of 100 oxygen + 100 hydrogen, whereas, since the vapour has never been weighed, it is just as likely to consist of 50 volumes of each; both gases unite with others in several instances in half volumes, so that, until the experiment has been made, all that can be stated is that 17 by weight of the peroxide consists of 1 hydrogen + 16 oxygen. As soon as we have ascertained the space occupied by the 17 of the peroxide in comparison to the 1 of the hydrogen we can state its composition by volume, but it is surely confusing the subject of volumes altogether to infer this from the combining weights alone.

In this Table the specific gravity of a simple gas in terms of hydrogen unity is taken as the value of its symbol (Col. 3, Part I.), and when this is inserted in parentheses (as from Nos. 10 to 20), it represents the value that has been assigned to the symbolical letter of the element in the compounds that follow, and indicates that it has not as yet been weighed in the simple vapour.

Thus the value of C is 12, of F 19, &c., throughout the Table wherever these letters occur, and at the side these numbers are taken as unity in denoting the several proportions with which they have been found to combine in a single volume of compound gases and vapours. Opposite hydrogen, for example, we see numbers from $\frac{1}{2}$ up to 16, which informs us that from $\frac{1}{2}$ a volume up to 16 volumes of this gas enters into one volume of its compounds, and the same with regard to the others. These indicate in some degree the molecular capabilities of the element. They are ratios that have been taken from vapours that have actually been weighed as well as analysed. A large proportion of them are of recent determination, and the original details of the experiments by DUMAS, MITSCHERLICH, REGNAULT, LAURENT, and BINEAU are to be found for the most part in the 'Annales de Chimie'; the reference to the volume and page of this invaluable work is given in parentheses after the name of the vapour, and the letter (m) is a reference to MITSCHERLICH'S 'Chemistry,' where several specific gravities of vapour are given that are not to be found elsewhere.

In the table of binary compounds a column is occupied with the chemical constitution of a single volume of each in terms of volumes of its elementary components. Thus, nitric acid is represented by $O_{2\frac{1}{2}}N$, which means that one volume of nitric acid vapour is composed of $2\frac{1}{2}$ volumes oxygen united to one volume nitrogen, and where a volume has not been weighed, although its constitution by weight is known, the symbol is within parentheses, thus [ ].

In the ternary and organic compounds the simple constitution of a volume is given in the first place, as with the binary, and in the next column the most probable arrangement of the constituents, when there is any ground for making a hypothesis.

Thus we have oxalic ether, No. 114, evidently composed of one volume oxalic acid and one volume of sulphuric ether condensed into one volume. This also allows us to infer with great probability the specific gravity of oxalic acid vapour. The next is nitrous ether, No. 115, which is quite a similar compound in the liquid form, but it will be remarked that in the act of rising into vapour it is decomposed, and what is one volume in the case of oxalic ether vapour is two volumes in the case of nitrous ether. Several other examples of this kind will be found. I have already referred to such facts as being favourable to the hypothesis of media which attributes a mechanical origin to the law of volumes, and have likewise referred to the remarkable circumstance that compounds which thus disunite in the act of vaporization nevertheless obey the general law of vapours (see Note B). As it seems, from the nature of the function that expresses this law, that a mechanical origin may be found for it also, the investigation of the subject might, perhaps, be made easier if the chart lines of vapour were determined for mixtures of pure alcohol and water in all proportions, and also for mixtures of alcohol and ether.

We should then, perhaps, discover the law of variation of the two constants G and H of the chart line, and this might provide us with a new condition or effect of the primary cause pointing to its origin from a new ground.

The two last columns contain the constants G and H, referred to in Note B.; where there are three places of decimals the numbers are nearly exact, when two only they are to be considered as approximate.

Column No. 1 contains the temperature on Fahrenheit scale at which the vapour in contact with its generating solid or liquid equilibrates a pressure of 30 inches of mercury.

Column No. 2 contains the specific gravity of the body in its usual liquid or solid form.

Column No. 3 contains, as before mentioned, the specific gravity of the body in its gaseous form in terms of hydrogen unity. It expresses the weight of a molecule of the hypothetical medium that answers to the gas in its physical relations.

Column No. 4 contains the inverse of the specific heat of the body in its usual liquid, solid, or gaseous form. The numbers are found by dividing the constant 3·2 by the specific heat. This constant is the product of the specific heat of air by its specific gravity in terms of hydrogen unity, and to the same product of all gases that conform to the law of equal volumes having the same specific heat. It is likewise the product of the specific heat of mercury in its liquid form by the specific gravity of its vapour (Note E). In other elementary bodies this product is a simple multiple of the same constant. In compounds the same product is also in most cases a simple multiple of the same constant. On the *vis viva* theory of heat the numbers in this column probably show the mean weight of the component parts of the gaseous molecule that have an independent motion when the body is in the liquid or solid form. Thus, No. 33, arsenious acid: the specific gravity of the vapour is 200 times that of hydrogen; but its specific heat in the solid form is 8 times what it is in the state of vapour, if in this form it obeyed the law of the specific heat of gases. Hence 25 is the number opposite in this column, which, since it goes 8 times in 200, shows that the molecule consists of about 8 parts, each of which has an independent motion. We see further that O_3As expresses the constitution of one volume of this acid, and referring to As, No. 7, we find that its specific heat in the solid form is nearly 4 times greater than it ought to be in the vapour.

The probable general inference is that such molecules consist of several parts, more or less free to move independently of each other, and that when they escape from the bonds of liquid cohesion and become free projectiles, these parts can no longer assume *vis viva* of their own, but are in subjection to the impressed condition of their common centre of gravity.

The great question in this department is, Do such compound bodies which have so great a specific heat in the solid form have it all at once reduced in so vast a proportion when raised to vapour? This interesting point, as remarked at length in Note B, has yet to be determined by an experiment made on the specific heat of sulphuric ether vapour, as being the most accessible, which, if these views are correct, ought to be only about $\frac{1}{6}$th of the specific heat of the liquid.

Column No. 5 contains the quotient of the gaseous specific gravity by the specific gravity of the liquid or solid, and represents the relative size of the molecules. The subject of atomic volume has recently been the subject of interesting discussion by M. Kopp ('Ann. de Chim.,' vol. 75, 1840, p. 406), and, doubtless, will increase in importance as science advances.

To these physical characteristics of gases it would be well if we could add the temperature of liquefaction, the latent heat of liquefaction, or the measure of the solid polar cohesion of molecules according to the *vis viva* theory, the differential of expansion through a range of temperature, and the latent heat of vapours or measure of liquid molecular cohesion.

December 24, 1845. J. J. Waterston.

This paper being the last in connection with the *vis viva* theory of gases that the writer is likely to have an opportunity of submitting to the Society, he begs, in taking leave of the subject, to express a hope that, although the nature of the fundamental hypothesis is likely to be repulsive to mathematicians, they will not reject it without a fair trial. The principle of the conservation of *vis viva* involves the indestructibility of force, and is a necessary consequence of the quality of perfect elasticity or reaction in the ultimate elements of matter: if this last is a universal property the first must also be of universal effect, and, as it does not admit of any diminution of force in nature, we may question whether, in such intense chemical action as the phenomena of combustion and explosion manifest, the sudden evolution of force is not merely an exhibition of its transference from one form of elastic matter to another.

Are not the properties of aeriform fluids and of the medium of light glaring proofs of the widely spread existence of this quality of perfect elasticity, whatever may be its essential mode of reaction; and do not they even demand of us a ready assent to its all-prevailing influence in the phenomena of Nature?

Although the utmost caution in adopting any hypothesis is the proper accompaniment of a sober spirit of inquiry, it does not appear inconsistent with such a spirit to advocate the trial of these principles as a foundation for mathematical research in the several departments of molecular physics.

It is the matured conviction of the writer that upon such foundation we shall have to build if we are destined ever to become acquainted with the secret mechanism of Nature. Would that his feeble voice could call attention to the subject, could direct upon it some portion of the vast mathematical talent that this country can now, more than at any former period, boast of.

J. J. W.

TABLE OF GASES AND VAPOURS.—I. Elementary Substances.

		Symbol.	Chemical equivalent.	Boiling temperature.	Specific gravity.	Gaseous specific gravity. H = 1.	Ditto inferred from specific heat.	Bulk of a molecule.	Combining proportions in volumes.	Vapour constants.*	
				1.	2.	3.	4.	5.		G.	H.
1	Oxygen	O	8	° ..	..	16	16	..	$\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, ($3\frac{1}{2}$?)		
2	Hydrogen	H	1	..	..	1	1	..	$\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, 3, 4, 5, 6, 8, 9, 16		
3	Nitrogen	N	14·1	..	..	14	14	..	$\frac{1}{2}$, 1		
4	Chlorine	Cl	35·4	6	1·33	36	..	27	$\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$	17·0	7·24
5	Bromine (55/31)	Br	78·4	116	3·00	80	24	27	$\frac{1}{2}$, 1, $1\frac{1}{2}$		
6	Iodine	I	126·3	350	4·95	125	35 39 60	25	$\frac{1}{2}$, 1, $1\frac{1}{2}$		
7	Arsenic (55/32)	As	37·7	356	5·88	152	40	26	$\frac{1}{4}$, 1		
8	Sulphur (50/170; 55/31)	S	16·1	600	1·99	96	16	48	$\frac{1}{6}$, $\frac{2}{6}$		
9	Phosphorus (49/211; 50/170)	P	15·7	500	1·71	64	16 8	38	$\frac{1}{4}$		
10	Mercury (55/32)	Hg	202	650	13·57	101	101	7·3	1	22·606	20·002
11	Fluorine	F	18·7	..	(0·73?)	[19]	..	..	$\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$		
12	Carbon	C	6·1	..	3·52 2·50	[12]	26 12 19	3·4	$\frac{1}{2}$, 1, 2, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, $7\frac{1}{2}$, 8, 10		
13	Boron	B	10·9	..	1·18 1·84	[11]	..	..	$\frac{1}{2}$		
14	Silicon	Si	7·5	..	..	[15]	..	..	1		
15	Selenium	Se	39·6	650	4·31	[40]	40	9·2	1		
16	Tin	Sn	58·9	..	7·29	[58]	58	8	1		
17	Antimony	Sb	64·6	..	6·70	[64]	64	9·5	1		
18	Titanium	Ti	24·3	..	5·30	[24]	..	4·5	1		
19	Tellurium	Te	32·3	..	6·11	[64]	64	10·5	1		
20	Chromium	Cr	28·0	..	5·90	[28]	..	5·5	1		

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* See note B.

II. BINARY Compounds.

		Boiling temperature.	Specific gravity.	Gaseous specific gravity. H = 1.	Ditto inferred from specific heat.	Bulk of a molecule.	Chemical analysis.	Chemical arrangement of molecules (hypothesis).	Vapour constants.	
		1.	2.	3.	4.	5.			G.	H.
21	Water	212°	1·00	9	3·2	9·0	$O_{\frac{1}{2}}H$	. . . .	19·492	10·830
22	Peroxide of hydrogen	..	1·45	[$8\frac{1}{2}$]	..	[6·0]	[$O_{\frac{1}{2}}H_{\frac{1}{2}}$]			
23	Binoxide of nitrogen	..	..	15	..	..	$O_{\frac{1}{2}}N_{\frac{1}{2}}$			
24	Nitrous oxide	..	..	22	..	..	$O_{\frac{1}{2}}N$			
25	Nitrous acid	82	1·45	46	..	32	O_2N			
26	Hyponitrous acid (55/32)	[0 ?]	[1·1 ?]	25·3	..	..	$ON_{\frac{2}{3}}$	$2(O_{\frac{1}{2}}N_{\frac{1}{2}}) + O_2N = 3$ vols.		
27	Nitric acid	..	1·70	54	..	32	$O_{2\frac{1}{2}}N$			
28	Quadrihydrate of nitric acid (68/417)	248	1·42	18	..	63	$O_{\frac{9}{10}}N_{\frac{1}{5}}H_{\frac{4}{5}}$	$O_{2\frac{1}{2}}N + 4(HO_{\frac{1}{2}}) = 5$ vols.		
29	Carbonic oxide	..	..	14	..	..	$O_{\frac{1}{2}}C_{\frac{1}{2}}$			
30	Carbonic acid	[−160 ?]	{·83 ·45}	22	22	..	$OC_{\frac{1}{2}}$	. . . .	10·09	10·62
31	Oxalic acid	[330]	[2·4 ?]	[36]	..	..	[$O_{1\frac{1}{2}}C$]			
32	Oxalic hydrate	..	..	[45]	..	..	[O_2HC]			
33	Arsenious acid (55/35)	380	3·7	200	25	54	O_3As			
34	Arsenic acid	..	[4·3 ?]	[232]	..	[54 ?]	[O_5As]			
35	Sulphurous acid	14	1·42	32	..	..	$OS_{\frac{1}{6}}$	. . . .	17·38	6·97
36	Sulphuric acid (55/32)	115	2·29	40	9·2	17	$O_{1\frac{1}{2}}S_{\frac{1}{6}}$			
37	Sulphuric hydrate	620	1·85	[49]	..	[27]				
38	Ammonia	− 46 ?	0·76	$8\frac{1}{2}$	..	11·3	$H_{1\frac{1}{2}}N_{\frac{1}{2}}$	. . . .	13·40	10·80
39	Arseniuretted hydrogen	..	..	$39\frac{1}{2}$	..	..	$H_{1\frac{1}{2}}As_{\frac{1}{4}}$			
40	Selenious acid (55/36)	..	..	56	..	..	SeO			
41	Sulphuretted hydrogen	− 90 ?	0·90	17	..	$18\frac{1}{2}$	$HS_{\frac{1}{6}}$			

II. Binary Compounds (continued).

		Boiling temperature.	Specific gravity.	Gaseous specific gravity. H = 1.	Ditto inferred from specific heat.	Bulk of a molecule.	Chemical analysis.	Chemical arrangement of molecules (hypothesis).	Vapour constants.	
		1.	2.	3.	4.	5.			G.	H.
42	Bisulphuretted hydrogen	° ..	1·77	[16½]	..	..	$[H_{\frac{1}{2}}S_{\frac{1}{6}}]$			
43	Phosphoretted hydrogen	..	..	17½	..	..	$H_{1\frac{1}{2}}P_{\frac{1}{4}}$			
44	Seleniuretted hydrogen (68/423)	..	..	41	..	..	HSe			
45	Tellurated hydrogen	..	..	65	..	..	HTe			
48	Muriatic acid	– 160 ?	2·30 ?	18½	..	[8]	$H_{\frac{1}{2}}Cl_{\frac{1}{2}}$			
49	Octohydrate of muriatic acid (68/423)	..	1·16	$10\frac{1}{18}$	..	..	$H_{\frac{17}{18}}Cl_{\frac{1}{18}}O_{\frac{4}{9}}$	$Cl_{\frac{1}{2}}H_{\frac{1}{2}} + 8(HO_{\frac{1}{2}}) = 9$ vols.		
50	Hydrobromic acid (55/9)	..	1·76	40½	..	23	$H_{\frac{1}{2}}Br_{\frac{1}{2}}$			
51	Hydroiodic acid	..	..	63	..	..	$H_{\frac{1}{2}}I_{\frac{1}{2}}$			
52	Hydrofluoric acid	60	1·06	[10]	..	[9]	$H_{\frac{1}{2}}F_{\frac{1}{2}}$			
53	Cyanogen	– 8	0·90	26	..	29	NC	. . . .	17·08	6·55
54	Bisulphuret of mercury (55/36)	..	..	80	..	..	$S_{\frac{1}{9}}M_{\frac{2}{3}}$	$S_{\frac{1}{6}}M + S_{\frac{1}{6}} + M = 3$ vols. ?		
55	Sulphuret of carbon	117·2	1·27	38	10	31	$S_{\frac{1}{3}}C_{\frac{1}{2}}$	. . . .	16·254	12·760
56	Protoxide of chlorine (euchlorine) (m)	..	..	35·2	..	..	$O_{\frac{2}{5}}Cl_{\frac{4}{5}}$	$[OCl_{\frac{1}{2}} + Cl] = 2$ vols. ?		
57	Peroxide of chlorine (m)	..	..	34·0	..	..	$OCl_{\frac{1}{2}}$			
58	Perchloric acid	392	1·65	[92]	..	..	$[O_{3\frac{1}{2}}Cl]$			
59	Quadrichloride of nitrogen	160	1·65	[79]	..	..	$[Cl_2N_{\frac{1}{2}}]$			
60	Sesquichloride of arsenic	..	..	92	..	..	$Cl_{1\frac{1}{2}}As_{\frac{1}{4}}$			
61	Sesquibromide of arsenic	..	..	158	..	..	$Br_{1\frac{1}{2}}As_{\frac{1}{4}}$			
62	Periodide of arsenic (55/35)	..	..	225	..	..	$I_{1\frac{1}{2}}As_{\frac{1}{4}}$			
63	Chloride of sulphur	147	1·62	52	..	32	$ClS_{\frac{1}{6}}$			
64	Dichloride of sulphur	280	1·69	68	..	40	$ClS_{\frac{2}{6}}$			

II. Binary Compounds (continued).

		Boiling temperature.	Specific gravity.	Gaseous specific gravity. H = 1.	Ditto inferred from specific heat.	Bulk of a molecule.	Chemical analysis.	Chemical arrangement of molecules (hypothesis).	Vapour constants.	
		1.	2.	3.	4.	5.			G.	H.
65	Sesquichloride of phosphorus (m)	205°	1·45	70	..	48	$Cl_{1\frac{1}{2}}P_{\frac{1}{4}}$			
66	Perchloride of phosphorus (m; 55/5)	180	..	71	..	..	$Cl_{\frac{5}{3}}P_{\frac{1}{6}}$	$Cl_{2\frac{1}{2}}P_{\frac{1}{4}} + Cl + Cl_{1\frac{1}{2}}P_{\frac{1}{4}} = 3$ vols.?		
67	Protochloride of mercury	..	7·14	118	..	16	$Cl_{\frac{1}{2}}Hg$			
68	Bichloride of mercury	..	5·42	136	..	26	$ClHg$			
69	Protobromide of mercury	..	7·31	140	..	19	$Br_{\frac{1}{2}}Hg$			
70	Bibromide of mercury	..	5·92	180	..	30	$BrHg$			
71	Biniodide of mercury	..	..	227	..	..	IHg			
72	Dichloride of carbon	400	..	[144]	..	..	$[Cl_3C_3]$			
73	Protochloride of carbon (71/374)	252	1·62	84	..	52	Cl_2C			
74	Perchloride of carbon	360	2·00	[120]	..	..	$[Cl_3C]$			
75	Perchloride of muriate of methyle (71/385)	173	1·60	78	..	49	$Cl_2C_{\frac{1}{2}}$			
76	Terchloride of boron (55/14)	..	..	59½	..	..	$Cl_{1\frac{1}{2}}B_{\frac{1}{2}}$			
77	Fluoride of boron	..	..	33	..	..	$F_{1\frac{1}{2}}B_{\frac{1}{2}}$			
78	Chloride of silicium	124	..	87	..	..	Cl_2Si			
79	Fluoride of silicium	..	..	62	..	..	$Fl_{2\frac{1}{2}}Si$			
80	Bichloride of tin	..	..	130	..	..	Cl_2Sn			
81	Sesquichloride of antimony (55/36)	..	..	118	..	..	$Cl_{1\frac{1}{2}}Sb$			
82	Bichloride of titanium (55/13)	220	..	96	..	..	Cl_2Ti			

III. TERNARY, &c., Compounds.

		Boiling temperature.	Specific gravity.	Gaseous specific gravity. H = 1.	Ditto inferred from specific heat.	Bulk of a molecule.	Chemical analysis.	Chemical arrangement of molecules (hypothesis).	Vapour constants.	
		1.	2.	3.	4.	5.			G.	H.
83	Hydrocyanic gas	80°	0·70	$13\frac{1}{2}$	..	19	$H_{\frac{1}{2}}N_{\frac{1}{2}}C_{\frac{1}{2}}$	$H_{\frac{1}{2}} + \frac{1}{2}(NC)$		
84	Chloride of cyanogen	12	1·32	31	..	$23\frac{1}{2}$	$Cl_{\frac{1}{2}}N_{\frac{1}{2}}C_{\frac{1}{2}}$	$Cl_{\frac{1}{2}} + \frac{1}{2}(NC)$	16·43	8·42
85	Terchloride of cyanogen (68/424) . .	374	4·0 ?	93	..	23 ?	$Cl_{1\frac{1}{2}}N_{1\frac{1}{2}}C_{1\frac{1}{2}}$	$3(Cl_{\frac{1}{2}} + \frac{1}{2}(NC))$		
86	Bromide of cyanogen	100	2·2 ?	53	..	23 ?	$Br_{\frac{1}{2}}N_{\frac{1}{2}}C_{\frac{1}{2}}$	$Br_{\frac{1}{2}} + \frac{1}{2}(NC)$		
87	Chlorocarbonic oxide gas	..	..	50	..	..	$ClO_{\frac{1}{2}}C_{\frac{1}{2}}$	$Cl + O_{\frac{1}{2}}C_{\frac{1}{2}}$		
88	Oxichloride of chromium	..	..	80	..	..	$ClOCr$	$Cl + OCr$		
89	Chlorosulphuric acid (71/445) . . .	170	1·66	68	..	41	$ClOS_{\frac{1}{6}}$	$Cl + OS_{\frac{1}{6}}$		
90	Anhydrous carbonate of ammonia (68/434)	..	..	13	..	..	$O_{\frac{1}{3}}C_{\frac{1}{6}}N_{\frac{1}{3}}H$	$OC_{\frac{1}{2}} + 2(N_{\frac{1}{2}}H_{1\frac{1}{2}}) = 3$ vols.		
91	Anhydrous hydrosulphate of ammonia .	180	..	$12\frac{3}{4}$	..	..	$S_{\frac{1}{12}}N_{\frac{1}{4}}H_{1\frac{1}{4}}$	$S_{\frac{1}{6}}H + N_{\frac{1}{2}}H_{1\frac{1}{2}} = 2$ vols.		
92	Anhydrous hydrocyanate of ammonia .	..	..	11	..	..	$N_{\frac{1}{2}}C_{\frac{1}{4}}H$	$H_{\frac{1}{2}}N_{\frac{1}{2}}C_{\frac{1}{2}} + N_{\frac{1}{2}}H_{1\frac{1}{2}} = 2$ vols.		
93	Anhydrous hydrotellurate of ammonia .	174	..	$36\frac{3}{4}$	..	..	$N_{\frac{1}{4}}Te_{\frac{1}{2}}H_{1\frac{1}{4}}$	$HTe + N_{\frac{1}{2}}H_{1\frac{1}{2}} = 2$ vols.		
94	Anhydrous muriate of ammonia (sal. am.)	..	..	$13\frac{1}{2}$	..	..	$Cl_{\frac{1}{4}}HN_{\frac{1}{4}}$	$Cl_{\frac{1}{2}}H_{\frac{1}{2}} + N_{\frac{1}{2}}H_{1\frac{1}{2}} = 2$ vols.		
95	Hydroiodate of phosphoretted hydrogen	174	..	..	..	..	$I_{\frac{1}{4}}HP_{\frac{1}{8}}$	$I_{\frac{1}{2}}H_{\frac{1}{2}} + P_{\frac{1}{4}}H_{1\frac{1}{2}} = 2$ vols.		
96	Hydrobromate of phosphoretted hydrogen	85	..	..	..	..	$Br_{\frac{1}{4}}HP_{\frac{1}{8}}$	$Br_{\frac{1}{2}}H_{\frac{1}{2}} + P_{\frac{1}{4}}H_{1\frac{1}{2}} = 2$ vols.		
97	Hydrochlorate of phosphoretted hydrogen	..	..	18	..	..	$Cl_{\frac{1}{4}}HP_{\frac{1}{8}}$	$Cl_{\frac{1}{2}}H_{\frac{1}{2}} + P_{\frac{1}{4}}H_{1\frac{1}{2}} = 2$ vols.		

IV. Organic Compounds.

		Boiling temperature.	Specific gravity.	Gaseous specific gravity. H = 1.	Ditto inferred from specific heat.	Bulk of a molecule.	Chemical analysis.	Chemical arrangement of molecules (hypothesis).	Vapour constants.	
		1.	2.	3.	4.	5.			G.	H.
98	Carburetted hydrogen	° ..	..	8	..	..	$H_2C_{\frac{1}{2}}$			
99	Olefiant gas or etherin or hydrocarbon .	..	(0·63)?	14	14	(22?)	H_2C	$\frac{1}{2}$, 1, 2, $4\frac{1}{2}$, 5, 8 (combining ratios)		
100	Etherine or bietherin	0	0·63	28	6·7	(45)	H_4C_2	$2\,(H_2C) = (H_2C)_2$		
101	Elaen	230	..	63	..	..	$H_9C_{4\frac{1}{2}}$	$4\frac{1}{2}\,(H_2C) = (H_2C)_{4\frac{1}{2}}$		
102	Ameline	320	0·85	70	..	82	$H_{10}C_5$	$5\,(H_2C) = (H_2C)_5$		
103	Ceten	527	..	112	..	..	$H_{16}C_8$	$8\,(H_2C) = (H_2C)_8$		
104	Benzin or bicarburet of hydrogen (55/45)	187	0·85	39	..	44	H_3C_3			
105	Mesitylene (70/103)	330	..	40	..	..	H_4C_3			
106	Naphtha	180	0·75	41	6·5	56	H_5C_3	. . . .	19·36	14·64
107	Naphthalin (50/183)	412	..	64	..	..	H_4C_5			
108	Paranaphthalin	572	..	96	..	..	H_6C_5			
109	Camphene or oil of turpentine (75/41) .	332	0·86	68	6·6	80	H_8C_5	. . . .	18·59	15·56
110	Caoutchin or citrene (70/103)	440	0·67	68	..	102	H_8C_5			
111	Petrolene or sweet oil of wine (71/411)	545	0·90	136	..	151	$H_{16}C_{10}$			
112	Alcohol	173	·795	23	5·0	29	$H_3CO_{\frac{1}{2}}$	$H_2C + HO_{\frac{1}{2}}$	19·20	10·00
113	Sulphuric ether	96	$\left\{\begin{matrix}·734 \\ ·632\end{matrix}\right\}$	37	5·8	58	$H_5C_2O_{\frac{1}{2}}$	$(H_2C)_2 + HO_{\frac{1}{2}}$	16·86	10·99
114	Oxalic ether (m)	400	1·09	73	..	66	$H_5C_3O_2$	$O_{1\frac{1}{2}}C + (H_2C)_2 + HO_{\frac{1}{2}}$		
115	Nitrous ether (m)	80	·89	$37\frac{1}{2}$	..	42	$H_{2\frac{1}{2}}ON_{\frac{1}{2}}C$	$\overline{O_{1\frac{1}{2}}N} + \overline{(H_2C)_2 + HO_{\frac{1}{2}}} = 2$ vols.		
116	Succinic ether (m)	515	1·04	87	..	84	$H_7C_4O_2$	$O_{1\frac{1}{2}}H_2C_2 + (H_2C)_2 + HO_{\frac{1}{2}}$		

IV. Organic Compounds (continued).

		Boiling temperature.	Specific gravity.	Gaseous specific gravity. H = 1.	Ditto inferred from specific heat.	Bulk of a molecule.	Chemical analysis.	Chemical arrangement of molecules (hypothesis).	Vapour constants.	
		1.	2.	3.	4.	5.			G.	H.
117	Formic ether (m)	132°	·92	37	..	80	$H_3C_{1\frac{1}{2}}O$	$\overline{O_{1\frac{1}{2}}HC} + \overline{(H_2C)_2 + HO_{\frac{1}{2}}} = 2$ vols.		
118	Acetic ether (m) (see 132 and 139)	205	·89	44	..	100	H_4C_2O	$\overline{O_{1\frac{1}{2}}H_3C_2} + \overline{(H_2C)_2 + HO_{\frac{1}{2}}} = 2$ vols.		
119	Mercaptan or hydrosulph. of etherine (71/391)	96 143	·83	31	..	38	$H_3CS_{\frac{1}{6}}$	$H_2C + HS_{\frac{1}{6}}$		
120	Hydrosulphuric ether or hydrosulph. of etherine (71/389)	160	·825	45	..	55	$H_5C_2S_{\frac{1}{6}}$	$(H_2C)_2 + HS_{\frac{1}{6}}$		
121	Benzoic acid or anhydrous carbonate of benzin (55/46)	528	·66	61	..	92	$H_3C_{3\frac{1}{2}}O$	$H_3C_3 + OC_{\frac{1}{2}}$		
122	Nitrobenzide or nitrite of benzin (57/87)	400	1·21	62	..	51	$H_3C_3ON_{\frac{1}{2}}$	$H_3C_3 + ON_{\frac{1}{2}}$		
123	Oxidized ether or aldehyde (m) (59/295)	72	·79	22	..	28	$H_2CO_{\frac{1}{2}}$	$(H_3C_3)_{\frac{1}{3}} + HO_{\frac{1}{2}}$ or $H_2C + O_{\frac{1}{2}}$		
124	Chloric ether (m)	220	1·07	32	..	30	$H_2CCl_{\frac{1}{2}}$	$H_2C + Cl_{\frac{1}{2}}$		
125	Chloride of hydrocarbon (58/307)	178 146	1·26 1·17	50	..	40	H_2CCl	$H_2C + Cl$		
126	Bromide of hydrocarbon (59/359)	322	2·16	94	..	43	H_2CBr	$H_2C + Br$		
127	Iodide of hydrocarbon	..	3·20	..	..	..	$[H_2CI]$			
128	Hydrochloric ether (Thenard)	57	0·84 0·77	$32\frac{1}{2}$	..	40	$H_{2\frac{1}{2}}Cl_{\frac{1}{2}}C$	$H_2C + Cl_{\frac{1}{2}}H_{\frac{1}{2}}$		
129	Hydrobromic ether	..	..	$[54\frac{1}{2}]$	..	..	$H_{2\frac{1}{2}}Br_{\frac{1}{2}}C$	$H_2C + Br_{\frac{1}{2}}H_{\frac{1}{2}}$		
130	Hydroiodic ether (Gay-Lussac and Thenard)	176 148	1·92	77	..	40	$H_{2\frac{1}{2}}I_{\frac{1}{2}}C$	$H_2C + I_{\frac{1}{2}}H_{\frac{1}{2}}$		
131	Hydrocyanic ether	216	0·79	$27\frac{1}{2}$	..	35	$H_{2\frac{1}{2}}N_{\frac{1}{2}}C_{1\frac{1}{2}}$	$H_2C + N_{\frac{1}{2}}C_{\frac{1}{2}}H_{\frac{1}{2}}$		
132	Acetic ether (49/209). Pyroacetic spirit (see 118 and 139)	132	0·79	29	..	37	$H_3C_{1\frac{1}{2}}O_{\frac{1}{2}}$	$H_2C + O_{\frac{1}{2}}C_{\frac{1}{2}}H$		

IV. Organic Compounds (continued).

		Boiling temperature.	Specific gravity.	Gaseous specific gravity. H = 1.	Ditto inferred from specific heat.	Bulk of a molecule.	Chemical analysis.	Chemical arrangement of molecules (hypothesis).	Vapour constants.	
		1.	2.	3.	4.	5.			G.	H.
133	Methylal (70/394)	107°	0·85	38	. .	44	$H_4C_{1\frac12}O$	$(H_2C+O_{\frac12}C_{\frac12}H)+HO_{\frac12}$		
134	Benzoic ether (m)	502	1·05	75	. .	71	$H_5C_{4\frac12}O$	$H_2C+(H_3C_3+C_{\frac12}O)$		
135	Pyroxilic spirit (58/11)	150	·80	16	. .	20	$H_2C_{\frac12}O_{\frac12}$	$(H_2C)_{\frac12}+HO_{\frac12}$ . .	18·06	11·00
136	Pyroxilic ether (58/21)	. .	·80	23	. .	29	$H_3CO_{\frac12}$	$H_2C+HO_{\frac12}$		
137	Nitrate of pyroxilic ether (58/42) . .	156	1·18	38½	. .	65	$H_{1\frac12}C_{\frac12}O_{1\frac12}N_{\frac12}$	$\overline{O_{2\frac12}N}+\overline{H_2C+HO_{\frac12}}=2$ vols.		
138	Sulphate of pyroxilic ether (58/34) . .	370	1·32	63	. .	48	$H_3CO_2S_{\frac16}$	$\overline{O_{1\frac12}S_{\frac16}}+\overline{H_2C+HO_{\frac12}}=2$ vols.		
139	Acetate of pyroxilic ether (49/209; 58/47) (see 118 and 132)	136	·92	37	. .	80	$H_3C_{1\frac12}O$	$\overline{O_{1\frac12}C_2H_3}+\overline{H_2C+HO_{\frac12}}=2$ vols.		
140	Formicate of pyroxilic ether (m) . . .	. .	·80	30	. .	38	H_2CO	$\overline{O_{1\frac12}CH}+\overline{H_2C+HO_{\frac12}}=2$ vols.		
141	Chloride of methyle (71/380)	85	1·34	43	. .	32	$HClC_{\frac12}$	$Cl+(H_2C)_{\frac12}$		
142	Muriate of methyle (58/27)	. .	[·82 ?]	25½	. .	[32]	$H_{1\frac12}Cl_{\frac12}C_{\frac12}$	$Cl_{\frac12}H_{\frac12}+(H_2C)_{\frac12}$		
143	Hydroiodate of methyle (58/30) . . .	110	2·24	70	. .	31	$H_{1\frac12}I_{\frac12}C_{\frac12}$	$I_{\frac12}H_{\frac12}+(H_2C)_{\frac12}$		
144	Hydrofluorate of methyle (61/194) . .	. .	. .	17	. .	. .	$H_{1\frac12}F_{\frac12}C_{\frac12}$	$F_{\frac12}H_{\frac12}+(H_2C)_{\frac12}$		
145	Benzoate of methyle (58/52)	380	1·10	68	. .	62	H_4C_4O	$(H_2C)_{\frac12}+(H_3C_3+C_{\frac12}O)$		
146	Formic acid (see 117 and 140 and 147)	. .	1·16	[37]	. .	[31]	$[HCO_{1\frac12}]$			
147	Hydrate of formic acid (68/421) . . .	. .	1·12	23	. .	20	$HC_{\frac12}O$	$\overline{HCO_{1\frac12}}+\overline{HO_{\frac12}}=2$ vols.		
148	Hydrate of ameline (56/317)	265	0·82	44	. .	54	$H_6C_{2\frac12}O_{\frac12}$	$(H_{10}C_5)_{\frac12}+HO_{\frac12}$		
149	Acetate of ameline (75/197)	257	. .	65	. .	. .	$H_7C_{3\frac12}O$	$H_2CO+(H_{10}C_5)_{\frac12}$		
150	Hydroiodate of ameline (71/95) . . .	248	. .	98	. .	. .	$H_{5\frac12}C_{2\frac12}I_{\frac12}$	$I_{\frac12}H_{\frac12}+(H_{10}C_5)_{\frac12}$		
151	Camphor or oxide of camphene (m) . .	400	1·00	76	. .	76	$H_8C_5O_{\frac12}$	$H_8C_5+O_{\frac12}$		

IV. ORGANIC Compounds (continued).

		Boiling temperature.	Specific gravity.	Gaseous specific gravity. H = 1.	Ditto inferred from specific heat.	Bulk of a molecule.	Chemical analysis.	Chemical arrangement of molecules (hypothesis).	Vapour constants.	
		1.	2.	3.	4.	5.			G.	H.
152	Bichlorhydrate of camphene (75/37)	° ..	1·02	[86]	..	..	$H_{8\frac{1}{2}}C_5Cl_{\frac{1}{2}}$	$H_8C_5 + Cl_{\frac{1}{2}}H_{\frac{1}{2}}$		
153	Bromohydrate of camphene	..	1·28	[$108\frac{1}{2}$]	..	..	$H_{8\frac{1}{2}}C_5Br_{\frac{1}{2}}$	$H_8C_5 + Br_{\frac{1}{2}}H_{\frac{1}{2}}$		
154	Hydroiodate of camphene	..	1·51	[131]	..	..	$H_{8\frac{1}{2}}C_5I_{\frac{1}{2}}$	$H_8C_5 + I_{\frac{1}{2}}H_{\frac{1}{2}}$		
155	Valeric acid (m)	344	·94	81	..	86	H_5C_5O	$H_3C_3 + H_2C + 2(O_{\frac{1}{2}}C_{\frac{1}{2}})$		
156	Chloride of benzide (m) (see 160)	400	1·46	$91\frac{1}{2}$	..	62	$H_{1\frac{1}{2}}C_3Cl_{1\frac{1}{2}}$	$(H_{\frac{1}{2}}Cl_{\frac{1}{2}})_3C_3$		
157	Chloroetheride or chloroforme (71/382; 56/119)	141	1·48	$60\frac{1}{2}$	..	41	$H_{\frac{1}{2}}Cl_{1\frac{1}{2}}C_{\frac{1}{2}}$	$ClC_{\frac{1}{2}}(Cl_{\frac{1}{2}}H_{\frac{1}{2}})$		
158	Bromoetheride or bromoforme	..	3·1 ?	$126\frac{1}{2}$	..	[41]	$H_{\frac{1}{2}}Br_{1\frac{1}{2}}C_{\frac{1}{2}}$	$BrC_{\frac{1}{2}}(Cl_{\frac{1}{2}}H_{\frac{1}{2}})$*		
159	Iodetheride or iodiforme	255	5·0 ?	[194]	..	[41]	[$H_{\frac{1}{2}}I_{1\frac{1}{2}}C_{\frac{1}{2}}$]	[$IC_{\frac{1}{2}}(Cl_{\frac{1}{2}}H_{\frac{1}{2}})$]*		
160	Bichloride of muriatic ether (71/365) (see 156)	167	1·37	$67\frac{1}{2}$	..	50	$H_{1\frac{1}{2}}Cl_{1\frac{1}{2}}C$	$(H_{\frac{1}{2}}Cl_{\frac{1}{2}})_3C$		
161	Trichloride of muriatic ether	216	1·53	85	..	55	Cl_2HC	$Cl(H_{\frac{1}{2}}Cl_{\frac{1}{2}})_2C$		
162	Quadrichloride of muriatic ether	295	1·64	$102\frac{1}{2}$	..	62	$Cl_{2\frac{1}{2}}H_{\frac{1}{2}}C$	$Cl_2(H_{\frac{1}{2}}Cl_{\frac{1}{2}})C$		
163	Chloral or carbonite of chloroforme (56/29)	200	1·50	$74\frac{1}{2}$	..	50	$Cl_{1\frac{1}{2}}H_{\frac{1}{2}}CO_{\frac{1}{2}}$	$Cl(H_{\frac{1}{2}}Cl_{\frac{1}{2}})C_{\frac{1}{2}} + C_{\frac{1}{2}}O_{\frac{1}{2}}$		
164	Bromal or carbonite of bromoforme	260 ?	3·35	..	..	42	$Br_{1\frac{1}{2}}H_{\frac{1}{2}}CO_{\frac{1}{2}}$	$Br(H_{\frac{1}{2}}Br_{\frac{1}{2}})C_{\frac{1}{2}} + C_{\frac{1}{2}}O_{\frac{1}{2}}$		
165	Hydrate of chloral (56/135)	..	..	$41\frac{3}{4}$	..	..	..	$\overline{Cl(H_{\frac{1}{2}}Cl_{\frac{1}{2}})C_{\frac{1}{2}}+C_{\frac{1}{2}}O_{\frac{1}{2}}}+\overline{HO_{\frac{1}{2}}}=2$ vols.		
166	Monochloride of methylitic ether (71/399)	220	1·31	60	..	46	$ClH_2CO_{\frac{1}{2}}$	$ClH_2C_{\frac{1}{2}} + C_{\frac{1}{2}}O_{\frac{1}{2}}$		
167	Bichloride of methylitic ether	265	1·61	93	..	58	$Cl_2HCO_{\frac{1}{2}}$	$Cl_2HC_{\frac{1}{2}} + C_{\frac{1}{2}}O_{\frac{1}{2}}$		

* [*Sic* in MS.]

IV. Organic Compounds (continued).

		Boiling temperature.	Specific gravity.	Gaseous specific gravity. H = 1.	Ditto inferred from specific heat.	Bulk of a molecule.	Chemical analysis.	Chemical arrangement of molecules (hypothesis).	Vapour constants.	
		1.	2.	3.	4.	5.			G.	H.
168	Perchloride of methylitic ether . . .	212°	1·59	64	..	40	$Cl_{1\frac{1}{2}}C_{\frac{1}{2}}O_{\frac{1}{4}}$	$\overline{Cl_2ClC_{\frac{1}{2}}} + \overline{C_{\frac{1}{2}}O_{\frac{1}{2}}} = 2$ vols.		
169	Muriate of benzin (58/319)	0	..	$31\frac{1}{2}$	..	..	$Cl_{\frac{1}{2}}H_{1\frac{1}{2}}C$	$Cl_{\frac{1}{2}}H_{\frac{1}{2}} + \frac{1}{3}(H_3C_3)$		
170	Hydrobromide of benzin (59/364) . .	95	1·52	$53\frac{1}{2}$	..	36	$Br_{\frac{1}{2}}H_{1\frac{1}{2}}C$	$Br_{\frac{1}{2}}H_{\frac{1}{2}} + \frac{1}{3}(H_3C_3)$		
171	Hydroiodate of benzin	135								
172	Oxichlorocarbonic ether (54/228) . .	200	1·13	$54\frac{1}{2}$	..	48	$Cl_{\frac{1}{2}}H_{2\frac{1}{2}}C_{1\frac{1}{2}}O$	$Cl_{\frac{1}{2}}H_{\frac{1}{2}} + H_2C + OC_{\frac{1}{2}}$		
173	Urethane (54/236)	226	1·11	$44\frac{1}{2}$	..	40	$N_{\frac{1}{2}}H_{3\frac{1}{2}}OC_{1\frac{1}{2}}$	$N_{\frac{1}{2}}H_{1\frac{1}{2}} + H_2C + OC_{\frac{1}{2}}$		

APPENDIX I.

EXTRACT from the Proceedings of the Royal Society (vol. 5, p. 604—March 5, 1846).

On the Physics of Media that are composed of free and perfectly elastic Molecules in a state of Motion. By J. J. WATERSON (*sic*), *Esq. Communicated by Captain* BEAUFORT, *R.N., F.R.S.*

THIS memoir contains the enunciation of a new theory of heat, capable of explaining the phenomena of its radiation and polarization, and the elasticity of various bodies, founded on the hypothesis of a medium consisting of a vast multitude of particles of matter endowed with perfect elasticity, and enclosed in elastic walls, but moving in all directions within that space, with perfect freedom, and in every possible direction. In the course of these motions, the particles must be supposed to encounter one another in every possible manner, during an interval of time so small as to allow of their being considered infinitesimal in respect to any sensible period; still, however, preserving the molecular *vis viva* constant and undiminished.

The author then enters into extensive analytical investigations; first, of the conditions that determine the equilibrium of such a homogeneous medium as is implied by the hypothesis, and of the laws of its elasticity; secondly, of the physical relations of media that differ from each other in the specific weight of their molecules; thirdly, of the phenomena that attend the condensing and dilating of media, and of the mechanical value of their molecular *vis viva;* fourthly, of the resistance of media to a moving surface; fifthly, of the vertical equilibrium of a medium surrounding a planet and constituting its atmosphere; and lastly, of the velocity with which impulses are transmitted through a medium so constituted.

In an Appendix, the author enters into a full explanation of a table of gases and vapours, drawn up with reference to the subjects discussed in his paper.

APPENDIX II.

EXTRACT from the Report of the 21st Meeting of the British Association, Ipswich, 1851. (Transactions of the Sections, p. 6.)

On a General Theory of Gases. By J. J. WATERSTON, *Bombay.*

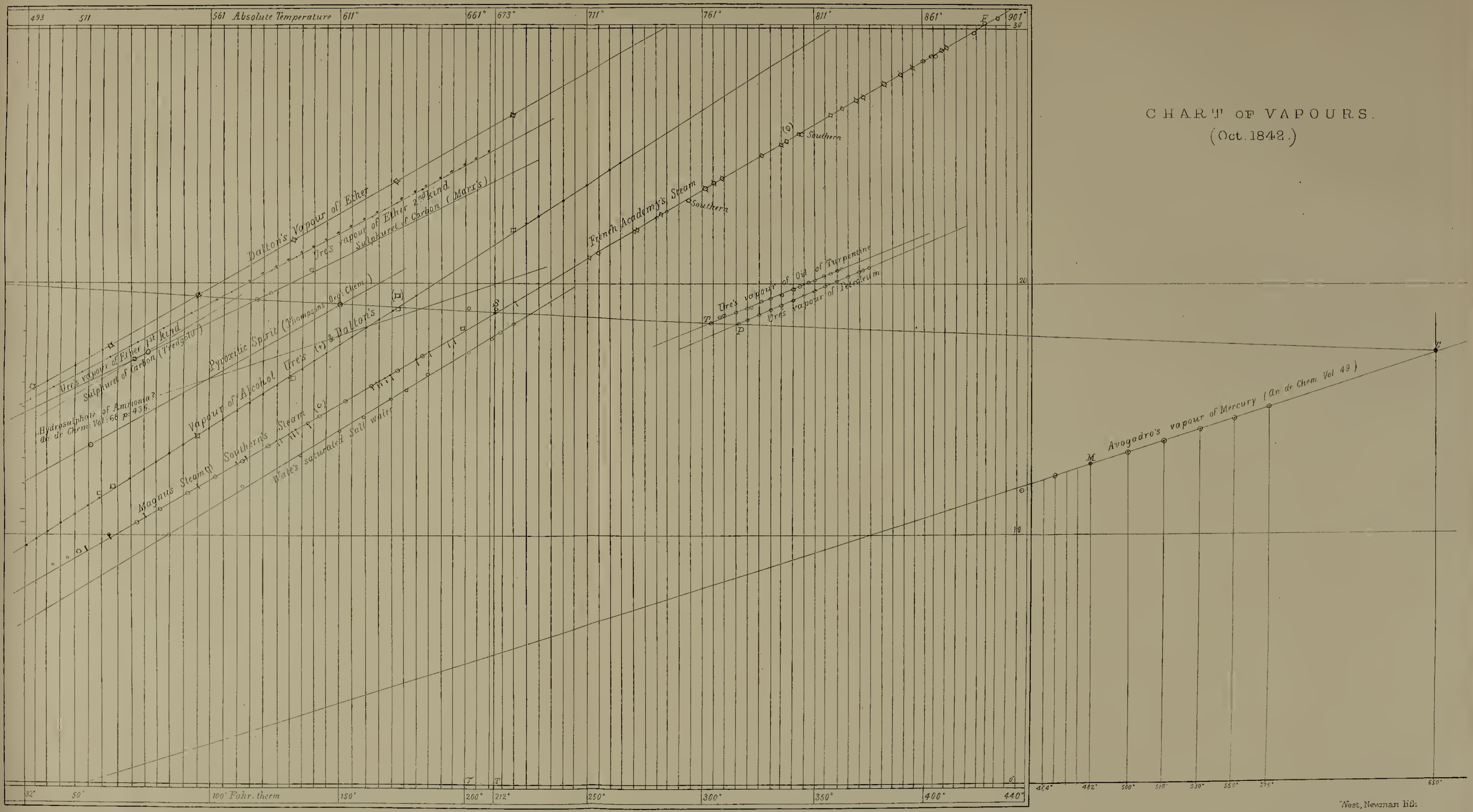
THE author deduces the properties of gases, with respect to heat and elasticity, from a peculiar form of the theory which regards heat as consisting in small but rapid motions of the particles of matter. He conceives that the atoms of a gas, being perfectly elastic, are in continual motion in all directions, being restrained within a limited space by their collisions with each other, and with the particles of surrounding bodies. The *vis viva* of those motions in a given portion of gas constitutes the quantity of heat contained in it.

He shows that the result of this state of motion must be to give the gas an elasticity proportional to the mean square of the velocity of the molecular motions, and to the total mass of the atoms contained in unity of bulk; that is to say, to the density of the medium. This elasticity, in a given gas, is the measure of temperature. Equilibrium of pressure and heat between two gases takes place when the number of atoms in unity of volume is equal, and the *vis viva* of each atom equal. Temperature, therefore, in all gases, is proportional to the mass of one atom multiplied by the mean square of the velocity of the molecular motions, being measured from an *absolute zero* 491° below the zero of Fahrenheit's thermometer.

If a gas be compressed, the mechanical power expended in the compression is transferred to the molecules of the gas, increasing their *vis viva;* and conversely, when the gas expands, the mechanical power given out during the expansion is obtained at the expense of the *vis viva* of the atoms. This principle explains the variations of temperature produced by the expansion and condensation of gases—the laws of their specific heat under different circumstances, and of the velocity of sound in them. The fall of temperature found on ascending in the atmosphere, if not disturbed by radiation and other causes, would correspond with the *vis viva* necessary to raise the atoms through the given height.

The author shows that the velocity with which gases diffuse themselves is proportional to that possessed by their atoms according to his hypothesis.

CHART OF VAPOURS.
(Oct. 1842.)



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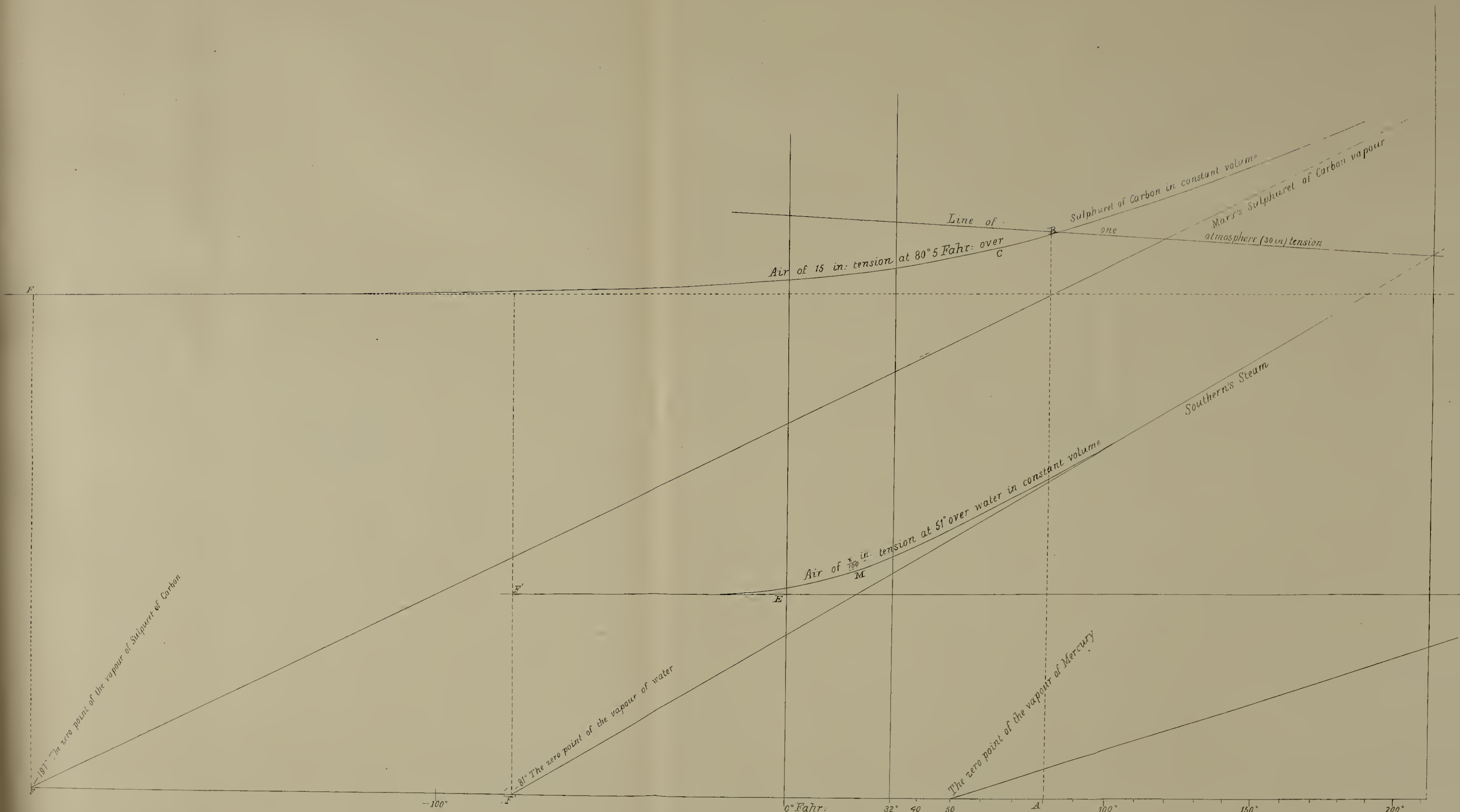

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II. *On the Time-relations of the Excursions of the Capillary Electrometer, with a Description of the Method of Using it for the Investigation of Electrical Changes of Short Duration.*

By George J. Burch, *B.A., Oxon.*

Communicated by Professor Bartholomew Price, *F.R.S.*

Received September 3,—Read November 19, 1891.

[Plates 3–6.]

In a preliminary note "On a Method of Determining the Value of Rapid Variations of a Difference of Potential by Means of the Capillary Electrometer," I indicated briefly the results of an investigation of the time-relations of the excursions of the capillary electrometer. The object of the present paper is to give further details concerning the method of measurement which I then put forward, and to describe the experiments which led to it.

The theory of the instrument has been treated by Lippmann, Hermann, and others, mainly from a mathematical stand-point, and the phenomena have been ascribed to polarization, but no adequate attempt appears to have been made to establish by actual measurement the accuracy of the formula for the time-relations, nor to ascertain whether there were any other influences at work besides those assumed in stating the problem. Poiseuille's experiments on the velocity of the flow of liquids through capillary tubes seem to have been overlooked, and no sufficient investigation made of the effect of the form and dimensions of the tubes upon the rate of movement of the meniscus.

It seemed, therefore, that there was scope for a thorough examination of the problem by the experimental method, with a view, not only to the further elucidation of the theory, but also to the discovery of the best manner of using the instrument so as to render available those special properties which make it unlike any other form of electrometer or galvanometer.

In the course of the investigation, which has occupied more than three years, over a hundred electrometers were made, besides many additional capillaries used in preliminary experiments.

General Data with respect to the Time-Relations of an Excursion.

(1.) Dilution of the acid beyond a certain point was found to lessen the rapidity of an excursion without altering the extent of it. This indicated that the velocity of the movement was affected by electrical resistance.

A similar result was also produced by the introduction of 100,000 ohms into the external circuit.

(2.) Shortening the capillary so as to reduce the length of the acid column made the instrument act more quickly, but in this case the mechanical friction, as well as the electrical resistance, was lessened.

(3.) The shape of the tube, where it tapers to form the capillary, was found to have a marked effect; the motion of the meniscus being much more sluggish if this part was made too long. In this case the increase of electrical resistance must be comparatively trifling, and the result is to be ascribed mainly to friction and eddy currents.

(4.) The same may be said of the orifice, a clean-cut capillary being usually much quicker in its action than a splintered one.

These things may be taken as indicating the action of both mechanical friction and electrical resistance in determining the rate of movement of the meniscus.

The next step was to ascertain whether there was any latent period before the commencement of the rise or fall. This was done by photography.

A difference of potential was suddenly communicated to the instrument by striking open a short-circuiting key, the end of which projected across the slit upon which the image of the mercury column was thrown, and so gave upon the sensitive plate, as it passed rapidly behind it, a record of the exact instant at which the current was allowed to act upon the electrometer. In no case could the smallest interval be detected between the opening of the key and the commencement of the excursion, even when the difference of potential employed was very small.

The converse of this was found to be true under ordinary circumstances, that is to say, the meniscus ceases to move the moment the source of electromotive force is withdrawn. In other words, the electrometer is practically dead-beat. It has been generally held to be perfectly so, but this is not the case. If the instrument is one that has been specially designed to act with great rapidity, it will be found, on communicating to it a fairly strong charge from a condenser of not more than one-third of the capacity of the electrometer, that the meniscus will start suddenly forward, and *then slowly return*, perhaps as much as one-tenth of the distance it has traversed. But this overshooting is entirely prevented by the introduction of an external resistance of, say, 50,000 ohms. Apparently the reason why this phenomenon has not been noticed is, that experimenters have contented themselves with observing the effect of suddenly breaking the circuit during an excursion; but in so doing, they have introduced an infinite resistance. In the case of the condenser experiment, there is, as it were, a reservoir on each side of the electrometer, which acts as a spring, permitting a certain oscillation of the charge. A far more severe test is to preserve the electrometer circuit unbroken, and to remove the source of electromotive force, by breaking the primary circuit of the rheochord. This I have done, and the resulting photographs show the effect of overshooting to be, for the same velocity of the meniscus, nearly twice as great. But it can still be overcome by external resistance, and is only noticeable when the difference of potential is two or three times as great as it ever is in physiological experiments, and even then I have not found it exceed 2 per cent. of the length of the complete excursion, with no external resistance.

It remains true, therefore, that with an ordinarily quick electrometer, under the conditions which obtain in practical work, *the meniscus ceases to move the instant the source of electromotive force is withdrawn.*

The velocity of the movement is also greatest at the first, and there is no portion of the normal photographic curve, even at its commencement, concave to the asymptote, except in those cases in which overshooting may occur. Under such circumstances, slight signs of increase of the velocity are sometimes just discernible. But this was only detected in three cases, when the currents used were as great as the electrometer would bear without injury.

These facts, implying the absence of acquired "momentum," seemed to indicate that in ordinary working the velocity of the meniscus at any moment must be some function of the accelerating force at that moment, in the sense that it is independent of any previous motion, and can contribute nothing—or practically nothing—to the velocity with which it moves during the next interval.

In order to determine experimentally the form of this function, I arranged a rheotome in the derived circuit of a rheochord in such a way that any desired difference of potential could be introduced into the electrometer circuit for an accurately measured period of from ·005 sec. to about ·6 sec. This rheotome, which was upon the principle of a drop-shutter, acted by first snatching open a short-circuiting key, and then after the required interval, breaking the circuit on both sides of the electrometer simultaneously—this arrangement being necessary, in order that the meniscus might stop suddenly without running back to zero. A specially sensitive electrometer was used, having a long range, and with no tendency to "creep." It was less rapid in its action than those usually employed in the laboratory for physiological work, being selected on this account so that the time-measurements might be more accurately made. The insulation was so good that no appreciable return of the meniscus took place during 10 minutes after the circuit was broken.

The experiments were conducted as follows:—Having found the total length of the excursion produced by a difference of potential derived from 150 millims. of the rheochord wire to be 126 divisions of the eye-piece micrometer, the rheotome was set so as to check the meniscus exactly in the middle of the excursion, namely, at 63 divisions from zero. This required a closure of ·414 sec. The rider of the rheochord was then shifted to 50 millims., when the complete excursion was found to be 42 divisions, and that given with the rheotome, 21 divisions. On further reducing the length of rheochord wire to 25 millims., the full excursion being 21 divisions, the meniscus shot up, when the rheotome was used, through 10·5 divisions of the eye-piece micrometer and there stopped. That is to say, with only one-sixth part of the original difference of potential, the middle point of the excursion was reached in exactly the same time, namely ·414 sec. In other words, the mean velocity of the meniscus during the first half of a normal excursion was found to be proportional to the total length of it. Further experiments showed that the same was true of other fractions besides the half, as for instance, one-fifth, one-third, and three-fourths, but in these cases the measurements were less reliable, owing to purely instrumental difficulties.

From these data I concluded that the velocity with which the meniscus moved at any instant during a normal excursion, must be proportional to its distance from the final position of rest at that time.

Fig. 1.

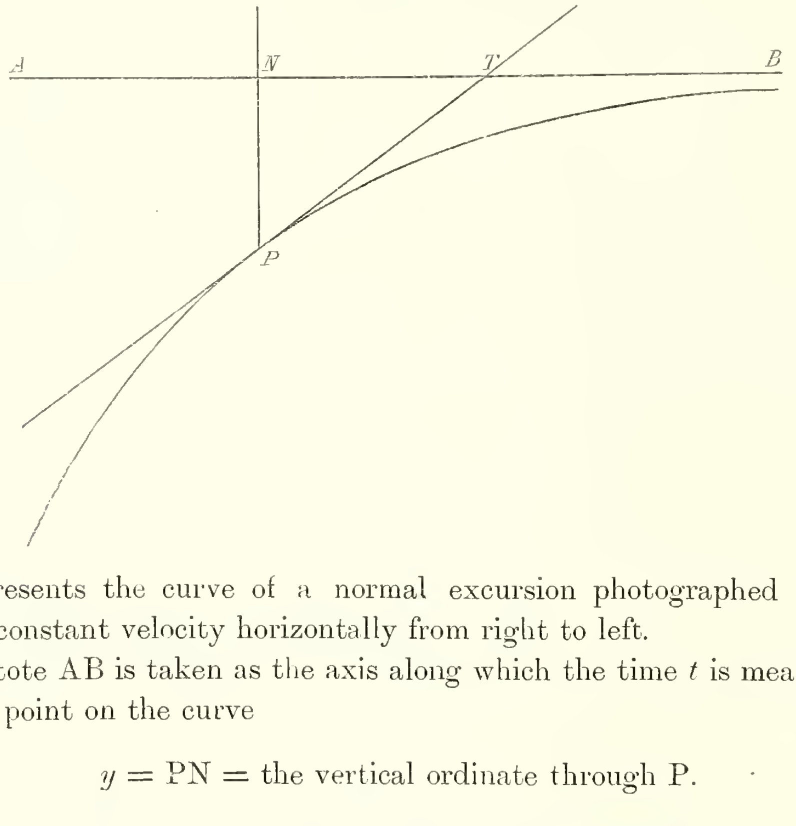


Fig. 1 represents the curve of a normal excursion photographed upon a plate moving with constant velocity horizontally from right to left.

The asymptote AB is taken as the axis along which the time t is measured.

Let P be a point on the curve

$$y = \text{PN} = \text{the vertical ordinate through P.}$$

Then, if u denote the velocity of the meniscus in the capillary

$$u = -\frac{dy}{dt}.$$

But, according to the experimental results,

$$u \text{ varies as } y,$$

so that

$$cy = -\frac{dy}{dt}, \text{ whence } -c\,dt = \frac{dy}{y}.$$

Integrating

$$\int \frac{dy}{y}\,dt = \log\frac{y}{a} = -ct,$$

or,

$$y = ae^{-ct}.$$

It is obvious that any curve having this equation can easily be recognised in either of two ways. The tabular logarithms of a series of ordinates corresponding to equal time-intervals, are in arithmetical progression. This method was used in verifying the hypothesis as to the time-relations of the normal excursion. The second is a graphic method depending on the fact that the subtangent NT (fig. 1) of a logarithmic curve, or intercept, upon the asymptote between the tangent and ordinate to any point upon the curve, is of constant length. This property affords a means of finding the position of the asymptote when only a portion of the excursion has been included in the photograph. Two points are chosen some distance apart, and the tangent and ordinate drawn to each. The level is then found by trial, at which the horizontal distance between the one tangent and its ordinate is equal to that between the other and its ordinate, or, in other words, the level is found at which the subtangents are constant.

The possibility of doing this in the case of the normal curve suggested the method of analyzing other curves propounded in my preliminary note. If, as my experiments have indicated, the velocity of the meniscus at any moment is due solely to the difference of potential between the terminals of the electrometer at that moment, and is not influenced in any way by its motion at any previous time, and adds nothing to its velocity at any subsequent time—in other words, if the electrometer is perfectly dead-beat—then the rate of movement at any given instant, during an irregular change of electromotive force, must be exactly the same as would be communicated to the meniscus by a permanent difference of potential equal to the electromotive force acting at that instant. That is to say, the tangent to a given point on *the curve produced by any excursion* must coincide in direction with that of the commencement of the *normal curve given by the difference of potential still existing between the terminals of the electrometer*. If, therefore, the length of the constant subtangent to the normal curve is known, the length of the ordinate for that particular inclination of the tangent can be determined, and it will show how far the meniscus has still to move before reaching the position in the capillary corresponding to the difference of potential to which its velocity is due at the time in question. If the mercury is at zero, the length of the ordinate thus determined will represent the actual electromotive force at that instant. If it is not at zero, the distance it has already travelled must be added to or subtracted from the ordinate, according as its sign, and that of the ordinate, is positive or negative.

In this way an excursion may be analyzed. Points are taken on the curve at intervals of ·005 or more, and the tangent and ordinate drawn to each and produced, upwards if the curve is rising, and downwards if it is falling, until the horizontal distance between them is equal to the subtangent of the normal curve taken through the same resistance. The ends of all the ordinates are joined, and the curve so produced represents, on the same scale as the ordinary readings of the electrometer, the total difference of potential during each instant of the experiment.

It may be useful to note that the ordinates of the original curve give the integral of the quantity of electricity that has passed at any instant.

Further Investigation of the Formula for the Time-Relations of the Normal Excursion.

Having ascertained, by measuring a number of photographed excursions, that the formula $y = ae^{-ct}$ would hold for some instruments and not for others, I proceeded to investigate the causes of these divergences.

There are two things which may modify the time-relations of the movement—the one accidental, and the other essential. The first is that the instrument may not be of equal sensitiveness throughout the part used. Obviously this difficulty may be got over by selecting a suitable electrometer. The second source of error cannot be so eliminated. It is that the internal resistance varies with the position of the meniscus in the capillary.

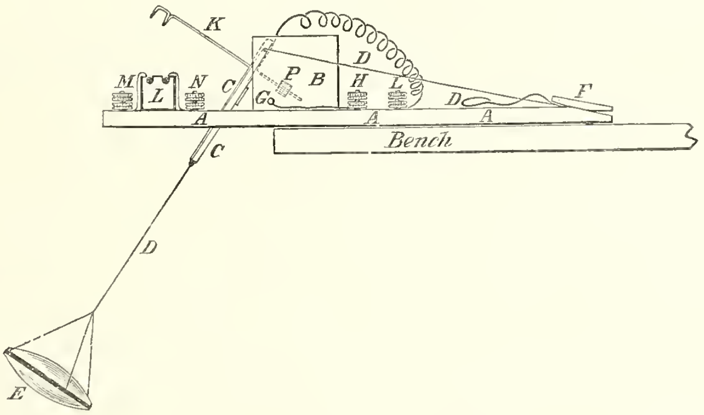
(1.) *Calibration Error.*—This has a twofold effect. It alters the electrical value of the scale-readings, and it has also a powerful influence upon the velocity of the movement. The greater the range of the excursion for a given difference of potential, the slower is the action of the instrument—not only relatively, but absolutely. The relation between the sensitiveness of an electrometer, and the time of half-charge, could not be determined directly by experiment, owing to the difficulty of making two capillaries sufficiently alike in all other respects; but from a large number of observations it is evident that the time of half-charge increases much more rapidly than the sensitiveness. I found by experiment that the electrical capacities of electrometers, with capillaries of the same internal diameter, are in direct proportion to the lengths of the excursions produced in them respectively by the same small difference of potential, and that the capacity is within wide limits independent of the difference of potential. As, therefore, the same change of surface-tension has to force the mercury column through distances proportional to the sensitiveness, and a quantity of electricity has to flow which is also proportional to the same thing, it seems probable that the time of half-charge may vary according to the square of the sensitiveness.

Further experiments will be made with regard to this. The important point, established by a number of observations, is that *the calibration error has a much greater effect upon the rapidity of the movement of the meniscus than upon the scale readings.* In most cases the sensitiveness of an electrometer increases as the meniscus approaches the tip of the capillary, and *its rapidity of action becomes less.* But this calibration error is entirely under the control of the maker of the instrument.

(2.) *Changes of Resistance.*—The principal seat of electrical resistance is in the slender column of dilute acid between the meniscus and the end of the capillary. But since the length of this varies continually during an excursion, it follows that the resistance must also vary, and with it, the velocity of the movement, *which must increase as the meniscus approaches the end of the capillary,* where the resistance is least. This variation was investigated in a series of experiments.

A special electrometer was prepared, with a very long capillary, of exactly equal sensitiveness for a considerable distance from the tip—twice as much as was required for the experiments. It was extremely slow in its action, but was otherwise a very fine instrument.

Fig. 2.



To measure the time-relations of the excursion the following simple and inexpensive rheotome was devised. Upon a base-board *A*, fig. 2, clamped to the bench so as to project beyond it, are fixed two upright pieces *B*, between which the bar *C* swings until it comes in contact, as it assumes a vertical position, with the cross-bar *G*, making connection in so doing between the terminals *H* and *I*, with which it is connected by a fine wire. As it is necessary that the circuit should be broken on both sides of the electrometer, to prevent "creeping," a light wire *K*, balanced by the counterpoise *P*, is attached to *C*, and moves with it. This wire is bent so as to dip into two pools of mercury, in the block of paraffin *L*, thus making a connection between the terminals *M* and *N*, about ·01 sec. before the bar *C* comes in contact with *G*. The weight *E*, which is a disc of lead weighing a couple of pounds, is attached, in the manner shown, to a string *D*, passing through a loop on the end of the bar *C*, over the rod on which it swings, to a cleat *F*. The weight *E* being drawn forward and then let go, carries the bar sharply up against the stop *G*, and then swings under the table, oscillating about the loop at the end of *C*, as a centre. On its return, as it passes the vertical, the contact is broken, first at *G*, which determines the period of closure, and immediately after, before the electrometer has time to creep, at *L*, on the other side of the circuit. It is then caught in the hand of the operator. The closure may be maintained during any odd number of swings by pressing the thumb upon the bar *C*, directly it has made contact, so as to hold it firmly against the stop, releasing it directly the weight has passed under the table for the last time, thus leaving it free to break contact on its return. The duration of a single swing is calculated by taking the time of 50 complete oscillations with the aid of a stop-watch. By suitably regulating the length of the string *D*, the time of closure can be adjusted to any desired period from ·4 sec. to 10 secs. The accuracy of the instrument was verified by comparing the results obtained with different lengths of string.

The object of the experiments was three-fold, viz. :—

(A.) To compare the time-relations of charge and discharge.

(B.) To ascertain the difference, if any, of the rate of movement when the meniscus was made to *rise*, *i.e.*, to advance towards the tip of the capillary, and when it was caused to *fall*, *i.e.*, to recede from the tip.

(C.) To measure the effect of the change of internal resistance during an excursion.

As the instrument was one of constant sensitiveness, this could be done by comparing the ratios of successive ordinates corresponding to the equal time-intervals given by 1, 3, 5, 7, etc., swings of the pendulum-rheotome, the mean of a number of observations being taken in each case.

The rheochord was set so as to give an excursion as near as possible of the full length of the scale of the eye-piece micrometer—50 divisions—and the capillary was fixed so that a mark 20 divisions from the tip coincided with the top of the scale. A low power was used, and the actual range of the movement was much greater than is employed in physiological work. One series of observations was taken with an added external resistance of 60,000 ohms. For measuring the rate of discharge, the rheotome was made to close a short circuit between the terminals of the electrometer.

In the following experiments,

l = the distance of the meniscus from the tip of the capillary, in divisions of the eye-piece micrometer.

y = the distance of the meniscus from its final position of rest after a closure of infinite duration.

The period of closure is expressed in swings of the pendulum rheotome.

The last column shows the ratio of each value of y to the one preceding it.

"Up," means an excursion towards the tip of the capillary.

"Down," signifies an excursion away from the tip of the capillary.

1. Charge. No External Resistance. Down.
Period of a single swing of the rheotome = ·36 sec.

Swings.	l.	y.	Ratio.
1	30	40	
3	41	29	1·38
5	48·5	21·5	1·35
7	54	16	1·34

2. Charge. No External Resistance. Up.
Period of a single swing of the rheotome = ·36 sec.

Swings.	l.	y.	Ratio.
1	63	44	
3	55	36	1·17
5	49	30	1·20
7	43	24	1·25

3. Discharge. No External Resistance. Down.
Period of a single swing of the rheotome = ·77 sec.

Swings.	l.	y.	Ratio.
1	34·5	35·5	
3	51·5	18·5	1·92
5	60	10	1·85
7	64	6	1·67

4. Discharge. No External Resistance. Up.
Period of a single swing of the rheotome = ·77 sec.

Swings.	l.	y.	Ratio.
1	56	36	
3	45	25	1·44
5	36	16	1·56
7	30	10	1·60
9	26	6	1·67

5. Charge. External Resistance = 60,000 ohms. Up.
Period of a single swing of the rheotome = ·77 sec.

Swings.	*l*.	*y*.	Ratio.
1	62·5	42·5	
3	51	31	1·37
5	44·4	22·4	1·38
7	36	16	1·40
9	31	11	1·45

N.B.—In the last three series of observations the period of the swing of the rheotome was made longer, in order to effect a better distribution of the measured points. On plotting the results of identical excursions, the curves were found to coincide with those obtained with a swing of ·36 sec., proving the accuracy of the method to be sufficient for the purpose of these experiments.

The above experiments indicate—

(*a*.) That the time-relations of charge and discharge are practically alike, as is shown also by the analysis of the photographed excursions.

(*b*.) That, in this particular instrument, the meniscus moved more quickly *away* from the tip of the capillary than *towards* it. No such difference could be detected in the case of electrometers of ordinary quickness of action, and the phenomenon appears to be peculiar to those of great sensitiveness and slow action. In some instruments it was reversed, and the mercury would only recede when assisted by tapping the stand, though it would advance freely with a small difference of potential.

(*c*.) That the change of the length of the acid column during an excursion *has* a measurable effect upon the time-relations of the movement, which, however, *is much less than it would be if the normal velocity were conditioned simply by the internal electrical resistance of the electrometer*. It was found by experiments with more than twenty instruments that the effect of an added external resistance in increasing the time of "half-charge" was such, that they might be assumed to have an internal resistance of the form

$$R = r(L + l),$$

where l = the distance of the meniscus at any moment from the tip of the capillary, and L is a constant of the particular instrument employed. In practice, L is many times greater than the maximum values of l, so that the variation of the speed-ratio from this cause is less than 3 per cent. over the entire width of the negatives referred to in this paper, and consequently not more than $\frac{1}{2}$ per cent. in the physiological curves to which the method of analysis was applied.

To sum up:—

The two causes which modify the time-relations of an excursion, act in opposite ways.

The decrease of internal electrical resistance as the meniscus approaches the tip of the capillary, tends to increase its velocity, while the increase of sensitiveness makes it move more slowly. But the latter has a much more powerful effect than the former, and may practically neutralize it, or even overpower it, without introducing a calibration error great enough to make an appreciable difference between the scale readings and the true value of an excursion.

The time-relations of the movement are conditioned partly by electrical resistance, but mainly by some other cause—probably mechanical friction.

The Production and Analysis of Photographs of the Excursions of the Capillary Electrometer.

The rapid movement of the sensitive plate required to bring out the details of the electric phenomena of muscle and nerve, necessitated an alteration in the form of the apparatus. The ingenious arrangement devised by the Rev. F. G. SMITH for producing a rectilinear motion of uniform velocity was inapplicable, owing to lack of space in the dark room. I therefore made an apparatus, the details of which will be described at length elsewhere, in which the sensitive plate was caused to describe an arc of a circle. The dark-slide containing it was attached to a kind of balanced pendulum, which carried it, at a uniform velocity, past the slit on which the magnified image of the column of mercury was thrown. The requisite velocity was given to the pendulum by a weight which, as in ATWOOD'S machine, was caught by a stop just before the plate reached the slit. The return of the pendulum after the exposure was prevented by a catch, and the key producing the excursion was actuated at the right moment by an arrangement of electromagnets. Time was recorded upon the plates by a magnetic vibrator placed in front of the slit, and driven by a tuning-fork in the usual way.

The exact instant of excitation was recorded by the signal-key before referred to, which was also placed in front of the slit.

With this arrangement the normal curve is most easily expressed in polar coordinates.

Time being recorded upon a circular arc,

$$t \text{ becomes } \theta.$$

Instead of the rectilinear asymptote, there is an asymptotic circle of radius $= \mathrm{R}$.

The expression for the radius vector is

$$r = \mathrm{R} \pm y,$$

the equation connecting y and θ being

$$y = ae^{-c\theta}.$$

This change necessitated a reconsideration of the method of analysis. The curve is a variety of the logarithmic spiral, but the intercept between the tangent and the radius vector on the asymptotic circle—the line corresponding to that which I had employed in the case of the rectilinear motion, is not constant, and not of a form convenient to use. There is, however, another property of polar coordinates which is at once available; namely, that the expression for the polar sub-normal is

$$r \cot \psi = \frac{dr}{d\theta} \cdot$$

In this case

$$\frac{dr}{d\theta} = - \frac{dy}{d\theta} = cy,$$

so that this property enabled me to analyze the curves with remarkable ease, and I propose to abandon rectilinear motion for the plates in future.*

Fig. 3.

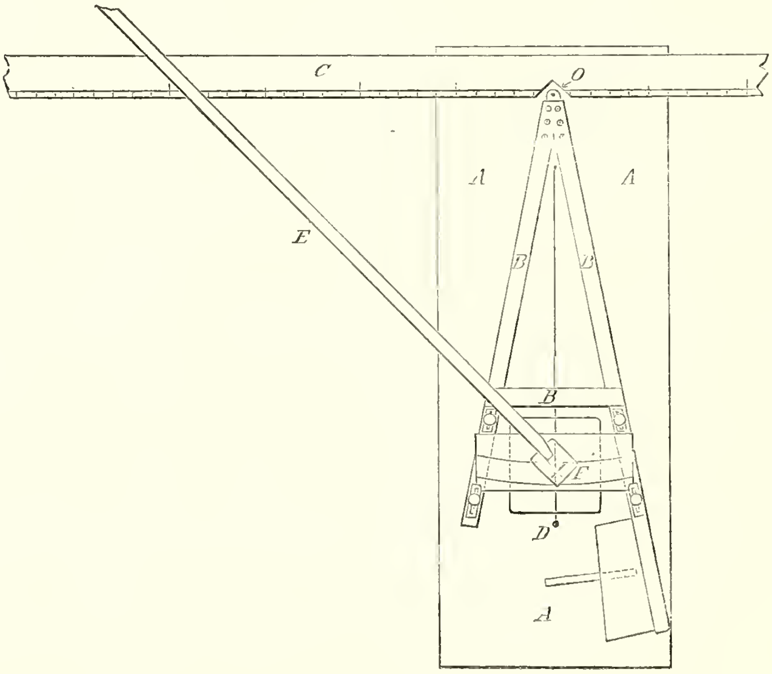
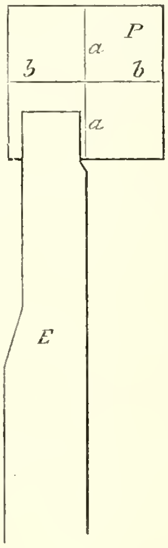


Fig. 4.



The problem of the analysis of a photographed excursion resolves itself therefore into the determination of the length of the subnormals of a sufficient number of points upon the curve, together with the measurement of the distances of these points from the circle corresponding to the position of the meniscus for zero potential. These measurements are most easily made upon the instrument represented in fig. 3.

The negative is fixed to a carrier *B*, pivoted at *O*, on the base-board *A*, in which a

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[*] See WILLIAMSON'S 'Differential Calculus,' 7th edition, p. 223, from which I got the clue while trying to elaborate a roundabout method of reproducing the polar curve in rectangular coordinates, so that I might analyze it.

hole is cut so that the photograph can be examined by transmitted light. The radius of the carrier is exactly equal to that of the pendulum on which the plate was exposed, and a fine wire or piece of horsehair is stretched from *D* to *O*, passing close underneath the glass without touching it, which serves as a radial line of reference. An index fixed to the carrier *B*, passing over a scale upon *A*, enables small angular displacements of the carrier to be accurately measured, thus determining the time intervals, and the ordinates are found by laying a graduated rule upon the negative just over the radial line *DO*, using as a reference circle the edge of the photograph, which is always well defined. The length of the subnormal is found by means of the flat rod, *E*, which carries at one end the glass plate, *F*, shown on a larger scale in fig. 4. On this plate is ruled a fine line, a, a_1, continuous with the edge of *E*, but broken for a distance of about 2 millims. at about the middle. Through the gap thus left passes the tangent line, b, exactly at right angles to a, a_1. When this line b, at the part where it is intersected by a, a_1, is placed as a tangent to the curve which is to be analyzed, at the point where it is cut by the radial line, *DO*, the length cut off by the rod, *E*, upon the graduated rule, *C*, which is permanently fixed at right angles to *DO*, is the subnormal to the curve at that point. The plate carrier is then shifted through an angle corresponding to a known interval of time, as determined from the time record upon the photograph, and the subnormal measured again for the part of the curve thus brought over the radial line, *DO*. This process is repeated at sufficiently close intervals throughout the curve, the corresponding ordinates being measured, taking the edge of the photograph as a reference circle.

For the application of this method it is first necessary to analyze a normal curve produced under the exact conditions, as to resistance, of the experimental photographs. With a suitable instrument, the subnormal is practically a constant multiple of y, the distance of the meniscus from its point of rest, and the value of this constant multiplier has to be found.

The following example of the analysis of the normal curve of the electrometer used in the physiological experiments described by Professor BURDON SANDERSON in a paper on the "Photographic Determination of the Time-relations of the Changes which take place in Muscle during the period of so-called Latent Stimulation," ('Proceedings,' vol. 48, p. 14), will serve to show how this is done. In this particular case, as the curve was obtained for the purpose of analyzing the physiological photographs which we had taken, the circuit was led through the non-polarizable electrodes and the muscle exactly as it had been arranged for those experiments. An equivalent metallic resistance was then substituted for the preparation, and several other photographs of the normal excursion were secured. But the resulting curves were found to give identical results and, therefore, the details of this one only are inserted here.

First, the ordinates corresponding to time-intervals of ·001 sec. were measured with an ivory rule, graduated very finely in millimetres, the tenth of a millimetre being estimated with the aid of a lens. As a check, three different reference lines were taken on successive days, and the results so obtained were found to be identical.

The next stage of the operation consisted in the determination of the asymptotic circle. In so short a period as that comprised between the passage across the slit of an eight-inch plate, the mercury has not time to reach its final position, which must therefore be found by calculation. Assuming for the time-relations of the excursion the simplest formula, given in my preliminary note, viz.:—

$$y = ae^{-ct},$$

we may write for the distances from any reference circle of three points corresponding to equal intervals of time,

$$\mathrm{A} = n + y,$$

$$\mathrm{B} = n + \mathrm{T}y,$$

$$\mathrm{C} = n + \mathrm{T}^2y,$$

where $n =$ the distance of the reference circle from the asymptotic circle, and T is some function of t.

Whence

$$\frac{\mathrm{C}-\mathrm{B}}{\mathrm{B}-\mathrm{A}} = \mathrm{T};$$

$$\frac{\mathrm{B}-\mathrm{A}}{\mathrm{T}-1} = y;$$

$$\mathrm{A} - y = n.$$

Selecting three points as far apart as possible, I found the position of the asymptotic circle and reduced the observed readings of the ordinates to the values of y measured from it, which were as follows:—

ANALYSIS of Normal Curve No. 289, fig. 5, Plate 4. Upward excursion equivalent to the difference of potential given by 83 centims. of the rheochord wire with 1 Callaud cell. Resistance in circuit, a gastrocnemius preparation with non-polarizable electrodes. Total length of the excursion = 31·7 millims.

Found.	Calculated.	Difference.
millims.	millims.	millim.
*$y =$ 28·000	28·800	
24·900	24·855	+ ·045
21·400	21·451	− ·051
18·500	18·513	− ·013
16·000	15·978	+ ·022
13·800	13·789	+ ·011
11·900	11·901	− ·001
10·200	10·271	− ·071
8·900	8·864	+ ·036
7·650	7·650	

The column of calculated values is a series in geometrical progression, inserted between the first and the last terms. It will be seen that the error is in all cases well within the limit of accuracy of the measurements.

Having thus determined the position of the asymptotic circle, *i.e.*, the position of the meniscus when $y = 0$, on the assumption that the formula $y = ae^{-ct}$ was exact, the next step was to try whether any traces could be discovered of the influence of change of resistance, or of the sensitiveness of the capillary, during the excursion. In other words, I had to determine whether these errors were due to inaccuracies of measurement or of the formula assigned to the curve. If the latter were the case, then there must be some value of n which would make T increase or decrease with some approach to regularity from one end of the curve to the other, and we should have approximately

$$\log(n + y_1) - \log(n + y_2) = \log \mathrm{T},$$
$$\log(n + y_2) - \log(n + y_3) = \log(\mathrm{T} + b),$$
$$\log(n + y_3) - \log(n + y_4) = \log(\mathrm{T} + 2b),$$

and so on. On assuming trial values for n, however, the discrepancies seemed only to increase, and, after a somewhat laborious investigation, I concluded that in this particular instrument the effect of diminished resistance was practically balanced by a slight increase of sensitiveness towards the tip of the capillary—the one tending to increase, and the other to diminish, the velocity of the meniscus, so that the equation $y = ae^{-ct}$ represents very closely the time-relations of the excursion. The process

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* These ten points correspond to intervals of ·01 sec., as given by the time-tracing. That is why the first one is some 3 millims. from the commencement of the curve.

above described is necessary with every new electrometer, in order to determine its constants. When the equation of the curve has thus been determined once for all the process is much more simple. The position of the asymptotic circle is found by direct measurement upon a screen placed in the position of the sensitive plate, and a single normal excursion of known value is taken through the resistance used in the experiments to which the method of analysis is to be applied. Two or three ordinates with their corresponding subnormals are then measured, and the ratio of each ordinate to its subnormal is determined.

The mean of these ratios is taken as the value of the constant multiplier.

Measurement of the Subnormal of Normal Curve No. 289.

The curve was spoilt for this purpose by the presence of small undulations in two places caused by some jar to the apparatus; these only interfered with the measurement of y at one point, where, however, it was easy to take the mean position between them. But in order accurately to place the fine line upon the glass plate of the measuring rod as a tangent to the curve, it was necessary that this should be perfect for some little distance on each side of the point of contact. The two most favourable positions were near the beginning and the end of the curve, and these gave for the value of C respectively

$$C = 8{\cdot}50,$$

$$C = 8{\cdot}51.$$

Two other positions, not so well situated, gave $C = 8{\cdot}40$ and $C = 8{\cdot}43$. The remaining readings were less reliable on account of the undulations referred to, and a short piece where the definition of the photograph was defective. The mean of the four measurements is $C = 8{\cdot}46$; but 8·50 is probably nearer the true reading.

With this instrument, therefore, a difference of potential due to 83 centims. of the rheochord wire gave an excursion of 31·7 millims. on the sensitive plate. With a resistance in circuit equal to that of an ordinary physiological preparation, the subnormal to the curve at its commencement was 26·945 centims.

Whence

$$1 \text{ centim. on the subnormal} = \frac{83}{26{\cdot}945} \text{ centims.} = 30{\cdot}8035 \text{ on the rheochord,}$$

and

$$1 \text{ centim. on the rheochord} = \frac{26{\cdot}945}{83} \text{ centims.} = {\cdot}32464 \text{ on the subnormal.}$$

Comparison with other Normal Excursions.

In order to show that the method may be relied on to give constant results, the following experiments are quoted. The circuit was the same as for Curve No. 289, and the value of each excursion, as measured by its subnormal, may be compared with that obtained by calculation.

No. 290.—Rider of the rheochord set at 67 centims. Muscle-current = 42 centims., giving an effective difference of potential of 25 centims. Subnormal at the commencement of the excursion = 8·19 centims. Calculated

83 : 25 : : 26·945 : 8·12.

No. 292.—The difference of potential was simply that due to the muscle-current, which had fallen to 37 centims. The excursion was therefore in the opposite direction. Subnormal at the commencement of the excursion = 11·955 centims. Calculated

83 : 37 : : 26·945 : 12·01.

These results agree fairly well. For greater accuracy, the actual measurement of the subnormal was made in all cases at a point from 2 millims. to 5 millims. from the commencement, and the difference of the ordinates of the measured point and of the zero line, multiplied by the constant factor 8·5, added to the result. Hence the number of places of decimals. (See Part II., p. 101.)

The muscle was then removed from the circuit, and the following experiments made.

No. 294.—Upward excursion (towards the tip of the capillary) given by 80 centims. of the rheochord wire. No external resistance. Subnormal at the commencement of the curve = 28·2 centims.

No. 295.—Upward excursion given by 80 centims. of the rheochord wire. External resistance = 1000 ohms. Subnormal at the commencement of the curve = 28·04 centims.

No. 296. Downward excursion given by 80 centims. of the rheochord wire. External resistance = 1000 ohms. Subnormal at the commencement of the curve = 28·14 centims.

These results show the identical character of the upward and downward excursions, and also that the introduction into the circuit of 1000 ohms makes no very marked difference; they do, however, indicate the effect of taking away the resistance of the muscle and its electrodes, in that the value of the constant multiplier has risen from 8·5 to about 9·2.

Artificial Spikes.

This name was given to the curves obtained by photographing the excursions produced by means of two currents in opposite directions, each lasting about ·005 sec. It was intended to imitate as nearly as possible the effect produced by the excitation of muscle, with differences of potential of known value and duration, so as to determine whether the capillary electrometer was capable of distinguishing between a current of definite strength suddenly thrown into it, and a more or less gradual rise of a difference of potential, extending over a period of equal duration. That it can do so was shown beyond a doubt. To produce these artificial spikes, the wires from the electrometer were joined up through the derived circuits of two rheochords, A and B, each of which was provided with a separate battery. These were connected with a rheotome furnished with four adjustable break-keys of the tip-over type. The first of these, when struck, opened a short circuit on the primary of A, allowing the current to act upon that rheochord. The second broke the battery circuit of A, and at the same instant the third in like manner opened a short circuit on the primary of the second rheochord B, in which the current circulated in the opposite direction. About ·005 sec. later the battery of B was thrown out of action by the fourth key, thus checking the excursion produced by the second current. It will be observed that the electrometer circuit remained unbroken throughout the experiment.

Unfortunately the short-circuit contacts were not good, so that a certain amount of current passed, rendering it impossible to ascertain the exact value of the differences of potential which acted upon the electrometer, in several cases—and, moreover, the Daniell's cells used were not sufficiently steady during the operations, which took a long time, to enable all the results to be compared with those previously obtained. The experiments were varied by reversing the direction of both currents, and sometimes omitting the second.

No. 306. Fig. 6, Plate 3.—"Artificial Spike," produced by two short currents of opposite direction. Up, ·0055 sec.; subnormal = 50 centims. Down, ·0045 sec.; subnormal = 60 centims.

It will be observed that the effect of the second current was to bring the meniscus back almost exactly to its former level, and that both the rise and fall commenced and ended suddenly, indicating, as do also the measurements of the subnormal at close intervals throughout the curve, that it was produced by the introduction of differences of potential which remained constant while they lasted. But the movement was in each case only about one-tenth of the full excursion for that difference of potential, and the quantity of electricity received as "charge," viewing the electrometer as a condenser, is approximately proportionate to the product of the time into the difference of potential, or what comes to the same thing, the subnormal, *i.e.*:—

$$\cdot 0055 \times 50 = \cdot 275 \text{ for the upward movement.}$$
$$\cdot 0045 \times 60 = \cdot 270 \text{ for the downward movement.}$$

But these "charges" may be measured in another way. The electrical capacity of an electrometer is perfectly definite in amount, being conditioned by the shape and size of the capillary, and perfectly independent of the difference of potential. The quantity received by an instrument from any current is therefore proportional to the product of its capacity (which is constant throughout the part of the capillary used) into the distance l, through which the meniscus is moved by it. The effect of two successive charges of equal quantity, but opposite in sign, must be to leave the mercury ultimately at zero, as was the case here. Not only was there this close correspondence between the values obtained in these two ways for the "charge," but the Total Indicated Differences of Potential, calculated from the subnormals, were within one per cent. of the values found by comparing the two rheochords employed, with that used for Curve No. 289.

The following curves are given as examples of the effects of overshooting. The actual difference of potential employed was not ascertained owing to the instrumental difficulties referred to, which were not noticed till the conclusion of the experiments.

No. 305. Fig. 7, Plate 3.—Artificial spike, given by an upward excursion lasting 0·0052 sec., followed by a downward excursion lasting 0·004 sec.

The second phase is slightly rounded at the beginning, showing that the full velocity was not attained immediately. The slight pause at the end of the fall, before the slow return to zero, is probably due to "stickiness" of the tube. (Compare also No. 299 where there is evidence of "stickiness" during the swiftest part of the movement).

No. 307. Fig. 8, Plate 3.—Artificial spike, as before, but in the reverse direction.

In this case, the second phase begins sharply at its full velocity, but at the end of it the mercury has continued to move, not so rapidly as when the current was on, but still, more quickly than it would have done had the electrometer been simply discharging itself.

No. 309. Fig. 9, Plate 3.—Excursion produced by a single current lasting 0·006 sec.

Here the meniscus overshot the mark to the extent of 0·7 millim. in an excursion, the full height of which would have measured 80 millims. on the photograph, and which had only risen 7 millims. when the current was checked. The analysis shows that the second rheochord somehow became connected and furnished a current in the opposite direction of about one-fourth of the difference of potential of the first—possibly one of the leads might have touched the rheotome.

It will be noticed that even this overshooting is under 1 per cent. of the total normal excursion, and that the time which elapsed between the withdrawal of the electromotive force and the cessation of the movement, was only about 0·002 sec.

These three photographs illustrate the form of overshooting, which I consider to be due to the elasticity of the meniscus, for, if it were true overshooting of the column, then the retardation of the return movement should be visible in the case of the reversed spike (No. 307, fig. 8); but this was only seen with much stronger currents. On the other hand, the sudden checking of an upward excursion, about three times the size of this, caused the meniscus to throw off a globule of mercury.

In the preceding experiments, there was no external resistance. The effect of introducing a coil of 10,000 ohms is shown in the next example.

No. 299. Fig. 10, Plate 3.—Downward excursion produced by a single current of short duration. External resistance = 10,000 ohms.

In spite of the difference of potential being greater than in any of the preceding experiments, there is practically no overshooting, and the curve of discharge is perfectly normal. The subnormal to the curve was too great to be measured on the machine. The slight notch about the middle of the descent, indicates a "sticky" place in the capillary. (Compare fig. 7.)*

A companion photograph, in which the excursion was upward, did not develop sufficiently dense to print, but was similar in character, the upward overshooting being very slight.

These photographs show, that *without external resistance in the circuit,* the effect of overshooting, and especially of that form of it which is due to the elasticity of the meniscus, interferes seriously with the application of this method of analysis, unless the differences of potential concerned are small, but that *these difficulties vanish on the introduction of a moderate resistance,* not sufficient to make the movements of the electrometer sluggish. With suitable precautions, the variations of a difference of potential may be determined for each two-thousandth part of a second, and under favourable circumstances, with an error of not more than one per cent.

PART II.

Application of the Method to the Study of the Electrical Variations of Muscle.

In a paper by Professor BURDON SANDERSON, entitled "Photographic Determination of the Time-relations of the Changes which take place in Muscle during the Period of so-called 'Latent Stimulation'" ('Roy. Soc. Proc.,' vol. 48, p. 14), it was stated that in the gastrocnemius of the Frog the electrical response to an instantaneous stimulus, as investigated with the aid of the capillary electrometer, is indicated by a sudden movement of the mercurial column of very short duration, and that the photographic expression of that movement shows that between the contacts two electrical changes of opposite sign, and not more than one two-hundredth of a second in duration, have immediately followed each other, or, more explicitly, that the spot excited became for about 0·005 sec. first negative, then for a similar period positive, to the other contact.

This statement relates exclusively to the case in which the effect is led off to the electrometer by two electrodes of which one (f) is placed on the tendon, and the other (m) is on the belly of the muscle. In this case, the photographic record of the electrical response to a single excitation of the nerve, resembles the record obtained, when in a circuit of the same resistance as the muscle, two currents of the same duration follow one another in opposite directions.

This resemblance between the two cases is, however, only general, for whereas in the artificial effect the difference of potential between the two terminals of the electrometer remains the same during each phase, changing sign at the moment that the first is followed by the second, the difference of potential between the two spots to

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* [This notch has not been reproduced in the figure. It can easily be seen in the negative even without a lens. March 3, 1892. G. J. B.]

which the electrodes are applied gradually rises to a negative maximum during the first phase, is reversed, and then, having become positive, gradually subsides.

My present purpose is to show in what way the method of analysis described can be applied to the photographic records of the electrical response (which I shall hereafter speak of as the record or curve), so as to determine the exact time-relations of the changes above described.

With this view I propose to present to the Society certain specimen records, and to state the results obtained by the analysis of each. This result will be best expressed by a graphic denoting the actual differences of potential between (m) and (f) at successive times during the period of electrical change.

It may be convenient at this point to describe in detail the method of analyzing such records as muscle curves, in which the electrical variations are of brief duration and not necessarily of constant intensity or of the same sign.

It has been shown in the first part of this paper that it is not difficult to procure an electrometer of which the time-relations may be expressed by the formula $y = ae^{-ct}$, and that, with a circuit of suitable resistance, the movement of the meniscus will commence the instant that a difference of potential is communicated to the terminals, and cease the instant it is withdrawn. With such an instrument under such conditions, as was stated in my preliminary note "On a Method of Determining the Value of Rapid Variations of a Difference of Potential by means of the Capillary Electrometer" ('Roy. Soc. Proc.,' vol. 48, p. 92), "the velocity with which the meniscus is moving at any instant is that with which it would start if the zero-line were moved to the level then occupied by the meniscus, and the difference of potential existing at the time between the terminals of the electrometer were suddenly introduced and made permanent. Thus, the *total indicated difference of potential* is made up of two parts, viz., the difference represented by the distance through which the meniscus has already moved, and that indicated by the velocity with which it is still moving."

In the analysis of muscle curves, as will be seen, the latter is usually many times greater than the former, except at the beginning, middle, and end of the curve where the tangent is at right angles to the radius vector, and the subnormal is zero.

The method of analysis set forth in my preliminary note is, in consequence of the change in the apparatus already referred to, whereby the sensitive plate is made to describe an arc of the circle instead of travelling in a straight line, superseded by the far simpler one given in the present paper. The subnormal to the normal curve—and, consequently, to any curve—is a constant multiple of y, the distance of the meniscus from the point of rest corresponding to the difference of potential existing at the time between the terminals of the electrometer.

As has been already said, the value of this constant multiplier is influenced by the resistance of the circuit in each experiment, and the first step is to determine it for the particular resistance used. This was done for the muscle curves of which the analysis is appended, the normal excursion photographed for the purpose being No. 289 (fig. 5), and the value of the constant multiplier was 8·5.

The next step is to place the glass negative of the muscle-curve upon the measuring instrument (fig. 3), carefully centreing it so as to ensure its occupying a position exactly corresponding to that in which it was placed on the pendulum of the photographic apparatus. If this is not done, the results will be falsified. The rate of movement has then to be determined. When, as in the cases given here, the angle subtended by the entire excursion is small, no appreciable error is introduced by employing a tangent scale instead of measuring the arc. This being so, a very simple method is available, whereby the hundredths of a second, given by the time-record on the photograph, can be subdivided into thousandths of a second. The index of the carrier (fig. 3), is a straight edge fixed radially, and the tangent scale, which is divided into millimetres, is placed at such a distance from the centre *O*, that ten of its divisions shall be passed over by the index when the carrier is shifted through a space corresponding to one-hundredth of a second, as indicated by the coincidence of two successive apices of the time-record on the photograph with the radial line *DO*.

The scale having been fixed in this position, the actual measurements are proceeded with. The instant of excitation, marked by the upward movement of the signal arm, is taken as the zero of time. This portion of the photograph is brought over the line *DO*, and the reading of the tangent scale is noted. The distance of the electrometer record at that time from the circular edge of the photograph is measured; this gives the position of zero potential. The next thing is to find the time at which the meniscus began to move. For this purpose, the rod *E* is laid along the line *DO*, and the line *aa*, on the glass plate *F*, placed accurately as a tangent to the curve. The carrier is then shifted until a point is reached when the curve commences to rise, and the time at which this takes place is noted on the scale.

Next, the carrier is further shifted in the same direction through a space corresponding to 0·0005 sec.

There are now two operations to be performed :—

(*a*.) To measure the subnormal of the curve at this point. The method of doing this has been already described. The result may be written

$$\text{Subnormal} = cy.$$

(*b*.) To measure how far the meniscus has moved from its original zero position for $t = 0$.

This is done by taking the distance of the curve at this point from the edge of the photograph, which is used as a reference circle, and subtracting the result from the zero distance already ascertained. The difference may be written

$$\text{Movement} = l.$$

It is most convenient to express the total indicated difference of potential in terms of the subnormal—reducing it subsequently to decimals of a volt. The total indicated difference of potential will therefore be

$$\pm cy \pm cl.$$

Attention must be paid to the *sign of both y and l.* In the examples given, an upward excursion is produced by the negative variation, so that the formula for the first phase will be

$$-cy - 8{\cdot}5l.$$

At the apex the subnormal passes through zero, but the total indicated potential is not yet zero, being indicated truly by the distance of the meniscus from the zero line, *i.e.*, the actual height of the excursion. This, expressed on the same scale as before, is

$$-8{\cdot}5l.$$

During the second phase, or descending portion of the curve, the subnormal is positive, and we must write

$$+cy - 8{\cdot}5l.$$

If, as sometimes happens, the meniscus should descend below the zero line, l also becomes positive, and we have

$$+cy + 8{\cdot}5l.$$

It is unnecessary to describe minutely the arrangement of the physiological preparations. The photographic curves of which the analysis is given are some of those obtained in the experiments in which I assisted Professor BURDON SANDERSON, to whose kindness I am indebted for permission to make use of them in this, the first application of the method of analysis which this paper describes.

Non-polarizable electrodes were employed, one (f) placed on the tendon, and the other (m) on the belly of the muscle, unless otherwise stated. The muscle was kept stretched by a weight, not greater than could be lifted by a moderate contraction. The exciting electrodes were applied to the nerve, the exact instant of the stimulus being marked by the upward movement of the arm of the signal-key, the shadow of which is recorded on the plate. This key, which has been already referred to, was placed in the primary circuit of the induction-coil, and by its use, all possibility of delay in registering the instant of the induction shock was eliminated, save that due to the induction coil itself.

No. 287. Fig. 11, *a*, Plate 4.—Gastrocnemius of *R. esculenta.* Exciting electrodes placed at the extreme end of the prepared nerve, as far as possible from the muscle.

The muscle current was balanced by the difference of potential derived from 42 centims. of the wire of the rheochord. The analysis of this curve is given in fig. 11. The excursion is not large, but the negative variation is sudden, and remains tolerably constant for some time. The positive variation is not great, but the return to the original zero is slow.

The relative value of the muscle current is shown by the dotted line, which represents the true zero potential.

Immediately after this experiment, without altering the electrometer circuit, the normal excursion, No. 289, was photographed for the determination of the constant multiplier used in this and the remaining illustrations.

No. 270. Fig. 12, *a*, Plate 5.—Gastrocnemius. Exciting electrodes as far as possible from the muscle.

No. 271. Fig. 12, *b*.—The same muscle. Exciting electrodes as close as possible to the muscle.

No. 272. Fig. 12, *c*.—The same muscle. Exciting electrodes as close as possible to the muscle as in No. 271, but the leading-off electrode (*m*) shifted from its previous position to a point 12 millims. nearer the tendon electrode (*f*).

These three curves were analyzed, and the results plotted together in fig. 12, Plate 5.

It will be observed that the two former, though somewhat different in the first phase, are fairly alike in the second, the delay of the electrical response, in consequence of the greater length of nerve through which the excitation was transmitted in the case of No. 270, being well shown.

It is interesting to note that the analysis of No. 272 approaches more nearly in shape to that of an "artificial spike." The rise of the electromotive force is fairly rapid, and it remains nearly constant for a considerable part of each phase. But neither the rise nor the fall is so sudden as in the case of Curve No. 306, although, as the difference of potential is less and the resistance of the circuit greater, any error due to overshooting of either kind is quite out of the question.

No. 274. Fig. 13, *a*.—Gastrocnemius. Exciting electrodes placed on the nerve, close to its insertion in the muscle. Leading-off electrode (*f*) attached to the tendon; electrode (*m*) on the side of the muscle, nearly at its thickest part.

No. 276. Fig. 13, *b*.—The same muscle. Exciting electrodes as before. Leading-off electrode (*m*) shifted into the normal position on the belly of the muscle.

No. 277. Fig. 13, *c*.—The same muscle. Exciting electrodes as before. Leading-off electrode (*m*) placed so close to the tendon electrode (*f*) as to give a scarcely visible excursion of the meniscus upon excitation.

The analyses of these three curves were plotted together in fig. 13, Plate 6.

They all cross the zero line within a remarkably small space. The first two differ slightly in the negative phase, but are tolerably similar in the positive, which is well developed. In both there is a second rise of the positive variation, reaching a maximum about 0·023 sec. after the excitation. This rise is, however, too small to be discernible from the photograph, and is made manifest only by the method of analysis.

It does not come within my province to discuss the physiological questions arising out of these results, which I desire to leave entirely in the hands of Professor BURDON SANDERSON, to whom I am deeply indebted for permission to make use of these photographs, taken in the course of the research in which I had the honour of assisting him, in order to illustrate the method of analysis which I now bring forward.

My sole object has been to show that the capillary electrometer can be employed to obtain measurements which hitherto could only be effected by a process open to some objections.

The repeating rheotome, introduced by Professor BERNSTEIN for the same purpose, although it has many advantages, can only give the average time-relations of a number of electrical variations due to a rapid succession of stimuli, and is inapplicable in any case in which changes may occur in the preparation from fatigue. Moreover, my experience of rheotomes leads me to believe that it is much more difficult than is sometimes supposed to obtain very short closures of constant duration. The photographic records given by the capillary electrometer are free from both these objections. They show the result of a single stimulus or of a number in rapid succession, and the circuit remains unbroken and therefore constant.

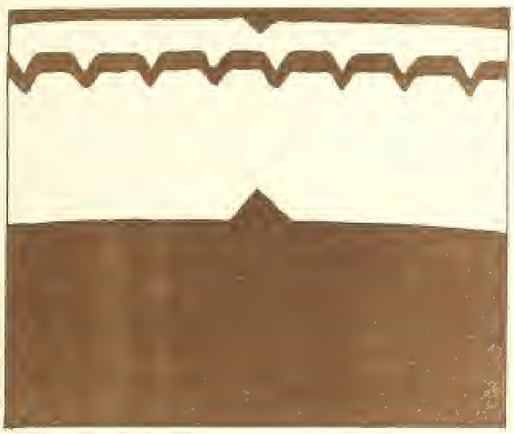
Hitherto no means by which these indications could be quantitatively interpreted, except in the case of slow changes, has been in the hands of experimenters. I have endeavoured to supply this lack, and in doing so, to represent the capillary electrometer in its true light as a measuring instrument, not indeed equal to the galvanometer in delicacy, nor in durability, nor, perhaps, in accuracy; but far excelling it and all other electrical measuring instruments in the one property which is peculiar to it, namely, rapidity of action.

Probably in this way the electrometer may be made to serve other purposes than those of physiological research; and I foresee that this method of analysis may be applicable to other instruments than the electrometer.

All the photographs and many of the experiments described in this paper were made in the Physiological Laboratory, Oxford, and my thanks are due to Professor Burdon Sanderson for his help and counsel, without which I could not have completed the research. I have also to thank Mr. C. Crump, of the Record Office, for his valuable assistance in discussing the mathematical parts of the problem. The complete method of applying the corrections for calibration error and change of resistance I had hoped to give with his aid, but it was found that some further experiments would be necessary to furnish the requisite data. The treatment of this part of the subject therefore must be deferred. In the instrument which was used for nearly all the photographs these errors were not sensible.

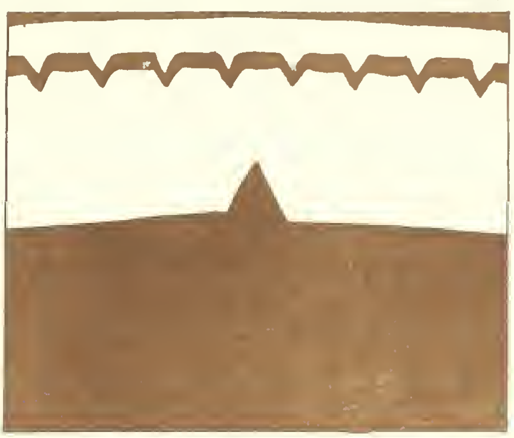
[Note.—The curves were analysed by measuring the negatives, and the results plotted from left to right. The photographs having been reproduced as prints, must be read in the reverse direction, from right to left. March 3, 1892. G. J. B.]

Fig 6.



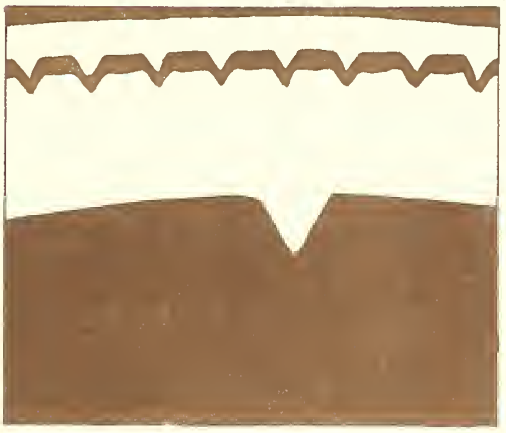
No. 306.

Fig 7.



No 305.

Fig 8.

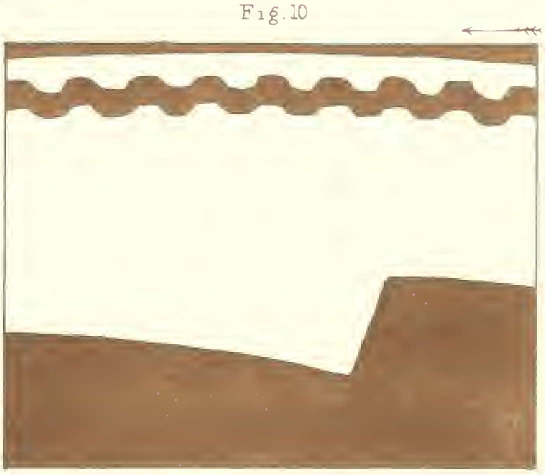


No 307.

Fig 9.

No. 309.

Fig. 10.

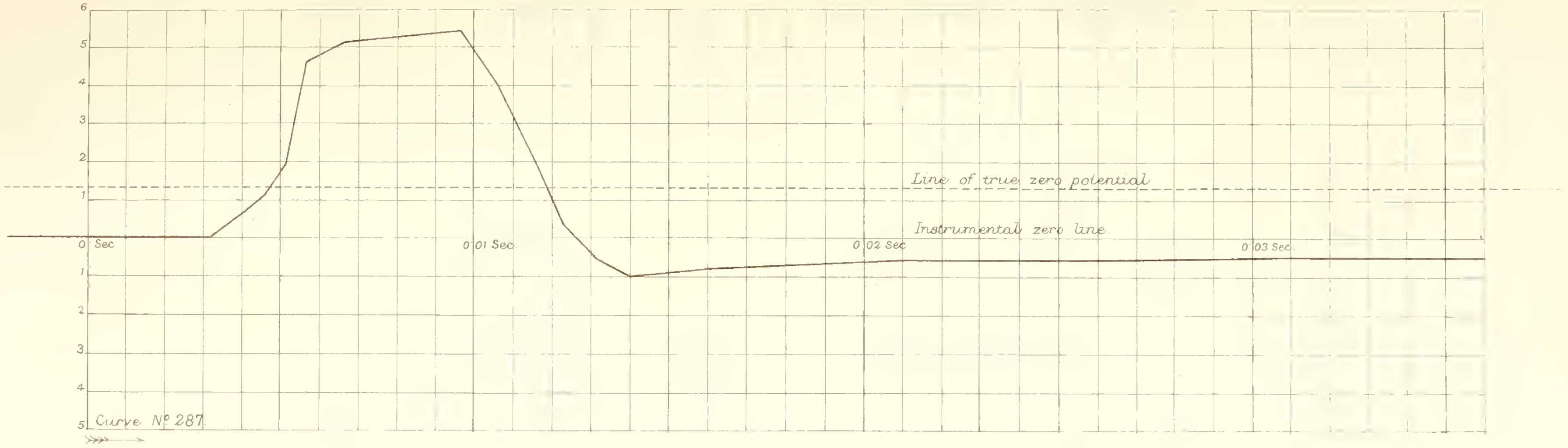


No 299.

West, Newman lith.

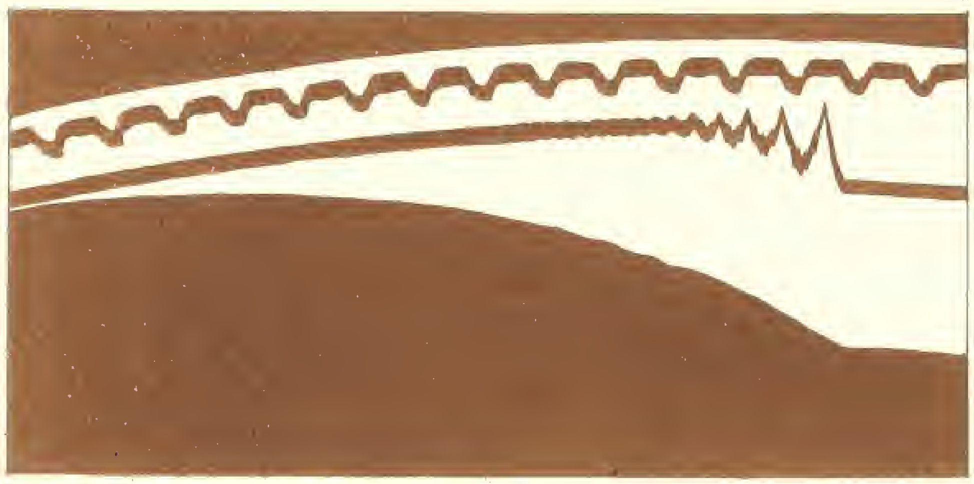
Fig. 11

(Vertical scale reduced to $\frac{1}{2}$ that of Fig. 11, a.)



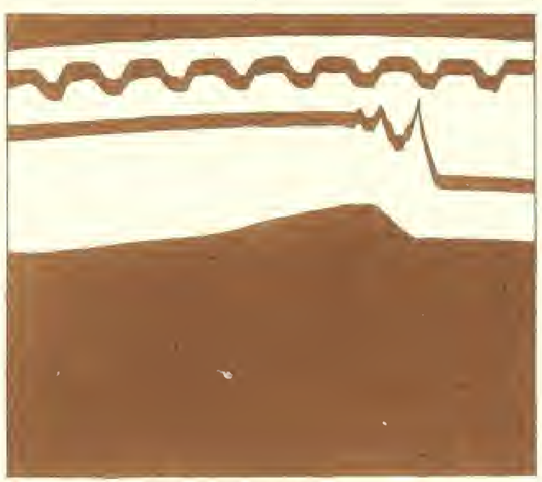
N.B. Fig 11, must be read from left to right, but Figs. 5 and 11, a, from right to left.

Fig. 5



No. 289

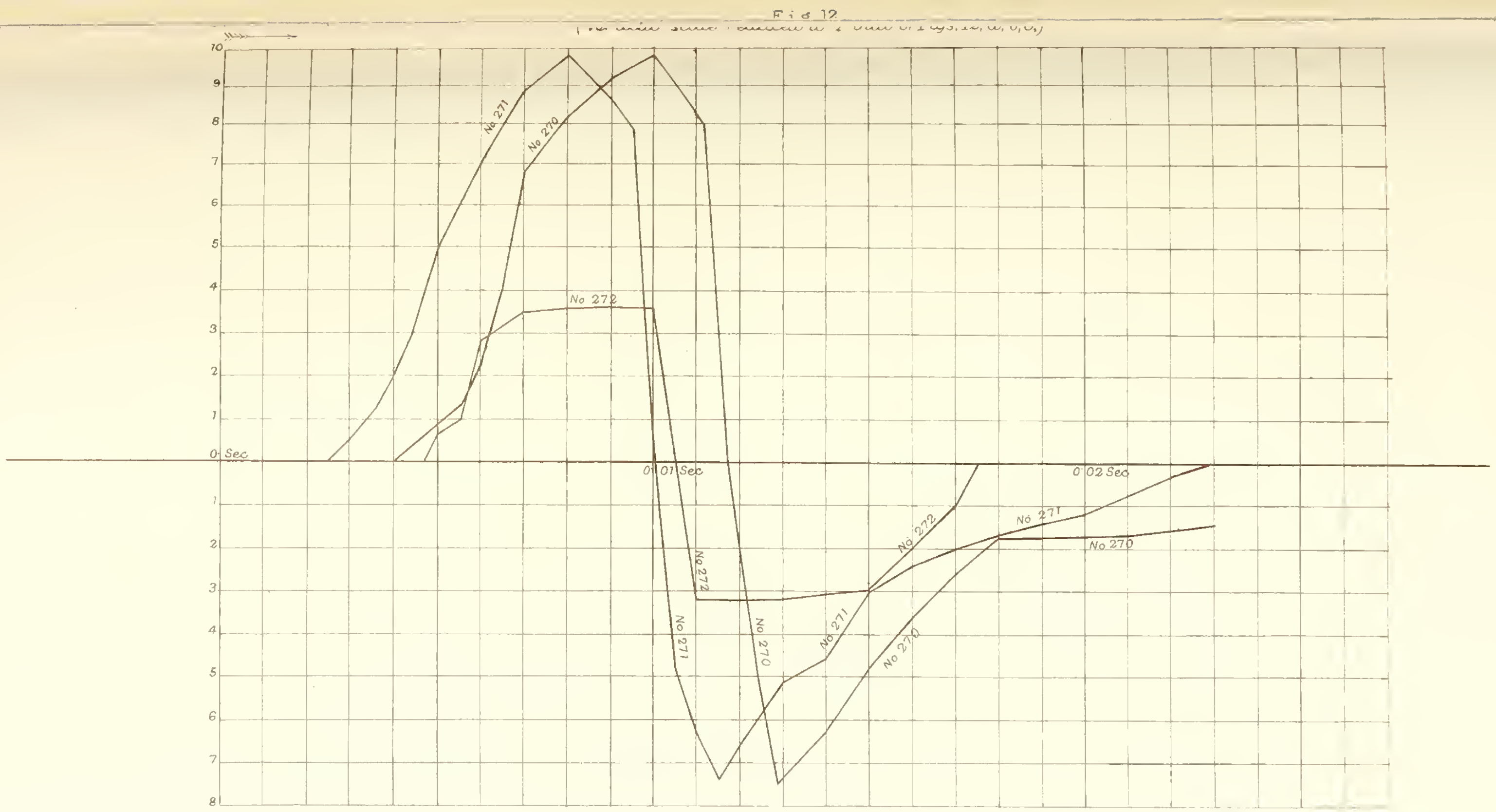
Fig 11a



No 287

West, Newman, lith.

Fig 12.



N.B. Fig 12 must be read from left to right, but Figs. 12, a, b, c, from right to left

Fig. 12 a.

Fig. 12 b.

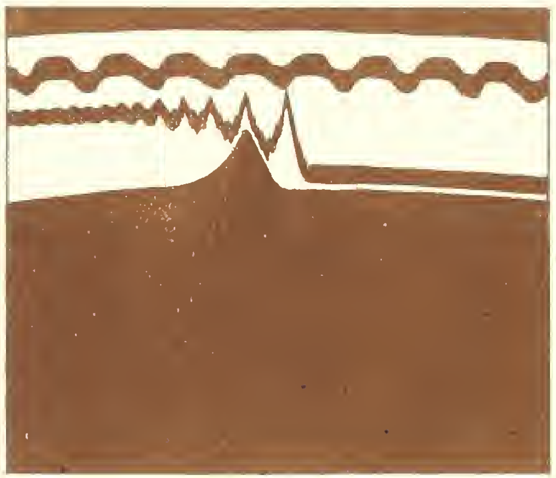
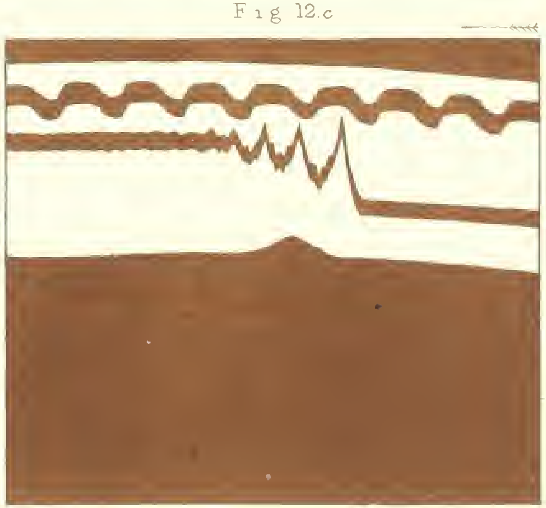


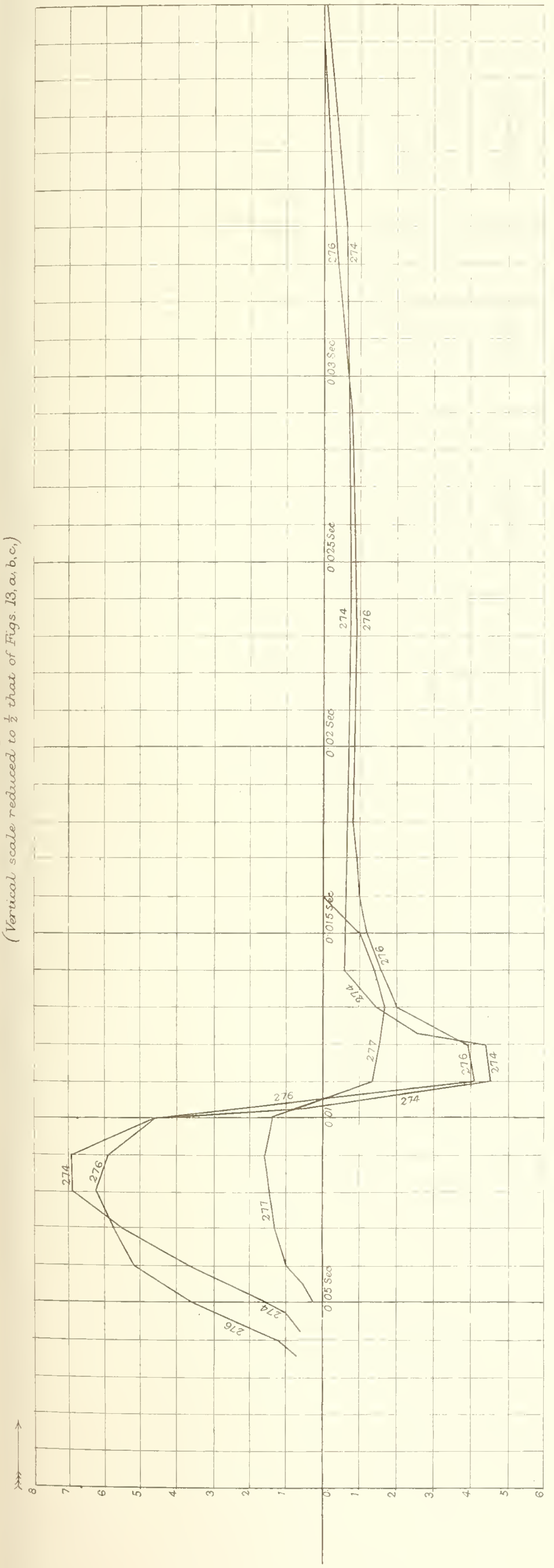
Fig. 12 c.



West Newman, lith.

Fig. 13.

(*Vertical scale reduced to ½ that of Figs 13, a, b, c,*)



N B Fig 13 must be read from left to right, but Figs 13, a, b, c from right to left

Fig 13 c

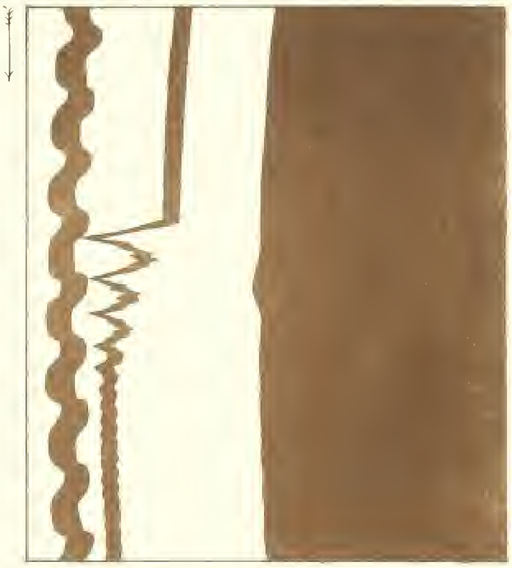
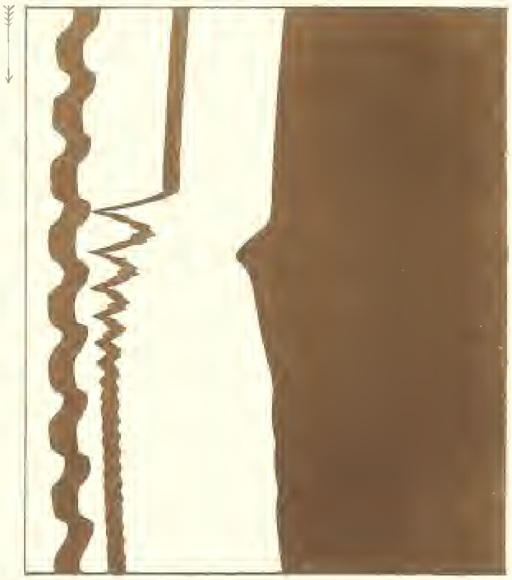


Fig 13 b



No 276

Fig 13 a

No 274

West Newman, lith

III. *On some of the Properties of Water and of Steam.*

By William Ramsay, *Ph.D.*, *F.R.S.*, *and* Sydney Young, *D.Sc.*

Received November 5,—Read December 10, 1891.

[Plate 7.]

The work of which an account is given in the following pages was carried out in the winter of 1887–88, and forms a sequel to much of a similar character already published by the authors, on methyl, ethyl, and isopropyl alcohols, on ethyl oxide, on acetic acid, and on a mixture of alcohol and ether ('Phil. Trans.,' 1886, Part I., p. 123 ; 1887, A., p. 57 ; 1887, A., p. 313 ; 1889, A., p. 137 ; 'Chem. Soc. Trans.,' 1886, p. 790 ; 1887, p. 755).

The apparatus with which it was carried out has been fully described in the memoir on ether (*loc. cit.*). The only important change was in the nature of the glass tubes used to contain the liquid under experiment. While, in the case of the alcohols, ether, &c., lead glass proved the best material for tubes, it is too easily attacked by water. At comparatively low temperatures it becomes etched, and it is impossible to take readings. Moreover, the water dissolves a not inconsiderable quantity of potassium silicate ; tubes of green boiler-gauge glass were, therefore, substituted for tubes of lead glass. This glass consists of a silicate of calcium and potassium, containing a trace of ferrous iron ($SiO_2 = 71·20$; $CaO = 14·99$; $K_2O = 13·19$; Total, 99·38.) They withstand a higher pressure than lead-glass tubes, and they are attacked with difficulty. Even after an exposure of several days to liquid water at 280° C., only 0·7 per cent. of residue remained on evaporating the water. Inasmuch as *some* of the material of the glass is dissolved, however, the water cannot be considered to be absolutely pure, but the results may be given as the best attainable with water in contact with glass.

It may be advisable to state here again, that the *pressures* were read from gauges containing dry air, kept at a known temperature by jackets of running water ; and that they are corrected for deviation from Boyle's law by help of experiments made by Amagat ('Compt. Rend.,' vol. 99, p. 1153) ; that the *temperatures* are those of an air-thermometer, and that they were secured by jacketing the tube containing the water with the vapours of pure chlorobenzene, bromobenzene, aniline, methyl-salicylate, or bromonaphthalene, of which the vapour-pressures corresponding to known temperatures have been determined by us ('Chem. Soc. Trans.,' vol. 47, p. 640). By causing one or other of these liquids to boil under definite known pressures, any desired temperature, within certain limits, was attainable; and, as a small alteration of temperature corresponds to a considerable alteration of pressure, it may be taken for certain that the temperatures are practically correct. The expansion of the glass made use of for the volume tube was carefully determined, and a correction was in all cases applied, so that the apparent volumes are reduced to real volumes. No correction, however, was applied for the alteration in the volume tube due to internal pressure, because an estimation of its amount showed it to be much within the error of reading volume.

The results may be classified under three heads :—

I. Expansion of liquid water.
II. Vapour-pressures.
III Density of unsaturated and saturated steam.

I. *Expansion of Liquid Water.*

For these experiments, the tube was two-thirds filled with water freed from air by boiling *in vacuo*. That no air was present was often proved during the experiments by condensing a bubble of steam; condensation always took place without appreciable rise of pressure. The weight of water taken was deduced from observations of its volume at the ordinary temperature, both before and after experiment. The results are as follows :—

Temperature.	Divisions of tube.	Volume at 4 and weight.
°		c.c. or grm.
12·25	246·75	1·1054
12·40	246·90	1·1060
15·7	246·20	1·1031
17·9	246·25	1·1035

The first pair of readings was taken before the commencement of the experiments; the second pair after the expansions had been measured. The small difference is probably due to adherence of liquid to the lower portions of the tube, which had become etched during the experiments.

The mean result, from which the following Table is calculated, is 1·1045 c.c. at 4°, or 1·1045 grm :—

Temperature.	Volume of 1 grm. in c.cs.		Temperature.	Volume of 1 grm. in c.cs.	
	Observed.	Smoothed.		Observed.	Smoothed.
°			°		
100	1·0432	1·0432	190	1·1412	1·1407
110	1·0510	1·0513	200	1·1567	1·1566
120	1·0594	1·0592	210	1·1734	1·1733
130	1·0678	1·0678	220	1·1915	1·1914
140	1·0791	1·0772	230	1·2122	1·2112
150	1·0880	1·0878	240	1·2319	1·2320
160	1·0996	1·0995	250	1·2516	1·2533
170	1·1124	1·1124	260	1·2750	1·2755
180	1·1260	1·1260	270	1·3000	1·2985

NOTE.—The pressures are here nearly those of the vapour at the respective temperatures.

The volumes of 1 grm. of water have been measured by MENDELÉEFF ('LIEBIG'S Annalen,' vol. 119, 1861, p. 10), and by HIRN ('Annales de Chimie' [4], vol. 10, 1866, p. 32). The following short Table shows that HIRN'S results agree very closely with those given by us :—

Temperature.	Volume of 1 grm.		
	MENDELÉEFF.	HIRN.	R. and Y.
°			
100	1·0428	1·04315	1·0432
120	..	1·05992	1·0594
131	1·0722	..	..
140	..	1·07949	1·0791
156·8	1·1016	..	..
160	..	1·10149	1·0996
180	..	1·12678	1·1260
200	..	1·15900	1·1567

The compressibility of water was measured at 190° and at higher temperatures. The results are as follows :—

Temperature.	Volume of 1 grm.	Pressures of mercury.	Temperature.	Volume of 1 grm.	Pressures of mercury.
	cub. centims.	millims.		cub. centims.	millims.
190°	1·1412	9,837	230°	1·2122	21,085
	1·1408	12,813		1·2101	31,745
	1·1386	30,029		1·2079	42,249
	1·1365	39,904			
			240	1·2319	25,276
200	1·1567	15,014		1·2300	32,452
	1·1559	18,695		1·2282	41,614
	1·1538	30,513			
	1·1516	43,901	250	1·2516	30,059
				1·2508	36,040
210	1·1734	14,404		1·2490	43,925
	1·1732	17,488			
	1·1710	31,467	260	1·2750	35,028
	1·1689	43,405		1·2735	40,210
220	1·1915	17,622			
	1·1905	27,251			
	1·1884	39,432			

II. *Vapour-pressures.*

The pressures of the vapour were read from experiments with a larger quantity of water in the tube, as well as when the quantity was small ; but uniform results were not obtained. The vapour-pressure of water, unlike that of other liquids, depends to some extent on the amount of substance present.* This is to be attributed to the fact, that with only a small amount of liquid present, a relatively large surface of the tube is exposed to the vapour, and, as there can be no doubt that the vapour adheres to the surface, the pressure is lowered to a certain extent. The results, with the larger quantity, are therefore given as probably more correct. Each pressure is the mean of some seven or eight readings in which the relative volumes of water and steam were altered within the greatest limits which the tube allowed.

The pressures are, as before stated, corrected for deviations from Boyle's law.

To facilitate comparison, the results obtained by Regnault and others are placed in juxtaposition with ours ; and also columns showing the differences between our results and those obtained in other ways. The columns are distinguished as follows :—

"R. and Y." stands for Ramsay and Young.

"Reg. curve" stands for Regnault's copper-plate curves, given in the 'Mémoires de l'Académie,' vol. 26.

"Reg. H." and "Reg. F." signify Regnault's results calculated by Biot's formula, $\log p = a + b\alpha^t + c\beta^t$, using the constants marked H and F respectively.

"Reg K." signifies Regnault's formula K, viz., $p = a\alpha^{x/(1+mx)}$

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* See Appendix, pp. 120 and 121.

"Unwin" ('Phil. Mag.,' 1886, p. 299). His formula is $\log p = 7{\cdot}5030^{-7{\cdot}579/t^{1{\cdot}25}}$.

"Antoine" refers to the last formula given by Ch. Antoine ('Annales de Chimie,' 1891, vol. 22,' p. 283). It is $\log p = 7{\cdot}921^{-1638/t+225}$. It has the merit of allowing temperature to be calculated from pressure, as easily as the converse; the results, however, calculated in each way, are not absolutely the same, but do not differ to any important extent.

"Buff" ('Liebig's Annalen,' Suppl. 2, 1862–3, p. 137) uses the formula

$$\log p = \log \frac{760}{373} + \log (273 + t) + \frac{\log (273 + t) - \log 373}{\alpha}.$$

$$\alpha = 0{\cdot}06479 + 0{\cdot}0001722\,t - 0{\cdot}0000001\,t^2.$$

Rankine" ('Steam Engine,' 9th Ed., p. 237) gives the formula

$$\log p = \mathrm{A} + \frac{\mathrm{B}}{\tau} + \frac{\mathrm{C}}{\tau^2}.$$

'Broch" ('Travaux et Mémoires du Bureau Internat. des Poids et Mesures,' vol. 1, p. 19, *et seq.*). His formula is

$$p = a \,.\, 10^{\,\frac{bt + ct^2 + dt^3 + et^4 + ft^5}{1 + \alpha t}}.$$

M. Broch has undertaken a most laborious investigation of the formulæ employed for the calculation of the vapour-pressures of water, adopting finally the one given in the table. The utmost care was taken to obtain the best possible results from the data taken; but, unfortunately, M. Broch has accepted Regnault's conclusion that the curve representing the vapour-pressures of ice is continuous with that obtained from the vapour-pressures of water, and he has employed the whole of the data from − 32° to 100°. By the method of calculation of the constants for the formula, this source of error has been to a certain extent eliminated, as is shown by the comparison (p. 31) of Regnault's individual observations with the pressures calculated from the formula. In this table it is seen that below − 5°, out of 31 comparisons, the calculated pressures are higher than those observed in every case but one. The error is, however, only partially eliminated, and this probably explains the fact that the formula, with the constants given, will not bear extrapolation above 100° even to 120°, though it is supposed to hold good through a range of 132°.

It is to be feared that the results between 0° and 100° may even be to some extent vitiated by this source of error—a most unfortunate circumstance, considering the enormous amount of labour bestowed on the work, and the fact that the pressures calculated from the formula have received the *imprimatur* of the Bureau International.

It is to be noticed that Regnault's formula K gives results nearest the truth, but that fairly approximate results are also obtainable by the use of Unwin's and of Antoine's formulæ. These formulæ, it need hardly be remarked, are all empirical.

TABLE of Vapour-pressures of Water.

T.	R. & Y.*	REG. Curve.	Δ_1	REG. H.	Δ_2	REG. F.	Δ_3	REG. K.	Δ_4	UNWIN.	Δ_5	AN-TOINE.	Δ_6	BUFF.	Δ_7	RAN-KINE.	Δ_8	BROCH.	Δ_9
120°	1,484	1,489	+ 5	1,491	+ 7	1,491	+ 7	1,487	+ 3	1,486	+ 2	1,491	+ 7	1,491	+ 7	1,491	+ 7	1,545	+ 61
130	2,019	2,029	10	2,030	11	2,030	11	2,023	4	2,020	1	2,030	11	2,029	10	2,030	11		
140	2,694	2,713	19	2,718	24	2,718	24	2,706	12	2,710	16	2,718	24	2,716	22	2,716	22		
150	3,568	3,572	4	3,581	13	3,582	14	3,564	− 4	3,564	− 4	3,581	13	3,578	10	3,578	10	4,244	+ 676
160	4,652	4,647	− 5	4,652	0	4,652	0	4,629	−23	4,633	− 19	4,652	0	4,646	− 6	4,647	− 5		
170	5,937	5,960	+21	5,962	25	5,959	22	5,933	− 4	5,942	+ 5	5,962	25	5,953	+ 16	5,955	+ 18		
180	7,487	7,545	67	7,546	68	7,537	59	7,513	+35	7,525	47	7,546	68	7,535	57	7,539	61		
190	9,403	9,428	25	9,443	40	9,425	22	9,407	4	9,423	20	9,443	40	9,429	26	9,432	29		
200	11,625	11,660	35	11,689	64	11,663	38	11,656	31	11,665	40	11,689	64	11,672	47	11,679	54	31,861	+20,237
210	14,240	14,308	68	14,325	85	14,297	57	14,302	62	14,316	76	14,325	85	14,310	70	14,319	79		
220	17,365	17,390	25	17,390	25	17,381	16	17,390	25	17,398	33	17,390	25	17,386	21	17,390	25		
230	20,936	20,915	−21	20,926	− 10	20,981	45	20,965	29	20,952	16	20,926	− 10	20,937	1	20,936	0		
240	25,019	..	..	24,973	− 46	25,176	157	25,074	55	25,046	27	25,027	+ 8	24,966	− 53	25,009	− 10		
250	29,734	..	..	29,571	−163	30,072	338	29,765	31	29,694	− 40	29,689	− 45	29,687	− 47	29,648	86		
260	35,059	..	..	34,757	−302	35,798	739	35,086	27	34,965	− 94	34,970	− 89	34,978	− 81	34,898	161		
270	41,101	..	..	40,570	−531	42,545	1444	41,088	−13	40,887	−214	40,926	−175	40,948	−153	40,794	307		

* Observed. The details of the observations are to be found in the Appendix, pp. 120 and 121.

III. *The Densities of the Saturated and Unsaturated Vapour of Water at Different Temperatures and Pressures.*

The method pursued was (1) to find the weight of the small quantity of water in the tube by ascertaining the values of the products of pressure and volume at large volumes and at high temperatures. If these products, for slight decrease of volume, and at any one temperature, were sensibly constant, and if the products at different temperatures bore to one another the same ratio as that between the absolute temperatures, it was assumed that the vapour-density of the steam was normal; if hydrogen be taken as unity, the density is nine. The weight could then be calculated by the equation given in the Appendix, p. 122. (2) The volume of the gas was then diminished by regular decrements at temperatures rising by 10° at a time, and the pressure corresponding to each volume was read. From these data the volume of 1 grm. in cubic centimetres could be calculated. When the volume was sufficiently diminished condensation ensued.

Two distinct sets of experiments were made: one with a large tube at low temperatures, and one with a strong pressure tube at high temperatures. The actual experimental data for high temperatures (120° to 270°) are given on pp. 124–127 of the Appendix; those at low temperatures on pp. 128–130. Reasons are there adduced to show that when the pressure approaches the vapour-pressure condensation on the glass occurs. This causes the pressure to be lower than it should be for a given volume, and renders the results very difficult to interpret. Such adhesion appears to be slight at large volumes where the vapour is unsaturated; but, as it occurs gradually, the effect is that on decreasing volume some water is left below the mercury, and as there is no clue to the amount thus trapped the weight is diminished by an unknown amount. But it is reckoned as undiminished, and hence in calculating the volume of 1 grm. from the actual measurements the volume ascribed is too small, and too small by an unknown quantity. At the same time the pressure is lowered by the adhesion to the glass, and so the curve is distorted as it approaches the pressure of the saturated vapour.

Under the circumstances it appeared best to follow the experimental results as closely as possible. However interesting from a theoretical point of view the absolute expansion of water-gas may be, in practice it is always in contact with a surface; and although it is probable that the material, as well as the extent of the surface, would influence the amount of adhesion, yet an indication of the behaviour of steam in contact with glass cannot fail to be of use in considering the practical case of steam in contact with iron. Moreover, the error is to a great extent eliminated by making use of the volumes of 1 grm. of saturated steam calculated from REGNAULT's heats of vaporization.

RANKINE ('Edinb. Trans.,' vol. 23, Part I., 1862, p. 147) discusses the want of concordance between the volumes of saturated steam observed by FAIRBAIRN and TATE, and those calculated from REGNAULT'S observations on the heats of vaporization; and suggests a difference in molecular condition between steam at rest, as in FAIRBAIRN and TATE'S experiments, and in motion, as in REGNAULT'S. It can scarcely be doubted that the true reason of the discrepancy is to be found in the adhesion of steam to glass. It is obvious that in REGNAULT'S experiments this source of error would be absent, inasmuch as the weight of water was simply that which entered the calorimeter.

Three sets of isothermal curves were drawn in which the experimental numbers were closely followed, the only adjustment being that of rendering the curves as smooth as possible, and of making use of the mutual relations between the curves, so as to smooth out the irregularities of each individual one. These curves were prolonged to the points calculated from REGNAULT'S observations by the formula

$$s_1 = \frac{LJ}{T\, dp/dt} + s_2,$$

where s_1 and s_2 are respectively the volumes of saturated vapour and liquid, L the heat of vaporization, J the mechanical equivalent of heat, T absolute temperature, and dp/dt the differential of pressure in degrees with respect to temperature.

By help of this formula the following table was calculated :—

Vapour-Densities (Saturated) of Water from Regnault's Heats of Vaporization.

t.° C.	T.°abs.	p.	dp/dt.	L (total).	Q.	Ht. of Vap. L.	$s_1 - s_2$.	s_1.	V.D.
0	273	4·600	0·329	606·5	0	606·5	211,130	211,131	8·74
10	283	9·165	0·609	609·5	10	599·5	108,730	108,731	8·83
20	293	17·39	1·074	612·6	20	592·6	58,867	58,868	8·89
30	303	31·55	1·81	615·7	30	585·7	33,384	33,385	8·94
40	313	54·91	2·93	618·7	40·1	578·6	19,722	19,723	8·98
50	323	91·98	4·58	621·7	50·1	571·6	12,078	12,079	9·04
60	333	148·8	6·91	624·8	60·1	564·7	7,671	7,672	9·06
70	343	233·1	10·11	627·8	70·2	557·6	5,026	5,027	9·10
80	353	354·6	14·40	630·9	80·3	550·6	3,386	3,387	9·13
90	363	525·4	20·01	633·9	90·4	543·5	2,339	2,340	9·18
100	373	760·0	27·14	637·0	100·5	536·5	1,657	1,658	9·20
110	383	1,075	36·27	640·0	110·6	529·4	1,191	1,192	9·29
120	393	1,491	47·35	643·1	120·8	522·3	877·4	878·5	9·32
130	403	2,030	60·90	646·1	131·0	515·1	656·1	657·2	9·39
140	413	2,718	77·06	649·2	141·2	508·0	499·0	500·1	9·44
150	423	3,581	96·17	652·2	151·5	500·7	384·7	385·8	9·52
160	433	4,652	118·5	655·3	161·7	493·6	300·7	301·8	9·58
170	443	5,962	144·1	658·3	172·1	486·2	238·1	239·2	9·65
180	453	7,546	173·4	661·4	182·4	479·0	190·6	191·7	9·73
190	463	9,443	206·5	664·4	192·8	471·6	154·2	155·3	9·81
200	473	11,689	243·4	667·5	203·2	464·3	126·1	127·3	9·88
210	483	14,325	284·4	670·5	213·7	456·8	104·0	105·2	9·96
220	493	17,390	329·4	673·6	224·2	449·4	86·50	87·70	10·05
230	503	20,926	378·5	676·6	234·7	441·9	72·54	73·76	10·13

p, Regnault's formula H; dp/dt from same formula; L (total), Q and L from Regnault.

As Regnault's measurements do not extend beyond 230° the higher isothermals were smoothed as well as possible and made to cut the vapour-pressure lines at certain points, to which we were guided to some extent by their own regularity, and to some extent by extending the curve drawn to pass through the points representing the volumes of 1 grm. of saturated vapour. The results cannot be far from the truth.

The next table shown was constructed by reading the pressures and temperatures corresponding to equal volumes.

Vol. c.c. per gramme.	T=140°.	150°.	160°.	170°.	180°.	190°.	200°.	210°.	220°.	230°.	240°.	250°.	260°.	270°.
650	2,128	2,190	2,250	2,310	2,370	2,425	2,485	2,540	2,600	2,660	2,720	2,780	2,840	2,900
600	2,285	2,365	2,425	2,490	2,552	2,615	2,685	2,745	2,810	2,880	2,965	3,015	3,070	3,135
550	2,480	2,560	2,640	2,710	2,775	2,850	2,925	2,995	3,070	3,145	3,215	3,290	3,350	3,415
500	2,720	2,810	2,895	2,975	3,045	3,125	3,210	3,295	3,375	3,455	3,535	3,615	3,685	3,760
450	..	3,110	3,205	3,300	3,390	3,475	3,565	3,650	3,740	3,830	3,920	4,010	4,095	4,175
400	..	3,465	3,600	3,705	3,800	3,905	4,000	4,095	4,205	4,305	4,400	4,505	4,600	4,690
350	..	..	4,085	4,215	4,325	4,450	4,560	4,660	4,780	4,905	5,020	5,140	5,260	5,375
300	..	..	..	4,920	5,040	5,170	5,315	5,460	5,595	5,725	5,865	5,995	6,125	6,255
250	..	..	..	5,760	6,020	6,190	6,360	6,530	6,700	6,870	7,030	7,200	7,360	7,510
200	..	..	..	..	7,300	7,660	7,875	8,080	8,300	8,505	8,710	8,930	9,140	9,360
175	..	..	..	..	..	8,580	8,990	9,175	9,430	9,675	9,900	10,120	10,375	10,600
150	..	..	..	..	..	..	10,230	10,570	10,850	11,100	11,370	11,650	11,930	12,185
125	..	..	..	..	..	..	..	12,490	12,860	13,180	13,490	13,840	14,150	14,480
100	..	..	..	..	..	..	..	..	15,660	16,160	16,545	16,920	17,300	17,700
90	..	..	..	..	..	..	..	..	17,010	17,740	18,180	18,590	19,000	19,400
80	..	..	..	..	..	..	..	..	..	19,550	20,150	20,650	21,060	21,520
70	..	..	..	..	..	..	..	..	..	..	22,470	23,340	23,750	24,200
60	..	..	..	..	..	..	..	..	..	..	..	26,350	27,060	27,650
50	..	..	..	..	..	..	..	..	..	..	..	..	31,120	32,000
45	..	..	..	..	..	..	..	..	..	..	..	..	33,650	34,780

The vapour-densities and volumes of a gramme of saturated vapour can be given at temperatures above 230°. They are as follows :—

Temperature.	Pressure.	Vol. of 1 grm.	Vapour-density (H = 1).		Vol. of 1 grm.
°	millims.	c.c.	read.	smoothed.	recalculated.
230	20936	73·00	10·23	10·22	73·06
240	25019	60·65	10·51	10·40	61·28
250	29734	50·30	10·87	10·63	51·43
260	35059	42·40	11·14	10·94	43·19
270	41100	37·20	11·04	11·36	36·15

There is very great difficulty in reconciling the results given by Regnault at the highest temperatures with our own direct measurements. Regnault's results interpreted as vapour-density obviously increase at too slow a rate above 210°, while ours appear to increase too rapidly at 240°, 250°, and 260°. We therefore constructed a curve showing the relationship of vapour-densities from all sources to temperature; the numbers given as "smoothed" are read from the curve, and the volumes of 1 grm. were again calculated.

In a series of papers published in the 'Philosophical Magazine' (1887, vol. 27, pp. 196 and 435), we showed that for ether, ethyl alcohol, and carbon dioxide, the pressures of the vapours of these substances for constant volumes are, within limits of experimental error, a rectilinear function of the temperature, or $p = bt - a$, where a and b are two constants, depending on the volume, and t is the absolute temperature. Carl Barus, as the result of experiments with other substances, has confirmed the statement, both for liquid and gas. He states ('Phil. Mag.,' 1890, vol. 30, p. 358) that "below 1000 atmospheres, the curves for ether, alcohol, thymol, diphenylamine, and toluidine, are so nearly linear that they may be accepted as such with an error no larger than 2 or 3 degrees at 1000 atmospheres." He also found that in the case of water the rectilinear relation does not hold.

If an isochoric diagram be constructed from the results given on p. 116, it will be seen that, except at small volumes, the isochoric lines are approximately straight, but curve considerably on approaching the condensation points. The lines should probably, as Barus found, be curved throughout, but no considerable error is introduced in considering them as straight during the major part of their course. Indeed, accepting Regnault's determinations as correct, it would appear that at temperatures above 140°, the curvature cannot be due to premature condensation, since the isothermal curves given in the diagram have been drawn, as already stated, to coincide with the results of Regnault's work on the heats of vaporization, and yet coincide with our experimental results, except at volumes very close to those at which liquefaction takes place; and at such small volumes, surface condensation is unmistakeable.

On reference to the Appendix, p. 128, an account of experiments at much lower temperatures will be found. These show that the relative effect of surface condensation is much more marked at low than at high temperatures. Thus, at about 30 millims. pressure the vapour-density calculated at 75° is 9·01 ; at 50°, 9·46 ; at 40°, 9·90 ; and at 30°, 10·70. An increase of pressure does not appear to raise the apparent vapour-density to the same extent as a fall of temperature. This may explain why an increase of pressure, at high temperatures, does not influence surface condensation appreciably, except very close to the point of condensation.

In addition to the work of REGNAULT, that of FAIRBAIRN and TATE on the densities of unsaturated and saturated steam may be mentioned. As their results were given in English units, we think it well to reproduce them here in the same form as has been adopted in this paper. It will be seen that the densities of the saturated vapour show a gradual rise with increase of temperature, although the results of individual experiments are far from regular.

FAIRBAIRN and TATE, 'Phil. Trans.,' 1860, pp. 218, 219. Quoted by RANKINE in Fahr. degrees and cubic feet per lb.

T. ° C.	T. ° Abs.	Volume of 1 grm.	Pressure.	Vapour-density.
		c.cs.	millims.	
58·20	331·20	8275·3	135·9	9·15
68·51	341·51	5333·5	218·9	9·09
70·75	343·75	4920·2	240·0	9·05
77·18	350·18	3722·6	316·7	9·23
77·49	350·49	3715·1	320·3	9·15
79·40	352·40	3438·1	346·0	9·21
83·50	356·50	3051·0	406·7	8·93
86·83	359·83	2623·4	466·4	9·14
92·65	365·65	2149·5	581·1	9·10
117·16	390·16	943·12	1361·7	9·44
118·23	391·23	908·03	1410·1	9·49
118·45	391·45	892·54	1419·6	9·60
124·16	397·16	759·44	1697·7	9·57
128·41	401·41	649·24	1935·4	9·93
130·67	403·67	635·30	2070·8	9·54
131·77	404·77	605·65	2138·9	9·71
134·86	407·86	584·44	2342·6	9·26
134·05	407·05	543·17	2287·9	10·18
137·45	410·45	514·98	2529·8	9·79
139·21	412·21	497·25	2655·2	9·70
141·80	414·80	458·30	2864·6	9·82
144·74	417·74	433·12	3105·1	9·65
142·36	415·36	449·62	2901·9	9·89

Fairbairn and Tate. Superheated Steam. 'Edinb. Trans.,' vol. 23, 1862, p. 147. Results arranged in isotherms, and Pressures and V.D.'s corrected for vapour pressure of mercury, using (1) Pressures used by Rankine, (2) Pressures by Ramsay and Young.

Temperature.	Temperature.	Weight.	Volume.	P_1.	P_2.	V. D. (1.)	V. D. (2.)
° F.	° C.	grms.	c.cs.	millims.	millims.		
302·88	150·5	0·01845	197·3	132·3	134·0	9·31	9·19
"	"	0·02625	197·3	184·6	186·3	9·48	9·39
"	"	0·03530	197·3	250·4	252·1	9·41	9·34
"	"	0·03435	197·3	242·0	243·7	9·46	9·40
"	"	0·05670	197·3	408·6	410·3	9·25	9·22
251·64	122·0	0·01845	197·2	123·8	124·7	9·29	9·22
"	"	0·02625	197·2	171·5	172·4	9·53	9·48
"	"	0·03530	197·2	234·0	234·9	9·39	9·36
"	"	0·03435	197·2	226·1	227·0	9·46	9·42
"	"	0·05670	197·2	381·6	382·5	9·25	9·23
200·74	93·75	0·01845	197·1	114·7	115·1	9·31	9·28
"	"	0·02625	197·1	159·9	160·3	9·49	9·47
"	"	0·03530	197·1	217·1	217·5	9·41	9·39
"	"	0·03435	197·1	210·0	210·4	9·46	9·44
"	"	0·05670	197·1	353·2	353·6	9·28	9·27
180·72	82·6	0·04670	197·0	340·2	340·5	9·35	9·34
165·45	74·15	0·03435	197·0	197·3	197·5	9·53	9·52
"	"	0·05670	197·0	259·3	259·5	11·98	11·97
150·18	65·65	0·01845	196·9	105·7	105·9	9·34	9·32
"	"	0·02625	196·9	147·6	147·8	9·50	9·49
"	"	0·03530	196·9	176·1	176·3	10·72	10·71
"	"	0·03435	196·9	177·1	177·3	10·37	10·35
100·0	37·75	0·01845	196·8	44·6	44·7	20·3	20·3

Hirn, in his 'Théorie Mécanique de la Chaleur,' vol. 1, 3rd Ed., p. 468, gives a table of the densities of saturated vapour at various temperatures, ranging from 98° to 196°. As these agree absolutely with the results calculated from Regnault's measurements, and as neither the method adopted by Hirn, nor his individual observations would appear to lead one to anticipate such concordance, we would suggest that it may be attributed to the "tracé de sentiment" with which he credits the assistant who interpreted his results.

In conclusion, it may be noted that Regnault made experiments on the densities of saturated steam at low temperatures for hygrometric purposes, and has shown that it is practically normal at temperatures between 0° and 27° ('Annales de Chimie' [3], vol. 15, 1845, p. 129, *et seq.*).

Note.—In Plate 7 the curves representing the relations of temperature, pressure, and volume of unsaturated and saturated steam are given on a scale large enough, we hope, to prove of practical use. For this purpose they have been divided into three sets, a different scale, however, being applied to each set. The pressures are given in metres of mercury; the volumes, in cubic centimetres per gramme. The heavy line shows the volumes of saturated vapour, and may be termed the orthobaric line.

APPENDIX.

VOLUMES (Vapour pressures) and Compressibility of Liquid Water at various Temperatures.

Temperature.	Divisions of tube.	Pressure.	Actual volumes.	Temperature.	Divisions of tube.	Pressure.	Actual volumes.
°		millims.	cub. centims.	°		millims.	cub. centims.
100	256·85	760	1·1503	210	288·00	17,488	1·2937
,,	256·50	19,651	1·1488	,,	287·50	31,467	1·2912
110	258·80	1,073	1·1546	,,	287·00	43,405	1·2888
120	260·95	1,484	1·1682	220	292·30	17,622	1·3138
130	263·10	2,019	1·1775	,,	292·00	27,251	1·3128
140	265·70	2,694	1·1900	,,	291·50	39,432	1·3105
150	268·25	3,568	1·1998	230	297·00	21,085	1·3367
160	271·05	4,652	1·2126	,,	296·50	31,745	1·3343
170	274·00	5,937	1·2267	,,	296·00	42,249	1·3319
180	277·10	7,478	1·2416	240	302·00	25,276	1·3584
190	280·60	9,837	1·2583	,,	301·50	32,452	1·3563
,,	280·50	12,813	1·2579	,,	301·00	41,614	1·3543
,,	280·00	30,029	1·2555	250	307·20	30,059	1·3801
,,	279·50	39,904	1·2532	,,	307·00	36,040	1·3793
200	284·20	15,014	1·2755	,,	306·50	43,925	1·3773
,,	284·00	18,695	1·2746	260	313·40	35,028	1·4059
,,	283·50	30,513	1·2723	,,	313·00	40,210	1·4043
,,	283·00	43,901	1·2698	270	319·50	43,433	1·4334
210	288·05	14,404	1·2942				

VAPOUR-PRESSURES of Water.—Individual Observations.

Temperature.	Pressure.	Mean corrected.*	Pressure.†	Temperature.	Pressure.	Mean corrected.*	Pressure.†
°	millims.	millims.	millims.	°	millims.	millims.	millims.
120	1,482			140	2,691		
	1,484				2,701		
	1,483	1,484	1,445		2,703	2,694	2,658
	1,493				2,695		
	1,483				2,700		
130	2,024			150 (Jacket C_6H_5 Br)	3,555		
	2,022				3,552		
	2,021	2,019	1,950		3,559	3,553	3.475
	2,018				3,558		
	2,022				3,569		

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* For deviation from BOYLE's law, and for vapour-pressure of mercury. In previous work the measurements of pressure were made rapidly; after the volume had been increased to its maximum, it was diminished by stages, and pressure was read at each stage. The correction for vapour-pressure of mercury is in such a case not admissible. But in this research the temperature was allowed to remain constant for at least an hour before taking the first reading.

† Observed with a much smaller quantity of liquid present—nearly all vapour.

VAPOUR-PRESSURES of Water.—Individual Observations—(continued).

Temperature.	Pressure.	Mean corrected.	Pressure.
°	millims.	millims.	millims.
150 (Jacket $C_6H_5NH_2$)	3,580		
	3,580		
	3,588	3,581	
	3,590		
	3,594		
160	4,656		
	4,648		
	4,660	4,652	4,609
	4,668		
	4,671		
170	5,958		
	5,958		
	5,939	5,937	5,889
	5,966		
	5,966		
180 (Aniline)	7,549		
	7,559		
	7,547		
	7,552		
	7,581	7,518	7,493
	7,536		
	7,519		
	7,480		
	7,539		
	7,578		
180 (Methyl Salicylate)	7,423		
	7,453		
	7,446		
	7,449		
	7,467	7,437	
	7,457		
	7,457		
	7,452		
	7,474		
190	9,425		
	9,418		
	9,431		
	9,425		
	9,456	9,403	9,374
	9,469		
	9,484		
	9,462		
200	11,679		
	11,685		
	11,674	11,625	11,564
	11,708		
	11,705		

Temperature.	Pressure.	Mean corrected.	Pressure.
°	millims.	millims.	millims.
210	14,338		
	14,329		
	14,320	14,241	14,197
	14,343		
	14,383		
210	14,337		
	14,346		
	14,342	14,255	
	14,382		
	14,377		
210	14,324		
	14,321	14,223	
	14,326		
220	17,484		
	17,468	17,329	17,472* 17,426
	17,490		
220 (End of work)	17,498		
	17,519		
	17,541	17,401	
	17,582		
	17,632		
230	21,059		
	21,175		
	21,163	20,936	20,879
	21,158		
	21,152		
240	25,296		
	25,294		
	25,340	25,019	24,803* 25,040
	25,399		
	25,393		
250	30,150		
	30,130		
	30,142	29,734	29,527* 29,687
	30,214		
	30,272		
260	35,644		
	35,630		
	35,649	35,059	35,036
	35,654		
	35,650		
270	41,913		
	41,890		
	41,869	41,101	40,956* 41,149
	41,864		
	41,861		
	41,821		

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* Determinations with two separate quantities, one of which was larger than the other.

Volumes of 1 *grm. of Steam, at Various Temperatures and Pressures.*

I. *High Temperatures.*

Two series of experiments were made. The weight of the water in the tube was ascertained by determining the products of pressure and volume, altering the volumes; and this was repeated at different temperatures. Assuming that if these products for any one temperature were constant, the density of the steam was constant, viz., the theoretical density, 9, the weight could be ascertained by the equation

$$W = \frac{\text{V.D.} \times p.v \times 273}{11{\cdot}1636 \times 1000 \times 760 \times (273 + t)}.$$

This expression simplifies to

$$\log W = \log p.v. + \bar{4}{\cdot}46179 - \log (273 + t).$$

During the progress of the experiments it happened that a trace of water passed up the tube, adding itself to that already present. This, of course, increased the weight, hence new measurements were made to determine the amount of the increase.

These weights were obtained from the following readings :—

Determination of the Weight of the Smallest Quantity of Water. (A.)

Temperature.	Volume.	Pressure.	p. v.	Weight.	Mean weight.
°	cub. centims.	millims.		grm.	grm.
220	1·7147	2528	4335	0·002546	
	1·5565	2770	4311	0·002533	
	1·4080	3060	4308	0·002531	
	1·2592	3435	4325	0·002541	
	1·1072	3893	4310	0·002532	0·002536
230	1·7151	2585	4434	0·002553	
	1·5569	2820	4391	0·002529	
	1·4084	3119	4393	0·002529	
	1·1579	3815	4417	0·002543	0·002538

General mean, 0·002537.

Here some water rose in the tube and increased the weight.

Temperature.	Volume.	Pressure.	p. v.	Weight.	Mean weight.
230	1·7151	2633	4516	0·002600	
	1·5569	2879	4482	0·002581	
	1·4084	3153	4441	0·002587	0·002579
240	1·7156	2697	4627	0·002612	
	1·5573	2948	4591	0·002598	
	1·4087	3253	4583	0·002587	
	1·1582	3964	4591	0·002592	0·002597
250	1·5064	3097	4665	0·002582	
	1·4091	3310	4664	0·002581	
	1·1585	4026	4664	0·002581	0·002581

General mean, 0·002587.

DETERMINATION of the Weight of the Smallest Quantity of Water. (A.)—(continued).

Temperature.	Volume.	Pressure.	p. v.	Weight.	Mean weight.
Here more water rose in the tube and increased the weight.					
°	cub. centims.	millims.		grms.	grm.
250	1·7160	2819	4837	0·002679	
	1·5577	3114	4851	0·002686	
	1·4091	3403	4795	0·002655	
	1·1585	4155	4813	0·002665	0·002671
260	1·7165	2872	4930	0·002678	
	1·5580	3144	4899	0·002662	
	1·4094	3478	4902	0·002663	
	1·1588	4220	4890	0·002657	0·002665
270	1·7169	2918	5010	0·002672	
	1·5585	3196	4981	0·002657	
	1·4099	3529	4975	0·002654	
	1·1591	4283	4964	0·002648	0·002656
General mean, 0·002664.					

These results can be verified by taking the weight at 270° as a standard, and calculating the smaller weights by the equation

$$\frac{p.v.}{273 + t} : \frac{p'v'}{273 + t'} :: \mathrm{W} : \mathrm{W}'$$

identical numbers with those found are obtained.

Hence the largest weight at 250°, 260°, and 270° is 0·002664 grm.;
the second weight at 250°, 240°, and 230° is 0·002588 grm.;
the first weight at 220° and 230° is 0·002547 grm.

The first weight holds also for lower temperatures.

These data are adduced as a proof of the relative accuracy of the determinations which follow.

VOLUMES of 1 grm. of Steam and Vapour-densities.

Temperature.	Pressure.	Volume of 1 grm.	p. v.	Vapour-density H = 1.	Remarks.
	millims.	c.c.			
120°	1,448	602·0	..	..	Condensed
	1,450	545·2	..	..	Condensed
130	1,930	622·7	..	..	Condensed
	1,953	563·7	..	..	Condensed
140	2,093	661·7	1,384,900	9·268	
	2,245	609·9	1,369,200	9·374	
	2,449	551·7	1,351,100	9·500	
	2,620	453·6	1,188,400	10·800	
	2,645	338·3	894,810	..	Condensed
	2,650	107·1	..	..	Condensed
150	2,147	661·8	1,420,900	9·251	
	2,309	610·0	1,408,500	9·333	
	2,550	551·8	1,407,100	9·343	
	3,068	453·7	1,392,000	9·445	
	3,280	415·2	1,361,900	9·654	
	3,442	353·9	..	..	Condensed
	3,475	241·5	..	..	Condensed
150	2,149	661·8	1,422,200	9·243	
	2,310	610·0	1,409,100	9·329	
	2,547	551·8	1,405,400	9·354	
160	2,211	662·0	1,463,700	9·193	
	2,382	610·2	1,453,500	9·258	
	2,633	552·0	1,453,400	9·259 ?	
	3,182	453·8	1,444,000	9·319	
	3,997	354·1	1,415,300	9·509	
	4,571	241·6	..	..	Condensed
170	2,267	662·2	1,501,200	9·171	
	2,447	610·3	1,493,400	9·219	
	2,705	552·1	1,493,500	9·219	
	3,271	453·9	1,484,700	9·273	
	4,172	354·1	1,477,300	9·364	
	5,603	252·7	1,415,900	9·722	
180	2,330	662·4	1,543,400	9·122	
	2,509	610·5	1,531,800	9·193	
	2,771	552·2	1,530,100	9·200	
	3,356	454·0	1,523,600	9·240	
	4,271	354·2	1,512,800	9·306	
	5,854	252·8	1,479,900	9·512	
	7,296	192·5	1,404,500	10·025	
	7,444	152·3	..	..	Condensed
	7,471	112·1	..	..	Condensed
190	2,387	662·6	1,581,600	9·098	
	2,576	610·7	1,573,200	9·147	
	2,842	552·4	1,569,900	9·172	
	3,441	454·2	1,562,900	9·208	
	4,385	354·3	1,553,600	9·261	

VOLUMES of 1 grm. of Steam and Vapour-densities—(continued).

Temperature.	Pressure.	Volume of 1 grm.	p. v.	Vapour density H = 1.	Remarks.
	millims.	c.c.			
190° (continued)	6,057	252·9	1,531,800	9·393	
	7,839	192·5	1,509,000	9·534	
	9,254	152·3	1,409,400	10·207	
	9,337	112·2	..	..	Condensed
200	2,458	662·7	1,628,900	9·024	
	2,655	610·8	1,621,700	9·065	
	2,930	552·5	1,618,800	9·080	
	3,546	454·2	1,610,600	9·126	
	4,516	354·4	1,600,400	9·185	
	6,276	253·0	1,587,800	9·260	
	8,107	192·6	1,561,400	9·738	
	10,004	152·4	1,524,600	10·246	
	11,505	112·2	..	..	Condensed
210	2,525	662·9	1,673,300	8·989	
	2,723	611·0	1,663,800	9·023	
	3,000	552·7	1,658,100	9·053	
	3,652	454·4	1,659,500	9·048	
	4,646	354·5	1,647,000	9·114	
	6,437	253·1	1,629,200	9·214	
	8,371	192·6	1,612,300	9·309	
	10,331	152·4	1,574,400	9·533	
	13,538	112·2	1,519,000	9·880	
	14,102	92·2	1,300,200	11·548	
220	2,585	663·0	1,713,800	8·939	
	2,782	611·1	1,700,100	9·012	
	3,072	552·8	1,698,200	9·022	
	3,734	454·5	1,697,100	9·029	
	4,758	354·6	1,687,300	9·082	
	6,601	253·1	1,670,700	9·171	
	8,580	192·7	1,653,400	9·268	
	10,649	152·4	1,622,900	9·438	
	14,073	112·2	1,579,000	9·699	
	16,458	92·2	1,517,400	10·098	
	17,165	72·2	..	..	Condensed
220	2,529	675·9	1,709,400	8·964	Weight altered
	2,770	613·5	1,699,400	9·016	
	3,060	555·0	1,698,300	9·022	
	3,435	496·3	1,704,800	8·986	
	3,894	436·4	1,699,400	9·016	
	4,493	376·2	1,690,300	9·064	
	5,685	294·3	1,673,100	9·136	
	7,114	235·8	1,677,500	9·134	
	8,575	193·4	1,658,400	9·237	
	10,638	153·0	1,627,600	9·411	
	14,032	112·7	1,581,400	9·690	
	16,641	92·6	1,541,000	9·948	
	17,476	72·5	1,267,000	10·059	
230	2,585	676·1	1,747,700	8·945	
	2,820	613·7	1,730,700	9·033	

VOLUMES of 1 grm. of Steam and Vapour-densities—(continued).

Temperature.	Pressure.	Volume of 1 grm.	p. v.	Vapour-density H = 1.	Remarks.
	millims.	c.c.			
230° (continued)	3,119	555·1	1,731,400	9·028	
	3,615	477·9	1,727,600	8·978	
	4,840	356·1	1,723,500	9·070	
	6,725	254·1	1,708,800	9·146	
	8,815	193·5	1,705,700	9·165	
	10,887	153·1	1,666,800	9·380	
	14,430	115·1	1,660,900	9·633	
	17,148	92·6	1,587,900	9·847	
	20,662	72·5	1,498,000	10·433	
	20,884	52·5	..	..	Condensed
230 (repeated)	2,633	664·5	1,749,600	8·934	Taken as normal; weight altered
	2,879	603·2	1,736,600	9·002	
	3,153	545·7	1,720,600	9·086	
240	2,697	664·7	1,792,700	8·691	Doubtful
	2,948	603·4	1,778,900	8·973	
	3,253	545·8	1,775,500	8·980	
	3,964	448·7	1,778,600	8·963	
	5,053	350·1	1,769,000	9·012	
	7,002	249·9	1,749,800	9·112	
	9,158	190·2	1,741,900	9·170	
	11,395	150·5	1,715,000	9·151	
	15,084	110·8	1,671,300	9·296	
	17,998	91·0	1,637,800	9·559	
	21,336	71·3	1,524,100	9·732	
	22,162	61·4	1,360,700	10·480	
	25,049	41·7	..	..	Condensed
250	3,097	583·6	1,807,400	8·992	
	3,310	545·9	1,806,900	8·995	
	4,027	448·8	1,807,300	8·993	
250 (repeated)	2,819	644·2	1,816,000	8·951	
	3,114	584·7	1,820,800	8·927	
	3,403	528·9	1,799,800	9·030	
	4,155	434·9	1,807,000	8·996	
	5,290	339·3	1,794,900	9·056	
	7,367	242·2	1,784,300	9·110	
	9,560	184·4	1,762,900	9·222	
	11,907	145·9	1,737,300	9·359	
	15,852	107·4	1,702,500	9·547	
	18,897	88·2	1,666,700	9·751	
	23,594	69·1	1,630,400	9·970	
	26,395	59·5	1,570,500	10·343	
	29,473	50·0	1,473,600	11·034	
	29,696	31·0	..	..	Condensed
260	2,872	644·3	1,850,400	8·952	
	3,144	584·9	1,838,900	9·008	
	3,479	529·1	1,840,800	9·000	
	4,225	435·0	1,837,900	9·013	
	5,409	339·4	1,835,800	9·045	
	7,521	242·2	1,821,600	9·093	

Volumes of 1 grm. of Steam and Vapour-densities—(continued).

Temperature.	Pressure.	Volume of 1 grm.	p. v.	Vapour density H = 1.	Remarks.
	millims.	c.c.			
260° (continued)	9,811	184·4	1,809,100	9·177	
	12,164	145·9	1,774,800	9·334	
	16,228	107·4	1,741,800	9·501	
	19,326	88·2	1,704,600	9·714	
	23,886	69·1	1,650,500	10·034	
	27,193	59·6	1,620,700	10·229	
	31,142	50·0	1,571,000	10·640	
	35,040	31·0	..	..	Condensed
270	2,918	644·5	1,880,700	8·973	
	3,196	585·0	1,869,700	9·025	
	3,529	529·2	1,867,600	9·036	
	4,288	435·1	1,865,700	9·045	
	5,486	339·5	1,862,600	9·062	
	7,664	242·3	1,857,000	9·088	
	9,974	184·4	1,839,200	9·173	
	12,392	145·9	1,808,000	9·331	
	16,519	107·5	1,775,800	9·507	
	19,729	88·3	1,742,100	9·691	
	24,427	69·1	1,687,900	9·993	
	27,813	59·6	1,657,700	10·185	
	31,958	50·0	1,597,900	10·560	
	37,862	40·5	1,533,400	11·010	
	40,854	29·1	..	..	Condensed

II. *Low Temperatures.*

(*a.*) Weight of water in the modified Hofmann's apparatus (described with a wood-cut in 'Phil. Trans.,' 1887 A., p. 59).

0·20087 grm.

Readings commenced at largest volume, and vapour then compressed.

Temperature.	Pressure.	Volume.	Vapour Density (H = 1).	Temperature.	Pressure.	Volume.	Vapour Density (H = 1).
°	millims.	c.c.		°	millims.	c.c.	
75	23·35	150·5	8·83	40	19·15	149·1	9·78
	23·40	148·5	8·93		21·2	135·7	9·71
	23·65	146·3	8·97		23·15	122·3	9·86
	25·80	134·9	8·92		26·0	109·0	9·85
	27·4	125·9	8·99		29·2	96·54	9·90
	29·6	116·4	9·01		33·75	83·07	9·96
	32·2	106·2	9·08		39·3	71·10	9·99
	35·3	96·36	9·13		48·45	55·90	10·31
	39·95	86·52	8·98		52·2	49·66	10·77
	45·7	74·78	9·08		54·1	42·05	12·27
				30	18·05	154·9	9·66
75 (repeated)	23·6	146·9	8·96		18·9	145·2	9·85
	24·1	141·2	9·12		20·1	134·5	10·00
	30·3	111·9	9·16		22·1	122·0	10·03
	47·65	72·17	9·03		24·45	109·0	10·14
	57·7	58·12	9·26		26·7	96·72	10·46
	79·2	42·76	9·17		29·9	84·50	10·70
					31·05	68·61	(12·69)
					31·35	55·0	Condensed
					31·45	43·0	"
50	20·35	153·0	9·25				
	21·8	140·5	9·40	25	17·25	154·2	9·99
	23·4	130·5	9·43		18·0	144·8	10·20
	26·95	113·2	9·45		19·25	134·4	10·27
	33·25	91·35	9·49		20·35	123·1	10·62
	42·1	71·82	9·53		21·8	111·6	10·93
	54·75	54·49	9·66		22·4	99·32	11·95
	70·75	42·52	9·58		23·35	86·5	(13·16)

It is to be noticed that four readings are required for each pressure, so that at the lowest pressure an error of 1 per cent., or say 0·2 millim., is not excessive.

In considering these results it is seen that, as a rule, the density increases on diminishing volume. This, however, cannot be ascribed to real increase of density, but to diminution of the actual weight of the water present in the state of vapour through adhesion of water to the surface of the tube. That this explanation is the correct one is seen from the following considerations.

At 75° very little rise of vapour-density is to be observed, although there is a slight tendency towards a higher value at the smallest volumes. The pressure was never near the vapour-pressure, which at this temperature is 288·5 millims.

At 50°, at which the vapour-pressure is 92 millims., the vapour-densities are all high, but do not rise rapidly after the first one. The highest pressure registered (70·75) approached the vapour-pressure.

At 40° the vapour-pressure is 54·9 ; the highest pressure read was 54·1. The rise of vapour-density is slow at first, and afterwards more rapid.

At 30° the vapour-pressure is 31·55 millims., while the highest pressure read was 31·05 millims. The vapour-densities are very high, and rise rapidly with decrease of volume; and at 25° the vapour-pressure is 23·55 millims., the highest pressure read being 23·35. Again, there is a rapid increase of density with diminishing volume.

These pressures are all somewhat lower than the values given by Regnault.

(*b*). Weight = 0·00295 grm.

The tube was heated to 100°.

Temperature.	Pressure.	Volume.	Vapour-density.
°	millims.	c.c.	
100	25·1	152·4	8·94
	28·7	131·3	9·08
	34·1	110·4	9·08
	41·45	91·13	9·05
	52·4	72·34	9·02

The tube was then cooled with the mercury at the same height as at the last reading, and then, without drawing down the mercury, the tube was heated to 40°. In this case there was no possibility of condensation of vapour on the lower part of the tube, while previously, when readings were begun at the largest volume, condensation might have taken place on the lower part of the tube. If, therefore, the high vapour-densities are due to adhesion and condensation, the value of this reading at 40° should be lower than it was found in the first series. This is in fact the case, as is shown by the following determinations :—

Temperature.	Pressure.	Volume.	Vapour-density.
°	millims.	c.c.	
40	42·05	71·7	9·52
	45·75	64·63	9·70
	51·6	55·41	10·04
	53·95	49·83	10·67
	54·8	42·0	Condensed.

In the first series, at volume 71·1 cub. centims., the vapour density was 9·99, as against 9·52 in the second at volume 71·7 cub. centims. The largest volume in the first series, 149·1, is a little more than twice as great as the largest volume in the second; therefore, as the tube was cylindrical, the surface on which condensation could take place was about twice as great. If the condensation is not much affected by the rise of pressure from 19·15 millims. at the largest volume of the first series to 42·05 millims. at the largest volume of the second, and if the high vapour-density is entirely due to condensation, we should expect the error to be about half as great in the second as in the first series, and this is approximately the case. It will be noticed that the weight of the water in both sets is nearly the same.

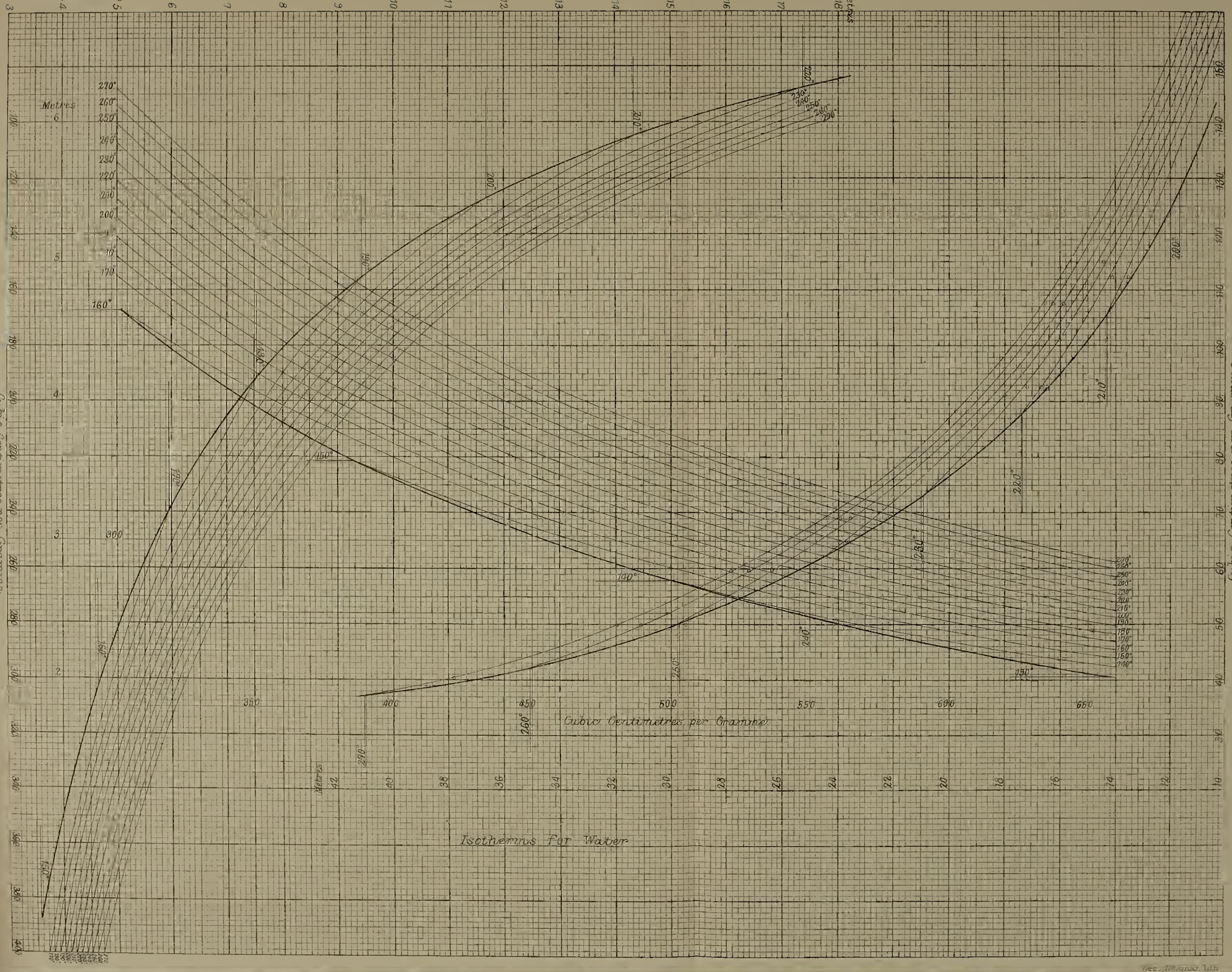
The tube was again heated to 100°, and the external pressure so arranged that the volume of vapour remained about 56 cub. centims. After cooling, the tube was again heated to 40°, and a reading was taken with the following results:—

Temperature.	Pressure.	Volume.	Vapour-density.
°	millims.	c.c.	
40	52·3	54·87	10·00

The result is not much lower than at the corresponding volume in the last experiment, but the gain of surface was not very great, and the pressure, 52·3 millims., was very near the condensing pressure, 54·9 millims.; and it is to be noticed that the vapour-density rises rapidly as this pressure is approached. At the same pressure in the first series, the vapour-density observed was 10·77—a much higher value.

The experiments show that the high vapour-densities are due to condensation on the walls of the tube, and that the error due to this cause may vary considerably, especially at pressures near the true vapour pressures.

Similar experiments were not made at high pressures, but on constructing isothermal curves, it is seen that as the volume diminishes, and as the vapour-pressure is approached, the curves instead of cutting the vapour-pressure line so as to form an angle, as is the case with the other liquids which we have examined, gradually turn and run nearly parallel to the vapour-pressure line at a somewhat lower pressure. When a considerable amount of liquid has condensed, the true vapour-pressure is reached. This behaviour is analogous to that of a mixture of a condensible vapour with a gas like air; but it cannot be attributed to such admixture, for the last bubble of steam left on raising pressure was easily and quickly absorbed, and without any perceptible rise of pressure. Its cause must, as already explained, lie in the adhesion of the vapour to the glass, causing condensation before the true vapour-pressure is reached.



Isotherms for Water

West, Newman lith.

IV. *Comparison of Simultaneous Magnetic Disturbances at Several Observatories.*

By *Professor* W. GRYLLS ADAMS, *D.Sc., F.R.S.*

Received June 11,—Read June 18, 1891.

[PLATES 8, 9.]

IN my former papers on Magnetic Disturbances and Earth Currents, which were read at the meetings of the British Association at Swansea in 1880 and at York in 1881, a comparison was made of the declination and the horizontal force traces given by the self-recording instruments at five European stations, also at one station in India, one in China, and one in Australia. An attempt was made to determine the relative amounts of the simultaneous changes at the several observatories by comparing them with one another by means of the scale values of the instruments employed, which were all of the pattern of the self-recording instruments at the Kew Observatory. It was found on comparison that there were great differences in the scale values of the instruments of the same kind at the different observatories, and in some cases there was great uncertainty as to the scale values, because no determination of them had recently been made. Hence great difficulty was found in arriving at the true meaning of the records which were taken regularly at the different observatories.

The comparison was sufficient to show the great importance of adopting the same scale values for the like instruments at all observatories. In my paper a recommendation was made that for horizontal force records a scale value of ·0005 millimetre-milligramme for a difference of scale reading of 1 mm. should be adopted as being the most convenient. The same scale value was recommended by Dr. WILD, of the St. Petersburg Observatory, and for the vertical force magnetometer the same scale value might conveniently be adopted. With this scale value the instruments would be sufficiently sensitive to give for a considerable magnetic disturbance changes which are capable of being measured, but yet would not be so sensitive as to send the spot of light off the photographic paper, even in a violent magnetic storm.

A violent magnetic storm was experienced in August, 1880, and the records from these observatories, viz., Kew, Stonyhurst, Lisbon, Vienna, St. Petersburg, Bombay, Zi-ka-wei, and Melbourne, as well as those from the Toronto Observatory in Canada, were compared and discussed in the paper read before the British Association at York in 1881, and printed among the Reports of that year.

During the last ten years the number of self-recording magnetic observatories where Kew instruments are used has increased to 8 in Europe, 3 in Asia, 1 at the Mauritius, 1 at Melbourne, and 2 in the United States of America, whilst there are self-recording instruments of a different type at Greenwich, Paris, Utrecht, and Toronto.

For the bifilar magnetometer or horizontal force instrument a scale value not greatly differing from ·0005 millimetre-milligrammes of horizontal force for 1 mm. of scale has been adopted at Kew, St. Petersburg, Vienna, Wilhelmshaven, Lisbon, San Fernando, Colaba, and Batavia, and the satisfactory nature of the results now obtained at these observatories clearly shows that it would greatly advance the study of terrestrial magnetism if the same scale value were adopted at other observatories. Nearly the same scale value is adopted for the balance or vertical force magnetometer at Kew, St. Petersburg, Lisbon, Batavia, and Mauritius, but at nearly all other observatories the instruments are considerably less sensitive, and their indications in ordinary magnetic disturbances are too small to be of much service for comparison or measurement of the amount of change of the magnetic forces or of the magnetic potential of the earth.

Thus in the disturbances on August 11th, 1880, at the beginning of the magnetic storm at 10.20 A.M., G.M.T., the disturbance of horizontal force is measured at Colaba (Bombay) by a rise of 3 mm., at Toronto by 7 mm., at Zi-ka-wei by 19 mm., and at Vienna by 13 mm.; but, as appears from Table I., the values of these deflections are very nearly the same, viz., a change of about ·005 millimetre-milligramme. With the scale value recommended, the measurements for this disturbance would have been from 10 to 12 mm.

TABLE I.

	Scale value for 1 mm.	Deflection at 10.20 A.M.	Value in mm.-mgr.	Deflection at 11.30 A.M.	Value in mm.-mgr.
		mm.		mm.	
Kew	·00127	..	..	− 7	·00889
Vienna	·00051	+13	·0066	−25	·01275
St. Petersburg	·00029	+24	·00696	−55	·01595
Colaba	·00167	+ 3	·00501	− 5	·00835
Melbourne	·00066	+ 6	·00396	−12	·00792
Zi-ka-wei	·00025	+19	·00475	−43	·0108
Toronto	·00072	+ 7	·00504	+13	·00936

Thus the sudden changes of force at stations widely separated over the Earth's surface do not differ very greatly from one another. A direct comparison of the several photographic records reduced to Greenwich mean time shows in a very marked manner that the character of a disturbance is the same over a very wide area of the Earth's surface, and that the different phases of the disturbances take place at the different stations at the same instant of time.

In the present paper a comparison is made of more recent magnetic disturbances in June, 1885, at ten other observatories in addition to the seven observatories whose records were previously compared.

Quite recently two additional sets of Kew magnetic self-recording instruments have been set up in the United States of America, at Washington and at Los Angeles, and the first Report from Washington, which has just been published and gives a record of work done in 1889, is very satisfactory, and should stimulate older magnetic observatories to bring their results into a state in which they will be more useful than at present and to publish them. It is satisfactory to find that the scale values adopted at Washington are nearly the same as the scale values which have been adopted at St. Petersburg, Vienna, Wilhelmshaven, Kew, and some other magnetic observatories. The importance of adopting as nearly as possible the same scale values for the similar instruments at different observatories cannot be too strongly enforced in the interest of the study of terrestrial magnetism.

An attempt has also been made to apply the Gaussian analysis to the simultaneous magnetic disturbances in order to discover the amount of change in the magnetic potential of the Earth, which would be sufficient to account for these sudden magnetic disturbances.

The large increase in the number of magnetic observatories with self-recording instruments seemed to warrant this attempt, but great difficulty has been experienced because we have scarcely yet arrived at the state foreshadowed by GAUSS fifty years ago, " when trustworthy and complete observations from all parts of the Earth shall be obtained."

For the complete solution of the problem, we should require records from Africa, from the continents of North and South America, and from Siberia, similar to those which are already obtained in Europe and Asia.

In order to obtain some fair approximation to the changes of magnetic potential, necessary to give rise to the magnetic disturbances, I have collected photographic records of the traces given by the self-recording instruments for June 24th and 25th, 1885, and for other more recent storms, from the observatories whose positions are given in Table II.

The following tables give (1) the positions of these observatories, (2) the absolute values of the magnetic elements, (3) the comparative scale-values of the self-recording instruments for June 24th, 1885, as far as it has been possible to arrive at them; (4) the values, in metric units, of certain magnetic disturbances. From these it will be seen that the instruments of the same type at the different observatories still have very different degrees of sensitiveness.

In some cases the scale-values of the magnetic instruments have not been determined, especially in the case of vertical force instruments. Hence the number of stations to which GAUSS's method is applicable is very limited.

On June 24th, 1885, a magnetic storm began quite suddenly at 10.32 P.M., Greenwich mean time, in which there were several well-marked features.

At 3.48 A.M., on June 25th, there was a sudden and very great disturbance; at 10.20 A.M., there was another characteristic, but rather small disturbance, and a larger disturbance at 12.15 P.M. The storm continued until about 8 A.M., on June 26th.

The values of certain sudden changes in the magnetic elements and the comparative scale-values of the self-recording instruments are given in Table III., except in cases where they are too small, or where there are no means of determining them. From this table it will be seen that at several European stations the sudden change in the horizontal force, at the beginning of the storm, is nearly the same; also the change in the horizontal force at 3.48 A.M., on June 25th, is nearly the same in amount at European stations, and at Colaba and Batavia.

The disturbance at Toronto is very abnormal in all the magnetic elements, and far greater than at any other station.

In Plates 8, and 9, the curves have been carefully traced from the photographic records, and set to Greenwich mean time, and grouped so as to bring out prominently the common features from widely distant stations.

TABLE II.—Latitude, Longitude, and Values of the Magnetic Elements in Metric Units.

	Latitude.	Longitude.	Difference of time.	Declination.	Dip.	H.F.	V.F.	Total force.
	° ′	° ′ ″	h. m. s.	° ′	° ′			
St. Petersburg	59 52 N.	+ 30 29 0	2 1 57	0 35	70 45	1·64	4·65	4·97
Stonyhurst.	53 51 N.	− 2 28 15	−0 9 53	19 47 W.	69 13	1·70	4·48	4·79
Wilhelmshaven	53 32 N.	+ 8 8 48	0 32 35	12 46 W.	67 38	1·80	4·37	4·73
Utrecht	52 5 N.	+ 5 7 48	0 20 31	18 48 W.	68 12	1·76	4·40	4·74
Kew	51 28 N.	− 0 18 45	−0 1 15	18 15 W.	67 36	1·81	4·39	4·75
Brussels	50 51 N.	+ 4 22 12	0 17 29	18 0 W.	67 0	1·82	4·29	4·66
Paris.	48 49 N.	+ 2 20 45	0 9 23	16 10 W.	65 17	1·94	4·21	4·64
Vienna	48 13 N.	+ 16 21 0	1 5 0	9 32 W.	63 23	2·06	4·13	4·60
Toronto	43 40 N.	− 79 23 39	−5 17 35	3 41 W.	74 52	1·65	6·10	6·33
Coimbra	40 12 N.	− 8 23 15	−0 33 33	19 18 W.	60 27	2·22	3·92	4·50
Lisbon	38 43 N.	− 9 8 45	−0 36 35	18 46 W.	58 54	2·29	3·80	4·44
San Fernando	36 28 N.	− 6 12 24	−0 24 50	22 0 W.	59 0	2·287	3·806	4·44
Zi-ka-wei	31 12·5 N.	+121 26 15	8 5 45	2 9 W.	46 18	3·29	3·44	4·76
Colaba	18 54 N.	+ 72 48 56	4 51 16	− 0 56 E.	19 43	3·74	1·34	3·97
Batavia	− 6 11 S.	+106 49 55	7 7 20	− 1 51 E.	− 27 30 S.	3·67	1·91	4·14
Mauritius	− 20 6 S.	+ 57 33 9	3 50 13	10 33 W.	− 55 5 S.	1·584	2·802	4·24
Melbourne	− 37 50 S.	+144 58 42	9 39 55	8 0 E.	− 67 9 S.	2·36	5·45	6·03

Table III.—Scale Values and Changes in the Magnetic Elements in Metric Units, June 24 and 25, 1885.

	Scale values for 1 mm.			Changes on June 24 at 10.32 p.m.		Changes on June 25 at 3.48 a.m.		
	Declination.	H.F.	V.F.	H.F.	V.F.	Declination.	H.F.	V.F.
				mm.	mm.		mm.	mm.
St. Petersburg . .	0′·96	·000497	·000544	12 = ·00596		44′	30 = ·0149	3·5 = ·0019
Stonyhurst . . .	1·128	·0024 or ·0022	·0002	3·5 = ·0084 or ·0077	− 4 = ·0008	13·54	7·5 = ·0180 or ·0166	5 = ·0010
Wilhelmshaven . .	1·15	·0005	·0011	13·5 = ·00675	..	13·8	30 = ·0150	
Utrecht.	1·15	..	..	15·5 = ·0082	− 13·6 = ·0142	19·55	..	− 30 =*
Kew	0·87	·0005	·0005	13·5 = ·00675	3 = ·0015	13·9	27 = ·0135	2 = ·0010
Brussels	1·1	..	..	3 =	2 =	..	6 =	7·5 =
Paris	1·39	·00093	·0008	7 = ·0065	2 = ·0016	..	13 = ·0121	3 = ·0024
Vienna	..	·00055 (?)	..	12 = ·0066	3 =	..	22 = ·0121	5 =
Toronto	1·279	·001083	·0011	11 = ·0119	3 = ·0033	53·72	43 = ·04657	43 = ·0473
Coimbra	1·13	·00079	·0007					
Lisbon	1·13	·00055	·000475	9·5 = ·00523	..	5·65	22 = ·0121	
San Fernando . .	1·146	·00041	..	7 = ·0029	3 =	3·43	17 = ·0070	5 =
Zi-ka-wei	0·63	..	..	6 =	..	..	18 =	
Colaba	2·9	·000514	·00127	7 = ·0036	..	..	22 = ·0113	
Batavia	1·14	·000475	·000569	7 = ·0033	4·5 = ·00256	..	24 = ·0114	8 = ·00455
Mauritius	1·116	·0017	·000535	1·5 = ·0026				
Melbourne	1·14	·0015	·00138	2·5 = ·00375	..	13·68	15 = ·0225	12 = ·01656

* The change in V.F. depends on the change in H.F., which at 3.48 a.m. was not recorded.

With regard to the scale values in Table III., we may note that the values of H.F. and V.F. in metric units, for 1 mm. of height of scale, and the values of H.F. and V.F. in centimetre-gramme-second units for 1 cm. of height of scale, are expressed by the same number. Thus, in the Kew curves, the values of H.F. and V.F. in metric units for 1 mm. of height of scale is ·0005 unit, and the values of H.F. and V.F., for 1 cm. of height of scale is ·0005 in c.g.s units.

Hence, in comparing past measurements in metric units, with 1 mm. for unit of height, with measurements in c.g.s. units, taking 1 cm. as unit of height, there will be very little trouble, as the scale values in the two sets of units are expressed by the same number.

Hence, in future, it will be advisable for the sake of comparison, and, at the same time, more in accordance with the c.g.s. system of units, to state the scale values under the form, 1 cm. = ·0005 c.g.s. units.

On comparing the traces for H.F. from different observatories (see Plate 8) it will be seen that the trace from Stonyhurst shows what may at first be regarded as a very slight disturbance, and, at 3.48 A.M., the maximum disturbance appears to be comparatively small, but from Table III., it will be seen that the actual disturbance is as large, or even larger, than at any other station except Toronto.

There is every reason to suppose that the trace at Stonyhurst would have shown the individual features of the storm quite as well as the Colaba, or the Kew, or the Lisbon traces, if the scale value, ·0005, had then been in use instead of the scale value ·0022, and the measurements could then have been made with much greater accuracy. On the other hand, the vertical force magnetometer at Stonyhurst is very much more sensitive than that at any other station, and the trace is very similar in character to the St. Petersburg V.F. trace. At 3.48 A.M. there is very little disturbance at Kew, and a small disturbance in V.F. at Stonyhurst, viz., a fall of ·001 metric unit.

The V.F. trace for Batavia is placed near to the Stonyhurst trace in Plate 8, but, as is shown in Table III., the sudden fall in V.F. at Batavia, at 10.32 P.M., on the 24th, is nearly four times as great; and at 3.48 A.M., on the 25th, is more than four times the corresponding fall in V.F. at Stonyhurst.

As Batavia is south of the equator, a diminution of the vertical force means a change in the direction of the needle corresponding to an increase of dip of the pole pointing towards the north; hence, for comparison with stations in the northern hemisphere, the Batavia V.F. trace inverted is also given in Plate 8.

The disturbance in V.F. is very much smaller than the disturbance in H.F., at all stations except Toronto, Batavia, and Melbourne. At Utrecht, the disturbance in V.F. nearly resembles the disturbance in declination, and is much greater *apparently* than the V.F. disturbances at Wilhelmshaven, or at any other observatory in the north-west of Europe. Unfortunately, the photographic paper on which the traces are taken is too narrow, and so the H.F. trace passes off the paper during the great magnetic disturbances, but there is a great resemblance between the traces given by the three Utrecht instruments. This resemblance is due to the fact that the horizontal and vertical force changes are given by means of deflecting magnets and iron induction bars according to LAMONT'S plan; hence, the horizontal and vertical force needles are affected by changes in declination, and the vertical force trace is also affected by changes in horizontal force.

Thus the change in horizontal force in metric units

$$\delta X = \cdot 000376\,(p_1 - p),$$

where p_1 and p are the changes in ordinates of horizontal force and declination expressed in millimetres.

If δZ be the increase of vertical force, and p_2 the change in the ordinate of the vertical force in millimetres, then

$$\delta Z = \cdot 00326\,(p_2 - p) + \cdot 00172\,(p_1 - p).$$

Thus, at 10·32 P.M. on June 24, $p = -\ 6{\cdot}4$ mm.,

$$p_1 = +\ 15{\cdot}5 \text{ mm. and } p_2 = -\ 13{\cdot}6 \text{ mm.}$$

Hence

$$\delta X = \cdot 000376\,(15{\cdot}5 + 6{\cdot}4) = \cdot 0082 \text{ metric unit,}$$

and

$$\delta Z = \cdot 00326\,(-\ 13{\cdot}6 + 6{\cdot}4) + \cdot 00172\,(15{\cdot}5 + 6{\cdot}4) = \cdot 0142 \text{ metric unit.}$$

Plate 9 contains only traces of horizontal force for June 25, 1885, from mid-day until 8 P.M. From 2.10 P.M. there is a large and continuous well-marked period of disturbance, beginning with an increase of force varying from ·006 to ·01 metric unit, followed by a decrease of very nearly the same amount at 3 P.M.

Another similar period of disturbance, well-marked at Melbourne, Batavia, and Bombay, as well as throughout Europe, occurs between 5.30 P.M. and 6.30 P.M. on the same day.

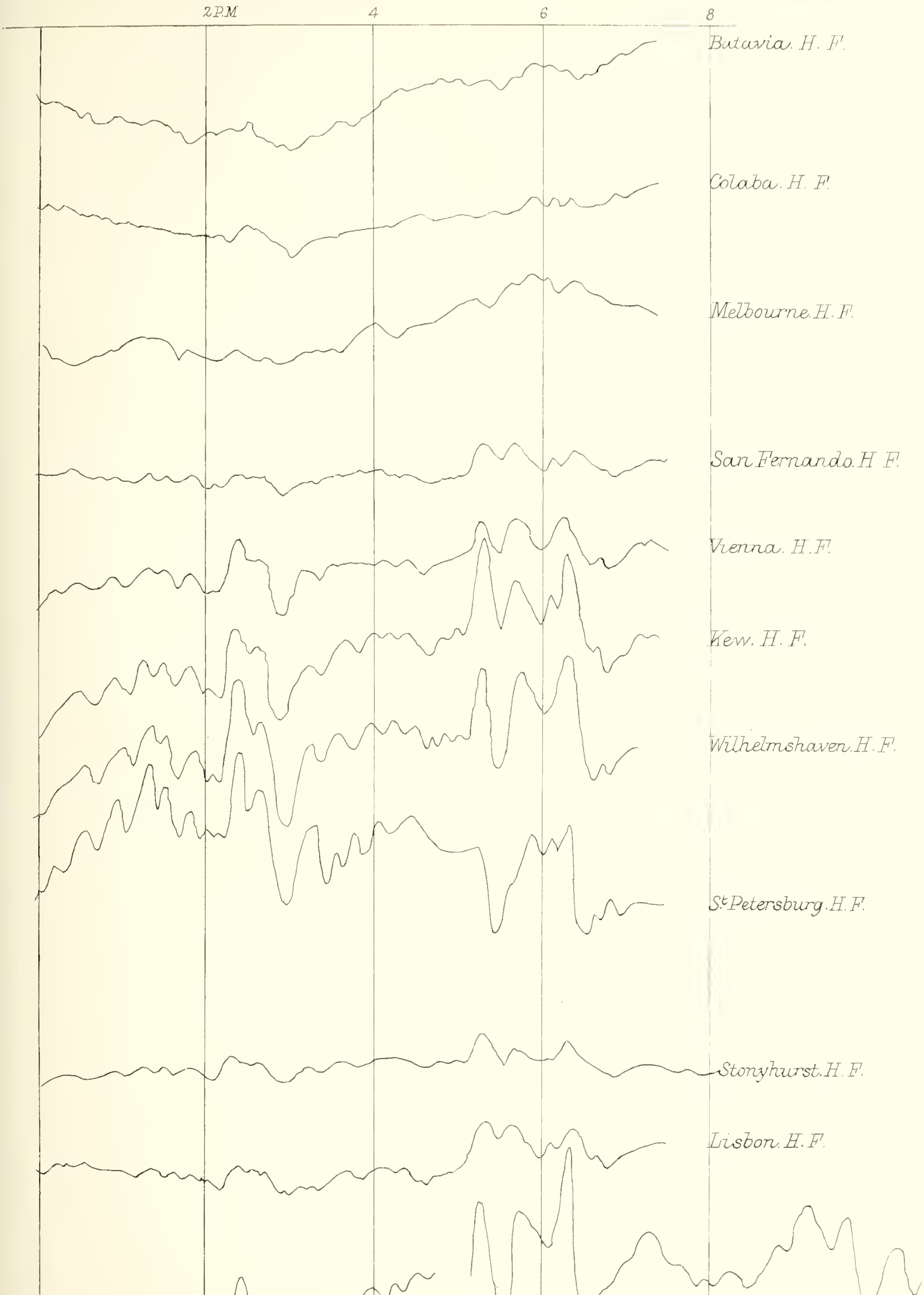
The values of these changes in metric units are given in Table IV.

TABLE IV.—Changes in Horizontal Force on June 25, 1885.

	At 2.10 P.M. (G.M.T.).	At 3 P.M.	At 5.10 P.M.	At 5.30 P.M.	At 6.20 P.M.
St. Petersburg	·0075	− ·0070	·0005	− ·008	− ·010
Stonyhurst	·0088	− ·0066	·011	− ·011	− ·009
Wilhelmshaven	·0090	− ·0075	·0062	− ·009	+ ·011
Utrecht	·0056	− ·0056	·0056	− ·003	− ·006
Kew	·0060	− ·0050	·0080	− ·008	− ·010
Vienna	·0055	− ·0050	·0033	− ·003	− ·005
Lisbon	·0027	− ·0022	·0033	− ·002	− ·003
San Fernando	·0012	− ·0012	·0025	− ·001	− ·002
Colaba	·0015	− ·0015	·0010	..	− ·001
Batavia	·0014	− ·0014	·0010	− ·001	− ·001
Melbourne	·0037	− ·0022	·0030	− ·003	− ·004

There is some little uncertainty as to the values at the four last stations in this table, because the measurements to be made are very small. This reduces the number of observatories at which the Gaussian analysis can be applied with advantage to too small a number to determine the Gaussian coefficients for this storm.


12
10
8
6
4
2
0
10 P.M.
West, Newman lith.
Toronto. H. F.
Zi-ka-Wei. H. F.
Batavia. H. F.
Colaba. H. F.
Toronto. V. F.
Melbourne. H. F.
San Fernando. H. F.
Vienna. H. F.
Kew. H. F.
Wilhelmshaven. H. F.
St Petersburg. H. F.
Stonyhurst. H. F.
Lisbon. H. F.
St Petersburg. Decl.n
Toronto. Decl.n
Wilhelmshaven. Decl.n
Kew. Decl.n
Utrecht. Decl.n
Utrecht. V. H. (inverted)
Batavia. V. F. (inverted)
Vienna. V. F.
Kew. V. F.
St Petersburg. V. F.
Stonyhurst. V. F.
Batavia. V. F.



West, Newman lith.

V. *On the Locus of Singular Points and Lines which occur in connection with the Theory of the Locus of Ultimate Intersections of a System of Surfaces.*

By M. J. M. HILL, *M.A.*, *Sc.D.*, *Professor of Mathematics at University College, London.*

Communicated by Professor HENRICI, *F.R.S.*

Received October 5,—Read November 19, 1891.

INTRODUCTION.

IN a paper "On the c- and p-Discriminants of Ordinary Integrable Differential Equations of the First Order," published in vol. 19 of the 'Proceedings of the London Mathematical Society,' the factors which occur in the c-discriminant of an equation of the form $f(x, y, c) = 0$, where $f(x, y, c)$ is a rational integral function of x, y, c, are determined analytically.

It is shown* that if $E = 0$ be the equation of the envelope locus of the curves $f(x, y, c) = 0$; if $N = 0$ be the equation of their node-locus; if $C = 0$ be the equation of their cusp-locus, then the factors of the discriminant are E, N^2, C^3.

The singularities considered are those whose forms depend on the terms of the second degree only, when the origin of coordinates is at the singular point.

The object of this paper is to extend these results to surfaces.

It is well known that if the equation of a system of surfaces contain arbitrary parameters, and if a locus of ultimate intersections exist, then there cannot be more than two independent parameters.

Hence the investigation falls naturally into two parts: the first is the case where there is only one independent parameter, and the second is the case where there are two.

The investigation given in this paper is limited to the case in which the equation is rational and integral, both as regards the coordinates and the parameters.†

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* The theorem was originally given by Professor CAYLEY in the 'Messenger of Mathematics,' vol. 2, 1872, pp. 6–12.

† An abstract of the contents of this paper has been printed in the Proceedings, vol. 50, pp. 180–186. A table of contents will be found below, pp. 274–278.

21.6.92.

PART I.—THE EQUATION OF THE SYSTEM OF SURFACES IS A RATIONAL INTEGRAL FUNCTION OF THE COORDINATES AND ONE ARBITRARY PARAMETER.

SECTION I. (Arts. 1–6).—THE FACTORS OF THE DISCRIMINANT WHICH IN GENERAL CORRESPOND TO ENVELOPE AND SINGULAR LINE LOCI.

Art. I.—*To show that if $E = 0$ be the equation of the Envelope Locus, the Discriminant contains E as a factor.*

Let the equation be

$$f(x, y, z, a) = 0 \quad \ldots\ldots\ldots\ldots \quad (1),$$

where x, y, z are the coordinates, a the parameter, and f is supposed to be a rational integral function of x, y, z, a.

Denoting partial differentiation when x, y, z, a are treated as independent variables by D, the locus of ultimate intersections can be obtained by eliminating a between (1) and

$$\frac{\mathrm{D}f(x, y, z, a)}{\mathrm{D}a} = 0 \quad \ldots\ldots\ldots\ldots \quad (2).$$

Let the roots of (2) treated as an equation in a be a_1, a_2, . . . , which will at first be supposed to be all different, so that they do not make

$$\frac{\mathrm{D}^2 f(x, y, z, a)}{\mathrm{D}a^2} = 0.$$

Then if R be a factor introduced to make the discriminant Δ, obtained by eliminating a between (1) and (2) of the proper order and weight,

$$\Delta = \mathrm{R} f(x, y, z, a_1) f(x, y, z, a_2) \ldots \quad (3).$$

Let $x = \xi$, $y = \eta$, $z = \zeta$ satisfy (1) and (2) when $a = \alpha$.

Suppose that a_1 becomes α, when $x = \xi$, $y = \eta$, $z = \zeta$, therefore,

$$f(\xi, \eta, \zeta, \alpha) = 0 \quad \ldots\ldots\ldots\ldots \quad (4),$$

$$\frac{\mathrm{D}f(\xi, \eta, \zeta, \alpha)}{\mathrm{D}\alpha} = 0 \quad \ldots\ldots\ldots\ldots \quad (5).$$

Now in Δ put $x = \xi$, $y = \eta$, $z = \zeta$; and consequently $a_1 = \alpha$, therefore $f(x, y, z, a_1)$ becomes $f(\xi, \eta, \zeta, \alpha)$ and consequently vanishes.

Therefore Δ vanishes when $x = \xi$, $y = \eta$, $z = \zeta$.

The next step is to show that the locus of ultimate intersections is the envelope.

Write, for brevity,

$$\Delta = \mathrm{Q} f(z, y, z, a_1) = \mathrm{Q} f_1 \quad \ldots\ldots\ldots\ldots \quad (6).$$

Now denoting differentiation when x, y, z are the only independent variables by ∂,

$$\frac{\partial\Delta}{\partial x} = \frac{\partial Q}{\partial x} f_1 + Q\left(\frac{Df_1}{Dx} + \frac{Df_1}{Da_1}\frac{\partial a_1}{\partial x}\right).$$

Hence, since $a = a_1$ satisfies (2)

$$\frac{\partial\Delta}{\partial x} = \frac{\partial Q}{\partial x} f_1 + Q\frac{Df_1}{Dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

assuming that $Df_1/Da_1\ \partial a_1/\partial x$ vanishes when Df_1/Da_1 vanishes.

Now when $x = \xi$, $y = \eta$, $z = \zeta$, $f_1 = 0$, therefore

$$\frac{\partial\Delta}{\partial x} = \left[Q\frac{Df_1}{Dx}\right]_{\substack{x=\xi\\ y=\eta\\ z=\zeta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Hence when $x = \xi$, $y = \eta$, $z = \zeta$

$$\frac{\partial\Delta}{\partial x}\bigg/\frac{Df_1}{Dx} = \frac{\partial\Delta}{\partial y}\bigg/\frac{Df_1}{Dy} = \frac{\partial\Delta}{\partial z}\bigg/\frac{Df_1}{Dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Now the tangent plane to the surface

$$f(x, y, z, \alpha) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

at the point $x = \xi$, $y = \eta$, $z = \zeta$ is

$$(X - \xi)\frac{Df}{D\xi} + (Y - \eta)\frac{Df}{D\eta} + (Z - \zeta)\frac{Df}{D\zeta} = 0 \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Now $Df/D\xi$ stands for $Df(x, y, z, \alpha)/Dx$, when $x = \xi$, $y = \eta$, $z = \zeta$. And the value of Df_1/Dx, *i.e.*, $Df(x, y, z, a_1)/Dx$, when $x = \xi$, $y = \eta$, $z = \zeta$, and, therefore, $a_1 = \alpha$ is the same as the value of $Df(x, y, z, \alpha)/Dx$, when $x = \xi$, $y = \eta$, $z = \zeta$.

This may be expressed thus :—

$$\frac{\partial\Delta}{\partial\xi}\bigg/\frac{Df}{D\xi} = \frac{\partial\Delta}{\partial\eta}\bigg/\frac{Df}{D\eta} = \frac{\partial\Delta}{\partial\zeta}\bigg/\frac{Df}{D\zeta}.$$

Hence the tangent planes to the surfaces $\Delta = 0$, $f(x, y, z, \alpha) = 0$ at the point ξ, η, ζ coincide.

This proves the envelope property in general for the locus of ultimate intersections.

Hence Δ vanishes, if in it x, y, z be made respectively equal to ξ, η, ζ, the coordinates of any point on the envelope-locus.

Therefore Δ contains E as a factor.

But the conclusion fails if

$$\frac{Df}{D\xi} = 0, \quad \frac{Df}{D\eta} = 0, \quad \frac{Df}{D\zeta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Hence the work itself suggests the examination of this exceptional case, *i.e.*, where a locus of singular points or lines exists.

Example 1.—*Envelope Locus.*

Let the surfaces be

$$[\phi(x, y, z) - a]^2 + \chi(x, y, z) = 0.$$

(A.) *The Discriminant.*

The discriminant is found by eliminating a between the above equation, and

$$-2[\phi(x, y, z) - a] = 0.$$

Hence the discriminant is $\chi(x, y, z)$.

Hence the locus of ultimate intersections is

$$\chi(x, y, z) = 0.$$

(B.) *The envelope locus is* $\chi(x, y, z) = 0$.

For let ξ, η, ζ be any point on $\chi(x, y, z) = 0$.

Let $a = \phi(\xi, \eta, \zeta)$, and consider the single surface

$$[\phi(x, y, z) - \phi(\xi, \eta, \zeta)]^2 + \chi(x, y, z) = 0.$$

Put $x = \xi + X$, $y = \eta + Y$, $z = \zeta + Z$.

Then the lowest terms in X, Y, Z are

$$X\frac{\partial\chi}{\partial\xi} + Y\frac{\partial\chi}{\partial\eta} + Z\frac{\partial\chi}{\partial\zeta}.$$

Hence the surface considered touches $\chi(x, y, z) = 0$ at ξ, η, ζ.

Hence $\chi(x, y, z) = 0$ is the envelope.

It touches the surface at every point of the curve

$$\chi(x, y, z) = 0,$$
$$\phi(x, y, z) = \phi(\xi, \eta, \zeta).$$

Hence this curve is the characteristic.

Art. 2.—*To prove that the Locus of Conic Nodes of the Surfaces $f(x, y, z, a) = 0$ is a Curve, not a Surface.*

At every point of the locus of conic nodes the equations

$$f(x, y, z, a) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

$$\frac{\mathrm{D}f(x, y, z, a)}{\mathrm{D}x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

$$\frac{\mathrm{D}f(x, y, x, a)}{\mathrm{D}y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

$$\frac{\mathrm{D}f(x, y, z, a)}{\mathrm{D}z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16),$$

are simultaneously satisfied.

In general these are satisfied by a finite number of values of x, y, z, a only. Hence there are only a finite number of conic nodes.

The next case is that in which equations (13)–(16) are equivalent to three independent equations only, and then it is possible to satisfy them by relations of the form

$$x = \phi(a), \qquad y = \psi(a), \qquad z = \chi(a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

In this case there is a curve locus of conic nodes. But as such a locus is defined by two equations, it cannot be determined by equating a factor of the discriminant to zero.

The next case is that in which equations (13)–(16) are equivalent to two independent equations only. Eliminating a between these, the equation of a surface is obtained. This is the case which will be further examined, and it will be shown that the tangent cone at every conic node must break up into two planes, *i.e.*, the conic node becomes a binode.*

Let ξ, η, ζ be the conic node on the surface (10).

Let $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ be the conic node on the consecutive surface

$$f(x, y, z, \alpha + \delta\alpha) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Then

$$f(\xi, \eta, \zeta, \alpha) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

$$\frac{\mathrm{D}f(\xi, \eta, \zeta, \alpha)}{\mathrm{D}\xi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

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* In this connection may be noticed Art. 11, in which it is proved that if a surface have upon it a line at every point of which there is a conic node, then the tangent cone at every conic node must break up into two planes so that the line is a binodal line.

$$\frac{Df(\xi, \eta, \zeta, \alpha)}{D\eta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

$$\frac{Df(\xi, \eta, \zeta, \alpha)}{D\zeta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

and the equations obtained from (19)–(22) by changing ξ, η, ζ, α into $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$, $\alpha + \delta\alpha$ respectively.

Denoting differential coefficients by brackets containing the independent variables, with regard to which the differentiations are performed, these last equations become by means of (19)-(22)

$$[\alpha](\delta\alpha) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23),$$

$$[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) + [\xi, \alpha](\delta\alpha) = 0 \quad . \quad . \quad . \quad (24),$$

$$[\eta, \xi](\delta\xi) + [\eta, \eta](\delta\eta) + [\eta, \zeta](\delta\zeta) + [\eta, \alpha](\delta\alpha) = 0 \quad . \quad . \quad . \quad (25),$$

$$[\zeta, \xi](\delta\xi) + [\zeta, \eta](\delta\eta) + [\zeta, \zeta](\delta\zeta) + [\zeta, \alpha](\delta\alpha) = 0 \quad . \quad . \quad . \quad (26).$$

By (23),

$$[\alpha] = 0$$

at every point of the conic node locus. Hence the co-ordinates of every point on the conic node locus satisfy the equation of the locus of ultimate intersections.*

Further, since $[\alpha] = 0$ at every point on the conic node locus, the corresponding equation is satisfied at the conic node on the surface (18).

Hence

$$[\alpha, \xi](\delta\xi) + [\alpha, \eta](\delta\eta) + [\alpha, \zeta](\delta\zeta) + [\alpha, \alpha](\delta\alpha) = 0 \quad . \quad . \quad . \quad (27).$$

Since equations (24)–(27) must give consistent values for $\delta\xi : \delta\eta : \delta\zeta : \delta\alpha$, it follows that the Jacobian

$$\frac{D\{[\xi], [\eta], [\zeta], [\alpha]\}}{D\{\xi, \eta, \zeta, \alpha\}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

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[*] I am indebted to Dr. FORSYTH for the following example:—

Let the surfaces be

$$(x - a)^2 + (y - a)^2 - \kappa^2(z - a)^2 = 0.$$

The discriminant is

$$\kappa^2(2z - x - y)^2 + (\kappa^2 - 2)(x - y)^2.$$

Hence the locus of ultimate intersections consists of the two planes

$$\kappa(2z - x - y) = \pm \sqrt{2 - \kappa^2}(x - y).$$

These planes intersect in the straight line $x = y = z$, which is the locus of conical points of the surfaces.

If this were the only relation between these equations, they would determine the ratios $\delta\xi : \delta\eta : \delta\zeta : \delta\alpha$.

Hence there would be a curve locus, not a surface locus of conic nodes.

If, then, there be a surface locus, equations (24)–(27) must be equivalent to two independent equations only.

Expressing that (24)–(26) are equivalent to two independent equations only, it follows that

$$\frac{\mathrm{D}\{[\xi], [\eta], [\zeta]\}}{\mathrm{D}\{\xi, \eta, \zeta\}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

But this is the condition that the tangent cone at the conic node should break up into two planes, and then the conic node becomes a binode.

Hence there cannot be a surface locus of conic nodes, unless the conic nodes become binodes.

Since equations (13)–(16) are equivalent to two independent equations only, every point on the intersection of the surfaces represented by (13) and (14) is a binode on the surface (13).

Hence the surface (13) has a binodal line.

The locus of these binodal lines is a surface at every point of which equations (1) and (2) are satisfied, hence it is a part of the locus of ultimate intersections, and its equation can be determined by equating a factor of the discriminant to zero.

Art. 3.—*To find the conditions which hold at every point on a Surface Locus of Binodal Lines.*

In this case (29) holds.

Hence, in order that (24)–(26) may give finite values for $\delta\xi : \delta\eta : \delta\zeta : \delta\alpha$,

$$\frac{\mathrm{D}\{[\xi], [\eta], [\zeta]\}}{\mathrm{D}\{\xi, \eta, \alpha\}} = 0 = \frac{\mathrm{D}\{[\xi], [\eta], [\alpha]\}}{\mathrm{D}\{\xi, \eta, \zeta\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

$$\frac{\mathrm{D}\{[\xi], [\eta], [\zeta]\}}{\mathrm{D}\{\xi, \zeta, \alpha\}} = 0 = \frac{\mathrm{D}\{[\xi], [\zeta], [\alpha]\}}{\mathrm{D}\{\xi, \eta, \zeta\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

$$\frac{\mathrm{D}\{[\xi], [\eta], [\zeta]\}}{\mathrm{D}\{\eta, \zeta, \alpha\}} = 0 = \frac{\mathrm{D}\{[\eta], [\zeta], [\alpha]\}}{\mathrm{D}\{\xi, \eta, \zeta\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

Now (30) shows that (27) depends on (24) and (25). Hence, in this case, the four equations (24–27) are equivalent to two independent equations only, which is obvious since (13–16) are equivalent to two independent equations only.

Art. 4.—*To find the conditions which hold at every point on a Surface Locus of Unodal Lines.*

At such a point the tangent cone, whose equation is

$$\begin{aligned}&[\xi, \xi](X-\xi)^2+[\eta, \eta](Y-\eta)^2+[\zeta, \zeta](Z-\zeta)^2\\ &2[\eta, \zeta](Y-\eta)(Z-\zeta)+2[\zeta, \xi](Z-\zeta)(X-\xi)+2[\xi, \eta](X-\xi)(Y-\eta)=0 \quad (33),\end{aligned}$$

breaks up into two coincident planes.

Hence

$$\begin{aligned}&[\xi, \xi]:[\xi, \eta]:[\xi, \zeta]\\ =&[\eta, \xi]:[\eta, \eta]:[\eta, \zeta]\\ =&[\zeta, \xi]:[\zeta, \eta]:[\zeta, \zeta] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).\end{aligned}$$

Of the four equations (24)–(27), which are satisfied when there is a surface locus of binodal lines, it has already been shown that only two are independent. The same equations hold when there is a locus of unodal lines.

Multiply (24) by $[\eta, \xi]$, (25) by $[\xi, \xi]$, subtract and use (34). Then,

$$(\delta\alpha)\{[\eta, \xi][\xi, \alpha]-[\xi, \xi][\eta, \alpha]\}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

therefore,

$$[\eta, \xi][\xi, \alpha]-[\xi, \xi][\eta, \alpha]=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).$$

Similarly

$$[\zeta, \xi][\xi, \alpha]-[\xi, \xi][\zeta, \alpha]=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

By (34) and (36) it follows that (25) depends on (24).

By (34) and (37) it follows that (26) depends on (24).

By (34), (36), (37), it follows that, if the values of $\delta\xi : \delta\eta : \delta\zeta : \delta\alpha$ satisfying (24)–(27) are finite, then (27) depends on (24), and the following ratios hold:—

$$\begin{aligned}&[\xi, \xi]:[\xi, \eta]:[\xi, \zeta]:[\xi, \alpha]\\ =&[\eta, \xi]:[\eta, \eta]:[\eta, \zeta]:[\eta, \alpha]\\ =&[\zeta, \xi]:[\zeta, \eta]:[\zeta, \zeta]:[\zeta, \alpha]\\ =&[\alpha, \xi]:[\alpha, \eta]:[\alpha, \zeta]:[\alpha, \alpha] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).\end{aligned}$$

In this case, then, (24)–(27) are equivalent to one independent equation only.

It may be noticed that in the case in which

$$[\alpha, \xi]=0, \ [\alpha, \eta]=0, \ [\alpha, \zeta]=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39),$$

in order that (27) may be satisfied,

$$[\alpha, \alpha] = 0 \quad \ldots \ldots \ldots \ldots \quad (40).$$

And now (27) becomes an identical equation. It does not depend on (24).

Art. 5.—*Examination of the Discriminant Δ, and its Differential Coefficients, when a Surface Locus of Binodal Lines exists. Proof that Δ contains B^2 as a factor.*

Let ξ, η, ζ be a point on the binodal line on the surface (10).

Then when $x = \xi$, $y = \eta$, $z = \zeta$,

$$f_1 = 0, \quad \frac{Df_1}{Dx} = 0.$$

Hence by (6) and (7), when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\Delta = 0, \quad \frac{\partial \Delta}{\partial x} = 0.$$

Similarly

$$\frac{\partial \Delta}{\partial y} = 0, \quad \frac{\partial \Delta}{\partial z} = 0.$$

Hence if $B = 0$ be the equation of the surface locus of binodal lines, Δ contains B^2 as a factor.

Example 2.—*Locus of Binodal Lines.*

Let the surfaces be

$$[\phi(x, y, z) - a]^2 + \chi(x, y, z)\,[\psi(x, y, z)]^2 = 0.$$

A. *The Discriminant.*

This is found by eliminating a between the above, and

$$-[2\phi(x, y, z) - a] = 0.$$

Hence the discriminant is $\chi(x, y, z)\,[\psi(x, y, z)]^2$.

Hence the locus of ultimate intersections is

$$\chi(x, y, z)\,[\psi(x, y, z)]^2 = 0.$$

B. *The Locus of Binodal Lines is $\psi(x, y, z) = 0$.*

Let ξ, η, ζ be any point on the surface $\psi(x, y, z) = 0$.

Take $a = \phi(\xi, \eta, \zeta)$ and consider the single surface

$$[\phi(x, y, z) - \phi(\xi, \eta, \zeta)]^2 + \chi(x, y, z)\,[\psi(x, y, z)]^2 = 0.$$

Put $x = \xi + \mathrm{X}$, $y = \eta + \mathrm{Y}$, $z = \zeta + \mathrm{Z}$.
Then the lowest terms in X, Y, Z are

$$\left[\mathrm{X}\frac{\partial\phi}{\partial\xi} + \mathrm{Y}\frac{\partial\phi}{\partial\eta} + \mathrm{Z}\frac{\partial\phi}{\partial\zeta}\right]^2 + \chi(\xi, \eta, \zeta)\left[\mathrm{X}\frac{\partial\psi}{\partial\xi} + \mathrm{Y}\frac{\partial\psi}{\partial\eta} + \mathrm{Z}\frac{\partial\psi}{\partial\zeta}\right]^2.$$

This breaks up into two factors of the first degree in X, Y, Z.

Hence ξ, η, ζ is a binode on the surface considered. Now the only relation satisfied by ξ, η, ζ is $\psi(\xi, \eta, \zeta) = 0$.

Hence any point ξ, η, ζ, on the surface $\psi(x, y, z) = 0$, is a binode on

$$[\phi(x, y, z) - \phi(\xi, \eta, \zeta)]^2 + \chi(x, y, z)\,[\psi(x, y, z)]^2 = 0.$$

Hence every point of intersection of the surfaces

$$\psi(x, y, z) = 0,$$

and

$$[\phi(x, y, z) - a]^2 + \chi(x, y, z)\,[\psi(x, y, z)]^2 = 0,$$

is a binode on the latter surface.

The equations of the binodal line of this surface are, therefore,

$$\psi(x, y, z) = 0,$$
$$\phi(x, y, z) = a.$$

This accounts for the occurrence of the factor $[\psi(x, y, z)]^2$ in Δ.

C. *The Envelope Locus is* $\chi(x, y, z) = 0$.

This may be proved as in Example 1.

Art. 6.—*Examination of the Discriminant* Δ, *and its Differential Coefficients, when a Surface Locus of Unodal Lines exists. Proof that* Δ *contains* U^3 *as a factor.*

Differentiating (7) with regard to x and y,

$$\frac{\partial^2\Delta}{\partial x^2} = f_1\frac{\partial^2 \mathrm{Q}}{\partial x^2} + 2\frac{\partial \mathrm{Q}}{\partial x}\left(\frac{\mathrm{D}f_1}{\mathrm{D}x} + \frac{\mathrm{D}f_1}{\mathrm{D}a_1}\frac{\partial a_1}{\partial x}\right) + \mathrm{Q}\left(\frac{\mathrm{D}^2 f_1}{\mathrm{D}x^2} + \frac{\mathrm{D}^2 f_1}{\mathrm{D}x\,\mathrm{D}a_1}\frac{\partial a_1}{\partial x}\right) \quad . \quad . \quad . \quad . \quad (41),$$

$$\frac{\partial^2\Delta}{\partial x\partial y} = f_1\frac{\partial^2 Q}{\partial x\partial y} + \frac{\partial Q}{\partial x}\left(\frac{Df_1}{Dy} + \frac{Df_1}{Da_1}\frac{\partial a_1}{\partial y}\right) + \frac{\partial Q}{\partial y}\frac{Df_1}{Dx} + Q\left(\frac{D^2f_1}{Dx\,Dy} + \frac{D^2f_1}{Dx\,Da_1}\frac{\partial a_1}{\partial y}\right) \quad . \quad (42).$$

To find $\partial a_1/\partial x$, $\partial a_1/\partial y$ it is necessary to use the equation

$$\frac{Df(x, y, z, a_1)}{Da_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43).$$

This gives

$$\frac{D^2f_1}{Dx\,Da_1} + \frac{D^2f_1}{Da_1^2}\frac{\partial a_1}{\partial x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44),$$

$$\frac{D^2f_1}{Dy\,Da_1} + \frac{D^2f_1}{Da_1^2}\frac{\partial a_1}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Now reserving the case, according to the remarks in the Abstract ('Proc. Roy. Soc., vol. 50, p. 180) and Art. 1, in which

$$\frac{D^2f_1}{Da_1^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46),$$

for further consideration, because, in this case, $\partial a_1/\partial x$, $\partial a_1/\partial y$, both become infinite or indeterminate, it follows that

$$\frac{\partial a_1}{\partial x} = -\frac{D^2f_1}{Dx\,Da_1}\Big/\frac{D^2f_1}{Da_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47),$$

$$\frac{\partial a_1}{\partial y} = -\frac{D^2f_1}{Dy\,Da_1}\Big/\frac{D^2f_1}{Da_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).$$

Hence

$$\frac{\partial^2\Delta}{\partial x^2} = \frac{\partial^2 Q}{\partial x^2}f_1 + 2\frac{\partial Q}{\partial x}\frac{Df_1}{Dx} + Q\left\{\frac{D^2f_1}{Dx^2}\frac{D^2f_1}{Da_1^2} - \left(\frac{D^2f_1}{Dx\,Da_1}\right)^2\right\}\Big/\frac{D^2f_1}{Da_1^2} \quad . \quad . \quad . \quad . \quad . \quad (49),$$

$$\frac{\partial^2\Delta}{\partial x\,\partial y} = \frac{\partial^2 Q}{\partial x\partial y}f_1 + \frac{\partial Q}{\partial x}\frac{Df_1}{Dy} + \frac{\partial Q}{\partial y}\frac{Df_1}{Dx} + Q\left\{\frac{D^2f_1}{Dx\,Dy}\frac{D^2f_1}{Da_1^2} - \frac{D^2f_1}{Dx\,Da_1}\frac{D^2f_1}{Dy\,Da_1}\right\}\Big/\frac{D^2f_1}{Da_1^2} \quad . \quad (50).$$

Hence, if ξ, η, ζ be a unode on the surface (10), and a_1 become equal to α when $x = \xi$, $y = \eta$, $z = \zeta$, then by means of (38),

$$\frac{\partial^2\Delta}{\partial x^2} = 0, \qquad \frac{\partial^2\Delta}{\partial x\,\partial y} = 0,$$

when $x = \xi$, $y = \eta$, $z = \zeta$.

Similarly all the other second differential coefficients of Δ with regard to x, y, z vanish when $x = \xi$, $y = \eta$, $z = \zeta$.

Hence, if $U = 0$ be the equation of the surface locus of unodal lines, Δ contains U^3 as a factor.

Example 3.—*Locus of Unodal Lines.*

Let the surfaces be

$$[\phi(x, y, z) - a]^2 + [\chi(x, y, z)]^3 = 0.$$

(A.) *The Discriminant.*

The discriminant is found by eliminating a between the above and

$$-2[\phi(x, y, z) - a] = 0.$$

Hence it is

$$[\chi(x, y, z)]^3.$$

Hence the locus of ultimate intersections is

$$[\chi(x, y, z)]^3 = 0.$$

(B.) *The Locus of Unodal Lines is* $\chi(x, y, z) = 0$.

Let ξ, η, ζ be any point on the locus $\chi(x, y, z) = 0$.
Let $a = \phi(\xi, \eta, \zeta)$, and consider the single surface

$$[\phi(x, y, z) - \phi(\xi, \eta, \zeta)]^2 + [\chi(x, y, z)]^3 = 0.$$

Put $x = \xi + X$, $y = \eta + Y$, $z = \zeta + Z$; then the lowest terms in X, Y, Z are

$$\left[X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta}\right]^2.$$

Hence ξ, η, ζ is a unode on the surface

$$[\phi(x, y, z) - \phi(\xi, \eta, \zeta)]^2 + [\chi(x, y, z)]^3 = 0.$$

Hence $\chi(x, y, z) = 0$ is the locus of unodal lines on the surfaces

$$[\phi(x, y, z) - a]^2 + [\chi(x, y, z)]^3 = 0.$$

The unodal line on any one of the surfaces is given by the equations

$$\chi(x, y, z) = 0,$$
$$\phi(x, y, z) = a.$$

This accounts for the occurrence of the factor $[\chi(x, y, z)]^3$ in the discriminant.

SECTION II. (Arts. 7–9).—CONSIDERATION OF THE CASES RESERVED IN WHICH TWO ROOTS OF THE EQUATION $Df/D\alpha = 0$ BECOME EQUAL AT ANY POINT ON THE LOCUS OF ULTIMATE INTERSECTIONS.

Art. 7.—*Consideration of the exceptional case of the Envelope Locus, in which two consecutive characteristics coincide.*

(A.) It will be shown that this is the case reserved in Art. 1, viz., where $D^2f_1/D\alpha_1{}^2 = 0$. The geometrical meaning of the condition will first of all be determined.

The surface

$$f(x, y, z, \alpha) = 0$$

intersects the surface

$$f(x, y, z, \alpha + \delta\alpha) = 0,$$

where $\delta\alpha$ is indefinitely small in the curve whose equations are

$$f(x, y, z, \alpha) = 0\,,$$

$$\frac{Df(x, y, z, \alpha)}{D\alpha} = 0\,.$$

This curve is called a characteristic. The equations of the next characteristic are obtained by changing α into $\alpha + \delta\alpha$ in the above. Hence they are

$$f(x, y, z, \alpha) + (\delta\alpha)\,\frac{Df(x, y, z, \alpha)}{D\alpha} = 0\,,$$

$$\frac{Df(x, y, z, \alpha)}{D\alpha} + (\delta\alpha)\,\frac{D^2f(x, y, z, \alpha)}{D\alpha^2} = 0\,.$$

Now, if the two consecutive characteristics coincide,

$$f(x, y, z, \alpha) = 0\,, \quad \frac{Df(x, y, z, \alpha)}{D\alpha} = 0\,, \quad \frac{D^2f(x, y, z, \alpha)}{D\alpha^2} = 0\,,$$

at every point of the coinciding characteristics.

Hence, the characteristic counts three times over as an intersection of the envelope and the surface, instead of twice as in the ordinary case.

(B.) It is now necessary to repeat the investigation in the case in which equation (2) has equal roots when $x = \xi$, $y = \eta$, $z = \zeta$, the co-ordinates of a point on the locus of ultimate intersections.

In this case, Δ may be written

$$\mathrm{R}f(x, y, z, a_1)\,f(x, y, z, a_2) = \mathrm{R}f_1 f_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (51)$$

where a_1, a_2 are the roots of (2) which become equal when $x = \xi$, $y = \eta$, $z = \zeta$.

Therefore,

$$\frac{\partial\Delta}{\partial x} = \frac{\partial \mathrm{R}}{\partial x} f_1 f_2 + \mathrm{R}f_2\left(\frac{\mathrm{D}f_1}{\mathrm{D}x} + \frac{\mathrm{D}f_1}{\mathrm{D}a_1}\frac{\partial a_1}{\partial x}\right) + \mathrm{R}f_1\left(\frac{\mathrm{D}f_2}{\mathrm{D}x} + \frac{\mathrm{D}f_2}{\mathrm{D}a_2}\frac{\partial a_2}{\partial x}\right) \quad . \quad . \qquad (52).$$

Now if it be assumed (see immediately below, under C) that the terms $f_2 \dfrac{\mathrm{D}f_1}{\mathrm{D}a_1}\dfrac{\partial a_1}{\partial x}$, $f_1 \dfrac{\mathrm{D}f_2}{\mathrm{D}a_2}\dfrac{\partial a_2}{\partial x}$ vanish, then when $x = \xi$, $y = \eta$, $z = \zeta$, it follows that $a_1 = a_2 = \alpha$, and, therefore, $f_1 = 0$, $f_2 = 0$, and $\dfrac{\partial\Delta}{\partial x} = 0$.

Similarly, $\dfrac{\partial\Delta}{\partial y} = 0$, $\dfrac{\partial\Delta}{\partial z} = 0$.

Therefore, Δ contains E^2 as a factor.

(C.) *Examination of the term* $f_2 \dfrac{\mathrm{D}f_1}{\mathrm{D}a_1}\dfrac{\partial a_1}{\partial x}$,

Taking $\partial a_1/\partial x$ from (47) this term becomes

$$- f_2 \frac{\mathrm{D}f_1}{\mathrm{D}a_1}\frac{\mathrm{D}^2 f_1}{\mathrm{D}a_1 \mathrm{D}x}\Big/\frac{\mathrm{D}^2 f_1}{\mathrm{D}a_1{}^2}.$$

Now f is of the form

$$\mathrm{A}\,(a - \alpha_1)\,(a - \alpha_2)\,(a - \alpha_3),$$

where α_1, α_2, α_3 all become equal to the same thing as a_1, a_2 when $x = \xi$, $y = \eta$, $z = \zeta$. Hence, taking as infinitesimal of the first order the difference in the values of the parameter a at the points ξ, η, ζ and $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$, it follows that f_2 is of the third order of small quantities, $\mathrm{D}f_1/\mathrm{D}a_1$ of the second, and $\mathrm{D}^2 f_1/\mathrm{D}a_1{}^2$ of the first. Hence, assuming that $\mathrm{D}^2 f_1/\mathrm{D}a_1\mathrm{D}x$ is not infinite, it follows that the term under investigation is of the fourth order, and therefore vanishes ultimately.

Example 4.—*Envelope Locus when two consecutive Characteristics coincide.*

Let the surfaces be

$$\phi\,(x, y, z) + [\psi\,(x, y, z) - a]^3 = 0.$$

(A). *The Discriminant.*

The discriminant is found by eliminating a between the above and

$$- 3\,[\psi\,(x, y, z) - a]^2 = 0.$$

Hence it is $[\phi\,(x, y, z)]^3$.

Hence the locus of ultimate intersections is

$$[\phi(x, y, z)]^2 = 0.$$

(B). *The Envelope Locus such that two consecutive Characteristics coincide is* $\phi(x, y, z) = 0$.

Let ξ, η, ζ be any point on $\phi(x, y, z) = 0$.

Take $a = \psi(\xi, \eta, \zeta)$, and consider the single surface

$$\phi(x, y, z) + [\psi(x, y, z) - \psi(\xi, \eta, \zeta)]^3 = 0.$$

Put $x = \xi + \mathrm{X}$, $y = \eta + \mathrm{Y}$, $z = \zeta + \mathrm{Z}$: then the lowest terms in X, Y, Z are

$$\mathrm{X}\frac{\partial\phi}{\partial\xi} + \mathrm{Y}\frac{\partial\phi}{\partial\eta} + \mathrm{Z}\frac{\partial\phi}{\partial\zeta}.$$

Hence the tangent plane to the surface at ξ, η, ζ is also the tangent plane to $\phi(x, y, z) = 0$.

Hence $\phi(x, y, z) = 0$ is the envelope, and the equations of the line of contact are $\phi(x, y, z) = 0$, $\psi(x, y, z) = \psi(\xi, \eta, \zeta)$.

Now the equations corresponding to $f = 0$, $\mathrm{D}f/\mathrm{D}a = 0$, $\mathrm{D}^2f/\mathrm{D}a^2 = 0$ are

$$\begin{aligned} \phi(x, y, z) + \quad [\psi(x, y, z) - a]^3 &= 0, \\ -\,3\,[\psi(x, y, z) - a]^2 &= 0, \\ 6\,[\psi(x, y, z) - a] &= 0. \end{aligned}$$

These are all satisfied by the coordinates of any point on the line of contact. Hence two consecutive characteristics coincide. This accounts for the factor $[\phi(x, y, z)]^2$ in the discriminant.

Art. 8.—*Consideration of Loci of Binodal Lines, which are also Envelopes.*

(A.) It will be shown that this is the case reserved in Art. 1, viz., where $\mathrm{D}^2f_1/\mathrm{D}a_1^2 = 0$.

The equation of the biplanes is

$$\begin{aligned} &[\xi, \xi](\mathrm{X} - \xi)^2 + [\eta, \eta](\mathrm{Y} - \eta)^2 + [\zeta, \zeta](\mathrm{Z} - \zeta)^2 \\ &+ 2[\eta, \zeta](\mathrm{Y} - \eta)(\mathrm{Z} - \zeta) + 2[\zeta, \xi](\mathrm{Z} - \zeta)(\mathrm{X} - \xi) + 2[\xi, \eta](\mathrm{X} - \xi)(\mathrm{Y} - \eta) = 0 \quad (53). \end{aligned}$$

This breaks up by (29) into the two planes

$$\begin{aligned}[t][\xi, \xi]\,\{[\eta, \xi](X - \xi) + [\eta, \eta](Y - \eta) + [\eta, \zeta](Z - \zeta)\} \\ + \{-[\xi, \eta] \pm \sqrt{[\xi, \eta]^2 - [\xi, \xi][\eta, \eta]}\}\,\{[\xi, \xi](X - \xi) + [\xi, \eta](Y - \eta) \\ + [\xi, \zeta](Z - \zeta)\} = 0 \quad . \quad (54).\end{aligned}$$

Now since equation (27) depends on (24) and (25), therefore

$$\begin{vmatrix} [\xi, \xi] & [\xi, \eta] & [\xi, \alpha] \\ [\eta, \xi] & [\eta, \eta] & [\eta, \alpha] \\ [\alpha, \xi] & [\alpha, \eta] & [\alpha, \alpha] \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55).$$

Putting $[\alpha, \alpha] = 0$, this becomes

$$[\alpha, \eta]^2 [\xi, \xi] - 2\,[\alpha, \eta][\alpha, \xi][\xi, \eta] + [\alpha, \xi]^2 [\eta, \eta] = 0 \quad . \quad . \quad . \quad (56).$$

Therefore

$$[\alpha, \eta]/[\alpha, \xi] = \{[\xi, \eta] \pm \sqrt{[\xi, \eta]^2 - [\xi, \xi][\eta, \eta]}\}/[\xi, \xi] \quad . \quad . \quad . \quad (57).$$

Hence the equation of *one* of the biplanes is

$$\begin{aligned}[t][\alpha, \xi]\,\{[\xi, \eta](X - \xi) + [\eta, \eta](Y - \eta) + [\zeta, \eta](Z - \zeta)\} \\ - [\alpha, \eta]\,\{[\xi, \xi](X - \xi) + [\xi, \eta](Y - \eta) + [\xi, \zeta](Z - \zeta)\} = 0 \quad . \quad . \quad (58).\end{aligned}$$

Now if $\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta$ be a point on the locus of binodal lines near to ξ, η, ζ, it follows by (24) and (25) that

$$\begin{aligned}[t][\alpha, \xi]\,\{[\xi, \eta](\delta\xi) + [\eta, \eta](\delta\eta) + [\zeta, \eta](\delta\zeta)\} \\ - [\alpha, \eta]\,\{[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta)\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59).\end{aligned}$$

Hence the tangent plane to the locus of binodal lines takes the same form as (58).

Hence the tangent plane to the locus of binodal lines is the same as one of the biplanes.

Hence the locus of binodal lines is also an envelope.

(B.) The converse proposition, viz., that if the locus of binodal lines be also an envelope, then $[\alpha, \alpha] = 0$, will now be proved.

As before, the equations of the biplanes are given by (54), and the tangent plane to the locus of binodal lines takes the same form as (58). If, then, the locus of binodal lines be also an envelope

$$\{-[\xi, \eta] \pm \sqrt{[\xi, \eta]^2 - [\xi, \xi][\eta, \eta]}\}/[\xi, \xi] = -[\alpha, \eta]/[\alpha, \xi] \quad . \quad . \quad (60),$$

therefore

$$[\alpha, \eta]^2 [\xi, \xi] - 2\,[\alpha, \eta][\alpha, \xi][\xi, \eta] + [\alpha, \xi]^2 [\eta, \eta] = 0 \quad . \quad . \quad . \quad (61).$$

The equation (61) is the same as (56).

Comparing it with (55), which holds when a locus of binodal lines exists, it follows that

$$\{[\xi, \xi][\eta, \eta] - [\xi, \eta]^2\}[\alpha, \alpha] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62).$$

Hence $[\alpha, a] = 0$, or

$$[\xi, \xi][\eta, \eta] - [\xi, \eta]^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63).$$

If (63) hold, then by (61)

$$[\alpha, \eta][\xi, \xi] = [\alpha, \xi][\xi, \eta],$$

therefore

$$[\xi, \xi]/[\xi, \eta] = [\xi, \eta]/[\eta, \eta] = [\alpha, \xi]/[\alpha, \eta].$$

Making use of these with (24) and (25), it follows that each of these fractions is equal to $[\xi, \zeta]/[\eta, \zeta]$.

Hence (24) and (25) are equivalent to one equation only.

But it was shown that (26) and (27) depend on (24) and (25). Hence (25), (26), (27) all depend on (24).

Hence the ratios (38) hold, and therefore there is a locus of unodal lines. But this is not the case under consideration, for it is supposed that there is a locus of binodal, not unodal, lines.

Hence (63) is not satisfied, and, therefore, $[\alpha, \alpha] = 0$.

(C.) In this case the values of $\partial a_1/\partial x$, $\partial a_1/\partial y$, given by (47) and (48), are really infinite, for $D^2 f_1/Da_1\, Dx$ does not vanish necessarily, but $D^2 f_1/Da_1^2 = 0$.

Consequently the differential coefficients of Δ require further examination.

Now

$$f_2 \frac{D^2 f_1}{Da_1\, Dx} \frac{\partial a_1}{\partial x} = - f_2 \left(\frac{D^2 f_1}{Da_1\, Dx}\right)^2 \bigg/ \frac{D^2 f_1}{Da_1^2}.$$

Since $D^2 f_1/Da_1\, Dx$ does not necessarily vanish, it must be shown that $f_2 \Big/ \frac{D^2 f_1}{Da_1^2} = 0$ at points on the locus of binodal lines; *i.e.*,

$$f(x, y, z, a_2) \bigg/ \frac{D^2 f(x, y, z, a_1)}{Da_1^2} = 0,$$

when $x = \xi$, $y = \eta$, $z = \zeta$, the coordinates of any point on the locus of binodal lines.

Now a_1, a_2 are the roots of

$$\frac{Df(x, y, z, a)}{Da} = 0,$$

which become equal when $x = \xi$, $y = \eta$, $z = \zeta$.

In this case $f(x, y, z, a) = 0$ is an equation for a, such that three values become equal when $x = \xi$, $y = \eta$, $z = \zeta$. (They become equal to the same value as a_1, a_2.)

Put, therefore,

$$f = \mathrm{R}\,(a - \alpha_1)(a - \alpha_2)(a - \alpha_3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (64),$$

where α_1, α_2, α_3 become equal when $x = \xi$, $y = \eta$, $z = \zeta$.

Therefore

$$\frac{\mathrm{D}f}{\mathrm{D}a} = \frac{\mathrm{DR}}{\mathrm{D}a}(a - \alpha_1)(a - \alpha_2)(a - \alpha_3)$$

$$+ \mathrm{R}\,\{(a - \alpha_2)(a - \alpha_3) + (a - \alpha_3)(a - \alpha_1) + (a - \alpha_1)(a - \alpha_2)\} \quad . \qquad (65),$$

$$\frac{\mathrm{D}^2 f}{\mathrm{D}a^2} = \frac{\mathrm{D}^2\mathrm{R}}{\mathrm{D}a^2}(a - \alpha_1)(a - \alpha_2)(a - \alpha_3)$$

$$+ 2\,\frac{\mathrm{DR}}{\mathrm{D}a}\{(a - \alpha_2)(a - \alpha_3) + (a - \alpha_3)(a - \alpha_1) + (a - \alpha_1)(a - \alpha_2)\}$$

$$+ 2\mathrm{R}\,\{(a - \alpha_1) + (a - \alpha_2) + (a - \alpha_3)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (66).$$

Now, at a point on the locus of binodal lines, the two equal values of a which make $\mathrm{D}f/\mathrm{D}a = 0$, become equal to the same thing as α_1, α_2, α_3.

Hence $f(x, y, z, a_2) = (\mathrm{R}')(a_2 - \alpha_1)(a_2 - \alpha_2)(a_2 - \alpha_3)$, where R' is what R becomes, when a is changed into a_2.

Hence $f(x, y, z, a_2)$ is of the third order of small quantities; but $\mathrm{D}^2 f(x, y, z, a_1)/\mathrm{D}a_1^2$ is of the first order, for the most important term in it is

$$2\,(\mathrm{R}')\,[(a_1 - \alpha_1) + (a_1 - \alpha_2) + (a_1 - \alpha_3)].$$

Hence

$$f(x, y, z, a_2)\Big/\frac{\mathrm{D}^2 f(x, y, z, a_1)}{\mathrm{D}a_1^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (67)$$

at points on the locus of binodal lines.

In like manner

$$f(x, y, z, a_2)\Big/\left\{\frac{\mathrm{D}^2 f(x, y, z, a_1)}{\mathrm{D}a_1^2}\right\}^2 = 0 \quad . \quad . \quad . \quad . \quad . \qquad (68),$$

but

$$f(x, y, z, a_2)\Big/\left\{\frac{\mathrm{D}^2 f(x, y, z, a_1)}{\mathrm{D}a_1^2}\right\}^3 \neq 0 \quad . \quad . \quad . \quad . \quad . \qquad (69)$$

at points on the locus of binodal lines.

(D). It should be noticed that in the preceding section (C), the infinitesimal of the first order is the increment in the value of α, a root of $\mathrm{D}f(x, y, z, \alpha)/\mathrm{D}\alpha = 0$, when ξ, η, ζ receive increments $\delta\xi, \delta\eta, \delta\zeta$ respectively; and in particular that it is not of the same order as $\delta\xi, \delta\eta, \delta\zeta$.

For if $\delta\alpha$ be the increment in the value α, then

$$\begin{array}{l} [\alpha, \xi](\delta\xi) + [\alpha, \eta](\delta\eta) + [\alpha, \zeta](\delta\zeta) + [\alpha, \alpha](\delta\alpha) \\ + \frac{1}{2}\left[\begin{array}{l} [\alpha, \xi, \xi](\delta\xi)^2 + [\alpha, \eta, \eta](\delta\eta)^2 + [\alpha, \zeta, \zeta](\delta\zeta)^2 + [\alpha, \alpha, \alpha](\delta\alpha)^2 \\ + 2[\alpha, \eta, \zeta](\delta\eta)(\delta\zeta) + 2[\alpha, \zeta, \xi](\delta\zeta)(\delta\xi) + 2[\alpha, \xi, \eta](\delta\xi)(\delta\eta) \\ + 2[\alpha, \alpha, \xi](\delta\alpha)(\delta\xi) + 2[\alpha, \alpha, \eta](\delta\alpha)(\delta\eta) + 2[\alpha, \alpha, \zeta](\delta\alpha)(\delta\zeta) \end{array}\right] \\ + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots = 0. \end{array}$$

Now because $[\alpha, \alpha] = 0$, this equation can be written in the form

$$u_1 + u_2 + 2v_1(\delta\alpha) + v_0(\delta\alpha)^2 = 0,$$

where the suffixes denote the order of the terms, when $(\delta\xi), (\delta\eta), (\delta\zeta)$ are taken to be of the first order.

Hence, if ϵ denote an infinitely small quantity of the first order in $(\delta\xi), (\delta\eta), (\delta\zeta)$, then $\delta\alpha$ is of the order $\epsilon^{1/2}$.

And now $f(x, y, z, a_2)$, when $x = \xi + \delta\xi$, $y = \eta + \delta\eta$, $z = \zeta + \delta\zeta$, and $a_2 = \alpha + \delta\alpha$, becomes

$$\begin{array}{l} f(\xi, \eta, \zeta, \alpha) + [\xi](\delta\xi) + [\eta](\delta\eta) + [\zeta](\delta\zeta) + [\alpha](\delta\alpha) \\ + \frac{1}{2}\left[\begin{array}{l} [\xi, \xi](\delta\xi)^2 + [\eta, \eta](\delta\eta)^2 + [\zeta, \zeta](\delta\zeta)^2 + [\alpha, \alpha](\delta\alpha)^2 \\ + 2[\eta, \zeta](\delta\eta)(\delta\zeta) + 2[\zeta, \xi](\delta\zeta)(\delta\xi) + 2[\xi, \eta](\delta\xi)(\delta\eta) \\ + 2[\alpha, \xi](\delta\alpha)(\delta\xi) + 2[\alpha, \eta](\delta\alpha)(\delta\eta) + 2[\alpha, \zeta](\delta\alpha)(\delta\zeta) \end{array}\right] \\ + \ldots \\ = \frac{1}{2}\left[\begin{array}{l} [\xi, \xi](\delta\xi)^2 + [\eta, \eta](\delta\eta)^2 + [\zeta, \zeta](\delta\zeta)^2 \\ + 2[\eta\zeta](\delta\eta)(\delta\zeta) + 2[\zeta, \xi](\delta\zeta)(\delta\xi) + 2[\xi, \eta](\delta\xi)(\delta\eta) \\ + 2[\alpha, \xi](\delta\alpha)(\delta\xi) + 2[\alpha, \eta](\delta\alpha)(\delta\eta) + 2[\alpha, \zeta](\delta\alpha)(\delta\zeta) \end{array}\right] \\ + \ldots \end{array}$$

Hence $f(x, y, z, a_2)$ is of the order $\epsilon^{3/2}$ when $x = \xi + \delta\xi$, $y = \eta + \delta\eta$, $z = \zeta + \delta\zeta$.

In like manner, when $x = \xi + \delta\xi$, $y = \eta + \delta\eta$, $z = \zeta + \delta\zeta$, $a_1 = \alpha + \delta\alpha'$,

$$\begin{aligned}\frac{\mathrm{D}^2 f(x, y, z, a_1)}{\mathrm{D}a_1^{\,2}} = {} & [\alpha, \alpha] \\ & + [\alpha, \alpha, \xi](\delta\xi) + [\alpha, \alpha, \eta](\delta\eta) + [\alpha, \alpha, \zeta](\delta\zeta) + [\alpha, \alpha, \alpha,](\delta\alpha') \\ & + \ldots\end{aligned}$$

Now $\delta\alpha'$ can be shown to be of the order $\epsilon^{1/2}$, in the same way that $\delta\alpha$ was shown to be of this order.

Hence $\mathrm{D}^2 f(x, y, z, a_1)/\mathrm{D}a_1^{\,2}$ is of the order $\epsilon^{1/2}$.

Hence the same results as those given in (67), (68), (69) follow.

(E.) *Examination of the Differential Coefficients of* Δ.

Differentiating (52) with regard to x,

$$\frac{\partial^2\Delta}{\partial x^2} = f_1 f_2 \frac{\partial^2 \mathrm{R}}{\partial x^2} + 2\frac{\partial \mathrm{R}}{\partial x}\left(f_2 \frac{\mathrm{D}f_1}{\mathrm{D}x} + f_1 \frac{\mathrm{D}f_2}{\mathrm{D}x}\right) + \mathrm{R}\left[\begin{aligned} & f_2\left(\frac{\mathrm{D}^2 f_1}{\mathrm{D}x^2} + \frac{\mathrm{D}^2 f_1}{\mathrm{D}x\,\mathrm{D}a_1}\frac{\partial a_1}{\partial x}\right) \\ & + 2\frac{\mathrm{D}f_1}{\mathrm{D}x}\frac{\mathrm{D}f_2}{\mathrm{D}x} \\ & + f_1\left(\frac{\mathrm{D}^2 f_2}{\mathrm{D}x^2} + \frac{\mathrm{D}^2 f_2}{\mathrm{D}x\,\mathrm{D}a_2}\frac{\partial a_2}{\partial x}\right)\end{aligned}\right] \quad \ldots\ldots\ldots\ldots \quad (70).$$

Hence the term

$$f_2 \frac{\mathrm{D}^2 f_1}{\mathrm{D}x\,\mathrm{D}a_1}\frac{\partial a_1}{\partial x} = -f_2\left(\frac{\mathrm{D}^2 f_1}{\mathrm{D}x\,\mathrm{D}a_1}\right)^2 \bigg/ \frac{\mathrm{D}^2 f_1}{\mathrm{D}a_1^{\,2}}$$

vanishes by (67) at points on the locus of binodal lines.

Hence $\partial^2\Delta/\partial x^2 = 0$ at points on the locus of binodal lines, if it be assumed that $\mathrm{D}^2 f_1/\mathrm{D}x\,\mathrm{D}a_1$ is finite.

This assumption can be made if a_1 be finite.

Again, differentiating (70) with regard to x,

$$\begin{aligned}\frac{\partial^3\Delta}{\partial x^3} = {} & \frac{\partial^3 \mathrm{R}}{\partial x^3} f_1 f_2 + 3\frac{\partial \mathrm{R}}{\partial x}\left(f_2\frac{\mathrm{D}f_1}{\mathrm{D}x} + f_1\frac{\mathrm{D}f_2}{\mathrm{D}x}\right) \\ & + 3\mathrm{R}\left[f_2\left(\frac{\mathrm{D}^2 f_1}{\mathrm{D}x^2} + \frac{\mathrm{D}^2 f_1}{\mathrm{D}x\,\mathrm{D}a_1}\frac{\partial a_1}{\partial x}\right) + 2\frac{\mathrm{D}f_1}{\mathrm{D}x}\frac{\mathrm{D}f_2}{\mathrm{D}x} + f_1\left(\frac{\mathrm{D}^2 f_2}{\mathrm{D}x^2} + \frac{\mathrm{D}^2 f_2}{\mathrm{D}x\,\mathrm{D}a_2}\frac{\partial a_2}{\partial x}\right)\right] \\ & + \mathrm{R}\left[\begin{aligned} & f_2\frac{\partial}{\partial x}\left(\frac{\mathrm{D}^2 f_1}{\mathrm{D}x^2} + \frac{\mathrm{D}^2 f_1}{\mathrm{D}x\,\mathrm{D}a_1}\frac{\partial a_1}{\partial x}\right) + 3\frac{\mathrm{D}f_2}{\mathrm{D}x}\left(\frac{\mathrm{D}^2 f_1}{\mathrm{D}x^2} + \frac{\mathrm{D}^2 f_1}{\mathrm{D}x\,\mathrm{D}a_1}\frac{\partial a_1}{\partial x}\right) \\ & + 3\frac{\mathrm{D}f_1}{\mathrm{D}x}\left(\frac{\mathrm{D}^2 f_2}{\mathrm{D}x^2} + \frac{\mathrm{D}^2 f_2}{\mathrm{D}x\,\mathrm{D}a_2}\frac{\partial a_2}{\partial x}\right) + f_1\frac{\partial}{\partial x}\left(\frac{\mathrm{D}^2 f_2}{\mathrm{D}x^2} + \frac{\mathrm{D}^2 f_2}{\mathrm{D}x\,\mathrm{D}a_2}\frac{\partial a_2}{\partial x}\right)\end{aligned}\right] \quad . \; . \quad (71).\end{aligned}$$

In this case the infinite quantity $\partial a_1/\partial x$ occurs in the term

$$f_2 \frac{\partial}{\partial x}\left(\frac{D^2 f_1}{Dx^2} + \frac{D^2 f_1}{Dx\,Da_1}\frac{\partial a_1}{\partial x}\right)$$

$$= f_2 \frac{\partial}{\partial x}\left\{\left(\frac{D^2 f_1}{Dx^2}\frac{D^2 f_1}{Da_1^2} - \left(\frac{D^2 f_1}{Dx\,Da_1}\right)^2\right)\Big/\frac{D^2 f_1}{Da_1^2}\right\}$$

$$= f_2 \left[\begin{array}{l} \dfrac{D}{Dx}\left\{\left(\dfrac{D^2 f_1}{Dx^2}\dfrac{D^2 f_1}{Da_1^2} - \left(\dfrac{D^2 f_1}{Dx\,Da_1}\right)^2\right)\Big/\dfrac{D^2 f_1}{Da_1^2}\right\} \\ - \dfrac{\dfrac{D^2 f_1}{Dx\,Da_1}}{\dfrac{D^2 f_1}{Da_1^2}}\dfrac{D}{Da_1}\left\{\left(\dfrac{D^2 f_1}{Dx^2}\dfrac{D^2 f_1}{Da_1^2} - \left(\dfrac{D^2 f_1}{Dx\,Da_1}\right)^2\right)\Big/\dfrac{D^2 f_1}{Da_1^2}\right\} \end{array}\right]$$

$$= f_2\frac{D^3 f_1}{Dx^3} - 3\frac{f_2}{\dfrac{D^2 f_1}{Da_1^2}}\frac{D^2 f_1}{Dx\,Da_1}\frac{D^3 f_1}{Dx^2\,Da_1}$$

$$+ 3\frac{f_2}{\left(\dfrac{D^2 f_1}{Da_1^2}\right)^2}\left(\frac{D^2 f_1}{Dx\,Da_1}\right)^2\frac{D^3 f_1}{Dx\,Da_1^2} - \frac{f_2}{\left(\dfrac{D^2 f_1}{Da_1^2}\right)^3}\left(\frac{D^2 f_1}{Dx\,Da_1}\right)^3\frac{D^3 f_1}{Da_1^3}.$$

Now it has been shown in (67)–(69) that $f_2\Big/\frac{D^2 f_1}{Da_1^2}$ and $f_2\Big/\left(\frac{D^2 f_1}{Da_1^2}\right)^2$ both vanish, but $f\Big/\left(\frac{D^2 f_1}{Da_1^2}\right)^3$ does not vanish at points on the locus of binodal lines.

Hence $f_2 \frac{\partial}{\partial x}\left(\frac{D^2 f_1}{Dx^2} + \frac{D^2 f_1}{Dx\,Da_1}\frac{\partial a_1}{\partial x}\right)$ does not vanish.

Similarly $f_1 \frac{\partial}{\partial x}\left(\frac{D^2 f_2}{Dx^2} + \frac{D^2 f_2}{Dx\,Da_2}\frac{\partial a_2}{\partial x}\right)$ does not vanish.

The order of the term

$$\frac{Df_2}{Dx}\frac{D^2 f_1}{Dx\,Da_1}\frac{\partial a_1}{\partial x}$$

cannot be determined in a perfectly general way, for although Df_2/Dx vanishes, yet it contains $\partial \alpha_1/\partial x$, $\partial \alpha_2/\partial x$, $\partial \alpha_3/\partial x$, which may be infinite, since α_1, α_2, α_3 are irrational functions of the coordinates.

These results point to the conclusion that $\partial^3\Delta/\partial x^3$ does not vanish at all points on the locus of binodal lines. This is readily proved in particular cases. (See Example 5 below.)

Hence, at points on the locus of binodal lines, $\Delta = 0$, $\partial\Delta/\partial x = 0$, $\partial^2\Delta/\partial x^2 = 0$; but $\partial^3\Delta/\partial x^3 \neq 0$.

Hence, if $B = 0$ be the equation of the locus of binodal lines, when that locus is also an envelope, Δ contains B^3 as a factor.

Example 5.—*Locus of Binodal Lines which is also an Envelope.*

Let the surfaces be

$$[\phi(x, y, z)]^2 + \phi(x, y, z)[\psi(x, y, z) - a] + [\psi(x, y, z) - a]^3 = 0.$$

(A.) *The Discriminant.*

The discriminant is found by eliminating a between the above and

$$\phi(x, y, z) + 3[\psi(x, y, z) - a]^2 = 0.$$

The last equation gives

$$\psi(x, y, z) - a = \pm\sqrt{\{-\tfrac{1}{3}\phi(x, y, z)\}}.$$

Hence the eliminant is

$$\{[\phi(x, y, z)]^2 + \tfrac{2}{3}\phi(x, y, z)\sqrt{-\tfrac{1}{3}\phi(x, y, z)}\}$$
$$\times\{[\phi(x, y, z)]^2 - \tfrac{2}{3}\phi(x, y, z)\sqrt{-\tfrac{1}{3}\phi(x, y, z)}\}$$
$$= [\phi(x, y, z)]^3[\phi(x, y, z) + \tfrac{4}{27}].$$

Hence the locus of ultimate intersections is

$$[\phi(x, y, z)]^3[\phi(x, y, z) + \tfrac{4}{27}] = 0.$$

(B.) *The locus of Binodal Lines (which is also an Envelope) is* $\phi(x, y, z) = 0$.

For let ξ, η, ζ be any point on $\phi(x, y, z) = 0$.

Take $a = \psi(\xi, \eta, \zeta)$, and consider the single surface

$$[\phi(x, y, z)]^2 + \phi(x, y, z)[\psi(x, y, z) - \psi(\xi, \eta, \zeta)] + [\psi(x, y, z) - \psi(\xi, \eta, \zeta)]^3 = 0.$$

Put $x = \xi + X$, $y = \eta + Y$, $z = \zeta + Z$; then the lowest terms in X, Y, Z are

$$\left(X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta}\right)\left\{\left(X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta}\right) + \left(X\frac{\partial\psi}{\partial\xi} + Y\frac{\partial\psi}{\partial\eta} + Z\frac{\partial\psi}{\partial\zeta}\right)\right\}.$$

Hence the origin is a binode.

Hence $\phi(x, y, z) = 0$ is the locus of binodal lines.

Further, because the biplanes are, when the origin is at the binode,

$$X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta} = 0,$$

$$\left(X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta}\right) + \left(X\frac{\partial\psi}{\partial\xi} + Y\frac{\partial\psi}{\partial\eta} + Z\frac{\partial\psi}{\partial\zeta}\right) = 0.$$

The first of these touches the locus of binodal lines. Hence the locus of binodal lines is also an envelope.

Hence the factor $[\phi(x, y, z)]^3$ of the discriminant is accounted for.

(C.) *The Surface* $\phi(x, y, z) + \frac{4}{27} = 0$ *is an Envelope.*

For seeking its intersection with

$$[\phi(x, y, z)]^2 + \phi(x, y, z,)[\psi(x, y, z) - a] + [\psi(x, y, z) - a]^3 = 0,$$

it follows that

$$[\psi(x, y, z) - a]^3 - \tfrac{4}{27}[\psi(x, y, z) - a] + \tfrac{16}{729} = 0.$$

Put

$$\psi(x, y, z) - a = \eta/9,$$

therefore

$$\eta^3 - 12\eta + 16 = 0,$$

therefore

$$(\eta - 2)^2(\eta + 4) = 0,$$

$$i.e., \{9[\psi(x, y, z) - a] - 2\}^2\{9[\psi(x, y, z) - a] + 4\} = 0.$$

Consider now any point ξ, η, ζ on the surface

$$[\phi(x, y, z)]^2 + \phi(x, y, z)[\psi(x, y, z) - a] + [\psi(x, y, z) - a]^3 = 0,$$

for which

$$\phi(\xi, \eta, \zeta) + \tfrac{4}{27} = 0,$$

and

$$\psi(\xi, \eta, \zeta) - a - \tfrac{2}{9} = 0.$$

The equation of the tangent plane at such a point is

$$[2\phi(\xi, \eta, \zeta) + \psi(\xi, \eta, \zeta) - a]\left[(X - \xi)\frac{\partial\phi}{\partial\xi} + (Y - \eta)\frac{\partial\phi}{\partial\eta} + (Z - \zeta)\frac{\partial\phi}{\partial\zeta}\right]$$
$$+ \{\phi(\xi, \eta, \zeta) + 3[\psi(\xi, \eta, \zeta) - a]^2\}\left[(X - \xi)\frac{\partial\psi}{\partial\xi} + (Y - \eta)\frac{\partial\psi}{\partial\eta} + (Z - \zeta)\frac{\partial\psi}{\partial\zeta}\right] = 0.$$

This reduces to

$$(X - \xi)\frac{\partial\phi}{\partial\xi} + (Y - \eta)\frac{\partial\phi}{\partial\eta} + (Z - \zeta)\frac{\partial\phi}{\partial\zeta} = 0,$$

which is equivalent to

$$\left\{(X - \xi)\frac{\partial}{\partial\xi} + (Y - \eta)\frac{\partial}{\partial\eta} + (Z - \zeta)\frac{\partial}{\partial\zeta}\right\}[\phi(\xi, \eta, \zeta) + \tfrac{4}{27}] = 0.$$

Hence it touches the surface $\phi(x, y, z) + \frac{4}{27} = 0$.

Hence $\phi(x, y, z) + \frac{4}{27} = 0$ is an envelope.

This accounts for the factor $\phi(x, y, z) + \frac{4}{27}$ in the discriminant.

(D). *Examination of the term* $\mathrm{D}^2 f_1/\mathrm{D}x\,\mathrm{D}a_1$ *for this example.*

The equation $\mathrm{D}f/\mathrm{D}a = 0$ is, in this case,

$$\phi(x, y, z) + 3[\psi(x, y, z) - a]^2 = 0.$$

Hence a_1 satisfies

$$\phi(x, y, z) + 3[\psi(x, y, z) - a_1]^2 = 0,$$

and $\partial a_1/\partial x$ is determined by

$$\frac{\partial\phi}{\partial x} + 6[\psi(x, y, z) - a_1]\left[\frac{\partial\psi}{\partial x} - \frac{\partial a_1}{\partial x}\right] = 0.$$

Hence, at a point on the locus of binodal lines, *i.e.*, where $\phi(x, y, z) = 0$, $\psi(x, y, z) = a_1$, it follows that $\partial a_1/\partial x$ is infinite.

Calculating $\mathrm{D}^2 f/\mathrm{D}x\,\mathrm{D}a$, it follows that it is

$$-\frac{\partial\phi}{\partial x} - 6[\psi(x, y, z) - a]\frac{\partial\psi}{\partial x}.$$

Hence, $\mathrm{D}^2 f_1/\mathrm{D}x\,\mathrm{D}a_1$ is equal to the value of $-\partial\phi/\partial x$ at the point on a locus of binodal lines. Hence it is finite.

(E). *Examination of the values of* $f_2\Big/\frac{\mathrm{D}^2 f_1}{\mathrm{D}a_1^2}$, $f_2\Big/\left(\frac{\mathrm{D}^2 f_1}{\mathrm{D}a_1^2}\right)^2$, $f_2\Big/\left(\frac{\mathrm{D}^2 f_1}{\mathrm{D}a_1^2}\right)^3$.

In this case a_1, a_2 are the roots of

$$\phi(\xi, \eta, \zeta) + 3[\psi(\xi, \eta, \zeta) - a]^2 = 0.$$

Therefore,

$$a_1 = \psi(\xi, \eta, \zeta) - \surd\{-\tfrac{1}{3}\phi(\xi, \eta, \zeta)\},$$

$$a_2 = \psi(\xi, \eta, \zeta) + \surd\{-\tfrac{1}{3}\phi(\xi, \eta, \zeta)\}.$$

Hence

$$\begin{aligned} f(x, y, z, a_2) &= [\phi(\xi, \eta, \zeta)]^2 + \phi(\xi, \eta, \zeta)[\psi(\xi, \eta, \zeta) - a_2] + [\psi(\xi, \eta, \zeta) - a_2]^3 \\ &= [\phi(\xi, \eta, \zeta)]^2 - \tfrac{2}{3}\phi(\xi, \eta, \zeta)\surd\{-\tfrac{1}{3}\phi(\xi, \eta, \zeta)\} \end{aligned}$$

$$\frac{\mathrm{D}^2 f(x, y, z, a_1)}{\mathrm{D}a_1^2} = 6[\psi(\xi, \eta, \zeta) - a_1] = 6\surd\{-\tfrac{1}{3}\phi(\xi, \eta, \zeta)\}.$$

Hence

$$f_2\Big/\frac{\mathrm{D}^2 f_1}{\mathrm{D}a_1^2} = \frac{\surd-3}{6}\Big[\phi(\xi, \eta, \zeta)\Big]^{3/2} - \tfrac{1}{9}\phi(\xi, \eta, \zeta)$$

$$f_2\Big/\left(\frac{\mathrm{D}^2 f_1}{\mathrm{D}a_1^2}\right)^2 = -\tfrac{1}{12}\phi(\xi, \eta, \zeta) + \tfrac{1}{18}\surd\{-\tfrac{1}{3}\phi(\xi, \eta, \zeta)\}.$$

$$f_2\Big/\left(\frac{\mathrm{D}^2 f_1}{\mathrm{D}a_1^2}\right)^3 = -\frac{\surd-3}{72}[\phi(\xi, \eta, \zeta)]^{1/2} + \tfrac{1}{108}.$$

Hence, at points on the locus of binodal lines, *i.e.*, when $\phi(\xi, \eta, \zeta) = 0$;

$$f_2 \Big/ \frac{D^2 f_1}{Da_1^2} = 0, \quad f_2 \Big/ \left(\frac{D^2 f_1}{Da_1^2}\right)^2 = 0, \quad J_2 \Big/ \left(\frac{D^2 f_1}{Da_1^2}\right)^3 = \tfrac{1}{108}.$$

Art. 9.—*Consideration of Loci of Unodal Lines which are also Envelopes.*

(A.) It will be shown that this is the case reserved in Art. 1, viz., where $D^2 f_1/Da_1^2 = 0$.

For if $D^2 f_1/Da_1^2 = 0$, *i.e.*, $[\alpha, \alpha] = 0$, be substituted in the ratios (38), it follows that

$$[\alpha, \xi] = 0, \quad [\alpha, \eta] = 0, \quad [\alpha, \zeta] = 0.$$

Substituting these in (24)-(26), it follows that

$$[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) = 0 \quad . \quad . \quad . \quad . \quad . \quad (72),$$

$$[\eta, \xi](\delta\xi) + [\eta, \eta](\delta\eta) + [\eta, \zeta](\delta\zeta) = 0 \quad . \quad . \quad . \quad . \quad . \quad (73),$$

$$[\zeta, \xi](\delta\xi) + [\zeta, \eta](\delta\eta) + [\zeta, \zeta](\delta\zeta) = 0 \quad . \quad . \quad . \quad . \quad . \quad (74).$$

Now (72)-(74) are equivalent to one equation only by (38). Hence the tangent plane to the locus of unodal lines is

$$[\xi, \xi](X - \xi) + [\xi, \eta](Y - \eta) + [\xi, \zeta](Z - \zeta) = 0.$$

Now the tangent cone at ξ, η, ζ is given by (53).

The left-hand side of its equation is by (38) a perfect square.

Hence the uniplane is

$$[\xi, \xi](X - \xi) + [\xi, \eta](Y - \eta) + [\xi, \zeta](Z - \zeta) = 0 \quad . \quad . \quad . \quad (75).$$

Hence the uniplane is the same as the tangent plane to the locus of unodal lines. Hence the locus of unodal lines is also an envelope.

(B.) The converse proposition, viz., that if the locus of unodal lines be also an envelope, then $[\alpha, \alpha] = 0$, will now be proved.

If $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ be a point on the locus of unodal lines near to ξ, η, ζ, then the equations (24)-(27) hold.

If the locus of unodal lines be also an envelope, then the equation of the uniplane (75) is satisfied by the values $X = \xi + \delta\xi$, $Y = \eta + \delta\eta$, $Z = \zeta + \delta\zeta$.

Therefore

$$[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) = 0.$$

Comparing this with (24), it follows that

$$[\alpha, \xi] = 0$$

Hence, by (38),

$$[\alpha, \alpha] = 0.$$

(C.) In this case, the values of $\partial a_1/\partial x$, $\partial a_1/\partial y$ given by (47), (48), are indeterminate. For, because $[\alpha, \alpha] = 0$, it follows by (38) that $[\alpha, \xi] = 0$, $[\alpha, \eta] = 0$, $[\alpha, \zeta] = 0$. Hence, to determine $\partial a_1/\partial x$, it is necessary to differentiate (44) with regard to x. Therefore

$$[\xi, \xi, \alpha] + 2\,[\xi, \alpha, \alpha]\frac{\partial a_1}{\partial x} + [\alpha, \alpha, \alpha]\left(\frac{\partial a_1}{\partial x}\right)^2 + [\alpha, \alpha]\frac{\partial^2 a_1}{\partial x^2} = 0 \quad . \quad . \quad (76).$$

Hence, because $[\alpha, \alpha] = 0$, and assuming that $\partial^2 a_1/\partial x^2$ is finite, $\partial a_1/\partial x$ satisfies the equation

$$[\xi, \xi, \alpha] + 2\,[\xi, \alpha, \alpha]\frac{\partial a_1}{\partial x} + [\alpha, \alpha, \alpha]\left(\frac{\partial a_1}{\partial x}\right)^2 = 0 \quad . \quad . \quad . \quad (77).$$

Similarly $\partial a_2/\partial x$ satisfies (77).

Hence, when $x = \xi$, $y = \eta$, $z = \zeta$, $a_1 = a_2 = \alpha$, it follows that $\partial a_1/\partial x$, $\partial a_2/\partial x$ are roots of the same quadratic.

They are finite provided

$$D^3 f_1/Da_1{}^3 \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78).$$

The case excluded is that in which $Df(x, y, z, a)/Da = 0$ is satisfied by three equal values of a, when $x = \xi$, $y = \eta$, $z = \zeta$. This case might be investigated in a similar manner to the case in which the above equation is satisfied by only two equal values of a, when $x = \xi$, $y = \eta$, $z = \zeta$.

D. *Examination of the Differential Coefficients of* Δ.

In this case Δ and its differential coefficients are given by equations (51), (52). (70), and (71). From these it can be seen, without solving the quadratic (77) for $\partial a_1/\partial x$, $\partial a_2/\partial x$, that Δ, $\partial\Delta/\partial x$, $\partial^2\Delta/\partial x^2$, $\partial^3\Delta/\partial x^3$ all vanish, when $x = \xi$, $y = \eta$, $z = \zeta$.

In like manner it can be shown that all the third differential coefficients of Δ vanish; and, therefore, if $U = 0$ be the equation of the locus of unodal lines which is also an envelope, Δ contains U^4 as a factor.

Example 6.—*Locus of Unodal Lines which is also an Envelope.*

Let the surfaces be

$$[\phi(x, y, z)]^2 - [\psi(x, y, z) - a]^3 = 0.$$

(A.) *The Discriminant.*

The discriminant is found by eliminating a between the above equation and

$$3\,[\psi(x, y, z) - a]^2 = 0.$$

Hence it is $[\phi(x, y, z)]^4$.

Hence the locus of ultimate intersections is

$$[\phi(x, y, z)]^4 = 0.$$

(B.) *The locus of Unodal Lines which is also an Envelope is* $\phi(x, y, z) = 0$.

For, let ξ, η, ζ be any point on the surface $\phi(x, y, z) = 0$. Take $a = \psi(\xi, \eta, \zeta)$, and consider the single surface

$$[\phi(x, y, z)]^2 - [\psi(x, y, z) - \psi(\xi, \eta, \zeta)]^3 = 0.$$

Put

$$x = \xi + \mathrm{X},\ y = \eta + \mathrm{Y},\ z = \zeta + \mathrm{Z}.$$

Then the lowest terms in X, Y, Z are

$$\left(\mathrm{X}\frac{\partial\phi}{\partial\xi} + \mathrm{Y}\frac{\partial\phi}{\partial\eta} + \mathrm{Z}\frac{\partial\phi}{\partial\zeta}\right)^2.$$

Hence ξ, η, ζ is a unode on the surface

$$[\phi(x, y, z)]^2 - [\psi(x, y, z) - \psi(\xi, \eta, \zeta)]^3 = 0.$$

Hence the locus of unodal lines of the surfaces under consideration is $\phi(x, y, z) = 0$.

Moreover, the uniplane

$$\mathrm{X}\frac{\partial\phi}{\partial\xi} + \mathrm{Y}\frac{\partial\phi}{\partial\eta} + \mathrm{Z}\frac{\partial\phi}{\partial\zeta} = 0$$

is also a tangent plane to the locus of unodal lines.

Hence the locus of unodal lines is also an envelope.

Hence the factor $[\phi(x, y, z)]^4$ in the discriminant is accounted for.

SECTION III. (Arts. 10–11).—SUPPLEMENTARY REMARKS.

Art. 10.—*Further remark on the case in which* $\mathrm{D}^2 f_1/\mathrm{D}a_1^2 = 0$.

This condition indicates in general that the equation $\mathrm{D}f_1/\mathrm{D}a_1 = 0$ has two equal roots, but if f_1 be of the second degree in a_1, $\mathrm{D}f_1/\mathrm{D}a_1$ is of the first degree in a_1, and hence it has either one root in a_1, or is satisfied by an infinite number of values of a_1.

It is desirable to notice the latter case, because it corresponds to an important case treated in Part II., Section IV., of this paper.

Let $f = Ua^2 + 2Va + W = 0$, where U, V, W are rational integral functions of x, y, z.

Then the conditions $f = 0$, $Df/Da = 0$, $D^2f/Da^2 = 0$ are equivalent to

$$\begin{aligned} Ua^2 + 2Va + W &= 0, \\ Ua + V &= 0, \\ U &= 0, \end{aligned}$$

at all points of the locus of ultimate intersections.

Hence $U = 0$, $V = 0$, $W = 0$ at all such points.

Hence unless U, V, W have a common factor, which could in that case be removed from the equation $f = 0$, the locus of ultimate intersections is not a surface, and hence its equation cannot be obtained by equating a factor of the discriminant to zero. Hence this case need not be further considered.

Art. 11.—*If the surface $f(x, y, z) = 0$ have upon it a curve at every point of which there is a conic node, then the tangent cones at the conic nodes must break up into two planes.*[*]

Let ξ, η, ζ; $\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta$; be neighbouring points on the curve.

Then since there is a conic node at ξ, η, ζ;

$$f(\xi, \eta, \zeta) = 0, \ Df/D\xi = 0, \ Df/D\eta = 0, \ Df/D\zeta = 0.$$

And since there is a conic node at $\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta$, four other equations hold, which by means of the above give

$$\begin{aligned} &[\xi, \xi](\delta\xi)^2 + [\eta, \eta](\delta\eta)^2 + [\zeta, \zeta](\delta\zeta)^2 \\ &+ 2[\eta, \zeta](\delta\eta)(\delta\zeta) + 2[\zeta, \xi](\delta\zeta)(\delta\xi) + 2[\xi, \eta](\delta\xi)(\delta\eta) + \ldots = 0, \\ &[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) + \ldots\ldots = 0, \\ &[\eta, \xi](\delta\xi) + [\eta, \eta](\delta\eta) + [\eta, \zeta](\delta\zeta) + \ldots\ldots = 0, \\ &[\zeta, \xi](\delta\xi) + [\zeta, \eta](\delta\eta) + [\zeta, \zeta](\delta\zeta) + \ldots\ldots = 0. \end{aligned}$$

Retaining only the principal terms in the last three equations, it follows that

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[*] The geometry of a surface of continuous curvature shows at once that there cannot be a curve of conical points on a surface.

$$[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) = 0,$$
$$[\eta, \xi](\delta\xi) + [\eta, \eta](\delta\eta) + [\eta, \zeta](\delta\zeta) = 0,$$
$$[\zeta, \xi](\delta\xi) + [\zeta, \eta](\delta\eta) + [\zeta, \zeta](\delta\zeta) = 0.$$

These equations must be consistent, and therefore

$$\frac{\mathrm{D}[\,[\xi], [\eta], [\zeta]\,]}{\mathrm{D}[\ \xi\ ,\ \eta\ ,\ \zeta\ ]} = 0.$$

This is the condition that the tangent cone at ξ, η, ζ, viz. :—

$$[\xi, \xi](\mathrm{X} - \xi)^2 + [\eta, \eta](\mathrm{Y} - \eta)^2 + [\zeta, \zeta](\mathrm{Z} - \zeta)^2$$
$$+ 2[\eta, \zeta](\mathrm{Y} - \eta)(\mathrm{Z} - \zeta) + 2[\zeta, \xi](\mathrm{Z} - \zeta)(\mathrm{X} - \xi) + 2[\xi, \eta](\mathrm{X} - \xi)(\mathrm{Y} - \eta) = 0,$$

may break up into two planes.

PART II.—THE EQUATION OF THE SYSTEM OF SURFACES IS A RATIONAL INTEGRAL FUNCTION OF THE COORDINATES AND TWO ARBITRARY PARAMETERS.

SECTION I. (Art. 1).—PRELIMINARY THEOREMS.

Art. 1. (A.) *If* ξ, η, ζ *are the coordinates of any point on the locus* $\phi(x, y, z) = 0$ *(where* ϕ *is a rational integral indecomposable function of* x, y, z*), and if the substitutions* $x = \xi$, $y = \eta$, $z = \zeta$ *make* $\psi(x, y, z)$ *and all its partial differential coefficients with regard to* x, y, z *up to the* n^{th} *order vanish, and if they also make any one of the partial differential coefficients of the* $(n + 1)^{th}$ *order vanish, they will also make all the partial differential coefficients of the* $(n + 1)^{th}$ *order vanish (*ψ *being a rational integral function of* x, y, z, *but not in general indecomposable).*

Suppose that when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\frac{\partial^{n+1}\psi}{\partial x^{r+1}\,\partial y^s\,\partial z^{n-r-s}} = 0,$$

where ∂ denotes partial differentiation when x, y, z are independent variables.

To prove that the same substitutions make

$$\frac{\partial^{n+1}\psi}{\partial x^r\,\partial y^{s+1}\,\partial z^{n-r-s}} = 0,$$

and

$$\frac{\partial^{n+1}\psi}{\partial x^r\,\partial y^s\,\partial z^{n-r-s+1}} = 0.$$

It is given that all the values of x, y, z which make $\phi(x, y, z) = 0$, also make

$$\frac{\partial^{n+1}\psi}{\partial x^{r+1}\,\partial y^{s}\,\partial z^{n-r-s}} = 0.$$

Now let $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ be a point near to ξ, η, ζ on $\phi(x, y, z) = 0$.

Therefore $\phi(\xi, \eta, \zeta) = 0$.

$$\phi(\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1).$$

$$\frac{\partial^{n+1}\psi}{\partial \xi^{r+1}\,\partial \eta^{s}\,\partial \zeta^{n-r-s}} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2).$$

And since ξ, η, ζ make

$$\frac{\partial^{n}\psi}{\partial x^{r}\,\partial y^{s}\,\partial z^{n-r-s}} = 0,$$

for this is a differential coefficient of the n^{th} order, $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ must also do the same.

Hence

$$(\delta\xi)\frac{\partial^{n+1}\psi}{\partial \xi^{r+1}\,\partial \eta^{s}\,\partial \zeta^{n-r-s}} + (\delta\eta)\frac{\partial^{n+1}\psi}{\partial \xi^{r}\,\partial \eta^{s+1}\,\partial \zeta^{n-r-s}} + (\delta\zeta)\frac{\partial^{n+1}\psi}{\partial \xi^{r}\,\partial \eta^{s}\,\partial \zeta^{n-r-s+1}} = 0.$$

Also from (1)

$$(\delta\xi)\frac{\partial\phi}{\partial\xi} + (\delta\eta)\frac{\partial\phi}{\partial\eta} + (\delta\zeta)\frac{\partial\phi}{\partial\zeta} = 0.$$

Since this is the only relation between $\delta\xi$, $\delta\eta$, $\delta\zeta$, it follows that

$$\frac{\left(\dfrac{\partial^{n+1}\psi}{\partial \xi^{r+1}\,\partial \eta^{s}\,\partial \zeta^{n-r-s}}\right)}{\left(\dfrac{\partial\phi}{\partial\xi}\right)} = \frac{\left(\dfrac{\partial^{n+1}\psi}{\partial \xi^{r}\,\partial \eta^{s+1}\,\partial \zeta^{n-r-s}}\right)}{\left(\dfrac{\partial\phi}{\partial\eta}\right)} = \frac{\left(\dfrac{\partial^{n+1}\psi}{\partial \xi^{r}\,\partial \eta^{s}\,\partial \zeta^{n-r-s+1}}\right)}{\left(\dfrac{\partial\phi}{\partial\zeta}\right)}.$$

Hence, by means of (2),

$$\frac{\partial^{n+1}\psi}{\partial \xi^{r}\,\partial \eta^{s+1}\,\partial \zeta^{n-r-s}} = 0, \qquad \frac{\partial^{n+1}\psi}{\partial \xi^{r}\,\partial \eta^{s}\,\partial \zeta^{n-r-s+1}} = 0.$$

In this way it is possible to pass from any one partial differential coefficient of order $(n + 1)$ by successive steps to any other of order $(n + 1)$; at each step always diminishing by one the number of differentiations with regard to one variable, and increasing by one the number of differentiations with regard to another variable.

Hence all the differential coefficients of the $(n + 1)^{\text{th}}$ order vanish when $x = \xi$, $y = \eta$, $z = \zeta$.

(B.) (i.) *If* ξ, η, ζ, *are the co-ordinates of any point on the locus* $\phi(x, y, z) = 0$ (*where* ϕ *is a rational integral function of* x, y, z *which contains no repeated factors*), *and if the substitutions* $x = \xi$, $y = \eta$, $z = \zeta$ *make* $\psi(x, y, z) = 0$ (*where* ψ *is a rational integral function of* x, y, z), *then* ψ *contains the first power of* ϕ *as a factor.*

(ii.) *If* $x = \xi$, $y = \eta$, $z = \zeta$ *make* $\psi = 0, \dfrac{\partial\psi}{\partial x} = 0$, *then* ψ *contains the second power of* ϕ *as a factor.*

(iii.) *If* $x = \xi$, $y = \eta$, $z = \zeta$ *make* $\psi = 0, \dfrac{\partial\psi}{\partial x} = 0, \ldots \dfrac{\partial^{m-1}\psi}{\partial x^{m-1}} = 0$, *then* ψ *contains* ϕ^m *as a factor.*

To prove (i.) suppose first that ϕ is indecomposable. It is obvious that ψ cannot be of lower dimensions than ϕ in any one of the variables; if it were, then all the values of x, y, z which make $\phi = 0$ would not make $\psi = 0$.

It may happen that ϕ does not contain all the variables x, y, z. But it must contain one of them; suppose it contains x.

If ϕ be not a factor of ψ, proceed as in the process for finding the common factor of highest dimensions in x of ϕ and ψ; and if, at any step of the process, fractional quotients in which the denominators are functions of y, z are obtained, let the denominators be removed in the usual way by multiplication throughout by a factor.

Then either the process will terminate, or there will at last be a remainder, which is a function of y, z only, not x.

In the first case ϕ and ψ will have a common factor, and ϕ will be decomposable, which is contrary to the hypothesis.

In the second case a relation of the form

$$A\psi = B\phi + C$$

exists, where A, B, C are rational integral functions of y, z only. In this case, since all the values of x, y, z which make $\phi = 0$, also make $\psi = 0$, therefore they make $C = 0$. But C is a function of y, z only, not x. Now, the values under consideration are values of x, y, z. This is impossible. Hence this alternative does not hold.

Hence ϕ must be a factor of ψ.

If ϕ be decomposable, its indecomposable factors may be taken separately, and shown as above to be factors of ψ.

As it is further supposed that ϕ contains no repeated factors, it follows that ψ contains ϕ as a factor.

To prove (ii.).

By the same argument as in (i.) it follows that ψ contains ϕ as a factor. Let $\psi = R\phi$.

Therefore,

$$\frac{\partial\psi}{\partial x} = \phi\frac{\partial \mathrm{R}}{\partial x} + \mathrm{R}\frac{\partial\phi}{\partial x}.$$

Now the substitutions $x = \xi$, $y = \eta$, $z = \zeta$ make $\phi = 0$, $\psi = 0$, $\partial\psi/\partial x = 0$. Therefore they make

$$\mathrm{R}\frac{\partial\phi}{\partial x} = 0.$$

Now all the values of x, y, z which make $\phi = 0$ cannot make $\partial\phi/\partial x = 0$. Hence $x = \xi$, $y = \eta$, $z = \zeta$ must make R vanish.

Therefore R is divisible by ϕ without remainder.

Therefore ψ is divisible by ϕ^2 without remainder.

To prove (iii.) proceed by induction.

Suppose that the theorem is true for a given value of m, viz., that if $x = \xi$, $y = \eta$, $z = \zeta$ make $\psi = 0$, $\partial\psi/\partial x = 0, \ldots \partial^{m-1}\psi/\partial x^{m-1} = 0$, then ψ contains ϕ^m as a factor.

Let it now be given that $x = \xi$, $y = \eta$, $z = \zeta$ also make $\partial^m\psi/\partial x^m = 0$.

Then by the assumption

$$\psi = \rho.\phi^m,$$

where ρ is some rational integral function of x, y, z.

Therefore

$$\begin{aligned}\frac{\partial^m\psi}{\partial x^m} &= \rho\frac{\partial^m\phi^m}{\partial x^m} + m\frac{\partial\rho}{\partial x}\frac{\partial^{m-1}\phi^m}{\partial x^{m-1}} + \ldots \\ &= \rho\frac{\partial^m\phi^m}{\partial x^m} + \phi\chi,\end{aligned}$$

where χ is some rational integral function of x, y, z.

Now

$$\frac{\partial^m\phi^m}{\partial x^m} = m\,!\left(\frac{\partial\phi}{\partial x}\right)^m + \phi.\sigma,$$

where σ is some rational integral function of x, y, z.

Therefore

$$\frac{\partial^m\psi}{\partial x^m} = m\,!\,\rho\left(\frac{\partial\phi}{\partial x}\right)^m + \phi\,(\rho\sigma + \chi).$$

Hence the substitutions $x = \xi$, $y = \eta$, $z = \zeta$ make

$$\rho\left(\frac{\partial\phi}{\partial x}\right)^m = 0,$$

but they do not make $\partial\phi/\partial x = 0$.

Therefore they make $\rho = 0$.

Therefore ρ contains ϕ as a factor.

Therefore ψ contains ϕ^{m+1} as a factor.

Hence, if the theorem is true for a special value of m, it is true for the next value. But it has been proved true when $m = 2$, hence it is true in general.

(C.) *If u, v be determined as functions of other quantities by the equations*

$$\phi(u, v) = 0, \quad \psi(u, v) = 0,$$

where ϕ and ψ are rational integral functions of u, v and the other quantities; then, if two systems of common values of u, v become equal, they will also satisfy the equation

$$\frac{\mathrm{D}[\phi, \psi]}{\mathrm{D}[u, v]} = 0.$$

Conversely, if values of u, v can be found to satisfy at the same time the three equations

$$\phi(u, v) = 0, \quad \psi(u, v) = 0, \quad \frac{\mathrm{D}[\phi, \psi]}{\mathrm{D}[u, v]} = 0,$$

then these values count twice over among the common solutions of the equations

$$\phi(u, v) = 0, \quad \psi(u, v) = 0,$$

except in the case where ϕ and ψ are of the first degree in u and v; and then the two equations have an infinite number of solutions in common.

To prove this, let u, v represent the coordinates of a point in a plane. Then $\phi(u, v) = 0$, $\psi(u, v) = 0$ are the equations of two algebraic curves.

The values of u, v which satisfy at the same time both equations are the coordinates of the points of intersection of the two curves.

Let $u = \alpha$, $v = \beta$ be the coordinates of one point of intersection. The tangents to the curves at α, β are

$$(\mathrm{U} - \alpha)\frac{\partial\phi(\alpha, \beta)}{\partial\alpha} + (\mathrm{V} - \beta)\frac{\partial\phi(\alpha, \beta)}{\partial\beta} = 0,$$

$$(\mathrm{U} - \alpha)\frac{\partial\psi(\alpha, \beta)}{\partial\alpha} + (\mathrm{V} - \beta)\frac{\partial\psi(\alpha, \beta)}{\partial\beta} = 0,$$

where U, V are current coordinates.

The two tangents will coincide, *i.e.*, the curves have two coincident points of intersection, if

$$\frac{\partial\phi(\alpha, \beta)}{\partial\alpha}\frac{\partial\psi(\alpha, \beta)}{\partial\beta} - \frac{\partial\phi(\alpha, \beta)}{\partial\beta}\frac{\partial\psi(\alpha, \beta)}{\partial\alpha} = 0,$$

i.e., if $u = \alpha$, $v = \beta$ satisfy

$$\frac{\mathrm{D}\,[\phi, \psi]}{\mathrm{D}\,[u, v]} = 0.$$

In the case where ϕ, ψ are each of the first degree in u, v, then the theorem requires to be specially interpreted, the interpretation corresponding to the fact that if two straight lines have two points in common, they have an infinite number of points in common. Hence, in this case, the two equations have an infinite number of solutions in common. (This particular case is of great importance in Section IV. of this investigation.)

(D.) *To determine the conditions that the equations* $\phi\,(u, v) = 0$, $\psi\,(u, v) = 0$, *may be satisfied by three coinciding systems of common values.*

In this case, considering as in (C) that u, v represent the coordinates of a point, the curves $\phi\,(u, v) = 0$, $\psi\,(u, v) = 0$ must have contact of the second order.

Now

$$\frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v}\frac{dv}{du} = 0$$

$$\frac{\partial^2\phi}{\partial u^2} + 2\frac{\partial^2\phi}{\partial u\,\partial v}\frac{dv}{du} + \frac{\partial^2\phi}{\partial v^2}\left(\frac{dv}{du}\right)^2 + \frac{\partial\phi}{\partial v}\frac{d^2v}{du^2} = 0.$$

Therefore

$$\frac{d^2v}{du^2} = -\left[\frac{\partial^2\phi}{\partial u^2}\left(\frac{\partial\phi}{\partial v}\right)^2 - 2\frac{\partial^2\phi}{\partial u\,\partial v}\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v} + \frac{\partial^2\phi}{\partial v^2}\left(\frac{\partial\phi}{\partial u}\right)^2\right]\left(\frac{\partial\phi}{\partial v}\right)^{-3}.$$

Hence, equating the values of d^2v/du^2 for the two curves, there is obtained the further condition

$$\left[\frac{\partial^2\phi}{\partial u^2}\left(\frac{\partial\phi}{\partial v}\right)^2 - 2\frac{\partial^2\phi}{\partial u\,\partial v}\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v} + \frac{\partial^2\phi}{\partial v^2}\left(\frac{\partial\phi}{\partial u}\right)^2\right]\left(\frac{\partial\phi}{\partial v}\right)^{-3}$$

$$= \left[\frac{\partial^2\psi}{\partial u^2}\left(\frac{\partial\psi}{\partial v}\right)^2 - 2\frac{\partial^2\psi}{\partial u\,\partial v}\frac{\partial\psi}{\partial u}\frac{\partial\psi}{\partial v} + \frac{\partial^2\psi}{\partial v^2}\left(\frac{\partial\psi}{\partial u}\right)^2\right]\left(\frac{\partial\psi}{\partial v}\right)^{-3}.$$

SECTION II. (Arts. 2–12).—THE FACTORS OF THE DISCRIMINANT, WHICH IN GENERAL CORRESPOND TO ENVELOPE AND SINGULAR POINT LOCI.

Art. 2.—*The Fundamental Equations.*

Let the equation of the system of surfaces be

$$f\,(x, y, z, a, b) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3),$$

where a, b are independent arbitrary parameters, and f is a rational integral indecomposable function of x, y, z, a, b.

The locus of ultimate intersections is obtained by eliminating a and b between (3) and

$$\frac{\mathrm{D}f(x, y, z, a, b)}{\mathrm{D}a} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$\frac{\mathrm{D}f(x, y, z, a, b)}{\mathrm{D}b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where D denotes partial differentiation when x, y, z, a, b are treated as independent variables.

Let the result of the elimination be $\Delta = 0$, then Δ is called the discriminant.

If x, y, z are chosen so as to make any factor of the discriminant vanish, it indicates *in general* that it is possible to satisfy equations (3), (4), (5) by the same values of a, b. Hence x, y, z can be expressed as functions of a, b.

In this case $x = \phi(a, b)$, $y = \psi(a, b)$, $z = \chi(a, b)$.

Eliminating a and b a surface locus is obtained.

This is the general case. The exceptional cases are noticed in Section VI., Art. 30.

Art. 3.—*The Loci of Singular Points of the System of Surfaces.*

The equation of the locus of singular points on the surfaces (3) can be obtained by eliminating a and b, between (3), and

$$\frac{\mathrm{D}f(x, y, z, a, b)}{\mathrm{D}x} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

$$\frac{\mathrm{D}f(x, y, z, a, b)}{\mathrm{D}y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

$$\frac{\mathrm{D}f(x, y, z, a, b)}{\mathrm{D}z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The singular points are in general conic nodes.

The locus of conic nodes is therefore a curve, whose equations are given by eliminating a, b between (3), (6), (7), (8).

It follows, also, by eliminating x, y, z between the same equations, that there is a definite relation between a, b.

If ξ, η, ζ be the coordinates of the conic node on the surface

$$f(x, y, z, \alpha, \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

and $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ the coordinates of the conic node on the surface

$$f(x, y, z, \quad \alpha + \delta\alpha, \quad \beta + \delta\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

then the following equations all hold at the same time,

$$f(\xi, \eta, \zeta, \alpha, \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

$$\frac{\mathrm{D}f(\xi, \eta, \zeta, \alpha, \beta)}{\mathrm{D}\xi} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

$$\frac{\mathrm{D}f(\xi, \eta, \zeta, \alpha, \beta)}{\mathrm{D}\eta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

$$\frac{\mathrm{D}f(\xi, \eta, \zeta, \alpha, \beta)}{\mathrm{D}\zeta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

and four other equations, which, by means of the above, become

$$\frac{\mathrm{D}f}{\mathrm{D}\alpha}(\delta\alpha) + \frac{\mathrm{D}f}{\mathrm{D}\beta}(\delta\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

$$[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) + [\xi, \alpha](\delta\alpha) + [\xi, \beta](\delta\beta) = 0 \quad . \quad (16),$$

$$[\eta, \xi](\delta\xi) + [\eta, \eta](\delta\eta) + [\eta, \zeta](\delta\zeta) + [\eta, \alpha](\delta\alpha) + [\eta, \beta](\delta\beta) = 0 \quad . \quad (17),$$

$$[\zeta, \xi](\delta\xi) + [\zeta, \eta](\delta\eta) + [\zeta, \zeta](\delta\zeta) + [\zeta, \alpha](\delta\alpha) + [\zeta, \beta](\delta\beta) = 0 \quad . \quad (18).$$

If $\delta\alpha$, $\delta\beta$ be eliminated from (15)–(18), the ratios $\delta\xi : \delta\eta : \delta\zeta$ are determined. These ratios determine the tangent line to the curve locus of conic nodes.

If β be determined as a function of α, so that (11)–(14) can be satisfied by the same values of ξ, η, ζ, then the equations (11) and (15) show that the locus of conic nodes is a curve lying on one of the *general integrals* of the partial differential equation of the surfaces (3).

Example 1. *Curve Locus of Conic Nodes.*

Let the surfaces be

$$x^2 + e(y - \tfrac{1}{2}a)^2 + (z - \tfrac{1}{2}b)^2 - cx + a + b = 0 \quad . \quad . \quad . \quad . \quad (19),$$

where e, c are fixed constants; a, b the arbitrary parameters.

(A.) *The locus of conic nodes is the straight line.*

$$x = \tfrac{1}{2}c, \quad y + z = \tfrac{1}{8}c^2.$$

To find this locus it is necessary to eliminate a, b between (19) and

$$\frac{Df}{Dx} = 2x - c = 0 \quad \dots\dots\dots\dots \quad (20),$$

$$\frac{Df}{Dy} = 2e\left(y - \tfrac{1}{2}a\right) = 0 \quad \dots\dots\dots\dots \quad (21),$$

$$\frac{Df}{Dz} = 2\left(z - \tfrac{1}{2}b\right) = 0 \quad \dots\dots\dots\dots \quad (22).$$

Therefore

$$x = \tfrac{1}{2}c, \quad y = \tfrac{1}{2}a, \quad z = \tfrac{1}{2}b, \quad y + z = \tfrac{1}{8}c^2.$$

Hence the locus of conic nodes is the straight line,

$$x = \tfrac{1}{2}c, \quad y + z = \tfrac{1}{8}c^2 \quad \dots\dots\dots\dots \quad (23).$$

(B.) *The locus of conic nodes lies on the general integral of the partial differential equation of the surfaces* (19) *obtained by putting* $b = \frac{1}{4}c^2 - a$.

To determine this general integral take the values of x, y, z from (20)–(22), and substitute in (19). This gives $a + b = \frac{1}{4}c^2$.

Hence the general integral is obtained by eliminating a from

$$x^2 + e\left(y - \tfrac{1}{2}a\right)^2 + \left(z + \tfrac{1}{2}a - \tfrac{1}{8}c^2\right)^2 - cx + \tfrac{1}{4}c^2 = 0,$$

and

$$-e\left(y - \tfrac{1}{2}a\right) + \left(z + \tfrac{1}{2}a - \tfrac{1}{8}c^2\right) = 0.$$

Hence it is

$$e\left(y + z - \tfrac{1}{8}c^2\right)^2 + (1 + e)\left(x - \tfrac{1}{2}c\right)^2 = 0.$$

It contains the locus of conic nodes, whose equations are given in (23).

(C.) *The locus of conic nodes does not lie on the locus of ultimate intersections.*

For the equation of the locus of ultimate intersections is obtained by eliminating a, b between

$$f = x^2 + e\left(y - \tfrac{1}{2}a\right)^2 + \left(z - \tfrac{1}{2}b\right)^2 - cx + a + b = 0,$$

$$\frac{Df}{Da} = -e\left(y - \tfrac{1}{2}a\right) + 1 = 0,$$

$$\frac{Df}{Db} = -\left(z - \tfrac{1}{2}b\right) + 1 = 0.$$

It is therefore

$$x^2 - cx + 2y + 2z - 1 - \frac{1}{e} = 0.$$

This does not contain the locus of conic nodes.

It is an envelope touching f at the two points

$$x = \frac{c}{2} \pm \left(\frac{c^2}{4} - 1 - \frac{1}{e} - a - b\right)^{\frac{1}{2}}.$$

$$y = \tfrac{1}{2}a + \frac{1}{e}.$$

$$z = \tfrac{1}{2}b + 1.$$

Art. 4.—*Investigation of the conditions which are satisfied at any point on the Locus of Conic Nodes.*

In the preceding article it was shown that the surfaces (3) have in general a curve locus of conic nodes.

If, however, *every one* of the surfaces (3) has a conic node, then equations (3), (6), (7), (8) are equivalent to three independent equations only, and the locus of conic nodes is a surface, whose equation is obtained by eliminating a and b between any three of the four equations (3), (6), (7), (8).

It will be proved that such a surface locus of conic nodes is a part at least of the locus of ultimate intersections.

With the notation of the last article, equations (11)–(18) hold; but now there is no relation between α and β.

There is a conic node on the surface

$$f(x, y, z, \alpha + \delta\alpha, \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

Hence (15) must hold when $\delta\beta = 0$.

Hence

$$\mathrm{D}f/\mathrm{D}\alpha = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

Similarly

$$\mathrm{D}f/\mathrm{D}\beta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

Since (11), (25), and (26) hold at all points of the conic node locus, it follows that the conic node locus is a part, at least, of the locus of ultimate intersections.

The position of the tangent plane to the conic node locus may be obtained from (16)–(18) by eliminating $\delta\alpha$, $\delta\beta$; and then using the relations

$$\frac{\delta\xi}{X-\xi} = \frac{\delta\eta}{Y-\eta} = \frac{\delta\zeta}{Z-\zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27),$$

where X, Y, Z are current coordinates.

Since (25) and (26) are satisfied at all points of the conic node locus, they are satisfied when ξ, η, ζ, α, β are replaced by $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$, $\alpha + \delta\alpha$, $\beta + \delta\beta$ respectively.

Hence

$$[\alpha, \xi](\delta\xi) + [\alpha, \eta](\delta\eta) + [\alpha, \zeta](\delta\zeta) + [\alpha, \alpha](\delta\alpha) + [\alpha, \beta](\delta\beta) = 0 \quad . \qquad (28).$$

$$[\beta, \xi](\delta\xi) + [\beta, \eta](\delta\eta) + [\beta, \zeta](\delta\zeta) + [\beta, \alpha](\delta\alpha) + [\beta, \beta](\delta\beta) = 0 \quad . \qquad (29).$$

Now (28) and (29) are not independent of (16)-(18).

For if $\delta\alpha$, $\delta\beta$ are definite infinitely small quantities, then (16)-(18) determine the values of $\delta\xi$, $\delta\eta$, $\delta\zeta$ corresponding to the conic node on the surface (10). Substituting these values in (28) and (29), and observing that $\delta\alpha$, $\delta\beta$ being independent may be supposed to vanish separately, the following relations are obtained (using the usual notation for Jacobians):—

$$\frac{D[[\xi], [\eta], [\zeta], [\alpha]]}{D[\ \xi\ ,\ \eta\ ,\ \zeta\ ,\ \alpha\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30),$$

$$\frac{D[[\xi], [\eta], [\zeta], [\beta]]}{D[\ \xi\ ,\ \eta\ ,\ \zeta\ ,\ \alpha\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31),$$

$$\frac{D[[\xi], [\eta], [\zeta], [\beta]]}{D[\ \xi\ ,\ \eta\ ,\ \zeta\ ,\ \beta\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32).$$

Other similar relations exist which may be found by taking any four of the equations (16), (17), (18), (28), (29), putting any one of the five quantities $\delta\xi$, $\delta\eta$, $\delta\zeta$, $\delta\alpha$, $\delta\beta$ equal to zero, and expressing that the equations give consistent values for the four quantities which remain.

Hence any minor of the fourth order of the Jacobian

$$\frac{D[[\xi], [\eta], [\zeta], [\alpha], [\beta]]}{D[\ \xi\ ,\ \eta\ ,\ \zeta\ ,\ \alpha\ ,\ \beta\ ]}$$

vanishes.

Art. 5.—*Investigation of the conditions which are satisfied at any point on the Locus of Biplanar Nodes.*

The equation of the tangent cone which, in this case, becomes the equation of the biplanes, at the singular point is

$$[\xi, \xi](X-\xi)^2 + [\eta, \eta](Y-\eta)^2 + [\zeta, \zeta](Z-\zeta)^2$$
$$+ 2[\eta,\zeta](Y-\eta)(Z-\zeta) + 2[\zeta,\xi](Z-\zeta)(X-\xi) + 2[\xi,\eta](X-\xi)(Y-\eta) = 0 \quad . \quad (33)$$

This breaks up into factors, linear with regard to $X - \xi$, $Y - \eta$, $Z - \zeta$.

Therefore

$$\frac{D[[\xi], [\eta], [\zeta]]}{D[\ \xi\ ,\ \eta\ ,\ \zeta\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

Now if in (16)–(18), $\delta\beta$ be put equal to zero (which is possible, since there is by hypothesis a biplanar node on the surface (24)), the values of $\delta\xi/\delta\alpha$, $\delta\eta/\delta\alpha$, $\delta\zeta/\delta\alpha$ must be finite.

But the denominators of the values of these expressions vanish by (34). Hence their numerators also vanish.

Therefore

$$\frac{D[[\xi], [\eta], [\alpha]]}{D[\ \xi\ ,\ \eta\ ,\ \zeta\ ]} \equiv \frac{D[[\xi], [\eta], [\zeta]]}{D[\ \xi\ ,\ \eta\ ,\ \alpha\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

$$\frac{D[[\xi], [\zeta], [\alpha]]}{D[\ \xi\ ,\ \eta\ ,\ \zeta\ ]} \equiv \frac{D[[\xi], [\eta], [\zeta]]}{D[\ \xi\ ,\ \zeta\ ,\ \alpha\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36),$$

$$\frac{D[[\eta], [\zeta], [\alpha]]}{D[\ \xi\ ,\ \eta\ ,\ \zeta\ ]} \equiv \frac{D[[\xi], [\eta], [\zeta]]}{D[\ \eta\ ,\ \zeta\ ,\ \alpha\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

And similarly it can be shown that the following equations obtained by changing into β in the above also hold good :—

$$\frac{D[[\xi], [\eta], [\beta]]}{D[\ \xi\ ,\ \eta\ ,\ \zeta\ ]} \equiv \frac{D[[\xi], [\eta], [\zeta]]}{D[\ \xi\ ,\ \eta\ ,\ \beta\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38),$$

$$\frac{D[[\xi], [\zeta], [\beta]]}{D[\ \xi\ ,\ \eta\ ,\ \zeta\ ]} \equiv \frac{D[[\xi], [\eta], [\zeta]]}{D[\ \xi\ ,\ \zeta\ ,\ \beta\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39),$$

$$\frac{D[[\eta], [\zeta], [\beta]}{D[\ \xi\ ,\ \eta\ ,\ \zeta\ ]} \equiv \frac{D[[\xi], [\eta], [\zeta]]}{D[\ \eta\ ,\ \zeta\ ,\ \beta\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40).$$

From these it follows that (16)–(18) are equivalent to two independent equations only in this case.

Now consider equations (16), (17), (28).

The equation (35) makes the determinant formed from the coefficients of $\delta\xi$, $\delta\eta$, $\delta\zeta$ in them vanish. Hence it bears to them the same relation that (34) bears to (16), (17), (18).

Therefore (16), (17), (28) are equivalent to two independent equations only.

In like manner (36) shows that (16), (18), (28) are equivalent to two independent equations only.

Also (37) shows that (17), (18), (28) are equivalent to two independent equations only.

Hence (28) and *any* two of the equations (16), (17), (18) are equivalent to two independent equations only.

Similarly, by means of (38)-(40), it can be shown that (29), and any two of the equations (16), (17), (18) are equivalent to two independent equations only.

Hence the five equations (16), (17), (18), (28), (29) are equivalent to two independent equations only in this case.

Since (16), (28), (29) are not independent, it follows that (amongst other relations)

$$\frac{\mathrm{D}[\,[\xi],\,[\alpha],\,[\beta]\,]}{\mathrm{D}[\ \xi\,,\ \alpha\,,\ \beta\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41),$$

and

$$\frac{\mathrm{D}[\,[\xi],\,[\alpha],\,[\beta]\,]}{\mathrm{D}[\ \eta\,,\ \alpha\,,\ \beta\ ]} \equiv \frac{\mathrm{D}[\,[\eta],\,[\alpha],\,[\beta]\,]}{\mathrm{D}[\ \xi\,,\ \alpha\,,\ \beta\ ]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42).$$

Art. 6.—*Investigation of the conditions which are satisfied at any point on the Locus of Uniplanar Nodes.*

In this case, the left-hand side of the equation (33) becomes a perfect square.

Therefore

$$\begin{aligned}&[\xi,\xi] : [\xi,\eta] : [\xi,\zeta]\\ =\ &[\eta,\xi] : [\eta,\eta] : [\eta,\zeta]\\ =\ &[\zeta,\xi] : [\zeta,\eta] : [\zeta,\zeta]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43).\end{aligned}$$

Now, multiplying (16) by $[\eta, \xi]$, (17) by $[\xi, \xi]$, and subtracting

$$(\delta\alpha)\ \{[\xi,\alpha]\,[\eta,\xi] - [\eta,\alpha]\,[\xi,\xi]\} + (\delta\beta)\ \{[\xi,\beta]\,[\eta,\xi] - [\eta,\beta]\,[\xi,\xi]\} = 0\,. \quad (44).$$

Now, there is a uniplanar node on the surface (24), hence $\delta\beta$ may be made to vanish.

Therefore

$$[\xi,\alpha]\,[\eta,\xi] - [\eta,\alpha]\,[\xi,\xi] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Similarly

$$[\xi,\beta]\,[\eta,\xi] - [\eta,\beta]\,[\xi,\xi] = 0.$$

Therefore

$$[\xi,\xi]\,/\,[\eta,\xi] = [\xi,\alpha]\,/\,[\eta,\alpha] = [\xi,\beta]\,/\,[\eta,\beta] \quad . \quad . \quad . \quad . \quad . \quad . \quad (46).$$

Now (43) and (46) show that (16) and (17) are equivalent to one independent equation only. Similarly (16) and (18) are equivalent to one independent equation only.

Hence the following ratios hold

$$\begin{aligned}&[\xi, \xi] : [\xi, \eta] : [\xi, \zeta] : [\xi, \alpha] : [\xi, \beta]\\ =\ &[\eta, \xi] : [\eta, \eta] : [\eta, \zeta] : [\eta, \alpha] : [\eta, \beta]\\ =\ &[\zeta, \xi] : [\zeta, \eta] : [\zeta, \zeta] : [\zeta, \alpha] : [\zeta, \beta] \quad . \quad . \quad . \quad . \quad . \quad . \quad (47).\end{aligned}$$

Applying (47) to (16) and (28), it follows that these two are equivalent to one independent equation only.

Similarly (16) and (29) are equivalent to one independent equation only.

Hence the five equations (16), (17, (18), (28), (29), are equivalent to one independent equation only, and, therefore, the following ratios hold

$$\begin{aligned}&[\xi, \xi] : [\xi, \eta] : [\xi, \zeta] : [\xi, \alpha] : [\xi, \beta]\\ =\ &[\eta, \xi] : [\eta, \eta] : [\eta, \zeta] : [\eta, \alpha] : [\eta, \beta]\\ =\ &[\zeta, \xi] : [\zeta, \eta] : [\zeta, \zeta] : [\zeta, \alpha] : [\zeta, \beta]\\ =\ &[\alpha, \xi] : [\alpha, \eta] : [\alpha, \zeta] : [\alpha, \alpha] : [\alpha, \beta]\\ =\ &[\beta, \xi] : [\beta, \eta] : [\beta, \zeta] : [\beta, \alpha] : [\beta, \beta] \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).\end{aligned}$$

Art. 7.—*Examination of the Form of the Discriminant, and Calculation of its Differential Coefficients of the First and Second Orders.*

Let a_1, b_1; a_2, b_2; . . . be the common roots of (4) and (5), and let it be supposed, in the first instance, that at points in the loci considered these sets of common roots are all *distinct.*

Then if

$$\Delta = \mathrm{A} f(x, y, z, a_1, b_1) f(x, y, z, a_2, b_2) \ldots \quad (49),$$

where A is a factor introduced to make the discriminant of the proper order and weight, the result of eliminating a and b between (3), (4), and (5) is

$$\Delta = 0.$$

Writing for brevity

$$\Delta = \mathrm{R} f(x, y, z, a_1, b_1) = \mathrm{R} f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

$$\frac{\partial \Delta}{\partial x} = \frac{\partial \mathrm{R}}{\partial x} f + \mathrm{R}\left(\frac{\mathrm{D}f}{\mathrm{D}x} + \frac{\mathrm{D}f}{\mathrm{D}a_1}\frac{\partial a_1}{\partial x} + \frac{\mathrm{D}f}{\mathrm{D}b_1}\frac{\partial b_1}{\partial x}\right).$$

To determine $\partial a_1/\partial x$, $\partial b_1/\partial x$ there are the equations

$$\frac{\mathrm{D}f}{\mathrm{D}a_1} = 0, \quad \frac{\mathrm{D}f}{\mathrm{D}b_1} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51).$$

These give

$$\left.\begin{aligned}[x, a_1] + [a_1, a_1]\frac{\partial a_1}{\partial x} + [a_1, b_1]\frac{\partial b_1}{\partial x} = 0\\ [x, b_1] + [a_1, b_1]\frac{\partial a_1}{\partial x} + [b_1, b_1]\frac{\partial b_1}{\partial x} = 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (52).$$

Similar equations exist for finding

$$\frac{\partial a_1}{\partial y}, \frac{\partial b_1}{\partial y} \text{ and } \frac{\partial a_1}{\partial z}, \frac{\partial b_1}{\partial z}.$$

These, however, are not yet required, it being seen at once that, in general,

$$\frac{\partial \Delta}{\partial x} = \frac{\partial \mathrm{R}}{\partial x} f + \mathrm{R}\frac{\mathrm{D}f}{\mathrm{D}x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

[For a case of exception, in which $\frac{\mathrm{D}f}{\mathrm{D}a_1}\frac{\partial a_1}{\partial x} + \frac{\mathrm{D}f}{\mathrm{D}b_1}\frac{\partial b_1}{\partial x}$ does not vanish, see Art. 19, Ex. 11, E.]

Differentiating again with regard to x,

$$\frac{\partial^2 \Delta}{\partial x^2} = \frac{\partial^2 \mathrm{R}}{\partial x^2} f + 2\frac{\partial \mathrm{R}}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}x} + \mathrm{R}\left[\frac{\mathrm{D}^2 f}{\mathrm{D}x^2} + \frac{\mathrm{D}^2 f}{\mathrm{D}x\,\mathrm{D}a_1}\frac{\partial a_1}{\partial x} + \frac{\mathrm{D}^2 f}{\mathrm{D}x\,\mathrm{D}b_1}\frac{\partial b_1}{\partial x}\right] \quad . \quad . \quad (54),$$

$$\frac{\partial^2 \Delta}{\partial x\,\partial y} = \frac{\partial^2 \mathrm{R}}{\partial x\,\partial y} f + \frac{\partial \mathrm{R}}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}y} + \frac{\partial \mathrm{R}}{\partial y}\frac{\mathrm{D}f}{\mathrm{D}x} + \mathrm{R}\left[\frac{\mathrm{D}^2 f}{\mathrm{D}x\,\mathrm{D}y} + \frac{\mathrm{D}^2 f}{\mathrm{D}x\,\mathrm{D}a_1}\frac{\partial a_1}{\partial y} + \frac{\mathrm{D}^2 f}{\mathrm{D}x\,\mathrm{D}b_1}\frac{\partial b_1}{\partial y}\right] \quad . \quad (55).$$

Hence, by (52),

$$\frac{\partial^2 \Delta}{\partial x^2} = \frac{\partial^2 \mathrm{R}}{\partial x^2} f + 2\frac{\partial \mathrm{R}}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}x} + \mathrm{R}\,\frac{\mathrm{D}\left[\frac{\mathrm{D}f}{\mathrm{D}x}, \frac{\mathrm{D}f}{\mathrm{D}a_1}, \frac{\mathrm{D}f}{\mathrm{D}b_1}\right]}{\mathrm{D}[x, a_1, b_1]} \Bigg/ \frac{\mathrm{D}\left[\frac{\mathrm{D}f}{\mathrm{D}a_1}, \frac{\mathrm{D}f}{\mathrm{D}b_1}\right]}{\mathrm{D}[a_1, b_1]} \quad . \quad . \quad (56),$$

$$\frac{\partial^2 \Delta}{\partial x\,dy} = \frac{\partial^2 \mathrm{R}}{\partial x\,\partial y} f + \frac{\partial \mathrm{R}}{\partial x}\frac{\mathrm{D}f}{\mathrm{D}y} + \frac{\partial \mathrm{R}}{\partial y}\frac{\mathrm{D}f}{\mathrm{D}x} + \mathrm{R}\,\frac{\mathrm{D}\left[\frac{\mathrm{D}f}{\mathrm{D}y}, \frac{\mathrm{D}f}{\mathrm{D}a_1}, \frac{\mathrm{D}f}{\mathrm{D}b_1}\right]}{\mathrm{D}[x, a_1, b_1]} \Bigg/ \frac{\mathrm{D}\left[\frac{\mathrm{D}f}{\mathrm{D}a_1}, \frac{\mathrm{D}f}{\mathrm{D}b_1}\right]}{\mathrm{D}[a_1, b_1]} \quad . \quad (57).$$

Art. 8.—*Proof of the Envelope Property.*

Let ξ, η, ζ be a point on the locus of ultimate intersections, and let the values of a, b satisfying the equations (3), (4), (5) when $x = \xi$, $y = \eta$, $z = \zeta$, be α, β. [It will be supposed first of all, that only one value of a exists, viz., α, and only one value of b exists, viz., β. But the following cases will afterwards be noticed, viz., (1) where more than one system of distinct values of a, b exist, and (2) the particular case of the preceding in which two systems of values of a, b exist which coincide.]

In this case equations (11), (25), (26) are satisfied.

Hence when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\Delta = 0$$

and

$$\frac{\partial \Delta}{\partial x} \quad \text{becomes} \quad \left[\mathrm{R} \frac{\mathrm{D}f}{\mathrm{D}x} \right]_{\substack{x = \xi \\ y = \eta \\ z = \zeta}}.$$

Hence when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\frac{\partial \Delta}{\partial x} \Big/ \frac{\mathrm{D}f}{\mathrm{D}x} = \frac{\partial \Delta}{\partial y} \Big/ \frac{\mathrm{D}f}{\mathrm{D}y} = \frac{\partial \Delta}{\partial z} \Big/ \frac{\mathrm{D}f}{\mathrm{D}z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

Hence the tangent planes to $\Delta = 0, f(x, y, z, \alpha, \beta) = 0$, coincide at ξ, η, ζ. This proves that the locus of ultimate intersections is generally an envelope.

It should be noticed that the proof shows that the locus of ultimate intersections touches in general at each point on it *one* of the infinite number of surfaces of the system passing through that point. This will be referred to in future, to distinguish it from more complicated cases, as a case of the *ordinary* envelope.

It should also be observed that the above conclusion cannot be drawn if $x = \xi$, $y = \eta$, $z = \zeta$ make $\mathrm{D}f/\mathrm{D}x = 0$, $\mathrm{D}f/\mathrm{D}y = 0$, $\mathrm{D}f/\mathrm{D}z = 0$.

Hence the investigation itself suggests the examination of the case in which a locus of singular points exists.

Art. 9.—*To prove that if* E = 0 *be the equation of the Envelope Locus,* Δ *contains* E *as a factor once and once only in general.*

(A). If $x = \xi$, $y = \eta$, $z = \zeta$ be a point on the envelope locus, then suppose that the values of a, b satisfying (11), (25), (26), are α, β.

Then one of the systems of values of a, b satisfying (4) and (5), must become equal to α, β when $x = \xi$, $y = \eta$, $z = \zeta$.

Suppose that a_1 becomes α, b_1 becomes β.

Hence Δ becomes $\mathrm{R}'f(\xi, \eta, \zeta, \alpha, \beta)$, where R' is what R becomes, and therefore Δ vanishes.

Hence by Art. 1, Preliminary Theorem B, Δ contains E as a factor.

Further, Δ does not contain E more than once as a factor in general, for the value of $\partial \Delta / \partial x$ given by (53) does not in general vanish. But it would vanish if Δ contained a power of E above the first as a factor: for suppose $\Delta = \mathrm{E}^m . \psi$, where m is a positive integer greater than unity, and ψ some rational integral function of x, y, z.

Therefore

$$\frac{\partial \Delta}{\partial x} = m\mathrm{E}^{m-1}\frac{\partial \mathrm{E}}{\partial x}\psi + \mathrm{E}^m\frac{\partial \psi}{\partial x}.$$

Hence when $x = \xi$, $y = \eta$, $z = \zeta$, $\partial\Delta/\partial x = 0$.

Consequently, Δ contains E once, and once only in general as a factor.

(B). It is necessary to examine the cases of exception.

(i). If equations (11), (25), (26) are satisfied by more than one set of distinct values of a, b; take, for example, the case where there are two sets of solutions, α_1, β_1; α_2, β_2.

Let a_1, b_1, a_2, b_2 become α_1, β_1, α_2, β_2 respectively, when $x = \xi$, $y = \eta$, $z = \zeta$.

Putting

$$\Delta = \mathrm{R}'f(x, y, z, a_1, b_1)\,f(x, y, z, a_2, b_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

$$\frac{\partial\Delta}{\partial x} = \frac{\partial \mathrm{R}'}{\partial x}f(x, y, z, a_1, b_1,)\,f(x, y, z, a_2, b_2) + \mathrm{R}'\frac{\mathrm{D}f(x, y, z, a_1, b_1)}{\mathrm{D}x}f(x, y, z, a_2, b_2)$$

$$+ \mathrm{R}'f(x, y, z, a_1, b_1)\frac{\mathrm{D}f(x, y, z, a_2, b_2)}{\mathrm{D}x} \quad . \quad . \quad . \quad (60).$$

Now, when $x = \xi$, $y = \eta$, $z = \zeta$,

$f(x, y, z, a_1, b_1)$ becomes $f(\xi, \eta, \zeta, \alpha_1, \beta_1)$ and vanishes,
$f(x, y, z, a_2, b_2)$ becomes $f(\xi, \eta, \zeta, \alpha_2, \beta_2)$ and vanishes.

Hence $\partial\Delta/\partial x$ vanishes. Similarly $\partial\Delta/\partial y$, $\partial\Delta/\partial z$ vanish. Therefore Δ contains E^2 as a factor.

Similarly if there be p distinct sets of values of a, b satisfying (11), (25), (26), it can be shown that all the partial differential coefficients of Δ up to the $(p - 1)^{\text{th}}$ order vanish. Hence Δ will contain E^p as a factor. (See examples 4 (C.), 5 (C. ii.), 6 (C.) in Arts. 10, 11, 12 respectively.)

(ii.) The case in which two of the systems of values of the parameters satisfying equations (11), (25), (26) coincide, is dealt with in Arts. (12)–(25). The case, in which more than two systems of values of the parameters satisfying equations (11), (25), (26) coincide, may be treated in a similar manner.

Example 2.—*Ordinary Envelope.*

Let the surfaces be

$$z + (x - a)(y - b) = 0.$$

(A.) *The Discriminant.*

$$\Delta = z.$$

(B.) *The Envelope Locus is* $z = 0$.

Every point on $z = 0$ is the point of contact of one only of the surfaces. Hence z occurs as a factor once only on the discriminant.

It may be noticed that $z = 0$ touches each of the surfaces at one point only, $x = a$, $y = b$. This result may be compared with the next example.

Example 3.—*Ordinary Envelope.*

Let the surfaces be

$$z + (x - a)(x^2 + y^2 - b) = 0.$$

(A.) *The Discriminant.*

$$\Delta = z.$$

(B.) *The Envelope Locus is* $z = 0$.

Every point on $z = 0$ is the point of contact of one only of the surfaces. Hence z occurs as a factor once only in the discriminant.

It may be noticed that $z = 0$ touches each of the surfaces at two points, viz., $x = a$, $y = \pm\sqrt{(b - a^2)}$.

Art. 10.—*To prove that if* C = 0 *be the equation of the Conic Node Locus,* Δ *contains* C² *as a factor in general.*

Let ξ, η, ζ be a point on the conic node locus, then equations (11)-(14) are satisfied.

Hence, by (50), the substitutions $x = \xi$, $y = \eta$, $z = \zeta$, make $\Delta = 0$; and, by (53), they also make $\partial\Delta/\partial x = 0$.

By symmetry they also make $\partial\Delta/\partial y = 0$, $\partial\Delta/\partial z = 0$.

Hence, by Art. 1, Preliminary Theorem B, Δ must contain C^2 as a factor.

Example 4.—*Locus of Conic Nodes.*

Let the surfaces be

$$\alpha(x - a)^3 + \beta(y - b)^3 + 6m(x - a)(y - b) + \gamma z^2 = 0,$$

where α, β, γ, m are fixed constants, a, b are the arbitrary parameters.

(A.) *The Discriminant.*

To eliminate a and b between $f = 0$, $Df/Da = 0$, $Df/Db = 0$, is, in this case, the same as eliminating $x - a$, $y - b$ between

$$f = 0, \quad \frac{Df}{D(x - a)} = 0, \quad \frac{Df}{D(y - b)} = 0;$$

i.e., making the equation homogeneous by putting

$$x - a = X/Z, \quad y - b = Y/Z,$$

it is necessary to find the discriminant of

$$\alpha X^3 + \beta Y^3 + 6mXYZ + \gamma z^2 Z^3 = 0.$$

The invariants will in this, and in several of the examples which follow, be calculated from the results given in SALMON'S 'Higher Plane Curves,' Second Edition, Arts. 217–224.

The invariant $S = m\alpha\beta\gamma z^2 - m^4$.
The invariant $T = \alpha^2\beta^2\gamma^2 z^4 - 20\, m^3\alpha\beta\gamma z^2 - 8m^6$.
Hence $\Delta = T^2 + 64S^3 = \alpha\beta\gamma z^2\,(\alpha\beta\gamma z^2 + 8m^3)^3$.

(B.) *The Conic Node Locus is* $z = 0$.

Transforming the equation to the new origin $a, b, 0$, the lowest terms are of the second degree.

Hence the new origin is a conic node on the surface.

Hence $z = 0$ is the conic node locus.

Hence Δ contains z^2 as a factor.

(C.) *Three non-consecutive Surfaces of the System touch each of the planes* $z = \pm\,(-\,8m^3/\alpha\beta\gamma)^{1/2}$ *at each point.*

To prove this, the tangent planes to the surfaces which are parallel to the plane $z = 0$ will be found.

The tangent plane to the surface

$$\alpha\,(x-a)^3 + \beta\,(y-b)^3 + 6m\,(x-a)\,(y-b) + \gamma z^2 = 0$$

at ξ, η, ζ is

$$(x-\xi)\,[3\alpha\,(\xi-a)^2 + 6m\,(\eta-b)] + (y-\eta)\,[3\beta\,(\eta-b)^2 + 6m\,(\xi-a)] + (z-\zeta)\,2\gamma\zeta = 0.$$

If it be parallel to $z = 0$, the coefficients of x and y must vanish, but the coefficient of z must not vanish.

Therefore,

$$\alpha\,(\xi-a)^2 + 2m\,(\eta-b) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61),$$

$$\beta\,(\eta-b)^2 + 2m\,(\xi-a) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62).$$

From these, and from the condition that ξ, η, ζ lies on the surface,

$$2m\,(\xi-a)\,(\eta-b) + \gamma\zeta^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63).$$

If these be satisfied, and ζ do not vanish, the tangent plane is $z = \zeta$.

The solutions of (61), (62), (63) are

$$\xi - a = 0, \qquad \eta - b = 0, \qquad \zeta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (64),$$

$$\xi - a = -\,2m\alpha^{-2/3}\beta^{-1/3}, \quad \eta - b = -\,2m\alpha^{-1/3}\beta^{-2/3}, \quad \zeta = \pm\,(-\,8m^3/\alpha\beta\gamma)^{1/2} \quad (65),$$

$$\xi - a = -\,2m\omega\alpha^{-2/3}\beta^{-1/3}, \quad \eta - b = -\,2m\omega^2\alpha^{-1/3}\beta^{-2/3}, \quad \zeta = \pm\,(-\,8m^3/\alpha\beta\gamma)^{1/2} \quad (66),$$

$$\xi - a = -\,2m\omega^2\alpha^{-2/3}\beta^{-1/3}, \quad \eta - b = -\,2m\omega\alpha^{-1/3}\beta^{-2/3}, \quad \zeta = \pm\,(-\,8m^3/\alpha\beta\gamma)^{1/2} \quad (67),$$

where ω is an imaginary cube root of unity.

The solution (64) corresponds to the locus of conic nodes.

In the case of the solutions (65)-(67) the tangent plane at ξ, η, ζ is $z = \zeta$.

Hence either of the planes $z = \pm (-8m^3/\alpha\beta\gamma)^{1/2}$ touches at any given point on it three of the surfaces of the system, viz., those whose parameters are given by the equations

$$a = \xi + 2m\alpha^{-2/3}\beta^{-1/3}, \quad b = \eta + 2m\alpha^{-1/3}\beta^{-2/3};$$
$$a = \xi + 2m\omega\alpha^{-2/3}\beta^{-1/3}, \quad b = \eta + 2m\omega^2\alpha^{-1/3}\beta^{-2/3};$$
$$a = \xi + 2m\omega^2\alpha^{-2/3}\beta^{-1/3}, \quad b = \eta + 2m\omega\alpha^{-1/3}\beta^{-2/3}$$

Hence by Art. 9, B (i.), each of the factors

$$z \pm (-8m^3/\alpha\beta\gamma)^{1/2}$$

may be expected to occur three times in the discriminant.

This accounts for the presence of the factors

$$\{z + (-8m^3/\alpha\beta\gamma)^{1/2}\}^3 \, . \, \{z - (-8m^3/\alpha\beta\gamma)^{1/2}\}^3$$

i.e.,

$$(z^2 + 8m^3/\alpha\beta\gamma)^3$$

in the discriminant.

Art. 11.—*To prove that if $B = 0$ be the equation of the Biplanar Node Locus, Δ contains B^3 as a factor in general.*

Let ξ, η, ζ be a point on the biplanar node locus. Then equations (11)-(14), (41), (42) are satisfied.

The argument of the preceding article applies so far as Δ and its first differential coefficients are concerned.

But further the values of $\partial^2\Delta/\partial x^2$, $\partial^2\Delta/\partial x\,\partial y$, given by (56), (57) vanish, in virtue of the above mentioned equations, except in the case (to be considered presently) where the substitutions $x = \xi$, $y = \eta$, $z = \zeta$ make

$$\frac{\mathrm{D}\left[\dfrac{\mathrm{D}f}{\mathrm{D}a_1}, \dfrac{\mathrm{D}f}{\mathrm{D}b_1}\right]}{\mathrm{D}\,[a_1, b_1]} = 0,$$

i.e.,

$$\frac{\mathrm{D}\,[[\alpha], [\beta]]}{\mathrm{D}\,[\alpha, \beta]} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68).$$

From the symmetry of the variables it follows that all the other second differential coefficients of Δ also vanish when $x = \xi$, $y = \eta$, $z = \zeta$ (or the same results follow by Art. 1, Preliminary Theorem A).

Hence by Art. 1, Preliminary Theorem B, it follows that Δ contains B^3 as a factor.

Example 5.—*Locus of Biplanar Nodes.*

Let the surfaces be

$$\alpha (x - a)^3 + \alpha (y - b)^3 + 6m (x - a - z)(y - b - z) = 0,$$

where α, m are fixed constants; a, b the arbitrary parameters.

(A.) *The Discriminant.*

With the notation of the last article, the discriminant is the same as that of the equation

$$\alpha X^3 + \alpha Y^3 + 6m (X - zZ)(Y - zZ) Z = 0.$$

Therefore

$$S = 2m^2\alpha^2z^2 - m^4.$$

$$T = 36m^2\alpha^4z^4 - 64m^3\alpha^3z^3 + 24m^4\alpha^2z^2 - 8m^6.$$

$$\Delta = 16m^4\alpha^3z^3 (9\alpha z + 4m)(3\alpha^2z^2 - 6m\alpha z + 4m^2)^2.$$

(B.) *The Biplanar Node Locus is* $z = 0$.

Transforming the equation by the substitutions $x = a + X$, $y = b + Y$, $z = Z$, it becomes

$$\alpha X^3 + \alpha Y^3 + 6m (X - Z)(Y - Z) = 0.$$

Hence the new origin is a biplanar node on the surface. Hence $z = 0$ is the locus of biplanar nodes.

(C.) (i.) *The Ordinary Envelope is* $9\alpha z + 4m = 0$.

(ii.) *The Envelope such that every point on it is the point of contact of two non-consecutive Surfaces is* $3\alpha^2z^2 - 6m\alpha z + 4m^2 = 0$.

To prove these statements it is necessary to find the tangent planes parallel to the plane $z = 0$.

Hence it is necessary to have

$$f = 0, \quad Df/Dx = 0, \quad Df/Dy = 0, \quad Df/Dz \neq 0,$$

i.e.,

$$\alpha (x - a)^3 + \alpha (y - b)^3 + 6m (x - a - z)(y - b - z) = 0 \quad . \quad . \quad (69).$$

$$\alpha (x - a)^2 + 2m (y - b - z) = 0 \quad . \quad . \quad (70).$$

$$\alpha (y - b)^2 + 2m (x - a - z) = 0 \quad . \quad . \quad (71).$$

$$x - a - z + y - b - z \neq 0 \quad . \quad . \quad (72).$$

From (70) and (71) either

$$x - a = y - b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73),$$

or

$$\alpha(x - a) + \alpha(y - b) = 2m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).$$

(i.) Taking (73), and eliminating z and $(y - b)$ from it and (69), (70), it follows that

$$(x - a)^3 (x - a + 4m/3\alpha) = 0.$$

Now, if $x - a = 0$, then $y - b = 0$ by (73), and $z = 0$ by (69).

Hence (72) is not satisfied. This solution corresponds to the biplanar node locus.

But if

$$x - a = - 4m/3\alpha,$$

$$y - b = - 4m/3\alpha \text{ by } (73),$$

and

$$z = - 4m/9\alpha \text{ by } (70).$$

These values satisfy (69)–(72).

Hence $z = - 4m/9\alpha$ is an envelope. Each point $\xi, \eta, - 4m/9\alpha$ on it is the point of contact of one surface of the system whose parameters are

$$a = \xi + 4m/3\alpha, \quad b = \eta + 4m/3\alpha.$$

Hence $9\alpha z + 4m = 0$ is an ordinary envelope.

(ii.) Taking (74), and eliminating $(x - a)$ and $(y - b)$ from it and (69), (70), it follows that

$$3\alpha^2 z^2 - 6m\alpha z + 4m^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75).$$

The corresponding values of $(x - a)$, $(y - b)$ are determined by (74) and

$$\alpha(x - a)^2 - 2m(x - a) - 2mz + 4m^2/\alpha = 0.$$

These values satisfy (69)–(73).

Hence each point ξ, η, ζ on the imaginary locus (75) is the point of contact of two surfaces of the system, whose parameters a, b are determined by the equations

$$\begin{aligned} \alpha^2(\xi - a)^2 - 2m\alpha(\xi - a) - 2m\alpha\zeta + 4m^2 &= 0, \\ \alpha(\xi - a) + \alpha(\eta - b) - 2m &= 0, \end{aligned}$$

where ζ is one of the roots of (75).

This accounts for the factor

$$(3\alpha^2 z^2 - 6m\alpha z + 4m^2)^2$$

in the discriminant.

Art. 12.—*To prove that if* U = 0 *be the equation of the Locus of Uniplanar Nodes,* Δ *contains* U^6 *as a factor in general.*

(A.) Amongst the conditions satisfied at every point of the uniplanar node locus, will be found the following, see the ratios (48) :—

$$[\alpha, \alpha]\,[\beta, \beta] - [\alpha, \beta]^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76).$$

Now, by Art. 1, Preliminary Theorem C, this means that the equations

$$\mathrm{D}f/\mathrm{D}a = 0, \qquad \mathrm{D}f/\mathrm{D}b = 0$$

are satisfied by two systems of values of the parameters which become equal when $x = \xi$, $y = \eta$, $z = \zeta$, the coordinates of a point on the uniplanar node locus.

[It must be remembered that the theorem has to be specially interpreted for the case in which $\mathrm{D}f/\mathrm{D}a$, $\mathrm{D}f/\mathrm{D}b$ are both of the first degree in a, b, *i.e.*, for the case in which f is of the second degree in a, b. This is done in Section IV.].

Now this is the case previously reserved in Art. 7, Art. 9 (the second case of exception), and Art. 11 (condition (68)).

As there are, in this case, two equal values of each of the parameters, a, b, it may be expected that there will be two (not necessarily equal) values of $\partial a/\partial x$, $\partial b/\partial x$.

It will appear presently that in some cases $\partial a/\partial x$, $\partial b/\partial x$ may become infinite, but this is not the case for the uniplanar node locus, in which the values of $\partial a_1/\partial x$, $\partial b_1/\partial x$ as given by (52), become indeterminate, because the conditions (48) are satisfied.

Differentiating (52) again with regard to x, it follows that

$$[x, x, a_1] + 2\,[x, a_1, a_1]\frac{\partial a_1}{\partial x} + 2\,[x, a_1, b_1]\frac{\partial b_1}{\partial x} + [a_1, a_1, a_1]\left(\frac{\partial a_1}{\partial x}\right)^2$$

$$+ 2\,[a_1, a_1, b_1]\left(\frac{\partial a_1}{\partial x}\right)\left(\frac{\partial b_1}{\partial x}\right) + [a_1, b_1, b_1]\left(\frac{\partial b_1}{\partial x}\right)^2 + [a_1, a_1]\frac{\partial^2 a_1}{\partial x^2} + [a_1, b_1]\frac{\partial^2 b_1}{\partial x^2} = 0 \quad . \quad (77),$$

$$[x, x, b_1] + 2\,[x, a_1, b_1]\frac{\partial a_1}{\partial x} + 2\,[x, b_1, b_1]\frac{\partial b_1}{\partial x} + [a_1, a_1, b_1]\left(\frac{\partial a_1}{\partial x}\right)^2$$

$$+ 2\,[a_1, b_1, b_1]\left(\frac{\partial a_1}{\partial x}\right)\left(\frac{\partial b_1}{\partial x}\right) + [b_1, b_1, b_1]\left(\frac{\partial b_1}{\partial x}\right)^2 + [a_1, b_1]\frac{\partial^2 a_1}{\partial x^2} + [b_1, b_1]\frac{\partial^2 b_1}{\partial x^2} = 0 \quad . \quad (78).$$

Multiplying (77) by $[b_1, b_1]$, (78) by $[a_1, b_1]$, subtracting, putting $x = \xi$, $y = \eta$, $z = \zeta$, and therefore $a_1 = \alpha$, $b_1 = \beta$, and using (76), it follows that

$$\begin{aligned}
&[\xi, \xi, \alpha]\,[\beta, \beta] - [\xi, \xi, \beta]\,[\alpha, \beta] \\
&\quad + 2\frac{\partial\alpha}{\partial\xi}\,\{[\xi, \alpha, \alpha,]\,[\beta, \beta] - [\xi, \alpha, \beta]\,[\alpha, \beta]\} + 2\frac{\partial\beta}{\partial\xi}\,\{[\xi, \alpha, \beta]\,[\beta, \beta] - [\xi, \beta, \beta]\,[\alpha, \beta]\} \\
&\quad + \left(\frac{\partial\alpha}{\partial\xi}\right)^2 \{[\alpha, \alpha, \alpha]\,[\beta, \beta] - [\alpha, \alpha, \beta]\,[\alpha, \beta]\} \\
&\quad + 2\left(\frac{\partial\alpha}{\partial\xi}\right)\left(\frac{\partial\beta}{\partial\xi}\right)\{[\alpha, \alpha, \beta]\,[\beta, \beta] - [\alpha, \beta, \beta]\,[\alpha, \beta]\} \\
&\quad + \left(\frac{\partial\beta}{\partial\xi}\right)^2 \{[\alpha, \beta, \beta]\,[\beta, \beta] - [\beta, \beta, \beta]\,[\alpha, \beta]\} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (79),
\end{aligned}$$

and this equation with either of the equations derived from (52) by changing therein x into ξ, y into η, z into ζ, and therefore a_1 into α, b_1 into β, determines in general two values for $\partial\alpha/\partial\xi$, and two corresponding values for $\partial\beta/\partial\xi$.

The second of equations (52) gives

$$[\beta, \xi] + [\alpha, \beta]\frac{\partial\alpha}{\partial\xi} + [\beta, \beta]\,\frac{\partial\beta}{\partial\xi} = 0 \quad . \; . \; . \; . \; . \; . \quad (80).$$

Eliminating $\partial\beta/\partial\xi$ between (79) and (80), it follows that

$$\begin{aligned}
&\left(\frac{\partial\alpha}{\partial\xi}\right)^2 \Big[[\alpha,\alpha,\alpha]\,[\beta,\beta]^3 - 3\,[\alpha,\alpha,\beta]\,[\beta,\beta]^2\,[\alpha,\beta] + 3\,[\alpha,\beta,\beta]\,[\beta,\beta]\,[\alpha,\beta]^2 - [\beta,\beta,\beta]\,[\alpha,\beta]^3\Big] \\
&+2\frac{\partial\alpha}{\partial\xi}\left[\begin{array}{l} [\xi, \alpha, \alpha]\,[\beta, \beta]^3 - 2\,[\xi, \alpha, \beta]\,[\beta, \beta]^2\,[\alpha, \beta] + [\xi, \beta, \beta]\,[\beta, \beta]\,[\alpha, \beta]^2 \\ -[\alpha, \alpha, \beta]\,[\beta, \beta]^2\,[\beta, \xi] + 2\,[\alpha, \beta, \beta]\,[\beta, \beta]\,[\alpha, \beta]\,[\beta, \xi] - [\beta, \beta, \beta]\,[\alpha, \beta]^2\,[\beta, \xi] \end{array}\right] \\
&+ \left[\begin{array}{l} \qquad [\xi, \xi, \alpha]\,[\beta, \beta]^3 - [\xi, \xi, \beta]\,[\beta, \beta]^2\,[\alpha, \beta] \\ -\,2\,[\xi, \alpha, \beta]\,[\xi, \beta]\,[\beta, \beta]^2 + 2\,[\xi, \beta, \beta]\,[\xi, \beta]\,[\beta, \beta]\,[\alpha, \beta] \\ +\quad [\xi, \beta]^2\,[\alpha, \beta, \beta]\,[\beta, \beta] - [\xi, \beta]^2\,[\beta, \beta, \beta]\,[\alpha, \beta] \end{array}\right] = 0 \quad . \; . \; . \; . \quad (81).
\end{aligned}$$

This is in general a quadratic for $\partial\alpha/\partial\xi$.

The two corresponding values of $\partial\beta/\partial\xi$ are given by (80).

The case of exception, when the quadratic for $\partial\alpha/\partial\xi$ reduces to an equation of the first degree, viz., when

$$[\alpha, \alpha, \alpha]\,[\beta, \beta]^3 - 3\,[\alpha, \alpha, \beta]\,[\beta, \beta]^2\,[\alpha, \beta] + 3\,[\alpha, \beta, \beta]\,[\beta, \beta]\,[\alpha, \beta]^2 - [\beta, \beta, \beta]\,[\alpha, \beta]^3 = 0,$$

will now be considered.

(B.) The meaning of the condition may be determined by means of Art. 1, Preliminary Theorem D.

Put therein $\phi(u, v) = \mathrm{D}f/\mathrm{D}u$, $\psi(u, v) = \mathrm{D}f/\mathrm{D}v$.

Then the condition of Preliminary Theorem C, which also holds, is

$$\frac{\mathrm{D}^2 f}{\mathrm{D}u^2}\frac{\mathrm{D}^2 f}{\mathrm{D}v^2} - \left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^2 = 0.$$

The condition of Preliminary Theorem D is equivalent to

$$\left[\frac{\mathrm{D}^3 f}{\mathrm{D}u^3}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^2 - 2\frac{\mathrm{D}^3 f}{\mathrm{D}u^2\,\mathrm{D}v}\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\frac{\mathrm{D}^2 f}{\mathrm{D}u^2} + \frac{\mathrm{D}^3 f}{\mathrm{D}u\,\mathrm{D}v^2}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u^2}\right)^2\right]\left(\frac{\mathrm{D}^2 f}{\mathrm{D}v^2}\right)^3$$
$$= \left[\frac{\mathrm{D}^3 f}{\mathrm{D}u^2\,\mathrm{D}v}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}v^2}\right)^2 - 2\frac{\mathrm{D}^3 f}{\mathrm{D}u\,\mathrm{D}v^2}\frac{\mathrm{D}^2 f}{\mathrm{D}v^2}\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v} + \frac{\mathrm{D}^3 f}{\mathrm{D}v^3}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^2\right]\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^3.$$

Dividing out by $(\mathrm{D}^2 f/\mathrm{D}u\,\mathrm{D}v)^2$ the last equation becomes, by the preceding,

$$\frac{\mathrm{D}^3 f}{\mathrm{D}u^3}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}v^2}\right)^3 - 3\frac{\mathrm{D}^3 f}{\mathrm{D}u^2\,\mathrm{D}v}\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}v^2}\right)^2 + 3\frac{\mathrm{D}^3 f}{\mathrm{D}u\,\mathrm{D}v^2}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^2\frac{\mathrm{D}^2 f}{\mathrm{D}v^2} - \frac{\mathrm{D}^3 f}{\mathrm{D}v^3}\left(\frac{\mathrm{D}^2 f}{\mathrm{D}u\,\mathrm{D}v}\right)^3 = 0.$$

Now in the former part of this article, α and β correspond to u and v.

Hence the condition that the equations $\mathrm{D}f/\mathrm{D}a = 0$, $\mathrm{D}f/\mathrm{D}b = 0$ may have three coinciding systems of common roots is

$$[\alpha, \alpha, \alpha][\beta, \beta]^3 - 3[\alpha, \alpha, \beta][\beta, \beta]^2[\alpha, \beta] + 3[\alpha, \beta, \beta][\beta, \beta][\alpha, \beta]^2 - [\beta, \beta, \beta][\alpha, \beta]^3 = 0.$$

This fact must be taken account of in forming the discriminant, and the whole of the work must be modified in accordance with it. But this case will not be further discussed.

(C.) It has now been shown how to determine the values of $\partial a/\partial x$, $\partial b/\partial x$, when $x = \xi$, $y = \eta$, $z = \zeta$, the coordinates of a point on the uniplanar node locus.

Now, when $x = \xi$, $y = \eta$, $z = \zeta$,

$$f(x, y, z, a_1, b_1), \quad \frac{\partial f(x, y, z, a_1, b_1)}{\partial x}$$

both vanish.

$$\frac{\partial^2 f(x, y, z, a_1, b_1)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f(x, y, z, a_1, b_1)}{\partial x}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\mathrm{D}f(x, y, z, a_1, b_1)}{\mathrm{D}x}\right)$$
$$= [x, x] + [x, a_1]\frac{\partial a_1}{\partial x} + [x, b_1]\frac{\partial b_1}{\partial x}.$$

Hence when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\frac{\partial^2 f(x, y, z, a_1, b_1)}{\partial x^2} \text{ becomes } [\xi, \xi] + [\xi, \alpha]\frac{\partial\alpha}{\partial\xi} + [\xi, \beta]\frac{\partial\beta}{\partial\xi}$$

i.e.,

$$\frac{[\xi, \xi]}{[\xi, \beta]}\left\{[\xi, \beta] + [\alpha, \beta]\frac{\partial\alpha}{\partial\xi} + [\beta, \beta]\frac{\partial\beta}{\partial\xi}\right\} \text{ by (48),}$$

therefore $\partial^2 f(x, y, z, a_1, b_1)/\partial x^2$ vanishes by (80).

In like manner $\partial^2 f(x, y, z, a_2, b_2)/\partial x^2$ also vanishes when $x = \xi$, $y = \eta$, $z = \zeta$.

Now writing $\Delta = \mathrm{R}f_1 f_2$ for brevity, and forming all the differential coefficients up to the fifth order inclusive, each term in any of these differential coefficients is the product of terms, one of which is f_1 or f_2 or a first or second differential coefficient of f_1 or f_2.

Hence when $x = \xi$, $y = \eta$, $z = \zeta$, all the differential coefficients up to the fifth order vanish, and, therefore, by Art. 1, Preliminary Theorem B, Δ contains U^6 as a factor.

Example 6.—*Locus of Uniplanar Nodes.*

Let the surfaces be

$$\alpha(x - a)^3 + \beta(y - b)^3 + 3[c(x - a) + gz]^2 = 0$$

where α, β, c, g are fixed constants; a, b the arbitrary parameters.

(A.) *The Discriminant.*

The discriminant is the same as that of the equation

$$\alpha\mathrm{X}^3 + \beta\mathrm{Y}^3 + 3\mathrm{Z}(c\mathrm{X} + gz\mathrm{Z})^2 = 0.$$

Hence

$$\mathrm{S} = 0$$
$$\mathrm{T} = \alpha\beta^2 g^3 z^3 (9\alpha gz - 4c^3)$$
$$\Delta = \alpha^2\beta^4 g^6 z^6 (9\alpha gz - 4c^3)^2.$$

(B.) *The Locus of Unodes is* $z = 0$.

For putting $x = a + \mathrm{X}$, $y = b + \mathrm{Y}$, $z = \mathrm{Z}$, the equation becomes

$$\alpha\mathrm{X}^3 + \beta\mathrm{Y}^3 + 3(c\mathrm{X} + g\mathrm{Z})^2 = 0.$$

Hence the new origin is a unode. There are no other singular points on the surface.

Hence the locus of unodes is $z = 0$.

(C.) *The Envelope such that every point on it is the point of contact of two non-consecutive Surfaces is* $9\alpha gz - 4c^3 = 0$.

To prove this it is necessary to find the tangent planes parallel to $z = 0$.

Hence it is necessary to find x, y, z, so that

$$f = \alpha(x-a)^3 + \beta(y-b)^3 + 3[c(x-a)+gz]^2 = 0 \quad . \quad . \quad . \quad (82),$$

$$\tfrac{1}{3}\frac{\mathrm{D}f}{\mathrm{D}x} = \alpha(x-a)^2 + 2c[c(x-a)+gz] \qquad = 0 \quad . \quad . \quad . \quad (83),$$

$$\tfrac{1}{3}\frac{\mathrm{D}f}{\mathrm{D}y} = \beta(y-b)^2 \qquad = 0 \quad . \quad . \quad . \quad (84),$$

$$\tfrac{1}{3}\frac{\mathrm{D}f}{\mathrm{D}z} = 2g[c(x-a)+gz] \qquad \neq 0 \quad . \quad . \quad . \quad (85).$$

Therefore

$$\begin{aligned} &y = b, \\ &\alpha(x-a)^3 + 3[c(x-a)+gz]^2 = 0, \\ &\alpha(x-a)^2 + 2c[c(x-a)+gz] = 0; \end{aligned}$$

therefore

$$[c(x-a)+gz][c(x-a)+3gz] = 0.$$

The solution $c(x-a)+gz = 0$ is inconsistent with (85).

Hence it is necessary to take

$$c(x-a) + 3gz = 0.$$

Substituting in (83),

$$\frac{gz}{c^2}(9g\alpha z - 4c^3) = 0.$$

The solution $z = 0$, gives $x = a$, $y = b$, and therefore belongs to the unode locus.

The solution $9g\alpha z - 4c^3 = 0$

gives

$$\alpha(x-a)^2 + 2c^2(x-a) + 8c^4/9\alpha = 0,$$

and therefore

$$x - a = -4c^2/3\alpha,$$

or

$$x - a = -2c^2/3\alpha.$$

Hence at every point ξ, η, $4c^3/9g\alpha$ on this locus, the locus is touched by two non-consecutive surfaces of the system, viz., those whose parameters are given by

$$a = \xi + 4c^2/3\alpha, \quad b = \eta;$$

and

$$a = \xi + 2c^2/3\alpha, \quad b = \eta.$$

This accounts for the factor

$$(9g\alpha z - 4c^3)^2$$

in the discriminant.

Example 7.—*Locus of Uniplanar Nodes.*

Let the surfaces be

$$\alpha (x - a)^3 + \beta (y - b)^3 - 3n (z - ax + by)^2 = 0,$$

where α, β, n are fixed constants, and a, b are the arbitrary parameters.

(A.) *The Discriminant.*

Putting $\zeta = z - x^2 + y^2$, the equation is

$$\alpha (x - a)^3 + \beta (y - b)^3 - 3n \{\zeta + x (x - a) - y (y - b)\}^2 = 0.$$

Hence the discriminant is the same as that of the equation

$$\alpha X^3 + \beta Y^3 - 3nZ (\zeta Z + xX - yY)^2 = 0.$$

Therefore

$$S = n^2 \alpha\beta xy\zeta^2$$

$$T = 9n^2\alpha^2\beta^2\zeta^4 + 4n^3\alpha\beta\zeta^3 (\beta x^3 - \alpha y^3).$$

Therefore

$$\Delta = n^4\alpha^2\beta^2\zeta^6 [\{9\alpha\beta\zeta + 4n (\beta x^3 - \alpha y^3)\}^2 + 64n^2\alpha\beta x^3y^3].$$

In order to show the way in which the factor ζ^6 arises in the discriminant, the discriminant will now be calculated.

It is known to be the result of eliminating X, Y, Z from $Df/DX = 0$, $Df/DY = 0$, $Df/DZ = 0$, *i.e.*, from

$$\alpha X^2 - 2nxZ (\zeta Z + xX - yY) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86),$$

$$\beta Y^2 + 2nyZ (\zeta Z + xX - yY) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87),$$

$$(\zeta Z + xX - yY) (3\zeta Z + xX - yY) = 0 \quad . \quad . \quad . \quad . \quad (88).$$

From (86) and (87)

$$Y = \pm X \sqrt{\left(-\frac{\alpha y}{\beta x}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89).$$

Substituting in (86)

$$\alpha X^2 - 2nxXZ \left\{x \mp y \sqrt{\left(-\frac{\alpha y}{\beta x}\right)}\right\} - 2nx\zeta Z^2 = 0.$$

Put

$$x - y \sqrt{\left(-\frac{\alpha y}{\beta x}\right)} = \xi, \quad x + y \sqrt{\left(-\frac{\alpha y}{\beta x}\right)} = \eta.$$

Then two of the values of X/Z are found from

$$\alpha X^2 - 2nx\xi XZ - 2nx\zeta Z^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90),$$

and then

$$Y = + X \sqrt{\left(-\frac{\alpha y}{\beta x}\right)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (91);$$

and the remaining two values from

$$\alpha X^2 - 2nx\eta XZ - 2nx\zeta Z^2 = 0 \quad . \; . \; . \; . \; . \; . \quad (92),$$

and then

$$Y = - X \sqrt{\left(-\frac{\alpha y}{\beta x}\right)} \quad . \; . \; . \; . \; . \; . \; . \quad (93).$$

Take the factors of the left-hand side of (88) separately, and form first of all the part of the discriminant depending on

$$\zeta Z + xX - yY,$$

Let the roots of (90) be X_1/Z_1 and X_2/Z_2; then the corresponding values of Y by (91) are

$$X_1 \sqrt{\left(-\frac{\alpha y}{\beta x}\right)} \quad \text{and} \quad X_2 \sqrt{\left(-\frac{\alpha y}{\beta x}\right)}$$

Hence

$$\begin{aligned}(\zeta Z_1 + xX_1 - yY_1)&(\zeta Z_2 + xX_2 - yY_2)\\ &= (\zeta Z_1 + \xi X_1)(\zeta Z_2 + \xi X_2)\\ &= Z_1Z_2\left(\zeta^2 + \zeta\xi\left(\frac{X_1}{Z_1} + \frac{X_2}{Z_2}\right) + \xi^2\frac{X_1X_2}{Z_1Z_2}\right)\\ &= Z_1Z_2(\zeta^2).\end{aligned}$$

In like manner, taking the roots corresponding to (92) and (93),

$$(\zeta Z_3 + xX_3 - yY_3)(\zeta Z_4 + xX_4 - yY_4) = Z_3Z_4\zeta^2.$$

Next, taking the other factor of the left-hand side of (88), the part of the discriminant corresponding to the roots of (90) and (91) is

$$\begin{aligned}(3\zeta Z_1 + xX_1 - yY_1)&(3\zeta Z_2 + xX_2 - yY_2)\\ &= (3\zeta Z_1 + \xi X_1)(3\zeta Z_2 + \xi X_2)\\ &= Z_1Z_2\left(9\zeta^2 + 3\zeta\xi\left(\frac{X_1}{Z_1} + \frac{X_2}{Z_2}\right) + \xi^2\frac{X_1X_2}{Z_1Z_2}\right)\\ &= Z_1Z_2\left(9\zeta^2 + \frac{4nx\zeta\xi^2}{\alpha}\right).\end{aligned}$$

In like manner the roots of (92) and (93) give rise to the following part of the discriminant :—

$$Z_3Z_4\left(9\zeta^2 + \frac{4nx\zeta\eta^2}{\alpha}\right).$$

Hence the roots of (90) and (91) give rise to the portion

$$(Z_1Z_2)^2\left(9\zeta+\frac{4nx\xi^2}{\alpha}\right)\zeta^3,$$

and the roots of (92) and (93) to

$$(Z_3Z_4)^2\left(9\zeta+\frac{4nx\eta^2}{\alpha}\right)\zeta^3.$$

Hence the discriminant is

$$\begin{aligned}(Z_1Z_2Z_3Z_4)^2\zeta^6&\left(81\zeta^2+\frac{36nx\zeta}{\alpha}(\xi^2+\eta^2)+\frac{16n^2x^2\xi^2\eta^2}{\alpha^2}\right)\\ &\doteq(Z_1Z_2Z_3Z_4)^2\zeta^6\left(81\zeta^2+\frac{72n\zeta}{\alpha\beta}(\beta x^3-\alpha y^3)+\frac{16n^2(\beta x^3+\alpha y^3)^2}{\alpha^2\beta^2}\right)\\ &=\left(\frac{Z_1Z_2Z_3Z_4}{\alpha\beta}\right)^2\zeta^6\left[\{9\alpha\beta\zeta+4n(\beta x^3-\alpha y^3)\}^2+64n^2\alpha\beta x^3y^3\right].\end{aligned}$$

This agrees with the result previously stated.

Returning now to the part of the discriminant arising from the two systems of roots of (90) and (91), it will be shown that the factor ζ^3 arises entirely from one of the systems only.

Consider, in fact,

$$(\zeta Z_1+xX_1-yY_1)(3\zeta Z_1+xX_1-yY_1),$$

which is the part of the discriminant due to the system of roots X_1, Y_1, Z_1.

It is equal to

$$(\zeta Z_1+\xi X_1)(3\zeta Z_1+\xi X_1)=Z_1{}^2\{3\zeta^2+4\xi\zeta(X_1/Z_1)+\xi^2(X_1/Z_1)^2\}$$

where

$$\alpha(X_1/Z_1)^2-2nx\xi(X_1/Z_1)-2nx\zeta=0.$$

Therefore

$$(\zeta Z_1+xX_1-yY_1)(3\zeta Z_1+xX_1-yY_1)=Z_1{}^2\left\{3\zeta^2+\frac{2nx\xi^2\zeta}{\alpha}+\frac{X_1}{Z_1}\left(4\xi\zeta+\frac{2nx\xi^3}{\alpha}\right)\right\}.$$

Now,

$$\begin{aligned}\frac{X_1}{Z_1}&=\frac{nx\xi}{\alpha}\pm\frac{\sqrt{(n^2x^2\xi^2+2nx\alpha\zeta)}}{\alpha}\\ &=\frac{nx\xi}{\alpha}\pm\frac{nx\xi}{\alpha}\left\{1+\frac{\alpha\zeta}{nx\xi^2}-\frac{1}{2}\frac{\alpha^2\zeta^2}{n^2x^2\xi^4}+\frac{1}{2}\frac{\alpha^3\zeta^3}{n^3x^3\xi^6}-\ldots\right\}.\end{aligned}$$

If the root corresponding to the positive sign be taken, then

$$(\zeta Z_1+xX_1-yY_1)(3\zeta Z_1+xX_1-yY_1)$$

is not divisible by ζ.

If, however, the root corresponding to the negative sign be taken, *i.e.*,

$$\frac{X_1}{Z_1} = -\left\{\frac{\zeta}{\xi} - \tfrac{1}{2}\frac{\alpha\zeta^2}{nx\xi^3} + \tfrac{1}{2}\frac{\alpha^2\zeta^3}{n^2x^2\xi^5} - \ldots\right\}$$

then

$$(\zeta Z_1 + xX_1 - yY_1)(3\zeta Z_1 + xX_1 - yY_1)$$
$$= Z_1^2\left\{\frac{\alpha\zeta^3}{nx\xi^2} + \text{terms containing higher powers of } \zeta\right\}$$

Hence the factor ζ^3 arises exclusively from the substitution in f of one system of values of the parameters satisfying both the equations $Df/Da = 0$, $Df/Db = 0$.

A similar demonstration shows that the solution

$$\frac{X_3}{Z_3} = -\left\{\frac{\zeta}{\eta} - \tfrac{1}{2}\frac{\alpha\zeta^2}{nx\eta^3} + \tfrac{1}{2}\frac{\alpha^2\zeta^3}{n^2x^2\eta^5} - \ldots\right\}$$

and

$$Y_3 = -X_3\sqrt{\left(-\frac{\alpha y}{\beta x}\right)},$$

will also give rise to the factor ζ^3.

Now it will be shown presently that $\zeta = 0$ is the unode locus. Hence at any point on the unode locus,

$$\frac{X_1}{Z_1} = 0,\quad \frac{Y_1}{Z_1} = 0\,;\quad \frac{X_3}{Z_3} = 0,\quad \frac{Y_3}{Z_3} = 0.$$

Hence, at such a point there are two values of $x - a$ and two of $y - b$ which vanish. Hence two systems of values of a, b, satisfying both the equations $Df/Da = 0$, $Df/Db = 0$ become equal; viz., the two values of the parameter a become equal to the x-coordinate of the point, and the two values of the parameter b become equal to the y-coordinate of the point.

(B.) *The Locus of Uniplanar Nodes is* $\zeta = 0$.

To find the singular points, it is necessary to find values of x, y, z satisfying all the equations

$$\begin{aligned}
\alpha(x-a)^3 + \beta(y-b)^3 - 3n(z - ax + by)^2 &= 0,\\
\alpha(x-a)^2 + 2na(z - ax + by) &= 0,\\
\beta(y-b)^2 - 2nb(z - ax + by) &= 0,\\
z - ax + by &= 0.
\end{aligned}$$

The only solutions of which are

$$x = a,\quad y = b,\quad z = a^2 - b^2.$$

Now, transforming the equation by means of the substitutions $x = a + X$, $y = b + Y$, $z = a^2 - b^2 + Z$, it becomes

$$\alpha X^3 + \beta Y^3 - 3n (Z - aX + bY)^2 = 0.$$

Hence, the new origin is a unode.

Hence the locus of unodes is $z = x^2 - y^2$, *i.e.*, $\zeta = 0$.

(C.) *The Envelope Locus is*

$$\{9\alpha\beta\zeta + 4n (\beta x^3 - \alpha y^3)\}^2 + 64\alpha\beta n^2 x^3 y^3 = 0 \quad . \quad . \quad . \quad . \quad (94).$$

On examining the manner in which the discriminant was formed it can be seen that the factor corresponding to the envelope focus is obtained from (86), (87), and $3\zeta Z + xX - yY = 0$.

Hence it is the result of eliminating X, Y, Z from

$$\left.\begin{aligned} \zeta Z + xX - yY &= -2\zeta Z \\ \alpha X^2 + 4nx\zeta Z^2 &= \quad 0 \\ \beta Y^2 - 4ny\zeta Z^2 &= \quad 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95).$$

To prove that (94) is an envelope locus, it is necessary to show that if ϕ be the left-hand side of (94), and $f = 0$ be the equation of the system of surfaces, then it is possible to find values of x, y, z which satisfy at the same time all the equations—

$$\left.\begin{aligned} f = 0, \quad \phi = 0 \\ \frac{Df}{Dx}\bigg/\frac{\partial\phi}{\partial x} = \frac{Df}{Dy}\bigg/\frac{\partial\phi}{\partial y} = \frac{Df}{Dz}\bigg/\frac{\partial\phi}{\partial z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96).$$

Changing the independent variables from x, y, z to x, y, ζ, then, if δ denote partial differentiation when x, y, ζ are independent variables, equations (96) are equivalent to

$$f = 0, \quad \phi = 0$$

$$\frac{\delta f}{\delta x}\bigg/\frac{\delta\phi}{\delta x} = \frac{\delta f}{\delta y}\bigg/\frac{\delta\phi}{\delta y} = \frac{\delta f}{\delta z}\bigg/\frac{\delta\phi}{\delta z},$$

i.e.

$$\frac{3\alpha (x - a)^2 - 6n (2x - a) [\zeta + (x - a) x - (y - b) y]}{24n\beta x^2 [9\alpha\beta\zeta + 4n (\beta x^3 - \alpha y^3)] + 192 n^2 \alpha\beta x^2 y^3}$$

$$= \frac{3\beta (y - b)^2 + 6n (2y - b) [\zeta + (x - a) x - (y - b) y]}{- 24n\alpha y^2 [9\alpha\beta\zeta + 4n (\beta x^3 - \alpha y^3)] + 192 n^2 \alpha\beta x^3 y^2}$$

$$= \frac{- 6n [\zeta + x (x - a) - y (y - b)]}{18\alpha\beta [9\alpha\beta\zeta + 4n (\beta x^3 - \alpha y^3)]};$$

or, putting $x - a = \mathrm{X}/\mathrm{Z}$, $y - b = \mathrm{Y}/\mathrm{Z}$,

$$\begin{aligned} & \frac{\alpha\mathrm{X}^2 - 2n(x\mathrm{Z} + \mathrm{X})(\zeta\mathrm{Z} + x\mathrm{X} - y\mathrm{Y})}{24n\beta x^2[9\alpha\beta\zeta + 4n(\beta x^3 + \alpha y^3)]} \\ = & \frac{\beta\mathrm{Y}^2 + 2n(y\mathrm{Z} + \mathrm{Y})(\zeta\mathrm{Z} + x\mathrm{X} - y\mathrm{Y})}{-24n\alpha y^2[9\alpha\beta\zeta - 4n(\beta x^3 + \alpha y^3)]} \\ = & \frac{-2n\mathrm{Z}(\zeta\mathrm{Z} + x\mathrm{X} - y\mathrm{Y})}{18\alpha\beta[9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)]}. \end{aligned}$$

It will now be shown that the values of x, y, z which satisfy (95) also satisfy these equations.

For substituting from (95), these equations become

$$\begin{aligned} \frac{\alpha\mathrm{X}^2 + 4n\zeta\mathrm{Z}(x\mathrm{Z} + \mathrm{X})}{4n\beta x^2[9\alpha\beta\zeta + 4n(\beta x^3 + \alpha y^3)]} & = \frac{\beta\mathrm{Y}^2 - 4n\zeta\mathrm{Z}(y\mathrm{Z} + \mathrm{Y})}{-4n\alpha y^2[9\alpha\beta\zeta - 4n(\beta x^3 + \alpha y^3)]} \\ & = \frac{4n\zeta\mathrm{Z}^2}{3\alpha\beta[9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)]} \end{aligned}$$

These reduce further to

$$\begin{aligned} \frac{4n\zeta\mathrm{Z}\mathrm{X}}{4n\beta x^2[9\alpha\beta\zeta + 4n(\beta x^3 + \alpha y^3)]} & = \frac{-4n\zeta\mathrm{Z}\mathrm{Y}}{-4n\alpha y^2[9\alpha\beta\zeta - 4n(\beta x^3 + \alpha y^3)]} \\ & = \frac{4n\zeta\mathrm{Z}^2}{3\alpha\beta[9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)]}, \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{\mathrm{X}}{4n\beta x^2[9\alpha\beta\zeta + 4n(\beta x^3 + \alpha y^3)]} & = \frac{\mathrm{Y}}{4n\alpha y^2[9\alpha\beta\zeta - 4n(\beta x^3 + \alpha y^3)]} \\ & = \frac{\mathrm{Z}}{3\alpha\beta[9\alpha\beta\zeta + 4n(\beta x^3 - \alpha y^3)]}. \end{aligned}$$

Now the relations (95) satisfy (94).

Further (94) can be written in either of the forms

$$[9\alpha\beta\zeta + 4n(\beta x^3 + \alpha y^3)]^2 = 144n\alpha^2\beta\zeta y^3,$$
$$[9\alpha\beta\zeta - 4n(\beta x^3 + \alpha y^3)]^2 = -144n\alpha\beta^2\zeta x^3.$$

Hence it is necessary to show that

$$\frac{\mathrm{X}}{48n\alpha\beta x^2 y\sqrt{(n\beta y\zeta)}} = \frac{\mathrm{Y}}{48n\alpha\beta xy^2\sqrt{(-n\alpha x\zeta)}} = \frac{\mathrm{Z}}{24n\alpha\beta xy\sqrt{(-\alpha\beta xy)}},$$

i.e.,

$$\mathrm{X}^2/\mathrm{Z}^2 = -4xn\zeta/\alpha$$
$$\mathrm{Y}^2/\mathrm{Z}^2 = 4ny\zeta/\beta,$$

and these are true by (95).

Further, $\phi = 0$, $f = 0$ are both consequences of (95).

Hence all the equations (96) are satisfied by the same values of x, y, z.

SECTION III. (Arts. 13–15).—CONSIDERATION OF THE CASES RESERVED IN THE PREVIOUS SECTION, IN WHICH TWO SYSTEMS OF VALUES OF THE PARAMETERS SATISFYING THE EQUATIONS, $Df/Da = 0$, $Df/Db = 0$, COINCIDE AT A POINT ON THE LOCUS OF ULTIMATE INTERSECTIONS.

The interpretation of the condition

$$[\alpha, \alpha][\beta, \beta] - [\alpha, \beta]^2 = 0,$$

which is marked (76), when the equation of the system of surfaces is of the second degree in the parameters, is different from its interpretation when it is of a higher degree.

It will be supposed, in this section, that the degree of the equation of the system of surfaces in the parameters is higher than the second.

Art. 13.—*To prove that if each Surface of the System have Stationary Contact with the Envelope, then* Δ *contains* E^2 *as a factor.*

(A.) It will be shown that when the condition (76) holds in the case of an envelope locus, the curve of intersection of the envelope with each surface of the system has a double point at the point of contact, such that the two tangents coincide. [Such contact is called stationary (see SALMON'S 'Geometry of Three Dimensions,' 3rd Edition, Arts. 204, 300).]

To prove this, it is necessary to find the direction of the tangents to the curve of intersection of the envelope and one of the surfaces of the system at the point of contact.

Let ξ, η, ζ, be a point on the envelope. Let the surface touching the envelope at this point be

$$f(x, y, z, \alpha, \beta) = 0,$$

which has been marked (9).

Then equations (11), (25), (26) hold.

Let $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ be a point near to ξ, η, ζ, which lies on the curve of intersection of the surface (9) and the envelope.

Since it is on the envelope, it will be the point of contact of one of the surfaces of the system.

Suppose it is the point of contact of the surface (10).

Then

$$f(\xi + \delta\xi,\ \eta + \delta\eta,\ \zeta + \delta\zeta, \alpha, \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (97),$$

and the equations obtained from (11), (25), (26) by changing ξ, η, ζ, α, β into $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$, $\alpha + \delta\alpha$, $\beta + \delta\beta$.

Then, from (97)

$$
\begin{aligned}
&[\xi]\,(\delta\xi) + [\eta]\,(\delta\eta) + [\zeta]\,(\delta\zeta) \\
&\quad + \tfrac{1}{2}\left\{\begin{aligned}&[\xi,\xi]\,(\delta\xi)^2 + [\eta,\eta]\,(\delta\eta)^2 + [\zeta,\zeta]\,(\delta\zeta)^2 \\ &\qquad + 2\,[\eta,\zeta]\,(\delta\eta)\,(\delta\zeta) + 2\,[\zeta,\xi]\,(\delta\zeta)\,(\delta\xi) + 2\,[\xi,\eta]\,(\delta\xi)\,(\delta\eta)\end{aligned}\right\} \\
&\quad + \text{terms of the third and higher orders} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (98),
\end{aligned}
$$

and by means of the substitutions in (11), (25), (26),

$$
\begin{aligned}
&\quad f(\xi, \eta, \zeta, \alpha, \beta) \\
&+ [\xi]\,(\delta\xi) + [\eta]\,(\delta\eta) + [\zeta]\,(\delta\zeta) + [\alpha]\,(\delta\alpha) + [\beta]\,(\delta\beta) \\
&+ \tfrac{1}{2}\left[\begin{aligned}&[\xi,\xi]\,(\delta\xi)^2 + [\eta,\eta]\,(\delta\eta)^2 + [\zeta,\zeta]\,(\delta\zeta)^2 \\ &\qquad + 2\,[\eta,\zeta]\,(\delta\eta)\,(\delta\zeta) + 2\,[\zeta,\xi]\,(\delta\zeta)\,(\delta\xi) + 2\,[\xi,\eta]\,(\delta\xi)\,(\delta\eta) \\ &+ 2\,[\xi,\alpha]\,(\delta\xi)\,(\delta\alpha) + 2\,[\eta,\alpha]\,(\delta\eta)\,(\delta\alpha) + 2\,[\zeta,\alpha]\,(\delta\zeta)\,(\delta\alpha) \\ &+ 2\,[\xi,\beta]\,(\delta\xi)\,(\delta\beta) + 2\,[\eta,\beta]\,(\delta\eta)\,(\delta\beta) + 2\,[\zeta,\beta]\,(\delta\zeta)\,(\delta\beta) \\ &+ [\alpha,\alpha]\,(\delta\alpha)^2 + 2\,[\alpha,\beta]\,(\delta\alpha)\,(\delta\beta) + [\beta,\beta]\,(\delta\beta)^2\end{aligned}\right] \\
&+ \text{terms of the third and higher orders} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (99),
\end{aligned}
$$

$$
\begin{aligned}
&[\alpha] + [\alpha,\xi]\,(\delta\xi) + [\alpha,\eta]\,(\delta\eta) + [\alpha,\zeta]\,(\delta\zeta) + [\alpha,\alpha]\,(\delta\alpha) + [\alpha,\beta]\,(\delta\beta) \\
&\qquad\qquad + \text{terms of the second and higher orders} = 0 \quad . \quad . \quad . \quad . \qquad (100),
\end{aligned}
$$

$$
\begin{aligned}
&[\beta] + [\beta,\xi]\,(\delta\xi) + [\beta,\eta]\,(\delta\eta) + [\beta,\zeta]\,(\delta\zeta) + [\beta,\alpha]\,(\delta\alpha) + [\beta,\beta]\,(\delta\beta) \\
&\qquad\qquad + \text{terms of the second and higher orders} = 0 \quad . \quad . \quad . \quad . \qquad (101).
\end{aligned}
$$

Making use of (11), (25), (26), (98), equations (99)–(101) become

$$
\begin{aligned}
&\quad [\xi,\alpha]\,(\delta\xi)\,(\delta\alpha) + [\eta,\alpha]\,(\delta\eta)\,(\delta\alpha) + [\zeta,\alpha]\,(\delta\zeta)\,(\delta\alpha) \\
&+ [\xi,\beta]\,(\delta\xi)\,(\delta\beta) + [\eta,\beta]\,(\delta\eta)\,(\delta\beta) + [\zeta,\beta]\,(\delta\zeta)\,(\delta\beta) \\
&+ \tfrac{1}{2}\,[\alpha,\alpha]\,(\delta\alpha)^2 + [\alpha,\beta]\,(\delta\alpha)\,(\delta\beta) + \tfrac{1}{2}\,[\beta,\beta]\,(\delta\beta)^2 \\
&+ \text{terms of the third and higher orders} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (102),
\end{aligned}
$$

$$
\begin{aligned}
&[\alpha,\xi]\,(\delta\xi) + [\alpha,\eta]\,(\delta\eta) + [\alpha,\zeta]\,(\delta\zeta) + [\alpha,\alpha]\,(\delta\alpha) + [\alpha,\beta]\,(\delta\beta) \\
&\qquad\qquad + \text{terms of the second and higher orders} = 0 \quad . \qquad (103),
\end{aligned}
$$

$$
\begin{aligned}
&[\beta,\xi]\,(\delta\xi) + [\beta,\eta]\,(\delta\eta) + [\beta,\zeta]\,(\delta\zeta) + [\beta,\alpha]\,(\delta\alpha) + [\beta,\beta]\,(\delta\beta) \\
&\qquad\qquad + \text{terms of the second and higher orders} = 0 \quad . \qquad (104).
\end{aligned}
$$

By (102), (103), and (104) it follows that

$$\tfrac{1}{2}[\alpha,\,\alpha]\,(\delta\alpha)^2 + [\alpha,\,\beta]\,(\delta\alpha)\,(\delta\beta) + \tfrac{1}{2}\,[\beta,\,\beta]\,(\delta\beta)^2$$
$$+ \text{ terms of the third and higher orders} = 0$$

Hence ultimately

$$[\alpha,\,\alpha]\,(\delta\alpha)^2 + 2\,[\alpha,\,\beta]\,(\delta\alpha)\,(\delta\beta) + [\beta,\,\beta]\,(\delta\beta)^2 = 0.$$

This determines the two values of the ratio $\delta\beta/\delta\alpha$.

Then to determine $\delta\xi$, $\delta\eta$, $\delta\zeta$ there are the following equations obtained from (98), (103), (104), by retaining only the principal terms.

$$[\xi]\,(\delta\xi) + [\eta]\,(\delta\eta) + [\zeta]\,(\delta\zeta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105),$$

$$[\alpha,\,\xi]\,(\delta\xi) + [\alpha,\,\eta]\,(\delta\eta) + [\alpha,\,\zeta]\,(\delta\zeta) + [\alpha,\,\alpha]\,(\delta\alpha) + [\alpha,\,\beta]\,(\delta\beta) = 0\,.\ (106),$$

$$[\beta,\,\xi]\,(\delta\xi) + [\beta,\,\eta]\,(\delta\eta) + [\beta,\,\zeta]\,(\delta\zeta) + [\beta,\,\alpha]\,(\delta\alpha) + [\beta,\,\beta]\,(\delta\beta) = 0\,.\ (107).$$

Hence the ratios $\delta\xi : \delta\eta : \delta\zeta$ can be determined.

Hence the directions of the tangents to each of the branches of the curve of intersection of the envelope and the surface (9) can be determined.

If, now, the condition (76) hold, the two values of $\delta\beta/\delta\alpha$ become equal, and, therefore, the two tangents at the double point of the curve of intersection coincide, and therefore, the contact is stationary.

Further, because in this case the values of $\delta\beta/\delta\alpha$ both become equal to

$$-\,[\alpha,\,\alpha]/[\alpha,\,\beta] = -\,[\alpha,\,\beta]/[\beta,\,\beta],$$

therefore (106) and (107) become

$$[\alpha,\,\xi]\,(\delta\xi) + [\alpha,\,\eta]\,(\delta\eta) + [\alpha,\,\zeta]\,(\delta\zeta) = 0,$$
$$[\beta,\,\xi]\,(\delta\xi) + [\beta,\,\eta]\,(\delta\eta) + [\beta,\,\zeta]\,(\delta\eta) = 0.$$

From these two equations and (105) it follows that the coinciding tangents at the double point of the curve of intersection lie in the tangent planes to the surfaces $\mathrm{D}f/\mathrm{D}\alpha = 0$, $\mathrm{D}f/\mathrm{D}\beta = 0$, $f = 0$.

(B.) In this case

$$\begin{aligned}\Delta &= \mathrm{R}f\,(x,\,y,\,z,\,a_1,\,b_1)\,f\,(x,\,y,\,z,\,a_2,\,b_2)\\ &= \mathrm{R}f_1f_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108).\end{aligned}$$

$$\frac{\partial\Delta}{\partial x} = \frac{\partial\mathrm{R}}{\partial x}f_1f_2 + \mathrm{R}\left(\frac{\mathrm{D}f_1}{\mathrm{D}x}f_2 + f_1\,\frac{\mathrm{D}f_2}{\mathrm{D}x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (109).$$

Hence $\Delta = 0$, $\partial\Delta/\partial x = 0$; for $f_1 = 0, f_2 = 0$ at every point on the envelope locus.

Hence Δ contains E^2 as a factor.

Example 8.—*Envelope Locus, each Surface of the System having Stationary Contact with the Envelope.*

Let the surfaces be

$$\alpha(x-a)^3+\beta(y-b)^3+3[\gamma(x-a)+\delta(y-b)]^2+z^2-c^2=0,$$

where $\alpha, \beta, \gamma, \delta, c$ are fixed constants; a, b the arbitrary parameters.

(A.) *The Discriminant.*

It is the same as that of the equation

$$\alpha X^3+\beta Y^3+3Z(\gamma X+\delta Y)^2+(z^2-c^2)Z^3=0.$$

Therefore

$$S=\alpha\beta\gamma\delta(z^2-c^2),$$
$$T=(z^2-c^2)[\alpha^2\beta^2(z^2-c^2)+4(\alpha\delta^3-\beta\gamma^3)^2].$$

Therefore

$$\Delta=(z^2-c^2)^2[\{\alpha^2\beta^2(z^2-c^2)+4(\alpha\delta^3-\beta\gamma^3)^2\}^2+64\alpha^3\beta^3\gamma^3\delta^3(z^2-c^2)].$$

(B.) *The Envelope, such that each Surface has Stationary Contact with it, is* $z^2-c^2=0$.

Transform the equation by means of the equations $x=a+X$, $y=b+Y$, $z=\pm c+Z$, and it becomes

$$\alpha X^3+\beta Y^3+3(\gamma X+\delta Y)^2+Z^2\pm 2cZ=0.$$

The tangent plane at the new origin is $Z=0$; it cuts the surface in the curve

$$\alpha X^3+\beta Y^3+3(\gamma X+\delta Y)^2=0,$$

which has a cusp at the origin.

Hence the contact is stationary.

Hence the factor $(z^2-c^2)^2$ in the discriminant is accounted for.

(C.) *The Locus*

$$\{\alpha^2\beta^2(z^2-c^2)+4(\alpha\delta^3-\beta\gamma^3)^2\}^2+64\alpha^3\beta^3\gamma^3\delta^3(z^2-c^2)=0$$

is an ordinary envelope.

This may be proved by finding the tangent planes parallel to the plane $z=0$.

It is necessary to satisfy at the same time

$$\alpha(x-a)^3+\beta(y-b)^3+3[\gamma(x-a)+\delta(y-b)]^2+z^2-c^2=0 \quad . \quad (110)$$

$$\left.\begin{aligned}\alpha(x-a)^2+2\gamma[\gamma(x-a)+\delta(y-b)]&=0\\ \beta(y-b)^2+2\delta[\gamma(x-a)+\delta(y-b)]&=0\\ 2z&\neq 0\end{aligned}\right\} . \quad (111).$$

From the above

$$[\gamma(x-a)+\delta(y-b)]^2+(z^2-c^2)=0,$$

therefore

$$\gamma(x-a)+\delta(y-b)=\sqrt{(c^2-z^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (112)$$

Hence by (111) and (112)

$$x-a=\pm\sqrt{\left\{-\frac{2\gamma}{\alpha}(c^2-z^2)^{1/2}\right\}}$$

$$y-b=\pm\sqrt{\left\{-\frac{2\delta}{\beta}(c^2-z^2)^{1/2}\right\}}.$$

Substituting in (112),

$$\pm\gamma\sqrt{\left\{-\frac{2\gamma}{\alpha}(c^2-z^2)^{1/2}\right\}}\pm\delta\sqrt{\left\{-\frac{2\delta}{\beta}(c^2-z^2)^{1/2}\right\}}=(c^2-z^2)^{1/2},$$

therefore

$$\left\{-2\left(\frac{\gamma^3}{\alpha}+\frac{\delta^3}{\beta}\right)\pm 2\gamma\delta\sqrt{\frac{4\gamma\delta}{\alpha\beta}}\right\}(c^2-z^2)^{1/2}=(c^2-z^2),$$

therefore

$$-2\left(\frac{\gamma^3}{\alpha}+\frac{\delta^3}{\beta}\right)\pm 2\gamma\delta\sqrt{\frac{4\gamma\delta}{\alpha\beta}}=(c^2-z^2)^{1/2},$$

therefore

$$4(\gamma^3\beta+\delta^3\alpha)^2+16\alpha\beta\gamma^3\delta^3\pm 16(\gamma^3\beta+\delta^3\alpha)\,\alpha^{1/2}\,\beta^{1/2}\,\gamma^{3/2}\,\delta^{3/2}=\alpha^2\beta^2(c^2-z^2).$$

Therefore

$$[\alpha^2\beta^2(z^2-c^2)+4(\gamma^3\beta+\delta^3\alpha)^2+16\alpha\beta\gamma^3\delta^3]^2=256\alpha\beta\gamma^3\delta^3(\gamma^3\beta+\delta^3\alpha)^2.$$

This reduces to

$$[\alpha^2\beta^2(z^2-c^2)+4(\alpha\delta^3-\beta\gamma^3)^2]^2+64\alpha^3\beta^3\gamma^3\delta^3(z^2-c^2)=0.$$

This accounts for the remaining factor in the discriminant. It corresponds to an ordinary envelope.

Art. 14.—*To prove that if the Conic Node Locus be also an Envelope, Δ contains C^3 as a factor.*

(A.) It will be shown that when the condition (76) holds, in the case of a conic node locus, then the conic node locus is also an envelope.

In this case, from (28) and (29), by means of (76), it follows that the equation of the tangent plane to the conic node locus is

$$\begin{aligned} &[\beta, \alpha]\,\{[\alpha, \xi](\mathrm{X} - \xi) + [\alpha, \eta](\mathrm{Y} - \eta) + [\alpha, \zeta](\mathrm{Z} - \zeta)\} \\ -&[\alpha, \alpha]\,\{[\beta, \xi](\mathrm{X} - \xi) + [\beta, \eta](\mathrm{Y} - \eta) + [\beta, \zeta](\mathrm{Z} - \zeta)\} = 0 \qquad (113). \end{aligned}$$

This will touch the tangent cone (33) at the conic node, if

$$\begin{vmatrix} [\xi,\xi] & [\xi,\eta] & [\xi,\zeta] & [\beta,\alpha][\alpha,\xi]-[\alpha,\alpha][\beta,\xi] \\ [\eta,\xi] & [\eta,\eta] & [\eta,\zeta] & [\beta,\alpha][\alpha,\eta]-[\alpha,\alpha][\beta,\eta] \\ [\zeta,\xi] & [\zeta,\eta] & [\zeta,\zeta] & [\beta,\alpha][\alpha,\zeta]-[\alpha,\alpha][\beta,\zeta] \\ \left\{\begin{matrix}[\beta,\alpha][\alpha,\xi] \\ -[\alpha,\alpha][\beta,\xi]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\eta] \\ -[\alpha,\alpha][\beta,\eta]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\zeta] \\ -[\alpha,\alpha][\beta,\zeta]\end{matrix}\right\} & 0 \end{vmatrix} = 0 \qquad (114).$$

This can be written

$$\begin{aligned} &[\beta,\alpha]\begin{vmatrix} [\xi,\xi] & [\xi,\eta] & [\xi,\zeta] & [\alpha,\xi] \\ [\eta,\xi] & [\eta,\eta] & [\eta,\zeta] & [\alpha,\eta] \\ [\zeta,\xi] & [\zeta,\eta] & [\zeta,\zeta] & [\alpha,\zeta] \\ \left\{\begin{matrix}[\beta,\alpha][\alpha,\xi] \\ -[\alpha,\alpha][\beta,\xi]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\eta] \\ -[\alpha,\alpha][\beta,\eta]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\zeta] \\ -[\alpha,\alpha][\beta,\zeta]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\alpha] \\ -[\alpha,\alpha][\beta,\alpha]\end{matrix}\right\} \end{vmatrix} \\ -&[\alpha,\alpha]\begin{vmatrix} [\xi,\xi] & [\xi,\eta] & [\xi,\zeta] & [\beta,\xi] \\ [\eta,\xi] & [\eta,\eta] & [\eta,\zeta] & [\beta,\eta] \\ [\zeta,\xi] & [\zeta,\eta] & [\zeta,\zeta] & [\beta,\zeta] \\ \left\{\begin{matrix}[\beta,\alpha][\alpha,\xi] \\ -[\alpha,\alpha][\beta,\xi]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\eta] \\ -[\alpha,\alpha][\beta,\eta]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\zeta] \\ -[\alpha,\alpha][\beta,\zeta]\end{matrix}\right\} & \left\{\begin{matrix}[\beta,\alpha][\alpha,\beta] \\ -[\alpha,\alpha][\beta,\beta]\end{matrix}\right\} \end{vmatrix} = 0 \qquad (115). \end{aligned}$$

For the constituent in the fourth row and column of the last determinant vanishes by (76); and the constituent in the fourth row and column of the preceding determinant is identically zero.

Hence the condition becomes

$$\begin{aligned}&[\beta, \alpha]^2 \frac{\mathrm{D}\{[\xi], [\eta], [\zeta], [\alpha]\}}{\mathrm{D}\{\xi, \eta, \zeta, \alpha\}} \\ &- 2[\beta, \alpha][\alpha, \alpha]\frac{\mathrm{D}\{[\xi], [\eta], [\zeta], [\alpha]\}}{\mathrm{D}\{\xi, \eta, \zeta, \beta\}} \\ &+ [\alpha, \alpha]^2 \frac{\mathrm{D}\{[\xi], [\eta], [\zeta], [\beta]\}}{\mathrm{D}\{\xi, \eta, \zeta, \beta\}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116).\end{aligned}$$

And this is satisfied, because (30)–(32) hold.

Hence if (76) hold, the tangent plane to the conic node locus touches the tangent cone at the conic node.

Therefore the conic node locus is also an envelope.

(B.) Conversely, if the tangent plane to the conic node locus always touch the tangent cone at the conic node, then the condition

$$[\alpha, \alpha][\beta, \beta] - [\alpha, \beta]^2 = 0$$

is satisfied at every point of the conic node locus.

To determine the position of the tangent plane to the conic node locus, it is sufficient to eliminate $\delta\alpha$, $\delta\beta$ from any three of the equations (16), (17), (18), (28), (29), and then to use the relations (27).

Suppose that the values of $\delta\xi$, $\delta\eta$, $\delta\zeta$, which satisfy (16), (17), (18), (28), (29), are

$$\left.\begin{aligned}\delta\xi &= \lambda_1\,\delta\alpha + \lambda_2\,\delta\beta \\ \delta\eta &= \mu_1\,\delta\alpha + \mu_2\,\delta\beta \\ \delta\zeta &= \nu_1\,\delta\alpha + \nu_2\,\delta\beta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117).$$

Then the tangent plane to the conic node locus is

$$(\mathrm{X} - \xi)(\mu_1\nu_2 - \mu_2\nu_2) + (\mathrm{Y} - \eta)(\nu_1\lambda_2 - \nu_2\lambda_1) + (\mathrm{Z} - \zeta)(\lambda_1\mu_2 - \lambda_2\mu_1) = 0 \quad . \quad (118).$$

The condition that this may touch the tangent cone (33) is

$$\begin{vmatrix} [\xi, \xi] & [\eta, \xi] & [\zeta, \xi] & \mu_1\nu_2 - \mu_2\nu_1 \\ [\xi, \eta] & [\eta, \eta] & [\zeta, \eta] & \nu_1\lambda_2 - \nu_2\lambda_1 \\ [\xi, \zeta] & [\eta, \zeta] & [\zeta, \zeta] & \lambda_1\mu_2 - \lambda_2\mu_1 \\ \mu_1\nu_2 - \mu_2\nu_1 & \nu_1\lambda_2 - \nu_2\lambda_1 & \lambda_1\mu_2 - \lambda_2\mu_1 & 0 \end{vmatrix} = 0 \,. \qquad (119).$$

It will now be proved that the same condition can be obtained by substituting the values of $(\delta\xi)$, $(\delta\eta)$, $(\delta\zeta)$ from (117) in

$$\begin{aligned}&[\xi,\xi](\delta\xi)^2 + [\eta,\eta](\delta\eta)^2 + [\zeta,\zeta](\delta\zeta)^2\\ &+ 2[\eta,\zeta](\delta\eta)(\delta\zeta) + 2[\zeta,\xi](\delta\zeta)(\delta\xi) + 2[\xi,\eta](\delta\xi)(\delta\eta) = 0 \quad . \quad . \quad (120),\end{aligned}$$

and then making the roots of the resulting quadratic in $\delta\alpha/\delta\beta$ equal.

For making the substitution, the result is

$$\begin{aligned}&(\delta\alpha)^2\{[\xi,\xi]\lambda_1^2 + [\eta,\eta]\mu_1^2 + [\zeta,\zeta]\nu_1^2 + 2[\eta,\zeta]\mu_1\nu_1 + 2[\zeta,\xi]\nu_1\lambda_1 + 2[\xi,\eta]\lambda_1\mu_1\}\\ &+ 2(\delta\alpha)(\delta\beta)\left[\begin{aligned}&[\xi,\xi]\lambda_1\lambda_2 + [\eta,\eta]\mu_1\mu_2 + [\zeta,\zeta]\nu_1\nu_2\\ &+ [\eta,\zeta](\mu_1\nu_2 + \mu_2\nu_1) + [\zeta,\xi](\nu_1\lambda_2 + \nu_2\lambda_1) + [\xi,\eta](\lambda_1\mu_2 + \lambda_2\mu_1)\end{aligned}\right]\\ &+ (\delta\beta)^2\{[\xi,\xi]\lambda_2^2 + [\eta,\eta]\mu_2^2 + [\zeta,\zeta]\nu_2^2 + 2[\eta,\zeta]\mu_2\nu_2 + 2[\zeta,\xi]\nu_2\lambda_2 + 2[\xi,\eta]\lambda_2\mu_2\} = 0.\end{aligned}$$

Now putting

$$\begin{aligned}L_1 &= [\xi,\xi]\lambda_1 + [\xi,\eta]\mu_1 + [\xi,\zeta]\nu_1,\\ M_1 &= [\eta,\xi]\lambda_1 + [\eta,\eta]\mu_1 + [\eta,\zeta]\nu_1,\\ N_1 &= [\zeta,\xi]\lambda_1 + [\zeta,\eta]\mu_1 + [\zeta,\zeta]\nu_1,\end{aligned}$$

and similar expressions for L_2, M_2, N_2, the condition for equal roots can be written

$$\begin{vmatrix}\lambda_1L_1 + \mu_1M_1 + \nu_1N_1 & \lambda_1L_2 + \mu_1M_2 + \nu_1N_2\\ \lambda_2L_1 + \mu_2M_1 + \nu_2N_1 & \lambda_2L_2 + \mu_2M_2 + \nu_2N_2\end{vmatrix} = 0.$$

It remains to show that this will be satisfied if (119) be satisfied.

Now

$$\begin{vmatrix}[\xi,\xi] & [\eta,\xi] & [\zeta,\xi] & \mu_1\nu_2 - \mu_2\nu_1\\ [\xi,\eta] & [\eta,\eta] & [\zeta,\eta] & \nu_1\lambda_2 - \nu_2\lambda_1\\ [\xi,\zeta] & [\eta,\zeta] & [\zeta,\zeta] & \lambda_1\mu_2 - \lambda_2\mu_1\\ \mu_1\nu_2 - \mu_2\nu_1 & \nu_1\lambda_2 - \nu_2\lambda_1 & \lambda_1\mu_2 - \lambda_2\mu_1 & 0\end{vmatrix} \times \begin{vmatrix}\lambda_1 & \mu_1 & \nu_1 & 0\\ \lambda_2 & \mu_2 & \nu_2 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix}^2$$

$$= \begin{vmatrix}L_1 & M_1 & N_1 & 0\\ L_2 & M_2 & N_2 & 0\\ [\zeta,\xi] & [\zeta,\eta] & [\zeta,\zeta] & \lambda_1\mu_2 - \lambda_2\mu_1\\ \mu_1\nu_2 - \mu_2\nu_1 & \nu_1\lambda_2 - \nu_2\lambda_1 & \lambda_1\mu_2 - \lambda_2\mu_1 & 0\end{vmatrix} \times \begin{vmatrix}\lambda_1 & \mu_1 & \nu_1 & 0\\ \lambda_2 & \mu_2 & \nu_2 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{vmatrix}$$

$$= \begin{vmatrix}\lambda_1L_1 + \mu_1M_1 + \nu_1N_1 & \lambda_1L_2 + \mu_1M_2 + \nu_1N_2 & N_1 & 0\\ \lambda_2L_1 + \mu_2M_1 + \nu_2N_1 & \lambda_2L_2 + \mu_2M_2 + \nu_2N_2 & N_2 & 0\\ N_1 & N_2 & [\zeta,\zeta] & \lambda_1\mu_2 - \lambda_2\mu_1\\ 0 & 0 & \lambda_1\mu_2 - \lambda_2\mu_1 & 0\end{vmatrix}$$

$$= -(\lambda_1\mu_2 - \lambda_2\mu_1)^2\begin{vmatrix}\lambda_1L_1 + \mu_1M_1 + \nu_1N_1 & \lambda_1L_2 + \mu_1M_2 + \nu_1N_2\\ \lambda_2L_1 + \mu_2M_1 + \nu_2N_1 & \lambda_2L_2 + \mu_2M_2 + \nu_2N_2\end{vmatrix}$$

Hence dividing out by $(\lambda_1\mu_2 - \lambda_2\mu_1)^2$, it follows that the left-hand side of equation (119) is equal to

$$-\begin{vmatrix} \lambda_1 L_1 + \mu_1 M_1 + \nu_1 N_1 & \lambda_1 L_2 + \mu_1 M_2 + \nu_1 N_2 \\ \lambda_2 L_1 + \mu_2 M_1 + \nu_2 N_1 & \lambda_2 L_2 + \mu_2 M_2 + \nu_2 N_2 \end{vmatrix}.$$

Hence if the left-hand side of equation (119) vanish, so also does

$$\begin{vmatrix} \lambda_1 L_1 + \mu_1 M_1 + \nu_1 N_1 & \lambda_1 L_2 + \mu_1 M_2 + \nu_1 N_2 \\ \lambda_2 L_1 + \mu_2 M_1 + \nu_2 N_1 & \lambda_2 L_2 + \mu_2 M_2 + \nu_2 N_2 \end{vmatrix},$$

which was to be shown.

Hence either method of proceeding will lead to the condition that the tangent plane to the conic node locus should touch the tangent cone at the conic node.

The second method being simpler in this case will now be adopted.

Multiply (16) by $\delta\xi$, (17) by $\delta\eta$, (18) by $\delta\zeta$, (28) by $-(\delta\alpha)$, (29) by $-\delta\beta$, and add.

Hence

$$[\xi,\xi](\delta\xi)^2 + [\eta,\eta](\delta\eta)^2 + [\zeta,\zeta](\delta\zeta)^2 + 2[\eta,\zeta](\delta\eta)(\delta\zeta) + 2[\zeta,\xi](\delta\zeta)(\delta\xi) + 2[\xi,\eta](\delta\xi)(\delta\eta)$$
$$= [\alpha,\alpha](\delta\alpha)^2 + 2[\alpha,\beta](\delta\alpha)(\delta\beta) + [\beta,\beta](\delta\beta)^2.$$

Hence the result of substituting the values of $\delta\xi$, $\delta\eta$, $\delta\zeta$, which satisfy (16), (17), (18), (28), (29) in (120) is

$$[\alpha,\alpha](\delta\alpha)^2 + 2[\alpha,\beta](\delta\alpha)(\delta\beta) + [\beta,\beta](\delta\beta)^2 = 0.$$

Forming the condition that the roots of this quadratic in $\delta\alpha/\delta\beta$ should be equal, it follows that

$$[\alpha,\alpha][\beta,\beta] - [\alpha,\beta]^2 = 0.$$

It will be proved in Art. 27 (see the equations (196)) that the common tangent line to the conic node and the conic node locus is in this case given by the equations

$$[\alpha,\xi](X-\xi) + [\alpha,\eta](Y-\eta) + [\alpha,\zeta](Z-\zeta) = 0,$$
$$[\beta,\xi](X-\xi) + [\beta,\eta](Y-\eta) + [\beta,\zeta](Z-\zeta) = 0.$$

Hence the common tangent line to the conic node and the conic node locus lies on the tangent planes to the surfaces $Df/D\alpha = 0$, $Df/D\beta = 0$; and it lies obviously on the tangent cone to the surface $f = 0$.

(C.) In this case equations (108) and (109) hold.

Also differentiating (109)

$$\frac{\partial^2\Delta}{\partial x^2} = \frac{\partial^2 R}{\partial x^2} f_1 f_2 + 2R\left[f_2\frac{Df_1}{Dx} + f_1\frac{Df_2}{Dx}\right]$$

$$+ R\left[\begin{array}{l} f_2\left(\dfrac{D^2 f_1}{Dx^2} + \dfrac{D^2 f_1}{DxDa_1}\dfrac{\partial a_1}{\partial x} + \dfrac{D^2 f_1}{DxDb_1}\dfrac{\partial b_1}{\partial x}\right) \\ + 2\dfrac{Df_1}{Dx}\dfrac{Df_2}{Dx} \\ + f_1\left(\dfrac{D^2 f_2}{Dx^2} + \dfrac{D^2 f_2}{DxDa_2}\dfrac{\partial a_2}{\partial x} + \dfrac{D^2 f_2}{DxDb_2}\dfrac{\partial b_2}{\partial x}\right) \end{array}\right] \quad . \; . \; . \; . \; . \; . \; . \quad (121).$$

Substituting the values of $\partial a_1/\partial x$, $\partial b_1/\partial x$ from (52), there is in $\partial^2\Delta/\partial x^2$ the term

$$Rf_2 \begin{vmatrix} [x, x] & [x, a_1] & [x, b_1] \\ [x, a_1] & [a_1, a_1] & [a_1, b_1] \\ [x, b_1] & [a_1, b_1] & [b_1, b_1] \end{vmatrix} \Bigg/ \begin{vmatrix} [a_1, a_1] & [a_1, b_1] \\ [a_1, b_1] & [b_1, b_1] \end{vmatrix}$$

which requires examination when $x = \xi$, $y = \eta$, $z = \zeta$, the coordinates of a point on the conic node locus.

Now in this case a_1, b_1 are roots of $Df_1/Da_1 = 0$, $Df_1/Db_1 = 0$.

Hence

$$[\alpha, \xi](\delta\xi) + [\alpha, \eta](\delta\eta) + [\alpha, \zeta](\delta\zeta) + [\alpha, \alpha](\delta\alpha) + [\alpha, \beta](\delta\beta)$$
$$+ \text{ terms of the second and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha, \delta\beta = 0 \quad . \quad (122)$$

$$[\beta, \xi](\delta\xi) + [\beta, \eta](\delta\eta) + [\beta, \zeta](\delta\zeta) + [\beta, \alpha](\delta\alpha) + [\beta, \beta](\delta\beta)$$
$$+ \text{ terms of the second and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha, \delta\beta = 0 \quad . \quad (123).$$

Multiply (122) by $[\alpha, \beta,]$, (123) by $[\alpha, \alpha]$ and subtract, the terms of the first order in $\delta\alpha$, $\delta\beta$ disappear, and the equation obtained is of the form :—

$$(\text{terms of the first order in } \delta\xi, \delta\eta, \delta\zeta)$$
$$+ (\text{terms of the second and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha, \delta\beta) = 0.$$

Hence if $\delta\xi$, $\delta\eta$, $\delta\zeta$ are of the order of the infinitely small quantity ϵ; then $\delta\alpha$, $\delta\beta$ are of the order of $\epsilon^{\frac{1}{2}}$.

Hence the principal terms in (122) and (123) are $[\alpha, \alpha](\delta\alpha) + [\alpha, \beta]\,\delta\beta$ and $[\beta, \alpha](\delta\alpha) + [\beta, \beta](\delta\beta)$ respectively.

Moreover by (122) and (123), although $\delta\alpha$, $\delta\beta$ are of the order $\epsilon^{\frac{1}{2}}$, yet $[\alpha, \alpha]\,\delta\alpha + [\alpha, \beta]\,\delta\beta$ being ultimately equal to $-\{[\alpha, \xi](\delta\xi) + [\alpha, \eta]\,\delta\eta + [\alpha, \zeta]\,\delta\zeta\}$ is of the order ϵ.

Similarly $[\beta, \alpha]\,\delta\alpha + [\beta, \beta]\,\delta\beta$ is of the order ϵ.

Next, when $x = \xi$, $y = \eta$, $z = \zeta$, $a_2 = \alpha + \delta\alpha'$, $b_2 = \beta + \delta\beta'$, $f(x, y, z, a_2, b_2)$ becomes

$$\begin{aligned}
& f(\xi, \eta, \zeta, \alpha, \beta) \\
& + [\xi](\delta\xi) + [\eta](\delta\eta) + [\zeta](\delta\zeta) + [\alpha](\delta\alpha') + [\beta](\delta\beta') \\
& + \tfrac{1}{2}\left[\begin{array}{l}
[\xi, \xi](\delta\xi)^2 + [\eta, \eta](\delta\eta)^2 + [\zeta, \zeta](\delta\zeta)^2 \\
+ 2[\eta, \zeta](\delta\eta)(\delta\zeta) + 2[\zeta, \xi](\delta\zeta)(\delta\xi) + 2[\xi, \eta](\delta\xi)(\delta\eta) \\
+ 2(\delta\alpha')\{[\xi, \alpha](\delta\xi) + [\eta, \alpha](\delta\eta) + [\zeta, \alpha](\delta\zeta)\} \\
+ 2(\delta\beta')\{[\xi, \beta](\delta\xi) + [\eta, \beta](\delta\eta) + [\zeta, \beta](\delta\zeta)\} \\
+ [\alpha, \alpha](\delta\alpha')^2 + 2[\alpha, \beta](\delta\alpha')(\delta\beta') + [\beta, \beta](\delta\beta')^2
\end{array}\right] \\
& + \text{terms of the third and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha', \delta\beta' \\
= \quad & (\text{terms of the second order in } \delta\xi, \delta\eta, \delta\zeta) \\
& + (\delta\alpha')(\text{terms of the first order in } \delta\xi, \delta\eta, \delta\zeta) \\
& + (\delta\beta')(\text{terms of the first order in } \delta\xi, \delta\eta, \delta\zeta) \\
& + \frac{1}{2[\alpha, \alpha]}\{[\alpha, \alpha](\delta\alpha') + [\alpha, \beta]\,\delta\beta'\}^2 \\
& + \text{terms of the third and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha', \delta\beta'.
\end{aligned}$$

Now, the terms of the second order in $\delta\xi$, $\delta\eta$, $\delta\zeta$ are of order ϵ^2.

The terms containing $\delta\alpha'$ or $\delta\beta'$, multiplied by terms of the first order in $\delta\xi$, $\delta\eta$, $\delta\zeta$, are of order $\epsilon^{3/2}$.

The terms $\dfrac{1}{2[\alpha, \alpha]}\{[\alpha, \alpha](\delta\alpha') + [\alpha, \beta]\,\delta\beta'\}^2$ are of order ϵ^2, since $[\alpha, \alpha](\delta\alpha') + [\alpha, \beta](\delta\beta')$ is of order ϵ, by the same argument as the one which was applied to show that $[\alpha, \alpha](\delta\alpha) + [\alpha, \beta](\delta\beta)$ was of order ϵ.

The most important terms of the third and higher orders in $\delta\xi$, $\delta\eta$, $\delta\zeta$, $\delta\alpha'$, $\delta\beta'$ are of order $\epsilon^{3/2}$.

Hence $f(x, y, z, a_2, b_2)$ is of order $\epsilon^{3/2}$, when $x = \xi$, $y = \eta$, $z = \zeta$, the coordinates of a point on the conic node locus.

Further

$$\begin{vmatrix} [a_1, a_1] & [a_1, b_1] \\ [a_1, b_1] & [b_1, b_1] \end{vmatrix}$$

becomes

$$\begin{aligned}
& \begin{vmatrix} [\alpha, \alpha] & [\alpha, \beta] \\ [\alpha, \beta] & [\beta, \beta] \end{vmatrix} \\
& + \left\{(\delta\xi)\frac{\mathrm{D}}{\mathrm{D}\xi} + (\delta\eta)\frac{\mathrm{D}}{\mathrm{D}\eta} + (\delta\zeta)\frac{\mathrm{D}}{\mathrm{D}\zeta} + (\delta\alpha)\frac{\mathrm{D}}{\mathrm{D}\alpha} + (\delta\beta)\frac{\mathrm{D}}{\mathrm{D}\beta}\right\}\begin{vmatrix} [\alpha, \alpha] & [\alpha, \beta] \\ [\alpha, \beta] & [\beta, \beta] \end{vmatrix} \\
& + \text{terms of the second and higher orders in } \delta\xi, \delta\eta, \delta\zeta, \delta\alpha, \delta\beta.
\end{aligned}$$

Hence, when $x = \xi$, $y = \eta$, $z = \zeta$,

$$\begin{vmatrix} [a_1, a_1] & [a_1, b_1] \\ [a_1, b_1] & [b_1, b_1] \end{vmatrix}$$

is of order $\epsilon^{1/2}$.

Hence

$$f_2 \Big/ \begin{vmatrix} [a_1, a_1] & [a_1, b_1] \\ [a_1, b_1] & [b_1, b_1] \end{vmatrix}$$

is of order ϵ, and therefore vanishes at points on the conic node locus.

Similarly it can be shown that the term

$$f_1 \Big/ \begin{vmatrix} [a_2, a_2] & [a_2, b_2] \\ [a_2, b_2] & [b_2, b_2] \end{vmatrix}$$

vanishes at points on the conic node locus.

Therefore Δ, $\partial\Delta/\partial x$, $\partial^2\Delta/\partial x^2$ all vanish on the conic node locus.

Therefore Δ contains C^3 as a factor.

Example 9.—*Conic Node Locus which is also an Envelope.*

Let the surfaces be

$$\alpha (x - a)^3 + 3\beta (y - b)^2 + 3\gamma (x - a) z + \delta z^2 = 0,$$

where α, β, γ, δ are fixed constants; a, b the arbitrary parameters.

(A.) *The Discriminant.*

It is the same as that of the equation

$$\alpha X^3 + 3\beta Y^2 Z + 3\gamma z X Z^2 + \delta z^2 Z^3 = 0.$$

Therefore

$$S = \alpha\beta^2\gamma z,$$

$$T = 4\alpha^2\beta^3 \delta z^2.$$

Therefore

$$\Delta = 16\alpha^3\beta^6 z^3 (\alpha\delta^2 z + 4\gamma^3).$$

(B.) *The Conic Node Locus, which is also an Envelope, is $z = 0$.*

To prove this, transform the equation by means of $x = a + X$, $y = b + Y$, $z = Z$. It becomes

$$\alpha X^3 + 3\beta Y^2 + 3\gamma XZ + \delta Z^2 = 0.$$

Hence the new origin is a conic node, and one of the tangent planes of the conic node is $Z = 0$.

Hence the conic node locus is $z = 0$, and it is also an envelope.

(C.) *The Locus* $\alpha\delta^2 z + 4\gamma^3 = 0$ *is an Ordinary Envelope.*

This can be proved by finding the tangent planes parallel to $z = 0$.

Hence it is necessary to satisfy at the same time

$$\begin{aligned} &\alpha(x-a)^3 + 3\beta(y-b)^2 + 3\gamma(x-a)z + \delta z^2 &&= 0, \\ &\alpha(x-a)^2 + \gamma z &&= 0, \\ &y - b &&= 0, \\ &\qquad 3\gamma(x-a) + 2\delta z &&\neq 0 \quad . \quad . \quad . \quad (124). \end{aligned}$$

Hence

$$y = b,$$

therefore

$$\begin{aligned} \alpha(x-a)^3 + 3\gamma(x-a)z + \delta z^2 &= 0, \\ \alpha(x-a)^2 + \gamma z &= 0. \end{aligned}$$

Hence

$$2\gamma(x-a)z + \delta z^2 = 0.$$

The solution $z = 0$ of the last equation makes $x = a$, and does not satisfy (124).

Hence it is necessary to take

$$2\gamma(x-a) + \delta z = 0.$$

This gives

$$\alpha\delta^2 z + 4\gamma^3 = 0.$$

Hence, when

$$x = a + 2\gamma^2/\alpha\delta, \quad y = b, \quad z = -4\gamma^3/\alpha\delta^2,$$

the tangent plane is parallel to the plane $z = 0$. It touches all the surfaces of the system.

Hence

$$z = -4\gamma^3/\alpha\delta^2$$

is an ordinary envelope.

Art. 15.—*To prove that if the Edge of the Biplanar Node always touch the Biplanar Node Locus, then* Δ *contains* B^4 *as a factor.*

(A.) It will be shown that when the condition (76) holds in the case of a biplanar node locus, then the edge of the biplanar node always touches the biplanar node locus.

The equation of the biplanes is given by (33).

Now, if the left-hand side of (33) break up into two linear factors, then the two planes, whose equations are given by equating the two linear factors to zero, will intersect in the straight line whose equations are given by any two of the three equations :—

$$\begin{aligned}&[\xi, \xi]\,(X-\xi)+[\xi, \eta]\,(Y-\eta)+[\xi, \zeta]\,(Z-\zeta)=0,\\ &[\eta, \xi]\,(X-\xi)+[\eta, \eta]\,(Y-\eta)+[\eta, \zeta]\,(Z-\zeta)=0,\\ &[\zeta, \xi]\,(X-\xi)+[\zeta, \eta]\,(Y-\eta)+[\zeta, \zeta]\,(Z-\zeta)=0 \quad . \quad . \quad . \quad (125).\end{aligned}$$

To find the tangent plane to the binode locus, proceed thus :—

The condition (41) gives

$$\begin{aligned}&[\xi, \xi]\,\{[\alpha, \alpha]\,[\beta, \beta]-[\alpha, \beta]^2\}\\ &-[\xi, \alpha]^2\,[\beta, \beta]+2\,[\xi, \alpha]\,[\xi, \beta]\,[\alpha, \beta]-[\xi, \beta]^2\,[\alpha, \alpha]=0,\end{aligned}$$

which, by means of (76), can be written

$$[\xi, \alpha]^2\,[\alpha, \beta]^2-2\,[\xi, \alpha]\,[\xi, \beta]\,[\alpha, \beta]\,[\alpha, \alpha]+[\xi, \beta]^2\,[\alpha, \alpha]^2=0.$$

Therefore

$$[\xi, \alpha]\,[\alpha, \beta]-[\xi, \beta]\,[\alpha, \alpha]=0.$$

Similarly

$$\begin{aligned}&[\eta, \alpha]\,[\alpha, \beta]-[\eta, \beta]\,[\alpha, \alpha]=0,\\ &[\zeta, \alpha]\,[\alpha, \beta]-[\zeta, \beta]\,[\alpha, \alpha]=0.\end{aligned}$$

Hence

$$\frac{[\xi, \alpha]}{[\xi, \beta]}=\frac{[\eta, \alpha]}{[\eta, \beta]}=\frac{[\zeta, \alpha]}{[\zeta, \beta]}=\frac{[\alpha, \alpha]}{[\alpha, \beta]}=\frac{[\beta, \alpha]}{[\beta, \beta]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (126).$$

Now, multiplying (16) by $[\eta, \alpha]$, (17) by $[\xi, \alpha]$ and subtracting

$$\begin{aligned}&[\eta, \alpha]\,\{[\xi, \xi]\,(\delta\xi)+[\xi, \eta]\,(\delta\eta)+[\xi, \zeta]\,(\delta\zeta)\}\\ &-[\xi, \alpha]\,\{[\eta, \xi]\,(\delta\xi)+[\eta, \eta]\,(\delta\eta)+[\eta, \zeta]\,(\delta\zeta)\}\\ &+(\delta\beta)\,\{[\eta, \alpha]\,[\xi, \beta]-[\eta, \beta]\,[\xi, \alpha]\}=0.\end{aligned}$$

Now, by (126), the coefficient of $\delta\beta$ vanishes, and the equation of the tangent plane to the binode locus is

$$\begin{aligned}&[\eta, \alpha]\,\{[\xi, \xi]\,(X-\xi)+[\xi, \eta]\,(Y-\eta)+[\xi, \zeta]\,(Z-\zeta)\}\\ &-[\xi, \alpha]\,\{[\eta, \xi]\,(X-\xi)+[\eta, \eta]\,(Y-\eta)+[\eta, \zeta]\,(Z-\zeta)\}=0 \quad . \quad (127).\end{aligned}$$

Hence the tangent plane to the binode locus passes through the intersection of two of the planes (125), and, therefore, through the edge of the binode.

Hence the edge of the binode always touches the binode locus.

It may be noticed, further, with respect to the edge of the binode, since equations (28) and (29) depend on (16) and (17), that since it is the intersection of the planes (125) it lies also on the planes

$$[\alpha, \xi](X - \xi) + [\alpha, \eta](Y - \eta) + [\alpha, \zeta](Z - \zeta) = 0,$$
$$[\beta, \xi](X - \xi) + [\beta, \eta](Y - \eta) + [\beta, \zeta](Z - \zeta) = 0.$$

(These planes, it may be noted, coincide in this case by (126).)

Hence the edge lies on the tangent planes to the surfaces $Df/D\alpha = 0$, $Df/D\beta = 0$, and it is obviously a tangent to the surface $f = 0$.

(B.) Conversely, if the edge of the binode always touches the binode locus, then the condition (76) holds at every point of the binode locus.

The equation of the edge is given by any two of the three equations (125).

Hence, if the edge is a tangent line to the locus of binodes, the equations (125) will be satisfied by putting

$$X = \xi + \delta\xi, \quad Y = \eta + \delta\eta, \quad Z = \zeta + \delta\zeta,$$

the coordinates of a binode near to ξ, η, ζ, which lies on the edge of the first binode and infinitely near to it.

Hence

$$\left.\begin{aligned}[\xi, \xi](\delta\xi) + [\xi, \eta](\delta\eta) + [\xi, \zeta](\delta\zeta) = 0,\\ [\eta, \xi](\delta\xi) + [\eta, \eta](\delta\eta) + [\eta, \zeta](\delta\zeta) = 0,\\ [\zeta, \xi](\delta\xi) + [\zeta, \eta](\delta\eta) + [\zeta, \zeta](\delta\zeta) = 0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (128).$$

But equations (16), (17), (18), (28), (29), also hold.

Hence, by (128) it follows that (16), (17), (18) become

$$[\xi, \alpha](\delta\alpha) + [\xi, \beta](\delta\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129),$$

$$[\eta, \alpha](\delta\alpha) + [\eta, \beta](\delta\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130),$$

$$[\zeta, \alpha](\delta\alpha) + [\zeta, \beta](\delta\beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

Hence

$$[\xi, \alpha]/[\xi, \beta] = [\eta, \alpha]/[\eta, \beta] = [\zeta, \alpha]/[\zeta, \beta] \quad . \quad . \quad . \quad . \quad . \quad . \quad (132).$$

Now only two of the five equations (16), (17), (18), (28), (29) are independent.

Suppose that (16), (17) are independent.

Then, since (28), (29) depend on these, relations exist of the form

$$\begin{aligned}
&[\xi, \alpha] = \lambda[\xi, \xi] + \mu[\xi, \eta], && [\xi, \beta] = \rho[\xi, \xi] + \sigma[\xi, \eta],\\
&[\eta, \alpha] = \lambda[\xi, \eta] + \mu[\eta, \eta], && [\eta, \beta] = \rho[\xi, \eta] + \sigma[\eta, \eta],\\
&[\zeta, \alpha] = \lambda[\xi, \zeta] + \mu[\zeta, \eta], && [\zeta, \beta] = \rho[\xi, \zeta] + \sigma[\eta, \zeta],\\
&[\alpha, \alpha] = \lambda[\xi, \alpha] + \mu[\alpha, \eta], && [\alpha, \beta] = \rho[\xi, \alpha] + \sigma[\eta, \alpha],\\
&[\alpha, \beta] = \lambda[\xi, \beta] + \mu[\beta, \eta] && [\beta, \beta] = \rho[\xi, \beta] + \sigma[\eta, \beta].
\end{aligned}$$

Hence, by (132),

$$\frac{\lambda[\xi,\xi]+\mu[\xi,\eta]}{\rho[\xi,\xi]+\sigma[\xi,\eta]}=\frac{\lambda[\xi,\eta]+\mu[\eta,\eta]}{\rho[\xi,\eta]+\sigma[\eta,\eta]}=\frac{\lambda[\xi,\zeta]+\mu[\eta,\zeta]}{\rho[\xi,\zeta]+\sigma[\eta,\zeta]}.$$

Therefore

$$[\lambda\sigma-\mu\rho]\{[\xi,\xi][\eta,\eta]-[\xi,\eta]^2\}=0,$$

and

$$[\lambda\sigma-\mu\rho]\{[\xi,\xi][\eta,\zeta]-[\xi,\eta][\xi,\zeta]\}=0.$$

Hence, unless $\lambda\sigma-\mu\rho=0$, it is necessary to have both

$$[\xi,\xi][\eta,\eta]-[\xi,\eta]^2=0,$$

and

$$[\xi,\xi][\eta,\zeta]-[\xi,\eta][\xi,\zeta]=0.$$

Hence

$$[\xi,\xi]/[\xi,\eta]=[\xi,\eta]/[\eta,\eta]=[\xi,\zeta]/[\eta,\zeta].$$

But if these results hold, the two equations taken to determine the edge of the binode would be the same, and would not determine it. Supposing then that those two equations have been selected, which are independent, this alternative cannot hold, and therefore

$$\lambda\sigma-\mu\rho=0.$$

Therefore

$$\frac{[\xi,\alpha]}{[\xi,\beta]}=\frac{[\eta,\alpha]}{[\eta,\beta]}=\frac{[\zeta,\alpha]}{[\zeta,\beta]}=\frac{[\alpha,\alpha]}{[\alpha,\beta]}=\frac{[\alpha,\beta]}{[\beta,\beta]}=\frac{\lambda}{\rho}.$$

Hence the equations (126) are satisfied, and in particular

$$[\alpha,\alpha][\beta,\beta]-[\alpha,\beta]^2=0.$$

Hence, if the edge of the binode always touch the binode locus, the condition (76) holds.

(C.) In this case Δ is given by (108).

Let ξ, η, ζ be any point on the binode locus.

Then when $x=\xi$, $y=\eta$, $z=\zeta$,

$a_1=a_2=a$ of surface having a binode at ξ, η, ζ,

$b_1=b_2=b$ of surface having a binode at ξ, η, ζ.

$\mathrm{D}f_1/\mathrm{D}x=0$, $\mathrm{D}f_1/\mathrm{D}y=0$, $\mathrm{D}f_1/\mathrm{D}z=0$, when $x=\xi$, $y=\eta$, $z=\zeta$.

$\mathrm{D}f_2/\mathrm{D}x=\mathrm{D}f_1/\mathrm{D}x$, when $x=\xi$, $y=\eta$, $z=\zeta$.

Therefore, $\mathrm{D}f_2/\mathrm{D}x=0$, and similarly $\mathrm{D}f_2/\mathrm{D}y=0$, $\mathrm{D}f_2/\mathrm{D}z=0$, when $x=\xi$, $y=\eta$, $z=\zeta$.

Now if each of the differential coefficients of Δ be formed up to the third order, then every term in the result will contain as a factor one of the following quantities :—

$$f_1 \text{ or } f_2 \text{ or a first differential coefficient of } f_1 \text{ or } f_2.$$

Hence, when $x = \xi$, $y = \eta$, $z = \zeta$, Δ and all its differential coefficients up to the third order vanish.

Hence, if $B = 0$ be the equation of the binode locus, such that the edge of the binode always touches the binode locus, Δ contains B^4 as a factor by Art. 1, Preliminary Theorem B.

Example 10.—*Locus of Biplanar Nodes such that the Edge of the Biplanar Node always touches the Biplanar Node Locus.*

Let the surfaces be

$$\alpha\,(x - a)^3 + \beta\,(y - b)^3 + 3\left[c\,(x - a) + e\,(y - b) + gz\right]^2 - h^2z^2 = 0,$$

where α, β, c, e, g, h are fixed constants ; a, b are the arbitrary parameters.

(A.) *The Discriminant.*

It is the same as that of the equation

$$\alpha X^3 + \beta Y^3 + 3Z\,(cX + eY + gzZ)^2 - h^2z^2Z^3 = 0.$$

Hence,

$$S = -\,\alpha\beta ce\,(g^2 + h^2)\,z^2,$$

$$T = \alpha^2\beta^2\,(3g^2 - h^2)^2\,z^4 + 4\alpha\beta g\,(\alpha e^3 + \beta c^3)\,(3h^2 - g^2)\,z^3 - 4h^2\,(\alpha e^3 - \beta c^3)^2\,z^2.$$

Therefore,

$$\Delta = z^4 \left[\begin{array}{c} \{(\alpha^2\beta^2)\,(3g^2 - h^2)^2\,z^2 + 4\alpha\beta g\,(\alpha e^3 + \beta c^3)\,(3h^2 - g^2)\,z - 4h^2\,(\alpha e^3 - \beta c^3)^2\}^2 \\ -\,64\alpha^3\beta^3c^3e^3\,(g^2 + h^2)^3\,z^2 \end{array}\right]$$

In order to show the way in which the factor z^4 arises, the method in which the discriminant is formed will now be examined.

It may be obtained by eliminating X, Y, Z from

$$(cX + eY + gzZ)\,(cX + eY + 3gzZ) - h^2z^2Z^2 = 0 \quad . \quad . \quad . \quad (133),$$

$$\alpha X^2 + 2cZ\,(cX + eY + gzZ) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134),$$

$$\beta Y^2 + 2eZ\,(cX + eY + gzZ) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135).$$

Hence,

$$Y = \pm\,X\,\sqrt{(e\alpha/c\beta)}.$$

Representing both values of Y by $Y = \lambda X$, it follows that

$$\alpha X^2 + 2c(c + e\lambda) XZ + 2cgzZ^2 = 0.$$

Therefore,

$$X/Z = -\frac{c}{\alpha}(c + e\lambda) \pm \frac{1}{\alpha}\{c^2(c + e\lambda)^2 - 2c\alpha gz\}^{1/2}.$$

Hence,

$$x - a = -\frac{c}{\alpha}(c + e\lambda) \pm \frac{1}{\alpha}\{c^2(c + e\lambda)^2 - 2c\alpha gz\}^{1/2} \quad . \quad . \quad . \quad . \quad (136),$$

$$y - b = \lambda\left\{-\frac{c}{\alpha}(c + e\lambda) \pm \frac{1}{\alpha}\{c^2(c + e\lambda)^2 - 2c\alpha gz\}^{1/2}\right\} \quad . \quad . \quad . \quad (137).$$

These give the values of a, b which, when substituted in the equation of the surfaces, give the discriminant.

The values of a, b corresponding to a point ξ, η, ζ on the binode locus, will now be found.

It will be shown presently that $z = 0$ is the binode locus.

Hence $\zeta = 0$, and therefore

$$\xi - a = -\frac{c}{\alpha}(c + e\lambda) \pm \frac{c}{\alpha}(c + e\lambda),$$

$$\eta - b = \lambda\left\{-\frac{c}{\alpha}(c + e\lambda) \pm \frac{c}{\alpha}(c + e\lambda)\right\}.$$

Hence, for *each* value of λ, one of the values of a is ξ, and one of the values of b is η.

Hence there are two sets of values of a, b satisfying $Df/Da = 0$, $Df/Db = 0$, which become equal when $x = \xi$, $y = \eta$, $z = 0$.

These two sets of values both give $a = \xi$, $b = \eta$.

It will now be shown that the substitution of each of these systems of values of a, b in f, will give rise to the factor z^2 in Δ.

Now

$$Y = \lambda X, \qquad X = \mu Z,$$

where

$$\lambda = \pm\sqrt{(e\alpha/c\beta)},$$

$$\mu = -\frac{c}{\alpha}(c + e\lambda) \pm \frac{1}{\alpha}\{c^2(c + e\lambda)^2 - 2c\alpha gz\}^{1/2}.$$

Substituting these in the left-hand side of (133), it becomes

$$[(c\mu + e\lambda\mu + gz)(c\mu + e\lambda\mu + 3gz) - h^2z^2]Z^2,$$

i.e.,

$$[\mu^2(c + e\lambda)^2 + 4g\mu(c + e\lambda)z + (3g^2 - h^2)z^2]Z^2.$$

Although, to find the discriminant in the usual way, it is necessary to substitute each set of values of λ, μ in this expression, and then to multiply the results together, it is possible to determine more readily which value of μ will lead to the factor z^2 by expanding μ in ascending powers of z.

Now

$$\mu = -\frac{c}{\alpha}(c + e\lambda) \pm \frac{c}{\alpha}(c + e\lambda)\left\{1 - \frac{\alpha g z}{c(c + e\lambda)^2} - \frac{\alpha^2 g^2 z^2}{2c^2(c + e\lambda)^4} - \dots\right\}.$$

Taking the upper sign

$$\mu = -\frac{gz}{c + e\lambda} - \frac{\alpha g^2 z^2}{2c(c + e\lambda)^3} - \dots$$

Substituting this value of μ in

$$[\mu^2(c + e\lambda)^2 + 4g\mu(c + e\lambda)z + (3g^2 - h^2)z^2]Z^2,$$

the coefficient of z^2 in the bracket is $(-h^2)$, there being no lower power of z.

This being true for each value of λ, the factor z^4 is accounted for.

The other value of μ will lead to a factor, in which there is a term independent of z.

The elimination will now be completed.

It is necessary to substitute the values of λ and μ from

$$\lambda = \pm\sqrt{(e\alpha/c\beta)},$$

and

$$\alpha\mu^2 + 2c(c + c\lambda)\mu + 2cgz = 0,$$

in

$$\mu^2(c + e\lambda)^2 + 4g\mu(c + e\lambda)z + (3g^2 - h^2)z^2.$$

Substituting first for μ^2, and multiplying by α, this becomes

$$\mu(c + e\lambda)[-2c(c + e\lambda)^2 + 4g\alpha z] + [-2cgz(c + e\lambda)^2 + \alpha z^2(3g^2 - h^2)].$$

Substituting both values of μ in this, multiplying the results together, and multiplying by α, the result is

$$\begin{aligned}&2cgz(c + c\lambda)^2[-2c(c + c\lambda)^2 + 4g\alpha z]^2\\ &\quad - 2c(c + e\lambda)^2[-2c(c + e\lambda)^2 + 4g\alpha z][-2cgz(c + c\lambda)^2 + \alpha z^2(3g^2 - h^2)]\\ &\quad + \alpha[-2cgz(c + c\lambda)^2 + \alpha z^2(3g^2 - h^2)]^2.\end{aligned}$$

This reduces to

$$\begin{aligned}&-4c^2\alpha h^2(c + c\lambda)^4 z^2\\ &+ 4cg\alpha^2(3h^2 - g^2)(c + e\lambda)^2 z^3\\ &+ \alpha^3(3g^2 - h^2)^2 z^4.\end{aligned}$$

Putting $\lambda^2 = e\alpha/c\beta$, and multiplying by β^2/α, this becomes

$$\begin{gathered}\alpha^2\beta^2(3g^2 - h^2)^2 z^4 + 4g\alpha\beta(3h^2 - g^2)(\alpha e^3 + \beta c^3) z^3 - 4h^2z^2(\alpha e^3 - \beta c^3)^2 \\ - 32\alpha e^3\beta c^3h^2z^2 + 8c^2e\beta\lambda\left[g\alpha\beta z^3(3h^2 - g^2) - 2h^2z^2(\alpha e^3 + \beta c^3)\right].\end{gathered}$$

Substituting for λ its two values $\pm\sqrt{(e\alpha/c\beta)}$, multiplying the results together, and reducing, it becomes

$$z^4\left[\begin{gathered}\{\alpha^2\beta^2(3g^2 - h^2)^2 z^2 + 4\alpha\beta g(\alpha e^3 + \beta c^3)(3h^2 - g^2) z - 4h^2(\alpha e^3 - \beta c^3)^2\}^2 \\ - 64\alpha^3\beta^3c^3e^3(g^2 + h^2)^3 z^2\end{gathered}\right]$$

This is the same value as before for the discriminant.

(B.) *The Surface $z = 0$ is a Binode Locus such that the Edge of the Binode touches the Binode Locus.*

Transforming the equation by means of $x = a + \mathrm{X}$, $y = b + \mathrm{Y}$, $z = \mathrm{Z}$, it becomes

$$\alpha\mathrm{X}^3 + \beta\mathrm{Y}^3 + 3(c\mathrm{X} + e\mathrm{Y} + g\mathrm{Z})^2 - h^2\mathrm{Z}^2 = 0.$$

Hence the new origin is a binode.

Hence the binode locus is $z = 0$.

The biplanes are

$$3^{\frac{1}{2}}(c\mathrm{X} + e\mathrm{Y} + g\mathrm{Z}) - h\mathrm{Z} = 0.$$
$$3^{\frac{1}{2}}(c\mathrm{X} + e\mathrm{Y} + g\mathrm{Z}) + h\mathrm{Z} = 0.$$

The equations of the edge are therefore

$$c\mathrm{X} + e\mathrm{Y} = 0, \quad \mathrm{Z} = 0.$$

It lies therefore in the plane $\mathrm{Z} = 0$, *i.e.*, in the plane $z = 0$.

Hence it may be considered to touch the binode locus.

The condition (76) is satisfied at every point on the binode locus.

Hence the factor z^4 is accounted for.

(C.) *The Surface*

$$\begin{gathered}\{\alpha^2\beta^2(3g^2 - h^2)^2 z^2 + 4g\alpha\beta(\alpha e^3 + \beta c^3)(3h^2 - g^2) z - 4h^2(\alpha e^3 - \beta c^3)^2\}^2 \\ - 64\,\alpha^3\beta^3c^3e^3(g^2 + h^2)^3 z^2 = 0\end{gathered}$$

is an ordinary Envelope.

This may be proved by finding the tangent planes parallel to the plane $z = 0$.

Hence it is necessary to satisfy at the same time

$$\alpha(x-a)^3+\beta(y-b)^3+3[c(x-a)+e(y-b)+gz]^2-h^2z^2=0 \quad (138),$$

$$\alpha(x-a)^2+2c[c(x-a)+e(y-b)+gz] = 0 \quad (139),$$

$$\beta(y-b)^2+2e[c(x-a)+e(y-b)+gz] = 0 \quad (140),$$

$$3g[c(x-a)+e(y-b)+gz]-h^2z \neq 0 \quad (141),$$

or putting $x-a=\mathrm{X}/\mathrm{Z}$, $y-b=\mathrm{Y}/\mathrm{Z}$, the equations (138)-(140) become the same as (133), (134), (135).

Hence the result of the elimination will be the same as in the previous case. It is only necessary to show that (141) is satisfied.

Multiplying (139) by $(x-a)$, (140) by $(y-b)$ and subtracting from (138), it follows that

$$[c(x-a)+e(y-b)+gz][c(x-a)+e(y-b)+3gz]-h^2z^2=0.$$

Therefore

$$[c(x-a)+e(y-b)+gz]^2+2gz[c(x-a)+e(y-b)+gz]-h^2z^2=0.$$

Therefore

$$c(x-a)+e(y-b)+gz=-gz\pm z\sqrt{(g^2+h^2)}.$$

Hence (141) is not satisfied unless $z=0$.

Now $z=0$ makes $c(x-a)+e(y-b)+gz=0$.

Therefore $x=a$ by (139) and $y=b$ by (140).

This solution corresponds to the binode locus.

It may therefore be excluded.

Hence the factor of the discriminant under discussion corresponds to an envelope locus, touching all the surfaces; it consists of four planes parallel to $z=0$, whose equations are independent of the arbitrary parameters.

SECTION IV. (Arts. 16–25).—CONSIDERATION OF CASES RESERVED FROM THE PREVIOUS SECTION. THE DEGREE OF $f(x, y, z, a, b)$ IN a, b IS NOW THE SECOND, AND THE EQUATIONS $\mathrm{D}f/\mathrm{D}a=0$, $\mathrm{D}f/\mathrm{D}b=0$ ARE INDETERMINATE EQUATIONS FOR THE PARAMETERS AT POINTS ON THE LOCUS OF ULTIMATE INTERSECTIONS.

It was supposed in the previous section that the degree of $f(x, y, z, a, b)$ in a, b was higher than the second; for if the degree were the second, and the analytical condition satisfied which expresses that at a point on the locus of ultimate intersections, two systems of values of the parameters, which satisfy $\mathrm{D}f/\mathrm{D}a=0$, $\mathrm{D}f/\mathrm{D}b=0$, become equal, then this analytical condition requires to be specially interpreted.

For now $\mathrm{D}f/\mathrm{D}a=0$, $\mathrm{D}f/\mathrm{D}b=0$ are two simple equations in a, b. Hence they are either satisfied by one value of a and one of b, or else are indeterminate. But since the condition

$$\frac{D^2f}{Da^2}\frac{D^2f}{Db^2} - \left(\frac{D^2f}{Da\,Db}\right)^2 = 0$$

holds, they are indeterminate.

In this case the discriminant cannot be formed as in the previous section.

There are not two coinciding systems of values of the parameters to consider. It is shown that there is one system which can be determined.

There is also the additional peculiarity that the rationalising factor introduced to make the discriminant of the proper order and weight always vanishes at a point on the locus of ultimate intersections. Hence, on account of it, the equation of an envelope or singular point locus may be expected to enter into the discriminant one or more times. As this number cannot be determined in a general way, it is better to express the equation of the system of surfaces as a quadric function of the parameters, and form the discriminant in the usual way.

Art. 16.—*The Discriminant and its Differential Coefficients as far as the third order.*

Let the equation of the system of surfaces be

$$ua^2 + 2Wab + vb^2 + 2Va + 2Ub + w = 0 \quad . \quad . \quad . \quad . \quad (142).$$

To find the discriminant, solve for a, b, the equations

$$ua + Wb + V = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143).$$

$$Wa + vb + U = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144),$$

obtaining hence

$$\left.\begin{aligned} a &= \frac{WU - vV}{uv - W^2} \\ b &= \frac{WV - uU}{uv - W^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145).$$

Now substitute these values of a, b in the left-hand side of (142).

The result is

$$\begin{vmatrix} u & W & V \\ W & v & U \\ V & U & w \end{vmatrix} \div \begin{vmatrix} u & W \\ W & v \end{vmatrix}.$$

The rationalising factor is $\begin{vmatrix} u & W \\ W & v \end{vmatrix}$.

Hence the discriminant

$$\Delta = \begin{vmatrix} u & W & V \\ W & v & U \\ V & U & w \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (146).$$

Therefore

$$\Delta_x = \begin{vmatrix} u_x & W_x & V_x \\ W & v & U \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W & v & U \\ V_x & U_x & w_x \end{vmatrix} \quad . \qquad (147)$$

$$\Delta_{xx} = \begin{vmatrix} u_{xx} & W_{xx} & V_{xx} \\ W & v & U \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_{xx} & v_{xx} & U_{xx} \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W & v & U \\ V_{xx} & U_{xx} & w_{xx} \end{vmatrix}$$

$$+ 2 \begin{vmatrix} u & W & V \\ W_x & v_x & U_x \\ V_x & U_x & w_x \end{vmatrix} + 2 \begin{vmatrix} u_x & W_x & V_x \\ W & v & U \\ V_x & U_x & w_x \end{vmatrix} + 2 \begin{vmatrix} u_x & W_x & V_x \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} . \qquad (148).$$

$$\Delta_{xy} = \begin{vmatrix} u_{xy} & W_{xy} & V_{xy} \\ W & v & U \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_{xy} & v_{xy} & U_{xy} \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W & v & U \\ V_{xy} & U_{xy} & w_{xy} \end{vmatrix}$$

$$+ \left\{ \begin{vmatrix} u_y & W_y & V_y \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_y & W_y & V_y \\ W & v & U \\ V_x & U_x & w_x \end{vmatrix} \right\}$$

$$+ \left\{ \begin{vmatrix} u_x & W_x & V_x \\ W_y & v_y & U_y \\ V & U & w \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_y & v_y & U_y \\ V_x & U_x & w_x \end{vmatrix} \right\}$$

$$+ \left\{ \begin{vmatrix} u_x & W_x & V_x \\ W & v & U \\ V_y & U_y & w_y \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_x & v_x & U_x \\ V_y & U_y & w_y \end{vmatrix} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (149).$$

$$\Delta_{xxx} = \begin{vmatrix} u_{xxx} & W_{xxx} & V_{xxx} \\ W & v & U \\ V & U & w \end{vmatrix} + \text{two similar terms}$$

$$+ 3 \left\{ \begin{vmatrix} u_{xx} & W_{xx} & V_{xx} \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_{xx} & W_{xx} & V_{xx} \\ W & v & U \\ V_x & U_x & w_x \end{vmatrix} \right\} + \text{two similar terms}$$

$$+ 6 \begin{vmatrix} u_x & W_x & V_x \\ W_x & v_x & U_x \\ V_x & U_x & w_x \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (150).$$

$$\Delta_{xxy} = \begin{vmatrix} u_{xxy} & W_{xxy} & V_{xxy} \\ W & v & U \\ V & U & w \end{vmatrix} + \text{two similar terms}$$

$$+ \left\{ \begin{vmatrix} u_{xx} & W_{xx} & V_{xx} \\ W_y & v_y & U_y \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_{xx} & W_{xx} & V_{xx} \\ W & v & U \\ V_y & U_y & w_y \end{vmatrix} \right\} + \text{two similar terms}$$

$$+ 2 \left\{ \begin{vmatrix} u_{xy} & W_{xy} & V_{xy} \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_{xy} & W_{xy} & V_{xy} \\ W & v & U \\ V_x & U_x & w_x \end{vmatrix} \right\} + \text{two similar terms}$$

$$+ 2 \begin{vmatrix} u_y & W_y & V_y \\ W_x & v_x & U_x \\ V_x & U_x & w_x \end{vmatrix} + 2 \begin{vmatrix} u_x & W_x & V_x \\ W_y & v_y & U_y \\ V_x & U_x & w_x \end{vmatrix} + 2 \begin{vmatrix} u_x & W_x & V_x \\ W_x & v_x & U_x \\ V_y & U_y & w_y \end{vmatrix} \quad . \quad . \quad (151).$$

$$\Delta_{xyz} = \begin{vmatrix} u_{xyz} & W_{xyz} & V_{xyz} \\ W & v & U \\ V & U & w \end{vmatrix} + \text{two similar terms}$$

$$+ \begin{vmatrix} u_y & W_y & V_y \\ W_x & v_x & U_x \\ V_z & U_z & w_z \end{vmatrix} + \begin{vmatrix} u_y & W_y & V_y \\ W_z & v_z & U_z \\ V_x & U_x & w_x \end{vmatrix} + \begin{vmatrix} u_z & W_z & V_z \\ W_x & v_x & U_x \\ V_y & U_y & w_y \end{vmatrix}$$

$$+ \begin{vmatrix} u_z & W_z & V_z \\ W_y & v_y & U_y \\ V_x & U_x & w_x \end{vmatrix} + \begin{vmatrix} u_x & W_x & V_x \\ W_y & v_y & U_y \\ V_z & U_z & w_z \end{vmatrix} + \begin{vmatrix} u_x & W_x & V_x \\ W_z & v_z & U_z \\ V_y & U_y & w_y \end{vmatrix}$$

$$+ \left\{ \begin{matrix} + \begin{vmatrix} u_{xy} & W_{xy} & V_{xy} \\ W_z & v_z & U_z \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_{xy} & W_{xy} & V_{xy} \\ W & v & U \\ V_z & U_z & w_z \end{vmatrix} + \begin{vmatrix} u_z & W_z & V_z \\ W_{xy} & v_{xy} & U_{xy} \\ V & U & w \end{vmatrix} \\ + \begin{vmatrix} u & W & V \\ W_{xy} & v_{xy} & U_{xy} \\ V_z & U_z & w_z \end{vmatrix} + \begin{vmatrix} u_z & W_z & V_z \\ W & v & U \\ V_{xy} & U_{xy} & w_{xy} \end{vmatrix} + \begin{vmatrix} u & W & V \\ W_z & v_z & U_z \\ V_{xy} & U_{xy} & w_{xy} \end{vmatrix} \end{matrix} \right\}$$

+ six determinants, which can be obtained from the last six by interchanging x and z

+ six determinants, which can be obtained from the same six determinants by interchanging y and z . . . . (152).

Art. 17.—*The relations which hold good at points on the Locus of Ultimate Intersections.*

(A.) The analytical condition (76) which holds, becomes with the notation of this section

$$uv - W^2 = 0 \qquad \ldots\ldots\ldots\ldots \qquad (153).$$

Hence the values of a, b given in (145) are either infinite or indeterminate.

Excluding the cases where they are infinite, it is necessary to have

$$\left.\begin{aligned} WU - vV &= 0 \\ WV - uU &= 0 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \qquad (154).$$

Again, by substituting from (143) and (144) in (142), it follows that

$$Va + Ub + w = 0 \qquad \ldots\ldots\ldots\ldots \qquad (155).$$

Solving (144) and (155) for a, b, it follows that

$$\left.\begin{aligned} a &= \frac{vw - U^2}{WU - Vv} \\ b &= \frac{UV - Ww}{WU - Vv} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \qquad (156).$$

Hence by (154) these values will be infinite unless

$$\left.\begin{aligned} vw - U^2 &= 0 \\ UV - Ww &= 0 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \qquad (157).$$

Hence by (153), (154), (157)

$$\left.\begin{aligned} & u : W : V \\ = {} & W : v : U \\ = {} & V : U : w \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \qquad (158).$$

Now if $P = 0$, $Q = 0$ represent any two of the five equations (153), (154), (157), then these are satisfied at every point of the locus of ultimate intersections.

Let ξ, η, ζ and $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ be neighbouring points on the locus of ultimate intersections.

Then

$$\frac{\partial P}{\partial \xi}\,\delta\xi + \frac{\partial P}{\partial \eta}\,\delta\eta + \frac{\partial P}{\partial \zeta}\,\delta\zeta = 0,$$

$$\frac{\partial Q}{\partial \xi}\,\delta\xi + \frac{\partial Q}{\partial \eta}\,\delta\eta + \frac{\partial Q}{\partial \zeta}\,\delta\zeta = 0$$

Now the only relation between $\delta\xi$, $\delta\eta$, $\delta\zeta$ is that which expresses that the point $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ is on the tangent plane to the locus of intersections at ξ, η, ζ.

Hence

$$\frac{\partial P}{\partial\xi}\Big/\frac{\partial Q}{\partial\xi} = \frac{\partial P}{\partial\eta}\Big/\frac{\partial Q}{\partial\eta} = \frac{\partial P}{\partial\zeta}\Big/\frac{\partial Q}{\partial\zeta} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (159).$$

It is now possible to determine the values of a, b which are indeterminate as given by (145).

For, representing the value of a in (145) by the equation

$$a = P/Q,$$

it follows that the true value of a is the limit to which the expression

$$\frac{P + \frac{\partial P}{\partial\xi}\delta\xi + \frac{\partial P}{\partial\eta}\delta\eta + \frac{\partial P}{\partial\zeta}\delta\zeta}{Q + \frac{\partial Q}{\partial\xi}\delta\xi + \frac{\partial Q}{\partial\eta}\delta\eta + \frac{\partial Q}{\partial\zeta}\delta\zeta}$$

approaches, when $\delta\xi$, $\delta\eta$, $\delta\zeta$ vanish.

Now $P = 0$, $Q = 0$; hence by (159) the true value of a is equal to any one of the three ratios in (159).

Besides the values of a, b given in (145), (156), other forms may be obtained from equations (143), (155).

Putting these together

$$\left.\begin{aligned} a &= \frac{WU - vV}{uv - W^2} = \frac{vw - U^2}{WU - Vv} = \frac{UV - Ww}{WV - Uu} \\ b &= \frac{WV - uU}{uv - W^2} = \frac{UV - Ww}{WU - Vv} = \frac{uw - V^2}{WV - Uu} \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (160).$$

All these values are indeterminate.

Now although the value of each of these fractions can be found by differentiating numerator and denominator with regard to any the same variable, yet they will not all lead to the true value of a, b, because the true values of a, b are found by solving the equations $ua + Wb + V = 0$, $Wa + vb + U = 0$, and finding what the values approach to as the coordinates approximate to the coordinates of a point on the locus of ultimate intersections. Now at points not on the locus of ultimate intersections, the values of a, b do not satisfy $Va + Ub + w = 0$. Hence the true values of a, b cannot in general be found by solving this last and either of the preceding equations, and then finding the values to which these approach as the coordinates approximate to the coordinates of a point on the locus of ultimate intersections.

The true values are obtainable only from the solutions (145).

(B.) If there be a conic node locus, then, besides equation (142), the following must be satisfied—

$$u_x a^2 + 2W_x ab + v_x b^2 + 2V_x a + 2U_x b + w_x = 0 \quad . \quad . \quad . \quad . \quad (161),$$

$$u_y a^2 + 2W_y ab + v_y b^2 + 2V_y a + 2U_y b + w_y = 0 \quad . \quad . \quad . \quad . \quad (162),$$

$$u_z a^2 + 2W_z ab + v_z b^2 + 2V_z a + 2U_z b + w_z = 0 \quad . \quad . \quad . \quad . \quad (163).$$

(C.) If there be a biplanar node locus, then the equations (126) are satisfied as well as the preceding.

In this case these are

$$\left.\begin{aligned} & 2(u_x a + W_x b + V_x) : 2(u_y a + W_y b + V_y) : 2(u_z a + W_z b + V_z) : 2u : 2W \\ = {} & 2(W_x a + v_x b + U_x) : 2(W_y a + v_y b + U_y) : 2(W_z a + v_z b + U_z) : 2W : 2v \end{aligned}\right\} . \quad (164).$$

From these, the following may be deduced.

Introducing a quantity λ, such that

$$u_x a + W_x b + V_x = \lambda u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (165),$$

it follows by (164) that

$$W_x a + v_x b + U_x = \lambda W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (166).$$

From (165) and (166)

$$u_x a^2 + 2\,W_x ab + v_x b^2 + aV_x + bU_x = \lambda\,(au + bW).$$

Hence by (143) and (161)

$$V_x a + U_x b + w_x = \lambda V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (167).$$

Similarly, quantities μ, ν exist, such that

$$\left.\begin{aligned} u_y a + W_y b + V_y &= \mu u \\ W_y a + v_y b + U_y &= \mu W \\ V_y a + U_y b + w_y &= \mu V \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168),$$

and

$$\left.\begin{aligned} u_z a + W_z b + V_z &= \nu u \\ W_z a + v_z b + U_z &= \nu W \\ V_z a + U_z b + w_z &= \nu V \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169).$$

Consider now the equations (143), (144), (155), (165), (166), (167); multiply (143) by $- v_x$, (144) by W_x, (165) by $- v$, (166) by W, and add.

Therefore

$$a\,(2WW_x - uv_x - vu_x) - Vv_x + UW_x - vV_x + WU_x = \lambda\,(W^2 - uv).$$

Hence, at points on the biplanar node locus,

$$a\frac{\partial}{\partial x}(uv - \mathrm{W}^2) = \frac{\partial}{\partial x}(\mathrm{UW} - \mathrm{V}v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170).$$

Again, multiplying the same equations by W_x, $-u_x$, W, $-u$ in order and adding, it follows that

$$b\frac{\partial}{\partial x}(uv - \mathrm{W}^2) = \frac{\partial}{\partial x}(\mathrm{VW} - \mathrm{U}u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171).$$

Again, multiplying (144), (155), (166), (167) by $-\mathrm{U}_x$, v_x, $-\mathrm{U}$, v in order and adding, it follows that

$$a(-\mathrm{WU}_x + \mathrm{V}v_x - \mathrm{UW}_x + v\mathrm{V}_x) + (vw_x + wv_x - 2\mathrm{UU}_x) = \lambda(v\mathrm{V} - \mathrm{UW}).$$

Hence

$$a\frac{\partial}{\partial x}(\mathrm{UW} - v\mathrm{V}) = \frac{\partial}{\partial x}(vw - \mathrm{U}^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172).$$

Again, multiplying the same equations by V_x, $-\mathrm{W}_x$, V, $-\mathrm{W}$ in order and adding, it follows that

$$b\frac{\partial}{\partial x}(\mathrm{UW} - v\mathrm{V}) = \frac{\partial}{\partial x}(\mathrm{UV} - w\mathrm{W}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (173).$$

Again, multiplying (143), (155), (165), (167) by U_x, $-\mathrm{W}_x$, U, $-\mathrm{W}$ in order and adding, it follows that

$$a(u\mathrm{U}_x - \mathrm{VW}_x + \mathrm{U}u_x - \mathrm{WV}_x) + (\mathrm{VU}_x - w\mathrm{W}_x + \mathrm{UV}_x - \mathrm{W}w_x) = \lambda(u\mathrm{U} - \mathrm{VW}).$$

Therefore

$$a\frac{\partial}{\partial x}(\mathrm{VW} - u\mathrm{U}) = \frac{\partial}{\partial x}(\mathrm{UV} - w\mathrm{W}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (174).$$

Again, multiplying the same equations by V_x, $-u_x$, V, $-u$ in order and adding, it follows that

$$b\frac{\partial}{\partial x}(\mathrm{VW} - u\mathrm{U}) = \frac{\partial}{\partial x}(uw - \mathrm{V}^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (175).$$

Further comparing the three equations (168) or the three equations (169) with equations (165)–(167), it is evident that it is possible in any one of the equations (170)–(175) to replace x by either y or z.

It will be noticed that in the case of the biplanar node locus, the true values of the parameters may be found from any one of the ratios in (160).

(D.) If there be a uniplanar node locus, then in addition to the results obtained in (B) and (C), it follows by (48) that

$$
\begin{aligned}
& u_{xx}a^2 + \ldots : u_{xy}a^2 + \ldots : u_{xz}a^2 + \ldots : 2(u_x a + \mathrm{W}_x b + \mathrm{V}_x) : 2(\mathrm{W}_x a + v_x b + \mathrm{U}_x) \\
&= u_{xy}a^2 + \ldots : u_{yy}a^2 + \ldots : u_{yz}a^2 + \ldots : 2(u_y a + \mathrm{W}_y b + \mathrm{V}_y) : 2(\mathrm{W}_y a + v_y b + \mathrm{U}_y) \\
&= u_{xz}a^2 + \ldots : u_{yz}a^2 + \ldots : u_{zz}a^2 + \ldots : 2(u_z a + \mathrm{W}_z b + \mathrm{V}_z) : 2(\mathrm{W}_z a + v_z b + \mathrm{U}_z) \\
&= 2(u_x a + \mathrm{W}_x b + \mathrm{V}_x) : 2(u_y a + \mathrm{W}_y b + \mathrm{V}_y) : 2(u_z a + \mathrm{W}_z b + \mathrm{V}_z) : 2u : 2\mathrm{W} \\
&= 2(\mathrm{W}_x a + v_x b + \mathrm{U}_x) : 2(\mathrm{W}_y a + v_y b + \mathrm{U}_y) : 2(\mathrm{W}_z a + v_z b + \mathrm{U}_z) : 2\mathrm{W} : 2v \qquad (176).
\end{aligned}
$$

(E.) (i.) The following equations will be useful in the case of biplanar and uniplanar node loci :—

$$
\begin{vmatrix} p & \mathrm{R} & \mathrm{Q} \\ \mathrm{W}_x & v_x & \mathrm{U}_x \\ \mathrm{V} & \mathrm{U} & w \end{vmatrix}
+ \begin{vmatrix} p & \mathrm{R} & \mathrm{Q} \\ \mathrm{W} & v & \mathrm{U} \\ \mathrm{V}_x & \mathrm{U}_x & w_x \end{vmatrix}
+ \begin{vmatrix} u_x & \mathrm{W}_x & \mathrm{V}_x \\ \mathrm{R} & q & \mathrm{P} \\ \mathrm{V} & \mathrm{U} & w \end{vmatrix}
$$

$$
+ \begin{vmatrix} u & \mathrm{W} & \mathrm{V} \\ \mathrm{R} & q & \mathrm{P} \\ \mathrm{V}_x & \mathrm{U}_x & w_x \end{vmatrix}
+ \begin{vmatrix} u_x & \mathrm{W}_x & \mathrm{V}_x \\ \mathrm{W} & v & \mathrm{U} \\ \mathrm{Q} & \mathrm{P} & r \end{vmatrix}
+ \begin{vmatrix} u & \mathrm{W} & \mathrm{V} \\ \mathrm{W}_x & v_x & \mathrm{U}_x \\ \mathrm{Q} & \mathrm{P} & r \end{vmatrix}
$$

$$
= (pa^2 + 2\mathrm{R}ab + qb^2 + 2\mathrm{Q}a + 2\mathrm{P}b + r)\frac{\partial}{\partial x}(uv - \mathrm{W}^2) \; . \quad . \quad . \qquad (177).
$$

For the first and second determinants

$$
= p\frac{\partial}{\partial x}(vw - \mathrm{U}^2) + \mathrm{R}\frac{\partial}{\partial x}(\mathrm{UV} - \mathrm{W}w) + \mathrm{Q}\frac{\partial}{\partial x}(\mathrm{WU} - \mathrm{V}v).
$$

The third and fourth determinants

$$
= \mathrm{R}\frac{\partial}{\partial x}(\mathrm{UV} - \mathrm{W}w) + q\frac{\partial}{\partial x}(uw - \mathrm{V}^2) + \mathrm{P}\frac{\partial}{\partial x}(\mathrm{WV} - \mathrm{U}u).
$$

The fifth and sixth determinants

$$
= \mathrm{Q}\frac{\partial}{\partial x}(\mathrm{UW} - \mathrm{V}v) + \mathrm{P}\frac{\partial}{\partial x}(\mathrm{VW} - \mathrm{U}u) + r\frac{\partial}{\partial x}(uv - \mathrm{W}^2).
$$

Hence in the case of biplanar and uniplanar node loci, these six determinants are by (170)-(175)

$$= (pa^2 + 2\mathrm{R}ab + qb^2 + 2\mathrm{Q}a + 2\mathrm{P}b + r)\frac{\partial}{\partial x}(uv - \mathrm{W}^2).$$

(ii.)

$$\begin{vmatrix} u_x & \mathrm{W}_x & \mathrm{V}_x \\ \mathrm{W}_x & v_x & \mathrm{U}_x \\ \mathrm{V}_x & \mathrm{U}_x & w_x \end{vmatrix} = -\lambda^2 u\frac{\partial}{\partial x}(uv - \mathrm{W}^2) \quad . \quad . \quad . \quad . \qquad (178)$$

For

$$\begin{vmatrix} u_x & \mathrm{W}_x & \mathrm{V}_x \\ \mathrm{W}_x & v_x & \mathrm{U}_x \\ \mathrm{V}_x & \mathrm{U}_x & w_x \end{vmatrix} = \begin{vmatrix} u_x & \mathrm{W}_x & au_x + b\mathrm{W}_x + \mathrm{V}_x \\ \mathrm{W}_x & v_x & a\mathrm{W}_x + bv_x + \mathrm{U}_x \\ \mathrm{V}_x & \mathrm{U}_x & a\mathrm{V}_x + b\mathrm{U}_x + w_x \end{vmatrix}$$

$$= \lambda\begin{vmatrix} u_x & \mathrm{W}_x & u \\ \mathrm{W}_x & v_x & \mathrm{W} \\ \mathrm{V}_x & \mathrm{U}_x & \mathrm{V} \end{vmatrix}$$

$$= \lambda\begin{vmatrix} u_x & \mathrm{W}_x & u \\ \mathrm{W}_x & v_x & \mathrm{W} \\ au_x + b\mathrm{W}_x + \mathrm{V}_x & a\mathrm{W}_x + bv_x + \mathrm{U}_x & au + b\mathrm{W} + \mathrm{V} \end{vmatrix}$$

$$= \lambda^2\begin{vmatrix} u_x & \mathrm{W}_x & u \\ \mathrm{W}_x & v_x & \mathrm{W} \\ u & \mathrm{W} & 0 \end{vmatrix}$$

$$= \lambda^2(-u_x\mathrm{W}^2 + 2u\mathrm{W}\mathrm{W}_x - u^2v_x)$$

$$= \lambda^2(-u_xuv + 2u\mathrm{W}\mathrm{W}_x - u^2v_x)$$

$$= -\lambda^2 u\frac{\partial}{\partial x}(uv - \mathrm{W}^2).$$

(iii.)

$$\begin{vmatrix} u_y & \mathrm{W}_y & \mathrm{V}_y \\ \mathrm{W}_x & v_x & \mathrm{U}_x \\ \mathrm{V}_z & \mathrm{U}_z & w_z \end{vmatrix} + \begin{vmatrix} u_y & \mathrm{W}_y & \mathrm{V}_y \\ \mathrm{W}_z & v_z & \mathrm{U}_z \\ \mathrm{V}_x & \mathrm{U}_x & w_x \end{vmatrix} + \begin{vmatrix} u_z & \mathrm{W}_z & \mathrm{V}_z \\ \mathrm{W}_x & v_x & \mathrm{U}_x \\ \mathrm{V}_y & \mathrm{U}_y & w_y \end{vmatrix}$$

$$+ \begin{vmatrix} u_z & \mathrm{W}_z & \mathrm{V}_z \\ \mathrm{W}_y & v_y & \mathrm{U}_y \\ \mathrm{V}_x & \mathrm{U}_x & w_x \end{vmatrix} + \begin{vmatrix} u_x & \mathrm{W}_x & \mathrm{V}_x \\ \mathrm{W}_y & v_y & \mathrm{U}_y \\ \mathrm{V}_z & \mathrm{U}_z & w_z \end{vmatrix} + \begin{vmatrix} u_x & \mathrm{W}_x & \mathrm{V}_x \\ \mathrm{W}_z & v_z & \mathrm{U}_z \\ \mathrm{V}_y & \mathrm{U}_y & w_y \end{vmatrix}$$

$$= -2\mu\nu u\frac{\partial}{\partial x}(uv - \mathrm{W}^2) - 2\nu\lambda u\frac{\partial}{\partial y}(uv - \mathrm{W}^2) - 2\lambda\mu u\frac{\partial}{\partial z}(uv - \mathrm{W}^2) \quad . \ (179).$$

For multiplying the first column in each of these determinants by a, the second by b, and adding to the last they become

$$\begin{vmatrix} u_y & W_y & \mu u \\ W_x & v_x & \lambda W \\ V_z & U_z & \nu V \end{vmatrix} + \begin{vmatrix} u_y & W_y & \mu u \\ W_z & v_z & \nu W \\ V_x & U_x & \lambda V \end{vmatrix} + \begin{vmatrix} u_z & W_z & \nu u \\ W_x & v_x & \lambda W \\ V_y & U_y & \mu V \end{vmatrix}$$

$$+ \begin{vmatrix} u_z & W_z & \nu u \\ W_y & v_y & \mu W \\ V_x & U_x & \lambda V \end{vmatrix} + \begin{vmatrix} u_x & W_x & \lambda u \\ W_y & v_y & \mu W \\ V_z & U_z & \nu V \end{vmatrix} + \begin{vmatrix} u_x & W_x & \lambda u \\ W_z & v_z & \nu W \\ V_y & U_y & \mu V \end{vmatrix}$$

The coefficient of λ is

$$\begin{vmatrix} u_y & W_y & 0 \\ W_z & v_z & W \\ V_z & U_z & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_z & v_z & W \\ V_y & U_y & 0 \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_y & v_y & 0 \\ V_z & U_z & V \end{vmatrix}$$

The coefficient of μ is

$$\begin{vmatrix} u_x & W_x & u \\ W_x & v_x & W \\ V_z & U_z & 0 \end{vmatrix} + \begin{vmatrix} u_x & W_x & u \\ W_z & v_z & 0 \\ V_x & U_x & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & 0 \\ W_x & v_x & W \\ V_x & U_x & V \end{vmatrix}$$

The coefficient of ν is

$$\begin{vmatrix} u_y & W_y & u \\ W_x & v_x & 0 \\ V_y & U_y & V \end{vmatrix} + \begin{vmatrix} u_y & W_y & u \\ W_y & v_y & W \\ V_x & U_x & 0 \end{vmatrix} + \begin{vmatrix} u_x & W_x & 0 \\ W_y & v_y & W \\ V_y & U_y & V \end{vmatrix}$$

The coefficient of μ can be obtained from that of λ by changing z into x and y into z.

The coefficient of ν can be obtained from that of λ by changing y into x and z into y.

Hence it is sufficient to calculate the coefficient of λ.

The coefficient of λ, viz. :—

$$\begin{vmatrix} u_y & W_y & 0 \\ W_z & v_z & W \\ V_z & U_z & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_y & v_y & 0 \\ V_z & U_z & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_z & v_z & W \\ V_y & U_y & 0 \end{vmatrix}$$

$$= \begin{vmatrix} u_y & W_y & 0 \\ W_z & v_z & W \\ V_z & U_z & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_y & v_y & 0 \\ V_z & U_z & V \end{vmatrix} + \begin{vmatrix} u_z & W_z & u \\ W_z & v_z & W \\ \mu u - au_y - bW_y & \mu W - aW_y - bv_y & 0 \end{vmatrix}$$

$$\begin{aligned} = \mu & \begin{vmatrix} u_z & W_z & u \\ W_z & v_z & W \\ u & W & 0 \end{vmatrix} \\ & + u_y \{Vv_z - WU_z - a(WW_z - uv_z)\} \\ & + W_y \{WV_z - 2VW_z + uU_z - b(WW_z - uv_z) - a(uW_z - Wu_z)\} \\ & + v_y \{Vu_z - uV_z - b(uW_z - Wu_z)\} \end{aligned}$$

$$\begin{aligned} = & - \mu u \frac{\partial}{\partial z}(uv - W^2) \\ & + u_y \{v_z(au + V) - W(aW_z + U_z)\} \\ & + W_y \{W(au_z + V_z) - W_z(au + bW + 2V) + u(bv_z + U_z)\} \\ & + v_y \{u_z(bW + V) - u(bW_z + V_z)\} \end{aligned}$$

$$\begin{aligned} = & - \mu u \frac{\partial}{\partial z}(uv - W^2) \\ & + u_y(-\nu W^2) + W_y(2\nu uW) + v_y(-\nu u^2) \end{aligned}$$

$$= - \mu u \frac{\partial}{\partial z}(uv - W^2) - \nu u \frac{\partial}{\partial y}(uv - W^2).$$

Hence the coefficient of μ obtained from this by changing z into x, and y into z; and, therefore, ν into λ, and μ into ν, is

$$- \nu u \frac{\partial}{\partial x}(uv - W^2) - \lambda u \frac{\partial}{\partial z}(uv - W^2).$$

And the coefficient of ν, obtained by changing, in the coefficient of λ, y into x, and z into y, and therefore, μ into λ, and ν into μ, is

$$- \lambda u \frac{\partial}{\partial y}(uv - W^2) - \mu u \frac{\partial}{\partial x}(uv - W^2).$$

From these the equation (179) follows.

Art. 18.—*To prove that under the conditions stated at the head of this Section, every Surface of the System touches the Locus of Ultimate Intersections along a Curve.*

Consider the surface (142), the values of a, b being now supposed to be fixed.

Consider any point ξ, η, ζ on the curve in which the surface (143) meets the locus of ultimate intersections; then, by (158), these coordinates also satisfy the surfaces (144), (155).

Multiplying (143) by a, (144) by b, (155) by 1, and adding, it follows that these coordinates also satisfy (142).

Hence any point on the curve of intersection of (143) with the locus of ultimate intersections lies on (142) and (144) also.

Hence the surfaces represented by the three fundamental equations meet the locus of ultimate intersections in the same curve.

It is necessary to prove that the surface of the system (142) will touch the locus of ultimate intersections along this curve.

Now,

$$\begin{aligned}
&\frac{\mathrm{D}}{\mathrm{D}x}(ua^2 + 2\mathrm{W}ab + vb^2 + 2\mathrm{V}a + 2\mathrm{U}b + w)\\
&= \frac{\mathrm{D}}{\mathrm{D}x}\left[\frac{1}{u}\{(ua + \mathrm{W}b + \mathrm{V})^2 + b^2(uv - \mathrm{W}^2) + 2b(\mathrm{U}u - \mathrm{VW}) + (uw - \mathrm{V}^2)\}\right]\\
&= -\frac{u_x}{u^2}\{(ua + \mathrm{W}b + \mathrm{V})^2 + b^2(uv - \mathrm{W}^2) + 2b(\mathrm{U}u - \mathrm{VW}) + (uw - \mathrm{V}^2)\}\\
&\quad + \frac{1}{u}\left\{2(ua + \mathrm{W}b + \mathrm{V})\frac{\mathrm{D}}{\mathrm{D}x}(ua + \mathrm{W}b + \mathrm{V})\right.\\
&\qquad\qquad \left. + b^2\frac{\mathrm{D}}{\mathrm{D}x}(uv - \mathrm{W}^2) + 2b\frac{\mathrm{D}}{\mathrm{D}x}(\mathrm{U}u - \mathrm{VW}) + \frac{\mathrm{D}}{\mathrm{D}x}(uw - \mathrm{V}^2)\right\}.
\end{aligned}$$

Hence, at a point on the locus of ultimate intersections, this is equal to

$$\frac{1}{u}\left\{b^2\frac{\partial}{\partial x}(uv - \mathrm{W}^2) + 2b\frac{\partial}{\partial x}(\mathrm{U}u - \mathrm{VW}) + \frac{\partial}{\partial x}(uw - \mathrm{V}^2)\right\}.$$

Hence the tangent plane to the surface at the point x, y, z is

$$\begin{aligned}
&(\mathrm{X} - x)\left\{b^2\frac{\partial}{\partial x}(uv - \mathrm{W}^2) + 2b\frac{\partial}{\partial x}(\mathrm{U}u - \mathrm{VW}) + \frac{\partial}{\partial x}(uw - \mathrm{V}^2)\right\}\\
&+ (\mathrm{Y} - y)\left\{b^2\frac{\partial}{\partial y}(uv - \mathrm{W}^2) + 2b\frac{\partial}{\partial y}(\mathrm{U}u - \mathrm{VW}) + \frac{\partial}{\partial y}(uw - \mathrm{V}^2)\right\}\\
&+ (\mathrm{Z} - z)\left\{b^2\frac{\partial}{\partial z}(uv - \mathrm{W}^2) + 2b\frac{\partial}{\partial z}(\mathrm{U}u - \mathrm{VW}) + \frac{\partial}{\partial z}(uw - \mathrm{V}^2)\right\} = 0.
\end{aligned}$$

Now this, by (159), reduces to

$$(\mathrm{X}-x)\frac{\partial}{\partial x}(uv-\mathrm{W}^2)+(\mathrm{Y}-y)\frac{\partial}{\partial y}(uv-\mathrm{W}^2)+(\mathrm{Z}-z)\frac{\partial}{\partial z}(uv-\mathrm{W}^2)=0,$$

which is the equation of the tangent plane to the locus of ultimate intersections, since $uv-\mathrm{W}^2=0$ at every point of the locus of ultimate intersections.

Hence each surface of the system touches the locus of ultimate intersections along a curve.

Art. 19.—*To prove that under the conditions stated at the head of this Section, there are in general at every point of the Locus of Ultimate Intersections two Conic Nodes; and if* $\mathrm{C}=0$ *be the equation of the Locus of these Conic Nodes,* Δ *contains* C^2 *as a factor.*

(A.) To prove that there are in general two conic nodes it is necessary to show that there are in general two distinct sets of values of a, b, which satisfy (142), (161), (162), (163).

These will be satisfied if (143), (161), (162), (163) be satisfied.

Eliminating b from (143) and (161) the result is

$$a^2(\mathrm{W}^2u_x-2u\mathrm{W}\mathrm{W}_x+v_xu^2)+2a(u\mathrm{V}v_x-\mathrm{V}\mathrm{W}\mathrm{W}_x+\mathrm{W}^2\mathrm{V}_x-\mathrm{W}u\mathrm{U}_x)$$
$$+(\mathrm{V}^2v_x-2\mathrm{W}\mathrm{V}\mathrm{U}_x+\mathrm{W}^2w_x)=0.$$

Hence by (153), (154), (157), after division by u, it follows that

$$a^2\frac{\partial}{\partial x}(uv-\mathrm{W}^2)+2a\frac{\partial}{\partial x}(\mathrm{V}v-\mathrm{U}\mathrm{W})+\frac{\partial}{\partial x}(vw-\mathrm{U}^2)=0 \quad . \quad . \quad (180).^*$$

And in like manner by eliminating a between the above equations

$$b^2\frac{\partial}{\partial x}(uv-\mathrm{W}^2)+2b\frac{\partial}{\partial x}(\mathrm{U}u-\mathrm{V}\mathrm{W})+\frac{\partial}{\partial x}(uw-\mathrm{V}^2)=0 \quad . \quad . \quad (181).^\dagger$$

Further, by means of (159), it is possible in these equations to change x into y or into z.

These equations will be called the parametric quadratics.

Hence choosing a and b to satisfy (143) and (161), they will also satisfy (143) and (162), and (143) and (163).

Hence it is possible in general to find two distinct systems of values of a and b which satisfy (142), (161), (162) and (163), at points on the locus of ultimate intersections.

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* The mean of the values of a satisfying (180) is the value of a given by (170).

† The mean of the values of b satisfying (181) is the value of b given by (171).

Hence there are in general two conic nodes at every point of the locus of ultimate intersections.

(B.) It follows from (146) and (147) by means of (158) that Δ, $\partial\Delta/\partial x$ both vanish at points on the locus of ultimate intersections.

By symmetry $\partial\Delta/\partial y$, $\partial\Delta/\partial z$ also vanish.

Hence Δ contains C^2 as a factor.

Example 11.—*Locus of two Conic Nodes.*

Let the surfaces be

$$(\alpha\zeta + \delta^2x^2)(x-a)^2 + 2(\beta\zeta + \delta\epsilon xy)(x-a)(y-b) + (\gamma\zeta + \epsilon^2y^2)(y-b)^2 \\ + 2g\zeta(x-a) + 2h\zeta(y-b) + k\zeta^n = 0,$$

where $\zeta = z - cx - dy$; and α, β, γ, δ, ϵ, c, d, g, h, k are fixed constants; a, b the arbitrary parameters; $n = 1$ or 2.

(A.) *The Discriminant.*

This can be formed by solving the equations

$$(\alpha\zeta + \delta^2x^2)(x-a) + (\beta\zeta + \delta\epsilon xy)(y-b) + g\zeta = 0,$$
$$(\beta\zeta + \delta\epsilon xy)(x-a) + (\gamma\zeta + \epsilon^2y^2)(y-b) + h\zeta = 0,$$

for a, b; and substituting in

$$g\zeta(x-a) + h\zeta(y-b) + k\zeta^n.$$

The values of a, b are (after removing the factor ζ which makes them indeterminate) given by

$$x - a = \frac{(h\beta - g\gamma)\zeta + \epsilon y(h\delta x - g\epsilon y)}{(\alpha\gamma - \beta^2)\zeta + (\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)},$$

$$y - b = \frac{(g\beta - h\alpha)\zeta - \delta x(h\delta x - g\epsilon y)}{(\alpha\gamma - \beta^2)\zeta + (\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)}.$$

Substituting these values, and multiplying by the rationalising factor

$$uv - w^2 = \zeta[(\alpha\gamma - \beta^2)\zeta + (\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)]$$

the result is

$$k(\alpha\gamma - \beta^2)\zeta^{n+2} + k(\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)\zeta^{n+1} \\ - (\alpha h^2 - 2\beta gh + \gamma g^2)\zeta^3 - (h\delta x - g\epsilon y)^2\zeta^2.$$

This might also have been obtained from the form (146).

(B.) *The Locus of two Conic Nodes is* $\zeta = 0$, *and if* $n = 2$ *there is also a Curve Locus of Conic Nodes.*

The singular points are determined by finding solutions of $f = 0$, $\mathrm{D}f/\mathrm{D}x = 0$, $\mathrm{D}f/\mathrm{D}y = 0$, $\mathrm{D}f/\mathrm{D}z = 0$.

But since $\mathrm{D}f/\mathrm{D}\zeta = \mathrm{D}f/\mathrm{D}z$, the equations

$$f = 0,\ \mathrm{D}f/\mathrm{D}x = 0,\ \mathrm{D}f/\mathrm{D}y = 0,\ \mathrm{D}f/\mathrm{D}\zeta = 0$$

may be used instead, where x, y, ζ are now the independent variables, so that the meaning of the symbol of differentiation D is changed.

The equations to be satisfied are

$$(\alpha\zeta + \delta^2x^2)(x-a)^2 + 2(\beta\zeta + \delta\epsilon xy)(x-a)(y-b) + (\gamma\zeta + \epsilon^2y^2)(y-b)^2 \\ + 2g\zeta(x-a) + 2h\zeta(y-b) + k\zeta^n = 0,$$

$$[\delta x(x-a) + \epsilon y(y-b)]\,\delta(2x-a) + \zeta[\alpha(x-a) + \beta(y-b) + g] = 0,$$

$$[\delta x(x-a) + \epsilon y(y-b)]\,\epsilon(2y-b) + \zeta[\beta(x-a) + \gamma(y-b) + h] = 0,$$

$$\alpha(x-a)^2 + 2\beta(x-a)(y-b) + \gamma(y-b)^2 + 2g(x-a) + 2h(y-b) + nk\zeta^{n-1} = 0.$$

From these it follows that

$$[\delta x(x-a) + \epsilon y(y-b)]^2 + k(1-n)\zeta^n = 0.$$

(i.) One method of solving the above equations is to take

$$\zeta = 0,$$

$$\delta x(x-a) + \epsilon y(y-b) = 0,$$

$$\alpha(x-a)^2 + 2\beta(x-a)(y-b) + \gamma(y-b)^2 + 2g(x-a) + 2h(y-b) + nk\zeta^{n-1} = 0.$$

Hence whether $n = 1$ or 2, there are two values of b, and two corresponding values of a. Hence there are two conic nodes. Hence $\zeta = 0$ is a locus of two conic nodes.

(ii.) Another method of solving the equations is to take

$$\frac{\delta(2x-a)}{\epsilon(2y-b)} = \frac{\alpha(x-a) + \beta(y-b) + g}{\beta(x-a) + \gamma(y-b) + h},$$

$$[\delta x(x-a) + \epsilon y(y-b)]\,\delta(2x-a) + \zeta[\alpha(x-a) + \beta(y-b) + g] = 0,$$

$$\alpha(x-a)^2 + 2\beta(x-a)(y-b) + \gamma(y-b)^2 + 2g(x-a) + 2h(y-b) + nk\zeta^{n-1} = 0,$$

$$[\delta x(x-a) + \epsilon y(y-b)]^2 + k(1-n)\zeta^n = 0.$$

If $n = 1$, then $\delta x(x - a) + \epsilon y(y - b) = 0$.

Hence $\zeta = 0$, and this is the same solution as in (i.).

If, however, $n = 2$, there are four equations to be satisfied by x, y, ζ. Eliminating x, y, ζ it is necessary that a certain relation should be satisfied by a, b, in order that the equations may be consistent.

From the above equations (when $n = 2$)

$$\delta x(x - a) + \epsilon y(y - b) = \sqrt{\kappa}\,\zeta$$
$$\sqrt{\kappa}\,\delta(2x - a) + \alpha(x - a) + \beta(y - b) + g = 0,$$

rejecting a solution $\zeta = 0$.

$$\sqrt{\kappa}\,\epsilon(2y - b) + \beta(x - a) + \gamma(y - b) + h = 0,$$

$$\alpha(x - a)^2 + 2\beta(x - a)(y - b) + \gamma(y - b)^2 + 2g(x - a) + 2h(y - b) + 2\sqrt{\kappa}\,[\delta x(x - a) + \epsilon y(y - b)] = 0.$$

Hence

$$g(x - a) + h(y - b) + \sqrt{\kappa}\,[\delta a(x - a) + \epsilon b(y - b)] = 0.$$

Hence

$$\begin{aligned}
&(x - a)(\alpha + 2\delta\sqrt{\kappa}) + (y - b)\,\beta + (g + a\delta\sqrt{\kappa}) = 0,\\
&(x - a)\,\beta + (y - b)(\gamma + 2\epsilon\sqrt{\kappa}) + (h + b\epsilon\sqrt{\kappa}) = 0,\\
&(x - a)(g + a\delta\sqrt{\kappa}) + (y - b)(h + b\epsilon\sqrt{\kappa}) = 0.
\end{aligned}$$

Hence

$$\begin{vmatrix} \alpha + 2\delta\sqrt{\kappa} & \beta & g + a\delta\sqrt{\kappa} \\ \beta & \gamma + 2\epsilon\sqrt{\kappa} & h + b\epsilon\sqrt{\kappa} \\ g + a\delta\sqrt{\kappa} & h + b\epsilon\sqrt{\kappa} & 0 \end{vmatrix} = 0,$$

i.e.,

$$(g + a\,\delta\sqrt{\kappa})^2(\gamma + 2\epsilon\sqrt{\kappa}) - 2\beta(g + a\,\delta\sqrt{\kappa})(h + b\epsilon\sqrt{\kappa}) + (h + b\epsilon\sqrt{\kappa})^2(\alpha + 2\,\delta\sqrt{\kappa}) = 0.$$

Hence only when this relation holds between a, b, will there be any conic node on the surface which is not also on the locus $\zeta = 0$.

As in the general theory explained in Art. 3, this leads to a curve locus of conic nodes. It need not therefore be further considered.

Hence the only locus of conic nodes that need be considered in the discussion of the discriminant is $\zeta = 0$.

Now, whether $n = 1$ or 2, the lowest power of ζ in the discriminant is ζ^2; hence this factor is accounted for.

(C.) *The Locus*

$$k\zeta^n(\alpha\gamma - \beta^2) + k\zeta^{n-1}(\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)$$
$$- \zeta(\alpha h^2 - 2\beta gh + \gamma g^2) - (h\delta x - g\epsilon y)^2 = 0$$

is an Ordinary Envelope.

The condition $uv - w^2 = 0$,

i.e.,

$$\zeta\,[(\alpha\gamma - \beta^2)\,\zeta + (\alpha\epsilon^2y^2 - 2\beta\,\delta\epsilon xy + \gamma\,\delta^2x^2)] = 0,$$

is not fulfilled at every point of this locus.

To prove that it is an envelope it will be sufficient to show that if x, y, ζ, be chosen so that

$$\begin{aligned}
(\alpha\zeta + \delta^2x^2)(x - a) + (\beta\zeta + \delta\epsilon xy)(y - b) + g\zeta &= 0,\\
(\beta\zeta + \delta\epsilon xy)(x - a) + (\gamma\zeta + \epsilon^2y^2)(y - b) + h\zeta &= 0,\\
g\zeta(x - a) \qquad + h\zeta(y - b) \qquad + k\zeta^n &= 0,
\end{aligned}$$

then the surface

$$k\zeta^n(\alpha\gamma - \beta^2) + k\zeta^{n-1}(\alpha\epsilon^2y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2x^2)$$
$$- \zeta(\alpha h^2 - 2\beta gh + \gamma g^2) - (h\delta x - g\epsilon y)^2 = 0$$

touches the surface

$$(\alpha\zeta + \delta^2x^2)(x - a)^2 + 2(\beta\zeta + \delta\epsilon xy)(x - a)(y - b) + (\gamma\zeta + \epsilon^2y^2)(y - b)^2$$
$$+ 2g\zeta(x - a) + 2h\zeta(y - b) + k\zeta^n = 0.$$

Calling the last two equations $\phi = 0$, $f = 0$ respectively, the conditions for contact may be expressed thus.

The same values of x, y, ζ, must satisfy

$$\phi = 0, \quad f = 0$$

$$\frac{D\phi}{Dx}\Big/\frac{Df}{Dx} = \frac{D\phi}{Dy}\Big/\frac{Df}{Dy} = \frac{D\phi}{D\zeta}\Big/\frac{Df}{D\zeta},$$

where x, y, ζ are the independent variables.

The values chosen for x, y, ζ obviously make $f = 0$.

Also eliminating $x - a$, $y - b$, the result is $\zeta^2\phi = 0$.

Hence the values of x, y, ζ can be chosen so as to make $\phi = 0$.

Next

$$\begin{aligned}
\frac{Df}{Dx} &= 2\delta(x - a)\,[\delta x(x - a) + \epsilon y(y - b)]\\
&\quad + 2\,[(\alpha\zeta + \delta^2x^2)(x - a) + (\beta\zeta + \delta\epsilon xy)(y - b) + g\zeta]\\
&= 2\,\delta(x - a)\,[\delta x(x - a) + \epsilon y(y - b)]
\end{aligned}$$

for the above values of x, y, ζ.

Similarly

$$\frac{\mathrm{D}f}{\mathrm{D}y} = 2\epsilon (y - b) \left[\delta x (x - a) + \epsilon y (y - b)\right],$$

$$\frac{\mathrm{D}\phi}{\mathrm{D}x} = 2\kappa\zeta^{n-1} (\gamma\delta^2 x - \beta\delta\epsilon y) - 2h\delta (h\delta x - g\epsilon y),$$

$$\frac{\mathrm{D}\phi}{\mathrm{D}y} = 2k\zeta^{n-1} (\alpha\epsilon^2 y - \beta\delta\epsilon x) + 2g\epsilon (h\delta x - g\epsilon y).$$

Hence

$$\frac{\mathrm{D}\phi}{\mathrm{D}x} \Big/ \frac{\mathrm{D}f}{\mathrm{D}x} = \frac{\mathrm{D}\phi}{\mathrm{D}y} \Big/ \frac{\mathrm{D}f}{\mathrm{D}y},$$

if

$$\begin{aligned}(x - a) &\left[k\zeta^{n-1} (\alpha\epsilon y - \beta\delta x) + g (h\delta x - g\epsilon y)\right] \\ = (y - b) &\left[k\zeta^{n-1} (\gamma\delta x - \beta\epsilon y) - h (h\delta x - g\epsilon y)\right].\end{aligned}$$

Making use of the values of $x - a$, $y - b$, which satisfy the equations which have been taken to determine them, and which are solved above in (A), it is necessary to show that

$$\begin{aligned}&\left[(h\beta - g\gamma) \zeta + \epsilon y (h\delta x - g\epsilon y)\right] \left[k\zeta^{n-1} (\alpha\epsilon y - \beta\delta x) + g (h\delta x - g\epsilon y)\right] \\ = &\left[(g\beta - h\alpha) \zeta - \delta x (h\delta x - g\epsilon y)\right] \left[k\zeta^{n-1} (\gamma\delta x - \beta\epsilon y) - h (h\delta x - g\epsilon y)\right],\end{aligned}$$

i.e., to show that

$$\begin{aligned}&k\zeta^{n} \left[(h\beta - g\gamma) (\alpha\epsilon y - \beta\delta x) - (g\beta - h\alpha) (\gamma\delta x - \beta\epsilon y)\right] \\ &+ k\zeta^{n-1} (h\delta x - g\epsilon y) \left[\alpha\epsilon^2 y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2 x^2\right] \\ &- \zeta (h\delta x - g\epsilon y) (\alpha h^2 - 2\beta gh + \gamma g^2) - (h\delta x - g\epsilon y)^3 = 0,\end{aligned}$$

i.e.,

$$(h\delta x - g\epsilon y) \phi = 0.$$

Hence this is satisfied.

Therefore

$$\frac{\mathrm{D}\phi}{\mathrm{D}x} \Big/ \frac{\mathrm{D}f}{\mathrm{D}x} = \frac{\mathrm{D}\phi}{\mathrm{D}y} \Big/ \frac{\mathrm{D}f}{\mathrm{D}y}.$$

It remains to prove that each of the equal quantities

$$\frac{(x - a) \left[\delta x (x - a) + \epsilon y (y - b)\right]}{k\zeta^{n-1} (\gamma\delta x - \beta\epsilon y) - h (h\delta x - g\epsilon y)}, \quad \frac{(y - b) \left[\delta x (x - a) + \epsilon y (y - b)\right]}{k\zeta^{n-1} (\alpha\epsilon y - \beta\delta x) + g (h\delta x - g\epsilon y)}$$

is equal to

$$\frac{\alpha (x - a)^2 + 2\beta (x - a) (y - b) + \gamma (y - b)^2 + 2g (x - a) + 2h (y - b) + nk\zeta^{n-1}}{nk\zeta^{n-1} (\alpha\gamma - \beta^2) + (n - 1) k\zeta^{n-2} (\alpha\epsilon^2 y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2 x^2) - (\alpha h^2 - 2\beta gh + \gamma g^2)}.$$

Multiply numerator and denominator of the first ratio by δx, of the second by ϵy, of the third by ζ, and form a new ratio by addition of the numerators and denominators.

Then each of these ratios must be equal to

$$\frac{\alpha\zeta(x-a)^2+2\beta\zeta(x-a)(y-b)+\gamma\zeta(y-b)^2+2g\zeta(x-a)+2h\zeta(y-b)+nk\zeta^n+[\delta x(x-a)+\epsilon y(y-b)]^2}{nk\zeta^n(\alpha\gamma-\beta^2)+nk\zeta^{n-1}(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)-\zeta(\alpha h^2-2\beta gh+\gamma g^2)-(h\delta x-g\epsilon y)^2}.$$

Hence, by means of the equations $f=0$, $\phi=0$, each of the ratios must be equal to

$$(n-1)k\zeta^n/[(n-1)k\zeta^n(\alpha\gamma-\beta^2)+(n-1)k\zeta^{n-1}(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)]$$

i.e.,

$$\zeta/[\zeta(\alpha\gamma-\beta^2)+(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)].$$

Hence it will be sufficient to prove

$$\frac{(x-a)[\delta x(x-a)+\epsilon y(y-b)]}{k\zeta^{n-1}(\gamma\delta x-\beta\epsilon y)-h(h\delta x-g\epsilon y)}=\frac{\zeta}{\zeta(\alpha\gamma-\beta^2)+(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)}.$$

Now using the values of $x-a$, $y-b$ given above in (A),

$$\begin{aligned}[\delta x(x-a)+\epsilon y(y-b)]\,[\zeta(\alpha\gamma-\beta^2)+(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)]\\ =\zeta[\delta x(h\beta-g\gamma)+\epsilon y(g\beta-h\alpha)].\end{aligned}$$

Hence, using the value of $(x-a)$, it is necessary to prove that

$$\begin{aligned}&[(h\beta-g\gamma)\zeta+\epsilon y(h\delta x-g\epsilon y)]\,[\delta x(h\beta-g\gamma)+\epsilon y(g\beta-h\alpha)]\\ =&[(\alpha\gamma-\beta^2)\zeta+(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)]\,[k\zeta^{n-1}(\gamma\delta x-\beta\epsilon y)-h(h\delta x-g\epsilon y)].\end{aligned}$$

Hence it is necessary to show that

$$\begin{aligned}&(\gamma\delta x-\beta\epsilon y)[k\zeta^n(\alpha\gamma-\beta^2)+k\zeta^{n-1}(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)]\\ &=\zeta[h(\alpha\gamma-\beta^2)(h\delta x-g\epsilon y)-g(h\beta-g\gamma)(\gamma\delta x-\beta\epsilon y)+h(h\beta-g\gamma)(\beta\delta x-\alpha\epsilon y)]\\ &\quad+(h\delta x-g\epsilon y)[h(\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2)-g\epsilon y(\gamma\delta x-\beta\epsilon y)+h\epsilon y(\beta\delta x-\alpha\epsilon y)],\end{aligned}$$

i.e.,

$$(\gamma\delta x-\beta\epsilon y)\phi=0.$$

Hence this is satisfied.

Hence the conditions for contact are satisfied.

Since $uv-w^2=0$ is not satisfied at all points of the locus $\phi=0$, the factor of the discriminant corresponding to it occurs only once.

(D.) It will be verified that the mean of the values of the parameter b, which correspond to the two surfaces having conic nodes, at a point on the locus $\zeta = 0$, is the same as the value of the parameter b, which is used to form the discriminant.

The values of the parameters corresponding to the conic node are given by

$$\zeta = 0, \quad \delta x (x - \alpha) + \epsilon y (y - b) = 0,$$
$$\alpha (x - a)^2 + 2\beta (x - a)(y - b) + \gamma (y - b)^2 + 2g (x - a) + 2h (y - b) + nk\zeta^{n-1} = 0.$$

Hence

$$(y - b)^2 (\alpha\epsilon^2 y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2 x^2) + 2 (y - b)\, \delta x (h\delta x - g\epsilon y) + nk\delta^2 x^2 \zeta^{n-1} = 0.$$

Hence the mean of the values of $y - b$ is

$$\delta x (g\epsilon y - h\delta x)/(\alpha\epsilon^2 y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2 x^2).$$

Now putting $\zeta = 0$ in the value of $y - b$, given above in (A), the same result is obtained.

(E.) This example is a case in which the assumption *equivalent* to that of Art. 7, viz., that

$$\frac{\mathrm{D}f}{\mathrm{D}a}\frac{\partial a}{\partial z} + \frac{\mathrm{D}f}{\mathrm{D}b}\frac{\partial b}{\partial z} = 0$$

at points on the locus of ultimate intersections cannot be made.

The equations $\mathrm{D}f/\mathrm{D}a = 0$, $\mathrm{D}f/\mathrm{D}b = 0$ are given in (A).

Hence $\partial a/\partial z$, $\partial b/\partial z$ are given by

$$(\alpha\zeta + \delta^2 x^2)\frac{\partial a}{\partial z} + (\beta\zeta + \delta\epsilon xy)\frac{\partial b}{\partial z} = \alpha (x - a) + \beta (y - b) + g,$$
$$(\beta\zeta + \delta\epsilon xy)\frac{\partial a}{\partial z} + (\gamma\zeta + \epsilon^2 y^2)\frac{\partial b}{\partial z} = \beta (x - a) + \gamma (y - b) + h.$$

Denoting for brevity

$\alpha (x - a) + \beta (y - b) + g$ by G,
$\beta (x - a) + \gamma (y - b) + h$ by H,
$\zeta^2 (\alpha\gamma - \beta^2) + \zeta (\alpha\epsilon^2 y^2 - 2\beta\delta\epsilon xy + \gamma\delta^2 x^2)$ by K,
$\delta x (x - a) + \epsilon y (y - b)$ by L,

it follows that

$$\frac{\partial a}{\partial z} = \frac{1}{\mathrm{K}}[\mathrm{G}(\gamma\zeta + \epsilon^2 y^2) - \mathrm{H}(\beta\zeta + \delta\epsilon xy)],$$
$$\frac{\partial b}{\partial z} = \frac{1}{\mathrm{K}}[\mathrm{H}(\alpha\zeta + \delta^2 x^2) - \mathrm{G}(\beta\zeta + \delta\epsilon xy)].$$

Therefore

$$-\tfrac{1}{2}\left(\frac{\mathrm{D}f}{\mathrm{D}a}\frac{\partial a}{\partial z}+\frac{\mathrm{D}f}{\mathrm{D}b}\frac{\partial b}{\partial z}\right)$$

$$=\frac{1}{\mathrm{K}}(\mathrm{G}\zeta+\mathrm{L}\delta x)\left[\mathrm{G}(\gamma\zeta+\epsilon^2y^2)-\mathrm{H}(\beta\zeta+\delta\epsilon xy)\right]$$

$$+\frac{1}{\mathrm{K}}(\mathrm{H}\zeta+\mathrm{L}\epsilon y)\left[\mathrm{H}(\alpha\zeta+\delta^2x^2)-\mathrm{G}(\beta\zeta+\delta\epsilon xy)\right].$$

Hence, dividing numerator and denominator by ζ, and then putting $\zeta = 0$,

$$-\tfrac{1}{2}\left(\frac{\mathrm{D}f}{\mathrm{D}a}\frac{\partial a}{\partial z}+\frac{\mathrm{D}f}{\mathrm{D}b}\frac{\partial b}{\partial z}\right)=\frac{(\mathrm{G}\epsilon y-\mathrm{H}\delta x)^2+\mathrm{L}\delta x(\mathrm{G}\gamma-\mathrm{H}\beta)+\mathrm{L}\epsilon y(\mathrm{H}\alpha-\mathrm{G}\beta)}{\alpha\epsilon^2y^2-2\beta\delta\epsilon xy+\gamma\delta^2x^2}.$$

Now, in the case $n = 2$, there is a conic node when $x = a, y = b, \zeta = 0$, and then $\mathrm{G} = g, \mathrm{H} = h, \mathrm{L} = 0$.

Hence

$$-\tfrac{1}{2}\left(\frac{\mathrm{D}f}{\mathrm{D}a}\frac{\partial a}{\partial z}+\frac{\mathrm{D}f}{\mathrm{D}b}\frac{\partial b}{\partial z}\right)=\frac{(g\epsilon b-h\delta a)^2}{\alpha\epsilon^2b^2-2\beta\delta\epsilon ab+\gamma\delta^2a^2}.$$

Hence

$$\frac{\mathrm{D}f}{\mathrm{D}a}\frac{\partial a}{\partial z}+\frac{\mathrm{D}f}{\mathrm{D}b}\frac{\partial b}{\partial z}$$

does not vanish.

Art. 20.—*To prove that under the conditions stated at the head of this Section, if the two Surfaces having Conic Nodes coincide, then they are replaced by a single Surface having a Biplanar or a Uniplanar Node.*

If the condition be expressed that the roots of either parametric quadratic be equal, then the roots of the other parametric quadratic must also in general be equal; for treating the parameters as coordinates of points in a plane, this amounts to expressing that the straight line (143) touches the conic (161).

In this case then, the two surfaces having conic nodes coincide, and if a, b be the values of the parameters corresponding to them, they may be found by finding the points of contact of the straight line (143) with the conic (161).

They are therefore given by the equations

$$\frac{u_xa+\mathrm{W}_xb+\mathrm{V}_x}{u}=\frac{\mathrm{W}_xa+v_xb+\mathrm{U}_x}{\mathrm{W}}=\frac{\mathrm{V}_xa+\mathrm{U}_xb+w_x}{\mathrm{V}}.$$

Now, since the equation (161) may be replaced by (162) or (163), it follows that in the above x may be changed into y or z. Hence

$$u_xa+\mathrm{W}_xb+\mathrm{V}_x : u_ya+\mathrm{W}_yb+\mathrm{V}_y : u_za+\mathrm{W}_zb+\mathrm{V}_z : u \quad : \mathrm{W}$$
$$=\mathrm{W}_xa+v_xb+\mathrm{U}_x : \mathrm{W}_ya+v_yb+\mathrm{U}_y : \mathrm{W}_za+v_zb+\mathrm{U}_z : \mathrm{W} : v.$$

Hence the conditions (164) are satisfied.

Hence there is in general a biplanar node.

But as a particular case there may be a uniplanar node.

Art. 21.—*If the two Conic Nodes are replaced by a single Biplanar Node, and if* $B = 0$ *be the equation of the Biplanar Node Locus, and if the Edge of the Biplanar Node touch the Biplanar Node Locus,* Δ *contains* B^3 *as a factor.*

It follows as in Art. 19 (B.) that Δ, $\partial\Delta/\partial x$, $\partial\Delta/\partial y$, $\partial\Delta/\partial z$ all vanish on the biplanar node locus.

Consider now $\partial^2\Delta/\partial x^2$ as given in (148).

The first three determinants vanish by (158).

To calculate the next three, put in (177)

$$p = u_x,\ q = v_x,\ r = w_x,\ P = U_x,\ Q = V_x,\ R = W_x.$$

Hence these three determinants

$$= (a^2u_x + 2abW_x + b^2v_x + 2aV_x + 2bU_x + w_x)\frac{\partial}{\partial x}(uv - W^2) = 0$$

by (161).

Next consider $\partial^2\Delta/\partial x\,\partial y$ as given in (149).

The first three determinants vanish by (158).

To obtain the next six, put in (177),

$$p = u_y,\ q = v_y,\ r = w_y,\ P = U_y,\ Q = V_y,\ R = W_y.$$

Hence their value is

$$(a^2u_y + 2abW_y + b^2v_y + 2aV_y + 2bU_y + w_y)\frac{\partial}{\partial x}(uv - W^2) = 0$$

by (162).

Hence by symmetry all the second differential coefficients of Δ vanish.

Therefore Δ contains B^3 as a factor.

Example 12.—*Locus of Biplanar Nodes, such that the Edges of the Biplanar Nodes always touch the Biplanar Node Locus, the equation of the Surfaces of the System being of the Second Degree in the Parameters.*

Let the surfaces be

$$(bx - ay + cz)^2 - g^2z^2 - 2mz(x - a)(y - b) = 0,$$

where c, g, m are fixed constants; a, b are the arbitrary parameters.

(A.) *The Discriminant.*

$$\Delta = \begin{vmatrix} y^2 & -xy-mz & (m-c)\,yz \\ -xy-mz & x^2 & (m+c)\,xz \\ (m-c)\,yz & (m+c)\,xz & (c^2-g^2)\,z^2 - 2m\,xyz \end{vmatrix}$$

$$= mz^3\,\{2g^2xy - (c^2-g^2)\,mz\}.$$

The way in which the factor z^3 arises will now be examined.

The discriminant is found by eliminating a, b between

$$(bx - ay + cz)^2 - g^2z^2 - 2m\,(x-a)\,(y-b)\,z = 0 \quad . \quad . \quad . \quad . \quad (\alpha),$$

$$-2y\,(bx-ay+cz) + 2m\,(y-b)\,z = 0 \quad . \quad . \quad . \quad . \quad (\beta),$$

$$2x\,(bx-ay+cz) + 2m\,(x-a)\,z = 0 \quad . \quad . \quad . \quad . \quad (\gamma).$$

By means of (β), (γ), it follows that (α) can be written

$$(bx-ay+cz)\,cz - g^2z^2 - mz\,(2xy - bx - ay) = 0 \quad . \quad . \quad . \quad . \quad (\delta).$$

The values of a, b, satisfying (β) and (γ) are

$$\frac{a}{\begin{vmatrix} -xy-mz & (m-c)\,yz \\ x^2 & (m+c)\,xz \end{vmatrix}} = \frac{b}{\begin{vmatrix} (m-c)\,yz & y^2 \\ (m+c)\,xz & -xy-mz \end{vmatrix}} = \frac{1}{\begin{vmatrix} y^2 & -xy-mz \\ -xy-mz & x^2 \end{vmatrix}}$$

Therefore

$$\frac{a}{-mxz\,\{2xy+(m+c)\,z\}} = \frac{b}{-myz\,\{2xy+(m-c)\,z\}} = \frac{1}{-mz\,(2xy+mz)}\,.$$

Now it will be shown that on the binode locus $z = 0$; therefore the values of a, b become indeterminate on the binode locus.

But they may be determined by dividing out by the factor z, which vanishes on the binode locus, and then

$$a = x\left(1 + \frac{cz}{2xy+mz}\right),$$

$$b = y\left(1 - \frac{cz}{2xy+mz}\right).$$

Hence if ξ, η, 0 be any point on the binode locus, then at this point the values of the parameters are $a = \xi$, $b = \eta$.

Hence there is a single set of values of the parameters satisfying the equations $\mathrm{D}f/\mathrm{D}a = 0$, $\mathrm{D}f/\mathrm{D}b = 0$ at points on the binode locus, which has been determined.

There is not a double set of equal values as in Art. 15 (see especially Example 10 of that article), where the degree of the equation of the system of surfaces in the parameters is higher than the second.

If the values given above for a, b be substituted in the left-hand side of (δ), and the result multiplied by the rationalising factor $uv - W^2$, which in this case is $-(2xymz + m^2z^2)$, the result is

$$-(2xymz + m^2z^2)\left\{\frac{mc^2z^3}{2xy + mz} - g^2z^2\right\} = mz^3\{2g^2xy - (c^2 - g^2)mz\},$$

which is the same value for the discriminant as before.

It will be noticed that the factor z enters once through the rationalising factor, and twice from the remaining part.

(B.) *The Node Locus is* $z = 0$.

Substituting $x = a + X$, $y = b + Y$, $z = Z$ in the equation, it becomes

$$(bX - aY + cZ)^2 - g^2Z^2 - 2mXYZ = 0.$$

Hence the new origin is a binode. There are no other singular points on the surface.

The biplanes are $bX - aY + cZ \pm gZ = 0$.

They intersect in the straight line $bX - aY = 0$, $Z = 0$.

Hence the binode locus is $z = 0$, and the edge of the binode, which lies in the binode locus, satisfies the condition for contact with the binode locus.

(C.) *The Locus* $(c^2 - g^2)mz - 2g^2xy = 0$ *is an Ordinary Envelope.*

To prove this it is necessary to satisfy at the same time

$$(bx - ay + cz)^2 - g^2z^2 - 2mz(x - a)(y - b) = 0 \quad . \quad . \quad . \quad . \quad (\alpha),$$

$$(c^2 - g^2)mz - 2g^2xy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\epsilon),$$

$$\frac{2b(bx - ay + cz) - 2m(y - b)z}{-2g^2y} = \frac{-2a(bx - ay + cz) - 2m(x - a)z}{-2g^2x}$$

$$= \frac{2c(bx - ay + cz) - 2g^2z - 2m(x - a)(y - b)}{m(c^2 - g^2)} \quad . \quad . \quad . \quad (\zeta).$$

Multiplying numerator and denominator of the first ratio in (ζ) by x, of the second by y, and of the third by z; adding the numerators to form a new numerator, and the denominators to form a new denominator, and reducing by (α) and (ϵ), each of the above ratios

$$= \frac{2(xy - ab)}{c^2 - g^2}.$$

Equating the third ratio of (ζ) to this, and substituting for z from (ϵ), and putting $a/x = \xi$, $b/y = \eta$, the result is

$$m(m-c)\xi + m(m+c)\eta + 2(g^2 - m^2) = 0 \quad . \quad . \quad . \quad . \quad . \quad (\eta).$$

In like manner, from the first and third ratios of (ζ),

$$m^2(c^2 - g^2)\eta^2 + 4m(m+c)g^2\eta + 4(g^4 - m^2g^2) \\ + \xi\{-m^2(c^2+g^2)\eta + 2mg^2(m-c)\} = 0 \quad . \quad . \quad . \quad (\theta).$$

Substituting for ξ from equation (η), this reduces to

$$mc\eta^2 + \eta(g^2 - cm) = 0.$$

Hence $\eta = 0$, $\eta = 1 - (g^2/cm)$.

Substituting in equation (η) the corresponding values of ξ are

$$\xi = \frac{2(m^2 - g^2)}{m(m-c)}, \quad \xi = 1 + (g^2/cm).$$

It remains to prove that one of the two systems of solutions will satisfy the equation obtained from the second and third ratios of (ζ).

This equation is

$$m^2(c^2 - g^2)\xi^2 + 4m(m-c)g^2\xi + 4(g^4 - m^2g^2) \\ + \eta[-m^2(c^2+g^2)\xi + 2mg^2(m+c)] = 0.$$

Substituting for η from equation (η), this reduces to

$$mc\xi^2 - \xi(cm + g^2) = 0.$$

Therefore

$$\xi = 0, \quad \xi = 1 + (g^2/cm).$$

Hence the solutions

$$x = acm/(cm + g^2),$$
$$y = bcm/(cm - g^2),$$
$$z = 2abc^2g^2m/\{(c^2m^2 - g^4)(c^2 - g^2)\},$$

satisfy all the equations.

Hence the surface (ϵ) is an envelope.

(D.) It will now be verified that the values of b given by (181), and the equations obtained by changing ξ into η and ζ, become equal in this case.

The equation

$$b^2 \frac{\partial}{\partial\zeta}(uv - \mathrm{W}^2) + 2b \frac{\partial}{\partial\zeta}(u\mathrm{U} - \mathrm{VW}) + \frac{\partial}{\partial\zeta}(uw - \mathrm{V}^2) = 0$$

is the only one that need be considered, because the others are identically satisfied.

In this case

$$\begin{aligned} uv - \mathrm{W}^2 &= -\,2xymz - m^2z^2 \\ u\mathrm{U} - \mathrm{VW} &= myz\,\{2xy + (m - c)\,z\} \\ uw - \mathrm{V}^2 &= (-\,m^2 + 2mc - g^2)\,y^2z^2 - 2mxy^3z. \end{aligned}$$

Hence if ξ, η, 0 be any point on the binode locus,

$$\begin{aligned} \frac{\partial}{\partial\zeta}(uv - \mathrm{W}^2) &= -\,2m\xi\eta, \\ \frac{\partial}{\partial\zeta}(u\mathrm{U} - \mathrm{VW}) &= 2m\xi\eta^2, \\ \partial\;(uw - \mathrm{V}^2) &= -\,2m\xi\eta^3. \end{aligned}$$

Hence the equation for b is

$$-\,2m\xi\eta\,(b - \eta)^2 = 0.$$

Hence both values of b become equal to η.

Art. 22.—*If the two Conic Nodes are replaced by a single Uniplanar Node, and if* $\mathrm{U} = 0$ *be the equation of the Uniplanar Node Locus, then* Δ *contains* U^4 *as a factor.*

It follows, as in Art. 19 (B), and Art. 21, that Δ and all its differential coefficients of the second order vanish.

Next take the value of $\partial^3\Delta/\partial x^3$ from (150).

The first three determinants vanish by (158).

To calculate the next set of terms, put in (177)

$$p = u_{xx}, \quad q = v_{xx}, \quad r = w_{xx}, \quad \mathrm{P} = \mathrm{U}_{xx}, \quad \mathrm{Q} = \mathrm{V}_{xx}, \quad \mathrm{R} = \mathrm{W}_{xx}.$$

Hence they are equal to

$$\begin{aligned} 3\,(a^2u_{xx} + 2ab\mathrm{W}_{xx} + b^2v_{xx} + 2a\mathrm{V}_{xx} + 2b\mathrm{U}_{xx} + w_{xx})\,\frac{\partial}{\partial x}(uv - \mathrm{W}^2) \\ = 6\lambda^2u\,\frac{\partial}{\partial x}(uv - \mathrm{W}^2). \end{aligned}$$

The last determinant by (178) is equal to

$$- 6\lambda^2 u \frac{\partial}{\partial x}(uv - \mathrm{W}^2).$$

Hence $\partial^3\Delta/\partial x^3 = 0$.

Next take the value of $\partial^3\Delta/\partial x^2 \partial y$ from (151).
The first three terms vanish by (158).
The next three terms by (177) are equal to

$$(a^2 u_{xx} + 2ab\mathrm{W}_{xx} + b^2 v_{xx} + 2a\mathrm{V}_{xx} + 2b\mathrm{U}_{xx} + w_{xx}) \frac{\partial}{\partial y}(uv - \mathrm{W}^2)$$
$$= 2\lambda^2 u \frac{\partial}{\partial y}(uv - \mathrm{W}^2).$$

The next three terms by (177) are equal to

$$2(a^2 u_{xy} + 2ab\mathrm{W}_{xy} + b^2 v_{xy} + 2a\mathrm{V}_{xy} + 2b\mathrm{U}_{xy} + w_{xy}) \frac{\partial}{\partial x}(uv - \mathrm{W}^2)$$
$$= 4\lambda\mu u \frac{\partial}{\partial x}(uv - \mathrm{W}^2).$$

The next three terms may be obtained from (179) by changing z into x, and, therefore, ν into λ. They are therefore equal to

$$- 4\lambda\mu u \frac{\partial}{\partial x}(uv - \mathrm{W}^2) - 2\lambda^2 u \frac{\partial}{\partial y}(uv - \mathrm{W}^2).$$

Hence $\partial^3\Delta/\partial x^2 \partial y = 0$.

Next take $\partial^3\Delta/\partial x\, \partial y\, \partial z$ from (152).
The first three determinants vanish by (158).
The next six are by (179)

$$= - 2\mu\nu u \frac{\partial}{\partial x}(uv - \mathrm{W}^2) - 2\nu\lambda u \frac{\partial}{\partial y}(uv - \mathrm{W}^2) - 2\lambda\mu u \frac{\partial}{\partial z}(uv - \mathrm{W}^2).$$

The next six are by (177)

$$= (a^2 u_{xy} + 2ab\mathrm{W}_{xy} + b^2 v_{xy} + 2a\mathrm{V}_{xy} + 2b\mathrm{U}_{xy} + w_{xy}) \frac{\partial}{\partial z}(uv - \mathrm{W}^2)$$
$$= 2\lambda\mu u \frac{\partial}{\partial z}(uv - \mathrm{W}^2).$$

Hence, the next six are

$$= 2\mu\nu u \frac{\partial}{\partial x}(uv - \mathrm{W}),$$

and the next six are

$$= 2\nu\lambda u \frac{\partial}{\partial y}(uv - W^2).$$

Hence, $\partial^3\Delta/\partial x\,\partial y\,\partial z = 0$.

Hence, by symmetry, all the third differential coefficients of Δ vanish.

Hence, Δ contains U^4 as a factor.

Example 13.—*Locus of Uniplanar Nodes when the equation of the System of Surfaces is of the Second Degree in the Parameters.*

Let the surfaces be

$$(bx - ay + z)^2 - z^3 - 2mz(x - a)(y - b) = 0.$$

(A.) *The Discriminant.*

It is

$$\begin{vmatrix} y^2 & -xy - mz & (m-1)yz \\ -xy - mz & x^2 & (m+1)xz \\ (m-1)yz & (m+1)xz & z^2 - z^3 - 2mxyz \end{vmatrix}$$
$$= mz^4(2xy + mz - m).$$

To show the origin of the factor z^4, the formation of the discriminant will be examined.

The equations $Df/Da = 0$, $Df/Db = 0$ are, in this case

$$ay^2 - b(xy + mz) + (m - 1)yz = 0,$$
$$-a(xy + mz) + bx^2 + (m + 1)xz = 0.$$

Therefore

$$\frac{a}{-mxz\{2xy + (m+1)z\}} = \frac{b}{-myz\{2xy + (m-1)z\}} = \frac{1}{-mz(2xy + mz)}.$$

Now, it will be shown presently that $z = 0$ is the uniplanar node locus. Hence, a, b become indeterminate on the uniplanar node locus. But, removing the factor $-mz$, which vanishes on this locus,

$$a = x\left(1 + \frac{z}{2xy + mz}\right), \quad b = y\left(1 - \frac{z}{2xy + mz}\right).$$

Hence, at any point, ξ, η, 0 on the uniplanar node locus, $a = \xi$, $b = \eta$.

Again, substituting the above values of a, b in

$$(bx - ay + z)^2 - z^3 - 2mz(x - a)(y - b),$$

the result is

$$\frac{mz^3}{2xy + mz} - z^3.$$

If this be expanded in ascending powers of z, the lowest is the third power.

But the rationalising factor applied to form the discriminant, viz. $-mz(2xy + mz)$ contains the factor z. Hence, the factor z^4 is accounted for.

The discriminant is as before

$$mz^4(2xy + mz - m).$$

(B.) *The Uniplanar Node Locus is* $z = 0$.

Put $x = a + \mathrm{X}$, $y = b + \mathrm{Y}$, $z = \mathrm{Z}$ in the equation. It becomes

$$(b\mathrm{X} - a\mathrm{Y} + \mathrm{Z})^2 - \mathrm{Z}^3 - 2m\mathrm{XYZ} = 0.$$

Hence the new origin is a uniplanar node.

Hence $z = 0$ is the uniplanar node locus.

(C.) *The Envelope Locus is* $2xy + mz - m = 0$.

The equation can be written

$$a^2y^2 - 2ab(xy + mz) + b^2x^2 + 2a(m-1)yz + 2b(m+1)xz + \rho$$
$$= \rho - z^2 + z^3 + 2mxyz.$$

Let ρ be determined as a function of x, y, z, so that the left-hand side of the equation may break up into factors linear with regard to a, b.

Then

$$\rho = z^2 - 2mxyz - \frac{mz^3}{2xy +}\cdot$$

It may then be verified that the equation can be written

$$y^2\left[\left\{a + \frac{1}{y^2}[-b(xy + mz) + (m-1)yz]\right\}^2\right.$$
$$\left. - \frac{m}{y^4}z(2xy + mz)\left\{b - y + \frac{yz}{2xy + mz}\right\}^2\right] = \frac{z^3(2xy + mz - m)}{2xy + mz}.$$

Hence it may be concluded that $2xy + mz - m = 0$ will touch the surface where both the factors of the left-hand side vanish, *i.e.*, where

$$a + \frac{1}{y^2}[-b(xy + mz) + (m-1)yz] = 0,$$

$$b - y + \frac{yz}{2xy + mz} = 0,$$

i.e., where

$$a = x\left(1 + \frac{z}{2xy + mz}\right), \qquad b = y\left(1 - \frac{z}{2xy + mz}\right).$$

Hence the points of contact are determined by

$$2xy + mz - m = 0,$$
$$a = x\left(1 + \frac{z}{m}\right),$$
$$b = y\left(1 - \frac{z}{m}\right).$$

Hence

$$(z^2 - m^2)(z - 1) - 2abm = 0.$$

Hence when a, b are given, there are three values of z, and three corresponding values of x, and three corresponding values of y. Hence each surface touches the envelope at three points. But each point on the envelope is the point of contact of only one surface of the system, since when the coordinates x, y, z of the point of contact are given, the values of a, b, the parameters of the surface touching the envelope there, are determined by the simple equations

$$a = x(1 + z/m), \quad b = y(1 - z/m).$$

The result may be verified thus:—

The values of x, y, z satisfying the equations

$$a = x(1 + z/m), \quad b = y(1 - z/m), \quad (z^2 - m^2)(z - 1) - 2abm = 0 \quad . \quad . \quad (\alpha),$$

will satisfy at the same time

$$\left.\begin{aligned}(bx - ay + z)^2 - z^3 - 2mz(x - a)(y - b) &= 0\\ 2xy + mz - m &= 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (\beta),$$

and

$$\frac{2(bx - ay + z)b - 2mz(y - b)}{2y} = \frac{-2(bx - ay + z)a - 2mz(x - a)}{2x}$$
$$= \frac{2(bx - ay + z) - 3z^2 - 2m(x - a)(y - b)}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma).$$

If x, y, z satisfy (α), then

$$bx - ay + z = \frac{z(z^2 + 2abm - m^2)}{z^2 - m^2},$$

and substituting in the first of equations (β), after making some reductions, the result is

$$\frac{z^2}{(z^2 - m^2)^2}(z^2 + 2abm - m^2)\,[2abm - (z^2 - m^2)\,(z - 1)] = 0,$$

which is satisfied by (α).

Hence the first of equations (β) is satisfied by the values of x, y, z given by (α).

Again substituting for x, y in terms of z from (α) in (γ) the ratios become equal to

$$\begin{aligned} &[z^3(-m-1) - m^2z^2 + (m^2 - 2abm)\,z]\,/\,m\,(m+z) \\ = &[z^3(-m+1) + m^2z^2 - (m^2 - 2abm)\,z]\,/\,m\,(m-z) \\ = &[3z^4 - 2z^3 + z^2\,(2mab - 3m^2) + z\,(2m^2 - 4abm)]\,/\,m\,(m^2 - z^2). \end{aligned}$$

Hence it is necessary to show that

$$\begin{aligned} &[z^2(-m-1) - m^2z + (m^2 - 2abm)]\,(m - z) \\ = &[z^2(-m+1) + m^2z - (m^2 - 2abm)]\,(m + z) \\ = &3z^3 - 2z^2 + z\,(2mab - 3m^2) + (2m^2 - 4abm) \quad . \quad . \quad . \quad . \quad (\delta). \end{aligned}$$

Equating the first and second quantities in (δ) it is necessary to prove that

$$z^3 - z^2 - m^2z + m^2 - 2abm = 0,$$

which holds by (α).

Equating the second and third quantities in (δ) and removing the factor ($m + 2$), the same result is obtained.

Hence the values of x, y, z given in (α) satisfy all the equations (β), (γ).

Art. 23.—*If the parameters of one of the two Surfaces having Conic Nodes become infinite, and if* $C = 0$ *be the equation of the Conic Node Locus,* Δ *contains* C^2 *as a factor.*

The conditions that one value of a and one value of b satisfying the parametric quadratics (180) and (181) should be infinite are that

$$\frac{\partial}{\partial x}(uv - W^2) = 0, \quad \frac{\partial}{\partial y}(uv - W^2) = 0, \quad \frac{\partial}{\partial z}(uv - W^2) = 0.$$

In this case the values of Δ and $\partial\Delta/\partial x$, as given by (146) and (147), both vanish. Hence Δ contains C^2 as a factor.

Example 14.—*Locus of one Conic Node.*

Let the surfaces be

$$[\alpha\,(x - a) + \beta\,(y - b)]^2 + 2gz\,(x - a) + 2hz\,(y - b) + kz^2 = 0.$$

(A.) *The Discriminant.*

It reduces to

$$\begin{vmatrix} \alpha^2 & \alpha\beta & gz \\ \alpha\beta & \beta^2 & hz \\ gz & hz & kz^2 \end{vmatrix} = -(h\alpha - g\beta)^2 z^2.$$

(B.) *The Conic Node Locus is* $z = 0$.

In this case equations (143), (161), (162), (163) are equivalent to the three equations

$$\begin{aligned} \alpha^2(x-a) + \alpha\beta(y-b) + gz &= 0, \\ \alpha\beta(x-a) + \beta^2(y-b) + hz &= 0, \\ g(x-a) + h(y-b) + kz &= 0, \end{aligned}$$

the only solutions of which (unless $g\beta - h\alpha = 0$) are

$$x = a, \quad y = b, \quad z = 0.$$

Hence there is now only one system of values of the parameters satisfying (143), (161), (162), (163).

The same value of the parameter b would be obtained from the equation (181) which becomes in this case, after changing x into ζ,

$$2b\frac{\partial}{\partial\zeta}(u\mathrm{U} - \mathrm{VW}) + \frac{\partial}{\partial\zeta}(uw - \mathrm{V}^2) = 0.$$

Now

$$\begin{aligned} u\mathrm{U} - \mathrm{VW} &= (g\beta - h\alpha)\,\alpha z, \\ uw - \mathrm{V}^2 &= 2(h\alpha - g\beta)\,\alpha yz + (k\alpha^2 - g^2)\,z^2\,; \end{aligned}$$

therefore,

$$\begin{aligned} \frac{\partial}{\partial z}(u\mathrm{U} - \mathrm{VW}) &= (g\beta - h\alpha)\,\alpha, \\ \frac{\partial}{\partial z}(uw - \mathrm{V}^2) &= 2(h\alpha - g\beta)\,\alpha y + 2(k\alpha^2 - g^2)\,z. \end{aligned}$$

On the conic node locus $z = 0$.

Therefore the equation for b is

$$2b(g\beta - h\alpha)\,\alpha + 2(h\alpha - g\beta)\,\alpha y = 0.$$

Therefore

$$b = y.$$

There is only one conic node, since $uv - \mathrm{W}^2 = 0$, and, therefore, equation (181) reduces to a simple equation for b.

Art. 24.—*If the parameters of both of the Surfaces having Conic Nodes become infinite, and* E = 0 *be the equation of the Envelope Locus, then* Δ *contains* E^3 *as a factor.*

In this case it is necessary that both roots of the parametric quadratics (180) and (181) should become infinite.

Hence the first differential coefficients of

$$uv - W^2, \quad Vv - UW, \quad Uu - VW,$$

with regard to any of the variables, must vanish on the envelope locus.

[It may be noted that if

$$\frac{\partial}{\partial x}(uv - W^2) = 0, \text{ and } \frac{\partial}{\partial x}(Vv - UW) = 0, \text{ then } \frac{\partial}{\partial x}(Uu - VW) = 0.$$

For

$$\left.\begin{aligned}\frac{\partial}{\partial x}(uv - W^2) &= u_x v + uv_x - 2WW_x = 0\\ \frac{\partial}{\partial x}(Vv - UW) &= V_x v + Vv_x - U_x W - UW_x = 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (182).$$

Multiplying these equations by V, u respectively, and subtracting

$$u_x Vv + U_x Wu - uvV_x - W_x(2WV - Uu) = 0\,;$$

therefore, using (158), after dividing by W,

$$u_x U + U_x u - WV_x - W_x V = 0\,;$$

therefore

$$\frac{\partial}{\partial x}(Uu - VW) = 0.]$$

Now Δ, $\partial\Delta/\partial x$ both vanish by (158).

Next consider $\partial^2\Delta/\partial x^2$ as given by (148).

The first three determinants vanish by (158).

The fourth and fifth determinants

$$= 2\left\{V_x\frac{\partial}{\partial x}(WU - Vv) + U_x\frac{\partial}{\partial x}(VW - Uu) + w_x\frac{\partial}{\partial x}(uv - W^2)\right\}$$

= 0 by the above conditions.

The sixth determinant is

$$2\begin{vmatrix} u_x & W_x & V_x \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix}$$

Substituting in it for u_x, v_x the values which can be obtained from

$$\frac{\partial}{\partial x}(\mathrm{V}v - \mathrm{UW}) = 0, \text{ and } \frac{\partial}{\partial x}(\mathrm{U}u - \mathrm{VW}) = 0,$$

it becomes

$$\frac{2}{\mathrm{UV}}\begin{vmatrix} \mathrm{WV}_x + \mathrm{VW}_x - u\mathrm{U}_x & \mathrm{UW}_x & \mathrm{UV}_x \\ \mathrm{VW}_x & \mathrm{U}_x\mathrm{W} + \mathrm{UW}_x - \mathrm{V}_x v & \mathrm{VU}_x \\ \frac{\mathrm{V}}{u}u & \frac{\mathrm{V}}{u}\mathrm{W} & \frac{\mathrm{V}}{u}\mathrm{V} \end{vmatrix}$$

$$= \frac{2}{u\mathrm{U}}\begin{vmatrix} \mathrm{WV}_x + \mathrm{VW}_x - u\mathrm{U}_x & \mathrm{UW}_x & \mathrm{UV}_x \\ \mathrm{VW}_x & \mathrm{UW}_x + \mathrm{U}_x\mathrm{W} - \mathrm{V}_x v & \mathrm{VU}_x \\ u & \mathrm{W} & \mathrm{V} \end{vmatrix}$$

$$= \frac{2}{u\mathrm{U}}\begin{vmatrix} \mathrm{WV}_x + \mathrm{VW}_x & \mathrm{UW}_x + \mathrm{WU}_x & \mathrm{UV}_x + \mathrm{VU}_x \\ \mathrm{VW}_x & \mathrm{UW}_x + \mathrm{WU}_x - \mathrm{V}_x v & \mathrm{VU}_x \\ u & \mathrm{W} & \mathrm{V} \end{vmatrix}$$

$$= \frac{2}{u\mathrm{U}}\begin{vmatrix} \mathrm{WV}_x & v\mathrm{V}_x & \mathrm{UV}_x \\ \mathrm{VW}_x & \mathrm{UW}_x + \mathrm{WU}_x - \mathrm{V}_x v & \mathrm{VU}_x \\ u & \mathrm{W} & \mathrm{V} \end{vmatrix}$$

$$= \frac{2\mathrm{V}_x}{\mathrm{U}u}\begin{vmatrix} \mathrm{W} & v & \mathrm{U} \\ \mathrm{VW}_x & \mathrm{UW}_x + \mathrm{WU}_x - \mathrm{V}_x v & \mathrm{VU}_x \\ u & \mathrm{W} & \mathrm{V} \end{vmatrix}$$

$= 0$ by (158).

Next take $\partial^2\Delta/\partial x\,\partial y$ from (149).

The first three determinants vanish by (158).

The fifth and seventh determinants

$$= \mathrm{V}_x\frac{\partial}{\partial y}(\mathrm{UW} - \mathrm{V}v) + \mathrm{U}_x\frac{\partial}{\partial y}(\mathrm{VW} - \mathrm{U}u) + w_x\frac{\partial}{\partial y}(uv - \mathrm{W}^2) = 0.$$

The eighth and ninth determinants

$$= \mathrm{V}_y\frac{\partial}{\partial x}(\mathrm{UW} - \mathrm{V}v) + \mathrm{U}_y\frac{\partial}{\partial x}(\mathrm{VW} - \mathrm{U}u) + w_y\frac{\partial}{\partial x}(uv - \mathrm{W}^2) = 0.$$

The fourth and sixth determinants are

$$\begin{vmatrix} u_y & W_y & V_y \\ W_x & v_x & U_x \\ V & U & w \end{vmatrix} + \begin{vmatrix} u_x & W_x & V_x \\ W_y & v_y & U_y \\ V & U & w \end{vmatrix}$$

$$= \frac{1}{UV}\begin{vmatrix} WV_y + VW_y - uU_y & UW_y & UV_y \\ VW_x & U_xW + UW_x - V_xv & VU_x \\ \frac{V}{u}u & \frac{V}{u}W & \frac{V}{u}V \end{vmatrix}$$

$$+ \frac{1}{UV}\begin{vmatrix} WV_x + VW_x - uU_x & UW_x & UV_x \\ VW_y & U_yW + UW_y - V_yv & VU_y \\ \frac{V}{u}u & \frac{V}{u}W & \frac{V}{u}V \end{vmatrix}$$

$$= \frac{1}{Uu}\begin{vmatrix} WV_y + VW_y & UW_y + WU_y & UV_y + VU_y \\ VW_x & U_xW + UW_x - V_xv & VU_x \\ u & W & V \end{vmatrix}$$

$$+ \frac{1}{Uu}\begin{vmatrix} WV_x + VW_x & UW_x + WU_x & UV_x + VU_x \\ VW_y & U_yW + UW_y - V_yv & VU_y \\ u & W & V \end{vmatrix}$$

Hence the coefficient of U_y is

$$\frac{1}{Uu}\begin{vmatrix} 0 & W & V \\ VW_x & U_xW+UW_x-V_xv & VU_x \\ u & W & V \end{vmatrix} + \frac{1}{Uu}\begin{vmatrix} WV_x+VW_x & UW_x+WU_x & UV_x+VU_x \\ 0 & W & V \\ u & W & V \end{vmatrix}$$

$$= \frac{1}{U}\begin{vmatrix} W & V \\ UW_x - V_xv & 0 \end{vmatrix} + \frac{1}{U}\begin{vmatrix} UW_x & UV_x \\ W & V \end{vmatrix}$$

$$= \frac{1}{U}(VvV_x - VUW_x + UVW_x - UWV_x) = 0.$$

The coefficient of V_y is

$$\frac{1}{Uu}\begin{vmatrix} W & 0 & U \\ VW_x & U_xW + UW_x - V_xv & VU_x \\ u & W & V \end{vmatrix} + \frac{1}{Uu}\begin{vmatrix} WV_x + VW_x & UW_x + WU_x & UV_x + VU_x \\ 0 & -v & 0 \\ u & W & V \end{vmatrix}$$

$$= \frac{1}{Uu}\begin{vmatrix} W & 0 & U \\ VW_x & U_xW + UW_x - V_xv & VU_x \\ 0 & W & 0 \end{vmatrix} - \frac{v}{Uu}(VWV_x + V^2W_x - uUV_x - uVU_x)$$

$$= -\frac{W}{Uu}(WVU_x - UVW_x) - \frac{v}{Uu}(V^2W_x - uVU_x)$$

$$= \frac{1}{Uu}\{U_x(uv - W^2)V + W_x(UW - vV)V\} = 0.$$

The coefficient of W_y is

$$\frac{1}{Uu}\begin{vmatrix} V & U & 0 \\ VW_x & U_xW + UW_x - V_xv & VU_x \\ u & W & V \end{vmatrix}$$

$$+ \frac{1}{Uu}\begin{vmatrix} WV_x + VW_x & UW_x + WU_x & UV_x + VU_x \\ V & U & 0 \\ u & W & V \end{vmatrix}$$

$$= \frac{V^2}{Uu}(U_xW - V_xv) + \frac{V}{Uu}(UWV_x - VWU_x)$$

$$= \frac{V}{Uu}\{V_x(UW - Vv)\} = 0.$$

Hence $\partial^2\Delta/\partial x\,\partial y = 0$.

Hence all the differential coefficients of the second order vanish.

Hence Δ contains E^3 as a factor.

Example 15.—*Envelope Locus, the parameters of both the Surfaces having Conic Nodes being infinite.*

Let the surfaces be

$$z^2a^2 + z + (ax + b + y)^2 = 0.$$

(A.) *The Discriminant.*

This is

$$\begin{vmatrix} z^2 + x^2 & x & xy \\ x & 1 & y \\ xy & y & y^2 + z \end{vmatrix}$$

$$= z^3.$$

(B.) *The Envelope Locus is* $z = 0$.

The tangent plane at ξ, η, ζ is

$$(\mathrm{X} - \xi)\, 2a\, (a\xi + b + \eta) + (\mathrm{Y} - \eta)\, 2\, (a\xi + b + \eta) + (\mathrm{Z} - \zeta)\, (2a^2\zeta + 1) = 0.$$

Hence at the point ξ, η, ζ, where

$$a\xi + b + \eta = 0,\ \zeta = 0,$$

the tangent plane is $\mathrm{Z} = 0$.

Hence the factor z^3 is accounted for.

(C.) *The Parameters of both Surfaces having Conic Nodes are infinite.*

In this case

$$u = z^2 + x^2,\ v = 1,\ w = y^2 + z,\ \mathrm{U} = y,\ \mathrm{V} = xy,\ \mathrm{W} = x.$$

Hence the equations

$$a^2 \frac{\partial}{\partial z}(uv - \mathrm{W}^2) + 2a \frac{\partial}{\partial z}(\mathrm{V}v - \mathrm{UW}) + \frac{\partial}{\partial z}(vw - \mathrm{U}^2) = 0,$$

and

$$b^2 \frac{\partial}{\partial z}(uv - \mathrm{W}^2) + 2b \frac{\partial}{\partial z}(\mathrm{U}u - \mathrm{VW}) + \frac{\partial}{\partial z}(uw - \mathrm{V}^2) = 0,$$

become, when $z = 0$,

$$(0)\, a^2 + 0\, (a) + 1 = 0,$$
$$(0)\, b^2 + 0\, (b) + x^2 = 0.$$

Hence both roots are infinite.

If the differential coefficients in the parametric quadratics had been taken with regard to x or y, the equations would have been wholly indeterminate.

Art. 25.—*If the parameters of both of the Surfaces having Conic Nodes become indeterminate, then at every point of the Locus of Ultimate Intersections there are an infinite number of Biplanar Nodes; each Surface of the system has a Binodal Line lying on the Locus of Ultimate Intersections, and if the locus of these Binodal Lines be* $\mathrm{B} = 0$, *then* Δ *contains* B^4 *as a factor.*

In order that the parametric quadratics may become wholly indeterminate, the first differential coefficients, with regard to each of the three variables, of $uv - \mathrm{W}^2$, $\mathrm{U}u - \mathrm{VW}$, $\mathrm{V}v - \mathrm{UW}$, $vw - \mathrm{U}^2$, $uw - \mathrm{V}^2$ must vanish. These involve the vanishing of the first differential coefficients of $\mathrm{UV} - \mathrm{W}w$.

It will be shown, first of all, that the ratios (164) are in this case equivalent only to the equation (143). [The same holds good in the previous article, but the condition (161) is not satisfied there.]

For consider the ratios

$$u_x a + W_x b + V_x : W_x a + v_x b + U_x = u : W.$$

Therefore

$$a(Wu_x - uW_x) + b(WW_x - uv_x) + (WV_x - uU_x) = 0.$$

This will be the same equation as (143) if

$$\frac{Wu_x - uW_x}{u} = \frac{WW_x - uv_x}{W} = \frac{WV_x - uU_x}{V}.$$

Hence if

$$u_x W^2 + u^2 v_x - 2uWW_x = 0,$$

and

$$VWW_x - Vuv_x - W^2 V_x + WuU_x = 0,$$

i.e., if

$$uvu_x + u^2 v_x - 2uWW_x = 0,$$

$$UuW_x - Vuv_x - uvV_x + WuU_x = 0,$$

i.e., if

$$\frac{\partial}{\partial x}(uv - W^2) = 0,$$

$$\frac{\partial}{\partial x}(UW - Vv) = 0,$$

which are satisfied.

Similarly the other ratios in (164) hold. Hence if any point be taken on the curve in which the surface (143) intersects the locus of ultimate intersections, that point is a binode on the surface (142). Hence the surface (142) has a binodal line situated on the locus of ultimate intersections. Hence each surface of the system has a binodal line situated on the locus of ultimate intersections.

It remains to show that if $B = 0$ be the locus of these binodal lines then Δ contains B^4 as a factor.

The proof in the last article will hold as far as the second differential coefficients of Δ are concerned.

Consider, therefore, the value of $\partial^3\Delta/\partial x^3$ given in (150).

The first three terms vanish by (158).

The next three are equal to three times

$$u_{xx}\frac{\partial}{\partial x}(vw - U^2) + 2W_{xx}\frac{\partial}{\partial x}(UV - Ww) + v_{xx}\frac{\partial}{\partial x}(uw - V^2)$$

$$+ 2V_{xx}\frac{\partial}{\partial x}(WU - Vv) + 2U_{xx}\frac{\partial}{\partial x}(WV - Uu) + w_{xx}\frac{\partial}{\partial x}(uv - W^2)$$

$$= 0.$$

The last determinant is by (178)

$$= -6\lambda^2 u \frac{\partial}{\partial x}(uv - W^2) = 0.$$

Therefore $\partial^3\Delta/\partial x^3 = 0$.

Next take $\partial^3\Delta/\partial x^2\,\partial y$ as given in (151).

The first three terms vanish by (158).

The next three

$$\begin{aligned}&= u_{xx}\frac{\partial}{\partial y}(vw - U^2) + 2W_{xx}\frac{\partial}{\partial y}(UV - Ww) + v_{xx}\frac{\partial}{\partial y}(uw - V^2)\\ &\quad + 2V_{xx}\frac{\partial}{\partial y}(WU - Vv) + 2U_{xx}\frac{\partial}{\partial y}(WV - Uu) + w_{xx}\frac{\partial}{\partial y}(uv - W^2)\\ &= 0.\end{aligned}$$

The next three

$$\begin{aligned}&= 2u_{xy}\frac{\partial}{\partial x}(vw - U^2) + 4W_{xy}\frac{\partial}{\partial x}(UV - Ww) + 2v_{xy}\frac{\partial}{\partial x}(uw - V^2)\\ &\quad + 4V_{xy}\frac{\partial}{\partial x}(WU - Vv) + 4U_{xy}\frac{\partial}{\partial x}(WV - Uu) + 2w_{xy}\frac{\partial}{\partial x}(uv - W^2)\\ &= 0.\end{aligned}$$

The next three may be calculated by means of (179) by putting $z = x$, and therefore $\nu = \lambda$.

Hence they are equal to

$$-4\lambda\mu u\frac{\partial}{\partial x}(uv - W^2) - 2\lambda^2 u\frac{\partial}{\partial y}(uv - W^2) = 0.$$

Hence $\partial^3\Delta/\partial x^2\,\partial y = 0$.

Next take $\partial^3\Delta/\partial x\,\partial y\,\partial z$ as given in (152).

The first three terms vanish by (158).

The next six terms by (179)

$$\begin{aligned}&= -2\mu\nu u\frac{\partial}{\partial x}(uv - W^2) - 2\nu\lambda u\frac{\partial}{\partial y}(uv - W^2) - 2\lambda\mu u\frac{\partial}{\partial z}(uv - W^2)\\ &= 0.\end{aligned}$$

The next six terms

$$\begin{aligned}&= u_{xy}\frac{\partial}{\partial z}(vw - U^2) + 2W_{xy}\frac{\partial}{\partial z}(UV - Ww) + v_{xy}\frac{\partial}{\partial z}(uw - V^2)\\ &\quad + 2V_{xy}\frac{\partial}{\partial z}(WU - Vv) + 2U_{xy}\frac{\partial}{\partial z}(WV - Uu) + w_{xy}\frac{\partial}{\partial z}(uv - W^2)\\ &= 0.\end{aligned}$$

The next six terms being obtainable from these last six by interchanging x and z vanish.

The remaining six vanish in like manner.

Hence all the third differential coefficients of Δ vanish.

Hence Δ contains B^4 as a factor.

Example 16.—*Locus of Binodal Lines.*

Let the surfaces be

$$z^2\{a^2 + \phi(x, y, z)\} - (ax + b + y)^2 = 0.$$

(A.) *The Discriminant.*

This is

$$\begin{vmatrix} z^2 - x^2 & -x & -xy \\ -x & -1 & -y \\ -xy & -y & -y^2 + z^2\phi(x, y, z) \end{vmatrix}$$
$$= -z^4\phi(x, y, z).$$

(B.) *The Locus of Binodal Lines is $z = 0$.*

For let ξ, η, ζ be any point on both the loci $z = 0$, $ax + b + y = 0$.

Then put $x = \xi + X$, $y = \eta + Y$, $z = \zeta + Z$, so that $\zeta = 0$, $a\xi + b + \eta = 0$.

Therefore

$$Z^2\left(a^2 + \phi(\xi, \eta, \zeta) + X\frac{\partial\phi}{\partial\xi} + Y\frac{\partial\phi}{\partial\eta} + Z\frac{\partial\phi}{\partial\zeta} + \ldots\right) - (a\xi + b + \eta + aX + Y)^2 = 0.$$

Hence the lowest terms in X, Y, Z are

$$Z^2\{a^2 + \phi(\xi, \eta, \zeta)\} - (aX + Y)^2 = 0.$$

These break up into two factors.

Hence the point ξ, η, ζ is a binode on the surface.

Hence the straight line $z = 0$, $ax + b + y = 0$ is a binodal line on the surface.

And $z = 0$ is the locus of binodal lines.

Hence the factor z^4 of the discriminant is accounted for.

(C.) *The Locus $\phi(x, y, z) = 0$ is connected with a Curve Locus, not a Surface Locus, of Ultimate Intersections.*

For the fundamental equations are in this case

$$z^2\{a^2 + \phi(x, y, z)\} - (ax + b + y)^2 = 0,$$
$$2z^2a - 2x(ax + b + y) = 0,$$
$$2(ax + b + y) = 0.$$

Hence, if $\phi(x, y, z) = 0$, then, in order that the above equations may be satisfied,

$$ax + b + y = 0,$$
$$z = 0.$$

The locus of these points is the curve

$$z = 0, \quad \phi(x, y, z) = 0.$$

This belongs to one of the exceptional cases enumerated in the Section VI. of this paper.

Example 17.—*This example shows the difference between the cases when the equation is of the Second Degree in the parameters and those in which it is of a Higher Degree, so far as regards Binode and Unode Loci.*

Let the surfaces be

$$\alpha(x-a)^3 + 3\beta(x-a)^2 z + cz^3 + 3d(y-b)^2 + ez^2 = 0.$$

(A.) *The Discriminant.*

It is the same as that of the equation

$$\alpha X^3 + 3\beta z X^2 Z + 3dY^2 Z + (cz^3 + ez^2) Z^3 = 0.$$

Therefore

$$S = -\beta^2 d^2 z^2,$$
$$T = 4d^3 z^2 \{\alpha^2(e + cz) + 2\beta^3 z\}.$$

Therefore

$$\Delta = 16 d^6 z^4 \{\alpha^4 (e + cz)^2 + 4\alpha^2\beta^3 z (e + cz)\}.$$

(B.) *The Locus of Biplanar Nodes is $z = 0$.*

For putting $x = a + X$, $y = b + Y$, $z = Z$, the equation becomes

$$\alpha X^3 + 3\beta X^2 Z + cZ^3 + 3dY^2 + eZ^2 = 0.$$

The edge of the biplanes is given by $Y = 0$, $Z = 0$.

Hence the edge of the biplanes lies in the biplanar node locus $z = 0$, and, therefore, satisfies the condition for contact with the biplanar node locus.

Hence the factor z^4 is accounted for (Art. 15).

(C.) *If $e = 0$, the Locus of Uniplanar Nodes is $z = 0$.*

In this case,

$$\Delta = 16 d^6 (\alpha^4 c^2 + 4\alpha^2\beta^3 c) z^6.$$

Hence the factor z^6 is accounted for (Art. 12).

(D.) *If $\alpha = 0$, Δ appears to vanish, but then the equation of the Surfaces,*

$$3\beta(x-a)^2z + cz^3 + 3d(y-b)^2 + ez^2 = 0,$$

is of the Second Degree in the parameters, and if the Discriminant be formed it does not really vanish.

For the discriminant required is not that of the cubic

$$3\beta zX^2Z + 3dY^2Z + (cz^3 + ez^2)Z^3,$$

but of the quadric

$$3\beta zX^2 + 3dY^2 + (cz^3 + ez^2)Z^2.$$

It is therefore

$$9\beta dz^3(cz + e).$$

(E.) *The Locus of Biplanar Nodes is now $z = 0$, the edge of the Biplanar Node being in the Biplanar Node Locus.*

The edge satisfies the condition for contact with the biplanar node locus. Hence the factor z^3 is accounted for (Art. 21).

(F.) *If $e = 0$, the Locus of Uniplanar Nodes is $z = 0$.*

In this case the discriminant is $9\beta cdz^4$. Hence the factor z^4 is accounted for (Art. 22).

SECTION V. (Arts. 26–29).—THE INTERSECTIONS OF CONSECUTIVE SURFACES.

It has been shown that when the analytical condition (76) is satisfied which expresses that the fundamental equations are satisfied by two coinciding systems of values, the number of factors in the discriminant corresponding to conic node, biplanar node, and uniplanar node loci, is less when the degree of the equation in the parameters is the second than when it is of a higher degree.

It has also been shown that, when (76) holds and the degree in the parameters is the second, each surface of the system, its consecutive surfaces, and the locus of ultimate intersections, intersect in a common curve.

It is desirable, therefore, to examine the nature of the intersections of consecutive surfaces in all other cases.

Art. 26.—*To prove that the Surfaces represented by the three fundamental equations intersect in one point on the Envelope Locus, unless the Envelope Locus have stationary contact with each Surface of the System, and then there are two points of intersection.*

(A.) First consider the case of an ordinary envelope.

Let ξ, η, ζ be a point of intersection of the surfaces

$$f(x, y, z, a, b) = 0,$$

$$\frac{Df}{Da} = 0,$$

$$\frac{Df}{Db} = 0.$$

Let $\xi + X$, $\eta + Y$, $\zeta + Z$ be a neighbouring point on the same three surfaces, so that the values of a, b are the same.

Therefore

$$f + [\xi]X + [\eta]Y + [\zeta]Z + \tfrac{1}{2}\{[\xi, \xi]X^2 + \dots\} \\ + \tfrac{1}{6}\{[\xi, \xi, \xi]X^3 + \dots\} + \dots = 0 \quad (183),$$

$$[\alpha] + [\xi, \alpha]X + [\eta, \alpha]Y + [\zeta, \alpha]Z + \tfrac{1}{2}\{[\xi, \xi, \alpha]X^2 + \dots\} + \dots = 0 \quad (184),$$

$$[\beta] + [\xi, \beta]X + [\eta, \beta]Y + [\zeta, \beta]Z + \tfrac{1}{2}\{[\xi, \xi, \beta]X^2 + \dots\} + \dots = 0 \quad (185).$$

Hence because $f = 0$, $[\alpha] = 0$, $[\beta] = 0$, the terms of lowest order in X, Y, Z in (183), (184), (185) are of the first degree in each case. Hence there is one solution $X = 0$, $Y = 0$, $Z = 0$. Hence there is one intersection at this point.

(B.) Next consider the case where the contact is stationary.

The equation of the tangent plane to the envelope locus is

$$(X - \xi)[\xi] + (Y - \eta)[\eta] + (Z - \zeta)[\zeta] = 0 \quad . \quad . \quad . \quad . \quad (186).$$

But also from (28) and (29)* by means of (76) the equation of the tangent plane can also be shown to be

$$[\alpha, \beta]\{(X - \xi)[\alpha, \xi] + (Y - \eta)[\alpha, \eta] + (Z - \zeta)[\alpha, \zeta]\} \\ - [\alpha, \alpha]\{(X - \xi)[\beta, \xi] + (Y - \eta)[\beta, \eta] + (Z - \zeta)[\beta, \zeta]\} = 0 \quad (187).$$

Hence

$$\frac{1}{[\xi]}\{[\alpha, \beta][\alpha, \xi] - [\alpha, \alpha][\beta, \xi]\} = \frac{1}{[\eta]}\{[\alpha, \beta][\alpha, \eta] - [\alpha, \alpha][\beta, \eta]\} \\ = \frac{1}{[\zeta]}\{[\alpha, \beta][\alpha, \zeta] - [\alpha, \alpha][\beta, \zeta]\}.$$

Hence the lowest terms in (183), (184), (185) are not independent of each other; and if the three fractions last written be each $= \mu$, it is possible by multiplying (183)

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* Equations (28) and (29) are satisfied at any point of an envelope locus. Equations (16), (17), (18) are not.

by μ, (184) by $-[\alpha, \beta]$, (185) by $[\alpha, \alpha]$, and adding, to form a new equation in which the lowest terms in X, Y, Z are of the second degree.

Hence the equations (183)–(185) are equivalent to three others in which the lowest terms in X, Y, Z are of degrees 2, 1, 1 respectively. Hence there are two sets of zero values of X, Y, Z. Hence there are two intersections.

Art. 27.—*To prove that the Surfaces represented by the three fundamental equations intersect in two points on the Conic Node Locus, unless it be also an Envelope Locus, and then there are three points of intersection.*

(A.) In the case of the Conic Node Locus $[\xi] = 0$, $[\eta] = 0$, $[\zeta] = 0$.

Hence the lowest terms in X, Y, Z in (183) are of the second degree, in (184) and (185) of the first degree.

Hence there are two intersections.

(B.) In the case where the conic node locus is also an envelope, it will be shown that the values of X, Y, Z, which make

$$[\alpha, \xi]\,X + [\alpha, \eta]\,Y + [\alpha, \zeta]\,Z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (188),$$

$$[\beta, \xi]\,X + [\beta, \eta]\,Y + [\beta, \zeta]\,Z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (189),$$

also make

$$[\xi, \xi]\,X^2 + [\eta, \eta]\,Y^2 + [\zeta, \zeta]\,Z^2 + 2\,[\eta, \zeta]\,YZ + 2\,[\zeta, \xi]\,ZX + 2\,[\xi, \eta]\,XY = 0 \quad (190),$$

so that the lowest terms in the equations, by which (183)–(185) may be replaced, are of degree 3, 1, 1 respectively, and hence there are three intersections.

Now the cone (190) touches the tangent plane to the conic node locus, viz. :—

$$\begin{aligned}&[\alpha, \beta]\,\{[\alpha, \xi]\,X + [\alpha, \eta]\,Y + [\alpha, \zeta]\,Z\}\\ &- [\alpha, \alpha]\,\{[\beta, \xi]\,X + [\beta, \eta]\,Y + [\beta, \zeta]\,Z\} = 0 \quad . \quad . \quad . \quad . \quad . \quad (191),\end{aligned}$$

this being the form for the tangent plane to the conic node locus which can be deduced from (28), (29), and (76).

Hence, to find the line of contact, whose equations are

$$X/X' = Y/Y' = Z/Z',$$

the origin of co-ordinates being taken at the singular point,

$$\begin{aligned}&\{[\xi, \xi]\,X' + [\xi, \eta]\,Y' + [\xi, \zeta]\,Z'\} \,/\, \{[\alpha, \beta]\,[\alpha, \xi] - [\alpha, \alpha]\,[\beta, \xi]\}\\ = &\{[\xi, \eta]\,X' + [\eta, \eta]\,Y' + [\eta, \zeta]\,Z'\} \,/\, \{[\alpha, \beta]\,[\alpha, \eta] - [\alpha, \alpha]\,[\beta, \eta]\}\\ = &\{[\xi, \zeta]\,X' + [\eta, \zeta]\,Y' + [\zeta, \zeta]\,Z'\} \,/\, \{[\alpha, \beta]\,[\alpha, \zeta] - [\alpha, \alpha]\,[\beta, \zeta]\} \quad . \quad (192).\end{aligned}$$

It will now be verified that

$$X'/\{[\alpha,\eta][\beta,\zeta]-[\alpha,\zeta][\beta,\eta]\} = Y'/\{[\alpha,\zeta][\beta,\xi]-[\alpha,\xi][\beta,\zeta]\}$$
$$= Z'/\{[\alpha,\xi][\beta,\eta]-[\alpha,\eta][\beta,\xi]\}.$$

For substituting these values in the first and second ratios of (192), they become

$$\frac{D[[\xi],[\eta],[\zeta]]}{D[\ \xi\ ,\ \alpha\ ,\ \beta\ ]}\Big/\frac{D[[\alpha],[\beta]]}{D[\ \xi\ ,\ \alpha\ ]}$$
$$=\frac{D[[\xi],[\eta],[\zeta]]}{D[\ \eta\ ,\ \alpha\ ,\ \beta\ ]}\Big/\frac{D[[\alpha],[\beta]]}{D[\ \eta\ ,\ \alpha\ ]} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (193).$$

Now, since the equations (16), (17), (18), (28), (29), are equivalent to three equations only, it follows that

$$\frac{D[[\xi],[\eta],[\zeta],[\alpha]]}{D[\ \xi\ ,\ \eta\ ,\ \alpha\ ,\ \beta\ ]}=0,$$

which may be written

$$[\alpha,\beta]\frac{D[[\xi],[\eta],[\zeta]]}{D[\ \xi\ ,\ \eta\ ,\ \alpha\ ]}-[\alpha,\alpha]\frac{D[[\xi],[\eta],[\zeta]]}{D[\ \xi\ ,\ \eta\ ,\ \beta\ ]}$$
$$+[\alpha,\eta]\frac{D[[\xi],[\eta],[\zeta]]}{D[\ \xi\ ,\ \alpha\ ,\ \beta\ ]}-[\alpha,\xi]\frac{D[[\xi],[\eta],[\zeta]]}{D[\ \eta\ ,\ \alpha\ ,\ \beta\ ]}=0 \quad . \quad . \quad . \quad (194).$$

Also from (16), (17), (18), (28), (29) may be deduced

$$\frac{D[[\xi],[\eta],[\zeta],[\beta]]}{D[\ \xi\ ,\ \eta\ ,\ \alpha\ ,\ \beta\ ]}=0,$$

which may be written

$$[\beta,\beta]\frac{D[[\xi],[\eta],[\zeta]]}{D[\ \xi\ ,\ \eta\ ,\ \alpha\ ]}-[\alpha,\beta]\frac{D[[\xi],[\eta],[\zeta]]}{D[\ \xi\ ,\ \eta\ ,\ \beta\ ]}$$
$$+[\beta,\eta]\frac{D[[\xi],[\eta],[\zeta]]}{D[\ \xi\ ,\ \alpha\ ,\ \beta\ ]}-[\beta,\xi]\frac{D[[\xi],[\eta],[\zeta]]}{D[\ \eta\ ,\ \alpha\ ,\ \beta\ ]}=0 \quad . \quad . \quad . \quad (195).$$

Multiplying (194) by $[\alpha,\beta]$, (195) by $[\alpha,\alpha]$, and subtracting,

$$\frac{D\{[\xi],[\eta],[\zeta]\}}{D\ \ [\xi,\ \alpha,\ \beta]}\{[\alpha,\eta][\alpha,\beta]-[\beta,\eta][\alpha,\alpha]\}$$
$$=\frac{D\{[\xi],[\eta],[\zeta]\}}{D\ \ [\eta,\ \alpha,\ \beta]}\{[\alpha,\xi][\alpha,\beta]-[\beta,\xi][\alpha,\alpha]\},$$

which proves (193).

This proves that the first ratio of (192) is equal to the second. By symmetry the first ratio is also equal to the third.

Hence the line of contact is the intersection of the planes

$$\left.\begin{array}{l}[\alpha, \xi]\mathrm{X} + [\alpha, \eta]\mathrm{Y} + [\alpha, \zeta]\mathrm{Z} = 0 \\ [\beta, \xi]\mathrm{X} + [\beta, \eta]\mathrm{Y} + [\beta, \zeta]\mathrm{Z} = 0\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (196).$$

Hence the values of X, Y, Z which satisfy (188) and (189) also satisfy (190), which was to be proved.

It may be noticed that the equations (196) are those of the tangent planes to the surfaces $\mathrm{D}f/\mathrm{D}\alpha = 0$, $\mathrm{D}f/\mathrm{D}\beta = 0$ at ξ, η, ζ.

Art. 28.—*To prove that the Surfaces represented by the three fundamental equations intersect in three points on the Biplanar Node Locus, unless the Edge of the Biplanar Node always touch the Biplanar Node Locus, and then there are four points of intersection.*

(A.) In this case

$$[\xi, \xi]\mathrm{X}^2 + [\eta, \eta]\mathrm{Y}^2 + [\zeta, \zeta]\mathrm{Z}^2 + 2[\eta, \zeta]\mathrm{YZ} + 2[\zeta, \xi]\mathrm{ZX} + 2[\xi, \eta]\mathrm{XY}$$

breaks up into the factors

$$[\xi, \xi]\{[\xi, \eta]\mathrm{X} + [\eta, \eta]\mathrm{Y} + [\zeta, \eta]\mathrm{Z}\}$$
$$-\{[\xi, \eta] \pm \surd([\xi, \eta]^2 - [\xi, \xi][\eta, \eta])\}\{[\xi, \xi]\mathrm{X} + [\xi, \eta]\mathrm{Y} + [\xi, \zeta]\mathrm{Z}\}.$$

Now, since equations (16), (17), (18), (28), (29) are equivalent to only two independent equations,

$$[\xi, \eta]\mathrm{X} + [\eta, \eta]\mathrm{Y} + [\zeta, \eta]\mathrm{Z} \quad \text{and} \quad [\xi, \xi]\mathrm{X} + [\xi, \eta]\mathrm{Y} + [\xi, \zeta]\mathrm{Z}$$

are linear functions of

$$[\xi, \alpha]\mathrm{X} + [\eta, \alpha]\mathrm{Y} + [\zeta, \alpha]\mathrm{Z} \quad \text{and} \quad [\xi, \beta]\mathrm{X} + [\eta, \beta]\mathrm{Y} + [\zeta, \beta]\mathrm{Z}.$$

Hence the equations (183)–(185) may be reduced to others in which the degrees of the lowest terms in X, Y, Z are 3, 1, 1 respectively. Hence there are three intersections.

(B.) If, however, the edge of the binode always touches the binode locus, then by (126) it follows that

$$[\xi, \alpha]\mathrm{X} + [\eta, \alpha]\mathrm{Y} + [\zeta, \alpha]\mathrm{Z} \text{ is a multiple of } [\xi, \beta]\mathrm{X} + [\eta, \beta]\mathrm{Y} + [\zeta, \beta]\mathrm{Z}.$$

In this case

$$[\xi, \eta]\mathrm{X} + [\eta, \eta]\mathrm{Y} + [\zeta, \eta]\mathrm{Z} \quad \text{and} \quad [\xi, \xi]\mathrm{X} + [\xi, \eta]\mathrm{Y} + [\xi, \zeta]\mathrm{Z}$$

are not, as in the last case, linear functions of

$$[\xi, \alpha]\mathrm{X} + [\eta, \alpha]\mathrm{Y} + [\zeta, \alpha]\mathrm{Z} \quad \text{and} \quad [\xi, \beta]\mathrm{X} + [\eta, \beta]\mathrm{Y} + [\zeta, \beta]\mathrm{Z},$$

for, if so, $[\xi, \xi] X + [\xi, \eta] Y + [\xi, \zeta] Z$ would be a multiple of $[\xi, \eta] X + [\eta, \eta] Y + [\eta, \zeta] Z$, and the biplanes would coincide, and there would be a uniplanar node.

Consequently, in this case the equations can be reduced as follows :—

The lowest terms in (183) to be of the second degree.

The lowest terms in (184) to be of the first degree.

The lowest terms in (185) by means of (184) to be of the second degree.

Hence the degrees are respectively 2, 1, 2.

Hence there are four intersections.

Art. 29.—*To prove that the Surfaces represented by the three fundamental equations intersect in six points on the Uniplanar Node Locus.*

In this case

$$[\xi, \xi] X^2 + [\eta, \eta] Y^2 + [\zeta, \zeta] Z^2 + 2 [\eta, \zeta] YZ + 2 [\zeta, \xi] ZX + 2 [\xi, \eta] XY$$

is a perfect square, and is proportional to the square of $[\xi, \xi] X + [\xi, \eta] Y + [\xi, \zeta] Z$; and this by means of the ratios (48) is proportional to $[\xi, \alpha] X + [\eta, \alpha] Y + [\zeta, \alpha] Z$ and also to $[\xi, \beta] X + [\eta, \beta] Y + [\zeta, \beta] Z$.

Hence the lowest terms in X, Y, Z may be reduced as follows :—

The lowest terms in (183) to be of the third degree ; the lowest terms in (184) to be of the first degree ; and the lowest terms in (185) by means of (184) to be of the second degree.

Hence the degrees are 3, 1, 2 respectively.

Hence there are six intersections.

SECTION VI. (Art. 30).—EXCEPTIONAL CASES.

Art. 30.

It remains to notice the exceptional cases in which the locus of ultimate intersections is not a surface.

An example is given of each, but the theory is not developed.

The general case which has been considered in this paper is that in which the fundamental equations are satisfied by values of the coordinates which are functions of *both* parameters.

The exceptional cases are :—

(I.) When the fundamental equations are satisfied by values of the coordinates which are functions of *one* parameter.

(II.) When the fundamental equations are satisfied by values of the coordinates which are functions of *neither* parameter, *i.e.*, are independent of the parameters.

(III.) When the fundamental equations cannot be satisfied by any values of the coordinates which make the discriminant or a factor of it vanish, the values of the parameters being finite.

(IV.) When the three fundamental equations, which contain the five quantities x, y, z, a, b are equivalent to two relations only between them.

(V.) When the three fundamental equations, which contain the five quantities x, y, z, a, b are equivalent to one relation only between them.

I. *The Fundamental Equations are satisfied by values of the coordinates which are functions of one parameter only.*

In this case, eliminating the parameter, two relations between the coordinates are obtained. Hence the locus of ultimate intersections is a curve.

Example 18.

Let the surfaces be

$$a^2x^2 - 2abxy + b^2y^2 - 2a(x+1) - 2by + z = 0 \quad . \quad . \quad . \quad (197).$$

(A.) *The Discriminant.*

It is

$$\begin{vmatrix} x^2 & -xy & -x-1 \\ -xy & y^2 & -y \\ -x-1 & -y & z \end{vmatrix}$$
$$= -y^2(2x+1)^2.$$

(B.) The coordinates of each point on the locus of ultimate intersections must satisfy (197) and

$$\left.\begin{aligned} ax^2 - bxy - (x+1) = 0 \\ -axy + by^2 - y = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (198).$$

From (197) and (198)

$$-a(x+1) - by + z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (199).$$

(i.) Now a solution of the second of equations (198) is

$$y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (200).$$

Substituting in the first of equations (198) and in (199)

$$ax^2 - (x+1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201),$$

$$-a(x+1) + z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (202).$$

From (201) and (202)

$$zx^2 - (x + 1)^2 = 0 \quad \ldots \quad (203).$$

A part of the locus of ultimate intersections is, therefore, given by (200) and (203). In this case x, y, z may be considered to be functions of a only. It will be noticed that if (200) be satisfied, $\Delta = 0$. But $\Delta = 0$ does not suffice to determine this part of the locus of ultimate intersections.

(ii.) Next take the other solution of the second of equations (198), viz. :—

$$-ax + by - 1 = 0 \quad \ldots \quad (204).$$

Combining this with the first of equations (198),

$$2x + 1 = 0 \quad \ldots \quad (205).$$

Hence, by (204),

$$y = (2 - a)/2b \quad \ldots \quad (206).$$

Therefore, by (199),

$$z = 1 \quad \ldots \quad (207).$$

Hence another portion of the locus of ultimate intersections is given by (205) and (207).

In this case the coordinates of any point on the locus of ultimate intersections may be regarded as functions of the single parameter $(2 - a)/2b$.

It will be noticed that if (205) be satisfied, $\Delta = 0$; but $\Delta = 0$ is not sufficient to determine this part of the locus of ultimate intersections.

II. *The Fundamental Equations are satisfied by values of the coordinates which are independent of the parameters.*

In this case all the surfaces of the system pass through a finite number of fixed points, or a fixed curve.

Example 19.

Let the surfaces be

$$\psi(x, y, z) + a\phi(x, y, z) + b\chi(x, y, z) = 0.$$

(A.) To find the locus of ultimate intersections, it is necessary to satisfy at the same time the above, and

$$\phi(x, y, z) = 0,$$
$$\chi(x, y, z) = 0.$$

Hence it is necessary to satisfy

$$\psi(x, y, z) = 0, \quad \phi(x, y, z) = 0, \quad \chi(x, y, z) = 0$$

The locus is, therefore, generally a finite number of points.

The values of x, y, z are independent of the parameters.

(B.) If two of the three expressions ψ, ϕ, χ, say ψ, ϕ, have a common factor θ, then the curve $\theta = 0$, $\chi = 0$ is a part of the locus of ultimate intersections.

(C.) If the equation of the system of surfaces be transformed to plane coordinates, then a point has an equation, and the locus of ultimate intersections would have an equation, which could be determined as a factor of the discriminant.

III. *The Fundamental Equations cannot be satisfied by any values of the coordinates which make the Discriminant vanish, the parameters being finite.*

Example 20.

Let the surfaces be the spheres

$$(z + c)(a^2 + b^2) - (c + d)\{2ax + 2by - x^2 - y^2 - (z - c)(z - d)\} = 0,$$

where c, d are fixed constants; a, b are the parameters.

They all touch the plane $z = d$, and the sphere $x^2 + y^2 + z^2 = c^2$.

(A.) *The Discriminant.*

It is

$$\begin{vmatrix} z + c & 0 & -(c + d)x \\ 0 & z + c & -(c + d)y \\ -(c + d)x & -(c + d)y & (c + d)\{x^2 + y^2 + (z - c)(z - d)\} \end{vmatrix}.$$

$$= (c + d)(z - d)(x^2 + y^2 + z^2 - c^2)(z + c).$$

(B.) *The Plane $z - d = 0$ is a part of the Envelope.*

(C.) *The Sphere $x^2 + y^2 + z^2 - c^2 = 0$ is a part of the Envelope.*

(D.) The remaining factor $z + c$ requires explanation. It is on account of this factor that this example is introduced.

If $z + c = 0$, the left-hand side of the equation of the system of surfaces, which is of the second degree in a, b breaks up into two factors, one of the first degree in a, b, the other of degree zero.

But the fundamental equations being equivalent to

$$\begin{aligned} a(z + c) \qquad\qquad - x(c + d) &= 0, \\ b(z + c) - y(c + d) &= 0, \\ -a(c + d)x - b(c + d)y + (c + d)[x^2 + y^2 + (z - c)(z - d)] &= 0, \end{aligned}$$

cannot be simultaneously satisfied by finite values of a, b when $z + c = 0$.

For if $z + c = 0$ and a, b be finite, the equations are equivalent to

$$x = 0,\ y = 0, \quad 2c(c + d)^2 = 0,$$

which equations cannot be satisfied.

Hence the values of a, b are infinite.

IV. *The Fundamental Equations are equivalent to only two relations between the coordinates and parameters.*

In such a case the discriminant must vanish identically.

Example 21.

Let the surfaces be

$$\alpha(x - a)^3 + 3\beta(y - b)^2 = 0,$$

where α, β are fixed constants; a, b the arbitrary parameters.

The other fundamental equations are

$$3\alpha(x - a)^2 = 0,$$
$$6\beta(y - b) = 0.$$

Hence the discriminant vanishes identically.

It may be noticed that in this case each surface of the system has a unodal line. Hence the singularity is of a higher order than when each surface has a single unode.

V. *The Fundamental Equations are equivalent to only one relation between the coordinates and parameters.*

In such a case the discriminant must vanish identically.

Analytically

$$[f(x, y, z, a, b)]^2 = 0$$

is an example.

But the left-hand side is resoluble.

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PART II.—THE EQUATION OF THE SYSTEM OF SURFACES IS A RATIONAL INTEGRAL FUNCTION OF THE COORDINATES AND TWO ARBITRARY PARAMETERS.

Section I. (Art. 1).—Preliminary Theorems.

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1.—(A.) If ξ, η, ζ are the coordinates of any point on the locus $\phi(x, y, z) = 0$ (where ϕ is a rational integral indecomposable function of x, y, z), and if the substitutions $x = \xi$, $y = \eta$, $z = \zeta$ make $\psi(x, y, z)$ and all its partial differential coefficients with regard to x, y, z up to the n^{th} order vanish, and if they also make any one of the partial differential coefficients of the $(n + 1)^{\text{th}}$ order vanish, they will also make all the partial differential coefficients of the $(n + 1)^{\text{th}}$ order vanish (ψ being a rational integral function of x, y, z, but not in general indecomposable) 169

(B.) (i.) If ξ, η, ζ are the coordinates of any point on the locus $\phi(x, y, z) = 0$ (where ϕ is a rational integral function of x, y, z which contains no repeated factors), and if the substitutions $x = \xi$, $y = \eta$, $z = \zeta$ make $\psi(x, y, z) = 0$ (where ψ is a rational integral function of x, y, z), then ψ contains the first power of ϕ as a factor.

(ii.) If $x = \xi$, $y = \eta$, $z = \zeta$ make $\psi = 0$, $\dfrac{\partial\psi}{\partial x} = 0$, then ψ contains the second power of ϕ as a factor.

(iii.) If $x = \xi$, $y = \eta$, $z = \zeta$ make $\psi = 0$, $\dfrac{\partial\psi}{\partial x} = 0$, . . . $\dfrac{\partial^{m-1}\psi}{\partial x^{m-1}} = 0$, then ψ contains ϕ^m as a factor 171

(C.) If u, v be determined as functions of other quantities by the equations

$$\phi(u, v) = 0, \quad \psi(u, v) = 0,$$

where ϕ and ψ are rational integral functions of u, v and the other quantities; then if two systems of common values of u, v become equal, they will also satisfy the equation

$$\frac{\mathrm{D}[\phi, \psi]}{\mathrm{D}[u, v]} = 0.$$

Conversely, if values of u, v can be found to satisfy at the same time the three equations

$$\phi(u, v) = 0, \quad \psi(u, v) = 0, \quad \frac{\mathrm{D}[\phi, \psi]}{\mathrm{D}[u, v]} = 0,$$

then these values count twice over among the common solutions of the equations

$$\phi(u, v) = 0, \quad \psi(u, v) = 0,$$

except in the case where ϕ and ψ are of the first degree in u and v; and then the two equations have an infinite number of solutions in common 173

(D.) To determine the conditions that the equations $\phi(u, v) = 0$, $\psi(u, v) = 0$ may be satisfied by three coinciding systems of common values 174

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Section IV. (*Arts.* 16–25).—*Consideration of cases reserved from the previous section. The degree of* $f(x, y, z, a, b)$ *in* a, b *is now the second, and the equations* $Df/Da = 0$, $Df/Db = 0$ *are indeterminate equations for the parameters at points on the Locus of Ultimate Intersections.*

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Section VI. (Art. 30).—Exceptional Cases.

VI. *Repulsion and Rotation produced by Alternating Electric Currents.*

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Communicated by Professor J. J. Thomson.

Received November 5,—Read December 10, 1891.

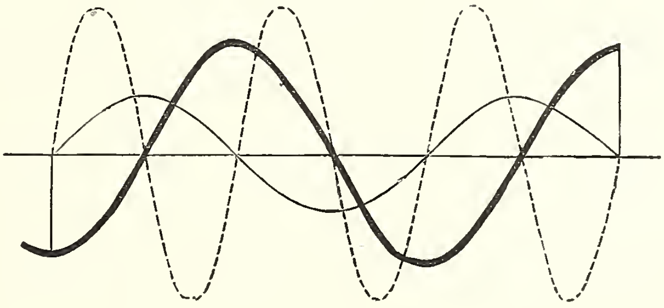
In the 'Electrical World,' May, 1887, p. 258, or the 'Electrical Engineer' (New York), June, 1887, p. 211, "Novel Phenomena of Alternating Currents," may be seen an account of some experiments by Professor Elihu Thomson on the mechanical force between conductors in which alternating currents are circulating.

In the case of a ring of metal in the presence of an electromagnet, in the coils of which an alternating current is passing, a force of repulsion is experienced by the ring, and this may be accounted for in the words of Professor Thomson as follows :—

"It may be stated as certainly true that were the induced currents in the closed conductor unaffected by any self-induction, the only phenomena exhibited would be alternate equal attractions and repulsions, because currents would be induced in opposite directions to that of the primary current when the latter current was changing from zero to maximum positive or negative current, so producing repulsion ; and would be induced in the same direction when changing from maximum positive or negative to zero, so producing attractions."

This may be illustrated by fig. 1. Here the strong line represents the primary and

Fig. 1.



the thin line the secondary, while of the dotted line any ordinate is the product of the ordinates of the lines representing the intensities of the currents and, hence, represents the mechanical force of attraction or repulsion.

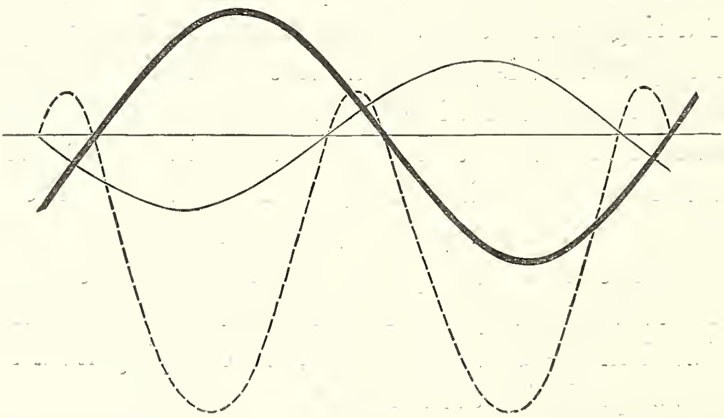
In the case of self-induction causing a lag, shift, or retardation of phase in the secondary current, the circumstances are described by Professor ELIHU THOMSON as follows :—

"It will be noticed that the period during which the currents are opposite, and during which repulsion can take place, is lengthened at the expense of the period during which the currents are in the right direction for attractive action.

"But far more important still in giving prominence to the repulsive effect than this difference of effective period, is the fact that, during the period of repulsion, both the induced and inducing currents have their greatest values, while, during the period of attraction, the currents are of small amounts comparatively. There is then a *repulsion due to the summative effects of strong opposite currents* for a *lengthened period* against an *attraction* due to the summative effects of *weak currents* of the *same direction* during a *shortened period*, the resultant effect being a greatly preponderating repulsion."

The diagram for this is given in fig. 2.

Fig. 2.



Professor THOMSON has proved experimentally that two circular coils, whose planes are perpendicular to the line joining their centres, repel one another when an alternating current traverses one of them.

If the coils consist of circular wires of radii A, a, and the planes be distant b, while the current traversing the primary is of strength I sin pt, then I have shown that the force of repulsion is

$$\mathrm{I}^2 \frac{2\pi^2 p^2 b \mathrm{N}}{\mathrm{S}^2 + p^2\mathrm{N}^2} [2\mathrm{E} - (1 + \cos^2\gamma)\,\mathrm{F}]\,[2\mathrm{F} - (1 + \sec^2\gamma)\,\mathrm{E}],$$

where

$$\mathrm{S} = \text{resistance of secondary circuit,}$$

$$\mathrm{N} = \text{its coefficient of self-induction,}$$

$$\sin\gamma = \frac{2\sqrt{(\mathrm{A}a)}}{\sqrt{\{(\mathrm{A}+a)^2 + b^2\}}},$$

and F, E are complete elliptic integrals to modulus sin γ.

This repulsion may easily be taken advantage of by using it as the basis of a meter for alternating currents.

If the coils consist of two circles of radii a and c (the former the greater), with their centres coincident and planes making an angle θ, and we send a current I sin pt through the larger, there will be a couple tending to increase the angle θ.

This has been proved experimentally by Professor THOMSON: its amount proves to be

$$\mathrm{I}^2 \frac{2\pi^4 p^2 \mathrm{N}}{\mathrm{S}^2 + p^2\mathrm{N}^2} \frac{c^4}{a^4} \left[\sin\theta\cos\theta + \tfrac{3}{8}\left(\frac{c}{a}\right)^2 \sin\theta\cos\theta\,(10\cos^2\theta - 3) + \ldots\right].$$

The positions $\theta = 0$ and $\theta = \pi/2$ are positions of equilibrium, the former being unstable and the latter stable.

By making the plane of the primary vertical and suspending the secondary inside so as to be capable of turning round a vertical axis by means of bifilar suspension with $\theta = 0$ as position of equilibrium, the deflection θ, when the alternating current is passing, will give the intensity of current.

We might also get the intensity by suspending the secondary by a single thread and observing the time of a small oscillation about $\theta = \pi/2$.

If the moment of inertia of the secondary about the vertical be mk^2 it is not difficult to show that the number of oscillations per second is

$$\mathrm{I} \frac{2\pi p c^2}{a\sqrt{\left\{\dfrac{2mk^2(\mathrm{S}^2 + \mathrm{N}^2 p^3)}{\mathrm{N}}\right\}}} \left[1 - \tfrac{9}{8}\left(\frac{c}{a}\right)^2 \ldots\right].$$

Professor ELIHU THOMSON has devised other interesting experiments of which the following is an example :—

A sheet of copper is placed so as to half cover an alternating magnetic pole. Upon this, near the pole, is laid a hollow sphere of copper. The electromagnetic action produces a couple so powerful that the friction of rotation is overcome and the sphere is spun round.

The mathematical analysis for this case being complicated I have evaluated the couples called into action in various combinations of hollow spherical and cylindrical shells.

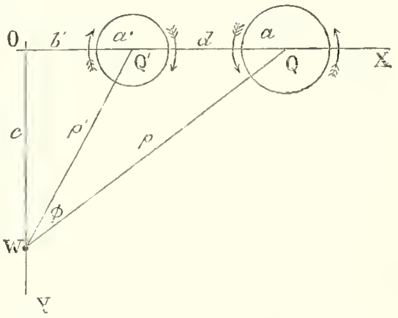
It is a known fact that in a spherical conductor no external field can give rise to induced currents that do not circulate in concentric spherical shells. After a preliminary theorem to the effect that there are no other families of surfaces which possess similar properties, the case has been considered of an infinitely long, thin, circular cylindrical shell in a field consisting of alternating currents parallel to its axis.

If the electrokinetic momentum of the primary field be expanded in harmonics over the cylinder, it turns out that if all the terms of each harmonic have the same phase, there will be no couple acting on the cylinder: as a particular case, if the external field have the same phase throughout there will be no couple.

The next case considered is that of two long cylindrical shells in the presence of an alternating current in a long parallel wire.

If the current in the wire be I cos pt, and σ, σ' be the resistances (across unit length of the surface) and a, a' the radii of the shells, while ρ, ρ' are their distances from the wire, d from one another, and c of the wire from the plane containing the two axes, then the couple on the a shell is in the direction represented in fig. 3, of amount

Fig. 3.



$$-\mathrm{I}^2\frac{8\pi^2p^2\sigma\sigma'a^3a'^3c}{\rho^2\rho'^2d}\left[\frac{1}{\mathrm{D}_1\mathrm{D}'_1}+\frac{4a^2(\rho'^2-d^2)}{\mathrm{D}'_1\mathrm{D}_2\rho^2d^2}+\ldots\right]$$

where

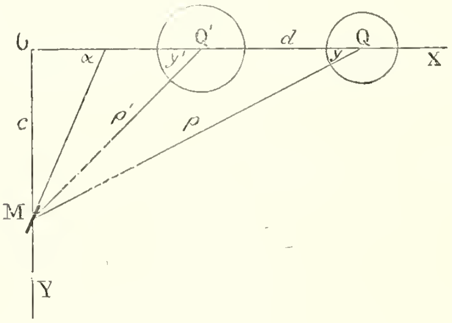
$$\mathrm{D}_n\equiv 4\pi^2a^2p^2+\sigma^2n^2,$$

$$\mathrm{D}'_n\equiv 4\pi^2a'^2p^2+\sigma'^2n^2.$$

In this case the couples on the two shells are, considering only the most important term, of equal magnitudes and of opposite signs.

If a second wire conveying a current of strength $-$ I cos pt be laid close to that already present, we obtain in effect a filament periodically magnetised in a direction perpendicular to its length. Let this direction make α with the plane containing the axes. Then, if QQ′ (the line of length d) subtend ϕ at the magnet M, the couple on the a shell is

Fig. 4.



$$- A^2 \cdot \frac{8\pi^2 p^2 \sigma\sigma' a^3 a'^3}{d^2 D_1 D'_1 \rho^2 \rho'^2} \sin 2\phi,$$

where A cos pt is the intensity of the magnet and terms of the eighth degree in the radii are omitted.

Thus, to this order of approximation, we have the following results :—

(α.) The couples exerted are independent of α, *i.e.*, of the direction of the axis of the electromagnet.

(β.) The shells have equal couples in opposite directions, the parts of the shells directed towards one another being attracted towards the electromagnet if ϕ is less than a right angle, and driven away if ϕ exceeds a right angle.

Fig. 5.

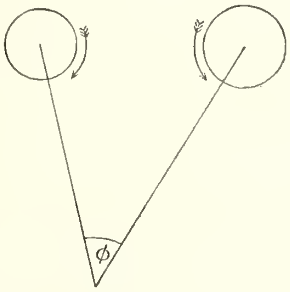
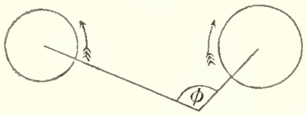


Fig. 6.



If ϕ be a right angle the couple is zero.

We next discuss the case of a thin spherical shell in any field and show that—

(α.) If the field be symmetrical round any diameter, there will be no couple about that diameter.

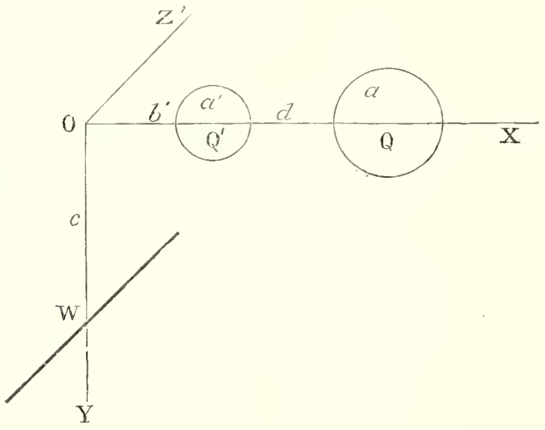
(β.) If the external field be completely in the same phase, or if, when the external magnetic potential is expanded in harmonics over the surface, the terms of the harmonic of any degree are in phases that are the same for the same degree, then the couple will vanish. From this it follows that if the primary field be in the same phase throughout, and any number of perfectly conducting bodies be introduced, their currents will be in the same phase as the primary field, and no couple will be produced.

It has been stated that in the case of Professor ELIHU THOMSON's sphere, spinning on a sheet of copper, the effect was due to the sheet acting as a "shield" and producing an unsymmetrical field. That this explanation is not satisfactory will be evident on considering that the field is unsymmetrical before the sheet is interposed, and that the better the sheet conducts the better the shielding effect; so that if it be a perfect conductor a large couple would be expected whereas in reality there is none whatever.

The effect must, I think, be traced to the fact that the currents induced in the sheet are caused by self-induction to lag, so that the field in action on the sphere does not alternate in one phase.

The results obtained in the preceding analysis are applied to the case of two thin spherical shells in the presence of an alternating current in a straight infinite wire.

Fig. 7.



If the line joining the centres of the spheres be taken as OX, and the axis OY be the shortest distance between OX and the wire W (the last two lines being taken perpendicular to one another), then calling the distances of the centres of the spheres from the origin b, b', and their radii a, a' (as with the cylinders), we have as the first terms of the couples upon the a, b and the a', b' shells—

$$\frac{144\pi^2p^2\sigma\sigma'a^4a'^4c}{\rho^2\rho'^2\,d^3\,\Delta_1\,\Delta'_1}\,(2b+b')\,\mathrm{I}^2,$$

and

$$\frac{144\pi^2p^2\sigma\sigma'a^4a'^4c}{\rho^2\rho'^2\,d^3\,\Delta_1\,\Delta'_1}\,(b+2b')\,\mathrm{I}^2,$$

where

$$\Delta_n \equiv 16\pi^2a^2p^2+(2n+1)^2\sigma^2,$$

and I cos pt, σ, σ', c, and d have the same meanings as before.

If we write h for $\frac{1}{2}(b+b')$, the couples are

$$\frac{72\pi^2p^2\sigma\sigma'a^4a'^4c}{\rho^2\rho'^2d^3\Delta_1\,\Delta'_1}\,(6h\pm d)\,\mathrm{I}^2.$$

Hence

(α.) If σ, σ' or c vanish, the couples vanish, as might be expected.

(β.) The signs of the couples fall into three cases :—

(i.) When $6h$ is positive and greater than d, both couples are positive.

Fig. 8.

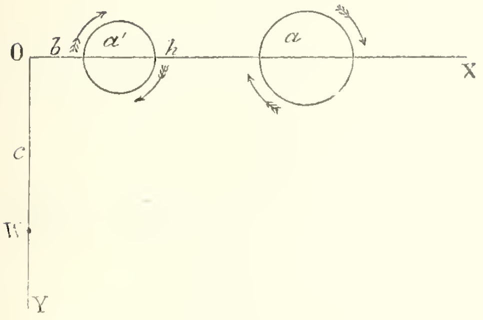
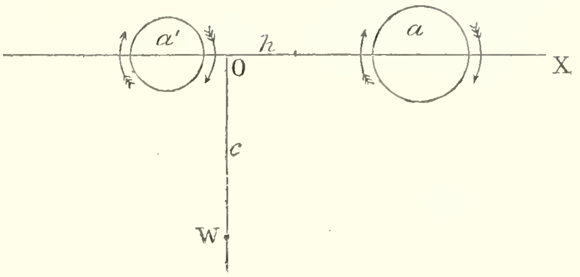
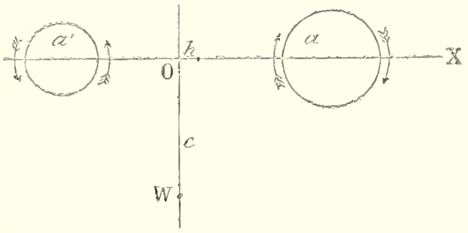


Fig. 9.



(ii.) When $6h$ is numerically less than d, the a, b couple is positive and the a', b' negative.

Fig. 10.



(iii.) When $6h$ is negative and numerically greater than d, both couples are negative.

Fig. 11.

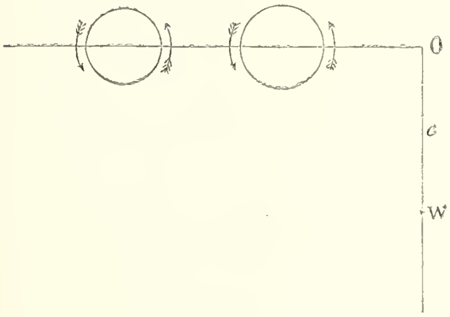
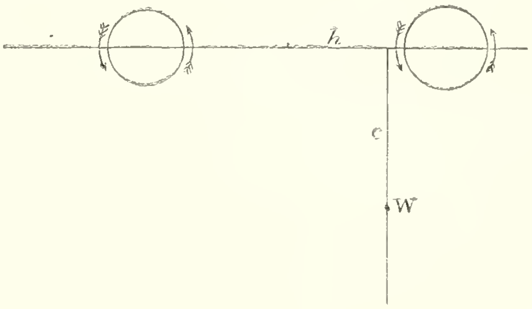


Fig. 12.



If we place another wire by the side of that already in position, and thus make a filament alternately magnetised in a direction perpendicular to its length, and making an angle α with OX (as with the cylinder), then the couple on the a, b sphere is

$$\mathrm{K}^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1 d^3} [\sin 2(\gamma' - \gamma) + 3 \sin 2(\alpha - \gamma - \gamma')],$$

and on the a', b' sphere

$$K^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1 d^3} [- \sin 2 (\gamma' - \gamma) + 3 \sin 2 (\alpha - \gamma - \gamma')],$$

where K cos pt is the magnetic moment of unit length.

To this order then

(α.) If the couples on the shells be equal and opposite

$$\alpha - \gamma - \gamma' = 0, \text{ or } \tfrac{1}{2}\pi.$$

(β.) The couples will not vanish when $c = 0$ (and $\gamma = \gamma' = 0$) unless, in addition, $\alpha = 0$ or $\frac{1}{2}\pi$.

In this case considerations of symmetry would give right results.

(γ.) For an example, take

$$\gamma = 30^\circ, \qquad \gamma' = 60^\circ,$$

and the couples will be

$$K^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1 d^3} \left[\pm \frac{\sqrt{3}}{2} - 3 \sin 2\alpha \right],$$

the upper sign referring to the a, b shell.

The sign of the bracket will be + when $\alpha = -\frac{1}{4}\pi$.
" " " ± " $\alpha = 0$ or $\frac{1}{2}\pi$.
" " " − " $\alpha = +\frac{1}{4}\pi$.

Fig. 13.

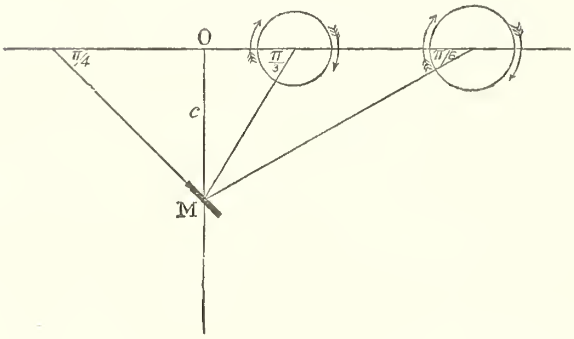


Fig. 14.

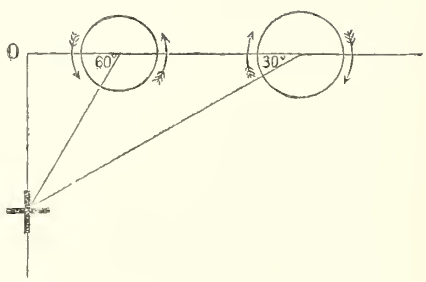
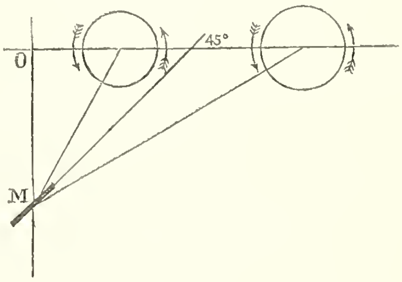


Fig. 15.



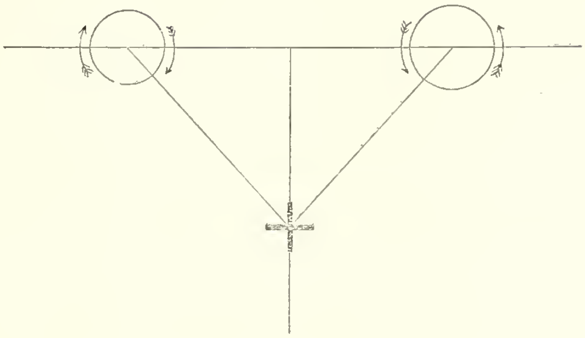
(δ.) In confirmation it may be noticed that when $\gamma = \pi - \gamma'$ (*i.e.*, when the spheres are equidistant from the origin), and α is zero or a right angle, the couples are equal and opposite.

Their magnitudes are

$$\mp K^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1\, d^3} \sin 2\gamma,$$

and are negative and positive on the a, b and a', b' shells as long as the former is to the right of the latter.

Fig. 16.



We now consider the couples on two spherical shells in the presence of a magnetic pole of strength H cos pt.

We take the same axes as before with the magnetic pole at (o, c, o).

The couple on the a, b shell proves to be

$$- H^2 \frac{36\pi^2 p^2 a^4 a'^4 c\sigma\sigma'}{\rho^3 \rho'^3 \Delta_1 \Delta'_1\, d^3} (b + 2b'),$$

and that on the a', b' shell

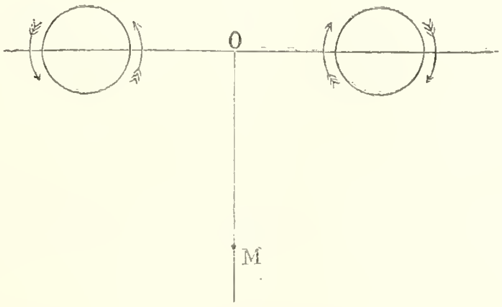
$$- H^2 \frac{36\pi^2 p^2 a^4 a'^4 c\sigma\sigma'}{\rho^3 \rho'^3 \Delta_1 \Delta'_1\, d^3} (2b + b').$$

Hence

(α.) The couples vanish, as they should, when $c = 0$, or $\sigma = 0$, or $\sigma' = 0$.

(β.) When $b = - b'$ and $a = a'$, the figure is symmetrical to the plane YOZ, and the couples are equal and opposite: that on the a, b shell to the right is positive.

Fig. 17.



(γ.) For other cases the discussion is similar to that for the cylinders with parallel current I cos pt.

Thus (i.), when $h\ \{\equiv \frac{1}{2}(b + b')\}$ is positive, and greater than $d/6$ the couples are both negative.

Fig. 18.

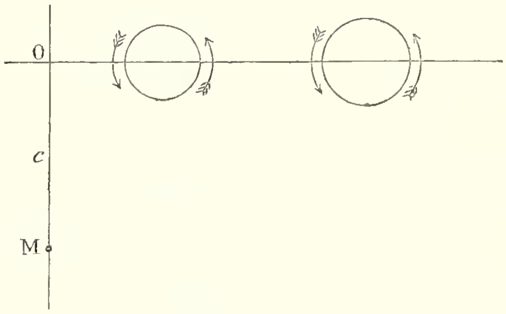
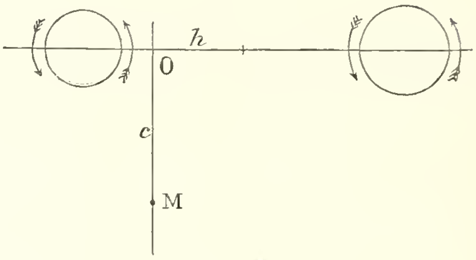
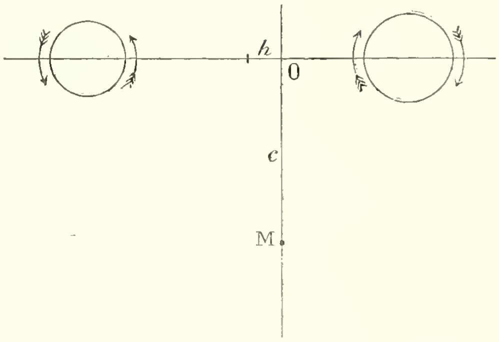


Fig. 19.



(ii.) When $6h$ is numerically less than d, the signs are $+, -$.

Fig. 20.



(iii.) When $6h$ is negative and numerically greater than d, the couples are both positive.

Fig. 21.

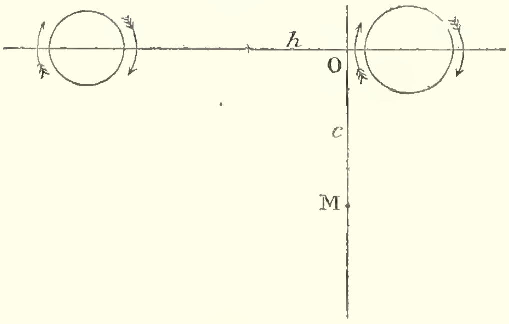
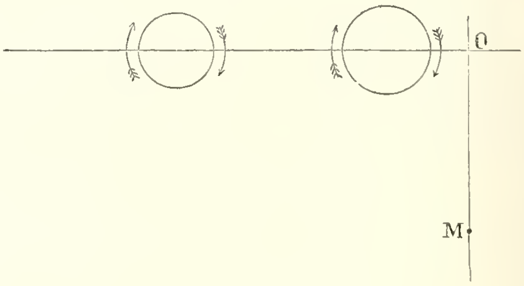


Fig. 22.



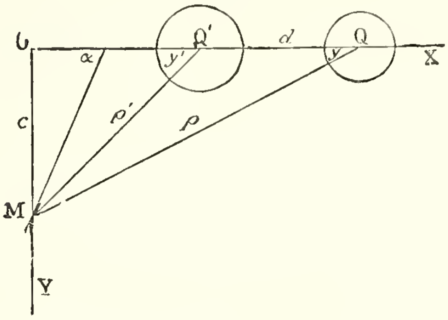
If we take two spherical shells in the presence of a small magnet of moment $K \cos pt$, whose axis cuts the line joining the centres of the spheres at angle α, we find for the couple on the a, b shell

$$-K^2 \frac{27\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{2\Delta_1 \Delta'_1 d^3 \rho^3 \rho'^3} \phi(\alpha, \gamma, \gamma'),$$

where $\phi(\alpha, \gamma, \gamma')$ is written for

$$\sin 2\alpha + 3(\sin 2\gamma + \sin 2\gamma') + \sin\overline{2\alpha - 2\gamma'} - \sin\overline{2\alpha - 2\gamma}$$
$$- 3\sin\overline{2\gamma' - 2\gamma} - 9\sin\overline{2\alpha - 2\gamma - 2\gamma'}.$$

Fig. 23.



For the other shell, we interchange γ and γ'.

For the couples to be equal and opposite when $\gamma' = \pi - \gamma$, we must have $\alpha = 0$ or $\frac{1}{2}\pi$. When $\gamma = \frac{1}{3}\pi$, the a, b couple is positive for both values of α, and when $\gamma = \frac{1}{6}\pi$, it is negative.

Fig. 24.

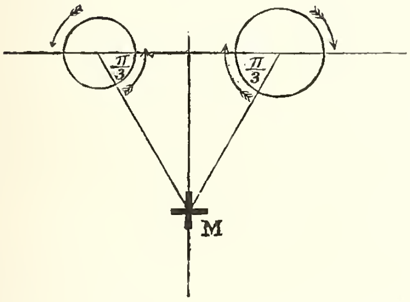
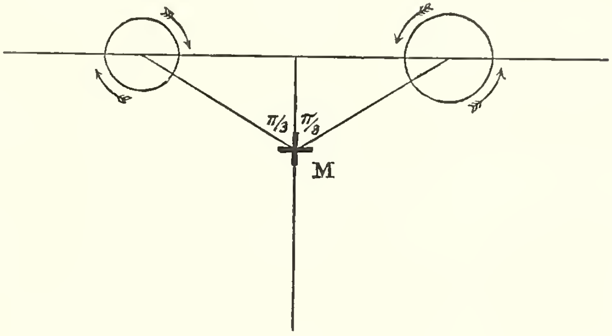


Fig. 25.



We now take as an example

$$\gamma = \pi/6, \qquad \gamma' = \pi/3,$$

and the a, b couple is negative from 0° to about 98° 23′, positive thence to 171° 37′ and negative to 180°. The a', b' couple is negative from 0° to 112° 34′, positive thence to 157° 26′ and negative to 180°.

In fig. 26, $\alpha = 45°$; in fig. 27, $\alpha = 105°$.

Fig. 26.

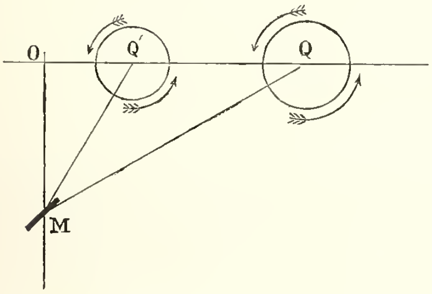
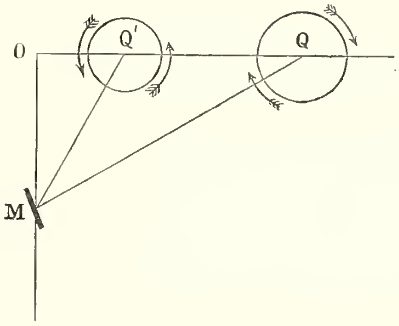


Fig. 27.



In fig. 28, $\alpha = 135°$; in fig. 29, $\alpha = 165°$.

Fig. 28.

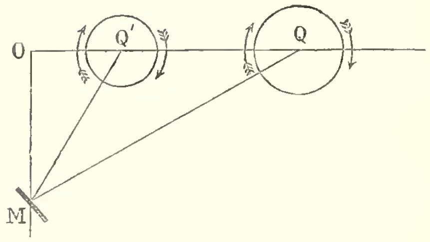
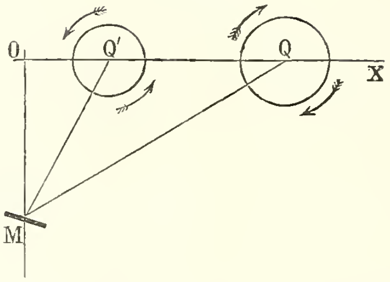


Fig. 29.



For general principles I have derived assistance from the paper by Professor C. Niven, in the 'Philosophical Transactions' of 1881, "On the Induction of Electric Currents in Infinite Plates and Spherical Shells."

The Mechanical Effects of the Currents Induced in One Coil by those in Another.

1. Adopting Maxwell's notation and taking $\dot{y}_1$, $\dot{y}_2$, as the currents in the two circuits, we have

$$2\mathrm{T} = \mathrm{L}\dot{y}_1^2 + 2\mathrm{M}\dot{y}_1\dot{y}_2 + \mathrm{N}\dot{y}_2^2.$$

If R and S be the resistance of the primary and secondary circuits, and the current in the former be I sin pt, then for the second circuit

$$\frac{d}{dt}(\mathrm{M}\dot{y}_1 + \mathrm{N}\dot{y}_2) + \mathrm{S}\dot{y}_2 = 0,$$

whence

$$\dot{y}_2 = \mathrm{A} \cos pt + \mathrm{B} \sin pt,$$

where

$$\frac{\mathrm{A}}{\mathrm{SM}p} = \frac{\mathrm{B}}{\mathrm{MN}p^2} = \frac{-\mathrm{I}}{p^2\mathrm{N}^2 + \mathrm{S}^2}.$$

Thus the "lag" due to self-induction in the secondary coil is $\tan^{-1} \mathrm{N}p/\mathrm{S}$, and the electromagnetic force tending to increase a coordinate x is

$$\dot{y}_1\dot{y}_2 \frac{d\mathrm{M}}{dx},$$

the mean value of which is

$$-\tfrac{1}{2}\frac{\mathrm{I}^2\mathrm{MN}p^2}{\mathrm{S}^2 + \mathrm{N}^2p^2}\frac{d\mathrm{M}}{dx}.$$

If the coils consist of circular wires of radii A and a, and their planes be perpendicular to the line of length b that joins their centres, then Maxwell shows (§ 701) that

$$\mathrm{M} = \frac{4\pi\sqrt{(\mathrm{A}a)}}{\sin\gamma}\{-(1+\cos^2\gamma)\,\mathrm{F} + 2\mathrm{E}\},$$

where

$$\sin\gamma = \frac{2\sqrt{(\mathrm{A}a)}}{\sqrt{\{(\mathrm{A}+a)^2+b^2\}}},$$

and F and E are complete elliptic integrals to modulus $\sin\gamma$.

Also

$$\frac{d\mathrm{M}}{db} = -\frac{\pi}{\sqrt{(\mathrm{A}a)}}\,b\sin\gamma\,\{2\mathrm{F} - (1+\sec^2\gamma)\,\mathrm{E}\},$$

so that the repulsion is

$$\mathrm{I}^2\frac{2\pi^2p^2b\mathrm{N}}{\mathrm{S}^2+p^2\mathrm{N}^2}[2\mathrm{E}-(1+\cos^2\gamma)\,\mathrm{F}]\,[2\mathrm{F}-(1+\sec^2\gamma)\,\mathrm{E}].$$

If the coils consist of two circles of radii a and c (the former the greater) with their centres coincident and planes inclined θ, we have from MAXWELL (§ 697),

$$\begin{aligned}\mathrm{M} &= 4\pi^2c\left\{\frac{1}{1.2}\frac{c}{a}[\mathrm{P}'_1(0)]^2\,\mathrm{P}_1(\theta) + \ldots + \frac{1}{r(r+1)}\left(\frac{c}{a}\right)^r[\mathrm{P}'_r(0)]^2\,\mathrm{P}_r(\theta) + \ldots\right\} \\ &= 4\pi^2c\left\{\frac{1}{1.2}\frac{c}{a}\mathrm{P}_1(\theta) + \frac{1}{3.4}\left(\frac{c}{a}\right)^3\left(\frac{3}{2}\right)^2\mathrm{P}_3(\theta) + \ldots\right. \\ &\qquad\left. + \frac{1}{r.r+1}\left(\frac{c}{a}\right)^r\left[\frac{3.5\ldots r}{2.4\ldots(r+1)}\right]^2\mathrm{P}_r(\theta)\ldots\right\}\end{aligned}$$

where r must now be odd.

The couple tending to increase θ is

$$-\tfrac{1}{2}\mathrm{I}^2\frac{\mathrm{M}\mathrm{N}p^2}{\mathrm{S}^2+p^2\mathrm{N}^2}\frac{d\mathrm{M}}{d\theta},$$

or

$$\begin{aligned}&\tfrac{1}{2}\mathrm{I}^2\frac{\mathrm{N}p^2}{\mathrm{S}^2+p^2\mathrm{N}^2}16\pi^4c^2\left[\frac{1}{1.2}\frac{c}{a}\cos\theta + \frac{1}{3.4}\left(\frac{c}{a}\right)\left(\frac{3}{2}\right)^2\frac{5\cos^3\theta-3\cos\theta}{2} + \ldots\right] \\ &\qquad\times\left[\frac{1}{1.2}\frac{c}{a} + \frac{1}{3.4}\left(\frac{c}{a}\right)^3\left(\frac{3}{2}\right)^2\frac{15\cos^2\theta-3}{2} + \ldots\right]\sin\theta,\end{aligned}$$

or

$$\mathrm{I}^2\frac{2\pi^4p^2\mathrm{N}c^2}{\mathrm{S}^2+p^2\mathrm{N}^2}\frac{c^2}{a^2}\sin\theta\cos\theta\left[1+\tfrac{3}{8}\left(\frac{c}{a}\right)^2(10\cos^2\theta-3)+\ldots\right].$$

2. In the course of the following work it will often be necessary to know what kind of distribution of electric currents is likely to be set up in conductors of various shapes on the introduction of external fields: it is known that in a sphere no external field can give rise to currents that do not circulate in concentric spherical surfaces, and it

might be thought that for some other surface an analogous property held, as that in an anchor ring the induced currents always lay in toroidal surfaces.

The question may be stated as follows :—

Given a system of orthogonal surfaces,

$$a = \text{constant},$$
$$b = \text{constant},$$
$$c = \text{constant},$$

and a uniform conductor whose bounding surface has a constant, what is the condition that, whatever be the nature of the external field, the induced currents may lie in the a surfaces.

If the length δs of the line joining the consecutive points a, b, c, and $a + \delta a$, $b + \delta b$, $c + \delta c$ be given by

$$\delta s^2 = A^2\, \delta a^2 + B^2\, \delta b^2 + C^2\, \delta c^2,$$

and if

$$\begin{matrix} u, & v, & w, \\ \alpha, & \beta, & \gamma, \\ F, & G, & H, \end{matrix}$$

denote the components of electric current, of magnetic force, and of electromagnetic momentum along the normals to the three orthogonal surfaces through any point, then MAXWELL's equations of electric currents become

$$4\pi BC\,.\,u = \frac{\partial}{\partial b}(C\gamma) - \frac{\partial}{\partial c}(B\beta)$$

$$4\pi CA\,.\,v = \frac{\partial}{\partial c}(A\alpha) - \frac{\partial}{\partial a}(C\gamma)$$

$$4\pi AB\,.\,w = \frac{\partial}{\partial a}(B\beta) - \frac{\partial}{\partial b}(A\alpha),$$

and if there be no magnetisable matter in the conductor, α, β, γ are components of magnetic induction, and are given by

$$BC\alpha = \frac{\partial}{\partial b}(CH) - \frac{\partial}{\partial c}(BG),$$

with two similar equations.

If σ denote the specific resistance of the conductor, and ψ the electrostatic potential, then within the conductor we have (it being at rest)

$$\sigma u = -\frac{\partial F}{\partial t} - \frac{\partial \psi}{A\,\partial a},$$

with two similar equations.

Remembering that the surface has a constant over it, we have as the surface condition (Professor C. NIVEN, 'Phil. Trans.,' 1881, p. 313)

$$\frac{\partial F}{\partial t} + \frac{\partial \psi}{A\,\partial a} = 0,$$

the normal outwards being in the direction of a increasing: and (from the same source) within the conductor

$$\nabla^2 \psi = 0.$$

Hence $\partial\psi/\partial n$ being known, ψ is uniquely determined (MAXWELL, vol. 5, § 100, *e*).

Let us write $\psi \equiv \partial\chi/\partial t$, and

$$F' \equiv F + \frac{\partial\chi}{A\,\partial a}$$

$$G' \equiv G + \frac{\partial\chi}{B\,\partial b}$$

$$H' \equiv H + \frac{\partial\chi}{C\,\partial c},$$

then our equations are typified by

$$\left.\begin{aligned} \sigma u &= -\frac{\partial F'}{\partial t} \\ BC\alpha &= \frac{\partial}{\partial b}(CH') - \frac{\partial}{\partial c}(BG') \\ 4\pi u BC &= \frac{\partial}{\partial b}(C\gamma) - \frac{\partial}{\partial c}(B\beta) \end{aligned}\right\}$$

If now the conductor be such that no external field can give rise to currents normal to a surfaces, then for all values of F′, G′, H′ consistent with

$$\frac{\partial}{\partial t} F' = 0$$

at the surface, and

$$\frac{\partial}{\partial a}(BCF') + \frac{\partial}{\partial b}(CAG') + \frac{\partial}{\partial c}(ABH') = 0$$

within the conductor, u must everywhere vanish; that is, inside,

$$\left.\begin{aligned} 0 &= \frac{\partial F'}{\partial t} \\ 0 &= \frac{\partial}{\partial b}(C\gamma) - {}^{\partial}\,(B\beta) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.}).$$

Now F′ is everywhere zero before the external field is brought into existence, hence by the former equation (1) it is everywhere zero permanently.

On substituting in the latter equation from the equations connecting β, γ with $\mathrm{F}', \mathrm{G}', \mathrm{H}'$, and remembering that $\mathrm{F}' = 0$, we get

$$\frac{\partial}{\partial b}\left\{\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}(\mathrm{BG}')\right\} + \frac{\partial}{\partial c}\left\{\frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{a}(\mathrm{CH}')\right\} = 0,$$

and this is to hold for all values of G', H' consistent with

$$\frac{\partial}{\partial b}(\mathrm{CAG}') + \frac{\partial}{\partial c}(\mathrm{ABH}') = 0.$$

This condition is replaced by taking

$$\mathrm{CAG}' \equiv \frac{\partial f}{\partial c}$$

$$\mathrm{ABH}' \equiv -\frac{\partial f}{\partial b},$$

and the former relation becomes

$$\frac{\partial}{\partial b}\left[\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial f}{\partial c}\right)\right] - \frac{\partial}{\partial c}\left[\frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial f}{\partial b}\right)\right] \equiv 0,$$

or

$$\frac{\partial}{\partial b}\left[\frac{\mathrm{C}}{\mathrm{AB}}f_c\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right) + \frac{1}{\mathrm{A}^2}f_{ac}\right] - \frac{\partial}{\partial c}\left[\frac{\mathrm{B}}{\mathrm{AC}}f_b\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{AB}}\right) + \frac{1}{\mathrm{A}^2}f_{ab}\right] \equiv 0.$$

therefore

$$f_{bc}\left[\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right) - \frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{\partial b}\left(\frac{\mathrm{C}}{\mathrm{AB}}\right)\right] + f_c\frac{\partial}{\partial b}\left[\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right)\right] - f_b\frac{\partial}{\partial c}\left[\frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{AB}}\right)\right]$$
$$+ f_{ac}\frac{\partial}{\partial b}\left(\frac{1}{\mathrm{A}^2}\right) - f_{ab}\frac{\partial}{\partial c}\left(\frac{1}{\mathrm{A}^2}\right) \equiv 0.$$

Now f being arbitrary the coefficients of f_{bc}, &c. must vanish, therefore

$$\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right) = \frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{AB}}\right) = \text{function of } a \text{ (or constant)} \quad . \quad . \quad . \quad . \qquad \text{(ii.)},$$

and

$$\frac{1}{\mathrm{A}^2} = \text{function of } a \text{ (or constant)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iii.)}.$$

On differentiation, the relation

$$\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right) = \frac{\mathrm{B}}{\mathrm{AC}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{AB}}\right)$$

becomes

$$\frac{\mathrm{C}}{\mathrm{A^2B}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{C}}\right) + \frac{1}{\mathrm{A}}\frac{\partial}{\partial a}\left(\frac{1}{\mathrm{A}}\right) = \frac{\mathrm{B}}{\mathrm{A^2C}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{B}}\right) + \frac{1}{\mathrm{A}}\frac{\partial}{\partial a}\left(\frac{1}{\mathrm{A}}\right).$$

Therefore

$$\frac{\mathrm{C}}{\mathrm{B}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{C}}\right) - \frac{\mathrm{B}}{\mathrm{C}}\frac{\partial}{\partial a}\left(\frac{\mathrm{C}}{\mathrm{B}}\right) = 0.$$

Therefore

$$\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{C}}\right) = 0.$$

Hence B/C is a function of b, c only.

Also it was proved that

$$\frac{\mathrm{C}}{\mathrm{AB}}\frac{\partial}{\partial a}\left(\frac{\mathrm{B}}{\mathrm{AC}}\right) = \text{function of } a \text{ (or constant)}.$$

Therefore

$$\frac{1}{\mathrm{A}}\frac{\partial}{\partial a}\left(\frac{1}{\mathrm{A}}\right) = \text{function of } a \text{ (or constant)}.$$

This, as well as the last condition (iii.), is satisfied by A = function of a. Geometrically this means that the normal distance between a and $a + \delta a$ is everywhere the same, *i.e.*, that the a surfaces are parallel.

Now the radius of curvature ρ_{ab} of the normal section of a constant along the normal to b is given by

$$\frac{1}{\rho_{ab}} = \frac{1}{\mathrm{AB}}\frac{\partial \mathrm{B}}{\partial a},$$

and so too

$$\frac{1}{\rho_{ac}} = \frac{1}{\mathrm{AC}}\frac{\partial \mathrm{C}}{\partial a}.$$

Hence, if B/C is independent of a,

$$\rho_{ab} = \rho_{ac},$$

i.e., the two curvatures at each point of an a surface are equal.

Thus, the property in question is satisfied by a sphere, and only by a sphere or spherical shell, including, as a special case, a plane slab infinite in both directions.

The Mechanical Effect of Currents set up in a Thin Circular Cylindric Shell of Infinite Length.

3. We consider the case of a periodic field consisting of currents parallel to the axis of the shell.

In this case the surface condition informs us that since there is no normal component of electromagnetic momentum, there will be no electrostatic potential; also, if the axis of the cylinder be taken as the axis of cylindric coordinates, we have

$$\sigma w = -\frac{d}{dt}(\mathrm{H} + \mathrm{H}_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{iv.}),$$

where H_0 is the momentum due to the external field and H that due to the induced currents, while σ and w represent the resistance and current across unit length of the surface.

Consider

$$\mathrm{H}_0 = e^{in\theta} \sin pt$$

at the surface of the shell, and let the induced currents give rise to momentum at the surface of

$$\mathrm{H} = e^{in\theta} [\mathrm{B} \sin pt + \mathrm{C} \cos pt].$$

Since H is the potential of a distribution on the cylinder of imaginary matter of surface density w, we have

$$w = \frac{n}{2\pi a} e^{in\theta} [\mathrm{B} \sin pt + \mathrm{C} \cos pt].$$

On substituting in (iv.) we see that

$$\frac{\sigma n}{2\pi a} [\mathrm{B} \sin pt + \mathrm{C} \cos pt] \equiv - p [\mathrm{B} \cos pt - \mathrm{C} \sin pt] - p\mathrm{A} \cos pt.$$

Therefore

$$\left.\begin{aligned} \frac{\sigma n}{2\pi a} \mathrm{B} - p\mathrm{C} &= 0 \\ p\mathrm{A} + p\mathrm{B} + \frac{\sigma n}{2pa} \mathrm{C} &= 0. \end{aligned}\right\}$$

Therefore

$$\frac{- 2\pi ap\mathrm{A}}{4\pi^2a^2p^2 + \sigma^2n^2} = \frac{\mathrm{B}}{2\pi ap} = \frac{\mathrm{C}}{\sigma n}.$$

Hence

$$w = - \mathrm{A} \frac{np \, (2\pi ap \sin pt + \sigma n \cos pt)}{4\pi^2a^2p^2 + \sigma^2n^2} e^{in\theta}$$

corresponding to a term

$$\mathrm{H}_0 = \mathrm{A}e^{in\theta} \sin pt.$$

On differentiating with respect to t, we see that if

$$\mathrm{H}_0 = \mathrm{A}'e^{in\theta} \cos pt,$$

then

$$w = - \mathrm{A}' \frac{np \, (2\pi ap \cos pt - \sigma n \sin pt)}{4\pi^2a^2p^2 + \sigma^2n^2} e^{in\theta}.$$

On separating real and imaginary parts, it is clear that when

$$\begin{aligned} \mathrm{H}_0 = {} & [\mathrm{M} \cos n\theta + \mathrm{N} \sin n\theta] \sin pt \\ & + [\mathrm{Q} \cos n\theta + \mathrm{R} \sin n\theta] \cos pt, \end{aligned}$$

then at the surface

$$
\begin{aligned}
\mathrm{H} = &- \frac{2\pi ap}{4\pi^2a^2p^2 + \sigma^2n^2}\left[2\pi ap \sin pt + \sigma n \cos pt\right]\left[\mathrm{M}\cos n\theta + \mathrm{N}\sin n\theta\right] \\
&- \frac{2\pi ap}{4\pi^2a^2p^2 + \sigma^2n^2}\left[2\pi ap \cos pt - \sigma n \sin pt\right]\left[\mathrm{Q}\cos n\theta + \mathrm{R}\sin n\theta\right]. \quad . \quad . \quad (\mathrm{v}),
\end{aligned}
$$

and

$$w = \frac{n}{2\pi a}\mathrm{H}.$$

4. We now proceed to find the components of magnetic force. When there is no magnetic matter (as we assume to be the case) the equations in cylindricals connecting α, β, γ with F, G, H, are

$$
\begin{aligned}
\varpi\alpha &= \frac{\partial \mathrm{H}}{\partial \theta} - \varpi\frac{\partial \mathrm{G}}{\partial z}, \\
\beta &= \frac{\partial \mathrm{F}}{\partial z} - \frac{\partial \mathrm{H}}{\partial \varpi}, \\
\varpi\gamma &= \frac{\partial}{\partial \varpi}(\varpi \mathrm{G}) - \frac{\partial \mathrm{F}}{\partial \theta}.
\end{aligned}
$$

With the distribution that we have chosen, $\mathrm{F} = 0$, $\mathrm{G} = 0$, and hence

$$
\left.
\begin{aligned}
\alpha &= \frac{1}{\varpi}\frac{\partial \mathrm{H}}{\partial \theta} \\
\beta &= -\frac{\partial \mathrm{H}}{\partial \varpi} \\
\gamma &= 0
\end{aligned}
\right\}.
$$

When the value of H_0 at the surface is

$$\mathrm{H}_0 = \mathrm{A}e^{in\theta}\sin pt,$$

then just outside,

$$\mathrm{H}_0 = \mathrm{A}\left(\frac{\varpi}{a}\right)^n e^{in\theta}\sin pt,$$

so that just outside,

$$
\left.
\begin{aligned}
\alpha_0 &= in\mathrm{A}\frac{\varpi^{n-1}}{a^n}e^{in\theta}\sin pt \\
\beta_0 &= -n\mathrm{A}\frac{\varpi^{n-1}}{a^n}e^{in\theta}\sin pt
\end{aligned}
\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{vi.}).
$$

Now when we consider the mechanical effects, since, on AMPÈRE's theory the mechanical effects of two elements of currents upon each other are equal and oppositely directed in the straight line joining them, the field, due to the induced currents, will have no direct resultant mechanical effect on the cylinder, and it will suffice to take the components of induction due to the external system.

The components of electromagnetic force along $\delta\varpi$, $\varpi\,\delta\theta$, δz are

$$-\beta w, \quad \alpha w, \quad 0,$$

and hence the mechanical force on the shell, parallel to the initial line from which θ is measured, is (per unit length)

$$a\int_0^{2\pi}\delta\theta\,.\,w\,(-\beta\cos\theta-\alpha\sin\theta).$$

The force in the perpendicular direction (perpendicular to the axis of the shell) is

$$a\int_0^{2\pi}\delta\theta\,.\,w\,(\alpha\cos\theta-\beta\sin\theta).$$

The couple about the axis is

$$a^2\int_0^{2\pi}\delta\theta\,.\,\alpha w$$

tending to increase θ.

Now, with the kind of field that we have taken,

$$\mathrm{F}_0 = 0,$$
$$\mathrm{G}_0 = 0,$$

and the period being $2\pi/p$, the value of H_0 over the surface may be expanded in the series,

$$\mathrm{H}_0 = \sum_{n=0}^{n=\infty}\{[\mathrm{M}_n\cos n\theta+\mathrm{N}_n\sin n\theta]\sin pt+[\mathrm{Q}_n\cos n\theta+\mathrm{R}_n\sin n\theta]\cos pt\}.$$

We have already shown (vi.) that when

$$\mathrm{H}_0 = \mathrm{A}e^{in\theta}\sin pt$$

at the surface, then at the surface

$$\alpha_0 = \frac{in}{a}\mathrm{A}e^{in\theta}\sin pt.$$

Hence, with the expansion of H_0, we get

$$\alpha_0 = \sum_{\substack{n=0\\ \text{or } 1}}^{n=\infty}\frac{n}{a}\{[-\mathrm{M}_n\sin n\theta+\mathrm{N}_n\cos n\theta]\sin pt+[-\mathrm{Q}_n\sin n\theta+\mathrm{R}_n\cos n\theta]\cos pt\}.$$

The corresponding value of w has been found to be (v.)

$$\sum_{n=1}^{n=\infty}\frac{-np}{4\pi^2a^2p^2+\sigma^2n^2}\{[2\pi ap\sin pt+\sigma n\cos pt][\mathrm{M}_n\cos n\theta+\mathrm{N}_n\sin n\theta]$$
$$+[2\pi ap\cos pt-\sigma n\sin pt][\mathrm{Q}_n\cos n\theta+\mathrm{R}_n\sin n\theta]\}$$

In finding the mean value of the couple, $a^2 \int_0^{2\pi} \alpha_0 w \, \delta\theta$, we bear in mind that the mean values of $\sin^2 pt$ and of $\cos^2 pt$ are $\frac{1}{2}$, and of $\sin pt \, . \cos pt$, zero. On picking out the coefficients of $\sin^2 pt$ and $\cos^2 pt$ in $\alpha_0 w$, we find for the mean value of the couple

$$\int_0^{2\pi} \delta\theta \sum_{n=1}^{n=\infty} \frac{-an^2p}{2(4\pi^2a^2p^2+\sigma^2n^2)} \left[\begin{aligned} &(-\mathrm{M}_n \sin n\theta + \mathrm{N}_n \cos n\theta)(2\pi ap\{\mathrm{M}_n \cos n\theta + \mathrm{N}_n \sin n\theta\} \\ &\qquad\qquad\qquad - \sigma n\{\mathrm{Q}_n \cos n\theta + \mathrm{R}_n \sin n\theta\}) \\ &+(-\mathrm{Q}_n \sin n\theta + \mathrm{R}_n \cos n\theta)(\sigma n\{\mathrm{M}_n \cos n\theta + \mathrm{N}_n \sin n\theta\} \\ &\qquad\qquad\qquad + 2\pi ap\{\mathrm{Q}_n \cos n\theta + \mathrm{R}_n \sin n\theta\}). \end{aligned}\right]$$

Also

$$\int_0^{2\pi} \sin^2 \mathrm{n}\theta \, \delta\theta = \int_0^{2\pi} \cos^2 n\theta \, \delta\theta = \pi,$$

and

$$\int_0^{2\pi} \sin n\theta \cos n\theta \, \delta\theta = 0.$$

Thus the mean value of the couple becomes

$$\sum_1^\infty \frac{\pi an^2p}{2(4\pi^2a^2p^2+\sigma^2n^2)} \left[\begin{aligned} &2\pi ap\{-\mathrm{M}_n\mathrm{N}_n + \mathrm{M}_n\mathrm{N}_n - \mathrm{Q}_n\mathrm{R}_n + \mathrm{Q}_n\mathrm{R}_n\} \\ &\qquad\qquad + \sigma n\{\mathrm{Q}_n\mathrm{N}_n - \mathrm{M}_n\mathrm{R}_n - \mathrm{M}_n\mathrm{R}_n - \mathrm{N}_n\mathrm{Q}_n\} \end{aligned}\right]$$

$$= \sum_1^\infty \frac{\pi an^3p\sigma}{4\pi^2a^2p^2+\sigma^2n^2} (\mathrm{N}_n\mathrm{Q}_n - \mathrm{M}_n\mathrm{R}_n) \quad . \ . \ . \ . \ . \ . \ . \quad \text{(vii.)}.$$

5. The couple vanishes when for all values of n,

$$\frac{\mathrm{M}_n}{\mathrm{Q}_n} = \frac{\mathrm{N}_n}{\mathrm{R}_n} = \tan \phi_n, \text{ say,}$$

so that H_0 is of the form

$$\Sigma \left[\mathrm{Q}_n \cos n\theta + \mathrm{R}_n \sin n\theta\right] \frac{\cos(pt - \phi_n)}{\cos \phi_n},$$

in other words, the couple vanishes when both parts of each harmonic are in the same phase, though that phase be not the same for all harmonics. As a particular case the remarkable result holds that whatever be the nature of the external field (it being made up, of course, of currents parallel to the axis), there will be no couple on the shell, provided the external field be altogether in the same phase.

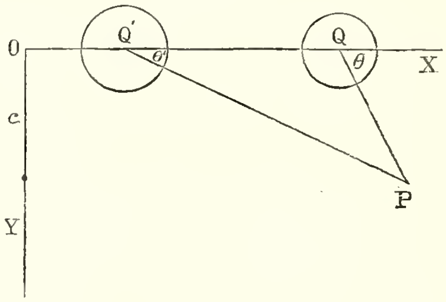
The Effect on an Infinite Cylindrical Shell, in the presence of an Alternating Current in a Parallel Wire, of the Interposition of a Parallel Cylindrical Shell.

6. Take the plane through the axes of the shells as that of ZOX, and a perpendicular plane through the wire as that of YOZ.

The shells are thin; let their radii be a, a', and their distances from the origin b and b'; let the distance of the wire from the origin be c, and the current in it $I \cos pt$.

Let the position of *a pt.*P in space be determined by its distances r, r' from the axes of the shells and the angles θ, θ'.

Fig. 30.



Then if $b - b' \equiv d$ we have

$$\left.\begin{array}{l} r \cos \theta = r' \cos \theta' - d \\ r \sin \theta = r' \sin \theta' \end{array}\right\}$$

Hence

$$re^{-i\theta} = r'e^{-i\theta} - d\,;$$

therefore

$$\frac{e^{ni\theta}}{r^n} = \frac{1}{(-d + r'e^{-i\theta'})^n},$$

and if $r' < d$,

$$\frac{\cos n\theta}{r^n} = \frac{(--)^n}{d^n}\left[1 + \frac{nr'}{d}\cos\theta' + \frac{n.\overline{n+1}}{2!}\frac{r'^2}{d^2}\cos 2\theta' + \ldots\right]$$

$$\frac{\sin n\theta}{r^n} = \frac{(-)^n}{d^n}\left[n\frac{n'}{d}\sin\theta' + \frac{n.\overline{n+1}}{2!}\frac{r'^2}{d^2}\sin 2\theta' + \ldots\right].$$

So, too, if $r < d$,

$$r'e^{-i\theta} = re^{i\theta} + d,$$

and

$$\frac{\cos n\theta'}{r'^n} = \frac{1}{d^n}\left[1 + \frac{n^r}{d}\cos\theta + \frac{n.\overline{n+1}}{2!}\frac{r^2}{d^2}\sin 2\theta + \ldots\right]$$

$$\frac{\sin n\theta'}{r'^n} = \frac{1}{d^n}\left[n\frac{r}{d}\sin\theta + \frac{n.\overline{n+1}}{2!}\frac{r^2}{d^2}\sin 2\theta + \ldots\right].$$

7. Due to the alternating current in the wire we have, over the a, b cylinder,

$$H_0 = -I \log \frac{(c - a \sin\theta)^2 + (b + a\cos\theta)^2}{D^2} \cos pt$$

where D is a constant.

On putting

$$\left.\begin{aligned} b &= \rho \cos \gamma \\ c &= \rho \sin \gamma \end{aligned}\right\}$$

we get when $a < \rho$,

$$\begin{aligned} \mathrm{H}_0 &= -\mathrm{I} \,.\, \log \frac{\rho^2 + 2\rho a \cos \overline{\gamma + \theta} + a^2}{\mathrm{D}^2} \cos pt \\ &= -\cos pt \left[\log \frac{\rho^2}{\mathrm{D}^2}\right] \\ &\quad - 2\mathrm{I} \cos pt \left[\frac{a}{\rho} \cos \overline{\theta + \gamma} - \frac{a^2}{2\rho^2} \cos \overline{2\theta + \gamma} + \frac{a^3}{3\rho^3} \cos \overline{3\theta + \gamma} \right. \\ &\qquad \left. - \frac{a^4}{4\rho^4} \cos \overline{4\theta + \gamma} \ldots \right]. \end{aligned}$$

On substituting for

$$\rho \cos \theta, \quad \rho \sin 2\theta, \quad \&\text{c.},$$

we get to the fourth power of a/ρ,

$$\begin{aligned} \mathrm{H}_0 = + \mathrm{I} \cos pt \Bigg[&- \log \frac{b^2 + c^2}{\mathrm{D}^2} - \frac{2a}{b^2 + c^2} (b \cos \theta - c \sin \theta) \\ &+ \frac{a^2}{(b^2 + c^2)^2} (\overline{- c^2 + b^2} \cos 2\theta - 2bc \sin 2\theta) \\ &+ \frac{2a^2}{(b^2 + c^2)^2} (\overline{3bc^2 - b^3} \cos 3\theta - \overline{c^3 - 3b^2c} \sin 3\theta) \\ &+ \frac{a^4}{2(b^2 + c^2)^4} (\overline{b^4 + c^4 - 6b^2c^2} \cos 4\theta - 4bc \, \overline{b^2 - c^2} \sin 4\theta) + \ldots \Bigg] \end{aligned}$$

8. The currents set up in both cylinders will clearly be parallel to their axes, and if those in the a, b shell produce at points on itself the momentum

$$\begin{aligned} \mathrm{H} = \mathrm{M}_0 \sin pt &+ \mathrm{Q}_0 \cos pt \\ &+ (\mathrm{M}_1 \cos \theta + \mathrm{N}_1 \sin \theta) \sin pt + (\mathrm{Q}_1 \cos \theta + \mathrm{R}_1 \sin \theta) \cos pt \\ &+ (\mathrm{M}_2 \cos 2\theta + \mathrm{N}_2 \sin 2\theta) \sin pt + (\mathrm{Q}_2 \cos 2\theta + \mathrm{R}_2 \sin 2\theta) \cos pt \\ &+ \text{higher harmonics,} \end{aligned}$$

and if the currents in the a', b' shell produce on itself a momentum, distinguished from the above by the dashing of the letters

$$\begin{aligned} \mathrm{H} = \mathrm{M}'_0 \sin pt &+ \mathrm{Q}'_0 \cos pt \\ &+ (\mathrm{M}'_1 \cos \theta + \ldots) \sin pt + (\ldots) \cos pt \\ &+ (\mathrm{M}'_2 \cos 2\theta + \ldots) \sin pt + \ldots \\ &+ \text{higher terms,} \end{aligned}$$

then in all we have, that on the a, b cylindrical surface, the momentum of the field that produces currents in that shell, is (to the 4th power of a, a', as will afterwards be explained)

$$- \mathrm{I} \log \frac{b^2 + c^2}{\mathrm{D}^2} \cos pt$$

$$- \mathrm{I} \frac{2a}{b^2 + c^2} (b \cos \theta - c \sin \theta) \cos pt$$

$$+ \mathrm{I} \frac{a^2}{(b^2 + c^2)^2} (\overline{b^2 - c^2} \cos 2\theta - 2bc \sin 2\theta) \cos pt$$

$$+ \mathrm{I} \frac{2a^3}{(b^2 + c^2)^3} (\overline{3bc^2 - b^3} \cos 3\theta - \overline{c^3 - 3b^2c} \sin 3\theta) \cos pt$$

$$+ \mathrm{I} \frac{a^4}{2(b^2 + c^2)^4} (\overline{b^4 + c^4 - 6b^2c^2} \cos 4\theta + 4bc\, \overline{c^2 - b^2} \sin 4\theta) \cos pt$$

$$+ \text{5th powers} \ldots$$

$$+ \mathrm{M}'_0 \sin pt + \mathrm{Q}'_0 \cos pt$$

$$+ (\mathrm{M}'_1 \sin pt + \mathrm{Q}'_1 \cos pt) \frac{a'}{d} \left(1 + \frac{a}{d} \cos \theta + \frac{a^2}{d^2} \cos 2\theta \ldots \right)$$

$$+ (\mathrm{N}'_1 \sin pt + \mathrm{R}_1 \cos pt) \left(\frac{a'}{d}\right) \left(\frac{a}{d} \sin \theta + \frac{a^2}{d^2} \sin 2\theta \ldots \right)$$

$$+ (\mathrm{M}'_2 \sin pt + \mathrm{Q}_2 \cos pt) \frac{a'^2}{d^2} (1 + \ldots)$$

$$+ \text{5th and higher powers.}$$

9. But if $\mathrm{D}_n \equiv 4\pi^2 a^2 p^2 + \sigma^2 n^2$, an external field of momentum

$$\mathrm{H}_0 = ({}^{\backprime}\mathrm{M}_n \cos n\theta + {}^{\backprime}\mathrm{N}_n \sin n\theta) \sin pt + ({}^{\backprime}\mathrm{Q}_n \cos n\theta + {}^{\backprime}\mathrm{R}_n \sin n\theta) \cos pt$$

at the surface of the a, b shell will produce in it currents whose momentum at the surface is by (v.)

$$\mathrm{H} = - \frac{2\pi ap}{\mathrm{D}_n} [2\pi ap \sin pt + \sigma n \cos pt][{}^{\backprime}\mathrm{M}_n \cos n\theta + {}^{\backprime}\mathrm{N}_n \sin n\theta]$$

$$- \frac{2\pi ap}{\mathrm{D}_n} [2\pi ap \cos pt - \sigma n \sin pt][{}^{\backprime}\mathrm{Q}_n \cos n\theta + {}^{\backprime}\mathrm{R}_n \sin n\theta].$$

Applying this to the values of H and H_0 that we have recently found, we see that to the 4th powers of a, a' (as we shall shortly explain)

$$\mathrm{M}_1 = \frac{2\pi ap\sigma}{\mathrm{D}_1}\left[-\frac{2\mathrm{I}ab}{b^2+c^2}+\frac{aa'}{d^2}\mathrm{Q'}_1\right]-\frac{4\pi^2a^2p^2}{\mathrm{D}_1}\frac{aa'}{d^2}\mathrm{M'}_1,$$

$$\mathrm{N}_1 = \frac{2\pi ap\sigma}{\mathrm{D}_1}\left[+\frac{2\mathrm{I}ac}{b^2+c^2}+\frac{aa'}{d^2}\mathrm{R'}_1\right]-\frac{4\pi^2a^2p^2}{\mathrm{D}_1}\frac{aa'}{d^2}\mathrm{N'}_1,$$

$$\mathrm{Q}_1 = -\frac{4\pi^2a^2p^2}{\mathrm{D}_1}\left[-\frac{2\mathrm{I}ab}{b^2+c^2}+\frac{aa'}{d^2}\mathrm{Q'}_1\right]-\frac{2\pi ap\sigma}{\mathrm{D}_1}\frac{aa'}{d^2}\mathrm{M'}_1,$$

$$\mathrm{R}_1 = -\frac{4\pi^2a^2p^2}{\mathrm{D}_1}\left[+\frac{2\mathrm{I}ac}{b^2+c^2}+\frac{aa'}{d^2}\mathrm{R'}_1\right]-\frac{2\pi ap\sigma}{\mathrm{D}_1}\frac{aa'}{d^2}\mathrm{N'}_1,$$

$$\mathrm{M}_2 = \frac{4\pi ap\sigma}{\mathrm{D}_2}\left[\frac{a^2.\overline{b^2-c^2}}{(b^2+c^2)^2}\mathrm{I}+\frac{a^2a'}{d^3}\mathrm{Q'}_1\right]-\frac{4\pi^2a^2p^2}{\mathrm{D}_2}\frac{a^2a'}{d^3}\mathrm{M'}_1,$$

$$\mathrm{N}_2 = \frac{4\pi ap\sigma}{\mathrm{D}_2}\left[-\frac{2a^2bc}{(b^2+c^2)^2}\mathrm{I}+\frac{a^2a'}{d^3}\mathrm{R'}_1\right]-\frac{4\pi^2a^2p^2}{\mathrm{D}_2}\frac{a^2a'}{d^3}\mathrm{N'}_1,$$

$$\mathrm{Q}_2 = -\frac{4\pi^2a^2p^2}{\mathrm{D}_2}\left[\frac{a^2.\overline{b^2-c^2}}{(b^2+c^2)^2}\mathrm{I}+\frac{a^2a'}{d^8}\mathrm{Q'}_1\right]-\frac{4\pi ap\sigma}{\mathrm{D}_2}\frac{a^2a'}{d^3}\mathrm{M'}_1,$$

$$\mathrm{R}_2 = \frac{4\pi^2a^2p^2}{\mathrm{D}_2}\left[-\frac{2a^2bc}{(b^2+c^2)^2}\mathrm{I}-\frac{a^2a'}{d^3}\mathrm{R'}_1\right]-\frac{4\pi ap\sigma}{\mathrm{D}_2}\frac{a^2a'}{d^3}\mathrm{N'}_1,$$

$$\mathrm{M}_3 = \frac{6\pi ap\sigma}{\mathrm{D}_3}\frac{2a^3b\,(3c^2-b^2)}{(b^2+c^2)^3}\mathrm{I},$$

$$\mathrm{N}_3 = -\frac{6\pi ap\sigma}{\mathrm{D}_3}\frac{2a^3c\,(c^2-3b^2)}{(b^2+c)^3}\mathrm{I},$$

$$\mathrm{Q}_3 = -\frac{4\pi^2a^2p^2}{\mathrm{D}_3}\frac{2a^3b\,(3c^2-b^2}{(b^2+c\;)^3}\mathrm{I},$$

$$\mathrm{R}_3 = \frac{4\pi^2a^2p^2}{\mathrm{D}_3}\frac{2a^3c\,(c^2-3b^2)}{(b^2+c^2)^3}\mathrm{I},$$

$$\mathrm{M}_4 = \frac{8\pi ap\sigma}{\mathrm{D}_4}\frac{a^4\,(b^4+c^4-6b^2c^2)}{2\,(b^2+c^2)^4}\mathrm{I},$$

$$\mathrm{N}_4 = -\frac{8\pi ap\sigma}{\mathrm{D}_4}\frac{2a^4bc\,(b^2-c^2)}{(b^2+c^2)^4}\mathrm{I},$$

$$\mathrm{Q}_4 = -\frac{4\pi^2a^2p^2}{\mathrm{D}_4}\frac{a^4\,(b^4+c^4-6b^2c^2)}{2\,(b^2+c^2)^4}\mathrm{I},$$

$$\mathrm{R}_4 = \frac{4\pi^2a^2p^2}{\mathrm{D}_4}\frac{2a^4bc\,(b^2-c^2)}{(b^2+c^2)^4}\mathrm{I}.$$

The equations that hold for the a', b' shell may be obtained from the above by interchanging dashed and undashed letters and altering the sign of d.

10. In order to justify the approximation we observe :—

(α). For ordinary values of the letters, $2\pi a p$ is comparable with σn ; in the case, for instance, of a copper shell of diameter 10 cm., and thickness $\frac{1}{3}$ cm.,

$$a = 5,$$
$$\sigma = 3 \times 1640 \text{ about ;}$$

and if

$$2\pi a p = \sigma n,$$

then $p = 160n$ about, which means that the number of reversals per second is about $55n$.

Hence, as far as we are concerned, we regard $4\pi a p \sigma / D_1$, or $8\pi^2 a^2 p^2 / D_2$ as of *no* degree in a.

(β). It only remains to remark that the degree in a, a' of the most important term of each of the coefficients N_1, Q_3, &c, is that of their suffix 1, 3, &c.

11. We have

$$M'_1 = -\frac{4\pi a' p \sigma'}{D'_1} \frac{a' b'}{(b'^2 + c^2)} I + \text{cubes.}$$

$$N'_1 = +\frac{4\pi a' p \sigma'}{D'_1} \frac{a' c}{b'^2 + c^2} I + \text{cubes.}$$

$$Q'_1 = +\frac{8\pi^2 a'^2 p^2}{D'_1} \frac{a' b'}{b'^2 + c^2} I + \ldots$$

$$R'_1 = -\frac{8\pi^2 a'^2 p^2}{D'_1} \frac{a' c}{b'^2 + c^2} I + \ldots$$

Hence, as far as the fourth degree, in a, a',

$$\left.\begin{aligned} M_1 &= \frac{2\pi a p \sigma}{D_1}\left[-\frac{2Iab}{b^2+c^2} + \frac{8\pi^2 a'^2 p^2}{D'_1 d^2}\frac{aa'^2 b'}{b'^2+c^2} I\right] + \frac{16\pi^3 a^2 a' p^3 \sigma'}{D_1 D'_1 d^2}\frac{aa'^2 b'}{b'^2+c^2} I \\ N_1 &= \frac{2\pi a p \sigma}{D_1}\left[+\frac{2Iac}{b^2+c^2} - \frac{8\pi^2 a'^2 p^2}{D'_1}\frac{aa'^2 c}{d^2(b'^2+c^2)} I\right] - \frac{16\pi^3 a^2 a' p^3 \sigma'}{D_1 D'_1}\frac{aa'^2 c}{d^2(b'^2+c^2)} I \end{aligned}\right\} \text{(viii.),}$$

with similar values for Q_1 and R_1.

$$M_2 = I \,.\, \frac{4\pi a p \sigma}{D_2}\left[\frac{a^2(b^2-c^2)}{(b^2+c^2)^2} + \frac{8\pi^2 a'^2 p^2}{D'_1}\frac{a^2 a'^2 b'}{d^3(b'^2+c^2)}\right] + \frac{16\pi^3 a^2 a' p' \sigma'}{D'_1 D_2}\frac{a^2 a'^2 b'}{d^3(b'^2+c^2)} I,$$

with similar values for N_2, Q_2, and R_2.

The values of the coefficients may be determined by the same method to a further degree of accuracy, and since the current w is given when the H due to it is given, *i.e.*,

$$w = \sum_1^\infty \frac{n}{2\pi a}\left[(\mathrm{M}_n \cos n\theta + \mathrm{N}_n \sin n\theta) \sin pt + (\mathrm{Q}_n \cos n\theta + \mathrm{R}_n \sin n\theta) \cos pt\right],$$

thus, the currents may be determined to any degree of closeness.

12. In order to find the couple that acts on the a, b shell, we use our previous result (vii.),

$$\Sigma \frac{\pi a p \sigma n^3}{\mathrm{D}_n}\left[{}^{\backprime}\mathrm{Q}_n{}^{\backprime}\mathrm{N}_n - {}^{\backprime}\mathrm{M}_n{}^{\backprime}\mathrm{R}_n\right].$$

As far as terms of the sixth degree in a, a' this is (writing $b^2 + c^2 \equiv \rho^2$, $b'^2 + c^2 \equiv \rho'^2$)

$$\frac{\pi a p \sigma}{\mathrm{D}_1}\left[\left(-\frac{2ab}{\rho^2}\mathrm{I} + \frac{aa'}{d^2}\mathrm{Q}'_1\right)\frac{aa'}{d^2}\mathrm{N}'_1 - \frac{aa'}{d^2}\mathrm{M}'_1\left(+\frac{2ac}{\rho^2}\mathrm{I} + \frac{aa'}{d^2}\mathrm{R}'_1\right)\right]$$
$$+ \frac{8\pi a p \sigma}{\mathrm{D}_2}\left[\frac{a^2 . \overline{b^2 - c^2}}{\rho^4}\mathrm{I} . \frac{a^2 a'}{d^3}\mathrm{N}'_1 + \frac{a^2 a'}{d^3}\mathrm{M}'_1 . \frac{2bca^2}{\rho^4}\mathrm{I}\right],$$

remembering that the third harmonic terms are (to this order) in the same phase and give no couple (${}^{\backprime}\mathrm{M}_3 = 0$, ${}^{\backprime}\mathrm{N}_3 = 0$).

This expression is

$$\frac{\pi a p \sigma}{\mathrm{D}_1} . \frac{aa'}{d^2}\left[-\left(\frac{2ab}{\rho^2}\mathrm{N}'_1 + \frac{2ac}{\rho^2}\mathrm{M}'_1\right)\mathrm{I} + \frac{aa'}{d^2}(\mathrm{Q}'_1\mathrm{N}'_1 - \mathrm{M}'_1\mathrm{R}'_1)\right]$$
$$+ \frac{8\pi a p \sigma}{\mathrm{D}_2}\frac{a^2 a'}{d^3}\frac{a^2\mathrm{I}}{\rho^4}\left[(b^2 - c^2)\,\mathrm{N}'_1 + 2bc\,\mathrm{M}'_1\right].$$

Now, $\mathrm{Q}'_1\mathrm{N}'_1 - \mathrm{M}'_1\mathrm{R}'_1$ has to be calculated to the second order only, and on reference to the values of these coordinates to the first order it will be seen that it is zero. Within the first bracket M'_1 and N'_1 are required to the third degree. We have (viii.)

$$\mathrm{M}'_1 = -\frac{2\pi a' p \sigma'}{\mathrm{D}'_1}\frac{2\mathrm{I}a'b'}{\rho'^2} + \frac{16\pi^3 a^3 a'^2 p^3 b}{\mathrm{D}_1\mathrm{D}'_1 d^2\rho^2}\mathrm{I}\,(a'\sigma + a\sigma'),$$

$$\mathrm{N}'_1 = +\frac{2\pi a' p \sigma'}{\mathrm{D}'_1}\frac{2\mathrm{I}a'c}{\rho'^2} - \frac{16\pi^3 a^3 a'^2 p^3 c}{\mathrm{D}_1\mathrm{D}'_1 d^2\rho^2}\mathrm{I}\,(a'\sigma + a\sigma').$$

Hence, for the couple we get

$$\frac{\pi a p \sigma}{\mathrm{D}_1}\frac{aa'}{d^2}\frac{2a}{\rho^2}\mathrm{I}^2\left[\frac{2\pi a' p \sigma'}{\mathrm{D}'_1}\frac{2a'}{\rho'^2}(b'c - bc) + \frac{16\pi^3 a^3 a'^2 p^3}{\mathrm{D}_1\mathrm{D}'_1 d^2\rho^2}(a'\sigma + a\sigma')(bc - bc)\right]$$
$$+ \frac{8\pi a p \sigma}{\mathrm{D}_2}\frac{a^2 a'}{d^3}\frac{a^2\mathrm{I}}{\rho^4}\frac{2\pi a' p \sigma'}{\mathrm{D}'_1}\frac{2\mathrm{I}a'}{\rho'^2}\left[-2bc . b' + (b^2 - c^2)\,c\right].$$

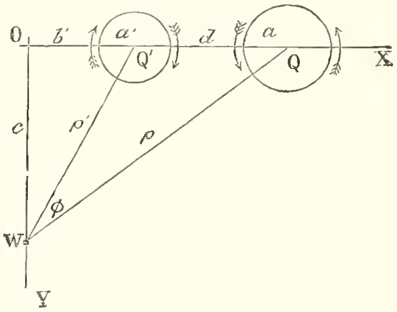
This consolidates finally into

$$-\mathrm{I}^2 . \frac{8\pi^2 p^2 \sigma\sigma' c a^3 a'^3}{\rho^2\rho'^2 d}\left[\frac{1}{\mathrm{D}_1\mathrm{D}_1'} + \frac{4a^2(\rho'^2 - d^2)}{\mathrm{D}'_1\mathrm{D}_2\rho^2 d^2}\right] \quad . \quad . \quad . \quad . \quad \text{(ix).}$$

13. It will be noticed that the more important term is symmetrical, except that it

changes sign if it be applied to the a', b' shell instead of the a, b; the direction of rotation is negative when $b - b'$ is positive, *i.e.*, is as indicated in the figure, and is *à priori* likely, since the repulsion is likelier to be stronger where the currents and field are stronger, *i.e.*, on the side nearest the wire of the shell to the left and on the unshielded side of that to the right. The couple changes sign with c as it should, and vanishes if σ or σ' vanish. The term of the eighth degree vanishes when $\rho' = d$, *i.e.*, $QQ' = Q'W$.

Fig. 31.



The Effects on two parallel Thin Infinite Circular Cylindrical Shells of a Thin Filament parallel to them, and alternately Magnetised in a direction perpendicular to its Length.

14. Let the same axes and coordinates be taken as before; the filament may be regarded as an electromagnet consisting of a current

$$\frac{A}{\kappa}\cos pt \quad \text{(where } \kappa \text{ is small)},$$

parallel to OZ at $x = 0$, $y = c$, and a current

$$-\frac{A}{\kappa}\cos pt,$$

parallel to OZ at $x = -\kappa\sin\alpha$, $y = c - \kappa\cos\alpha$.

The direction of magnetisation will then make an angle of α with OX, and the strength will be A cos pt per unit length.

If $I \cdot f(b, b', c, d)$ were a coefficient in the expansion of the electromagnetic momentum in the case of a current $I\cos pt$ at $x = 0$, $y = c$, then, with the electromagnet, that coefficient will be the limit of

$$\frac{A}{\kappa}f(b, b', c, d) - \frac{A}{\kappa}f(b + \kappa\sin\alpha,\ b' + \kappa\sin\alpha,\ c - \kappa\cos\alpha,\ d),$$

i.e.,

$$A\left[-\sin\alpha \,.\left(\frac{\partial f}{\partial b} + \frac{\partial f}{\partial b'}\right) + \cos\alpha\frac{\partial f}{\partial c}\right],$$

but the same plan will not avail when applied to find the new couple (by directly differentiating the couple due to the current I cos pt), owing to the interaction of the currents set up by the two elementary currents of the electromagnet.

We shall have to use our previous result, that if the external field on the surface of a shell has momentum

$$\Sigma\left[(`\mathrm{M}_n \cos n\theta + `\mathrm{N}_n \sin n\theta) \sin pt + (`\mathrm{Q}_n \cos n\theta + `\mathrm{R}_n \sin n\theta) \cos pt\right],$$

the couple is by (vii.),

$$\Sigma \frac{\pi a p \sigma n^3}{\mathrm{D}_n} [`\mathrm{N}_n `\mathrm{Q}_n - `\mathrm{M}_n `\mathrm{R}_n].$$

On reference it will be seen that due to the current I cos pt, we had

$$`\mathrm{M}_1 = - \frac{aa'}{d^2} \frac{4\pi a' p \sigma'}{\mathrm{D}'_1} \frac{a'b'}{\rho'^2} \mathrm{I} + \text{cubes.}$$

$$`\mathrm{N} \ = + \frac{aa'}{d^2} \frac{4\pi a' p \sigma'}{\mathrm{D}'_1} \frac{a'c}{\rho'^2} \mathrm{I} \ldots$$

$$`\mathrm{Q}_1 = - \frac{2ab}{\rho^2} \mathrm{I} + \frac{aa'}{d^2} \frac{8\pi^2 a'^2 p^2}{\mathrm{D}'_1} \frac{a'b'}{\rho'^2} \mathrm{I} \ldots$$

$$`\mathrm{R}_1 = + \frac{2ac}{\rho^2} \mathrm{I} - \frac{aa'}{d^2} \frac{8\pi^2 a'^2 p^2}{\mathrm{D}'_1} \frac{a'c}{\rho'^2} \mathrm{I} \ldots$$

Now, with the electromagnet instead of the single current (remembering $\rho \cos \gamma = b$, $\rho \sin \gamma = c$), $\frac{b}{\rho^2}$ I will become

$$+ \mathrm{A}\left[- \frac{\sin \alpha}{\rho^2} + \frac{2b}{\rho^2} (+ b \sin \alpha + c \cos \alpha)\right]$$

or

$$\mathrm{A}\left[- \frac{\sin \alpha}{\rho^2} + \frac{2 \cos \gamma}{\rho^2} \sin \overline{\alpha + \gamma}\right].$$

Also $\frac{c}{\rho^2}$ I will become

$$\frac{\mathrm{A}}{\rho^2} [\cos \alpha - 2 \sin \gamma \sin \overline{\alpha + \gamma}]$$

and for $\frac{b'}{\rho'^2}$ I, $\frac{c}{\rho'^2}$ I, we have merely to add dashes to ρ and γ.

If we omit terms of the *eighth* degree in $a_1\ a'$, the value of the couple will be

$$\frac{\pi a p \sigma}{\mathrm{D}_1} [``\mathrm{N}_1 ``\mathrm{Q}_1 - ``\mathrm{M}_1 ``\mathrm{R}_1],$$

where

$$\text{``}M_1 = -\frac{4\pi p\sigma' a a'^3}{d^2 D'_1 \rho'^2} A\,(-\sin\alpha + 2\cos\gamma'\sin\overline{\alpha+\gamma'}),$$

$$\text{``}N_1 = -\frac{4\pi p\sigma' a a'^3}{d^2 D'_1 \rho'^3} A\,(\cos\alpha - 2\sin\gamma'\sin\overline{\alpha+\gamma'}),$$

$$\text{``}Q_1 = -\frac{2aA}{\rho^2}(-\sin\alpha + 2\cos\gamma\sin\overline{\alpha+\gamma}) + \frac{8\pi^2 a a'^4 p^2}{d^2 D'_1 \rho'^2} A\,(-\sin\alpha + 2\cos\gamma'\sin\overline{\alpha+\gamma'}),$$

$$\text{``}R_1 = -\frac{2aA}{\rho^2}(\cos\alpha - 2\sin\gamma\sin\overline{\alpha+\gamma}) + \frac{8\pi^2 a a'^4 p^2}{d^2 D'_1 \rho'^2} A\,(\cos\alpha - 2\sin\gamma'\sin\overline{\alpha+\gamma'}).$$

On multiplying up, we notice that the second terms of $\text{``}Q_1$ and $\text{``}R_1$ cancel, when multiplied respectively by $\text{``}N_1$ and $\text{``}M_1$, and we get for the couple

$$\frac{\pi a p\sigma}{D_1}\,\frac{2aA}{\rho^2}\,\frac{4\pi p\sigma' a a'^3}{d^2 D'_1 \rho'^2} A$$
$$\times\,[(\cos\alpha - 2\sin\gamma'\sin\overline{\alpha+\gamma'})(-\sin\alpha + 2\cos\gamma\sin\overline{\alpha+\gamma})$$
$$-(\cos\alpha - 2\sin\gamma\sin\overline{\alpha+\gamma})(-\sin\alpha + 2\cos\gamma'\sin\overline{\alpha+\gamma'})]$$

or

$$A^2\,\frac{8\pi^2 p^2 \sigma\sigma' a^3 a'^3}{d^2 D_1 D'_1 \rho^2 \rho'^2}\,[2\sin\overline{\alpha+\gamma}\,(\cos\gamma\cos\alpha - \sin\gamma\sin\alpha)$$
$$-2\sin\overline{\alpha+\gamma'}\,(\cos\gamma'\cos\alpha - \sin\gamma'\sin\alpha)$$
$$-4\sin\overline{\alpha+\gamma}\sin\overline{\alpha+\gamma'}\,(\cos\gamma\sin\gamma' - \cos\gamma'\sin\gamma)].$$

The expression within the bracket is

$$\sin\overline{2\alpha+\gamma} - \sin\overline{2\alpha+\gamma'} - 4\sin\overline{\alpha+\gamma}\sin\overline{\alpha+\gamma'}\sin\overline{\gamma'-\gamma}$$

or

$$-2\sin\overline{\gamma'-\gamma}\,[\cos\overline{2\alpha+\gamma+\gamma'} + 2\sin\overline{\alpha+\gamma}\sin\overline{\alpha+\gamma'}]$$

or

$$-2\sin\overline{\gamma'-\gamma}\cos\overline{\gamma'-\gamma}.$$

Hence, omitting terms of the *eighth* degree, the couple on the a, b shell is

$$-A^2\,\frac{8\pi^2 p^2 \sigma\sigma' a^3 a'^3}{d^2 D_1 D'_1 \rho^2 \rho'^2}\sin 2\phi,$$

where ϕ is $(\gamma' - \gamma)$, the angle subtended by the axes of the shells at the electromagnetic filament.

15. To this order then we have the following results :—

(α.) The couple exerted on a shell is independent of α, *i.e.*, of the direction of the axis of the electromagnet.

(β.) The shells have equal couples in opposite directions, the parts of the shells directed towards one another being driven towards the electromagnet if ϕ is less than a right angle and driven away from it if ϕ exceeds a right angle. If ϕ be a right angle the couple vanishes.

Fig. 32.

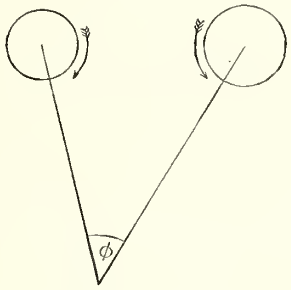
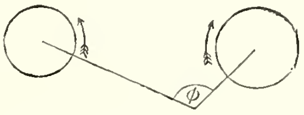


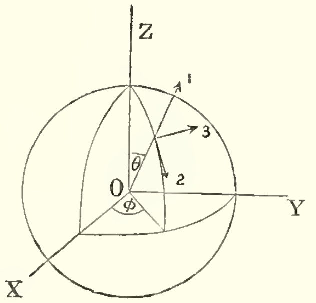
Fig. 33.



The Mechanical Effect of Currents set up in a Thin Spherical Shell.

16. In obtaining the currents set up by a given field we follow Professor C. Niven: If, using spherical coordinates and the ordinary notation, Φ be the current function of the distribution in the shell, P the potential of imaginary matter spread over it with surface density Φ, then

Fig. 34.



$$\Omega = -\frac{1}{a}\frac{d}{dr}(\mathrm{P}r), \quad \mathrm{F} = 0,$$

$$\mathrm{G} = \frac{1}{a}\frac{d\mathrm{P}}{\sin\theta\, d\phi}, \quad \mathrm{H} = -\frac{d\mathrm{P}}{a\, d\theta}.$$

The components of the current are

$$u = 0, \quad v = -\frac{\partial\Phi}{a\sin\theta\,\partial\phi}, \quad w = \frac{\partial\Phi}{a\,\partial\theta}.$$

If the external field have magnetic potential Ω_0, and if

$$\Omega_0 = -\frac{1}{a}\frac{d}{dr}(P_0 r)$$

(ever distinguishing quantities that refer to the externally applied system by a zero suffix), then the equations connecting the components of the vector potential with the components of induction are satisfied by

$$F_0 = 0, \quad G_0 = \frac{1}{a}\frac{\partial P_0}{\sin\theta\,\partial\phi}, \quad H_0 = -\frac{\partial P_0}{a\,\partial\theta}.$$

The equations giving the currents are

$$-\sigma\frac{\partial\Phi}{a\sin\theta\,\partial\phi} = -\frac{d}{dt}\left[\frac{1}{a\sin\theta}\frac{\partial}{\partial\phi}(P + P_0)\right] - \frac{\partial\psi}{a\,\partial\theta},$$

$$+\sigma\frac{\partial\Phi}{a\,\partial\theta} = -\frac{d}{dt}\left[-\frac{1}{a}\frac{\partial}{\partial\theta}(P + P_0)\right] - \frac{\partial\psi}{a\sin\theta\,\partial\phi},$$

and are satisfied by

$$\psi = 0,$$

$$\sigma\Phi = \frac{d}{dt}(P + P_0).$$

Let us consider the case

$$\Omega_0 = A\left(\frac{r}{a}\right)^n Y_n e^{ipt},$$

where a is the radius of the shell, and Y_n is a spherical harmonic of degree n.

We have

$$P_0 = -A\frac{r^n}{a^{n-1}}\frac{Y_n}{n+1}e^{ipt} = -A\frac{a}{n+1}Y_n e^{ipt} \text{ at the surface.}$$

If due to this

$$\Phi = BY_n e^{ipt};$$

then

$$P = \frac{4\pi a}{2n+1}BY_n e^{ipt}$$

at the surface, and the equation for B is

$$\sigma B = ip\left[\frac{4\pi a}{2n+1}B - \frac{a}{1+n}A\right],$$

therefore

$$\mathrm{B} = \frac{-\mathrm{A}\dfrac{iap}{n+1}}{\dfrac{4i\pi pa}{2n+1}+\sigma}$$

$$= \mathrm{A}\,\frac{\overline{2n+1}\,ap}{n+1}\,\frac{4\pi pa - i\,\overline{2n+1}\,\sigma}{16\pi^2a^2p^2+\overline{2n+1}^2\sigma^2}.$$

Denoting then

$$16\pi^2a^2p^2+\overline{2n+1}^2\sigma^2$$

by Δ_n, we shall have corresponding to

$$\Omega_0 = \mathrm{AY}_n\cos pt$$

at the surface, the value

$$\Phi = \mathrm{A}\,\frac{\overline{2n+1}\,pa}{(n+1)\,\Delta_n}\,\mathrm{Y}_n\,(4\pi pa\cos pt + \overline{2n+1}\,\sigma\sin pt),$$

and corresponding to

$$\Omega_0 = \mathrm{BZ}_n\sin pt$$

$$\Phi = \mathrm{BZ}_n\,\frac{\overline{2n+1}\,pa}{\overline{n+1}\,\Delta_n}\,(4\pi pa\sin pt - \overline{2n+1}\,\sigma\cos pt.)$$

17. If the components of magnetic induction or magnetic force along the directions

$$\delta r, \qquad r\,\delta\theta, \qquad r\sin\theta\,\delta\phi$$

be α, β, γ, the components of electromagnetic force in these directions will be

$$\begin{array}{lcr}
\gamma v - \beta w, & \text{or} & -\gamma\dfrac{\partial\Phi}{a\sin\theta\,\partial\phi} - \beta\dfrac{\partial\Phi}{a\,\partial\theta} \\[2ex]
\alpha w, & \text{or} & \alpha\dfrac{\partial\Phi}{a\,\partial\theta} \\[2ex]
-\alpha v, & \text{or} & \alpha\dfrac{\partial\Phi}{a\sin\theta\,\partial\phi}.
\end{array}$$

The couple about the axis of z will be the integral over the surface of

$$-\alpha v\,.\,a\sin\theta,$$

where in α it will be sufficient to take into account the external field (as in the case of the cylinder), for the shell will not exert any couple on itself. This couple is

$$\iint d\mathrm{S}\,.\,\alpha\frac{\partial\Phi}{\partial\phi}\,.$$

Now if Ω_0 be periodic in the time $2\pi/p$, it can be expanded near to the surface of the sphere in the series

$$\sum_{n=1}^{n=\infty} \left(\frac{r}{a}\right)^n [Y_n \cos pt + Z_n \sin pt]$$

where Y_n, Z_n include only harmonics of the n^{th} degree.

Due to this value of Ω_0 we shall have at the surface

$$\alpha = -\Sigma \frac{n}{a} [Y_n \cos pt + Z_n \sin pt] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{x.}),$$

and

$$\Phi = \Sigma \frac{(2n+1)pa}{(n+1)\Delta_n} [4\pi pa (Y_n \cos pt + Z_n \sin pt) + \overline{2n+1}\, \sigma (Y_n \sin pt - Z_n \cos pt)] \quad . \quad . \quad . \quad (\text{xi.}).$$

The mean value of $\sin^2 pt$ and $\cos^2 pt$ being $\frac{1}{2}$, and of $\sin pt \cos pt$ zero, that of

$$\iint dS \,.\, \alpha \frac{\partial \Phi}{\partial \phi}$$

will be, denoting differentiation to ϕ by dashes,

$$\tfrac{1}{2}\iint dS \Big[-\Sigma \left(\frac{n}{a} Y_n\right) \Sigma \frac{\overline{2n+1}\, pa}{n+1\, \Delta_n} (4\pi pa Y'_n - \overline{2n+1}\, \sigma Z'_n) - \Sigma \left(\frac{n}{a} Z_n\right) \Sigma \frac{\overline{2n+1}\, pa}{n+1\, \Delta_n} (4\pi pa Z'_n + \overline{2n+1}\, \sigma Y'_n) \Big]$$

Now Y'_n, Z'_n are harmonics of the n^{th} degree and will give zero when multiplied by harmonics of other degrees than n^{th} and integrated over the sphere: hence it is sufficient to write for the couple

$$\tfrac{1}{2}\iint dS \sum_1^\infty \frac{n \,.\, \overline{2n+1}\, p}{(n+1)\Delta_n} [(2n+1)\sigma (Y_n Z'_n - Y'_n Z_n) - 4\pi pa (Y_n Y'_n + Z_n Z'_n)].$$

Also $Y_n (\partial Y_n / \partial \phi)$ being a perfect differential, vanishes when integrated to ϕ from 0 to 2π: thus

$$\iint dS \,.\, Y_n Y'_n = 0,$$

$$\iint dS \,.\, Z_n Z'_n = 0,$$

and the couple is

$$\sum_1^\infty \frac{n \,.\, (2n+1)^2 p\sigma}{2(n+1)\Delta_n} \iint dS \,.\, \left[Y_n \frac{\partial Z_n}{\partial \phi} - Z_n \frac{\partial Y_n}{\partial \phi} \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xii.}).$$

[It might be objected that being given the external field, we have no right to take $F_0 = 0$ at the surface; we have proved, however, in our introductory work, that if Ω_0 be given, an alteration in the normal component of the vector potential only introduces an alteration in the electrostatic potential.]

18. Hence it follows that—

(α.) If the external field be symmetrical round the axis of Z, so that Ω is independent of ϕ, the couple will be zero;

(β.) if the external field be completely in the same phase, or if when expanded in harmonics over the surface its form is

$$\Omega_0 = \Sigma\, U_n \cos (pt + \epsilon_n),$$

where ϵ_n is independent of the coordinates θ, ϕ, then the couple vanishes.

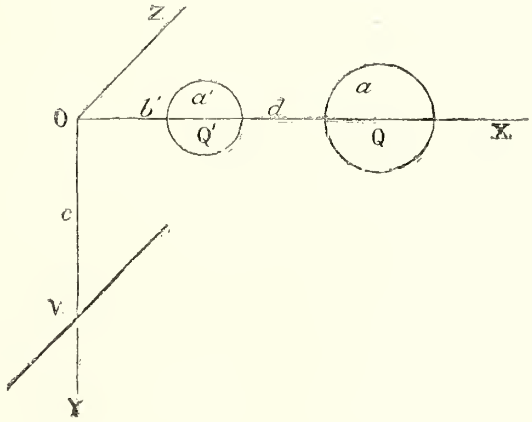
From this it follows that if we have an exciting field in the same phase, and introduce any number of perfectly conducting bodies, the couple will still be zero.

The Couples on two Thin Spherical Shells due to the presence of an Alternating Current in a Straight Infinite Wire.

19. Let the line joining the centres of the shells be taken as OX and the shortest distance between this line and the infinite wire as OY.

Let the radii of the shells be a, a', and their central distances from O, b and b': let the distance from O of the wire be c.

Fig. 35.



If a point in space have polar coordinates r, θ, ϕ when referred to Q as origin and axes parallel to OX, OY, OZ, and coordinates r', θ', ϕ' to Q' as origin, its Cartesian coordinates will be

$$\begin{aligned} x &= b + r \sin\theta \cos\phi \\ &= b' + r' \sin\theta' \cos\phi' \end{aligned}$$

$$\begin{aligned} y &= r \sin\theta \sin\phi \\ &= r' \sin\theta' \sin\phi' \end{aligned}$$

$$\begin{aligned} z &= r \cos\theta \\ &= r' \cos\theta'. \end{aligned}$$

Thus (if $b - b' \equiv d$)

$$\left.\begin{aligned} r' \sin\theta' \cos\phi' &= r \sin\theta \cos\phi + d \\ r' \sin\theta' \sin\phi' &= r \sin\theta \sin\phi \\ r' \cos\theta' &= r \cos\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiii.)},$$

so that

$$r'^2 = r^2 + 2\,dr \sin\theta \cos\phi + d^2,$$

and when $r = a$ and is less than d,

$$\frac{1}{r'} = \frac{1}{d}\left[1 - \frac{a}{d} \sin\theta \cos\phi \ldots\right],$$

so that, omitting second harmonics,

$$\begin{aligned} \frac{\sin\theta' \cos\phi'}{r'^2} &= \frac{1}{d^3}\left(1 - \frac{3a}{d} \sin\theta \cos\phi\right)(d + a \sin\theta \cos\phi) \\ &= \frac{1}{d^2} - \frac{2a}{d^3} \sin\theta \cos\phi + \text{higher harmonics.} \end{aligned}$$

$$\frac{\sin\theta' \sin\phi'}{r'^2} = \frac{a \sin\theta \sin\phi}{d^3} \ldots$$

$$\frac{\cos\theta'}{r'^2} = \frac{a \cos\theta}{d^3} \ldots$$

20. Hence if the currents in the a, b surface produce upon that surface a magnetic potential

$$\begin{aligned} \Omega = {} & (\text{A} \sin\theta \cos\phi + \text{B} \sin\theta \sin\phi + \text{C} \cos\theta) \cos pt \\ & + (\text{D} \sin\theta \cos\phi + \text{E} \sin\theta \sin\phi + \text{F} \cos\theta) \sin pt \\ & + \text{harmonics of second and higher orders,} \end{aligned}$$

and if the currents in the a', b' shell produce upon itself a potential Ω', whose value is distinguished by dashes from the above, then the value of Ω' upon the a, b shell will be

$$\begin{aligned}&\frac{a'^2}{d^2}(\mathrm{A}'\cos pt + \mathrm{D}'\sin pt)\\ &+\frac{aa'^2}{d^3}[-2\mathrm{A}'\sin\theta\cos\phi + \mathrm{B}'\sin\theta\sin\phi + \mathrm{C}'\cos\theta]\cos pt\\ &+\frac{aa'^2}{d^3}[-2\mathrm{D}'\sin\theta\cos\phi + \mathrm{E}'\sin\theta\sin\phi + \mathrm{F}'\cos\theta]\sin pt\\ &+\text{higher harmonics.}\end{aligned}$$

The magnetic potential at x, y, z of the current in the infinite wire is

$$2\mathrm{I}\tan^{-1}\frac{c-y}{x}\cos pt,$$

and hence at θ, ϕ on the a, b sphere is

$$2\mathrm{I}\left[\tan^{-1}\frac{c-a\sin\theta\sin\phi}{b+a\sin\theta\cos\phi}\right]\cos pt,$$

or

$$2\mathrm{I}\cos pt\left[\tan^{-1}\frac{c}{b}-\frac{ab\sin\theta\sin\phi}{b^2+c^2}-\frac{ac\sin\theta\cos\phi}{b^2+c^2}\ldots\right],$$

omitting harmonics of second and higher degrees.

Hence the first harmonic of the magnetic potential of the system external to the a, b shell is upon its surface,

$$\begin{aligned}\Omega_0 = ({}^{\backprime}\mathrm{A}\sin\theta\cos\phi + {}^{\backprime}\mathrm{B}\sin\theta\sin\phi + {}^{\backprime}\mathrm{C}\cos\theta)\cos pt\\ + ({}^{\backprime}\mathrm{D}\sin\theta\cos\phi + {}^{\backprime}\mathrm{E}\sin\theta\sin\phi + {}^{\backprime}\mathrm{F}\cos\theta)\sin pt,\end{aligned}$$

where, if $b^2 + c^2 = \rho^2$,

$$\begin{aligned}{}^{\backprime}\mathrm{A} &= -\frac{2\mathrm{I}a}{\rho^2} - \frac{2aa'^2}{d^3}\mathrm{A}'\\ {}^{\backprime}\mathrm{B} &= -\frac{2\mathrm{I}ab}{\rho^2} + \frac{aa'^2}{d^3}\mathrm{B}'\\ {}^{\backprime}\mathrm{D} &= \qquad\quad - \frac{2aa'^2}{d^3}\mathrm{D}'\\ {}^{\backprime}\mathrm{E} &= \qquad\quad + \frac{aa'^2}{d^3}\mathrm{E}'\\ {}^{\backprime}\mathrm{C} &= \qquad\quad + \frac{aa'^2}{d^3}\mathrm{C}'\\ {}^{\backprime}\mathrm{F} &= \qquad\quad + \frac{aa'^2}{d^3}\mathrm{F}'.\end{aligned}$$

Now we have shown (xi.) that a field

$$\Omega_0 = \mathrm{Y}_n e^{ipt}$$

produces a current function

$$\Phi = \frac{\overline{2n+1}\,ap}{(n+1)\,\Delta_n}\,(4\pi pa - i\,.\,\overline{2n+1}\,\sigma)\,\mathrm{Y}_n e^{ipt}.$$

The P of this is

$$\frac{4\pi a}{2n+1}\,.\,\Phi\,.\left(\frac{a}{r}\right)^{n+1},$$

and its magnetic potential at the surface

$$\frac{4\pi n}{(2n+1)}\,\Phi.$$

Hence the magnetic potential of the currents induced in the a, b shell by the system

$$\Omega_0 = ({}^{\backprime}\mathrm{A}\sin\theta\cos\phi + {}^{\backprime}\mathrm{B}\sin\theta\sin\phi + {}^{\backprime}\mathrm{C}\cos\theta)\,pt$$
$$+ ({}^{\backprime}\mathrm{D}\sin\theta\cos\phi + {}^{\backprime}\mathrm{E}\sin\theta\sin\phi + {}^{\backprime}\mathrm{F}\cos\theta)\sin pt$$

will be

$$\Omega = \frac{2\pi pa}{\Delta_1}\,[({}^{\backprime}\mathrm{A}\sin\theta\cos\phi + {}^{\backprime}\mathrm{B}\sin\theta\sin\phi + {}^{\backprime}\mathrm{C}\cos\theta)\,(4\pi pa\cos pt + 3\sigma\sin pt)$$
$$+ ({}^{\backprime}\mathrm{D}\sin\theta\cos\phi + {}^{\backprime}\mathrm{E}\sin\theta\sin\phi + {}^{\backprime}\mathrm{F}\cos\theta)\,(4\pi pa\sin pt - 3\sigma\cos pt)].$$

Comparing this with

$$(\mathrm{A}\sin\theta\cos\phi + \mathrm{B}\sin\theta\sin\phi + \mathrm{C}\cos\theta)\cos pt$$
$$+ (\mathrm{D}\sin\theta\cos\phi + \mathrm{E}\sin\theta\sin\phi + \mathrm{F}\cos\theta)\sin pt,$$

we have, on giving ${}^{\backprime}\mathrm{A}$. their values,

$$\mathrm{A} = \frac{2\pi pa}{\Delta_1}\left[4\pi pa\left(-\frac{2\mathrm{I}ac}{\rho^2} - \frac{2aa'^2}{d^3}\,\mathrm{A}'\right) - 3\sigma\left(-\frac{2aa'^2}{d^3}\,\mathrm{D}'\right)\right],$$

$$\mathrm{B} = \frac{2\pi pa}{\Delta_1}\left[4\pi pa\left(-\frac{2\mathrm{I}ab}{\rho^2} + \frac{aa'^2}{d^3}\,\mathrm{B}'\right) - 3\sigma\left(\frac{aa'^2}{d^3}\,\mathrm{E}'\right)\right],$$

$$\mathrm{C} = \frac{2\pi pa}{\Delta_1}\left[4\pi pa\left(\frac{aa'^2}{d^3}\,\mathrm{C}'\right) - 3\sigma\,\frac{aa'^2}{d^3}\,\mathrm{F}'\right],$$

$$\mathrm{D} = \frac{2\pi pa}{\Delta_1}\left[3\sigma\left(-\frac{2\mathrm{I}ac}{\rho^2} - \frac{2aa'^2}{d^3}\,\mathrm{A}'\right) + 4\pi pa\left(-\frac{2aa'^2}{d^3}\,\mathrm{D}'\right)\right],$$

$$\mathrm{E} = \frac{2\pi pa}{\Delta_1}\left[3\sigma\left(-\frac{2\mathrm{I}ab}{\rho^2} + \frac{aa'^2}{d^3}\,\mathrm{B}'\right) + 4\pi pa\left(\frac{aa'^2}{d^3}\,\mathrm{E}'\right)\right],$$

$$\mathrm{F} = \frac{2\pi pa}{\Delta_1}\left[3\sigma\left(\frac{aa'^2}{d^3}\,\mathrm{C}'\right) + 4\pi pa\,\frac{aa'^2}{d^3}\,\mathrm{F}'\right].$$

21. There are, of course, six similar equations obtained from consideration of the currents in the a', b' shell: these equations are accurate, since the introduction of further approximations gives rise to harmonics of higher orders than the first.

From the four equations giving C, F, C′ and F′, it follows that all these quantities are accurately zero, as might have been expected, since the system is unaltered on taking $-z$ for $+z$, *i.e.*, putting $(\pi - \theta)$ for θ.

From these equations it follows, as in the case of the cylinder, that the principal terms of the first harmonics are of the first degree in the radii, but their next are of the fourth (not the third, as for the cylinder).

It will also be obvious that the principal terms of the harmonics of the second order will be of the second degree in the radii.

The values of the coefficients can be calculated with ease, as with the cylinder, but we wait to see which of them are involved in the couple.

Writing the external field on the a, b shell in the form

$$\Omega_0 = \mathrm{Y}_1 \cos pt + \mathrm{Z}_1 \sin pt,$$

the couple will be (xii.)

$$\frac{1.3^2 p\sigma}{2.2\,\Delta_1} \iint d\mathrm{S} \,.\left[\mathrm{Y}_1 \frac{\partial \mathrm{Z}_1}{\partial \phi} - \mathrm{Z}_1 \frac{\partial \mathrm{Y}_1}{\partial \phi}\right],$$

and the most important term omitted (that from the second harmonic) is of at least two degrees higher in powers of the radii.

Also,

$$\mathrm{Y}_1 = {}^{\backprime}\mathrm{A} \sin\theta \cos\phi + {}^{\backprime}\mathrm{B} \sin\theta \sin\phi,$$

$$\mathrm{Z}_1 = {}^{\backprime}\mathrm{D} \sin\theta \cos\phi + {}^{\backprime}\mathrm{E} \sin\theta \sin\phi,$$

so that the couple is

$$\frac{9p\sigma}{4\Delta_1} \iint d\mathrm{S}\,[({}^{\backprime}\mathrm{A} \cos\phi + {}^{\backprime}\mathrm{B} \sin\phi)(-{}^{\backprime}\mathrm{D} \sin\phi + {}^{\backprime}\mathrm{E} \cos\phi)$$
$$- (-{}^{\backprime}\mathrm{A} \sin\phi + {}^{\backprime}\mathrm{B} \cos\phi)({}^{\backprime}\mathrm{D} \cos\phi + {}^{\backprime}\mathrm{E} \sin\phi)] \sin^2\theta,$$

or

$$\frac{9p\sigma}{4\Delta_1} \int_0^\pi \delta\theta \,.\, a^2 \sin^3\theta \int_0^{2\pi} \delta\phi \,\{({}^{\backprime}\mathrm{A}{}^{\backprime}\mathrm{E} - {}^{\backprime}\mathrm{B}{}^{\backprime}\mathrm{D})(\sin^2\phi + \cos^2\phi)\},$$

or

$$\frac{9p\sigma}{4\Delta_1} \,.\, \tfrac{4}{3} a^2 \,.\, 2\pi ({}^{\backprime}\mathrm{A}{}^{\backprime}\mathrm{E} - {}^{\backprime}\mathrm{B}{}^{\backprime}\mathrm{D}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(xiv.)}.$$

On reference to the values of `A, &c., it will be seen that to fourth powers of radii,

$${}^{\backprime}\mathrm{A}{}^{\backprime}\mathrm{E} - {}^{\backprime}\mathrm{B}{}^{\backprime}\mathrm{D} = -\frac{2\mathrm{I}a}{\rho^2} [c\mathrm{E}' + 2b\mathrm{D}'] \frac{aa'^2}{d^3},$$

also

$$\mathrm{D}' = -\frac{12\pi p a'^2 \sigma' c}{\rho'^2 \Delta'_1}\,\mathrm{I} + \text{terms three degrees higher,}$$

$$\mathrm{E}' = -\frac{12\pi p a'^2 \sigma' b'}{\rho'^2 \Delta'_1}\,\mathrm{I}\ldots$$

Hence the couple increasing ϕ is

$$\frac{6\pi p a^2 \sigma}{\Delta_1} \cdot \frac{2\mathrm{I} a^2 a'^2}{\rho^2 d^3} \cdot \frac{12\pi p a'^2 \sigma'}{\rho'^2\,\Delta'_1}\,\mathrm{I}\,(cb' + 2bc)$$

or

$$\frac{144\pi^2 p^2 \sigma\sigma' a^4 a'^4 c}{\rho^2 \rho'^2\, d^3\, \Delta_1\, \Delta'_1}\,(2b + b')\,\mathrm{I}^2.$$

22. In obtaining the couple on the a', b' shell it will not do merely to interchange dashed and undashed letters, for the equations (xiii.) give when $r' = a'$ and $< d$,

$$r^{-1} = \frac{1}{d}\left[1 + \frac{a'}{d}\sin\theta'\cos\phi'\ldots\right],$$

so that

$$\frac{\sin\theta\cos\phi}{r^2} = \frac{1}{d^8}\left[1 + \frac{3a'}{d}\sin\theta'\cos\phi'\ldots\right]\left[-d + a'\sin\theta'\cos\phi'\right]$$

$$= -\frac{1}{d^2} - \frac{2a'}{d^3}\sin\theta'\cos\phi',$$

$$\frac{\sin\theta\sin\phi}{r^2} = \frac{a'}{d^3}\sin\theta'\sin\phi',$$

$$\frac{\cos\theta}{r^2} = \frac{a'}{d^3}\cos\theta'.$$

These equations may be obtained from (xiii.) by interchanging dashed and undashed letters, *leaving the sign of d unaltered*, with the exception of the term $-(1/d^2)$ in $(\sin\theta\cos\phi)/r^2$, and as this term does not appear afterwards (being constant over the sphere), the exception is negligible.

The subsequent work does not introduce d afresh, it only makes use of the formulæ we have obtained, and thus it will be seen that the final couple on the a', b' shell is got by the changing of dashed and undashed letters, leaving the sign of d unaltered; it is

$$\frac{144\pi^2 p^2 \sigma\sigma' a^4 a'^4 c}{\rho^2 \rho'^2\, d^3\, \Delta_1\, \Delta'_1}\,(b + 2b')\,\mathrm{I}^2.$$

If we write h for $\frac{1}{2}(b + b')$, the mean of distances of the centres from 0, the couples are

$$\frac{72\pi^2 p^2 \sigma\sigma' a^4 a'^4 c}{\rho^2 \rho'^2\, d^3\, \Delta_1\, \Delta'_1}\,(6h \pm d)\,\mathrm{I}^2.$$

23. Hence

(α.) If σ, σ' or c vanish, the couple vanishes, as might have been expected.

(β.) The signs of the couples fall into three cases :—

(i.) When $6h$ is positive and $> d$, both couples are positive, and tend to increase ϕ.

Fig. 36.

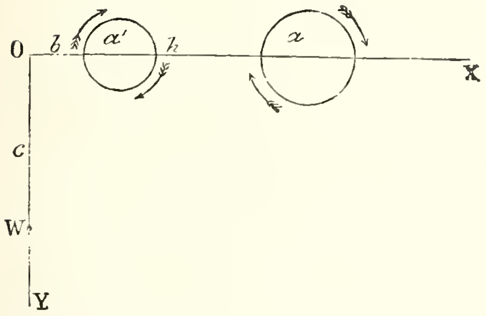
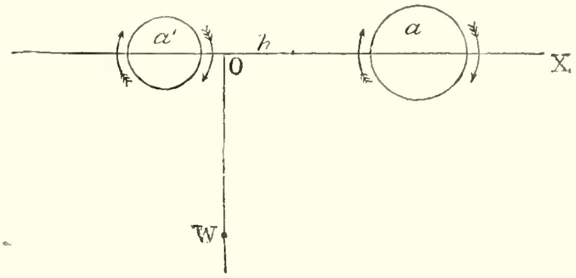
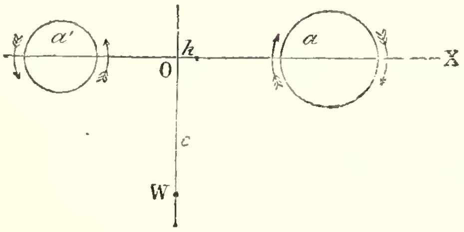


Fig. 37.



(ii.) When $6h$ is numerically less than d, the a, b couple is positive, and the a', b' negative.

Fig. 38.



(iii.) When $6h$ is negative and numerically $> d$, both couples are negative.

Fig. 39.

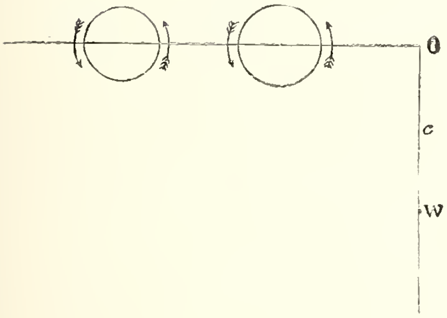
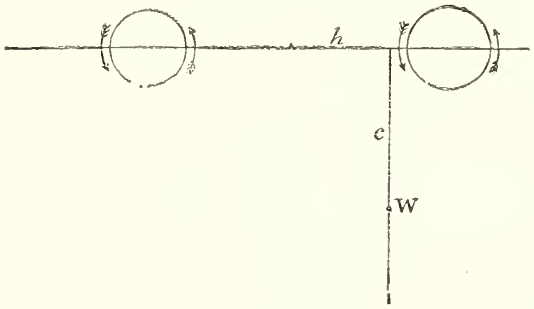
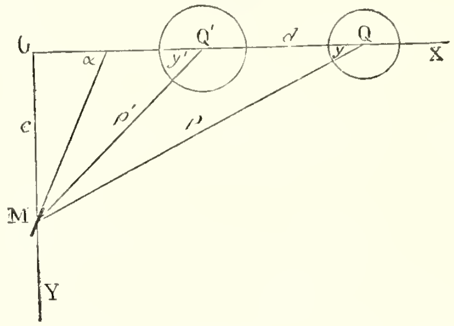


Fig. 40.



The Couples on two Spherical Shells, produced by a Filament of infinite length, perpendicular to the Line joining their Centres, and alternately Magnetised in a direction at right angles to its length.

Fig. 41.



24. Let the same axes be taken as before, and let the direction of magnetisation make α with the plane XOZ. The filament is equivalent to an electromagnet formed by currents

$$\frac{\mathrm{K}}{\kappa}\cos pt,$$

parallel to OZ at $x = 0$, $y = c$, and

$$-\frac{\mathrm{K}}{\kappa}\cos pt$$

at $x = -\kappa\sin\alpha$, $y = c - \kappa\cos\alpha$, in the limit when κ is small; the strength of magnetisation will then be $\mathrm{K}\cos pt$.

As before, a coefficient of magnetic potential

$$\mathrm{I}f(b, b', c)$$

due to the current $\mathrm{I}\cos pt$, will become

$$\mathrm{K}\left[-\sin\alpha\left(\frac{\partial f}{\partial b} + \frac{\partial f}{\partial b'}\right) + \cos\alpha\frac{\partial f}{\partial c}\right]$$

due to the electromagnet, and if

$$\left.\begin{aligned}\rho\cos\gamma &= b\\ \rho\sin\gamma &= c\end{aligned}\right\}$$

then on reference to previous work it will be found that whereas we had for ${}^{\backprime}\mathrm{A}$ due to $\mathrm{I}\cos pt$,

$$-\frac{2\mathrm{I}ac}{\rho^2} - \text{fourth powers},$$

now we shall have

$${}^{\backprime}_{,}\mathrm{A} = -\frac{2\mathrm{K}a}{\rho^2}\left[\cos\alpha + 2\sin\gamma\sin\overline{\alpha - \gamma}\right] \quad . \quad . \quad .$$

So, too, we shall have instead of

$${}^{\backprime}\mathrm{B} = -\frac{2\mathrm{I}ab}{\rho^2} \ldots,$$

$${}^{\backprime}_{,}\mathrm{B} = -\frac{2\mathrm{K}a}{\rho^2}\left[-\sin\alpha + 2\cos\gamma\sin\overline{\alpha-\gamma}\right],$$

and

$${}^{\backprime}_{,}\mathrm{E} = \frac{aa'^2}{d^3}\,{}_{,}\mathrm{E}'$$

(denoting by suffix ($_{,}$) that we have an electromagnet, not a current).

Also

$$\mathrm{E}' = -\frac{12\pi pa'^2\sigma'b'}{\Delta'_1\rho'^2}\,\mathrm{I}.$$

Therefore,

$${}_{,}\mathrm{E}' = -\frac{12\pi pa'^2\sigma'}{\Delta'_1\rho'^2}\,\mathrm{K}\left[-\sin\alpha + 2\cos\gamma'\sin\overline{\alpha-\gamma'}\right],$$

and similarly,

$${}^{\backprime}_{,}\mathrm{D} = -\frac{2aa'^2}{d^2}\,{}_{,}\mathrm{D}',$$

and

$${}_{,}\mathrm{D}' = -\frac{12p\pi a'^2\sigma'}{\Delta'_1\rho'^2}\,\mathrm{K}\left[\cos\alpha + 2\sin\gamma'\sin\overline{\alpha-\gamma'}\right].$$

But the couple tending to increase ϕ on the a, b shell is, by (xiv.),

$$\frac{6\pi pa^2\sigma}{\Delta_1}\left({}^{\backprime}_{,}\mathrm{A}\,{}^{\backprime}_{,}\mathrm{E} - {}^{\backprime}_{,}\mathrm{B}\,{}^{\backprime}_{,}\mathrm{D}\right),$$

or

$$\frac{6\pi pa^2\sigma}{\Delta_1}\,\frac{aa'^2}{d^3}\left({}^{\backprime}_{,}\mathrm{A}\,{}_{,}\mathrm{E}' + 2\,{}^{\backprime}_{,}\mathrm{B}\,{}_{,}\mathrm{D}'\right),$$

or

$$+\frac{6\pi pa^3a'^2}{\Delta_1 d^3}\cdot\frac{2\mathrm{K}a}{\rho^2}\cdot\frac{12p\pi a'^2\sigma'}{\Delta'_1\rho'^2}\,\mathrm{K}\cdot f(\alpha,\gamma,\gamma'),$$

where

$$\begin{aligned} f(\alpha,\gamma,\gamma') &= \left[\cos\alpha + 2\sin\gamma\sin\overline{\alpha-\gamma}\right]\left[-\sin\alpha + 2\cos\gamma'\sin\overline{\alpha-\gamma'}\right] \\ &\quad + 2\left[-\sin\alpha + 2\cos\gamma\sin\overline{\alpha-\gamma}\right]\left[\cos\alpha + 2\sin\gamma'\sin\overline{\alpha-\gamma'}\right] \\ &= -3\sin\alpha\cos\alpha + \sin\overline{\alpha-\gamma}\left[4\cos\alpha\cos\gamma - 2\sin\alpha\sin\gamma\right] \\ &\quad + \sin\overline{\alpha-\gamma'}\left[2\cos\alpha\cos\gamma' - 4\sin\alpha\sin\gamma\right] \\ &\quad + 2\sin\overline{\alpha-\gamma}\sin\overline{\alpha-\gamma'}\left[2\sin\gamma\cos\gamma' + 4\cos\gamma\sin\gamma'\right] \\ &= -\tfrac{3}{2}\sin 2\alpha + \sin\overline{\alpha-\gamma}\left[\cos\overline{\alpha-\gamma} + 3\cos\overline{\alpha+\gamma}\right] \\ &\qquad\qquad + \sin\overline{\alpha-\gamma'}\left[3\cos\overline{\alpha+\gamma'} - \cos\overline{\alpha-\gamma'}\right] \\ &\quad + \left[\cos\overline{\gamma-\gamma'} - \cos 2\overline{\alpha-\gamma-\gamma'}\right]\left[3\sin\overline{\gamma+\gamma'} + \sin\overline{\gamma'-\gamma}\right]. \end{aligned}$$

On multiplying out and replacing products of sines and cosines by sines and cosines of added or subtracted angles most of the terms cancel, and we are left with

$$\tfrac{1}{2}\sin 2(\gamma' - \gamma) + \tfrac{3}{2}\sin 2(\alpha - \gamma - \gamma').$$

Thus the couple on the a, b sphere is

$$\mathrm{K}^2\frac{72\pi^2p^2a^4a'^4\sigma\sigma'}{\rho^2\,\rho'^2\Delta_1\Delta'_1\,d^3}\left[\sin 2\,\overline{\gamma' - \gamma} + 3\sin 2(\alpha - \gamma - \gamma')\right],$$

and on the a', b' sphere,

$$\mathrm{K}^2\frac{72\pi^2p^2a^4a'^4\sigma\sigma'}{\rho^2\rho'^2\Delta_1\Delta'_1\,d^3}\left[-\sin 2\,\overline{\gamma' - \gamma} + 3\sin 2(\alpha - \gamma' - \gamma)\right].$$

25. From this we see that

(α.) If the couples on the two shells be equal and opposite

$$(\alpha - \gamma - \gamma') = 0 \quad \text{or} \quad \pm\tfrac{1}{2}\pi,$$

i.e.,

$$\alpha = \gamma + \gamma' \quad \text{or} \quad \gamma + \gamma' \pm \tfrac{1}{2}\pi.$$

(β.) The couples will not vanish when $c = 0$ (and $\gamma = \gamma' = 0$), unless in addition $\alpha = 0$ or $\pm\frac{1}{2}\pi$ (in which case there is by symmetry obviously no couple).

(γ.) We may take as an example

$$\gamma = 30^\circ, \qquad \gamma' = 60^\circ,$$

and the couples will be

$$\frac{72\pi^2p^2a^4a'^4\sigma'\sigma}{\rho^2\rho'^2\Delta_1\Delta'_1\,d^3}\left[\pm\frac{\sqrt{3}}{2} - 3\sin 2\alpha\right],$$

the upper sign referring to the a, b shell.

The bracket will be positive when $\alpha = -45^\circ$ say.

The bracket will be positive and negative when $\alpha = 0$ or 90°.

Fig. 42.

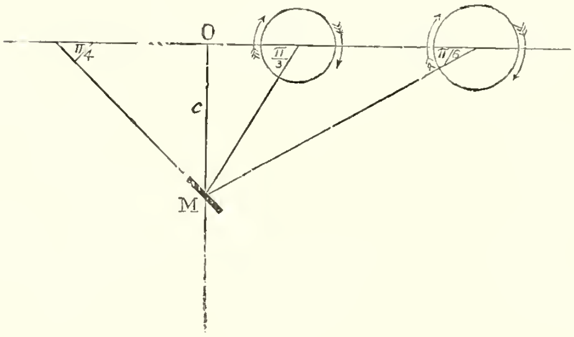
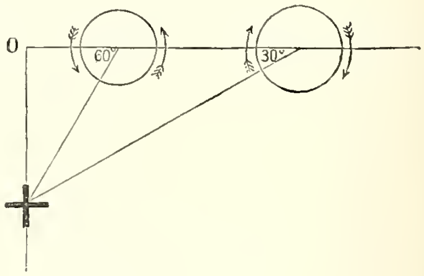
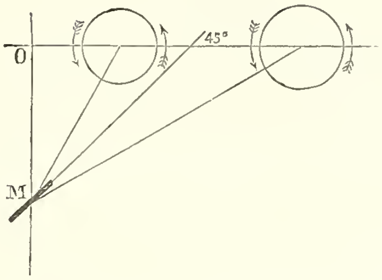


Fig. 43.



The bracket will be negative when $\alpha = +45°$.

Fig. 44.



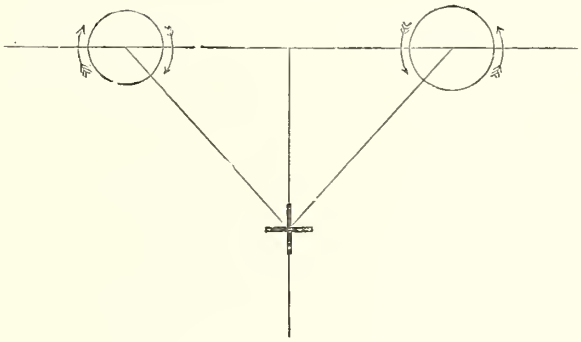
(δ.) In confirmation it may be noticed that if $\gamma = \pi - \gamma'$ (*i,e.*, when the system is symmetrical to OY), and $\alpha = 0°$ or $90°$, the couples are equal and opposite.

The couples are then

$$\mp \mathrm{K}^2 \frac{72\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{\rho^2 \rho'^2 \Delta_1 \Delta'_1 d^3} \sin 2\gamma,$$

and are negative and positive on the a, b and a', b' shells respectively, for all possible values of γ, for which the a, b shell is to the right of a', b'.

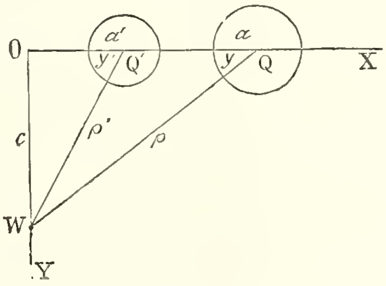
Fig. 45.



The Couples on Two Spherical Shells in the presence of a Magnetic Pole of strength H cos *pt.*

26. Take OX through the centres of the spheres and OY through the pole, whose distance from O is c.

Fig. 46.



Then, with the same notation as before, the magnetic potential due to the single pole will be, on the *a*, *b* shell,

$$\Omega_o = \frac{\mathrm{H}\cos pt}{[\rho^2 + a^2 + 2\rho a \sin\theta \cos\overline{\phi + \gamma}]^{\frac{1}{2}}}$$
$$= \frac{\mathrm{H}\cos pt}{\rho}\left[1 - \frac{a}{\rho}\sin\theta\cos\overline{\gamma + \phi} + \text{second harmonics}\right].$$

The equations (xiii.) now give

$$\left.\begin{aligned} {}^{\backprime}\mathrm{A} &= -\frac{\mathrm{H}a}{\rho^2}\cos\gamma + \text{fourth powers of } a \\ {}^{\backprime}\mathrm{B} &= \quad \frac{\mathrm{H}a}{\rho^2}\sin\gamma + \dots \\ {}^{\backprime}\mathrm{D} &= -\frac{2aa'^2}{d^3}\,\mathrm{D}' \\ {}^{\backprime}\mathrm{E} &= +\frac{aa'^2}{d^3}\,\mathrm{E}' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (\text{xv.}),$$

and since in Ω_0 there was no term cos θ, the coefficients ‘C, ‘F will be zero.

Also

$$\mathrm{D} = \frac{2\pi p a}{\Delta_1}[3\sigma\,.\,{}^{\backprime}\mathrm{A}] + \text{fourth powers},$$
$$\mathrm{E} = \frac{2\pi p a}{\Delta_1}[3\sigma\,.\,{}^{\backprime}\mathrm{B}]\dots$$

Therefore

$$\mathrm{D}' = \frac{2\pi p a'}{\Delta'_1}\,3\sigma'\,.\left[-\frac{\mathrm{H}a'}{\rho'}\cos\gamma'\right]\dots,$$
$$\mathrm{E}' = \frac{2\pi p a'}{\Delta'_1}\,3\sigma'\,.\left[+\frac{\mathrm{H}a'}{\rho'^2}\sin\gamma'\right]\dots$$

The couple on the *a*, *b* shell has been proved to be (xiv.)

$$\frac{6\pi p a^2\sigma}{\Delta_1}({}^{\backprime}\mathrm{A}\,{}^{\backprime}\mathrm{E} - {}^{\backprime}\mathrm{B}\,{}^{\backprime}\mathrm{D}).$$

This is equal to

$$\frac{6\pi p a^2\sigma}{\Delta_1}\,\frac{aa'^2}{d^3}\,[{}^{\backprime}\mathrm{A}\,.\,\mathrm{E}' + 2\,{}^{\backprime}\mathrm{B}\,.\,\mathrm{D}'],$$

or

$$\frac{6\pi p a^2\sigma}{\Delta_1}\,\frac{aa'^2}{d^3}\,\frac{\mathrm{H}a}{\rho^2}\,\frac{6\pi p a'\sigma'}{\Delta'_1}\,\frac{\mathrm{H}a'}{\rho'^2}\,[-\cos\gamma\sin\gamma' - 2\sin\gamma\cos\gamma'],$$

or

$$-\,\mathrm{H}^2\,\frac{36\pi^2 p^2 a^4 a'^4 c\sigma\sigma'}{\rho^3\rho'^3\,\Delta_1\,\Delta'_1\,d^3}\,(b + 2b').$$

The couple on the a', b' shell is

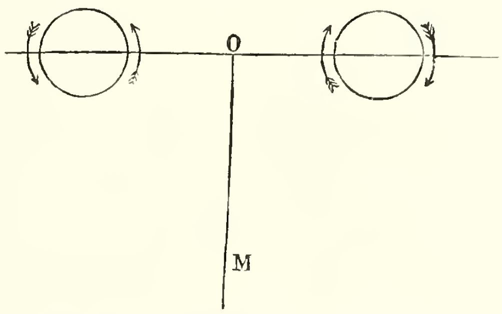
$$-\mathrm{H}^2 \frac{36\pi^2 p^2 a^4 a'^4 c\sigma\sigma'}{\rho^3 \rho'^3 \Delta_1 \Delta'_1 d^3} (2b + b').$$

27. Thus we see that

(α.) The couples vanish (as they should) when $c = 0$.

(β.) When $b = -b'$ and $a = a'$, the figure is symmetrical to plane YOZ, and the couples will be equal and opposite (as they should): that on the a, b shell being positive if d be positive.

Fig. 47.



(γ.) For other cases the discussion of sign is similar to that for the current I cos pt, and there are three cases:—

I. When b and b' are positive, the couples are both negative, as also if b' be negative and $\frac{1}{2}(b + b') > d/6$.

Fig. 48.

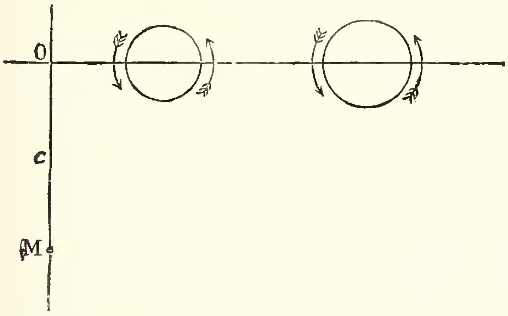
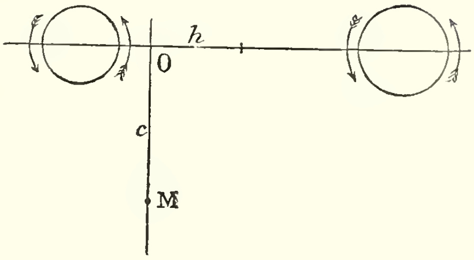
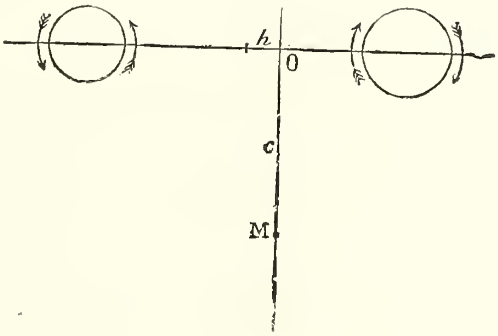


Fig. 49.



II. When $\frac{1}{2}(b + b') < d/6$ numerically, the signs are +, −.

Fig. 50.



III. When $\frac{1}{2}(b + b')$ is negative, and is numerically greater than $d/6$, the couples are both positive.

Fig. 51.

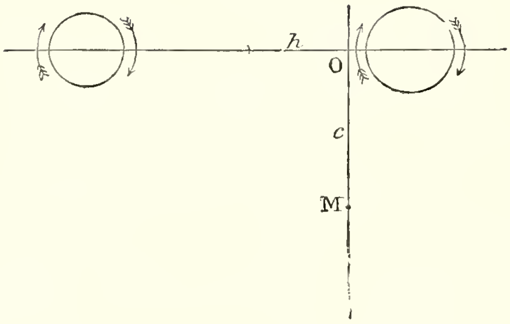
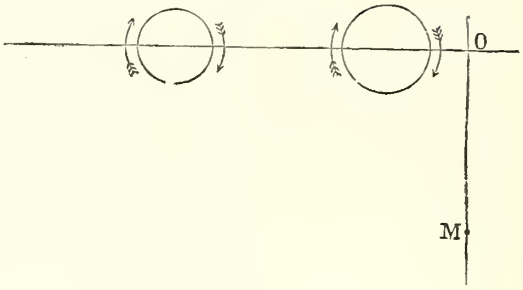


Fig. 52.

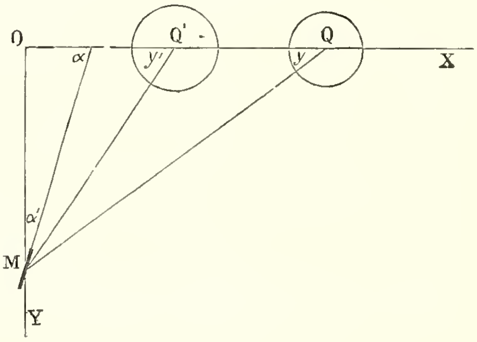


The Couples on two Spherical Shells in the presence of a small Magnet of Moment K cos *pt, whose Axis cuts the Line joining the Centres of the Spheres at an angle* α.

28. We regard the magnetic particle as having a pole of strength $(\mathrm{K}/\kappa)\cos pt$ at $x = 0$, $y = c$, and a pole of strength $-(\mathrm{K}/\kappa)\cos pt$ at

$$x = -\kappa \sin \alpha', \quad y = c + \kappa \cos \alpha', \quad \text{where} \quad \alpha' = (\pi/2) - \alpha.$$

Fig. 53.



If a coefficient of magnetic potential due to a single pole at $(0, c)$ of strength $\mathrm{H}\cos pt$, be

$$\mathrm{H}f(b, b', c)$$

that due to the magnet will be the limit when κ is small of

$$(\mathrm{K}/\kappa) f(b, b', c) - (\mathrm{K}/\kappa) f(b + \kappa \sin \alpha', b' + \kappa \sin \alpha', c + \kappa \cos \alpha'),$$

i.e., will be

$$-\mathrm{K}\left[\left(\frac{\partial f}{\partial b} + \frac{\partial f}{\partial b'}\right)\sin \alpha' + \frac{\partial f}{\partial c}\cos \alpha'\right].$$

Now we had (xv.)

$$ {}^{\backprime}\mathrm{A} = -\frac{\mathrm{H}a}{\rho^2}\cos\gamma = -\frac{\mathrm{H}ab}{\rho^3}\,. $$

Thus, distinguishing coefficients arising from the magnet by a suffix $({}_{,})$ as distinct from the magnetic pole, we have

$$ \begin{aligned} {}^{\backprime}_{,}\mathrm{A} &= -\,\mathrm{K}a\left[\left(-\frac{1}{\rho^3}+\frac{3b^2}{\rho^5}\right)\sin\alpha' + \frac{3bc}{\rho^5}\cos\alpha'\right] \\ &= -\,\frac{\mathrm{K}a}{\rho^3}\,[\,\sin\alpha' + 3\cos\gamma\,\sin\overline{\alpha'+\gamma}] \end{aligned} $$

So too from ${}^{\backprime}\mathrm{B} = \mathrm{H}ac/\rho^3$ (xv.)

$$ {}^{\backprime}_{,}\mathrm{B} = \frac{\mathrm{K}a}{\rho^3}\,[-\cos\alpha' + 3\sin\gamma\,\sin\overline{\alpha'+\gamma}]\,. $$

We had also

$$ \begin{aligned} {}^{\backprime}\mathrm{D} &= \frac{2aa'^2}{d^3}\cdot\frac{6\pi pa'\sigma'}{\Delta'_1}\,\frac{\mathrm{H}a'b'}{\rho'^3} + \text{higher powers}, \\ {}^{\backprime}\mathrm{E} &= \frac{aa'^2}{d^3}\cdot\frac{6\pi pa'\sigma'}{\Delta_1'}\,\frac{\mathrm{H}a'c}{\rho'^3}\,. \end{aligned} $$

Hence

$$ \begin{aligned} {}^{\backprime}_{,}\mathrm{D} &= \mathrm{K}\,\frac{12\pi paa'^4\sigma'}{d^3\Delta'_1\rho'^3}\,[-\sin\alpha' + 3\cos\gamma'\,\sin\overline{\alpha'+\gamma'}]\ldots \\ {}^{\backprime}_{,}\mathrm{E} &= \mathrm{K}\,\frac{6\pi paa'^4\sigma'}{d^3\Delta'_1\rho'^3}\,[-\cos\alpha' + 3\sin\gamma'\,\sin\overline{\alpha'+\gamma'}]\ldots \end{aligned} $$

Thus the couple

$$ \begin{aligned} &= \frac{6\pi pa^2\sigma}{\Delta_1}\,({}^{\backprime}_{,}\mathrm{A}\,{}^{\backprime}_{,}\mathrm{E} - {}^{\backprime}_{,}\mathrm{B}\,{}^{\backprime}_{,}\mathrm{D}), \\ &= \frac{6\pi pa^2\sigma}{\Delta_1}\cdot\frac{\mathrm{K}^2a}{\rho^3}\cdot\frac{6\pi paa'^4\sigma'}{d^3\Delta'_1\rho'^3}\,f(\alpha'\,\gamma,\gamma'), \end{aligned} $$

where

$$ \begin{aligned} f(\alpha',\gamma,\gamma') \equiv &-[-\sin\alpha' + 3\cos\gamma\,\sin\overline{\alpha'+\gamma}]\,[-\cos\alpha' + 3\sin\gamma'\,\sin\overline{\alpha'+\gamma'}] \\ &-[-\cos\alpha' + 3\sin\gamma\,\sin\overline{\alpha'+\gamma}]\,[-2\sin\alpha' + 6\cos\gamma'\,\sin\overline{\alpha'+\gamma'}] \\ = &-\tfrac{1}{4}\{(\sin\alpha' + 3\sin\overline{\alpha'+2\gamma})\,(\cos\alpha' - 3\cos\overline{\alpha'+2\gamma'}) \\ &+ (\cos\alpha' - 3\cos\overline{\alpha'+2\gamma})\,(\sin\alpha' + 3\sin\overline{\alpha'+2\gamma'})\}. \end{aligned} $$

On multiplying and continuing the practice of replacing products by sines and cosines of sums or differences, we get

$$ \begin{aligned} = -\tfrac{3}{8}\,[\sin 2\alpha' &+ 3\,(\sin 2\gamma + \sin 2\gamma') + (\sin 2\,\overline{\alpha'+\gamma'} - \sin 2\overline{\alpha'+\gamma}) \\ &- 3\sin 2\,(\gamma'-\gamma) - 9\sin\overline{2\alpha'+2\gamma+2\gamma'}]. \end{aligned} $$

Expressed in terms of α, the couple on the two shells will be

$$- \mathrm{K}^2 \frac{27\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{2\Delta_1\Delta'_1\, d^3\rho^3\rho'^3}\, \phi(\alpha, \gamma, \gamma')$$

$$- \mathrm{K}^2 \frac{27\pi^2 p^2 a^4 a'^4 \sigma\sigma'}{2\Delta_1\Delta'_1\, d^3\rho^3\rho'^3}\, \phi(\alpha, \gamma', \gamma),$$

where

$$\phi(\alpha, \gamma, \gamma') \equiv \sin 2\alpha + 3(\sin 2\gamma + \sin 2\gamma') + \sin \overline{2\alpha - 2\gamma'}$$
$$- \sin \overline{2\alpha - 2\gamma} - 3 \sin \overline{2\gamma' - 2\gamma} - 9 \sin \overline{2\alpha - 2\gamma - 2\gamma'}.$$

29. Hence we have

(α.) If $\gamma' = \pi - \gamma$,

$$\phi(\alpha, \gamma, \gamma') = 3 \sin 4\gamma + 2 \sin 2\gamma \cos 2\alpha - 8 \sin 2\alpha.$$

For the couples to be equal and opposite we must have $\alpha = 0$ or $\frac{1}{2}\pi$, and then

$$\phi(\alpha, \gamma, \gamma') = 3 \sin 4\gamma \pm 2 \sin 2\gamma.$$

If $\gamma = \frac{1}{3}\pi$ (fig. 54), the values of $\phi(\alpha, \gamma, \gamma')$ are negative, both for $\alpha = 0$ and $\alpha = \frac{1}{2}\pi$; if $\gamma = \frac{1}{6}\pi$ (fig. 55), both values of $\phi(\alpha, \gamma, \gamma')$ are positive.

Fig. 54.

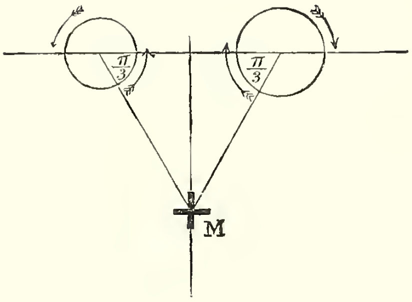
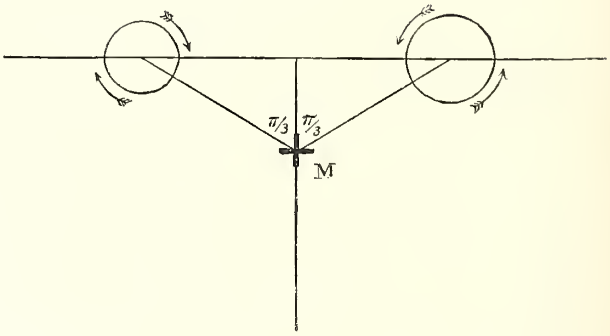


Fig. 55.



(β.) If we take $\gamma = 30°$, $\gamma' = 60°$, we find that

$$f(\alpha, \gamma, \gamma') = -\tfrac{3}{8}\left[\frac{3\sqrt{3}}{2} + 9 \sin 2\alpha\right]$$

$$f(\alpha, \gamma', \gamma) = -\tfrac{3}{8}\left[\frac{9\sqrt{3}}{2} + 11 \sin 2\alpha\right].$$

The former is negative when α increases from 0° to about 98° 23′, and positive thence to 171° 37′, being negative to 180°. The latter is negative from 0° to 112° 34′, positive thence to 157° 26′, and afterwards negative to 180°.

In fig. 56, $\alpha = 45°$; in fig. 57, $\alpha = 105°$; in fig. 58, $\alpha = 135°$; and in fig. 59, $\alpha = 165°$.

Fig. 56.

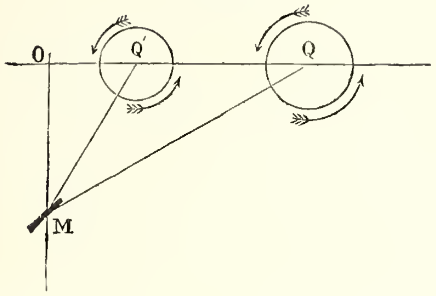


Fig. 57.

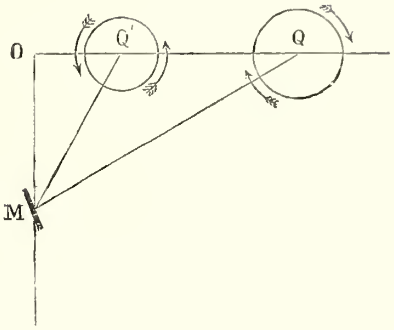


Fig. 58.

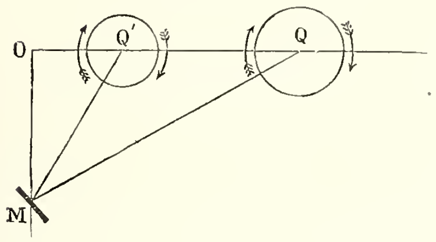
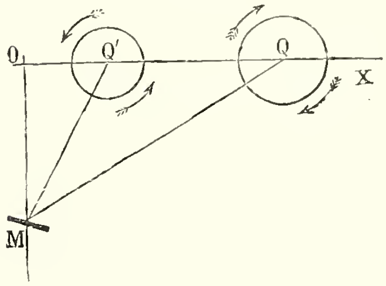


Fig. 59.



VII. *Re-determination of the Mass of a Cubic Inch of Distilled Water.*

By H. J. Chaney.

Communicated by Sir G. Gabriel Stokes, Bart., *F.R.S.*

Received February 4,—Read June 19, 1890,—Revised January 16, 1892.

The evaluation of the mass of a cubic inch of distilled water, as at present accepted, was based on weighings made in 1798 by Sir G. Shuckburgh ('Phil. Trans.,' 1798, p. 133); and on measurements made in 1821 by Captain Kater ('Phil. Trans.,' 1821, pp. 316 and 326). Subsequent researches however, particularly those in relation to the mass of a cubic decimetre, show that it is desirable to re-determine the mass of the cubic inch of distilled water.

The result of Shuckburgh's experiments was that the cubic inch of distilled water at the temperature of 66° Fahr., the barometer being at 29·74 inches, weighed 252·422 grains, and this value as corrected by Captain Kater, became 252·458 grains at the temperature of 62° Fahr., the barometer being at 30 inches; or *in vacuo* ($t. = 62°$ Fahr.) the cubic inch of distilled water weighed 252·724 grains. This corrected value, 252·458 grains, has been adopted in various legislative enactments;—for instance the Weights and Measures Act, 5 Geo. IV., c. 74, section 5 (1824), declared that "a cubic inch of distilled water, weighed in air by brass weights, at the temperature of sixty-two degrees of Fahrenheit's thermometer, the barometer being at thirty inches, is equal to two hundred and fifty-two grains and four hundred and fifty-eight thousandth parts of a grain, of which the imperial standard Troy pound contains five thousand seven hundred and sixty."

Owing to doubt as to the true mass of a cubic inch of distilled water, the above section of the Act of 1824 was repealed in 1878, and has not been re-enacted. No re-determination of the mass of a cubic inch of distilled water in terms of the present imperial pound has yet been made; the above value, 252·458 grains, being based on the old Troy pound of 1758. Dr. Wild's investigations[*] appear indeed to show that later results obtained in the evaluation of a definite volume of distilled water, differ appreciably from those obtained in terms of the old Troy pound. If, for instance, the mass of a cubic decimetre of distilled water at 4° C., as originally determined by Lefèvre-Gineau and Trallès in 1799, is taken as 1000·000 grms.

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[*] 'Bericht über die Arbeiten zur Reform der Schweizerischen Urmaasse,' Zürich, 1868.

Then the equivalent value of the cubic decimetre, as deduced from the SHUCKBURGH and KATER value of the cubic inch, would be . . . . 1000·480 grms.

Other investigations have given :—

(1825)	BERZELIUS SVANBERG AKERMANN	1000·296	,,
(1834)	STAMPFER	999·653	,,
(1841)	KUPFFER	999·989	,,

The arithmetical mean of these five results is 1000·084 grms., and there is a difference of 0·827 grm. between the highest and lowest result. The difference between the results appears to arise mainly from inaccuracy in the measurement of temperature, and uncertainty as to the condition of the water used.

Dr. BROCH[*] revised a report by TRALLÈS on LEFÈVRE-GINEAU'S original work, and arrived at the conclusion that probably the true mass of a cubic decimetre of distilled water at its maximum density, and in a vacuum, is from 90 to 120 mgrms. less than 1000 grms.

Methods and Instruments.

It has been the object of the present investigation to ascertain the weight of water displaced by a body or gravimeter, whose masses, in air and *in vacuo*, and linear dimensions, had been carefully ascertained, rather than to determine the theoretical mass of a cubic inch of water. It has been considered that the mass of a definite volume of distilled water cannot well be ascertained by weighing the water contained in a vessel of a given capacity; or by the use of gravimeters, made only of one material (as brass), and of one particular form (as a sphere). Three gravimeters (or hydrometers) therefore, of the following forms were now adopted :—

Two gravimeters, C and Q, of cylindrical form, one hollow and the other solid, the third gravimeter S being hollow, and of spherical form. The hollow gun-metal circular cylinder C (fig. 1, page 340) was nearly nine inches in height and diameter, and was adopted as being a body the weight and dimensions of which might give the least probable error in the several operations of weighing in air and in water. The gravimeter C was protected form oxidation by platinising, it having been first made air-tight. For the purpose of linear measurement, there were traced on the gravimeter a series of lines, as indicated in fig. 1. These lines were cut rather deeper than was desirable, but the space taken up by them on the surface of the cylinder has been considered.

[*] 'Procès-Verbaux—Commission Internationale du Mètre. Réunions des Membres Français. 1873.'

The cylinder Q was adopted as a solid gravimeter because the density and expansion of quartz are well known, as well as on account of the hardness of quartz, its capability of receiving a high polish and absence of hygroscopic properties. Q is a pure crystal, and was originally adjusted under the directions of Dr. VOIT, its coefficient of linear expansion for 1° C. being taken as 0·00000781 in a direction parallel to the principal axis of the crystal.

The third gravimeter S, is a hollow brass sphere, having on its surface two engraved lines for the purpose of linear measurement, and its approximate diameter is six inches.

Water.

Expansion of Water.—As the weighings of the gravimeters in water could not be actually made at the normal temperature of 62° Fahr., corrections were necessary for the density of water at various temperatures. There has not yet been made any determination of the rate of expansion of water which might alone be accepted, and we have therefore to adopt the mean result of several selected determinations. ROSSETTI has stated the result of his experiments ('Atti del Istituto Veneto,' 1867–8), as well as those of KOPP (1847), PIERRE (1845, 1852), DESPRETZ (1839), HAGEN (1855), and MATTHIESSEN (1866), in a table which gives 1·001121 as the ratio of the density of water at 62° Fahr. (16°·6 C.) to its maximum density at 4° C. It does not appear whether ROSSETTI corrected the earlier results by more recent determinations of the rates of expansion of mercury and of glass; which corrections at some temperatures would affect the last place of decimals in the above expression (1·001121). The differences between the above determinations would also affect the present investigation to ± 0·0009 grain in the mass of the cubic inch of water.

In 1870 Dr. FOERSTER,[*] after a critical examination of the results above referred to, as well as of those of JOLLY (1864) and W. H. MILLER (1856),[†] adopted the mean results of MILLER and ROSSETTI (after SCHIAPARELLI, 1868); which give 1·001118 as the ratio at 62° Fahr.; or if the maximum density of water is taken at the temperature of 4° C., then at 62° Fahr. the density of water may be expressed by 0·998881; and these mean results have, therefore, been followed in the present investigation.

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[*] 'Metronomische Beiträge,' No. 1, Berlin, 1870.

[†] 'Phil. Trans.,' 1857 (Part III. for 1856).

TABLE I.—Density of Water.

Logarithms of Ratios of the Maximum Density of Water to its Density at *t*.

t.	Log. W_t.	Diff.	*t*.	Log. W_t.	Diff.
4° C.	0·0000000		17° C.	0·0005098	
5	36	36	18	5867	769
6	136	100	19	6682	815
7	299	163	20	7544	862
8	524	225	21	8451	907
9	808	284	22	9401	950
10	1152	344	23	10393	992
11	1553	401	24	11426	1033
12	2011	458	25	12498	1072
13	2523	512	26	13609	1111
14	3090	567	27	14757	1148
15	3708	618	28	15941	1184
16	4378	670	29	17158	1217
17	0·0005098	720	30	0·0018409	1251

Condition of the Water.—The water was twice distilled in pure tin and glass stills, and was found to be free from any impurities likely to affect the weighings. It was deprived of air by boiling, and no correction for the absorption of air was therefore made. It was, however, found that at the temperature of 62° Fahr. (B. = 30 inches) a cubic foot of distilled water, freed from air, weighed about 321 grains more than when nearly "saturated" with air.

Standards of Length and Comparators.

Standards of Length and Comparators.—For the purpose of measuring the external dimensions of the gavimeters in inches of the present imperial yard, four 9-inch end standards of length were used; as well as a steel 6-inch and a steel 3-inch standard (1884). At 62° Fahr., the true mean length of the four 9-inch standards was 8·99975 inches ± 0·00001 inch; the 6-inch and 3-inch standards having a true length of 6·00020 and 2·99975 inches respectively. For intervals between 0·1 and 0·01 inch there was also used a subdivided inch "D," the accuracy of the subdivisions of which had been measured by the Standards Commission in 1868 to ± 0·000005 inch.

For the comparison of the dimensions of the gravimeters with the standards of length, there were used two comparators of well-known forms; one a Whitworth contact comparator, the other a micrometer-microscope comparator designed by Mr. J. SIMMS. By means of the former comparator measurements by touch might be made to 0·0001 inch; and optically to 0·000025 inch, by means of the microscopes.

When two standards or measures of extension are to be compared by means of the Whitworth comparator, first one standard and then the other is placed in a horizontal position between two contact points ; one of which is a fixed point, the other being a movable point or the termination of a micrometer-screw. Any difference, in parts of an inch, between the two standards can then be measured by means of the screw, the value in parts of an inch of one revolution or whole turn of the screw being known. In this comparator, the appreciation of the moment of contact is ascertained by means of a "gravity-piece." When the gravimeter was placed in position, one of its sides was in contact with the fixed point, its other side being in contact with the gravity-piece, which was interposed between the gravimeter and the screw, and the moment of contact between the gravimeter and the screw was then ascertained by gradually turning or releasing the screw until the gravity-piece fell by its own weight.

In the general form of the ordinary micrometer-microscope comparator, or optical beam-compass, there are two fixed microscopes, the distance between which is determined by comparison with a standard of length. The micrometer heads of the microscopes are divided into 100 divisions, the mean value of one division being equal to nearly 0·00003 inch; and the linear standards or measures of extension are compared by being alternately placed under the microscopes, a microscope being fixed over either end of the standard under observation.

As the comparators were only required for the purpose of measuring small differences between the standards and the gravimeters, it was unnecessary to verify the whole run of the micrometer screws ; but the particular parts of the screws used were verified by comparison with the subdivided inch D.

There were no defining lines marked on the gravimeter Q by which its dimensions could be measured, although certain lines, as previously mentioned, were engraved on C and S, for the purpose of indicating generally the particular parts of the surface of the gravimeter brought under measurement. Therefore, in the microscopic comparisons of Q, it became necessary to develop defining lines, visible through the microscope, and this was done after the method of FIZEAU and CORNU.

When the gravimeter was placed in position under the fixed microscopes, two pencil points, made of polished silver, were brought nearly into contact with the gravimeter, within 0·001 inch of the gravimeter at each end. The "points" did not therefore actually touch the sides of the gravimeter, but were so reflected as to form an apparent line of contact, visible through the microscope, on either side of the gravimeter. The actual distance between the two apparent lines formed by the "points" and their reflections, as measured by the fixed microscopes, was then compared with a standard of length which was placed under the microscopes after the removal of the gravimeter.

Rates of Expansion of the Gravimeters by Heat.—The rates of expansion by heat of the three gravimeters were not ascertained by actual experiment, as the probable error which would arise in ascertaining the rate of expansion of a body having the particular form of either one of the gravimeters would be greater than the probable error of experiments on the dilatation of a body of the simple form of a rod or bar.

The gravimeter C, for instance, is made of a gun-metal alloy, the coefficient of linear expansion of which alloy was found by SHEEPSHANKS in 1848 to be 0·00000917 for 1° Fahr. It was subsequently found by CLARKE (1860) to be 0·00000986; FIZEAU, also, has shown that SHEEPSHANKS' coefficient was too low, owing possibly to the form of the mercurial thermometer adopted in 1848. For the gravimeter C, CLARKE'S coefficient of linear expansion has been therefore taken. For the gravimeter S the coefficient of linear expansion for 1° Fahr., 0·0000104, has been taken; and for Q the cubic expansion 0·00001924 for 1° Fahr.

Thermometers.

Six standard mercurial thermometers were used—

Centigrade	4,517
,,	4,518
,,	4,575
Fahrenheit	430
,,	12,765
,,	20,065

The errors of these thermometers were originally determined in relation to the hydrogen thermometer at the Bureau International des Poids et Mesures, Paris, and were re-determined after the experiments. Thermometers 4517 and 4518 were made of a hard glass (including 71·5 silica, 14·5 lime, 11·2 soda, 1·3 alumina, 0·7 sulphuric acid, and 0·3 potash), and were highly sensitive.

Thermometer No. 4517.—Divided into 0°·1 C. from 4°·3 to + 103°·4. Distance from middle of reservoir to 0° C. is 53 mm. Total length of thermometer is 702 mm. Length of a degree is 5·290 mm.

CORRECTION for Calibration.

Divisions.	Corrections.	Divisions.	Corrections.
°	°	°	°
0	0·0000	50	+ 0·1668
+ 2	0·0068	60	+ 0·1599
8	0·0393	62	+ 0·1520
16	+ 0·0651	70	+ 0·1163
18	+ 0·0639	80	− 0·0082
20	+ 0·0635	90	− 0·0257
30	+ 0·1039	100	± 0·0000
40	+ 0·1246		

Divisions equidistant. Probable error of correction ± 0°·0010 C.

Coefficient of pressure (or pressure on reservoir) for one millimetre of mercury, as applied when thermometer was used in a vertical position is 0·0001025.

Boiling-point in a horizontal position is 100°·0242 ± 0·000011. The value of a division in "normal" degrees being 0°·999758 ± 0°·000011.

Error of zero-point, when the thermometer was placed in a horizontal position, was 0°·074 C.

Thermometer No. 4518, is a Centigrade thermometer of similar form to 4517.

Boiling-point in a horizontal position is 100°·0665. The value of a division being 0°·999335 ± 0·000017.

Zero-point; horizontal position, − 0°·087.

Thermometer No. 4575.—Divided into 0°·1 C. from − 4°·1 to 55°·1. Distance from middle of reservoir to 0° C. is 64 mm. Total length of thermometer, 482 mm. Divisions not equidistant, but by comparison with a standard thermometer the following corrections were obtained:—

Scale.	Corrections.
°	°
5·1	+ 0·014
10·7	+ 0·007
15·2	+ 0·011
20·2	− 0·006
24·9	− 0·024
30·1	− 0·045
35·0	− 0·051

Coefficient of pressure is 0·0002264.

Zero-point; horizontal position, + 0°·314.

Thermometer; Kew, 430.—Divided into 1° Fahr. from − 10° to + 217° Fahr. Nos. 12,765 and 20,065. Divided into tenths of a degree from 20° to 84° Fahr.

	430.	12,765.	20,065.
	mm.	mm.	mm.
Distance in millimetres from middle of the reservoir to 32° Fahr.	228·3	134	80·3
Length of a degree	1·908	2·75	4·27
Total length of the thermometer	620	161	333

It was found that the construction of the above thermometers, Kew, 12,765, and 20,065, did not permit of exact calibration, and the following corrections were determined in a vertical position, by comparison therefore with a standard thermometer.

Kew No. 430.		No. 12,765.		No. 20,065.	
Scale readings.	Corrections.	Scale readings.	Corrections.	Scale readings.	Corrections.
° F.	° F.	° F.	° F.	° F.	° F.
41·5	+ 0·013	41·6	+ 0·121	41·7	– 0·001
50·2	+ 0·047	50·4	+ 0·045	50·4	+ 0·071
59·0	+ 0·060	59·3	– 0·003	59·2	+ 0·124
68·1	+ 0·101	68·5	– 0·037	68·4	+ 0·017
76·8	+ 0·013	77·0	+ 0·027	76·9	+ 0·104
86·1	+ 0·080	86·4	+ 0·065		
94·7	+ 0·130	95·0	+ 0·082		

Estimated probable error of correction is 0°·01 Fahr.

The coefficients of interior pressure at 100° C. for one millimetre of mercury is 0·0004334, for No. 430 ; and for the other thermometers the coefficients of pressure were stated as :—

12,765	0·0002950 ± 0·0000014.
20,065	0·0002039 ± 0·0000008.

Zero-points :—

Kew, 430	32°·212 Fahr.
12,765	32°·442 ,,
20,065	32°·418 ,,

Weighings.

Weighings.—For the weighings three balances were used, by means of which differences of $\frac{1}{100}$th, $\frac{1}{2000}$th, and $\frac{1}{10000}$th of a grain might be ascertained respectively ; for weighings in water it was possible, however, to weigh only to $\frac{1}{100}$th of a grain. The weighings were made by BORDA'S method of counterpoise, the position of equilibrium of the beam, and the weight of water displaced by the cylinder being calculated after the methods given by MILLER and BROCH.

The gravimeter was suspended in a glass vessel by a platinum wire hook, and was surrounded by at least two inches of water, the depth of the water being regulated so that it always rose to the same height in the vessel, whether the gravimeter was suspended therein or the wire hook only. The wire hook was kept polished, any water found to adhere to its upper surface being either wiped off or dried off by a blowpipe before the weighings were taken.

In such weighings in water minute bubbles of air are found to be carried down by the gravimeter or by the wire by which it is suspended, but by repeated immersions, and examination with a small telescope and a glow-lamp, it is possible to avoid appreciable error in this direction. When not in use the gravimeters were kept in air of similar temperature to that at which the weighings were made.

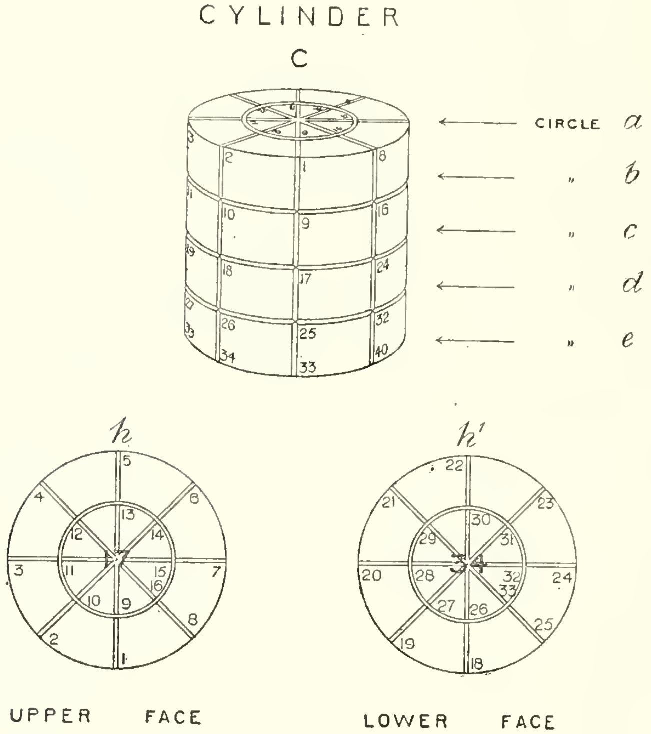
It is, of course, difficult, during such weighings, to find the precise temperature of a comparatively large volume of water, and as any uncertainty of 0°·2 Fahr. in the reading of the thermometer during the weighing of the cylinder C in water would amount to nearly three grains, it was desirable not only to use sensititive thermometers, but to place their bulbs at proper depths, and to read them quickly by means of cathetometers during the periods of weighing.

All weighings in air were reduced to "normal air," or air which at the temperature of 62° Fahr., the barometer being at 30 inches at 32° Fahr., reduced to latitude 45° and at sea level, contains four volumes of carbonic anhydride in every 10,000 volumes of air, and also contains two-thirds of the amount of aqueous vapour contained in saturated air. This is the average condition of the air at Westminster (latitude 51° 29′ 53″, at 16 feet above sea level), where the weighings were made. A litre of such air, if dry, but containing the above proportion of carbonic anhydride, would weigh 1·293934 grammes; and, if dry, but containing no carbonic anhydride, would weigh 1·293519 grammes.

Details of Measurements and Weighings.

The following are details of the measurements and weighings of the three gravimeters C, S, and Q; together with statements of the results obtained.

Fig. 1.



Height of Cylinder C.

Date, 1888.	t. Fahr.	Height (h − h′, fig. 1). In divisions of the Whitworth micrometer. Outer circle.									Mean reading of Whitworth micrometer for standard 9-inch bars.
		1–18.	2–19.	3–20.	4–21.	5–22.	6–23.	7–24.	8–25.	Mean.	
		div.	div.	div.	div.	div.	div.	div.	div.	div.	div.
November 1	62°·5	491·3	503·5	501·6	502·4	497·3	502·6	498·9	503·5	500·1	486·0
,, 5	62·5	497·6	501·9	501·4	501·8	501·3	502·7	502·4	502·2	501·4	484·5
,, 7	63·2	496·6	504·1	502·5	506·5	501·1	505·2	506·4	502·7	503·1	484·9
,, 8	62·0	485·7	503·2	502·2	502·2	500·3	501·9	499·1	502·1	499·6	483·8
,, 13	63·4	488·6	504·6	503·7	505·0	501·1	505·2	506·3	504·4	502·3	484·5
Mean =	62·72	491·96	503·46	502·28	503·58	500·22	503·52	502·62	502·98	501·3	484·74

In the measurement of C, the results are stated in divisions of the Whitworth micrometer, one division of which is equal to 0·0001 inch. Each result is the mean of three readings of the micrometer.

Date. 1888.	t. Fahr.	Height (h − h′, fig. 1). Inner circle.									Mean reading of Whitworth micrometer for standard 9-inch bars.
		9–26.	10–27.	11–28.	12–29.	13–30.	14–31.	15–32.	16–33.	Mean.	
		div.	div.	div.	div.	div.	div.	div.	div.	div.	
November 1	63°·0	505·0	504·4	505·1	511·4	509·2	518·5	514·2	515·6	510·4	486·0
,, 5	63·5	501·2	502·0	498·5	511·3	510·9	514·3	508·6	509·7	507·0	484·5
,, 7	62·0	505·6	507·6	502·8	513·1	512·3	514·1	513·6	512·4	510·2	484·9
,, 8	62·1	501·2	503·6	499·8	508·6	508·6	511·4	508·1	509·0	506·3	483·8
,, 13	63·4	505·7	507·7	505·2	513·4	512·3	514·2	513·7	512·5	510·6	484·5
Mean =	62·8	503·74	505·06	502·28	511·56	510·66	514·5	511·64	511 84	508·9	484·74

Height ($h - h'$, fig. 1). Centre.			
Date. 1888.	t. Fahr.	17–34.	Mean reading of Whitworth micrometer for standard 9-inch bars.
	°	div.	div.
November 1	62·5	509·5*	486·0
,, 5	63·2	506·7	484·5
,, 7	61·5	512·1	484·9
,, 8	62·1	506·9	483·8
,, 13	63·0	506·5	484·5
Mean =	62·46	508·34	484·74

At 62° Fahr. the mean height of the cylinder C was therefore taken at 9·00202 inches.

---

* Ten consecutive micrometer readings on November 1 ($t = 62·5°$), gave a result by one observer as follows:—

509·8 div.
·7
·0
·5
·2
·6
·6
·6
·5
509·4

509·5 Mean.

Diameter of Cylinder C.

Date. 1888.	*t.* Fahr.	Circle *a*.—Number of diameter measured.									Mean reading of Whitworth micrometer for standard 9-inch bars.
		1–5.	2–6.	3–7.	4–8.	5–1.	6-2.	7–3.	8–4.	Mean.	
	°	div.	div.	div.	div.	div.	div.	div.	div.	div.	div.
November 1	64·5	449·7	500·6	497·5	498·3	498·8	501·2	500·5	497·9	499·1	486·3
„ 3	62·8	501·2	501·2	495·6	501·2	496·7	495·7	496·1	501·7	498·6	484·5
„ 6	61·5	501·5	500·8	501·0	501·5	502·3	501·4	501·7	501·8	501·5	484·4
„ 9	62·3	501·3	500·7	501·1	501·2	502·0	501·4	501·8	502·2	501·5	484·1
„ 12	62·4	493·2	491·8	491·8	492·3	492·6	492·6	492·6	493·2	492·5	483·2
Mean =	62·70	499·36	499·02	497·4	498·9	498·48	498·46	498·54	499·36	497·46	484·5

The measurements from (1–5) to (4–8), for instance, were taken with the cylinder C resting on its lower face. The position of the cylinder was then reversed, so that it rested on its upper face, in which latter position the measurements from (5–1) to (8–4) were then taken, the relative positions of the points on the cylinder brought into contact with the Whitworth micrometer-screw being thus reversed. Although, therefore, "1–5" and "5–1" are apparently the same straight line or diameter, yet the points between which the line would fall would be slightly different in each case.

Date. 1888.	*t.* Fahr.	Circle *b.*—Number of the diameter measured.									Mean reading of Whitworth micrometer for standard 9-inch bars.
		9–13.	10–14.	11–15.	12–16.	13–9.	14–10.	15–11.	16–12.	Mean.	
	°	div.	div.	div.	div.	div.	div.	div.	div.	div.	div.
November 1	64·1	501·5	495·9	495·5	494·0	495·4	493·7	492·8	494·5	495·9	486·3
,, 3	64·0	496·3	494·4	494·6	496·6	498·0	492·7	493·0	501·5	495·9	484·5
,, 6	61·7	492·7	490·8	490·5	491·0	490·8	491·3	491·5	492·3	491·4	484·4
,, 9	62·0	492·5	490·7	490·5	491·2	491·0	491·3	491·6	492·5	491·4	484·1
,, 12	62·4	492·4	492·0	492·1	491·8	492·6	492·5	491·7	492·5	492·2	483·2
Mean =	62·84	495·08	492·76	492·64	492·92	493·56	492·3	492·12	494·66	493·26	484·5

Date. 1888.	*t.* Fahr.	Circle *c.*—Number of the Diameter Measured.									Mean reading of Whitworth micrometer for standard 9-inch bars.
		17–21.	18–22.	19–23.	20–24.	21–17.	22–18.	23–19.	24–20.	Mean.	
	°	div.	div.	div.	div.	div.	div.	div.	div.	div.	div.
November 1	64·0	492·8	494·3	492·3	492·2	494·8	491·4	493·6	492·0	492·9	486·3
,, 3	63·2	494·7	492·0	493·7	493·4	494·9	494·2	495·6	494·6	494·1	484·5
,, 6	61·9	492·2	491·7	493·2	492·1	492·4	491·1	493·8	493·4	492·5	484·4
,, 9	62·2	492·3	491·7	493·0	492·1	492·3	491·0	493·6	493·2	492·4	484·1
,, 12	64·0	494·0	493·8	494·4	493·5	494·5	494·5	494·6	494·1	494·2	483·2
Mean =	63·06	493·2	492·7	493·32	492·66	493·78	492·44	494·24	493·46	493·22	484·5

Date. 1888.	*t.* Fahr.	Circle *d.*—Number of the Diameter Measured.									Mean reading of Whitworth micrometer for standard 9-inch bars.
		25–19.	26–30.	27–31.	28–32.	29–25.	30–26.	31–27.	32–28.	Mean.	
	°	div.	div.	div.	div.	div.	div.	div.	div.	div.	div.
November 1	62·8	498·2	497·3	498·0	498·1	497·5	494·2	498·9	498·5	497·6	486·3
,, 3	62·3	487·7	488·8	489·0	491·5	488·9	488·3	491·6	492·6	489·8	484·5
,, 6	62·1	487·7	488·6	489·2	491·4	489·1	488·3	488·6	487·3	488·8	484·4
,, 9	62·3	487·6	488·4	489·1	491·4	489·1	488·2	488·6	487·4	488·7	484·1
,, 12	64·0	491·6	491·6	491·5	492·3	490·7	492·3	491·9	491·5	491·7	483·2
Mean =	62·7	490·56	490·94	491·36	492·94	491·06	490·26	491·92	491·46	491·32	484·5

Date. 1888.	*t.* Fahr.	Circle *e.*—Number of the Diameter Measured.									Mean reading of Whitworth micrometer for standard 9-inch bars.
		33–37.	34–38.	35–39.	36–40.	33–37.	34–38.	35–39.	36–40.	Mean.	
	°	div.	div.	div.	div.	div.	div.	div.	div.	div.	div.
November 1	62·5	507·8	507·8	510·3	511·2	511·1	508·1	512·0	511·6	509·9	486·3
,, 3	64·0	505·6	502·0	510·0	507·5	509·0	505·6	504·4	506·7	506·4	484·5
,, 6	64·0	500·6	498·7	502·7	502·7	502·6	502·4	505·0	503·6	502·3	484·4
,, 9	62·3	500·6	498·6	502·7	502·7	502·5	502·4	505·0	503·6	502·3	484·1
,, 12	67·5	503·4	502·3	506·4	506·4	506·3	505·5	508·7	506·7	505·7	483·2
Mean =	64·06	503·6	501·88	506·42	506·5	506·1	504·8	507·02	506·44	505·32	484·5

The lines traced on C have the effect of diminishing its cubic capacity; these lines are V-shaped, and have a depth of nearly 0·018-inch, with an average breadth at the top of 0·01-inch. There are 24 vertical lines, each nearly 9 inches long; three circular lines, each nearly 28 inches long; besides two circular lines nearly 14 inches long. The edges also of the cylinder were not quite sharp.

If, therefore, the diameter of C is taken at 9·00115 inches, then the cubic contents of C may be taken at 572·80365 cubic inches.

WEIGHING of Cylinder C in Air.

Date, 1888.	Description of Weights used (*w*).	Result.	Mean *t*. Fahr.	Barometer. Inches.
	Before immersion.		°	in.
Nov. 16 . .	14, 7, 4, 1 lb., R;* 2, 1 oz.; 8 dr.; 100, 30, 10, 5 grains	$C \bumpeq w + 0{\cdot}086$ grain . . Or $C = 183676{\cdot}25 + 0{\cdot}086$ grains	57·5	29·98
,, 19 . .	14, 7, 4, 1 lb., R;* 2, 1 oz.; 8 dr.; 100, 30, 10, 5 grains	$C \bumpeq w + 0{\cdot}009$ grain . . $C = 183676{\cdot}25 + 0{\cdot}009$ grains	57·3	29·98
	After immersion.			
,, 22 . .	14 lb. . . . . 8 dr.; 100, 30, 10, 5 grains	$C \bumpeq w + 0{\cdot}998$ grain . . Or $C = 183675{\cdot}25 + 0{\cdot}998$ grains.	60·1	30·22
,, 29 . .	14 lb. . . . . 8 dr.; 100, 30, 10, 5 grains	$C \bumpeq w + 1{\cdot}052$ grain . . $C = 183675{\cdot}25 + 1{\cdot}052$ grains	58·2	29·34

The following are the weights of air displaced by C and *w* respectively.

The density of the weights used (*w*) was 8·0298, but their values were always corrected to the density of 8·1430, or to the density (Δ) of the brass representative of the Imperial Standard Pound adopted in this country.

	C displaces	*w* displaces
1888, Nov. 16 . .	177·03 grains . .	28·20 grains.
,, ,, 19 . .	177·63 ,, . .	28·10 ,,
,, ,, 22 . .	178·66 ,, . .	28·27 ,,
,, ,, 29 . .	173·50 ,, . .	27·45 ,,

In normal air, C displaces 176·08, and *w* 27·47 grains, whence the weight, in grains, of the cylinder was taken as follows:—

	In vacuo.	In normal air.
C =	183797·198 grains . . .	183676·066 grains.

* R, Reference Avoirdupois Standards; having small errors in relation to the present Imperial Pound, which errors were allowed for.

As the weighings of the gravimeters were made against standard weights, which were parts and multiples of the Imperial Standard Pound, it became necessary, in order to find the true masses of the gravimeters, to reduce the weighings to a vacuum, as the Imperial Standard Pound itself has its true mass stated *in vacuo*.

WEIGHING of C in water.

Date, 1888.	Weight of counterpoise (w^{-1}).	Mean t. water.	Mean t. air.	B.
	grains.	°	°	in.
Nov. 20	39091·84	58·87	56·0	30·14
„ 21	39109·85	58·10	57·2	30·15
„ 22	39109·87	56·38	56·5	30·22
„ 24	39141·86	61·42	59·5	30·30
„ 27	39090·50	57·61	56·8	29·38
„ 29	39125·00	58·00	58·2	29·34
	39111·49	58·40	57·37	29·92

The differences in the above weighings of C on the several days were considered to be owing partly to thermometric variations, and partly to the presence of minute bubbles of air carried down by this large cylindrical body. Results, however, could not be rejected merely because they did not always closely agree.

At 32° Fahr., to which temperature densities are reduced :

$$\begin{aligned}\Delta C_{32^\circ} = {} & 1{\cdot}27029 \\ & 1{\cdot}27050 \\ & 1{\cdot}27061 \\ & 1{\cdot}27055 \\ & 1{\cdot}27036 \\ & 1{\cdot}27064 \\ \hline & 1{\cdot}27049\end{aligned}$$

or log $\Delta C_{32^\circ} = 0{\cdot}1040169$.

The logarithm of the volume of C in grains at 62° Fahr., may be expressed as follows :—

$$\begin{aligned}\text{Log vol. C} = {} & 5{\cdot}1599338 \\ & 8626 \\ & 8272 \\ & 8476 \\ & 9102 \\ & 5{\cdot}1598148\end{aligned}$$

or $vC = 144499{\cdot}48$ grains weight of water at 62° Fahr.

As the cubic contents of C = 572·80365 cubic inches, the mass of one cubic inch of distilled water (Δ 0·998881) at 62° Fahr., B. = 30 inches, has been taken as 252·267 grains.

DIAMETER and Height of Q.

The diameter of Q was measured at 25 different points, each measurement being the mean of three readings. The method of FIZEAU was followed, the true distance between the "points" $(p - p')$ being ascertained after the diameter of Q had been measured.

1889.	Mean readings in inches in terms of $p - p'$.	t.
	inches.	°
(1) Jan. 8 . . .	3·083513	61·8
(2) ,, . . .	3·083546	
(3) ,, 9 . . .	3·083684	61·8
(4) ,, . . .	3·083792	
(5) ,, 10 . . .	3·083846	62·0
(6) ,, . . .	3·083921	
(7) ,, . . .	3·084083	61·9
(8) ,, . . .	3·084123	
(9) ,, . . .	3·084086	61·7
(10) ,, . . .	3·083924	
(11) ,, . . .	3·083910	
(12) ,, 11 . . .	3·083923	61·6
(13) ,, . . .	3·083964	
(14) ,, . . .	3·083982	
(15) ,, . . .	3·084013	
(16) ,, . . .	3·084214	61·5
(17) ,, . . .	3·084267	
(18) ,, . . .	3·084285	
(19) ,, . . .	3·084235	61·8
(20) ,, 15 . . .	3·084108	
(21) ,, . . .	3·084124	
(22) ,, . . .	3·084086	61·9
(23) ,, . . .	3·084093	
(24) ,, . . .	3·084004	
(25) ,, . . .	3·083902	

At t. 62° the true distance been $p - p'$ was found to be 3·08234 inches by the 3-inch steel measure (2·999749 inch). Hence at 62° Fahr. the diameter of Q was taken at 3·08399 inches.

HEIGHT of Q as measured at 25 different points by Whitworth Micrometer.

No.	1889.	Mean reading.	t.
		Divisions of micrometer.	°
1	Jan. 7	+4555	61·8
2	,,	+4500	
3	,,	+3998	
4	,,	+3842	
5	,,	+3459	
6	,, 8	+4335	61·7
7	,,	+4257	
8	,,	+3980	
9	,,	+3938	
10	,,	+3935	
11	,,	+4203	
12	,,	+4200	
13	,,	+4000	
14	,,	+4000	
15	,, 10	+4000	62·0
16	,,	+4038	
17	,,	+3925	
18	,,	+3875	
19	,,	+3928	
20	,,	+4054	
21	,, 12	+3695	61·8
22	,,	+3700	
23	,, 15	+4000	
24	,,	+4358	
25	,,	+4450	
		Mean +4049	

For the standard 3-inch measure, the mean reading was found to be = 4799·5 divisions, hence the height of Q was taken as 3·088485 inches.

By the above measurements of the height of Q it may be seen that the quartz gravimeter is not of true cylindrical form, and that its height could only be approximately ascertained. After allowing for bevels on the upper and lower edges of the cylinder, as measured by the microscopic comparator, its cubic contents were taken as 23·04015 cubic inches.

WEIGHING of Q in Air.

Date, 1888.	Description of weights used (w).	Result.	Mean t. Fahr.	Barometer inches.
Jan. 3, 10 A.M.	Gilt lbs. 31, 32, + gilt grains 1000, 300, 100, 20, 10, 1, 0·5	grains 15426·68	57°·35	30·81
	After immersion.			
,, 5 P.M.	.. ..	15426·68	57·33	30·81

The following are the weights of air displaced by Q (Δ 2·6505) and *w* (Δ 8·4924) respectively :—

Q.	*w*.
7·3496 gr.	2·2944 gr.
7·3495	2·2943

In normal air Q displaces 7·0865 gr. and *w* displaces 2·2123 gr., whence, *in vacuo* and in normal air respectively, the weight in grains of the sphere would be :—

	In vacuo.	In normal air.
Q =	15429·55515	15426·95495 gr.

Weighing of Q in Water.

Date, 1888.	Weight of counterpoise *w*′.	Mean *t*. water.	Mean *t*. air.	Barometer inches.
Jan. 3 . . .	Gilt lb. 32 + 4 oz. + 500, 300, 50, 10, and 1 grains	49°·92	57°·57	30·82
„ 4 . . .	Gilt lb. 32 + 4 oz. + 500, 300, 50, 10, and 0·3 grains	62·75	57·62	30·75
„ 5 . . .	Gilt lb. 32 + 4 oz. + 500, 300, 50, 10, 1, ·5, 0·2, 0·1 grains	53·03	57·60	30·55

Then at 62° Fahr., B = 30 inches,

$$\text{Log vol. Q in grains} = 3\cdot7643360, \text{ or}$$

$$v_{\text{Q}} = 5812\cdot14 \text{ grains.}$$

The density of the quartz cylinder being :—

At 62°.
2·265425

Therefore, as the cubic contents of Q are equal to 23·04015 cubic inches, the mass of one cubic inch at 62° Fahr. (B. 30 inches) may, by these experiments with Q, be taken as 252·261 grains.

DIAMETER of Sphere S.

Date. 1888.	$t.$ Fahr.	Diameter of S at eight different positions, as compared with a standard distance between two contact points, P–P′.								Mean diameter.	Standard distance between contact points P—P′.
		1.	2.	3.	4.	5.	6.	7.	8.		
	°	inches.	inches.	inches.	inches.	inches.	inches.	inches.	inches.	inches.	inches.
Nov. 15 . . .	62·2	0·05013	0·05024	0·05025	0·05025	0·00026	0·05026	0·05023	0·05025	0·050234	0·05796
,, 16 . . .	64·0	0·05013	0·05014	0·05013	0·05023	0·05026	0·05027	0·05024	0·05027	0·050209	0·05796
,, 17 . . .	63·2	0·05014	0·05018	0·05025	0·05017	0·05025	0·05026	0·05027	0·05026	0·050225	0·05795
,, 19 . . .	63·3	0·05014	0·05018	0·05013	0·05018	0·05025	0·05026	0·05027	0·05026	0·050209	0·05795
,, 21 . . .	62·2	0·05013	0·05024	0·05013	0·05025	0·05026	0·05027	0·05025	0·05026	0·050224	0·05796
,, 23 . . .	63·0	0·05013	0·05024	0·05013	0·05024	0·05026	0·05027	0·05027	0·05026	0·050225	0·05796
Mean . . .	62·98	0·05013	0·05020	0·05017	0·05022	0·050257	0·050265	0·050255	0·05026	0·050221	0·057957

The measurements were made by means of a pair of micrometer microscopes. It was found that the sphere was 0·00774 inch shorter in diameter than the steel 6-inch bar ($t.$ 62°·98 Fahr.).

After allowing for the difference between the rates of expansion of the steel standard 6-inch and the brass sphere, and the error of the steel standard in relation to the imperial yard, the diameter of the sphere is 5·992444 inches. On the sphere, however, two lines had been traced, for which a correction is to be made. These lines are triangular in section, being nearly 0·01 inch wide and 0·005 inch deep, therefore the actual contents of S would be at 62° Fahr. = 112·6694096 cubic inches.

WEIGHING of Sphere S in Air.

Date, 1888.	Description of weights used. (*w*.) $\Delta w = 8{\cdot}143$.	Result.	*t*. Fahr.	B. inches.
	Before immersion.			
Dec. 10 . . .	2 lb. + gilt lbs. Nos. 31 + 32 ; + gilt grains 300, 100, 10 .	S = 28410·113 grains . .	61°·65	in. 30·17
„ 11 . . .	2 lb. + gilt lbs. Nos. 31 + 32 ; + gilt grains 300, 100, 10 .	S = 28409·813 „ . .	57·70	30·26
„ 12 . . .	2 lb. + gilt lbs. Nos. 31 + 32 ; + gilt grains 300, 100, 10 .	S = 28409·802 „ . .	60·28	30·37
„ 17 . . .	2 lb. + gilt lbs. Nos. 31 + 32 ; + gilt grains 300, 100, 10 .	S = 28409·917 „ . .	58·32	30·35
	After immersion.			
„ 18 . . .	2 lb. + gilt lbs. Nos. 31 + 32 ; + gilt grains 300, 100, 10 .	S = 28409·918 „ . .	58·10	30·18

The following are the weights of air displaced by S and *w* respectively :—

	S displaces	*w* displaces
Dec. 10	34·810 grains.	4·198 grains.
„ 11	35·249 „	4·245 „
„ 12	35·200 „	4·239 „
„ 17	35·193 „	4·238 „
„ 18	35·129 „	4·230 „

In normal air S displaces 34·643 and *w* 4·171 grains respectively ; whence *in vacuo*, and in normal air, the weights in grains of the sphere are as follows :—

In vacuo.	In normal air.
28,440·725 grains.	28,410·253 grains.
28,440·817 ,,	28,410·345 ,,
28,440·763 ,,	28,410·291 ,,
28,440·772 ,,	28,410·300 ,,
28,440·810 ,,	28,410·345 ,,
28,440·777 grains.	28,410·307 grains.

Or the logarithms of the weights in grains are :—

In vacuo.	In air.
4·4539414	4·4534759

WEIGHING of S in Water.

Date, 1888.	Difference of equipoise *w'*.		Mean *t*. water.	Mean *t*. air.	B.
		gr.	°	°	inches.
Dec. 12	Gilt grains weights added,	27·02	54·58	60·22	30·37
,, 12	,, ,, ,,	27·01	54·72	60·50	30·37
,, 13	,, ,, ,,	24·70	56·63	59·20	30·30
,, 17	,, ,, ,,	23·72	57·60	58·48	30·35
,, 18	,, ,, ,,	23·65	57·48	58·00	30·18

S just floats in water and was suspended and kept in position by a "sinker." The weights of water displaced by the sinker, with and without the gravimeter attached to the sinker, were ascertained by separate equipoises, the small grain weights added each day representing the differences of such equipoises.

The density of the weights *w'* used was 8·143 ; hence the density of S is—

$$\begin{aligned}\Delta_S &= 0·999333\\ &= 0·999331\\ &= 0·999302\\ &= 0·999177\\ &= 0·999181\\ \hline &\ 0·999265\end{aligned}$$

or log $\Delta_S = \bar{1}·9996553$.

As the weight of S in air is 28410·307 grains, the volume of S in grains at 62° Fahr. will be 28426·61 grains. As the cubic contents of S have been taken as 112·6694096 cubic inches, the mass of one cubic inch of water ($t. = 62°$) would be 252·301 grains.

Final Result.

The measurements of the sphere S appear to afford more accurate results than the measurements of the cylinders. The measurements of the larger cylinder C afford a more accurate result than those of the cylinder Q; although, on the other hand, there appears to be less uncertainty as to the volume of the solid quartz cylinder at the various temperatures.

If all the weighings and measurements of the three gravimeters had been of equal value, then a mean (252·276) of the three results:—

C. 252·267 grains
S. 252·301 ,,
Q. 252·261 ,,

might have been taken as the final result κ of the present experiments; but, for the considerations above stated, it was thought desirable to take as the final result $\kappa = C + 3S + Q/5$, or 252·286 grains ± 0·0002 grain, as the mass of the cubic inch of distilled water freed from air, weighed in air at the temperature of 62° Fahr., the barometer being reduced to 30 inches, against brass weights of the density of 8·143.

A cubic foot of such water under the above conditions would weigh 435950·208 grains, or 62·278601 lbs. avoirdupois.

VIII. *On the Mechanical Stretching of Liquids: an Experimental Determination of the Volume-Extensibility of Ethyl-Alcohol.*

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Communicated by Professor Poynting, *F.R.S.*

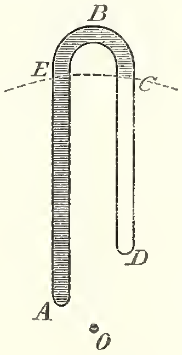
Received February 1,—Read February 4, 1892.

[Plate 10.]

Three methods are known by which a liquid may be subjected to a bodily tension.

(1) *The method of the inverted barometer*, familiar to most physicists, by which, with care, a mercury column of many times the barometric height may be supported by its adhesion to the top of the tube. In such a column the hydrostatic pressure is negative above the barometric height, or the liquid above this level is in a state of tension. This tension increases with the height and is propagated in all directions to the walls of the tube. When the upper part of the tube is made elliptical in cross-section and of thin glass, its yielding to the inward pull may be easily observed.

(2) *The centrifugal method*, devised by Professor Osborne Reynolds, in which a U-tube, ABCD, of glass, closed at both ends, contains air-free liquid, ABC, and vapour, CD. This tube is fixed to a suitable board and whirled about an axis, O, a little beyond the end, A, and perpendicular to the plane of the board. If CE (see figure) be the arc of a circle described about O, then while rotation continues the liquid between E and A is in a state of tension, increasing from zero (if we ignore the vapour-pressure) at E to a maximum at A. By this method Professor Osborne Reynolds has subjected water to a tension of about 5 atmospheres or 72·5 pounds per square inch, while the author, experimenting in the Cavendish Laboratory in



1886, succeeded in reaching, with alcohol, a tension of 7·9 atmospheres or 116 pounds per square inch, and, with strong sulphuric acid, 11·8 atmospheres or 173 pounds per square inch.

(3) *The method of cooling*, discovered by M. Berthelot, and described by him in a paper entitled "Sur la Dilatation Forcée des Liquides," published in 1850 ('Ann. de Chimie,' vol. 30, 1850, pp. 232–237), by which he succeeded in obtaining a great variety of liquids in a state of very considerable mechanical extension, the amount of which he appears, however, rather to have estimated than to have measured, but which, according to his estimate, was as much as $\frac{1}{120}$ of the whole volume in the case of water, $\frac{1}{93}$ in the case of alcohol, and $\frac{1}{59}$ in that of ether.

In M. Berthelot's experiments the liquid, freed of air by long boiling, nearly filled a straight, thick-walled glass tube, the small residual space being occupied by its vapour. When slightly heated the liquid expanded and filled the whole tube, but on being again cooled remained extended, still filling the tube, of which it at last let go its hold with a loud metallic click, when the bubble of vapour re-appeared. It was from the length of this bubble that the extension was calculated.

It will be observed that methods (1) and (2) afford measures of the tensile stress, but not of the strain or extension; while, on the other hand, the method (3) affords a measure of the strain but not of the stress. The object of the present paper is to describe the process by which, after a great variety of trials made during the past six years, I have succeeded in what, so far as I am aware, has not been previously attempted, viz., in obtaining simultaneous measures of stress and strain in a liquid under tension. The measures are not, indeed, as numerous as could be wished, for reasons that will appear in the sequel, but they are fairly consistent, and mark a stage in an investigation with which I hope to proceed further, and to which I am anxious to attract the attention of more skilful experimenters.

That a liquid can pass into and exist stably in a state of tension without any breach whatever of physical continuity has been denied or questioned by eminent physicists,* and the contrary is commonly asserted by writers on Hydrodynamics.† The experiments to be described will be found, I think, to remove the last possibility of doubt upon the matter.

Method of Experimenting.

The mode of subjecting the liquid to tensile stress is essentially that of Berthelot, and may be briefly described as follows:—The liquid is contained in a strong closed glass vessel which it nearly fills at the ordinary temperature of the air. The small space not filled by liquid is occupied only by its vapour. Dissolved air, and especially the film of air which at first lies between the liquid and the walls of the vessel, is got rid of, as far as possible, by prolonged boiling before the vessel is sealed up.

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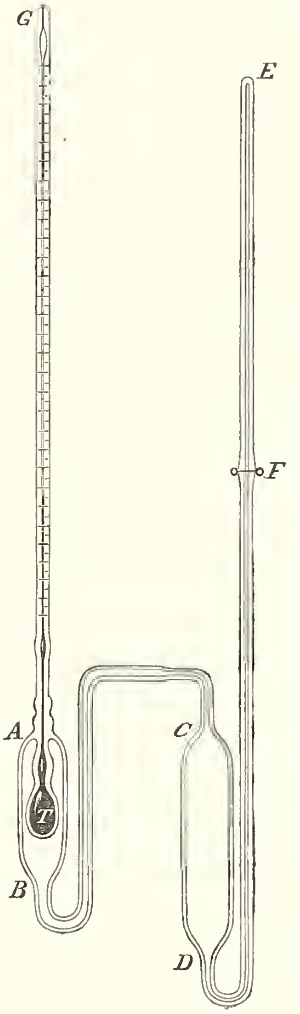
* See Balfour Stewart, 'Elementary Physics,' § 69, p. 70.

† See Lamb, 'Treatise on the Motion of Fluids,' § 9, p. 7.

The temperature of part of the vessel and its contents is now raised. This causes the liquid to expand and fill the whole; if the rise of temperature continued the vessel would burst; but before the pressure becomes dangerously great the liquid is cooled again by means of ice-cold water. It now, however, adheres so firmly to the walls of the vessel that, though cooled, it cannot contract, and it remains extended or stretched, tugging at the walls until at last, as the cooling proceeds, the tension becomes so great that the liquid lets go its hold, and releases the glass walls with a loud metallic click, and itself springs back to the smaller volume appropriate to the temperature and to the pressure of its saturated vapour.

Fig. 1.



Whether the cohesion of the liquid for itself or its adhesion to the glass is first overcome is not decisively determined, but all indications point rather to the latter than to the former supposition.

The form and dimensions of the vessel used in my experiments are given by the accompanying diagram, which is drawn to the scale of one quarter.

The vessel consists of two very strong cylindrical bulbs, AB and CD, with rounded ends, connected by a tube of narrow bore bent in the manner shown.

The walls of the bulbs were about 2 millims. thick and the bore of the connecting tube was about 2·5 millims. internal diameter. The diagram being drawn to scale gives any other necessary information as to dimensions.

DE is a narrow tube of very uniform bore (2.2 mm.), across which, at F, was sealed, with as little disturbance of the bore of the tube as possible, a very fine platinum wire.

Measurement of the Tension.

In order to measure the tension of the liquid at any instant, an ellipsoidal bulb T filled with mercury, and provided with a narrow graduated capillary stem AE is sealed into the bulb AB. This bulb had been previously subjected to various measured pressures in an hydraulic press up to 60 atmospheres, and the corresponding rise of the mercury in the stem noted. This was found, within the limits of observational error, to be proportional to the pressure applied. This rise is due to the diminution in capacity of the bulb, which becomes less spherical under the external pressure.

On the other hand, when the surrounding liquid is in a state of *tension*, it tugs at the walls of this bulb and makes it more spherical and of greater capacity, and since the bulb is thick-walled and fairly rigid, and the alteration of volume only a very small fraction of the whole, it appears quite safe to assume that the enlargement produced by a given tension is equal to the diminution of volume that is produced by an equal pressure applied *over the same surface.*

This instrument for measuring the tension I call the Tonometer. In that actually used the relation between bulb and tube was such that a pressure of 1 atmosphere of 15 lbs. to the square inch, as measured by a Bourdon gauge, gave a rise of 3·296 millims. The actual observations from which this value was determined were the following:—

Date.	Pressure employed (Bourdon gauge).	Resulting rise in tonometer reading for 1 atmosphere of 15 pounds per sq. inch.
	atmospheres.	millims.
Nov. 29, 1889	8	3·25
" "	8	3·375
" "	8	3·3125
Dec. 13, 1889	8	3·125
" "	. .	3·187
" "	10	3·450
" "	10	3·333
April 29, 1891	12	3·333
	Mean .	3·296

The Bourdon gauge used was employed for other observations to be described later, and when all were completed it was re-tested for me by Mr. C. F. Casella, by means of a mercury column, at the beginning of December, 1891, up to 14 atmospheres, with the following results :—

Bourdon gauge.		Mercurial column.
atmospheres.	pounds per sq. inch.	pounds per sq. inch.
0	0	0
2	30	30
4	60	62
6	90	89
8	120	121·5
10	150	151·5
12	180	179
14	210	209

The deviations from the original calibration up to this range will be observed almost to fall within the errors of reading, and by an independent comparison, which I made myself in the same month, of this gauge with two official standard gauges, by E. Bourdon, belonging to the Admiralty, I could not detect with any certainty any appreciable divergence of the readings, which I think may therefore be taken as correct within 1 per cent.

It may be mentioned, that in May, 1889, pressures of 36 gauge-atmospheres had been applied to the tonometer, giving a mean rise of 3·56 millims. per atmosphere, but in December, 1891, the higher readings of the gauge were found to be as much as 8 per cent. too low. The application of this correction reduced the reading to 3·275 millims. which accords well with the results already quoted, but on account of the doubt as to whether the gauge was ever correct at this part of the scale, it seemed better to exclude these observations altogether in determining the tonometer calibration, and to rely only on those within the range that I had myself tested.

It was afterwards observed by using the tonometer as a thermometer that the rise of the mercury due to a rise of temperature of 1°·85 C. was 2 centims., showing that the deformation of the bulb due to the maximum tension afterwards obtained, viz. 17 atmospheres, amounted to only about 1/1000th of its whole volume.

Measurement of the Strain or Extension.

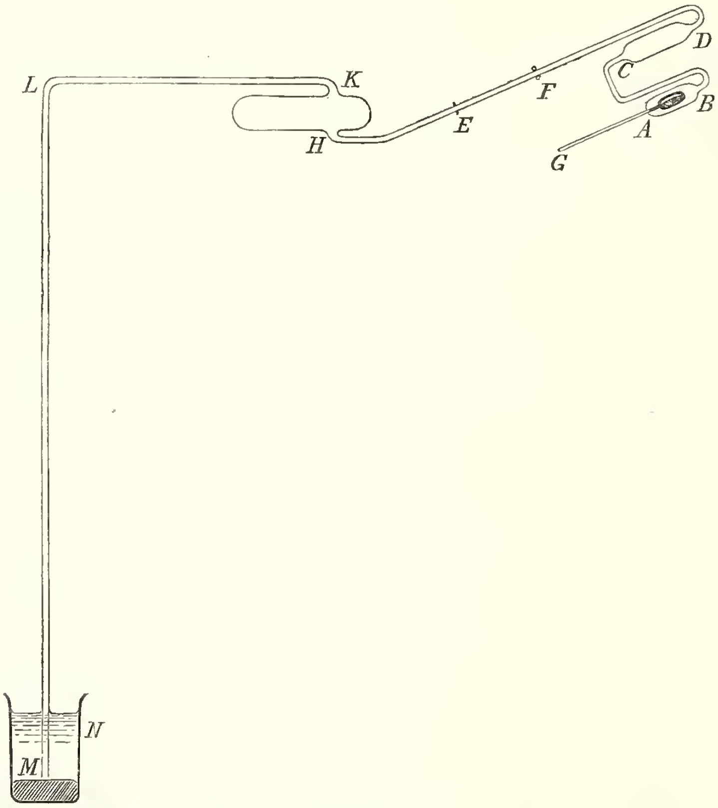
In order to ascertain the extension at any instant, the liquid was caused to let go its hold, and thus spring back to its unstretched volume, and then the volume of the space left empty of all but vapour was measured.

For this purpose the straight tube AB was traversed at F by the fine platinum wire, already mentioned, running across the centre of the tube. One of the projecting ends of this wire was connected by a stouter copper wire to one pole of an electric storage cell, and the other could at pleasure be touched by the bare end of a similar stout wire from the other pole. By this means the fine platinum wire could be heated suddenly to redness, when the liquid which was tugging at it immediately let go its hold. The bubble of vapour thus caused to appear had its upper end at the wire, and extended below to a distance along the fine tube, which could be accurately measured against a scale placed, to avoid parallax, on a mirror behind the tube.

The relation of the bore of the tube to the whole internal volume of the apparatus had been previously ascertained, and thus the fraction of the whole volume which the extension represents was known.

This extension is, however, only *apparent,* and it is necessary to determine and to subtract from it the amount by which the volume of the containing vessel has been diminished by the inward pull. The separate operation for determining this correction will be described later. I shall now explain the mode of filling and sealing the apparatus, and of conducting an experiment, with the precautions and minor corrections necessary for securing good measures of the extension.

Fig. 2.



Method of Filling the Apparatus

After thoroughly cleaning the apparatus with, first, a solution of potassium hydrate, then dilute hydrochloric acid, and then finally with distilled water, it is sealed at E to a second glass bulb HK, with arms HE, KLM, as shown in fig. 2.

The branch LM, which is more than the length of the mercury barometer, hangs vertically. For the sake of greater clearness of representation the diagram shows the bulbs HK, CD, and AB with their axes in the same plane, but in practice it is more convenient when making the connection at E to turn the apparatus that is to be filled through a right angle about EF as axis, since in this position liquid contained in either bulb can be more easily boiled by means of a small Bunsen flame.

The liquid to be experimented on is placed in a large beaker N over a layer of quite clean mercury 2 or 3 centims. deep, and into this beaker the open end M of the tube ZM dips, not quite reaching the mercury. After the liquid in N has been boiled to expel dissolved air, it is allowed to enter the bulbs HK, CD, and AB, and is then boiled in each simultaneously by suitable flames.* Thus, on the removal of the flame from beneath AB, air-free liquid from CD enters and fills it ; and when that in CD ceases to boil, air-free liquid from GHK enters and fills CD, and so on.

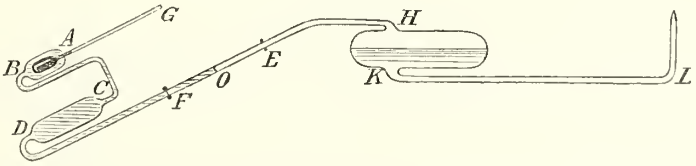
The process of alternately boiling out and filling is continued, with pauses, for several hours, till the residual bubbles disappear very quickly on cooling, and the tendency to boil by bumping threatens to endanger the apparatus.

Then while the liquid in HK is kept gently boiling, the tube LM is heated nearly to redness, just below L, in order to drive off attached air which is carried off in the stream of passing vapour.

Then when this tube has again cooled, and the vessel HK is about half-full, the vessel N is raised so as to submerge the open end M below the mercury, which, when all is cool, rises to nearly barometric height. Then after any liquid that has condensed above the mercury in ML has been, by judicious heating, driven over into the bulb GHK, the tube LM is heated just below L to the softening point, and allowed to close under atmospheric pressure. This process of sealing invariably liberates gas either from the surface of glass or by decomposition of the vapour of the liquid. It appears, however, to be less in quantity when the tube near the sealing point has been previously subjected to strong heating in the manner described.

The portion of the apparatus thus cut off from the barometer tube is now turned about a horizontal axis along GE, through 180°, into the position shown in fig. 3,

Fig. 3.



and by careful heating the liquid filling the tube EF is boiled over into the bulb HK till when all has cooled to the temperature of the room, the surface stands at some

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* It is important that the stem AG of the tonometer shall be provided with an enlargement at the end in order that the instrument may not burst when the mercury in the ellipsoidal bulb is raised to the temperature of the surrounding hot liquid.

suitable place O between E and F, 2 or 3 centims. from the extremity E of the uniform capillary tube. Then the liquid in the bulbs AB, CD is cooled by ice or cold water, so that the surface recedes towards, or even beyond F. Next the liquid in HK is cooled so as to diminish the vapour-pressure and cause a draught of vapour from the surface of the warmer liquid in the tube, and while this draught continues, the tube EF is heated just between O and E, and allowed to close under atmospheric pressure. Gases formed or liberated in this operation are for the most part carried over into the large bulb GHK by the draught of vapour.

The length of tube which it is desirable to leave empty above the surface of the liquid at O when all is at the temperature of the room again, should be just, but not more than, sufficient to secure that the instrument shall not be in danger of bursting when left to itself and exposed to unavoidable changes of temperature in the room where it is placed.

The trace of gas that is liberated in sealing is readily dissolved in the liquid, but on this account the liquid in the tube EF becomes unduly charged with gas, and does not, at first, adhere very well to the walls. It is well, therefore, by raising the temperature of the bulbs, to compress the residual vapour into a bubble small enough to be floated along the tube into the bulbs, where any residual gas it may contain is dispersed through a large mass of liquid.

The apparatus is now ready for experimenting. For this it is set in the erect position shown in fig. 1. A beaker of water of the temperature of the room is placed round the bulb containing the tonometer to secure it from temperature changes. A second beaker, containing warm water, is placed round the other bulb, while a third beaker, which can be quickly substituted for this, contains ice and ice-cold water. Plane mirrors are placed behind the two tubes to prevent parallactic errors.

It is convenient now to have two observers, O_1 and O_2, one to watch the tonometer, and the other to manipulate the beakers of warm and of ice-cold water, and the wire terminals of the storage cell, and to observe the bubble. Let us suppose that this is lying in the tube with its upper end at the platinum wire. On setting the warm water Y in place, the bubble closes in, and, just before it disappears, will begin to float up the tube, disappearing, however, before it has risen more than 1 or 2 centims. At the moment of its disappearance, *and not before*, does the observer of the tonometer notice a sudden rise of the mercury; this shows the freedom from residual undissolved gas. The warm liquid is quickly removed, and the ice-cold liquid substituted; this causes the mercury in the tonometer to fall, and its observer O_1 calls aloud the divisions as it passes them; when the tension is approached at which it is desired to observe the extension, the ice-cold water is removed, so that the cooling proceeds more slowly, and while O_1 is carefully watching and calling the readings, O_2 makes a momentary completion of the electric circuit, which results in the sudden reappearance of a bubble of definite length, which he reads off while observer O_1 notes the height to which the mercury in the tonometer has again risen.

It is not always possible in practice to prevent the liquid from letting go its hold too soon, and it often happens that the bubble forms in one of the bulbs, and not in the tube, where only its volume can be read off, but this fact need not prevent an observation of the volume from being made, for, if the observer O_2 only heats the platinum wire for an instant, the liquid below it, which has been clinging to it, separates, owing to its weight, and at once a bubble of hot vapour is formed, which, by its greater pressure, causes the bubble in the bulb, which also is at a lower level, to close up and vanish. Thus the bubble can be instantly transferred to the place where it can be measured.

The method of liberating the bubble where it was wanted by means of the heated platinum wire was only hit upon after many other methods had been tried unsuccessfully, but I had not foreseen that it would carry with it the advantage of enabling the observer to transfer the bubble in the manner just mentioned. This was a piece of good luck.

Corrections to be applied to the Measure of the Bubble.

(1.) *Correction for Capillary Curvature.*—This was applied in the usual way, the ends being considered as hemispheres of the diameter of the tube, which was 2·20 millims.

(2.) *Correction for Inequality of the Bore near the Wire.*—This was determined by floating the bubble when very small to a point in the tube 3 or 4 centims. below the platinum wire, and then allowing the whole liquid to cool to a steady temperature, thus forming a long bubble in a uniform part of the tube. The length of this bubble was then carefully measured, and then this bubble was "transferred" in the manner mentioned, so that its upper end was at the wire. The difference of the lengths gives the correction to be applied. The manner in which the upper end of the bubble was entangled with the platinum wire was not, however, always the same, but appeared to be always one or other of two configurations, and the correction to be applied for the variation of bore was + 1·55 millims., or + ·015 millims., according to the configuration which was observed and recorded on each occasion.

Sources of Error in Determining the Length of the Bubble.

(1.) It was often impossible so to regulate the influx or efflux of heat as to secure that a stationary stage had been reached when the bubble was formed. Thus the liquid generally continued to shrink by cooling after having let go the walls, and on this account the bubble required to be read immediately.

(2.) On the other hand, if the reading was made too soon there was not time for the liquid still clinging to the side of the tube to drain down.

Each of these two sources of error tends to make the bubble appear too long, *i.e.*, the measure of the strain too great; and it is, without doubt, to their action that the irregularity of the observations must be chiefly attributed. Fortunately, the degree of steadiness of the tonometer just before the liquid is made to let go enables the observer to discriminate somewhat between "good" and "less certain" observations.

Difficulties and General Observations.

It is in this connection, and before passing to the measurements themselves, that the difficulties attendant on the experiments may be alluded to. In the first place, considerable tensions (above 12 atmospheres) are not easy to attain, the liquid letting go its hold unexpectedly and too soon; sometimes this will continue for an hour or two, the bubble appearing perpetually in the same place in one or other of the bulbs, or at one of the bends. This is, perhaps, attributable to the liberation of air from minute crevices in the glass, which air has to be dispersed before adhesion can be re-established. For this purpose I have often found the application of ice or of a freezing mixture to the affected part to be efficacious, especially when accompanied by judicious sharp tapping of the apparatus, so as to secure an impulsive pressure of the liquid against the surface at the non-adherent part of the glass, whereby I imagine that the minute bubble or film of air is the better dispersed. It is also useful, by manipulation and variation of the temperature of different parts of the liquid, to produce currents which sweep away from the narrow tubes into the wider bulbs portions of the liquid that may have become somewhat charged with dissolved gas. Yet at all times the behaviour of the liquid, as regards its adhesion, is somewhat uncertain. Thus, there may be no difficulty in reaching, time after time, a tension, say, of 9 atmospheres, the liquid always separating from the wall at some particular place; and when, after an hour or more of patient manipulation, the observer is beginning to think the disease incurable, and that the apparatus must be reopened and the whole process of filling and boiling-out gone through again, the difficulty will unexpectedly disappear, or change.

Meanwhile, and especially when high tensions have been reached, there is constant danger that the suddenness of the release may break the whole apparatus; and this is the more to be feared when the release is an accidental one, taking place in one of the bulbs, than when it is intentionally effected at the wire in the narrow tube, for in the latter case the friction of the long column of liquid against the walls of the tube appears to act as a brake, and the shock to the apparatus is less sudden. Four pieces of apparatus similar to that described broke in this way, under the sudden release, before I was able to obtain any determinate measures. I have endeavoured to diminish this danger by connecting the two stems, AG and DE (fig. 1), and the cross-connecting tube as rigidly as possible to a light board when experimenting, so as to prevent these parts acquiring any considerable momentum.

It must not, however, be supposed that the condition of the liquid, when in a state of tension, is necessarily nearly unstable. Thus, before the method of detachment by heating the platinum wire was hit upon, it was often very difficult to effect a release. With the tonometer indicating a pull of from 8 to 12 atmospheres, the liquid could not be made to let go its hold by moving the apparatus about, nor by means of taps and jars as violent as it seemed safe to give, nor by strong local heating of the narrow tube. Meanwhile, specks of suspended impurity could be seen floating about in the interior, proving the liquidity of the substance; and, when the release did take place, the immediate rise of the mercury in the tonometer to its normal position showed that no mistake had been made in the measurement of the tension to which the liquid was all the time subjected.

The Observations.

The accompanying diagram (Plate 10) is a graphic record of all the observations that I succeeded in obtaining.

The first set were taken on May 17, 1890, and are distinguished thus ⊕. It was in making these that I became fully aware of the importance of aiming at steadiness of temperature before liberating the bubble, so as to permit of a little waiting till the liquid on the sides of the tube had drained down. Hence these measures are, with apparently one exception, probably all a little too high. The observations recorded + were made on May 19, with all possible care; two of them low down in the diagram were recorded as doubtful, and are marked (?). Those marked ⊕ correspond to cases in which the bubble formed at the top of the tube FE (fig. 1), from which position it could not be "transferred" to the wire. This upper portion of the tube had not been calibrated with special attention, since its employment had not been anticipated, and the subsequent breakage of the apparatus prevented this being done afterwards. Nevertheless, I have thought it best to introduce the observations into the diagram. An eye estimate had to be made of the contraction of the tube due to the sealing at E, but in long bubbles an error on this score must have been comparatively small. It will be observed, however, that all the observations in question fall rather below the mean curve. The observations marked ⊞ were made on May 22, in order to fill in regions about which information was still needed. All of these were recorded as "good" observations.

I interpret these observations as representing the straight line through the origin drawn in the diagram and making an angle with the axis of stress, whose tangent is 0·700.

The vibrations set up, and the shock sustained by the apparatus on the sudden release from the highest tension reached (over 17 atmospheres) were so violent that it seemed unsafe to proceed further before making such observations as were necessary to eliminate the effect of the yielding of the glass itself, and which could not be made if the vessel were broken.*

The vessel was therefore opened, and the alcohol in it subjected to a pressure of 12·38 atmospheres in excess of the external atmospheric pressure, corresponding to 40 tonometer divisions, and the *apparent* compression of the liquid was observed. The measures of the retreat of the end of the column along the tube were

	millims.
	28·19
	27·53
	28·36
	28·02
	27·05
	27·6
(Specially trustworthy)	27·78
Mean .	27·79

The observed apparent extension *in the same vessel* under an equal tension, as given by the line of the diagram, is 27·8 millims. The closeness of the agreement is, of course, in part fortuitous, for the line of the diagram cannot be placed with extreme precision; but the practical coincidence of the two numbers is a very satisfactory confirmation of the view that the observations correspond to a straight line, and, since the small yielding of the nearly rigid glass vessel must be proportional to the stress, permits us to draw the conclusion that *in the neighbourhood of the zero pressure the absolute coefficient of volume elasticity of alcohol is the same for extension as for compression, and so far as the observations show is constant between a pressure of* + 12 *and* − 17 *atmospheres.*

The best way to obtain the *absolute* value of the coefficient is probably by direct experiments on the compressibility of alcohol in the neighbourhood of the zero point. The value given in LUPTON's tables is $\frac{1}{11\cdot4}$ per 10^6 grms. per sq. centim., or ·0000906 per atmosphere of 1033·3 grms. Mr. SKINNER by recent experiments made in the Cavendish Laboratory on alcohol at 13°·5 C. between 1 atmosphere and 1·3 atmospheres obtains the value ·000093 per atmosphere.

It seemed desirable, if only to serve as a check on the experiments, to mention the attempt that I made to determine directly the yielding of the glass vessel. For this purpose the alcohol was boiled out until only a very small residue was left, when mercury was allowed to enter which completely filled up the whole instrument. This mercury, whose compressibility is some fifty times less than that of alcohol, was then

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* As a matter of fact, the stem DE was unfortunately broken in setting up the apparatus for exhibition in the rooms of the Royal Society, at the Soirée of June 18, 1890, but this was afterwards repaired without any interference with the neighbouring bulb.

subjected to pressures of 10 and 12 atmospheres (above the external atmospheric pressure), and the retreat of the end of the mercury column along the tube was noted. The following measures of this recession were obtained for a pressure of 10 gauge-atmospheres of 15 pounds per sq. inch.

	millims.
Direct observation at 10 atmospheres . . .	7·31
	7·33
From observation at 12 atmospheres . . .	7·40
	7·40
	7·1
Mean	7·31

The mean value is equivalent to 7·22 millims. per 10 atmospheres of 1033 grms. per sq. centim.

The measures were somewhat unsatisfactory on account of (1) the smallness of the total length to be measured, (2) the deformation of the meniscus due partly to a certain want of cleanness at the surface which dragged along the glass, (3) a tendency to entangle bubbles of alcohol between the mercury and the sides of the tube owing to the churning as the end of the column moved to and fro. This ultimately caused a separation of the column into segments, and must have had the effect of diminishing the apparent recession. (4) It should also be mentioned that the end of the column was situate in the upper part of the tube FE, which, owing to the subsequent breakage of the apparatus, I was not able to specially calibrate, but which, if we may judge from the measures of the bubbles, was probably somewhat wider than the part below the wire. On this account also the measured result is probably rather too small.

If the mercury were quite incompressible, and if there were no residual alcohol in the vessel, this number would represent, in millims. of the tube, the yielding of the glass per 10 atmospheres, but by reason of the corrections required on these two accounts this length is reduced to 6·626 millims.*

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* The calculation of these corrections was made as follows :—Let the change required in the volume of the glass per 10 atmospheres, be v_g millims. of the tube, and let the corresponding changes in the volume of the mercury, and of the residual alcohol be v_m and v_a millims. of the tube respectively, and let the observed apparent alteration of volume be (a) millims. of tube. Then

$$v_g + v_m + v_a = a \qquad \text{(i.)}.$$

Let the vessel, when full, contain n times as much alcohol as this residue. Then admitting extension and compression under numerically equal stresses to have been proved equal (p. 367), we have

$$v_g + nv_a = \mathrm{A} \qquad \text{(ii.)},$$

Subtracting this amount from the ordinate of the diagram corresponding to 10 atmospheres, the remainder, of length 16·17 divisions, gives the corrected value of the absolute strain, and is equivalent to 10·67 ten-thousandths of the whole volume, per stress of 10 atmospheres, or to ·0001067 per atmosphere, a result which is about 15 per cent. higher than that of Mr. SKINNER. From the nature of my observations on this point, I do not think that much importance is to be attached to this discrepancy, which represents a possible error of 2 millims. to be divided between the measure of the length of the bubble on the release from 10 atmospheres tension, and the measure of the recession of the mercury in the experiments just quoted.

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where A is the observed apparent alteration of volume in millims. of tube, due to 10 atmospheres tension, when the vessel was full.

Hence

$$v_a = \frac{A}{n} - \frac{v_g}{n}.$$

Substituting this value for v_a in (i.), and multiplying up

$$(n-1)\,v_g + nv_m = na - A,$$

or

$$v_g = \frac{n}{n-1}\,a - \frac{n}{n-1}\,v_m - \frac{A}{n-1},$$

in which expression $\frac{n}{n-1}\,v_m$ is a small corrective term due to the compressibility of the mercury, while $\frac{A}{n-1}$ is a small corrective term due to the compressibility of the residual alcohol present with the mercury in the bulb.

The numerical values required were obtained as follows:—

	grms.
The weight of mercury and alcohol together was found to be . . .	769·3
,, ,, alone	768·38
Therefore, weight of residual alcohol alone =	0·92

Hence the volume of the the mercury alone = 56·5 cub. centims., and the volume of the residual alcohol alone = 1·15 cub. centims.

The volume of the alcohol used in the stretching experiments was 58·5 cub. centims., therefore

$$n = \frac{58\cdot5}{1\cdot15} = 50\cdot6, \text{ and } n - 1 = 49\cdot6.$$

The observed recession of the mercury (a) = 7·22 millims. of tube, as already stated in the text.

A = 22·7 millims. of tube, as read off from the line in the diagram.

v_m = ·273 millims. of tube (deduced from the value of the compressibility of mercury given in LUPTON's tables, viz., $\frac{1}{552\cdot5}$ per 10^6 grms. per sq. centim., and the fact that $\frac{1}{10000}$ of the volume of mercury employed = $\frac{56\cdot5}{58\cdot5}$ of 1·513 millim. of the tube).

Hence

$$v_g = \frac{50\cdot6}{49\cdot6} \times 7\cdot22 - \frac{50\cdot6}{49\cdot6} \times \cdot273 - \frac{22\cdot7}{49\cdot6}.$$

$$= 7\cdot36 - 0\cdot278 - 0\cdot456.$$

$$= 6\cdot626 \text{ millims. of tube.}$$

General Remarks on the Experiments.

It will be noticed that the manner of experimentation has the disadvantage that it does not permit of the whole of the liquid under tension being at the same temperature. The mean temperature was probably at no time more than 5° removed from 16° C. I am now endeavouring to make an apparatus in which it shall be possible to maintain the portion of the liquid, whose stretching is to be measured, at any desired constant temperature, whereby also the chief difficulty in the way of more exact measurements will be overcome.

As regards the conclusion reached, that the changes of volume of a mass of liquid are equal for numerically equal increments of pressure, whether positive or negative, it may be justly observed that this was only to be expected. No one, however, could predict that the coefficient of extensibility would remain practically constant up to tensions of 17 atmospheres, and nothing but further experiment can decide what changes may take place in its value as the tension is increased.

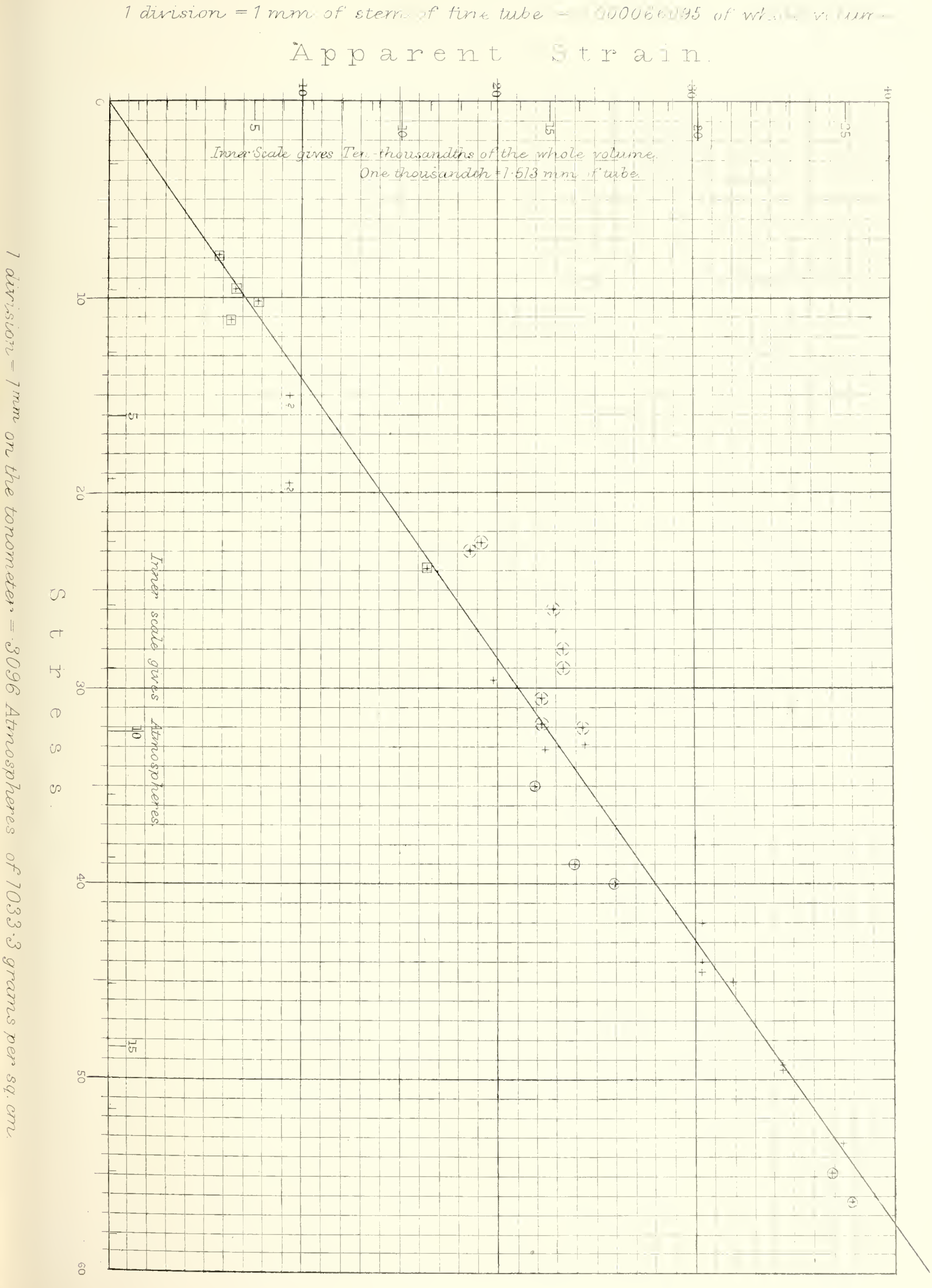
Bearing of the Observations on the Theory of Surface-Tension.

The unequivocal proof that a liquid can exist in stable equilibrum, in a state of isotropic tensile strain, has a bearing on the theory of surface forces in fluids. For it can be shown to be necessary for equilibrium that a compressible liquid shall be, close to the free surface, less dense than in the interior; in other words, the surface layers are in a condition to which interior liquid could be brought by stretching it, and are, therefore, a seat of energy in precisely the same way that stretched liquid is a seat of energy. A theory, such as that of LAPLACE, which assumes uniform density precludes us from admitting in the material itself any such modification correspondent to the surface energy, and drives us to seek it in the condition of the superjacent ether film.

Note on a Curious Phenomenon of Adhesion between two Solids immersed in a Stretched Liquid.

Desiring to ascertain whether an air-free liquid would adhere under tension, as well to a metal as to glass, I enclosed a small piece of folded sheet copper in a glass bulb, which was then filled with boiled-out, air-free alcohol. Experiments with this showed strong adhesion to the copper, as well as to the glass, provided the vessel was kept still, but any agitation at once caused the stretched liquid to let go its hold at the place of contact of the copper and the glass. Close attention showed that the copper seemed to "grow to the glass" at the points of contact, when the surrounding liquid was in a state of tension. This led to experiments on bulbs, with smaller bulbs of glass inside, and in all cases the same phenomenon was observed: when the liquid was stretched, the loose bulb attached itself to the side of the vessel. The equilibrium was, however, very unstable. The release of the liquid took place on the slightest jar, the bubble always appearing at the contact of the solid with the wall, and the loose piece being generally tossed up when the rupture took place. I succeeded best with one small irregular bulb with a projecting stem; this could be gently waved about in the stretched liquid, while the foot of the stem adhered to a point on the side of the containing vessel (showing incidentally that considerable currents may exist in a stretched liquid). The explanation of this phenomenon, which at first puzzled me, may, I think, be given as follows.

At the surface of glass, any liquid, such as alcohol, which wets it, is condensed. Over the area of contact of the loose piece with the side of the vessel, this condensed film is probably somewhat thinner than elsewhere, being squeezed out (as by hydrostatic pressure) by the cohesive attraction between the two solids. When the liquid is in a state of tension, there is everywhere a demand for liquid to stretch, which is met by any approach of the cohering surfaces, for such approach will increase both the closeness and the area of contact, and yield a supply of hitherto condensed liquid, by diminution of the surface over which it has been condensed. Any displacement in the direction of further approach will therefore be resisted only by the elasticity of the solids, called into play by the deformation at the area of contact. Under ordinary circumstances such displacement is resisted also by the hydrostatic pressures. To understand the instability, it is only necessary to remember that, with a comparatively rigid substance like glass, small relative motion of the parts may generate very large impulsive stresses at points where the relative motion is prevented.



Stretching of ETHYL ALCOHOL.

West, Newman sc.

IX. *The Thermal Emissivity of Thin Wires in Air.*

By W. E. Ayrton, *F.R.S.*, *and* H. Kilgour.

Received July 2,—Read November 19, 1891.

[Plates 11–15.]

I. *General Character of the Experiments.*

In 1884 it was observed experimentally that whereas the electric current required to maintain a *thick* wire of given material, under given conditions, at a given temperature, was roughly proportional to the diameter of the wire raised to the power three-halves, the current was more nearly proportional to the first power of the diameter if the wire were *thin.* When this difference in the behaviour of a thick and a thin wire was first noticed it was regarded as quite unexpected. But, as pointed out by one of us in the course of a discussion at a meeting of the Royal Society, the unexpected character of the result was due to people having assumed that the loss of heat from radiation and convection per square centimetre of surface per 1° excess temperature was a constant for a given kind of surface and independent of the size and shape of the cooling body, although as early as 1868 Box had drawn attention to the great difference that existed between the rate of loss of heat from unit area of a horizontal cylinder and per unit area of a sphere. The interchange of heat between unit area of a body and the enclosure might be independent of the shape of the body as far as radiation alone was concerned, but it seemed nearly obvious that the cooling by convection must be materially affected by the shape of the cooling body.

The very valuable investigations that have been made on emissivity by Mr. Macfarlane, Professor Tait, Mr. Crookes, Mr. J. T. Bottomley, and by Mr. Schleiermacher, had for their object the determination of the variation of the emissivity with changes of the surface and with change in the density of the gas surrounding the cooling body, but it was not part of these investigations to determine the change in the emissivity that is produced by change in the shape and size of the cooling body. Indeed, so little has been the attention devoted to the very large change that can be brought about in the value of the emissivity by simply changing the dimensions of the cooling body, that in Professor Everett's very valuable book on Units and Physical Constants, the absolute results obtained by Mr. Macfarlane are given as the "results of experiments on the loss of heat from blackened and polished copper in air at atmosphere pressure," and no reference is made either to the shape or to the size of the cooling body.

[November 19, 1891.—Since this paper was sent in to the Royal Society, a new edition of this book has appeared, and, in consequence of a suggestion made to Professor EVERETT, the word "balls" has been added after the word "copper" in this new edition, as well as the following paragraph :—

"*Influence of Size.*

"According to Professor AYRTON, who quotes a table in 'Box on Heat,' the coefficient of emission increases as the size of the emitting body diminishes, and for a blackened sphere of radius r cm. may be stated as

$$\text{“}\, 0{\cdot}0004928 + \frac{0{\cdot}0003609}{r}.$$

"The value in MACFARLANE'S experiments was 2."]

The laws which govern the loss of heat from thin cylindrical conductors have not only considerable scientific interest in showing how the shape of a body affects the convection currents, but they are of especial importance to the electrical engineer in connection with glow-lamps, hot-wire-voltmeters, fuses, &c. We, therefore, thought it desirable to ascertain the way in which the law of cooling for thick wires, which involved the diameter raised to the power three-halves, passed into the law for the cooling of thin wires, involving only the first power of the diameter. For this object the investigation described in the present communication was commenced at the beginning of 1888.

A considerable number of preliminary experiments having been conducted for the purpose of arriving at the best conditions to be adopted in the investigation, we finally, in April, 1888, decided to measure the emissivity at different temperatures from nine platinum wires, having respectively the diameters of 1, 2, 3, 4, 6, 8, 10, 12, and 15 mils, or thousandths of an inch. Wires of these sizes having been ordered from Messrs. JOHNSON, MATTHEY, and Co., they were found, when received, to have the following diameters at 15° C. :—

Mils.	Millimetres.
1·2	0·031
2·0	0·051
2·9	0·074
4·0	0·102
6·0	0·152
8·1	0·206
9·3	0·236
11·1	0·282
14·0	0·356

Throughout this paper we have given the diameters of the wires both in mils and in millimetres. It may seem unscientific to mix up dimensions in thousandths of an inch with dimensions in centimetres, but, while it is convenient, for the purposes of comparison, to use one square centimetre as the unit of area in experiments on emissivity, it is also convenient to have some concrete conception of the wires spoken of. Now, fine wires are practically known in this country as wires of 1, $1\frac{1}{2}$, 2 mils, &c., diameter, a wire of 1 mil diameter being, for example, the thinnest that has been practically used in the construction of electrical apparatus. A better idea is, therefore, obtained from stating that the diameters of wires are 1, 2, or 3 mils, than from saying that they are 0·025, 0·051, or 0·074 millim. Attached to several of the curves are the diameters of the wires expressed in millimetres. These numbers are stated to four decimal places, but it would have been better to have given, as in the above Table, only three significant figures, this being the probable limit to the accuracy of the measurement of the diameters.

Suspecting that some of the results of published experiments on the currents required to fuse wires had been much influenced by the cooling action of the blocks to which the ends of the wires were attached, we started by making a calculation on the length necessary to give to our wires so that the loss of heat by conduction should not introduce any important error into the determination of the emissivity. To do this it was necessary to calculate the distribution of temperature along a wire through which a steady current was flowing, and from which heat was lost by radiation, convection, and conduction, and it was further necessary to improve on the calculation one of us had published on this subject in the 'Electrician,' for 1879, by now taking into account the fact that the emissivity, as well as the thermal and electric conducting powers of the wire, were different at different points in consequence of the difference of temperature. Such a calculation has not, as far as we are aware, been hitherto made, it having been assumed in all previous investigations that the effect due to the variation of the thermal and electric conducting power of the material with temperature, as well as the variation of the emissivity per square centimetre with temperature and with the diameter of the wire, could be neglected.

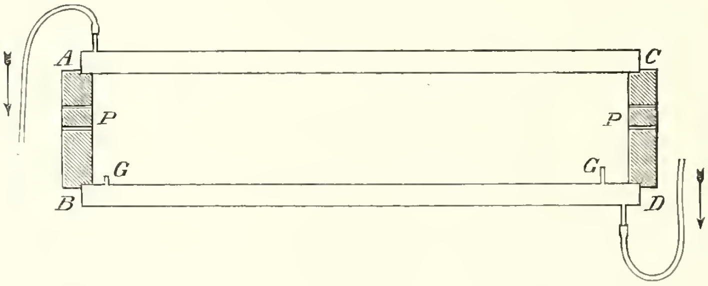
Until we had completed the experiments described in this paper we could of course only employ, in this calculation, values that we had guessed at as something near the truth for the emissivity of platinum wire for different diameters and at different temperatures. Hence, after the completion of the experiments, we took up the mathematical investigation again, substituting for the emissivity such a function of the diameter of the wire and the temperature of the point as we had experimentally found it to be. The investigation by which we finally arrived at the calculated distribution of temperature along the wire is given in § V. of the paper.

The rate at which heat was lost by any one of the wires was measured by the product of the current passing through it into the P. D. (potential difference) maintained between its ends, while the ratio of the P. D. to the current gave the resistance of the wire, and therefore its temperature. As the variation of resistance with temperature of different specimens of platinum is known to differ, it was not considered sufficiently accurate to deduce the temperature of the wire experimented on by using some supposed temperature coefficient for platinum; consequently the variation of resistance, with temperature of each piece of platinum wire employed, was experimentally determined up to 300° C. The investigation, therefore, divided itself into two distinct parts, viz. :—

A, the measurement of the power required to be given to platinum wires of various diameters, so as to maintain them at various temperatures above that of the enclosure.

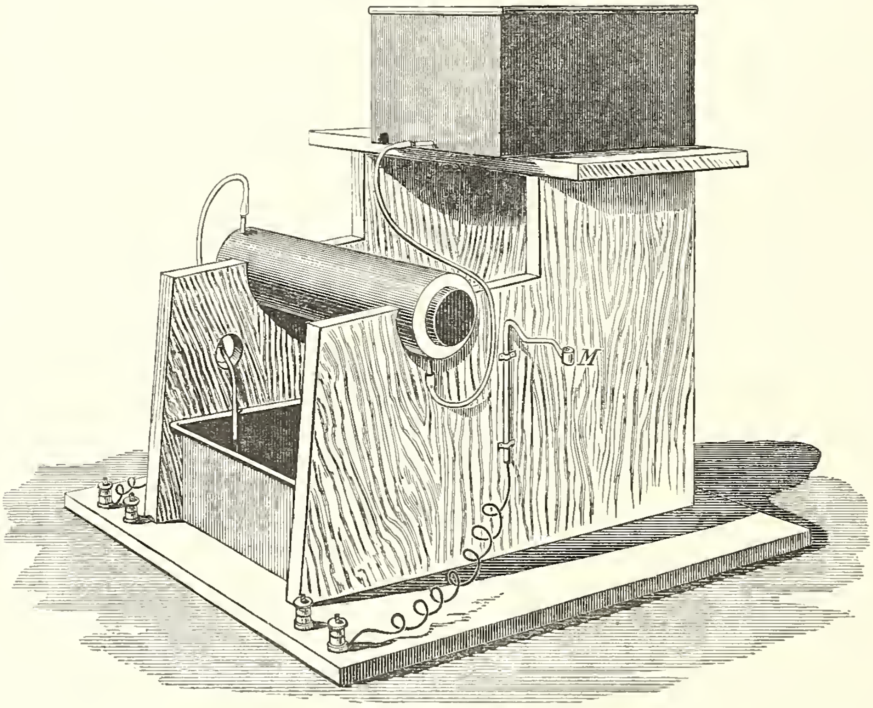
B, the determination of the law connecting the resistance with the temperature for each piece of wire employed in A.

Fig. 1.



And the second investigation also consisted of two parts, since a very considerable time had to be finally spent in determining the errors of the thermometers that had been employed in measuring the variation of the resistance of the wires with temperature.

Fig. 2.



For carrying out the first part of the investigation, each wire was placed horizontally along the axis of a water-jacketed cylinder shown in section *A*, *B*, *C*, *D* (fig. 1), 32·5 centims. long, 5·08 centims. internal, and 7·62 centims. external diameter, the interior surface of the cylinder being coated with dull lamp black. A stream of water of constant temperature entering by the pipe at *D*, and leaving by the pipe at *A*, flowed through the water jacket and prevented the surface of the cylinder from becoming warmed by radiation and convection from the hot wire under experiment. Fig. 2 shows the water-jacketed cylinder in position.

For inserting the wire to be tested into the enclosure-cylinder, it was attached at its two ends to a lamp-blacked brass carrier *E*, *F* (fig. 3), which could be slid, between guiding pins *G*, *G* (fig. 1), along the bottom of the inner cylinder into a definite position. The two ends of the wire were soldered to the tips α, α (fig. 3), of two wires $\alpha\beta$, $\alpha\beta$, 1 millim. in diameter, which were made of platinum-silver so as to have but a small heat conductivity. Two short pieces of platinum wire, 0·025 millim. in diameter, to be used as terminals for the voltmeter, were soldered with the slightest touch of solder to two points about 6 millims. distant from the ends α, α, of the wire under test, it having been determined that at this distance from the terminals α, α, the temperature would not be much lower than the average temperature of the wire. In the case of the two finest wires tested, having diameters of 0·025 and 0·051 millim. respectively, it was not found necessary to adopt this arrangement, and the ends of the wires were themselves used as the voltmeter terminals, as seen in fig. 4; the platinum-silver wires, $\alpha\beta$, $\alpha\beta$, were, however, filed down quite thin to prevent the fine wire being cooled by heat being conducted away from its ends. The shaded portion of figs. 3 and 4 represents an ebonite cap used to insulate one end of the wire α, α, from the brass carrier *E*, *F*.

Fig. 3.

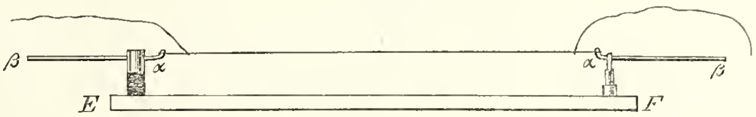


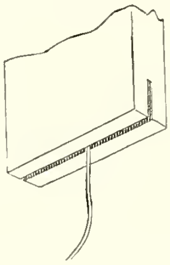
Fig. 4.

To prevent draughts entering the enclosure-cylinder, each of its ends was closed with an ebonite plug, *P*, *P* (fig. 1) through two holes in which passed the platinum-silver wires, $\alpha\beta$, $\alpha\beta$, used as the terminals for the main current, and the fine platinum wire used for measuring the P. D. Connection was made between these latter and the wires which led to the voltmeter itself by their dipping into two mercury cups *M*, *M* (fig. 2), carried from the stand of the apparatus, and which were turned into position close to the holes in the ebonite plugs, after the latter had been inserted into position.

For the experiments on the variation of resistance of the wires with temperature which were conducted during May, June, and July, 1888, the two ends of each wire were fastened to two thick rectangular copper bars, about 7 millims. thick by 20 millims. wide. The fastening was effected by making a fine saw cut about 6 millims.

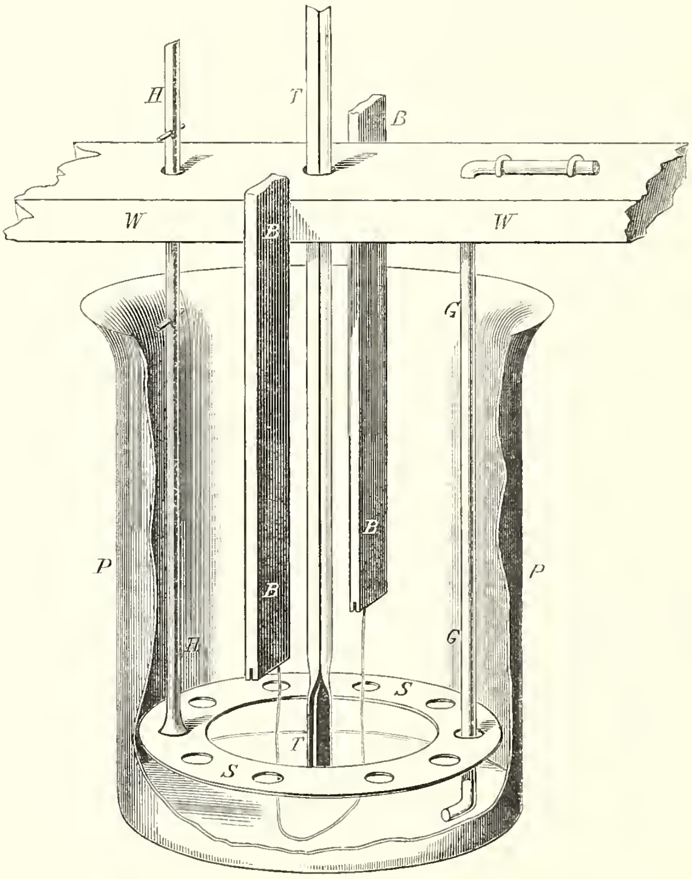
deep in the end of the bar (fig. 5) inserting the end of the wire to be tested into the cut and gently hammering the copper so as to grip the wire. The copper was then heated in a blowpipe, and solder having a higher melting point than 350° C., run in so as to fill up all interstices.

Fig. 5.



These copper bars, *B*, *B* (fig. 6), dipped into oil contained in an iron pot *P P* wrapped round with asbestos cloth to prevent loss of heat, and were supported from a wooden cross-piece *W W*, which carried the thermometer *T T*, the guide *G G* for the stirrer *S S*, and itself guided the handle *H H* of the stirrer, the up and down motion of which was limited by two pins.

Fig. 6.



Four thermometers of different ranges were used in these tests, and each of them was subsequently compared with a Kew standard thermometer. Considerable difficulty was introduced into the carrying out of these comparisons from the fact that it is, or, at any rate, was not three years ago, possible to obtain a standard thermometer from the Kew Observatory to read, say, from 200° to 300° C., with a short wide chamber at the base in which the mercury expanded below 200° C. All that could be obtained was a long thermometer which had been carefully tested between 0° C. and 100° C., and the remainder of whose tube had been simply calibrated for uniformity of bore. The consequence was that when we desired to compare one of our thermometers reading say, from 200° C. to 300° C., with the Kew standard thermometer, their bulbs were very far apart when they were both immersed in oil so that the top of the mercury column of each was just at the top of the oil; secondly, whereas we had kept each of our thermometers with its bulb close to the wire whose resistance was being tested, and, therefore, at a fixed distance below the surface of the oil while it was being used, the Kew standard thermometer had to be continually lowered further and further into the oil as the temperature rose in order that correct readings could be obtained.

By adopting, however, the following device, a satisfactory, although very laborious, comparison between the four thermometers and the Kew standard thermometer was finally carried out in the autumn of 1889, by three of the students of the Central Institution, Messrs. MÜLLER, STEPHENS, and WIGHTMAN. The thermometer to be tested was placed in a deep oil bath in exactly the same position, relatively to the surface of the oil, as that in which it had previously been used. To obtain uniformity of temperature throughout the oil bath, the oil was kept constantly agitated by means of a stirrer driven by an electromotor, and further, the heat was applied not merely at the bottom of the bath but along the whole of its long vertical sides, which were covered with several layers of asbestos cloth to prevent the flames warming one part of the surface more than another. The bulb of the thermometer was surrounded by a bobbin of wire, the resistance of which was very carefully measured for many readings of the thermometer. Then the Kew standard thermometer was inserted in place of the thermometer to be tested, but now, as the temperature rose the coil and the standard thermometer were depressed together so as to keep the coil always surrounding the bulb of the thermometer, and so that the level of the mercury in the thermometer tube was always only just above the level of the oil. The temperatures were then read off on the standard thermometer which caused the coil of wire to have exactly the same resistances as before, and which were, therefore, the true temperatures corresponding with the readings of the thermometer to be tested. In consequence of the unwieldy length of a Kew standard thermometer reading to 300° C., it was necessary to use an oil bath 80 centims. deep for this experiment.

Much thought and labour was given to the piece of apparatus, and many devices were introduced into it which it is not necessary to describe here, first because after we had completed this part of the investigation we learnt that much more suitable standard thermometers might have been obtained from abroad than could be purchased from the Kew Observatory; secondly, because if one were making such an investigation again, one would use as standards of comparison the coils of accurately known resistances at high temperatures which can now be purchased, but which we were advised three years ago had not then the necessary degree of accuracy.

II. *Variation of Resistance of Platinum Wires with Temperature.*

We will now return to the main part of the investigation B, and it is to be understood that all temperatures mentioned are the *true* temperatures as corrected by using the results obtained from the comparison of the thermometers with the standard thermometer.

The first point was to determine the variation of resistance with temperature of the copper bars *BB, BB* (fig. 6), and the thick copper wires that went to the Wheatstone's bridge. The total resistance of the bars and wires at 0° C. was 0·0050 ohm, and, as the wires were protected by an asbestos screen from the action of the heat, the total variation in the resistance of the copper bars and wires combined was very small, being only 0·0003 ohm between 0° C. and 325° C. This variation was, therefore, almost negligible compared with the variation of the resistance of even the thickest platinum wire experimented on, for which the following are the results.

TABLE I.—Piece of the Platinum Wire that had been used in the Experiments on Emissivity.

May 10th, 1888. Diameter 14 mils, or 0·356 millim. Length unknown, but about 23 centims.

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		° C.		° C.	
17·5	0·262	72·7	0·310	148·3	0·374
17·9	0·262	81·8	0·318	154·4	0·382
23·9	0·268	93·3	0·328	170·5	0·392
37·5	0·280	105·3	0·338	182·0	0·402
46·8	0·288	122·0	0·352	194·2	0·412
61·1	0·300	135·1	0·362		

Although the exact length of the above wire was unknown, the results are not the less useful for giving the law of variation of resistance with temperature, for, as pointed out in § III., "Results of Emissivity Experiments," the observations contained in Tables I. to V. were employed to give the ratios of the resistances of any one of the wires at any two temperatures, and not the specific resistance at any one temperature. Hence it was unnecessary to know the lengths of the wires used in the experiments, the results of which we recorded in Tables I. to V., and the lengths where mentioned are only approximate in some cases.

The next piece of wire used was of the same diameter, and was also a bit of the actual wire that had been used in the emissivity experiments. The length of this piece was, however, accurately known.

TABLE II.—Piece of Platinum Wire that had been used in the Experiments on Emissivity.

May 14th and 15th, 1888. Diameter 14 mils, or 0·356 millim. Length 22·76 centims. Curve No. 1, fig. 7 (Plate 11.)

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		° C.		° C.	
15·8	0·262	145·1	0·372	242·6	0·452
16·1	0·262	145·1	0·372	255·2	0·462
52·3	0·294	157·3	0·382	256·9	0·462
55·0	0·296	168·8	0·392	270·1	0·472
63·1	0·303	171·6	0·394	271·1	0·472
74·5	0·313	180·4	0·402	277·8	0·477
85·8	0·323	193·1	0·412	278·8	0·478
97·5	0·333	206·1	0·422	284·6	0·482
	0·343	207·3	0·422	291·0	0·487
	0·343	218·6	0·432	298·5	0·492
121·4	0·352	230·3	0·442	306·1	0·497
132·9	0·362	231·9	0·442		

The equation connecting the resistance and the temperature of the preceding specimens was determined in two distinct ways. First, a curve (No. 1, fig. 7) was drawn connecting all the values of temperature and resistance obtained on May 14th and 15th, then three points on this curve were selected, viz., those corresponding with

Temperature.	Resistance.
15°·0 C.	0·262
153°·3 C.	0·380
300°·0 C.	0·493

From these, on the assumption that r, the resistance at temperature t° C., could be expressed in terms of r_0, the resistance at the temperature 0° C. by the equation

$$r = r_0 (1 + \alpha t + \beta t^2)$$

the values of r_0, α, and β were calculated. The values thus obtained were

$$\left.\begin{array}{rll} r_0 = & 0{\cdot}2487 & \text{ohm} \\ \alpha = & 0{\cdot}00358 & ,, \\ \beta = & -\ 0{\cdot}000000101 & ,, \end{array}\right\} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (1).$$

Next, with the object of ascertaining the magnitude of the error that would be introduced into the values of r_0, α, and β by a small error having been made in reading the temperature, the preceding calculation was repeated on the assumption that the second temperature instead of being 153°·3 C. was 155° C. The values then obtained were

$$\left.\begin{array}{rll} r_0 = & 0{\cdot}2488 & \text{ohm} \\ \alpha = & 0{\cdot}00354 & \text{,,} \\ \beta = & -\ 0{\cdot}000000896 & \text{,,} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thirdly, the method of least squares was applied to all the observations made on May 10th, and given in Table I. In this way there were obtained the values

$$\left.\begin{array}{rll} r_0 = & 0{\cdot}246987 & \text{ohm} \\ \alpha = & 0{\cdot}003560 & \text{,,} \\ \beta = & -\ 0{\cdot}000000645 & \text{,,} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Lastly, the method of least squares was applied to all the observations made on May 14th and 15th, and given in Table II., the values thus obtained being

$$\left.\begin{array}{rll} r_0 = & 0{\cdot}247338 & \text{ohm} \\ \alpha = & 0{\cdot}003650 & \text{,,} \\ \beta = & -\ 0{\cdot}0000001091 & \text{,,} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

To compare the values of r_0, α, and β obtained from the curve recording the observations of May 14th and 15th with the values obtained by applying the method of least squares to the same observations, we may examine the value of the difference between r_{100}, the resistance of the wire at 100° C., and r_0, the resistance at 0° C.; using the values of r_0, α, and β given in (1), we find

$$r_{100} - r_0 = 0{\cdot}0862,$$

whereas, using the values of r_0, α, and β given in (4) we have

$$r_{100} - r_0 = 0{\cdot}086325.$$

Again the mean coefficient of increase of resistance per 1° C., between 15° C. and 85° C. when obtained from the curve alone, is 0·00350, whereas the mean coefficient per 1° C., between 0° C. and 100° C., using the means of the values of r_0, α, and β given in (1), (3), and (4), is 0·00348.

We may, therefore, conclude that it is not necessary to use the lengthy method of least squares to obtain the values of r_0, α, and β, and that the values obtained by using three points on a curve which graphically records the results of the experiments on temperature and resistance are accurate enough for practical purposes. The three-point method was, therefore, alone adopted for the remaining wires. Indeed, for the purpose of determining the temperatures of the wires in the emissivity experiments which corresponded with the various observed resistances, it was found most easy and most accurate to simply read off the temperature at once from the curves which recorded the results of the experiments made on the variation of the resistance with temperature for the particular wires. Specimens of such curves are seen in curves 1, 2, 3, 4, 5, 6, figs. 7 and 8. (Plates 11 and 12.)

We were, however, led to study the various formulæ that had been published by different investigators on the connection between the resistance and temperature of different metals, and this led us to carry out a considerable amount of calculation which brought to light some interesting results. Since these results, however, are not specially connected with this present investigation on emissivity, we purpose presenting them in a separate communication later on.

Table III.—Piece of the Platinum Wire that had been used for the Experiments on Emissivity.

June 11th and 12th, 1888. Diameter, 11·1 mils, or 0·282 millim. Length, 14·02 centims. Curve No. 2, fig. 7 (Plate 11)

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		° C.		° C.	
15·3	0·2953	102·2	0·365	214·0	0·455
15·6	0·2959		0·375	225·6	0·465
16·3	0·2962		0·375	226·8	0·465
31·8	0·3096	141·5	0·395	226·8	0·465
39·1	0·3150	164·9	0·415	238·8	0·475
39·1	0·3150	176·2	0·425	239·2	0·475
51·6	0·325	176·9	0·425	239·2	0·475
64·9	0·335	188·8	0·435	251·8	0·485
75·8	0·345	200·4	0·445	252·7	0·485
77·3	0·345	200·8	0·445	264·3	0 495
87·8	0·355	201·8	0·445	276·8	0·505
89·4	0·355	212·8	0·455		
99·8	0·365	213·3	0·455		

The following results (Table IV.) were not obtained with a piece of wire that had been actually employed in the emissivity experiments, but with a piece of wire cut off the same reel from which was taken the wire used in the emissivity experiments. Before, however, making the measurements given in Table IV., the piece of wire used was heated by a strong current, so as to be brightly luminous for 40 minutes, in order to bring it to the same state as that of the piece that had been used in the emissivity experiments.

TABLE IV.

July 16th, 1888. Diameter, 11·1 mils, or 0·282 millim. Length unknown. Curve No. 3, fig. 7 (Plate 11).

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		° C.		° C.	
15·9	0·4496		0·577	222·1	0·711
40·9	0·483	141·9	0·609	238·2	0·731
59·9	0·507	150·3	0·620	255·2	0·752
68·2	0·518	162·6	0·635	272·0	0·771
79·1	0·532	175·4	0·650	291·4	0·792
89·2	0·545	187·6	0·665		0·809
100·5	0·560	200·7	0·6801	292·5	0·792
	0·569	209·6	0·695	273·1	0·771

The curves recording the results given in Tables III. and IV. are shown in Curves 2 and 3, fig. 7.

TABLE V.—Piece of Platinum Wire that had been used in Experiments on Emissivity.

July 27th, 1888. Diameter, 9·3 mils, or 0·236 millim. Length, 20·19 centims.

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		° C.		° C.	
19·2	0·6844	150·8	0·905	276·5	1·104
48·5	0·736	175·0	0·945	293·3	1·130
79·2	0·789	207·2	0·996	304·0	1·145
97·4	0·820	232·4	1·035	July	30th.
113·0	0·847	256·9	1·065	17·3	0·678

JULY 20th, 1888. Diameter, 8·1 mils, or 0·206 millim. Length unknown.

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		C.		° C.	
18·1	0·6191	112·5	0·805	301·0	1·138
43·7	0·670	205·6	0·971	295·8	1·130
62·8	0·707	232·9	1·020	265·9	1·079
78·1	0·737	262·9	1·075	254·8	1·059
94·8	0·770	284·3	1·110		
104·7	0·790	295·2	1·128		

JULY 26th, 1888. Piece of the preceding wire. Length, 18·54 centims.

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		° C.		° C.	
19·0	0·6184	106·3	0·790	216·9	0·990
42·4	0·665	114·0	0·805	233·3	1·020
58·9	0·697	149·2	0·865	251·9	1·055
69·1	0·717	160·0	0·885	274·7	1·095
79·2	0·737	170·7	0·905	284·1	1·110
89·4	0·757	187·0	0·935		
96·1	0·770	206·5	0·971		

JULY 12th, 1888. Diameter, 6 mils, or 0·152 millim. Length, 17·07 centims.
Fig. 14.

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		° C.		° C.	
14·8	1·047	131·1	1·445	266·7	1·885
32·7	1·110	153·7	1·520	273·2	1·905
45·0	1·153	160·1	1·541	280·0	1·925
53·85	1·183	176·8	1·595	289·0	1·950
61·8	1·210	183·2	1·616	294·8	1·967
73·0	1·248	195·2	1·657	302·0	1·988
81·0	1·276	202·5	1·680	306·0	2·000
93·3	1·318	213·7	1·715	309·0	2·010
102·8	1·352	224·7	1·750	311·0	2·015
110·2	1·377	231·3	1·772	305·0	1·995
112·5	1·385	240·0	1·800		
114·0	1·390	250·5	1·835	149·2	1·0475
126·7	1·426	257·0	1·855		

JULY, 19th, 1888. Diameter, 4 mils, or 0·102 millim. Length, 15·37 centims.
Curve 4, fig. 8 (Plate 11).

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		° C.		° C.	
17·3	4·127	113	4·645	233·5	5·245
42·4	4·265	147·7	4·815	253·4	5·345
76·0	4·445	163·5	4·895	273·8	5·445
94·5	4·545	183·1	4·995	280·3	5·475
103·6	4·595	207·8	5·115		

The smallness of the variation of resistance of the last wire with temperature shows that it is not a pure platinum wire, and that it possibly contains a trace of silver or iridium.

July 5th, 1888. Diameter, 2·9 mils, or 0·074 millim. Length, 18·08 centims.
Curve 5, fig. 8 (Plate 12).

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		° C.		° C.	
17	4·630	181·5	7·115	282·3	8·535
35·1	4·915	194·0	7·295		
49·8	5·145	202·3	7·415	15·8	4·611
59·6	5·295	211·7	7·545	69·2	5·445
69·5	5·445	215·9	7·605	105·3	5·995
79·1	5·595	224·2	7·725		
89·0	5·745	232·5	7·845	9·3	4·515
88·2	5·735	239·3	7·945	8·1	4·495
98·7	5·895	249·8	8·095	5·1	4·445
105·2	5·995	257·1	8·195	0·0	4·365
111·7	6·095	261·1	8·255	− 0·8	4·355
125·9	6·305	264·5	8·305	− 5·2	4·285
132·1	6·395	268·9	8·365	− 7·8	4·275
154·0	6·715	273·0	8·415	−11·0	4·225
159·8	6·805	274·9	8·445	−11·75	4·215
166·1	6·895	277·7	8·475	−12·5	4·205
173·3	6·995	279·0	8·495		
180·0	7·095	280·5	8·515		

The last set of results was obtained by placing the wire in alcohol and surrounding it with a freezing mixture.

July 3rd, 1888. Diameter, 2 mils, or 0·051 millim. Length about 10·31 centims.
Curve No. 6, fig. 8 (Plate 12).

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		° C.		° C.	
16·4	5·375	94·0	6·795	160·1	7·945
66·1	6·295	99·6	6·895	168·7	8·095
71·7	6·395	105·4	6·995	175·1	8·195
77·2	6·495	107·8	7·045	184·1	8·345
82·3	6·595	111·0	7·095	192·5	8·495
88·4	6·695	155·3	7·865	203·3	8·645

July 11th, 1888. Diameter, 1·2 mil, or 0·031 millim. Length about 12·17 centims.

Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.	Temperature.	Resistance, in ohms.
° C.		° C.		° C.	
15·15	15·169	85·7	18·52	157·3	21·81
37·6	16·25	96·2	19·01	163·7	22·09
53·2	16·99	102·5	19·31	174·9	22·59
66·1	17·59	109·0	19·62		
74·6	17·99	129·9	20·56		

III. *Results of Emissivity Experiments.*

To determine the temperature of any one of the wires used in the emissivity experiments from its observed resistance, we might proceed as follows : calculate the specific resistance and then determine from the preceding curves, 1, 2, 3, &c. (figs. 7 and 8), the temperature at which a piece of the same wire had the same specific resistance. To do this it would be, of course, necessary to know the lengths of the two pieces of the same wire used in the two sets of experiments.

Now, whereas in each of the emissivity experiments a straight piece of wire of about 28 centims. was employed, the length of which could be measured with considerable accuracy, it was only possible to use shorter pieces for the experiments on the variation of resistance with temperature, for the latter pieces were parts of the wires that had actually been used in the emissivity experiments, and detaching a piece of wire from the apparatus seen in figs. 1 to 4, and attaching it to the clamp shown in figs. 5 and 6, necessarily shortened its length.

Since, therefore, there was this greater difficulty in measuring the exact lengths of the wires used in the experiments recorded in Tables I. to V., curves 1, 2, 3, &c., figs. 7 and 8, it was decided to regard these experiments as giving the relative resistances of each particular wire at different temperatures, but not the specific resistance at any particular temperature.

In order then to use these curves of relative resistance with temperature 1, 2, 3, &c., for the purpose of determining the temperature of the wires in the emissivity experiments, we must know the resistance of each of the latter wires at some one temperature. This might have been ascertained by means of the Wheatstone's bridge, but, as a relative resistance rather than absolute resistance was required, the following method was employed instead, since it avoided the possible introduction of an error that might have arisen from some want of agreement in the unit of resistance of the Wheatstone's bridge and the units of current and P. D. The ratio of P. D. to current, or the resistance of the wire, having been measured for a number of currents, a curve was plotted connecting the resistance of the wire for various currents flowing through it, and by continuing this curve until it cut the axis along which resistance was measured we obtained the limiting value of the ratio P. D. to current for current nought, that is, the resistance of the wire when at the temperature of the enclosure. Fig. 9 (Plate 13) shows the curve thus obtained for the thickest wire, viz., that of 14 mils, and similar curves were drawn for all the other wires for the purpose of ascertaining their respective resistances when at the temperature of the enclosure.

Let l and d be the length and diameter of the wire in centimetres, t_0 be the temperatures of the water jacket surrounding the enclosure cylinder *ABCD* (figs. 1 and 2); t be the temperature of the wire when A amperes are passing through it, and when V volts is the P. D. between the ends of the wire, e be the emissivity, that is the number of calories (gramme C.) lost per second per square centimetre of surface per 1° C. excess temperature, then

$$e = \frac{0{\cdot}239\ \mathrm{AV}}{\pi\,.\,l\,.\,d\,(t - t_0)},$$

l and d are not, however, absolute constants for a given wire, since as the wire gets hotter and hotter its length and diameter become larger and larger. If γ be the coefficient of linear increase of platinum per 1° C., which, according to KOHLRAUSCH, is 0·000009, then allowing for the lengthening and thickening of the wire as it increases in temperature, we have, if l and d be the values at 15° C.—

$$e = \frac{(1 + 15\gamma)^2}{(1 + \gamma t)^2} \cdot \frac{0{\cdot}239\ \mathrm{AV}}{\pi\,.\,l\,.\,d\,(t - t_0)}$$

$$= \frac{0{\cdot}076094}{l\,.\,d} \cdot \frac{\mathrm{AV}}{(1 + \gamma t)^2\,(t - t_0)}.$$

Now $(1 + \gamma t)^2 (t - t_0)$ may be regarded as equal to $(t - t_0) + \Delta\,(t - t_0)$ where $\Delta\,(t - t_0)$ is a correction that must be added to the observed excess temperature $(t - t_0)$, the amount of this correction depending on the value of t. The values of $(1 + \gamma t)^2\ (t - t_0)$ and the amounts of the correction for different values of t are given in the following Table VI., and are shown plotted in fig. 10 (Plate 13). The corrections are but small, but they have nevertheless been taken into account in determining the values of the emissivity given in the following Tables VII., VIII., and IX.

TABLE VI.

Temperature t.	$(1 + \gamma t)^2\ (t - t_0)$.	$\Delta\ (t - t_0)$.
° C.		
15	0·500065	0·00014
30	15·5084	0·008
50	35·532	0·032
70	55·57	0·07
85	70·61	0·11
100	85·65	0·15
120	105·73	0·23
140	125·82	0·32
160	145·92	0·42
180	166·04	0·54
200	186·17	0·67
240	226·48	0·98
280	266·84	1·34
300	287·04	1·54

TABLE VII.

Diameter, 14 mils, or 0·356 millim. Length, 28·09 centims. Water-jacket, 14°·5 C.
May 3rd, 1888. Fig. 11. (Plate 14.)

Current, in amperes.	P.D., in volts.	Resistance, in ohms.	Equivalent temperature of wire.	Emissivity.
			° C.	
0·05	0·01584	0·3168		
0·075	0·02395	0·3195		
0·10	0·03198	0·3198		
0·15	0·04853	0·3235		
0·20	0·06386	0·3193		
0·25	0·07900	0·3160		
0·30	0·09464	0·3155		
0·40	0·1268	0·3170		
0·50	0·1602	0·3204	17·4	·002104
0·60	0·1939	0·3232	19·7	·001705
0·80	0·2627	0·3284	24·6	·001585
1·00	0·3349	0·3349	30·0	·001645
1·20	0·4091	0·3409	35·7	·001764
1·50	0·5334	0·3556	50·4	·001698
1·80	0·6667	0·3700	64·5	·001825
2·10	0·8261	0·3934	87·0	·001820
2·40	1·004	0·4182	111·0	·001899
2·70	1·201	0·4449	137·4	·002005
3·00	1·439	0·4796	172·5	·002074
3·40	1·815	0·5338	226·6	·002207
3·80	2·291	0·6029	304·0	·002278

The first eight observations in the above Table were made to enable the early part of the curve in fig. 9 to be plotted, so as to find the resistance of the wire that corresponded with current nought; and this the curve shows to be 0·318 ohm. The wire, then, at the temperature of the water-jacket, viz., 14°·5 C., has this resistance, 0·318 ohm.

Table VIII.

Diameter, 11·1 mils, or 0·282 millim. Length, 28·35 centims. Water-jacket, 15° C
May 2nd, 1888. Fig. 11.

Current, in amperes.	P.D., in volts.	Resistance, in ohms.	Equivalent temperature of wire.	Emissivity.
			° C.	
0·05	0·02817	0·5634		
0·075	0·04241	0·5655		
0·10	0·05655	0·5655		
0·15	0·08532	0·5688		
0·20	0·1133	0·5665		
0·25	0·1415	0·5660		
0·30	0·1694	0·5647		
0·40	0·2260	0·5650		
0·50	0·2867	0·5734		
0·60	0·3479	0·5798	25·6	·002054
0·80	0·4782	0·5978	37·3	·001705
1·00	0·6148	0·6148	48·7	·001787
1·20	0·7583	0·6319	60·0	·001965
1·50	1·007	0·6713	84·9	·002081
1·80	1·307	0·7261	121·1	·002123
2·10	1·667	0·7938	166·0	·002211
2·40	2·088	0·8700	215·4	·002379
2·70	2·587	0·9581	273·7	·002563

As in the case of Table VII., the first nine observations in Table VIII. were made only for the purpose of enabling the early part of the curve in fig. 9 to be drawn. In the following Table IX., which gives the results for the remaining wires, the first eight or nine observations that were made in each case are omitted, since, although these observations were of value in enabling the early part of the curve, such as is shown in fig. 9, to be drawn for each wire, the difference between the temperature of the wire and of the water-jacket was too small to enable the corresponding emissivities to be accurately calculated.

Table IX.

Diameter, 9·3 mils, or 0·236 millim. Length, 27·86 centims. Water-jacket, 15°·9 C.
April 28th, 1888. Fig. 11.

Current, in amperes.	P.D., in volts.	Resistance, in ohms.	Equivalent temperature of wire.	Emissivity.
			° C.	
0·60	0·5193	0·866	40·9	·001441
0·80	0·7078	0·885	49·9	·001926
1·00	0·9206	0·921	66·9	·002087
1·20	1·1554	0·963	86·1	·002281
1·40	1·423	1·016	113·6	·002353
1·70	1·895	1·115	164·8	·002495
2·00	2·484	1·242	232·4	·002642
2·30	3·186	1·385	310·0	·002865

TABLE IX.—continued.

Diameter, 8·1 mils, or 0·206 millim. Length, 27·86 centims. Water-jacket, 12°·3 C. April 26th, 1888. Fig. 11.

Current, in amperes.	P.D., in volts.	Resistance, in ohms.	Equivalent temperature of wire.	Emissivity.
			° C.	
0·50	0·5200	1·040	24·8	·002761
0·60	0·6366	1·061	31·2	·002682
0·70	0·7589	1·084	38·6	·002681
0·80	0·8922	1·115	48·0	·002654
0·90	1·0336	1·148	58·0	·002696
1·00	1·1855	1·186	69·5	·002747
1·20	1·5423	1·285	101·3	·002755
1·40	1·971	1·408	141·8	·002822
1·60	2·478	1·549	188·7	·002974
1·80	3·093	1·718	242·0	·003206
2·00	3·792	1·896	309·0	·003375

Diameter, 6 mils, or 0·152 millim. Length, 27·46 centims. Water-jacket, 12° C. April 24th, 1888. Fig. 11.

Current, in amperes.	P.D., in volts.	Resistance, in ohms.	Equivalent temperature of wire.	Emissivity.
			° C.	
0·075	0·1283	1·711	15·2	·000547
0·10	0·1716	1·716	16·4	·000709
0·15	0·2600	1·734	19·3	·000972
0·20	0·3488	1·744	20·8	·001442
0·25	0·4390	1·756	23·0	·001814
0·30	0·5334	1·778	26·6	·001993
0·40	0·7238	1·810	32·6	·002556
0·50	0·9414	1·883	45·9	·002525
0·60	1·179	1·965	60·2	·002663
0·70	1·449	2·069	78·2	·002781
0·80	1·744	2·180	97·8	·002954
0·90	2·093	2·326	125·4	·003013
1·00	2·466	2·466	150·8	·003222
1·20	3·398	2·832	219·5	·003560
1·40	4·620	3·300	314·0	·003873

TABLE IX.—continued.

Diameter, 4 mils, or 0·102 millim. Length, 27·69 centims. Water-jacket 13° C.
April 14th, 1888. Fig. 12. (Plate 14.)

Current, in amperes.	P.D., in volts.	Resistance, in ohms.	Equivalent temperature of wire.	Emissivity.
			° C.	
0·04102	0·30934	7·540	17·9	·000701
0·06154	0·46555	7·565	20·4	·001047
0·08205	0·61868	7·540	17·9	·002803
0·10256	0·77591	7·565	20·4	·002909
0·12307	0·93418	7·590	23·0	·003110
0·14358	1·0894	7·587	22·4	·004502
0·17435	1·3319	7·639	27·8	·004245
0·20512	1·5600	7·605	24·4	·007593
0·24614	1·8951	7·699	33·8	·006067
0·28717	2·2301	7·766	40·9	·006210
0·32819	2·5877	7·884	52·4	·005831
0·36922	2·9464	7·980	61·9	·006006
0·41024	3·3246	8·104	74·6	·005980
0·46152	3·8179	8·272	91·9	·006034
0·5128	4·347	8·477	113·3	·006000
0·5641	4·892	8·655	132·0	·006258
0·6154	5·498	8·935	162·6	·006010
0·6666	6·141	9·211	193·0	·006132
0·7179	6·809	9·483	222·7	·006282
0·8205	8·340	10·164	298·9	·006441

We have already drawn attention to the fact that the above wire, of 4 mils or 0·102 millim. in diameter, has a much smaller variation of resistance with temperature than the other wires, and therefore that it is probable that this wire has some iridium or silver in its composition. From the preceding table we see that the variation of emissivity with temperature is also much smaller with this wire than with the others.

TABLE IX.—continued.

Diameter, 2·9 mils, or 0·074 millim. Length, 28·85 centims. Water-jacket, 15°·6 C.
May 4th, 1888. Fig. 12.

Current, in amperes.	P.D., in volts.	Resistance, in ohms.	Equivalent temperature of wire.	Emissivity.
			° C.	
0·020	0·1614	8·070	19·1	·000330
0·030	0·2426	8·087	19·7	·000635
0·040	0·3268	8·170	22·8	·000650
0·050	0·4120	8·241	25·3	·000760
0·075	0·6266	8·266	26·2	·001587
0·100	0·8471	8·471	33·5	·001694
0·150	1·312	8·748	44·1	·002472
0·200	1·817	9·083	56·5	·003181
0·250	2·369	9·476	72·0	·003753
0·300	2·989	9·965	89·5	·004339
0·350	3·699	10·569	112·5	·004774
0·400	4·574	11·435	146·2	·005001
0·450	5·587	12·415	185·4	·005283
0·500	6·817	13·634	234·2	·005559
0·550	8·271	15·038	293·5	·005829

Diameter, 2 mils, or 0·051 millim. Length, 24·46 centims. Water-jacket, 13° C.
April 4th, 1888. Fig. 12.

Current, in amperes.	P.D., in volts.	Resistance, in ohms.	Equivalent temperature of wire.	Emissivity.
			° C.	
0·06155	0·7904	12·842	19·0	·004965
0·08206	1·0690	13·026	23·6	·005068
0·10258	1·3619	13·277	29·4	·005217
0·13335	1·8296	13·710	39·4	·005660
0·16413	2·3281	14·185	50·2	·006291
0·19490	2·8615	14·682	61·7	·006999
0·22568	3·4639	15·349	77·2	·007445
0·25645	4·1443	16·160	96·3	·007804
0·29748	5·1115	17·183	121·1	·008598
0·33851	6·532	19·296	173·4	·008416
0·38980	8·351	21·424	227·0	·009268
0·44109	10·834	24·561	306·0	·009934

TABLE IX.—continued.

Diameter, 1·2 mil, or 0·031 millim. Length, 25·10 centims. Water-jacket, 10°·5 C
April 13th, 1888. Fig. 12.

Current, in amperes.	P.D., in volts.	Resistance, in ohms.	Equivalent temperature of wire.	Emissivity.
			° C.	
0·01026	0·31863	31·062	13·0	·001301
0·02052	0·64035	31·212	14·54	·003236
0·03077	0·96618	31·396	16·5	·004930
0·04103	1·3054	31·814	20·8	·005173
0·06155	2·0177	32·782	30·4	·006208
0·08206	2·7598	33·630	39·3	·007823
0·10285	3·5564	34·669	49·5	·009306
0·12310	4·4660	36·281	66·1	·009819
0·14361	5·4784	38·148	85·0	·010492
0·16413	6·6502	40·518	109·3	·010968
0·18464	7·9247	42·919	135·4	·011627
0·20516	9·4305	45·967	168·0	·012182
0·23593	12·0567	51·102	224·5	·013168
0·26671	15·1916	56·960	291·2	·014284

Using the values of temperature and emissivity given in the preceding Tables VII., VIII., and IX., the curves shown in fig. 11 have been drawn for the five thicker wires, viz., those having diameters of 14 mils, or 0·356 millim. ; 11·1 mils, or 0·282 millim. ; 9·3 mils, or 0·236 millim. ; 8·1 mils, or 0·206 millim. ; and 6 mils, or 0·152 millim., respectively ; and the curves shown in fig. 12 have been similarly drawn for the four finer wires, viz., those having diameters of 4 mils, or 0·102 millim. ; 2·9 mils, or 0·074 millim. ; 2 mils, or 0·051 millim. ; and 1·2 mil, or 0·031 millim., respectively.

The curves for the four finest wires on fig. 12 and for two of the thicker wires on fig. 11, appear to be fairly regular throughout their whole length, while the curves for the 14, the 11·1, and the 8·1 mils wires appear doubtful for temperatures below 60° C.

On examining the curves we see that :—

1. For any given temperature the emissivity is the higher the finer the wire.

2. For each wire the emissivity increases with the temperature, and the rate of increase is the greater the finer the wire. For the finest wire the rate of increase of emissivity with temperature is very striking.

3. Hence the effect of surface on the total loss of heat (by radiation and convection) per second per square centim. per 1° C. excess temperature increases as the temperature rises.

The wire of 4 mils diameter appears to be an exception to these rules ; but we have already seen reasons for believing that this wire was not drawn from pure platinum like the rest, but possibly from platinum-iridium or platinum-silver.

On comparing the loss of heat from the wire of 1·2 mil diameter when at 300° C. with that from the wire of 6 mils diameter when at 15° C., both being in an enclosure at 10° C., we see that the former loses per square centimetre per second, not

$$\frac{300 - 10}{15 - 10} \text{ or } 58 \text{ times}$$

as much heat as the latter (which it would do if the emissivity were the same) but instead,

$$60 \times 58 \text{ or } 3480 \text{ times}$$

as much heat; arising from the fact that the emissivity, that is, the number of calories (gramme C.°) lost per second per square centim. of surface per 1° C. excess temperature of the 1·2 mil wire at 300° C., is 60 times as great as that of the 6 mils wire at 15°, the latter varying very rapidly with the temperature near 15° C.

From the curves on figs. 11 and 12 the following table has been drawn up giving the emissivities of the various wires at eight useful even temperatures.

TABLE X.—Emissivities at—

Diameter of wire in		40° C.	60° C.	80° C.	100° C.	150° C.	200° C.	250° C.	300° C.
Mils.	Millims.								
1·2	0·031	0·00823	0·00956	0·01030	0·01085	0·01187	0·01278	0·01362	0·01440
2·0	0·051	0·00595	0·00686	0·00750	0·00790	0·00860	0·00907	0·00948	0·00985
2·9	0·074	0·00219	0·00334	0·00409	0·00455	0·00509	0·00538	0·00563	0·00584
6·0	0·152	0·00246	0·00266	0·00280	0·00293	0·00321	0·00346	0·00367	0·00384
8·1	0·206	..	..	..	0·00280	0·00294	0·00308	0·00322	0·00335
9·3	0·236	..	..	..	0·00230	0·00245	0·00259	0·00272	0·00284
11·1	0·282	..	..	..	0·00205	0·00222	0·00236	0·00249	0·00261
14·0	0·356	..	..	..	0·00189	0·00203	0·00214	0·00222	0·00229

The next step taken was to express mathematically the law connecting the emissivity of a wire with its diameter for a fixed temperature, and this was done for three fixed temperatures, viz., 100° C., 200° C., and 300° C. Using the values of the emissivity contained in the three columns headed with these temperatures in Table X. we obtained :—

$$\text{For } 100^\circ \text{ C.}, \; e = 0·001036 + 0·012078\, d^{-1} \quad . \quad . \quad . \quad . \quad . \quad (5),$$

$$,, \quad 200^\circ \text{ C.}, \; e = 0·001111 + 0·014303\, d^{-1} \quad . \quad . \quad . \quad . \quad . \quad (6),$$

$$,, \quad 300^\circ \text{ C.}, \; e = 0·001135 + 0·016084\, d^{-1} \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where d is the diameter of the wire in mils or thousandths of an inch.

The following Tables, XI., XII., and XIII., give the actual emissivities, the emissitives calculated by means of the three preceding formulæ respectively, the differences, and the percentage differences.

TABLE XI.—Wire at 100° C.

Diameter of wire.		Actual emissivity.	Calculated emissivity.	Difference.	Percentage difference.
mils.	millims.				
1·2	0·031	0·01085	0·01104	+ 0·00019	+ 1·8
2·0	0·051	0·00790	0·00707	− 0·00083	− 11·7
2·9	0·074	0·00455	0·00520	+ 0·00065	+ 12·5
6·0	0·152	0·00293	0·00305	+ 0·00012	+ 3·9
8·1	0·206	0·00280	0·00253	− 0·00027	− 10·9
9·3	0·236	0·00230	0·00233	+ 0·00003	+ 1·6
11·1	0·282	0·00205	0·00212	+ 0·00007	+ 3·3
14·0	0·356	0·00189	0·00190	+ 0·00001	+ 0·5

TABLE XII.—Wire at 200° C.

Diameter of wire.		Actual emissivity.	Calculated emissivity.	Difference.	Percentage difference.
mils.	millims.				
1·2	0·031	0·01278	0·01303	+ 0·00025	+ 1·9
2·0	0·051	0·00907	0·00826	− 0·00081	− 9·8
2·9	0·074	0·00538	0·00604	+ 0·00066	+ 11·0
6·0	0·152	0·00346	0·00349	+ 0·00003	+ 1·0
8·1	0·206	0·00308	0·00288	− 0·00020	− 6·9
9·3	0·236	0·00259	0·00265	+ 0·00006	+ 2·4
11·1	0·282	0·00236	0·00240	− 0·00004	+ 1·5
14·0	0·356	0·00214	0·00213	− 0·00001	+ 0·4

TABLE XIII.—Wire at 300° C.

Diameter of wire.		Actual emissivity.	Calculated emissivity.	Difference.	Percentage difference.
mils.	millims.				
1·2	0·031	0·01440	0·01454	+ 0·00014	+ 1·0
2·0	0·051	0·00985	0·00918	− 0·00067	− 7·3
2·9	0·074	0·00584	0·00668	+ 0·00084	+ 12·6
6·0	0·152	0·00384	0·00382	− 0·00002	− 0·5
8·1	0·206	0·00335	0·00312	− 0·00023	− 7·4
9·3	0·236	0·00284	0·00286	+ 0·00002	+ 0·7
11·1	0·282	0·00261	0·00258	− 0·00003	− 1·2
14·0	0·352	0·00229	0·00228	− 0·00001	− 0·4

The values of the emissivity given in the last table are plotted in the curve shown in fig. 13 (Plate 15).

The statement not unfrequently made that the current required to maintain a wire of a given material at a given temperature above that of the surrounding envelope is proportional to the diameter of the wire raised to the power three halves, is equivalent to stating that the emissivity is independent of the diameter. Now if we may assume that the three formulæ (5), (6), (7) given above for e may be used not merely for the platinum wires when fine, but also as giving at any rate a rough approximation of the value of e when the wire is as much as 4 or 5 millims. thick, we may conclude that for a temperature of 100° C. the value of d in the formula

$$e = 0·001036 + 0·012078\ d^{-1}$$

must be something of the order 220 mils, or 5·6 millims., so that the neglect of the second term may not make an error in e of more than 5 per cent., and something of the order 1·15 inch, or 29·3 millims., if the error is not to exceed 1 per cent.; that for a temperature of 200° C., the value of d in the formula

$$e = 0·001111 + 0·014303\ d^{-1}$$

must be something of the order 244 mils, or 6·2 millims., so that the neglect of the second term may not make an error in e of more than 5 per cent., and something of the order 1·28 inch, or 32·5 millims., if the error is not to exceed 1 per cent. ; and that for a temperature of 300° C., the value of d in the formula

$$e = 0·001135 + 0·016084\ d^{-1}$$

must be something of the order 267 mils, or 6·8 millims., so that the neglect of the second term may not make an error in e of more than 5 per cent, and something of the order 1·39 inch, or 35·3 millims., if the error is not to exceed 1 per cent.

Generally, then, although it may be possible to obtain only very rough approximations of the values of the emissivities of thick wires by using the three formulæ that we have deduced from the experiments on thin wires, still it follows that to assume that the emissivity is a constant for wires whose diameters vary from a small value up to 1 inch is to make a large error in the case of the greater number of the wires, and an error of hundreds per cent. in the case of some of them.

The formulæ (5), (6), (7) given above have been calculated from the results of experiments made on wires varying from 1·2 to 14 mils, and the method of calculation employed makes the percentage difference between the observed and calculated emissivity very small at each end of the range as well as in the middle. We may, therefore, use the formula to obtain some idea of what the emissivity is likely to be for a wire somewhat smaller than that used in the experiments, say, of 0·75 mil in diameter. Using formula (7) to obtain the emissivity at 300° C., we find it to be 0·02258. We can now make an approximate estimate of the current density, or amperes per square centimetre, it would be necessary to employ with a platinum wire of 0·75 mil in order to keep it at a temperature of 300° C., when the enclosure was, say, at 15° C.

From the tables that we have given of the resistance of platinum wire at different temperatures, we see that the resistance of 17 centims. of wire 0·152 millim. in diameter is about 2 ohms at 300° C. Hence the resistance of 17 centims. of wire 0·75 mil., or 0·0191 millim. in diameter, will be about

$$2\left(\frac{0{\cdot}152}{0{\cdot}0191}\right)^2, \text{ or } 128{\cdot}0 \text{ ohms.}$$

If, therefore, x be the required current density, the rate of production of heat in calories (gramme-centigrade) per second will be

$$\left\{x \times \pi \times \left(\frac{0{\cdot}00191}{2}\right)^2\right\}^2 \times 128{\cdot}0 \times 0{\cdot}239,$$

which must equal the number of calories lost per second, viz.,

$$17 \times \pi \times 0{\cdot}00191 \times 0{\cdot}02258 \times 285, \text{ or } 0{\cdot}6546,$$

therefore, $x = 51330$ amperes per square centimetre. If the wire, instead of being platinum, had been made of copper, and if the emissivity for a copper wire, 0·75 mil in diameter, were the same as that of a platinum wire of the same diameter, the current density that would be required to keep the copper wire at a temperature of 300° C., when the enclosure was at 15° C., would be about

$$51330\sqrt{5{\cdot}7},$$

that is, 122,600 amperes per square centimetre, or 790,500 amperes per square inch.

[It may be useful to give the general formula which we have arrived at for calculating the current A amperes, required to be sent through any long thin platinum wire d mils in diameter to maintain it at a temperature of 300° C. when the enclosure (having about the dimensions shown in fig. 1) is maintained at a temperature of 15° C.

From the various measurements given in the paper it follows that the resistance per cubic mil of platinum at 300° C. is about 0·0086 ohm. Hence, equating the rate of production and loss of heat per mil length of the wire, we have

$$\frac{A^2 \times 0{\cdot}0086 \times 4 \times 0{\cdot}24}{\pi d^2} = \frac{\pi d \times 285}{(393{\cdot}7)^2}(0{\cdot}001135 + 0{\cdot}016084\, d^{-1}),$$

therefore,

$$A = 1{\cdot}483\sqrt{(0{\cdot}001135\, d^3 + 0{\cdot}016084\, d^2)} \text{ about.}$$

Also, if the wire having the conditions described above be l inches long, the watts that must be expended on it equal approximately $(0{\cdot}3869 + 0{\cdot}02732\, d)\, l$.

IV. *Comparison of the Results with those obtained by MM.* DULONG *and* PETIT.

All the emissivity curves in figs. 11 and 12 (Plate 14) have the same general shape, being concave to the axis of temperature, and it is interesting to see how far this general shape agrees with the results obtained in the classical researches of MM. DULONG and PETIT for the loss of heat from thermometer bulbs in air.

The formulæ they developed for the loss of heat by radiation and convection in air lead to the following expression for the emissivity :—

$$e = \frac{1}{t}\{Ha^{\theta}(a^{t} - 1) + Kp^{0·45}t^{1·2333}\},$$

where θ is the temperature of the enclosure, t the excess temperature of the cooling body, a a constant having the value 1·0077 if the temperature be measured in degrees Centigrade, p the pressure of the gas, H a constant depending on the nature of the surface of the cooling body and of the enclosure, and K a constant depending mainly on the nature of the gas surrounding the cooling body and but very slightly on the nature of the surface of the cooling body.

Substituting the value of a and expanding in powers of t we have

$$\begin{aligned} e = {} & 0·0076705 \times 1·0077^{\theta}.H.\{1 + 0·00383526\, t + 0·000009806\, t^{2} \\ & + 0·0000000188\, t^{3} + 0·0000000000029\, t^{4} + \ldots\} \\ & + K.p^{0·45}.t^{0·233}. \end{aligned}$$

If y be used to stand for the expression in the brackets and y' for $t^{0·233}$, calculation shows that y and y' have the following values for the different values of t :—

Values of		
t.	y.	y'.
0	1	0
10	1·03935	1·7100
100	1·50067	2·9242
200	2·31430	3·4367
300	3·56405	3·7772

And using for H and K the values found by Mr. HOPKINS for polished limestone cooling in air at 760 millims. pressure, contained in an enclosure at 0° C., the expression given above for e becomes

$$e = \lambda(0\cdot069\,y + 0\cdot038\,y'),$$

hence, substituting the values given in the last table for y and y', we have

Values of	
t.	e/λ.
0	0·069
10	0·137
100	0·222
200	0·291
300	0·389

Consequently, if the law connecting the loss of heat from thermometer bulbs in air with the difference of temperature between the bulb and the enclosure is of the same nature as that connecting the loss of heat from very fine horizontal wires in air with the difference of temperature, we should expect to find that the curve connecting the values of e/λ and t in the last table would, when plotted, be everywhere *concave* to the axis of t. But this is not the case, for we find on plotting this curve that while it resembles our curves in being concave to the axis of t, for values of t less than about 200° C. it changes its curvature at about this point and becomes distinctly convex.

It is, of course, to be remembered the maximum value of t in the experiments of MM. DULONG and PETIT was 240° C., while in some of ours t exceeded 300° C. We are, however, inclined to attribute the inability of the formulæ of MM. DULONG and PETIT to give even the general shape of the curves which we have obtained to the fact that the convection which occurs with thermometer bulbs hardly suggests the very great convective cooling that experiments show to occur with very fine wires at high temperatures.—May 31, 1892.]

V. *Calculation of the Distribution of Temperature along a Platinum Wire Heated by an Electric Current.*

Let d be the diameter of the wire in centimetres.

t ,, temperature in degrees centigrade at any point of the wire distant x centimetres from the nearer of the two supports to which the ends of the wire are attached.

t_0 ,, temperature of the supports.

$f(t)$,, electrical resistance, in ohms, of a cubic centimetre of the wire at a temperature t.

$\phi(t)$,, thermal resistance of a cubic centimetre of the wire at a temperature t.

$\psi(t)$ be the emissivity at a temperature t for a wire of diameter d.
K ,, number of calories corresponding with one watt-second.
A ,, current in amperes flowing through the wire;

then following the method of reasoning employed with such problems, we have

$$\frac{\pi d^2}{4}\frac{d}{dx}\left(\frac{\frac{dt}{dx}}{\phi(t)}\right)+\frac{4}{\pi d^2}\mathrm{KA}^2 f(t)=\pi d\,(t-t_0)\,\psi(t) \quad . \quad . \quad . \quad . \quad . \quad (8).$$

$\phi(t)$ is in reality the reciprocal of the amount of heat in calories that would flow per second across one cubic centimetre of the material for 1° C. difference of temperature between the opposite faces and for a mean temperature of the material of $t°$.

$\psi(t)$ is the number of calories lost per second on account of radiation and convection for 1° C. excess temperature from a square centimetre of the platinum wire of diameter d and at a temperature t.

In the preceding equation d is strictly the diameter of each part of the wire at the particular temperature it is at. As, however, an increase of the temperature from 0° C. up to 300° C. only increases the linear dimensions of platinum by about 0·26 per cent., d may be taken as the diameter of the wire at 15° C.

$f(t)$ and $\psi(t)$ are known from the various curves connecting resistance with temperature and emissivity with temperature for each of the various wires experimented on. The variation of the thermal resistance of platinum with temperature has not, as far as we can learn, been experimentally examined, nor does it appear to be even known whether the thermal resistance of platinum increases with temperature, as does the thermal resistance of iron, or diminishes with the rise of temperature, as does the thermal resistance of copper and German silver. Under these circumstances we decided to assume that the thermal resistance of platinum was a constant, and had the value it is known to possess at ordinary temperatures, and to see what sort of result a mathematical assumption based on this result would lead to.

As a matter of fact, even the thermal conductivity of platinum at ordinary temperatures is not stated explicitly in books, but it can be easily arrived at indirectly. For we find that WIEDEMANN and FRANZ determined that

$$\frac{\text{Thermal conductivity of platinum}}{\text{Thermal conductivity of copper}}=\frac{8{\cdot}2}{77{\cdot}2},$$

and in the article "Heat" in the 'Encyclopædia Britannica,' the value of the thermal conductivity of copper, 0·96 as determined by ÅNGSTRÖM, is stated to be trustworthy. From these numbers we deduce that the thermal resistance of platinum, at ordinary temperatures, is 9·858, which is the value we have taken for $\phi(t)$.

Equation (8) can then be written in the form

$$\frac{d^2t}{dx^2} = \mathrm{P}\,(t - t_0)\,\psi\,(t) - \mathrm{Q}f\,(t)\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where

$$\mathrm{P} = \frac{4 \times 9{\cdot}858}{d},$$

and

$$\mathrm{Q} = \frac{4\mathrm{KA}^2}{\pi d^2} \cdot \frac{4 \times 9{\cdot}858}{\pi d^2},$$

P and Q are therefore constants for a particular wire with a particular current flowing through it.

We next tried to expand $\psi(t)$ and $f(t)$ as functions of t from the curves which we had experimentally obtained connecting emissivity with temperature and resistance with temperature, but we found that it was not possible to express these functions of t in any such simple shape as would allow the integration of equation (9) to be effected analytically, and a result obtained suitable to be used for easily determining the value of t for any value of x. We, therefore, consulted Professor HENRICI regarding the integration of equation (9) in a practical form, and we have to express our thanks to him for the warm interest that he has taken in this mathematical problem, and for the many suggestions that he has kindly made, and which have enabled us to arrive at the following solution. We have also to thank one of our assistants, Mr. WALKER, for carrying out the graphical and numerical calculations contained in this section of the paper.

It is clear that the law of distribution of temperature along the wire will depend on the diameter of the wire among other things, also on the current passing through it, the variation of temperature with length of wire being the more rapid the thicker the wire and the greater the current passing through it. We therefore selected for consideration a wire of mean diameter, namely, that of 6 mils or 0·152 millim., and we took the case when 1·4 ampere was passing through it, which is the greatest current that was passed through this wire in the emissivity experiments.

Having selected this wire and current, the next step consisted in calculating $\mathrm{P}\,(t - t_0)\,\psi\,(t)$ and $\mathrm{Q}f\,(t)$ for different values of t. t_0 is 12° C., P is 2587·4, and the value of $\psi\,(t)$ may be obtained from the emissivity curve for the 6 mils or 0·152 millim. wire given in fig. 11. Instead of calculating $f(t)$ the resistance per cubic centimetre of the material for different values of (t) it is more convenient to write

$$\mathrm{Q}f(t) \text{ as } \frac{\mathrm{KA}^2}{l} \cdot \frac{4 \times 9{\cdot}858}{\pi d^2}\,\mathrm{F}\,(t),$$

where l is the length, 17·07 centims., of this 6-mil wire that was used in the experiments from the results of which the curve in fig. 14 (Plate 15) has been drawn, and

F (t) is the resistance in ohms of this 17·07 centims. of this 6-mil wire at any particular temperature, t. Now, when A equals 1·4 ampere, and d equals 0·152 millim., and l is 17·07 centims.,

$$\frac{\mathrm{KA}^2}{l} \cdot \frac{4 \times 9{\cdot}858}{\pi d^2} = 1491{\cdot}7\ ;$$

therefore, the equation (9) becomes

$$\frac{d^2t}{dx^2} = 2587{\cdot}4\,(t - 12)\,\psi(t) - 1491{\cdot}7\ \mathrm{F}(t) \quad . \quad . \quad . \quad . \quad . \quad (10).$$

To obtain a curve representing the first term on the right-hand side of this equation, the curve for $\psi(t)$, given for the 6-mil wire in fig. 11, was altered by simple geometrical construction so as to give a curve for $t \times \psi(t)$. The ordinates of this curve were then altered, graphically, so as to give to a convenient scale a curve for $2587{\cdot}4\ t\,.\,\psi(t)$, and, lastly, the ordinates of this curve were reduced, graphically, in the proportion of $t - 12$ to t.

To obtain the curve representing the second term on the right-hand side of equation (10), the ordinates of the curve on fig. 14 were multiplied by 1491·7, and then the curve was re-drawn to the same scale as the curve representing $2587{\cdot}4\,(t - 12)\,\psi(t)$. The ordinates of the two curves were then subtracted from one another, and a curve, having for its ordinates the difference of the ordinates of the last two curves, was drawn, which gives the values of d^2t/dx^2 for any value of t. This curve is seen in fig. 15 (Plate 15), it is parabolic in shape, cuts the axis along which temperature is reckoned at t equals 315° C., and has its vertex approximately in the line along which d^2t/dx^2 is reckoned.

A similar investigation was made for the same wire for a current of 0·6 ampere, and it was found that the curve for d^2t/dx^2 in this case cut the axis of temperature at 60°·3 C.

The fact that d^2t/dx^2 is nought at a particular temperature tells us, of course, mathematically nothing about the actual value of dt/dx, but from our general knowledge of temperature curves, we know that when a current is passing through a fine wire, as in our experiments, the temperature will rise rapidly along the wire in the neighbourhood of the supports, then rise more slowly, and at no great distance from the supports the temperature curve will become nearly flat, and will be absolutely flat over the middle portion of the wire.

We are, therefore, justified in assuming that d^2t/dx^2 and dt/dx are nought at about the same point of the wire.

On examining the numbers in Table V., which refer to the 6-mil wire, it will be noticed that the mean temperatures of the wire for currents of 1·4 and 0·6 ampere respectively, were 314° C. and 60°·2 C., which are almost exactly the temperatures for which d^2t/dx^2 are nought for these two currents in question. It is certainly surprising that a calculation based on the assumption that the thermal resistance of platinum is the same at all temperatures between 0° C. and 300° C. as it is at ordinary temperatures, should have led to the result that, both for a current of 1·4 ampere and a current of 0·6 ampere passing through this 6-mil wire, d^2t/dx^2 should be nought, that is, the temperature curve should be flat at almost exactly the mean temperature that the wire had in each case. It would, therefore, appear that the assumption regarding the thermal resistance of platinum having a constant value 9·858 at different temperatures has not introduced any serious error.

If we assume that the vertex of the parabolic curve for d^2t/dx^2 is in the line along which d^2x/dt^2 is reckoned, which is very nearly the case, then

$$\frac{d^2t}{dx^2} = t^2 \cdot \frac{a}{\mathrm{T}} - a \qquad \text{(11).}$$

where a is minus the value of d^2t/dx^2 when x equals nought, and T is the value of t when d^2t/dx^2 equals nought.

At the point of the wire where the temperature is the highest the temperature curve will be flat, that is, dt/dx will be nought, and at the end of the wire where it is attached to the support the temperature will have some definitive value, T_0, in other words when

$$t = \mathrm{T}, \quad \frac{dt}{dx} = 0,$$

and when

$$x = 0, \quad t = \mathrm{T}_0.$$

With these two conditions only and without any reference to the length of the wire, it is possible to integrate the equation (11), and the result we arrive at is

$$x = \pm \sqrt{\frac{\mathrm{T}}{2a}} \cdot \log \frac{\sqrt{(t + 2\mathrm{T})} - \sqrt{3\mathrm{T}}}{\sqrt{(t + 2\mathrm{T})} + \sqrt{3\mathrm{T}}} \cdot \frac{\sqrt{(\mathrm{T}_0 + 2\mathrm{T})} + \sqrt{3\mathrm{T}}}{\sqrt{(\mathrm{T}_0 + 2\mathrm{T})} - \sqrt{3\mathrm{T}}} \qquad \text{(12).}$$

An examination of this equation shows that in order that t may equal T, x must equal infinity, and, therefore, in obtaining this integral we have tacitly assumed that the maximum temperature of the wire is only obtained at an infinite distance from its end. But while great simplicity is obtained by this hypothesis, no more error is introduced by its employment than is met with by the use of the ordinary equation for the conduction of heat, and which leads to the result that two bodies initially at a different temperature take an infinite time to arrive at thermal equilibrium, even when connected together by a good conductor of heat.

On examining the curve for d^2t/dx^2 on fig. 15, which as already stated has been calculated for the 6-mil wire for the maximum current used, viz., 1·4 ampere, we find that

$$a = 1592,$$

$$T = 315^\circ \text{ C.}$$

The temperature of the enclosure in the emissivity experiments with this wire was 12° C., and as the temperature T_0 of the point of the wire where it was joined to the support would be somewhat higher than this, we may take T_0 as about 15° C. It is to be noticed that the higher this temperature actually was the more uniform must have been the temperature of the whole wire and the more accurate will be our experiments on emissivity.

Substituting these values for a, T and T_0 in equation (12), we have

$$x = \pm 3148 \log \left(- 10{\cdot}58 \frac{\sqrt{(t + 630)} - \sqrt{945}}{\sqrt{(t + 630)} + \sqrt{945}} \right),$$

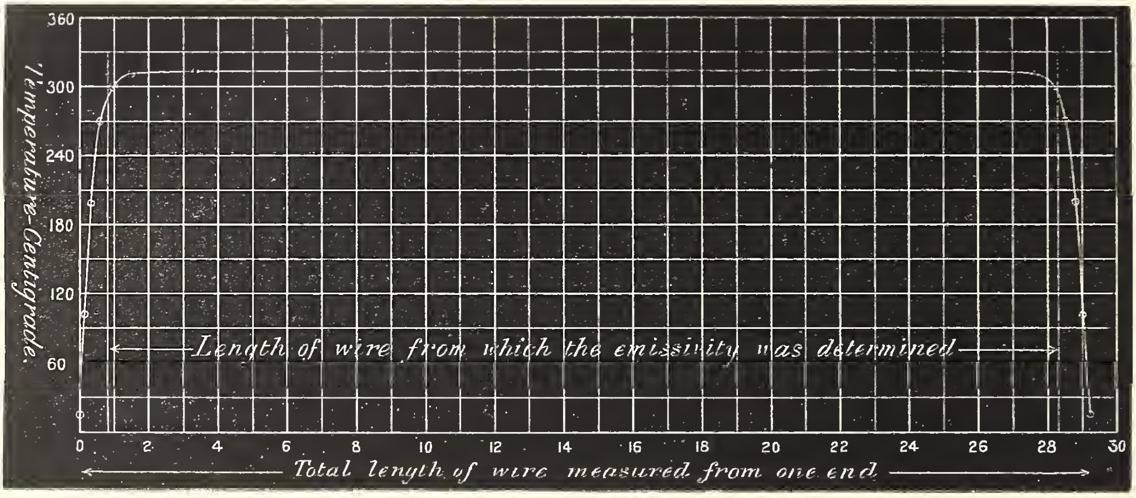
and the following are the values of t obtained for the corresponding values of x :—

Temperature, in degrees Centigrade.	Distance in centimetres from the end of the wire.
15	0
100	0·123
200	0·339
270	0·646
300	0·980
312	1·498
314	3·296

The actual length of this 6-mil wire between the points of support was 29·18 centims., and the maximum temperature was 315° C. Therefore, as the calculated temperature is 300° C. at a distance of 0·98 centim. from the support, it follows that in 3·3 per cent. of the wire, reckoning from the support, the temperature rises to about 95 per cent. of the value it has at the middle of the wire; while at a distance of 1·498 centim. from the support, that is, in about 5 per cent. of the whole length, the temperature has reached about 99 per cent. of its value at the middle.

The calculated distribution of temperature along this 6-mil wire, when traversed by the largest current employed with this wire, is shown in fig. 16. The distance between the voltmeter wires was 27·46 centims., as indicated in the figure; and it is with the distribution of temperature over this portion of the wire only that we have to deal in the emissivity experiments. Of this length, about 25·70 centims. is seen to be at a temperature a little over 314° C.; while 1·76 centim. has a mean temperature of about 308° C. The error, therefore, which we have made in assuming that the whole length of 27·46 was at 314° C. was not serious.

Fig. 16.



And, *à fortiori*, the lengths which were employed with all the finer wires in the emissivity experiments and the position of the voltmeter wires, were such that the temperature was practically constant over all the length of the wire used in the emissivity calculations.

In the case of the wires thicker than 6 mils, the error arising from the conduction was necessarily greater; and the calculations that we have made on the distribution of temperature along the 6-mil wire have led us to the conclusion that, if we were going to repeat the experiments, we should, in the case of the thickest wires employed, attach the voltmeter wires at about 2 centims. from the supports. We do not, however, think that any serious error has been introduced by attaching them at only about 7 millims., except, perhaps, in the case of the largest currents with the largest wires.

When plotting the curve for d^2t/dx^2 given in fig. 15, it was noticed that this curve, where it cuts the axis along which d^2t/dx^2 is reckoned, was not strictly parabolic. Professor HENRICI suggested that an idea might be obtained of the error introduced, by assuming the lower portion of this curve to be parabolic, if a tangent were drawn to the true curve for d^2t/dx^2 at about t equals 315° C., and if calculations were made on the assumption that this tangent line were itself the curve for d^2t/dx^2.

Doing this, we obtain the following equation by integration for the temperature curve

$$x = \sqrt{\frac{T}{b}} \log \frac{t - T}{T - T_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where T, as before, is 315° C., and where b is minus the value of the ordinate of this tangent when t equals nought. When measured, b is found to be 2388.

Solving equation (13) for various values of t, we find

Temperature, in degrees Centigrade.	Distance, in centimetres, from the end of the wire.
15	0
100	0·123
200	0·339
270	0·649
320	0·980
312	1·498
314	3·296

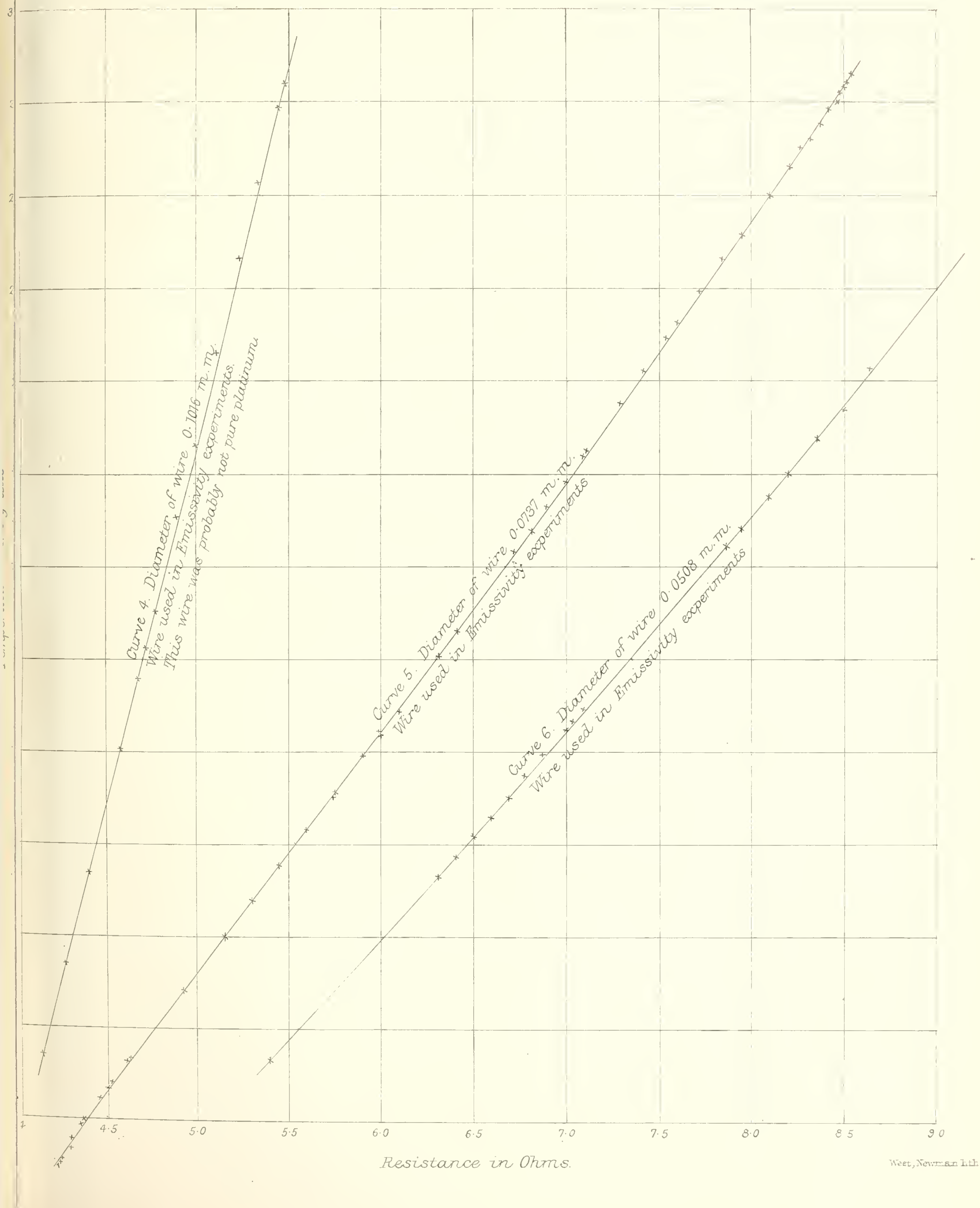
If now a distribution of temperature curve be drawn, similar to that seen in fig. 15, but using these values of t and x instead of those found from equation (9), it is found that the two curves showing the distribution of temperature along the wire are not very different. We may therefore conclude that the exact shape of the curve for d^2t/dx^2 for small values of t, has but little effect on the resultant distribution of temperature curve, and therefore that no important error is made by assuming, as we have done, that the true curve for d^2t/dx^2 is a parabola, with its vertex on the line along which d^2t/dx^2 is reckoned in fig. 15.

Fig. 7.

West, Newman lith.

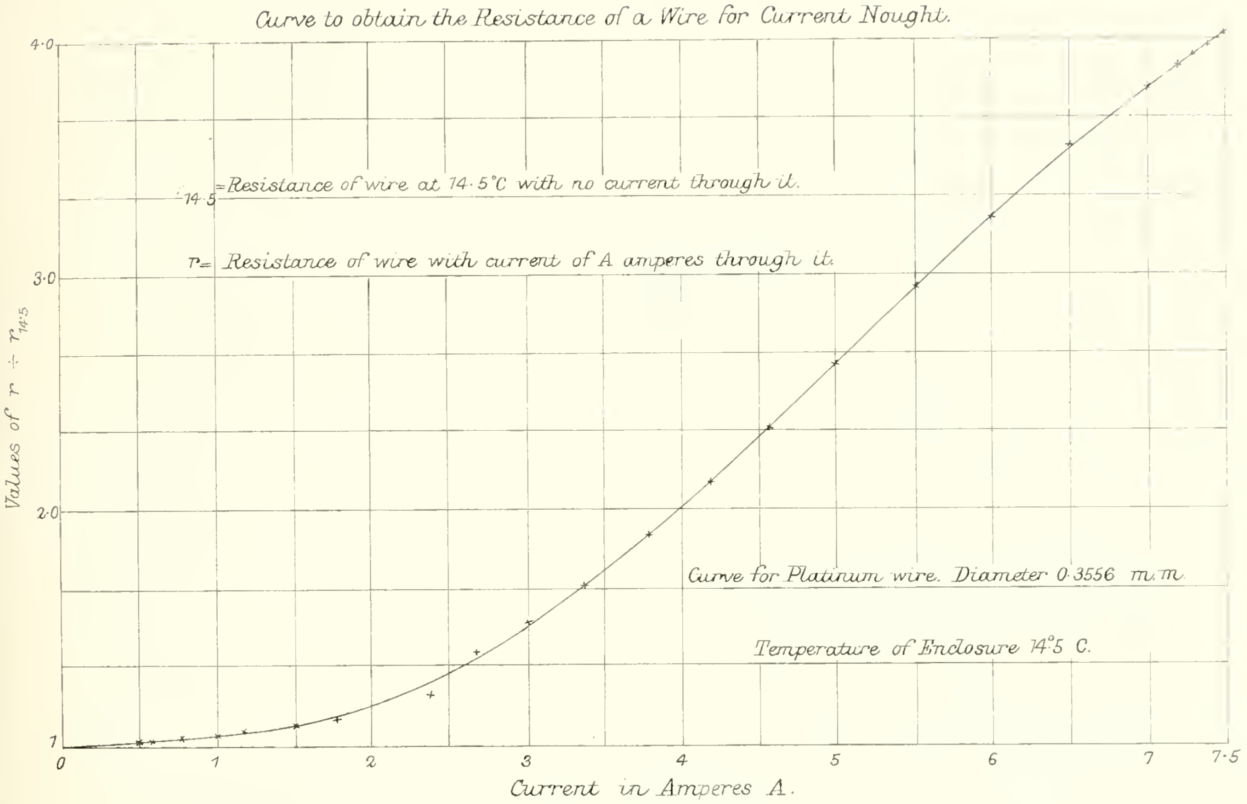
Fig. 8.

Variation of Temperature & Resistance of Wires.



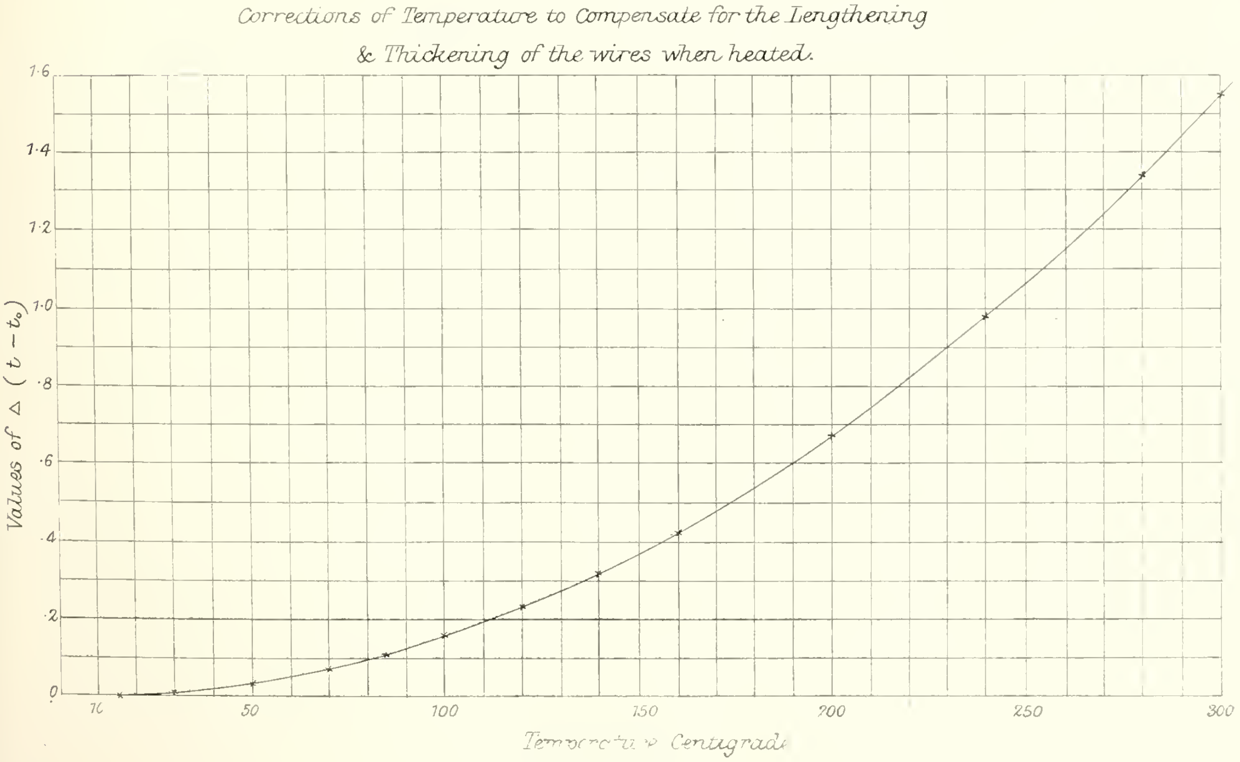
West, Newman lith.

Fig. 9.



Curve to obtain the Resistance of a Wire for Current Nought.

Fig. 10.



Corrections of Temperature to Compensate for the Lengthening & Thickening of the wires when heated.

Fig. 11.

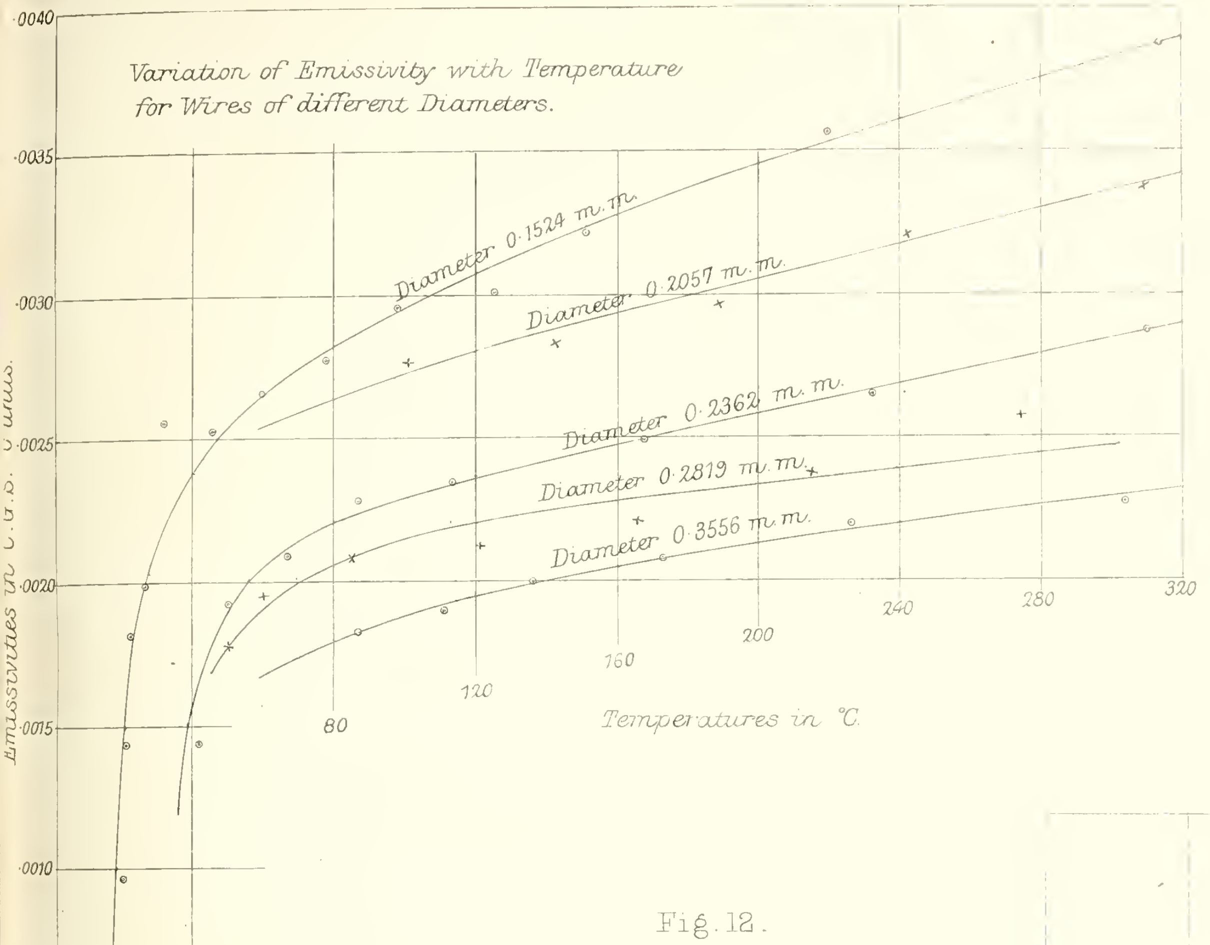
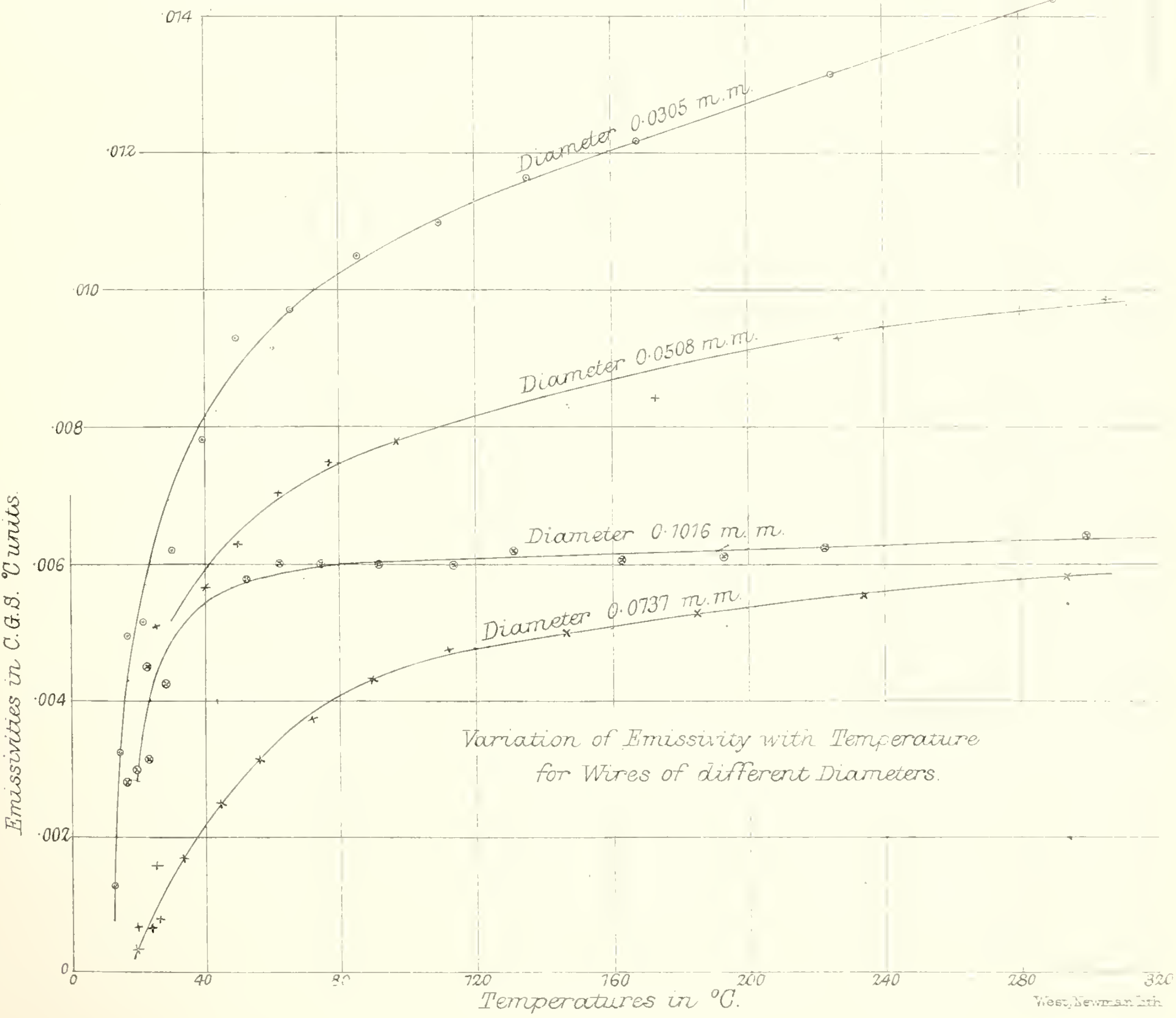


Fig. 12.



West, Newman lith.

Fig. 13.

Variation of Emissivity with Diameter at 300° C.

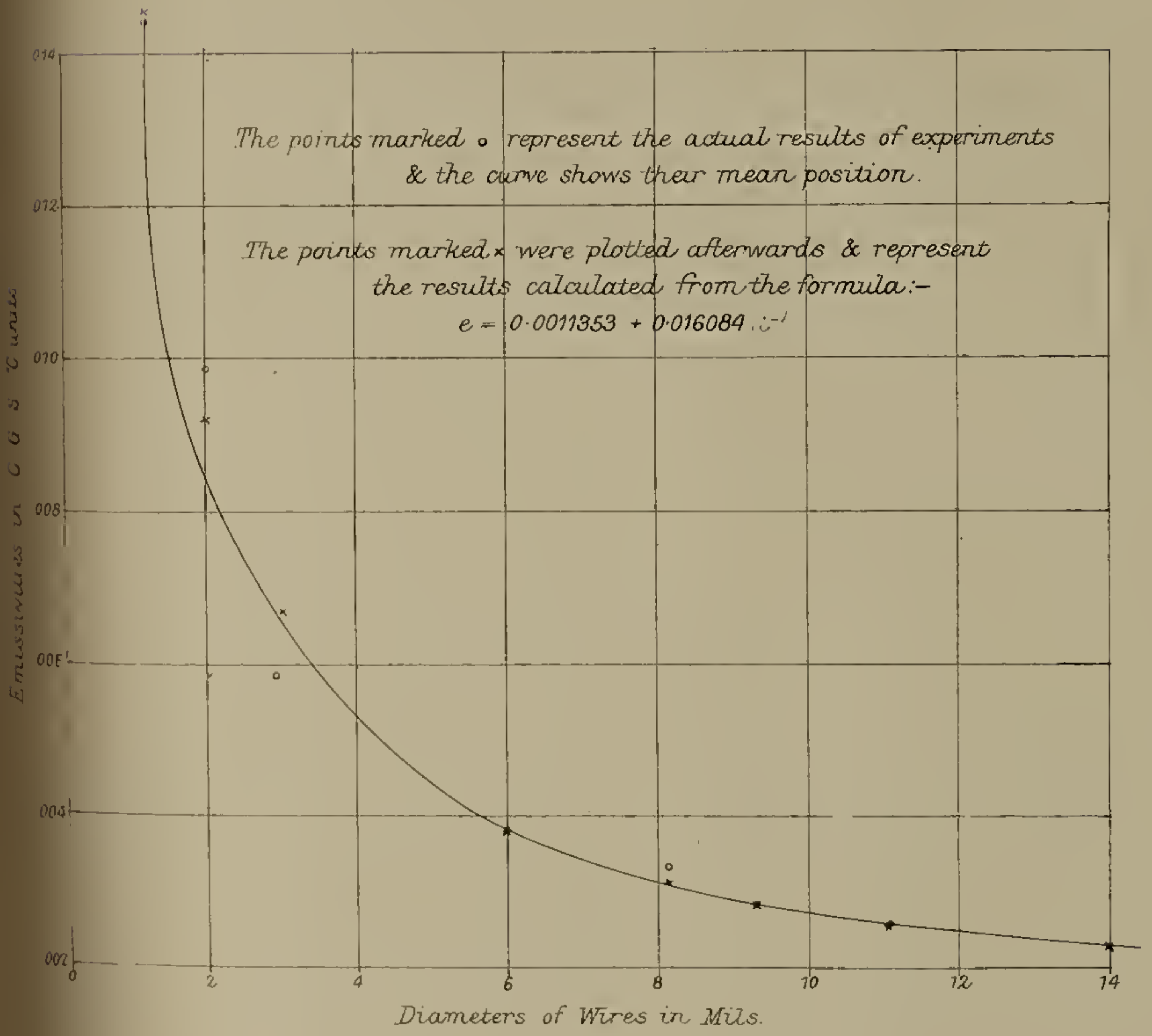
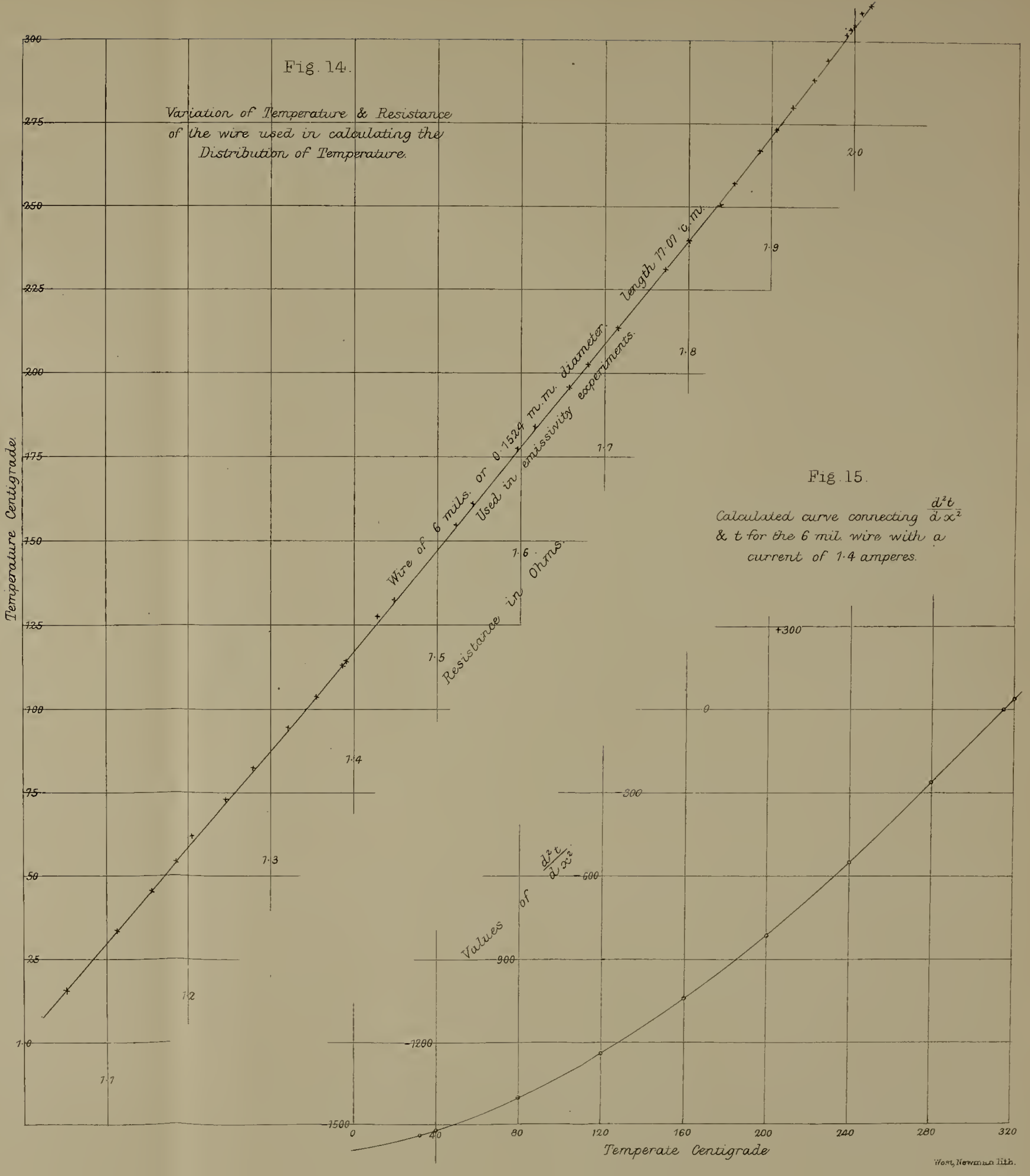


Fig. 14.

Variation of Temperature & Resistance of the wire used in calculating the Distribution of Temperature.

Fig. 15.

Calculated curve connecting $\frac{d^2t}{dx^2}$ & t for the 6 mil wire with a current of 1·4 amperes.



West, Newman lith.

X. *On the Collision of Elastic Bodies.*

By S. H. Burbury, *F.R.S.*

Received October 24,—Read November 19, 1891.
Revised June 25, 1892.

In a paper read before the Society on June 11, Sir William Thomson expressed a doubt as to the general truth of the Maxwell-Boltzmann doctrine concerning the distribution of energy among a great number of mutually acting bodies, and suggested that certain test cases should be investigated. The test that he proposed on that occasion was a number of hollow elastic spheres, each of mass M, and each containing a smaller elastic sphere of mass m, free to move within a larger one. This pair he calls a doublet. This case is within the general proof of the doctrine given below. It is, however, I think amenable to a simpler treatment, which has been applied to the case of elastic spheres external to one another.

1. Every doublet has a centre of inertia of the sphere M and its imprisoned m. Let V be the velocity of that centre of inertia, R the relative velocity of M and m. If V and R be given in magnitude, R given in direction, V may have any direction, and in Maxwell's distribution, for given direction of R, all directions of V are equally probable. Conversely, if, whatever be the values of V and R, for given direction of R all directions of V are equally probable, Maxwell's law prevails. Now consider a very great number of doublets, all having their relative velocity and the velocity of centre of inertia within limits R, $R + dR$, and V, $V + dV$. Consider them before and after collisions between M and m. Nothing is changed by collision except the direction of R, and that change of direction is independent of the direction of V. Therefore after collision for given direction of R all directions of V are equally probable, and therefore Maxwell's distribution prevails after as well as before collision, and is therefore not affected by collisions.

To proceed to more general cases.

2. The characteristic property of collisions of conventional elastic bodies is that with continuous variation of the coordinates, and without variation of the kinetic energy, the velocities at a certain instant change discontinuously. The general treatment adapted to systems of this kind is as follows:—Let there be a system defined by n coordinates $p_1 \ldots p_n$, the corresponding velocities being $\dot{p}_1 \ldots \dot{p}_n$, and the generalised components of momentum $q_1 \ldots q_n$. At a certain instant a collision, *i.e.*, discontinuity of $\dot{p}_1 \ldots \dot{p}'_n$ occurs. After the collision, let the velocities and

3.8.92

momenta be denoted by $\dot{p}'_1 \ldots \dot{p}'_n$, $\dot{q}'_1 \ldots \dot{q}'_n$. Let U be the potential. In the configuration $p_1 \ldots p_n$ there are $n - 1$ independent linear functions of the n forces, $-\frac{d\mathrm{U}}{dp_1} \ldots -\frac{d\mathrm{U}}{dp_n}$, each equal to zero. We might then find n new coordinates $c_1 \ldots c_n$, such that $\frac{d\mathrm{U}}{dc_1} = 0 \ldots \frac{d\mathrm{U}}{dc_{n-1}} = 0$, and therefore if S_1, $\mathrm{S}_2 \ldots \mathrm{S}_n$ be the components of momentum corresponding to $c_1 \ldots c_n$, $\frac{d\mathrm{S}_1}{dt} = \frac{d\mathrm{T}}{dc_1}$ &c., and $\frac{d\mathrm{S}_1}{dt} \ldots \frac{d\mathrm{S}_{n-1}}{dt}$ are finite or zero. In the limit when the remaining force becomes infinite and acts for an infinitely short time t, $\int_0^t \frac{d\mathrm{S}_1}{dt}\,dt = 0$ or $\mathrm{S}'_1 - \mathrm{S}_1 = 0 \ldots \mathrm{S}'_{n-1} - \mathrm{S}_{n-1} = 0$, and restoring the original coordinates.

$$\left.\begin{array}{ll} a_1(\dot{p}_1 - \dot{p}'_1) + b_1(\dot{p}_2 - \dot{p}'_2) + \ldots + k_1(\dot{p}_n - \dot{p}'_n) & = 0 \\ a_2(\dot{p}_1 - \dot{p}'_1) + \ldots & = 0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \\ a_{n-}\ (\dot{p}_1 - \dot{p}'_1) + \ldots & = 0 \end{array}\right\} \quad . \quad . \quad . \quad (\mathrm{A}),$$

in which the coefficients are functions of the coordinates.

And therefore $n - 1$ linear functions of the $\dot{p}$'s are unaltered by the collision, namely :—

$$\left.\begin{array}{ll} a_1\dot{p}_1 + b_1\dot{p}_2 + \ldots + k_1\dot{p}_n = a_1\dot{p}'_1 + b_1\dot{p}'_2 + \ldots + k_1\dot{p}'_n = \mathrm{S}_1 & \text{suppose} \\ a_2\dot{p}_1 + b_2\dot{p}_2 + \ldots + k_2\dot{p}_n = a_2\dot{p}'_1 + a_2\dot{p}'_2 + \ldots + k_2\dot{p}'_n = \mathrm{S}_2 & \\ \cdot\quad\cdot & \\ a_{n-1}\dot{p}_1 + b_{n-1}\dot{p}_2 + \ldots + k_{n-1}\dot{p}_n = a_{n-1}\dot{p}'_1 + b_{n-1}\dot{p}'_2 + \ldots + k_{n-1}\dot{p}'_n = \mathrm{S}_{n-1} & \end{array}\right\} . \quad (\mathrm{B}).$$

Again, since the kinetic energy is unchanged,

$$\Sigma\,\dot{p}q = \Sigma\dot{p}'q',$$

and by the properties of generalised coordinates

$$\Sigma\,\dot{p}'q = \Sigma\dot{p}q'.$$

Therefore

$$\Sigma\,(q + q')(\dot{p} - \dot{p}') = 0 \quad . \quad . \quad . \quad . \quad\quad . \quad . \quad . \quad (\mathrm{C}).$$

The last equation forms, with the $n - 1$ equations (A), a system of n equations, all linear as regards $\dot{p}_1 - \dot{p}'_1$, &c. Since $\dot{p}_1 - \dot{p}'_1$, &c., are not all zero, we must equate the determinant of the system (A) and (C) to zero. That gives us a linear equation in $q_1 + q'_1$, $q_2 + q'_2$, &c. We can now substitute for the q's their values in terms of $\dot{p}_1 \ldots \dot{p}_n$, and so obtain a linear equation

$$\alpha(\dot{p}_1 + \dot{p}'_1) + \beta(\dot{p}_2 + \dot{p}'_2), \text{ \&c.} = 0,$$

or

$$\alpha\dot{p}_1 + \beta\dot{p}_2 + \ldots = \mathrm{R} = -(\alpha\dot{p}'_1 + \beta\dot{p}'_2 \ldots),$$

where α, β, &c. are functions of the coordinates and constants.

That is, we have in all $n - 1$ linear functions of the velocities, namely, $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$, which remain unaltered by the collision, and one other linear function, R, which remains unaltered in value, but changes sign. That must be the case on every collision of elastic bodies.

3. The kinetic energy may be expressed in terms of the n variables $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$, and R, in lieu of the n velocities $\dot{p}_1 \ldots \dot{p}_n$, and since it is not altered by the collision, which changes the sign of R, leaving $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ unaltered, it must be of the form

$$2\mathrm{E} = f(\mathrm{S}_1 \ldots \mathrm{S}_{n-1}) + \lambda\mathrm{R}^2,$$

in which $f(\mathrm{S}_1 \ldots \mathrm{S}_{n-1})$ is a quadratic function of $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ with coefficients functions of the coordinates and constants of the system, and λ is a function of the coordinates and constants.

4. The system, after collision, has velocities $\dot{p}'_1$, &c., which we will call the second state. We may conceive a system with the same coordinates having velocities $-\dot{p}'_1$, $-\dot{p}'_2$, &c., and call this the second state with reversed velocities. In this state, $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ will have opposite signs to those they have in the first state, and R has the same sign as in the first state. The system retraces its course, and a collision occurs changing $-\dot{p}'_1$ into $-\dot{p}_1$, &c., leaving $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ unaltered, and changing R into $-$ R.

5. To define a collision, we may suppose that a certain function, ψ, of the coordinates and constants is prevented by the physical conditions of the system from becoming positive. When ψ becomes zero, $d\psi/dt$ from being positive becomes discontinuously negative, and a collision is said to take place. We may take ψ for one of our generalised coordinates in lieu of p_n, and $\dot{\psi}$, or $d\psi/dt$, for the corresponding component of velocity. The kinetic energy is a function of $\dot{p}_1 \ldots \dot{p}_{n-1}$, $\dot{\psi}$, and we may express it as a function of $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$, and $\dot{\psi}$, where $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ are the constants found above. Since the kinetic energy is not altered by the discontinuous change in $\dot{\psi}$, whatever the values of $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$, it must be of the form $f(\mathrm{S}_1 \ldots \mathrm{S}_{n-1}) + \frac{1}{2}\lambda\dot{\psi}^2$. That is $\dot{\psi}$ is reversed in sign, but unaltered in magnitude by the collision, and is, therefore, equal or proportional to R found above.

6. What we have proved for a system is of course true if for system we write pair of systems. For instance, let there be two sets of systems: (1) systems M defined by coordinates $p_1 \ldots p_r$, and velocities $\dot{p}_1 \ldots \dot{p}_r$, and (2) systems m defined by coordinates and velocities $p_{r+1} \ldots p_n$ and $\dot{p}_{r+1} \ldots \dot{p}_n$. If ψ is a function of $p_1 \ldots p_n$ such

that when $\psi = 0$ a collision—*i.e.*, a discontinuity of the velocities—occurs, we may treat the pair of systems M, m in all respects as a single system within the preceding investigation.

7. All those systems, or pairs of systems, for which at any instant ψ lies between zero and $-(d\psi/dt)\,\delta t$, $d\psi/dt$ being positive, will undergo collision within the time δt after that instant. We may, therefore, take $d\psi/dt$ or R as measuring the frequency of collisions for given values of $\dot{p}_1$, &c.

8. From the linear equations (B) above given we can find any of the velocities, for instance, $\dot{p}_1$, as a linear function of $S_1 \ldots S_{n-1}$ R, and $\dot{p}'_1$ will be the same function with $-$ R written for R. Hence $\dot{p}_1^2 - \dot{p}'^2_1 = 4R\Sigma\mu S$ where

$$\Sigma\mu S = \mu_1 S_1 + \mu_2 S_2 + \text{\&c.},$$

and the μ's are functions of the coordinates and constants.

Also $(\dot{p}_1^2 - \dot{p}'^2_1)\,R = 4R^2\Sigma\mu S$.

Now, without altering E or R, or the coordinates, let us make $S_1 \ldots S_{n-1}$ pass through the whole range of values consistent with

$$2E = \lambda^2 R^2 + f(S_1 \ldots S_{n-1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (E).$$

Also let $\phi(S_1 \ldots S_{n-1})\, dS_1 \ldots dS_{n-1}$ be the number of systems for which these variables lie between the limits

$$S_1 \text{ and } S_1 + dS,$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$S_{n-1} \text{ and } S_{n-1} + dS_{n-1},$$

E and R and the coordinates being constant.

Then

$$\iint \ldots (\dot{p}_1^2 - \dot{p}'^2_1)\,R\,.\,\phi(S_1 \ldots S_{n-1})\, dS_1 \ldots dS_{n-1}$$
$$= 4R^2 \iint \ldots \phi(S_1 \ldots S_{n-1})\,\Sigma\mu S\, dS_1 \ldots dS_{n-1},$$

the integrations being over all values consistent with (E).

Now, in the Maxwell-Boltzmann distribution $\phi(S_1 \ldots S_{n-1})$ is a function of the kinetic energy only, and is therefore constant throughout this integration. Therefore the right-hand member of the last equation is zero, because for any given set of values of $S_1 \ldots S_{n-1}$ satisfying (E), there is a corresponding set with the opposite signs also satisfying it. Therefore

$$\iint \ldots R\phi(S_1 \ldots S_{n-1})(\dot{p}_1^2 - \dot{p}'^2_1)\, dS_1 \ldots dS_{n-1} = 0,$$

and, therefore, the average value of $\dot{p}_1^2 - \dot{p}'^2_1$ for all collisions given E and R is zero. The same is true for $\dot{p}_2^2 - \dot{p}'^2_2$, &c.

And, therefore, since E and R are arbitrary, $\dot{p}_1^2$, &c., are not altered by collision at all, that is, the Maxwell-Boltzmann distribution, given existing, is not altered by collisions.

The above proof also shows that $\overline{\dot{p}^2} - \overline{\dot{p}'^2_1}$ is zero without the factor R, that is, the average value for all *systems* is zero, as well as for all *collisions*; and in proving that $\overline{\dot{p}^2} = \overline{\dot{p}'^2}$, it does not matter whether we introduce the factor R or not.

9. We will give certain examples of the functions $S_1 \ldots S_{n-1}$ R.

I. Elastic spheres of masses m and M respectively. Here a colliding pair, which corresponds to a system in the general treatment, has six degrees of freedom. There should, therefore, be five linear functions of the velocity unaltered by collision. They are x, y, X, Y, the tangential components of velocity at the instant of collision, and

$$mu + \mathrm{MU} = (\mathrm{M} + m)\,\mathrm{V} = mu' + \mathrm{MU}',$$

where u, U are the normal components.

We have from the last equation

$$m\,(u - u') + \mathrm{M}\,(\mathrm{U} - \mathrm{U}') = 0,$$

and by the equation of energy

$$m\,(u^2 - u'^2) + \mathrm{M}\,(\mathrm{U}^2 - \mathrm{U}'^2) = 0,$$

whence

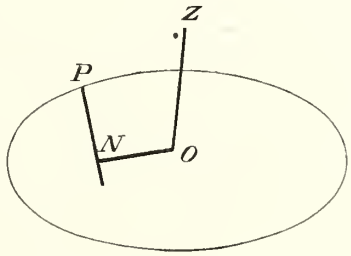
$$u + u' = \mathrm{U} + \mathrm{U}',$$

or

$$u - \mathrm{U} = -\,(u' - \mathrm{U}') = \mathrm{R}.$$

II. The system consists of a sphere of mass m colliding with a spheroid of mass M. It is assumed that the spheroid will acquire no rotation about its axis of figure, but may have rotation about any other principal axis. It has then five degrees of freedom, and the system of sphere and spheroid has eight.

We require, then, seven linear functions of the velocities to be invariable.



Let O be the centre of the spheroid, OZ in the plane of the paper, its axis of figure, P the point of collision, PN normal at P, ON perpendicular to PN, and ON $= c$, A the moment of inertia of the spheroid round an axis through O perpendicular to OZ. Then our seven constants are

(1), (2), (3), (4). Four tangential components of velocity.

(5). The angular velocity θ of the spheroid round an axis perpendicular to OZ in the plane of the paper.

And if u, U be normal components of velocity, and ω the angular velocity round an axis through O perpendicular to the plane of the paper, the following two, viz. :—

By conservation of momentum,

$$(6.)\ mu + \mathrm{MU} = (\mathrm{M} + m)\,\mathrm{V} = mu' + \mathrm{MU}'.$$

By conservation of moment of momentum round the axis through O perpendicular to the plane of the paper,

$$(7.)\ -\mathrm{M}c\mathrm{U} + \mathrm{A}\omega = \mathrm{S} = -\mathrm{M}c\mathrm{U}' + \mathrm{A}\omega'.$$

We then form the equations

$$\begin{aligned} m(u - u') + \mathrm{M}(\mathrm{U} - \mathrm{U}') \qquad\qquad\qquad &= 0, \\ -\mathrm{M}c(\mathrm{U} - \mathrm{U}') + \mathrm{A}(\omega - \omega') &= 0, \end{aligned}$$

and by conservation of energy,

$$m(u + u')(u - u') + \mathrm{M}(\mathrm{U} + \mathrm{U}')(\mathrm{U} - \mathrm{U}') + \mathrm{A}(\omega + \omega')(\omega - \omega') = 0\,;$$

and equating the determinant to zero we obtain, neglecting common factors,

$$\begin{aligned} u - \mathrm{U} - c\omega &= \rho, \\ u' - \mathrm{U}' - c\omega' &= -\rho, \end{aligned}$$

and, using V, S, and ρ for $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ and R in the general equations, we see that if the MAXWELL-BOLTZMANN distribution of energy exist, it is not disturbed by collisions between spheres and spheroids.

III. Professor BURNSIDE's problem (see his paper, Roy. Soc. Edinburgh, July 18, 1887). He supposes a number of similar and equal spheres, each of unit mass, but each sphere, instead of being homogeneous, has its centre of inertia at a distance c from its centre, c being supposed very small compared with the radius. The principal moments of inertia are for each sphere A, B, C, and the direction cosines of c referred to the principal axes through the centre of inertia are for each sphere the same, viz., α, β, γ. The direction cosines of the line of centres at impact referred to the principal axes are for one sphere L, M, N, and for the other l, m, n.

Further the normal velocities are U, u, and the angular velocities round the principal axes are Ω_1, Ω_2, Ω_3, ω_1, ω_2, ω_3 for the two spheres respectively.

Finally we write,

$$\begin{aligned} &\mathrm{N}\beta - \mathrm{M}\gamma = \mathrm{P} \qquad && n\beta - m\gamma = p. \\ &\mathrm{L}\gamma - \mathrm{N}\alpha = \mathrm{Q} && l\gamma - n\alpha = q. \\ &\mathrm{M}\alpha - \mathrm{L}\beta = \mathrm{R} && m\alpha - l\beta = r \end{aligned}$$

The system of two spheres has twelve degrees of freedom. We require, therefore, eleven linear functions of the velocities to be invariable. They are as follows, viz., four components of velocity in the common tangent plane, x, y, X, y, and seven others, viz.,

$$\begin{aligned}
u &+ \mathrm{U} &&= \mathrm{V} &&= u' &&+ \mathrm{U}',\\
cpu &+ \mathrm{A}\omega_1 &&= s_1 &&= cpu' &&+ \mathrm{A}\omega'_1,\\
cqu &+ \mathrm{B}\omega_2 &&= s_2 &&= cqu' &&+ \mathrm{B}\omega'_2,\\
cru &+ \mathrm{C}\omega_3 &&= s_3 &&= cru' &&+ \mathrm{C}\omega'_3,\\
c\mathrm{PU} &- \mathrm{A}\Omega_1 &&= \mathrm{S}_1 &&= c\mathrm{PU}' &&- \mathrm{A}\Omega'_1,\\
c\mathrm{QU} &- \mathrm{B}\Omega_2 &&= \mathrm{S}_2 &&= c\mathrm{QU}' &&- \mathrm{B}\Omega'_2,\\
c\mathrm{RU} &- \mathrm{C}\Omega_3 &&= \mathrm{S}_3 &&= c\mathrm{RU}' &&- \mathrm{C}\Omega'_3.
\end{aligned}$$

As before, we form the equations,

$$\begin{aligned}
&u - u' + \mathrm{U} - \mathrm{U}' &&= 0,\\
&cp\,(u - u') + \mathrm{A}\,(\omega_1 - \omega_{\ 1}) &&= 0,\\
&cq\,(u - u') + \mathrm{B}\,(\omega_2 - \omega'_2) &&= 0,\\
&cr\,(u - u') + \mathrm{C}\,(\omega_3 - \omega'_3) &&= 0,\\
&c\mathrm{P}\,(\mathrm{U} - \mathrm{U}') - \mathrm{A}\,(\Omega_1 - \Omega'_1) &&= 0,\\
&c\mathrm{Q}\,(\mathrm{U} - \mathrm{U}') - \mathrm{B}\,(\Omega_2 - \Omega'_2) &&= 0,\\
&c\mathrm{R}\,(\mathrm{U} - \mathrm{U}') - \mathrm{C}\,(\Omega_3 - \Omega'_3) &&= 0,
\end{aligned}$$

and by the conservation of energy,

$$(u + u')(u - u') + (\mathrm{U} + \mathrm{U}')(\mathrm{U} - \mathrm{U}') + \mathrm{A}\,(\omega_1 + \omega'_1)(\omega_1 - \omega'_1) + \&c. = 0$$

and equating the determinant to zero,

$$\begin{aligned}
u - \mathrm{U} - c\,(p\omega_1 + q\omega_2 + r\omega_3 + \mathrm{P}\Omega_1 + \mathrm{Q}\Omega_2 + \mathrm{R}\Omega_3) &= \rho,\\
u' - \mathrm{U}' - c\,(p\omega'_1 + q\omega'_2 + r\omega'_3 + \mathrm{P}\Omega'_1 + \mathrm{Q}\Omega'_2 + \mathrm{R}\Omega'_3) &= -\,\rho
\end{aligned}$$

We can now substitute V, s_1, s_2, s_3, S_1, S_2, S_3, ρ for $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}\mathrm{R}$ in the general equation, and we obtain, as before, the result that the MAXWELL-BOLTZMANN distribution, given existing, is not affected by collisions.

10. Professor BURNSIDE obtains the same eight equations as above given, and I acknowledge my obligation to him, but he originally deduced the result that the energy of rotation is twice the energy of translation, instead of equal to it, as, according to the theory, it should be. He has since seen reason to change his views with regard to this problem.

The equations given by Professor BURNSIDE can easily be modified so as to meet the case of elastic bodies of any shape.

11. It would not be difficult to extend the method of (8) and show that the MAXWELL-BOLTZMANN distribution is a necessary, as well as a sufficient condition for stationary motion. But that is more completely done by following or extending BOLTZMANN'S method.

Let there be a set of systems which we will call systems M, whose co-ordinates and velocities are $p_1 \ldots p_r$ and $\dot{p}_1 \ldots \dot{p}_r$.

Let the number of such systems for which at any instant the co-ordinates lie between the limits

$$\left.\begin{array}{l} p_1 \text{ and } p_1 + dp_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ p_r \text{ and } p_r + dp_r \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (\text{A}),$$

and the velocities between the limits

$$\left.\begin{array}{l} \dot{p}_1 \text{ and } \dot{p}_1 + d\dot{p}_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \dot{p}_r \text{ and } \dot{p}_r + d\dot{p}_r \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (\text{A}'),$$

be

$$\text{F}\,(p_1 \ldots p_r\, \dot{p}_1 \ldots \dot{p}_r)\, dp_1 \ldots dp_r\, d\dot{p}_1 \ldots d\dot{p}_r,$$

or, shortly,

$$\text{F}dp_1 \ldots dp_r\, d\dot{p}_1 \ldots d\dot{p}_r.$$

Let there be another set of systems, which we will call systems m, whose co-ordinates are $p_{r+1} \ldots p_n$ and velocities $\dot{p}_{r+1} \ldots \dot{p}_n$.

And let the number of systems m for which at any instant the coordinates lie between the limits

$$\left.\begin{array}{l} p_{r+1} \text{ and } p_{r+1} + dp_{r+1} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ p_n \text{ and } p_n + dp_n \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (\text{B}),$$

and the velocities between the limits

$$\left.\begin{array}{l} \dot{p}_{r+1} \text{ and } \dot{p}_{r+1} + d\dot{p}_{r+1} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \dot{p}_n \text{ and } \dot{p}_n + d\dot{p}_n \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (\text{B}'),$$

be

$$f\,(p_{r+1} \ldots p_n\, \dot{p}_{r+1} \ldots \dot{p}_n)\, dp_{r+1} \ldots dp_n\, d\dot{p}_{r+1} \ldots d\dot{p}_n$$

or, shortly,

$$f\,.\,dp_{r+1}\,.\,.\,.\,dp_n\,d\dot{p}_{r+1}\,.\,.\,.\,d\dot{p}_n.$$

The number of pairs of systems, each consisting of one system from each set, whose coordinates and velocities at any instant lie within the above limits, is

$$dp_1\,.\,.\,.\,d\dot{p}_r\,dp_{r+1}\,.\,.\,.\,d\dot{p}_n\,\mathrm{F}f.$$

Now let ψ be a function of the coordinates which cannot become positive, and such that when $\psi = 0$ the velocities change discontinuously, and a collision occurs.

We may use ψ for one of our coordinates, expressing p_n in terms of $p_1\,.\,.\,.\,p_{n-1}$ and ψ, and in like manner $\dot{p}_n$ in terms of $\dot{p}_1\,.\,.\,.\,\dot{p}_{n-1}$ and $\dot{\psi}$. All those pairs for which at the given instant ψ lies between zero and $-\dot{\psi}\delta t$, ψ being positive, will undergo collision within the time δt after that instant. And so the number of such collisions in unit time is, writing R for $\dot{\psi}$,

$$dp_1\,.\,.\,.\,dp_{n-1}\,d\dot{p}_1\,.\,.\,.\,d\dot{p}_{n-1}\,d\mathrm{R}\,.\,\mathrm{F}f\mathrm{R}.$$

Here F is a function of $p_1\,.\,.\,.\,p_r$ only, but f, by virtue of the elimination of p_n and $\dot{p}_n$, is now a function $p_1\,.\,.\,.\,p_{n-1}$ and $\dot{p}_1\,.\,.\,.\,\dot{p}_{n-1}$ R.

Each of these collisions changes $\dot{p}_1$ into p'_1, &c., that is, changes the system from the first into the second state. The number in unit time of collisions, which with the same coordinates change the system from the second state with reversed velocities into the first state with reversed velocities, is

$$dp_1\,.\,.\,.\,dp_{n-1}\,d\dot{p}'_1\,.\,.\,.\,d\dot{p}'_{n-1}\,d\mathrm{R}\,\mathrm{F}'f'\mathrm{R},$$

in which F′, f' are the same functions of $\dot{p}'_1$, &c., as F, f are of $\dot{p}_1$, &c.

By a known theorem

$$d\dot{p}'_1\,.\,.\,.\,d\dot{p}'_{n-1} = d\dot{p}_1\,.\,.\,.\,d\dot{p}_{n-1},$$

and in MAXWELL'S distribution $\mathrm{F}f = \mathrm{F}'f'$. And so the number of reverse collisions is in that distribution equal to the number of direct collisions. And this insures the permanence of the distribution. It is assumed that there are always as many systems with any given set of velocities as with those velocities reversed. And so we speak of the second state with reversed velocities as equivalent to the second state.

12. If $\mathrm{F}'f' \neq \mathrm{F}f$, the number of reverse collisions is not equal to the number of direct collisions. And, therefore, more (or fewer) pairs of systems pass out of the first state into the second than *vice versâ*. In that case the number of systems M, whose coordinates and velocities are within the limits A, A′, is increased by collisions

with systems m, whose coordinates and velocities lie within the limits B, B′, by the quantity

$$dp_1 \ldots dp_{n-1}\, d\dot{p}_1 \ldots dp_{n-1}\, d\mathrm{R}\,(\mathrm{F}'f' - \mathrm{F}f)\,\mathrm{R} \text{ per unit of time,}$$

and is increased per unit of time by collisions with systems m for all values of $\dot{p}_{r+1} \ldots p_n$ by the quantity

$$dp_1 \ldots dp_r\, d\dot{p}_1 \ldots d\dot{p}_r \iint \ldots (\mathrm{F}'f' - \mathrm{F}f)\,\mathrm{R}\, dp_{r+1} \ldots d\dot{p}_{n-1}\, d\mathrm{R},$$

all values of p_{r+1}, &c., being included in the integration.

We will now assume (see p. 418, *post*) that the velocities of M and m are not on average altered except by collision between M and m. And so the above-mentioned increments are the only increments by which the class of M systems within the limits A, A′ is affected.

In that case

$$\frac{d\mathrm{F}}{dt}\, dp_1 \ldots d\dot{p}_r = dp_1 \ldots d\dot{p}_r \iint \ldots (\mathrm{F}'f' - \mathrm{F}f)\,\mathrm{R}\, dp_{r+1} \ldots d\dot{p}_{n-1}\, d\mathrm{R},$$

and, therefore,

$$\iint \ldots \frac{d\mathrm{F}}{dt} \log \mathrm{F}\, dp_1 \ldots d\dot{p}_r \text{ (over all values of } p_1 \ldots \dot{p}_r)$$

$$= \iint \ldots (\mathrm{F}'f' - \mathrm{F}f)\,\mathrm{R} \log \mathrm{F}\, dp_1 \ldots d\dot{p}_{n-1}\, d\mathrm{R}.$$

By symmetry, as the right-hand member includes all possible collision between M and m,

$$\iint \ldots \frac{df}{dt} \log f\, dp_{r+1} \ldots d\dot{p}_n$$

$$= \iint \ldots (\mathrm{F}'f' - \mathrm{F}f)\,\mathrm{R} \log f\, dp_1 \ldots d\dot{p}_{n-1}\, d\mathrm{R},$$

and, therefore,

$$\iint \ldots \frac{d\mathrm{F}}{dt} \log \mathrm{F}\, dp_1 \ldots d\dot{p}_r + \iint \ldots \frac{df}{dt} \log f\, dp_{r+1} \ldots d\dot{p}_n$$

$$= \iint \ldots (\mathrm{F}'f' - \mathrm{F}f)\,\mathrm{R} \log (\mathrm{F}f)\, dp_1 \ldots d\dot{p}_{n-1}\, d\mathrm{R}.$$

By symmetry, as we may interchange the accents,

$$\iint \ldots \frac{d\mathrm{F}}{dt} \log \mathrm{F}\, dp_1 \ldots d\dot{p}_r + \iint \ldots \frac{df}{dt} \log f\, dp_{r+1} \ldots d\dot{p}_n$$

$$= \iiint \ldots (\mathrm{F}f - \mathrm{F}'f')\,\mathrm{R} \log (\mathrm{F}'f')\, dp_1 \ldots d\dot{p}_{n-1}\, d\mathrm{R},$$

and, therefore,

$$= \tfrac{1}{2}\iiint \ldots (F'f' - Ff)\, R \log \frac{Ff}{F'f'}\, dp_1 \ldots d\dot{p}_{n-1}\, dR.$$

Now let

$$H = \iiint \ldots F(\log F - 1)\, dp_1 \ldots d\dot{p}_r + \iiint \ldots f(\log f - 1)\, dp_{r+1} \ldots d\dot{p}_n,$$

and, therefore,

$$\frac{dH}{dt} = \iiint \ldots \frac{dF}{dt} \log F\, dp_1 \ldots d\dot{p}_r + \iiint \ldots \frac{df}{dt} \log f\, dp_{r+1} \ldots d\dot{p}_n$$

$$= \tfrac{1}{2}\iiint \ldots (F'f' - Ff)\, R \log \frac{Ff}{F'f'}\, dp_1 \ldots d\dot{p}_{n-1}\, dR,$$

as we have seen.

Now, this expression is necessarily negative, unless $F'f' = Ff$ whenever the pair of systems having coordinates and velocities $p_1 \ldots \dot{p}_n$ can pass by collision, and, therefore, with unchanged kinetic energy, into the state in which they are $p_1 \ldots \dot{p}'_n$, that is, unless the MAXWELL-BOLTZMANN distribution exist, and is then zero. H therefore tends to a minimum which it reaches when $Ff = F'f'$.

We will now make

$$H = H_1 + K$$

where H_1 is the minimum value assumed by H when $F'f' = Ff$. Then $dH/dt = 0$, and $dK/dt = dH/dt$, and is always negative. We may define the function K to be *the disturbance*, and $(1/K)\,(dK/dt)$ to be the *rate of subsidence* of the disturbance by collision. In certain cases, we can calculate the rate.

13. We have assumed that f varies only as the result of collisions. That is, if $\partial f/\partial t$ denote the time variation of f due to causes other than collisions, and $\partial H/\partial t$ be formed from $\partial f/\partial t$ as dH/dt from df/dt, then $\partial H/\partial t = 0$, on average. It is worth while to consider on what condition this may be safely assumed.

Let

$$\frac{\partial H}{\partial t} = \iiint \ldots \frac{\partial f}{\partial t} \log f\, dp_1 \ldots d\dot{p}_n.$$

As we are dealing with rigid elastic bodies under the action of no forces, we may treat f as a function of three translation velocities, and three angular velocities, w_1, w_2, w_3, about three principal axes of the solids. Let A, B, C, be the principal moments of inertia. Evidently, there being no forces, the translation velocities cannot vary except as the result of collisions. But for each solid, w_1, w_2, w_3 may vary, the law of their variations being EULER'S equations. We may, therefore, in calculating $\partial H/\partial t$ treat f as a function of w_1, w_2, w_3 only. Then

$$\frac{\partial f}{\partial t} = \frac{df}{dw_1}\frac{dw_1}{dt} + \frac{df}{dw_2}\frac{dw_2}{dt} + \frac{df}{dw_3}\frac{dw_3}{dt}$$

$$= \frac{\mathrm{B} - \mathrm{C}}{\mathrm{A}}\frac{df}{dw_1} w_2 w_3 + \frac{\mathrm{C} - \mathrm{A}}{\mathrm{B}}\frac{df}{dw_2} w_1 w_3 + \frac{\mathrm{A} - \mathrm{B}}{\mathrm{C}}\frac{df}{dw_3} w_1 w_2,$$

and

$$\frac{\partial \mathrm{H}}{\partial t} = \frac{\mathrm{B} - \mathrm{C}}{\mathrm{A}} \iiint \frac{df}{dw_1} \log f \,.\, w_2 w_3 \,.\, dw_1\, dw_2\, dw_3,$$

with two other corresponding terms, the limits being in each $\pm\infty$. Now, with these limits,

$$\iiint \frac{df}{dw_1} \log f \,.\, w_2 w_3 \,.\, dw_1\, dw_2\, dw_3$$

$$= \iint dw_2\, dw_3\, w_2 w_3 \left\{ f \log f_{w_1 = \infty} - f \log f_{w_1 = -\infty} \right\}.$$

Now, we may assume $f = 0$ and $f \log f = 0$, when any one of the three variables is infinite, whether positive or negative. And this assumption is sufficient to justify the statement $\partial \mathrm{H}/\partial t = 0$.

14. It is possible to calculate, in a simple case, the rate at which a disturbance subsides by collisions. For example, two sets of elastic spheres, N of mass M, and n of mass m, in unit of volume. In the normal state, the number in unit of volume, whose velocities are represented by lines drawn from an origin to points within an element of volume $\mathrm{U}^2 \sin\theta\, d\theta\, d\phi\, d\mathrm{U}$ is for the M spheres

$$\mathrm{N}\left(\frac{h\mathrm{M}}{\pi}\right)^{\frac{3}{2}} \epsilon^{-h\mathrm{MU}^2}\, \mathrm{U}^2 \sin\theta\, d\theta\, d\phi\, d\mathrm{U},$$

where U, θ, ϕ are usual spherical coordinates; or, let us say,

$$\mathrm{F}(\mathrm{U}) = \mathrm{N}\left(\frac{h\mathrm{M}}{\pi}\right)^{\frac{3}{2}} \epsilon^{-h\mathrm{MU}^2}.$$

Similarly, for the m spheres,

$$f(u) = n\left(\frac{hm}{\pi}\right)^{\frac{3}{2}} \epsilon^{-hmu^2}$$

expresses the law of distribution of velocities in the undisturbed state. We will write F and f for these expressions.

We will now suppose there is a small disturbance consisting in h having different values for the two sets. Let h be written $h(1 + \mathrm{D})$ for the M spheres, and $h(1 + d)$ for the m spheres. We shall neglect third and higher powers of D, d. Then F becomes in the disturbed state

$$\mathrm{N}\left(\frac{h\mathrm{M}}{\pi}\right)^{\frac{3}{2}}(1 + \mathrm{D})^{\frac{3}{2}} \epsilon^{-h(1+\mathrm{D})\mathrm{MU}^2},$$

that is,

$$N\left(\frac{hM}{\pi}\right)^{\frac{3}{2}}(1+D)^{\frac{3}{2}}\epsilon^{-hMU^2}\{1-hDMU^2+\text{terms in }D^2\}.$$

(It will appear that terms in D^2, &c., are not required.)

Similarly f becomes

$$n\left(\frac{hm}{\pi}\right)^{\frac{3}{2}}(1+d)^{\frac{3}{2}}\epsilon^{-hmu^2}(1-dhmu^2+d^2\ \&c.).$$

We will further suppose that the disturbance is introduced without changing the total energy. That gives the relation

$$N\cdot\frac{3}{2h(1+D)}+n\frac{3}{2h(1+D)}=\frac{3}{2h}(N+n),$$

or

$$\frac{N}{1+D}+\frac{n}{1+d}=N+n.$$

15. The disturbance will subside by collisions between M and m. And we will treat of the case in which it subsides in such manner that the above values of F and f apply at every instant with the values that D and d at that instant have. Such a mode of subsidence is possible, at all events if our equations lead (as they do) to a relation of the form $(1/K)(dK/dt)=$ constant.

Let us then form the function

$$H=\int_0^\infty\int_0^\pi\int_0^{2\pi}.\ F(\log F-1)\,U^2\sin\alpha\,d\alpha\,d\beta\,dU,$$

where U, α, and β are usual spherical coordinates,

$$+\int_0^\infty\int_0^\pi\int_0^{2\pi}f(\log f-1)\,u^2\sin\alpha\,d\alpha\,d\beta\,du,$$

where F and f have the values above given.

That is

$$H=\tfrac{3}{2}\log(1+D)\iiint F.U^2\sin\alpha\,d\alpha\,d\beta\,dU$$

$$+\tfrac{3}{2}\log(1+d)\iiint f.u^2\sin\alpha\,d\alpha\,d\beta\,du$$

$$+\text{ terms independent of D and }d,$$

or

$$H=N\tfrac{3}{2}\log(1+D)+n\tfrac{3}{2}\log(1+d),$$
$$+\text{ terms independent of D and }d,$$

because

$$\iiint FU^2\sin\alpha\,d\alpha\,d\beta\,dU=N$$

$$\iiint fu^2\sin\alpha\,d\alpha\,d\beta\,du=n.$$

The terms independent of D and d are equivalent to H_1, the minimum value of H when there is no disturbance.

And so

$$K = H - H_1 = \tfrac{3}{2} N \log (1 + D) + \tfrac{3}{2} n \log (1 + d).$$

That is

$$K = \tfrac{3}{2} n \left(d - \frac{d^2}{2}\right) + \tfrac{3}{2} N \left(D - \frac{D^2}{2}\right).$$

Now since

$$\frac{N}{1 + D} + \frac{n}{1 + d} = N + n,$$

$$D = - \frac{nd}{N + (N + n) d},$$

and

$$K = \tfrac{5}{2} n \left(d - \frac{d^2}{2}\right) - \tfrac{3}{2} N \,.\, \frac{nd}{N + (N + n) d} - \tfrac{3}{4} \,.\, \frac{n^2 d^2}{\{N + (N + n) d\}^2}$$

$$= \tfrac{3}{2} n \left(d - \frac{d^2}{2}\right) - \tfrac{3}{2} nd \,.\left(1 - \frac{N + n}{N} d\right) - \tfrac{3}{4} \frac{n^2 d^2}{\{N + (N + n) d\}^2}$$

$$= \tfrac{3}{2} nd^2 - \tfrac{3}{4} nd^2 + \tfrac{3}{2} \frac{n^2}{N} d^2 - \tfrac{3}{4} \frac{n^2}{N} d^2$$

$$= \tfrac{3}{4} \frac{n}{N} (N + n) d^2.$$

In order to find dK/dt we will transform our coordinates thus: Let V denote the velocity of the centre of inertia of a pair of spheres M and m, ρ their relative velocity, θ the angle between V and ρ. Then

$$U^2 = V^2 + \left(\frac{m}{M + m} \rho\right)^2 + \frac{2m}{M + m} V\rho \cos \theta$$

$$u^2 = V^2 + \left(\frac{M}{M + m} \rho\right)^2 - \frac{2M}{M + m} V\rho \cos \theta$$

and

$$MU^2 + mu^2 = (M + m) V^2 + \left(\frac{Mm}{M + m}\right) \rho^2$$

and so

$$Ff = Nn \left(\frac{hM}{\pi}\right)^{\frac{3}{2}} \left(\frac{hm}{\pi}\right)^{\frac{3}{2}} (1 + D)^{\frac{3}{2}} (1 + d)^{\frac{3}{2}} \epsilon^{-h \left\{(M + m) V^2 + \frac{Mm}{M + m} \rho^2\right\}} (1 - DhMU^2 - dhmu^2)$$

$$F'f' = Nn \left(\frac{hM}{\pi}\right)^{\frac{3}{2}} \left(\frac{hm}{\pi}\right)^{\frac{3}{2}} (1 + D)^{\frac{3}{2}} (1 + d)^{\frac{3}{2}} \epsilon^{-h \left\{(M + m) V^2 + \frac{Mm}{M + m} \rho^2\right\}} (1 - DhMU'^2 - dhmu'^2)$$

where U′, u' are the values of U, u after collision.

Therefore

$$\mathrm{F}f - \mathrm{F}'f' = \mathrm{N}n\left(\frac{h\mathrm{M}}{\pi}\right)^{\frac{3}{2}}\left(\frac{hm}{\pi}\right)^{\frac{3}{2}}(1+\mathrm{D})^{\frac{3}{2}}(1+d)^{\frac{3}{2}}\,\epsilon^{-h\left\{(\mathrm{M}+m)\mathrm{V}^2+\frac{\mathrm{M}m}{\mathrm{M}+m}\rho^2\right\}}$$
$$\left\{\mathrm{D}h\mathrm{M}(\mathrm{U}'^2-\mathrm{U}^2)+dhm\,(u'^2-u^2)\right\}$$

Now, if θ' be what θ becomes after collision,

$$\mathrm{U}'^2 - \mathrm{U}^2 = 2\mathrm{V}\rho\,.\,\frac{m}{\mathrm{M}+m}(\cos\theta' - \cos\theta)$$

$$u'^2 - u^2 = -\,2\mathrm{V}\rho\,\frac{\mathrm{M}}{\mathrm{M}+m}(\cos\theta' - \cos\theta)$$

Also by the relation

$$\frac{\mathrm{N}}{1+\mathrm{D}} + \frac{n}{1+d} = \mathrm{N} + n$$

$$\mathrm{D} = -\,\frac{nd}{\mathrm{N}+(\mathrm{N}+n)\,d},$$

and making these substitutions

$$\mathrm{F}'f' - \mathrm{F}f = \mathrm{N}n\left(\frac{h\mathrm{M}}{\pi}\right)^{\frac{3}{2}}\left(\frac{hm}{\pi}\right)^{\frac{3}{2}}(1+\mathrm{D})^{\frac{3}{2}}(1+d)^{\frac{3}{2}}\,\epsilon^{-h\left\{(\mathrm{M}+m)\mathrm{V}^2+\frac{\mathrm{M}m}{\mathrm{M}+m}\rho^2\right\}}$$
$$\left(h\,\frac{\mathrm{M}m}{\mathrm{M}+m}\,2\,.\,\mathrm{V}\rho\,.\,\frac{\mathrm{N}+n}{\mathrm{N}+(\mathrm{N}+n)\,d}(\cos\theta' - \cos\theta)\,d\right).$$

Also

$$\log\frac{\mathrm{F}f}{\mathrm{F}'f'} = h\,\frac{\mathrm{M}m}{\mathrm{M}+m}\,2\mathrm{V}\rho\,\frac{\mathrm{N}+n}{\mathrm{N}+(\mathrm{N}+n)\,d}(\cos\theta - \cos\theta')\,d.$$

In forming $(\mathrm{F}'f' - \mathrm{F}f)\log(\mathrm{F}f/\mathrm{F}'f')$ we see that the last factor is squared, and so the product contains the factor d^2. We may, therefore, now write 1 for $(1+\mathrm{D})^{\frac{3}{2}}$ and $(1+d)^{\frac{3}{2}}$, and also write $(\mathrm{N}+n)/\mathrm{N}$ for $(\mathrm{N}+n)/\{\mathrm{N}+(\mathrm{N}+n)\,d\}$, otherwise we should have terms in d^3.

Therefore

$$(\mathrm{F}'f' - \mathrm{F}f)\log\frac{\mathrm{F}f}{\mathrm{F}'f'} = -\,\mathrm{N}n\left(\frac{h\mathrm{M}}{\pi}\right)^{\frac{3}{2}}\left(\frac{hm}{\pi}\right)^{\frac{3}{2}}\epsilon^{-h\left\{(\mathrm{M}+m)\mathrm{V}^2+\frac{\mathrm{M}m}{\mathrm{M}+m}\rho^2\right\}}$$
$$h^2\,d^2\left(\frac{\mathrm{M}m}{\mathrm{M}+m}\right)^2 4\mathrm{V}^2\rho^2\,\frac{(\mathrm{N}+n)^2}{\mathrm{N}^2}(\cos\theta' - \cos\theta)^2.$$

Again, $d\mathrm{K}/dt$ contains the factor ψ denoting the frequency of collision. Now the number of collisions given V and ρ in unit of volume and time is proportional to $\pi s^2\rho$, where s is the sum of the radii of M and m. Also all directions of ρ before collision are equally probable, and given the direction before collision, all directions after collision are equally probable. Therefore, given V and ρ, the number in unit of volume and time of colliding pairs for which the angle between V and ρ before collision is between θ and $\theta + d\theta$ is $\frac{1}{2}\pi s^2 \rho \sin\theta\, d\theta$, and the number for which after collision it lies between θ' and $\theta' + d\theta''$ is $\frac{1}{2}\pi s^2 \rho \sin\theta'\, d\theta'$. Hence we have to multiply $(\cos\theta' - \cos\theta)^2$ by $\frac{1}{4}\pi s^2 \rho \sin\theta\, d\theta \sin\theta'\, d\theta'$, and integrate in each case from π to 0. The result is $\frac{2}{3}\pi s^2 \rho$.

And so we get

$$\frac{d\mathrm{K}}{dt} = -\tfrac{2}{3}\,\mathrm{N}n\frac{(\mathrm{N}+n)^2}{\mathrm{N}^2}\,h^2 d^2 \left(\frac{\mathrm{M}m}{\mathrm{M}+m}\right)^2 \left(\frac{h\mathrm{M}}{\pi}\right)^{\frac{3}{2}} \left(\frac{hm}{\pi}\right)^{\frac{3}{2}}$$

$$\times \int_0^\infty\!\!\int_0^\pi\!\!\int_0^{2\pi} \mathrm{V}^2\, d\mathrm{V} \sin\alpha\, d\alpha\, d\beta \int_0^\infty d\rho\; 4\pi\rho^2\, \epsilon^{-h\left\{(\mathrm{M}+m)\mathrm{V}^2 + \frac{\mathrm{M}m}{\mathrm{M}+m}\rho^2\right\}}\; 4\mathrm{V}^2\rho^3 \,.\, \pi s^2,$$

where V, α, β are usual spherical coordinates

$$= -\frac{4}{\sqrt{\pi}}\frac{n}{\mathrm{N}}(\mathrm{N}+n^2)\frac{\sqrt{(\mathrm{M}m)}}{(\mathrm{M}+m)^{\frac{3}{2}}}\frac{\pi s^2}{\sqrt{h}}d^2.$$

Also, as we have seen

$$\mathrm{K} = \tfrac{3}{4}\frac{n}{\mathrm{N}}(\mathrm{N}+n)\,d^2,$$

and therefore

$$\frac{1}{\mathrm{K}}\frac{d\mathrm{K}}{dt} = -\frac{16}{3\sqrt{\pi}}(\mathrm{N}+n)\frac{\sqrt{(\mathrm{M}m)}}{(\mathrm{M}+m)^{\frac{3}{2}}}\frac{\pi s^2}{\sqrt{h}}$$

$$= -\mathrm{C} \text{ suppose.}$$

And if K_0, D_0, d_0 be initial values,

$$\mathrm{K} = \mathrm{K}_0\epsilon^{-ct}, \qquad \mathrm{D} = \mathrm{D}_0\epsilon^{-\frac{1}{2}ct}, \qquad d = d_0\epsilon^{-\frac{1}{2}ct},$$

the rate of subsidence is directly proportional to the density and to the square root of the absolute temperature.*

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* Since writing the above I find that this result has already been obtained for the case of elastic spheres by Professor TAIT, by an independent method.

XI. *On the Forces, Stresses, and Fluxes of Energy in the Electromagnetic Field.*

By OLIVER HEAVISIDE, *F.R.S.*

Received June 9,—Read June 18, 1891.*

General Remarks, especially on the Flux of Energy.

§ **1.** THE remarkable experimental work of late years has inaugurated a new era in the development of the Faraday-Maxwellian theory of the ether, considered as the primary medium concerned in electrical phenomena—electric, magnetic, and electromagnetic. MAXWELL'S theory is no longer entirely a paper theory, bristling with unproved possibilities. The reality of electromagnetic waves has been thoroughly demonstrated by the experiments of HERTZ and LODGE, FITZGERALD and TROUTON, J. J. THOMSON, and others; and it appears to follow that, although MAXWELL'S theory may not be fully correct, even as regards the ether (as it is certainly not fully comprehensive as regards material bodies), yet the true theory must be one of the same type, and may probably be merely an extended form of MAXWELL'S.

No excuse is therefore now needed for investigations tending to exhibit and elucidate this theory, or to extend it, even though they be of a very abstract nature. Every part of so important a theory deserves to be thoroughly examined, if only to see what is in it, and to take note of its unintelligible parts, with a view to their future explanation or elimination.

§ **2.** Perhaps the simplest view to take of the medium which plays such a necessary part, as the recipient of energy, in this theory, is to regard it as continuously filling all space, and possessing the mobility of a fluid rather than the rigidity of a solid. If whatever possess the property of inertia be matter, then the medium is a form of matter. But away from ordinary matter it is, for obvious reasons, best to call it as usual by a separate name, the ether. Now, a really difficult and highly speculative question, at present, is the connection between matter (in the ordinary sense) and ether. When the medium transmitting the electrical disturbances consists of ether and matter, do they move together, or does the matter only partially carry forward the ether which immediately surrounds it? Optical reasons may lead us to conclude, though only tentatively, that the latter may be the case; but at present, for the purpose of fixing the data, and in the pursuit of investigations not having specially

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* Typographical troubles have delayed the publication of this paper. The footnotes are of date May 11, 1892.

optical bearing, it is convenient to assume that the matter and the ether in contact with it move together. This is the working hypothesis made by H. HERTZ in his recent treatment of the electrodynamics of moving bodies; it is, in fact, what we tacitly assume in a straightforward and consistent working out of MAXWELL'S principles without any plainly-expressed statement on the question of the relative motion of matter and ether; for the part played in MAXWELL'S theory by matter is merely (and, of course, roughly) formularised by supposing that it causes the etherial constants to take different values, whilst introducing new properties, that of dissipating energy being the most prominent and important. We may, therefore, think of merely one medium, the most of which is uniform (the ether), whilst certain portions (matter as well) have different powers of supporting electric displacement and magnetic induction from the rest, as well as a host of additional properties; and of these we can include the power of supporting conduction current with dissipation of energy according to JOULE'S law, the change from isotropy to eolotropy in respect to the distribution of the several fluxes, the presence of intrinsic sources of energy, &c.*

§ **3.** We do not in any way form the equations of motion of such a medium, even as regards the uniform simple ether, away from gross matter; we have only to discuss it as regards the electric and magnetic fluxes it supports, and the stresses and fluxes of energy thereby necessitated. First, we suppose the medium to be stationary, and examine the flux of electromagnetic energy. This is the POYNTING flux of energy. Next we set the medium into motion of an unrestricted kind. We have now necessarily a convection of the electric and magnetic energy, as well as the POYNTING flux. Thirdly, there must be a similar convection of the kinetic energy, &c., of the translational motion; and fourthly, since the motion of the medium involves the working of ordinary (Newtonian) force, there is associated with the previous a flux of energy due to the activity of the corresponding stress. The question is therefore a complex one, for we have to properly fit together these various fluxes of energy in harmony with the electromagnetic equations. A side issue is the determination of the proper measure of the activity of intrinsic forces, when the medium moves; in another form, it is the determination of the proper meaning of "true current" in MAXWELL'S sense.

§ **4.** The only general principle that we can bring to our assistance in interpreting electromagnetic results relating to activity and flux of energy, is that of the per-

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* Perhaps it is best to say as little as possible at present about the connection between matter and ether, but to take the electromagnetic equations in an abstract manner. This will leave us greater freedom for future modifications without contradiction. There are, also, cases in which it is obviously impossible to suppose that matter in bulk carries on with it the ether in bulk which permeates it. Either, then, the mathematical machinery must work between the molecules; or else, we must make such alterations in the equations referring to bulk as will be practically equivalent in effect. For example, the motional magnetic force $\mathbf{VDq}$ of equations (88), (92), (93) may be modified either in $\mathbf{q}$ or in $\mathbf{D}$, by use of a smaller effective velocity $\mathbf{q}$, or by the substitution in $\mathbf{D}$ or $c\mathbf{E}$ of a modified reckoning of c for the effective permittivity.

sistence of energy. But it would be quite inadequate in its older sense referring to integral amounts; the definite localisation by MAXWELL, of electric and magnetic energy, and of its waste, necessitates the similar localisation of sources of energy; and in the consideration of the supply of energy at certain places, combined with the continuous transmission of electrical disturbances, and therefore of the associated energy, the idea of a flux of energy through space, and therefore of the continuity of energy in space and in time, becomes forced upon us as a simple, useful, and necessary principle, which cannot be avoided.

When energy goes from place to place, it traverses the intermediate space. Only by the use of this principle can we safely derive the electromagnetic stress from the equations of the field expressing the two laws of circuitation of the electric and magnetic forces; and this again becomes permissible only by the postulation of the definite localisation of the electric and magnetic energies. But we need not go so far as to assume the objectivity of energy. This is an exceedingly difficult notion, and seems to be rendered inadmissible by the mere fact of the relativity of motion, on which kinetic energy depends. We cannot, therefore, definitely individualise energy in the same way as is done with matter.

If ρ be the density of a quantity whose total amount is invariable, and which can change its distribution continuously, by actual motion from place to place, its equation of continuity is

$$\mathrm{conv}\ \mathbf{q}\rho = \dot{\rho}, \qquad (1)$$

where $\mathbf{q}$ is its velocity, and $\mathbf{q}\rho$ the flux of ρ. That is, the convergence of the flux of ρ equals the rate of increase of its density. Here ρ may be the density of matter. But it does not appear that we can apply the same method of representation to the flux of energy. We may, indeed, write

$$\mathrm{conv}\ \mathbf{X} = \dot{\mathrm{T}}, \qquad (2)$$

if $\mathbf{X}$ be the flux of energy from all causes, and T the density of localisable energy. But the assumption $\mathbf{X} = \mathrm{T}\mathbf{q}$ would involve the assumption that T moved about like matter, with a definite velocity. A part of T may, indeed, do this, viz., when it is confined to, and is carried by matter (or ether); thus we may write

$$\mathrm{conv}\ (\mathbf{q}\mathrm{T} + \mathbf{X}) = \dot{\mathrm{T}}, \qquad (3)$$

where T is energy which is simply carried, whilst $\mathbf{X}$ is the total flux of energy from other sources, and which we cannot symbolise in the form $\mathrm{T}\mathbf{q}$; the energy which comes to us from the Sun, for example, or radiated energy. It is, again, often impossible to carry out the principle in this form, from a want of knowledge of how energy gets to a certain place. This is, for example, particularly evident in the case of gravitational energy, the distribution of which, before it is communicated to matter, increasing its kinetic energy, is highly speculative. If it come from the ether (and where else *can* it come from ?), it should be possible to symbolise this in **X**, if not in **q**T; but in default of a knowledge of its distribution in the ether, we cannot do so, and must therefore turn the equation of continuity into

$$\mathrm{S} + \mathrm{conv}\,(\mathbf{q}\mathrm{T} + \mathbf{X}) = \dot{\mathrm{T}}, \quad \ldots \ldots \quad (4)$$

where S indicates the rate of supply of energy per unit volume from the gravitational source, whatever that may be. A similar form is convenient in the case of intrinsic stores of energy, which we have reason to believe are positioned within the element of volume concerned, as when heat gives rise to thermoelectric force. Then S is the activity of the intrinsic sources. Then again, in special applications, T is conveniently divisible into different kinds of energy, potential and kinetic. Energy which is dissipated or wasted comes under the same category, because it may either be regarded as stored, though irrecoverably, or passed out of existence, so far as any immediate useful purpose is performed. Thus we have as a standard practical form of the equation of continuity of energy referred to the unit volume,

$$\mathrm{S} + \mathrm{conv}\,\{\mathbf{X} + \mathbf{q}\,(\mathrm{U} + \mathrm{T})\} = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}, \quad \ldots \ldots \quad (5)$$

where S is the energy supply from intrinsic sources, U potential energy and T kinetic energy of localisable kinds, **q** (U + T) its convective flux, Q the rate of waste of energy, and **X** the flux of energy other than convective, *e.g.*, that due to stresses in the medium and representing their activity. In the electromagnetic application we shall see that U and T must split into two kinds, and so must **X**, because there is a flux of energy even when the medium is at rest.

§ **5.** Sometimes we meet with cases in which the flux of energy is either wholly or partly of a circuital character. There is nothing essentially peculiar to electromagnetic problems in this strange and apparently useless result. The electromagnetic instances are paralleled by similar instances in ordinary mechanical science, when a body is in motion and is also strained, especially if it be in rotation. This result is a necessary consequence of our ways of reckoning the activity of forces and of stresses, and serves to still further cast doubt upon the " thinginess " of energy. At the same time, the flux of energy is going on all around us, just as certainly as the flux of matter, and it is impossible to avoid the idea ; we should, therefore, make use of it and formularise it whenever and as long as it is found to be useful, in spite of the occasional failure to obtain readily understandable results.

The idea of the flux of energy, apart from the conservation of energy, is by no means a new one. Had gravitational energy been less obscure than it is, it might have found explicit statement long ago. Professor POYNTING* brought the principle

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* POYNTING, 'Phil. Trans.,' 1884.

into prominence in 1884, by making use of it to determine the electromagnetic flux of energy. Professor LODGE[*] gave very distinct and emphatic expression of the principle generally, apart from its electromagnetic aspect, in 1885, and pointed out how much more simple and satisfactory it makes the principle of the conservation of energy become. So it would, indeed, could we only understand gravitational energy ; but in that, and similar respects, it is a matter of faith only. But Professor LODGE attached, I think, too much importance to the identity of energy, as well as to another principle he enunciated, that energy cannot be transferred without being transformed, and conversely ; the transformation being from potential to kinetic energy or conversely. This obviously cannot apply to the convection of energy, which is a true flux of energy ; nor does it seem to apply to cases of wave motion in which the energy, potential and kinetic, of the disturbance, is transferred through a medium unchanged in relative distribution, simply because the disturbance itself travels without change of type ; though it may be that in the unexpressed internal actions associated with the wave propagation there might be found a better application.

It is impossible that the ether can be fully represented, even merely in its transmissive functions, by the electromagnetic equations. Gravity is left out in the cold ; and although it is convenient to ignore this fact, it may be sometimes usefully remembered, even in special electromagnetic work ; for, if a medium have to contain and transmit gravitational energy as well as electromagnetic, the proper system of equations should show this, and, therefore, include the electromagnetic. It seems, therefore, not unlikely that in discussing purely electromagnetic speculations, one may be within a stone's throw of the explanation of gravitation all the time. The consummation would be a really substantial advance in scientific knowledge.

On the Algebra and Analysis of Vectors without Quaternions. Outline of Author's System.

§ **6.** The proper language of vectors is the algebra of vectors. It is, therefore, quite certain that an extensive use of vector-analysis in mathematical physics generally, and in electromagnetism, which is swarming with vectors, in particular, is coming and may be near at hand. It has, in my opinion, been retarded by the want of special treatises on vector analysis adapted for use in mathematical physics, Professor TAIT's well-known profound treatise being, as its name indicates, a treatise on Quaternions. I have not found the HAMILTON-TAIT notation of vector operations convenient, and have employed, for some years past, a simpler system. It is not, however, entirely a question of notation that is concerned. I reject the quaternionic basis of vector-analysis. The anti-quaternionic argument has been recently ably stated by Professor WILLARD GIBBS.[†] He distinctly separates this from the question

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[*] LODGE, 'Phil. Mag.,' June, 1885, "On the Identity of Energy."

[†] Professor GIBBS's letters will be found in 'Nature,' vol. 43, p. 511, and vol. 44, p. 79 ; and Professor

of notation, and this may be considered fortunate, for whilst I can fully appreciate and (from practical experience) endorse the anti-quaternionic argument, I am unable to appreciate his notation, and think that of HAMILTON and TAIT is, in some respects, preferable, though very inconvenient in others.

In HAMILTON's system the quaternion is the fundamental idea, and everything revolves round it. This is exceedingly unfortunate, as it renders the establishment of the algebra of vectors without metaphysics a very difficult matter, and in its application to mathematical analysis there is a tendency for the algebra to get more and more complex as the ideas concerned get simpler, and the quaternionic basis forms a real difficulty of a substantial kind in attempting to work in harmony with ordinary Cartesian methods.

Now, I can confidently recommend, as a really practical working system, the modification I have made. It has many advantages, and not the least amongst them is the fact that the quaternion does not appear in it at all (though it may, without much advantage, be brought in sometimes), and also that the notation is arranged so as to harmonise with Cartesian mathematics. It rests entirely upon a few definitions, and may be regarded (from one point of view) as a systematically abbreviated Cartesian method of investigation, and be understood and practically used by any one accustomed to Cartesians, without any study of the difficult science of Quaternions. It is simply the elements of Quaternions without the quaternions, with the notation simplified to the uttermost, and with the very inconvenient *minus* sign before scalar products done away with.*

TAIT's in vol. 43, pp. 535, 608. This rather one-sided discussion arose out of Professor TAIT stigmatising Professor GIBBS as "a retarder of quaternionic progress." This may be very true; but Professor GIBBS is anything but a retarder of progress in vector analysis and its application to physics.

* §§ 7, 8, 9 contain an introduction to vector-analysis (without the quaternion), which is sufficient for the purposes of the present paper, and, I may add, for general use in mathematical physics. It is an expansion of that given in my paper "On the Electromagnetic Wave Surface," 'Phil. Mag.,' June, 1885. The algebra and notation are substantially those employed in all my papers, especially in "Electromagnetic Induction and its Propagation," 'The Electrician,' 1885.

Professor GIBBS's vectorial work is scarcely known, and deserves to be well known. In June, 1888, I received from him a little book of 85 pages, bearing the singular imprint NOT PUBLISHED, Newhaven, 1881–4. It is indeed odd that the author should not have published what he had been at the trouble of having printed. His treatment of the linear vector operator is specially deserving of notice. Although "for the use of students in physics," I am bound to say that I think the work much too condensed for a first introduction to the subject.

In 'The Electrician' for Nov. 13, 1891, p. 27, I commenced a few articles on elementary vector-algebra and analysis, specially meant to explain to readers of my papers how to work vectors. I am given to understand that the earlier ones, on the algebra, were much appreciated; the later ones, however, are found difficult. But the vector-algebra is identically the same in both, and is of quite a rudimentary kind. The difference is, that the later ones are concerned with analysis, with varying vectors; it is the same as the difference between common algebra and differential calculus. The difficulty, whether real or not, does not indicate any difficulty in the vector-algebra. I mention this on account of the great prejudice which exists against vector-algebra.

§ 7. Quantities being divided into scalars and vectors, I denote the scalars, as usual, by ordinary letters, and put the vectors in the plain black type, known, I believe, as Clarendon type, rejecting MAXWELL'S German letters on account of their being hard to read. A special type is certainly not essential, but it facilitates the reading of printed complex vector investigations to be able to see at a glance which quantities are scalars and which are vectors, and eases the strain on the memory. But in MS. work there is no occasion for specially formed letters.

Thus **A** stands for a vector. The tensor of a vector may be denoted by the same letter plain; thus A is the tensor of **A**. (In MS. the tensor is A_0.) Its rectangular scalar components are A_1, A_2, A_3. A unit vector parallel to **A** may be denoted by $\mathbf{A}_1$, so that $\mathbf{A} = A\mathbf{A}_1$. But little things of this sort are very much matters of taste. What is important is to avoid as far as possible the use of letter prefixes, which, when they come two (or even three) together, as in Quaternions, are very confusing.

The scalar product of a pair of vectors **A** and **B** is denoted by **AB**, and is defined to be

$$\mathbf{AB} = A_1B_1 + A_2B_2 + A_3B_3 = AB\cos \overset{\wedge}{\mathbf{AB}} = \mathbf{BA}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The addition of vectors being as in the polygon of displacements, or velocities, or forces; *i.e.*, such that the vector length of any closed circuit is zero; either of the vectors **A** and **B** may be split into the sum of any number of others, and the multiplication of the two sums to form **AB** is done as in common algebra; thus

$$\left.\begin{aligned}(\mathbf{a} + \mathbf{b})\,(\mathbf{c} + \mathbf{d}) &= \mathbf{ac} + \mathbf{ad} + \mathbf{bc} + \mathbf{bd},\\ &= \mathbf{ca} + \mathbf{da} + \mathbf{cb} + \mathbf{db}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If **N** be a unit vector, **NN** or $\mathbf{N}^2 = 1$; similarly $\mathbf{A}^2 = A^2$ for any vector.

The reciprocal of a vector **A** has the same direction; its tensor is the reciprocal of the tensor of **A**. Thus

$$\mathbf{AA}^{-1} = \frac{\mathbf{A}}{\mathbf{A}} = 1;$$

and

$$\mathbf{AB}^{-1} = \mathbf{B}^{-1}\mathbf{A} = \frac{\mathbf{A}}{\mathbf{B}} = \frac{A}{B}\cos \overset{\wedge}{\mathbf{AB}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The vector product of a pair of vectors is denoted by V**AB**, and is defined to be the vector whose tensor is AB sin $\overset{\wedge}{\mathbf{AB}}$, and whose direction is perpendicular to the plane of **A** and **B**. Or

$$\mathrm{V}\mathbf{AB} = \mathbf{i}\,(A_2B_3 - A_3B_2) + \mathbf{j}\,(A_3B_1 - A_1B_3) + \mathbf{k}\,(A_1B_2 - A_2B_1) = -\,\mathrm{V}\mathbf{BA}, \quad . \quad . \quad . \quad (9)$$

where **i, j, k**, are any three mutually rectangular unit vectors. The tensor of V**AB** is $\mathrm{V}_0\mathbf{AB}$; or

$$\mathrm{V}_0\mathbf{AB} = AB\sin \overset{\wedge}{\mathbf{AB}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Its components are $\mathbf{i}\mathrm{V}\mathbf{AB}$, $\mathbf{j}\mathrm{V}\mathbf{AB}$, $\mathbf{k}\mathrm{V}\mathbf{AB}$.

In accordance with the definitions of the scalar and vector products, we have

$$\left.\begin{array}{lll} \mathbf{i}^2 = 1, & \mathbf{j}^2 = 1, & \mathbf{k}^2 = 1\,; \\ \mathbf{ij} = 0, & \mathbf{jk} = 0, & \mathbf{ki} = 0\,; \\ \mathrm{V}\mathbf{ij} = \mathbf{k}, & \mathrm{V}\mathbf{jk} = \mathbf{i}, & \mathrm{V}\mathbf{ki} = \mathbf{j}\,; \end{array}\right\} \qquad (11)$$

and from these we prove at once that

$$\mathrm{V}(\mathbf{a} + \mathbf{b})(\mathbf{c} + \mathbf{d}) = \mathrm{V}\mathbf{ac} + \mathrm{V}\mathbf{ad} + \mathrm{V}\mathbf{bc} + \mathrm{V}\mathbf{bd},$$

and so on, for any number of component vectors. The order of the letters in each product has to be preserved, since $\mathrm{V}\mathbf{ab} = -\mathrm{V}\mathbf{ba}$.

Two very useful formulæ of transformation are

$$\begin{aligned} \mathbf{A}\mathrm{V}\mathbf{BC} &= \mathbf{B}\mathrm{V}\mathbf{CA} = \mathbf{C}\mathrm{V}\mathbf{AB} \\ &= \mathrm{A}_1(\mathrm{B}_2\mathrm{C}_3 - \mathrm{B}_3\mathrm{C}_2) + \mathrm{A}_2(\mathrm{B}_3\mathrm{C}_1 - \mathrm{B}_1\mathrm{C}_3) + \mathrm{A}_3(\mathrm{B}_1\mathrm{C}_2 - \mathrm{B}_2\mathrm{C}_1)\,; \end{aligned} \qquad (12)$$

and

$$\left.\begin{aligned} \mathrm{V}\mathbf{A}\mathrm{V}\mathbf{BC} &= \mathbf{B}.\mathbf{CA} - \mathbf{C}.\mathbf{AB}, \\ &= \mathbf{B}(\mathbf{CA}) - \mathbf{C}(\mathbf{AB}). \end{aligned}\right\} \qquad (13)$$

Here the dots, or the brackets in the alternative notation, merely act as separators, separating the scalar products $\mathbf{CA}$ and $\mathbf{AB}$ from the vectors they multiply. A space would be equivalent, but would be obviously unpractical.

As $\frac{\mathbf{A}}{\mathbf{B}}$ is a scalar product, so in harmony therewith, there is the vector product $\mathrm{V}\frac{\mathbf{A}}{\mathbf{B}}$. Since $\mathrm{V}\mathbf{AB} = -\mathrm{V}\mathbf{BA}$, it is now necessary to make a convention as to whether the denominator comes first or last in $\mathrm{V}\frac{\mathbf{A}}{\mathbf{B}}$. Say therefore, $\mathrm{V}\mathbf{AB}^{-1}$. Its tensor is

$$\mathrm{V}_0\frac{\mathbf{A}}{\mathbf{B}} = \frac{\mathrm{A}}{\mathrm{B}}\sin\widehat{\mathbf{AB}}. \qquad (14)$$

§ **8.** Differentiation of vectors, and of scalar and vector functions of vectors with respect to scalar variables is done as usual. Thus,

$$\left.\begin{aligned} \dot{\mathbf{A}} &= \mathbf{i}\dot{\mathrm{A}}_1 + \mathbf{j}\dot{\mathrm{A}}_2 + \mathbf{k}\dot{\mathrm{A}}_3. \\ \frac{d}{dt}\mathbf{AB} &= \mathbf{A}\dot{\mathbf{B}} + \mathbf{B}\dot{\mathbf{A}}. \\ \frac{d}{dt}\mathbf{A}\mathrm{V}\mathbf{BC} &= \dot{\mathbf{A}}\mathrm{V}\mathbf{BC} + \mathbf{A}\mathrm{V}\dot{\mathbf{B}}\mathbf{C} + \mathbf{A}\mathrm{V}\mathbf{B}\dot{\mathbf{C}}. \end{aligned}\right\} \qquad (15)$$

The same applies with complex scalar differentiators, *e.g.*, with the differentiator

$$\frac{\partial}{\partial t} = \frac{d}{dt} + \mathbf{q}\nabla,$$

used when a moving particle is followed, q being its velocity. Thus,

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{AB} &= \mathbf{A}\frac{\partial \mathbf{B}}{\partial t} + \mathbf{B}\frac{\partial \mathbf{A}}{\partial t}\\ &= \mathbf{A}\dot{\mathbf{B}} + \mathbf{B}\dot{\mathbf{A}} + \mathbf{A}.\mathbf{q}\nabla.\mathbf{B} + \mathbf{B}.\mathbf{q}\nabla.\mathbf{A}. \quad (16)\end{aligned}$$

Here $\mathbf{q}\nabla$ is a scalar differentiator given by

$$\mathbf{q}\nabla = q_1\frac{d}{dx} + q_2\frac{d}{dy} + q_3\frac{d}{dz}, \quad (17)$$

so that $\mathbf{A}.\mathbf{q}\nabla.\mathbf{B}$ is the scalar product of $\mathbf{A}$ and the vector $\mathbf{q}\nabla.\mathbf{B}$; the dots here again act essentially as separators. Otherwise, we may write it $\mathbf{A}\,(\mathbf{q}\nabla)\,\mathbf{B}$.

The fictitious vector ∇ given by

$$\nabla = \mathbf{i}\nabla_1 + \mathbf{j}\nabla_2 + \mathbf{k}\nabla_3 = \mathbf{i}\frac{d}{dx} + \mathbf{j}\frac{d}{dy} + \mathbf{k}\frac{d}{dz} \quad (18)$$

is *very* important. Physical mathematics is very largely the mathematics of ∇. The name Nabla seems, therefore, ludicrously inefficient. In virtue of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the operator ∇ behaves as a vector. It also, of course, differentiates what follows it.

Acting on a scalar P, the result is the vector

$$\nabla P = \mathbf{i}\nabla_1 P + \mathbf{j}\nabla_2 P + \mathbf{k}\nabla_3 P, \quad (19)$$

the vector rate of increase of P with length.

If it act on a vector **A**, there is first the scalar product

$$\nabla\mathbf{A} = \nabla_1 A_1 + \nabla_2 A_2 + \nabla_3 A_3 = \mathrm{div}\,\mathbf{A}, \quad (20)$$

or the divergence of **A**. Regarding a vector as a flux, the divergence of a vector is the amount leaving the unit volume.

The vector product $\mathrm{V}\nabla\mathbf{A}$ is

$$\begin{aligned}\mathrm{V}\nabla\mathbf{A} &= \mathbf{i}\,(\nabla_2 A_3 - \nabla_3 A_2) + \mathbf{j}\,(\nabla_3 A_1 - \nabla_1 A_3) + \mathbf{k}\,(\nabla_1 A_2 - \nabla_2 A_1),\\ &= \mathrm{curl}\,\mathbf{A}. \quad (21)\end{aligned}$$

The line-integral of **A** round a unit area equals the component of the curl of **A** perpendicular to the area.

We may also have the scalar and vector products $\mathbf{N}\nabla$ and $\mathrm{V}\mathbf{N}\nabla$, where the vector **N** is not differentiated. These operators, of course, require a function to follow them on which to operate; the previous $\mathbf{q}\nabla.\mathbf{A}$ of (16) illustrates.

The Laplacean operator is the scalar product ∇^2 or $\nabla\nabla$: or

$$\nabla^2 = \nabla_1^2 + \nabla_2^2 + \nabla_3^2; \quad (22)$$

and an example of (13) is

$$\mathrm{V}\nabla\mathrm{V}\nabla\mathbf{A} = \nabla.\nabla\mathbf{A} - \nabla^2\mathbf{A},$$

or

$$\mathrm{curl}^2\mathbf{A} = \nabla\ \mathrm{div}\ \mathbf{A} - \nabla^2\mathbf{A}, \quad \dots \quad (23)$$

which is an important formula.

Other important formulæ are the next three.

$$\mathrm{div}\ \mathrm{P}\mathbf{A} = \mathrm{P}\ \mathrm{div}\ \mathbf{A} + \mathbf{A}\nabla.\mathrm{P}, \quad \dots \quad (24)$$

P being scalar. Here note that $\mathbf{A}\nabla.\mathrm{P}$ and $\mathbf{A}\nabla\mathrm{P}$ (the latter being the scalar product of $\mathbf{A}$ and $\nabla\mathrm{P}$) are identical. This is not true when for P we substitute a vector. Also

$$\mathrm{div}\ \mathrm{V}\mathbf{AB} = \mathbf{B}\ \mathrm{curl}\ \mathbf{A} - \mathbf{A}\ \mathrm{curl}\ \mathbf{B}; \quad \dots \quad (25)$$

which is an example of (12), noting that both **A** and **B** have to be differentiated. And

$$\mathrm{curl}\ \mathrm{V}\mathbf{AB} = \mathbf{B}\nabla.\mathbf{A} + \mathbf{A}\ \mathrm{div}\ \mathbf{B} - \mathbf{A}\nabla.\mathbf{B} - \mathbf{B}\ \mathrm{div}\ \mathbf{A}. \quad \dots \quad (26)$$

This is an example of (13).

§ **9.** When one vector **D** is a *linear* function of another vector **E**, that is, connected by equations of the form

$$\left.\begin{aligned} \mathrm{D}_1 &= c_{11}\mathrm{E}_1 + c_{12}\mathrm{E}_2 + c_{13}\mathrm{E}_3, \\ \mathrm{D}_2 &= c_{21}\mathrm{E}_1 + c_{22}\mathrm{E}_2 + c_{23}\mathrm{E}_3, \\ \mathrm{D}_3 &= c_{31}\mathrm{E}_1 + c_{32}\mathrm{E}_2 + c_{33}\mathrm{E}_3, \end{aligned}\right\}, \quad \dots \quad (27)$$

in terms of the rectangular components, we denote this simply by

$$\mathbf{D} = c\mathbf{E}, \quad \dots \quad (28)$$

where c is the linear operator. The conjugate function is given by

$$\mathbf{D}' = c'\mathbf{E}, \quad \dots \quad (29)$$

where $\mathbf{D}'$ is got from **D** by exchanging c_{12} and c_{21}, &c. Should the nine coefficients reduce to six by $c_{12} = c_{21}$, &c., **D** and $\mathbf{D}'$ are identical, or **D** is a self-conjugate or symmetrical linear function of **E**.

But, in general, it is the sum of **D** and $\mathbf{D}'$ which is a symmetrical function of **E**, and the difference is a simple vector-product. Thus

$$\left.\begin{aligned} \mathbf{D} &= c_0\mathbf{E} + \mathrm{V}\boldsymbol{\epsilon}\mathbf{E}, \\ \mathbf{D}' &= c_0\mathbf{E} - \mathrm{V}\boldsymbol{\epsilon}\mathbf{E}, \end{aligned}\right\} \quad \dots \quad (30)$$

where c_0 is a self-conjugate operator, and $\boldsymbol{\epsilon}$ is the vector given by

$$\boldsymbol{\epsilon} = \mathbf{i}\frac{c_{32} - c_{23}}{2} + \mathbf{j}\frac{c_{13} - c_{31}}{2} + \mathbf{k}\frac{c_{21} - c_{12}}{2} \quad \ldots \ldots \ldots \quad (31)$$

The important characteristic of a self-conjugate operator is

$$\left.\begin{aligned} \mathbf{E}_1\mathbf{D}_2 &= \mathbf{E}_2\mathbf{D}_1, \\ \text{or} \qquad \mathbf{E}_1c_0\mathbf{E}_2 &= \mathbf{E}_2c_0\mathbf{E}_1, \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (32)$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are any two $\mathbf{E}$'s, and $\mathbf{D}_1$, $\mathbf{D}_2$ the corresponding $\mathbf{D}$'s. But when there is not symmetry, the corresponding property is

$$\left.\begin{aligned} \mathbf{E}_1\mathbf{D}_2 &= \mathbf{E}_2\mathbf{D}'_1, \\ \text{or} \qquad \mathbf{E}_1c\mathbf{E}_2 &= \mathbf{E}_2c'\mathbf{E}_1. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (33)$$

Of these operators we have three or four in electromagnetism connecting forces and fluxes, and three more connected with the stresses and strains concerned. As it seems impossible to avoid the consideration of rotational stresses in electromagnetism, and these are not usually considered in works on elasticity, it will be desirable to briefly note their peculiarities here, rather than later on.

On Stresses, irrotational and rotational, and their Activities.

§ **10.** Let $\mathbf{P}_{\mathrm{N}}$ be the vector stress on the $\mathbf{N}$ plane, or the plane whose unit normal is $\mathbf{N}$. It is a linear function of $\mathbf{N}$. This will fully specify the stress on any plane. Thus, if $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ are the stresses on the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ planes, we shall have

$$\left.\begin{aligned} \mathbf{P}_1 &= \mathbf{i}P_{11} + \mathbf{j}P_{12} + \mathbf{k}P_{13}, \\ \mathbf{P}_2 &= \mathbf{i}P_{21} + \mathbf{j}P_{22} + \mathbf{k}P_{23}, \\ \mathbf{P}_3 &= \mathbf{i}P_{31} + \mathbf{j}P_{32} + \mathbf{k}P_{33}. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (34)$$

Let, also, $\mathbf{Q}_{\mathrm{N}}$ be the conjugate stress; then, similarly,

$$\left.\begin{aligned} \mathbf{Q}_1 &= \mathbf{i}P_{11} + \mathbf{j}P_{21} + \mathbf{k}P_{31}, \\ \mathbf{Q}_2 &= \mathbf{i}P_{12} + \mathbf{j}P_{22} + \mathbf{k}P_{32}, \\ \mathbf{Q}_3 &= \mathbf{i}P_{13} + \mathbf{j}P_{23} + \mathbf{k}P_{33}. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (35)$$

Half the sum of the stresses $\mathbf{P}_{\mathrm{N}}$ and $\mathbf{Q}_{\mathrm{N}}$ is an ordinary irrotational stress; so that

$$\left.\begin{aligned} \mathbf{P}_{\mathrm{N}} &= \phi_0\mathbf{N} + \mathrm{V}\boldsymbol{\epsilon}\mathbf{N} \\ \mathbf{Q}_{\mathrm{N}} &= \phi_0\mathbf{N} - \mathrm{V}\boldsymbol{\epsilon}\mathbf{N}, \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (36)$$

where ϕ_0 is self-conjugate, and

$$2\mathbf{\epsilon} = \mathbf{i}(P_{23} - P_{32}) + \mathbf{j}(P_{31} - P_{13}) + \mathbf{k}(P_{12} - P_{21}). \qquad (37)$$

Here $2\mathbf{\epsilon}$ is the torque per unit volume arising from the stress **P**.

The translational force, **F**, per unit volume is (by inspection of a unit cube)

$$\mathbf{F} = \nabla_1\mathbf{P}_1 + \nabla_2\mathbf{P}_2 + \nabla_3\mathbf{P}_3 \qquad (38)$$

$$= \mathbf{i}\,\mathrm{div}\,\mathbf{Q}_1 + \mathbf{j}\,\mathrm{div}\,\mathbf{Q}_2 + \mathbf{k}\,\mathrm{div}\,\mathbf{Q}_3; \qquad (39)$$

or, in terms of the self-conjugate stress and the torque,

$$\mathbf{F} = (\mathbf{i}\,\mathrm{div}\,\phi_0\mathbf{i} + \mathbf{j}\,\mathrm{div}\,\phi_0\mathbf{j} + \mathbf{k}\,\mathrm{div}\,\phi_0\mathbf{k}) - \mathrm{curl}\,\mathbf{\epsilon}, \qquad (40)$$

where $-\mathrm{curl}\,\mathbf{\epsilon}$ is the translational force due to the rotational stress alone, as in Sir W. THOMSON's latest theory of the mechanics of an "ether."*

Next, let **N** be the unit normal drawn outward from any closed surface. Then

$$\Sigma\mathbf{P}_{\mathrm{N}} = \Sigma\mathbf{F}, \qquad (41)$$

where the left summation extends over the surface and the right summation throughout the enclosed region. For

$$\begin{aligned}\mathbf{P}_{\mathrm{N}} &= N_1\mathbf{P}_1 + N_2\mathbf{P}_2 + N_3\mathbf{P}_3 \\ &= \mathbf{i}.\mathbf{N}\mathbf{Q}_1 + \mathbf{j}.\mathbf{N}\mathbf{Q}_2 + \mathbf{k}.\mathbf{N}\mathbf{Q}_3; \qquad (42)\end{aligned}$$

so the well-known theorem of divergence gives immediately, by (39),

$$\Sigma\mathbf{P}_{\mathrm{N}} = \Sigma(\mathbf{i}\,\mathrm{div}\,\mathbf{Q}_1 + \mathbf{j}\,\mathrm{div}\,\mathbf{Q}_2 + \mathbf{k}\,\mathrm{div}\,\mathbf{Q}_3) = \Sigma\mathbf{F}. \qquad (43)$$

Next, as regards the equivalence of rotational effect of the surface-stress to that of the internal forces and torques. Let **r** be the vector distance from any fixed origin. Then $\mathrm{V}\mathbf{r}\mathbf{F}$ is the vector moment of a force, **F**, at the end of the arm **r**. Another (not so immediate) application of the divergence theorem gives

$$\Sigma\mathrm{V}\mathbf{r}\mathbf{P}_{\mathrm{N}} = \Sigma\mathrm{V}\mathbf{r}\mathbf{F} + \Sigma 2\mathbf{\epsilon}, \qquad (44)$$

Thus, any distribution of stress, whether rotational or irrotational, may be regarded as in equilibrium. Given any stress in a body, terminating at its boundary, the body will be in equilibrium both as regards translation and rotation. Of course, the boundary discontinuity in the stress has to be reckoned as the equivalent of internal divergence in the appropriate manner. Or, more simply, let the stress fall off continuously from the finite internal stress to zero through a thin surface-layer. We

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* 'Mathematical and Physical Papers,' vol. 3, Art. 99, p. 436.

then have a distribution of forces and torques in the surface-layer which equilibrate the internal forces and torques.

To illustrate; we know that MAXWELL arrived at a peculiar stress, compounded of a tension parallel to a certain direction, and an equal lateral pressure, which would account for the mechanical actions apparent between electrified bodies, and endeavoured similarly to determine the stress in the interior of a magnetised body to harmonise with the similar external magnetic stress of the simple type mentioned. This stress in a magnetised body I believe to be thoroughly erroneous; nevertheless, so far as accounting for the forcive on a magnetised body is concerned, it will, when properly carried out with due attention to surface-discontinuity, answer perfectly well, not because it *is* the stress, but because *any* stress would do the same, the only essential feature concerned being the external stress in the air.

Here we may also note the very powerful nature of the stress-function, considered merely as a mathematical engine, apart from physical reality. For example, we may account for the forcive on a magnet in many ways, of which the two most prominent are by means of forces on imaginary magnetic matter, and by forces on imaginary electric currents, in the magnet and on its surface. To prove the equivalence of these two methods (and the many others) involves very complex surface- and volume-integrations and transformations in the general case, which may be all avoided by the use of the stress-function instead of the forces.

§ **11.** Next as regards the activity of the stress $\mathbf{P}_N$ and the equivalent translational, distortional, and rotational activities. The activity of $\mathbf{P}_N$ is $\mathbf{P}_N\mathbf{q}$ per unit area, if $\mathbf{q}$ be the velocity. Here

$$\mathbf{P}_N\mathbf{q} = q_1.\mathbf{N}\mathbf{Q}_1 + q_2.\mathbf{N}\mathbf{Q}_2 + q_3.\mathbf{N}\mathbf{Q}_3, \quad \ldots\ldots\ldots\ldots \quad (45)$$

by (42); or, re-arranging,

$$\begin{aligned}\mathbf{P}_N\mathbf{q} &= \mathbf{N}\,(q_1\mathbf{Q}_1 + q_2\mathbf{Q}_2 + q_3\mathbf{Q}_3) = \mathbf{N}\Sigma\mathbf{Q}q,\\ &= \mathbf{N}q\mathbf{Q}_q, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (46)\end{aligned}$$

where $\mathbf{Q}_q$ is the conjugate stress on the $\mathbf{q}$ plane. That is, $q\mathbf{Q}_q$ or $\Sigma\mathbf{Q}q$ is the negative of the vector flux of energy expressing the stress-activity. For we choose P_{NN} so as to mean a pull when it is positive, and when the stress $\mathbf{P}_N$ works in the same sense with $\mathbf{q}$ energy is transferred against the motion, to the matter which is pulled.

The convergence of the energy-flux, or the divergence of $q\mathbf{Q}_q$, is therefore the activity per unit volume. Thus

$$\mathrm{div}\,(\mathbf{Q}_1q_1 + \mathbf{Q}_2q_2 + \mathbf{Q}_3q_3) = \mathbf{q}\,(\mathbf{i}\,\mathrm{div}\,\mathbf{Q}_1 + \mathbf{j}\,\mathrm{div}\,\mathbf{Q}_2 + \mathbf{k}\,\mathrm{div}\,\mathbf{Q}_3) + (\mathbf{Q}_1\nabla q_1 + \mathbf{Q}_2\nabla q_2 + \mathbf{Q}_3\nabla q_3)\,. \quad (47)$$

$$= \mathbf{q}\,(\nabla_1\mathbf{P}_1 + \nabla_2\mathbf{P}_2 + \nabla_3\mathbf{P}_3) + \mathbf{P}_1\nabla_1\mathbf{q} + \mathbf{P}_2\nabla_2\mathbf{q} + \mathbf{P}_3\nabla_3\mathbf{q} \quad \ldots\ldots \quad (48)$$

where the first form (47) is generally most useful. Or

$$\mathrm{div}\,\Sigma\mathbf{Q}q = \mathbf{F}\mathbf{q} + \Sigma\mathbf{Q}\nabla q\,; \tag{49}$$

where the first term on the right is the translational activity, and the rest is the sum of the distortional and rotational activities. To separate the latter introduce the strain velocity vectors (analogous to $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$)

$$\mathbf{p}_1 = \tfrac{1}{2}(\nabla q_1 + \nabla_1\mathbf{q}), \qquad \mathbf{p}_2 = \tfrac{1}{2}(\nabla q_2 + \nabla_2\mathbf{q}), \qquad \mathbf{p}_3 = \tfrac{1}{2}(\nabla q_3 + \nabla_3\mathbf{q})\,; \tag{50}$$

and generally

$$\mathbf{p}_{\mathrm{N}} = \tfrac{1}{2}(\nabla.\mathbf{q}\mathbf{N} + \mathbf{N}\nabla.\mathbf{q}). \tag{51}$$

Using these we obtain

$$\begin{aligned}\Sigma\mathbf{Q}\nabla\mathbf{q} &= \mathbf{Q}_1\mathbf{p}_1 + \mathbf{Q}_2\mathbf{p}_2 + \mathbf{Q}_3\mathbf{p}_3 + \mathbf{Q}_1\frac{\nabla q_1 - \nabla_1\mathbf{q}}{2} + \mathbf{Q}_2\frac{\nabla q_2 - \nabla_2\mathbf{q}}{2} + \mathbf{Q}_3\frac{\nabla q_3 - \nabla_3\mathbf{q}}{2} \\ &= \Sigma\mathbf{Q}\mathbf{p} + \tfrac{1}{2}\,\mathbf{Q}_1\mathrm{V}\mathbf{i}\ \mathrm{curl}\ \mathbf{q} + \tfrac{1}{2}\,\mathbf{Q}_2\mathrm{V}\mathbf{j}\ \mathrm{curl}\ \mathbf{q} + \tfrac{1}{2}\,\mathbf{Q}_3\mathrm{V}\mathbf{k}\ \mathrm{curl}\ \mathbf{q} \\ &= \Sigma\mathbf{Q}\mathbf{p} + \boldsymbol{\epsilon}\ \mathrm{curl}\ \mathbf{q}. \end{aligned} \tag{52}$$

Thus $\Sigma\mathbf{Q}\mathbf{p}$ is the distortional activity and $\boldsymbol{\epsilon}$ curl $\mathbf{q}$ the rotational activity. But since the distortion and the rotation are quite independent, we may put $\Sigma\mathbf{P}\mathbf{p}$ for the distortional activity; or else use the self-conjugate stress, and write it $\frac{1}{2}\Sigma(\mathbf{P}+\mathbf{Q})\,\mathbf{p}$.

§ **12.** In an ordinary "elastic solid," when isotropic, there is elastic resistance to compression and to distortion. We may also imaginably have elastic resistance to translation and to rotation; nor is there, so far as the mathematics is concerned, any reason for excluding dissipative resistance to translation, distortion, and rotation; and kinetic energy may be associated with all three as well, instead of with the translation alone, as in the ordinary elastic solid.

Considering only three elastic moduli, we have the old k and n of THOMSON and TAIT (resistance to compression and rigidity), and a new coefficient, say n_1, such that

$$\boldsymbol{\epsilon} = n_1\ \mathrm{curl}\ \mathbf{D}, \tag{53}$$

if $\mathbf{D}$ be the displacement and $2\boldsymbol{\epsilon}$ the torque, as before.

The stress on the $\mathbf{i}$ plane (any plane) is

$$\begin{aligned}\mathbf{P}_1 &= n(\nabla\mathrm{D}_1 + \nabla_1\mathbf{D}) + \mathbf{i}\,(k - \tfrac{2}{3}n)\ \mathrm{div}\ \mathbf{D} + n_1\ \mathrm{V}\ \mathrm{curl}\ \mathbf{D}.\mathbf{i} \\ &= (n + n_1)\,\nabla_1\mathbf{D} + (n - n_1)\,\nabla\mathrm{D}_1 + (k - \tfrac{2}{3}n)\ \mathbf{i}\ \mathrm{div}\ \mathbf{D}\,; \end{aligned} \tag{54}$$

and its conjugate is

$$\begin{aligned}\mathbf{Q}_1 &= n(\nabla\mathrm{D}_1 + \nabla_1\mathbf{D}) + \mathbf{i}\,(k - \tfrac{2}{3}n)\ \mathrm{div}\ \mathbf{D} - n_1(\nabla_1\mathbf{D} - \nabla\mathrm{D}_1) \\ &= (n - n_1)\,\nabla_1\mathbf{D} + (n + n_1)\,\nabla\mathrm{D}_1 + \mathbf{i}\,(k - \tfrac{2}{3}n)\ \mathrm{div}\ \mathbf{D}\,; \end{aligned} \tag{55}$$

from which

$$\mathrm{F}_1 = \mathrm{div}\ \mathbf{Q}_1 = (n - n_1 + k - \tfrac{2}{3}n)\,\nabla_1\ \mathrm{div}\ \mathbf{D} + (n + n_1)\,\nabla^2\mathrm{D}_1 \tag{56}$$

is the $\mathbf{i}$ component of the translational force; the complete force $\mathbf{F}$ is therefore

$$\mathbf{F} = (n + n_1)\,\nabla^2\mathbf{D} + (k + \tfrac{1}{3}n - n_1)\,\nabla\,\mathrm{div}\,\mathbf{D}\,; \quad \text{. . . . .} \quad (57)$$

or, in another form, if

$$\mathrm{P} = -\,k\,\mathrm{div}\,\mathbf{D},$$

P being the isotropic pressure,

$$\mathbf{F} = -\,\nabla\mathrm{P} + n\,(\nabla^2\mathbf{D} + \tfrac{1}{3}\,\nabla\,\mathrm{div}\,\mathbf{D}) - n_1\,\mathrm{curl}^2\,\mathbf{D}, \quad \text{. . . . .} \quad (58)$$

remembering (23) and (53).

We see that in (57) the term involving div **D** may vanish in a compressible solid by the relation $n_1 = k + \frac{1}{3}\,n$; this makes

$$n + n_1 = k + \tfrac{4}{3}n, \qquad n_1 - n = k - \tfrac{2}{3}n, \quad \text{. . . . .} \quad (59)$$

which are the moduli, longitudinal and lateral, of a simple longitudinal strain; that is, multiplied by the extension, they give the longitudinal traction, and the lateral traction required to prevent lateral contraction.

The activity per unit volume, other than translational, is

$$\begin{aligned}\Sigma\mathbf{Q}\nabla q &= (n - n_1)\,\big(\nabla_1\mathbf{D}.\nabla q_1 + \nabla_2\mathbf{D}.\nabla q_2 + \nabla_3\mathbf{D}.\nabla q_3\big)\\ &\quad + (n + n_1)\,\big(\nabla\mathrm{D}_1.\nabla q_1 + \nabla\mathrm{D}_2.\nabla q_2 + \nabla\mathrm{D}_3.\nabla q_3\big)\\ &\quad + (k - \tfrac{2}{3}n)\,\mathrm{div}\,\mathbf{D}\,\mathrm{div}\,\mathbf{q}\\ &= n\,\big(\nabla_1\mathbf{D}.\nabla q_1 + \nabla_2\mathbf{D}.\nabla q_2 + \nabla_3\mathbf{D}.\nabla q_3 + \nabla\mathrm{D}_1\nabla q_1 + \nabla\mathrm{D}_2\nabla q_2 + \nabla\mathrm{D}_3\nabla q_3\big)\\ &\quad + (k - \tfrac{2}{3}n)\,\mathrm{div}\,\mathbf{D}\,\mathrm{div}\,\mathbf{q} + n_1\,\mathrm{curl}\,\mathbf{D}\,\mathrm{curl}\,\mathbf{q}\,; \quad \text{. . . . .} \quad (60)\end{aligned}$$

or, which is the same,

$$\begin{aligned}\Sigma\mathbf{Q}\nabla q = \frac{d}{dt}\Big[&\tfrac{1}{2}k\,(\mathrm{div}\,\mathbf{D})^2 + \tfrac{1}{2}n_1\,(\mathrm{curl}\,\mathbf{D})^2\\ &+ \tfrac{1}{2}n\{(\nabla\mathrm{D}_1)^2 + (\nabla\mathrm{D}_2)^2 + (\nabla\mathrm{D}_3)^2 + \nabla\mathrm{D}_1.\nabla_1\mathbf{D} + \nabla\mathrm{D}_2.\nabla_2\mathbf{D} + \nabla\mathrm{D}_3.\nabla_3\mathbf{D}\} - \tfrac{1}{3}n\,(\mathrm{div}\,\mathbf{D})^2\Big], \quad . \quad (61)\end{aligned}$$

where the quantity in square brackets is the potential energy of an *infinitesimal* distortion and rotation. The italicised reservation appears to be necessary, as we shall see from the equation of activity later, that the convection of the potential energy destroys the completeness of the statement

$$\Sigma\mathbf{Q}\nabla q = \dot{\mathrm{U}},$$

if U be the potential energy.

In an elastic solid of the ordinary kind, with $n_1 = 0$, we have

$$\left.\begin{aligned}\mathbf{P}_{\mathrm{N}} &= n\,\big(2\,\mathrm{curl}\,\mathrm{V}\mathbf{D}\mathbf{N} + \mathrm{V}\mathbf{N}\,\mathrm{curl}\,\mathbf{D}\big),\\ \mathbf{F} &= -\,n\,\mathrm{curl}^2\,\mathbf{D}.\end{aligned}\right\} \quad \text{. . . . .} \quad (62)$$

In the case of a medium in which n is zero but n_1 finite (Sir W. THOMSON's rotational ether),

$$\left.\begin{aligned}\mathbf{P}_{\mathrm{N}} &= n_1 \mathrm{V}\,\mathrm{curl}\,\mathbf{D}.\mathbf{N},\\ \mathbf{F} &= -\,n_1\,\mathrm{curl}^2\,\mathbf{D}.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (63)$$

Thirdly, if we have both $k = -\frac{4}{3}n$ and $n = n_1$, then

$$\left.\begin{aligned}\mathbf{P}_{\mathrm{N}} &= 2n\,\mathrm{curl}\,\mathrm{V}\mathbf{D}\mathbf{N},\\ \mathbf{F} &= -\,2n\,\mathrm{curl}^2\,\mathbf{D},\end{aligned}\right\} (\mathbf{E} = n\,\mathrm{curl}\,\mathbf{D}), \quad \ldots\ldots \quad (64)$$

i.e., the sums of the previous two stresses and forces.

§ **13.** As already observed, the vector flux of energy, due to the stress, is

$$-\Sigma\mathbf{Q}q = -\mathbf{Q}_q q = -(\mathbf{Q}_1 q_1 + \mathbf{Q}_2 q_2 + \mathbf{Q}_3 q_3). \quad \ldots\ldots\ldots \quad (65)$$

Besides this, there is the flux of energy

$$\mathbf{q}\,(\mathrm{U} + \mathrm{T})$$

by convection, where U is potential and T kinetic energy. Therefore,

$$\mathbf{W} = \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma\mathbf{Q}q \quad \ldots\ldots\ldots\ldots \quad (66)$$

represents the complete energy flux, so far as the stress and motion are concerned. Its convergence increases the potential energy, the kinetic energy, or is dissipated. But if there be an impressed translational force **f** its activity is **fq**. This supply of energy is independent of the convergence of **W**. Hence

$$\mathbf{fq} + \mathrm{Q} + \dot{\mathrm{U}} = \dot{\mathrm{T}} + \mathrm{div}\left[\mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma\mathbf{Q}q\right] \quad \ldots\ldots\ldots \quad (67)$$

is the equation of activity.

But this splits into two parts at least. For (67) is the same as

$$(\mathbf{f} + \mathbf{F})\,\mathbf{q} + \Sigma\mathbf{Q}\nabla q = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathrm{div}\,\mathbf{q}\,(\mathrm{U} + \mathrm{T}), \quad \ldots\ldots\ldots \quad (68)$$

and the translational portion may be removed altogether. That is,

$$(\mathbf{f} + \mathbf{F})\,\mathbf{q} = \mathrm{Q}_0 + \dot{\mathrm{U}}_0 + \dot{\mathrm{T}}_0 + \mathrm{div}\,\mathbf{q}\,(\mathrm{U}_0 + \mathrm{T}_0), \quad \ldots\ldots\ldots \quad (69)$$

if the quantities with the zero suffix are only translationally involved. For example, if

$$\mathbf{f} + \mathbf{F} = \rho\frac{\partial\mathbf{q}}{\partial t}, \quad \ldots\ldots\ldots\ldots \quad (70)$$

as in fluid motion, without frictional or elastic forces associated with the translation, then

$$(\mathbf{f} + \mathbf{F})\,\mathbf{q} = \rho \mathbf{q} \frac{\partial \mathbf{q}}{\partial t} = \dot{\mathrm{T}} + \operatorname{div} \mathbf{q}\mathrm{T}, \qquad (71)$$

if $\mathrm{T} = \frac{1}{2}\rho q^2$, the kinetic energy per unit volume. The complete form (69) comes in by the addition of elastic and frictional resisting forces. So deducting (69) from (68) there is left

$$\Sigma \mathbf{Q} \nabla q = \mathrm{Q}_1 + \dot{\mathrm{U}}_1 + \dot{\mathrm{T}}_1 + \operatorname{div} \mathbf{q}\,(\mathrm{U}_1 + \mathrm{T}_1), \qquad (72)$$

where the quantities with suffix unity are connected with the distortion and the rotation, and there may plainly be two sets of dissipative terms, and of energy (stored) terms. Thus the relation

$$\boldsymbol{\epsilon} = \left(n_1 + n_2 \frac{d}{dt} + n_3 \frac{d^2}{dt^2}\right) \operatorname{curl} \mathbf{D} \qquad (73)$$

will bring in dissipation and kinetic energy, as well as the former potential energy of rotation associated with n_1.

That there can be dissipative terms associated with the distortion is also clear enough, remembering STOKES's theory of a viscous fluid. Thus, for simplicity, do away with the rotating stress, by putting $\boldsymbol{\epsilon} = 0$, making $\mathbf{P}_{\mathrm{N}}$ and $\mathbf{Q}_{\mathrm{N}}$ identical. Then take the stress on the **i** plane to be given by

$$\mathbf{P}_1 = \left(n + \mu \frac{d}{dt} + \nu \frac{d^2}{dt^2}\right)(\nabla \mathrm{D}_1 + \nabla_1 \mathbf{D}) - \mathbf{i}\left\{\mathrm{P} + \tfrac{2}{3}\left(n + \mu \frac{d}{dt} + \nu \frac{d^2}{dt^2}\right) \operatorname{div} \mathbf{D}\right\}, \qquad (74)$$

and similarly for any other plane; where $\mathrm{P} = -k \operatorname{div} \mathbf{D}$.

When $\mu = 0$, $\nu = 0$, we have the elastic solid with rigidity and compressibility. When $n = 0$, $\nu = 0$, we have the viscous fluid of STOKES. When $\nu = 0$ only, we have a viscous elastic solid, the viscous resistance being purely distortional, and proportional to the speed of distortion. But with n, μ, ν, all finite, we still further associate kinetic energy with the potential energy and dissipation introduced by n and μ.

We have

$$\Sigma \mathbf{P} \nabla q = \mathrm{Q}_2 + \dot{\mathrm{U}}_2 + \dot{\mathrm{T}}_2$$

for infinitesimal strains, omitting the effect of convection of energy; where

$$\mathrm{T}_2 = \tfrac{1}{2}\nu\left[-\tfrac{2}{3}(\operatorname{div} \mathbf{q})^2 + \nabla q_1(\nabla q_1 + \nabla_1 \mathbf{q}) + \nabla q_2(\nabla q_2 + \nabla_2 \mathbf{q}) + \nabla q_3(\nabla q_3 + \nabla_3 \mathbf{q})\right], \qquad (75)$$

$$\mathrm{Q}_2 = \mu\left[-\tfrac{2}{3}(\operatorname{div} \mathbf{q})^2 + \nabla q_1(\nabla q_1 + \nabla_1 \mathbf{q}) + \nabla q_2(\nabla q_2 + \nabla_2 \mathbf{q}) + \nabla q_3(\nabla q_3 + \nabla_3 \mathbf{q})\right], \qquad (76)$$

$$\mathrm{U}_2 = \tfrac{1}{2}n\left[\left(\frac{k}{n} - \tfrac{2}{3}\right)(\operatorname{div} \mathbf{D})^2 + \nabla \mathrm{D}_1(\nabla \mathrm{D}_1 + \nabla_1 \mathbf{D}) + \nabla \mathrm{D}_2(\nabla \mathrm{D}_2 + \nabla_2 \mathbf{D}) + \nabla \mathrm{D}_3(\nabla \mathrm{D}_3 + \nabla_3 \mathbf{D})\right]. \qquad (77)$$

Observe that T_2 and Q_2 only differ in the exchange of μ to $\frac{1}{2}\nu$; but U_2, the potential energy, is not the same function of n and $\mathbf{D}$ that T_2 is of ν and $\mathbf{q}$. But if we take $k = 0$, we produce similarity. An elastic solid having no resistance to compression is also one of Sir W. THOMSON'S ethers.

When $n = 0$, $\mu = 0$, $\nu = 0$, we come down to the frictionless fluid, in which

$$\mathbf{f} - \nabla P = \rho \frac{\partial \mathbf{q}}{\partial t}, \qquad (78)$$

and

$$\Sigma \mathbf{P} \nabla q = - P \operatorname{div} \mathbf{q}, \qquad (79)$$

with the equation of activity

$$\mathbf{fq} = \dot{U} + \dot{T} + \operatorname{div} (U + T + P) \mathbf{q}, \qquad (80)$$

the only parts of which are not always easy to interpret are the $P\mathbf{q}$ term, and the proper measure of U. By analogy, and conformably with more general cases, we should take

$$P = - k \operatorname{div} \mathbf{D}, \qquad \text{and} \qquad U = \tfrac{1}{2} k (\operatorname{div} \mathbf{D})^2,$$

reckoning the expansion or compression from some mean condition.

The Electromagnetic Equations in a Moving Medium.

§ **14.** The study of the forms of the equation of activity in purely mechanical cases, and the interpretation of the same is useful, because in the electromagnetic problem of a moving medium we have still greater generality, and difficulty of safe and sure interpretation. To bring it as near to abstract dynamics as possible, all we need say regarding the two fluxes, electric displacement **D** and magnetic induction **B**, is that they are linear functions of the electric force **E** and magnetic force **H**, say

$$\mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{D} = c\mathbf{E}, \qquad (81)$$

where c and μ are linear operators of the symmetrical kind, and that associated with them are the stored energies U and T, electric and magnetic respectively (per unit volume), given by

$$U = \tfrac{1}{2} \mathbf{ED}, \qquad T = \tfrac{1}{2} \mathbf{HB}, \qquad (82)$$

In isotropic media c is the permittivity, μ the inductivity. It is unnecessary to say more regarding the well-known variability of μ and hysteresis than that a magnet is here an ideal magnet of constant inductivity.

As there may be impressed forces, **E** is divisible into the force of the field and an impressed part; for distinctness, then, the complete **E** may be called the "force of the flux" **D**. Similarly as regards **H** and **B**.

There is also waste of energy (in conductors, namely) at the rates

$$Q_1 = \mathbf{EC}, \qquad Q_2 = \mathbf{HK}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

where the fluxes **C** and **K** are also linear functions of **E** and **H** respectively; thus

$$\mathbf{C} = k\mathbf{E}, \qquad \mathbf{K} = g\mathbf{H}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

where, when the force is parallel to the flux, and k is scalar, it is the electric conductivity. Its magnetic analogue is g, the magnetic conductivity. That is, a magnetic conductor is a (fictitious) body which cannot support magnetic force without continuously dissipating energy.

Electrification is the divergence of the displacement, and its analogue, magnetification, is the divergence of the induction; thus

$$\rho = \operatorname{div} \mathbf{D}, \qquad \sigma = \operatorname{div} \mathbf{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

are their volume densities. The quantity σ is probably quite fictitious, like **K**.

According to MAXWELL'S doctrine, the true electric current is always circuital, and is the sum of the conduction current and the current of displacement, which is the time rate of increase of the displacement. But, to preserve circuitality, we must add the convection current when electrification is moving, so that the true current becomes

$$\mathbf{J} = \mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

where **q** is the velocity of the electrification ρ. Similarly

$$\mathbf{G} = \mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

should be the corresponding magnetic current.

§ **15.** MAXWELL'S equation of electric current in terms of magnetic force in a medium at rest, say,

$$\operatorname{curl} \mathbf{H}_1 = \mathbf{C} + \dot{\mathbf{D}},$$

where $\mathbf{H}_1$ is the force of the field, should be made, using **H** instead,

$$\operatorname{curl} (\mathbf{H} - \mathbf{h}_0) = \mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho,$$

and here $\mathbf{h}_0$ will be the force of intrinsic magnetisation, such that $\mu\mathbf{h}_0$ is the intensity of intrinsic magnetisation. But I have shown that when there is motion, another impressed term is required, viz., the motional magnetic force

$$\mathbf{h} = \mathrm{V}\mathbf{Dq}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

making the first circuital law become

$$\text{curl}\,(\mathbf{H}-\mathbf{h}_0-\mathbf{h})=\mathbf{J}=\mathbf{C}+\dot{\mathbf{D}}+\mathbf{q}\rho. \quad \ldots\ldots \quad (89)$$

MAXWELL'S other connection to form the equations of propagation is made through his vector-potential $\mathbf{A}$ and scalar potential Ψ. Finding this method not practically workable, and also not sufficiently general, I have introduced instead a companion equation to (89) in the form

$$-\text{curl}\,(\mathbf{E}-\mathbf{e}_0-\mathbf{e})=\mathbf{G}=\mathbf{K}+\dot{\mathbf{B}}+\mathbf{q}\sigma, \quad \ldots\ldots \quad (90)$$

where $\mathbf{e}_0$ expresses intrinsic force, and $\mathbf{e}$ is the motional electric force given by

$$\mathbf{e}=\mathrm{V}\mathbf{q}\mathbf{B}, \quad \ldots\ldots \quad (91)$$

which is one of the terms in MAXWELL'S equation of electromotive force. As for $\mathbf{e}_0$, it includes not merely the force of intrinsic electrification, the analogue of intrinsic magnetisation, but also the sources of energy, voltaic force, thermoelectric force, &c.

(89) and (90) are thus the working equations, with (88) and (91) in case the medium moves; along with the linear relations before mentioned, and the definitions of energy and waste of energy per unit volume. The fictitious $\mathbf{K}$ and σ are useful in symmetrizing the equations, if for no other purpose.

Another way of writing the two equations of curl is by removing the $\mathbf{e}$ and $\mathbf{h}$ terms to the right side. Let

$$\left.\begin{array}{ll}\text{curl}\,\mathbf{h}=\mathbf{j}, & \mathbf{J}+\mathbf{j}=\mathbf{J}_0, \\ -\text{curl}\,\mathbf{e}=\mathbf{g}, & \mathbf{G}+\mathbf{g}=\mathbf{G}_0.\end{array}\right\} \quad \ldots\ldots \quad (92)$$

Then (89) and (90) may be written

$$\left.\begin{array}{r}\text{curl}\,(\mathbf{H}-\mathbf{h}_0)=\mathbf{J}_0=\mathbf{C}+\dot{\mathbf{D}}+\mathbf{q}\rho+\mathbf{j}, \\ -\text{curl}\,(\mathbf{E}-\mathbf{e}_0)=\mathbf{G}_0=\mathbf{K}+\dot{\mathbf{B}}+\mathbf{q}\sigma+\mathbf{g}.\end{array}\right\} \quad \ldots\ldots \quad (93)$$

So far as circuitality of the current goes, the change is needless, and still further complicates the make-up of the true current, supposed now to be $\mathbf{J}_0$. On the other hand, it is a simplification on the left side, deriving the current from the force of the flux or of the field more simply.

A question to be settled is whether $\mathbf{J}$ or $\mathbf{J}_0$ should be the true current. There seems only one crucial test, viz., to find whether $\mathbf{e}_0\mathbf{J}$ or $\mathbf{e}_0\mathbf{J}_0$ is the rate of supply of energy to the electromagnetic system by an intrinsic force $\mathbf{e}_0$. This requires, however, a full and rigorous examination of all the fluxes of energy concerned.

The Electromagnetic Flux of Energy in a stationary Medium.

§ **16.** First let the medium be at rest, giving us the equations

$$\mathrm{curl}\,(\mathbf{H}-\mathbf{h}_0)=\mathbf{J}=\mathbf{C}+\dot{\mathbf{D}}, \qquad (94)$$

$$-\,\mathrm{curl}\,(\mathbf{E}-\mathbf{e}_0)=\mathbf{G}=\mathbf{H}+\dot{\mathbf{B}}. \qquad (95)$$

Multiply (94) by $(\mathbf{E}-\mathbf{e}_0)$, and (95) by $(\mathbf{H}-\mathbf{h}_0)$, and add the results. Thus,

$$(\mathbf{E}-\mathbf{e}_0)\,\mathbf{J}+(\mathbf{H}-\mathbf{h}_0)\,\mathbf{G}=(\mathbf{E}-\mathbf{e}_0)\,\mathrm{curl}\,(\mathbf{H}-\mathbf{h}_0)-(\mathbf{H}-\mathbf{h}_0)\,\mathrm{curl}\,(\mathbf{E}-\mathbf{e}_0),$$

which, by the formula (25), becomes

$$\mathbf{e}_0\mathbf{J}+\mathbf{h}_0\mathbf{G}=\mathbf{E}\mathbf{J}+\mathbf{H}\mathbf{G}+\mathrm{div}\,\mathrm{V}\,(\mathbf{E}-\mathbf{e}_0)\,(\mathbf{H}-\mathbf{h}_0)\,;$$

or, by the use of (82), (83),

$$\mathbf{e}_0\mathbf{J}+\mathbf{h}_0\mathbf{G}=\mathrm{Q}+\dot{\mathrm{U}}+\dot{\mathrm{T}}+\mathrm{div}\,\mathbf{W}, \qquad (96)$$

where the new vector **W** is given by

$$\mathbf{W}=\mathrm{V}\,(\mathbf{E}-\mathbf{e}_0)\,(\mathbf{H}-\mathbf{h}_0). \qquad (97)$$

The form of (96) is quite explicit, and the interpretation sufficiently clear. The left side indicates the rate of supply of energy from intrinsic sources. These $(\mathrm{Q}+\dot{\mathrm{U}}+\dot{\mathrm{T}})$ shows the rate of waste and of storage of energy in this unit volume. The remainder, therefore, indicates the rate at which energy is passed out from the unit volume; and the flux **W** represents the flux of energy necessitated by the postulated localisation of energy and its waste, when **E** and **H** are connected in the manner shown by (94) and (95).

There might also be an independent circuital flux of energy, but, being useless, it would be superfluous to bring it in.

The very important formula (97) was first discovered and interpreted by Professor Poynting, and independently discovered and interpreted a little later by myself in an extended form. It will be observed that in my mode of proof above there is no limitation as to homogeneity or isotropy as regards the permittivity, inductivity, and conductivity. But c and μ should be symmetrical. On the other hand, k and g do not require this limitation in deducing (97).*

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* The method of treating Maxwell's electromagnetic scheme employed in the text (first introduced in "Electromagnetic Induction and its Propagation," 'The Electrician,' January 3, 1885, and later)

It is important to recognize that this flux of energy is not dependent upon the translational motion of the medium, for it is assumed explicitly to be at rest. The vector **W** cannot, therefore, be a flux of the kind $\mathbf{Q}_q q$ before discussed, unless possibly it be merely a rotating stress that is concerned.

The only dynamical analogy with which I am acquainted which seems at all satisfactory is that furnished by Sir W. THOMSON'S theory of a rotational ether. Take the case of $\mathbf{e}_0 = 0$, $\mathbf{h}_0 = 0$, $k = 0$, $g = 0$, and c and μ constants, that is, pure ether uncontaminated by ordinary matter. Then

$$\text{curl}\,\mathbf{H} = c\dot{\mathbf{E}}, \qquad (98)$$

$$-\,\text{curl}\,\mathbf{E} = \mu\dot{\mathbf{H}}. \qquad (99)$$

Now, let **H** be velocity, μ density; then, by (99), $-$ curl **E** is the translational force due to the stress, which is, therefore, a rotating stress; thus,

$$\mathbf{P}_N = \mathrm{V}\mathbf{EN}, \qquad \mathbf{Q}_N = \mathrm{V}\mathbf{NE}; \qquad (100)$$

and $2\mathbf{E}$ is the torque. The coefficient c represents the compliancy or reciprocal of the quasi-rigidity. The kinetic energy $\frac{1}{2}\mu \mathrm{H}^2$ represents the magnetic energy, and the potential energy of the rotation represents the electric energy; whilst the flux of energy is $\mathrm{V}\mathbf{EH}$. For the activity of the torque is

$$2\mathbf{E}.\frac{\text{curl}\,\mathbf{H}}{2} = \mathbf{E}\,\text{curl}\,\mathbf{H},$$

and the translational activity is

$$-\,\mathbf{H}\,\text{curl}\,\mathbf{E}.$$

Their sum is

may, perhaps, be appropriately termed the Duplex method, since its characteristics are the exhibition of the electric, magnetic, and electromagnetic relations in a duplex form, symmetrical with respect to the electric and magnetic sides. But it is not merely a method of exhibiting the relations in a manner suitable to the subject, bringing to light useful relations which were formerly hidden from view by the intervention of the vector-potential and its parasites, but constitutes a method of working as well. There are considerable difficulties in the way of the practical employment of MAXWELL'S equations of propagation, even as they stand in his treatise. These difficulties are greatly magnified when we proceed to more general cases, involving heterogeneity and eolotropy and motion of the medium supporting the fluxes. The duplex method supplies what is wanted. Potentials do not appear, at least initially. They are regarded strictly as auxiliary functions which do not represent any physical state of the medium. In special problems they may be of great service for calculating purposes; but in general investigations their avoidance simplifies matters greatly. The state of the field is settled by **E** and **H**, and these are the primary objects of attention in the duplex system.

As the papers to which I have referred are not readily accessible, I may take this opportunity of mentioning that a Reprint of my 'Electrical Papers' is in the press (MACMILLAN and Co.), and that the first volume is nearly ready.

$$-\operatorname{div} \mathbf{VEH},$$

making **VEH** the flux of energy.*

All attempts to construct an elastic solid analogy with a distortional stress fail to give satisfactory results, because the energy is wrongly localised, and the flux of energy incorrect. Bearing this in mind, the above analogy is at first sight very enticing. But when we come to remember that the d/dt in (98) and (99) should be $\partial/\partial t$, and find extraordinary difficulty in extending the analogy to include the conduction current, and also remember that the electromagnetic stress has to be accounted for (in other words, the known mechanical forces), the perfection of the analogy, as far as it goes, becomes disheartening. It would further seem, from the explicit assumption that $\mathbf{q} = 0$ in obtaining **W** above, that no analogy of this kind can be sufficiently comprehensive to form the basis of a physical theory. We must go altogether beyond the elastic solid with the additional property of rotational elasticity. I should mention, to avoid misconception, that Sir W. THOMSON does not push the analogy even so far as is done above, or give to μ and c the same interpretation. The particular meaning here given to μ is that assumed by Professor LODGE in his "Modern Views of Electricity," on the ordinary elastic solid theory, however. I have found it very convenient from its making the curl of the electric force be a Newtonian force (per unit volume). When impressed electric force $\mathbf{e}_0$ produces disturbances, their real source is, as I have shown, not the seat of $\mathbf{e}_0$, but of curl $\mathbf{e}_0$. So we may with facility translate problems in electromagnetic waves into elastic solid problems by taking the electromagnetic source to represent the mechanical source of motion, impressed Newtonian force.

Examination of the Flux of Energy in a moving Medium, and Establishment of the Measure of "True" Current.

§ **17.** Now pass to the more general case of a moving medium with the equations

$$\operatorname{curl} \mathbf{H}_1 = \quad \operatorname{curl} (\mathbf{H} - \mathbf{h}_0 - \mathbf{h}) = \mathbf{J} = \mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho, \quad \ldots \ldots \quad (101)$$

$$-\operatorname{curl} \mathbf{E}_1 = -\operatorname{curl} (\mathbf{E} - \mathbf{e}_0 - \mathbf{e}) = \mathbf{G} = \mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma, \quad \ldots \ldots \quad (102)$$

where $\mathbf{E}_1$ is, for brevity, what the force **E** of the flux becomes after deducting the intrinsic and motional forces; and similarly for $\mathbf{H}_1$.

From these, in the same way as before, we deduce

$$(\mathbf{e}_0 + \mathbf{e}) \mathbf{J} + (\mathbf{h}_0 + \mathbf{h}) \mathbf{G} = \mathbf{EJ} + \mathbf{HG} + \operatorname{div} \mathbf{VE}_1\mathbf{H}_1; \quad \ldots \ldots \quad (103)$$

and it would seem at first sight to be the same case again, but with impressed forces

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* This form of application of the rotating ether I gave in 'The Electrician,' January 23, 1891, p. 360.

$(\mathrm{e} + \mathrm{e}_0)$ and $(\mathrm{h} + \mathrm{h}_0)$ instead of e_0 and h_0, whilst the POYNTING flux requires us to reckon only $\mathbf{E}_1$ and $\mathbf{H}_1$ as the effective electric and magnetic forces concerned in it.*

But we must develop $(\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}})$ plainly first. We have, by (86), (87), used in (103),

$$\mathrm{e}_0\mathbf{J} + \mathrm{h}_0\mathbf{G} = \mathbf{E}\,(\mathbf{C} + \dot{\mathbf{D}} + \mathrm{q}\rho) + \mathbf{H}\,(\mathbf{K} + \dot{\mathbf{B}} + \mathrm{q}\sigma) - (\mathrm{e}\mathbf{J} + \mathrm{h}\mathbf{G}) + \operatorname{div} \mathrm{V}\mathbf{E}_1\mathbf{H}_1. \quad . \quad . \quad (104)$$

Now here we have

$$\left.\begin{aligned} \dot{\mathrm{U}} = \frac{d}{dt}\tfrac{1}{2}\mathbf{ED} &= \tfrac{1}{2}\mathbf{E}\dot{\mathbf{D}} + \tfrac{1}{2}\mathbf{D}\dot{\mathbf{E}} \\ &= \mathbf{E}\dot{\mathbf{D}} + \tfrac{1}{2}(\mathbf{D}\dot{\mathbf{E}} - \mathbf{E}\dot{\mathbf{D}}) \\ &= \mathbf{E}\dot{\mathbf{D}} - \tfrac{1}{2}\mathbf{E}\dot{c}\mathbf{E} \\ &= \mathbf{E}\dot{\mathbf{D}} - \dot{\mathrm{U}}_c. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

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* It will be observed that the constant 4π, which usually appears in the electrical equations, is absent from the above investigations. This demands a few words of explanation. The units employed in the text are rational units, founded upon the principle of continuity in space of vector functions, and the corresponding appropriate measure of discontinuity, viz., by the amount of divergence. In popular language, the *unit* pole sends out *one* line of force, in the rational system, instead of 4π lines, as in the irrational system. The effect of the rationalisation is to introduce 4π into the formulæ of central forces and potentials, and to abolish the swarm of 4π's that appear in the practical formulæ of the practice of theory on FARADAY-MAXWELL lines, which receives its fullest and most appropriate expression in the rational method. The rational system was explained by me in 'The Electrician,' in 1882, and applied to the general theory of potentials and connected functions in 1883. (Reprint, vol. 1, p. 199, and later, especially p. 262.) I then returned to irrational formulæ because I did not think, then, that a reform of the units was practicable, partly on account of the labours of the B. A. Committee on Electrical Units, and partly on account of the ignorance of, and indifference to, theoretical matters which prevailed at that time. But the circumstances have greatly changed, and I do think a change is now practicable. There has been great advance in the knowledge of the meaning of MAXWELL's theory, and a diffusion of this knowledge, not merely amongst scientific men, but amongst a large body of practicians called into existence by the extension of the practical applications of electricity. Electricity is becoming, not only a master science, but also a very practical one. It is fitting, therefore, that learned traditions should not be allowed to control matters too greatly, and that the units should be rationalised. To make a beginning, I am employing rational units throughout in my work on "Electromagnetic Theory," commenced in 'The Electrician,' in January, 1891, and continued as fast as circumstances will permit; to be republished in book form. In Section XVII. (October 16, 1891, p. 655), will be found stated more fully the nature of the change proposed, and the reasons for it. I point out, in conclusion, that as regards theoretical treatises and investigations, there is no difficulty in the way, since the connection of the rational and irrational units may be explained separately; and I express the belief that when the merits of the rational system are fully recognised, there will arise a demand for the rationalisation of the practical units. We are, in the opinion of men qualified to judge, within a measurable distance of adopting the metric system in England. Surely the smaller reform I advocate should precede this. To put the matter plainly, the present system of units contains an absurdity running all through it of the same nature as would exist in the metric system of common units were we to define the unit area to be the area of a circle of unit diameter. The absurdity is only different in being less obvious in the electrical case. It would not matter much if it were not that electricity is a practical science.

Comparison of the third with the second form of (105) defines the generalised meaning of $\dot{c}$ when c is not a mere scalar. Or thus,

$$\dot{\mathrm{U}}_c = \mathbf{E}^2\dot{c} = \tfrac{1}{2}\frac{d}{dt}(\mathbf{ED})_c$$

$$= \tfrac{1}{2}\dot{c}_{11}\mathrm{E}_1^2 + \tfrac{1}{2}\dot{c}_{22}\mathrm{E}_2^2 + \tfrac{1}{2}\dot{c}_{33}\mathrm{E}_3^2 + \dot{c}_{12}\mathrm{E}_1\mathrm{E}_2 + \dot{c}_{23}\mathrm{E}_2\mathrm{E}_3 + \dot{c}_{31}\mathrm{E}_1\mathrm{E}_3, \qquad (106)$$

representing the time-variation of U due to variation in the $\dot{c}$'s only.

Similarly

$$\dot{\mathrm{T}} = \mathbf{H}\dot{\mathbf{B}} - \tfrac{1}{2}\mathbf{H}\dot{\mu}\mathbf{H} = \mathbf{H}\dot{\mathbf{B}} - \dot{\mathrm{T}}_\mu, \qquad (107)$$

with the equivalent meaning for $\dot{\mu}$ generalised.

Using these in (104) we have the result

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{q}(\mathbf{E}\rho + \mathbf{H}\sigma) + (\tfrac{1}{2}\mathbf{E}\dot{c}\mathbf{E} + \tfrac{1}{2}\mathbf{H}\dot{\mu}\mathbf{H}) - (\mathbf{eJ} + \mathbf{hG}) + \operatorname{div} \mathrm{V}\mathbf{E}_1\mathbf{H}_1. \qquad (108)$$

Here we have, besides $(\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}})$, terms indicating the activity of a translational force. Thus $\mathbf{E}\rho$ is the force on electrification ρ, and $\mathbf{Eq}\rho$ its activity. Again,

$$\frac{\partial c}{\partial t} = \dot{c} + \mathbf{q}\nabla.c;$$

so that we have

and, similarly,

$$\left.\begin{aligned} \dot{c} &= \frac{\partial c}{\partial t} - \mathbf{q}\nabla.c, \\ \dot{\mu} &= \frac{\partial \mu}{\partial t} - \mathbf{q}\nabla.\mu, \end{aligned}\right\} \qquad (109)$$

the generalised meaning of which is indicated by

$$-\frac{\partial \mathrm{U}_c}{\partial t} + \tfrac{1}{2}\mathbf{E}\dot{c}\mathbf{E} = -\tfrac{1}{2}\mathbf{E}(\mathbf{q}\nabla.c)\mathbf{E} = -\mathbf{q}\nabla\mathrm{U}_c; \qquad (110)$$

where, in terms of scalar products involving $\mathbf{E}$ and $\mathbf{D}$,

$$-\mathbf{q}\nabla\mathrm{U}_c = -\tfrac{1}{2}(\mathbf{E}.\mathbf{q}\nabla.\mathbf{D} - \mathbf{D}.\mathbf{q}\nabla.\mathbf{E}). \qquad (111)$$

This is also the activity of a translational force. Similarly,

$$-\frac{\partial \mathrm{T}_\mu}{\partial t} + \tfrac{1}{2}\mathbf{H}\dot{\mu}\mathbf{H} = -\mathbf{q}\nabla\mathrm{T}_\mu \qquad (112)$$

is the activity of a translational force. Then again

$$-(\mathbf{eJ} + \mathbf{hG}) = -\mathbf{J}\mathrm{V}\mathbf{qB} - \mathbf{G}\mathrm{V}\mathbf{Dq} = \mathbf{q}(\mathrm{V}\mathbf{JB} + \mathrm{V}\mathbf{DG}) \qquad (113)$$

expresses a translational activity. Using them all in (108) it becomes

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{q}\left(\mathbf{E}\rho + \mathbf{H}\sigma - \nabla\mathrm{U}_c - \nabla\mathrm{T}_\mu + \mathrm{V}\mathbf{JB} + \mathrm{V}\mathbf{DG}\right)$$
$$+ \operatorname{div} \mathrm{V}\mathbf{E}_1\mathbf{H}_1 + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu). \quad . \quad . \quad . \quad . \qquad (114)$$

It is clear that we should make the factor of $\mathbf{q}$ be the complete translational force. But that has to be found; and it is equally clear that, although we appear to have exhausted all the terms at disposal, the factor of $\mathbf{q}$ in (114) is not the complete force, because there is no term by which the force on intrinsically magnetised or electrized matter could be exhibited. These involve $\mathbf{e}_0$ and $\mathbf{h}_0$. But as we have

$$\mathbf{q}\,(\mathrm{V}\mathbf{j}_0\mathbf{B} + \mathrm{V}\mathbf{D}\mathbf{g}_0) = -\,(\mathbf{e}\mathbf{j}_0 + \mathbf{h}\mathbf{g}_0), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (115)$$

a possible way of bringing them in is to add the left member and subtract the right member of (115) from the right member of (114); bringing the translational force to $\mathbf{f}$, say, where

$$\mathbf{f} = \mathbf{E}\rho + \mathbf{H}\sigma - \nabla\mathrm{U}_c - \nabla\mathrm{T} \;+ \mathrm{V}(\mathbf{J} + \mathbf{j}_0)\,\mathbf{B} + \mathrm{V}\,(\mathbf{G} + \mathbf{g}_0)\,\mathbf{D}. \quad . \quad . \quad . \quad . \quad . \qquad (116)$$

But there is still the right number of (115) to be accounted for. We have

$$-\operatorname{div}(\mathrm{V}\mathbf{e}\mathbf{h}_0 + \mathrm{V}\mathbf{e}_0\mathbf{h}) = \mathbf{e}\mathbf{j}_0 + \mathbf{h}\mathbf{g}_0 + \mathbf{e}_0\mathbf{j} + \mathbf{h}_0\mathbf{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (117)$$

and, by using this in (114), through (115), (116), (117), we bring it to

$$\mathbf{e}_0\mathbf{J} + \mathbf{h}_0\mathbf{G} = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{f}\mathbf{q} - (\mathbf{e}_0\mathbf{j} + \mathbf{h}_0\mathbf{g}) + \operatorname{div}\left(\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{e}\mathbf{h}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}\right) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu); \qquad (118)$$

or, transferring the $\mathbf{e}_0$, $\mathbf{h}_0$ terms from the right to the left side,

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathbf{f}\mathbf{q} + \operatorname{div}\left(\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{e}\mathbf{h}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}\right) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) \quad . \quad . \qquad (119)$$

Here we see that we have *a* correct form of activity equation, though it may not be *the* correct form. Another form, equally probable, is to be obtained by bringing in $\mathrm{V}\mathbf{e}\mathbf{h}$; thus

$$\operatorname{div} \mathrm{V}\mathbf{e}\mathbf{h} = \mathbf{h}\operatorname{curl}\mathbf{e} - \mathbf{e}\operatorname{curl}\mathbf{h} = -\,(\mathbf{e}\mathbf{j} + \mathbf{h}\mathbf{g}) = \mathbf{q}\,(\mathrm{V}\mathbf{j}\mathbf{B} + \mathrm{V}\mathbf{D}\mathbf{g}), \quad . \quad . \quad . \quad . \qquad (120)$$

which converts (119) to

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathbf{F}\mathbf{q} + \operatorname{div}\left(\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{e}\mathbf{h} - \mathrm{V}\mathbf{e}\mathbf{h}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}\right) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) \quad . \qquad (121)$$

where $\mathbf{F}$ is the translational force

$$\mathbf{F} = \mathbf{E}\rho + \mathbf{H}\sigma - \nabla\mathrm{U}_c - \nabla\mathrm{T}_\mu + \mathrm{V}\operatorname{curl}\mathbf{H}.\mathbf{B} + \mathrm{V}\operatorname{curl}\mathbf{E}.\mathbf{D}, \quad . \quad . \quad . \quad . \qquad (122)$$

which is perfectly symmetrical as regards $\mathbf{E}$ and $\mathbf{H}$, and in the vector products utilises the fluxes and their complete forces, whereas former forms did this only partially. Observe, too, that we have only been able to bring the activity equation to a correct form (either (119) or (122)) by making $\mathbf{e}_0\mathbf{J}_0$ be the activity of intrinsic force $\mathbf{e}_0$, which requires that $\mathbf{J}_0$ should be the true electric current, according to the energy criterion, not $\mathbf{J}$.

§ **18.** Now, to test (119) and (121), we must interpret the flux in (121), or say

$$\mathbf{Y} = \mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{eh} - \mathrm{V}\mathbf{eh}_0 - \mathrm{V}\mathbf{e}_0\mathbf{h}, \quad \ldots \ldots \ldots \ldots \quad (123)$$

which has replaced the POYNTING flux $\mathrm{V}\mathbf{E}_1\mathbf{H}_1$ when $\mathbf{q} = 0$, along with the other changes. Since $\mathbf{Y}$ reduces to $\mathrm{V}\mathbf{E}_1\mathbf{H}_1$ when $\mathbf{q} = 0$, there must still be a POYNTING flux when $\mathbf{q}$ is finite, though we do not know its precise form of expression. There is also the stress flux of energy and the flux of energy by convection, making a total flux

$$\mathbf{X} = \mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma \mathbf{Q}q + \mathbf{q}\,(\mathrm{U}_0 + \mathrm{T}_0), \quad \ldots \ldots \ldots \quad (124)$$

where $\mathbf{W}$ is the POYNTING flux, and $- \Sigma \mathbf{Q}q$ that of the stress, whilst $\mathbf{q}\ (\mathrm{U}_0 + \mathrm{T}_0)$ means convection of energy connected with the translational force. We should therefore have

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + (\mathrm{Q}_0 + \dot{\mathrm{U}}_0 + \dot{\mathrm{T}}_0) + \operatorname{div} \mathbf{X} \quad \ldots \ldots \quad (125)$$

to express the continuity of energy. More explicitly

$$\begin{aligned} \mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} &+ \operatorname{div}\left[\mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T})\right] \\ + \mathrm{Q}_0 + \dot{\mathrm{U}}_0 + \dot{\mathrm{T}}_0 &+ \operatorname{div}\left[- \Sigma \mathbf{Q}q + \mathbf{q}\,(\mathrm{U}_0 + \mathrm{T}_0)\right] \quad \ldots \ldots \ldots \quad (126) \end{aligned}$$

But here we may simplify by using the result (69) (with, however, $\mathbf{f}$ put $= 0$), making (126) become

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{Fq} + \mathbf{Sa} + \operatorname{div}\left[\mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma \mathbf{Q}q\right], \quad \ldots \quad (127)$$

where $\mathbf{S}$ is the torque, and $\mathbf{a}$ the spin.

Comparing this with (121), we see that we require

$$\mathbf{W} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) - \Sigma \mathbf{Q}q = \mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{eh} - \mathrm{V}\mathbf{e}_0\mathbf{h} - \mathrm{V}\mathbf{eh}_0, \quad \ldots \ldots \quad (128)$$

with a similar equation when (119) is used instead; and we have now to separate the right member into two parts, one for the POYNTING flux, the other for the stress flux, in such a way that the force due to the stress is the force $\mathbf{F}$ in (121), (122), or the force $\mathbf{f}$ in (119), (116); or similarly in other cases. It is unnecessary to give the failures; the only one that stands the test is (121), which satisfies it completely.

I argued that

$$\mathbf{W} = \mathrm{V}\,(\mathbf{E} - \mathbf{e}_0)\,(\mathbf{H} - \mathbf{h}_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

was the probable form of the POYNTING flux in the case of a moving medium, not $\mathrm{V}\mathbf{E}_1\mathbf{H}_1$, because when a medium is endowed with a *uniform* translational motion, the transmission of disturbances through it takes place just as if it were at rest. With this expression (129) for $\mathbf{W}$, we have, identically,

$$\mathrm{V}\mathbf{E}_1\mathbf{H}_1 - \mathrm{V}\mathbf{eh} - \mathrm{V}\mathbf{e}_0\mathbf{h} - \mathrm{V}\mathbf{eh}_0 = \mathbf{W} - \mathrm{V}\mathbf{eH} - \mathrm{V}\mathbf{Eh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

Therefore, by (128) and (130), we get

$$\Sigma\mathbf{Q}q = \mathrm{V}\mathbf{eH} + \mathrm{V}\mathbf{Eh} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

to represent the negative of the stress flux of energy, so that, finally, the fully significant equation of activity is

$$\begin{aligned}\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \mathbf{Fq} + \mathbf{Sa} + \operatorname{div}\left[\mathrm{V}\,(\mathbf{E} - \mathbf{e}_0)\,(\mathbf{H} - \mathbf{h}_0) + \mathbf{q}\,(\mathrm{U} + \mathrm{T})\right] \\ - \operatorname{div}\left[\mathrm{V}\mathbf{eH} + \mathrm{V}\mathbf{Eh} + \mathbf{q}\,(\mathrm{U} + \mathrm{T})\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)\end{aligned}$$

This is, of course, an identity, subject to the electromagnetic equations we started from, and is only one of the multitude of forms which may be given to it, many being far simpler. But the particular importance of this form arises from its being the only form apparently possible which shall exhibit the principle of continuity of energy without outstanding terms, and without loss of generality; and this is only possible by taking $\mathbf{J}_0$ as the proper flux for $\mathbf{e}_0$ to work upon.*

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* In the original an erroneous estimate of the value of $(\partial/\partial t)(\mathrm{U}_c + \mathrm{T}_\mu)$ was used in some of the above equations. This is corrected. The following contains full details of the calculation. We require the value of $(\partial/\partial t)\mathrm{U}_c$, or of $\frac{1}{2}\mathbf{E}\,(\partial c/\partial t)\mathbf{E}$, where $\partial c/\partial t$ is the linear operator whose components are the time-variations (for the same matter), of those of c. The calculation is very lengthy in terms of these six components. But vectorially it is not difficult. In (27), (28), we have

$$\left.\begin{aligned}\mathbf{D} = c\mathbf{E} &= \mathbf{i}.\mathbf{c}_1\mathbf{E} + \mathbf{j}.\mathbf{c}_2\mathbf{E} + \mathbf{k}.\mathbf{c}_3\mathbf{E} \\ &= (\mathbf{i}.\mathbf{c}_1 + \mathbf{j}.\mathbf{c}_2 + \mathbf{k}.\mathbf{c}_3)\mathbf{E},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132a)$$

if the vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, are given by

$$\mathbf{c}_1 = \mathbf{i}c_{11} + \mathbf{j}c_{12} + \mathbf{k}c_{13}, \qquad \mathbf{c}_2 = \mathbf{i}c_{21} + \mathbf{j}c_{22} + \mathbf{k}c_{23}, \qquad \mathbf{c}_3 = \mathbf{i}c_{31} + \mathbf{j}c_{32} + \mathbf{k}c_{33}.$$

We, therefore, have

$$\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{E}\left(\frac{\partial\mathbf{i}}{\partial t}.\mathbf{c}_1 + \frac{\partial\mathbf{j}}{\partial t}.\mathbf{c}_2 + \frac{\partial\mathbf{k}}{\partial t}.\mathbf{c}_3\right)\mathbf{E} + \mathbf{E}\left(\mathbf{i}.\frac{\partial\mathbf{c}_1}{\partial t} + \mathbf{j}.\frac{\partial\mathbf{c}_2}{\partial t} + \mathbf{k}.\frac{\partial\mathbf{c}_3}{\partial t}\right)\mathbf{E} \quad . \quad . \quad . \quad . \quad (132b)$$

The part played by the dots is to clearly separate the scalar products.

Now suppose that the eolotropic property symbolised by c is intrinsically unchanged by the shift of the matter. The mere translation does not, therefore, affect it, nor does the distortion; but the rotation

Derivation of the Electric and Magnetic Stresses and Forces from the Flux of Energy.

§ 19. It will be observed that the convection of energy disappears by occurring twice oppositely signed; but as it comes necessarily into the expression for the stress flux of energy, I have preserved the cancelling terms in (132). A comparison of the stress flux with the POYNTING flux is interesting. Both are of the same form, viz., vector products of the electric and magnetic forces with convection terms; but whereas in the latter the forces in the vector product are those of the field (*i.e.*, only intrinsic forces deducted from **E** and **H**), in the former we have the motional forces **e** and **h** combined with the complete **E** and **H** of the fluxes. Thus the stress depends

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does. For if we turn round an eolotropic portion of matter, keeping **E** unchanged, the value of **U** is altered by the rotation of the principal axes of c along with the matter, so that a torque is required.

In equation (132*a*), then, to produce (132*b*), we keep **E** constant, and let the six vectors, **i**, **j**, **k**, $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ rotate as a rigid body with the spin $\mathbf{a} = \frac{1}{2}\,\text{curl}\,\mathbf{q}$. But when a vector magnitude **i** is turned round in this way, its rate of time-change $\partial\mathbf{i}/\partial t$ is $\mathrm{V}\mathbf{ai}$. Thus, for $\partial/\partial t$, we may put $\mathrm{V}\mathbf{a}$ throughout. Therefore, by (132*b*),

$$\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{E}(\mathrm{V}\mathbf{ai}.\mathbf{c}_1 + \mathrm{V}\mathbf{aj}.\mathbf{c}_2 + \mathrm{V}\mathbf{ak}.\mathbf{c}_3)\mathbf{E} + \mathbf{E}(\mathbf{i}.\mathrm{V}\mathbf{ac}_1 + \mathbf{j}.\mathrm{V}\mathbf{ac}_2 + \mathbf{k}.\mathrm{V}\mathbf{ac}_3)\mathbf{E}. \quad . \quad . \quad (132c)$$

In this use the parallelepipedal transformation (12), and it becomes

$$\begin{aligned}\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} &= \mathrm{V}\mathbf{Ea}(\mathbf{i}.\mathbf{c}_1 + \mathbf{j}.\mathbf{c}_2 + \mathbf{k}.\mathbf{c}_3)\mathbf{E} + \mathbf{E}(\mathbf{i}.\mathbf{c}_1 + \mathbf{j}.\mathbf{c}_2 + \mathbf{k}.\mathbf{c}_3)\mathrm{V}\mathbf{Ea} \\ &= (\mathrm{V}\mathbf{Ea})c\mathbf{E} + \mathbf{E}c(\mathrm{V}\mathbf{Ea}) = (\mathbf{D} + \mathbf{D}')\mathrm{V}\mathbf{Ea}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132d)\end{aligned}$$

by (132*a*), if $\mathbf{D}'$ is conjugate to **D**; that is, $\mathbf{D}' = c'\mathbf{E} = \mathbf{E}c$. So, when $c = c'$, as in the electrical case, we have

$$\left.\begin{aligned}\frac{\partial \mathrm{U}_c}{\partial t} &= \tfrac{1}{2}\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{D}\mathrm{V}\mathbf{Ea} = \mathbf{a}\mathrm{V}\mathbf{DE}, \\ \text{and similarly} \qquad \frac{\partial \mathrm{T}_\mu}{\partial t} &= \tfrac{1}{2}\mathbf{H}\frac{\partial \mu}{\partial t}\mathbf{H} = \mathbf{B}\mathrm{V}\mathbf{Ha} = \mathbf{a}\mathrm{V}\mathbf{BH}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132e)$$

Now the torque arising from the stress is (see (139))

$$\mathbf{S} = \mathrm{V}\mathbf{DE} + \mathrm{V}\mathbf{BH},$$

so we have

$$\frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) = \mathbf{Sa} = \text{torque} \times \text{spin}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132f)$$

The variation allowed to **i**, **j**, **k** may seem to conflict with their constancy (as reference vectors) in general. But they merely vary for a temporary purpose, being fixed in the matter instead of in space. But we may, perhaps better, discard **i**, **j**, **k** altogether, and use any independent vectors, **l**, **m**, **n** instead, making

$$\mathbf{D} = (\mathbf{l}.\mathbf{c}_1 + \mathbf{m}.\mathbf{c}_2 + \mathbf{n}.\mathbf{c}_3)\mathbf{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132g)$$

wherein the **c**'s are properly chosen to suit the new axes. The calculation then proceeds as before, half entirely on the fluxes, however they be produced, in this respect resembling the electric and magnetic energies.

To exhibit the stress, we have (131), or

$$\mathbf{Q}_1 q_1 + \mathbf{Q}_2 q_2 + \mathbf{Q}_3 q_3 = \mathrm{V}\mathbf{eH} + \mathrm{V}\mathbf{Eh} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (133)$$

In this use the expressions for $\mathbf{e}$ and $\mathbf{h}$, giving

$$\begin{aligned} \Sigma \mathbf{Q}q &= \mathrm{V}\mathbf{H}\mathrm{V}\mathbf{Bq} + \mathrm{V}\mathbf{E}\mathrm{V}\mathbf{Dq} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) \\ &= \mathbf{B}.\mathbf{Hq} - \mathbf{q}.\mathbf{HB} + \mathbf{D}.\mathbf{Eq} - \mathbf{q}.\mathbf{ED} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}) \\ &= (\mathbf{B}.\mathbf{Hq} - \mathbf{q}\mathrm{T}) + (\mathbf{D}.\mathbf{Eq} - \mathbf{q}\mathrm{U})\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (134) \end{aligned}$$

where observe the singularity that $\mathbf{q}\,(\mathrm{U} + \mathrm{T})$ has changed its sign. The first set belongs to the magnetic, the second to the electric stress, since we see that the complete stress is thus divisible.

The divergence of $\Sigma \mathbf{Q}q$ being the activity of the stress-variation per unit volume, its $\mathbf{N}$ component is the activity of the stress per unit surface, that is,

$$(\mathbf{NB}.\mathbf{Hq} - \mathbf{Nq}.\mathrm{T}) + (\mathbf{ND}.\mathbf{Eq} - \mathbf{Nq}.\mathrm{U}) = \mathbf{q}\,(\mathbf{H}.\mathbf{BN} + \mathbf{E}.\mathbf{DN} - \mathbf{N}\mathrm{U} - \mathbf{N}\mathrm{T}) = \mathbf{P}_{\mathrm{N}}\mathbf{q}. \quad . \qquad (135)$$

The stress itself is therefore

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the value of $\partial \mathrm{U}_c/\partial t$ arising from the variation of $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$, and the other half from the $\mathbf{c}$'s, provided c is irrotational.

Or we may choose the three principal axes of c in the body, when $\mathbf{l}$, $\mathbf{m}$, $\mathbf{n}$ will coincide with, and therefore move with them.

Lastly, we may proceed thus :—

$$\mathbf{E}\frac{\partial c}{\partial t}\mathbf{E} = \mathbf{E}\frac{\partial \mathbf{D}}{\partial t} - \mathbf{D}\frac{\partial \mathbf{E}}{\partial t} = \mathbf{E}\mathrm{V}\mathbf{aD} - \mathbf{D}\mathrm{V}\mathbf{aE} = 2\mathbf{a}\mathrm{V}\mathbf{DE}. \quad . \quad . \quad . \quad . \quad . \qquad (132h)$$

That is, replace $\partial/\partial t$ by $\mathrm{V}\mathbf{a}$ when the operands are $\mathbf{E}$ and $\mathbf{D}$. This is the correct result, but it is not easy to justify the process directly and plainly; although the clue is given by observing that what we do is to take a difference, from which the time-variation of $\mathbf{E}$ disappears.

If it is $\mathbf{D}$ that is kept constant, the result is $2\mathbf{a}\mathrm{V}\mathbf{ED}$, the negative of the above.

It is also worth noticing that if we split up $\mathbf{E}$ into $\mathbf{E}_1 + \mathbf{E}_2$ we shall have

$$\left.\begin{aligned} \mathbf{E}_1\frac{\partial c}{\partial t}\mathbf{E}_2 &= \mathbf{a}\,[\mathrm{V}(\mathbf{E}_1 c)\mathbf{E}_2 - \mathrm{V}\mathbf{E}_1(c\mathbf{E}_2)], \\ \mathbf{E}_2\frac{\partial c}{\partial t}\mathbf{E}_1 &= \mathbf{a}\,[\mathrm{V}(\mathbf{E}_2 c)\mathbf{E}_1 - \mathrm{V}_2(c\mathbf{E}_1)]. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (132i)$$

These are only equal when $c = c'$, or $\mathbf{E}c = c\mathbf{E}$; so that, in the expansion of the torque,

$$\mathrm{V}\mathbf{DE} = \mathrm{V}\mathbf{D}_1\mathbf{E}_1 + \mathrm{V}\mathbf{D}_2\mathbf{E}_2 + \mathrm{V}\mathbf{D}_2\mathbf{E}_1 + \mathrm{V}\mathbf{D}_1\mathbf{E}_2,$$

the cross-torques are not $\mathrm{V}\mathbf{D}_2\mathbf{E}_1$ and $\mathrm{V}\mathbf{D}_1\mathbf{E}_2$, which are unequal, but are each equal to half the sum of these vector-products.

$$\mathbf{P}_{\mathrm{N}} = (\mathbf{E.DN} - \mathbf{N}\mathrm{U}) + (\mathbf{H.BN} - \mathbf{N}\mathrm{T}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (136)$$

divided into electric and magnetic portions. This is with restriction to symmetrical μ and c, and with persistence of their forms as a particle moves, but is otherwise unrestricted.

Neither stress is of the symmetrical or irrotational type in case of eolotropy, and there appears to be no getting an irrotational stress save by arbitrary assumptions which destroy the validity of the stress as a correct deduction from the electromagnetic equations. But, in case of isotropy, with consequent directional identity of **E** and **D**, and of **H** and **B**, we see, by taking **N** in turns parallel to, or perpendicular to **E** in the electric case, and to **H** in the magnetic case, that the electric stress consists of a tension U parallel to **E** combined with an equal lateral pressure, whilst the magnetic stress consists of a tension T parallel to **H** combined with an equal lateral pressure. There are, in fact, MAXWELL'S stresses in an isotropic medium homogeneous as regards μ and c. The difference from MAXWELL arises when μ and c are variable (including abrupt changes from one value to another of μ and c), and when there is intrinsic magnetisation, MAXWELL'S stresses and forces being then different.

The stress on the plane whose normal is V**EH**, is

$$\frac{\mathbf{E.D}\mathrm{V}\mathbf{EH} + \mathbf{H.B}\mathrm{V}\mathbf{EH} - (\mathrm{U} + \mathrm{T})\,\mathrm{V}\mathbf{EH}}{\mathrm{V}_0\mathbf{EH}},$$

$$= \frac{\mathbf{E.H}\mathrm{V}\mathbf{DE} + \mathbf{H.E}\mathrm{V}\mathbf{HB} - (\mathrm{U} + \mathrm{T})\,\mathrm{V}\mathbf{EH}}{\mathrm{V}_0\mathbf{EH}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (137)$$

reducing simply to a pressure (U + T) in lines parallel to V**EH** in case of isotropy.

§ **20.** To find the force **F**, we have

$$\begin{aligned} \mathbf{FN} &= \mathrm{div}\ \mathbf{Q}_{\mathrm{N}} = \mathrm{div}\left(\mathbf{D.EN} - \mathbf{N}\mathrm{U} + \mathbf{B.HN} - \mathbf{N}\mathrm{T}\right) \\ &= \mathbf{EN}.\rho + \mathbf{D}\nabla.\mathbf{EN} - \tfrac{1}{2}\mathbf{E.N}\nabla.\mathbf{D} - \tfrac{1}{2}\mathbf{D.N}\nabla.\mathbf{E} + \&\mathrm{c}. \\ &= \mathbf{EN}.\rho + \mathbf{D}\left(\nabla.\mathbf{EN} - \mathbf{N}\nabla.\mathbf{E}\right) + \tfrac{1}{2}\left(\mathbf{D.N}\nabla.\mathbf{E} - \mathbf{E.N}\nabla.\mathbf{D}\right) + \&\mathrm{c}. \\ &= \mathbf{N}\left[\mathbf{E}\rho + \mathrm{V}\ \mathrm{curl}\ \mathbf{E.D} - \nabla\mathrm{U}_c + \&\mathrm{c}.\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (138) \end{aligned}$$

where the unwritten terms are the similar magnetic terms. This being the **N** component of **F**, the force itself is given by (122), as is necessary.

It is V curl $\mathbf{h}_0 . \mathbf{B}$ that expresses the translational force on intrinsically magnetised matter, and this harmonises with the fact that the flux **B** due to any impressed force $\mathbf{h}_0$ depends solely upon curl $\mathbf{h}_0$.

Also, it is $-\nabla\mathrm{T}_\mu$ that explains the forcive on elastically magnetised matter, *e.g.*, FARADAY'S motion of matter to or away from the places of greatest intensity of the field, independent of its direction.

If $\mathbf{S}$ be the torque, it is given by

$$\mathrm{V}\mathbf{SN} = \mathbf{P}_{\mathrm{N}} - \mathbf{Q}_{\mathrm{N}} = \mathbf{E.DN} - \mathbf{D.EN} + \&c.$$
$$\mathrm{V}\mathbf{N}\,(\mathrm{V}\mathbf{ED} + \mathrm{V}\mathbf{HB});$$

therefore

$$\mathbf{S} = \mathrm{V}\mathbf{DE} + \mathrm{V}\mathbf{BH} \quad \dots\dots\dots\dots \quad (139)$$

But the matter is put more plainly by considering the convergence of the stress flux of energy and dividing it into translational and other parts. Thus

$$\operatorname{div} \Sigma \mathbf{Q}q = \mathbf{Fq} + \big(\mathbf{E.D}\nabla.\mathbf{q} - \mathrm{U} \operatorname{div} \mathbf{q}\big) + \big(\mathbf{H.B}\nabla.\mathbf{q} - \mathrm{T} \operatorname{div} \mathbf{q}\big), \quad \dots\dots \quad (140)$$

where the terms following $\mathbf{Fq}$ express the sum of the distortional and rotational activities.

Shorter Way of going from the Circuital Equations to the Flux of Energy, Stresses, and Forces.

§ **21.** I have given the investigation in §§ 17 to 19 in the form in which it occurred to me before I knew the precise nature of the results, being uncertain as regards the true measure of current, the proper form of the Poynting flux, and how it worked in harmony with the stress flux of energy. But knowing the results, a short demonstration may be easily drawn up, though the former course is the most instructive. Thus, start now from

$$\left.\begin{aligned} \operatorname{curl}\,(\mathbf{H} - \mathbf{h}_0) &= \mathbf{J}_0, \\ -\operatorname{curl}\,(\mathbf{E} - \mathbf{e}_0) &= \mathbf{G}_0, \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (141)$$

on the understanding that $\mathbf{J}_0$ and $\mathbf{G}_0$ are the currents which make $\mathbf{e}_0\mathbf{J}_0$ and $\mathbf{h}_0\mathbf{G}_0$ the activities of $\mathbf{e}_0$ and $\mathbf{h}_0$ the intrinsic forces. Then

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathbf{EJ}_0 + \mathbf{HG}_0 + \operatorname{div} \mathbf{W}, \quad \dots\dots\dots \quad (142)$$

where

$$\mathbf{W} = \mathrm{V}\,(\mathbf{E} - \mathbf{e}_0)\,(\mathbf{H} - \mathbf{h}_0); \quad \dots\dots\dots\dots \quad (143)$$

and we now take this to be the proper form of the Poynting flux. Now develop $\mathbf{EJ}_0$ and $\mathbf{HG}_0$ thus :—

$$\begin{aligned} \mathbf{EJ}_0 + \mathbf{HG}_0 &= \mathbf{E}\big(\mathbf{C} + \dot{\mathbf{D}} + \mathbf{q}\rho + \operatorname{curl} \mathbf{h}\big) + \mathbf{H}\big(\mathbf{K} + \dot{\mathbf{B}} + \mathbf{q}\sigma - \operatorname{curl} \mathbf{e}\big), \quad \text{by (93)};\\ &= \mathrm{Q}_1 + \dot{\mathrm{U}} + \dot{\mathrm{U}}_c + \mathbf{E}\mathbf{q}\rho + \mathbf{E} \operatorname{curl} \mathrm{V}\mathbf{Dq} \\ &\quad + \mathrm{Q}_2 + \dot{\mathrm{T}} + \dot{\mathrm{T}}_\mu + \mathbf{H}\mathbf{q}\sigma + \mathbf{H} \operatorname{curl} \mathrm{V}\mathbf{Bq}, \quad \text{by (88) and (91)};\\ &= \mathrm{Q}_1 + \dot{\mathrm{U}} + \dot{\mathrm{U}}_c + \mathbf{E}\mathbf{q}\rho + \mathbf{E}\big(\mathbf{D} \operatorname{div} \mathbf{q} + \mathbf{q}\nabla.\mathbf{D} - \mathbf{q} \operatorname{div} \mathbf{D} - \mathbf{D}\nabla.\mathbf{q}\big) \\ &\quad + \mathrm{Q}_2 + \dot{\mathrm{T}} + \dot{\mathrm{T}}_\mu + \mathbf{H}\mathbf{q}\sigma + \mathbf{H}\big(\mathbf{B} \operatorname{div} \mathbf{q} + \mathbf{q}\nabla.\mathbf{B} - \mathbf{q} \operatorname{div} \mathbf{B} - \mathbf{B}\nabla.\mathbf{q}\big), \quad \text{by (26)}.\\ &= \mathrm{Q}_1 + \dot{\mathrm{U}} + \dot{\mathrm{U}}_c + 2\mathrm{U} \operatorname{div} \mathbf{q} + \mathbf{E.q}\nabla.\mathbf{D} - \mathbf{E.D}\nabla.\mathbf{q} \\ &\quad + \text{magnetic terms},\\ &= \big(\mathrm{Q}_1 + \dot{\mathrm{U}} + \operatorname{div} \mathbf{q}\mathrm{U}\big) + \big(\mathrm{U} \operatorname{div} \mathbf{q} - \mathbf{E.D}\nabla.\mathbf{q}\big) + \big(\dot{\mathrm{U}}_c - \mathbf{q}\nabla.\mathrm{U} + \mathbf{E.q}\nabla.\mathbf{D}\big) \\ &\quad + \text{magnetic terms.} \quad \dots\dots\dots\dots\dots\dots\dots \quad (144) \end{aligned}$$

Now here

$$\mathrm{q}\nabla.\mathrm{U} = \tfrac{1}{2}\mathbf{E}.\mathrm{q}\nabla.\mathbf{D} + \tfrac{1}{2}\mathbf{D}.\mathrm{q}\nabla.\mathbf{E},$$

so that the terms in the third pair of brackets in (144) represent

$$\dot{\mathrm{U}}_c + \mathrm{q}\nabla.\mathrm{U}_c = \frac{\partial \mathrm{U}_c}{\partial t} = \tfrac{1}{2}\,\mathbf{E}\,\frac{\partial c}{\partial t}\,\mathbf{E},$$

with the generalised meaning before explained. So finally

$$\begin{aligned}\mathbf{EJ}_0 + \mathbf{HG}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div} \mathrm{q}(\mathrm{U} + \mathrm{T}) + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) \\ + \big(\mathrm{U} \operatorname{div} \mathrm{q} - \mathbf{E.D}\nabla.\mathrm{q}\big) + \big(\mathrm{T} \operatorname{div} \mathrm{q} - \mathbf{H.B}\nabla.\mathrm{q}\big), \quad . \quad . \quad . \quad (145)\end{aligned}$$

which brings (142) to

$$\begin{aligned}\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div}\{\mathbf{W} + \mathrm{q}\,(\mathrm{U} + \mathrm{T})\} \\ + \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) + \big(\mathrm{U} \operatorname{div} \mathrm{q} - \mathbf{E.D}\nabla.\mathrm{q}\big) + \big(\mathrm{T} \operatorname{div} \mathrm{q} - \mathbf{H.B}\nabla.\mathrm{q}\big), \quad . \quad . \quad . \quad (146)\end{aligned}$$

which has to be interpreted in accordance with the principle of continuity of energy.

Use the form (127), first, however, eliminating **Fq** by means of

$$\operatorname{div} \Sigma \mathbf{Q}q = \mathbf{Fq} + \Sigma \mathbf{Q}\nabla \mathrm{q},$$

which brings (127) to

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 = \mathbf{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div}\{\mathbf{W} + \mathrm{q}\,(\mathrm{U} + \mathrm{T})\} - \Sigma\mathbf{Q}\nabla q + \mathbf{Sa}\,; \quad . \quad . \quad . \quad . \quad . \quad (147)$$

and now, by comparison of (147) with (146) we see that

$$\begin{aligned}-\,\mathbf{Sa} + \Sigma\mathbf{Q}\nabla q = \big(\mathbf{E.D}\nabla.\mathrm{q} - \mathrm{U} \operatorname{div} \mathrm{q}\big) - \frac{\partial \mathrm{U}_c}{\partial t} \\ + \big(\mathbf{H.B}\nabla.\mathrm{q} - \mathrm{T} \operatorname{div} \mathrm{q}\big) - \frac{\partial \mathrm{T}_\mu}{\partial t}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)\end{aligned}$$

from which, when μ and c do not change intrinsically, we conclude that

$$\left.\begin{aligned}\mathbf{Q}_\mathrm{N} = \mathbf{B.HN} - \mathbf{N}\mathrm{T} + \mathbf{D.EN} - \mathbf{N}\mathrm{U}, \\ \mathbf{P}_\mathrm{N} = \mathbf{H.BN} - \mathbf{N}\mathrm{T} + \mathbf{E.DN} - \mathbf{N}\mathrm{U},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (149)$$

as before. In this method we lose sight altogether of the translational force which formed so prominent an object in the former method as a guide.

Some Remarks on HERTZ'S *investigation relating to the Stresses.*

§ **22.** Variations of c and μ in the same portion of matter may occur in different ways, and altogether independently of the strain variations. Equation (146) shows how their influence affects the energy transformations; but if we consider only such changes as depend on the strain, *i.e.*, the small changes of value which μ and c undergo as the strain changes, we may express them by thirty-six new coefficients each (there being six distortion elements, and six elements in μ, and six in c), and so reduce the expressions for $\partial U_c/\partial t$ and $\partial T_\mu/\partial t$ in (148) to the form suitable for exhibiting the corresponding change in $\mathbf{Q}_N$ and in the stress function $\mathbf{P}_N$. As is usual in such cases of secondary corrections, the magnitude of the resulting formula is out of all proportion to the importance of the correction terms in relation to the primary formula to which they are added.

Professor H. HERTZ[*] has considered this question, and also refers to VON HELMOLTZ's previous investigation relating to a fluid. The c and μ can then only depend on the density, or on the compression, so that a single coefficient takes the place of the thirty-six. But I cannot quite follow HERTZ's stress investigation. First, I would remark that in developing the expression for the distortional (*plus* rotational) activity, he assumes that all the coefficients of the spin vanish identically; this is done in order to make the stress be of the irrotational type. But it may easily be seen that the assumption is inadmissible by examining its consequence, for which we need only take the case of c and μ intrinsically constant. By (139) we see that it makes $\mathbf{S} = 0$, and therefore (since the electric and magnetic stress are separable), $\mathbf{VHB} = 0$, and $\mathbf{VED} = 0$; that is, it produces directional identity of the force $\mathbf{E}$ and the flux $\mathbf{D}$, and of the force $\mathbf{H}$ and the flux $\mathbf{B}$. This means isotropy, and, therefore, breaks down the investigation so far as the eolotropic application, with six μ and six c coefficients, goes. Abolish the assumption made, and the stress will become that used by me above.

Another point deserving of close attention in HERTZ's investigation, relates to the principle to be followed in deducing the stress from the electromagnetic equations. Translating into my notation it would appear to amount to this, the *à priori* assumption that the quantity

$$\frac{1}{v}\frac{\partial}{\partial t}\left(\mathrm{T}v\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150)$$

where v indicates the volume of a moving unit element undergoing distortion, may be taken to represent the distortional (*plus* rotational) activity of the magnetic stress. Similarly as regards the electric stress.

Expanding (150) we obtain

$$\frac{\partial \mathrm{T}}{\partial t} + \frac{\mathrm{T}}{v}\frac{\partial v}{\partial t} = \mathbf{H}\frac{\partial \mathbf{B}}{\partial t} + \mathrm{T}\,\mathrm{div}\,\mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

Now the second circuital law (90) may be written

$$-\,\mathrm{curl}\,(\mathbf{E} - \mathbf{e}_0) = \mathbf{K} + \frac{\partial \mathbf{B}}{\partial t} + \left(\mathbf{B}\,\mathrm{div}\,\mathbf{q} - \mathbf{B}\nabla.\mathbf{q}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

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[*] 'WIEDEMANN's Annalen,' v. 41, p. 369.

Here ignore $\mathbf{e}_0$, $\mathbf{K}$, *and ignore the curl of the electric force*, and we obtain, by using (152) in (151),

$$\mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - \mathbf{HB} \operatorname{div} \mathbf{q} + \mathrm{T} \operatorname{div} \mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t} = \mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - \mathrm{T} \operatorname{div} \mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t}, \quad . \quad . \quad . \quad (153)$$

which represents the distortional activity (my form, not equating to zero the coefficients of curl $\mathbf{q}$ in its development). We *can*, therefore, derive the magnetic stress in the manner indicated, that is, from (150), with the special meaning of $\partial\mathbf{B}/\partial t$ later stated, and the ignorations or nullifications.

In a similar manner, from the first circuital law (89), which may be written

$$\operatorname{curl} (\mathbf{H} - \mathbf{h}_0) = \mathbf{C} + \frac{\partial \mathbf{D}}{\partial t} + (\mathbf{D} \operatorname{div} \mathbf{q} - \mathbf{D}\nabla.\mathbf{q}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

we can, by ignoring the conduction current and the curl of the magnetic force, obtain

$$\frac{1}{v}\frac{\partial}{\partial t}(v\mathrm{U}) = \mathbf{E}.\mathbf{D}\nabla.\mathbf{q} - \mathrm{U} \operatorname{div} \mathbf{q} - \frac{\partial \mathrm{U}_c}{\partial t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

which represents the distortional activity of the electric stress.

The difficulty here seems to me to make it evident *à priori* that (150), with the special reckoning of $\partial\mathbf{B}/\partial t$ *should* represent the distortional activity (*plus* rotational understood); this interesting property should, perhaps, rather be derived from the magnetic stress when obtained by a safe method. The same remark applies to the electric stress. Also, in (150) to (155) we overlook the POYNTING flux. I am not sure how far this is intentional on Professor HERTZ's part, but its neglect does not seem to give a sufficiently comprehensive view of the subject.

The complete expansion of the magnetic distortional activity is, in fact,

$$\mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - \mathrm{T} \operatorname{div} \mathbf{q} - \frac{\partial \mathrm{T}_\mu}{\partial t} = \mathrm{Q}_2 + \dot{\mathrm{T}} + \operatorname{div} \mathbf{q}\mathrm{T} - \mathbf{HG}_0 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (156)$$

and similarly, that of the electric stress is

$$\mathbf{E}.\mathbf{D}\nabla.\mathbf{q} - \mathrm{U} \operatorname{div} \mathbf{q} - \frac{\partial \mathrm{U}_c}{\partial t} = \mathrm{Q}_1 + \dot{\mathrm{U}} + \operatorname{div} \mathbf{q}\mathrm{U} - \mathbf{EJ}_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

It is the last term of (156) and the last term of (157), together, which bring in the POYNTING flux. Thus, adding these equations,

$$\Sigma \mathbf{Q}\nabla q - \frac{\partial}{\partial t}(\mathrm{U}_c + \mathrm{T}_\mu) = \mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}} + \operatorname{div} \mathbf{q}\, (\mathrm{U} + \mathrm{T}) - (\mathbf{EJ}_0 + \mathbf{HG}_0), \quad . \quad . \quad . \quad (158)$$

where

$$(\mathbf{EJ}_0 + \mathbf{HG}_0) = (\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0) - \operatorname{div} \mathbf{W} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

and so we come round to the equation of activity again, in the form (146), by using (159) in (158).

Modified Form of Stress-vector, and Application to the Surface separating two Regions.

§ **23.** The electromagnetic stress, $\mathbf{P}_{\mathrm{N}}$ of (149) and (136) may be put into another interesting form. We may write it

$$\mathbf{P}_{\mathrm{N}} = \tfrac{1}{2}(\mathbf{E}.\mathbf{ND} + \mathrm{V}.\mathrm{V}\mathbf{NE}.\mathbf{D}) + \tfrac{1}{2}(\mathbf{H}.\mathbf{NB} + \mathrm{V}.\mathrm{V}\mathbf{NH}.\mathbf{B}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (160)$$

Now, **ND** is the surface equivalent of div **D** and **NB** of div **B**; whilst V**NE** and V**NH** are the surface equivalents of curl **E** and curl **H**. We may, therefore, write

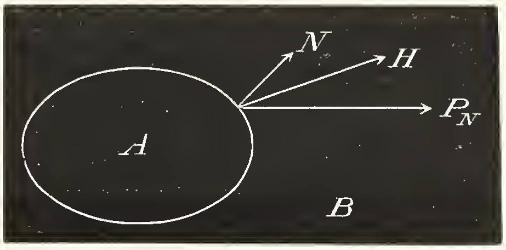
$$\mathbf{P}_{\mathrm{N}} = \tfrac{1}{2}(\mathbf{E}\rho' + \mathrm{V}\mathbf{DG}') + \tfrac{1}{2}(\mathbf{H}\sigma' + \mathrm{V}\mathbf{J}'\mathbf{B}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (161)$$

and this is the force, reckoned as a pull, on unit area of the surface whose normal is **N**. Here the accented letters are the surface equivalents of the same quantities unaccented, which have reference to unit volume.

Comparing with (122) we see that the type is preserved, except as regards the terms in **F** due to variation of c and μ in space. That is, the stress is represented in (101) as the translational force, due to **E** and **H**, on the fictitious electrification, magnetification, electric current, and magnetic current produced by imagining **E** and **H** to terminate at the surface across which $\mathbf{P}_{\mathrm{N}}$ is the stress.

The coefficient $\frac{1}{2}$ which occurs in (161) is understandable by supposing the fictitious quantities (" matter " and " current ") to be distributed uniformly within a very thin layer, so that the forces **E** and **H** which act upon them do not then terminate quite abruptly, but fall off gradually through the layer from their full values on one side to zero on the other. The mean values of **E** and **H** through the layer, that is, $\frac{1}{2}\mathbf{E}$ and $\frac{1}{2}\mathbf{H}$ are thus the effective electric and magnetic forces on the layer as a whole, per unit volume density of matter or current; or $\frac{1}{2}\mathbf{E}$ and $\frac{1}{2}\mathbf{H}$ per unit surface density when the layer is indefinitely reduced in thickness.

Considering the electric field only, the quantities concerned are electrification and magnetic current. In the magnetic field only they are magnetification and electric current. Imagine the medium divided into two regions A and B, of which A is internal, B external, and let **N** be the unit normal from the surface into the external region. The mechanical action between the two regions is fully represented by the stress $\mathbf{P}_{\mathrm{N}}$ over their interface, and the forcive of B upon A is fully represented by the **E** and **H** in B acting upon the fictitious matter and current produced on the boundary of B, on the assumption that **E** and **H** terminate there. If the normal and $\mathbf{P}_{\mathrm{N}}$ be drawn the other way, thus negativing them both, as well as the fictitious matter and current on the interface, then it is the forcive of A on B that is represented by the action of **E** and **H** in A on the new interfacial matter and current. That is, the **E** and **H** in the region A may be done away with altogether, because their abolition will immediately introduce the fictitious matter and current equivalent, so far as B is concerned, to the influence of the region A. Similarly **E** and **H** in B may be abolished without altering them in A. And, generally, any portion of the medium may be taken by itself and regarded as being subjected to an equilibrating system of forces, when treated as a rigid body.



§ **24.** When c and μ do not vary in space, we do away with the forces $-\frac{1}{2}\mathbf{E}^2\nabla c$ and $-\frac{1}{2}\mathbf{H}^2\nabla\mu$, and make the form of the surface and volume translational forces agree. We may then regard every element of ρ or of σ as a source sending out from itself displacement and induction isotropically, and every element of **J** or **G** as causing induction or displacement according to AMPÈRE'S rule for electric current and its analogue for magnetic current. Thus

$$\mathbf{E} = \Sigma\frac{\rho/c + \mathrm{V}\mathbf{r}_1\mathbf{G}}{4\pi r^2}, \qquad (162)$$

$$\mathbf{H} = \Sigma\frac{\sigma/\mu + \mathrm{V}\mathbf{J}\mathbf{r}_1}{4\pi r^2}, \qquad (163)$$

where $\mathbf{r}_1$ is a unit vector drawn from the infinitesimal unit volume in the summation to the point at distance r where **E** or **H** is reckoned. Or, introducing potentials,

$$\mathbf{E} = -\nabla\Sigma\frac{\rho/c}{4\pi r} - \mathrm{curl}\,\Sigma\frac{\mathbf{G}}{4\pi r}, \qquad (164)$$

$$\mathbf{H} = -\nabla\Sigma\frac{\sigma/\mu}{4\pi r} + \mathrm{curl}\,\Sigma\frac{\mathbf{J}}{4\pi r}. \qquad (165)$$

These apply to the whole medium, or to any portion of the same, with, in the latter case, the surface matter and current included, there being no **E** or **H** outside the region, whilst within it **E** and **H** are the same as due to the matter and current in the whole region ("matter," ρ and σ; "current," **J** and **G**). But there is no known general method of finding the potentials when c and μ vary.

We may also divide **E** and **H** into two parts each, say $\mathbf{E}_1$ and $\mathbf{H}_1$ due to matter and current in the region A, and $\mathbf{E}_2$, $\mathbf{H}_2$ due to matter and current in the region B surrounding it, determinable in the isotropic homogeneous case by the above formulæ. Then we may ignore $\mathbf{E}_1$ and $\mathbf{H}_1$ in estimating the forcive on the matter and current in the region A ; thus,

$$\Sigma\,(\mathbf{H}_2\sigma_1 + \mathrm{V}\mathbf{J}_1\mathbf{B}_2) + \Sigma\,(\mathbf{E}_2\rho_1 + \mathrm{V}\mathbf{D}_2\mathbf{G}_1), \qquad (166)$$

where $\sigma_1 = \operatorname{div} \mathbf{B}_1 = \operatorname{div} \mathbf{B}$, and $\mathbf{J}_1 = \operatorname{curl} \mathbf{H}_1 = \operatorname{curl} \mathbf{H}$ in region A, is the resultant force on the region A, and

$$\Sigma\,(\mathbf{H}_1\sigma_2 + \mathrm{V}\mathbf{J}_2\mathbf{B}_1) + \Sigma\,(\mathbf{E}_1\rho_2 + \mathrm{V}\mathbf{D}_1\mathbf{G}_2), \qquad (167)$$

is the resultant force on the region B ; the resultant force on A due to its own $\mathbf{E}$ and $\mathbf{H}$ being zero, and similarly for B. These resultant forces are equal and opposite, and so are the equivalent surface-integrals

$$\Sigma\,(\mathbf{H}_2\sigma_1' + \mathrm{V}\mathbf{J}_1'\mathbf{B}_2) + \Sigma\,(\mathbf{E}_2\rho_1' + \mathrm{V}\mathbf{D}_2\mathbf{G}_1'), \qquad (168)$$

and

$$\Sigma\,(\mathbf{H}_1\sigma_2' + \mathrm{V}\mathbf{J}_2'\mathbf{B}_1) + \Sigma\,(\mathbf{E}_1\rho_2' + \mathrm{V}\mathbf{D}_1\mathbf{G}_2'), \qquad (169)$$

taken over the interface. The quantity summed is that part of the stress-vector, $\mathbf{P}_{\mathrm{N}}$, which depends upon products of the $\mathbf{H}$ of one region and the $\mathbf{B}$ of the other, &c. Thus, for the magnetic stress only,

$$\begin{aligned}\mathbf{H.BN} - \mathbf{N}.\tfrac{1}{2}\mathbf{HB} &= \left(\mathbf{H}_1.\mathbf{B}_1\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_1\mathbf{B}_1\right) + \left(\mathbf{H}_1.\mathbf{B}_2\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_1\mathbf{B}_2\right)\\ &\quad + \left(\mathbf{H}_2.\mathbf{B}_2\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_2\mathbf{B}_2\right) + \left(\mathbf{H}_2.\mathbf{B}_1\mathbf{N} - \mathbf{N}.\tfrac{1}{2}\mathbf{H}_2\mathbf{B}_1\right), \qquad (170)\end{aligned}$$

and it is the terms in the second and fourth brackets (which, be it observed, are not equal) which together make up the magnetic part of (168) and (169) or their negatives, according to the direction taken for the normal ; that is, since $\mathbf{H}_1\mathbf{B}_2 = \mathbf{H}_2\mathbf{B}_1$,

$$\begin{aligned}\Sigma\,\mathbf{P}_{\mathrm{N}} &= \Sigma\left(\mathbf{H}_1.\mathbf{B}_2\mathbf{N} + \mathbf{H}_2.\mathbf{B}_1\mathbf{N} - \mathbf{N}.\mathbf{H}_1\mathbf{B}_2\right) = \Sigma\left(\mathbf{H.BN} - \mathbf{N}.\tfrac{1}{2}\mathbf{HB}\right)\\ &= \Sigma\left(\mathbf{H}_1\sigma_2' + \mathrm{V}\mathbf{J}_2'\mathbf{B}_1\right) = \Sigma\left(\mathbf{H}_2\sigma_1' + \mathrm{V}\mathbf{J}_1'\mathbf{B}_2\right) = \dot{\Sigma}\left(\mathbf{H}\sigma' + \mathrm{V}\mathbf{J}'\mathbf{B}\right)\\ &= \Sigma\,\mathbf{F} = \Sigma\left(\mathbf{H}_1\sigma_2 + \mathrm{V}\mathbf{J}_2\mathbf{B}_1\right) = \Sigma\left(\mathbf{H}_2\sigma_1 + \mathrm{V}\mathbf{J}_1\mathbf{B}_2\right) = \Sigma\left(\mathbf{H}\sigma + \mathrm{V}\mathbf{JB}\right), \qquad (171)\end{aligned}$$

where the first six expressions are interfacial summations, and the four last summations throughout one or the other region, the last summation applying to either region. No special reckoning of the sign to be prefixed has been made. The notation is such that $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$, $\sigma = \sigma_1 + \sigma_2$, &c., &c.

The comparison of the two aspects of electromagnetic theory is exceedingly curious ; namely, the precise mathematical equivalence of "explanation" by means of instantaneous action at a distance between the different elements of matter and current, each according to its kind, and by propagation through a medium in time at a finite velocity. But the day has gone by for any serious consideration of the former view other than as a mathematical curiosity.

Quaternionic Form of Stress-Vector.

§ **25.** We may also notice the Quaternion form for the stress function, which is so vital a part of the mathematics of forces varying as the inverse square of the distance, and of potential theory. Isotropy being understood, the electric stress may be written

$$\mathbf{P}_\mathrm{N} = \tfrac{1}{2}c\,[\mathbf{E}\mathbf{N}^{-1}\mathbf{E}], \quad \ldots \ldots \ldots \quad (172)$$

where the quantity in the square brackets is to be understood quaternionically. It is, however, a pure vector. Or,

$$\left[\frac{\mathbf{P}_\mathrm{N}}{\mathbf{E}}\right] = \frac{c}{2}\left[\frac{\mathbf{E}}{\mathbf{N}}\right], \quad \ldots \ldots \ldots \quad (173)$$

that is, not counting the factor $\frac{1}{2}c$, the quaternion $\left[\frac{\mathbf{P}_\mathrm{N}}{\mathbf{E}}\right]$ is the same as the quaternion $\left[\frac{\mathbf{N}}{\mathbf{E}}\right]$; or the same operation which turns **N** to **E** also turns **E** to $\mathbf{P}_\mathrm{N}$. Thus, **N**, **E**, and $\mathbf{P}_\mathrm{N}$ are in the same plane, and the angle between **N** and **E** equals that between **E** and $\mathbf{P}_\mathrm{N}$; and **E** and $\mathbf{P}_\mathrm{N}$ are on the same side of **N** when **E** makes an acute angle with **N**. Also, the tensor of $\mathbf{P}_\mathrm{N}$ is U, so that its normal and tangential components are $\mathrm{U}\cos 2\theta$ and $\mathrm{U}\sin 2\theta$, if $\theta = \widehat{\mathbf{NE}}$.

Otherwise

$$\mathbf{P}_\mathrm{N} = -\tfrac{1}{2}c\,[\mathbf{ENE}], \quad \ldots \ldots \ldots \quad (174)$$

since the quaternionic reciprocal of a vector has the reverse direction. The corresponding volume translational force is

$$\mathbf{F} = -c\mathrm{V}\,[\mathbf{E}\nabla\mathbf{E}], \quad \ldots \ldots \ldots \quad (175)$$

which is also to be understood quaternionically, and expanded, and separated into parts to become physically significant. I only use the square brackets in this paragraph to emphasise the difference in notation. It rarely occurs that any advantage is gained by the use of the quaternion, in saying which, I merely repeat what Professor WILLARD GIBBS has been lately telling us; and I further believe the disadvantages usually far outweigh the advantages. Nevertheless, apart from practical application, and looking at it from the purely quaternionic point of view, I ought to also add that the invention of quaternions must be regarded as a most remarkable feat of human ingenuity. Vector analysis, without quaternions, could have been found by any mathematician by carefully examining the mechanics of the Cartesian mathematics; but to find out quaternions required a genius.

Remarks on the Translational Force in Free Ether.

§ **26.** The little vector $\mathbf{Veh}$, which has an important influence in the activity equation, where $\mathbf{e}$ and $\mathbf{h}$ are the motional forces

$$\mathbf{e} = \mathrm{V}\mathbf{qB}, \qquad \mathbf{h} = \mathrm{V}\mathbf{Dq},$$

has an interesting form, viz., by expansion,

$$\mathrm{V}\mathbf{eh} = \mathbf{q}.\mathbf{q}\mathrm{V}\mathbf{DB} = \frac{\mathbf{q}}{v^2}\cdot \mathbf{q}\mathrm{V}\mathbf{EH}, \quad \text{. . . . .} \quad (176)$$

if v be the speed of propagation of disturbances. We also have, in connection therewith, the equivalence

$$\mathbf{eD} = \mathbf{hB}, \quad \text{. . . . .} \quad (177)$$

always.

The translational force in a non-conducting dielectric, free from electrification and intrinsic force, is

$$\mathbf{F} = \mathrm{V}\mathbf{JB} + \mathrm{V}\mathbf{DG} + \mathrm{V}\mathbf{jB} + \mathrm{V}\mathbf{Dg},$$

or, approximately,

$$= \mathrm{V}\dot{\mathbf{D}}\mathbf{B} + \mathrm{V}\mathbf{D}\dot{\mathbf{B}} = \frac{d}{dt}\,\mathrm{V}\mathbf{DB} = \frac{1}{v^2}\,\frac{d}{dt}\,\mathrm{V}\mathbf{EH} = \frac{\dot{\mathbf{W}}}{v^2}. \quad \text{. . . . .} \quad (178)$$

The vector $\mathrm{V}\mathbf{DB}$, or the flux of energy divided by the square of the speed of propagation, is, therefore, the momentum (translational, not magnetic, which is quite a different thing), provided the force $\mathbf{F}$ *is* the complete force from all causes acting, and we neglect the small terms $\mathrm{V}\mathbf{jB}$ and $\mathrm{V}\mathbf{Dg}$.

But have we any right to safely write

$$\mathbf{F} = m\frac{\partial \mathbf{q}}{\partial t}, \quad \text{. . . . .} \quad (179)$$

where m is the density of the ether? To do so is to assume that $\mathbf{F}$ is the only force acting, and, therefore, equivalent to the time-variation of the momentum of a moving particle.*

Now, if we say that there is a certain forcive upon a conductor supporting electric current; or, equivalently, that there is a certain distribution of stress, the magnetic stress, acting upon the same, we do not at all mean that the accelerations of momentum of the different parts are represented by the translational force, the "electromagnetic force." It is, on the other hand, a dynamical problem in which the electromagnetic force plays the part of an impressed force, and similarly as regards the magnetic

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* Professor J. J. THOMSON has endeavoured to make practical use of the idea, 'Phil. Mag.,' March, 1891. See also my article, 'The Electrician,' January 15, 1886.

stress; the actual forces and stresses being only determinable from a knowledge of the mechanical conditions of the conductor, as its density, elastic constants, and the way it is constrained. Now, if there is any dynamical meaning at all in the electromagnetic equations, we must treat the ether in precisely the same way. But we do not know, and have not formularised, the equations of motion of the ether, but only the way it propagates disturbance through itself, with due allowance made for the effect thereon of given motions, and with formularisation of the reaction between the electromagnetic effects and the motion. Thus the theory of the stresses and forces in the ether and its motions is an unsolved problem, only a portion of it being known so far, *i.e.*, assuming that the Maxwellian equations do express the known part.

When we assume the ether to be motionless, there is a partial similarity to the theory of the propagation of vibrations of infinitely small range in elastic bodies, when the effect thereon of the actual translation of the matter is neglected.

But in ordinary electromagnetic phenomena, it does not seem that the ignoration of $\mathbf{q}$ can make any sensible difference, because the speed of propagation of disturbances through the ether is so enormous, that if the ether were stirred about round a magnet, for example, there would be an almost instantaneous adjustment of the magnetic induction to what it would be were the ether at rest.

Static Consideration of the Stresses.—Indeterminateness.

§ **27.** In the following the stresses are considered from the static point of view, principally to examine the results produced by changing the form of the stress function. Either the electric or the magnetic stress alone may be taken in hand. Start then, from a knowledge that the force on a magnetic pole of strength m is $\mathbf{R}m$, where $\mathbf{R}$ is the polar force of any distribution of intrinsic magnetisation in a medium, the whole of which has unit inductivity, so that

$$\operatorname{div} \mathbf{R} = m = \operatorname{conv} \mathbf{h}_0 \quad \dots \dots \dots \dots \quad (180)$$

measures the density of the fictitious "magnetic" matter; $\mathbf{h}_0$ being the intrinsic force, or, since here $\mu = 1$, the intensity of magnetisation. The induction is $\mathbf{B} = \mathbf{h} + \mathbf{R}$. This rudimentary theory locates the force on a magnet at its poles, superficial or internal, by

$$\mathbf{F} = \mathbf{R} \operatorname{div} \mathbf{R}. \quad \dots \dots \dots \dots \quad (181)$$

The $\mathbf{N}$ component of $\mathbf{F}$ is

$$\mathbf{FN} = \mathbf{RN}.\operatorname{div} \mathbf{R} = \operatorname{div} \{\mathbf{R}.\mathbf{RN} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^2\}, \quad \dots \dots \dots \quad (182)$$

because $\operatorname{curl} \mathbf{R} = 0$. Therefore

$$\mathbf{P}_{\mathrm{N}} = \mathbf{R}.\mathbf{RN} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^2 \quad \dots \dots \dots \dots \quad (183)$$

is the appropriate stress, of irrotational type. Now, however uncertain we may be about the stress in the interior of a magnet, there can be no question as to the possible validity of this stress in the air outside our magnet, for we know that the force $\mathbf{R}$ is then a polar force, and that is all that is wanted, m and $\mathbf{h}$ being merely auxiliaries, derived from $\mathbf{R}$.

Now consider a region A, containing magnets of this kind, enclosed in B, the rest of space, also containing magnets. The mutual force between the two regions is expressed by $\Sigma\mathbf{P}_\mathrm{N}$ over the interface, which we may exchange for $\Sigma\mathrm{R}m$ through either region A or B, still on the assumption that $\mathbf{R}$ remains polar.

But if we remove this restriction upon the nature of $\mathbf{R}$, and allow it to be arbitrary, say in region B or in any portion thereof, we find

$$\mathbf{NF} = \operatorname{div}\mathbf{P}_\mathrm{N} = \mathbf{RN}\operatorname{div}\mathbf{R} + \mathbf{N}\mathrm{V}\operatorname{curl}\mathbf{R}\,.\,\mathbf{R}\,;$$

or

$$\mathbf{F} = \mathrm{R}m + \mathrm{V}\mathbf{JR},$$

if $\mathbf{J} = \operatorname{curl}\mathbf{R}$. This gives us, from a knowledge of the external magnetic field of polar magnets only, the mechanical force exerted by a magnet on a region containing $\mathbf{J}$, whatever that may be, provided it be measurable as above; and without any experimental knowledge of electric currents, we could now predict their mechanical effects in every respect by the principle of the equality of action and reaction, not merely as regards the mutual influence of a magnet and a closed current, but as regards the mutual influence of the closed currents themselves; the magnetic force of a closed current, for instance, being the force on unit of m, is equivalently the force exerted by m on the closed current, which, by the above, we know. Also, we see that according to this magnetic notion of electric current, it is necessarily circuital.

At the same time, it is to be remarked that our real knowledge must cease at the boundary of the region containing electric current, a metallic conductor for instance; the surface over which $\mathbf{P}_\mathrm{N}$ is reckoned, on one side of which is the magnet, on the other side electric current, can only be pushed up as far as the conductor. The stress $\mathbf{P}_\mathrm{N}$ may therefore cease altogether on reaching the conductor, where it forms a distribution of surface force fully representing the action of the magnet on the conductor. Similarly, we need not continue the stress into the interior of the magnet. Then, so far as the resultant force on the magnet as a whole, in translating or rotating it, and, similarly, so far as the action on the conductor, is concerned, the simple stress $\mathbf{P}_\mathrm{N}$ of constant tensor $\frac{1}{2}\mathbf{R}^2$, varying from a tension parallel to $\mathbf{R}$ to an equal pressure laterally, acting in the medium between the magnet and conductor, accounts, by its terminal pulls or pushes, for the mechanical forces on them. The lateral pressure is especially prominent in the case of conductors, whilst the tension goes more or less out of sight, as the immediate cause of motion. Thus, when parallel currents appear to attract one another, the conductors are really pushed together by the lateral pressure on each conductor being greater on the side remote from the other than on the near side: whilst if the currents are oppositely directed, the pressure on the near sides is greater than on the remote sides, and they appear to repel one another.

The effect of continuing the stress into the interior of a conductor of unit inductivity, according to the same law, instead of stopping it on its boundary, is to distribute the translational force bodily, according to the formula $\Sigma \mathbf{VJB}$, instead of superficially, according to $\Sigma \mathbf{P}_N$. In either case, of course, the conductor must be strained by the magnetic stress, with the consequent production of a mechanical stress. But the strain (and associated stress) will be different in the two cases, the applied forces being differently localised. The effect of the stress on a straight portion of a wire supporting current, due to its own field only, is to compress it laterally, and to lengthen it. Besides this, there will be resultant force on it arising from the different pressures on its opposite sides due to the proximity of the return conductor or rest of the circuit, tending to move it so as to increase the induction through the circuit per unit current, that is, the inductance of the circuit.

§ **28.** If now, we bring an elastically magnetisable body into a magnetic field, it modifies the field by its presence, causing more or less induction to go through it than passed previously in the air it replaces, according as its inductivity exceeds or is less than that of the air. The forcive on it, considered as a rigid body, is completely accounted for by the simple stress $\mathbf{P}_N$ in the air outside it, reckoned according to the changed field, and supposed to terminate on the surface of the disturbing body. This is true whether the body be isotropic or heterotropic in its inductivity; nor need the induction be a linear function of the magnetic force. It is also true when the body is intrinsically magnetised; or is the seat of electric current. In short, since the external stress depends upon the magnetic force outside the body, when we take the external field as we may find it, that is, as modified by any known or unknown causes within the body, the corresponding stress, terminated upon its boundary, fully represents the forcive on the body, as a whole, due to magnetic causes. This follows from the equality of action and reaction; the force on the body due to a unit pole is the opposite of that of the body on the pole.

If we wish to continue the stress into the interior of the body, surrounded on all sides by the unmagnetised medium of unit inductivity, as we must do if we wish to arrive ultimately at the mutual actions of its different parts, and how they are modified by variations of inductivity, by intrinsic magnetisation, and by electric current in the body, we may, so far as the resultant force and torque on it are concerned, do it in any way we please, provided we do not interfere with the stress outside. For the internal stress, of any type, will have no resultant force or torque on the body, and there is merely left the real external stress.

Practically, however, we should be guided by the known relations of magnetic force, induction, magnetisation, and current, and not go to work in a fanciful manner; furthermore, we should always choose the stress in such a way that if, in its expression, we take the inductivity to be unity, and the intrinsic magnetisation zero, it must reduce to the simple Maxwellian stress in air (assumed to represent ether here).

But as we do not know definitely the forcive arising from the magnetic stress in the interior of a magnet, there are several formulæ that suggest themselves as possible.

Special Kinds of Stress Formulæ statically suggested.

§ **29.** Thus, first we have the stress (183); let this be quite general, then

$$(1)\begin{cases}\mathbf{P}_{\mathrm{N}} = \mathbf{R}.\mathbf{RN} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^2, & (184)\\ \mathbf{F} \;= \mathbf{R}\operatorname{div}\mathbf{R} + \mathrm{V}\mathbf{JR}. & (185)\end{cases}$$

Here **R** is the magnetic force of the field, not of the flux **B**. If $\mu = 1$, div **R** is the density of magnetic matter, the convergence of the intrinsic magnetisation, but not otherwise. In general, it is the density of the matter of the magnetic potential, calculated on the assumption $\mu = 1$. The force on a magnet is located in this system at its poles, whether the magnetisation be intrinsic or induced. The second term in (185) represents the force on matter bearing electric current ($\mathbf{J} = \operatorname{curl}\mathbf{R}$), but has to be supplemented by the first term, unless div $\mathbf{R} = 0$ at the place.

§ **30.** Next, let the stress be μ times as great for the same magnetic force, but be still of the same simple type, μ being the inductivity, which is unity outside the body, but having any positive value, which may be variable, within it. Then we shall have

$$(2)\begin{cases}\mathbf{P}_{\mathrm{N}} = \mathbf{R}.\mathbf{N}\mu\mathbf{R} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}\mu\mathbf{R}, & (186)\\ \mathbf{F} \;= \mathbf{R}m + \mathrm{V}\mathbf{J}\mu\mathbf{R} - \tfrac{1}{2}\mathbf{R}^2\nabla\mu, & (187)\end{cases}$$

where $m = \operatorname{conv}\mu\mathbf{h}_0 = \operatorname{div}\mu\mathbf{R}$ is the density of magnetic matter, $\mu\mathbf{h}_0$ being the intensity of intrinsic magnetisation.

The electromagnetic force is made μ times as great for the same magnetic force; the force on an intrinsic magnet is at its poles; and there is, in addition, a force wherever μ varies, including the intrinsic magnet, and not forgetting that a sudden change in μ, as at the boundary of a magnet, has to count. This force, the third term in (187), explains the force on inductively magnetised matter. It is in the direction of most rapid decrease of μ.

§ **31.** Thirdly, let the stress be of the same simple type, but taking **H** instead of **R**, **H** being the force of the flux $\mathbf{B} = \mu\mathbf{H} = \mu(\mathbf{R} + \mathbf{h}_0)$, where $\mathbf{h}_0$ is as before. We now have

$$(3)\begin{cases}\mathbf{P}_{\mathrm{N}} = \mathbf{H}.\mathbf{NB} - \mathbf{N}.\tfrac{1}{2}\mathbf{HB}, & (188)\\ \mathbf{F} \;= \mathrm{V}\mathbf{JB} + \mathrm{V}\mathbf{j}_0\mathbf{B} - \tfrac{1}{2}\mathbf{H}^2\nabla\mu, & (189)\end{cases}$$

where $\mathbf{j}_0 = \operatorname{curl}\mathbf{h}_0$ is the distribution of fictitious electric current which produces the same induction as the intrinsic magnetisation $\mu\mathbf{h}_0$, and **J** is, as before, the real current.

It is now *quasi*-electromagnetic force that acts on an intrinsic magnet, with, however, the force due to $\nabla\mu$, since a magnet has usually large μ compared with air.

The above three stresses are all of the simple type (equal tension and perpendicular pressure), and are irrotational, unless μ be the eolotropic operator. No change is, in the latter case, needed in (186), (188), whilst in the force formulæ (187), (189), the only change needed is to give the generalised meaning to $\nabla\mu$. Thus, in (189), instead of $\mathbf{H}^2\nabla\mu$, use $2\nabla\mathrm{T}_{\mu}$,

or

$$\nabla_{\mu}\left(\mathbf{H}\mu\mathbf{H}\right),$$

or

$$\mathrm{i}\left(\mathbf{H}\frac{d\mu}{dx}\mathbf{H}\right)+\mathrm{j}\left(\mathbf{H}\frac{d\mu}{dy}\mathbf{H}\right)+\mathrm{k}\left(\mathbf{H}\frac{d\mu}{dz}\mathbf{H}\right),$$

or

$$\left(\nabla_{\mathrm{B}}-\nabla_{\mathrm{H}}\right)\mathbf{HB},$$

or

$$\mathrm{i}\left(\mathbf{H}\nabla_1\mathbf{B}-\mathbf{B}\nabla_1\mathbf{H}\right)+\mathrm{j}\left(\mathbf{H}\nabla_2\mathbf{B}-\mathbf{B}\nabla_2\mathbf{H}\right)+\mathrm{k}\left(\mathbf{H}\nabla_3\mathbf{B}-\mathbf{B}\nabla_3\mathbf{H}\right),$$

showing the **i**, **j**, **k** components.

Similarly in the other cases occurring later.

The following stresses are not of the simple type, though all consist of a tension parallel to **R** or **H** combined with an isotropic pressure.

§ **32.** Alter the stress so as to locate the force on an intrinsic magnet bodily upon its magnetised elements. Add $\mathbf{R}.\mu\mathbf{h}_0\mathbf{N}$ to the stress (186), and therefore $\mu\mathbf{h}_0.\mathbf{RN}$ to its conjugate; then the divergence of the latter must be added to the **N**-component of the force (187). Thus we get, if $\mathbf{I}=\mu\mathbf{h}_0$,

$$(4)\quad\begin{cases}\mathbf{P}_{\mathrm{N}}=\mathbf{R}.\mathbf{BN}-\mathbf{N}.\tfrac{1}{2}\,\mathbf{R}\mu\mathbf{R}, & \qquad (190)\\ \mathbf{F}\ =\mathbf{I}\nabla.\mathbf{R}+\mathrm{V}\mathbf{J}\mu\mathbf{R}-\tfrac{1}{2}\mathbf{R}^2\nabla\mu. & \qquad (191)\end{cases}$$

But here the sum of the first two terms in **F** may be put in a different form. Thus,

$$\begin{aligned}\mathbf{I}\nabla.\mathbf{R}&=\mathrm{I}_1\nabla_1\mathbf{R}+\mathrm{I}_2\nabla_2\mathbf{R}+\mathrm{I}_3\nabla_3\mathbf{R}\\&=\mathbf{i}.\mathbf{I}\nabla\mathrm{R}_1+\mathbf{j}.\mathbf{I}\nabla\mathrm{R}_2+\mathbf{k}.\mathbf{I}\nabla\mathrm{R}_3.\end{aligned}$$

Also

$$\mathbf{I}\nabla\mathrm{R}_1=\mathbf{I}\nabla_1\mathbf{R}+\mathbf{I}\left(\nabla\mathrm{R}_1-\nabla_1\mathbf{R}\right)=\mathbf{I}\nabla_1\mathbf{R}+\mathbf{i}\mathrm{V}\mathbf{JI}.$$

These bring (191) to

$$\mathbf{F}=\left(\mathbf{i}.\mathbf{I}\nabla_1\mathbf{R}+\mathbf{j}.\mathbf{I}\nabla_2\mathbf{R}+\mathbf{k}.\mathbf{I}\nabla_3\mathbf{R}\right)+\mathrm{V}\mathbf{JB}-\tfrac{1}{2}\,\mathbf{R}^2\nabla\mu, \qquad (192)$$

where the first component (the bracketted part) is MAXWELL'S force on intrinsic magnetisation, and the second his electromagnetic force. The third, as before, is required where μ varies.

§ **33.** To the stress (190) add $-\,\mathbf{N}.\tfrac{1}{2}\,\mathbf{RI}$, without altering the conjugate stress making

$$(5)\left\{\begin{array}{ll}\mathbf{P}_{\mathrm{N}} = \mathbf{R}.\mathbf{BN} - \mathbf{N}.\tfrac{1}{2}\,\mathbf{RB}, & \qquad (193)\\ \mathbf{F} = \mathrm{V}\mathbf{JB} - \tfrac{1}{2}\left\{\mathbf{i}\left(\mathbf{R}\nabla_1\mathbf{B} - \mathbf{B}\nabla_1\mathbf{R}\right) + \mathbf{j}\left(\mathbf{R}\nabla_2\mathbf{B} - \mathbf{B}\nabla_2\mathbf{R}\right) + \mathbf{k}\left(\mathbf{R}\nabla_3\mathbf{B} - \mathbf{B}\nabla_3\mathbf{R}\right)\right\}. & \qquad (194)\end{array}\right.$$

$$= \mathrm{V}\mathbf{JB} - \left(\nabla_{\mathrm{B}} - \nabla_{\mathrm{R}}\right)\tfrac{1}{2}\mathbf{RB}.$$

This we need not discuss, as it is merely a transition to the next form.

§ **34.** To the stress (193) add $\mathbf{h}_0.\mathbf{NB}$; we then get

$$(6)\left\{\begin{array}{ll}\mathbf{P}_{\mathrm{N}} = \mathbf{H}.\mathbf{NB} - \mathbf{N}.\tfrac{1}{2}\mathbf{RB}, & \qquad (195)\\ \mathbf{F} = \mathrm{V}\mathbf{JB} + \left\{\mathbf{i}.\mathbf{B}\nabla h_1 + \mathbf{j}.\mathbf{B}\nabla h_2 + \mathbf{k}.\mathbf{B}\nabla h_3\right\} & \\ \quad - \tfrac{1}{2}\left\{\mathbf{i}.\left(\mathbf{R}\nabla_1\mathbf{B} - \mathbf{B}\nabla_1\mathbf{R}\right) + \mathbf{j}.\left(\mathbf{R}\nabla_2\mathbf{B} - \mathbf{B}\nabla_2\mathbf{R}\right) + \mathbf{k}.\left(\mathbf{R}\nabla_3\mathbf{B} - \mathbf{B}\nabla_3\mathbf{R}\right)\right\}, & \qquad (196)\end{array}\right.$$

$$= \mathrm{V}\mathbf{JB} + \mathbf{B}\nabla.\mathbf{h}_0 - \left(\nabla_{\mathrm{B}} - \nabla_{\mathrm{R}}\right)\tfrac{1}{2}\mathbf{RB},$$

where h_1, h_2, h_3 are the components of $\mathbf{h}_0$.

Now if to this last stress (195) we add $-\,\mathbf{N}.\tfrac{1}{2}\mathbf{h}_0\mathbf{B}$, we shall come back to the third stress, (188), of the simple type.

Perhaps the most instructive order in which to take the six stresses is (1), (2), (4), (5), (6), and (3); merely adding on to the force, in passing from one stress to the next, the new part which the alteration in the stress necessitates.

To the above we should add MAXWELL'S general stress, which is

$$(7)\left\{\begin{array}{ll}\mathbf{P}_{\mathrm{N}} = \mathbf{R}.\mathbf{NB} - \mathbf{N}.\tfrac{1}{2}\mathbf{R}^2, & \qquad (197)\\ \mathbf{F} = \mathrm{V}\mathbf{JB} + \left\{\mathbf{i}.\mathbf{I}\nabla_1\mathbf{R} + \mathbf{j}.\mathbf{I}\nabla_2\mathbf{R} + \mathbf{k}.\mathbf{I}\nabla_3\mathbf{R}\right\} & \\ \qquad + \left\{\mathbf{i}.\mathbf{M}\nabla_1\mathbf{R} + \mathbf{j}.\mathbf{M}\nabla_2\mathbf{R} + \mathbf{k}.\mathbf{M}\nabla_2\mathbf{R}\right\}, & \qquad (198)\end{array}\right.$$

$$= \mathrm{V}\mathbf{JB} + \nabla_{\mathrm{R}}\left[\mathbf{R}\left(\mathbf{I} + \mathbf{M}\right)\right],$$

where $\mathbf{M} = (\mu - 1)\,\mathbf{R}$ = intensity of induced magnetisation. There is a good deal to be said against this stress; some of which later.

Remarks on MAXWELL'S *General Stress.*

§ **35.** All the above force formulæ refer to the unit volume; whenever, therefore, a discontinuity in the stress occurs at a surface, the corresponding expression per unit surface is needed; *i.e.*, in making a special application, for it is wasted labour else. It might be thought that as MAXWELL gives the force (198), and in his treatise usually gives surface expressions separately, so none is required in the case of this his force system (198). But this formula will give entirely erroneous results if carried out literally. It forms no exception to the rule that all the expressions require surface additions.

MAXWELL'S general stress has the apparent advantage of simplicity. It merely requires an alteration in the tension parallel to **R**, from $\mathbf{R}^2$ to **RB**, whilst the lateral pressure remains $\tfrac{1}{2}\mathbf{R}^2$, when we pass from unmagnetised to magnetised matter. The force to which it gives rise is also apparently simple, being merely the sum of two forces, one the electromagnetic, $\mathrm{V}\mathbf{JB}$, the other a force on magnetised matter whose i component is $(\mathbf{I} + \mathbf{M})\nabla_1\mathbf{R}$, both per unit volume, the latter being accompanied (in case of eolotropy) by a torque. Now **I** is the intrinsic and **M** the induced magnetisation, so the force is made irrespective of the proportion in which the magnetisation exists as intrinsic or induced. In fact, MAXWELL'S "magnetisation" is the sum of the two without reservation or distinction. But to unite them is against the whole behaviour of induced and intrinsic magnetisation in the electromagnetic scheme of MAXWELL, as I interpret it. Intrinsic magnetisation (using Sir W. THOMSON'S term) should be regarded as impressed ($\mathbf{I} = \mu\mathbf{h}_0$, where $\mathbf{h}_0$ is the equivalent impressed magnetic force); on the other hand, "induced" magnetisation depends on the force of the field $\{\mathbf{M} = (\mu - 1)\mathbf{R}\}$. Intrinsic magnetisation keeps up a field of force. Induced magnetisation is kept up by the field. In the circuital law **I** and **M** therefore behave differently. There may be absolutely no difference whatever between the magnetisation of a molecule of iron in the two cases of being in a permanent or a temporary magnet. That, however, is not in question. We have no concern with molecules in a theory which ignores molecules, and whose element of volume must be large enough to contain so many molecules as to swamp the characteristics of individuals. It is the resultant magnetisation of the whole assembly that is in question, and there is a great difference between its nature according as it disappears on removal of an external cause, or is intrinsic. The complete amalgamation of the two in MAXWELL'S formula must certainly, I think, be regarded as a false step.

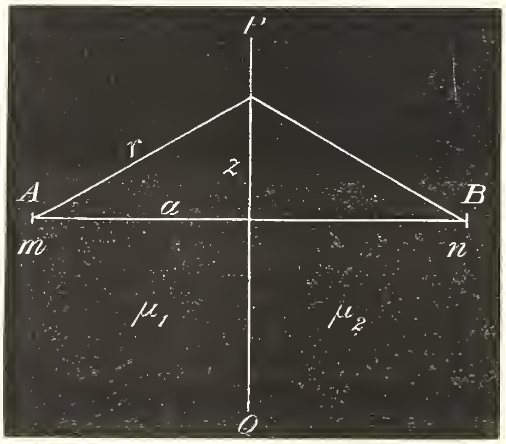
We may also argue thus against the probability of the formula. If we have a system of electric current in an unmagnetisable ($\mu = 1$) medium, and then change μ everywhere in the same ratio, we do not change the magnetic force at all, the induction is made μ times as great, and the magnetic energy μ times as great, and is similarly distributed. The mechanical forces are, therefore, μ times as great, and are similarly distributed. That is, the translational force in the $\mu = 1$ medium, or $\mathrm{V}\mathbf{JR}$, becomes $\mathrm{V}\mathbf{J}\mu\mathbf{R}$ in the second case in which the inductivity is μ, without other change. But there is no force brought in on magnetised matter *per se*.

Similarly, if in the $\mu = 1$ medium we have intrinsic magnetisation **I**, and then alter μ in any ratio everywhere alike, keeping **I** unchanged, it is now the induction that remains unaltered, the magnetic force becoming μ^{-1} times, and the energy μ^{-1} times the former values, without alteration in distribution (referring to permanent states, of course). Again, therefore, we see that there is no translational force brought in on magnetised matter merely because it is magnetised.

Whatever formula, therefore, we should select for the stress function, it would certainly not be MAXWELL'S, for cumulative reasons. When, some six years ago, I had occasion to examine the subject of the stresses, I was unable to arrive at any very definite results, except outside of magnets or conductors. It was a perfectly indeterminate problem to find the magnetic stress inside a body from the existence of a known, or highly probable, stress outside it. All one could do was to examine the consequences of assuming certain stresses, and to reject those which did not work well. After going into considerable detail, the only two which seemed possible were the second and third above (those of equations (186) and (188) above). As regards the seventh (MAXWELL'S stress equation (198) above), the apparent simplicity produced by the union of intrinsic and induced magnetisation, turned out, when examined into its consequences, to lead to great complication and unnaturalness. This will be illustrated in the following example, a simple case in which we can readily and fully calculate all details by different methods, so as to be quite sure of the results we ought to obtain.

A worked-out Example to Exhibit the Forcives contained in Different Stresses.

§ **36**. Given a fluid medium of inductivity μ_1, in which is an intrinsic magnet of the same inductivity. Calculate the attraction between the magnet and a large solid mass of different inductivity μ_2. Here it is only needful to calculate the force on a single pole, so let the magnet be infinitely thin and long, with one pole of strength m at distance a from the medium μ_2, which may have an infinitely extended plane boundary. By placing a fictitious pole of suitable strength at the optical image in the second medium of the real pole in the first, we may readily obtain the solution.



Let PQ be the interface and the real pole be at A and its image at B. We have first to calculate the distribution of **R**, magnetic force, in both media due to the pole m, as disturbed by the change of inductivity. We have div $\mu_1\mathbf{R}_1 = m$ in the first medium, and div $\mu_2\mathbf{R}_2 = 0$ in the second, therefore **R** has divergence only on the interface. Let σ be the surface density of the fictitious interfacial matter to correspond; its force goes symmetrically both ways; the continuity of the normal induction therefore gives, at distance r from A, the condition

$$\mu_1\left(\frac{ma}{4\pi\mu_1 r^3} - \tfrac{1}{2}\sigma\right) = \mu_2\left(\frac{ma}{4\pi\mu_1 r^3} + \tfrac{1}{2}\sigma\right), \qquad (199)$$

because $m/4\pi\mu_1 r^2$ is the tensor of the magnetic force due to m in the μ_1 medium when of infinite extent. Therefore

$$\sigma = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{ma}{2\pi\mu_1 r^3}. \qquad (200)$$

The magnetic potential Ω, such that $\mathbf{R} = -\nabla\Omega$ is the polar force in either region, is therefore the potential of m/μ_1 at A and of σ over the interface.

But if we put matter n at the image B, of amount

$$n = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{m}{\mu_1}, \qquad (201)$$

the normal component of $\mathbf{R}_1$ on the μ_1 side due to n and the pole m will be

$$\frac{ma}{4\pi\mu_1 r^3} - \frac{na}{4\pi r^3} = \frac{ma}{4\pi\mu_1 r^3} - \tfrac{1}{2}\sigma, \qquad (202)$$

the same value as before; the force $\mathbf{R}_1$ on the μ_1 side is, therefore, the same as that due to matter m/μ_1 at A and matter n at B; whilst on the μ_2 side the force $\mathbf{R}_2$ is that due to matter m/μ_1 at A and matter n also at A, that is, to matter $\dfrac{2m}{\mu_1 + \mu_2}$ at A. Thus in the μ_2 medium the force $\mathbf{R}_2$ is radial from A as if there were no change of inductivity, though altered in intensity.

The repulsion between the pole m and the solid mass is not the repulsion between the matters m/μ_1 and n of the potential, but is

= m × magnetic force at A due to matter n at B,

= n × magnetic force at B due to matter m/μ_1 at A,

$$= \frac{mn}{4\pi(2a)^2} = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \frac{m^2}{4\pi\mu_1 (2a)^2}, \qquad (203)$$

becoming an attraction when $\mu_2 > \mu_1$, making n negative. When $\mu_2 = 0$, the repulsion is

$$\frac{m^2}{4\pi\mu_1 (2a)^2};$$

when $\mu_2 = \infty$, it is turned into an attraction of equal amount.

Similarly, if we consider the attraction to be the resultant force between m and the interfacial matter σ, we shall get the same result by

$$\Sigma \frac{\sigma ma}{4\pi r^3}, \qquad (204)$$

the quantity summed (over the interface) being σ × normal component of magnetic force due to matter m in a medium of unit inductivity, or the normal component of induction due to m in its own medium. For this is

$$\int\frac{ma}{4\pi r^3}\cdot\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\cdot\frac{ma}{2\pi\mu_1 r^3}\,2\pi r\,dr=\frac{m^2a^2}{4\pi\mu_1}\cdot\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\int\frac{dr}{r^5}=\text{(203) again.}$$

Another way is to calculate the variation of energy made by displacing either the pole m or the μ_2 mass. The potential energy is expressed by

$$\tfrac{1}{2}(\mathrm{P}+p)\,m=\tfrac{1}{2}\mathrm{P}m+\tfrac{1}{2}\Sigma\mathrm{P}\sigma\mu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (205)$$

where $\mathrm{P}=m/4\pi\mu_1 r$ and $p=\Sigma\,\sigma/4\pi r$, the potentials of matter m/μ_1 and σ, where r is the distance from m or from σ to the point where P and p are reckoned.

The value of the second part in (205), depending upon σ, comes to

$$\tfrac{1}{2}\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\cdot\frac{m^2}{4\pi\mu_1\,.\,2a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (206)$$

and its rate of decrease with respect to a expresses the repulsion between the pole and the μ_2 region. This gives (203) again.

A fourth way is by means of the *quasi*-electromagnetic force on fictitious interfacial electric current, instead of matter, the current being circular about the axis of symmetry AB. The formula for the attraction is

$$\Sigma\mathrm{V}\ \mathrm{curl}\ \mathbf{B}.\mathbf{R}_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (207)$$

if $\mathbf{R}_0$ be the radial magnetic force from m in its own medium, tensor $m/4\pi\mu_1 r^2$. Here the curl of $\mathbf{B}$ is represented by the interfacial discontinuity in the tangential induction, or

$$\frac{2zm}{4\pi r^3}\,\frac{\mu_1-\mu_2}{\mu_1+\mu_2}.$$

Also the tangential component of $\mathbf{R}_0$ is $mz/4\pi\mu_1 r^3$. Therefore the repulsion is

$$\int\frac{2mz}{4\pi r^3}\cdot\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\cdot\frac{mz}{4\pi\mu_1 r^3}\,2\pi r\,dr=\frac{m^2}{4\pi\mu_1}\cdot\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\int_a^\infty\frac{r^2-a^2}{r^5}\,dr=\frac{m^2}{4\pi\mu_1}\cdot\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\cdot\frac{1}{4a^2}, \quad . \qquad (208)$$

as before, equation (203). This method (207) is analogous to (204).

§ **37.** There are several other ways of representing the attraction, employing fictitious matter and current; but now let us change the method, and observe how the attraction between the magnetic pole and the iron mass is accounted for by a stress distribution, and its space-variation. The best stress is the third, equation (188), § 31. Applying this, we have simply a tension of magnitude $\frac{1}{2}\mu_1\mathrm{R}_1{}^2=\mathrm{T}_1$ in the first medium and $\frac{1}{2}\mu_2 R_2^2 = T_2$ in the second, parallel to $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively, each combined with an equal lateral pressure, so that the tensor of the stress vector is constant.

But, so far as the attraction is concerned, we may ignore the stress in the second medium altogether, and consider it as the $\Sigma \mathbf{P}_N$ of the stress-vector in the first medium over the surface of the second medium. The tangential component summed has zero resultant; the attraction is therefore the sum of the normal components, or $\Sigma T_1 \cos 2\theta_1$, where θ_1 is the angle between $\mathbf{R}_1$ and the normal. This is the same as $\Sigma \frac{1}{2}\mu_1 (R_N^2 - R_T^2)$, if R_N and R_T are the normal and tangential components of $\mathbf{R}_1$; or

$$\int_a^\infty 2\pi r\, dr\, \tfrac{1}{2}\mu_1 \left[\left(\frac{ma}{4\pi\mu_1 r^3}\frac{2\mu_2}{\mu_1 + \mu_2}\right)^2 - \left(\frac{mz}{4\pi\mu_1 r^3}\frac{2\mu_1}{(\mu_1 + \mu_2)}\right)\right]; \quad \ldots \ldots \quad (209)$$

which on evaluation gives the required result (203).

But this method does not give the true distribution of translational force due to the stresses. In the first medium there is no translational force, except on the magnet. Nor is there any translational force in the second μ_2 medium. But at the interface, where μ changes, there is the force $-\frac{1}{2}R^2\nabla\mu$ per unit volume, and this is represented by the stress-difference at the interface. It is easily seen that the tangential stress-difference is zero, because

$$T \sin 2\theta = \mu R_N R_T, \quad \ldots \ldots \quad (210)$$

and both the normal induction and the tangential magnetic force are continuous. The real force is, therefore, the difference of the normal components of the stress-vectors, and is, therefore, normal to the interface. This we could conclude from the expression $-\frac{1}{2}R^2\nabla\mu$. But since the resultant of the interfacial stress in the second medium is zero, we need not reckon it, so far as the attraction of the pole is concerned. The normal traction on the interface, due to both stresses, is of amount

$$\frac{m^2}{8\pi^2 r^6}\frac{\mu_2 - \mu_1}{(\mu_1 + \mu_2)^2}\left(r^2 + a^2\frac{\mu_2 - \mu_1}{\mu_1}\right) \quad \ldots \ldots \quad (211)$$

per unit area. Summed up, it gives (203) again.

That (211) properly represents the force $-\frac{1}{2}R^2\nabla\mu$ when μ is discontinuous, we may also verify by supposing μ to vary continuously in a very thin layer, and then proceed to the limit.

The change from an attraction to a repulsion as μ_2 changes from being greater to being less than μ_1, depends upon the relative importance of the tensions parallel to the magnetic force and the lateral pressures operative at different parts of the interface. In the extreme case of $\mu_2 = 0$, we have $\mathbf{R}_1$ tangential, with, therefore, a pressure everywhere. For the other extreme, $\mathbf{R}_1$ is normal, and there is a pull on the second medium everywhere. When μ_2 is finite there is a certain circular area on the interface within which the translational force due to the stress in the medium containing the pole m is towards that medium, whilst outside it the force is the other way. But when both stresses are allowed for, we see that when $\mu_2 > \mu_1$ the pull is towards the first medium in all parts of the interface, and that this becomes a push in all parts when $\mu_1 > \mu_2$.

A definite Stress only obtainable by Kinetic Consideration of the Circuital Equations and Storage and Flux of Energy.

§ **38.** We see that the stress considered in the last paragraph gives a rationally intelligible interpretation of the attraction or repulsion. The same may be said of other stresses than that chosen. But the use of MAXWELL'S stress, or any stress leading to a force on inductively magnetised matter as this stress does, leads us into great difficulties. By (198) we see that there is first a bodily force on the whole of the μ_2 medium, because it is magnetised, unless $\mu_2 = 1$. When summed up, the resultant does not give the required attraction. For, secondly, the μ_1 medium is also magnetised, unless $\mu_1 = 1$, and there is a bodily force throughout the whole of it. When this is summed up (not counting the force on the magnet), its resultant added on to the former resultant still does not make up the attraction (*i.e.*, equivalently, the force on the magnet). For, thirdly, the stress is discontinuous at the interface (though not in the same manner as in the last paragraph). The resultant of this stress-discontinuity, added on to the former resultants, makes up the required attraction. It is unnecessary to give the details relating to so improbable a system of force.

Our preference must naturally be for a more simple system, such as the previously considered stress. But there is really no decisive settlement possible from the theoretical statical standpoint, and nothing short of actual experimental determination of the strains produced and their exhaustive analysis would be sufficient to determine the proper stress-function. But when the subject is attacked from the dynamical standpoint, the indeterminateness disappears. From the two circuital laws of variable states of electric and magnetic force in a moving medium, combined with certain distributions of stored energy, we are led to just one stress-vector, viz. (136). It is, in the magnetic case, the same as (188); that is, it reduces to the latter when the medium is kept at rest, so that $\mathbf{J}_0$ and $\mathbf{G}_0$ become $\mathbf{J}$ and $\mathbf{G}$.

It is of the simple type in case of isotropy (constant tensor), but is a rotational stress in general, as indeed are all the statically probable stresses that suggest themselves. The translational force due to it being divisible conveniently into (*a*) the electromagnetic force on electric current, (*b*) the ditto on the fictitious electric current taking the place of intrinsic magnetisation, (*c*) force depending upon space-variation of μ; we see that the really striking part is (*b*). Of all the various ways of representing the forcive on an intrinsic magnet it is the most extreme. The magnetic "matter" does not enter into it, nor does the distribution of magnetisation; it is where the intrinsic force $\mathbf{h}_0$ has curl that the translational force operates, usually on the sides of a magnet. From actual experiments with bar magnets, needles, &c., one would naturally prefer to regard the polar regions as the seat of translational force. But the equivalent forcive $\Sigma \mathbf{j}_0\mathbf{B}$ has one striking recommendation (apart from the dynamical method of deducing it), viz., that the induction of an intrinsic magnet is determined by curl $\mathbf{h}_0$, not by $\mathbf{h}_0$ itself; and this, I have shown, is true when $\mathbf{h}_0$ is imagined to vary, the whole varying states of the fluxes **B**, **D**, **C** due to impressed force being determined by the curls of $\mathbf{e}_0$ and $\mathbf{h}_0$, which are the sources of the disturbances (though not of the energy).

The rotational peculiarity in eolotropic substances does not seem to be a very formidable objection. Are they not solid?

As regards the assumed constancy of μ, a more complete theory must, to be correct, reduce to one assuming constancy of μ, because, as Lord RAYLEIGH[*] has shown, the assumed law has a limited range of validity, and is therefore justifiable as a preparation for more complete views. Theoretical requirements are not identical with those of the practical engineer.

But, for quite other reasons, the dynamically determined stress might be entirely wrong. Electric and magnetic "force" and their energies are facts. But it is the total of the energies in concrete cases that should be regarded as the facts, rather then their distribution; for example, that, as Sir W. THOMSON proved, the "mechanical value" of a simple closed current C is $\frac{1}{2}LC^2$, where L is the inductance of the circuit (coefficient of electromagnetic capacity), rather than that its distribution in space is given by $\frac{1}{2}\mathbf{HB}$ per unit volume. Other distributions may give the same total amount of energy. For example, the energy of distortion of an elastic solid may be expressed in terms of the square of the rotation and the square of the expansion, if its boundary be held at rest; but this does not correctly localise the energy. If, then, we choose some other distribution of the energy for the same displacement and induction, we should find quite a different flux of energy. But I have not succeeded in making any other arrangement than MAXWELL'S work practically, or without an immediate introduction of great obscurities. Perhaps the least certain part of MAXWELL'S scheme, as modified by myself, is the estimation of magnetic energy as $\frac{1}{2}\mathbf{HB}$ in intrinsic magnets, as well as outside them, that is, by $\frac{1}{2}\mathbf{B}\mu^{-1}\mathbf{B}$, however **B** may be caused. Yet, only in this way are thoroughly consistent results apparently obtainable when the electromagnetic field is considered comprehensively and dynamically.

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[*] 'Phil. Mag.,' January, 1887.

APPENDIX.

Received June 27, 1891.

Extension of the Kinetic Method of arriving at the Stresses to cases of Non-linear Connection between the Electric and Magnetic Forces and the Fluxes. Preservation of Type of the Flux of Energy Formula.

§ **39**. It may be worth while to give the results to which we are led regarding the stress and flux of energy when the restriction of simple proportionality between "forces" and "fluxes," electric and magnetic respectively, is removed. The course to be followed, to obtain an interpretable form of the equation of activity, is sufficiently clear in the light of the experience gained in the case of proportionality.

First assume that the two circuital laws (89) and (90), or the two in (93), hold good generally, without any initially stated relation between the electric force **E** and its associated fluxes **C** and **D**, or between the magnetic force **H** and its associated fluxes **K** and **B**. When written in the form most convenient for the present application, these laws are

$$\text{curl}\ (\mathbf{H} - \mathbf{h}_0) = \mathbf{J}_0 = \mathbf{C} + \frac{\partial \mathbf{D}}{\partial t} + (\mathbf{D}\ \text{div}\ \mathbf{q} - \mathbf{D}\nabla.\mathbf{q}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (212)$$

$$-\ \text{curl}\ (\mathbf{E} - \mathbf{e}_0) = \mathbf{G}_0 = \mathbf{K} + \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{B}\ \text{div}\ \mathbf{q} - \mathbf{B}\nabla.\mathbf{q}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (213)$$

Now derive the equation of activity in the manner previously followed, and arrange it in the particular form

$$\begin{aligned} &\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 + \text{conv}\ \text{V}\ (\mathbf{E} - \mathbf{e}_0)\ (\mathbf{H} - \mathbf{h}_0) \\ &\quad = (\mathbf{EC} + \mathbf{HK}) + \left(\mathbf{E}\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}\right) + (\mathbf{E}.\mathbf{D}\nabla.\mathbf{q} - \mathbf{ED}\ \text{div}\ \mathbf{q}) + (\mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - \mathbf{HB}\ \text{div}\ \mathbf{q}), \quad . \quad (214) \end{aligned}$$

which will best facilitate interpretation.

Although independent of the relation between **E** and **D**, &c., of course the dimensions must be suitably chosen so that this equation may really represent activity per unit volume in every term.

Now, guided by the previous investigation, we can assume that $(\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0)$ represents the rate of supply of energy from intrinsic sources, and also that $\text{V}\ (\mathbf{E} - \mathbf{e}_0)\ (\mathbf{H} - \mathbf{h}_0)$, which is a flux of energy independent of **q**, is the correct form in general. Also, if there be no other intrinsic sources of energy than $\mathbf{e}_0$, $\mathbf{h}_0$, and no other fluxes of energy besides that just mentioned except the convective flux and that due to the stress, the equation of activity should be representable by

$$
\begin{aligned}
&(\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0) + \operatorname{conv}\left[\mathrm{V}\,(\mathbf{E} - \mathbf{e}_0)\,(\mathbf{H} - \mathbf{h}_0) + \mathbf{q}\,(\mathrm{U} + \mathrm{T})\right] \\
&= (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \mathbf{F}\mathbf{q} + \operatorname{conv}\mathbf{Q}_q q \\
&= (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}) + \Sigma\mathbf{Q}\nabla q, \qquad (215)
\end{aligned}
$$

where $\mathbf{Q}$ is the conjugate of the stress vector, $\mathbf{F}$ the translational force, and Q, U, and T the rate of waste and the stored energies, whatever they may be.

Comparing with the preceding equation (214), we see that we require

$$
\begin{aligned}
\Sigma\mathbf{Q}\nabla q = (\mathrm{Q} - \mathbf{EC} - \mathbf{HK}) &+ \left(\frac{\partial \mathrm{U}}{\partial t} - \mathbf{E}\frac{\partial \mathbf{D}}{\partial t}\right) + \left(\frac{\partial \mathrm{T}}{\partial t} - \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}\right) \\
&+ \left[\mathbf{E}.\mathbf{D}\nabla.\mathbf{q} - (\mathbf{ED} - \mathrm{U})\operatorname{div}\mathbf{q}\right] + \left[\mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - (\mathbf{HB} - \mathrm{T})\operatorname{div}\mathbf{q}\right]. \qquad (216)
\end{aligned}
$$

Now assume that there is no waste of energy except by conduction ; then

$$\mathrm{Q} = \mathbf{EC} + \mathbf{HK}. \qquad (217a)$$

Also assume that

$$\frac{\partial \mathrm{U}}{\partial t} = \mathbf{E}\frac{\partial \mathbf{D}}{\partial t}, \qquad \frac{\partial \mathrm{T}}{\partial t} = \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}. \qquad (217b)$$

These imply that the relation between $\mathbf{E}$ and $\mathbf{D}$ is, for the same particle of matter, an invariable one, and that the stored electric energy is

$$\mathrm{U} = \int_0^{\mathbf{D}} \mathbf{E}\, d\mathbf{D}, \qquad (218)$$

where $\mathbf{E}$ is a function of $\mathbf{D}$. Similarly,

$$\mathrm{T} = \int_0^{\mathbf{B}} \mathbf{H}\, d\mathbf{B}, \qquad (219)$$

expresses the stored magnetic energy, and $\mathbf{H}$ must be a definite function of $\mathbf{B}$.

On these assumptions, (216) reduces to

$$\Sigma\mathbf{Q}\nabla q = \left[\mathbf{E}.\mathbf{D}\nabla.\mathbf{q} - (\mathbf{ED} - \mathrm{U})\operatorname{div}\mathbf{q}\right] + \left[\mathbf{H}.\mathbf{B}\nabla.\mathbf{q} - (\mathbf{HB} - \mathrm{T})\operatorname{div}\mathbf{q}\right], \qquad (220)$$

from which the stress-vector follows, namely,

$$\mathbf{P}_{\mathrm{N}} = \left[\mathbf{E}.\mathbf{DN} - \mathbf{N}\,(\mathbf{ED} - \mathrm{U})\right] + \left[\mathbf{H}.\mathbf{BN} - \mathbf{N}\,(\mathbf{HB} - \mathrm{T})\right]. \qquad (221)$$

Or,

$$\mathbf{P}_{\mathrm{N}} = (\mathrm{V}\mathbf{D}\mathrm{V}\mathbf{EN} + \mathbf{N}\mathrm{U}) + (\mathrm{V}\mathbf{B}\mathrm{V}\mathbf{HN} + \mathbf{N}\mathrm{T}). \qquad (222)$$

Thus, in case of isotropy, the stress is a tension U parallel to $\mathbf{E}$ combined with a lateral pressure ($\mathbf{ED}$ − U) ; and a tension T parallel to $\mathbf{H}$ combined with a lateral pressure ($\mathbf{HB}$ − T).

The corresponding translational force is

$$
\begin{aligned}
\mathbf{F} = \mathbf{E}\operatorname{div}\mathbf{D} &+ \mathbf{D}\nabla.\mathbf{E} - \nabla\,(\mathbf{ED} - \mathrm{U}) \\
&+ \mathbf{H}\operatorname{div}\mathbf{B} + \mathbf{B}\nabla.\mathbf{H} - \nabla\,(\mathbf{HB} - \mathrm{T}), \qquad (223)
\end{aligned}
$$

which it is unnecessary to put in terms of the currents.

Exchange $\mathbf{E}$ and $\mathbf{D}$, and $\mathbf{H}$ and $\mathbf{B}$, in (221) or (222) to obtain the conjugate vector $\mathbf{Q}_N$; from which we obtain the flux of energy due to the stress,

$$\begin{aligned} -q\mathbf{Q}_q &= \mathbf{D}.\mathbf{E}\mathbf{q} - \mathbf{q}\,(\mathbf{E}\mathbf{D} - \mathrm{U}) + \mathbf{B}.\mathbf{H}\mathbf{q} - \mathbf{q}\,(\mathbf{H}\mathbf{B} - \mathrm{T}) \\ &= \mathrm{V}\mathbf{E}\mathrm{V}\mathbf{D}\mathbf{q} + \mathrm{V}\mathbf{H}\mathrm{V}\mathbf{B}\mathbf{q} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}), \quad \ldots\ldots\ldots\ldots \quad (224) \end{aligned}$$

or

$$-q\mathbf{Q}_q = \mathrm{V}\mathbf{e}\mathbf{H} + \mathrm{V}\mathbf{E}\mathbf{h} + \mathbf{q}\,(\mathrm{U} + \mathrm{T}), \quad \ldots\ldots\ldots\ldots \quad (225)$$

where $\mathbf{e}$ and $\mathbf{h}$ are the motional electric and magnetic forces, of the same form as before (88) and (91); so that the complete form of the equation of activity, showing the fluxes of energy and their convergence, is

$$\mathbf{e}_0\mathbf{J}_0 + \mathbf{h}_0\mathbf{G}_0 + \mathrm{conv}\big[\mathrm{V}(\mathbf{E}-\mathbf{e}_0)(\mathbf{H}-\mathbf{h}_0) + \mathbf{q}(\mathrm{U}+\mathrm{T})\big] - \mathrm{conv}\big[\mathrm{V}\mathbf{e}\mathbf{H} + \mathrm{V}\mathbf{E}\mathbf{h} + \mathbf{q}(\mathrm{U}+\mathrm{T})\big] = \mathbf{F}\mathbf{q} + (\mathrm{Q} + \dot{\mathrm{U}} + \dot{\mathrm{T}}), \quad (226)$$

where $\mathbf{F}$ has the above meaning.

There is thus a remarkable preservation of form as compared with the corresponding formulæ when there is proportionality between force and flux. For we produce harmony by means of a POYNTING flux of identical expression and a flux due to the stress, which is also of identical expression, although U and T now have a more general meaning, of course.*

Example of the above, and Remarks on Intrinsic Magnetisation when there is Hysteresis.

§ **40.** In the stress-vector itself (for either the electric or the magnetic stress) the relative magnitude of the tension and the lateral pressure varies unless the curve

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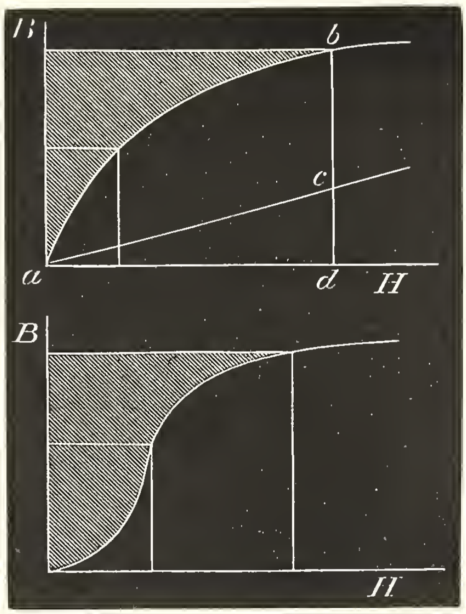
* As the investigation in this Appendix has some pretensions to generality, we should try to settle the amount it is fairly entitled to. No objection is likely to be raised to the use of the circuital equations (212), (213), with the restriction of strict proportionality between $\mathbf{E}$ and $\mathbf{H}$ and the fluxes $\mathbf{D}$ and $\mathbf{B}$, or $\mathbf{C}$ and $\mathbf{K}$ entirely removed; nor to the estimation of $\mathbf{J}_0$ and $\mathbf{G}_0$ as the "true" currents; nor to the use of the same form of flux of electromagnetic energy when the medium is stationary. For these things are obviously suggested by the preceding investigations, and their justification is in their being found to continue to work, which is the case. But the use in the text of language appropriate to linear functions, which arose from the notation, &c., being the same as before, is unjustifiable. We may, however, remove this misuse of language, and make the equation (226), showing the flux of energy, rest entirely upon the two circuital equations. In fact, if we substitute in (226) the relations (217*a*), (217*b*), it becomes merely a particular way of writing (214).

It is, therefore, to (217*a*), (217*b*) that we should look for limitations. As regards (217*a*), there does not seem to be any limitation necessary. That is, there is no kind of relation imposed between $\mathbf{E}$ and $\mathbf{C}$, and $\mathbf{H}$ and $\mathbf{K}$. This seems to arise merely from Q meaning energy wasted for good, and having no further entry into the system. But as regards (217*b*), the case is different. For it seems necessary, in order to exclude terms corresponding to $\mathbf{E}(\partial c/\partial t)\mathbf{E}$ and $\mathbf{H}(\partial\mu/\partial t)\mathbf{H}$ in the linear theory, when there is connecting the force and the induction be a straight line. Thus, if the curve be of the type shown in the first figure, the shaded area will represent the stored energy and the tension, and the remainder of the rectangle will represent the lateral pressure. They are equal when **H** is small; later on the pressure preponderates, and more and more so the bigger **H** becomes.



But if the curve be of the type shown in the second figure, then, after initial equality the tension preponderates; though, later on, when **H** is very big the pressure preponderates.

To obtain an idea of the effect, take the concrete example of an infinitely long rod, uniformly axially inductized by a steady current in an overlapping solenoid, and consider the forcive on the rod. Here both **H** and **B** are axial or longitudinal; and so, by equation (223), the translational force would be a normal force on the surface of the rod, acting outwards, of amount

$$(\mathbf{HB} - \mathrm{T}) - \tfrac{1}{2}\mathbf{H}_0\mathbf{B}_0$$

per unit area; this being the excess of the lateral pressure in the rod over $\frac{1}{2}\mathrm{H}_0\mathrm{B}_0$, the lateral pressure just outside it.

In case of proportionality of force to flux, the first pressure is $\frac{1}{2}\mathrm{HB}$, and if there is no intrinsic magnetisation H and H_0 are equal. The outward force is therefore positive for paramagnetic, and negative for diamagnetic substances, and the result would be lateral expansion or contraction, since the infinite length would prevent elongation.

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rotation, that **E** and **D** should be parallel, and likewise **H** and **B**. At any rate, if such terms be allowed, some modification may be required in the subsequent reckoning of the mechanical force. In other respects, it is merely implied by (217*b*) that **E** and **D** are definitely connected, likewise **H** and **B**, so that there is no waste of energy other than that expressed by Q.

But if the curve in the rod be of the type of the first figure, and the straight line *ac* be the air curve to correspond, it is the area *abc* that now represents the outward force per unit area when the magnetic force has the value *ad*. If the straight line can cross the curve *ab*, we see that by sufficiently increasing H we can make the external air pressure preponderate, so that the rod, after initially expanding, would end by contracting.

If the rod be a ring of large diameter compared with its thickness, the forcive would be approximately the same, viz., an outward surface force equal to the difference of the lateral pressures in the rod and air. The result would then be elongation, with final retraction when the external pressure came to exceed the internal.

Bidwell found a phenomenon of this kind in iron, but it does not seem possible that the above supposititious case is capable of explaining it, though of course the true explanation may be in some respects of a similar nature. But the circumstances are not the same as those supposed. The assumption of a definite connexion between H and B, and elastic storage of the energy T, is very inadequate to represent the facts of magnetisation of iron, save within a small range.

Magneticians usually plot the curve connecting $H - h_0$ and B, not between H and B, or which would be the same, between $H - h_0$ and $B - B_0$, where B_0 is the intrinsic magnetisation. Now when an iron ring is subjected to a given gaussage (or magneto motive force), going through a sequence of values, there is no definite curve connecting $H - h_0$ and B, on account of the intrinsic magnetisation. But, with proper allowance for h_0, it might be that the resulting curve connecting H and B in a given specimen, would be approximately definite, at any rate, far more so than that connecting $H - h_0$ and B. Granting perfect definiteness, however, there is still insufficient information to make a theory. The energy put into iron is not wholly stored; that is, in increasing the coil current we increase B_0 as well as B, and in doing so dissipate energy; and although we know, by Ewing's experiments, the amount of waste in cyclical changes, it is not so clear what the rate of waste is at a given moment. There is also the further peculiarity that the energy of the intrinsic magnetisation at a given moment, though apparently locked up, and really locked up temporarily, however loosely it may be secured, is not wholly irrecoverable, but comes into play again when H is reversed. Now it may be that the energy of the intrinsic magnetisation plays, in relation to the stress, an entirely different part from that of the elastic magnetisation. It is easy to make up formulæ to express special phenomena, but very difficult to make a comprehensive theory.

But in any case, apart from the obscurities connected with iron, it is desirable to be apologetic in making any application of Maxwell's stresses or similar ones to practice when the actual strains produced are in question, bearing in mind the difficulty of interpreting and harmonising with Maxwell's theory the results of Kerr, Quincke, and others.

XII. *On the Thermal Conductivities of Crystals and other Bad Conductors.*

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Communicated by Professor ARTHUR SCHUSTER, *F.R.S.*

Received January 22,—Read February 4, 1892.

Introduction.

OUR knowledge of the Thermal Conductivities of Crystals is derived mainly from the experiments of DE SENARMONT, VON LANG, and JANNETAZ,[*] who, using the wax melting or analogous methods, have determined the ratios of what may be called the "principal conductivities" and the positions of the axes of conductivity within a number of crystals belonging to the simple systems. According to their experiments the isothermal surfaces about a heated point in a crystal are, in general, ellipsoids, having their axes parallel to the optical axes. In the case of a uniaxal crystal, this ellipsoid becomes a spheroid of revolution about the axis, and is, as a rule, oblate or prolate according as the wave-surface for the extraordinary ray is oblate or prolate. Although this rule has a number of exceptions, it is sufficiently general to render it probable that there may be some relation between the passage of light and of heat through a crystal. The recent determinations of the refractive indices of metals by KUNDT have shown that they stand in the same order as conductors of heat, and as to the velocity of propagation of light through them, and this fact brings again into prominence the old determinations with respect to crystals. That the comparison which KUNDT has made for the metals cannot be carried to other bodies is at once seen from the fact that the index of refraction of iron differs little from those of glass and several commoner crystals, the conductivities of which are shown to be very small compared to that of iron. A comparison may, however, be possible among transparent bodies themselves, and the following experiments were made with the object of furnishing data for this comparison, the results given by previous observers differing greatly from each other. They have, however, been extended to embrace non-transparent bodies commonly in use in a physical laboratory, and about the conductivity of which we have had a very meagre or absolutely no knowledge.

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[*] SENARMONT, 'Ann. de Chim. et de Phys.,' (3), vols. 21, 22, and 23 (1848); VON LANG, 'Pogg. Ann.,' vol. 135 (1868); JANNETAZ, 'Ann. de Chim. et de Phys.,' (4), vol. 29 (1873).

Outline of Method.

The most important consideration in determining the method to be used is the fact that it is difficult to get large pieces of the crystals to be experimented on. This excludes methods requiring large plates, such as that of WEBER and TUSCHMID, or large spheres or cubes, such as that of KIRCHHOFF or THOMSON. A method which seemed to present several advantages was the one first suggested by LODGE,[*] and which may be called the "divided bar" method, and after some preliminary experiments had been made to determine its suitability, it was finally adopted. It consists in observing the temperature along a bar heated at one end and cooled at the other, and divided halfway between the two ends by a plane perpendicular to its axis, when (1) the divided ends are together, (2) a disc of the crystal or other body is between.

If temperature observations are taken at several points in each half of the bar, the corrections to be applied to the second set of observations for the distances of the points of observation from the contacts can be determined from the first set, and thus the temperature at each side of the crystal disc can be found. If, in addition, we know the thermal conductivity of the bar used, the amount of heat flowing through the disc can be found from the temperature slope; and thence we have the thermal conductivity of the crystal.

It may be noticed that there would be considerable uncertainty as to the nature of the contacts between bar and crystal unless some special precautions were taken. LODGE proposed to use pads of tinfoil to obviate this difficulty, but this method is not satisfactory. The difficulty has, however, been completely overcome by using bars of a material which would amalgamate, and making the contacts by means of mercury.

Description of Apparatus. (Fig. 1.)

The bar used in the experiments was one of brass, which presents several advantages.

(1) It readily amalgamates, and therefore enables good contacts to be made.

(2) Its conductivity is not so high as to make comparison between it and that of crystals, &c., impossible.

(3) Its conductivity, according to LORENZ, increases with the temperature, a fact which, as will be seen later, partly neutralises the deviation of the cooling from NEWTON'S law.

The diameter of the bar was 1·93 cm., and its length 67 cm. To each end of the bar was soldered a can, through one of these cans steam was sent and through the other water. The whole was arranged in a wooden frame, so that the bar was horizontal and exposed to the air over a covered tank through which a current of water could be sent to keep the temperature constant. The apparatus was surrounded by paper screens to protect the bar as far as possible from air currents in the room.

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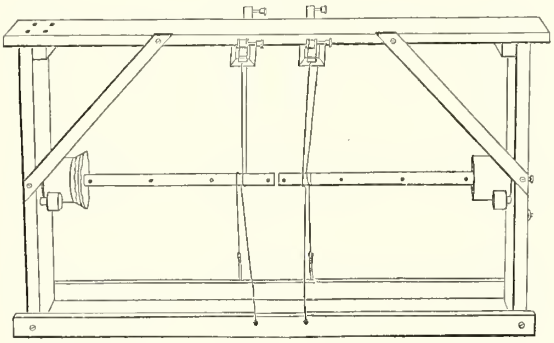
[*] 'Phil. Mag.,' (5), vol. 5, p. 110 (1878).

The cans at the extremities of the bar were supported on slides attached to the framework in such a way that the bar could be moved parallel to itself by screws passing through the framework and bearing in conical indentations in the backs of the cans in the prolongation of the axis of the bar. These supports were sufficient while the bar was entire, and the experiments to determine its conductivity were being made. After the bar was cut for the insertion of the discs, it became necessary to support the cut ends in such a way as (1) to enable the motion of the bar parallel to its axis still to take place; (2) to enable the cut faces to be adjusted parallel to each other; (3) to cause as small as possible a disturbance of the isothermal surfaces. These objects were secured by supporting each free end in a loop of thin string, hanging from the upper part of the framework. The ends of this loop were attached to screws for the purpose of raising or lowering it as required. The sides of the loop made an angle of 60° or 70° with each other at the bar. These loops were sufficient to support the bars properly if the arrangement was kept horizontal. It was, however, found that the mercury contacts could be made with much greater certainty if the bars were vertical and the mercury surfaces horizontal. On this account two other loops were attached to the lower part of the framework and passed over the bars. These were elastic, and served merely to keep the upper loops taut, and the bars in the same positions with respect to each other, whether the framework was horizontal, as in the experiments, or vertical, as in making the mercury contacts.

Fig. 1.



Measurement of Temperature.

The most direct method of measuring the temperature along the bar is to sink thermometers into holes in the bar, but this method, although it has been used by FORBES, TAIT, and MITCHELL for their large bars, is very objectionable, as it diminishes the available area of flow, and therefore makes the change of temperature along the bar more rapid than it would be for a continuous bar. The method first used by Wiedemann and Franz of measuring the temperature by means of a thermo-junction brought into contact with the bar is free from this objection, and will produce no appreciable effect on the distribution of temperature in the bar if the mass of the junction is not large.

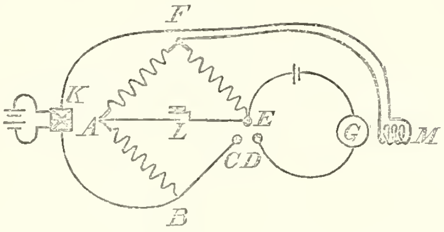
The bars used by Wiedemann and Franz had a silvered surface, and it was sufficient to bring the thermo-couple into contact with the surface in order to get the temperature. In the later of the following experiments the bars were painted, so that contact could not be made in this way. Small conical holes about ·5 mm. deep were, however, made at opposite extremities of horizontal diameters of the bars, and at regular distances along. Eight such diameters were taken, the two nearest the centre of the uncut bar being 1 cm. apart, and the rest 10·5 cms. apart. Each of these small holes was amalgamated, and sufficient mercury left in them to make good contact with the ends of the two wires which were used as a thermo-couple. The thermo-circuit would thus consist at the bars of—first wire of couple; mercury of first hole; brass of bar; mercury of hole at opposite extremity of diameter; second wire of couple. The mercury in the holes was constantly cleaned by touching the surface with dilute nitric acid, washing and drying, and the ends of the thermo-couple wires were kept bright. Under these conditions the arrangement worked satisfactorily.

The wires used originally for the thermo-couple were of brass and iron, but these gave unsatisfactory readings, mainly on account of the poor contact between iron and mercury. As there is an advantage in using copper as one of the elements, a few combinations of copper with other metals and alloys were tried, and eventually the platinum-silver alloy used for resistance coils was selected as the other element. This couple gives an E.M.F. of about ·00016 volt for 70° difference of temperature, and its constant only increases about 7 per cent. for 100° C. It is, therefore, well suited for thermo-electric measurement of temperature. The wires used for the thermo-couple were about 20 cms. long and ·2 mm. diameter. The ends which served to make contact with the galvanometer circuit dipped into mercury cups in blocks of wood which could be moved along guides parallel to the bars in order that the whole of the temperature observations could be made with one couple. The cup into which the platinum-silver wire dipped was provided with a thermometer graduated in $\frac{1}{10}$° C.

The galvanometer was a low resistance one of the Wiedemann type, with Siemens' bell-magnet and copper damping sphere. The resistance of the whole circuit was only about 1·5 ohm, and as the resistances of the contacts at the holes in the bar might vary, it was necessary to take measurements of the resistance of the circuit when a galvanometer deflection was taken. For this purpose Thomson's modification of the bridge method was used. The diagram (fig. 2) shows the arrangement of the circuit.

G is the galvanometer in series with the thermo-element. C, D, E are three mercury cups, arranged so that when E and D are connected, the galvanometer and thermo-element are in circuit alone, and when C and D are connected the galvanometer and element form one side of the bridge. *K* is a reversing switch, and *L* a key. *AB* and *AF* are each 100 ohms from post-office box; *FE* is a multiple arc box capable of adjustment to $\frac{1}{500}$ ohm. The resistance *R* of the galvanometer circuit is found by adjusting *FE* so that there is no effect on the galvanometer on pressing down *L*, first with *K* in one direction and then in the other. The coil *M* is introduced in the battery circuit to counteract the effect of the current through the galvanometer while making these observations. It will be noticed that the current in it is reversed by the operation which reverses the current in the galvanometer coils, and that the galvanometer needle is therefore kept close to the zero of the scale during the whole of the observations.

Fig. 2.



Let the values of the resistance *EF* determined as above be R_1 and R_2, and let P be the resistance of each of the arms *AF, AB*, W that of the coil *M*, E the E.M.F. of the battery, *e* that of the thermo-element. Then writing down the condition that A and E have the same potential in each case, we have

$$(E + e)\frac{1 + e}{P + R + 2W} = \frac{E}{P + R_1 + 2W}$$

$$(E - e)\frac{E - e}{P + R + 2W} = \frac{E}{P + R_2 + 2W}.$$

Hence

$$\frac{2}{P + R + 2W} = \frac{1}{P + R_1 + 2W} + \frac{1}{P + R_2 + 2W},$$

which gives

$$R = \frac{R_1 + R_2}{2} - \frac{\{\frac{1}{2}(R_1 - R_2)\}^2}{P + \frac{1}{2}(R_1 + R_2) + 2W}.$$

The last term of the right member of this equation never exceeds $\frac{1}{500}$, and may therefore be neglected, and we have $R = \frac{1}{2}(R_1 + R_2)$.

The constants of the thermo-couple were determined by comparison with a thermometer reading to $\frac{1}{30}$ degree Centigrade. An auxiliary iron bar of 80 cms. length and 2·5 cms. square cross-section, was placed horizontal and heated at one end and cooled at the other by the waste steam and water respectively from the principal bar. The upper surface of this auxiliary bar was tinned and amalgamated. On this surface two copper mercury cups of 1 cm. diameter and 1·8 cm. height were placed. The under surfaces of these cups were amalgamated so as to make good contact with the iron bar. The insides of the cups were amalgamated, and each cup contained sufficient mercury to fill it entirely when the thermometer was placed in it. At opposite extremities of a diameter, about half way down each cup, two small holes like those in the divided bar were punched, and into these holes, which were amalgamated, the wires of the thermo-couple could be inserted. The constants of the couple are in this way determined under exactly the same conditions as when they are in use. The iron bar was buried in sawdust, and the copper mercury cups placed in such positions on the upper surface that the thermometer read in them 90° C. and 55° C. respectively. After the insertion of the wires of the thermo-couple into the holes of the mercury cups, the cups were surrounded with sawdust, and 10 minutes allowed for the temperature of the cups to get as nearly as possible uniform before observations were taken.

In an experiment made in the early part of the work the following observations were taken :—

(*a*) Temperature of hot junction, 85°·90 C.
Temperature of cold junction, 15°·67.
Deflection of galvanometer in scale divisions, 296.
Resistance of circuit, mean of observations taken, as previously described 1·431 ohm.
E.M.F. of element in units used, 423·6.

(*b*) Temperature of hot junction, 55°·60.
Temperature of cold junction, 15°·74.
Deflection of galvanometer, 162·3.
Resistance of circuit, 1·448 ohms.
E.M.F. of element in units used, 235·0.

These observations are sufficient to determine the constants in the formula $e = a(t_2 - t_1)(1 + b\,\overline{t_2 + t_1})$ where t_2 and t_1 are the temperatures of the hot and cold junctions respectively. It was found more convenient for the reduction of observed E.M.F.'s to temperature, to draw up a small table of values of $a(1 + b\,\overline{t_2 + t_1})$ for different values of $t_2 + t_1$. Thus, for the above observations, we have :—

$t_2 + t_1$.	$a(1 + b\,\overline{t_2 + t_1})$.	Differences.	
101	6·030	+ 2°	+ ·009
91	5·985	4	·018
81	5·940	6	·027
71	5·895	8	·036
61	5·850		
51	5·805		
41	5·760		
31	5·715		

By inspection, we have the divisor for converting E.M.F. observations to temperature difference, since in every case $t_1 + t_2$ is known approximately to within one or two degrees.

The value of $a\,(1 + b\,\overline{t_2 + t_1})$ is seen from the above table to change only about 7 per cent. for 100° C. of change of $\overline{t_2 + t_1}$; the platinum-silver-copper element is therefore well suited for thermo-electric observations of temperature.

The determination of the constants of the element was repeated every few days during the progress of the work. The value of a was found to diminish slowly, the total change during the three months during which the couple was used being about 1·5 per cent.

The temperature of the enclosure in which the divided bar was suspended was determined by means of a thermometer graduated in $\frac{1}{10}$ degrees, placed about 7 cms. above the top of the water-tank and about 15 cms. below the bar. The bulb of the thermometer was protected from radiation from the hot bar by a small sheet of paper, just large enough to screen the bulb.

Determination of the Thermal Conductivity of the Brass Bar.

The thermal conductivity of the bars was determined, following the method of FORBES, by two series of experiments, the first to determine the law connecting the rate of loss of heat from a heated surface, with the temperature of that surface, and the second to determine the law of steady distribution of temperature along a bar heated at one end, and exposed to the air along its entire length. These two observations furnish data for determining the thermal conductivity required.

Cooling Experiments.

It is customary in making these experiments to use a short length of the bar used in the steady experiments. I have, however, shown that the law of cooling is the same for bars of different material, but of the same cross-section and surface; and as a knowledge of the specific heat of the bar is required, it seemed better to make use of a bar of a material the specific heat of which is well known, rather than a brass bar in these experiments. As the values found by different observers for the specific heat of copper are practically identical, this metal was chosen. As it is a better conductor than brass, it also enables the assumption that the temperature throughout the bar is the same to be made without much chance of error. The bar was of the same diameter as the brass bar, 1·93 cm. and 26 cms. long. It had a polished nickel-plated surface. To make the cooling experiment correspond more closely to that of an infinitely long bar, the ends of the bar were covered with a layer of sulphur about 5 mms. thick. This was done by placing the bar vertical, wrapping a sheet of paper round the upper end so as to project about 1 cm. above the end, and then pouring melted sulphur into the paper cylinder. Since sulphur is an extremely poor conductor of heat, it may be assumed that the loss of heat from the ends of the bar is very small.

The bar was suspended in the position occupied afterwards by the divided bar by means of threads. The temperature was determined by means of the platinum-silver-copper wires inserted into small holes amalgamated and filled with mercury, on opposite sides of a diameter at the middle of the bar. Observation of current, resistance, temperature of the cool junction, and of the enclosure, were made every two minutes during the experiments. From these observations the temperature of the bar at any time is determined, and curves drawn for each experiment showing the connection between time and temperature of bar and of enclosure. Of these curves that which corresponded most nearly with the mean of all was selected, and is referred to in what follows.

If m is the mass of the bar, s its surface, c the specific heat of the material of the bar, v the excess of the temperature of the bar over that of the enclosure, which will be supposed constant, we have, as the connection between v and time,

$$mc\frac{dv}{dt} + sh_v . v = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where h_v is a quantity which is generally assumed to be constant (NEWTON'S law), but which I have shown increases for a nickel-plated bar of the size used 70 or 80 per cent. for 100° C. rise of v.* In the paper referred to this rise is expressed by writing $h_v = hv^n$ where n is a small fraction $= \frac{1}{7}$ about. In that paper v was always positive. As in the following work v is sometimes negative, in which case v^n would require special interpretation, it is preferable to write $h_v = h(1 + bv)$ where b is a small positive quantity. The foregoing equation becomes, then,

$$mc\frac{dv}{dt} + sh(1 + bv)\,v = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

the solution of which is

$$\frac{1}{v} + b = \mathrm{A}e^{(sh/cm)t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where A is an arbitrary constant.

The following table shows that this equation represents the cooling with a fair amount of accuracy for $b = {\cdot}008$. The temperature of the enclosure has been assumed constant $= 15{\cdot}2°$ C., but in reality it varied from 15·3° at the commencement to 15·4° at $t = 10$ minutes, and then down to 15·2 at 40 minutes, 15·18 at 60 minutes, 15·1 at 70 minutes. On this account no attempt has been made to determine b with greater accuracy; the value given above is determined by trial.

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* 'Phil. Mag.,' vol. 28, p. 429 (1889)

COOLING of Nickel-plated Bar.

t in minutes.	Temperature. − 15·2° C. = v.	$1/v$.	$\frac{1/v_t + b}{1/v_{t+2} + b} = e^{-(sh/cm)120}$.	
				Means.
0	75·4	·01326	. .	·9468
2	69·2	1445	·9469	
4	64·03	1562	504	
6	58·67	1704	434	
8	54·36	1840	485	
10	50·14	1994	448	
12	46·58	2147	482	·9467
14	43·30	2309	479	
16	40·16	2490	450	
18	37·37	2676	466	
20	34·80	2874	460	
22	32·43	3084	459	·9462
24	30·31	3299	475	
26	28·28	3536	454	
28	26·43	3784	459	
30	24·72	4045	463	
32	23·16	4318	467	·9466
34	21·77	4593	490	
36	20·44	4892	474	
38	19·10	5236	430	
40	17·93	5576	467	
43	16·22	6165	428	·9474
46	14·95	6689	528	
49	13·67	7315	479	
52	12·49	8006	470	
55	11·41	·08764	465	
60	9·88	·1013	·9454	·9454
65	8·51	1175		
70	7·21	1387		
75	6·26	1597		

The mean of the values of $e^{-(sh/cm)120}$ is ·9466.

Hence we have

$$e^{120(sh/cm)} = \frac{1}{\cdot 9466} = 1\cdot 0564.$$

Therefore

$$\frac{sh}{cm} = \frac{\cdot 02383}{\cdot 4343} \times \frac{1}{120} = \cdot 0004572.$$

Now, $m = 676$ grms., $s = 157\cdot 6$ sq. cms., $c = \cdot 093$. Hence we have

$$h = \cdot 0001804.$$

The heat lost from a nickel-plated surface exposed to air is, therefore,

$$\cdot 0001804v\,(1 + \cdot 008v) \text{ gram degrees per sq. cm. per second,}$$

where v is the excess of temperature of the surface over that of the air.

Statical Experiments.

These experiments were made with the brass bar before it was cut. One end was heated by steam, and the other cooled by water. Eight observations of temperature were taken at different points of the bar in the way previously described.

Let v be the excess of temperature at point x of bar over that of surrounding air,

p = perimeter of cross-section of bar,
q = area of cross-section of bar,
k_v = thermal conductivity at temperature excess = v.

Then, at any point of the surface of the bar, we have

$$k_v \frac{dv}{dn} + hv\,(1 + bv) = 0,$$

where dv/dn is the rate of change of v along the normal to the surface, s, h, and b having the meanings assigned to them in last section.

At $v = 60°$ C. this gives

$$k_v \frac{dv}{dn} = - \cdot 00018 \times 60 \times 1 \cdot 5$$
$$= - \cdot 0162.$$

Now k_v will be seen later to have the value ·27 approximately. Hence, at a point of surface, at the temperature of 60° C. excess, $dv/dn = - \cdot 06$.

At the same point on the surface $dv/dx = 2 \cdot 5$ approximately. Hence, the inclination of the normal to the isothermal surface at this point to the axis of the bar is arctan $\cdot 06/2 \cdot 5$ = arctan $\cdot 024 = 1° \, 25'$. If the isothermal surface be assumed to be part of a sphere, the radius of curvature is about 42 cms. At 15° C. excess, a similar calculation gives the radius of curvature of the isothermal surface to be 60 cms.

Hence we may assume the isothermal surfaces to be planes perpendicular to the axis of the bar.

The equation for the motion of heat in the bar is under these conditions

$$q \frac{d}{dx}\left(k_v \frac{dv}{dx}\right) = phv\,(1 + bv). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The conductivity k_v has generally been considered constant in treating this equation, but this is scarcely justifiable, as most experimenters find changes of

± 20 per cent. in conductivity for 100° C. change of temperature. Taking therefore k_v as equal to $k(1 + av)$ the above equation becomes

$$(1 + av)\frac{d^2v}{dx^2} + a\left(\frac{dv}{dx}\right)^2 = \frac{ph}{qk}v(1 + bv). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Multiplying through by $2(1 + av)(dv/dx)$, this becomes

$$2(1 + av)^2\frac{d^2v}{dx^2}\frac{dv}{dx} + 2a(1 + av)\left(\frac{dv}{dx}\right)^2\frac{dv}{dx} = 2\frac{ph}{qk}v(1 + bv)(1 + av)\frac{dv}{dx},$$

or,

$$\frac{d}{dv}\left(\overline{1 + av}\,\frac{dv}{dx}\right)^2 = 2\frac{ph}{qk}(v + \overline{a + b}\,v^2 + abv^3)\frac{dv}{dx}\cdot$$

From which by integration we obtain

$$\left(\overline{1 + av}\,\frac{dv}{dx}\right)^2 = \frac{ph}{qk}\left(v^2 + 2\frac{a + b}{3}v^3 + \frac{ab}{2}v^4\right) + \text{constant.}$$

If the value of dv/dx when $v = 0$ be dv_0/dx, the above equation becomes

$$\left(\overline{1 + av}\,\frac{dv}{dx}\right)^2 - \left(\frac{dv_0}{dx}\right)^2 = \frac{ph}{qk}v^2\left(1 + 2\frac{a + b}{3}v + \frac{ab}{2}v^2\right). \quad . \quad . \quad . \quad . \quad (6)$$

If $dv_0/dx = 0$, the integral of this equation can be expressed in terms of logarithmic and circular functions. If $dv_0/dx \neq 0$, the integration introduces circular functions and elliptic integrals of the 1st and 3rd kind. If the problem under discussion were —given the conductivities—to determine the distribution of temperature throughout the bar, this integration would be necessary, but as we are given the distribution and have to find the conductivity, the problem can be solved without further integration. Although this method is probably not as accurate as that depending on the integrated equation, its accuracy is sufficient for the present purpose, where the value of the conductivity is only required over a small range (25°–37° C.) of temperature.

To determine dv/dx at the points of observation v is represented by an empirical function of x and the differential coefficient with respect to x taken. The known conditions which v satisfies, fix, to some extent, the function to be used; we see, *e.g.*, that it vanishes for some value of x, and that at that point the first differential coefficient is finite; that it increases in one direction with x in an approximately exponential manner.

These considerations lead at once to the function $A \sinh(\alpha x + \beta)$, and this function has been used. It evidently cannot express v accurately throughout the bar, since it is the solution of a linear differential equation, but it can be made by a proper choice of the constants to represent the main feature of the curve, the differences between its values and those observed being afterwards represented by an additional expression, which has generally only small values.

For the curve which corresponded best with the mean of the three sets of observations taken, the value of v is represented along the cooler part of the bar by $\cdot 795 \cosh \cdot 0378\, x + 11 \cdot 86 \sinh \cdot 0378\, x$.

x.	v calculated.	v observed.	Difference.	$\frac{dv}{dx}$.
0	·8	·8	0	·448
10·56	5·72	5·72	0	·497
21·04	11·52	11·53	+ ·01	·624
31·55	19·17	19·18	+ ·01	·852
32·59	20·07	20·09	+ ·02	·881
43·15	31·23	31·20	− ·03	1·262
53·59	47·24	47·88	+ ·64	1·971
64·03	70·64	73·47	+2·83	3·002

The differences, with the exception of the last two, are within the limits of error of observation, and may be neglected. The last two may be taken into account by adding to the above expression for v, between $x = 43$ and $x = 64$, an expression of the form

$$a\left(\frac{x-43}{10 \cdot 5}\right)^2 + b\left(\frac{x-43}{10 \cdot 5}\right)^3, \text{ where } a = \cdot 572,\ b = \cdot 067.$$

The values of the differential coefficient of this expression at $x = 43 \cdot 15$, $53 \cdot 59$, $64 \cdot 03$ are 0, ·128, ·294 respectively. Differentiating the hyperbolic expression for v we have $dv/dx = \cdot 03005 \sinh \cdot 0378\, x + \cdot 4483 \cosh \cdot 0378\, x$, which gives, on substituting for x the coordinates of the points of observation, the required values of dv/dx, to which, at $x = 53 \cdot 59$ and $64 \cdot 03$, we must add the quantities ·128 and ·294 to get the true temperature slope. The values thus obtained are given in the above table.

Having thus got the values of v and dv/dx at eight points on the bar, we can by trial determine a so that the equation (6) holds. A few trials show that a is positive and equal approximately to ·002. It is not necessary to determine a with any great accuracy as it affects very little the subsequent work. Assuming $a = \cdot 002$ we have the following table of results :—

x.	v.	$v^2\left(1 + 2\frac{a+b}{3}v + \frac{ab}{2}v^2\right)$	$\left(\overline{1+av}\,\frac{dv}{dx}\right)^2$	(5.)	(6.)
0	·8	·643	·201	·001	·0016
10·56	5·72	33·96	·253	·053	·0015
21·04	11·53	143·3	·406	·206	·00143
31·55	19·18	415	·781	·581	·00140
32·59	20·09	459	·839	·639	·00139
43·15	31·20	1174	1·796	1·596	·00136
53·59	47·80	3067	4·666	4·466	·00145
64·03	73·47	8269	11·860	11·660	·00141

Column (5) gives the values of $\left(\overline{1 + av\frac{dv}{dx}}\right)^2 - \left(\frac{dv_0}{dx}\right)^2$, and column (6) gives the values of

$$\frac{\left(\overline{1 + av\frac{dv}{dx}}\right)^2 - \left(\frac{dv_0}{dx}\right)^2}{v^2\left(1 + 2\frac{a+b}{3}v + v^2\right)},$$

which by equation (6) $= ph/qk$.

Neglecting the first two numbers in column (6), which are uncertain on account of the smallness of the quantity (5) at the cool end of the bar, we have, as the mean value of ph/qk, ·00141.

Hence

$$k = \frac{ph}{\cdot 00141\, q} = \frac{2\cdot 093 \times \cdot 0001804}{\cdot 00141} = \frac{\cdot 0003777}{\cdot 00141} = \cdot 268\,\frac{\text{gram.}}{\text{cm. sec.}}.$$

The temperature of the air in this experiment was 17° C. hence we have for the thermal conductivity of the brass bar used*

$$k = \cdot 268\,(1 + \cdot 002\,\overline{v - 17^\circ})\,\frac{\text{gram.}}{\text{cm. sec.}}.$$

Experiments on Crystals, &c.

Before cutting the bar for the insertion of the discs the conductivities of which were to be measured, several observations were made of the distribution of temperature along the bar after it had been painted with Aspinall's enamel. It is necessary to have the radiating surfaces of bar and discs the same, and this is most easily secured by painting both. The loss of heat from a point of the surface at any temperature is about 40 per cent. greater than the loss at the same temperature for the nickel-plated surface, but it does not increase as rapidly with rise of temperature. On this account it is possible to express the distribution of temperature along the bar in the empirical form $v = \text{A}\sinh(\alpha x + \beta)$, and A, α, and β can be determined so that this equation holds with a closer approximation than in the case of the unpainted bar, throughout the whole length of the bar. It is assumed to hold in what follows for each half of the divided bar, α having the same, but A, β having different values for each half.

The bar was divided in the middle, between the contact holes 1 cm. apart, and the ends ground down to be as nearly as possible planes perpendicular to the axis of the bar. To secure this a vertical hole, a little larger than the bar, was drilled in a prism

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* The following values of k for brass have been found by different experimenters :—NEUMANN, 'Ann. de Chim. et de Phys.' (III.), vol. 66 (1862), ·302 ; WEBER, 'Monatsber. Berlin Akad.,' for 1880, p. 457, ·150 ; LORENZ, 'Wied. Ann.,' vol. 13 (1881) ; 'red brass,' ·252 $(1 + \cdot 0018\, t.)$; 'yellow brass,' ·212 $(1 + \cdot 0020\, t.)$.

of wood, 30 cms. × 20 cms. base, and 15 cms. height. This was done in a drilling machine the table of which was found to be perpendicular to the spindle. One of the bars was then fixed by wedges into this hole in such a way that a point, which when the bar was in position would be the highest point of the cut surface, came in contact with a mark on the block of wood. The block and bar were then moved about on a stone and afterwards on a slate slab, till the surface was plane and smooth. The other bar was then fixed so that the lowest point of the cut surface came into contact with the same mark, and the surface rubbed down. This process secured that the cut surfaces should be parallel to each other when the bars were coaxal. To secure and test that the surfaces were parallel when the bars were in position in the framework, two test gauges were made, one for testing the upper surface and the other the side of the bars. The bars were taken out of the frame and placed vertical, the upper bar resting with its cut surface accurately on the cut surface of the lower bar, and not being supported by any other means. Four screws on each gauge were then adjusted so that they would touch four points, two on each bar near its extremities. By applying these gauges to the bars when in position, either in contact with each other or with a disc between, it is therefore easy to make the required adjustment of the cut surfaces.

The surfaces were now amalgamated, the amalgamated surfaces brought together, and experiments made on the distribution of temperature throughout the bar, to test the efficiency of the mercury contact. When the bar was uncut, the difference of temperature observed between the points in the middle, 1·05 cm. apart, was, as a mean of three experiments, ·76° C. With the cut bar and the mercury contact, the distribution of temperature in the other parts of the bars being the same, the difference of temperature observed was ·64° C. The cutting and grinding of the bar had reduced the distance between the points of observation to ·88 cm., hence if the mercury contact has no appreciable resistance the fall should be $\frac{\cdot 76 \times \cdot 88}{1 \cdot 05} = \cdot 65^\circ$, which is, within the limits of error, what was observed. Hence, as far as the distribution of temperature along the bar is concerned, the mercury contact secures practical continuity. Occasional tests were made during the course of the work to see if this continued to hold. The resistance of the contact was found to increase slightly. Three tests made at the end of the work gave ·78° as the difference of temperature. This is equivalent to an increase of distance between the points of observation from ·88 cm. to 1·06 cm., and is probably caused by the gradual soaking of the mercury into the brass. Account is taken of this in the following calculations.

The conditions of the above experiments are not quite the same as in experiments with crystals, &c., for while in the above there is only one mercury film, in the crystal experiments there are two. To investigate the effect of this, a thin film of mica, about ·0003 cm. thick, was inserted. Although there are then two films of mercury and a film of mica between the bars, no change could be detected in the difference of temperature of the observing points. The same held on insertion of a piece of platinum foil ·0025 cm. thick. The corrections for the short length of bar and mercury contact between the points at which the temperatures are observed and the surfaces of the discs, can, therefore, be determined from the observation with the amalgamated ends of the bars in contact.

In preparing the apparatus for an experiment, the bars and framework were placed so that the bars were vertical, with the cool bar at the top. The bars were taken out of the frame and the holes and contacts cleaned with dilute nitric acid, provided with mercury, washed, and then dried by filter paper. The hot bar was then inserted into lower part of the frame, and if a disc was to be experimented on, it was placed on the mercury surface and the excess of mercury forced from under it. In the case of a transparent disc it was possible to see that the mercury formed a perfect mirror. The cool bar was then put into the upper part of the frame, and held up, in contact with the centre screw at the top of the frame, by means of a spring. The amalgamated end would be thus brought about ·5 cm. above the upper surface of the disc. It was then supplied with mercury, which hung down as a pendent drop. The upper screw was then turned till the end of the upper bar was about ·5 mm. from the upper surface of the disc. In this position the gauges were applied to the bars, and the loops so adjusted by means of the screws at their ends, that the four points of each gauge touched the bars. The upper screw was then rotated till the upper bar came into contact with the disc, and the excess of mercury was forced out. The frame was then placed over the water tank with the bars horizontal and connections made to the steam and water supplies. The apparatus was allowed to stand about one and a half hours, in order that the distribution of temperature might become steady before observations were taken. Almost invariably on taking the bars apart at the conclusion of an experiment, the discs were found to adhere to the cool bar, and in the case of transparent discs this enabled it to be seen if the contact between the cool bar and the disc had been good. It was generally found as good a mirror as the contact between the hot bar and the disc. Some experiments were however made to try the effect of making contacts which were poor as far as could be judged by the appearance of the mirrors. No difference could be detected in the observations between an optically good and a bad mirror, so that a few experiments, in which the contacts were optically defective, have been included in the results given.

Reduction of Observations with Discs.

We have seen that the temperature throughout any short length of the bar may be represented by the empirical equation $v = \mathrm{A}\cosh \alpha x + \mathrm{B}\sinh \alpha x$. We assume that this equation holds for the first three observations on each side of the disc, and thus have, if x_1, x_2, x_3 are the coordinates of the points of observation, measured from the surface of the disc, and v_1, v_2, v_3 the observed temperatures,

$$\left.\begin{aligned} v_1 &= \mathrm{A}\cosh \alpha x_1 + \mathrm{B}\sinh \alpha x_1 \\ v_2 &= \mathrm{A}\cosh \alpha x_2 + \mathrm{B}\sinh \alpha x_2 \\ v_3 &= \mathrm{A}\cosh \alpha x_3 + \mathrm{B}\sinh \alpha x_3 \end{aligned}\right\}$$

These three equations are sufficient to fix the values of the constants A, B, α, but their solution is difficult unless $x_3 - x_2 = x_2 - x_1$. This relation is very nearly satisfied by the points of observation, in the hot bar, $x_2 - x_1 = 10{\cdot}56$ cms. $x_3 - x_2 = 10{\cdot}44$ cms.; in the cool bar, $x_2 - x_1 = 10{\cdot}51$ cms. $x_3 - x_2 = 10{\cdot}48$ cms.

The error introduced by assuming each of these intervals $= 10{\cdot}5$ cms. is small enough to be neglected in the cool bar, but a small correction is necessary in the case of the hot bar.

If dv_2/dx be the value of dv/dx at the point x_2 we have for the temperature v_2' at the point $x_1 + 10{\cdot}5$,

$$v_2' = v_2 - {\cdot}06\,\frac{dv_2}{dx}.$$

Now dv_2/dx is found to differ little from 1 for any of the experiments,* and we may, therefore, with sufficient accuracy, take $v_2' = v_2 - {\cdot}06$.

Making use of v_2' we have from the three temperatures, v_1, v_2', v_3

$$\cosh \alpha l = \frac{v_1 + v_3}{2v_2'},$$

where $l = x_2 - x_1 = x_3 - x_2$, &c. This equation determines α from any three observations of temperature. The mean of the values thus determined for different points along the bars and for different experiments is used in the subsequent work.

We have then the equations

$$\left.\begin{aligned} v_1 &= \mathrm{A}\cosh \alpha x_1 + \mathrm{B}\sinh \alpha x_1 \\ v_2 &= \mathrm{A}\cosh \alpha x_2 + \mathrm{B}\sinh \alpha x_2 \end{aligned}\right\}$$

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* The value of dv_2/dx may be determined as follows. By TAYLOR'S Theorem we have—

$$f(x + l) - f(x) = lf'(x) + \frac{l^2}{\underline{|2}}f''(x) + \frac{l^3}{\underline{|3}}f'''(x) + \text{\&c.}$$

$$f(x) - f(x - l) = lf'(x) - \frac{l^2}{\underline{|2}}f''(x) + \frac{l^3}{\underline{|3}}f'''(x) + \text{\&c.}$$

Therefore

$$\frac{f(x + l) - f(x - l)}{2l} = f'(x) + \frac{l^2}{\underline{|3}}f'''(x).$$

Putting $v = f(x)$ we have, since for any short length of the bar $\frac{d^2v}{dx^2} = \alpha^2 v$,

$$f'''(x) = \alpha^2 f'(x).$$

Therefore

$$f'(x) = \frac{f(x + l) - f(x - l)}{2l\left(1 + \frac{l^2\alpha^2}{6}\right)}.$$

The value of $\frac{1}{6}(l^2\alpha^2)$ for this point of the bar is in all the experiments approximately ·04.

where $\cosh \alpha x_1$, $\sinh \alpha x_1$, $\cosh \alpha x_2$, $\sinh \alpha x_2$ are known quantities, to determine the values of A and B.

Differentiating the equation $v = \mathrm{A} \cosh \alpha x + \mathrm{B} \sinh \alpha x$ and putting $\alpha = 0$, we have at the surface in contact with the disc, $v = \mathrm{A}$, $dv/dx = \alpha \mathrm{B}$.

The isothermal surfaces in the discs themselves may be assumed to be planes, for a small calculation, like that made (p. 490) for the bar itself, shows that the radius of curvature of these surfaces is about 40 cms. Writing a_1, a_2 for the coordinates of the surfaces of the disc, A_1, A_2 the temperatures determined as above, Q_1, Q_2 the values of $k_1 \alpha \mathrm{B}$ in the bars at the surfaces in contact with the disc, k_1 being the conductivity of bars, and k that of the disc, we have, for the temperature in the disc itself:

$$v = \frac{\mathrm{A}_2 \sinh \sqrt{\frac{ph}{qk}} \cdot \overline{x - a_1} + \mathrm{A}_1 \sinh \sqrt{\frac{ph}{qk}} \,.\, \overline{a_2 - x}}{\sinh \sqrt{\frac{ph}{qk}} \overline{a_2 - a_1}} .$$

Differentiating and writing down the expressions for the flow of heat into and out of the disc we have the equations—

$$\mathrm{Q}_1 q_1 = qk \sqrt{\frac{ph}{qk}} \cdot \frac{\mathrm{A}_2 - \mathrm{A}_1 \cosh \sqrt{\frac{ph}{qk}} \overline{a_2 - a_1}}{\sinh \sqrt{\frac{ph}{qk}} \overline{a_2 - a_1}} ,$$

and

$$\mathrm{Q}_2 q_1 = qk \sqrt{\frac{ph}{qk}} \frac{\mathrm{A}_2 \cosh \sqrt{\frac{ph}{qk}} \overline{a_2 - a_1} - \mathrm{A}_1}{\sinh \sqrt{\frac{ph}{qk}} \cdot \overline{a_2 - a_1}}$$

where p, h, q, k refer to the disc and have the usual meanings, q being $= q_1$ in most cases, differing only slightly in others. Now $a_2 - a_1$, the thickness of the discs, is small enough to make $\sqrt{\frac{ph}{qk}} \overline{a_2 - a_1}$ small. Hence, writing $a_2 - a_1 = t$, and expanding the hyperbolic functions, we have, as a close approximation—

$$q_1 \mathrm{Q}_1 = qk \sqrt{\frac{ph}{qk}} \frac{\mathrm{A}_2 - \mathrm{A}_1 \left(1 + \frac{ph}{qk} \cdot \frac{t^2}{2}\right)}{\sqrt{\frac{ph}{qk}} \cdot t \left(1 + \frac{ph}{qk} \cdot \frac{t^2}{6}\right)} .$$

Therefore

$$k = \left(\frac{q_1}{q}\right) \cdot \frac{\mathrm{Q}_1 t \left(1 + \frac{ph}{qk} \cdot \frac{t^2}{6}\right)}{\mathrm{A}_2 - \mathrm{A}_1 \left(1 + \frac{ph}{qk} \cdot \frac{t^2}{2}\right)} ;$$

similarly

$$k = \left(\frac{q_1}{q}\right) \cdot \frac{\mathrm{Q}_2 t \left(1 + \frac{ph}{qk} \cdot \frac{t^2}{6}\right)}{\mathrm{A}_2 \left(1 + \frac{ph}{qk} \cdot \frac{t^2}{2}\right) - \mathrm{A}_1} ;$$

either of which equations determines k, the conductivity of the disc, if its approximate value is substituted in the terms involving t^2. The value of h used is that for a painted surface, at the mean temperature of the disc, and the value of $\frac{ph}{qk}t^2$ is given with sufficient accuracy for the above substitution by the equation

$$\frac{ph}{qk}t^2 = \cdot 00056\,\frac{A_2 - A_1}{Q}\,t.$$

Observations.[*]

I TRANSPARENT ISOTROPIC BODIES.—The temperatures given are the means of three experiments.

CROWN Glass Disc, 1·93 cm. diam., ·169 cm. thick. Air, 16°·2 C.

Cool bar.	Hot bar.
v_1 8·58 A_1 8·83	v_1 18·44 A_2 18·25
v_2 4·20 Q_1 ·1365	v_2 26·54 Q_2 ·1336
$\therefore k_1 = \frac{\cdot 1365 \times \cdot 169 \times 1\cdot 001}{18\cdot 25 - 8\cdot 83 \times 1\cdot 003} = \cdot 00246$	$k_2 = \frac{\cdot 1336 \times \cdot 169 \times 1\cdot 001}{18\cdot 25 \times 1\cdot 003 - 8\cdot 83} = \cdot 00241$

k mean = ·00244.

FLINT Glass Disc, 1·93 cm. diam., ·177 cm. thick. Air, 16°·6 C.

Cool bar.	Hot bar.
v_1 7·60 A_1 7·83	v_1 19·02 A_2 18·84
v_2 3·54 Q_1 ·1254	v_2 26·80 Q_2 ·1246
$\therefore k_1 = \frac{\cdot 1254 \times \cdot 177 \times 1\cdot 002}{18\cdot 84 - 7\cdot 83 \times 1\cdot 006} = \cdot 00203$	$k_2 = \frac{\cdot 1246 \times \cdot 177 \times 1\cdot 002}{18\cdot 84 \times 1\cdot 006 - 7\cdot 83} = \cdot 00199$

k mean = ·00201.

ROCK Salt Disc, 1·95 cm. diam., ·83 cm. thick. Air, 15°·0 C.

Cool bar.	Hot bar.
v_1 9·24 A_1 9·50	v_1 18·47 A_2 18·22
v_2 4·61 Q_1 ·1457	v_2 27·31 Q_2 ·1515
$k_1 = (\cdot 98)\,\frac{\cdot 1457 \times 83 \times 1\cdot 005}{18\cdot 22 - 9\cdot 50 \times 1\cdot 014} = \cdot 0138$	$k_2 = (\cdot 98)\,\frac{\cdot 1515 \times \cdot 83 \times 1\cdot 005}{18\cdot 22 \times 1\cdot 014 - 9\cdot 50} = \cdot 0138$

k mean = ·0138.

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[*] The results are collected in a table on page 506.

II. Crystals.—Temperatures, mean of three experiments.

Quartz Disc, 1·91 cm. diam., 1·005 cm. thick. Thermal stream lines parallel to optical axis. Air, 15°·1 C.

Cool bar.	Hot bar.
v_1 10·09 A_1 10·50	v_1 16·69 A_2 16·40
v_2 4·64 Q_1 ·1680	v_2 26·28 Q_2 ·1763
$k_1 = (1{\cdot}02)\frac{{\cdot}168 \times 1{\cdot}005 \times 1{\cdot}003}{16{\cdot}40 - 10{\cdot}5 \times 1{\cdot}010} = {\cdot}0298$	$k_2 = (1{\cdot}02)\frac{{\cdot}1763 \times 1{\cdot}005 \times 1{\cdot}003}{16{\cdot}40 \times 1{\cdot}010 - 10{\cdot}5} = {\cdot}0299$

k mean = ·0299.

Quartz Disc, 1·93 cm. diam., ·811 cm. thick. Thermal stream lines perpendicular to optical axis. Air, 16°·2 C.

Cool bar.	Hot bar.
v_1 9·09 A_1 9·39	v_1 17·71 A_2 17·45
v_2 4·07 Q_1 ·1545	v_2 26·86 Q_2 ·1616
$k_1 = \frac{{\cdot}1545 \times {\cdot}811 \times 1{\cdot}004}{17{\cdot}45 - 9{\cdot}36 \times 1{\cdot}011} = {\cdot}01574$	$k_2 = \frac{{\cdot}1616 \times {\cdot}811 \times 1{\cdot}004}{17{\cdot}45 \times 1{\cdot}011 - 9{\cdot}36} = {\cdot}01589$

k mean = ·0158.

Iceland Spar Disc, 1·91 cm. diam., ·70 cm. thick. Thermal stream lines parallel to optical axis. Air, 16°·4 C.

Cool bar.	Hot bar.
v_1 7·82 A_1 8·10	v_1 18·73 A_2 18·49
v_2 3·36 Q_1 ·1436	v_2 27·41 Q_2 ·1478
$k_1 = (1{\cdot}02)\frac{{\cdot}1436 \times {\cdot}7 \times 1{\cdot}004}{18{\cdot}49 - 8{\cdot}10 \times 1{\cdot}013} = {\cdot}01001$	$k_2 = (1{\cdot}02)\frac{{\cdot}1478 \times {\cdot}7 \times 1{\cdot}004}{18{\cdot}49 \times 1{\cdot}013 - 8{\cdot}10} = {\cdot}00998$

k mean = ·0100.

Iceland Spar Disc, 1·93 cm. diam., ·602 cm. thick. Thermal stream lines perpendicular to optical axis. Air, 16°·3 C.

Cool bar.	Hot bar.
v_1 7·95 A_1 8·12	v_1 18·61 A_2 18·37
v_2 3·33 Q_1 ·1405	v_2 27·31 Q_2 ·1475
$k_1 = \frac{{\cdot}1405 \times {\cdot}602 \times 1{\cdot}004}{18{\cdot}37 - 8{\cdot}12 \times 1{\cdot}012} = {\cdot}00837$	$k_2 = \frac{{\cdot}1475 \times {\cdot}602 \times 1{\cdot}004}{18{\cdot}37 \times 1{\cdot}012 - 8{\cdot}12} = {\cdot}00852$

k mean = ·00845.

Mica, large plate, ·18 cm. thick. Thermal stream lines perpendicular to cleavage planes. Air, 15°·6 C.

Cool bar.	Hot bar.
v_1 6·36 A_1 6·58 v_2 2·77 Q_1 ·1093 $k_1 = \frac{\cdot 1093 \times \cdot 18 \times 1\cdot 002}{18\cdot 67 - 6\cdot 58 \times 1\cdot 05} = \cdot 00164$	v_1 18·90 A_2 18·67 v_2 27·44 Q_2 ·1436 $k_2 = \frac{\cdot 1436 \times \cdot 18 \times 1\cdot 002}{18\cdot 67 \times 1\cdot 05 - 6\cdot 58} = \cdot 00199$

k mean = ·00181.

As the mica was in the form of a large plate, the above method of finding k is not strictly correct, and this accounts for the difference between the calculated values of k_1 and k_2. It was found impossible to use a disc, as in the other experiments, on account of the mica splitting. The value of k given must be taken as approximate only.

III. Rock Specimens.

Marble Disc, 1·93 cm. diam., ·26 cm. thick. The marble was the white variety used for statues. Air, 15°·5 C.

Cool bar.	Hot bar.
v_1 10·15 A_1 10·40 v_2 4·38 Q_1 ·1767 $k_1 = \frac{\cdot 1767 \times \cdot 26 \times 1\cdot 001}{17\cdot 01 - 10\cdot 4 \times 1\cdot 003} = \cdot 00699$	v_1 17·31 A_2 17·01 v_2 27·27 Q_2 ·1839 $k_2 = \frac{\cdot 1839 \times \cdot 26 \times 1\cdot 001}{17\cdot 01 \times 1\cdot 003 - 10\cdot 4} = \cdot 00718$

k mean = ·00709.

Slate Disc, 1·93 cm. diam., ·31 cm. thick, cut from an ordinary school slate. Air, 14°·9 C.

Cool bar.	Hot bar.
v_1 8·80 A_1 9·08 v_2 4·27 Q_1 ·1508 $k_1 = \frac{\cdot 1508 \times \cdot 31 \times 1\cdot 002}{19\cdot 15 - 9\cdot 08 \times 1\cdot 006} = \cdot 00468$	v_1 19·40 A_2 19·15 v_2 28·58 Q_2 ·1581 $k_2 = \frac{\cdot 1581 \times \cdot 31 \times 1\cdot 002}{19\cdot 15 \times 1\cdot 006 - 9\cdot 08} = \cdot 00482$

k mean = ·00475.

IV. VARIOUS BODIES CONSTANTLY USED IN PHYSICAL WORK.

Where no diameter is given the plate is of irregular shape, and rather larger than the section of the bars.

SHELLAC Plate, ·068 cm. thick. Air, 14°·2 C.

This plate was made between two microscope slides having plane surfaces. One slide was placed horizontal, and the upper surface covered with a smooth sheet of tinfoil. On the foil was laid a sufficient number of the thin films of shellac supplied by varnishmen to make a plate of the requisite size and thickness. The slide was then warmed gently from underneath till the shellac was soft, and the cold slide brought down on to the soft mass. In order to get the requisite thickness, and the surfaces of the plate parallel to each other, the two slides were kept apart by two short lengths of wire ·068 cm. diameter laid across the lower slide near its ends, the upper slide being pressed against these. The tinfoil which attaches itself to the plate is readily dissolved off by mercury. This method is very convenient and gives excellent plates.

Cool bar.	Hot bar.
v_1 7·16 A_1 7·40	v_1 20·95 A_2 20·76
v_2 3·32 Q_1 ·1183	v_2 28·74 Q_2 ·1159
$k_1 = \frac{\cdot 1183 \times \cdot 068 \times 1\cdot 001}{20\cdot 76 - 7\cdot 40 \times 1\cdot 002} = \cdot 000604$	$k_2 = \frac{\cdot 1159 \times \cdot 068 \times 1\cdot 001}{20\cdot 76 \times 1\cdot 002 - 7\cdot 40} = \cdot 000589.$

k mean $= \cdot 000596.$

PARAFFIN Plate, ·038 cm. thick. Air, 13°·4 C.

This plate was made by dropping a little melted paraffin on to the cold upper surface of the lower slide, and placing the upper slide on it as described above. The cold surfaces are sufficient to ensure the paraffin being detachable from the glass.

Cool bar.	Hot bar.
v_1 8·90 A_1 9·19	v_1 18·10 A_2 17·87
v_2 4·49 Q_1 ·1378	v_2 26·48 Q_2 ·1407
$k_1 = \frac{\cdot 1378 \times \cdot 038 \times 1\cdot 0002}{17\cdot 87 - 9\cdot 19 \times 1\cdot 0006} = \cdot 000604$	$k_2 = \frac{\cdot 1407 \times \cdot 038 \times 1\cdot 0002}{17\cdot 87 \times 1\cdot 0006 - 9\cdot 19} = \cdot 000615$

k mean $= \cdot 000610.$

PARA RUBBER, pure. Sheet ·0264 cm. thick. Air, 13°·8 C.

Cool bar.	Hot bar.
v_1 9·00 A$_1$ 9·26 v_2 5·01 Q$_1$ ·1401 $k_1 = \dfrac{\cdot 1401 \times \cdot 0264 \times 1\cdot 0002}{19\cdot 00 - 9\cdot 26 \times 1\cdot 0005} = \cdot 000380$	v_1 19·23 A$_1$ 19·00 v_2 27·87 Q$_2$ ·1433 $k_2 = \dfrac{\cdot 1433 \times \cdot 0264 \times 1\cdot 0002}{19\cdot 00 \times 1\cdot 0005 - 9\cdot 26} = \cdot 000388$

k mean = ·000384.

SULPHUR Plate, ·0584 cm. thick. Air, 14°·2 C.

This plate was made in the same way as the plate of shellac, except that it was found better to use a sheet of tinfoil on the surface of each slide.

Cool bar.	Hot bar.
v_1 6·60 A$_1$ 6·83 v_2 3·08 Q$_1$ ·1082 $k_1 = \dfrac{\cdot 1082 \times \cdot 0584 \times 1\cdot 0007}{20\cdot 74 - 6\cdot 83 \times 1\cdot 002} = \cdot 000455$	v_1 20·92 A$_2$ 20·74 v_2 28·42 Q$_2$ ·1085 $k_2 = \dfrac{\cdot 1085 \times \cdot 0584 \times 1\cdot 0007}{20\cdot 74 \times 1\cdot 002 - 6\cdot 83} = \cdot 000455$

k mean = ·000455.

EBONITE Disc,* 1·93 cm. diam., ·0414 cm. thick. Air, 14°·9 C.

Cool bar.	Hot bar.
v_1 7·35 A$_1$ 7·61 v_2 3·36 Q$_1$ ·1227 $k_1 = \dfrac{\cdot 1227 \times \cdot 0414 \times 1\cdot 0004}{20\cdot 10 - 7\cdot 61 \times 1\cdot 0013} = \cdot 000407$	v_1 20·30 A$_2$ 20·10 v_2 28·16 Q$_2$ ·1202 $k_2 = \dfrac{\cdot 1202 \times \cdot 0414 \times 1\cdot 0004}{20\cdot 10 \times 1\cdot 0013 - \cdot 701} = \cdot 000398$

k mean = ·000403.

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* Kindly supplied to me by Mr. C. H. GRAY, of the Silvertown Company.

Gutta Percha Sheet, ·0617 cm. thick. Air, 14°·4 C.

This sheet was made of the material used for insulating wire. A small quantity was cut from a wire, melted, and pressed into a sheet between the two cold microscope slides.

Cool bar.	Hot bar.
v_1 5·97 A_1 6·19 v_2 2·35 Q_1 ·1087 $k_1 = \frac{\cdot 1087 \times \cdot 0617 \times 1\cdot 0006}{20\cdot 76 - 6\cdot 19 \times 1\cdot 002} = \cdot 000461$	v_1 20·94 A_2 20·76 v_2 28·40 Q_2 ·1076 $k_2 = \frac{\cdot 1076 \times \cdot 0617 \times 1\cdot 0006}{20\cdot 76 \times 1\cdot 002 - 6\cdot 19} = \cdot 000456$

k mean = ·000458.

Paper Disc, 1·93 cm. diam., ·019 cm. thick. Air, 15°·4 C.

This was cut from an ordinary visiting card.

Cool bar.	Hot bar.
v_1 9·42 A_1 9·73 v_2 4·56 Q_1 ·1515 $k_1 = \frac{\cdot 1515 \times \cdot 019 \times 1\cdot 0001}{18\cdot 81 - 9\cdot 73 \times 1\cdot 0003} = \cdot 000317$	v_1 19·05 A_2 18·81 v_2 27·91 Q_2 ·1496 $k_2 = \frac{\cdot 1496 \times \cdot 019 \times 1\cdot 0001}{18\cdot 81 \times 1\cdot 0003 - 9\cdot 73} = \cdot 000313$

k mean = ·000315.

"Asbestos Paper" Disc, 1·93 cm. diam., ·047 cm. thick. Air, 14°·9 C.

This was the ordinary asbestos millboard of commerce. It consists of paper to the pulp of which sufficient asbestos has been added to render it incombustible.

Cool bar.	Hot bar.
v_1 8·40 A_1 8·68 v_2 4·02 Q_1 ·1358 $k_1 = \frac{\cdot 1358 \times \cdot 047 \times 1\cdot 0004}{19\cdot 93 - 8\cdot 68 \times 1\cdot 001} = \cdot 000568.$	v_1 20·16 A_2 19·93 v_2 28·67 Q_2 ·1368 $k_2 = \frac{\cdot 1368 \times \cdot 047 \times 1\cdot 0004}{19\cdot 93 \times 1\cdot 001 - 8\cdot 68} = \cdot 000571$

k mean = ·000570.

MAHOGANY Disc, 1·93 cm. diam., ·056 cm. thick, cut from a sheet of "veneer" used by cabinet makers. Air, 15°·9 C.

Cool bar.	Hot bar.
v_1 6·67 A_1 6·89 v_2 3·06 Q_1 ·1107 $k_1 = \frac{\cdot1107 \times \cdot056 \times 1\cdot0006}{20\cdot57 - 6\cdot89 \times 1\cdot002} = \cdot000454$	v_1 20·75 A_2 20·07 v_2 28·51 Q_2 ·1167 $k_2 = \frac{\cdot1167 \times \cdot056 \times 1\cdot0006}{20\cdot57 \times 1\cdot002 - 6\cdot89} = \cdot000477$

k mean = ·000465.

WALNUT Disc, 1·93 cm. diam., ·069 cm. thick, cut from a sheet of veneer. Air, 16°·0 C.

Cool bar.	Hot bar.
v_1 5·11 A_1 5·29 v_2 2·14 Q_1 ·0917 $k_1 = \frac{\cdot0917 \times \cdot069 \times 1\cdot001}{22\cdot46 - 5\cdot29 \times 1\cdot004} = \cdot000363$	v_1 22·60 A_2 22·46 v_2 29·52 Q_2 ·0888 $k_2 = \frac{\cdot0888 \times \cdot069 \times 1\cdot001}{22\cdot46 \times 1\cdot004 - 5\cdot29} = \cdot000358$

k mean = ·000360.

CORK Disc, 1·93 cm. diam., ·05 cm. thick, cut by a section-cutting machine from a sound cork. Air, 15°·1 C.

Cool bar.	Hot bar.
v_1 2·22 A_1 2·33 v_2 ·34 Q_1 ·0530 $k_1 = \frac{\cdot0530 \times \cdot05 \times 1\cdot002}{23\cdot30 - 2\cdot33 \times 1\cdot005} = \cdot000127$	v_1 23·39 A_2 23·30 v_2 29·10 Q_2 ·0547 $k_2 = \frac{\cdot0547 \times \cdot05 \times 1\cdot002}{23\cdot30 \times 1\cdot005 - 2\cdot33} = \cdot000130$

k mean = ·000129.

SILK Sheet, ·0084 cm. thick. This was cut from a dress piece of plain brown silk. Air, 16°·4 C.

Cool bar.	Hot bar.
v_1 10·06 A_1 10·39 v_2 4·95 Q_1 ·1600 $k_1 = \frac{\cdot16 \times \cdot0084}{16\cdot76 - 10\cdot39} = \cdot000210$	v_1 17·03 A_2 16·76 v_2 26·35 Q_2 ·1690 $k_2 = \frac{\cdot169 \times \cdot0084}{16\cdot76 - 10\cdot39} = \cdot000223$

k mean = ·000216.

SILK Sheet, ·017 cm. thick, cut from a piece of ribbed silk ribbon. The thickness given is the mean of that across ribs and across grooves. Air, 16°·6 C.

Cool bar.	Hot bar.
v_1 8·03 A_1 8·30 v_2 3·72 Q_1 ·1333 $k_1 = \frac{\cdot1333 \times \cdot017}{18{\cdot}65 - 8{\cdot}30} = \cdot00022$	v_1 18·88 A_2 18·65 v_2 27·37 Q_2 ·1415 $k_2 = \frac{\cdot1415 \times \cdot017}{18{\cdot}65 - 8{\cdot}30} = \cdot00023$

k mean = ·00023.

COTTON Sheet, ·085 cm. thick. A piece of thick tape. Air, 16°·4 C.

Cool bar.	Hot bar.
v_1 5·23 A_1 5·43 v_2 1·95 Q_1 ·100 $k_1 = \frac{\cdot100 \times \cdot085 \times 1{\cdot}001}{21{\cdot}24 - 5{\cdot}43 \times 1{\cdot}004} = \cdot000539$	v_1 21·41 A_2 21·24 v_2 28·79 Q_2 ·1044 $k_2 = \frac{\cdot1044 \times \cdot085 \times 1{\cdot}001}{21{\cdot}24 \times 1{\cdot}004 - 5{\cdot}43} = \cdot000557$

k mean = ·00548.

FLANNEL Sheet, ·1 cm. thick. Air, 15°·1 C.

Cool bar.	Hot bar.
v_1 2·34 A_1 2·44 v_2 ·59 Q_1 ·0504 $k_1 = \frac{\cdot0504 \times \cdot1 \times 1{\cdot}006}{24{\cdot}58 - 2{\cdot}44 \times 1{\cdot}018} = \cdot000229$	v_1 24·67 A_2 24·58 v_2 30·39 Q_2 ·0505 $k_2 = \frac{\cdot0505 \times \cdot1 \times 1{\cdot}006}{24{\cdot}58 \times 1{\cdot}018 - 2{\cdot}44} = \cdot000225$

k mean = ·000227.

As the silk, cotton, and flannel yield somewhat, the measurements of thickness are not very accurate in these three cases. The error will be greatest in the cases of flannel and cotton, and probably small in the case of silk.

TABLE of Results.

Material and direction of stream lines.	Thermal conductivity between 25 and 35° C. C.G.S. units.		Refractive Index.
Copper	..	·7 to ·8 Lorenz, F. Weber, &c.	
Brass	..	·25 to ·3 Lorenz, Neumann, &c.	
Brass bar used	·27		
Bismuth	..	·017 Lorenz	
Mercury	..	·018 Ångström	
Crown glass	·00243	·0016 H. Meyer	1·53
Flint glass	·00201	·0014 ,,	1·64
Glass	..	·0021 Peclet ·0005 G. Forbes	
Rock salt	·0138	·016 Tuschmidt	1·55
Quartz along axis	·0299	·026 Tuschmidt ·001 G. Forbes	1·55
,, perpendicular to axis	·0158	·016 Tuschmidt ·004 G. Forbes	1·56
Iceland spar along axis	·0100	·016 Tuschmidt	1·66
,, perpendicular to axis	·0084	·0086 ,,	1·49
Mica perpendicular to cleavage	·0018		1·57
White marble	·0071	·0075 Peclet ·0073 Yamagawa ·0011 G. Forbes	
Slate	·0047	·0008 ,,	
Water	..	·0015 Winkelmann	
Glycerine	..	·0007 ,,	
Olive oil	..	·0004 F. Weber	
Shellac	·00060		
Paraffin	·00061	·00014 G. Forbes	
Pure Pará rubber	·00038		
Rubber	..	·00046 Peclet	
Vulcanised rubber	..	·00009 G. Forbes	
Sulphur	·00045		
Ebonite	·00040	·00008 ,,	
Gutta percha	·00046		
Paper	·00031	·00045 G. Forbes	
Asbestos paper	·00057		
Mahogany across fibre	·00047		
Walnut across fibre	·00036		
Cork	·00013		
Silk	·00022		
Cotton	·00055		
Flannel	·00023		

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G. Forbes, 'Roy. Soc. Proc. Edin.,' vol. 9, p. 64 (1873).
Tuschmidt, 'Beiblätter,' 1884, p. 490.
Yamagawa, 'Journal, College of Tokio,' vol. 2. p. 263 (1888).
Winkelmann, 'Poggendorff's Annalen,' vol. 153, p. 481 (1874).

In the foregoing table the conductivities of a few metals and liquids are given in order to show the position of the bodies experimented on amongst other conductors.

Where experiments have been previously made by other experimenters their results are given for comparison.

Such comparison shows a fair degree of concordance between the present results and those obtained by totally different methods, with the exception of the results of G. Forbes, which were obtained by the following method. A can about 9 cms. diameter was filled with a freezing mixture, placed on a sheet of the material the conductivity of which was to be determined, and the lower surface of the sheet brought down on to a surface of water at 0° C. Ice is formed on the surface of the sheet, and Forbes works out an expression for the thickness of this ice in terms of the temperature of the freezing mixture (which was observed by a thermometer placed in it), the thickness of the material, the time, and the conductivities of ice and the material. In working out this expression, however, he assumes the conductivity of the materials between the thermometer in the freezing mixture and the upper surface of the sheet to be infinitely great, which, when one considers the layers of liquid, metal, and air present, is scarcely justifiable. On this account almost all Forbes's results are low.

Peclet's results for metals were all found to be low, and this led to the assumption that all his results were the same. It is, however, evident from the present experiments that his results for bad conductors are correct. This is probably owing to the fact that the layers of still water which, in his experiments, adhered to the surfaces of the sheet experimented on, are of much less importance when the sheet is a bad conductor.

Tuschmidt's results are obtained by a method which Weber used for liquids. The sheet of liquid is simply replaced by one of the crystal, the contacts being made by means of glycerine. As glycerine has a conductivity which is only about $\frac{1}{20}$th of those of the crystal plates, the glycerine layers have a great effect on the flow of heat through the plates. This is a defect of the method, and, in addition, the plates themselves must be large, and are therefore expensive. Tuschmidt's results and those of the present paper agree fairly, the only great difference being in the case of Iceland spar along the axis.

MEYER's determinations depend on the rate of rise of the temperature of a calorimeter into which a heated cube of the material is plunged. It is questionable whether the assumption which he makes, that the temperature of the surface of the cube at any instant is that given by the thermometer in the calorimeter, is justifiable on account of the difficulty of preventing a layer of water adhering to the surface. The same objection applies to the method which he used to confirm the results obtained by his first method.

Remarks.

From the Table of Results it is at once evident that, for transparent bodies, no such comparison between thermal conductivity and velocity of propagation of light can be made, as has been made by KUNDT for the metals. The thermal conductivities vary enormously for very small change of refractive index, and the variations are sometimes in the same direction as the variation of the index of refraction, sometimes in opposite. It is remarkable that the bodies quartz and rock salt, which are diathermanous bodies, should also be good thermal conductors, quartz being a better conductor than bismuth. To prove that the high value of the conductivity is not due to heat passing through these bodies by radiation from the hot bar to the cold, several experiments were made with the apparatus arranged as it was during the experiment in quartz, but the quartz disc was removed, thus allowing the hot bar to radiate heat through the intervening air space to the cool bar. Under these conditions, no change of temperature at the point of observation near the end of the cool bar, could be detected when the hot bar was suddenly cooled. As air is a more diathermanous body than quartz, it is thus evident that the amount of heat radiated through the quartz from one bar to the other, is too small to affect the above results.

The high conductivity of quartz would render the use of fused quartz (the conductivity of which does not probably differ much from that of quartz crystal) advantageous for vessels subject to sudden change of temperature, and for delicate thermometers.

The results for Iceland spar and marble seem to indicate that the irregular arrangement of the crystals in marble interferes with the passage of heat.

Water and salt solutions have conductivities about equal to that of glass.

The solid insulations used in electrical work rank with glycerine and the oils, so far as thermal conductivity is concerned.

Silk has about one-third the conductivity of shellac. Hence, if a silk covered wire has the covering saturated with shellac, it will have its heat conducted away to surrounding bodies much faster than previously. The practice of soaking galvanometer coils in shellac is, therefore, good from a thermal point of view, as it enables the heat generated in the wire to pass more rapidly to the exterior of the coils, and be radiated away.

On account of the low conductivity of paraffin, standard resistance coils with temperature coefficients ought not to be embedded in paraffin, as the B.A. standard is.

Cork is the worst conductor experimented on.

Almost all the substances experimented on are insulators electrically. Although tables of their electrical conductivities are available, the values given seem to depend more on the condition of the surface of the body experimented on than on its material, and, on this account, it is of little use to make any comparisons between the electrical and the thermal conductivities of these substances.

The above experiments were carried on in the Physical Laboratory of the Owens College.

XIII. *Researches on Turacin, an Animal Pigment containing Copper.*—II.

By A. H. Church, *M.A., F.R.S., Professor of Chemistry in the Royal Academy of Arts, London.*

Received April 2,—Read April 28, 1892.

Contents.

§ 1. *Touracos, Turacin, and Copper in Animals.*

Since the publication of my paper on Turacin,[*] read before the Society in May, 1869, several interesting facts have come to light in reference to the Touracos, and to the occurrence of copper in the animal kingdom. Before giving the results of my own further researches, I would first of all mention the book on "Angola and the River Congo" by the late J. J. Monteiro.[†] In the second volume of this work, pp. 75 to 79, will be found some curious particulars about two species of *Turacus* (*Corythaix*), as well as an account of Mr. H. Bassett's experiments with turacin, to which further reference will presently be made. Mr. Monteiro describes a singular trait manifested by one of his tame plantain-eaters, a *Turacus schalowi*, which showed marked delight in gaily coloured dresses and pictures. Several instances are given of Touracos in captivity having moulted and then renewed their crimson plumage with all its original richness of colour. Some feathers from these birds, which Mr. Monteiro handed to me for examination, were found, by optical tests, to contain a turacin identical with that which colours them in their native countries. Yet these feathers had been produced in captivity, and after the birds had been for several years in England, where they were fed upon imported bananas and other vegetable foods. Certainly they had not had the opportunity of picking up "the grains of malachite and of other copper minerals to which, it has been suggested, they may have had access in

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[*] 'Phil. Trans.,' vol. 152, Part II., pp. 627–636 (1870).

[†] London: Macmillan and Co., 1875.

their African homes, and from which they might have derived the copper necessary for the formation of their cupreous pigment. Such a suggestion appears, however, of doubtful value, for the plantain-eaters are arboreal feeders; and it is unnecessary, since copper is now known to be very widely distributed in plants, and I have shown it to be present in decided traces in bananas, the chief food of many species of Touraco

In 1881 Dr. M. GIUNTI published a paper entitled "Ricerche sulla Diffusione del Ramo nel Regno Animale."* This memoir gives a *résumé* of the work previously done on this subject, but also contains many new observations. Dr. GIUNTI found in the ash of the animals and animal products named below the following percentages of copper oxide (CuO) :—

Chrysomela americana	1·293
Lizard skins, *Podarcis muralis*	1·049
Blatta orientalis	0·826
Bat-guano (from Calabria)	0·817
Catonia hirtella	0·661
Julus terrestris	0·221
Swallows	0·217
Armadillidium vulgare	0·197
Helix pisana	0·089
Hedgehog	0·016

These results are of value not only as confirming the view that copper is widely diffused in the animal kingdom, but because they render probable further discoveries as to the occurrence of definite organic cupreous compounds in nature.

The researches of Dr. C. F. W. KRUKENBERG, on "Die Farbstoffe der Federn,"† must now be considered. This investigator has described a green colouring matter obtained from the green feathers of *Turacus corythaix*, and of other plantain-eaters, by the employment of a 2 per cent. caustic soda-solution as the solvent. He calls this pigment "turacoverdin," and, although he did not obtain enough of it for quantitative analysis, states that it contains "much iron, but no great quantity of copper and manganese," and that "perhaps, like turacin, it is free from sulphur and nitrogen." Here I must observe that Dr. KRUKENBERG, on referring to my paper on turacin, with which he was acquainted, would have found that turacin contains between 6 and 7 per cent. of nitrogen. The Author goes on to say: "This colouring matter (turacoverdin) becomes of high interest through the results of the experiments made by CHURCH on turacin." And then he proceeds as follows—I quote the original German of Dr. KRUKENBERG—"In seiner ausgezeichneten Arbeit über das Turacin—einem, auf dem Continente durchaus unbekannt gebliebenen und von mir in der grossen Katakombe des 'Philosophical Transactions' deshalb auch erst so spät aufgefundenen Meisterwerke auf dem Felde der physiologischen Chemie—mit welcher

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* 'Accademia Reale delle Scienze,' Napoli, 1881.

† Heidelberg, 'Vergl.-physiol. Studien,' 1881.

ich leider erst zu einer Zeit bekannt geworden bin, wo diese Untersuchungen bereits abgeschlossen waren und ausgearbeitet vorlagen, theilt CHURCH Folgendes mit: 'Turacin, by long exposure to air and moisture, or by continued ebullition with water or alkaline liquids, acquires a colour closely resembling that of chlorophyll.'" Dr. KRUKENBERG decides, after comparing the spectrum of altered turacin, as given in my memoir (*loc. cit.*, fig. 4), with the spectrum of turacoverdin, that the presence of the absorption band on the less refrangible side of line D indicates the presence of his turacoverdin in my "altered turacin," and proves that I had, in 1869, transformed turacin into turacoverdin, although there remained, as I had at the time conjectured, some unaltered turacin in the preparation examined. Although I cannot but appreciate very highly the handsome way in which Dr. KRUKENBERG speaks of my work and his confirmation of my results, I am unable to agree with Mr. F. E. BEDDARD* when he states that Dr. KRUKENBERG has added important details to those furnished by its discoverer with regard to turacin. And I may add in this place the remark that, if turacin be transformable in the way above described into turacoverdin, it is impossible that the latter can contain as essential elements either iron or manganese.

The numerous observations which have been made from time to time as to the occurrence of copper in certain Mollusks and Arthropods have finally resulted in the detection of a definite cupreous pigment in the blood of certain members of these two groups. This body exists in two states, and possesses, like hæmoglobin, respiratory functions: oxidized it is blue, but in the reduced condition colourless. Its spectrum has no definite absorption bands. It was named hæmocyanin by LÉON FREDERICQ,† to whom we owe some admirable researches on this important compound. It has been recognised in several genera of Crustacea, of Arachnida, of Gastropoda, and of Cephalopoda. It is a coagulable proteid belonging to the globulin group, and contains a very small proportion of copper—a proportion which may be just high enough to be comparable with the proportion of iron present in hæmoglobin.‡ Hæmocyanin is thus widely removed by its chemical and physical characters, as well as by its physiological *rôle*, from turacin. It is a far less stable body, and contains a comparatively insignificant percentage of the characteristic metallic element. It would, I think, be desirable to learn if there could be obtained from hæmocyanin a cupreous derivative bearing to its source a relation analogous with that borne by hæmatin to hæmoglobin. At present, hæmocyanin and turacin are the only definite animal pigments containing copper which have been recognised.

In this place reference may be made to a letter§ on the subject of copper in Birds

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* 'Nature,' December 19, 1889.

† 'Bulletin de l'Académie Royale de Belgique,' 2^me^ série, vol. 46 (1878).

‡ [F. HEIM has recently ('Comptes Rendus,' vol. 114, pp. 771–4) controverted several of the statements made by FREDERICQ as to the composition and properties of hæmocyanin; he denies that copper is an essential constituent of this compound.—Postscript, September 1, 1892.]

§ 'Chemical News,' vol. 28, p. 212.

—a letter written by Mr. SYDNEY LUPTON in 1873. Mr. LUPTON detected copper in the ash of seven or eight feathers of an Australian love-bird, *Melopsittacus undulatus.* He suggested that the plumage of this bird might contain a green cupreous pigment.

§ 2. *Occurrence of Turacin.*

In my previous paper I named twelve species of Touracos, in which I had recognised, by chemical or optical methods, the occurrence of turacin; from four of these species I had actually extracted this pigment. Since the publication of that memoir several new species have been discovered, and old species split, while the nomenclature of the whole family of the Musophagidæ has been revised. The order Picariæ, to which that family belongs, contains two sub-orders of equal rank, Scansores and Coccyges. The Coccyges comprise four families, namely, Indicatoridæ, Capitonidæ, Cuculidæ, and Musophagidæ. The Musophagidæ are arranged under five genera, and comprise twenty-five species. I am now able to state that turacin occurs in eighteen species, namely, in all the fourteen species of *Turacus,* in the two species of *Gallirex,* and in the two species of *Musophaga.* It is absent from the seven species comprised in the genera *Corythæola, Schizorhis,* and *Gymnoschizorhis.* It is of interest to note that the zoological classification of these Birds is now in accord with what may be called their "chemical" sequence. Formerly, a single Bird destitute of turacin, and now constituting the solitary species of *Corythæola,* was included in *Corythaix,* that is, *Turacus*; the anomaly of its presence among the turacin-bearers has now been removed. I ought to say that I have had the opportunity of examining specimens of 17 out of the 18 known species of *Turacus, Gallirex,* and *Musophaga* with the spectroscope, and have recognised the presence of turacin by means of its characteristic spectrum in them all. I have, moreover, actually separated the pigment from the wing feathers of eleven different species of Touraco; the species not available for this experiment were: *Turacus reichenowi, T. livingstoni, T. schuetti, T. fischeri, T. leucolophus, T. hartlaubi,* and *Musophaga rossæ,* most of which are very rare Birds.*

The following conspectus of the genera and species of Musophagidæ is taken from the account of this family recently written by Captain G. E. SHELLEY;† the seven African ornithological sub-regions, in which the plantain-eaters are distributed, are those which the same most competent authority has suggested to me in a private communication, dated 26th October, 1891; they may be thus defined:—

W. The whole coast from Senegal to the Quanza River, and inland to 30° E. long.
S.W. South of the Quanza to the Orange River, and inland to Lake Ngami.

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* [Turacin occurs not only in the wing-feathers of *Turacus meriani,* but in those of the crest, the tips of which are crimson; these tips yield up their pigment to dilute ammonia with some difficulty. I have extracted turacin also from the red head-feathers of *Musophaga violacea.* It should be stated that two species of *Schizorhis* (*S. africana* and *S. zonura*) show on their wing-feathers white patches destitute of pigment but corresponding in position with the red tracts of the vanes in the turacin-bearers.—Postscript, September 1, 1892.]

† 'Catalogue of the Birds in the British Museum,' vol. 19, pp. 435-456 (1891).

S. Cape Colony and north to the Orange River.
S.E. Natal, Transvaal, Zambesi, with the Shiré tributary, and north to 15° S. lat.
E. The coast from Mozambique (15° S. lat.) to 5° N. lat., and inland to 35° E. long.
C. The Central or Lakes sub-region, comprising Victoria Nyanza, Albert Nyanza, south-east portion of Niam-Niam country, and extending south to Tanjanyika Lake.
N.E. The northern half of Niam-Niam, the whole Nile district from the Albert Nyanza, east through Gala land and Somali, and north to the mouths of the Nile.

The countries north of the Sahara desert to the Mediterranean coast belong, not to the Ethiopic, but to the European region.

It must be noted that our acquaintance with the African avi-fauna is too imperfect to admit of the construction of a distribution-chart of the Musophagidæ which shall be exhaustive and final.

TABLE Showing Distribution of the Touracos in West, South-West, South, South-East, East, Central, and North-East Africa.

	Genera and species.	W.	S.W.	S.	S.E.	E.	C.	N.E.
1	*Turacus leucotis* (RÜPP.)							—
2	*T. persa* (L.)	—						
3	*T. buffoni* (V.)	—						
4	*T. schalowi* (REICHEN.)		—					
5	*T. livingstoni* (GRAY)				—			
6	*T. reichenowi* (FISCHER)					—		
7	*T. corythaix* (WAGL.)			—				
8	*T. schuetti* (CAB.)	—					—	
9	*T. macrorhynchus* (FRASER)	—						
10	*T. meriani* (RÜPP.)	—						
11	*T. fischeri* (REICHEN.)					—		
12	*T. erythrolophus* (V.)	—						
13	*T. leucolophus* (HENGL.)							—
14	*T. hartlaubi* (FISCHER and REICHEN.)					—		
15	*Gallirex porphyreolophus* (VIG.)			—	—			
16	*G. chlorochlamys* (SHELLEY)					—		
17	*Musophaga violacea* (ISERT.)	—						
18	*M. rossae* (GOULD)	—					—	
. . .	. . . .							
19	*Corythæola cristata* (V.)	—					—	
20	*Schizorhis africana* (LATH.)	—						
21	*S. zonura* (RÜPP.)							—
22	*S. leucogaster* (RÜPP.)					—		—
23	*S. concolor* (SMITH)		—	—				
24	*Gymnoschizorhis personata* (RÜPP.)							—
25	*G. leopoldi* (SHELLEY)					—		
25		10	2	3	2	6	3	5

NOTE.—The 18 species above the dotted line constitute 3 genera, and may be considered to contain turacin, which is certainly absent from the remaining 7 species of the family.

With regard to the formation of turacin in the organism of these birds we have much to learn. That the pigment is not always present in the blood is certain. I could not detect a trace of it in a specimen of *Turacus persa*, which was examined immediately after death, although I did recognise the presence of a minute quantity of copper in the ash of its flesh. It is desirable that the blood of the birds should be tested for turacin during the moulting period.

§ 3. *Isolation of Turacin.*

The mode of extracting turacin from Touraco feathers has been slightly improved. The red parts of the vane are first thoroughly washed with distilled water, and then, after drying, treated successively with absolute alcohol and with ether. The material is allowed to dry before being extracted, not, as previously, with very dilute sodium hydrate solution, but with exceedingly weak aqueous ammonia. The crimson solution thus obtained is filtered, and then precipitated by pouring it into a large excess of pure strong hydrochloric acid diluted with twice its bulk of water. It is curious to observe the immense bulk assumed by freshly-precipitated turacin. Thus, in an operation in which about 4 grms. of this pigment were dealt with, it was necessary to increase the volume of the mixture of turacin, hydrochloric acid, and water to nearly one gallon before its thick consistency could be so reduced as to permit of its being brought upon the filter. Filters of the finest calico were found to be far preferable to those of paper; they were first thoroughly cleansed by means of boiling distilled water: the method of vacuum filtration was adopted. All the operations should be conducted quickly; exposure of the moist turacin to light should be avoided as far as possible.

§ 4. *Characters of Turacin.*

The account given in my previous paper of the properties of turacin and of its behaviour with several reagents needs one addition. When powdered turacin is dissolved in cold oil of vitriol* it is profoundly changed, a part of its copper being removed and a new colouring matter produced. On pouring the sulphuric acid solution into a considerable volume of water, a flocculent precipitate of a reddish-chocolate colour appears. Collected on a filter and thoroughly washed with hot water, this precipitate is found to be still rich in copper, although a part of the metal has been withdrawn by the action of the sulphuric acid, and is found in the filtrate. The precipitate dissolves in dilute ammonia-water, with a crimson hue.† Dr. C. A. MacMunn, of Wolverhampton, has kindly examined this solution for me, and states

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* Oil of vitriol (containing 96 per cent. of H_2SO_4) appears to exert the same action on turacin, and to yield the same derivative, at temperatures much higher than 14° C., at which the first experiment was tried.

† [The ammoniacal solution of this substance, which I have called turacoporphyrin, exhibits a bright red fluorescence which is not seen in a similar solution of turacin; the solution in dilute sulphuric acid also fluoresces red.—Postscript, September 1, 1892.]

that its spectrum bears a very close resemblance to that of alkaline hæmatoporphyrin. It has four well-marked absorption-bands (shown in fig. A), the wave-lengths of which, in millionths of a millimetre, are—1st band, λ 619 to 601 ; 2nd band, 577 to 550·5 ; 3rd band, 545 to 526 ; 4th band, 512·5 to 488. These bands differ in position and relative intensity and breadth from those of a similar solution of turacin (compare figs. 6 and 7). The resemblance of this new derivative of turacin to hæmatoporphyrin is strongly accentuated by the further observation, that the acid filtrate from the precipitate above described possesses a spectrum which Dr. MacMunn says has "a most remarkable resemblance to that of acid hæmatoporphyrin." It has a purple rather than a crimson hue ; and, though its spectrum has four bands (shown in fig. B), they differ in position and in relative intensity from those of the alkaline solution just described, and from those of an alkaline solution of turacin. Their wave-lengths are—1st band, 601 to 587 ; 2nd band, 579 to 569 ; 3rd band, 562 to 535 ; 4th band, 523 to 499 ; the 2nd band is faint, the 4th a slight shading only. This acid solution, when very nearly neutralised with ammonia, becomes colourless, depositing a flocculent precipitate, which proves to be identical with that previously separated on pouring the oil of vitriol solution into cold water. It is, in fact, soluble in some dilute acids, but not in pure water nor in solutions of neutral salts.

Fig. A

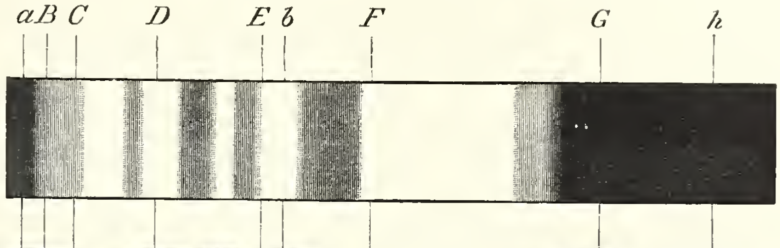
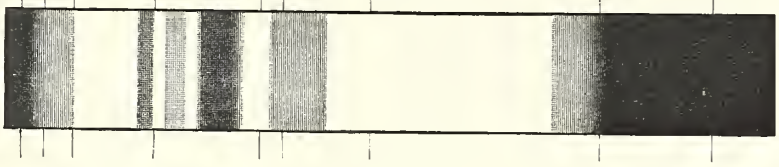


Fig. B



From the analyses which will be presently given, it seemed likely that the atomic ratio of metal to nitrogen in turacin might prove to be 1 : 4, as in hæmatin ; and it was, indeed, the resemblance between these two pigments which induced me to try and ascertain whether turacin would yield, by solution in oil of vitriol, a coloured metal-free derivative similar to that which hæmatin produces under the same circumstances. My expectation has, it will be seen, been partially realised ; but the new coloured derivative of turacin still retains much copper. Its solubility in dilute sulphuric acid and the spectra of its alkaline and acid solutions afford, however, strong corroborative evidence in favour of the view that the new derivative of turacin is nearly related to hæmatoporphyrin, and that turacin itself, though nearly related to hæmatin, has a somewhat more complex constitution.

My former description of the effects of heat upon turacin needs one important addition. The violet, or rather purple vapour given off when dry turacin is strongly and suddenly heated, consists of, or contains, an organic compound of which copper is an essential constituent. This vapour condenses on cooling into a partly crystalline sublimate, which, unlike turacin, is soluble in ether and in alcohol, but insoluble in dilute ammonia. It is of a reddish colour, and is obtained in the crystalline state by the spontaneous evaporation, in dry air, of its etherial solution. It well deserves further investigation, for the complete analysis of this derivative could hardly fail to throw light upon the constitution of turacin itself. For the same reason, the derived pigment obtained by means of sulphuric acid merits extended study; for this latter body I propose the name of turacoporphyrin.

If the very limited amount of turacin at my command has prevented me from making with it certain chemical and physical experiments which might have helped to elucidate its nature, and to determine its molecular weight, yet the study of its characteristic spectra has afforded invaluable aid in its recognition, especially in the case of the rarer species of *Turacus*. The chief features of these spectra were given in my former paper. The two well-marked absorption bands in the spectrum of turacin as it exists in the feather, closely resemble those of oxyhæmoglobin; the two chief bands in an alkaline turacin solution are not unlike those of CO-hæmoglobin. But it was desirable to investigate the spectroscopy of this unique pigment more thoroughly by examining solutions of different strengths, and prepared in different ways.

For the series of drawings of turacin-spectra reproduced on p. 519, I am indebted to the skill and kindness of Dr. C. A. MacMunn, of Wolverhampton. This able experimenter, whose acquaintance with the spectroscopy of animal pigments is unrivalled, gives the following explanatory notes concerning the chart of spectra.

The spectra were mapped by means of a one-prism chemical spectroscope made by Mr. Adam Hilger. The solutions were examined in a Preyer's hæmatinometer, which is provided with plane parallel glass sides one centimetre apart; the layer of solution examined was thus one centimetre thick.

Fig. 1. Solar spectrum with some of the Fraunhofer lines, the scale at the top being the arbitrary one of the instrument.

Fig. 2. Spectrum of the crimson web of a feather of *Musophaga violacea* with transmitted light. The feather was mounted in front of the slit, the light of a so-called "Sun" Argand gas-burner being condensed upon it it by means of a bull's-eye lens. All the remaining spectra figured were observed by means of the same light source. The following are the approximate measurements, in millionths of a millimetre, of the wave-lengths of the absorption-bands in the spectrum of this feather:—1st band, named α: shading begins at λ 599, is dark at 597, extends dark to 571, shades off to 567·5; centre is at 585 or 583. 2nd band, named β: is feebly shaded at 557, becomes dark at 553·5, extends dark to 529, is shaded off to 521·5; centre is at 538. The shading at the violet end begins about 510.

Fig. 3. Spectrum of a freshly-prepared aqueous and faintly ammoniacal solution, obtained directly from a feather. The measurements, in wave-lengths, of the bands in this spectrum were:—1st band, or α: shading begins at 579, is black at 575, extends back to 550·5, is shaded off to 548. 2nd band, or β: is shaded feebly at 540, is dark at 536·5, extends dark to 511, is shaded off to 505; centre of band $\alpha = 562$; centre of band $\beta = 523$. Note, also, the faintly-shaded band δ on either side of F; this extends from 496 to 475.

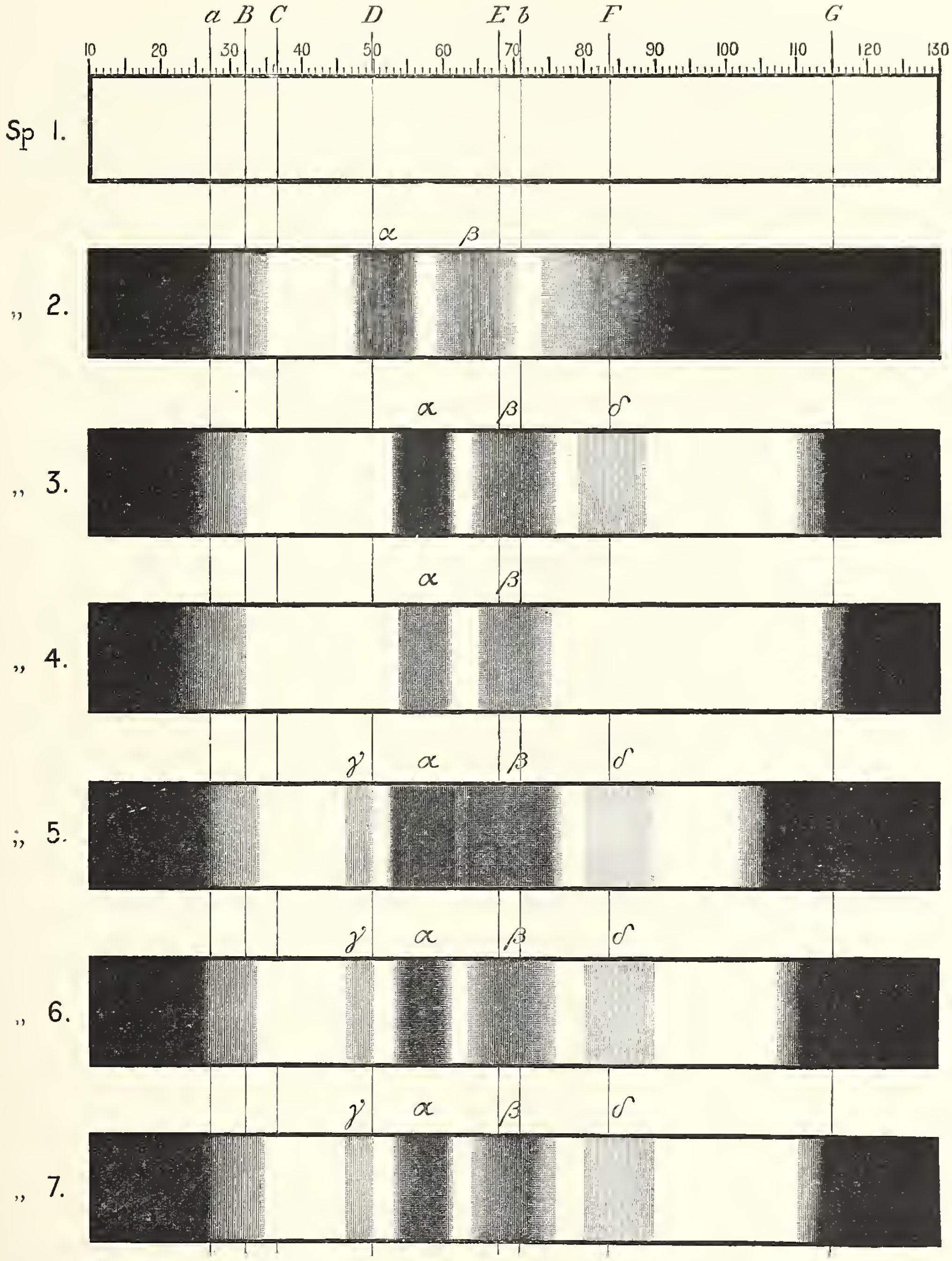


Fig. 4. Spectrum of a weaker solution prepared as above. 1st band, or α: shading begins at 577, is dark at 573, extends dark to 553·5, is shaded off to 549; centre of band $\alpha = 562$. 2nd band, or β: begins to be shaded at 538, is dark at 533, extends dark to 514, is shaded of to 506; centre of band $\beta = 523$.

Fig. 5. Spectrum of a strong solution of pure isolated turacin in faintly ammoniacal water. Note the occurrence, beside the band δ already described, of a fourth and stronger, though narrower, band γ near the line D. Measurements:—Band γ: 605 to 589: centre 597. Band α: shading begins at 581, is black 577, extends black to 547; there is a shaded space from 547 to 540. Band β; extends from 540 to 506, is shaded to 503. Band δ extends from 494 to 473.

Fig. 6. Spectrum of a weaker solution of turacin than that shown in fig. 5, but prepared in the same way. Measurements:—Band γ: 605 to 589. Band α: begins to be shaded at 577, is black at 573, extends black to 552, is shaded off to 547; centre at 562. Band β: begins to be shaded at 540, is dark at 533·5, extends dark to 511, is shaded off to 506; centre at 523. Band δ extends from 494 to 473.

Fig. 7. Spectrum of a solution, prepared in the year 1868, of turacin in weak ammonia water. Measurements: Band γ, 605 to 589. Band α: begins to be shaded at 579, is black at 573, extends black to 552, is shaded off to 549; centre at 562. Band β: begins to be shaded at 540, is dark at 535, extends dark to 514, is shaded off to 506; centre at 523. Band δ extends from 494 to 473.

[The bands γ and δ in the spectra are shown too dark in the figures—δ is, in fact, barely perceptible.]

The general results of these spectrum observations may be summarised thus:—Turacin, as it exists in the feathers, possesses two absorption bands, one of which, α, is rather darker and less wide than the other, and is situated about the line D, extending some distance on its more refrangible side; the other band, β, lies between the first-named and the line *b*. Turacin in alkaline solution always shows the same two bands, the first being, as before, darker and narrower than the second, but both bands are shifted towards the more refrangible end of the spectrum. When an alkaline solution of turacin is weak, it shows the above-named pair of bands only; when its strength is increased, a third band, δ, faint but broad, appears near the line F. But if some isolated and dried turacin be dissolved in weak ammonia water, a liquid is obtained which, if strong enough to show the band δ, will also show a fourth band, γ, darker and much narrower than δ, and situated on the less refrangible side of D. The recognition of the bands δ and γ is due to Dr. MacMunn; the remaining features of these spectra were figured in my previous memoir, but with less completeness and exactitude. A question arises as to whether the band γ—which is not seen in the spectra of turacin solutions freshly and directly prepared from the feathers, but only in the spectra of solutions of such turacin as had been previously isolated and dried—belongs to the unaltered pigment, or whether its occurrence is a sign of the presence of a decomposition or oxidation-product of turacin. The spectral position of the band is, I think, identical with that of the characteristic band (as figured in my previous paper) of "altered turacin," and with the band of KRUKENBERG'S turacoverdin. But the band γ is so much fainter and so much narrower than the bands in question, that it can betoken the existence of no more than a trace of an alteration-product in the carefully prepared sample of turacin with which the majority of the analyses here recorded were performed; the presence of such a trace can scarcely have affected in an appreciable degree the figures obtained. In connection with the probable presence of traces of an alteration-product of turacin in solutions made from the isolated pigment it is important to compare the spectra 6 and 7 in the cut, p. 519, they are virtually identical. There is no increase in the prominence of the suspicious γ band in the spectrum of a solution of isolated turacin in ammonia water after the lapse of twenty-three years. The particular specimen of turacin solution examined had been kept in a white glass bottle nearly full, but from which the stopper had been several times withdrawn; it had been exposed to moderately strong diffused daylight during fifteen years, but had been preserved in the dark for the eight years which immediately preceded its spectroscopic examination. Such a result is in striking contrast with that which is obtained when turacin, precipitated in flocculi by an acid from an alkaline solution, is allowed to remain for some time in contact with distilled water and atmospheric air. The precipitate loses its red hue, becoming dull and dark at first, and then acquiring a decided green colour. The same changes occur when a red Touraco feather is repeatedly wetted with pure water and dried, or when a piece of paper, stained with an ammoniacal solution of turacin, is kept for some time under the varying conditions of ordinary atmospheric moisture. The green turacin derivative produced in these experiments shows the spectral band γ with great distinctness, but it is always accompanied by the three bands proper to pure turacin in alkaline solution; this observation is true, even when every endeavour is made to complete the process of change.

Although the two spectra of turacoporphyrin described at the beginning of this section closely resemble the spectra of hæmatoporphyrin (obtained from hæmatin by the same treatment), yet the presence of a metallic constituent in the former derivative, and its absence from the latter is sufficient to prove that the two bodies are not exact analogues.

§ 5. *Nitrogen in Turacin.*

The production of the volatile derivative of turacin, when that pigment is strongly heated, is probably the cause of the difficulty experienced in determining correctly its percentage of nitrogen, this volatile product being formed in varying proportions, and being very difficult of combustion. The mean percentage of nitrogen given in my former paper was 6·38, a figure which, on a careful recalculation of the analytic data, comes out rather lower, and which was founded on three determinations by

DUMAS' method. A fourth determination by the same process had given me 7·01 per cent., a figure which I rejected at the time as not being in accord with the three other analyses, which agreed well with one another. Subsequently I made a large number of other "absolute" nitrogen determinations, with more discrepant results, some yielding figures a good deal higher, and others, again, a good deal lower than those previously obtained. Many of these analyses were performed with the same sample of turacin; in one instance only did the percentage obtained, 6·75, show a fair agreement with the earlier numbers. I then secured the aid of two expert analysts, accustomed to the use of all the modern refinements in apparatus and processes for the determination of nitrogen by the absolute method, and constantly performing analyses of the most varied and complex nitrogenous bodies. But their results, six in number, though all obtained with the same sample of turacin, agreed neither with one another nor with any of my own previous determinations, the highest figure in the whole set being only 6·15 per cent. I had previously tried once more the soda-lime combustion method, making two analyses, in a current of dry hydrogen, with the following results:—

	i.	ii.
	grm.	grm.
Turacin taken, corrected for ash	·2448	·478
$(NH_4)_2PtCl_6$ obtained	·2713	·5265
Platinum obtained from above salt	·1222	·233
Nitrogen percentage, calculated from platinum found (Pt = 194·8)	7·17	7 01

These results, it will be observed, tended to confirm the higher figures obtained in some of the DUMAS' analyses. I therefore made two more determinations by the soda-lime method, conducting the combustion, as before, in a current of hydrogen, but receiving the evolved ammonia in a standard sulphuric acid. On subsequent titration of the contents of the bulbs, the percentages of nitrogen obtained were 7·23 and 7·38. This volumetric estimation was difficult to complete, owing to the hue of the indicator being interefered with by that of some coloured distillation-products in the bulbs, and I do not attach much importance to these very high figures, the mean of which is no less than 7·3. It seemed, however, worth while to make further experiments with the modified soda-lime method, and Dr. F. E. MATTHEWS, of Cooper's Hill College, very kindly undertook to determine the nitrogen by this process, which he had brought to great perfection, and had practised, with undoubted success, in the case of substances very difficult to analyse satisfactorily. Dr. MATTHEWS purposely employed very small quantities of turacin, and he used the NESSLER method of estimating the evolved ammonia, but I have every reason to believe that his results are the most exact obtainable under the conditions named. He used part of the same sample of turacin (the whole weighed 4 grms.)

which had been employed in the six estimations of nitrogen by DUMAS' method, to which reference has already been made, as discrepant and low. The results obtained by Dr. MATTHEWS were as follows:—1 centigrm. of turacin, dried for 15 hours at 100° C., was mixed with soda-lime and burnt in a stream of hydrogen. The contents of the nitrogen-bulbs (previously faintly acidulated with hydrochloric acid), were diluted to 250 c.c., and then Nesslerized.

i. 50 c.c. required 13·5 c.c. of a solution of NH_4Cl, 1 c.c. of which = ·00001 gram of nitrogen; result = 6·75 per cent. N.
ii. 10 c.c. required 2·75 c.c.; result = 6·825 per cent. N.

A second combustion was made of the same amount of turacin, the same solutions being employed.

iii. 50 c.c. required 13·55 c.c. = 6·775 per cent. N.
iv. 10 c.c. required 2·75 c.c. = 6·87 per cent. N.

The mean of these four percentages is 6·8, or, when corrected for ash, 6·87. Before considering the value of this number it will be advisable to give some results obtained by means of KJELDAHL'S process. My friend, Professor KINCH, of Cirencester, having had great experience with this method, was good enough to make two estimations of nitrogen in turacin by its means, using the identical sample which had been employed by Dr. MATTHEWS. Here are the results:—

i. ·1718 grm. turacin gave ·01335015 grm. NH_3 = 6·399 per cent. N.
ii. ·3075 grm. turacin gave ·0236034 grm. NH_3 = 6·321 per cent. N.

The mean of these two results, when corrected for ash, becomes 6·419. In the face of the higher percentages obtained by other methods, and of the known difficulty in oxidising turacin completely, I incline to regard this figure as too low.

We are now in possession of five pairs of nitrogen determinations, all made with great care and by methods differing more or less in character. The percentages obtained, duly corrected for ash, are arranged in this table.

Series A.	Series B.	Series C.	Series D.	Series E.
By DUMAS' method.	By soda-lime and platinum.	By soda-lime and standard acid.	By soda-lime and Nessler.	By KJELDAHL'S method.
i. 7·01 ii. 6·82	i. 7·17 ii. 7·01	i. 7·23 ii. 7·38	i. 6·855 ii. 6·895	i. 6·453 ii. 6·384
Mean 6·92	7·09	7·30	6·87	6·419

The mean percentage deduced from the five series is 6·92. If Series C be omitted the percentage becomes 6·82; if Series E, it is 7·05; if both these series be excluded the percentage stands at 6·96. These three numbers are so near together that it seems hardly material which we accept, but after having given due weight to the various circumstances likely to affect the validity of the several results obtained, I incline to adopt the figure 6·96. I may here mention that this percentage of nitrogen corresponds to 7·7 per cent. of copper, if the ratio $N_4 : Cu$ be assumed, or to 7 per cent. of copper in the case of the ratio $N_9 : Cu_2$.

§ 6. *Copper in Turacin.*

For the copper-determinations recorded in my first paper (*loc. cit.*, p. 632) I was not able, through scarcity of material, to employ adequate amounts of turacin. But the differences between my earlier results and those which will be given presently arose mainly from a strange and unsuspected peculiarity of turacin. When the pure dry pigment is quickly and strongly heated it gives off, as I mentioned in my previous memoir, "a violet vapour resembling that of iodine." Now this vapour contains an organic copper compound, capable of being condensed into a red crystalline solid, which is soluble in ether and insoluble in alkalies—unlike the original body which yields it. If the turacin be slowly heated it gives off no visible or coloured vapours, even when the temperature is subsequently raised to a dull red heat. Here, then, is to be found the cause of the discrepancy between the six per cent. of copper found in my earlier determinations and the seven per cent. recorded below. In all the older analyses, the turacin, after having been dried at 100° C., was heated alone in a crucible, whereby it lost about one-sixth or one-seventh of its metallic constituent in the form of the violet (or rather, purple) vapour mentioned above. When it was found that turacin, if first submitted for ten minutes to the temperature of boiling mercury, gave off no visible vapours when afterwards more strongly heated or even incinerated, an easy method for estimating both the copper and the accidental ash-constituents of this pigment seemed available; accordingly, two quantitative experiments were carried out in the following way. The turacin was dried at 100° C. until constant in weight; it was then heated for one hour in a bath of boiling mercury, and finally incinerated over a Bunsen burner until the weight of the residue became constant. The total ash was weighed, and then the copper present in it was determined, first, by titration with potassium iodide and sodium thiosulphate, according to **Brown**'s method; and, secondly, by a gravimetric process, the iodine and the organic matter from the starch-indicator having been removed. The copper was precipitated as sulphide, and afterwards converted into and weighed as oxide. The total ash in the sample having been first of all ascertained in these analyses, an opportunity was afforded for the estimation of the fixed matter other than copper oxide left on the incineration of turacin. Here are the results:—

	i.	ii.
	grm.	grm.
Turacin taken, dried at 100° C.	·4627	·6832
Total ash obtained	·0454	·066
Turacin taken, corrected for ash	·4581	·6764
Copper, volumetrically determined	·0322088	·0464738
Copper oxide, gravimetrically determined	·0413	·0588
Copper, gravimetrically determined	·03297	·046936

These results correspond to the following percentages :—

	i.	ii.
Total ash	9·81	9·66
Ash other than CuO	1·0	1·09
Copper, volumetrically determined	7·03	6·87
Copper, gravimetrically determined	7·19	6·94

In the above calculations it has been assumed that the proportion of adventitious ash in turacin amounts to 1 per cent. This figure has been amply confirmed by other determinations, and has been adopted throughout the present paper as a correction.

The mean corrected percentage of copper in turacin, as deduced from the four determinations just given, is 7·01.* This figure, which corresponds to 8·79 per cent. of CuO, though higher than that obtained in my early experiments, is lower than that found by Mr. H. Bassett,† who by incinerating turacin with nitre and sodium carbonate obtained, in two analyses, oxide of copper equal to 7·6 and 8·0 per cent. of the metal. As no correction was made for ash these numbers are probably 1 per cent. below the truth, and may be safely taken to correspond to a mean percentage of 7·78. Now this number is confirmed by a determination which I made so long ago as September 14, 1874. The turacin employed was prepared from *Turacus corythaix,* and was purified and dried in the usual way. It was oxidised by long warming with concentrated nitric acid in a flask. The solution became at last of a clear green colour; it was cautiously evaporated to dryness and the residue incinerated. The residue was treated with nitric acid, and the solution filtered to remove a trace of silica. Finally the copper was precipitated by caustic potash and weighed as oxide. The figure obtained was :—

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* In another experiment ·2318 grm. turacin, oxidized by repeated treatments with fuming nitric acid, gave, by Brown's thiosulphate method, 6·87 per cent. of copper—a figure which when corrected for ash becomes 6·94 per cent. During the progress of this analysis and of another determination made in the same way, it was observed that turacin, if treated at first with an insufficient amount of nitric acid, yields an intermediate product having a "beetle-wing" lustre, and very difficult to oxidise by subsequent treatment with fresh portions of acid.

† 'Chemical News,' vol. 28, p. 201 (1873).

Turacin taken, corrected for ash	·3178 grm.
CuO obtained	·0323 ,,

This result corresponds to 8·11 per cent. of metallic copper. But on testing the black oxide of copper it was found to contain calcium phosphate and traces of iron and manganese.

A sample of turacin from *Musophaga violacea* was examined for copper by Professor Kinch, in November, 1891, the weighed substance being incinerated with every precaution in a crucible with sodium carbonate and potassium nitrate. The cupric oxide formed was dissolved in nitric acid, and precipitated by sodium hydrate, the precipitate being again dissolved, to remove silica, and re-precipitated.

The figures in this analysis were—

Turacin taken, corrected for ash	·1635 grm.
CuO obtained	·0168 ,,

This result corresponds to 8·2 per cent. of copper.

The mean percentage of copper in turacin, as deduced from the four determinations last recorded is 8·03. But there is a very good reason for not accepting this figure, for we have no guarantee that the precipitate produced by a fixed alkali in the solutions of turacin ash was pure. I conclude, indeed, that in Mr. Bassett's two analyses and in the pair just recorded, the apparent percentage of copper has been raised, because the substance weighed really included the calcium phosphate and the traces of oxides of manganese and iron known to be present in turacin ash, and regarded as accidental or adventitious constituents thereof. This conclusion is strengthened by the following argument:—If turacin contains 8 per cent. of copper it must yield, when so incinerated that no loss occurs, quite 10 per cent. of CuO, which, added to the 1 per cent. of extraneous ash ascertained to be present, amounts to 11 per cent. But the true percentage of total ash is just under 10—a figure based on numerous determinations, and confirmed by a recent result of Professor Kinch, who, by oxidising ·1522 grm. of turacin with nitric acid, evaporating the solution to dryness and incinerating the residue, obtained ·0149 grm. of total ash, equal to 9·79 per cent. If, then, turacin be a definite compound possessing but one essential metallic constituent, there cannot be more than about 7 per cent. of copper in it. It might perhaps be argued that the iron and manganese are not accidental impurities, and that turacin may, after all, contain 8 per cent. of metal, namely, about 7 parts of copper and 1 part of iron and manganese. This assumption is, however, untenable, for the oxides of iron and manganese found in the ash of all preparations of turacin constitute but a small part of the 1 per cent. of so-called *accidental* ash, and are accompanied by other bodies such as calcium phosphate. I conclude, therefore, that the volumetric determinations of copper are to be accepted in preference, and that the higher figures obtained by precipitation with sodium hydrate arose from the presence of impurities.

It may be well to state here that there is no evidence, chemical or spectroscopic, that turacin suffers any loss of copper when treated with the somewhat diluted hydrochloric acid used in its preparation, or even when boiled with a mixture of one volume of the fuming acid with one volume of water.

§ 7. *Carbon and Hydrogen in Turacin.*

The percentages of these elements, as deduced from my previously published analyses, were, respectively, 54·63 and 5·22. The results of combustions of the very carefully prepared sample of turacin with which the greater number of the analyses recorded in the present paper were made gave somewhat lower figures. The smaller proportion of carbon found may be due to the precautions taken to remove the last traces of fatty matter from the sample, and to decompose any oxides of nitrogen formed in the combustion; the great care exercised in excluding accidental moisture may have reduced the percentage of hydrogen. These are the analytical numbers :—

COMBUSTIONS of Turacin.

	Substance taken.	H_2O obtained.	CO_2 obtained.
	grm.	grm.	grm.
i.	·1362	·055	·2653
ii.	·2085	·0834	·4076
iii.	·1435	·0611	·2792

These numbers correspond to the following percentages :—

	Hydrogen.	Hydrogen corrected for ash.	Carbon.	Carbon corrected for ash.
i.	4·48	4·53	53·12	53·65
ii.	4·44	4·48	53·31	53·84
iii.	4·73	4·78	53·06	53·59

The mean corrected percentages are :—

Carbon	53·69
Hydrogen	4·60

§ 8. *Summary of Analytical Results.*

On the assumption that the only metallic constituent of turacin is copper its centesimal composition may be given as :—

Carbon	53·69
Hydrogen	4·60
Copper	7·01
Nitrogen	6·96
Oxygen	27·74

It is somewhat rash to suggest an empirical formula for a body which can neither be crystallised nor distilled, and to which few of the ordinary criteria of purity can be applied. The experimental percentages do not agree closely with any formula in which the ratio Cu : N_4 occurs—a ratio which analogy with hæmatin (which has Fe : N_4) would lead one to prefer, but they do correspond very nearly with the following more complex expression :—

Theory, $C_{82}H_{81}Cu_2N_9O_{32}$.		Experiment.
	Per cent.	Per cent.
$C_{82} = 984$	$= 53·78$	53·69
$H_{81} = 81$	$= 4·43$	4·60
$Cu_2 = 126·6$	$= 6·92$	7·01
$N_9 = 126$	$= 6·89$	6·96
$O_{32} = 512$	$= 27·98$	27·74

The agreement of these theoretical and experimental numbers is satisfactory. Although I should have preferred a simpler empirical formula, and the atomic ratio between metal and nitrogen of 1 : 4 rather than the ratio 2 : 9, yet there exist two arguments in favour of a more complex expression. One of these arguments is based on the *partial* loss of copper which turacin suffers when strongly heated—a loss which seems to indicate that the whole of the copper present does not exist in the same state of combination. The other argument is founded on the *partial* retention of copper by turacoporphyrin. This shows that its formation does not proceed upon exactly the same lines as those of hæmatoporphyrin, which retains none of the metallic constituent present in its parent-substance, hæmatin.

I wish it to be understood that I lay no stress upon the formula which I have suggested, regarding it merely as a mode of expressing the results of analysing a substance which I believe to be constant in composition and very nearly pure. And, in the absence of sufficient data of control, I do not think it worth while to compare the experimental percentages with those demanded by such simpler formulæ as—$C_{40}H_{40}CuN_4O_{16}$; $C_{40}H_{40}CuN_4O_{15}$; $C_{40}H_{38}CuN_4O_{16}$; and $C_{40}H_{38}CuN_4O_{15}$.

§ 9. *Summary and Conclusions.*

The more important positions established by the present research are these :—

I. The constant occurrence, in 18 out of 25 known species of Musophagidæ, of a definite organic pigment containing as an essential constituent about 7 per cent. of metallic copper.

II. The "turacin-bearers" comprise all the known species of the three genera, *Turacus, Gallirex,* and *Musophaga,* while from all the species of the three remaining genera of the family Musophagidæ, namely *Corythæola, Schizorhis,* and *Gymnoschizorhis,* turacin is absent. Furthermore, the zoological arrangement of the genera constituting this family is in accord with that founded on the presence of turacin.

III. The spectrum of turacin in alkaline solution shows, besides the two dark absorption bands previously figured, a faint, broad band on either side of line F, and extending from λ 496 to λ 475.

IV. The spectrum of *isolated* turacin in ammoniacal solution shows, besides the three bands already named, a narrow fourth band, lying on the less refrangible side of line D, and extending from λ 605 to λ 589. It probably arises from the presence of traces of the green alteration-product of turacin formed during the preparation of that pigment in the isolated condition, an alteration-product which is likely to prove identical with KRUKENBERG'S turacoverdin.

V. Turacin in ammoniacal solution remains unchanged after the lapse of 23 years.

VI. Turacin in the dry state, when suddenly and strongly heated, yields a volatile, copper-containing red derivative, which, though undissolved by weak ammonia-water, is not only soluble in, but may be crystallised from ether.

VII. Turacin in the dry state, when heated in a tube surrounded by the vapour of boiling mercury, becomes black, gives off no visible vapour, is rendered insoluble in alkaline liquids, and is so profoundly changed that it evolves no visible vapour when afterwards strongly heated.

VIII. The percentage composition of turacin is probably carbon 53·69, hydrogen 4·6, copper 7·01, nitrogen 6·96, and oxygen 27·74; turacin may possibly be represented by the empirical formula, $C_{82}H_{81}Cu_2N_9O_{32}$.

IX. Turacin presents some analogies with hæmatin, and yields by solution in oil of vitriol a coloured derivative having a spectrum much resembling that of hæmatoporphyrin, the corresponding derivative of hæmatin, but retaining, unlike hæmatoporphyrin, part of the metallic constituent of the parent-substance.

The rarity of turacin and the singular difficulty experienced in burning it completely may, I hope, be regarded as furnishing a legitimate apology for offering to the Society a long discussion of analytical results, for the lack of absolute decisiveness in the evidence brought forward as to the centesimal composition of the pigment, for the limited information obtained as to the behaviour of turacin with reagents, and as to its relationships and derivatives. When one looks back upon the steps by which the true formula of so definite and abundant a crystalline pigment as alizarin was finally established; when one recalls the mystery still shrouding hæmoglobin and chlorophyll, some excuse may perhaps be allowed for my failure to accomplish more towards the elucidation of a colouring matter so anomalous and costly as turacin. It is, however, certain that the chemical and physiological interest belonging to this pigment amply warrant its further and more thorough investigation.

In conclusion I have to express my thanks to the Royal Society for a handsome money grant in aid of the prosecution of this inquiry. My acknowledgments are also due to the several friends whose names have been recorded in the present paper, especially to Dr. McMunn, whose help in the spectroscopy of turacin has been invaluable.

XIV. *Colour Photometry.*—Part III.

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Received December 14, 1891,—Read January 28, 1892.

§ XLIV.—*Measurement of Luminosity.*[*]

In the paper on Colour Photometry (Bakerian Lecture, 1886) a curve of luminosity of the spectrum of the light from the "crater" of the positive pole of the electric (arc) light was given.

The apparatus used for measuring the luminosity was described in that paper, and certain modifications afterwards made were described in the appendix, and in a further paper with the same title in 'Phil. Trans.,' 1888.

Shortly, the arrangement of the apparatus was as follows : a collimator, two prisms, and lens were used to form a spectrum ; a second lens, placed a little obliquely, re-combined the rays so as to form a white patch 3 inches square on a screen. A slide, having a slit in it, being placed in the spectrum, any ray could be selected and made to fall on the patch.

The beam of white light reflected from the surface of the first prism was, by an arrangement of mirror and lens, made to fall on the same patch. By placing an upright rod in the path of the two beams, one half of the patch was illuminated by the monochromatic ray, and the other by the beam of white light, which, for convenience, we call the "reference" beam, as it has been used throughout our late observations as the standard of reference. The relative luminosity of the two beams could then be compared by reducing one or other by the rotating sectors until the two halves of the patch appeared of equal brightness, the aperture of the sectors being a measure of the proportional brightness of the two beams.

The patch of light was viewed at a distance of somewhere about three feet, its image thus occupied an angular field on the retina of 5°. As all the observations referred to in both papers, whether taken by ourselves or by others, were made with the same apparatus and under similar circumstances, they were strictly comparable

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[*] The numbering of the paragraphs and figures in this paper is a continuation of that of Parts I. and II., 'Phil. Trans.,' 1886 and 1888.

with each other, and the angular dimensions of the patch had not to be taken into account.

An extension of the measurements to embrace a part of the physiological aspect of colour has, however, necessitated a slight modification of the apparatus with which a new series of observations have been made. The length of the spectrum, before re-combination, has been more than doubled by using a lens of greater focal length than formerly for collecting the rays proceeding from the prisms. The size of the white surface on which the shadows are received has been reduced to $1\frac{1}{4}$ inch square, and the observations now recorded were made from a distance of 4 feet from it. This allows the image of the whole of the patch, when viewed direct, to fall on the middle of the yellow spot of the eye, which occupies a central position in the retina, and has an approximate angular aperture of from 6° to 8°. As the absorption of the yellow spot diminishes towards its boundary, it follows that, within certain limits, the smaller the patch that is viewed the greater will be the loss of luminosity in that part of the spectrum where the absorption takes place. From observations made in the manner shortly to be described, it was found that, for our eyes, there was no sensible difference in the results when the two shadows fell on the white square of $1\frac{1}{2}$-inch side, or a square of 1-inch side, but that, if the side were increased to 2 inches, the measures differed slightly but unmistakably. This will account for the fact that the original luminosity curve slightly differs from that now recorded for the centre of the eye, as part of the image of the larger patch must have fallen on the less absorbing part of the yellow spot. It is not quite apparent why the eye should not distinguish between the differing luminosities of the different parts of the shadows, but it is probable that the average luminosity was observed.

Table I., Col. IV., gives the observed measurements, and fig. 33 gives the curve A plotted from them.

Equalisation of the luminosities of the coloured and white shadows was effected by opening and closing the rotating sectors (which were described in the former paper) in the white beam of light, and confirmation of the measurement was obtained by setting the sectors at fixed angles, thus cutting off definite proportions of white light and shifting the slide carrying the slit which traverses the spectrum till equality of illumination was obtained at each angle so set.

From our experience we believe that the most accurate measurements are those made by altering the angular aperture of the sectors during rotation; as to produce a certain difference of luminosity a greater motion of the hand is required on the lever of the rotating sectors than on the slit in the spectrum. Near the place of maximum luminosity the latter plan fails, as pointed out in our previous paper.

§ XLV.—*Absorption of the Yellow Spot.*

Though it was not the first inquiry which was undertaken, it will be well thus early to record the method by which the character and amount of the absorption of the yellow spot was ascertained.

A white spot, very feebly illuminated, was placed six inches from the patch on which the beams to be compared were thrown. One eye was closed and the other directed centrally to the white spot, the observer being at a distance of 4 feet from it. The image of the patch was thus received on a part of the retina beyond the boundary of the yellow spot. It may appear strange to others, as it did to ourselves at first, that the luminosities of the two shadows could be compared with almost greater facility than they could be when looked at centrally. When a comparison was to be made, the presence of colour often appeared not exactly to vanish but to offer no difficulty to the reading. The luminosity was thus determined, and it was found practically that the same curve was obtained in whatever angular position the white spot was placed, so long as it was six inches from the patch. The luminosities of the colours on the patch were also measured when looking directly at them. Any difference between the readings by the eye in the two cases showed a lessened or increased sensitiveness of the central part of the retina under observation for the particular colour. Table I, Col. III., and fig. 33, curve B, gives the results of these observations.

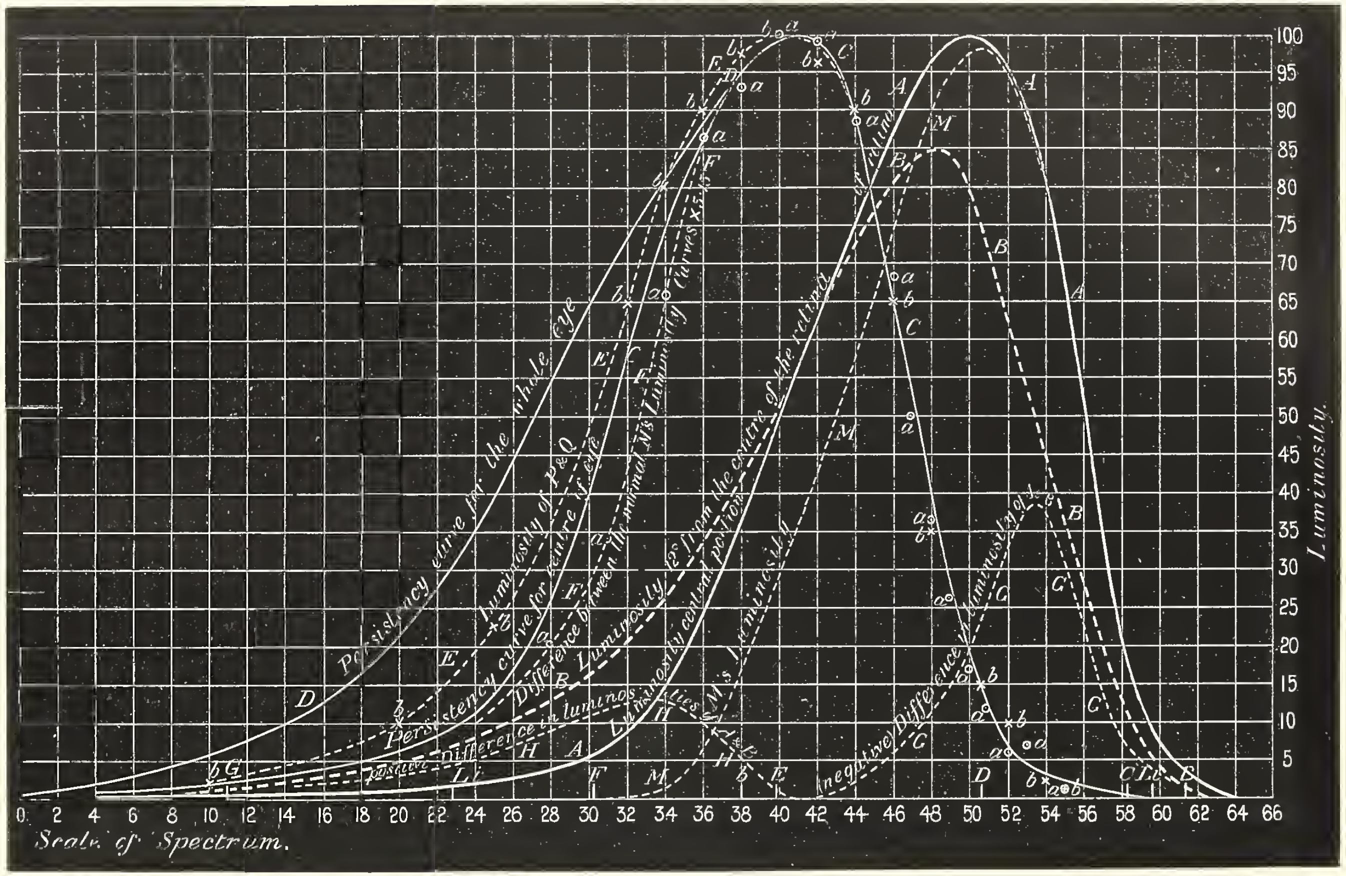
If two square patches of $1\frac{1}{2}$-inch side are placed six inches apart, and illuminated with white light of the same intensity, and one be looked at centrally, the image of the other will fall outside the yellow spot.

By diminishing the illumination of one or other, the two may be rendered equally bright to the parts of the retina used, and by first looking at one centrally then at the other, two sets of observations can be obtained. Adopting this plan, and after a large number of estimations (it was necessary to take a large number owing to the difficulty of the observation), it was found that the relative sensitiveness for white light of the centre of the retina to that of the outer part was approximately as 37 to 33. The areas of the curves of luminosity plotted from the readings are in the ratio of 167 to 156, which is so nearly the same ratio that each of their ordinates may be taken to indicate the relative amounts of light seen by either part of the retina in the different parts of the spectrum.

Whilst there is, as might be expected, an increase in the luminosity to the outer part of the retina of the portion of the spectrum from about E to the violet end, over that to the central part of the retina, it is remarkable that the reverse is the case with respect to the portion from the green to the red. Evidently, therefore, the outer part of the retina is less sensitive than the central part to the less refrangible rays of the spectrum. The curve for this part of the retina is very similar to that obtained from the observations made with the centre of the eye by persons who have a slightly shortened spectrum, and who are, therefore, what is termed partially red-blind.

It should be noted that the luminosity curve given in our former paper, and which

Fig. 33.



a.a. are observations derived from the luminosity curves of the normal eye and of *M*.
b.b. are actual observations of luminosity by *P* and *Q*.
A, the luminosity of the spectrum as seen when using the yellow spots.
B, " " " " by the retina 10° outside from the centre of the retina.
C, the persistency curve of the normal eye from observations taken with the centre of the retina.
D, " " " " using all the retina.
E, the luminosity curve of *P* and *Q*.
F, the difference between *A* and *M* magnified 5·15 times.
M, the luminosity curve of *M*.
H, the absorption by the yellow spot.
G. the excess of luminosity of the least refrangible end of the spectrum when viewed by the central part of the retina over that when viewed 10° from it.

was made from observations in which the image of the colour patch covered the yellow spot and some of the outer part of the retina as well, lies between the curves A, B, of fig. 33.

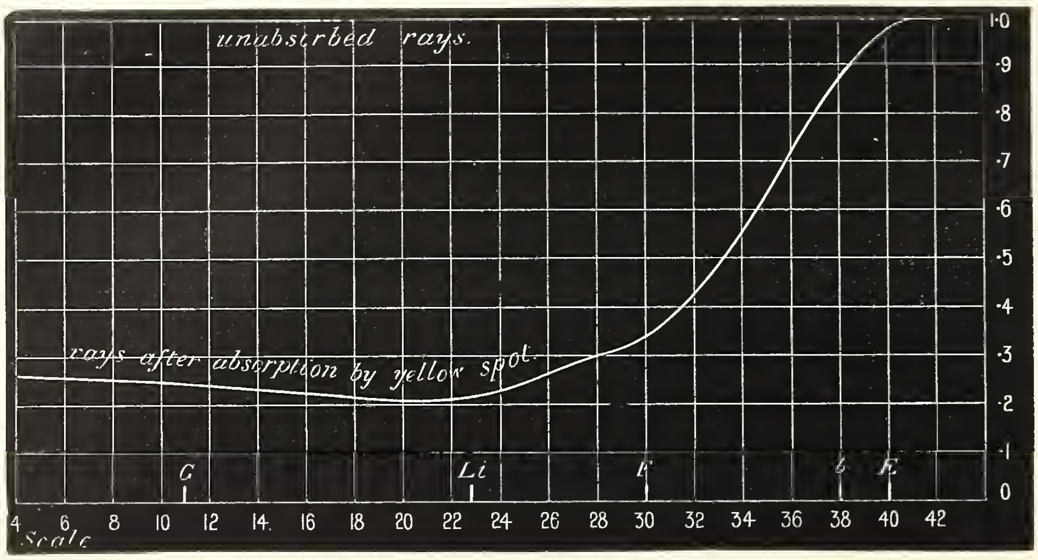
The curve H in fig. 33 shows the absolute absorption of the rays between the green and violet by the yellow spot. Fig. 34 gives the proportionate absorption of the same. The colour of the absorbing medium in the yellow spot can be shown on the screen by using a template cut out in the way described in Colour Photometry, Part II., § XXXV.

Table I.—Luminosity Curves.

I.	II.	III.	IV.	V.	I.	II.	III.	IV.	V.
Scale number.	Wave-length.	Outside yellow spot.	Yellow spot.	Fovea centralis.	Scale number.	Wave-length.	Outside yellow spot.	Yellow spot.	Fovea centralis.
64	7217				32	4924	21	8·5	6·5
63	7082	..	1		31	4885	18·5	7·0	5·5
62	6957	1	2	2	30	4848	16·5	5·5	4·0
61	6839	2	4	4	29	4812	14·5	4·7	3·5
60	6728	3·5	7	8	28	4776	13·0	4·0	3·0
59	6621	7·5	12·5	15·5	27	4742	11·5	3·5	2·0
58	6520	12·5	21	24	26	4707	10·5	2·8	2·4
57	6423	19	33	37·5	25	4675	9·4	2·3	2·1
56	6330	27·5	50	60	24	4639	8·2	1·82	1·9
55	6242	35	65	77	23	4608	7·3	1·6	1·5
54	6152	43	80	90	22	4578	6·3	1·4	
53	6074	52·5	90	97	21	4548	5·7	1·2	
52	5996	61·0	96	100	20	4517	5·0	1·08	1·0
51	5219	71·0	99	100	19	4488	4·5	·94	
50	5850	79·0	100	98	18	4459	4·0	·86	
49	5873	84	99	95	17	4437	3·6	·78	
48	4720	85	97	90	16	4404	3·1	·70	
47	5658	83·5	92·5	85	15	4377	2·7	·62	·62
46	5596	81·0	87	79	14	4349	2·3	·56	
45	5538	77·0	81	72·5	13	4323	2·1	·50	
44	5481	72·5	75	66	12	4296	1·9	·45	
43	5427	68·0	69	59	11	4271	1·65	·40	
42	5373	62·5	62·5	51	10	4245	1·4	·34	
41	5321	57	57	45	9	4221	1·2	·30	
40	5270	52	50	40	8	4197	1·0	·26	
39	5221	46	42·5	32	7	4174	·88	·22	
38	5172	41·5	36	27·5	6	4151	·75	·18	
37	5128	37·5	29·5	22·0	5	4131	·63	·16	
36	5085	33·5	24	18	4	4106	·50	·14	
35	5043	30·0	18·2	14	3				
34	5002	26·5	14·2	10	2				
33	4963	24	10·5	8·4	1				

The following are the scale numbers of the different fiduciary Fraunhofer and bright lines :—B 61·3, Li 59·7, C 58·1, D 50·6, E 39·8, *b* 28, F 30·2, Li (blue) 22·8, G 11·1.

Fig. 34.



Absorption by the macula lutea.

§ XLVI.—*The Fovea Centralis.*[*]

The question of visual sensation at the fovea centralis, if that be admitted to be coincident with the visual axis of the eye, has occupied our attention, and we have thought it worth while to give the measures of luminosity when the images of the illuminated shadows fell on this portion of the retina. A cube of $\frac{1}{4}$-inch side was prepared, and the beams of light allowed to fall on it in the usual manner. The luminosities of the white and coloured shadows were equalised when they were observed at a distance of 60 inches from the eye. One eye was closed during the observations. The measures made are given in Table I., Col. V. It will be noticed the fovea is rather more sensitive to the red rays than the macula lutea, and is in general much less sensitive to the green rays. A calculation of the areas of the curves of luminosity shows that the fovea is $\frac{1}{6}$ more sensitive to D light than the macula lutea as a whole. It is somewhat remarkable that the sensitiveness to green and blue of the fovea is not greater, and is even less at certain places, than of the macula lutea, considering the almost entire absence of pigment from the former.

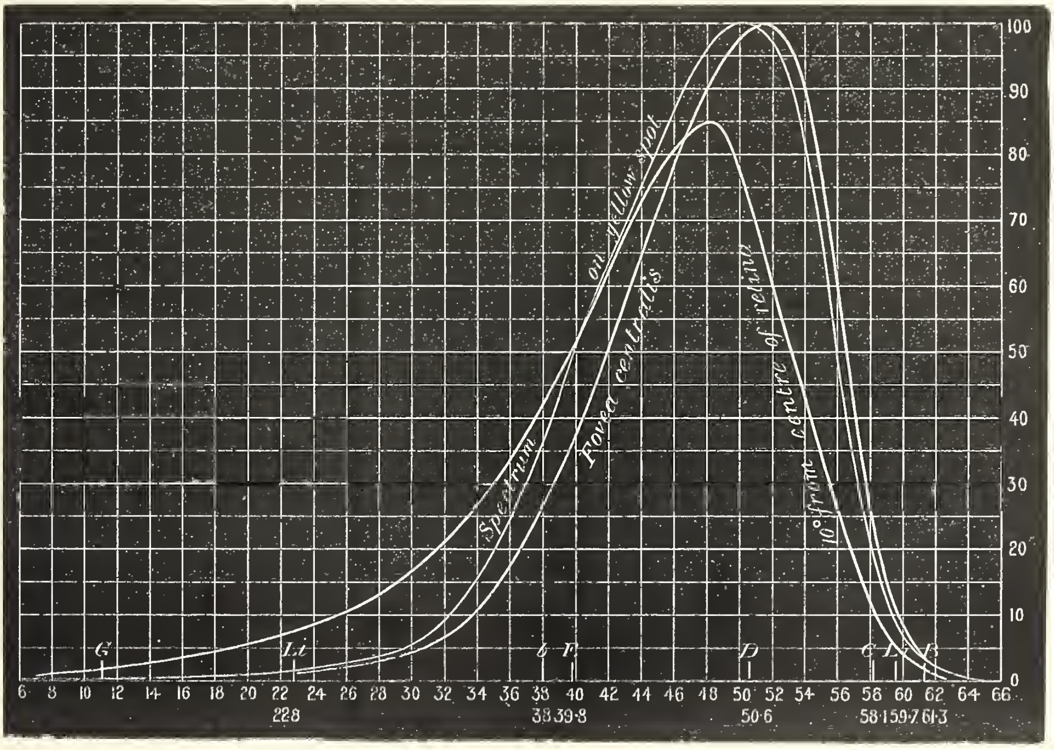
If the small cube be examined at still further distances there is a still further increase in the luminosity of the red and a further decrease in that of the green. What the limit may be where no further change takes place we are not at present prepared to say. If a star or a distant light be observed in the point where the visual axis of the eye cuts the retina, and then on the part of the retina slightly removed from this point, the different colour of the images will be evident.

[*] Added July 20.

§ XLVII.—*The Limit of Colour Vision.*

It is well-known that as light of any colour becomes enfeebled the eye fails to see colour, though it can recognise the presence of light. From a physical as well as from a physiological point of view, it appeared to be of interest to ascertain the amount of illumination of a screen at which all appreciation of colour in the different rays of the spectrum disappeared, leaving a sensation of what, for want of a better word, we may call *grey* light. In order to ascertain this, an apparatus (fig. 36) was devised as a supplement to that already described, by which a white light of very low intensity could be compared with the spectrum colours.

Fig. 35.

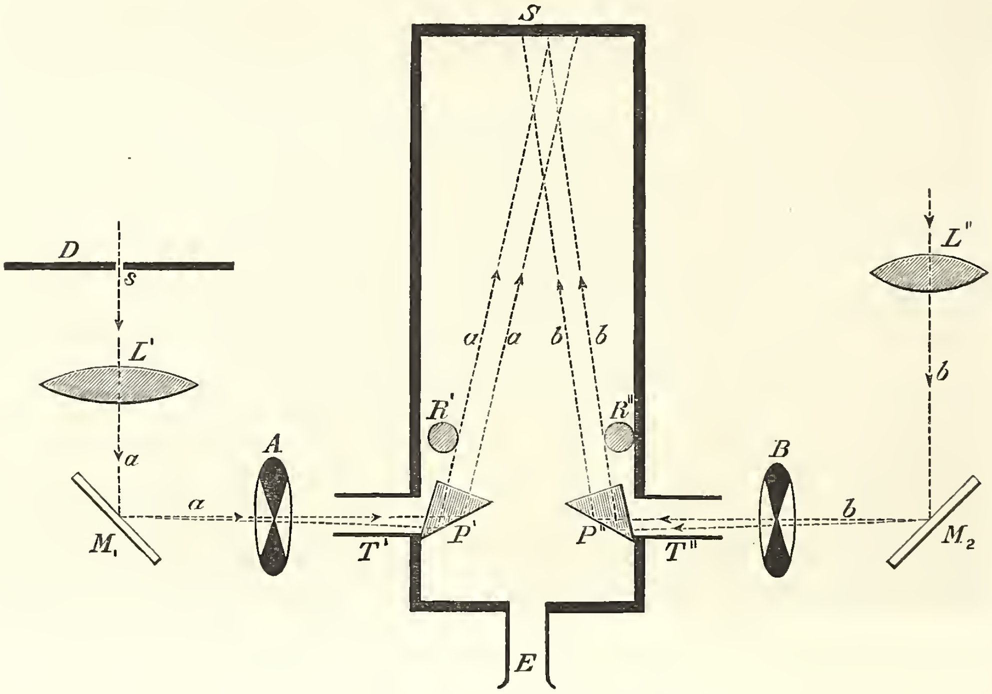


Curves showing the luminosity of the spectrum when measured (1) with the fovea centralis, (2) with the area of the yellow spot or macula lutea, (3) with the retina 10° from the fovea centralis and outside the macula lutea.

At one end of a box, shown in plan, is an eye-piece E. The other end has at its centre a patch S, $1\frac{1}{2}$ inches square, whitened with zinc oxide, the rest of the inside of the box being blackened. The monochromatic beam a coming from the spectrum through the side slit, and the reference beam b, are reflected by *plain* glass mirrors M_1M_2 to apertures in opposite sides of the box, and from just inside these apertures, by right-angled prisms P_1P_2 so as to fall on and cover S. Rods R_1R_2 are inserted in the box in the paths of the beams so that they illuminate opposite halves of S.

Diaphragms inside the box cut off any stray rays of light, and rotating sectors placed at *A* and *B* regulate the strength of the beams. The room containing the apparatus is darkened. The sectors *A* are closed until no colour is discernible in the monochromatic beam, whilst the intensity of the white beam regulated by the sectors *B* gives the standard of whiteness to which the coloured beam is to be reduced. It is worthy of notice that when the white beam is entirely cut off, or made very feeble, colour often seems absent from the monochromatic light, but is again perceived when the beam is brightened. This is especially the case with the red part of the spectrum. The strength of the coloured beam was therefore always reduced to the point that no colour was apparent whatever was the strength of the white beam. The aperture of the sector *A* was noted for each colour. The direct measurement of such a feeble light would be very difficult, the luminosity was therefore determined in the following manner. The box and sectors were removed, and a white screen was placed at the same distance from *M* that *S* was. The card carrying the slit in the spectrum was also removed so that a patch of white light was received on the screen, the luminosity of this was measured by direct comparison with an amyl-acetate lamp. The mirror M_1 was next removed, and the beam then fell on the screen of the original apparatus. Its luminosity was then compared with the reference beam. The slit slide being put back in the spectrum, the luminosity of the D light was measured against the same comparison light. The proportion that the luminosity of the D light bore to the

Fig. 36.



Apparatus to measure the limit of colour vision.

re-combined white patch was thus determined. As the value of the white light reflected from M to the end of the box was known from the first observation, the luminosity of the D light so reflected was calculated. The luminosity of the D light having been found, that of all the other rays was calculated from the luminosity curve derived from observations made with the central portion of the retina (see fig. 33, A), as it was with this part that the observations now being described were made.

The actual value of each ray when the colour disappeared was calculated from the aperture of the sectors.

TABLE II.—Limit of Colour Vision.

Scale number.	Wave-length.	Mean reading of the colour limit of the spectrum D, being 1 amyl lamp in $\frac{1}{10000}$ths.	Luminosity of the ordinary spectrum.	Luminosity of the rays when each colour disappears, each ray having the original luminosity of 1 amyl lamp in $\frac{1}{100000}$ths.
61	6839	120	4	48·0
60	6728	67	7	46·9
58	6520	26	21	54·6
56	6330	13	50	65·0
54	6152	9·5	80	76·0
52	5996	9·0	96	86·4
50	5850	9·0	100	90·0
48	5720	9·0	97	87·3
44	5481	9·5	75	71·3
40	5270	10·5	50	52·5
36	5085	12·5	24	30·0
32	4924	18	8·5	15·3
28	4776	32	4·0	12·8
24	4639	55	1·8	12·0
20	4517	90	1·08	9·7
16	4404	160	·70	11·2
12	4296	250	·45	11·0
8	4197	400	·26	10·4
4	4106	700	·14	9·8

In fig. 37 the continuous curve is constructed from these observations,* and the dotted curve B is that derived from curve A, supposing that each ray had an original luminosity of one amyl light at the distance of 1 foot.

It will be seen that the colour of the central portion of the spectrum is discernible with much greater reduction of light than is that of the extremities. This accounts for the fact that objects illuminated by moonlight appear of a greenish hue. The light from the full moon, as is well known, is somewhere about half a million times less bright than that of the sun, or about $\frac{1}{100}$ of an amyl lamp at 1 foot. The figure

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* The extreme left end being plotted to a different scale so as to bring it within the paper.

shows that at this point most of the spectrum has lost its colour. The faint spectrum when re-combined will be pale green, mixed, of course, with a fair proportion of grey, due to those portions of the spectrum from which the colour has disappeared.

Fig. 37.

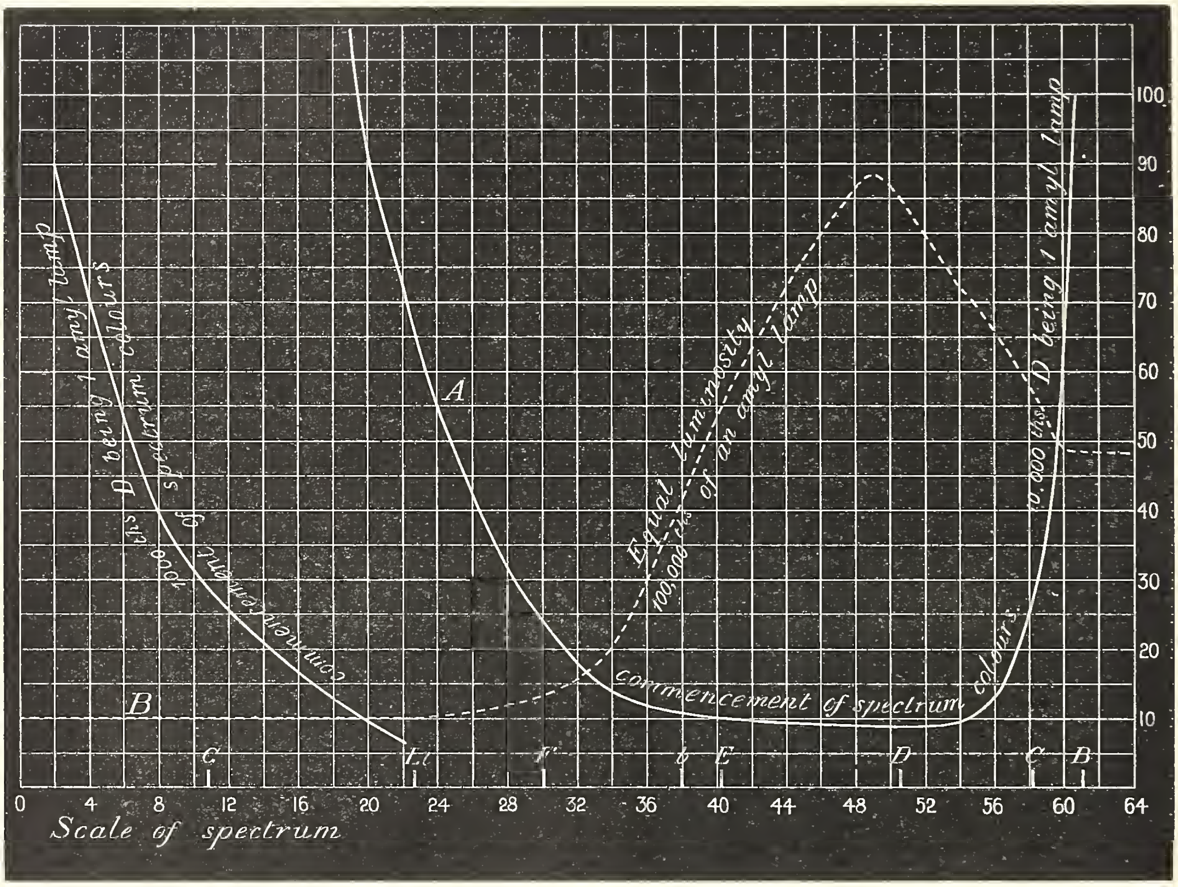


Diagram of limit of colour vision.

§ XLVIII.—*Extinction of the Light of Different Parts of the Spectrum.*

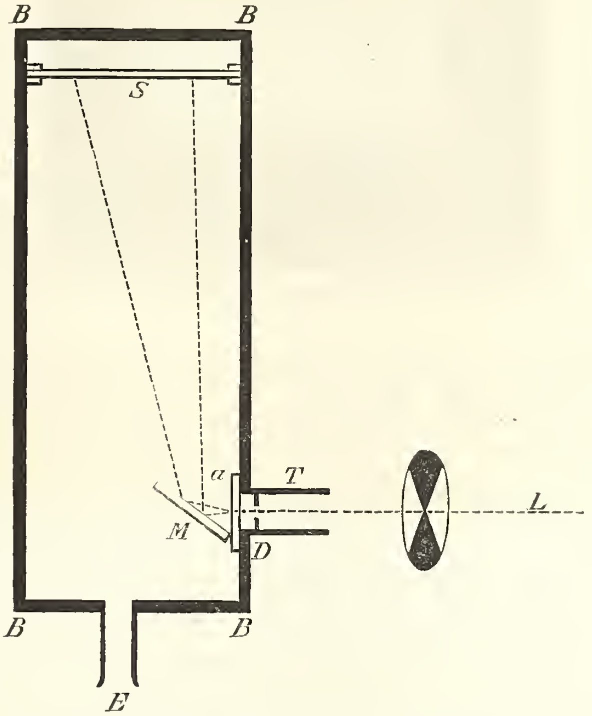
A preliminary note has already been published by one of us (Captain Abney) on this subject. It was expected that the results then given might be liable to correction, as in work of this kind it is only after repeated experiment that sources of error can be discovered and eliminated. The errors which have since been discovered by comparing the results of some hundreds of observations are not serious, though their correction alters the extinction curve to some small extent.

In the first experiments the readings of the scale had to be made by the observer, and it has been found that the light used to illuminate the scale, small though it was, fatigued the eye sufficiently to vitiate, to a small but sensible extent, the observations. This source of error has, in the latter experiments, been eliminated.

The necessity of resting the eye for some time in darkness in order to give it the full sensitiveness to feeble light was soon recognised. When extinction has been made and the instrument left untouched, if the eye was exposed to the light of day for some time, and then an observation was made, even after two minutes rest, no light from any ray of the spectrum was visible in the extinction box, even with the sectors removed; after a further rest of two minutes the rays last to be extinguished could be perceived; and finally, after about ten minutes' rest, the eye became of the same sensitiveness as before it was exposed. When several successive extinction readings of the same ray agreed, it was considered that the eye was in a fit state to commence a series of observations.

The apparatus used was usually of the form described below, but variations in its arrangement and in the methods of observations were made from time to time, in order to track out any possible error.

Fig. 38.

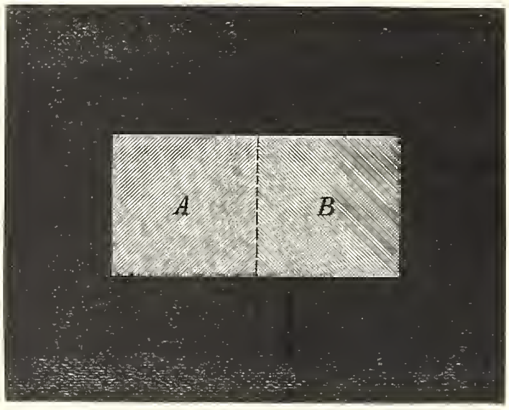


Apparatus to measure extinction of light.

BB (fig. 38), is a closed box 3 feet long and about 1 foot wide and 1 foot high, having two circular apertures $1\frac{1}{2}$ inches in diameter in the positions shown. The aperture at the side is covered on the inside by a piece of glass, *a*, finely ground on both sides, and a tube, *T*, is inserted in which diaphragms, *D*, of any required aperture can be inserted. *E* is a tube fixed into the other aperture, and should for comfort be fitted with an end shaped to receive the eye, as the observations are made through it. *S* is a cardboard screen inserted from the top of the box, the aperture being rendered light-tight by a batten. The screen is black except one circular patch which can be altered at pleasure in colour or size, but which in the experiments now to be described was white and $\frac{3}{4}$ inch in diameter.

When using this instrument the beam to be extinguished was directed through the tube *T* and diaphragm *D* on to the ground glass by which it was diffused. A portion of the diffused beam was reflected by the mirror *M* to the white patch on the screen at *S*. By altering the diaphragm *D* the amount of light falling on *S* can be varied at pleasure, and it can be still further regulated by putting the rotating sectors in the path of the incident beam outside *T*.

Fig. 39.



Screen for measuring illumination.

The point of extinction was observed as follows. The slits of the collimator and of the slide were closed to convenient widths, and the light was subsequently diminished by inserting diaphragms. Two methods of extinction were tried, (1) The slit traversing the spectrum was moved until the ray was found which was just extinguished with each diaphragm; and (2) after placing the slit in fixed positions in the spectrum at a known ray the light was diminished by the rotating sectors as well as by the diaphragms. The latter is evidently the more convenient plan, but both were fully tried in order to determine whether the method of reducing the light by the rotating sectors could be relied on in experiments of this nature. The agreement between the results obtained, which was as close as could be expected in such experiments, convinced us of the trustworthiness of the latter method.

A means had to be devised by which a beam of sufficient intensity to be easily measured could be reduced to the point of extinction, and the proportion in which it had been reduced ascertained. The slit slide was taken away from the spectrum, and the intensity of the re-combined beam determined in terms of the light of the amyl-acetate lamp as follows. A card (fig. 39) was pierced with a square aperture *B*, as shown, and a piece of Saxe paper pasted over the whole. A black paper mask was then applied, so as to leave *B* and an equal area *A* visible. From one side the paper appeared as a white oblong, though *B* was translucent and *A* opaque. The re-combined beam was then allowed to fall on the back of the card, and *B* became illuminated. On the other side of the card an amyl-acetate lamp illuminated *A*, *B* being screened by a rod in the path of the light. The brightness of *A* and *B* were made equal by placing the rotating sectors in the path of the beam of the amyl lamp, and thus the proportion of light passing through the paper was measured in terms of that reflected from a white surface. The card then replaced the screen *S* in the box, the end of the box was opened, and *A* and *B* were exposed to view. The light of the re-combined beam was directed on to *T* so as to illuminate *S*. *A* was made equally bright by the amyl-acetate lamp and the illumination calculated; knowing from the former observation what proportion of the light falling on *B* is visible from the other side, the amount of light falling on the screen, and, therefore, its proportion to that received at *T* could be determined. Measurements were taken with each diaphragm, and the illumination of the screen in terms of the light received at *T*, was found to be proportional to the areas of the apertures, as might be expected, and, as follows, for the diaphragms used :—

No. 0, $\frac{1}{90}$; No. 1, $\frac{1}{155}$; No. 2, $\frac{1}{208}$; No. 3, $\frac{1}{270}$; No. 4, $\frac{1}{478}$; No. 5, $\frac{1}{620}$; No. 6, $\frac{1}{956}$; No. 7, $\frac{1}{2430}$.*

The method of diminishing the illumination of the screen by ground-glass was found to be most effective. A beam of monochromatic light from the brightest part of the spectrum can be diminished to such an extent as to come within the limits of extinction by the rotating sectors, with the apertures of such an angular dimension as to be properly read (say more than 6°).

The D light coming through the spectrum slit was measured against an amyl lamp by placing a white opaque screen at the aperture *a* (the tube *T* being removed). The luminosity of the D light being thus known, that of any other ray could be calculated from the curve *A* in fig. 33. Another method of observation was as follows: a diaphragm with a small circular aperture was placed in front of the last prism of the apparatus. The patch of light on the screen was now a small circular disc, instead of being square, as before. A similar box was prepared to that of fig. 38, but the ground-glass was omitted. The ray of light now falling on *M* formed a circular patch on the screen *S*, but the beam of light so formed is too powerful to be extinguished by any readable aperture of the rotating sectors, it was therefore further reduced by placing in its path, and at an angle of 45° to it, two parallel mirrors *A*, *B* (see fig. 40). Each mirror can be either silvered or plain glass; three combinations of different reducing powers are therefore possible, viz. : (*a*) both mirrors silvered, (*b*) one plain and one silvered, (*c*) both plain.

The proportion of the light reflected with each combination can be readily deter-

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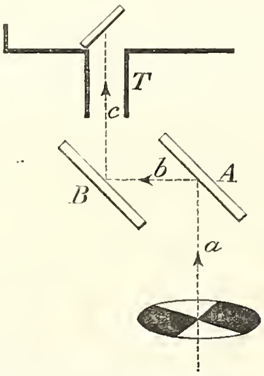
* [This method of measuring the ratio of light falling on the screen to that on the ground glass has been subsequently modified, the two being directly compared one with the other.—July 20.]

mined. When the last was used the intensity of *c* was almost exactly $\frac{1}{100}$ of that of *a*. As the rotating sectors gave a further extreme reduction of, say, $\frac{1}{15}$, *a* could be used of a manageable intensity.

Fig. 40.



Method of double reflection into extinction box.

When employing this method the collecting lens in front of the spectrum was so adjusted, that the re-combined beam from the whole spectrum formed a circular spot on *S*, the position of the spot of light on *S* was, therefore, the same for all parts of the spectrum.

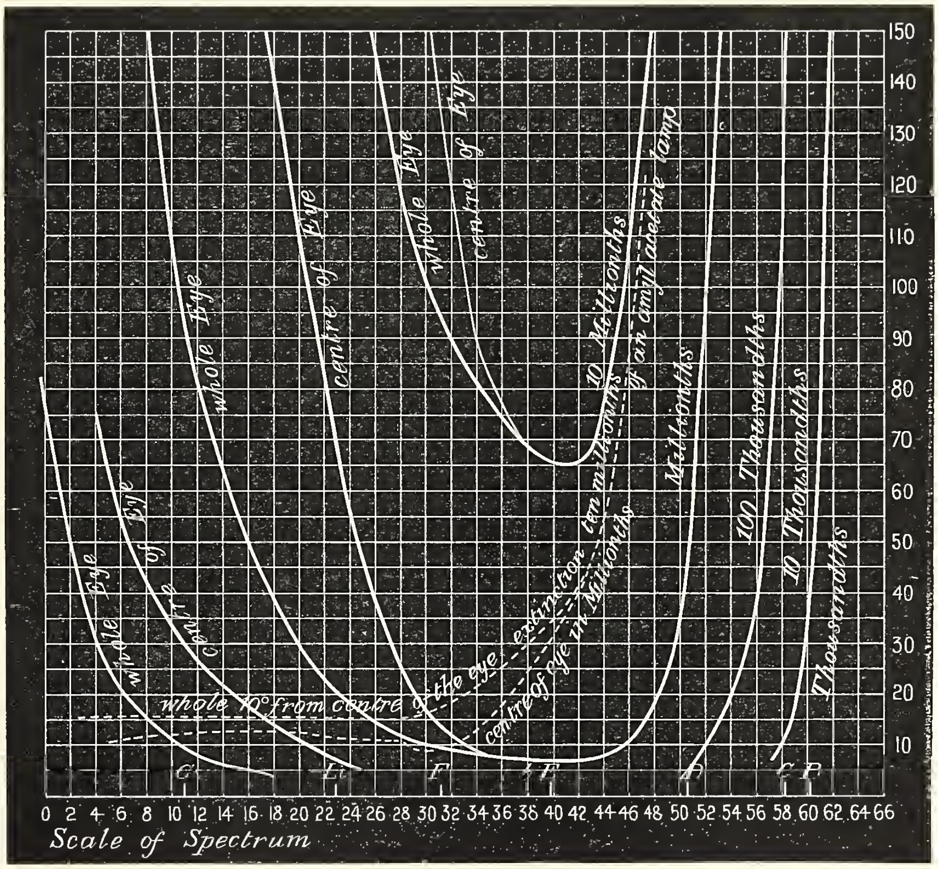
The absolute luminosity of the beam from D of the spectrum was measured by placing an open screen at the same distance from the mirror *M* (fig. 38) that *S* was, two silvered mirrors being used at *A* and *B*, and using the amyl-acetate lamp for comparison. The absolute luminosities of beams from other parts of the spectrum were then calculated from this by means of the luminosity curves, fig. 33.

The results obtained by using the rotating sectors with this apparatus were also tested by the method before described, and were found to be perfectly trustworthy.

From the observations made, a curve was plotted showing what was the proportion of the beam from each part of the spectrum which was just not visible. The absolute luminosity of each part of the spectrum having been determined in the way explained above, a second curve was plotted of which the ordinates represent the absolute luminosity of each part of the spectrum at the extinction point, or, in other words, the proportion which would be just not visible, supposing that each part had been originally of the uniform luminosity of, say, one candle. This curve rose from the blue-green towards the red, when, after reaching a maximum, it tended to drop again. There appeared to be a similar irregularity at the violet end. It was suspected that these irregularities might be caused by some admixture of white light due to want of perfect transparency of the prisms, and further investigation showed that this was the case, and that when this stray white light was eliminated the curve became of the form shown by the dotted line, fig. 41.

Helmholtz's plan of dispersing this white light was first tried. A prism was placed in the path of the beam from the collecting lens at such a distance that the beam filled the prism, and by using a second lens the faint, continuous spectrum so formed was cut off. This plan was found, however, to be too complicated, and was abandoned for the simpler one of using absorbing media. A combination of "cobalt blue" and "signal green" glass was used for the violet end of the spectrum, and "stained-red" glass—*i.e.*, glass flashed on one side with copper, and on the other with gold—for the red end.

Fig. 41.



Extinction curves of normal eye.

The continuous line curves show the proportion of the beam from each part of the spectrum which is just not visible, the illumination by the beam from D when unreduced being equal to that of one amyl-acetate lamp at one foot from a screen.

The dotted curves show the proportion, supposing that all beams had equal intensity to that of D.

The luminosity of each beam after passing the medium was determined, also the proportion left when it was reduced so as just to extinguish the light, the product of the numbers representing these quantities would evidently represent the absolute luminosity at the point of extinction, or, in other words, the proportion left on the supposition of a uniform luminosity for all parts of the spectrum.

Tables III. and IV. give the results of these observations with which the observations of Table V. are combined. The figures in the third column represent the proportion to which each beam was reduced at extinction, those in the fourth column the absolute luminosity of the beam. The last column gives the products of these two quantities, which are the luminosities at the extinction point.

TABLE III.—Extinction by Central Portion of Normal Eye.

I.	II.	III.	IV.	V.	VI.
Scale number.	Wave-length.	E. Reduction of original luminosity in millionths to cause extinction.	L. Luminosity of original beam.	$\frac{E \times L}{100}$.	Persistency curve $\frac{650}{E}$ (Maximum = 100).
64	7,217	55,000			
63	7,082	30,000	1	300·0	
62	7,957	15,000	2	300·0	
61	6,839	7,500	4	300·0	
60	6,728	3,750	7	262·5	
59	6,621	1,900	12·5	237·5	·34
58	6,520	1,050	21	220·5	·62
57	6,423	650	33	214·5	1·0
56	6,330	380	50	190·0	1·71
55	6,242	272	65	176·8	2·38
54	6,152	196	80	156·0	3·32
53	6,074	140	90	126·0	4·64
52	5,996	97	96	93·12	6·70
51	5,919	57	99	56·43	11·40
50	5,850	35	100	35·0	18·6
49	5,783	24	99	23·76	27·1
48	5,720	17	97	16·49	38·2
47	5,658	12·6	92·5	11·65	51·6
46	5,596	10·2	87	8·87	63·7
45	5,538	8·6	81	6·97	75·6
44	5,481	7·4	75	5·55	87·8
43	5,427	6·7	69	4·62	97·0
42	5,373	6·55	62·5	4·09	99·5
41	5,321	6·5	57	3·705	100
40	5,270	6·55	50	3·27	98·5
39	5,221	6·65	42·5	2·83	97·5
38	5,172	6·85	36	2·46	95·0
37	5,128	7·2	29·5	2·12	90·0
36	5,085	7·6	24	1·82	81·3
35	5,043	8·15	18·2	1·48	80·0
34	5,002	8·8	14·2	1·25	74·0
33	4,963	10·2	10·5	1·07	63·0
32	4,924	11·6	8·5	·988	56·0
31	4.885	13·6	7·0	·952	47·7
30	4,848	16·3	5·5	·896	40·0
29	4,812	20·5	4·7	·963	31·7
28	4,776	26·0	4·0	1·040	25·0
27	4,742	31·0	3·5	1·085	20·9
26	4,707	38·5	2·8	1·078	16·9
25	4,674	46·0	2·3	1·058	14·1
24	4,639	56·0	1·82	1·019	11·6
23	4,608	67·0	1·6	1·072	9·7

TABLE III. (continued).

I.	II.	III.	IV.	V.	VI.
Scale number.	Wave-length.	E. Reduction of original luminosity in millionths to cause extinction.	L. Luminosity of original beam.	$\frac{E \times L}{100}$.	Persistency curve $\frac{650}{E}$ (Maximum = 100).
22	4578	80	1·4	1·120	8·41
21	4548	95	1·2	1·140	7·22
20	4517	107	1·08	1·156	6·1
19	4488	124	·94	1·165	5·23
18	4459	140	·86	1·204	4·64
17	4437	160	·78	1·228	4·1
16	4404	180	·70	1·260	3·60
15	4377	200	·62	1·240	3·25
14	4349	220	·56	1·232	2·95
13	4323	240	·50	1·200	2·7
12	4296	270	·45	1·215	2·4
11	4271	300	·40	1·200	2·18
10	4245	335	·34	1·139	1·94
9	4221	375	·30	1·125	1·73
8	4197	430	·26	1·118	1·51
7	4174	490	·22	1·078	1·32
6	4151	510	·18	·918	1·27
5	4131	640	·16	1·024	1·01
4	4106	750	·14	1·050	0·86

TABLE IV.—Extinction by Whole Eye.

I.	II	III.	IV.	V.	VI.
Scale number.	Wave-length.	E. Reduction of original luminosity in millionths to cause extinction.	L. Luminosity of original beam.	$\frac{E \times L}{160}$.	Persistency curve $\frac{650}{E}$ (Maximum = 100).
38	5172	6·9	41·5	2·86	94·2
37	5128	7·1	37·5	2·66	91·6
36	5085	7·4	33·5	2·48	87·8
35	5043	7·7	30·0	2·31	84·4
34	5002	8·0	26·5	2·12	81·2
33	4963	8·4	24·0	2·02	77·5
32	4924	8·8	21·0	1·85	73·8
31	4885	9·4	18·5	1·74	69·2
30	4848	10·0	16·5	1·65	65·0
29	4812	10·7	14·5	1·55	60·6
28	4776	11·5	13·0	1·49	56·5
27	4742	13·0	11·5	1·49	50·0
26	4707	14·5	10·5	1·52	44·8
24	4639	18·5	8·2	1·52	34·1
22	4578	23·0	6·3	1·45	28·3
20	4517	30·0	5·0	1·50	21·7
18	4459	39·0	4·0	1·56	16·7
16	4404	51	3·1	1·59	12·3
14	4349	66	2·3	1·52	9·85
12	4296	80	1·9	1·52	8·12
10	4245	110	1·4	1·54	5·91
8	4197	154	1·0	1·54	4·22
6	4151	204	·75	1·54	3·18
4	4106	307	·5	1·54	2·11
2	4063	513	·3	1·54	1·26
0	4020	770	·2	1·54	·84

Table V.—Extinction Curves of Light transmitted through Blue and Red Glasses.

	Scale number.	Wave length.	Reduction to produce extinction, E. (Sector readings).	Comparative Luminosities of unreduced beams, L.	E × L.
through blue glass.	14·5	4363	4	148	592
	13·5	4336	4·5	132	594
	12·5	4309	5	120	600
	11·5	4284	5·5	108	594
	10·5	4258	6·2	96	595
	9·5	4233	7·0	86	602
	8·5	4209	8·0	76	608
	7·5	4186	8·8	68	594
	6·5	4163	9·8	62	617
	5·5	4141	11·0	56	616
	4·5	4118	12·0	50	600
	3·5	4094	13	45	585
	2·5	4070	16	37	592
	1·5	4047	19	35·5	598
	·5	4025	25	24	600
	– ·5	4023	35	17	595
through red glass.	64	7212	73	8	584
	63·5	7147	45	13	585
	62·5	7022	25	23	575
	61·5	6900	12	49	588
	60·5	6785	5·5	101	555
	59·5	6675	3	168	504

From Table V. it is seen that from the extreme violet end of the spectrum, to No. 14·5, the luminosities of extinction are practically the same; in fact the curve at this part is horizontal; the same is the case with regard to the part between scale No. 61·5 and the extreme red end of the spectrum.

This seems to confirm the view that the colour sensation of the eye for each of these parts is a simple one. These results have been, as already said, incorporated in Tables III. and IV.

In the diagrams, fig. 41, two curves of extinction are given. One shows what proportion of the beam, at different parts of the spectrum, is just not visible to the central portion of the eye, the other, the curve with regard to the *whole* eye. These curves correspond with each other, except where the absorption by the yellow spot takes place. The part of the retina which appears most sensitive to the light of this part of the spectrum is about 20° below the centre, and about 45° from the vertical line. The light is certainly most persistent at this point.

If the reciprocals of the ordinates of either of the curves just referred to—that is to say, of the reduction of the beam at different points—be taken as ordinates, the curve so constructed may be called a "persistency" curve, and should relate to some colour sensation in our eyes. Such curves (with ordinates so reduced that the maximum is 100) were constructed for the centre of the eye, and for the whole eye (fig. 33, C, D). Column VI. in the tables gives the ordinates.

§ XLIX.—*Extinction of Light to the Colour Blind.*

In a paper in the 'Proc. Roy. Soc.' (1891) one of us has given a curve of luminosity of the spectrum as seen by two brothers whose sensation was monochromatic. On comparing this with our persistency curve for the centre of the eye (which must represent the luminosity of the spectrum to some sensation which we have), we were surprised to find that the curves corresponded except in the yellow spot absorption portion, when ours fell below theirs, see fig. 33. It, therefore, appears that the sensation of the two brothers very nearly corresponds with what must be the dominant sensation in our eyes.

Before commenting on this, it will be of interest to give a further confirmation of the existence of this one sensation. A gentleman, whom we will call M., had his vision tested. His is a case we have never tested before, and is most remarkable. The only two colours he saw are what he called red and black. He called all green and blue black, green however he called bright black, blue being described as a darker black. Yellow he called white. At 52 on the scale he saw a "little red," at 50 "no colour"; his neutral point—if it may be so called—or the point where he saw the spectrum colourless, would be about 495 or about 5800.

His luminosity curve is given at M, fig. 33. The following is the table from which it was plotted. The mean readings being multiplied by 1·8, the curve of luminosity of the red part of the spectrum almost exactly coincides with that of the authors.

Column I. gives the scale number, II. the wave-length, III. the actual mean reading, IV. the last × 1·8, V. the ordinates of the normal luminosity curve of the central part of the eye, VI. the difference between M.'s curve and the normal luminosity curve, whilst VII. gives the difference multiplied by 5·15 to bring the maximum to 100 for comparison with other curves. It will be seen that this curve (F., fig. 33) very nearly coincides with our persistency curve, except in the part of the spectrum affected by the yellow spot.

No measures were taken to ascertain if the eye of M. had any central absorption, and, therefore, we do not know what correction should be made to the curve to make it comparable with the others; but taken as it is, it is remarkable how closely this, which represents the deficiency in M.'s sensations, corresponds with the one sensation of either of the brothers. In fact, it seems as if the eyes of M. and P. together would make up a pair of normal eyes.

TABLE VI.—M.'s Luminosity Curve compared with the Normal.

I.	II.	III.	IV.	V.	VI.	VII.
Scale number.	Wave-length.	Mean reading.	Mean reading × 1·8.	Normal luminosity curve, centre of eye.	Difference of last two columns.	Difference × 5·15.
61	6839	2	3·6	4	·4	2·57
59	6621	7	12·6	12·5	−·1	·51
57	6423	18	32·4	33	+·6	3·09
55	6242	36	64·8	65	·2	1·03
53	6074	49	88·2	89·5	1·3	6·71
52	5996	52	95·4	96·5	1·1	5·66
51	5919	54	97·2	99·5	2·3	11·8
50	5850	54	97·2	100	2·8	14·4
49	5782	52·5	94·5	99·5	5·0	25·7
48	5720	50	90	97	7·0	36·0
47	5658	46	82·8	92·5	9·7	49·9
46	5596	41	73·8	87	13·2	68·0
44	5481	32	57·6	75	17·4	89
42	5373	23	43·2	62·5	19·3	99
40	5270	17	30·6	50	19·4	100
38	5172	10	17·5	35·5	18	93
36	5085	4	7·2	24	16·8	86·5
34	5002	1·0	1·8	14·5	12·7	65·5
31	4885	·5	·7	6·5	5·8	37·7
28	4776	0	0	4	4	20·6

A further examination into cases of colour-blindness cannot fail to be interesting, and appears to us to throw considerable light on the subject of colour vision.

Several red and green colour-blind people have been tested in the manner described, but the difficulty in many cases of inducing them to note whether the observations of extinction were made with the whole eye or the central part only, was very great, and there has, therefore, been some uncertainty as to the results.

We give, however, the results in three cases which may be considered typical, and in which the observations appear to have been extremely well made. The first (H. R.) is red blind, the second (V. H.) green blind, the third (P.) has monochromatic vision. The first two were educated men who understood exactly what they had to look for, the last (one of the brothers P. and Q.) was an excellent observer, sharp and intelligent, and anxious to help on the experimenter. (See Tables VII., VIII., and IX.)

We are aware that the YOUNG-HELMHOLTZ theory of vision is open to criticism from certain points of view, but we adopt it tentatively as being at least convenient.

On this theory we should expect, if the monochromatic vision of the third was supposed to consist of the blue (or violet) sensation, that all three of the observers would give approximately the same curves in the most refrangible part of the spectrum, since in the red blind and green blind this same sensation may be supposed to be existent. That the monochromatic sensation is blue, and corresponds to the dominant sensation in the normal eye seems to be fairly probable. A glance at the extinction curves of the spectrum shows how similar in many respects P.'s is to those of H. R. and V. H., as well as to that of the normal eye. It is also worth remarking that if a bright red and blue be mixed together by rays coming through two slits placed in the spectrum, so as to form a reddish purple, the red sensation is extinguished some time before the blue pales to any great degree, and in that part of the spectrum where the existence of this monochromatic sensation is evident, the last colour visible is always bluish even when very faint, whilst in the yellow part of the spectrum there seems a tendency before extinction for a greenish hue to appear, whereas, almost to the moment of extinction in the extreme red, the colour is of a ruddy grey. The strongest evidence, however, is to be found in the persistency curves of the red and green blind, which only slightly differ from one another and from that of the normal eye, and to an extent which might be expected from the nature of the observations. The persistency curve (fig. 43) of V. H. differs but little in any respect from the normal, and this tends to show that the persistency is far greater in the blue sensation than in the green, in other words, that the green part of the spectrum excites the blue sensation in the normal eye after the light has been so much reduced that the green sensation has ceased to be excited. V. H. is the first case in which the total absence of a green sensation is an established fact, and the probable luminosity of such green sensation is derived by subtracting the ordinates of his curve from those of the curve of the normal eye. That H. R. is not *totally* red blind, we have on several occasions had the opportunity of proving. He has a slight perception of red, and hence the difference between his curve and that of the normal eye cannot be treated in the same manner, as it would not represent the luminosity of the red sensation in its entirety. Tables VI., VII., VIII. give the observations made by P., V. H., and H. R. respectively, and figs. 43, 44, 45 give their luminosity and extinction curves, together with the normal luminosity curve for the centre of the eye.

The persistency of the blue sensation, or, we might say, perhaps, of the sensation which is confined principally to the most refrangible part of the visible spectrum, is very remarkable, and affords some clue to the reason of the disappearance of the red and green before the blue in cases of colour-blindness induced by disease.

We believe it probable that, adopting the YOUNG-HELMHOLTZ theory, the three colour sensations obtained from these observations by colour-blind people can be made to form the luminosity curve of the normal eye, and, at the same time, to be in accordance with the colour equations which have been found by CLERK MAXWELL, as well as by ourselves.

M.'s observations of extinction were sometimes erratic, and we therefore cannot make much use of the results. But they appear to afford proof that his dominant sensation is not more than $\frac{1}{180}$ as powerful as that of the normal eye, and may even be considerably less. An inspection of his results also shows that the extinction of the part of the spectrum in the red very closely resembles that of the normal eye in character and in amount. What is M.'s dominant sensation is not quite apparent, for, although he described the light between E and D as "white" when of ordinary intensity, yet he averred that it always appeared ruddy at the moment of extinction.

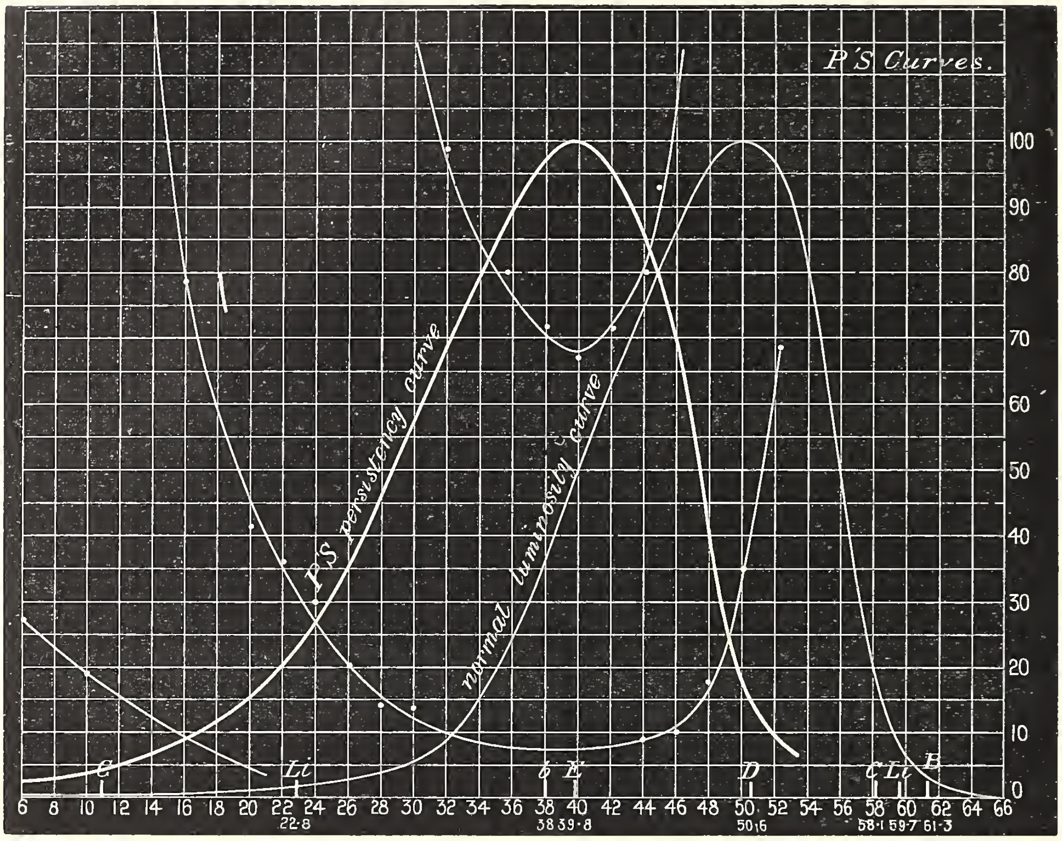
TABLE VII.—P.'s Curves.*

I.	II.	III.	IV.	V.	VI.	VII.
Scale number.	Wave-length.	Mean reading of extinction in millionths of original luminosity.	Adopted reading in millionths of original luminosity.	Persistency curve $\frac{680}{\text{ad. reading}}$.	P.'s luminosity curve.	Absolute luminosity of extinction $\frac{\text{IV.} \times \text{VI.}}{14}$.
52	5996	68	68	10	7·	34
50	5850	35	35	19·4	19·	47·5
48	5720	17	17	40	39	47·3
46	5596	10·2	10	68	65	46·4
45	5538	9·3	9·0	76	76	48·8
44	5481	8·0	8·1	84	90	52·8
42	5373	7·2	7·2	94·5	98	50·3
40	5270	6·7	6·8	100	99	48·1
38	5172	7·2	7·0	97	97·5	48·7
36	5085	8·05	7·7	90	90	49·5
34	5002	8·05	8·4	81	80	47·9
32	4924	9·9	9·8	69	65	45·5
30	4848	13·2	12·5	54	50	44·6
28	4776	13·9	15·0	45·3	36	38·6
27	4742	16·8	17·0	40	31·5	38·2
26	4707	21·6	20·5	32	26·5	38·8
24	4639	30	27	25	19·5	37·6
22	4578	36	35	19	14	35
20	4517	42	45	15·5	10	32·2
16	4404	79	79	8·5	5·5	31·2
10	4245	180	190	3·6	2·5	32·2
6	4151	270	270	2·7		

---

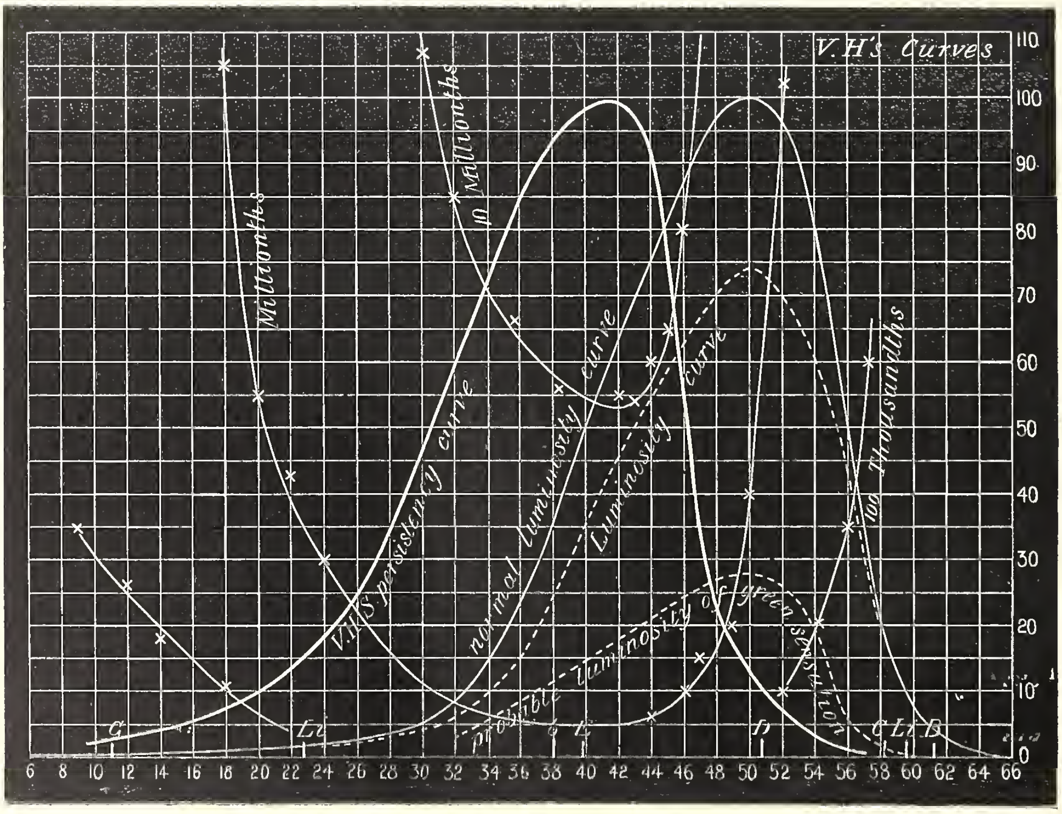
* In this and the next two Tables the intensity of the illumination of the D ray before reduction is equal to that of an amyl-acetate lamp at one foot from a screen. The figures in Col. VII. are in millionths of the illumination of an amyl-acetate lamp at one foot distant, every ray being made of that intensity.

Fig. 42.



Extinction curve of monochromatic vision.

Fig. 43



Extinction and luminosity curves of a green blind.

Table VIII.—V. H.'s Curves.

I.	II.	III.	IV.	V.	VI.	VII.
Scale number.	Wave-length.	Mean reading of extinction in millionths of original luminosity.	Adopted reading in millionths of original luminosity.	Persistency curve $\frac{530}{\text{ad. reading}}$.	Luminosity curve.	Absolute luminosity of extinction $\frac{\text{IV.} \times \text{VI.}}{75}$.
57	6423	500	500	1·1	31	206
56	6330	350	350	1·5	43	200
54	6152	200	180	2·9	61	146·4
52	5996	100	100	5·3	70	93·3
50	5850	40	40	13·3	73	38·9
48	5720	..	25	21·2	69	23
46	5596	10	10	53·0	63	8·4
45	5538	6·5	6·5	81·6	58	5·0
44	5481	6·0	5·7	93	54	4·1
42	5373	5·5	5·3	100	46	3·3
40	5270	5·5	5·4	98·2	36	2·6
38	5172	5·7	5·7	93	24	1·8
36	5085	6·7	6·5	81·6	15	1·3
34	5002	7·0	7·0	75·7	9·5	·89
32	4924	8·5	8·5	62·3	7·0	·79
30	4848	10·7	10·5	50·5	5·0	·70
28	4776	16	16	33·1	3·7	·79
26	4707	..	22·5	23·5	2·7	·81
24	4639	30	31	17·1	1·82	·75
22	4578	42·5	42	12·6	1·4	·78
20	4517	55	55	9·6	1·0	·73
16	4404	105	100	5·3	·7	·93
12	4296	175	170	3·1	·45	1·02
10	4245	200	200	2·7	·34	·91

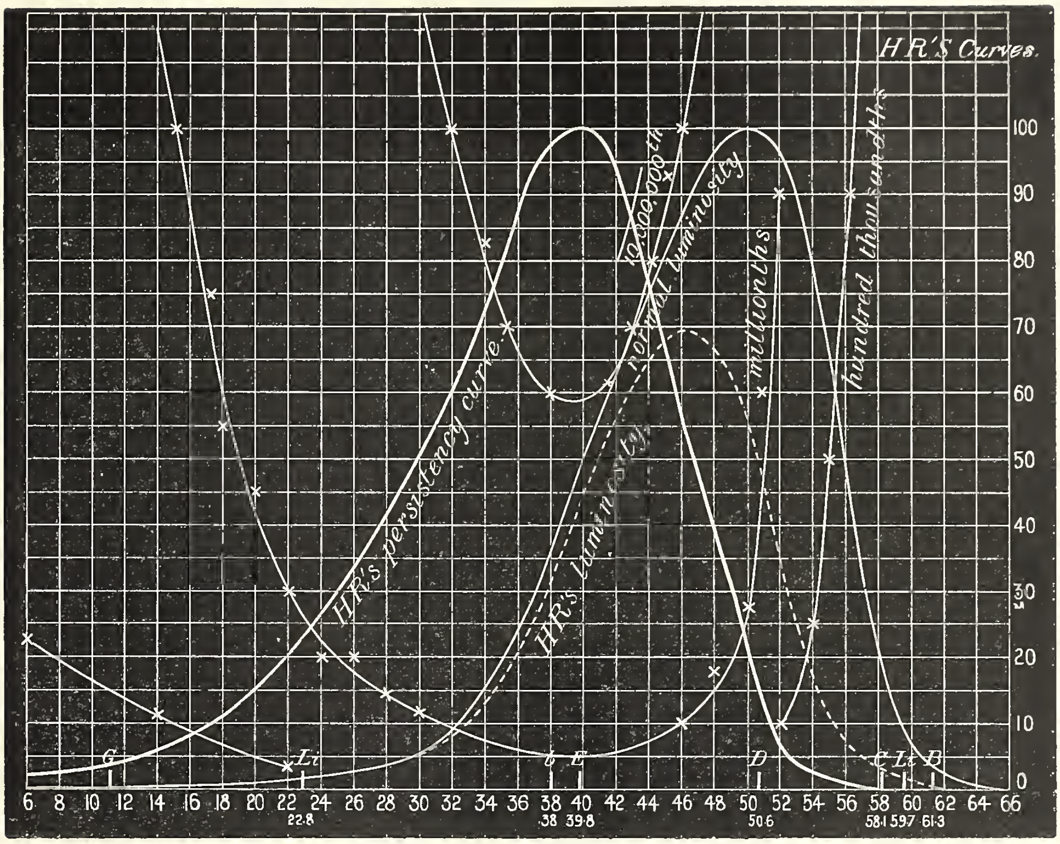
TABLE IX.—H. R.'s Curves.

I.	II.	III.	IV.	V.	VI.	VII.
Scale number.	Wave-length.	Mean reading of extinction in millionths of original luminosity.	Adopted reading in millionths of original luminosity.	Persistency curve $\frac{590}{\text{ad. reading}}$.	Luminosity curve.	Absolute luminosity of extinction $\frac{\text{IV.} \times \text{VI.}}{48}$.
57	6423	1200	1200	·49	5	125
56	6330	900	850	·69	7	124
55	6242	500	550	1·07	10	115
54	6152	250	250	2·36	17	88
53	6074	..	150	3·93	25	78
52	5996	90	90	6·56	35	66
51	5919	60	45	13·1	47	44
50	5850	27	27	21·8	57	32
48	5720	18	15	39·3	66	21
46	5596	10	10	59	69	14
44	5481	9·3	8	73·8	64	11
42	5373	6·5	6·2	95·1	56·5	7
40	5270	5·9	5·9	100	45	5·5
38	5172	6	6	98·3	32	4
36	5085	..	6·6	89·4	20	2·7
35	5043	7	7·2	81·9	16	2·4
34	5002	..	8	73·8	12·5	2·1
32	4924	10	9·6	61·5	8	1·6
30	4848	11·5	12	49·2	6	1·5
28	4776	14·5	14·5	40·7	5	1·5
26	4707	20	17·5	33·7	4	1·5
24	4639	20	22	26·8	3	1·4
22	4578	..	30	19·7	2·4	1·5
18	4459	55	57	10·4	1·3	1·5
14	4349	115	115	5·1	·7	1·7
10	4245	..	160	3·7	·5	1·7
6	4151	200	200	2·9	·4	1·7

§ L.—*Luminosity of a Spectrum produced by Feeble Light.*

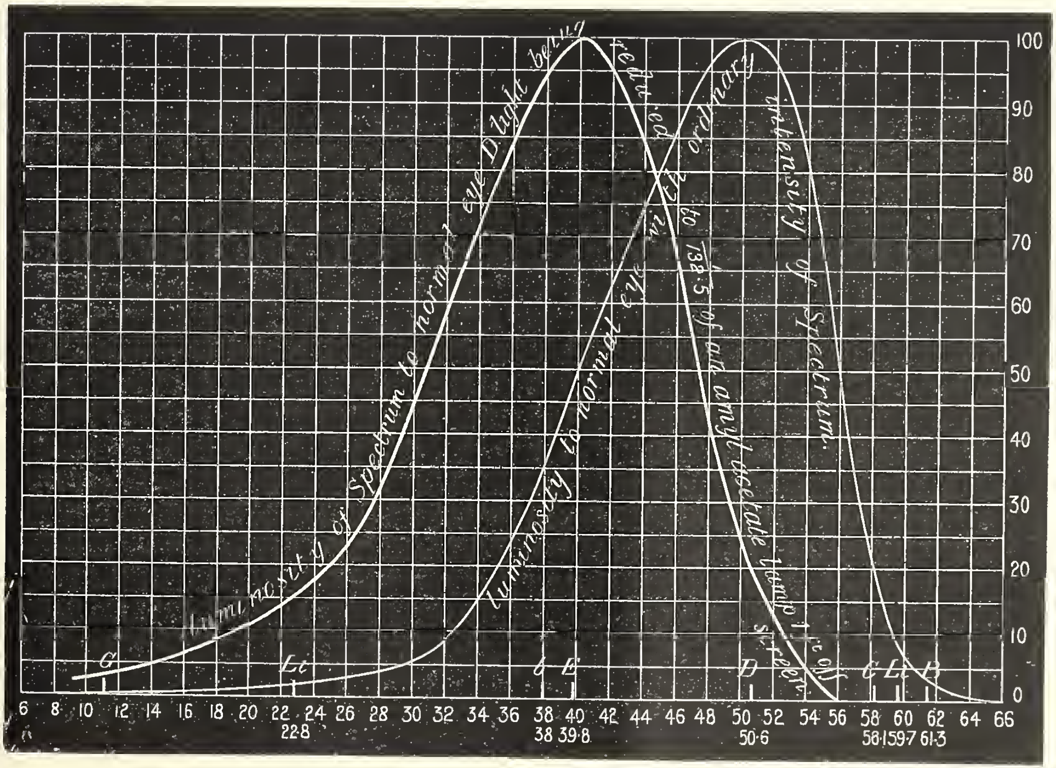
Having found that the persistency curve was apparently and presumably the same as the luminosity curve of the two persons who had but one colour sensation, it almost followed that if the beam of light producing the spectrum were sufficiently reduced its curve of luminosity would approach these. An experiment was therefore made. The reference beam was introduced into the measuring box already described (fig. 36), and, when uninterrupted by the sectors, had a luminosity of $\frac{1}{132\cdot5}$ of an amyl lamp at one foot off. The beams from the spectrum of the colour patch apparatus were also introduced into the apparatus so that they fell as before on *S*. The luminosity of the different rays was taken in the ordinary manner interposing the rotating sectors in the reference beam. The following results were obtained (see Table XI), the mean of the readings being given. This mean in Column III. is multiplied by 1·25

Fig. 44.



Extinction and luminosity curves of a red blind.

Fig. 45.



Luminosity to a normal eye of a feeble spectrum.

to bring the maximum to 100. This curve (fig. 45), that of the monochromatic sensation, and the persistency curve are tabulated together in Table X., to show how closely they agree. Here we have a proof that the normal eye is as little sensitive to the red end of the spectrum formed from a very much reduced light as those of the two brothers when ordinary light is used. It must be remarked, however, that all colour was not entirely absent, though it was very considerably reduced in saturation. The measurements were made with some trouble at first, owing to the inclination of the eye to direct its axis to some point other than the centre of the patch where the white strip and the colour strip touch one another. The diversion of the axis of the eye in some cases made the colour more luminous, and in other cases less so, than it did when the eye was properly directed, as might be surmised from the luminosity curves of light of ordinary intensity. By reducing the light in less degrees it became possible to obtain curves of luminosity which agreed very closely with those of the different degrees of red blindness (that is, where the spectrum is shortened at the red end), of which we have had many cases to try. As already pointed out, the outer part of the retina of our own eyes is really in one stage of red blindness, having a slightly shortened spectrum.

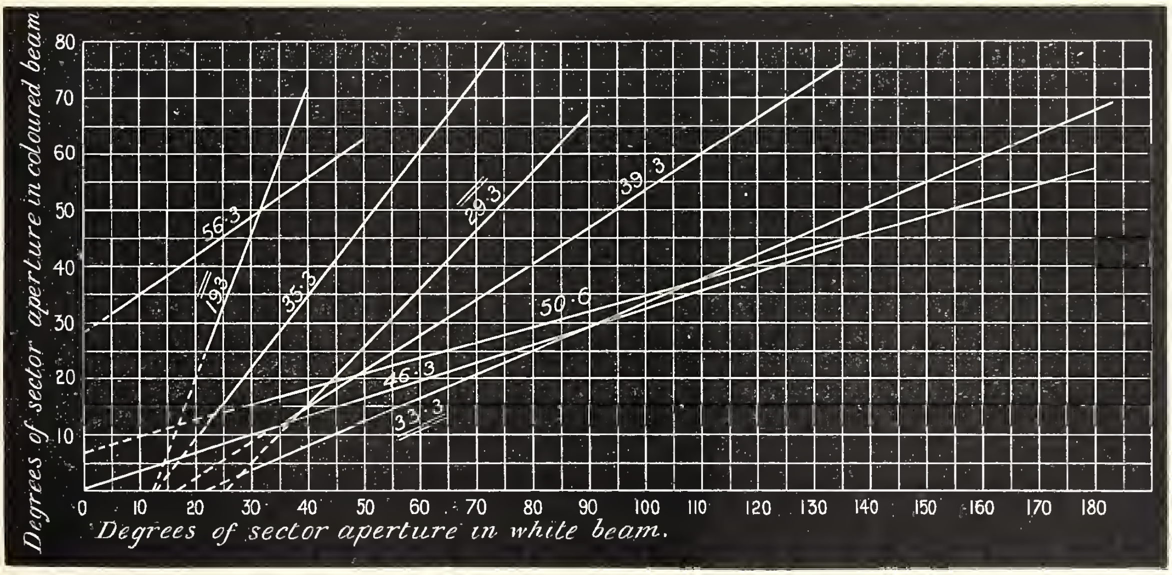
TABLE X.—Luminosity of Spectrum Reduced in Intensity so that $D = \frac{1}{132\cdot5}$ amyl lamp 1 foot distant.

Scale number.	Mean reading.	Mean reading, reduced to 100 max.	P. and Q.'s readings, 100 max.	Persistency curve for the centre of the eye.
55·6	·5	·6	2	2
53·6	5·5	7·0	3·6	3·6
51·6	13	16·7	8	8
49·6	23	29·7	22	22
47·6	40	50·0	44	44
45·6	57	71·2	69	69
43·6	70	87·5	93	93
41·6	79	98·7	100	99·5
39·6	78	97·5	99·5	98·5
37·6	74	92·5	96	93
35·6	66	82·5	89	84
33·6	55	68·7	77·5	71
31·6	44·5	55·2	61	53·5
29·6	35	43·7	45·5	36·5
27·6	24	30·0	33·5	24
25·6	17	21·7	25	16
23·6	13	16·7	18	10
21·6	10	12·5	13	8
19·6	8	10·0	9·5	6
13·6	3	3·7	4·2	3
9·6	2	2·5	2·5	2

§ LI.—*Relative Luminosity of Rays for Different Spectrum Intensity.*

Having found that the curves of luminosity of a spectrum when feeble and when bright differed, it became a matter of some importance to ascertain in what manner the relative luminosity of the rays varied when the intensity of the light which formed the spectrum was altered in a definite ratio. Evidently the most satisfactory method of ascertaining this was to throw a patch of white light on the screen and then to diminish its luminosity to known amounts, and having selected some ray of the spectrum, to equalize their luminosities. The box already described (fig. 36) was brought into requisition, and a beam of white light was caused to illuminate one half of the white patch on the screen at the end of the box, and the other half was illuminated by the ray whose luminosity was to be tried. Rotating sectors were placed in each beam; the apertures of those in the white were fixed at different angles, whilst those of the sectors in the coloured beam were opened or closed till the luminosities appeared the same to the eye, a series of readings being taken for each ray. The results thus obtained were plotted, and some typical ones are shown in fig. 46. The ordinates are the apertures of the sectors in the monochromatic rays, and the abscissæ the apertures of the sectors in the white beam. The tangent of the inclination to the vertical of the curve at any point, therefore, represents the ratio of the luminosity of the coloured to that of the white beam for a certain intensity of light. If this ratio were the same for all intensities the curve would become a straight line starting from the origin. This is the case, it will be seen, with one ray only, that at scale number 46·3, or about λ 5618. This ray and

Fig. 46.



Relative luminosities of rays with different intensities of the spectrum.

white light would therefore be extinguished together. It may be more than a coincidence that this ray does not differ much in wave length from that ray which, as stated by one of us in a paper on the Transmission of Sunlight through the Earth's Atmosphere (see 'Phil. Trans.,' present volume) was found to be affected to the same degree as the integrated light of the whole spectrum, no matter what was the thickness of the atmosphere through which it had passed.

It will be seen, however, from the diagram, that the other curves become straight lines when certain degrees of intensity, different in each case, are reached; and if these straight lines are produced to cut the axis the ordinates of the rays which lie towards the blue end of the spectrum above 46·3 have a negative value at the zero of white light, whilst those which lie toward the red side of 46·3 have a positive value; showing that the blue part of the spectrum is extinguished last, and the red part first, as we have already seen to be the case.

It is, moreover, evident, and this has been demonstrated by experiments described above, that for low intensities the luminosity curve of the spectrum will vary with difference of intensity, but that a degree of intensity is soon reached, when all the curves have become straight lines, and that the distances from the origin at which they cut the axis are so small compared with the distance where the curves of all the rays become straight, that the relative luminosities of the different rays in spectra of ordinary intensity are practically the same. In the experiments last described, the D light on the screen when not reduced by the sectors was equivalent to ·027 of an amyl lamp at one foot. This would bring it far beyond the point where its curve, and indeed those of all other rays, would become straight.*

The following table shows the agreement of the results of these last measurements with those of the observations, from which the luminosity curve for the central part of the eye was constructed. The quotient of the difference of two abscissæ in the straight part of each curve divided by the difference of the corresponding ordinates evidently is the tangent of the inclination to the vertical, which, as stated above, is a measure of the luminosity of the corresponding ray. In Column V. of the table the first five of these quotients are multiplied by 28·2 in order to make them easily comparable with the ordinates of the normal curve which are given in Column VI. In the case of the last three entries in the table, the beam of white light was necessarily diminished in intensity before it passed through the sectors, the quotients have therefore to be multiplied by 5·03.

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* It must be remembered that we are only dealing with light reflected from a white screen, and it does not follow that the lines may continue straight indefinitely when the light is of the brilliancy seen when looking direct at a bright spectrum, such as that of the sun, with a fairly wide slit to the collimator.

TABLE XI.—-Relative Luminosities of Rays.

I.	II.	III.	IV.	V.	VI.
Scale number.	Wave-lengths.	$\frac{\text{Diff. abscissæ}}{\text{Diff. ordinates}}$.	Tan. of inclination.	Tan. × 28·2 or 5·03.	Luminosity of normal curve.
56·3	6358	$\frac{50-20}{62-42}$	1·5	42·3	43·5
50·6	5889	$\frac{180-40}{58-18}$	3·5	98·6	99·5
46·3	5618	$\frac{180}{57}$	3·16	88	88
39·3	5246	$\frac{135-45}{76-18}$	1·55	43·7	44·5
35·3	5066	$\frac{75-25}{80-15}$	·77	21·7	20·2
33·3	4975	$\frac{105}{48-4}$	2·39	12	12
29·3	4822	$\frac{50}{66-15}$	·96	4·9	5
19·3	4497	$\frac{10}{80-50}$	·33	1·68	1·5

ADDENDUM.

(Added July 20, 1892.)

§ LII.—*A Case of Green Monochromatic Vision.*

Since the foregoing paper was read a very phenomenal case of colour-blindness has been investigated by us for the Colour Vision Committee of the Royal Society. It is the case of a type so rare that we have not hesitated to publish it at the earliest opportunity. The patient (B. C.) had been examined by Mr. NETTLESHIP, who kindly secured his attendance at South Kensington for the purpose of being examined by the spectrum and other tests. B. C. is a youth of 19, who has served as an apprentice at sea. His form vision is perfect, and he is not night blind. He can see well at all times, though he states that on a cloudy day his vision seemed to be slightly more acute than in sunshine. He was first requested to make matches with the Holmgren wools in the usual manner, with the result that he was found to possess monochromatic vision. He matched reds, greens, blues, dark yellows, browns, greys, and purples together; and it was a matter of chance if he selected any proper match for any of the test colours. Finally, when pressed, he admitted that the whole of the heap of wools were "blue" to him, any one only differing from another in brightness. The brighter colours he called "dirty" or "pale" blue, terms which eventually proved to be synonymous. We then examined him with patches of monochromatic spectrum colours by means of the colour patch apparatus. He designated every colour as "blue," except a bright yellow, which he called white, but when the luminosity of this colour was reduced he pronounced it a good blue. So with white, as the illumination was decreased, he pronounced it to pass first into dirty blue, and then into a full blue.

MAXWELL's discs were then brought into requisition, and it was hard at first to know how to make the necessary alterations, owing to the terms he employed to express the difference which existed between the inner disc and the outer grey ring. By noting that a pale "blue" passed into a pure blue when the amount of white in the outer ring was diminished, and that the inner disc was described as "pale" or "dirty" when the outer ring was described as a "a very full blue," we were enabled to make him match accurately a red, a green, and a blue disc separately with mixtures of black and white.

The following are the equations :—

360 red = 315 black + 45 white.
360 green = 258 black + 102 white.
360 blue = 305 black + 55 white.

With these proportions he emphatically stated that all were good blues, and that the inner disc and outer ring were identical in brightness and in colour.

It may be remarked that this is a case of congenital colour blindness, and that there is reason to believe that some of his ancestors were colour blind.

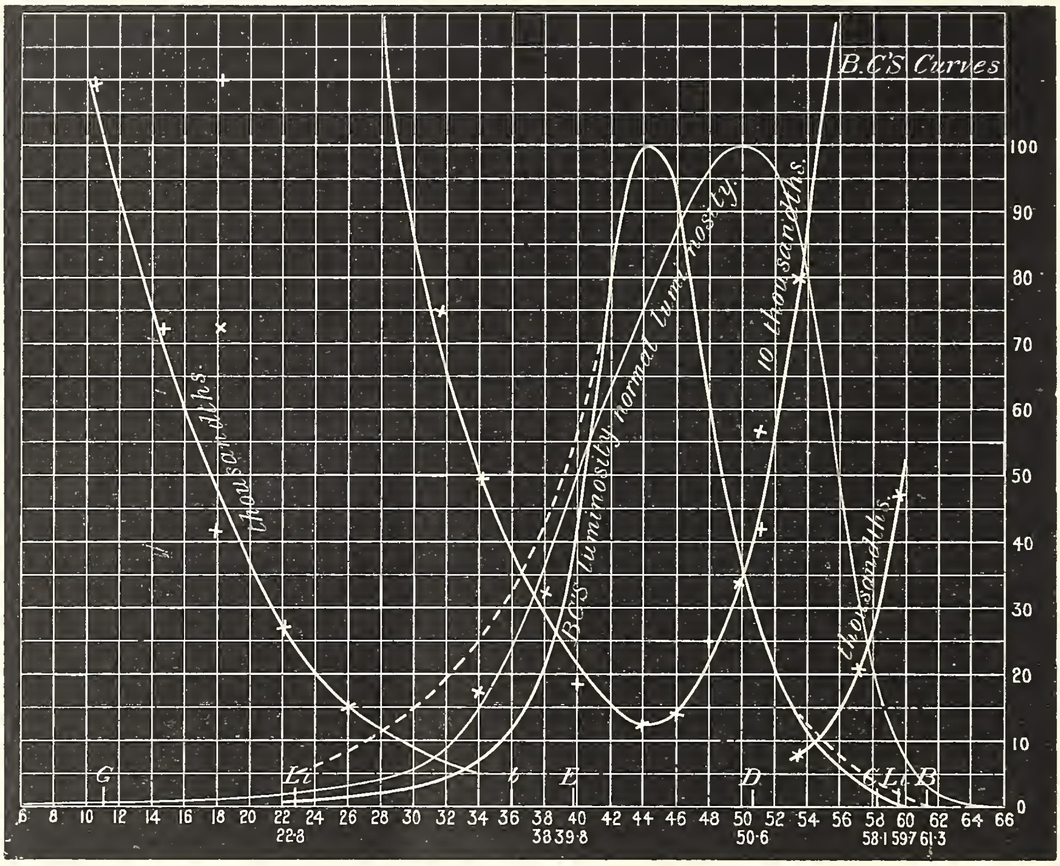
Before using the discs an attempt was made to ascertain the luminosity of the spectrum as it appeared to him. His readings, however, were so erratic that nothing could be made out from these first observations, except to fix the place of maximum luminosity, the terms "pale" and "dirty" puzzling us as to their real meanings. After the experience with the discs we had a clue as to what he wished to express by pale or dirty blue, which only meant that the colour or white was too bright, and on making a second attempt he matched the luminosities of the two shadows as easily as did P. and Q., the other cases of monochromatic vision. The method adopted was to diminish the white light illuminating one shadow to the point at which he pronounced it a good blue, when a slight alteration in the intensity was always sufficient to secure to his eye equality of luminosity between it and the coloured shadow without his perceiving any alteration in the saturation.

The curve of luminosity, fig. 47, is a very remarkable one, being different in character to that of P. and Q., the maximum being well on the D side of E. A great falling off in the luminosity when compared with that measured by the normal eye will be noticed both in the blue and in the red. The evidence was, therefore, presumptive that B. C.'s colour sensation was neither red nor blue, but probably a green.

The next test was made to throw light on this point. He made observations of the extinction of the different parts of the spectrum (see § XLVIII.). His observations were very fair, except on the violet side of F, where they became slightly erratic, but by requesting him to use all parts of his retina to obtain the last glimpse of light, a very concordant curve resulted, as shown in fig. 47. Some of his observations at this part were evidently made with the centre of the retina, for they gave readings which, when the "persistency" curve was calculated, and these observations treated as part of the extinction, agreed with the luminosity curve. We may, therefore, conclude that B. C. has a region in the retina in which there is an absorbing medium corresponding to the yellow spot of the normal eyed. This is diagrammatically shown in fig. 47 by the difference in height of ordinates in the persistency and the luminosity curves. On the red side of the maximum the two curves are practically identical, except from Scale number 54. At this point for similar reasons, as given in § XLVIII., it is probable that the white light which illuminated the prism vitiated the readings to some degree, as Column VI. of the following table shows. At the violet end something similar, doubtless, occurs, but it is masked by the difference in extinction by the central part of the retina and that of the whole eye.

It must, however, be remarked that the amount of reduction of the intensity of a ray to produce extinction is very different for B. C. and for the normal eyed, or for the red- and green-blind or for P. and Q. B. C. can bear nearly 200 times less reduction for the rays near E. We have already pointed out that the same is practically the case with M., whom we presume to be violet blind. We may therefore deduce the fact that the monochromatic vision in this case is of a totally different type to that of P. and Q., and that the last sensation to be lost is the same as that of M. If any violet sensation were present in either, the fact would be made evident by the order of the extinction. The sensation of B. C. is thus apparently the green sensation, though that this particular sensation is exactly the same as that absent in the green blind is not certain; his curve agrees very closely in form and position with that deduced by KŒNIG by different methods as that of the green sensation.

Fig. 47.



B. C.'s luminosity and extinction curves.

TABLE XII.—B. C.'s Curves.

I.	II.	III.	IV.	V.	VI.
Scale number.	Wave-length.	Adopted reading in hundred thousandths.	Persistency curve, $\frac{12{,}500}{\text{readings in V.}}$	Luminosity of original beam.	Absolute luminosity of extinction III. and V.
61	6839	7,500	1·6		
60	6728	5,500	2·3	·5	27·5
59	6622	4,000	3·1	1	40
58	6520	2,800	4·5	2	56
57	6423	2,000	6·2	4	80
56	6330	1,500	8·3	6	90
55	6242	1,150	10·8	8	92
54	6152	950	13·1	11·5	109·2
53	6074	750	16·6	16	120
52	5996	580	21·6	21·5	125
51	5919	430	29	28·5	122·5
50	5850	350	36	37	129·5
49	5783	275	45·5	47	129·2
48	5720	215	58	60	129
47	5658	170	73·4	76	129·2
46	5596	140	89·3	92	129
45	5538	125	100	98	122·5
44	5481	125	100	100	125
43	5427	130	96·1	97	126
42	5373	150	83	85	127·5
41	5321	180	69·4	65	117
40	5270	215	59	45	96·7
39	5221	250	50	30	75
38	5172	290	43	21·5	72·3
37	5128	335	37	16	53·6
36	5055	380	33	11·5	43·7
34	5002	500	25	7	35
32	4994	650	19	4	26
30	4848	850	14	2·5	23·3
28	4776	1,100	11·4	2	22
26	4707	1,500	8 3	1·5	22
24	4639	2,000	6·2	1	20
22	4578	2,700	4·6	5	13·5
18	4459	4,750			
14	4349	7,500			
10	4245	11,000			

XV. *On the Clark Cell as a Standard of Electromotive Force.*

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Received February 17,—Read March 17, 1892.

[PLATE 16.]

Summary of Contents.

PART I.—THE ABSOLUTE ELECTROMOTIVE FORCE OF A CLARK CELL.

PART II.—COMPARISON OF VARIOUS FORMS OF CELLS AND OF THE MATERIALS USED.

Part I.—On the Absolute Electromotive Force of a Clark Cell.

§ 1. *Introduction.*

The experiments recorded in the following paper were undertaken with the object of testing various points in the manufacture and use of the Clark cell as a standard of electromotive force, specially with a view of investigating the most simple method of setting up a number of such cells which should have, within certain limits, a definite electromotive force, and of determining the limits within which, it is safe to say, that the E.M.F. of such a cell will lie.

We have also determined afresh the relation between the E.M.F. of a Clark cell and the electrochemical equivalent of silver. The experiments have been conducted in the main on the lines of those described in Lord Rayleigh's papers "On the Electrochemical Equivalent of Silver and on the Absolute Electromotive Force of Clark Cells" ('Phil. Trans.,' Part II., 1884), and "On the Clark Cell as a Standard of Electromotive Force" ('Phil. Trans.,' Part II., 1885).

The investigation was undertaken in connection with the work of the Committee appointed by the Board of Trade on Standards for the Measurement of Electricity.

Resolution No. 14 of the Report of that Committee is as follows:—

"That the electrical pressure at a temperature of 62° F. between the poles or electrodes of the voltaic cell known as Clark's cell may be taken as not differing from a pressure of 1·433 volt by more than an amount which will be determined by a Sub-Committee appointed to investigate the question, who will prepare a specification for the construction and use of the cell."

The following paper contains an account of some of the experiments made during the investigation.

In the same Report, Resolution 10 states that an unvarying current, which, when passed through a solution of nitrate of silver in water, in accordance with the specification attached to this report, deposits silver at the rate of 0·001118 of a gramme per second, may be taken as a current of 1 ampere.

The specification referred to is as follows:—

In the following specification the term silver voltameter means the arrangement of apparatus by means of which an electric current is passed through a solution of nitrate of silver in water. The silver voltameter measures the total electrical quantity which has passed during the time of the experiment, and by noting this time the time-average of the current, or if the current has been kept constant, the current itself can be deduced.

In employing the silver voltameter to measure currents of about 1 ampere the following arrangements should be adopted. The kathode on which the silver is to be deposited should take the form of a platinum bowl not less than 10 cm. in diameter, and from 4 to 5 cm. in depth.

The anode should be a plate of pure silver some 30 square cm. in area, and 2 or 3 millims. in thickness.

This is supported horizontally in the liquid near the top of the solution by a platinum wire passed through holes in the plate at opposite corners. To prevent the disintegrated silver which is formed on the anode from falling on to the kathode, the anode should be wrapped round with pure filter paper, secured at the back with sealing-wax.

The liquid should consist of a neutral solution of pure silver nitrate, containing about 15 parts by weight of the nitrate to 85 parts of water.

The resistance of the voltameter changes somewhat as the current passes. To prevent these changes having too great an effect on the current, some resistance, besides that of the voltameter, should be inserted in the circuit. The total metallic resistance of the circuit should not be less than 10 ohms.

Method of making a Measurement.

The platinum bowl is washed with nitric acid and distilled water, dried by heat, and then left to cool in a desiccator. When thoroughly dry it is weighed carefully.

It is nearly filled with the solution, and connected to the rest of the circuit by being placed on a clean copper support to which a binding screw is attached. This copper support must be insulated.

The anode is then immersed in the solution so as to be well covered by it and supported in that position; the connections to the rest of the circuit are made.

Contact is made at the key, noting the time of contact. The current is allowed to pass for not less than half-an-hour, and the time at which contact is broken is observed. Care must be taken that the clock used is keeping correct time during this interval.

The solution is now removed from the bowl, and the deposit is washed with distilled water and left to soak for at least six hours. It is then rinsed successively with distilled water and absolute alcohol, and dried in a hot-air bath at a temperature of about 160° C. After cooling in a desiccator it is weighed again. The gain in weight gives the silver deposited.

To find the current in amperes, this weight, expressed in grammes, must be divided by the number of seconds during which the current has been passed, and by ·001118.

The result will be the time-average of the current, if during the interval the current has varied.

In determining by this method the constant of an instrument the current should be kept as nearly constant as possible, and the readings of the instrument taken at frequent observed intervals of time. These observations give a curve from which the reading corresponding to the mean current (time-average of the current) can be found. The current, as calculated by the voltameter, corresponds to this reading.

In our experiments the method above described, which is that adopted by Lord Rayleigh, was carefully adhered to.

While our work was in progress, a paper on "The Causes of Variation of Clark Standard Cells" was communicated to the British Association by Mr. J. Swinburne ('Electrical Review,' August 28th, 1891). The results of our investigations confirm his conclusions in many respects.

In the first part of the paper we shall deal with the experiments on the absolute electromotive force of a Clark cell, and, in the second, with the comparisons between various cells and the conclusions to be drawn from them.

§ 2. *Theory of the Method.*

The theory of the method is very simple. A fairly constant current is passed through a silver voltameter and a known resistance. The value of the current is given by the voltameter, and from this the potential difference between the terminals of the resistance is found. This potential difference is compared by a potentiometer method with the E.M.F. of a Clark cell, and thus an absolute value is obtained for the E.M.F. of the cell.

We shall describe in turn the various parts of the apparatus and the method of experimenting.

§ 3. *The Standard Cell.*

Lord RAYLEIGH left at Cambridge, in the charge of one of us (R. T. G.), three of the cells he constructed during the year 1883. One of these has since dried up. The other two, which have been carefully kept, have been compared from time to time, and differ very slightly in E.M.F. (see Part II.). We took one of these, which we call Rayleigh No. 1, as our standard. Rayleigh No. 1 is a cell of the pattern originally adopted by LATIMER CLARK. According to the data in Lord RAYLEIGH's paper, "the saturated solution of zinc sulphate was nearly neutral. The metallic zinc was bought as pure from Messrs. HOPKIN and WILLIAMS. The mercurous sulphate was from the same source, and the metallic mercury was redistilled in the laboratory."

It seems probable that the cell Rayleigh No. 1, is No. 4 of Lord RAYLEIGH's paper, though it is difficult to be quite certain of this point. In his paper, 'Phil. Trans.,' 1885, Lord RAYLEIGH says: "Cells (4), (8), (9), were, I think, left at Cambridge." Of these it is clear ('Phil. Trans.,' 1884, p. 442), that No. 4 was of the original pattern, the others were made at a later date. Our cell, No. 1, is of the pattern originally devised by LATIMER CLARK, while No. 2 is of the form used later by Lord RAYLEIGH. We therefore infer that our No. 1 is Lord RAYLEIGH's No. 4.

In the earlier experiments, the E.M.F. of this cell was compared directly with the difference of potential between the ends of the resistance. As in this procedure there was some slight risk of passing an appreciable current through the cell, it was modified. A large cell, denoted in what follows by No. 90, or the bottle cell, was constructed in a glass bottle, and the E.M.F. of this was determined. This cell was from time to time compared with the standard.

§ 4. *The Voltameters.*

Five different platinum bowls were used.

I. The large bowl employed by Lord RAYLEIGH, in shape approximately the segment of a sphere, diameter across the top 10 cms., depth 4·5 cms.

II. A bowl, kindly lent us by Professor LIVEING, 10 cms. in diameter, 3·5 cms. in depth, the bottom of this is flatter than that of I., and the sides more nearly vertical.

III. A bowl, very similar to I., and of about the same dimensions, also lent us by Professor LIVEING.

IV. A third bowl, similar to I., and of about the same size, though much lighter, lent us by Mr. F. H. NEVILLE.

V. A deep cup-shape bowl, 7·25 cms. in diameter, 8 cms. deep, belonging to the Laboratory.

Of these, No. II. had been used a good deal for chemical work. Its surface was somewhat dull, and we never succeeded in obtaining an adhesive deposit with it. Traces of the silver always came away in the washing, and when weighed, the silver was always less than that deposited in the other bowl. We have not used the deposits from this bowl.

In each experiment two bowls were used in series.

The anode was usually a circular piece of silver plate, about 7 or 8 cms. in diameter, and ·12 cm. in thickness, supplied by Messrs. JOHNSON and MATTHEY as "Ordinary Fine Silver Sheet," price 5*s.* per oz. This was supported horizontally in the liquid by three platinum wires, the whole being covered with filter paper secured at the back with sealing wax.

In the case of Bowl V., the anode was bent into the form of a cylinder, which was immersed in the liquid with its axis vertical.

The bowls rested on clean copper plates insulated on ebonite and paraffin blocks. The supports which carried the anodes were insulated in the same way. The insulation was tested by connecting up the voltameters before they had been filled with the liquid with a battery of six Leclanché cells and a delicate high-resistance galvanometer. No trace of a leak could be found.

The liquid used was a solution of silver nitrate, containing 85 parts by weight of distilled water and 15 parts of pure recrystallised silver nitrate, supplied by Messrs. JOHNSON and SONS, as containing 63·5 per cent. of fine silver, or $99\frac{2}{3}$ per cent. of pure anhydrous nitrate of silver.

A coil of wire, having about the same resistance as two of the voltameters, was connected up in parallel with them, in such a way that, by means of a key consisting of mercury cups cut in a block of paraffin, the current could be sent either through the voltameters or through the coil. The adjustments were made with the current flowing through the wire, when all was complete the key was shifted so as to send the current through the voltameters, and the time taken.

§ 5. *The Standard Resistance.*

It was necessary that this should not be seriously heated by a current of about 1 ampere.

It was, therefore, constructed of a strip of platinoid about 1 cm. wide and ·05 cm. thick. This was wound on an open frame about 50 cms. long and 50 cms. wide by 6 cms. deep. The frame was of wood covered with strips of ebonite which insulated the metal from the wood.

The whole was immersed horizontally in a bath of paraffin, containing about 60 litres of oil. A stirrer, in the form of a T-shaped piece of wood, passed under the coil between it and the bottom of the bath, and by means of this the liquid could be kept in a state of agitation. This stirrer was worked frequently during an experiment. A delicate thermometer, placed with its bulb very close to the strip, showed an immediate slight rise in temperature, one or two tenths of a degree, if the oil was not stirred. If the stirring was maintained the rise during an experiment was usually only slight.

The ends of the platinoid strip were soldered to thick copper rods, which dipped into two mercury cups insulated on a paraffin block.

By means of copper rods, these cups could be put into communication with the B.A. bridge, and the resistance of the strip measured *in situ* in terms of the standard coil "Flat." This was done without shifting the coil both before and after each experiment; the two results never differed by more than ·0007 B.A. unit, and the mean has been taken in calculating the result. The resistance of the strip at 17° C. was 1·0011 B.A. unit.

For some of the experiments this resistance might, with advantage, have been higher, though the fact that it was nearly equal to the standard secured accuracy in the comparison with it, and it was thought desirable, from this point of view, to use the same resistance throughout.

In order to reduce the effects due to variations in the resistance of the voltameters, a resistance of thick platinoid wire, usually some twelve or fifteen ohms in amount, was included in the circuit. This was varied until the current was of the right amount.

§ 6. *The Balance.*

A short beam balance by OERTLING was used for the weighings. This and the weights were the same as those employed in the determination of the specific resistance of mercury by GLAZEBROOK and FITZPATRICK ('Phil. Trans.,' A., 1888). The weights had been compared with the standard, and the correction found to be inappreciable. Some of the weighings were done in both pans in order to determine the ratio of the arms of the beam. This was found to be 1·00002. The correction to the weight *in vacuo* varied between 1·000030 and 1 − ·000013, according to the number of platinum weights used; both these are too small to affect our results, and have not been introduced into the calculations.

§ 7. *The Clock.*

The Laboratory chronometer, beating half-seconds, was used to give the time during which the current flowed, and the rate of the chronometer was determined by a series of comparisons with a clock belonging to the Cambridge Philosophical Society, which is rated from the Observatory.

It was found that the chronometer was losing at the rate of about ·16 seconds per hour. A small correction for this has been introduced in the final result.

§ 8. *The Resistance Boxes and Apparatus for the Comparison of Electromotive Forces.*

For the comparison of the electromotive forces two resistance boxes by ELLIOTT BROS. were used. The coils in these boxes agreed together sufficiently nearly to introduce no serious error. The current from two Leclanché cells ran continuously through 10,000 ohms, taken out of the two boxes in series. The standard Clark or the bottle cell was connected with one of the boxes through a sensitive galvanometer and a resistance coil of 10,000 ohms, and plugs transferred from one box to the other until there was no deflection on closing the galvanometer circuit. The Leclanchés remained very steady during the whole course of the experiments. The resistance required in the main circuit to balance the Clark changed in about a fortnight from 5722 to 5784, or about 1 per cent. This comparison was made from time to time during each individual experiment, to check the constancy of the Leclanchés, with satisfactory results. On August 7 an extreme variation of 6 ohms occurred. On other occasions the change during the time an experiment lasted was only 2 or 3 ohms. In making the reductions the mean value was taken.

The galvanometer, battery, and resistance boxes were all insulated on separate paraffin blocks.

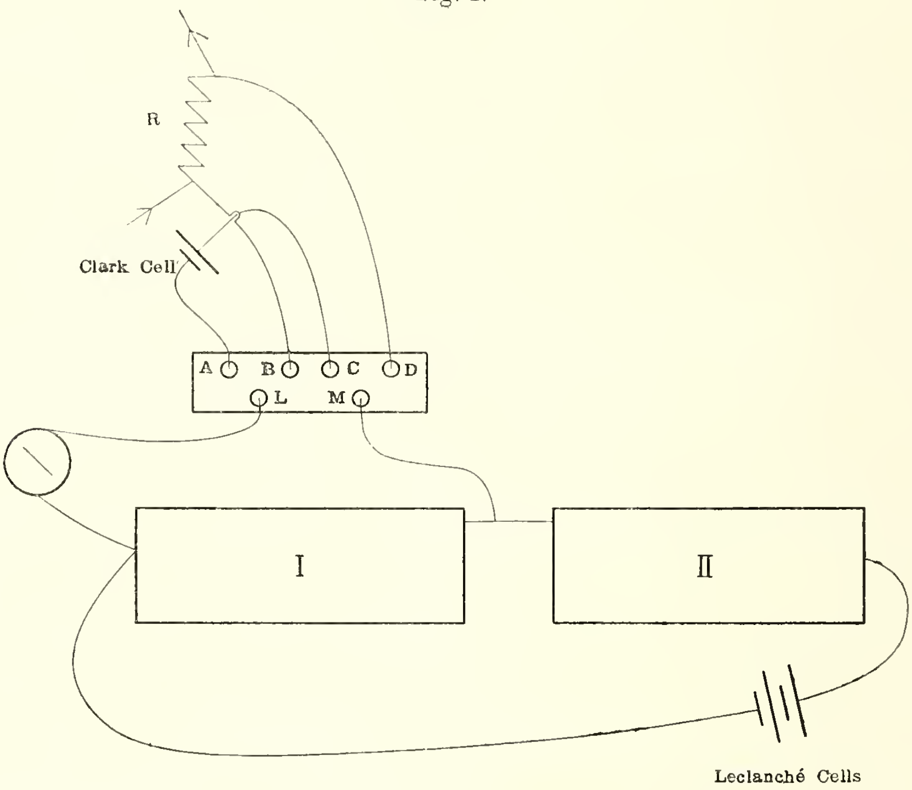
The temperature of the bath containing the Clark was observed from time to time during the experiments. It usually rose slightly by about 0°·1 C.

The thermometers employed had been tested at Kew, and the necessary corrections are introduced.

Fig. 1 gives a diagrammatic representation of this part of the apparatus; I. and II. are the two resistance boxes through which the current from the two Leclanchés passes.

A, B, C, D, L, M are six mercury cups in a block of paraffin. Of these L is connected through the galvanometer G with one end of the box I., while M is in connection with the junction of the two boxes. The poles of the Clark are connected to A and C, and well insulated wires join B and D to the ends of the strip resistance R. Connections can be made as needed among the cups by means of pieces of wire bent into the required form and attached to pieces of sealing wax as insulating handles. When a current, i, is maintained in the resistance R, the difference of potential between B and D is Ri. By connecting L to B and M to D, and adjusting the plugs in I. and II., keeping their sum constant until there was no deflection of the galvanometer, this difference of potential was expressed in terms of the E.M.F. of the Leclanchés, while by connecting A to L and C to M the ratio between the E.M.F. of the Clark and that of the Leclanchés was found. This method was adopted in some of the experiments. In other experiments, C and D were connected together, while A was joined to L and B to M. In this arrangement, the electromotive force between A and B was the difference between that of the Clark and R*i*. The difference was expressed in terms of the Leclanchés, and then compared as before with that of the Clark. By employing a proper current in the main circuit through the voltameters, the difference between the E.M.F. of the Clark and R*i* could be made small, and thus the result was more nearly independent of changes in the Leclanchés.

Fig. 1.



The sensitiveness of the galvanometer was such that an alteration of the resistance in I. by 1 ohm produced a deflection of about 3 cms. on the scale, this corresponds to a variation in the E.M.F. between L and M of about one six-thousandth of a Clark cell. Thus variations of ·0001 volt could be detected without difficulty.

§ 9. *Details of the Method.*

The theory of the experiment just outlined supposes that the current *i* in the main circuit remains constant. In practice this is not the case; two methods were adopted for meeting the difficulty.

Let V be the resistance taken out of the Box I. when the ends of the resistance R are connected to L and M. Then as the current varies V varies also, being always proportional to i. During the time the current was passing a series of observations of the value of V, regulated to produce a balance, was made and the corresponding times noted. These observations were plotted on squared paper, and a curve formed with the times for abscissæ and the values of V for ordinates; the area of this curve is proportional to the quantity of electricity which has passed, and it is easy to calculate from the curve the mean value of the current and the corresponding value of V. This method was followed in the observations on July 31, August 5, and August 7.

On July 31 the value of V changed considerably, viz., from 3876 to 3917·5. With one exception, however, the observations lie on a very regular curve, and the mean value, 3904·5, got from the curve is probably accurate to 1 or 2 ohms.

On August 5 the changes were very small, and, as shown by the corresponding curve for the Clark cells, are almost entirely due to alterations in the Leclanché cells. The curves are given in Plate 16, figs. 1, 2, 3.

In all the other experiments an adjustable mercury resistance was introduced into the main circuit with the voltameters. The plugs in the boxes were adjusted until on making the galvanometer key there was no deflection. The galvanometer circuit was then kept closed, and the mercury resistance adjusted so that no deflection should take place. Occasionally it happened that the changes needed in the resistance of the main circuit to maintain i constant were outside the limits of the mercury rheostat. When this was the case a suitable change was made in the plugs in I. and II. and in the rheostat simultaneously, and the time noted. The change needed, except on the morning of August 12, was always small, some 4 or 5 ohms at most, and the mean value of V was readily determined with all the accuracy required. On August 12, after the current had been flowing steadily for 25 minutes, a sudden drop of over 1 per cent. took place. As during the next five minutes the current continued somewhat unsteady, the experiment was concluded at the end of 30 minutes 45 seconds, a much shorter time than in any of the other experiments.

In this second method of making the observations the very slight heating effect produced in the strip coil when the bath was not stirred was just observable. A balance was obtained, the bath being unstirred for a few moments. On moving the stirrer the spot was very slightly deflected in the direction indicating a fall in the resistance R, and the rheostat required a slight readjustment to restore the balance. The readjustment necessary, however, was extremely slight, being equivalent to a change of only a small fraction of an ohm in the Box I. As 1 ohm in the box means an alteration of ·00024 volt in the E.M.F., the error produced by heating in the strip was quite negligible.

§ 10. *Method of Carrying out an Experiment.*

The usual method of conducting the observations was as follows :—

The platinum bowls were washed with nitric acid and distilled water, dried by heating in a gas flame, and left to cool, usually overnight, in a desiccator. They were then weighed and placed in position, being filled with the solution, the anodes were immersed and the necessary connections made ; while this was being done the current was allowed to flow in the main circuit through the alternative wire, and the value of the resistances required in the boxes to balance the Clark and the potential difference then existing between the ends of the strip coil approximately determined.

In cases in which the bottle cell No. 90 was used the difference between its E.M.F. and that of the standard Rayleigh No. 1 was found by the usual opposition method.

The main current was then broken and the difference in resistance between the strip coil and the standard "Flat" determined by CAREY-FOSTER's method, the necessary temperatures being observed.

The main current was then made, still through the alternative wire, and the resistances in the boxes required to balance the Clark accurately taken. The plugs were then set to the approximate value required to balance the potential difference between the ends of the strip coil, and the key in the main circuit shifted so as to send the current through the voltameters, the time being noted on the chronometer. Some small changes in the resistance boxes or in the mercury rheostat were usually required to obtain a balance in the galvanometer circuit. These were made and the spot of light watched, as already described, the necessary changes being made from time to time to keep it at rest.

After the experiment had proceeded for some time a comparison between the Clark and Leclanchés was usually again made as rapidly as possible. This was done by shifting the connections from L and M into the cups A and C, the current in the main circuit not being interfered with. The connections were again restored to their original position and the experiment proceeded as before. After the current had passed for some time, usually from forty minutes to an hour, it was again broken, the time being noted.

A comparison between the Clark and the Leclanchés was again made, and the differences between the electromotive forces of the bottle cell No. 90 and the standard and between the resistances of the strip coil and flat were observed.

The temperatures of the bath containing the strip coil and of the bottle and standard cells were noted from time to time during the experiment.

While the comparisons of E.M.F. and of resistance were in progress the bowls were rinsed with distilled water and then left to soak overnight in distilled water. The next morning they were again rinsed with distilled water and alcohol and then dried in a hot air bath, at a temperature of from 160° to 180° C.

After cooling for some hours in a desiccator they were weighed. The weighings were repeated on the following day, the bowls having usually been left for the interval in the balance case. In some cases during the interval the bowls were again heated. In none of the observations was there any difference sufficient to affect the result between the weighings.

Temperature corrections had to be introduced into the comparison between the bottle cell and the standard, for these could not conveniently be put into the same bath, and consequently differed slightly in temperature.

This was done by the aid of Lord RAYLEIGH's value for the coefficient, viz., ·00078—a value which our own experiments (see p. 615) sufficiently confirmed—in the following way. Since the plugs required out of the Box I. to balance the Clark were about 5800 ohms, a change of ·00078 in the E.M.F. of the Clark will mean a change in the resistance of 5800 × ·00078, or about 4·5 ohms in the box. Thus an increase of 1° in the temperature of the cell means a fall of 4·5 ohms in the resistance required to balance it. The actual change in E.M.F. corresponding to this will be 1·43 × ·00078, or ·00112 volt, and the change corresponding to one ohm of the box is found by dividing this by 4·5. This gives ·00025 volt.

The temperature of the bottle cell and of the Leclanchés varied slightly during the progress of each experiment, and part of the variations observed in the ratios of the two are no doubt due to temperature changes.

§ 11. *Details of Experiments.*

We proceed now to the details of each experiment. Most of these can be best given in tabular form, and this is done in Table I. (p. 581).

Some explanation is needed, however, of the method by which the values of some of the tabulated numbers, specially those in columns 3, 6, 10, and 15, are arrived at, and of the notation employed.

Let V be the resistance out of the box required to balance the difference of potential Ri between the ends of the strip coil, W that required to balance the Clark, E the E.M.F. of the Clark which was compared with Ri at the temperature of the observation.

R is, as above, the resistance of the strip, i the current. Let M be the mass of silver deposited, T the time the current has passed, and γ the electrochemical equivalent of silver in grammes per ampere per second.

Then

$$\gamma = \cdot 001118.$$

Then we have

$$M = i\gamma T,$$

$$E/Ri = W/V,$$

therefore,

$$E = R \,.\, \frac{W}{V} \cdot \frac{M}{\gamma \,.\, T} \cdot$$

The values of E thus found require correcting (1) to reduce them to a common temperature of 15° C. (2) for the difference between the bottle cell and the standard Rayleigh 1.

Experiment 1, July 31, 1891.

The observed values of V and W are shown on Plate 16. In these one vertical division represents 5 ohms, and one horizontal division 5 minutes. The curves from which the numbers were calculated were drawn to a larger scale. It will be seen that the value of i, as indicated by the values of V, rose gradually for about 25 minutes, then remained fairly steady for about 15 minutes, and afterwards began to fall. The mean value for V, calculated as already described, is $3875 + 29\cdot5$, or $3904\cdot5$.

Three observations were taken for W; the mean value is 5722·5. Bowls I. and II. were used.

Experiment 2, August 3, 1891.

The observations are given in Plate 16. The current remained fairly steady for about 25 minutes, then suddenly increased considerably, the resistance changing in five minutes from 4585 to 4600, dropping in the next 15 minutes to 4595. The mean value of V is 4591 ohms, while that of W is 5756·5. Bowls I. and II. were used.

After these two experiments precautions were taken to work only when no other electrical work was going on in the laboratory. The current was supplied from storage cells in the battery room of the laboratory, and the wires conveying the current into the room in which the experiment was conducted were not satisfactorily insulated from other wires conveying a current from other cells into a different room. Fluctuations in the current were caused when these other cells were working.

Experiment 3, August 5.

The curves are shown in Plate 16. It will be seen that the variations are much less; in fact, the curves for V and W are very approximately parallel straight lines, showing that the changes are almost entirely due to the alterations in the Leclanchés. The mean values are—

$$V = 3691\cdot2,$$
$$W = 5751\cdot5.$$

Bowls I. and II. used.

In each of these three experiments some of the deposit in Bowl No. II. was lost in the washing, and the results from this bowl have not been used.

After this the mercury rheostat was used. This consists of two parallel grooves in a piece of dry mahogany which were filled with mercury. A movable copper bridge put the two columns into electrical connection. The current entered at one end of one column, passed along it, and after traversing the bridge emerged from the similar end of the second column. By sliding the bridge, the length of column traversed by the current was varied. A scale of centimetres was fitted parallel to the grooves, and it was found that with a current of about 1 ampere, shifting the bridge by 50 divisions of the scale produced an alteration in the current, which required a change of 1 ohm in the Box No. 1 to balance it; in other words, the difference of potential between the ends of the strip coil was changed by ·00025 volt, and the current by about ·00025 ampere.

Experiment 4, August 7.

Values for W 5773, 5775, 5770; mean, 5772·5. The difference between the E.M.F. of the Clark and R*i* was observed, and the resistance out, which was 2149, remained unchanged.

Thus

$$W - V = 2149,$$

therefore

$$V = 3623{\cdot}5.$$

The slide of the rheostat was moved over only about 10 divisions, so that the current was very constant. Bowls I. and II. used.

Experiment 5, August 10.

In this experiment a very large current, about 1·6 ampere, was used. The silver was deposited in four bowls, Nos. I. and IV. being arranged in parallel and then in series with them, Nos. III. and V. also in parallel. The value of W remained steady at 5784, and the value of W − V was also steady. By an oversight the plugs out for W − V were not noted till after the experiment was concluded. They lay certainly between 220 and 225, but there is some doubt as to the exact unit; we incline to think it was 223. With this value we have

$$V = 5561.$$

The slide was shifted about 24 divisions, so that the variation in the current was small.

Experiment 6, August 12, morning.

The value of W varied from 5756 to 5754; mean W = 5755. The value of W − V remained steady at 1176 for 25 minutes, when a sudden change took place. The main current was stopped, the time being noted on the chronometer, and the plugs altered to 1238. The current was again started, the time being taken, and remained fairly steady for 2·5 minutes, when the plugs were altered to 1233 without breaking the main current, and the slide correspondingly adjusted. The current remained steady for a time, but at the end of 3·25 minutes more showed signs of variation, so the experiment was ended and the time taken.

The current lasted, in the first instance, for 25 minutes, and in the second, for 5 minutes 45 seconds, thus we have for T the value 1845 seconds.

The value of W — V was 1176 for 25 minutes, 1238 for 2·5 minutes, and 1233 for 3·25 minutes.

From these we obtain, as the mean value of W — V, 1187, and hence V = 4568.

The fact that there was a break in the middle so that four readings of the chronometer are involved, renders the time measurement less trustworthy than in the other experiments.

Bowls I. and IV. were used.

Experiment 7, August 12, afternoon.

W changed from 5753 to 5756 during the experiment, the mean value was 5754·5. The value of W — V was changed after the current had been passing for 5 minutes from 3349 to 3344, but without breaking the current, the slide being readjusted to suit. It remained very steady at this value while the experiment lasted. The whole time of the experiment was 50 minutes 5 seconds.

Thus the mean value of W — V is 3344·5, therefore

$$V = 2410.$$

Bowls III. and V. were used.

Experiment 8, August 14, morning.

The value of W changed from 5738 to 5737; mean value, 5737·5. The value of W — V was steady all the time at 1120, and the slide was very little altered; thus the current was very steady, and we have

$$V = 4617{\cdot}5.$$

Bowls I. and IV. were used.

Experiment 9, August 14, afternoon.

The value of W changed from 5730 to 5729·5; mean value, 5729·8. The current ran for 1 hour and 5 seconds.

The values of W — V were 3290 for 6 minutes, then 3288 for 26 minutes, changing during the next minute to 3280, and after that being steady at 3280; for the remaining 27 minutes the current was not broken, but the slide of the rheostat was altered as the resistances were changed. From this we have as the mean value of W — V, 3284·5, whence

$$V = 2445{\cdot}3.$$

Bowls III. and V. were used.

It should be noticed, in the cases in which the current varied, that an accurate value of the time at which the changes in the position of the slide took place, is not needed, provided the whole time is known accurately.

These are all the experiments made.

TABLE I.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
No. of experiment.	Date.	Value of R, resistance of strip.	Temp. R.	Value of W.	Value of V.	Temp. of Clark.	Mass of silver in first bowl.	Mass of silver in second bowl.	Value of M adopted.	Value of T, time of flow.	Value of E at temperature of Clark.	Correction to 15°.	Value of E at 15°.	Corrections to Rayleigh 1.	E.M.F. of Rayleigh 1., at 15°.
1	July 31 .	·9877	17°	5722·5	3904·5	16·5	2·6071	*2·6003**	2·6071	2357	1·4322	·0019	1·4341	0	1·4341
2	Aug. 3 .	·9877	17	5756·5	4591	16·4	3·2070	*3·2043**	3·2070	2480	1·4320	·0016	1·4336	0	1·4336
3	Aug. 5 .	·9877	17	5751·5	3691·2	16·45	2·8095	*2·8031**	2·8095	2700	1·4325	·0016	1·4341	0	1·4341
4	Aug. 7 .	·9877	17	5772·5	3623·5	17	2·7935	*2·7905**	2·7935	2745	1·4323	·0022	1·4345	—·0005	1·4340
5	Aug. 10	·9878	17·4	5784	5561	16·5	4·6951	4·6926	4·6939	3010	1·4328	·0017	1·4345	—·0005	1·4340
6	Aug. 12 morning	·9879	17·9	5755	4568	17	2·3745	2·3732	2·3739	1845	1·4325	·0022	1·4347	—·0005	1·4342
7	Aug. 12 afternoon	·9880	18·1	5754·5	2410	17·3	2·0399	2·0385	2·0392	3005	1·4321	·0026	1·4347	—·0005	1·4342
8	Aug. 14 morning	·9882	18·7	5737·5	4617·5	17·8	3·9104	3·9104	3·9104	3000	1·4316	·0031	1·4347	—·0007	1·4340
9	Aug. 14 afternoon	·9883	19·2	5729·5	2445·3	18·3	2·4908	2·4924	2·4916	3605	1·4318	·0036	1·4354	—·0009	1·4345

* In all these observations Bowl II. was used and some silver was visibly lost in the washing. The numbers therefore have not been employed.

§ 12. *Discussion of Results.*

The mean of the above values is 1·4341 volt at 15° C. The variation from the mean is small. The smaller result in Experiment 2 may be due to the unsteadiness of the current during the experiment. It will be remembered it rose in value suddenly by about one part in 300 during the observations. The result in Experiment 9 is too high. This is, no doubt, due to the fact that the temperature of the bottle cell was varying somewhat. The temperature of the bath changed between the morning and afternoon observations on August 14 by 0°·5 C. Now, our observations have shown us that the E.M.F. of the cell lags behind the temperature when this is changing. This lag is due to the time taken in changing the state of concentration of the zinc sulphate solution. It is practically certain that on the afternoon of August 14 the cell had not reached the steady state corresponding to 18°·3, and that, therefore, the correction to 15° of ·0036 volt is too high.

It will be remembered that some uncertainty of one or two units is attached to the value of V in Experiment 5, August 10.

The weighings of the second bowl, No. 2, in Experiments 1, 2, 3, 4 have not been used because some small amount of silver was visibly lost in the washings.

The weighings in the other experiments agree with each other by about one part in 2000. This agreement is rather less good than that attained in Lord Rayleigh's experiments. It must be remembered that the rate of deposition was about three times as great as in his work, and this affects the deposit. The value of the E.M.F. thus found needs increasing by about one part in 20,000 for the error of the clock, and we thus obtain as the E.M.F. of the standard cell, Rayleigh No. 1, at a temperature of 15° C. the value

1·4342 volt.

The English standard legal temperature is 62° Fahr., or $16\frac{2}{3}$° C.

Reducing to this temperature we find the value

1·4324 volt at 62° Fahr.

The value found by Lord Rayleigh for his standard No. 1, was 1·435 volt at 15°, if we include another figure, the value given in Lord Rayleigh's Table becomes 1·4348. Now, his cell No. 4, which is probably our No. 1, was two parts in 10,000 below his No. 1; and again we have, in accordance with the resolution of the Board of Trade Committee, taken ·9866 as the value of the B.A. unit in ohms, Lord Rayleigh used ·9867. On this account our result needs to be raised by one part in 10,000 to compare with his; thus altogether our result needs to be raised by three parts in 10,000, or by about ·0004 volt, to give the E.M.F. of his original cell, No. 1, in his units. We thus get for this cell the value 1·4346 volt, as against 1·4348 found by Lord Rayleigh, and the agreement is within the errors of the observations.

A reference to the table shows that our bottle cell No. 90 is about three parts in 10,000 above the cell Rayleigh 1. It agrees more nearly, therefore, with Lord Rayleigh's original standard.

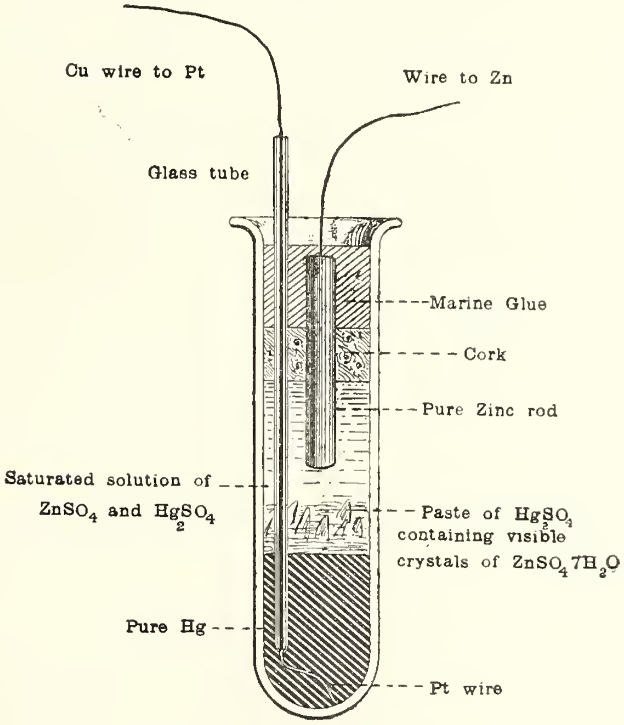
In comparing our result with that of other experimenters it must be remarked that it is given in "volts." To reduce it to the "legal volts" of the Paris Congress it must be multiplied by the ratio 1063/1060, and it then becomes at 15° C. 1·4382 "legal volt."

Part II.—Comparison of Various forms of Cells and of the Materials used.

§ 13. *Introduction. General Remarks. Methods of Testing.*

A normal Clark cell is one prepared according to certain definite directions, and the electromotive force of such a cell has been determined in Part I. above. Various causes may lead to a deviation from the normal value for the E.M.F. of any given cell, and it becomes important to consider these. We have in the course of the work met with at least one batch of cells which agree very closely among themselves, but which differ very considerably in E.M.F. from the standard. Directions for the construction of a normal Clark cell have been provisionally drawn up by a sub-committee of the Electrical Standards Committee of the Board of Trade, and have been issued to various laboratories for the purpose of gaining experience and enabling the Committee to issue at some future date a definite memorandum. These instructions in

Fig. 2.



their latest form, June 23, 1891, are given here. Fig. 2 shows the arrangements of the parts of the cell as described in the memorandum.

MEMORANDUM ON THE PREPARATION OF THE CLARK'S STANDARD CELL.

Definition of the Cell.

The cell consists of zinc and mercury in a saturated solution of zinc sulphate and mercurous sulphate in water, prepared with mercurous sulphate in excess, and is conveniently contained in a cylindrical glass vessel.

Preparation of the Materials.

1. *The Mercury.*—To secure purity it should be first treated with acid in the usual manner, and subsequently distilled in vacuo.

2. *The Zinc.*—Take a portion of a rod of pure zinc, solder to one end a piece of copper wire, clean the whole with glass paper, carefully removing any loose pieces of the zinc. Just before making up the cell dip the zinc into dilute sulphuric acid, wash with distilled water, and dry with a clean cloth or filter paper.

3. *The Zinc Sulphate Solution.*—Prepare a saturated solution of pure ("pure re-crystallised") zinc sulphate by mixing in a flask distilled water with nearly twice its weight of crystals of pure zinc sulphate, and adding a little zinc carbonate to neutralise any free acid. The whole of the crystals should be dissolved with the aid of gentle heat, *i.e.*, not exceeding a temperature of 30° C., and the solution filtered, while still warm, into a stock bottle. Crystals should form as it cools.

4. *The Mercurous Sulphate.*—Take mercurous sulphate, purchased as pure, and wash it thoroughly with cold distilled water by agitation in a bottle; drain off the water, and repeat the process at least twice. After the last washing drain off as much of the water as possible.

Mix the washed mercurous sulphate with the zinc sulphate solution, adding sufficient crystals of zinc sulphate from the stock bottle to ensure saturation, and a small quantity of pure mercury. Shake these up well together to form a paste of the consistence of cream. Heat the paste sufficiently to dissolve the crystals, but not above a temperature of 30°. Keep the paste for an hour at this temperature, agitating it from time to time, then allow it to cool. Crystals of zinc sulphate should then be distinctly visible throughout the mass; if this is not the case, add more crystals from the stock bottle, and repeat the process.

This method ensures the formation of a saturated solution of zinc and mercurous sulphates in water.

The presence of the free mercury throughout the paste preserves the basicity of the salt, and is of the utmost importance.

Contact is made with the mercury by means of a platinum wire about No. 22 gauge. This is protected from contact with the other materials of the cell by being sealed into a glass tube. The ends of the wire project from the ends of the tube; one end forms the terminal, the other end and a portion of the glass tube dip into the mercury.

To set up the Cell.

The cell may conveniently be set up in a small test tube of about 2 cm. diameter, and 6 or 7 cm. deep. Place the mercury in the bottom of this tube, filling it to a depth of, say, 1·5 cm. Cut a cork about ·5 cm. thick to fit the tube; at one side of the cork bore a hole through which the zinc rod can pass tightly; at the other side bore another hole for the glass tube which covers the platinum wire; at the edge of the cork cut a nick through which the air can pass when the cork is pushed into the tube. Pass the zinc rod about 1 cm. through the cork.

Clean the glass tube and platinum wire carefully, then heat the exposed end of the platinum red hot and insert it in the mercury in the test tube, taking care that the whole of the exposed platinum is covered.

Shake up the paste and introduce it without contact with the upper part of the walls of the test tube, filling the tube above the mercury to a depth of rather more than 2 cm.

Then insert the cork and zinc rod, passing the glass tube through the hole prepared for it. Push the cork gently down until its lower surface is nearly in contact with the liquid. The air will thus be nearly all expelled, and the cell should be left in this condition for at least 24 hours before sealing, which should be done as follows.

Melt some marine glue until it is fluid enough to pour by its own weight, and pour it into the tes tube above the cork, using sufficient to cover completely the zinc and soldering. The glass tube should project above the top of the marine glue.

The cell thus set up may be mounted in any desirable manner. It is convenient to arrange the mounting so that the cell may be immersed in a water bath up to the level of, say, the upper surface of the cork. Its temperature can then be determined more accurately than is possible when the cell is in air.

It is clearly desirable to determine whether these instructions are sufficient to enable different makers to produce cells having the same E.M.F., and to investigate in what points a slight departure from the instructions may be made without materially affecting the E.M.F. of the cell.

The following paper contains an attempt to answer the questions :—

(1.) How far cells made with different samples of materials have the same E.M.F. ?

(2.) How far batches of cells from different makers agree with our standard ?

(3.) What are the chief sources of variation in a Clark cell ?

We have examined over 100 cells, of which we shall now give the details.

The values have generally been given in terms of the cell No. 1, constructed by Lord Rayleigh over eight years ago.

The constancy of this cell has been ascertained by comparison with other standards, and by the fact that it was our ultimate standard of reference for the absolute value of the E.M.F. in the investigation already described. The result of this agrees almost exactly with Lord Rayleigh's.

The method of comparison already described in Part I. has been employed in all the later comparisons. Two Leclanché cells are allowed to work through two resistance boxes, with a total resistance of 10,000 ohms, in circuit. One of the Clarks, usually the bottle cell, is connected through a galvanometer and a high resistance to two points on this circuit, and the resistances adjusted until the potential difference between these points just balances the E.M.F. of the Clark. This Clark is then connected in turn in opposition with the other cells of which the E.M.F. is required, and the difference between the electromotive forces of the two determined in terms of the fall of potential along the Leclanché circuit.

We found that this fall of potential changed very slightly throughout our experiments. The fall corresponding to 1 ohm is very approximately ·00025 volt, and we have expressed the differences between the various cells examined in terms of this unit. We will collect, in a series of Tables, the results obtained, together with descriptions of the method of construction.

§ 14. *Tests on various Cells used as Standards. Cells set up by Lord* RAYLEIGH, *Dr.* SCHUSTER, *Mr.* ELDER, *and Mr.* CALLENDAR.

Table II. refers to cells of which most were not made by ourselves, but which have either been in the Laboratory for some time or have been lent us by other makers.

In some cases comparisons of an early date made by other experimenters have been included. In such cases the standard used is not always quite definite, but details will be given. We proceed to describe the cells.

Cell No. 1.—This is our standard cell, made in 1883 by Lord RAYLEIGH, probably the No. 4 of his paper, 'Phil. Trans.,' 1884: "It was prepared generally according to directions given by Dr. ALDER WRIGHT ('Phil. Mag.,' July, 1883). The saturated solution of zinc sulphate was nearly neutral. The metallic zinc was bought as pure from Messrs. HOPKIN and WILLIAMS. The mercurous sulphate was from the same source, and the mercury was re-distilled in the laboratory."

Cell No. 2.—Two other cells, probably Nos. 8 and 9 of Lord RAYLEIGH's paper, were left by him in Cambridge. They were set up in January, 1884. One of these has since dried up, the other is our Cell No. 2. The same materials as for the Cell No. 1 were used. According to Lord RAYLEIGH's tests, when first made, they were slightly higher—by some two or three of our units, than our Cell No. 1.

Cells Nos. 3*a*, 3, 4, 5.—These are cells of the H form suggested by Lord RAYLEIGH, made in 1886, at Wellington College, by Mr. H. M. ELDER. Originally there were five cells; the other has been destroyed through the bursting of the tube containing the zinc amalgam. They were prepared according to the descriptions given by Lord RAYLEIGH in his paper, from materials obtained from various sources, and when tested soon after being made, were fairly close to his standard.

The first set of comparisons quoted were made by Mr. E. H. GRIFFITHS, who, at the same date, tested the Rayleigh cells Nos. 1 and 2.

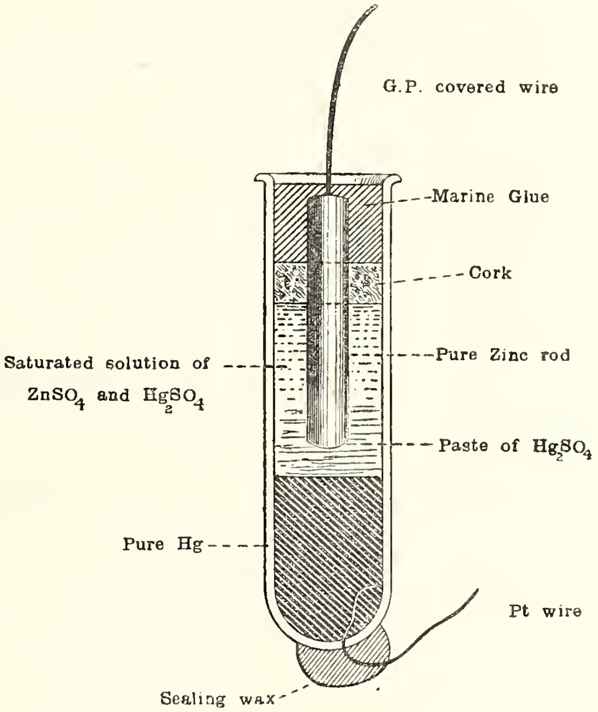
Cells Nos. 6, 7, 8, 9, 10, are five cells made in 1886, by Mr. H. L. CALLENDAR, marked by him as I., II., III., IV., V. They are of the pattern described by Lord RAYLEIGH in his Second Paper, 'Phil. Trans.,' 1885. It will be convenient to quote here his directions, as they will be frequently referred to.

"The zinc sulphate is prepared in a flask by mixing distilled water with about twice its weight of crystals. A little carbonate of zinc is added to neutralize the free acid, and the solution is effected with the aid of *gentle heat*. If time can be afforded, it is a good plan to let the solution stand, as a good deal of iron is usually deposited, even when 'pure' zinc sulphate is used. The solution may then be filtered in a warm place, into the stock-bottle.

"When it is intended to charge H cells, or to prepare paste, the bottle should be exposed to gentle warmth for a few hours, and the solution drawn with a pipette from near the crystals at *the bottom of the bottle.* Otherwise there is no security that the liquid used will be saturated.

"To prepare paste we may rub up together in a mortar 150 grms. mercurous sulphate, as purchased, 5 grms. zinc carbonate, and as much of the saturated solution as is required to make a thick paste. Carbonic anhydride is liberated, and must be allowed a sufficient time to separate. I have found it convenient to leave the paste in a mortar for two or three days, rubbing it up at intervals with additions of the zinc sulphate solution, until the gas has escaped. By the addition of a small crystal, and by evaporation, we have security that the paste is saturated, and will remain so, notwithstanding such moderate elevation of temperature as the cells are expected to bear. The paste may then be transferred to a tightly corked bottle, and, so far as my experience extends, will remain available for many months at least. Before pouring the bottle of paste should be well shaken up."

Fig. 3.



The cell takes the form of a small tube with the platinum wire sealed in at its lower end. (See fig. 3.) "In charging the cells the first step is to pour in sufficient *pure* mercury to cover the platinum effectively. The paste is introduced with the aid of a small funnel, care being taken not to soil the sides above the proper level. The rods, cut from rods of pure zinc, as supplied by HOPKIN and WILLIAMS and not re-cast, are soldered to copper wires and cleaned in the lathe. Just before use they are dipped in dilute sulphuric acid, washed in distilled water, and dried with a clean cloth or filter paper. Each zinc is mounted in a short piece of cork fitting the tube (but not too tightly) and nicked in order to allow of the passage of air. The cork is pushed gradually down until its lower face is almost in contact with the paste. The object is to leave but little air, and at the same time to avoid squeezing up the paste between the cork and the glass. The whole is now made tight by pouring marine glue over the cork, high enough to cover the zinc and soldering and leave only the wire projecting. The tube should rise high enough to receive the glue and thus secure a good adhesion.

In the operation of pouring in the marine glue the glass is heated by the glue sufficiently for adhesion ; but this heat does not extend appreciably below the cork.

Neither in this nor in any other stage of the process of charging is heat applied to the paste."

Mr. CALLENDAR's cells were prepared in this way, except that the zinc sulphate solution was boiled with zinc oxide. Lord RAYLEIGH recommends the use of "gentle heat," not boiling. The area of the zinc immersed is various in the different cells, being small in No. 6, large in Nos. 9 and 8. These cells have always been very close to the standard. They were compared with the standard No. 1 by Mr. GRIFFITHS, in August, 1890. In February, 1891, they were taken to London and compared there with the cells in the possession of the Board of Trade.

Cell No. 90.—This is a large cell made in a wide-mouthed bottle, about 10 cm. high and 7 or 8 cm. in diameter, by one of us (S. S.) in July, 1891. The paste was prepared according to the Board of Trade memorandum, and is the same as that used in cells Nos. 71–79. (See later, p. 603.)

The zinc was ordinary commercial plate zinc not amalgamated. At first the cell was too high by 25 of our units ; crystals of zinc sulphate were dropped in on July 9, and on July 10 it was 14 units too high. From this date onwards the tests are given in the Table.

Cell No. 97.—A large cell set up by Professor SCHUSTER, and brought by R.T.G. from Manchester, on July 11. The cell is set up in a kind of small WOULFE's bottle about 5 cm. across, and some 7 or 8 cm. high. The central opening at the top contains a thermometer ; the zinc and platinum pass in glass tubes through the other openings, being secured with sealing wax into glass tubes. There is also a mercury valve which permits of the escape of gas if any is evolved.

Dr. SCHUSTER writes as follows of the cell :—" The cell was made with mercurous sulphate prepared and treated as follows :—Mercurous nitrate was prepared by dissolving mercury in nitric acid and boiling up the solution thus formed with mercury in a flat-bottomed flask, the mercury completely covering the bottom of the flask. The salt was crystallised out from the solution and dried with filter paper. The mercurous nitrate was then dissolved in an 8 per. cent. solution of nitric acid and precipitated with sodium sulphate. The precipitate was filtered, washed, and strained by pressure, then once washed with a saturated solution of zinc sulphate. The sulphate was then boiled for half-an-hour in a conical flat-bottomed flask containing enough mercury to cover the bottom, some zinc sulphate solution and crystals being added before treating ; the mercurous sulphate was then transferred directly to the cells, the E.M.F. directly after setting up (April 30) was -28, the unit being 10^{-4}

of a Clark cell. On May 5, it was − 12, and on May 25, ± 0, the standard being an old carefully prepared small cell agreeing closely with a number of other cells. About a dozen comparisons were made between the end of May and July 11, and it was within a few parts in 10,000 of the standard always."

These last two cells could be immersed in a water bath, and were always so used, side by side. The two Rayleigh cells Nos. 1 and 2, were enclosed in two large test tubes, and these were immersed in water. In some of the earlier experiments they could not conveniently be put into the same bath as Nos. 90 and 97 ; in the later experiments the four cells were side by side. Even when this was done there was some slight uncertainty as to the temperatures of Nos. 1 and 2, on occasions when the temperature of the air differed by 1° or so from that of the bath ; for then the temperature given by a thermometer placed in the test tubes with the cells, differed somewhat from that of the water. In these cases the temperature of the thermometer in the tube was taken as the temperature of the cell. The other cells could not be immersed, and their temperatures were taken by a thermometer supported close to them. As in all cases, with one exception noted below, in which experiments were made, the air temperature varied very slowly, the uncertainty introduced was not large. In the numbers given in the Table, allowance has been made for the difference in temperature, if any, between the cells and the standard. The numbers are reduced on the assumption that the standard has the same temperature as the cell, and that its E.M.F decreases by 4·3 of our units per 1° C., and give the excess of the E.M.F. of the cell in question over our Standard No. 1 in all cases except those in which there is a special note.

It will be seen from the Table that the Cells 1, 2, 90, and 97, remained very close together throughout, and that too, over a considerable range of temperature. An exception must be made for No. 97 on November 3, when it was unaccountably low. Finding it remain so, we removed the zinc by withdrawing the glass stopper, and rubbed it on a clean silk handkerchief. We also dried and rubbed with emery cloth the upper part of the cell and the ebonite terminals to which the wires were attached. After this the cell recovered its value.

The experiments on November 19 were made with the object of testing the effect of a rather sudden change of temperature. The heating apparatus in the room had been turned off for about a week previous to November 18. After the observations of that day it was turned on, and the temperature of the bath rose about 5° C. in 24 hours. It will be seen that the cells are all wrong. The large cells 90 and 97 are too high (assuming the standard right). They have the E.M.F. corresponding to a lower temperature, lower by about 2°·5 C. The liquid which was saturated at 8°·5 has not had time to reach the state of saturation corresponding to the new condition. But the standard is itself too high, for No. 2 has fallen with respect to it by some seven divisions.

Now No. 2 is a smaller cell than No. 1 and contains more liquid. The zinc sulphate in No. 1 is very full of crystals, and it is reasonable to suppose that it takes

TABLE II.

Date	August, 1890.	February, 1891.	May 21, 1891.	July 20, 1891.	August 6, 1891.	August 10, 1891.	August 22, 1891.	November 1, 1891.	November 3, 1891.	November 14, 1891.	November 18, 1891.	November 19, 1891.	November 23, 1891.	November 30, 1891.	December 2, 1891.	December 7, 1891.	December 16, 1891.	December 22, 1891.	March 9, 1892.	January 11, 1892.
Temperature of No. 1 .	15	..	15·2	18	16·5	16·4	16·4	11·1	14·2	9·2	8·5	13·8	14·1	13·3	14·7	16	15·6	0	11·5	8·2
,, of other cells	..	..	15·2	18	16·4	16·4	16·4	11·1	14·2	9·2	8·5	13·8	13·1	12·9	14·5	15·5	15·4	0		
Cell No 2	..	..	0	..	−2	−2	..	..	−3	−2	−2	−9	0	−1	−2	0	−1	−2		
,, 90	..	..	..	5	2	1	0	4	1	2	3	14	1	0	2	0	0	0		
,, 97	..	..	..	4	1	1	−1	−2	−7	−1	0	11	1	1	3	2	1	—	..	*−1
Temperature of cells .	15	16	15·2	..	17·6	17·4	..	..	..	11·1	..	..	..	..	..	..	16·2			
Cell No. 3A	..	..	−3	..	−3	..	..	..	..	−8										
,, 3	−2	..	−6	..	−3	..	..	..	..	−20										
,, 4	−4	..	−5	..	−7	..	..	..	..	−9										
,, 5	−5	..	−5	..	−6	..	..	..	..	−7										
,, 6	−2	−1	..	..	1	3	..	..	..	3	..	..	..	..	..	..	3	..	*—	
,, 7	−2	−3	..	..	1	3	..	..	..	5	..	..	..	..	..	..	0	..	2	
,, 8	−1	0	..	..	2	4	..	..	..	5	..	..	..	..	..	..	3	..	3	
,, 9	0	0	..	..	1	4	..	..	..	5	..	..	..	..	..	..	3	..	4	
,, 10	0	0	..	..	1	3	..	..	..	3	..	..	..	..	..	..	2	..	3	

* Comparison with the Board of Trade Standard.

this semi-solid mass longer to reach the equilibrium condition than is required for the liquid in No 2.

To obtain the observations on December 22, the heating apparatus had been turned off and the windows of the room left open on the two previous nights. On December 21 there was a thick coating of ice on the bath which continued for one or two days; longer observations were made on December 22. Dr. SCHUSTER'S cell, No. 97, had been taken to London a few days previously.

A number of observations on the difference between the Cell No. 90 and the standard were made during the absolute determination of the E.M.F. Several of these confirm the fact that No. 90 gained its temperature equilibrium more slowly than No. 1. Thus, between the morning and evening of August 14, the temperature of the bath rose by 1°. The difference between No. 90 and No. 1 increased from 2 to 5 units.

Other observations again showed that it takes some time for No. 1 to change its value. Thus, on August 15, the bath was at $18^{\circ}\cdot1$, and the air in the test tube round No. 1 was about $0^{\circ}\cdot5$ higher; the difference in E.M.F. between Nos. 90 and 1 was 5. No. 1 was then taken out of the test tube and placed in the bath with No. 90. In about three hours the difference had fallen to 3, at which it remained for some time.

Similar effects were noted again during the early part of this year. On January 19 the room had been cooled down to about 9° C. About a week later the difference between No. 90 and No. 1 was 11, the temperature being 13°. The next day the difference was 8, and it was noticed that there was a thick compact layer of crystals at the bottom of the clear liquid. The cell was slightly shaken so as to mix up the sulphate of zinc which was clearly saturated below, but not above. The next day, Tuesday, January 26, the cells were together. On January 28, the temperature having risen to 15°, the difference had increased to 4; this increased to 5 the next day, the temperature of both cells being 16°, and some time more elapsed before the cells again came together.

It will be seen that the other cells are fairly near the standard.

Mr. ELDER'S H cells, 3*a*–5, are slightly too low, and it would appear as if their temperature coefficients were different.

The observation for No. 3, on November 14, shows distinct change, but we cannot attach great weight to the one isolated observation. It must be remembered, also, that it is difficult to read the temperature with great exactness.

It would appear as if Mr. CALLENDAR'S cells, Nos. 6–10, had risen in value between February and August. This is quite possible, for the insulation was originally somewhat defective—the wires from both poles of each cell were fastened down to a piece of deal.

It is probable, however, that their temperature, specially on November 14, is not very accurately known. The temperature of the room on that day, as indicated by a thermometer placed near the cells, rose by about 0°·4 in about an hour previous to the experiment, it is probable, therefore, that the E.M.F. of the cells corresponds to a temperature some few tenths lower than that given, and that the numbers are slightly too high; in any case the differences are small.

§ 15. *Tests on Old Cells in the Cavendish Laboratory.*

The cells of this set have been made at various dates by members of the advanced class in practical electricity at the Cavendish Laboratory, and are all of the pattern described by Lord Rayleigh ('Phil. Trans.,' 1885). Their comparison was begun in November, 1889, and has been continued. The standard against which the earlier comparisons up to December, 1889, were made is slightly doubtful, and also the temperature at which these were made.

The cells are interesting, for they show the kind of accuracy reached by experimenters putting up one or two cells from printed directions, with only such help as is usually given by the demonstrator. The materials have been purchased from the same sources as they were at the time when Lord Rayleigh was at Cambridge, and so they may be taken to be of approximately the same quality. Several cells belonging to this set have unfortunately been broken. Cell No. 23, which shows a very high E.M.F., is known to have been unsaturated with $ZnSO_4$.

Table III.

Date.	Nov. 7, 1889.	Nov. 13, 1889.	Dec. 3, 1889.	Mar. 16, 1891.	Mar. 21, 1891.	April 21, 1891.	May 2, 1891.	May 21, 1891.	Aug. 11, 1891.
Temperature.	..	..	..	13°	10°	13°	16°·4	15°·2	17°·6
Number									
11	+ 4	+ 0	..	− 4	+ 1	− 3	−3	−5	−1
12	+ 2	+ 8	..	− 5	− 0·5	− 5	−4	−6	−4
13	+ 8	+ 8	..	− 1	− 1	− 2	−3	−5	−1
14	− 4	+ 2	..	− 9·5	+ 4	−11	−4	−5	+2
16	+ 8	+16	..	− 0·5	− 2	+ 1·5	+7	−3	+3
17	+64	+16	..	− 5	−10	− 5·5	+2	−3	+6
19	..	..	+ 6	+ 0	− 6	− 1			
20	..	..	..	+ 0	+ 1	− 2·5			
21	− 4	+ 6	+ 0	− 2	− 7	− 5			
22	..	..	− 2	+ 0	+ 2	− 3			
23	+32	+36	+28	+31					
24	..	+ 8	+ 2	+ 2	+ 0	+ 1			

In these cells the paste was neutralised with zinc carbonate.

We come now to the observations on cells set up for the purposes of this enquiry.

§ 16. *Treatment of the Mercurous Sulphate.*

In studying the effect of variations in the materials, the impurities usually found in the mercurous sulphate appeared to be of primary importance. It is not easy to purchase this substance in a state of purity. The common preparations usually met with are either grey or very white. The grey powder contains mercury in excess; this does not appear to be hurtful to the cell; the white contains a considerable quantity of mercuric sulphate. This is shown by the substance turning yellow on adding water, with the formation of turpeth mineral or basic mercuric sulphate. This also is not harmful in small quantities, but its presence shews that the original powder has contained mercuric sulphate. This has been resolved by the water with the formation of the yellow substance and a soluble acid mercuric sulphate, and unless this process is complete and all traces of the original mercuric salt have been removed, error will be introduced.* Thus good cells may be set up with paste which is yellow at starting, provided the change is complete, but if the paste of a cell turns yellow after the cell has been made, it means that the mercuric salt is present, and this affects the E.M.F. On treating a sample containing mercuric sulphate with mercury, the mercuric sulphate is converted into mercurous, and may be used with safety.

The fact that the mercury mixes shows, as Mr. SWINBURNE has remarked, that the sulphate is bad, but the result of the mixing is that the evil is cured. This, then, is the *rationale* of the method of preparing the paste given in the memorandum. The washing with water resolves most, if not all, of the mercuric salt present into the yellow basic sulphate $3HgO.SO_3$, and the soluble sulphate $HgSO_4.2SO_3$; the latter is dissolved and removed. The shaking with mercury at a temperature near 30° removes any further traces of the acid mercuric salt. The temperature should not exceed 30°, for, when cooling from a temperature in excess of this, crystals may form which are not of the proper composition $ZnSO_4.7H_2O$.

§ 17. *Cells set up by Mr.* L. R. WILBERFORCE, *March*, 1891.

This set of cells was made by Mr. L. R. WILBERFORCE during March, 1891 (probably on March 19), and are in short test tubes with platinum wires fused through their bases for terminals. They cannot, therefore, be directly immersed in water. They were constructed with a view of finding the alterations in E.M.F. which would result from slight modifications in the materials and in the modes of preparing the cells. The last tests show them to be all in very close agreement amongst themselves and with cell No. 1.

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* According to WATTS' 'Dictionary' the mercuric sulphate in presence of water becomes

$$\begin{cases} HgSO_4.2SO_3 \text{ soluble.} \\ 3HgO.SO_3 \text{ turpeth mineral.} \end{cases}$$

We shall now describe the materials of which these cells were prepared.

The zinc rods in all were of pure stick zinc from Messrs. JOHNSON and MATTHEY. They were cleaned with a file and sand-paper, and generally washed in a little dilute sulphuric acid. They were allowed to stand in distilled water until required for use, when they were carefully dried.

The nature of each of the materials used in setting up the cells is indicated by Greek letters $\alpha \ldots \kappa$ in the earlier columns of the Table (p. 596).

α is a saturated solution of zinc sulphate prepared by boiling a strong solution with zinc oxide, filtering whilst hot to remove excess of zinc oxide, and again when cold to remove the basic zinc sulphate $ZnO.ZnSO_4$, which had then crystallised. Afterwards it was evaporated to such a strength that crystals of $ZnSO_4.7H_2O$ separated at the temperature of the room. The solution was so thoroughly neutralised that it only turned a blue litmus paper to a port-wine colour. Although great care had been thus taken to remove all basic zinc sulphate, it is very probable that the solution still contained some traces of this compound.

β is a saturated solution of zinc sulphate prepared entirely below 30° C. Zinc carbonate was added to remove any excess of acid. This solution, when cooled to the temperature of the room, deposited crystals which were, without doubt, $ZnSO_4.7H_2O$. It turned a blue litmus to a port-wine colour. The reason for limiting the heating to 30° only was to ensure that the crystals separated on cooling should contain the right number of molecules of water of crystallisation. Solutions which deposit crystals above this temperature may give hydrates containing less water. The zinc sulphate had been prepared according to the directions of the 'British Pharmacopœia,' and was marked (Brit. Pharm.).

For the mercurous sulphate five different specimens were tried.

γ and ϵ are two samples of paste prepared in the same way from a specimen of mercurous sulphate supplied by Messrs. HARRINGTON, of Cork. This substance was not pure, it contained some mercuric and basic mercuric sulphates, which was clearly shown by its turning bright yellow on adding water. These pastes were prepared by mixing at the temperature of the room some of the mercurous sulphate, a little pure mercury, and sufficient of the saturated zinc sulphate β to make a cream. This was vigorously shaken at intervals for two days before use. In general appearance it was white, with minute grey globules, which were collected in a more dense layer in the lower part of the bottle.

δ and ζ. These pastes were prepared at the temperature of the room from very pure mercurous sulphate and the zinc sulphate solutions α and β. The mercurous sulphate was prepared at the laboratory by precipitating pure mercurous nitrate (which was obtained by allowing an excess of pure mercury to stand with nitric acid), with sodium or potassium sulphate, when mercurous sulphate falls down as a white crystalline powder, and the nitrate of the alkaline metal remains in solution. This white precipitate is collected on a filter paper and washed with cold distilled water until the washings no longer give any reaction for a nitrate. It is useless to test the wash water for sulphate as mercurous sulphate is slightly soluble in water. The moist precipitate, after proper washing, is dried in a current of air at about 17° C. Prepared in this way, the powder, which is dull white, will not become yellow when moistened with water, nor will the water give an acid reaction. There is no mercuric sulphate present. It will slowly become grey if, whilst moist, it be exposed to light. ζ is a paste containing mercurous sulphate prepared according to this method. In the preparation of the mercurous sulphate for δ, however, very dilute solutions were used so that no immediate precipitate of sulphate came down. After four or five days white crystals about $\frac{1}{2}$ to 1 millim. in diameter separated. This is the purest mercurous sulphate we have used.*

η. The sample of mercurous sulphate in these cells was prepared from mercuric sulphate by reducing it with an excess of pure mercury. The white mercuric salt is placed in a mortar, and some pure mercury added, with sufficient water to convert the whole into a paste ; it is then ground until the whole becomes grey. The end of the reaction is easily recognised by the appearance of the surface of the globules of mercury, for, during the time that chemical action is going on, their surfaces will appear tarnished. The sulphate prepared in this way is grey with excess of mercury.

The various samples of mercury are denoted by θ, ι, and κ.

θ. Pure mercury distilled under reduced pressure in the laboratory.

ι. Mercury which had been used for general purposes, and was only filtered before being used for the cells.

κ. Mercury distilled in a small glass retort at atmospheric pressure.

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* A very useful suggestion is made by Mr. SWINBURNE in his paper before the British Association, 1891, in which he recommends the use of zinc sulphate solution for the purpose of precipitating the mercurous sulphate, and washing with saturated zinc sulphate to remove zinc nitrate. The sulphate then does not require drying.

TABLE IV.

Date.	Zinc sulphate.	Mercurous sulphate.	Mercury.	March 21.	April 21.	May 2.	May 21.	August 7.	August 11.	November 14.	December 15.	
Temperature				10	13	17	15	16·8	17·8	10·8	16·5	
Number.												
25	α	γ	θ	− 2	1	2	1	−1	2	2	0	
26	α	γ	θ	− 3	4	8	5	0	4	4	2	
27	α	δ	θ	1	6	7	6	0	3	2	2	
28	α	δ	θ	0	5	10	3	0	4	3	6	
29	α	ϵ	θ	− 2	− 1	1	3	1	2	1	−1	No crystals added
30	α	ϵ	θ	− 8	1	5	0	−2	2	2	4	Much crystals added
31	β	ζ	ι	−40	−16	−12	4	−1	3	6	3	
32	β	ζ	ι	−51	−28	−19	−25	−1	4	6	3	
33	β	ζ	κ	− 3	1	3	2	1	5	4	2	
34	β	η	κ	− 9	2	3	0	−1	4	4	0	
35	β	η	κ	− 5	− 8	− 2	− 6	−2	3	4	2	

The results are of much interest. All the cells are now within about 1 in 1000 of the standard; for, as we have said, one of our units is ·00025 volt; thus six of these units is ·0015 volt, or rather over 1 in 1000. The first six cells have been, it will be seen, nearly right from the beginning; those with the good mercurous sulphate δ were, two days after being set up, extremely close to the standard, while 25 and 26, 29 and 30, in which the bad sulphates, γ and ϵ, were used after treatment with the mercury, were rather too low. Nos. 31 and 32 were originally far too low; this would appear to be due to impurity in the mercury, for the zinc and mercurous sulphates were good, while the mercury was only cleaned by filtering. Moreover No. 33, which only differs from No. 32 in having distilled mercury, is nearly right.

The mercury used in Nos. 33, 34, 35 was distilled at atmospheric pressure, and the somewhat low results given by these cells may be due to this; in the case of Nos. 34 and 35 it would appear more likely that they have a similar source to the low numbers found in Nos. 25, 26, 27, and 30. The differences of E.M.F. had, however, been considerably reduced, except in the case of No. 32, some two months after making the cells, while at the next observation, after an interval of three months more, the differences had practically vanished. In the first three comparisons the reference to the standard was somewhat indirect; the absolute values of the numbers given cannot be trusted to very high accuracy, and the temperature is a little uncertain. It appears as if the values of the E.M.F. of Nos. 31 and 32 are still rising.

This set then shows the tendency of Clark cells made with different materials to settle down to the same value in time.

§ 18. *Cells set up by S.S. in March and April,* 1891.

This set of cells are all in short test tubes of the RAYLEIGH pattern, and were set up by one of us (S.S.) in March and April, 1891. They contain materials which are indicated as above by Greek letters. Some of these are the same as were used by Mr. WILBERFORCE; particulars of the others are given. λ. This paste was prepared from HARRINGTON'S specimen of mercurous sulphate, which was washed by decantation and on a filter paper until the washings were no longer acid. During this operation the paste became yellow from the conversion of the mercuric sulphate present into basic mercuric sulphate and sulphuric acid. The basic mercuric sulphate or turpeth mineral being insoluble, remained with the mercurous sulphate, whilst the acid passed away with the washings. After washing, pure mercury was mixed with the paste, and heated with it in a pan of boiling water, with the view of attacking any yet undecomposed mercuric sulphate. The paste was allowed to remain with the mercury for two or three days, and then it was dried at the room temperature. To prepare it for use in the cells it was rubbed with saturated zinc sulphate to the consistence of a cream. The paste, therefore, contained mercurous sulphate, zinc sulphate, mercury, and traces of basic mercuric sulphate. This was an attempt to prepare the paste in accordance with instructions given verbally to one of us (R.T.G.) by Dr. ALEXANDER MUIRHEAD, to whose great knowledge of the subject we are much indebted.

Different mercury was used in some of these cells. This (μ) was a sample treated with nitric acid for some days, then washed and afterwards distilled *in vacuo*.

It will be seen from Table V. that the cells all start by being too low, but that in from three to four weeks of the time they were set up, the first seven, those made in March, have attained a steady value not far from the standard. The observation of 52, on August 7, is abnormal. The first six cells were lent for some other experiments in August.

The cells 50 and 51, made of the same materials as 25 and 26, agree with them in being too low. At first, the cells in which the mercury κ was used, are a good deal too low. Nos. 48 and 49 are of the same material as 27 and 28.

The first two tests recorded on these cells, made immediately after the cells were set up, were very rough, being performed by joining the cell up in opposition with another Clark, and passing a current from the two through a galvanometer and 20,000 ohms. They were merely intended to see that there was no great error. They give us, however, some important information, for the E.M.F. sank rapidly between March 17 and March 21, but regained its normal value by April 12.

The observations, given in Table IV., on Mr. WILBERFORCE'S cells, Nos. 25–35, were made two days after they were set up, and we do not possess for them in such

TABLE V.

Date.	Zinc sulphate.	Mercurous sulphate.	Mercury.	Date of construction.	March 17.	March 18.	March 21.	April 12.	April 21.	April 25.	April 28.	May 2.	May 21.	August 7.	August 11.	November 14.	November 18.	November 19.	December 15.
Temperature.					..	..	10	..	13	..	16·5	16·5	16·6	16·8	16·6	11·1	11·3	15	17·2
Number																			
48	a	δ	θ	March 17	− 1	− 4	−15	0	2	..	3	5	..	6					
49	a	δ	θ	„	− 2	− 7	−11	0	2	..	4	4	..	4					
50	a	γ	θ	„	− 6	− 8	− 3	1	3	..	2	3	..	0					
51	a	γ	θ	„	−18	−12	− 9	− 5	− 5	..	− 9	− 8	..	− 5					
52	a	ζ	κ	„ 18	..	− 5	−24	− 5	− 4	..	1	0	..	9 ?					
53	a	ζ	κ	„	..	−10	− 4	− 4	− 2	..	4	3	..	1					
56	a	η	κ	„	..	..	−16	− 7	− 5	..	− 2	− 3	..	− 5	− 2	2	3	*16	2
57	a	η	μ	April 25	..	..	..	..	..	− 2	14	..	..	18	21	4	4	16	17
58	a	η	μ	„	..	..	..	..	..	− 2	14	..	..	− 4	− 5	−15	−12	3	2
59	a	λ	μ	„ 26	..	..	..	..	..	..	− 1	3	− 2	−23	−16	−30	−36	− 5	−39
60	a	λ	μ	„	..	..	..	..	..	..	− 1	2	− 4	− 3	0	1	3	15	− 3

* These observations are introduced to show the uncertainty introduced when sudden changes of temperature occur.

full detail the same information as to the changes which occur immediately after setting up. These do not occur in the cells 50, 51, and 53, at least to so marked an extent, but the effect is shown in 52.

Now, Nos. 50 and 51 are made of the same materials but crystals of zinc sulphate were added to 50, and the zinc rod was pushed a good deal further down so as to be well immersed in the paste which settles above the mercury. This may produce an important difference.

Nos. 57 and 58 were made simultaneously on April 25. The liquid used was purposely not warmed at all, and had been standing for some time in the room. It was saturated at the temperature of the room. The two cells agree in having a smaller temperature coefficient than the standard, but they differ in E.M.F. There is probably some slight difference in the temperatures at which the two are saturated. The results are only of interest as emphasising the importance often insisted on by others of introducing crystals of zinc sulphate, and making sure that they are in excess at all the temperatures at which the cell will be used.

In cells 59 and 60 the treatment was the same, except that in 59 the zinc rod was washed as usual with dilute sulphuric acid, and in 60 this was omitted. No. 60 has agreed with the standard throughout, in No. 59 there was a sudden and unexplained drop in E.M.F. between May 21 and August 7. Of the above cells, Nos. 57 and 58 are not Clark cells as defined in the specification. The others, with the exception of No. 59, attained their normal value within a month of being set up, and have retained it since.

§ 19. *Cells of Modified Pattern set up by* L.R.W. *and* S.S. *in March,* 1891.

The next set of cells were put up in the form indicated in the Board of Trade specification, p. 601, though at some time previous to its issue. The test tubes used, however, were rather larger, being about 10 cm. high and 2 to 2·5 in diameter. The platinum wires making contact with the mercury are enclosed in glass tubes, and were heated to a red heat before being placed in the mercury, so that proper contact was ensured.

Nos. 36 to 41 were put up by Mr. WILBERFORCE on March 18. In the first three the platinum wire is sealed into the glass tube which encloses it, in the last three this has not been done. On April 21 Nos. 36, 37, 40, and 41 were much too low, and it was noticed that there was a thick grey deposit on the zincs, the zincs from these cells and from No. 38 were withdrawn and scraped, they were then replaced and again tested, the results are somewhat too high, but from that time on, with the exception of No. 41, they have been fairly consistent. Nos. 61, 62, 63 were cells of the same pattern, made on March 18, by one of us (S.S.), at the same time as No. 56 and of the same materials, while No. 64 was made in the same way as No. 60 on April 24.

TABLE VI.

Date.	Zinc sulphate.	Mercurous sulphate.	Mercury.	Date of construction.	March 20.	March 21.	April 12.	April 21.	April 21.	April 23.	April 28.	May 2.	May 21.	May 25	August 11.	August 12.	August 15.	November 14.	November 23.	November 26.	December 2.	December 7.
Temperature.					..	10	..	13	16	14	16·5	16·6	15·2	..	16·8	16·9	18	9·4	13·3	13·8	14·9	16
Number.																						
36	β	ζ	κ	March 18	−43	−51	..	−15	4	3	4	− 1	− 4	..	1	− 1	2	3	..	..	..	1
37	β	ζ	κ	,,	− 6	− 5	..	−15	5	5	8	7	Broken									
38	β	ζ	κ	,,	− 5	− 2	..	0	3	3	7	5	3	..	1	− 1	3	3	..	..	..	3
39	β	ϵ	κ	,,	− 8	− 6	..	− 3	− 2	− 1	0	0	1	..	− 1	0	− 1	0	..	..	..	− 3
40	β	ϵ	κ	,,	−43	−57	..	−12	5	0	2	1	− 4	..	0	0	0	1	..	..	..	1
41	β	ζ	κ	,,	−43	−53	..	−35	5	−13	− 6	− 5	−10	..	− 9	0	− 9	− 8	..	..	..	..
61	α	η	κ	March 18	−12	−25	−13	−15	..	−14	− 8	−11	−18	−19	* ..	..	..	−16	− 2	− 2	1	3
62	α	η	κ	,,	− 6	−10	− 1	0	..	− 2	2	1	0	− 1	..	..	..	−17	−10	−10	− 9	− 3
63	α	η	κ	,,	−11	−33	− 7	−10	..	− 8	− 3	− 5	−14	−11	..	Broken						
64	α	λ	μ	April 26..	..	..	..	..	..	..	− 1	2	1	1	..	..	..	− 5	2	0	0	2

* The cells were used in the laboratory on one occasion early in June when 63 was broken. It is possible that 61 and 62 were interchanged then. The record after November 14 must be treated as a fresh one

The Table shows that while Nos. 62 and 64 are about right, Nos. 61 and 63 were decidedly too low, though at the time of the latest tests they are all fairly close together. Rough tests of the cells had been made on March 20, one or two days after they were set up. It will be seen that they were then too low, but that they afterwards got worse.

It thus appears that many of the cells we have examined up to this date began by being low, and that this fault increases at first; a grey deposit is also formed on the zinc. After a time, however, in most cases a change takes place, and in about a month or so the cells have ordinarily come right. In the case of this last set the natural development was stopped, and the fault corrected by scraping the zincs and removing the deposit.

It should also be noticed that the solutions are very free from acid.

§ 20. *First Form of Board of Trade Specification and Cells Set up in accordance with it.*

Early in May a provisional memorandum on the cell was prepared, and on May 25 six cells, Nos. 65 to 70, were set up by one of us (S.S.) in accordance with it.

HOPKIN and WILLIAMS' mercurous sulphate was used and treated as prescribed in the memorandum annexed.

MEMORANDUM ON THE PREPARATION OF THE CLARK'S STANDARD CELL.

Definition of the Cell.

The cell consists of zinc and mercury in a saturated solution of zinc sulphate and mercurous sulphate in water, prepared with mercurous sulphate in excess, and is conveniently contained in a cylindrical glass vessel.

Preparation of the Materials.

1. *The Mercury.*—To secure purity it should first be treated with acid in the usual manner, and subsequently distilled *in vacuo.*

2. *The Zinc.*—Take a portion of a rod of pure zinc, solder to one end a piece of copper wire, clean the whole with glass paper, carefully removing any loose pieces of the zinc. Just before making up the cell dip the zinc into dilute sulphuric acid, wash with distilled water, and dry with a clean cloth or filter paper.

3. *The Zinc Sulphate Solution.*—Prepare a zinc sulphate solution of pure ("pure re-crystallised") zinc sulphate by mixing in a flask distilled water with about twice its weight of crystals of pure zinc sulphate, and adding a little zinc carbonate to neutralise any free acid. The whole of the crystals should be dissolved with the aid of gentle heat; and the solution filtered, while still warm, into a stock bottle. Crystals will form as it cools, and when the solution is to be used a little should be taken from the bottom of the bottle near the crystals with a pipette. While this is being done the liquid should be at a decidedly higher temperature than that at which the cells will ordinarily be used.

4. *The Mercurous Sulphate.*—Take mercurous sulphate, purchased as pure, and wash it with distilled water. Drain off as much of the water as possible. Mix the washed mercurous sulphate in a mortar with the zinc sulphate solution, adding sufficient crystals of zinc sulphate from the stock bottle and a small quantity of pure mercury. Mix these well together to form a paste of the consistence of cream,

and then heat the paste sufficiently to dissolve the crystals, afterwards allowing the mixture to cool, and thus to form a saturated solution of mercurous sulphate in a saturated solution of zinc sulphate. The presence of the free mercury throughout the paste preserves the basicity of the salt, and is of the utmost importance.

Contact is made with the mercury by means of a platinum wire about No. 22 gauge. This is protected from contact with the other materials of the cell by being sealed into a glass tube. The ends of the wire project from the end of the tube; to one end a piece of copper wire is soldered, the other end and a portion of the glass tube dip into the mercury.

When tested it was found that the E.M.F. of all the cells was about 1 per cent. too high, though they agreed fairly among themselves. The cells were not saturated with the zinc sulphate, and were probably acid as well.

On May 26, Nos. 65 and 69 were opened, and crystals of zinc sulphate were inserted, while two days later 66 and 70 were also opened. Crystals were dropped into these latter cells and stirred into the paste.

The history of the four cells thus treated is given in Table VII. On August 14 Nos. 65, 69, and 70, with some other cells, were lent to Mr. E. H. Griffiths. The history of these cells is continued in Table IX. Nos. 66, 67, and 68 have been tested from time to time since, with the result that No. 66 is about right, while Nos. 67 and 68 are not far apart, but are about 30 too high.

Table VII.

Date.	May 25.	May 26.	May 26.	May 27.	May 28.	May 28.	May 29.	May 30.	August 6.	August 10.	August 12.	August 14.	December 7.
Temperature . .	..	..	..	..	14·5	..	..	15	16·2	16·3	17·3	17·5	15
Number													
65	67	*	30	33	19	..	17	12	− 3	− 2	− 2	0	..
66	67	..	..	73	66	†16	14	21	10	8	7	9	− 4
67	67	..	..	71	65	..	60	59	60	61	..	..	30
68	62	..	..	69	61	..	56	61	58	60	..	..	30
69	72	*	15	26	10	..	8	5	− 2	− 2	− 2	0	..
70	68	..	..	68	59	†24	18	19	5	3	2	4	..

It will be noticed that these unsaturated cells were fairly close together from the first, and show every sign of continuing to give results which, though wrong, are consistent among themselves.

* Crystals of zinc sulphate were inserted.

† Crystals of zinc sulphate were inserted and stirred into the paste.

§ 21. *Cells set up in accordance with the Second Form of the Specification.*

Of the cells described in Table VIII., Nos. 71, 72, 73 were made on May 30. The paste was prepared according to Lord RAYLEIGH'S instructions, already quoted, from mercurous sulphate, supplied by HOPKIN and WILLIAMS. The zinc rod was supplied by HARRINGTONS, and is stated to contain a trace of iron. Before using the paste was shaken up with mercury. The mercury used in 72 and 73 was supplied by HOPKIN and WILLIAMS as " pure redistilled."

No. 71 is MUIRHEAD'S portable form; the platinum wire is bent into a spiral, and dipped while red-hot into mercury. There is no free mercury at the bottom of the test tube.

Nos. 74, 75, 76 were made on June 3, and Nos. 77, 78, 79 on June 11, following the directions of the second form of the Board of Trade memorandum, which was issued June 23 (see p. 584). Mercury distilled in the laboratory was used; the zincs were from HARRINGTONS, and the mercurous sulphate from HOPKIN and WILLIAMS. The zinc sulphate, marked "Brit. Pharm.," from HOPKIN and WILLIAMS, was used in all the cells. The cells Nos. 71–76 were taken to the Board of Trade towards the end of June, and brought back to Cambridge on July 18. Nos. 77, 78, 79 were lent to Mr. GRIFFITHS on August 14. Details of their further history are given in Table IX.

It will be seen that the above cells, with the exception of No. 76, agreed with the standard to within one in a thousand from the beginning.

No. 73 was somewhat low during three tests in June, resembling in this respect the cells already put up; in all the other cases the agreement is very close.

Two comparisons with the Board of Trade standards are given at the end of the Table. These will be discussed later. The cells were brought back to Cambridge on July 20. Two other cells, Nos. 86, 87, were made on July 20 in the same way, using mercurous sulphate from SCHUCKHARTS. They were both within one of the standard.

Table IX. gives the tests on the cells lent to Mr. GRIFFITHS. He made careful daily comparisons, using No. 43 as a standard. The results of two of his observations are given in the first two columns; the others were quite consistent. Nos. 69 and 70 were also treated by him as standards. The other four, Nos. 65, 77, 78, 79, were allowed to produce current through a resistance of 20,000 ohms.

On September 13 the E.M.F. of No. 43 fell suddenly by nearly two per cent., and gradually recovered, regaining its normal value on September 17. On September 18 No. 70 was too high, and continued so till September 23, when the tests ceased. On September 21 and 23 No. 43 was too low.

TABLE VIII.

Date . . .	May 30.	June 2.	June 3.	June 4.	June 6.	June 9.	June 11.	July 20.	August 6.	August 10.	August 12.	August 14.	August 22.	November 2.	November 14.	December 7.	Compared at the Board of Trade.	
																	July 7.	July 18.
Temperature . .	..	..	16	16	16	14·5	14·5	18	16·2	16·4	17·4	17·5	16·4	14·4	9·2	15	15	15
Number.																		
71	− 5	− 4	..	− 4	− 1	− 1	..	0	0	0	..	0	0	− 1	− 2	− 2	2	− 4
72	− 5	− 4	..	− 3	− 1	− 1	..	1	1	0	..	0	0	1	1	1	3	2
73	− 4	− 4	..	− 8	− 8	− 8	..	1	1	0	..	1	0	1	1	0	1	2
74	..	..	− 3	− 2	1	2	..	2	1	0	.	0	0	1	1	− 1	4	2
75	..	..	− 6	− 5	− 1	0	..	0	1	0	..	1	− 1	1	1	− 1	4	2
76	..	..	− 10	− 3	2	1	..	0	− 1	0	..	− 2	− 1	1	0	0	2	3
77	..	..	..	..	..	..	− 2	..	− 5	− 2	− 1	1						
78	..	..	..	..	..	..	− 2	..	0	− 2	− 1	1						
79	..	..	..	..	..	..	− 4	..	0	− 2	− 1	1						

Date . .		July 20.	July 21.	January, 1892.
Temperature.		18	16·2	17·4
Number.				
86	Made July 20, from SCHUCKART's sulphate	− 1	0	− 2
87		− 1	0	− 3

TABLE IX.

Date.	Sept. 12.	Sept. 17.	Nov. 11.	Nov. 23.	Nov. 25.	Dec. 2.	Dec. 7.
Temperature.	14·6	14·4	11·1	13·2	12·8	14·4	15·5
65	0	0	0	−4	0	0	−2
69	1	1	−6	−2	−1	−1	−3
70	−3	−3	−2	−3	−2	0	−1
77	−1	−4	3	−2	1	−1	−1
78	−1	−1	0	−5	−3	−1	−3
79	−1	−1	−1	−4	−2	−1	−3
43	S	S	1	−1	1	1	0

The cells were returned at the beginning of November. The tests given in Table IX. show that they have retained their E.M.F. with practically no change. Thus we may say that all the cells set up by us after the last date of the Board of Trade circular agree among themselves and with our standard to about one part in three thousand.

§ 22. *Errors Arising at the Zinc Pole. Effect of Amalgamation of the Zinc.*

After a cell has been made, the paste, which, when poured into the test tube is fairly homogeneous, and of the consistency of cream, separates into two portions. The more solid part, the mercurous sulphate, settles to the bottom, resting on the mercury, and a clear liquid remains on the top. The zinc rods used varied somewhat in length ; in some cells they were entirely in the clear liquid, in others they extended partly into the more solid portion below. Various observations had shown us that this might produce some difference in the behaviour of the cell.

The differences appear to arise from the fact that when the zinc rod touches the mercurous sulphate it becomes properly amalgamated. A small trace of iron in the zinc, or in the zinc sulphate solution, does not, then, produce any serious effect. If, however, the zinc is only in the clear liquid, the amalgamation does not always take place. The zinc, instead, frequently becomes covered with a grey deposit. This grey deposit consists in the main of mercury in very fine globules, and when it is formed all over the zinc the E.M.F. of the cell falls, and it becomes unsteady.

The experiments recorded in Table X. show the effects of this deposit and of the method of treating it.

The cells in question were six, Nos. 42–47, constructed towards the end of April by Mr. L. R. WILBERFORCE. The materials of the cells are indicated as above, by Greek letters. They are of the Board of Trade pattern, but rather larger, being put up in test tubes about 10 cm. high and 2·5 cm. in diameter. It will be seen from the Table that while on April 27 they were nearly right, on May 2 they had most of them fallen considerably, and by May 26 they were, on the average, some 35 units too low. It

TABLE X.

Date.	Zinc sulphate.	Mercurous sulphate.	Mercury.	April 27.	April 28.	May 2.	May 26.	July 22.	July 23.	July 24.	July 30.	August 6.	August 7.	August 12.	November 11.	December 7.
Temperature.				17	16·5	..	..	..	..	..	..	16·2	16·2	17·6	11·1	15·8
Number.																
42	β	ϵ	μ	0	− 2	− 33	− 43	..	..	− 37†	− 30	− 11	− 10	− 4	− 5	− 1
43	β	ϵ	μ	− 2	− 30	− 40	− 43	− 34*	0	0	0	0	3	..	‡	..
44	β	ϵ	μ	− 1	− 4	− 34	− 42	..	..	− 32	..	− 30	− 32	− 21	− 3	1
45	β	λ	μ	3	− 1	− 8	− 31	..	..	− 21	..	− 29	− 26	− 23	1	2
46	β	λ	μ	3	− 8	− 27	− 32	..	..	− 23	..	− 31	− 26	− 23	− 17	0
47	β	λ	μ	− 1	− 31	− 38	− 35	− 32	..	− 34	..	− 43	− 36	− 35	0	4

* Amalgamated rod introduced, fell to − 8 at once.

† Fresh zinc inserted (? amalgamated).

‡ Amalgamated rods inserted in August, cells left unsealed till November.

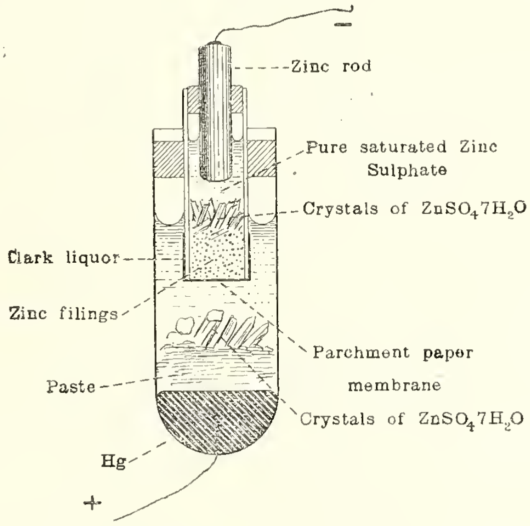
will be remembered that a good many other cells showed a fall at first, but most of them recovered in time.

The observations on July 22 show a tendency to recover here also.

The mercury used in the cells was the purest attainable, and the cells contained visible crystals of $ZnSO_4.7H_2O$ of the proper form. It was not possible, therefore, to attribute the fall of E.M.F. to impurity of the mercury or to supersaturation, and some other source had to be sought for the difficulty.

It was noticed that the grey deposit had formed on the zincs, and we had seen (see Table VI., April 20) that by removing this deposit a rise in the E.M.F. was produced. From this, and from some experiments which were being made on two cells, Nos. 80 and 81, it appeared probable that the fault lay at the zinc pole; that, in order that the cell should have its proper E.M.F., some action between the zinc and the soluble mercurial compounds was necessary, and that this action did not occur when the grey deposit was formed all over the zinc, but only when part, at least, of the zinc surface became bright. It is not clear whether the necessity for this action arises from impurities or from some other unexplained cause.

Fig. 4.



The diagram (fig. 4) shows how Nos. 80 and 81 were constructed. There is inside the cell surrounding the zinc rod a small glass tube, closed at its lower end by a membrane of parchment paper. On this is a layer of zinc filings. The tube surrounding the zinc is filled with saturated solution of zinc sulphate with an excess of crystals, and dips into the ordinary Clark liquor in the outer cell. Thus none of the soluble mercurial compounds in the liquor can reach the zinc pole, for in diffusing into the inner cell they have to pass the zinc filings which would react with them.

The cell is not really a Clark cell for the liquid in the glass tube is not a saturated solution of zinc sulphate *and mercurous sulphate.*

Observations on these cells showed that they were always considerably lower than the standard. After they had been compared the zinc of No. 80 was removed and amalgamated; the cell then became correct. This is shown in Table XI.

TABLE XI.

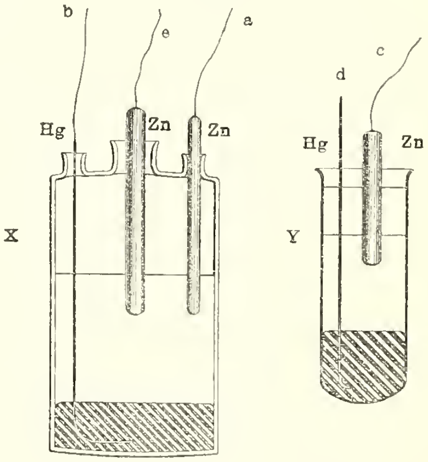
Number.	July 8.	July 9.	July 10.	July 11.	July 20.		July 21.	July 22.	July 23.	July 30.
80	− 25	− 24	− 26	− 23	− 19	Zinc amalgamated	+ 2	+ 2	+ 2	+ 8
81	0	− 53	− 29	− 23	− 21		− 16	− 15	− 13	− 20

Since the result of amalgamation had been to bring No. 80 into agreement with the standard it was thought worth while to try the effect of amalgamating the zinc of one of the cells Nos. 42–47 now under consideration. On July 22 the zinc rod of No. 43 which, as has been remarked, was covered with the grey deposit, was removed and amalgamated. After careful washing and drying it was restored to the cell which, since that date, has agreed well with our standard (see Table X.).

On July 24 the zinc rod from No. 42 was removed and a new rod inserted in its place. There is some doubt as to whether this rod was amalgamated or not; the entry in the note book is not quite clear; the cell was still too low, although better than before. The rod which had been taken out was placed in some fresh Clark liquor, and the difference of potential between it and a well amalgamated rod measured. It proved to be 8 or 9 units lower than the amalgamated rod.

Experiments were commenced on August 17 to test this point with a special form of cell.

Fig. 5.



A Clark cell was set up in a three-necked Woulfe's bottle. Two of the apertures, *a* and *b*, contained the ordinary zinc and platinum of the cell. The third, *e*, was left free for the insertion of the zincs to be tested. In fig. 5, X represents this cell and Y the Clark cell, the contents of which are to be examined, *d* being the platinum and *c* the zinc pole of Y. The plan of the experiment was to insert a well-amalgamated zinc in *e*, and compare the E.M.F. between *a* and *b*, *e* and *b*, respectively. These were practically the same, and equal to the standard Clark. The differences between their values and those of the standard are given in the first two columns of Table XII. In column 5 is given $c-d$, the difference between the E.M.F. of the Clark Y and the standard; it is negative, as Y had too small an E M.F. The zinc, c, was then transferred to the Woulfe's bottle, e being placed in Y. If the error is due to the zinc, and it retains its properties in the new liquid, the E.M.F. between b and c will be less than that of the standard, that between d and e will have its normal value, and there will be a difference of potential between the two zincs, a and c. On making the transference these effects took place, as is shown by the last three columns of Table XII., but the zinc c, when transplanted into the new liquor, rapidly changed, and the E.M.F. between b and c soon became nearly normal in amount. Columns 6 and 7 give the differences between the electromotive forces of $c-b$ and $e-d$, respectively, and the Clark; while column 8 gives the difference of potential actually observed between a and c. It is clear that some effect had taken place in the cell Y on the zinc which produced the low E.M.F. observed. The liquid of X acted in a short time on this zinc in such a way as to counteract this effect.

TABLE XII.

Cell.	Temperature.	Before changing zincs.			After changing zincs.		
		$a-b$.	$e-b$.	$c-d$.	$c-b$.	$e-d$.	Difference between a and c.
No. 47, on Aug. 19	17·2°	−1	0	−33	−11	−14	9
					− 9	− 4	8
						− 3	
						− 3	
						− 2	
					− 8	− 2	8
No. 46, on Aug. 19	17·6	−1	0	−23	−15	− 1	23
							10
							6
					− 6	− 2	6
No. 45, on Aug. 21	16·9	0	0	−23	−21		
					−20		
					−16		
					−15		
					− 8	− 3	
No. 44, on Aug. 21	16·8	0	0	−36	−50 } Immediately		
					−45 } after insertion		
					−35 } In a few		
					−30 } minutes		
					−25		
					−18		
					−10	− 1	
					− 7		

Since with the new zincs the cells Y give normal readings, we infer that the rest of their contents were not at fault, though there probably was some difference between the solutions in X and Y, for the zinc from Y soon became right when put into X.

Since the new zincs were inserted the cells have remained nearly normal.

It will be remembered that the fall noted in the cells described in Table VI. was also corrected by the removal of the grey deposit.

The evidence we have collected on these points is not conclusive, and we are still engaged in experiment. There is no doubt that there is a tendency in many cases for cells set up with good materials to fall in E.M.F. at first. This drop appears to reach a maximum, and in good cells to correct itself fairly soon. In other cells the recovery is very slow; indeed, the experiments must go on for some time longer before we can say in all cases whether it will be complete. In such cells a grey deposit is formed on the zinc. This deposit is mainly mercury.* On removing it from the zinc the E.M.F. rises; the same is the case on substituting an amalgamated zinc for the original one; but it is not clear whether this happens because some action of the nature of amalgamation is required to prevent slight impurities in the zinc sulphate from having an injurious effect, or because even with a pure zinc and mercurous sulphate solution an amalgamated surface is required, or again, because the action which has led to the formation of the grey deposit has removed from the solution injurious impurities. These points are still under examination.

We are also inclined to think that a trace of acidity in the solutions is useful in preventing the formation of the grey deposit; on this point also we are continuing the work.† The slight acidity in time corrects itself by its action on the zinc. The important fact remains that the formation of this grey deposit must be, as far as possible, prevented, and, if it does form, the deposit must be removed.

The experience of Dr. SCHUSTER with regard to this grey deposit somewhat resembles our own. He has kindly sent us the results of measurements of a set of rather large cells prepared from commercial sulphate.

Soon after being set up these cells were tested against some cells made two years since. Taking the E.M.F. of this set as unity, the new cells were as follows:—

---

* It is only necessary to dip the grey covered zinc rods for a moment in a dilute acid to obtain a bright amalgamated surface; no fresh mercury is required. Dr. SCHUSTER informs us in a letter that he has analysed the grey deposit, and found it to be nearly pure mercury.

† The solutions used in the bad cells Nos. 42–47 were, as tested by litmus paper, distinctly less acid than those employed in the good cells Nos. 72–79. While the solution in the cell No. 90 is now less acid than that in the bottle from which the cell was originally filled.

Date.	January 23.	January 30.	February 6.	March 11.
Number.				
5	·9777	·9587	·9633	1·0005
6	·9823	·9590	·9658	1·0072
7	·9848	·9627	·9629	1·0159
8	·9969	·9516	·9626	·9943
9	·9892	·9632	·9711	
10	..	·9695	·9459	

Writing in May, 1891, Dr. SCHUSTER says :—

"My present opinion is that the cell only arrives at its steady state after some action has taken place between the zinc and the solution, and it is necessary that the zinc surface exposed should not be too small compared with the quantity of liquid. At any rate, most of my large cells prepared with good sulphate seem to settle down at an E.M.F. about $\frac{1}{3}$ per cent. too low, and the lower the smaller the surface of zinc exposed. Amalgamation seems a mistake, for it is just the action of the zinc previous to amalgamation which brings the cell into a steady state."

Writing at a later date, February, 1892, Dr. SCHUSTER expresses the opinion that the action required to bring the cell to a steady state is the formation of a basic zinc sulphate. This process may be facilitated by the introduction of zinc oxide into the cell, and in that case amalgamation will probably be an advantage, but if the cell is left to settle down of itself, the process of amalgamation helps the formation of the basic zinc sulphate, and it is desirable, therefore, that this process should go on in the cell.

He also tells us that in the cells made in the Reichsanstalt at Berlin the zinc sulphate is boiled with zinc to secure the formation of the basic salt.

On this point, however, our own view would be somewhat different from that expressed by Dr. SCHUSTER, for it has been observed by one of us (S.S.), and by Mr. E. H. GRIFFITHS, and is in accordance with the chemical theory of the interchange of bases, that the mercurous sulphate which is used in preparing the paste reacts with the basic zinc sulphate, producing zinc sulphate and probably basic mercurous sulphate ; thus, if the zinc oxide used to neutralize the zinc sulphate does form with it a basic salt, the mercurous sulphate clears this out of the solution.

Dr. SCHUSTER has also confirmed our results as to the danger of the grey deposit and the rise in E.M.F. produced by removing it.

Mr. SWINBURNE, in his paper already quoted, also recommends amalgamation. He found differences of potential amounting to ·4 per cent. of the E.M.F. of a Clark cell between different specimens of zinc immersed in saturated zinc sulphate. These he assigns partly to "impurities, probably iron, in the zinc sulphate." They disappeared, practically entirely, when the zincs were amalgamated.

It may be noted that Lord RAYLEIGH did not meet with this source of error, for his zincs, being generally plunged into the more solid part of the paste, were immediately amalgamated.

§ 23. *Effect of Shaking on the Cells.*

The history of these cells illustrates another point. Dr. HOPKINSON called our attention to the fact that he had observed that while slight mechanical shaking did not affect the E.M.F. of a good cell it produced considerable change in the E.M.F. of one which was too low. This fact we have repeatedly verified.

Thus, on August 12, Nos. 41, 45, 46, and 47 were all to low. Their E.M.F. in each case rose considerably when the cell was tapped, but fell back towards the old value when the tapping ceased. Cell No. 40 was right, and no effect was produced by tapping it.

The effect of the shaking would appear to take place through its bringing fresh zinc surfaces into contact with the solution.

A similar result took place with some cells which were too high, only, in this case, the E.M.F. was lowered. Thus, on August 6th, No. 68 was 56 too high ; on tapping, its E.M.F. fell till it was only 23 too high ; it then rose to about its original value ; and hence, on further tapping, fell to within 5 of the standard. Cells No. 71 and 72 were quite unaffected by the tapping, remaining within 1 or 2 of the standard throughout. Another cell sent us to test was about 20 too high ; on slight tapping it fell to about 10.

Thus, it appears that slight tapping produces a change in bad cells ; we have never found it produce any change in good ones.

§ 24. *Mr.* E. H. GRIFFITHS' *H Cells.*

Table XIII. contains details of a set of H cells, constructed in February, 1888, by Mr. E. H. GRIFFITHS, which have been subject to much careful examination. They were tested against the Board of Trade cells. The results are important, because the cells are very consistent among themselves, and, at the same time, they are undoubtedly wrong. They appear to be unsaturated, and have, in consequence, a lower temperature coefficient than our cells, and agree with them more nearly at low than at high temperatures. It is clear, then, that a set of cells may be made which will agree among themselves, and yet be wrong. Mr. GRIFFITHS writes as follows of the materials used :—

"Re-distilled zinc and mercurous sulphate were used. The solution of zinc sulphate was allowed to stand and not used unless it deposited crystals of the proper form. The mercury was some which I had purified and distilled myself."

The first three series of observations given in the table were made by Mr. GRIFFITHS. The temperature was in each case nearly 15°, and the observations were corrected to 15° by the use of Lord RAYLEIGH's coefficient. This coefficient is clearly wrong for these cells, as the observations show, but the error produced will be small. The change per 1° relative to our standard is about 2·8 of our units. Our standard changes by about 4·5 units, so that the change in the Griffiths cells is only about 4·5 − 2·8 or 1·7, and this implies a coefficient of about one-third that of the standard cell. The last column gives the values found at the Board of Trade in terms of their standards.

This, and the cells described in Table VII. are, so far as our experience has gone, the only sets of cells which have been quite wrong and yet have remained consistent among themselves for some time, and that over a considerable range of temperature.

Table XIII.

Date	August, 1888.	December, 1888.	August, 1890.	August 8, 1891.	August 10, 1891.	November 14, 1891.	December 16, 1891.	March 9, 1891.*
Temperature	15	15	15	16·8	17·4	11·1	16·2	11·5
Number.								
91	26	27	26	26	29	12	23	12
92	28	29	28	28	31	15	26	15
93	24	24	24	25	27	10	22	11
94	28	28	26	26	28	12	24	12
95	28	28	26	27	29	12	24	13
96	28	28	28	28	31	14	25	14

§ 25. *Comparison with the Board of Trade Standard Cells.*

A means of comparing the E.M.F. of our cells with those of the standards of the Board of Trade at Westminster, which were set up by Dr. Muirhead, is afforded by the last columns of Tables II., VIII., and XIII., which give, in terms of the unit we have adopted, viz., ·00025 volt, the differences between our cells and the standards of the Board of Trade. The comparisons were kindly made for us by Mr. Rennie, Major Cardew's assistant. It will be seen at once that the agreement is very close. For the cells Nos. 7, 8, 9, 10, Table II., constructed by Mr. Callendar, the differences observed are rather greater than those found by us either before the cells were sent or soon after they returned.

The last two columns of Table VIII. give the results for the five cells Nos. 71–76 No. 71 appears to have been low at one of these observations, for the rest we see that the differences are rather greater than those with our own cells.

---

* Comparison with the Board of Trade standards.

A similar comparison is also afforded by the last column on Table XIII., and here there is almost exact agreement between our results on November 14 and the Board of Trade measurements on March 9.

The standards would appear to be exactly equal, but these observations hardly deserve as much weight as the others, for the temperature of the large H cells is not very easy to be sure of.

It would seem then the Board of Trade standards are lower in E.M.F. than our standard by, possibly, one or two of our units, that is, by about ·0003 volt. This clearly is very small. The Board of Trade standards are a set of 72 cells constructed by Dr. MUIRHEAD. These cells are from time to time compared against each other, and Mr. RENNIE states that the greatest variation among them is less than ·00068 volt, that is, distinctly under three of our units. The cells used were known to be good average cells.

Dr. SCHUSTER'S cell No. 97 was taken to London on December 21, and on January 11 it was compared, with the following results given by Mr. RENNIE.

At 8°·0 C. the E.M.F. of B.T. 822/35 exceeds that of SCHUSTER'S cell by 0·00021 volt.

At 8°·1 C. the excess was 0·00017 volt.

At 8°·3 C. the E.M.F. of B.T. 822/34 was less than that of SCHUSTER'S cell by 0·00005 volt.

Thus SCHUSTER'S cell lies between the two Board of Trade standards, and falls short in E.M.F. of the mean of the two by less than one of our units.

§ 26. *The Temperature Coefficient of the Cells.*

We have also made some experiments with a view of investigating the temperature coefficient of our cells.

It has already appeared from Table II. extending over a range of temperature from 0° to 18° that our more important cells, including the standard Rayleigh 1, have practically the same coefficient. When we were engaged in the experiments in August we did not expect them to be continued so long, and therefore a cell was fitted up which could be packed in ice.

It was a cell of the ordinary form, but contained a delicate thermometer, the bulb of which was in the paste. The cell was placed inside a long test tube, packed with glass-wool and asbestos; the thermometer passed through a cork closing the test tube. The stem of the thermometer was so long that most of its graduations were outside the cork. It could be read, therefore, without seriously disturbing the cell when packed in ice. With this cell a number of measurements were taken.

The results of these are given in the accompanying Table XIV.

TABLE XIV.

Date.	Temperature of cell.	Difference from No. 1.	Temperature of No. 1.	Notes.
	°		°	
August 10	16·4	0	16·4	
,, 2.30 . .	16·7	0	16·7	Packed in ice.
,, 3.50 . .	. .	51		
,, 4.30 . .	. .	63		
,, 5.30 . .	. .	71	16·7	Readings still changing.
August 12, 10.30 . .	0	74	16·9	Repacked in ice at 10.32.
,, 10.55 . .	0	74	16·9	
,, 1.15 . .	0	74	17	
,, 4.30 . .	0	75	17	Repacked at 5.10.
,, 5.15 . .	0	75+	17·1	
,, 7. 0 . .	0	76	17·4	
August 13, 10. 0 A.M. .	0	74	16·7	Placed in water of temperature 16°·7 at 10.5.
,, 10.30 . .	13	30		
,, 11. 5 . .	16·6	11		
,, 6.10 . .	17·5	6	16·7	
August 14, 9.40 . .	17·8	7	17·7	
,, 4.50 . .	18·8	7	18·5	
August 15, 10. 0 . .	18·4	6	18·4	Placed in water at 32° at 12.30 on August 17, and allowed to cool.
August 18	18·3	2	18·3	
,, 19	17·2	2	17·2	
November 2 . . .	14·4	1+	14·4	Cells all at one temperature.
,, 14 . . .	9·2	1+	9·2	
December 22 . . .	0	2−	0	

When the cell was left on August 10 the readings were still changing though it had been in ice for three hours.

On August 11 it appeared that some moisture had got among the glass-wool, the cell was taken out and repacked, taking care to insulate the leads thoroughly with gutta percha. After this the difference rose to 66, but was still increasing at 5.40. Throughout the next day and up to 10 A.M. on August 13 it was steady, the mean reading being 74·6, and the average temperature of the standard 17°. After the last observation the cell was placed in water at 16°·7. At first there was a rapid fall, but after a time the fall ceased and a steady difference of some 6 or 7 of our units persisted all through the next day.

On August 17 the cell was placed in warm water at 32° and allowed to cool. On August 18 the difference at about 18° was 2, and this continued through the next day. Before the cell was placed in ice the difference between it and the standard was 0. The mean gives us as the difference when the two cells are at the same temperature the value 1, so that the change for 17° is 74·6 — 1, and the change per 1° C. is 4·33.

On August 12 the E.M.F. of the Clark was found to correspond to 5736 of our units, and we have then as the coefficient of change of E.M.F. with temperature the value 4·33/5736 or ·000755. Lord RAYLEIGH gives as the value of the coefficient from about 0° to about 15° ·000765, for another it is ·00073, and for a third ·00083.

The last three observations in Table XIV. show that for a range of temperature from 14°·4 to 0° our cell and Lord RAYLEIGH's cell No. 1 have the same E.M.F.

When the cell was put into the warm water on August 17 readings were taken of the E.M.F. and temperature.

About half-an-hour after being put into the bath the E.M.F. was 45 units below the standard, the temperature was 31°, that of the standard 17°·3. The readings remained steady at this for five minutes.

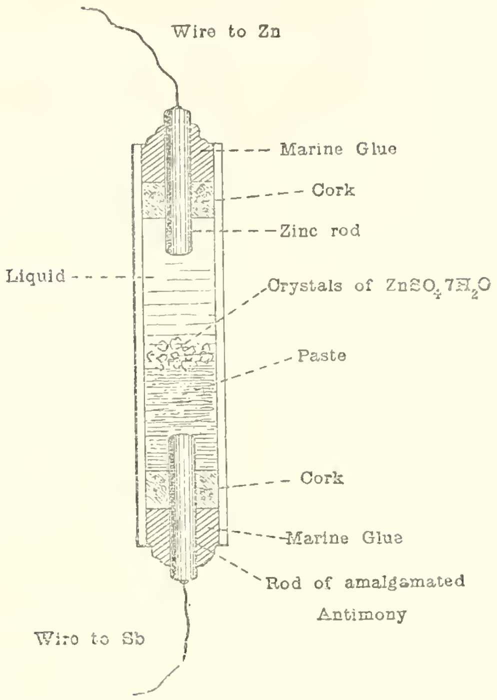
This would correspond to an increase of only about 3·3 units per 1° C. instead of 4·3, and again emphasizes the fact that the E.M.F. of the cell is not a fixed quantity at a given temperature unless the temperature has remained steady for some time previously.

Most of our cells were compared with the standard at various temperatures between 9° and 18°. The observations show that all the cells except those known to be unsaturated have the same coefficient, and the average value of that coefficient between 0° and 15° is ·00076. Our experiments do not show whether the relation between E.M.F. and temperature is linear or not.

§ 27. *Portable Form of Cells.*

Cells which contain much mercury are not suitable for carrying about for testing purposes, as the mercury may be thrown in contact with the zinc rod and become contaminated with zinc. It has been shown that a very small percentage of zinc dissolved in mercury causes that to behave as though it were zinc. To avoid the use of much mercury cells have been constructed with merely an amalgamated platinum wire for their negative metal. The platinum being more negative than mercury does not interfere with the E.M.F. of the cell. Cell No. 71, described in Section 20, is constructed with an amalgamated platinum wire. Since it is only necessary to use a metal more negative than mercury for the negative pole and to amalgamate its surface, we have tried the use of small antimony rods. A small rod of antimony is cast and boiled in mercury to amalgamate its surface. It is then fitted into the cell, as shown in the diagram, fig. 6. It is of importance that the antimony be pure, for should it contain metals more positive than mercury the electromotive force of the cell will be reduced by their solution in the mercury. For this purpose, therefore, antimony metal reduced from pure tartar emetic should be used.

Fig. 6.



With a large stout antimony plate cells of low resistance could easily be made. The tests made on this cell are shown in the following table.

TABLE XV.

No.	July 9.	July 10.	July 11.	July 20.	July 21.	July 22.	July 30.	Aug. 6.	Aug. 17.	Nov. 14.	Dec. 16.
			18°	18°	19°	18°·5	17°	16°·6	17°·2	11°·1	16°·2
82	−32	−5	−9	−3	−2	−3	−3	−5	−3	2	−2
83	..	..	..	..	−6	−4	−4	−5	−4	3	−3

Cell 82 was made on July 9th, 1891.
Cell 83 was made on July 21st, 1891.

During several of the tests the cells were inverted top to bottom without producing any effect. It is clear that at temperatures of 16° to 18° they are rather less than the standard, while at 11° they appear to be greater. Thus their temperature coefficient is rather greater than that of the usual form. Over the range worked with the difference between their E.M.F. and that of the standard is well within ·1 per cent.

§ 28. VON HELMHOLTZ *Standards.*

In the 'Sitzber. der Akad. der Wiss., Berlin,' 1882, p. 26, VON HELMHOLTZ has described an electromotive force standard consisting of mercury, mercurous chloride, zinc chloride, and zinc. Similar cells have been prepared and used by OSTWALD ('Zeitschrift für Physikalische Chemie,' vol. 1.) We have prepared a set of six cells like these, and have tested them at intervals. They are put up in small test tubes, with platinum wires sealed through the bases. The solution of zinc chloride was prepared from solid sticks of zinc chloride, and therefore probably contains some oxychloride. It had a specific gravity 1·445 at about 15°. Since the solution alters in strength if a current be taken from these cells, it is necessary to be very careful that they do not become short-circuited. These cells had an E.M.F. of about ·98 volt; if the solution be taken slightly more dilute they may be adjusted to be exactly 1 volt at 15°. OSTWALD has attempted to do this, and states that it is secured by using a solution of specific gravity 1·41.

Three series of observations were made of these cells, and the following table gives the results in terms of the E.M.F. of a Clark as unity at the temperatures mentioned.

TABLE XVI.

Date	February 17.	March 16.	August 22.
Temperature .	16	13	17·4
Number.			
101	6890	6879	6892
102	6891	6882	6894
103	6892	6882	6895
104	6890	6879	6892
105	6891	6882	6892
106	6891	6882	6894
Mean	6891	6881	6893
Value in volts .	·9876	·9881	·9870

The last line gives the value in volts, assuming the E.M.F. of a Clark at 15° C. to be 1·4242 volt.

The absolute values at the earlier dates are somewhat uncertain because of the uncertainty attaching to the actual value of the Clark used as standard. In this case the values, on February 17, were referred to CALLENDAR's cells, but it is clear that the cells cannot have changed much, and that they retain unaltered their relative values.

§ 29. Gouy's *Cells.*

Another modification of the cells has been suggested by M. Gouy ("Sur une Pile Etalon," 'Journal de Physique,' 1888.)

It consists of zinc, zinc sulphate, mercuric oxide, and mercury. According to M. Gouy, the E.M.F. at any temperature, t, is given in legal volts by

$$E = 1{\cdot}390 - {\cdot}0002\ (t - 12).$$

The zinc sulphate solution is prepared by boiling a saturated solution for an hour with a few thousandths of its own weight of oxide of silver. The liquid, when cooled and filtered, has sticks of zinc put into it, which precipitate any silver remaining. Four cells were made; of these two, viz., Nos. 54 and 55, contain the yellow oxide, the other two, Nos. 88 and 89, the red oxide of mercury. The values of the E.M.F. have been reduced to 12° by M. Gouy's coefficient.

Table XVII.

Numbers . . .	54	55	88	89
Values . . .	1·3794 1·3803 1·3806 1·3809	1·3793 1·3806 1·3869 ..	1·3906 1·3867 1·3890 1·3839	1·3896
Mean	1·3803	1·3823	1·4875	1·3896
Mean of two . .	1·3810		1·3879	

To reduce them to "legal volts," and so to compare them with M. Gouy's results, we must multiply these results by the ratio 1063/1060, we have thus the values 1·385 and 1·392. The latter is nearly 2 parts in 1000 above M. Gouy's result.

APPENDIX.

(August 18, 1892.)

§ 30. *Further Comparisons of the Cells already described.*

Since the foregoing paper was communicated, the experiments have been continued, and numerous other results have been obtained. The more important of these are described in the following Appendix. The cells already described have been repeatedly compared with the standard.

TABLE XVIII.

CELLS described in Table II.

	No. of cell.										
	2	90	3*a*	3	4	5	6	7	8	9	10
Difference, Dec., 1891 ..	−2	0	−3*	−3*	−7*	−6*	3	0	3	3	2
,, July, 1892 ..	−2	1	−1	−2	−4	−1	1	3	3	3	3

CELLS described in Table IV.

	No. of cell.										
	25	26	27	28	29	30	31	32	33	34	35
Difference, Dec., 1891 ..	0	2	2	6	−1	4	3	3	2	0	2
,, July, 1892 ..	3	broken	2	2	1	1	3	3	3	3	3

CELLS described in Tables V. and VI.

	No. of cell.														
	56	57	58	59	60	36	37	38	39	40	41	61	62	63	64
Difference, Dec., 1891..	2	17	2	−39	−3	1	..	3	−3	1	..	3	−3	..	2
,, July, 1892..	0	17	6	−36	−6	..	..	1	−1	0	0	2	2	..	−8

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* These observations were made in August, 1891.

Cells described in Tables VIII. and IX.

	No. of cell.											
	71	72	73	74	75	76	65	69	70	77	78	79
Difference, Dec., 1891 ..	−2	1	0	−1	−1	0	−2	−3	−1	−1	−3	−3
,, July 14, 1892..	−1	−1	−1	−1	−3	−5						
,, ,, 21, 1892..	−2	−1	−1	−1	−1	−2	0	−3	−1	−1	−1	−1
,, ,, 22, 1892..	−3	−1	−1	−1	−1	−2	1	−4	−1	−2	−2	−3

Cells described in Tables X. and XV.

	No. of cell.							
	42	43	44	45	46	47	82	83
Difference, Dec., 1891..	−1	1	1	2	0	4	−2	− 3
,, July, 1892..	−9	−1	−4	−2	−1	−1	−2	−16

The results of comparisons made at the end of 1891, and of the latest comparisons we have made, are given in Table No. XVIII. The Table shows that, with few exceptions, the relative values of the cells remain closely the same as before.

It will be remembered that Nos. 57 and 58 were known to be unsaturated, while No. 59 dropped suddenly in value between May and August, 1891, and has remained low ever since.

The behaviour of No. 76 requires some notice. Some comparisons were made on July 13 and it was observed that it was then five units too low. This continued on July 14; but when the cells were next tested, on July 21, it had recovered, and has since remained right, having been frequently re-tested. The cause of this is not quite clear, but it is probable that it was due to the following circumstance:—The binding screw which connected No. 76 to the circuit was allowed by accident to fall into the bath; the water was dirty, having been standing for some time; and possibly some of the moisture adhered to the screw, and set up a small E.M.F. at the junction. It was found by experiment on July 21 that, by wetting this binding screw, small changes could be produced in the E.M.F. observed.

With this exception, the sets Nos. 71 to 76 and 65 to 79 show very steady agreement. These, it will be recollected, are cells set up in accordance with the second form of the Board of Trade Memorandum.

The set Nos. 42 to 47 shows some change. These are the cells from which the grey deposit was removed. They were left unsealed for some time, and after the treatment to which they have been subjected, can hardly be considered standards.

In June the portable cell No. 82 was sent by parcel post to London, and thence to Manchester, where it was compared by Dr. SCHUSTER with his standards. About the same time No. 83 was taken to Manchester by one of us (R. T. G.) in a handbag. The cells were brought back to Cambridge in a portmanteau which travelled as passenger's luggage. Dr. SCHUSTER found that No. 82 was less than his standard by two of our units, while No. 83 was less by six units. The Table shows that on their return No. 82 had not changed relative to our standard, while No. 83 had varied considerably. Other comparisons confirmed these results.

A large number of comparisons have been made from time to time between the large bottle cell, No. 90, and the standard. The two remain practically equal so long as the temperature remains steady for some time, but some experiments in March and April emphasize the importance of maintaining this very constant, at any rate, for large cells.

Towards the end of March the weather was very cold. On March 30 the room temperature was 6°; it had been lower, and the baths in the room were as low as 3°·5; the cells were not examined at this date. On March 31, when the bath was at 10°·8 C., No. 90 exceeded the standard by nine units. On April 1, the temperature being 11°·2, the difference was five units. A rod was then put through a hole in the cork of the bottle cell, and the zinc sulphate was stirred up. The difference fell immediately to less than one unit, the temperature remaining unchanged. Thus, on March 31, the E.M.F of No. 90 corresponded to a temperature more than 2° below the temperature of the cell, and on April 1 it was still too high by an amount equivalent to the change due to 1 degree. The crystals had formed in a hard mass on the top of the paste; the zinc was some little way above these, and the E.M.F remained too high until they were broken up.

§ 31. *Cells set up by Mr.* E. H. GRIFFITHS.

In January and February of the present year, Mr. E. H. GRIFFITHS set up a number of cells, following in the main the directions of the second issue of the Board of Trade Memorandum. These cells are numbered from 131 to 160. In one batch of these, No. 149 to 154, the zinc sulphate was treated with mercurous sulphate in the manner described on page 611.

We give the description of these cells, kindly furnished to us by Mr. GRIFFITHS, to whom our best thanks are due.

Cells Nos. 131–142, made January 26, 1892. Chemicals supplied by HARRINGTONS. The $ZnSO_4$ was saturated at 33° C. The Hg_2SO_4 was used as supplied, except that it was shaken up with mercury, but not washed.

Cells Nos. 143–148. January 31, 1892. The Board of Trade directions followed implicitly.

Cells Nos. 149–154. February 6, 1892. To the $ZnSO_4$ solution saturated at 33° C., neutralised with ZnO and filtered at 30° C., a small quantity of Hg_2SO_4 was added, and the black deposit was filtered off. In other respects the cells were the same as 143–148, the Board of Trade directions being followed. These cells assumed the standard E.M.F. immediately on formation.

Cells Nos. 155–157, made according to Lord RAYLEIGH's directions from paste which had been in stock since Michaelmas, 1889.

Cell No. 158 the same as Nos. 131–142, except that the zinc was amalgamated.

Cell No. 159 the same as Nos. 143–148, except that the zinc was amalgamated.

Cell No. 160 the same as Nos. 149–154, except that the zinc was amalgamated.

Cells Nos. 131–136 were compared with our standards repeatedly for about a fortnight after being set up, and showed very close agreement.

The other cells were compared by Mr. GRIFFITHS with our cell No. 65; the set of cells described in Table IX., viz., Nos. 65, 69, 70, 77, 78, 79 had again been lent to him for this purpose. It appears from our Tables that No. 65 is in very close agreement with our standard, and we may take, therefore, this comparison as giving the E.M.F. of Mr. GRIFFITHS' cells in terms of the standard.

TABLE XIX.—Cells constructed by Mr. E. H. GRIFFITHS.

	Date.	Number of cell.														
		131	132	133	134	135	136	137	138	139	140	141	142	158	143	144
Differences	Feb., 1892	1	1	1	1	2	2	0	1	1	1	2	2	3	2	1
,,	July, 1892	0	0	1	0	−1	0	1	1	1	1	0	0	−1	3	2
		145	146	147	148	159	149	150	151	152	153	154	160	155	156	157
Differences	Feb., 1892	1	1	2	0	−1	−1	−1	0	−1	0	−1	0	−4	−4	−4
,,	July, 1892	2	3	3	2	2	0	0	0	0	0	0	0	−4	−4	−4

Table XIX. shows the results; the unit is slightly less than in our measurements, being about ·0002 volt.

In February the temperature was about 13°, in July it was 14·4.

It will be seen that the cells are all fairly close to the standard. Nos. 131–148 are on the whole a little too high. A comparison between Nos. 158, 159, and 160, with the batches immediately preceding them, respectively shows that amalgamating the zinc has in these cases no distinct effect; the mercurous sulphate in the solution of itself insures amalgamation in the cell.

The series, Nos. 149–154, in which Mr. GRIFFITHS' method of clearing the zinc oxide or basic zinc sulphate from the solution by the action of mercurous sulphate and filtration was employed, is most closely in agreement with our standard, and this agreement was attained immediately after the cells were set up. It will be noticed, however, that the cells, Nos. 155–157, in which the old paste was used, are too low by about ·0008 volt, though they agree well together.

§ 32. *Professor* AYRTON's *Standards.*

By the kindness of Professor AYRTON we have been able to compare two cells set up Dr. MUIRHEAD, and used as standards at the Central Institute, with ours. These two cells have a thermometer sealed into the case which contains them. This case was packed in sawdust, and one of our own thermometers verified at Kew last November was placed in the sawdust. The thermometer in the cell was found on some five or six different occasions to read 0°·4 or 0°·5 below our own. The Kew correction for our standard is − 0°·16, at about 16°, the temperature of the observations. According to this there is a correction of about + 0°·3 required to the thermometer in the cell.

Making this correction, the following values were found for the excess of E.M.F. of Professor AYRTON's cell over our own reduced to the same temperature :—

TABLE XX.

Cells.	July 13.	July 14.	July 21.
Number.			
1	− 1·5	0	− 2
2	·5	1	− 1

The differences, it will be seen, are within the errors arising from uncertainty of temperature.

§ 33. *Cells set up by Dr.* KAHLE *in Berlin.*

In July last four cells were brought to Cambridge from Berlin, by Dr. LINDECK. Two of these, Nos. 69 and 70, were cells of the pattern devised by Dr. FEUSSNER and described by Dr. KAHLE, ('Electrotechnische Zeitschrift,' July 22, 1892.)

The positive pole is a piece of amalgamated platinum sheet. This and the mercurous sulphate are contained in a porous pot. The outer vessel surrounding the porous pot contains saturated zinc sulphate solution and crystals of zinc sulphate. The negative pole is a rod of solid zinc amalgam, bent at right angles so that one part is vertical, the other horizontal. The vertical portion is surrounded by a tightly fitting tube of glass, the horizontal part is among the crystals, so that the effective portion of the zinc is always surrounded by saturated liquid.

The other two cells, Nos. 29 and 12, are H cells, of the form devised by Lord

Rayleigh. The cells had been compared with each other and with other standards before leaving Berlin, the differences being determined in hundred-thousandths of a volt.

Table XXI. gives the differences between these cells and our standards in hundred-thousandths of a volt.

Table XXI.

Date	July 26.	July 27.	July 29.
Temperature .	16°·1	15°·9	15°·9
Number.			
69	− 15	− 18	− 14
71	− 11	− 15	Taken away
29	− 36	− 35	− 36
12	− 39	− 38	− 39

In comparing these results with the numbers given in other tables, it must be borne in mind that the unit here is only one-twenty-fifth of that previously employed.

It appears then that the standard form in use at Berlin has a slightly lower E.M.F. than that of our standard, while the H-cells are distinctly lower, differing by nearly ·0004 volt.

Dr. Kahle has made an important series of observations, 'Zeitschrift für Instrumentenkunde,' April, 1892, on the effect of impurities on the E.M.F. of an H cell, and has determined absolutely their E.M.F., and that of Dr. Feussner's form of cell.

On reducing his numbers to the units employed by us, we find for the E.M.F. of the Feussner cell the value 1·4339 volt, and for that of the H-cell 1·4337 volt. The value obtained by us for the E.M.F. of our standard is 1·4342 volt. Thus Dr. Kahle's experiments agree with our own in making the E.M.F. of our cells rather higher than those constructed by him, though the difference is small.

As we have already mentioned, the differences between the Cell No. 69 and the others were found at Berlin; the same differences can be obtained from our experiments. They are shown in the following :—

TABLE XXII.

Place of experiment.	Values of E.M.F. of		
	69–71	69–29	69–12
Berlin	− 2	29	29
Berlin	− 4	23	25
Berlin	− 4	25	26
Cambridge	− 4	21	24
Cambridge	− 3	19	20
Cambridge	..	22	25

The differences between the cells at Berlin and at Cambridge are practically the same. Thus, we may fairly suppose they have not been affected by the journey.

Dr. KAHLE states that it is his experience that an H cell always has a rather lower E.M.F. than one with a porous pot put up with the same materials. On this point we have made no direct experiments.

§ 34. *Third form of the Board of Trade Memorandum.*

As experience has been gained in the construction of the cells, the Board of Trade Memorandum has been somewhat modified. We give it in its third form here.

The cells constructed in accordance with it are not readily portable, and may possibly in time get damaged by the accidental contact of the mercury and zinc. To avoid this the porous pot of the Feussner cell, or the cork diaphragm used by Professor CARHART, may with advantage be employed.

MEMORANDUM ON THE PREPARATION OF THE CLARK STANDARD CELL.

Definition of the Cell.

The cell consists of zinc and mercury in a saturated solution of zinc sulphate and mercurous sulphate in water, prepared with mercurous sulphate in excess, and is conveniently contained in a cylindrical glass vessel.

Preparation of the Materials.

1. *The Mercury.*—To secure purity it should be first treated with acid in the usual manner, and subsequently distilled in vacuo.

2. *The Zinc.*—Take a portion of a rod of pure redistilled zinc, solder to one end a piece of copper wire, clean the whole with glass paper, carefully removing any loose pieces of the zinc. Just before making up the cell dip the zinc into dilute sulphuric acid, wash with distilled water, and dry with a clean cloth or filter paper.

3. *The Zinc Sulphate Solution.*—Prepare a saturated solution of pure ("pure re-crystallised") zinc sulphate by mixing in a flask distilled water with nearly twice its weight of crystals of pure zinc sulphate, and adding about 2 per cent. of zinc oxide to neutralize any free acid.* The crystals should be dissolved with the aid of gentle heat, but the temperature to which the solution is raised should not exceed 30° C.; about 12 per cent. of mercurous sulphate treated as described in 4 should be added, and the solution filtered, while still warm, into a stock bottle. Crystals should form as it cools.

4. *The Mercurous Sulphate.*—Take mercurous sulphate, purchased as pure, and wash it thoroughly with cold distilled water by agitation in a bottle; drain off the water, and repeat the process at least twice.* After the last washing drain off as much of the water as possible.

Mix the washed mercurous sulphate with the zinc sulphate solution, adding sufficient crystals of zinc sulphate from the stock bottle to ensure saturation, and a small quantity of pure mercury. Shake these up well together to form a paste of the consistence of cream. Heat the paste, but not above a temperature of 30°. Keep the paste for an hour at this temperature, agitating it from time to time, then allow it to cool; continue to shake it occasionally while it is cooling. Crystals of zinc sulphate should then be distinctly visible, and should be distributed throughout the mass; if this is not the case, add more crystals from the stock bottle, and repeat the whole process.

This method ensures the formation of a saturated solution of zinc and mercurous sulphates in water.

Contact is made with the mercury by means of a platinum wire about No. 22 gauge. This is protected from contact with the other materials of the cell by being sealed into a glass tube. The ends of the wire project from the ends of the tube; one end forms the terminal, the other end and a portion of the glass tube dip into the mercury.

To set up the Cell.

The cell may conveniently be set up in a small test tube of about 2 centimetres diameter, and 6 or 7 centimetres deep. Place the mercury in the bottom of this tube, filling it to a depth of, say, 1·5 centimetre. Cut a cork about ·5 centimetre thick to fit the tube; at one side of the cork bore a hole through which the zinc rod can pass tightly; at the other side bore another hole for the glass tube which covers the platinum wire; at the edge of the cork cut a nick through which the air can pass when the cork is pushed into the tube. Pass the zinc rod about 1 centimetre through the cork.

Clean the glass tube and platinum wire carefully, then heat the exposed end of the platinum red hot, and insert it in the mercury in the test tube, taking care that the whole of the exposed platinum is covered.

Shake up the paste and introduce it without contact with the upper part of the walls of the test tube, filling the tube above the mercury to a depth of rather more than 2 centimetres.

Then insert the cork and zinc rod, passing the glass tube through the hole prepared for it. Push the cork gently down until its lower surface is nearly in contact with the liquid. The air will thus be nearly all expelled, and the cell should be left in this condition for at least 24 hours before sealing, which should be done as follows:—

Melt some marine glue until it is fluid enough to pour by its own weight, and pour it into the test tube above the cork, using sufficient to cover completely the zinc and soldering. The glass tube should project above the top of the marine glue.

The cell thus set up may be mounted in any desirable manner. It is convenient to arrange the mounting so that the cell may be immersed in a water bath up to the level of, say, the upper surface of the cork. Its temperature can then be determined more accurately than is possible when the cell is in air.

In using the cell sudden variations of temperature should as far as possible be avoided.

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* See notes at the end of the memorandum.

NOTES TO THE MEMORANDUM ON CLARK'S CELL.

The Zinc Sulphate Solution.—The object to be attained is the preparation of a neutral solution of pure zinc sulphate saturated with $ZnSO_4 . 7 H_2O$.

At temperatures above 30° C. the zinc sulphate may crystallise out in another form; to avoid this, 30° C. should be the upper limit of temperature. At this temperature water will dissolve about 1·9 times its weight of the crystals. If any of the crystals put in remain undissolved they will be removed by the filtration.

The amount of zinc oxide required depends on the acidity of the solution, but 2 per cent. will, in all cases which will arise in practice with reasonably good zinc sulphate, be ample. Another rule would be to add the zinc oxide gradually until the solution became slightly milky. The solution, when put into the cell, should not contain any free zinc oxide; if it does then, when mixed with the mercurous sulphate, zinc sulphate and mercurous oxide are formed; the latter may be deposited on the zinc and affect the E.M.F. of the cell. The difficulty is avoided by adding as described about 12 per cent. of mercurous sulphate before filtration; this is more than sufficient to combine with the whole of the zinc oxide originally put in, if it all remains free; the mercurous oxide formed together with any undissolved mercurous sulphate is removed by the filtration.

The treatment of the mercurous sulphate has for its object the removal of any mercuric sulphate which is often present as an impurity.

Mercuric sulphate decomposes in the presence of water into an acid and a basic sulphate. The latter is a yellow substance—turpeth mineral—practically insoluble in water; its presence at any rate in moderate quantities has no effect on the cell. If, however, it is formed, the acid sulphate is formed also. This is soluble in water and the acid produced affects the E.M.F. The object of the washings is to dissolve and remove this acid sulphate, and for this purpose the three washings described in the memorandum will in nearly all cases suffice. If, however, a great deal of the turpeth mineral is formed, it shows that there is a great deal of the acid sulphate present, and it will then be wiser to obtain a fresh sample of mercurous sulphate rather than to try by repeated washing to get rid of all the acid.

The free mercury helps in the process of removing the acid, for the acid mercuric sulphate attacks it, forming mercurous sulphate and acid which is washed away.

§ 35. *Conclusion.*

We conclude, then, as the result of these experiments, that Clark cells put up at very different times and in very different manners have very approximately the same E.M.F.

1·434 volts at 15°,

and that they may very conveniently be used as standards of electromotive force.

X = Observations for Current

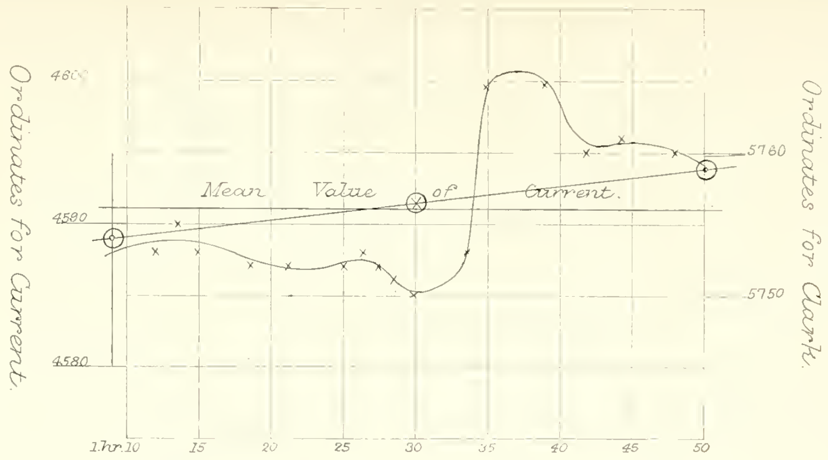
⊙ = Observations for Clark

⊗ = Mean Value of Clark

The Zeros from which the Ordinates for the Current and Clark respectively are measured are different.

Mean Value of Current 4591.

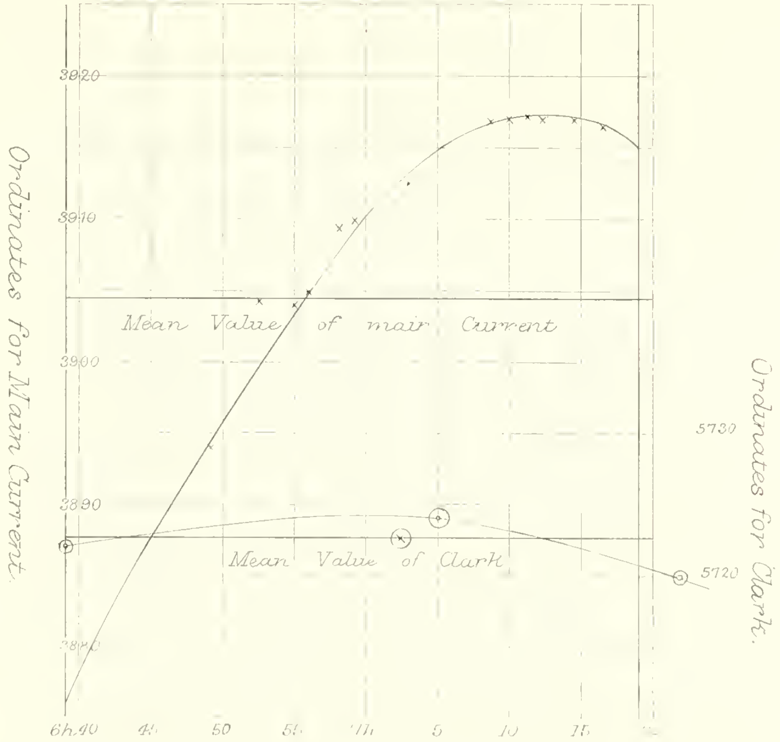
Mean Value of Clark 5756·5



Curve of E.M.F. Aug. 3rd

Mean Value of Current 3904·5

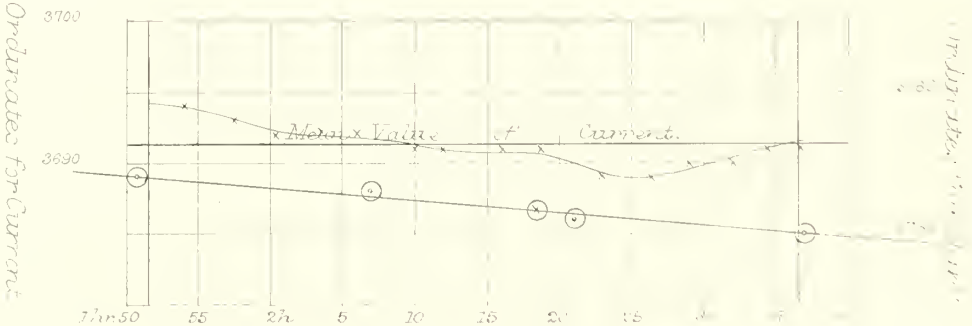
Mean Value of Clark 5722·5



Curve of E.M.F. July 3rd

Mean Value of Current 3691·2.

Mean Value of Clark 5751·5.



Curve of E.M.F. Aug. 5th

West Newman sc.

XVI. *On the Liquation of Metals of the Platinum Group.*

By Edward Matthey, *F.S.A., F.C.S., Assoc. Roy. Sch. of Mines.*

Communicated by Sir G. G. Stokes, *Bart., F.R.S.*

Received March 3,—Read May 5, 1892.

In the present paper, which is a continuation of a former one which has already been submitted to the Royal Society and has been published in the "Proceedings,"[*] an attempt is made to ascertain whether liquation occurs in alloys of the rarer metals, notably in those of platinum, with palladium, and with rhodium, and of gold with aluminium.

In the paper communicated to the Royal Society in 1890 upon this subject, I stated that ninety parts of fine gold melted with ten parts of pure platinum, and cast into a spherical mould showed a liquation of the platinum to the centre of the sphere, as shown in Diagram C in that paper.

Subjoined are the full details proving that such liquation occurred.

A.	Gold	900 parts.
	Platinum	100 ,,

Parts in 1000 of gold :—

Outside.	Intermediate.	Centre.
903	888	845
908	888·5	
905		
902·5		
905		
900		
Average . 903·9		

Maximum difference in the gold between centre and mean of outside, 58·9 per thousand.

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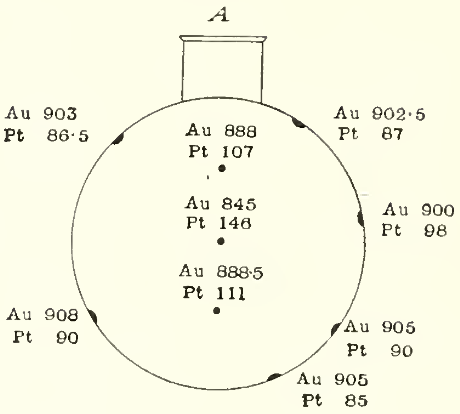
[*] 'Proceedings of the Royal Society,' 1890, vol. 47, pp. 180–186.



In order to prove incontestably this liquation of platinum to the centre of the mass, I again melted together ninety parts of fine gold with ten parts of pure platinum.

This alloy was melted in a plumbago crucible in the intense heat of an oil furnace, and when melted several times and stirred, was cast into a spherical mould of cast-iron. The weight of the sphere was about 5 kilogs. This, cut into two hemispheres, showed the results given in Diagram B.

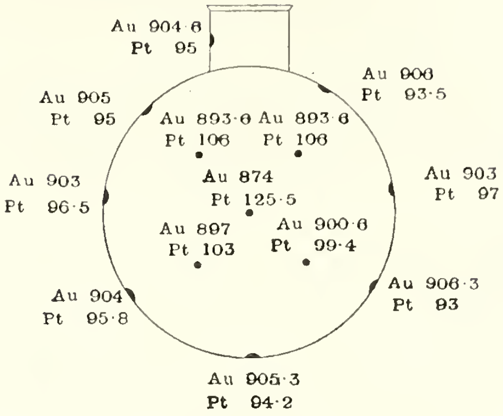
B.	Gold	900 parts.
	Platinum	100 ,,

Parts of gold in 1000 :—

Outside.	Intermediate.	Centre.
904·6	893·6	874
905	893·6	
903	897	
904	900·6	
905·3		
906·3		
903		
906		
Average . 904·6		

Maximum difference in the gold between centre and mean of outside, 30·6 per thousand.

B.



The re-arrangement in this instance (B) is less than in the case of (A).

I re-melted this metal, and again made analyses, with a view to determining the amount of the gold at the points indicated on one of the hemispheres obtained (BB) and these results go very far to confirm those immediately preceding.

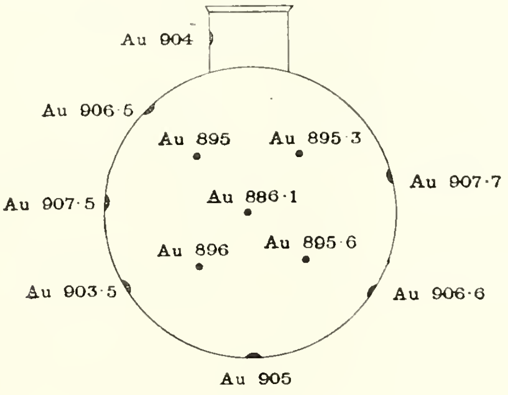
BB. Gold	. . . .	900 parts.
Platinum	. . . .	100 ,,

Parts of gold in 1000 :—

	Outside.	Intermediate.	Centre.
	904 906·5 907·5 903·5 905 906·6 907·7	895 895·3 896 895·6	886·1
Average .	905·8		

Maximum difference in the gold between centre and mean of outside, 19·7 per thousand.

BB.



Having obtained these castings by melting in a crucible in a wind-oil furnace, I now re-fused this metal (B, BB) by means of the oxyhydrogen flame in a lime furnace at the most intense white heat.

The metal was cast into a spherical mould and divided by cutting into two hemispheres in the usual way. The relative distribution of the gold is shown in diagram C.

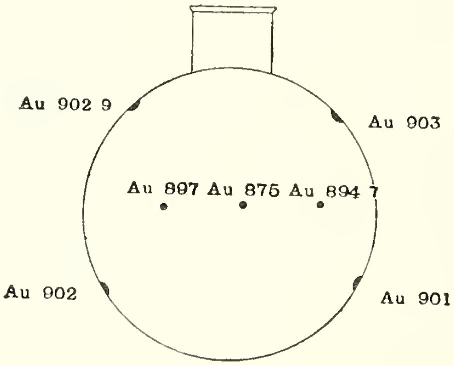
C.	Gold	900 parts.
	Platinum	100 ,,

Parts of gold in 1000 :—

Outside.	Intermediate.	Centre.
902·9 902 901 903	897 894·7	875
Average . 902·2		

Maximum difference in the gold between centre and mean of outside, 27·2 per thousand.

C.



It is evident that the temperature at which the metal is cast, materially affects the extent to which the platinum liquates towards the centre.

In the next experiment, an alloy consisting of pure gold ten parts, and pure platinum ninety parts, was fused together.

This was conducted by melting the two metals in a lime furnace, by means of the oxyhydrogen flame. When the alloy was thoroughly liquid, it was poured into a mould and again melted and cast, to ensure a thorough mixture of the two metals.

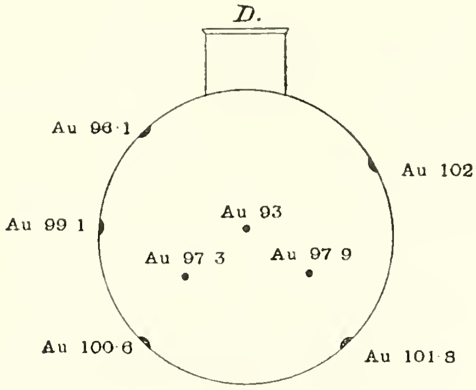
The alloy was melted for a third time, and then poured into a spherical mould made of lime. The quantity operated upon was about $5\frac{1}{2}$ kilogs. The sphere showed, when cut into two halves, a brilliant white crystalline structure (D). The gold in the portions of metal removed for examination was very carefully determined by the method which will be described subsequently; the platinum will of course be the difference.

D.	Gold	100 parts.
	Platinum	900 ,,

Parts of gold in 1000 :—

Outside.	Intermediate.	Centre.
96·1 99·1 100·6 101·8 102	97·3 97·9	93
Average . 99·9		

Maximum difference in the gold between centre and mean of outside, 6·9 per thousand.



To follow this up: an alloy of gold twenty-five per cent. and of platinum seventy-five per cent. was then made in a similar manner to the alloy D, and the utmost care was taken to ensure a thorough mixture of the two metals. This alloy was also melted in a lime furnace by the oxyhydrogen flame and cast into a spherical lime mould. The sphere weighed between 5 and 6 kilogs., and the alloy was hard and brittle, resembling somewhat one of grey cast-iron.

The following are the results obtained from the hemispheres into which the sphere was divided :—

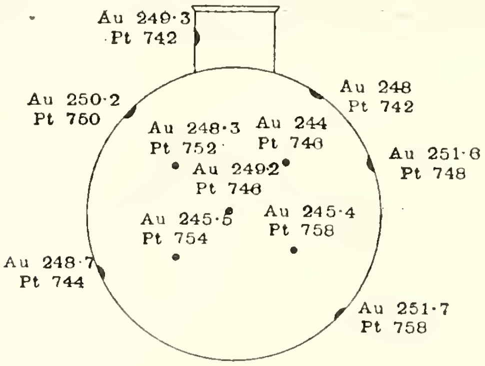
E. Gold . . . . 250 parts.
Platinum . . . . 750 „

Parts of gold in 1000 :—

Outside.	Intermediate.	Centre.
249·3 250·2 248·7 251·7 251·6 248	245·5 245·4 248·3 244	249·2
Average . 249·9		

Maximum difference in the gold between centre and outside, 2·5 per thousand.
Maximum difference in the gold between intermediate and outside, 7·7 per thousand.

E.



It will be observed that in the spheres D and E, in which platinum constitutes the bulk of the mass, the platinum is still driven to the centre. This point requires further investigation; but it may be pointed out that the same effect occurs in the case of standard silver. The most fusible alloy of the copper-silver series is the one which contains about 650 parts of silver in 1000, and it might, therefore, be expected that the *centre* of an alloy containing more silver than 65 per cent. would be poorer in silver than the outside; nevertheless, the authorities, LEVOL, ROBERTS-AUSTEN, and in fact, all who have worked with silver-copper alloys know the contrary to be the case. Silver is driven inwards notwithstanding the fact that an alloy poor in silver remains fluid long after the mass as a whole has become comparatively solid. The platinum-gold alloys now in question behave like the silver-copper series.

Palladium-Gold Alloys.

It is not an easy matter to obtain trustworthy assays from alloys of gold and palladium, and this fact first called my attention to the liquation of this metal from gold.

I melted an alloy of ninety parts pure gold with ten parts of pure palladium in a plumbago crucible in a wind-oil furnace (several times to ensure mixture), and then cast this into a spherical mould, which gave a sphere of about 5 kilogs. of the alloy.

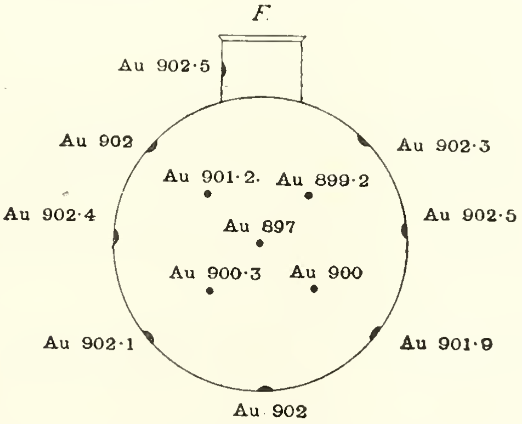
This sphere, which I call F, was cut into halves. The following are the determinations of the gold results from various parts of the hemisphere—the palladium can safely be taken by the difference :—

F.	Gold	900 parts.
	Palladium	100 ,,

Parts of gold in 1000 :—

Outside.		Intermediate.	Centre.
	902·5	901·2	
	902	899·2	
	902·4	900·3	897
	902·1	900	
	902		
	901·9		
	902·5		
	902·3		
Average .	902·2		

Maximum difference in the gold between centre and mean of outside, 5·2 per thousand.



This same metal was now re-melted under the oxyhydrogen flame in a lime furnace, and re-cast into the same spherical iron mould as before, the alloy therefore was at a much higher temperature than in the previous case.

The sphere, which I call G, being cut equally into two, the following are the results from the various indicated portions of the hemisphere :—

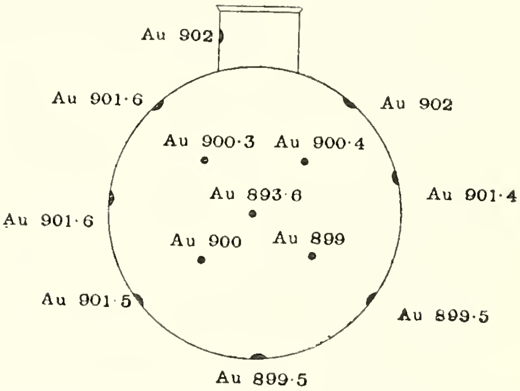
G.	Gold	900 parts.
	Palladium.	100 „

Parts of gold in 1000 :—

Outside.		Intermediate.	Centre.
	902	900·3	893·6
	901·6	900·4	
	901·6	900	
	901·5	899	
	899·5		
	899·5		
	901·4		
	902		
Average .	901·1		

Maximum difference in the gold between centre and mean of outside, 7·5 per thousand.

G.



It seems therefore clearly demonstrated by these experiments that the metal palladium is driven towards the centre of the mass in alloys of gold and palladium in a similar manner, though not perhaps to the same extent, as platinum-gold alloys, and the experiments shown by the diagrams given prove this fact incontestably.

The methods adopted for the determination of the gold in the respective platinum and palladium alloys were the following :—

In the alloys of gold, 10 per cent., and platinum, 90 per cent. Two carefully weighed portions of the alloys to be analysed were weighed and cupelled with two and three-quarter times their weight of fine silver—in identically the same manner as in conducting the process of gold assaying—one of the resulting buttons being then "parted" in nitric acid, and the other by sulphuric acid. In the first instance, the whole of the platinum dissolves with the silver by employing two distinct treatments with nitric acid; in the latter, the silver only separates, leaving the whole of the gold and of the platinum together, the proportion of platinum in the alloy being the difference. The two resulting products weighed against each other yield the gold and the platinum. This is a simple but very accurate process, and is the recognised method adopted by the professional assayer when determining gold alloys containing small proportions of platinum.

With each *set* of analyses I employed standards synthetically made with ten parts pure platinum and ninety parts pure gold, by means of which any possible error which might arise was completely checked and controlled.

In the case of the estimation of the gold in the alloys of 90 per cent. platinum, and 10 per cent. of gold, the foregoing process was obviously unavailable, and I was compelled to try several methods in order to obtain trustworthy results. The following, however, proved to be an exceedingly satisfactory process, and was therefore adopted.

Exact weighings of 50 grains each, in duplicate, were taken of each of the portions of the hemispheres removed for examination. These were dissolved in *aqua regia*, evaporated to ensure the elimination of all free acid, and then diluted with distilled water to about 20 c.c. capacity, which was ascertained by experiment to be the best for the complete precipitation of the gold; the gold was precipitated by crystals of pure oxalic acid, washed and weighed. All the duplicate results agreed most satisfactorily. To corroborate these gold results, I precipitated the platinum by means of pure zinc, boiled the precipitated platinum in hydrochloric acid, and after washing and drying weighed the resulting pure platinum. With every set of analyses of metal from their respective spheres, I made up standards synthetically, each of 37·5 grains pure platinum and 12·5 grains pure gold to ensure accuracy, it being practically impossible to extract the whole of the constituent metals within $\frac{2}{100}$ths per cent. I tried other known processes with a view to obtain absolutely correct results, but I consider that the methods employed were very accurate.

I am confirmed in this view by the experience gained in similar experiments by Mr. W. BETTEL, in a paper contributed to the 'Chemical News,' vol. 56, No. 1452, which is evidently the result of much careful work.

With alloys of gold and palladium it is usual to determine the amount of gold by quartation with pure silver, and parting by nitric acid.

Carefully weighed portions of the alloys are cupelled in pure lead with two and three-quarter times their own weight of fine silver.

These cupelled buttons are laminated and annealed, and then treated by boiling in nitric acid three distinct times. By these means the palladium becomes dissolved, as well as the silver, leaving the pure gold, which, after washing and annealing, is weighed. The difference is, of course, the palladium.

With alloys of gold and palladium, say of gold 90 parts and of palladium 10 parts in the hundred, it is preferable to re-cupel and re-part the gold obtained with a further proportion of fine silver, as sometimes, and where there is as much as 10 per cent. of palladium, the whole of the palladium is not removed in one parting operation.

It is also of importance to work side by side with the portions of metal under examination checks or standards, made of pure gold and pure palladium as nearly to the composition of the alloy to be tested as possible.

The results by this process of quartation are very accurate.

Platinum and Palladium.

The first alloy operated upon in this next series was one of palladium and platinum, in proportions of—

H.	Palladium	85 per cent.
	Platinum	15 ,,

To ensure accuracy, the alloy was made from the pure metals in the spongy condition, fused separately, and then alloyed in the foregoing proportions.

The alloy was melted and re-melted in a lime furnace with oxyhydrogen gas, and cast into a mould, to ensure a complete mixture of the two metals.

The alloy was then again remelted, and cast into a lime sphere of about 3 inches diameter. About 4 kilogs. of the alloy was made for this experiment.

The sphere so produced was afterwards cut, and divided into two hemispheres. The alloy, which had been melted and cast at a temperature of about 2000° C., was decidedly brittle and of a dark grey colour, and the casting was slightly honeycombed from its centre to the tube, or "gate," through which the metal had been poured into the mould. The alloy was very hard.

In the lower portion of the sphere, shown by the dotted line, the peculiar bluish-purple characteristic of palladium showed itself as a distinct feature.

Pieces cut from the places indicated showed the results respectively of palladium and of platinum.

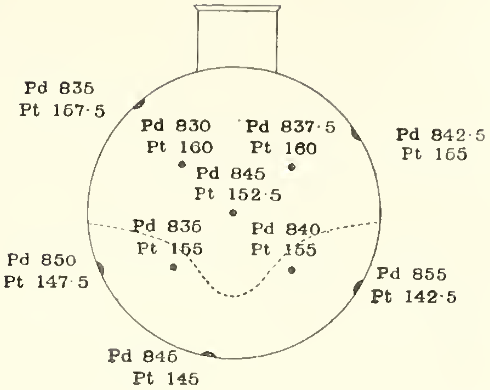
Alloy H.	Palladium	850 parts.
	Platinum	150 ,,

Parts in 1000 :—

Outside.		Intermediate.		Centre.	
Palladium.	Platinum.	Palladium.	Platinum.	Palladium.	Platinum.
835	157·5	830	160	845	152·5
850	147·5	835	155		
845	145	840	155		
855	142·5	837·5	160		
842·5	155				
Average . 845·5	149·5	835·6	157·5		

Maximum difference between centre and mean of outside: Palladium, 5 per thousand; Platinum, 30 per thousand

II.



These results distinctly prove that, although there is not much evidence of true "liquation"—that is, rejection of a constituent of an alloy—the palladium does become concentrated by gravity towards the bottom of the "freezing" mass. It is noteworthy that this alloy, when heated to bright redness *in vacuo* by means of a Sprengel pump, did not yield hydrogen. It was, in fact, singularly free from occluded gas.

Following the matter up, the proportions of the same metals were reversed, and an alloy of—

I.	Platinum	85 per cent.
	Palladium	15 ,,

was then operated on.

The constituent metals for this alloy were, as in the case of the former one, twice melted in a lime furnace by means of the oxyhydrogen flame, and cast to ensure perfect homogeneity, then re-melted, and cast into a lime sphere of 3 inches diameter.

About 5 kilogs. was the quantity of the alloy made for this experiment.

The alloy, when cut into two hemispheres showed a bright white crystalline structure, not brittle, but tough from the peculiar crystallization, which consisted of interlaced planes.

The casting was perfectly solid, but toward the gate or tube the casting was slightly honeycombed.

The following proportions of platinum and palladium were found at the points indicated :—

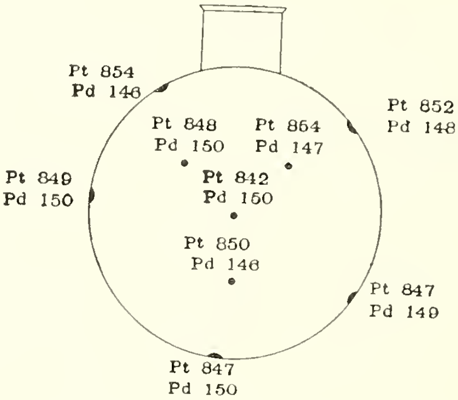
Alloy I. Platinum . . . . 850 parts.
Palladium . . . . 150 „

Parts in 1000 :—

Outside.		Intermediate.		Centre.	
Platinum.	Palladium.	Platinum.	Palladium.	Platinum.	Palladium.
854 849 847 847 852	146 150 150 149 148	848 854 850	150 147 146	842	150
Average . 849·8	148·6	850·6	147·6		

Maximum difference between centre and mean of outside :—Platinum, 7·8 per thousand; Palladium, 1·4 per thousand.

I.



The foregoing results distinctly prove that there is still a slight concentration of palladium at the centre.

H and I.

As the results of the two foregoing alloys were non-malleable, the first being decidedly brittle, I proceeded to investigate this point further by intimately mixing pure sponge platinum and pure sponge palladium in the same proportions as in these two alloys, subjecting the mixed sponge metals to hydraulic pressure in a steel mould, heating them and forging the alloys so heated, in the same manner as with ingot platinum made from sponge platinum. In both cases these alloys, which had been prepared by welding and not fusion, were perfectly tough and malleable, proving that the high temperature necessary for melting these alloys enables them to pass into a crystalline state.

I then melted these two malleable alloys, H and I, by means of the oxyhydrogen flame in lime crucibles, and allowed them to cool in the crucibles. I employed about 65 grms. for each of these experiments.

Upon examination I found both of these alloys to be malleable, so that the difference between these results, as regards brittleness, is probably to be accounted for by the fact that the slow cooling enables a fine network of crystals to form with resulting toughness.

The process adopted for the determinations in the platinum-palladium series of alloys was the following :—

Of each portion of the alloy to be analysed fifty grain determinations were taken and dissolved in nitrohydrochloric acid and evaporated nearly to dryness, these resulting chlorides re-acidified with hydrochloric acid and again evaporated nearly to dryness.

The re-evaporated chlorides then dissolved up in water and the respective solutions of each determination diluted to about 150 c.c. liquid capacity ; when cold the palladium was precipitated by mercury cyanide. The precipitate thus obtained was allowed to stand for twenty-four hours in a slightly warmed atmosphere, as I have found by experience that the precipitate of palladium cyanide comes down better under these circumstances.

This precipitate, collected on filters, washed, dried, and ignited, gives the whole quantity of pure metallic palladium in the alloy.

To obtain the platinum from the mother liquors, from which the palladium cyanide had been precipitated, these solutions were in each determiatnion evaporated to about 30 c.c. capacity, and the platinum was then precipitated by ammonium chloride.

The precipitate of the double salt of platinum ammonium chloride was then collected on filters, dried, ignited, and weighed, the result being pure platinum.

The small proportion of platinum remaining in the mother liquors was precipitated by means of pure metallic zinc as metallic platinum, which, washed and digested in weak hydrochloric acid, was collected, ignited, and weighed, the results being added to those obtained by the ammonium chloride precipitation.

All these results were checked by standards of pure palladium and of pure platinum in the proportion of—

Palladium	85	in the case of the alloys marked H.
Platinum	15	

and also of—

Platinum	85	in those marked I.
Palladium	15	

the results of which confirmed the accuracy of the process employed.

Platinum and Rhodium.

Much attention has lately been drawn to an alloy of pure platinum, with 10 per cent. of rhodium, which has become important from the excellent service it has rendered in the determination of high temperatures. The alloy of platinum with 10 per cent. of rhodium is used with pure platinum as a thermo-couple, and it is, therefore, interesting to be able to set at rest any doubt which might arise as to this alloy being uniform in composition when melted and drawn into wire.

For this purpose I prepared an alloy, J, of :—

Platinum	90 per cent.
Rhodium	10 ,,

by fusing together in a lime furnace by the oxyhydrogen gas flame pure melted platinum with pure melted rhodium in the above proportions.

About one and a half kilogs. of this alloy was made, and after three times melting and casting, the alloy was re-melted and poured into a lime sphere of two inches diameter.

Upon cutting the sphere into hemispheres I found that a beautiful white malleable alloy was the result, with very slight evidence of shrinkage, care being taken to provide against this by an extra long gate or pour.

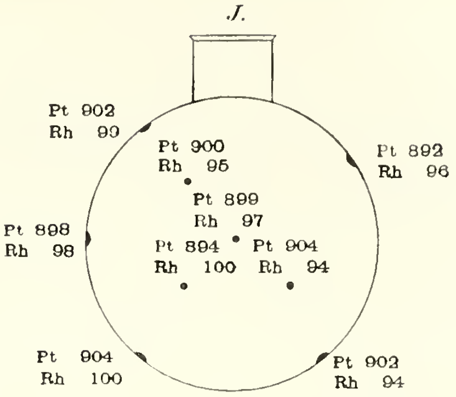
The following are the proportions of platinum and rhodium found by analysis at the points indicated :—

Alloy J.	Platinum	900 parts.
	Rhodium	100 ,,

Parts in 1000 :—

Outside.		Intermediate.		Centre.	
Platinum.	Rhodium.	Platinum.	Rhodium.	Platinum.	Rhodium.
902	99	900	95	899	97
898	98	894	100		
904	100	904	94		
902	94				
892	96				
Average . 899·6	97·4				

Maximum difference between centre and mean of outside :—Platinum ·6 per thousand; Rhodium, ·4 per thousand.



This result proves that the alloy is not subject to liquation, and fully justifies the high opinion that H. LE CHATELLIER and ROBERTS-AUSTEN have formed as to its suitability for thermometric measurements.

The process adopted for the separation and determination of the rhodium and platinum in this alloy is one that I have found by experience to be exceedingly accurate if conducted with care. (It is described by TURNER, 'Elements of Chemistry,' 5th Ed., 1834, p. 652.)

Fifty grains of each determination were dissolved in nitrohydrochloric acid, and evaporated to dryness. The dry chlorides were then dissolved up in a warm solution of sodium chloride, and again evaporated to dryness. The resulting dry mass was then taken from the evaporating dishes and triturated in a porcelain mortar, from which each determination was then transferred to closed flasks, and digested in absolute alcohol in a very slightly warmed atmosphere for 24 hours, the bottles being shaken from time to time.

At the end of this period the whole of the platinum chloride is taken up in solution by the alcohol, and the rhodium chloride left combined with the sodium chloride as a double salt of rhodium and sodium chloride insoluble in alcohol. This latter salt was washed with alcohol to remove all platinum chloride, and then dissolved in water.

From this solution the metallic rhodium was obtained by direct precipitation with pure zinc, washed digested in weak hydrochloric acid, collected, dried, ignited, and weighed.

The platinum from the alcoholic chlorides when evaporated and re-dissolved in water was determined by precipitation with ammonium chloride and pure zinc, as in the palladium-platinum series, and weighed as metallic platinum.

With these analyses accurately weighed portions of pure rhodium and pure platinum, in the proportions of platinum 900, and rhodium 100 parts, were dissolved and treated as standards to ensure accuracy in the results of the analyses.

It will be observed that, in some cases of members of the platinum group, the results of the analyses add up in excess of 1000.

This is to be explained by the exceptional difficulty which attends the accurate determination of the metals of this group.

Gold and Aluminium.

The interest of the problems connected with the liquation of alloys of the rare and precious metals, suggested the desirability of investigating the behaviour of a metal with a very high specific gravity alloyed with a metal with a very low one.

For this purpose gold, which has a specific gravity of 19·3, and aluminium, whose specific gravity is 2·6, were selected.

In order to show an absolute contrast, two alloys were adopted, viz.: —

K.	Aluminium	90 parts,
	Gold	10 ,,

and

L.	Gold	90 parts.
	Aluminium	10 ,,

The alloy K was the first one experimented with. The ten parts of gold were first melted in a crucible, and the 90 parts of aluminium gradually introduced, the alloy being stirred as each portion of aluminium was added. When the whole quantity of aluminium had been added to the melted gold, the crucible was withdrawn from the fire, the contents thoroughly stirred and cast into a spherical mould. This alloy was white and tough, but the fracture at the lower part of the hemisphere exhibited a purple-pink mottled appearance, of which more will be said subsequently.

The specific gravity of this alloy was 2·89. The sphere was then cut into two equal halves from top to bottom. About 1 kilog. of the alloy was employed for this experiment.

Alloy K.	Aluminium	900 parts.
	Gold	100 ,,

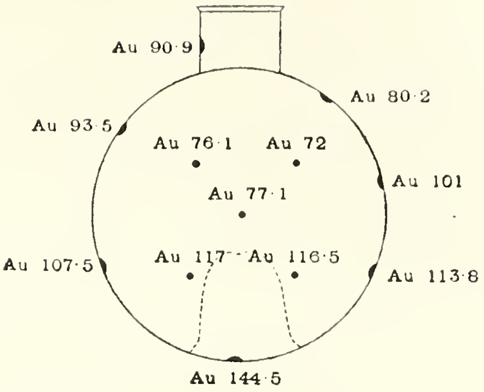
Parts in 1000 of gold—

Outside.	Intermediate.	Centre.
90·9 93·5 107·5 144·5 113·8 101·0 80·2	76·1 72 117 116·5	77·1
Average . 104·4		

Maximum difference in the gold between centre and mean of outside, 27·3 per thousand.

Maximum difference in the gold between centre and mean of bottom of casting, 67·4 per thousand.

K.



The above figures show the proportion of gold found by analysis at the different points of the hemisphere K. The result is a complex one, a rich alloy of gold and aluminium falls to the bottom of the mass by gravity, about 8 per cent. remaining in solution, and, on freezing, the gold becomes concentrated externally from the mother liquor.

The alloy L, of gold 90 per cent. and aluminium 10 per cent., was then made by first melting the gold and subsequently adding the 10 per cent. of aluminium, well stirring, and casting into a similar spherical mould as with the other alloy (K).

This alloy was very fluid, its fracture hard, white, crystalline, and very brittle. The whole of the interior, when the sphere was cut in two halves from top to bottom, showed purple-pink spots throughout; but from the bottom of the hemisphere to the centre a pyramidal cone of white crystalline metal showed itself distinctly through the mass, in the direction shown by the dotted line. Its specific gravity was 11·96.

About 4½ kilogs. were employed for this experiment.

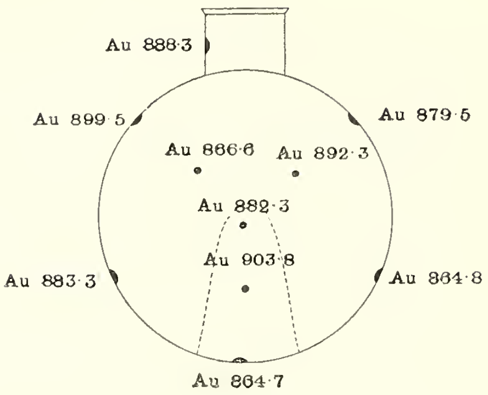
Alloy L.	Gold	900 parts.
	Aluminium	100 ,,

Parts in 1000 of gold :—

Outside.	Intermediate.	Centre.
888·3 899·5 883·3 864·7 864·8 879·5	866·6 892·3 903·8	882·3
Average . 880		

Maximum difference in the gold between centre and mean of outside, 2·3 per thousand.

L.



The above are the proportions of gold found at the different points of the hemisphere.

It will be evident that the cooling of the alloy has been attended with much re-arrangement, but it is difficult to say that there has been true liquation or concentration of any given constituent of the alloy, as the mass seems to be a congeries of complex alloys of aluminium and gold.

The beautiful rose-purple alloy of aluminium and gold, prepared in accordance with Professor ROBERTS-AUSTEN'S instructions in 'Nature,' June 21, 1891—viz., 786 parts of pure gold and 214 parts of aluminium—were melted together, thoroughly mixed, and cast into a 3-inch sphere.

This alloy, which I call M, is extremely brittle, but the beauty of its magnificent fracture was most striking; and the purple-pink appearances in the fractured spheres K and L are evidently reproductions of this alloy in some other proportions. Evidences of this colouring have been alluded to as showing in the alloys K and L.

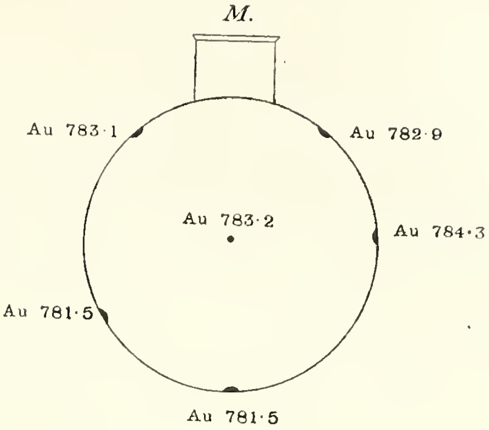
Trials made from various parts of the sphere, when broken up, showed proportions of gold as under.

Alloy M.	Gold	786 parts.
	Aluminium	214 ,,

Parts in 1000 of gold:—

Outside.	Centre.
783·1 781·5 781·5 784·3 782·9	783·2
Average . . 782·6	

Maximum difference in the gold between centre and mean of outside, ·6 per thousand.



This result is of special interest. The alloy has a composition which corresponds to the formula Al_2Au, the constituent metals therefore exist in atomic proportions; it would appear to be a true chemical compound. There is little or no evidence of liquation, the maximum difference between any two assays being only 2·8 parts in a thousand. As this is the most highly coloured alloy yet known, the simple atomic relations between its constituents is of much theoretical importance. It would appear to be as uniform in its composition as any alloy known.

Several methods were tried for the determination of the gold in these various alloys of gold and aluminium.

Cupellation with excess of lead to remove the aluminium proved quite useless.

The removal of the aluminium by digestion in hydrochloric acid, and collecting the residual gold did not yield satisfactory results.

The process adopted, therefore, was as follows :—

Accurately weighed portions of fifty grains each of the alloys under examination were fused with litharge, under a flux of potassium carbonate and borax with a small proportion of powdered charcoal, and the resulting slag re-fused with a further small quantity of litharge and powdered charcoal.

The lead buttons containing all the gold (the aluminium having combined with the fluxes employed) were cupelled, and the resulting gold cupelled with silver and parted with nitric acid in the usual manner. All these determinations were worked with checks or standards of fine gold and pure aluminium.

This process for the determination of gold was employed for every one of the foregoing aluminium-gold alloys.

Conclusions.

I regret that time has not enabled me to examine more members of each particular series of alloys, so as to present results in fuller detail; in fact, the silver-copper series is the only one upon which anything like exhaustive work has been done. No doubt, in every series of alloys there is one definite alloy which would yield a uniform mass on cooling, and it is known that in the silver-copper series this alloy (LEVOL'S) contains 718 parts of silver per thousand. It is not certain, however, that this is the *euctectic* alloy of the series—that is, the one with the lowest melting point—but it is well known that when silver-copper alloys which contain more silver than 718 parts per thousand are cooled, the centre of the solidified mass is richer than the exterior. This is the case with standard silver for instance, which contains 925 parts of silver per thousand, and it is safe to conclude that an alloy rich in copper is the first to fall out from the mass, and that this alloy sets round the inner surface of the mould, driving a still fluid alloy—rich in silver—to the centre. The general rule in the present results, seems to be that *in the cooling of a fluid mass of two united metals, an alloy rich in the more fusible constituent of the mass falls out first driving the less fusible constituent to the centre.* The gold-platinum alloys (A, B, C, D, and E) seem to be always rich in gold externally.

It is remarkable that the metals of the platinum group do not show much liquation among themselves, but, on the other hand, when gold is united to the members of the platinum group there is evidence of liquation.

The gold-palladium one (F), follows the above rule.

There is evidence that the alloy E, containing 750 parts of platinum and 250 of gold, is near the composition of a true compound, as it shows but little sign of liquation, and is, moreover, hard and brittle, differing materially from the rest of the series. The purple alloy of gold and aluminium M, $AuAl_2$, is almost certainly a true chemical compound, the solidified mass being as nearly uniform in composition as may be. The uniformity of the alloy (J) of platinum with 10 per cent. of rhodium is of much interest in view of the important part which the alloy is playing in pyrometric work.

Conducting the experiments, the results of which are embodied in the present paper, has been very laborious, and although, as already stated, no complete series of the alloys of any two metals has been examined, quite sufficient data have been collected to afford valuable guidance to the metallurgist, who will now know what behaviour may be expected from the other members of the groups of the alloys in question. The gold-platinum series of alloys are of industrial importance, as native gold is so often associated with platinum, and it is somewhat surprising to find that assays made on pieces of metal cut from the exterior of an ingot cannot be trusted to represent the composition of the mass. The aim of the investigation has been to show that notwithstanding the great difficulty which attends the preparation of alloys of metals with very high melting points, it is possible to elicit from them the same kind of information which has proved to be so useful in the case of the more ordinary and tractable alloys.

Methods adopted for the Analysis of the several Alloys Experimented upon.

(Added June 13, 1892.)

For alloys of gold 100, and platinum 900 parts.—Two carefully weighed portions of the alloys to be analysed were weighed and cupelled in pure lead with two and three-quarter times their weight of pure silver, in identically the same manner as in conducting the process of gold assaying; one of the buttons so obtained was then, after lamination and annealing, parted in nitric acid, which by two distinct treatments removes the whole of the silver and the platinum, leaving the gold in a pure state, which, when washed, annealed and weighed accurately, gives the proportion of pure gold in the alloy.

The other button, after being laminated and annealed in precisely the same way, was parted in sulphuric acid, which removes the silver only, leaving the whole of the gold and the platinum intact. By weighing the gold obtained by parting in nitric acid against the gold and platinum left by the sulphuric acid parting, the proportion of platinum obtained is the difference. This is a simple but very accurate method, and is universally recognised and employed by professional assayers when determining alloys of gold containing platinum. By working checks or standards of gold and platinum, made synthetically of platinum ten and gold ninety parts, any error which might arise is completely checked and controlled.

In the case of the alloys of platinum 900, and gold 100 parts, the foregoing process was obviously unavailable. After many different experiments, the following method was one which I found to be accurate and trustworthy; and duplicate analyses, each on 50 grains of metal, were carefully made by this process on each of the portions of the hemispheres removed for examination.

Exact weighings of fifty grains each were taken cf each of the alloys under examination and treated with an excess of nitrohydrochloric acid, which gradually dissolved the whole. The resulting solutions of platinum-gold chloride were then evaporated nearly to dryness to ensure the elimination of all free acid, so as to obtain perfectly neutral solutions. These chloride solutions were then diluted with distilled water to about 20 cub. centims. capacity, a degree of strength which was ascertained by experiment to be the best for ensuring complete precipitation of the gold. The metallic gold was thrown down by means of crystals of oxalic acid and was carefully washed, dried, and weighed.

From the mother liquors the metallic platinum was then precipitated by means of pure metallic zinc, and the resulting precipitated platinum was thoroughly washed, and boiled in diluted hydrochloric acid. The platinum thus purified was then washed, dried, and weighed.

Side by side, with each set of analyses, standards synthetically prepared were used, each of 37·5 grains pure platinum and 12·5 grains pure gold. This was necessary to ensure accuracy, as in practice it is found impossible to obtain the whole of the constituent metals from these alloys within $\frac{2}{100}$ths per cent.

I tried other known processes for these determinations, but none gave such accurate results ; and in this view I am confirmed by Mr. W. BETTEL whose paper in the 'Chemical News,' vol. 56, No. 1452, shows that he has devoted much patient attention to the subject.

The method of analysis adopted in the case of the alloys of gold 900 and palladium 100 parts, consisted in removing the palladium by the process of "quartation" with fine silver. Carefully weighed portions of the alloys under examination were cupelled in pure lead with two-and-three-quarter times their weight of fine silver. Each cupelled button so produced was then laminated, annealed, and treated with and "parted" by boiling in nitric acid three distinct times. By these means the palladium is dissolved out as well as the silver, the gold being left. The resulting gold, after being washed and annealed, was again cupelled in lead with a further proportion of two-and-three-quarter times its weight of fine silver, and again parted by boiling in nitric acid three distinct times, to ensure the removal of the whole of the palladium, the gold so obtained after being washed, annealed, and weighed, yielding the percentage of pure gold in the alloy. The proportion of palladium was then found by the difference

To ensure accuracy, checks or standards, made up of pure gold 900 parts and palladium 100 parts, were worked side by side with all the analyses made.

All the results by this process are trustworthy and accurate.

The process adopted for the determinations of the platinum and palladium in the series of alloys made of these metals was as follows :—

Of each portion of the alloy to be analysed, accurately weighed portions of 50 grains each were attacked by nitrohydrochloric acid, which gradually dissolved the whole of the alloy under digestion. The resulting chlorides were then carefully evaporated nearly to dryness, then re-acidified with hydrochloric acid, and again evaporated almost to dryness. These re-evaporated chlorides were then dissolved in distilled water, and the respective solutions of each analysis were diluted to a volume of about 5 ounces, and allowed to become quite cold.

Cyanide of mercury was then added in slight excess to precipitate the palladium, and the whole allowed to remain for twenty-four hours in a warm atmosphere, experience showing that this precipitate of palladium cyanide comes down better under these circumstances. At the end of this time the precipitate of palladium cyanide was collected on filters, washed, dried, ignited, and weighed as pure palladium.

The whole contents of the palladium in the alloys is precipitated by these means.

To obtain the platinum from the mother liquors, from which the palladium had been removed, they were evaporated to a capacity of about 30 cub. centim. Ammonium chloride was then added, and the platinum precipitated as the double salt of platinum-ammonium chloride. This precipitate was then carefully collected on filters, dried, ignited, and weighed, the result being pure platinum.

The small proportion of platinum remaining in the mother liquors was precipitated by means of pure metallic zinc as metallic platinum, which, when washed and digested in diluted hydrochloric acid, was collected, ignited, and weighed, the results being added to those obtained by the ammonium chloride precipitation. All these results were checked by standards of pure palladium and pure platinum in the proportions of—

Palladium	85	in the case of the alloys marked H,
Platinum	15	

and of—

Platinum	85	in those marked I,
Palladium	15	

and the results obtained by analysing these standards confirmed my views as to the accuracy of the process employed.

Alloy J.	Platinum	90 parts.
	Rhodium	10 „

The process adopted for the separation and determination of the rhodium and platinum in this alloy is one that I have found by experience to be exceedingly accurate if conducted with care. (It is but little known, although it was described many years ago in TURNER's 'Elements of Chemistry, 5th edition, 1834, page 652.)

Fifty grains of each of the alloys under examination were dissolved in nitro-hydrochloric acid, and evaporated to dryness. The dry chlorides were then dissolved in a warm solution of sodium chloride, and again evaporated to dryness. The resulting dry mass was then taken from the evaporating dishes and triturated in a porcelain mortar, from which the mass was then transferred to closed flasks and digested in absolute alcohol in a slightly warm atmosphere for 24 hours, the bottles being shaken from time to time. At the end of this period the whole of the platinum chloride is taken up in solution by the alcohol, and the rhodium chloride is left combined with the sodium chloride as a double chloride of rhodium and sodium insoluble in alcohol. This latter salt was washed with alcohol to remove any platinum chloride, and then dissolved in water. From this solution the metallic rhodium was obtained by direct precipitation with pure zinc; it was washed, digested in weak hydrochloric acid, collected, dried, ignited, and weighed.

The alcoholic chlorides containing the platinum chloride were evaporated and re-dissolved in distilled water. The platinum was then thrown down by means of ammonium chloride and by pure zinc, as in the palladium-platinum series, and weighed as metallic platinum.

In these analyses, accurately weighed portions of pure rhodium and pure platinum, in the proportions of platinum 900, and rhodium 100 parts, were dissolved and treated as standards to ensure accuracy in the results of the analyses.

In the case of the gold-aluminium alloys, several methods were tried for the determination of the gold in these various alloys, direct cupellation with lead, and the removal of the aluminium by dissolution in hydrochloric acid, but neither yielded satisfactory results. The process adopted, therefore, was to ascertain accurately the proportion of gold in the various portions of the alloys removed for analysis by the following method, the aluminium being found by the difference.

Determinations of 50 grains each from the various portions of the alloys removed for examination were carefully fused in small clay crucibles with litharge under a flux of borax and potassium carbonate, with a small proportion of powdered charcoal. The button of lead so obtained was removed, and the slag re-fused with a further small quantity of litharge and powdered charcoal. The resulting lead buttons were then cupelled, and yielded the gold contained in the alloy (the whole of the aluminium having combined with the fluxes employed). The gold so obtained was then cupelled in pure lead with two-and-three-quarter times their weight of fine silver, and parted in nitric acid. The gold thus purified was washed, dried, annealed, and weighed.

This process for the determination of the gold was employed for every one of the aluminium gold alloys.

All these analyses were checked by means of synthetical standards made up of gold and aluminium in the same proportions as in the alloys for examination.

XVII. BAKERIAN LECTURE.—*On the Grand Currents of Atmospheric Circulation.*

By JAMES THOMSON, *LL.D., F.R.S., Emeritus Professor of Civil Engineering and Mechanics in the University of Glasgow.*

Received March 10,—Read March 10, 1892.
[Professor THOMSON died on May 8, 1892, before this Lecture was printed.]

IN the early times of the Royal Society (a little more than 200 years ago) a spirit of inquiry and of speculation as to the causes of the Trade Winds arose among its members. The papers which we may presume to have first brought the subject into special notice in the Society, and which were published in the 'Transactions,' offered views which, in the light of subsequent knowledge and theory, show themselves as being untenable, and in part even grotesque. But those papers were soon followed by, and probably had an effect in leading to, a much more important paper by the eminent astronomer EDMUND HALLEY; and this was followed 49 years later by one, more important still, by GEORGE HADLEY, in which we may with confidence judge that a substantially true theory of a large part of the system of Atmospheric Circulation in its grandest and most dominant conditions was for the first time offered to the world through the pages of the 'Philosophical Transactions.'

Further speculations on the subject and advances in our knowledge of it have been made in later times and have been brought into notice in various ways. I believe that I have myself arrived at some improved considerations which are to a large extent trustworthy and go far towards completing the true theory of the grand currents of atmospheric circulation, and I entertain the ambition to have my views placed on record by this Society—the Society in which the subject had its most important beginnings.

With this in view it appears indispensable that some historical recital should be adduced of the progress made by others previously: but still, for those who may at any time wish to direct their attention specially to the physical conditions irrespective of the history of the progress of thought or of discovery on the subject, it appears desirable that an exposition of the resultant theory which I have devised should be presented without being itself encumbered by historical details of the courses through which it has been ultimately arrived at. I propose, therefore, to present, in a first section, a historical sketch of all the speculations and theories which, as far as known to me, have conduced in any important way towards the resulting theory that I have to offer as being tenable and trustworthy; and then to set forth that new theory itself divested as far as possible of historical or personal

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references; and to conclude with some considerations as to the reasons for or against the views put forward by various persons.

The first opening up of considerations and discussions in the Royal Society on the subject of Atmospheric Circulation appears to have been made in a paper submitted to the Society, in 1684, by Dr. MARTIN LISTER, Doctor of Physic of the University of Oxford, and published in the 'Transactions.'* As an illustration of the scanty and crude condition of knowledge and of thought on this great subject at that time—the middle period of the life of Sir ISAAC NEWTON—I may be permitted to cite the views of Dr. LISTER in his own words as offered briefly in that paper:—

"Among the known *Sea Plants,* the *Sargosse,* or *Lenticula Marina,* is not to be forgot; this grows in vast quantities from 36 to 18 *Degrees Northern Latitude,* and elsewhere, upon the deepest Seas. And I think (to say something by the by of that great *Phenomenon* of the *Winds*) from the daily and constant breath of that *Plant,* the *Trade* or *Tropick Winds* do in great part arise: because the matter of that *Wind,* coming (as we suppose) from the breath of only one *Plant* it must needs make it constant and uniform: Whereas the great variety of *Plants* and *Trees* at Land must needs furnish a confused matter of *Winds*: Again the *Levant Breezes*† are briskest about *Noon,* the *Sun* quickening the *Plant* most then, causing it to *breathe* faster, and more vigorously; and that *Plants* mostly languish in the *night* is evident from many of them which contract themselves and close at that time; also from the effects of our winters upon them, which cause them to cast both fruit and leaves too; whereas they are said (the same *Plants* for kind) universally to flourish all the year alike within the *Tropicks.*

"As for the *direction* of this *Breeze* from *East* to *West,* it may be owing to the *General current* of the *Sea,* for a gentle *Air* will still be led with the *stream* of our *Rivers,* for example. Again every Plant is in some measure an Heliotrope, and bends itself, and moves after the *Sun,* and consequently emits its vapours thitherward, and so its direction is in that respect also owing in some measure to the *Course* of the *Sun.*"

[NOTE.—The above is the whole passage given by Dr. LISTER about Trade Winds. The rest of his paper relates to entirely different subjects, chiefly to salt springs and brines.]

In scrutinizing these utterances of Dr. LISTER, we may notice that he must have been in possession of some information, more or less vague, to the effect that over extensive regions of the great oceans between the Tropics, or near to them, winds blowing from east towards west are prevalent; and that he has attempted to explain this prevalence by attributing it to the breath of a plant floating on the sea and turning "as an heliotrope" so as to blow its breath westward according to the

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* 'Phil. Trans.,' No. 156, p. 494. Date February, 1683–84.

† By "Levant Breezes," here Dr. LISTER obviously means breezes from the east, in fact, the Trade Winds of the tropics.—JAMES THOMSON.

direction of the Sun's diurnal relative motion through the sky from its rising in the east to its setting in the west. He does not indicate any knowledge of the fact that on the two sides of the Equator in tropical regions there are two Trade-Wind zones, one on each side, in each of which the wind prevails from east to west, with an accompanying motion in each case towards the Equator.

We may, indeed, suppose, that such knowledge was only gradually acquired, chiefly by mariners, and was but vaguely and imperfectly intercommunicated among them, and was spread very little among others during a long period of time. I do not suppose that any remarkable step in the discovery and promulgation of knowledge of the prevalent courses of the winds in those seas in and about the Torrid Zone is to be attributed to any one person in particular, nor that there was, indeed, any very important and clear promulgation of the floating knowledge on the subject until the time when the astronomer, Halley, collected and systematized a large amount of valuable information, and presented it to the Royal Society, in his paper in the 'Transactions' of 1686, to which I shall make particular reference a little further on.

It may be well at the present stage, before going further into the history of speculations, to draw attention to the chief features of the Trade Winds and other perennially prevalent air currents, as they present themselves very manifestly to the notice of mariners.

The mariners on board a ship at sea, it is to be observed, however, have direct cognizance only of the wind blowing at the spot on the ocean's expanse where for the time being their ship is situated. They can make no observations on the winds blowing at the same moment 100 miles away, and the vault of the sky above them presents to their eyes no adequate indication of the upper currents, or of the places whence these come or whither they are going in their circuits. But even long ago, by the collation among navigators of facts contemporaneously observed by various seamen, important knowledge was acquired gradually as to the general character of contemporaneously existing air currents at the surface of the sea, without the aid of any trustworthy theory as to the continuations of such currents in circulation through the upper regions of the atmosphere. It is further to be noticed that the geographical distribution of sea and land, presenting as it does great regions of ocean, and large continents themselves varied with mountain ranges and low-lying plains, introduces great local variations in the conditions determining the courses of winds, and prevents the institution of any complete uniformity in the character of the air currents all round the Equator, or throughout zones between any parallels of latitude.

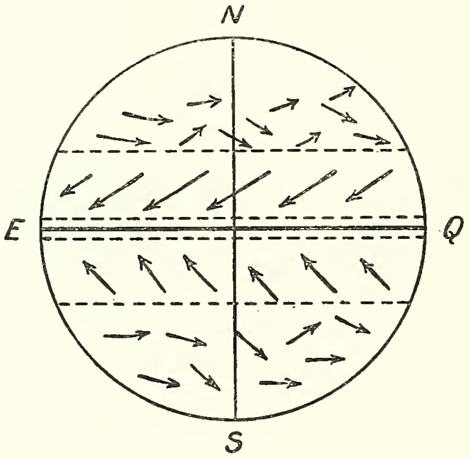
But in the Atlantic and Pacific Oceans there are extensive regions within, and adjacent to, the Torrid Zone, in which the winds blow with remarkable constancy from the east while converging also from north and south at the two sides, towards a medial belt of calms and rains which is situated along, or very near to, the Equator.

These remarkably persistent winds blowing in the northern hemisphere from the north-east, and in the southern from the south-east, are called the Trade Winds. The outer limits of the two Trade Winds vary in different seasons of the year, and are affected by casually varying conditions of the atmosphere in other parts of the world, and by the geographical configurations of the surrounding continents affecting them unequally in different parts; but, without minute exactitude, they may be regarded as occupying some such breadth as perhaps 25° or 30° on each side of the Equator.

It was also found by mariners in those early times previous to the development of theories of atmospheric circulation, that in the great oceans, in the higher latitudes, outside of the trade-wind bands, west winds are prevalent in frequency and strength over winds in other directions. It became the practice of traders when going on a voyage from east to west to make their way into the trade-wind region, where they were sure of finding favouring breezes, and on their return voyage to get into higher latitudes, so as to take advantage of the prevailing west winds there.

Until recent years no information was definitely gathered from observations or otherwise as to whether or not there be any prevalent general average tendency in those west winds to blow in their variations more towards the Pole or towards the Equator; and I avoid entering on any statements on the subject at the present historical stage, as that matter will be better associated with the subsequent progress of theories than with the early history.

Fig. 1.



The explanations just given in words as to the chief features of the trade winds and of the west winds of higher latitudes may be supplemented so as to come more vividly before the imagination by aid of fig. 1.

This figure is sketched without regard to the disturbing influences of continents and mountain ranges. It may be regarded as being suggestive of the most remarkable features which would probably present themselves in the winds if the surface of the world were all ocean, or were ocean mottled very uniformly with small islands.

Now, to revert to the historical sketch already entered on, of speculations and theories as to perennially prevalent winds, and to variable winds which manifest perennial prevalence in special directions, the next theory to which I have to refer, is that of Dr. GARDEN of Aberdeen, which, about one year after that of Dr. LISTER, was placed on record in the 'Transactions of the Royal Society.' Dr. GARDEN, in his paper,* attributes the east to west motion of the Trade Winds of the Atlantic and Pacific Oceans to the supposed vortices of a supposed ether, or all-pervading atmosphere, which, according to the planetary system proposed by DES CARTES, and at that period still believed in by some, were imagined to be the agents carrying on or sustaining the revolutions of the Planets round the Sun, and of the Moon round the Earth, and of the Earth round its own axis. Dr. GARDEN'S paper gives indication of his having some knowledge, not only of prevalence of winds from the east within the tropics, but also of prevalence of winds from the west in higher latitudes outside of the tropical regions. He gives no indication of knowledge of the Trade Winds having, along with their westward motion, also motions towards the Equator from both sides; and is, in this respect, apparently on an equality with Dr. LISTER. They had both made praiseworthy exertions in collecting and bringing into notice important results from the observations of mariners and other travellers. When, however, Dr. GARDEN offers explanations of his supposed reasons for the blowing from east to west within the Tropics, and from west to east in latitudes higher than those of the tropical regions, his statements, in their meaninglessness, quite transcend the inadequacy of the explanations in the amusing attempt of Dr. LISTER.

The papers of these two men may probably have had a beneficial effect in instigating HALLEY to prepare, for the Royal Society, a paper presenting the results of his researches as to the observable facts of the winds, and his speculations to account for the prevalent directions of their motions. In 1686, about one year after Dr. GARDEN'S paper, HALLEY, then at the age of thirty, submitted to the Society an elaborate and very clear account of the information as to the winds in different parts of the world which he could collect from numerous sources, including observations carefully made by himself on voyages and on land between the Tropics. The title of his paper in the 'Transactions,' is "An Historical Account of the Trade Winds, and Monsoons observable in the Seas between and near the Tropicks, with an attempt to assign the Physical Causes of the said Winds."† His description of the observed facts and his theoretical considerations on the subject, have constituted an important step in the development of the science of that subject, even though his theory in its most important part—that which relates to the east to west motion of the Trade Winds—turns out to be fundamentally untenable. He adduced, no doubt, in his explanation, an important part of the real truth as to causes of the wind, a part which, if not first suggested by him, was clearly either not generally known or not generally adopted at the time.

This true element in his theory consisted in his assigning as the primary motive cause of the winds, the expansion of the air of hot regions, accompanied by its outflow in its upper parts from those regions towards places of less heat and entailing a

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* 'Phil. Trans.,' vol. 15, No. 175. September and October, 1685.

† 'Phil. Trans.,' No. 183, p. 153.

diminished pressure at the base of the ascending heated current, and consequently entailing an influx at bottom from the lower part of the atmosphere at the colder places where descending currents are generated. In applying this general principle further to the explanation of the observed winds, he rightly explained the influx of the air from both sides towards the Equator or some medial part of the trade-wind region as being due to the more intense heating effects of the Sun in the Equatorial regions. But in the more important, because less obvious, element for explanation of the Trade Winds and of atmospheric circulation generally—that which is requisite for explaining the east to west motions of the Trade Winds, and the prevalence of winds from west to east in higher latitudes—he quite missed the true explanation. He attributed the east to west flow of the Trade Winds to the diurnal revolution round the equatorial zone from east to west of the maximum of accumulation of heating effect from the daily sunshine, which gives an accumulation of heat in the afternoon in each successive locality. Briefly, he said to the effect that as the maximum of accumulated heat runs round the Torrid Zone from east to west, passing each place at a few hours after noon of that place, and as the maximum of heat in travelling round always causes an indraught towards itself, so the atmosphere of the Torrid Zone must be brought into flowing round from east to west likewise. But this conclusion from the submitted premises is really quite inconsequential.

In reference to this speculation, and treating for the present the direction which we will call the forward direction round the Torrid Zone as being that of the Sun's progress from east to west, we may entertain considerations such as the following:—That consequent on the indraught from all sides towards the hot region, where the barometric pressure is most reduced, the backward-tending forces acting on the air in front of the maximum may be acting as much in respect to time and duration backward on the air in front of the maximum as do the forward-tending forces on the air behind that maximum, and that, through this consideration by itself, we might not be entitled to suppose that any resultant tendency to the generation of a current round the Torrid Zone one way or other, east to west, or west to east, would be produced. But when we further consider the unsymmetrical character of the conditions of the two influxes towards the maximum region from before and from behind, and the to us very unknown accompanying frictional conditions between these unsymmetrically conditioned currents of air and the surface of the earth or sea over which they pass, we may be led to think it very unlikely that the forwarding and backwarding influences would exactly counteract one another; and I certainly think they would not do so, and I think some resultant flow from east to west, or from west to east would be produced, but in which way, east to west or west to east, it would occur I am quite unprepared to say.*

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* As a matter of curiosity I think it might be interesting in a time of comparative leisure for some person to make experiments with a spirit lamp or other heater kept revolving slowly round in a circular path under a circular tray filled with water, the path being of a little smaller radius than the tray. The

The theory or speculation in the terms in which it was set forth by its author makes no reference to the inertial conditions of the atmosphere concerned in its diurnal revolution along with the Earth, to which, as a matter of fact, it clings so as to have at all times and all places almost the same revolutional speed or angular velocity of diurnal rotation as the Earth has. In fact, HALLEY's theory would be equally applicable to the case of the world being non-rotative and having the Sun, or an equivalent source of heat, revolving round it from east to west.

But, in view of the very powerfully influencing conditions subsequently brought to light in the theory of HADLEY which will next be adduced, any such feeble causes as those relied on by HALLEY must fall practically into insignificance, the indubitable cause shown in HADLEY's theory being such as to be dominant.

In 1735 GEORGE HADLEY (brother of the JOHN HADLEY who invented the instrument commonly known as HADLEY's Quadrant) submitted to the Royal Society the paper of which I have made mention already as supplying for the first time a substantially true theory of the primarily dominant conditions of atmospheric circulation.* The paper is entitled " Concerning the Cause of the General Trade-Winds," and it is right here to notice that HADLEY applied the name General Trade-Winds, not merely to those winds of equatorial regions to which the name Trade Winds is ordinarily restricted, but uses it as including also the west to east winds known to be prevalent in higher latitudes, and used in trade by mariners for ocean passages from west to east. Thus the scope of his theory must be understood as being much wider than what would be conveyed in ordinary nomenclature by the name, Theory of the Trade-Winds.

In his paper, HADLEY commences by adopting, as a part of the whole truth, the view already in his time currently held by others, that the Sun's heat, intensely applied and greatly accumulated in the equatorial regions of the Earth, conjointly with the cooler temperatures of the regions in higher latitudes, is the main and primary cause of the Trade Winds and other currents of the atmosphere. In this way he supposes that at the Equator or near to it there is a belt of air ascending because of its high temperature and consequent rarefaction, and an influx from both sides towards a zonal region of diminished pressure at its base ; and that from its upper part currents float away to both sides, northward and southward, and that these continue in the upper regions of the atmosphere advancing pole-ward until, by cooling in the higher latitudes, their substance gradually becoming less buoyant sinks down gradually and returns towards the equatorial regions as a lower current along the Earth's surface, thence to renew the circulation by ascent again in the equatorial region. While indicating virtually that such atmospheric circulation would be generated, whether in an irrotative world with a source of heat revolving round it

question being, would or would not the water be set into revolutional motion, and if so would it revolve in the same direction as the lamp or other source of heat does ?

* 'Phil. Trans.,' vol. 39, No. 437, for April, May, and June, 1735, p. 58.

corresponding to the Sun in its apparent diurnal revolution, or in a world revolving on its axis as does the Earth, he shows that in the latter case—the case, namely, of the revolving Earth—in addition to such circulation as has just been described, east-to-west and west-to-east motions relative to the Earth's surface would necessarily come into being for reasons which may be stated or suggested as follows :—

If we consider the air in a nearly calm region at the outer limits of the trade-wind zone, and regard the air at that place as being at rest relatively to the Earth's surface, and if we consider it to be drawn over the surface by indraught towards the Equator without application to it of any other force than that of the indraught, except what it may receive by friction from the surface of the Earth, be that land or ocean, this air in arriving at places always lower and lower in latitude (and consequently further and further out from the Earth's axis) is coming to places in succession each moving eastward quicker than the previous one; and thus the air is arriving successively at places each going quicker eastward than the air itself was going when at the previous place; consequently the air in arriving at each new place must obviously have a slower motion eastward than the Earth's surface at that place has.

Thus throughout that course the Earth must be rushing forward under the air eastward quicker than the air goes, and that is the same as to say that the air must be blowing westward over the surface of the Earth.

In connection with this part of his theory he brings into notice that, while the surface of the Earth at the outer edges of the Trade Winds has much less of absolute velocity eastward in diurnal revolution round the Earth's axis than the surface at or near the Equator has, yet the trade-wind air, on arriving at the foot of the equatorial belt of rising air after its course from those outer parts in higher latitudes, has become imbued with eastward velocity little less than that of the equatorial surface of the Earth, the only deficiency in this eastward velocity from that of the equatorial surface being what is manifested as wind blowing westward over the Earth's surface, or having in relation to that surface a moderate westward velocity. He shows, for an example, that the eastward velocity of the Earth at either of the tropic circles is less than that at the Equator by about 87 miles per hour, but yet that the air which comes from calm regions near the tropic circles to the equatorial belt has, on its arrival at that belt, an eastward absolute velocity which is only a few miles per hour in defect of the velocity of the Earth there, the actual defect being manifested in the relative velocity with which the wind at the equatorial parts blows westward over the surface of the land or sea. He explains that this result is brought about by reason that the air, during its course from the outer edge of the trade-wind zone to the foot of the equatorial rising belt, is perpetually being dragged forward eastward by the quicker-moving land or sea below it, and so its velocity is kept nearly assimilated to that of the part of the Earth over which, for the time being, it exists, and is allowed only to be a little less than that velocity.

Such, then, is HADLEY'S theory, in so far as it relates to the origin of the Trade

Winds of the equatorial regions on both sides of the Equator. His theory further extends to explain the cause of the prevalence of winds from west to east in latitudes higher than those of the winds of equatorial regions, to which, except in the nomenclature of HADLEY himself, the name *Trade Winds* has been usually restricted; and this part of his theory may be represented as follows:—

The equatorial surface of the Earth has a velocity of diurnal revolution from west to east of about 1000 miles per hour. The air of the land and sea at and near the Equator participates nearly in the same velocity. The ascending equatorial belt of heated air retains as it ascends an absolute velocity from west to east nearly the same as that of the equatorial surface of the Earth. He supposes, then, in his theory, that the air floating out from the upper part of the rising belt to north and south over the equatorial zones of Trade Winds, and thence, still in the upper parts of the atmosphere, spreading over extensive regions of land and sea in latitudes higher than those of the Trade Winds, will, on reaching those regions whose velocities of diurnal revolution are much slower, be rushing forward from west to east quicker than do the portions of the Earth's surface over which it successively arrives in floating poleward; that greater speed of eastward motion of the air than of the Earth beneath being, however (as he indicates with a fair approach to clearness), kept in moderation by influences from the surface of the land or sea offering resistance to relative motions of the air above it. Further, he supposes that this upper air, while moving eastward quicker than does the Earth below it, gradually loses a great part of its previously acquired heat, and becomes less buoyant, and consequently descends gradually towards the surface of the Earth, the supply above being always maintained by fresh arrivals from the equatorial regions; and he supposes that the descending air brings from aloft perpetually new supplies of west-to-east motion relative to the surface, and so maintains winds blowing over the surface from west to east. The air then, after its descent from the sky towards the surface throughout extensive regions, must, I think, necessarily, under his theory—although he does not explicitly mention this—be supposed to flow gradually back in the lower levels of the atmosphere towards the Equator, while also blowing prevalently from west to east, till it reaches again the outer border of the trade-wind region, thence to go forward repeating such a circulation as has just been described.

HADLEY concludes his paper with a short passage which, considered in reference to the crude condition of progressive opinions prevalent in respect to atmospheric circulation up to the time of the promulgation of his theory, is to be regarded as suggesting, though in somewhat vague and not entirely correct expression, a very notable and important principle.

The passage is as follows:—"That the N.E. and S.E. Winds within the Tropicks must be compensated by as much N.W. and S.W. in other Parts, and generally all Winds from any one Quarter must be compensated by a contrary wind somewhere or other; otherwise some Change must be produced in the Motion of the Earth round its Axis."

The really important idea which it appears to me is suggested in this passage, is that in respect to the Earth's rotation round its axis the sum of all the forward turning-force-influences applied by the winds to the surface of the Earth, land and sea included, must be equal to the sum of all the backward turning-force-influences likewise applied to the Earth's surface; so that these force influences may be such as conjointly to produce no acceleration or retardation in the revolution of the Earth round its axis.

In putting forward this idea he was doubtless assuming as a principle that we are not to attribute to the thermal influence of the Sun any effects in altering the rotation of the Earth by producing winds blowing upon the Earth more effectually on the whole forward than backward, or the reverse. He did not, nor probably did anyone else till long after his time, notice the now known principle that the Sun and Moon can, by their attractions, apply to fluids on the Earth—to the sea or to the atmosphere—turning forces* which these fluids must communicate to the solid earth, and which must, in very long periods of time, make changes on the Earth's rotation. Such influences, however, are certainly so very small comparatively to those HADLEY had under consideration as occurring in the action of the equatorial Trade Winds from the east, and the winds of higher latitudes from the west, that his not knowing of them is not to be regarded as derogating from the practical or substantial truth and validity of his Theory of the Winds in its main features.

In the account I have given of HADLEY'S theory of the primarily important perennial features of atmospheric circulation, I have endeavoured faithfully to give a fair and favourable account of the truths which he brought to light. I have not held it as a duty to bring under review every statement or phrase to which objection might be taken by an adverse critic. There is one mistake, however, into which HADLEY fell, and which is too important to be passed over without notice. This error, although incorporated by himself along with his true explanations in respect to the causes of the equatorial Trade Winds and of prevalent westerly winds of higher latitudes is quite separable from those true explanations; and its elimination does not make any break down in any essential part of his reasoning as to the real conditions of the atmospheric motions. His error pertained not to his suppositions as to the actual motions of the real air, but to supposed motions and behaviour of air in an ideal case which he adduced as a simplified illustration intended to be helpful to the consideration of the more complex conditions of the real case. The two cases—the ideal and the real—are not explicitly and distinctively specified by himself, but they are brought implicitly under consideration in his statements to the following effect:—

Firstly.—That air having been in an approximately calm condition at one of the Tropic Circles, and having moved thence in the Trade Wind to the Equator, will, on arriving at the Equator, retain still the same absolute eastward velocity that it had

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* Any system of forces which can be balanced by what under the nomenclature of POINSOT is called a *couple*, may be described as a turning-force-influence, and may now with advantage be called a *torque*.

when at the Tropic, and so will at the Equator have less velocity of absolute eastward motion than the Earth there has, by 2083 miles per day, or 87 miles per hour, and that so it will be moving relatively to the Earth there as a wind blowing at the rate of 2083 miles per day from east to west.

And *Secondly*:—That as an amendment on the previous statement, it is to be considered that " before the air from the Tropicks can arrive at the Equator, it must have gained some motion eastward from the surface of the earth or sea, whereby its relative motion will be diminished, and in several successive circulations may be supposed to be reduced to the strength it is found to be of."

In reading these two statements conjointly we may with confidence judge that the first of them is not meant to convey the actual truth in respect to the real behaviour of the atmosphere, but that it is only a theoretical utterance as to an ideal case, in which the frictional drag between the surface of the ocean and the atmosphere is left out of account, and that the second is that which is meant to convey the real truth. Now the important error into which he has here fallen, consists in his supposing that in an ideal case, in which the trade-wind air is regarded as frictionless and free from receiving any eastward or westward force-influences from the ocean below it, or as I will add, from the atmosphere immediately above it, it ought to be expected on arriving at the Equator, from a calm at the tropic circle, to retain the same amount of eastward absolute motion which it had when at the tropic. Instead of that, in the ideal case, if fully specified with due limitations, such as we may suppose were tacitly contemplated, without being fully thought out, the true averment would have been that the air on arriving at the Equator would have a velocity of eastward absolute motion less than that at the tropic, in ratio inverse of that of the distances of the two places respectively from the Earth's axis. What I mean here to say, may, perhaps, without elaborate definitions and specifications, be tolerably well suggested in brief words, by saying that, in a vortex of free mobility, with circular motions round an axis, the velocities at different distances from the axis must be inversely as those distances.

But now, in truth, the ideal case which HADLEY touched upon, was quite outside of the scope of the real conditions of the atmospheric motions, which he professed to explain better than had been done in the attempts of others before his time. He had amply sufficient reason for his averment, to the effect that the real trade-wind air in its approach from the tropic to the Equator, under the influence of indraught towards the Equator, should be expected at each new place nearer to the Equator than the previous one, to have a less velocity of eastward absolute motion than the surface of the sea has at that new place, and that the frictional drag eastward applied to the air by the sea surface will only act towards assimilation of the eastward velocity of the air to that of the water, while still in principle, as in fact, leaving the air to go slower eastward than does the water—that is, to blow as a westward wind relative to the ocean. If what he professed to do had been to bring into notice a special variety of vortex motion, constituting what we may call a vortex of free mobility in a frictionless fluid, and to offer a dynamic theory of its motions, and if his theory had included such an error as the one in question; then his theory would have been fundamentally erroneous. But such was not at all what he professed to do. He proposed to explain certain large and very remarkable phenomena of the observed winds. This he did well, and in doing so he made a very important advance in development of true theory in respect to atmospheric motions.

I have touched in some detail on these matters, because I think that remarks making inadequate recognition of the importance of HADLEY'S true discoveries have sometimes been put forward in our own times.

During a period of more than a century from the time of the promulgation of HADLEY'S theory, in 1735, there was, I consider, little if any remarkable progress made in development of new speculations for better or for worse in respect to the grand or perennial currents of atmospheric circulation. A long time elapsed, in which there seems to have been little or no vigorous spirit of investigation into the significances or the relative merits of the speculations which had been propounded, or of effort to amend the existing theories, or to discover new truths on the subject. In confirmation of this it may be noticed that we find that 58 years after the publication of HADLEY'S paper, DALTON arrived independently at substantially the same theory as that part of HADLEY'S which dealt with the equatorial Trade Winds, and in his book entitled "Meteorological Observations and Essays,"* which in 1793 he was preparing for publication, he gave an account of his theory, supposing it to be original, but he discovered, before the book was issued to the public, that he had been completely anticipated by HADLEY'S paper, of the existence of which he had not been previously aware. In his preface to that book, after making recognition of HADLEY'S priority, he goes on to say:—"I cannot help observing here, that the following fact appears to be one of the most remarkable that the history of the progress of natural philosophy could furnish.—Dr. HALLEY published in the 'Philosophical Transactions' a theory of the trade-winds which was quite inadequate and immechanical, as will be shown, and yet the same has been almost universally adopted; at least I could name several modern productions of great repute in which it is found and do not know of one that contains any other." . . . "On the other hand G. HADLEY, Esq., published in a subsequent volume of the said 'Transactions' a rational and satisfactory explanation of the trade-winds, but where else shall we find it?"

It is right here to remark further that DALTON in his own speculations did not touch at all upon the prevalence of west winds in extra-tropical regions, either as to its explanation or even as to its existence: and that he does not seem to have noticed or appreciated the great importance of HADLEY'S theory in this respect.

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* "Meteorological Observations and Essays," by JOHN DALTON, D.C.L., F.R.S., 1793. Of this work there is also a second edition, which is a verbatim reprint issued by Dr. DALTON himself, in 1834.

Not only before, but also after this episode of DALTON'S speculations and researches so published, the theory of HADLEY must certainly have remained but little read in its author's original paper.

Within the first half of the present century writings on the winds, including the Trade Winds and general circulation of the atmosphere, have been very numerous, some of these have appeared in our encyclopædias, and others in works on meteorology and navigation, and have been widely diffused in atlases containing maps and charts on physical geography.

In such ways many sketches have been presented to the public as explanations of the Trade Winds and other currents of the atmosphere related to them, embodying more or less of the fundamental principles of HADLEY'S theory, but often without reference to his name, and usually without due appreciation of the meaning and importance of his theory. In many of these cases we may suppose that the authors had never seen his own original paper, but had obtained their information indirectly through the writings of others.

On the other hand, within the period just mentioned—the first half of the present century—real progress was made in many ways, in the gaining of new knowledge and the making of a few new discoveries, chiefly in connection with the temporary and local disturbances of the atmosphere, and in the bringing together of information of various kinds to help in the elucidation of the subject of the winds. The influence of moisture in air of any given temperature and pressure in rendering the fluid more buoyant was brought effectually into consideration.

The attainment of information from the practical observations of mariners and travellers, and especially explorers of the polar regions, and also from meteorological observatories, was making gradual but important progress. Considerable progress was made in the collecting and correlating by many persons of observational results as to winds and weather and barometric pressures in various latitudes, and in the presenting for practical use among navigators and others of the generalized conclusions so derived.

In that course of progressive labours there were included various speculations or theories as to great storms, commonly designated as hurricanes, tornadoes, or cyclones. In beginning to touch on this subject I have to mention that from among the many persons who may have taken part in researches and speculations regarding cyclones, those whom I deem the most noteworthy are CAPPER, DOVE, REDFIELD, THOM, REID, and PIDDINGTON.

Now, within the period which we have at present under consideration—the first half of the present century—by a very gradual course of experience, chiefly maritime, and of speculation based on such experience, it came to be promulgated that violent storms were generally great whirlwinds; and so the old name *tornado*, of Portuguese origin, suggestive of turning, and the new name *cyclone*, used in the sense not merely of circular form, but also of revolving motion, came to be accepted as well suited for the designation.

Also it was found that in the centre of a cyclone there is a region of comparative calm, and that the centre does not remain stationary, but travels at some moderate speed, taking generally a curved course over the surface of sea or land.

The discovery also was established beyond room for doubt that cyclones in the northern hemisphere revolve in the direction opposite to that of the hands of a watch situated in their locality with its face up; while in the southern hemisphere they revolve in the same direction as do the hands of a watch situated in their locality with its face up.

Also it was discovered and promulgated that in the central region of a cyclone the barometric pressure is remarkably diminished as compared with that of the general surrounding atmosphere, and that this condition must necessarily subsist as a concomitant of the centrifugal tendency or "centrifugal force" of the revolving air, but whether the diminished pressure was to be regarded as a result of the centrifugal force of the revolving air, or as one of the primary causes of the institution of the cyclonic revolution, seems commonly to have been left unnoticed or to have been adverted to under erroneously imperfect views.

Dove, for instance, when discussing the tremendously violent whirling motion which is met with in the inner part of a cyclone immediately around the central region of remarkable calm, says, "*the diminution of barometrical pressure is not the cause of the violent disturbance of the air, but rather a secondary effect of it*,"* and through that passage with its context it seems doubtless that, while entertaining the view that the rapid revolving motion of the air somehow instituted maintains by centrifugal tendency the diminished pressure in the central region, he fails to notice the more complete truth, that without the actual occurrence of centripetal motion caused by predominating influence of inward suction the rapid revolving motion would not institute itself at all.

This being said, however, there is yet, of the whole truth, another element which must be brought into notice, and which I here briefly describe, with some perhaps new ideas that have occurred to myself.

It is, that while for a beginning an accumulation of buoyant air at bottom elongates itself upwards into a shape approaching to a columnar form, and so effects an abatement of pressure at its base; and this abatement of pressure (or suction) induces a centripetal flow towards that place from outer regions where some slight, though it may be almost imperceptible, motions having revolutional momentum (or, in other words, moment of momentum) round that place may already exist, and the revolving mass of air through the action of the centreward forces applied to it, takes an increasingly rapid revolving motion; and further, this rapid motion reacts on the buoyant central column, *keeping that from scattering through the air around it*, and so institutes a very lofty continuous column of the buoyant air.

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* Dove, 'Law of Storms,' English translation by Robert H. Scott, M.A., p. 198.

To make this clearer we may notice that if a buoyant central column were for a moment existing surrounded by non-rotative air having greater pressure in its lower parts than that in the column at the same level, that column could not continue its existence. The outer air with its greater pressure would press in on the column, and would increase the pressure in its substance instantly, but the weight of the upper portion of the buoyant column would be inadequate to resist the upward thrust so produced in the lower part, and so the lower parts would shoot those above them upwards with violently accelerating motion. Through the rushing upwards so generated a breaking up of the column would supervene, and its substance would scatter itself in rolling masses among the surrounding air; and the two commingling would ascend gradually, and at the same time the pressure of the surrounding air would communicate itself to the region where the base of the column had been.

But now, on the other hand, if the mass of air around the central buoyant column be whirling, it will keep itself out by the centrifugal tendency accompanying its own rapid revolution, and so will not press in upon and break up that central column of air of diminished pressure, and thus the abatement of pressure at the foot of the column will be maintained and will become further intensified.

To REDFIELD is due much credit for his able and long-sustained labours in collecting and correlating observed facts as to cyclones and the smaller kinds of whirlwinds. He gathered and published* a very interesting collection of accounts of violent columnar whirlwinds which formed themselves over large fires of circular masses of brushwood, the flame and smoke in each case ascending as a lofty rotating column; and this has had part in suggesting to me some elements in the theoretical considerations here briefly sketched out. In his remarks on these whirlwinds he emphatically brought into contrast the distinction between the flames and smoke ascending without whirling motion from hot furnaces and various ordinary fires, on the one hand, and, on the other hand, the revolving columns of flame and smoke often met with in those great fires of brushwood in the open air. By his various researches into the actions and effects of great storms, REDFIELD contributed more, perhaps, than any other man to the advancement of observationally-derived knowledge of their cyclonic character and features.

Wild and fantastic notions were, however, afloat in those times as to the origin of cyclones. Thus PIDDINGTON, in his well-known work entitled the 'Sailor's Hornbook,' even in the edition so late as 1860,† in stating his resultant opinions and conclusions, makes such statements as the following:—

That he considers cyclones to be flat circular disks which may be formed at the sides and upper and lower surfaces of clouds, and which, once formed, may either rise

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* "Some account of Violent Columnar Whirlwinds which appear to have resulted from the action of large Circular Fires," by W. C. REDFIELD. Read before the Connecticut Academy of Arts and Sciences, Jan. 22, 1839. Printed in the 'American Journal of Science and Arts" (SILLIMAN'S), 1839, vol. 36. p. 50.

† The 'Sailor's Hornbook,' third edition.

higher or descend downwards, and may extend themselves greatly or contract in diameter, and which may be "parallel to the surface of the globe" or "inclined forwards;" he goes on to say: "It appears to me that a simple flattened spiral stream of electric fluid generated above in a broad disk, and descending to the surface of the Earth, may amply, and simply, account for the commencement of a Cyclone."*

After making careful search through numerous writings on the subject of cyclones, I have to say that I have no reason to think that the investigators who took part in the discovery of the directions of turning of cyclones in the northern and southern hemispheres had generally, or that any of them in particular had, any clear dynamic theory explanatory of the connection between these modes of turning and the rotation of the Earth, nor even of the origin of the very rapid whirling motion itself, but I have found strong indications of deficiency of such knowledge. Even HERSCHEL, so late as 1857, in his article on "Meteorology," in the 'Encyclopædia Britannica,'† stated that a complete account of the phenomena of cyclones had been afforded "by HADLEY's theory as developed by DOVE in his 'Law of Rotation,' and applied to this specific class of aërial movements by Professor TAYLOR," and then went on to give what we may presume to be that explanation, but the explanation he gives, although containing enough of truth to prove the connection between the direction of the Earth's rotation and that of the mode of turning of cyclones in each hemisphere, is incomplete, and is vitiated by important errors of principle.

Mr. WM. FERREL, of Nashville, Tennessee, in a paper of date 1856 (to be referred to further on in connection with other matters), adduced dynamic considerations of more advanced character for explanation of causes of the gyratory motions of cyclones; but his treatment, although in some respects usefully suggestive and indicating sufficient reason for the direction of turning in each hemisphere, I cannot regard as being on the whole to very good effect.

Also, as a further result of the researches and scrutinies and efforts towards generalization told of already, it came gradually into notice and into acceptance as an established truth that in the latitudes outside the limits of the Trade Winds extending far towards the poles, sometimes for brevity called the middle latitudes, the wind, while prevailing from the west as had been long previously known, prevails also for each hemisphere more from the Equator towards the Pole than from the Pole towards the Equator, so that, on the whole, to take for simplicity the case of the northern hemisphere, the prevalent average atmospheric current at the surface of the Earth in those latitudes was judged to be from the south-west; or, rather, without particularizing one exact point of the compass, and with allowance for great variations in different localities, and at different times, we may better say from south of west towards north of east.

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* 'Sailor's Hornbook,' third edition, p. 338, section 410.

† 'Encyc. Brit.,' eighth edition, vol. 14, p. 650.

To account for the component from the south in these westerly winds of our middle latitudes, it came to be supposed, for instance, by LEOPOLD VON BUCH[*] prominently, as also by many others, that the air departing for the northern hemisphere from the top of the Equatorial Belt of buoyant air, while flowing northward still in the lofty regions of the atmosphere and over the Trade-wind zone, soon becomes a current from the south-west, and continues after descending to the Earth's surface at the northern border of the trade-wind region still to move forward in continuation of its old course as a current from the south-west. But why in the lower regions a pole-ward motion should be maintained rather than a return flow towards the Equator, and how the return from higher to lower latitudes to compensate for this supposed pole-ward surface current should be accomplished, are questions which appear to have been scarcely mooted or to have been left enshrouded in vagueness.

Many examples might be cited indicating the wide currency which such conclusions attained to, but one or two may suffice. Thus, for instance, in JOHNSTON'S 'National Atlas,' of date 1843, we have a map of the winds by Dr. HEINRICH BERGHAUS, of Berlin, on which the zone of south-westerly winds of middle latitudes is described in mysteriously poetic words more captivating to the imagination than satisfying to the reason, as "*Region of South-Westerly Currents of Air, or of the downward returning North-Eastern Trade Wind in Triumphal Conflict with the Northern Polar Currents.*"

HERSCHEL, in his 'Astronomy,'[†] of date 1850, gives an account for explanation of the south-west winds of middle latitudes substantially to the same effect as that of LEOPOLD VON BUCH and BERGHAUS, and with like vagueness as to the return currents from higher towards lower latitudes.

But from the shelter of that prevalent vagueness, MAURY, in 1855, stepped out and boldly offered a scheme of the general currents of atmospheric circulation which he supposed to prevail, in courses extending from Pole to Pole, and traversing in different ways the lower regions of the atmosphere next the surface of the Earth, and the upper regions which present themselves less directly to the observation of men. Fig. 2 is a copy of his diagram which, in conjunction with his printed explanations, sets forth his scheme of supposed circulation.[‡]

That figure shows a hemisphere of the Earth's surface taken from Pole to Pole.

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[*] LEOPOLD VON BUCH, 'Gesammelte Schriften,' vol. 3, Berlin, 1877, where there is to be found his 'Physikalische Beschreibung der Canarischen Inseln,' Berlin, 1825, chapter 2, 'Bemerkung über das Klima der Canarischen Inseln,' pp. 288, 289, and 290. A slightly abbreviated translation of the passage in question is given in DOVE'S 'Law of Storms,' SCOTT'S translation, 1862, p. 39, in a chapter entitled "The Upper Return Trade Wind."

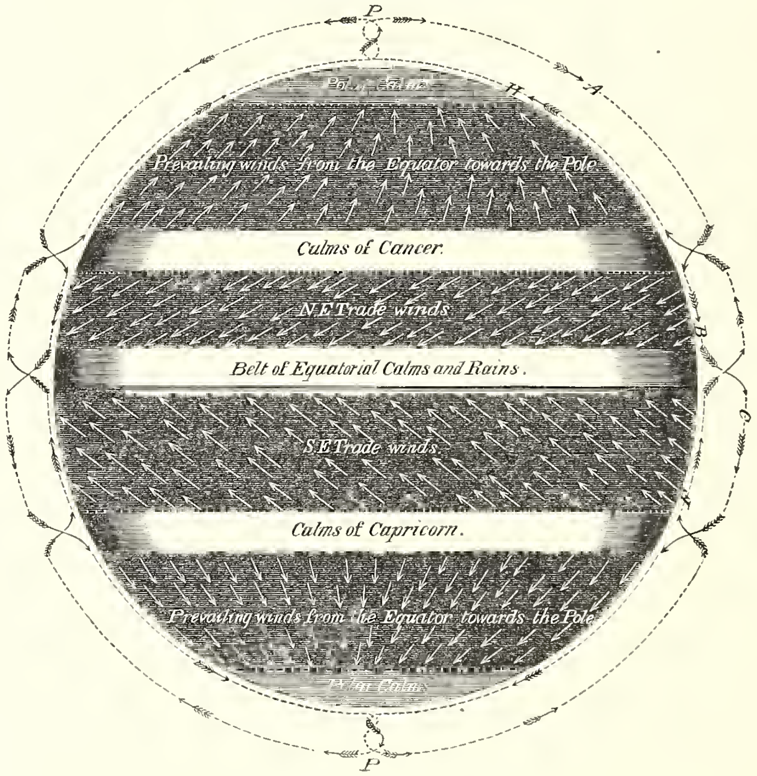
[†] Third edition.

[‡] MAURY'S 'Physical Geography of the Sea.' The first and second editions appeared in 1855. The statements here made apply alike to his 2nd and 6th editions, and presumably also to other editions.

The continents and lands generally are not exhibited, and their disturbing effects on the atmospheric motions are left almost entirely out of consideration. The circulation imagined and described by Maury in connection with the diagram is meant to be a fair representation of what he would suppose likely to be realised in case of local and temporary disturbances and irregularities being only in a small degree effective. His supposed circulation may be thus described :—He supposes that the air entering the Belt of Equatorial Calms from the southern hemisphere rises there to the lofty regions of the atmosphere, and flows thence as an upper current to the Belt of the Calms of Cancer where it descends to the bottom, from whence it travels on as a south-west wind over the surface of the sea to the high latitudes round the Pole; and that then ascending at and near the Pole, it flows as an upper current out to the Calms of Cancer, where it sinks again to the bottom of the atmosphere crossing the current already mentioned as descending there, and then passes along the surface of the sea as a bottom current forming the north-east Trade Wind, and then enters the Belt of Equatorial Calms, rises there, crossing the previously mentioned rising current there, and thence departs as an upper current towards the Calms of Capricorn, to go through a circulation in the Southern Hemisphere which is an exact counterpart of that already described for the Northern Hemisphere. The supposed currents are further indicated by arrows in the diagram, which, on inspection, may easily be

Fig. 2.

MAURY—1855.



understood. It is to be understood that the diagram shows a hemisphere of the surface of the Earth with the two Trade Wind Zones exhibited one on each side of the Equator, and separated by the Equatorial Belt of Calms and Rains, which is often also called the Doldrum Belt. And that it also shows the two Border Belts, or Calms of Cancer and Capricorn; and also, in the Northern Hemisphere, the zonal region of wind prevailing from south of west, and, in the Southern Hemisphere, the corresponding zone of prevalent winds from north of west. The arrows shown on the surface of the globe throughout these various zones indicate the directions of motion of the bottom currents of the atmosphere constituting the winds blowing on the surface of the sea. Around the representation of the globe the atmosphere is shown in section with arrows to indicate the north and south, and up and down motions in the circulation, which has just now been described in words.

In offering this scheme of atmospheric circulation, MAURY himself, in respect to the part of it which he propounds as taking place in the regions between the Trade Wind Zones and the Poles, confesses that it is "for some reason which does not appear to have been very satisfactorily explained by philosophers" that the currents he supposes do take place instead of their contraries. In short, he admits that he does not think reason has been found why in those regions the lower current should be towards the Pole and the upper towards the Equator, instead of what we might more obviously expect—namely, a flow towards the Pole in the upper regions of the atmosphere, and a return current towards the Equator in the lower regions close upon the surface of the sea. He even describes the known prevalent motion of the bottom layers of the atmosphere towards the Pole in extra-tropical latitudes as being seemingly paradoxical as to its reason, and although he offers an argument for abatement of the paradox, that argument on the slightest consideration may readily be seen to be futile.

In 1856—the year following after the publication of MAURY's scheme of circulation in his book entitled 'The Physical Geography of the Sea'—quite a new theory was put forward by FERREL in a paper on "The Winds and the Currents of the Ocean," published in the 'Nashville Journal of Medicine and Surgery.'* The scheme of circulation which he then proposed and upheld by mathematical reasonings is illustrated in his paper by a diagram, from which fig. 3 here is taken as a copy. This scheme, as may be noticed by reference to the diagram, and as may be further ascertained by reference to the original paper, includes for each hemisphere three zonal rings of atmosphere, making six in all, each having a separate circulation for itself, except that some small amount of commingling would necessarily take place at each narrow annular interface of meeting between two contiguous zonal rings. For either hemisphere one of these zonal rings of atmosphere covers the trade-wind region

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* October and November, 1856. This essay is to be found reprinted in 'Professional Papers of the U.S.A. Signal Service,' No. 12, published by authority of the Secretary of War, Washington, Office of the Chief Signal Officer, 1882.

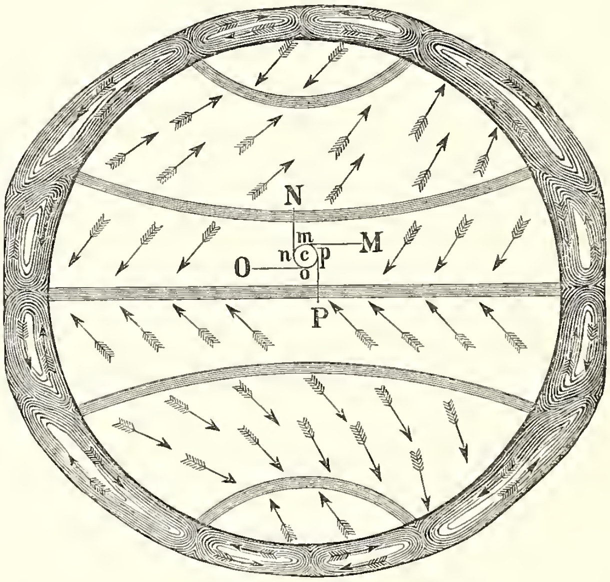
of that hemisphere, another covers the middle latitudes in which winds prevail from south of west in the northern hemisphere, and north of west in the southern, and the third covers the polar region.

Now, attention for simplicity being confined to the northern hemisphere, explanations of the scheme may be continued as follows :—

In the trade-wind zonal ring the bottom current flows from the Calms of Cancer as the Trade Wind to the Equatorial Belt and rises there, and flows then in the upper regions of the atmosphere till it comes to a situation aloft nearly over the Calms of Cancer, and thence it descends obliquely to the Calms at bottom to flow again towards the Equator, and so to begin another circuit alike in character to the one now described. Next in the zonal ring of the middle latitudes, according to the scheme, the current of air taken as beginning at the Calms of Cancer advances in the lower regions over the surface of the sea as a wind from south of west till it comes to about the Arctic Circle where it ascends to the upper regions, to begin a return course proceeding southwards as an upper current till it comes to places aloft nearly over the Calms of Cancer, thence to descend to those Calms below, and so to complete its circulation from some part of that belt back again to the same belt. Next as to the supposed circulation in the zonal ring of the Arctic Regions, it may suffice to say briefly that the lower current is asserted to be from the Pole and the upper current towards the Pole, the ascent from the lower to the upper being at or near to the Arctic Circle, and the descent being in a region closely surrounding the Pole, all as may be seen by inspection of the diagram.

Fig. 3.

FERREL—1856.



Ferrel, in setting forth in his paper his scheme of circulation and his theoretical reasonings on the subject, introduces as a fundamental principle in it the assertion that there must be a heaping up of the top layers of the atmosphere to a maximum height at about the parallel of 28° and a "depression" of them over the Equator, and also a "depression" of them at and around the poles and in high latitudes generally, and his diagram is purposely drawn to represent these features.

What has now been said is enough to give a good general idea of FERREL's scheme of Atmospheric Circulation of 1856. His assumptions, his reasonings, and his conclusions are, I may say with confidence, pervaded by impossibilities and incongruities. But notwithstanding this his paper is deserving of credit for the praiseworthy efforts it manifests towards a more complete consideration of important principles bearing on the subject, which had previously been unknown or neglected or imperfectly touched upon by others.

While I have told of this paper by Mr. FERREL at the present stage in order of dates, yet I deem it right to explain here that I had no knowledge of its existence, nor of any of its author's views, until some years after the publication of the new theory by myself, about which I have to tell forthwith in the present paper.

Through a paper* read before the Natural History and Philosophical Society of Belfast in 1856, by Mr. JOSEPH JOHN MURPHY, of that town, interest was strongly aroused in my mind, in the question of what ought to be supposed to be the true state of the case as to the courses of atmospheric circulation in the zonal regions situated between the trade-wind zone and the Pole in each hemisphere. In that paper Mr. MURPHY brought under notice of the Society the scheme of currents of atmospheric circulation set forth by MAURY, as the truth; and gave a theory or course of reasoning formed by himself, for explaining on dynamic principles how those supposed motions should be accounted for.

On the subject so presented for consideration, I had to judge that Mr. MURPHY's course of reasoning was not valid for sustaining MAURY's theory of the atmospheric motions, and I had to judge moreover, that MAURY's theory was itself, in so far as it dealt with the circulation outside of the trade-wind zones, entirely untenable and impossible.

Mr. MURPHY's course of reasoning, however, included within it one important element not limited in its scope to the application made of it in that particular course of reasoning. It was the supposition that the low barometric pressure of Polar regions and other high latitudes, already discovered as a fact, through observations of voyagers and others, was to be regarded as due to the centrifugal force of the air revolving from west to east throughout the great cap of atmosphere covering the middle and high latitudes.

Having rejected MAURY's theory, and having got the benefit of the valuable suggestion just referred to in Mr. MURPHY's paper, I succeeded in framing a new theory for the circulation in the regions outside of the trade-wind zones. That new theory I put forward in a paper read by me at the meeting of the British Association,

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* On the 'Circulation of the Atmosphere,' by Mr. JOSEPH JOHN MURPHY, Belfast Natural History and Philosophical Society, 27th February, 1856.

held at Dublin, in the following year, 1857; and a clear account of it is to be found in the Abstract of the paper published in the British Association volume for that year.

The verbal explanations given in the reading of that paper before the meeting were illustrated by a drawing showing the scheme of circulation described in the paper. Fig. 4, here given, is an accurate copy of that drawing, differing from it only in some unimportant matters, such as in the number of arrows shown, and in its being drawn with abatement of some exaggerations which were made in the original in order to render small features more readily visible at a distance in a large room. The full significance of the original in all respects is retained unchanged in the copy here.

In endeavouring to penetrate the mystery as to what the courses of circulation might be in the middle and higher latitudes, I was in preliminary ways fully satisfied that HADLEY's theory[*] in its main features—those, in fact, which in the present paper I have already described with commendation—must be substantially true, and must form the basis of any tenable theory that could be devised.

Now, under HADLEY's theory, when we come to consider what may be the courses of circulation that we should attribute to the atmosphere in the latitudes outside of the trade-wind zones, we should naturally be led to expect (as I have pointed out in some detail in an earlier part of the present paper in describing his theory) that the great sheet of air floating out from the Equator in the upper regions of the atmosphere towards either Pole, while having a motion towards the east also, would gradually cool in advancing to higher latitudes, and would therefore descend in middle and high latitudes to the Earth's surface and would next, as a bottom current, flow back towards the Equator while also flowing eastward, and so would be a current towards the Equator, not towards the Pole. But, on the other hand, it had been brought out through accumulated observational results that the winds of middle latitudes while blowing towards the east, which so far is in agreement with HADLEY's theory, do, in opposition to what would be expected under that theory, blow more towards the Pole than from the Pole. Thus the facts and theory seemed to be at variance. It then occurred to me that facts and theory could be reconciled by supposing that the great circulation brought into probability under HADLEY's theory does actually occur, but occurs subject to this modification, that a thin stratum of air on the surface of the Earth in the latitudes higher than about 30°—a stratum in which the inhabitants of those latitudes have their existence, and of which the movements constitute the observed winds of those latitudes—being by friction and impulses on the surface of

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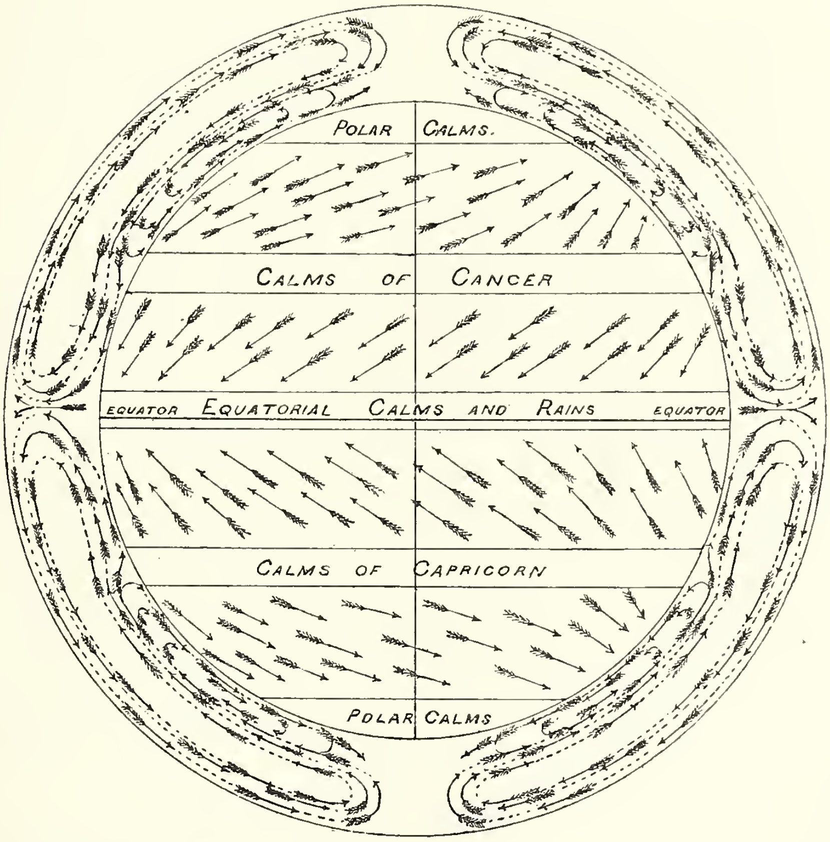
[*] Reference having been made in the text here to my paper read at the British Association Meeting for 1857, on the "Grand Currents of Atmospheric Circulation," and to the Abstract of it printed in the volume of the Association for that year, I have to mention as a correction that the theory here described and correctly designated as HADLEY's Theory, was in the printed Abstract erroneously named as HALLEY's Theory. I was led into that mistake as to authorship of the commonly accepted explanation of the trade winds, through my finding it designated as "HALLEY's theory of the trade winds" by MAURY, in his "Physical Geography of the Sea," to whose newly proposed views in that book my attention was at the time specially applied.

the Earth retarded with reference to the rapid whirl or vortex motion from west to east of the great mass of air above it, tends to flow towards the Pole, and actually does so flow under the indrawing influence of the partial void in the central parts of that vortex, due to the centrifugal force of its revolution. Thus it appeared to me that in temperate latitudes there are three currents at different heights :—That the uppermost moves towards the Pole and is part of a grand primary circulation between Equatorial and Polar Regions ;—that the lowermost moves also towards the Pole, but is only a thin stratum forming part of a secondary circulation ;—that the middle current moves from the Pole and constitutes the return current for both the preceding ;—and that all these three currents have a prevailing motion from west to east in advance of the Earth. This was the substance of the new theory which I framed and which, in 1857, I submitted to the British Association at its Dublin meeting. The atmospheric currents supposed under this theory are indicated by arrows in the diagram, fig. 4, and may be traced out readily on inspection. This drawing it is to be understood is not intended to offer any indications of supposed variations in height from bottom to top of the atmosphere in different latitudes.

Fig. 4.

THOMSON – 1857.



I exhibited at the meeting, as an illustration, a simple experiment easily extemporizable on any occasion. It is mentioned in the printed abstract briefly in the following words :—"If a shallow circular vessel with flat bottom, be filled to a moderate depth with water, and if a few small objects, very little heavier than water, and suitable for indicating to the eye the motions of the water in the bottom, be put in, and if the water be set to revolve by being stirred round, then, on the process of stirring being terminated, and the water being left to itself, the small particles in the bottom will be seen to collect in the centre. They are evidently carried there by a current determined towards the centre along the bottom in consequence of the centrifugal force of the lowest stratum of the water being diminished in reference to the strata above, through a diminution of velocity of rotation in the lowest stratum by friction on the bottom. The particles being heavier than the water, must, in respect of their density, have more centrifugal force than the water immediately in contact with them ; and must, therefore, in this respect have a tendency to fly outwards from the centre, but the flow of water towards the centre overcomes this tendency and carries them inwards ; and thus is the flow of water towards the centre in the stratum in contact with the bottom palpably manifested."

The general hydraulic principle intended thus to be illustrated by the exhibition of an easily conducted simple case of it is, that if water were lying on a revolving flat-bottomed circular plate or tray, and were revolving at each part quicker than the tray immediately below that part, a flow would institute itself in the bottom layer towards the centre, and that this would occur alike for different speeds of revolution of the tray, and would still take place, likewise, in the case of the speed of revolution of the tray being abated to zero. The case of the non-rotative tray was taken for illustration of the more general proposition simply because of the facility which that particular case presents for being brought into visible manifestation, so as to form to an intelligent mind a help to the imagination in considering the action of the great cap of air lying on the middle and higher latitudes, and revolving prevalently at each part quicker than the Earth below that part does. I offer these explanatory remarks here because in a paper by Mr. FERREL, to be told of a little further on, my illustration by means of the non-revolving tray has been made a point of adverse criticism as to both the nature and the value of the theory I had offered.

Now, before passing quite away from the subject of the original framing of my own theory ; I feel it right to make special reference to two considerations which were put forward by Mr. FERREL in his paper of October, 1856.

Firstly.—Ideas were put forward in that paper by Mr. FERREL to the effect, that the low barometric pressure found observationally to exist in polar regions and other high latitudes, is due to the centrifugal force or tendency of the air of the surrounding middle latitudes revolving from west to east quicker than does the earth below : but his views on the matter being unknown to Mr. MURPHY and to myself, did not happen to influence my considerations.

And *secondly,* Mr. FERREL in that paper adduced in connection with other suppositions an idea which, taking it in a wider scope than that in which he applied it, and with congruity in application not pertaining to the case for which he adduced it, I may describe as implying considerations to the effect that in an atmosphere covering a zonal region such as that of the middle latitudes, and having eastward motion relative to the Earth's surface or, what is the same, having a speed of eastward revolution quicker than that of the Earth below it, a layer at bottom retarded by friction on the Earth's surface, and so having less centrifugal tendency than has the quicker eastward-going air above will be caused to take, along with its eastward motion, a motion also towards the Pole.

The principle is an important one in its applicability to atmospheric circulation; but Mr. FERREL did not apply it to good account. He applied it only in reference to a system of motions already assumed by him, but which in the actual atmosphere are impossible as to causes for their origin and maintenance, and are incongruous in their mutual relations. His purpose in this matter was to show reason for the bottom current flowing towards the Pole while he had the upper current assumed as flowing towards the Equator. He assumed throughout the whole depth from bottom to top in his zonal ring of the atmosphere a motion eastward relative to the Earth, and thereby explained that the frictionally resisted bottom part should flow towards the Pole. But now we have to observe that the only reason why under his theory he can be entitled to assume eastward motion in the lower portion is because of that portion having been previously assumed to flow towards the Pole; and as to the upper portion which he assumes to flow from the Pole, that reason does not hold at all, and the upper portion should rather be supposed, under his theory, to flow westward than eastward. Thus it comes out that he explained the motion towards the Pole in the lower part of the atmosphere by first assuming, for no valid reason, a motion towards the Pole of that lower part. But now, for the primary assumption of that motion towards the Pole in the lower portion of the atmosphere, the reason which he assigned, and which I have just now treated as being not valid, was his supposed heaping up of the atmosphere at top, and consequent increased pressure at bottom at about the parallel of 28°; but, for the heaping up of the atmosphere there he needs in the upper region of the atmosphere over the middle latitudes a speed of revolutional motion greater than that of the Earth's surface immediately below, briefly a relative eastward motion, so that there may be the necessary centrifugal tendency for producing the heaping up, and that is incongruous with the flow in those upper regions taking place, as under his theory he made it do from higher to lower latitudes—from the Arctic Circle to about the parallel of 28°.

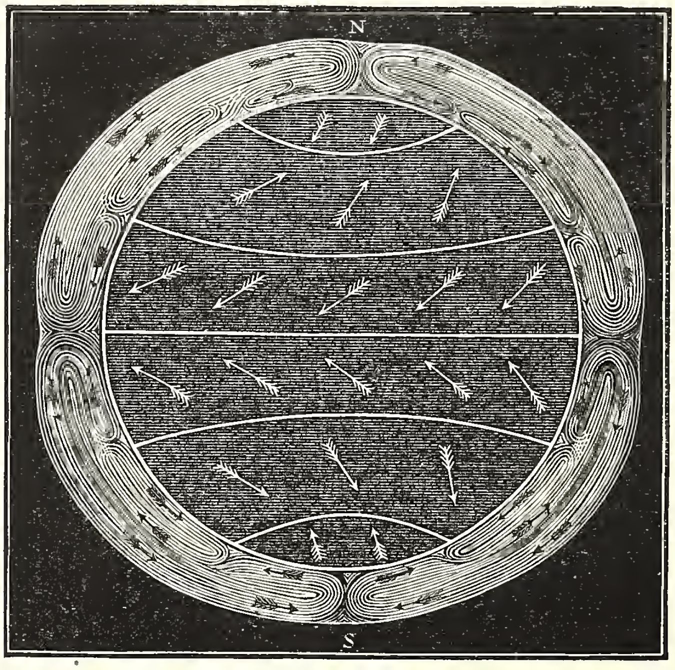
He has not thereby anticipated the new and, I think I may say, the true theory offered by me, in which the great body of the lower half of the atmosphere is already shown for good reason to have motion towards the Equator along with motion from west to east, but that a comparatively thin lamina at bottom of it, in virtue of frictional retardation of its eastward motion and consequent abatement of centrifugal tendency in it as compared to the air above, is caused to reverse what would otherwise be its motion towards the Equator, and to take its course towards the Pole instead.

The next publication to which I have to advert is a second paper by Mr. FERREL. It is entitled "The Motions of Fluids and Solids, relative to the Earth's Surface; comprising Applications to the Winds and the Currents of the Ocean,"* and is dated at its close, "Cambridge, Mass., February, 1860," and is noted on its title-page as being "Taken from the First and Second Volumes of the 'Mathematical Monthly.'"

In that paper he offered a scheme of atmospheric circulation totally different from his previous one of 1856, and entailing a fundamentally altered theory. In fact, he there abandoned his arrangement of six zonal vortex rings of circulation, three for the northern hemisphere and three for the southern, and, instead, he adopted really the scheme that had been put forward by me in 1857 with its two great currents of primary circulation, one flowing from equatorial to polar regions above, and the other flowing as a great return current from polar to equatorial regions below; together with the bottom subordinate current close on the surface of the Earth in middle latitudes or middle and higher latitudes, flowing pole-ward on account of the frictional retardation by the Earth's surface of its eastward relative motion and consequent diminution of centrifugal tendency.

Fig. 5.

FERREL—1860.



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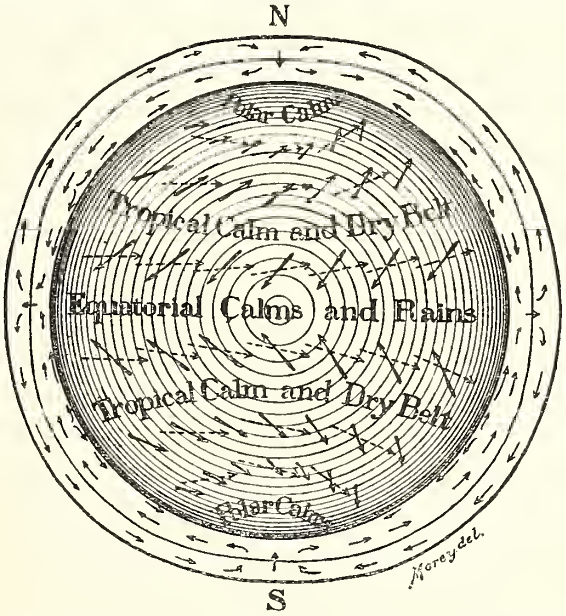
* New York, IVISON, PHINNEY, and Company. London, TRÜBNER and Co., 1860. It appears that this paper was subsequently republished by the United States Signal Service in 'Professional Papers,' No. VIII., with extensive notes giving the mathematical processes in detail by Professor FRANK WALDO.

These currents are shown distinctly by arrows in his diagram, notwithstanding some puzzling confusion introduced by lines which present the appearance of being meant to indicate average current lines, but which, in some parts, would suggest impossible courses, and which show signs of their having been put in without deliberate care. Fig. 5 here is a copy of that diagram.*

The diagram retains two vestiges of his original scheme. Thus it exhibits distinctly a depression of the top of the atmosphere at the Equator making a place of minimum height for the atmosphere there; and it retains systems of arrows throughout the polar regions representing winds having, relatively to the Earth's surface, motion towards the west together with motion towards the Equator; and so in the polar region of the northern hemisphere representing north-east winds. Both of these features I regard as having been introduced through mistaken apprehension. In a later work, indeed, by Mr. Ferrel, entitled, 'A Popular Treatise on the Winds,' 1889,† both these features of his former scheme of circulation are completely eliminated from his scheme and theory as there presented. This is shown by his diagram‡ taken in connection with the printed explanations by which it is accompanied. Fig 6 is a copy of this diagram. The depression of the top of the atmosphere, or more strictly speaking, the depression of any isobaric interface in the very lofty

Fig. 6.

FERREL—1889.



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* The same diagram exhibiting his scheme of the winds is repeated in a subsequent paper by Ferrel of date 1861, which he offered as being more popular and less mathematical. It is to be found reprinted in 'Professional Papers of the United States Signal Service,' No. XII.

† London, Macmillan and Co.

‡ Ferrel's 'Popular Treatise on the Winds,' 1889, § 105, p. 155.

regions to a minimum height over the Equator remains in the diagram, but it is expressly eliminated by words in the accompanying text. This diagram, when corrected according to Mr. FERREL's printed words is, as may readily be seen, essentially the same as my own.

In the closing passage of his second paper, 1860, FERREL made mention of the theory given by me at the British Association meeting in Dublin, 1857, but he did this with erroneous representation of the theory, and with inadequate recognition of its importance and of the fundamental changes he had made from his own previous theory in adopting the main features of mine and incorporating them with some remnants of his own previous views or modes of consideration.

I proceed next to offer some considerations which, I think, may be of intrinsic interest in themselves, besides helping towards the development and elucidation of true theory in regard to atmospheric motions and other conditions.

I have to mention, at this stage, that it may sometimes be convenient, as an aid towards brevity and clearness in expression, to characterize air which has no eastward or westward motion relative to the Earth's surface as having *par*, or being *at par* of revolutional velocity and, likewise, to use the designation *over par* of revolutional velocity to signify eastward relative motion, and *under par* to signify westward relative motion.

(*a*.) Recalling to notice the theory of MAURY and the first theory of FERREL given in his 1856 paper, and drawing attention to the confluence supposed, under both these theories, of two great upper currents of the atmosphere meeting aloft over the belt called the Calms of Cancer in the Northern Hemisphere, and of other two currents likewise meeting over the belt called the Calms of Capricorn in the Southern, I think it is well to remark that, if such a confluence were to take place of two currents, one coming from higher latitudes and the other from lower to a zonal belt of meeting, the current from the higher latitudes would have a rapid westward motion relatively to the Earth below, that is, a revolutional velocity greatly under par, and the current from lower latitudes would have a rapid relative motion eastward, or, in other words, a revolutional velocity greatly over par. They would meet one another obliquely with a velocity of each relative to the other very great because of its having had no frictional mitigating resistance such as the Earth's surface would afford to currents meeting in like manner at bottom of the atmosphere. Thus the belt of meeting aloft would be a place of extraordinary commotion, and this commotion would be propagated with the two descending currents down to the surface of the Earth below; and thus, instead of the Calms of Cancer or Capricorn we ought to expect to find there a belt of wild and varying storms. This very simple, and, I think, very obvious principle, is one of the numerous objections which might singly or conjointly have checked both MAURY and FERREL in the early inception of their theories, and might reasonably have prevented them from propagating views so fallacious.

(*b*.) Next we may raise questions, and proceed to solve them more or less completely, as to what must be the general character of the motions of the air at various places in the Trade-Wind Zone, both in the lower great current approaching to the Equatorial or Doldrum Belt,* and in the upper great current departing from that Equatorial Belt and flowing aloft over the Trade-Wind Zone to pass over the Border Belt and thence into what, for want of a better name, we may for the present call the middle latitudes. In doing this, we shall have to consider and bring to light some features of the motions of the atmosphere in the middle latitudes more fully in detail than hitherto in the present paper.

Let us accompany in thought the progress of a point advancing with the current along an average stream line, or rather an average current course in the great under-current from Polar to Equatorial Regions. For simplicity, let us confine attention to the Northern Hemisphere. Let us begin the course somewhere within the middle latitudes. To help imagination we may fix on a point of commencement situated vertically over New York. The moving point may, if we please, be idealized as being a small balloon constrained by frictionless guidance to keep in an average current course while being propelled along that average course by the more or less varying motions of the surrounding air. Now, during the progress of the travelling point in its course making way both eastward and southward, so long as the bottom lamina close to the surface of the Earth directly beneath the travelling point is blowing eastward with over-par revolutional velocity, the air above the bottom lamina there must be going forward with still greater over-par velocity. The reason for this statement is, that the only cause for maintenance of eastward relative motion in the frictionally restrained bottom lamina is, that the air above in virtue of revolutional momentum brought from equatorial regions, and not yet exhausted, is blowing with over-par revolutional velocity, and driving forward the resisted lamina below. Also, as long as the over-par velocity is existing in the frictionally resisted bottom lamina under our travelling point, a flow pole-ward also must exist in that bottom lamina at the place, for the time being, directly below the travelling point. This is for reasons fully explained in the account already given of my own theory. When further, the travelling point, in making its way southward, arrives at a stage where that eastward bottom over-par motion no longer exists directly below, and the travelling point then goes on making progress further south, it comes to places where the bottom lamina at the place then below it is moving equator-ward because of indraught thither, and because all reason for that lamina's going northward has ceased. Thereafter in the

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* This equatorial belt of rising air may also well be called the Medial Belt, while the Calms of Cancer and Capricorn may be referred to as the Border Belts, this last especially when it is wished to speak of either of these two indifferently, without distinction as to whether it be in the northern or southern hemisphere. Also, either of these border belts may very well be described, or may be named when desirable, as the Belt of Offturn Parting or briefly as the Offturn Parting. The reason for this name will be seen readily by inspection of the diagram, where the great return current towards the Equator is parted into two currents, one going on southwards, and the other turning off towards the north.

Trade-Wind region now entered upon the surface of the Earth is dragging the bottom air forward revolutionally, and so is helping it briskly towards the Equator through increasing its centrifugal tendency.

Then we have to notice that the air, during its course equatorwards and back again through the Trade-Wind Zone, receives forward revolutional momentum through the frictional forward drag applied to it by the Earth's surface, and it loses no revolutional momentum, as the vacuum above the atmosphere can take none from it. So in departing northwards, as the grand upper current, it must carry with it far more revolutional momentum than it had in entering, as the great under current from the north across the Border Belt; but that great under current in entering was either at par, or partly at par, and partly at over-par, of revolutional velocity; consequently the grand upper current must depart across the Border Belt with great over-par of revolutional velocity.

It follows from this, as a corollary, that the top of the atmosphere, or any isobaric interface near the top, must have a declivity in approaching the Border Belt from the top of the Equatorial Belt; and the Border Belt must not have a maximum height with declivity thence to a minimum at the Equator.

The foregoing demonstration seems also likely to give help towards the proper interpretation to be put on observational results recorded by the Krakatoa Committee, and to render highly improbable any suggestions such as seem to be conveyed in some parts of the report, to the effect that in the very lofty regions of the atmosphere—at such elevations as 13 miles above the sea level—a velocity such as 70 miles per hour from east to west has been indicated in the atmosphere, through the phenomena manifested after the great Krakatoa eruption.

(*c.*) In connection with the reasoning or demonstration I have just given, there is another element which I regard as forming part of the whole truth, and which must, I think, form an important element towards the development of the theory more completely. I have already indicated in the demonstration just now offered, that the bottom lamina of the atmosphere in the trade-wind region, is especially helped to advance towards the Equator by the increased absolute centrifugal tendency superimposed on it by the forward revolutional drag it there receives from the Earth's surface; and which communicates to it, throughout its course towards the Equator, new accessions of revolutional momentum, and prevents it from getting into under-par of revolutional velocity, so much as does the air above it in the great under-current towards the Equator. For ready apprehension of this, it is well to notice that the under-par and increased under-par of revolutional velocity imply westward relative motion in the bottom lamina, and quicker westward relative motion in the air next above within the great under current towards the Equator.

This greater abatement of absolute revolutional velocity below par, or increase of relative velocity westward, constitutes a condition opposing flow towards the Equator in the main body of the great under current, and we may reasonably suppose that the principal flow towards the Equator takes place in the bottom lamina—the lamina whose motions constitute the winds noticeable by action on the sails of ships. So we may suppose that the main body of that great under-current blows nearly due westward with only a small component of motion equator-ward. I could not venture, through theoretical considerations alone, to form an opinion as to the velocity of westward relative motion which might thus be attained to in the main body of the great under-current, or the velocity of westward relative motion which might remain in some parts of the upper current proceeding from the Equator before it has made much advance in latitude to places importantly nearer the Earth's axis. The complications involved in the frictional conditions attendant on the flow of sheets of air with others below and with others above going at very different velocities render the question practically unsolvable by theory alone. But I have to point out emphatically that the Doldrum air, deadened as it is to the condition commonly spoken of as equatorial calm, is very approximately at par of revolutional velocity, and when it rises to the top, or to the very high regions, of the atmosphere, it will have scarcely any westward relative motion, and therefore will not be able to make its way thence as an upper current pole-ward except by flowing as we may say down hill, or as we may better say, among isobaric interfaces down-sloping forward. The lower part of this sheet of deadened air departing aloft pole-ward, and which lower part is much below the top of the atmosphere, and is in close contiguity with the current of westward relatively moving air (already just now mentioned) commencing to move pole-ward without ever having attained to par of revolutional velocity, will we may suppose, by buffeting and commingling between it and that westward relatively moving air, be dragged forward from the Equator, even among up-sloping isobaric interfaces, in a manner that may be likened to being dragged up hill.

I might at present extend the explanations and reasonings on this matter somewhat further, but I abstain from doing so in order not to prolong unduly the present paper. I prefer to leave the subject over for further consideration and exposition by myself, perhaps, and probably by others.

(*d.*) I propose next to offer some considerations in respect to the atmosphere of the polar regions. For simplicity of expression I shall speak, in particular, of the polar regions of the Northern Hemisphere; and I intend that in this, as indeed throughout nearly all I have said in the present paper, the complications introduced into atmospheric motions by local distinctions of the Earth's surface into land and sea are to be, primarily at least, disregarded.

I consider that we should take as one element of our theories the principle that we have to suppose a stagnation of impounded air around the Pole over a great extent of the Polar Regions, this impounded air being maintained by the influx along the surface of the Earth of air frictionally deprived of the over-par of revolutional velocity which is possessed by the great cap of air higher up above the surface of the Earth. This impounded air lags, I affirm, in the Polar Regions, being unable, for want of revolutional momentum, and accompanying want of centrifugal inertial tendency, to take part readily in the great circulation between polar and equatorial regions. In fact, it cannot get out from its imprisonment there except by being dragged away through gradual entanglement with the comparatively rapidly revolving air arriving by the great upper current from regions having more rapid revolutional motion and passing away in the great middle return current towards the Equator.

(*e*.) Further, I may now offer some considerations as to whether, according to theory, we should expect very clear skies to prevail in the Polar Region of impounded deadened air. I think we must suppose the great upper atmospheric current converging towards the Pole and having over-par of revolutional velocity must be already very dry, owing to its greatly reduced pressure and cold temperature. So, when its air descends in level to return towards the Equator, that air must, I think, be greatly under its saturation point with water-substance ; or, in other words, must be far from ready to form clouds, or to precipitate rain or snow. We have to recollect that descending air is generally very rainless.

On the other hand the bottom flow along the surface of the land and sea converging towards the Pole I affirm to be moist. It will be from lower latitudes and generally warmer climates, and will carry moisture with it from sea and land. This bottom current will supply water-substance for cloud and snow in the impounded deadened polar air. The cold of radiation out to interstellar space, coupled with expansion in ascending before it can join the great middle current of return towards the Equator, will cause clouds and snow.

I will now conclude this paper by offering a sketch of a contemplated experimental apparatus for affording practical illustration of the theory of Atmospheric Circulation which I have propounded.

The apparatus would consist mainly of a horizontal circular tray kept revolving round a vertical axis through its centre. The tray would be filled to some suitable depth with water. Heat would be applied round its circumference at bottom, and cold would be applied or cooling would be allowed to proceed in and around the central part at or near the surface. Under these circumstances I would expect that motions would institute themselves, which would be closely allied to those of the great general currents supposed under the theory to exist in either hemisphere of the Earth's atmosphere. The motions of the water, I would propose, should be rendered perceptible to the eye by dropping in small particles of aniline dye, and perhaps by other contrivances. Great variations would be available in respect to the velocity of rotation given to the tray, and in respect to the depth of water used, and the intensity of the heating and cooling influences applied. By various trials with variations in these respects I think it likely that the phenomena expected could be made manifest.

XVIII. *On the Mathematical Theory of Electromagnetism.*

By ALEX. MCAULAY, *M.A., Ormond College, Melbourne.*

Communicated by the Rev. N. M. FERRERS, *D.D., F.R.S.*

Received January 8—Read April 28, 1892.
Revised October, 1892.

I. INTRODUCTION.

A *Electromagnetic Coordinates.*

1. IT has been thought advisable to reserve an account of the general aims and scope of the following paper till a few preliminary matters have been disposed of.

2. Consider the following statement, of the truth of which probably no one will doubt. If a body on being moved from a position A to a position B were found thereby to have lost a charge of electricity, physicists would not be content to explain the circumstance on the mere ground that it had left its charge behind. They would hold that processes had gone on, precisely similar to such as would have been required to divest it of its charge, had it remained in its first position A.

This has an important bearing on the way in which the "electric displacement" is related to matter. The polarisation thus called is some sort of *polarisation of matter, and this polarisation is carried about by the matter when it moves.* There certainly is no lack of evidence that electric actions go on in space where there is, to the best of our knowledge, no matter. In this space, however, is a medium of some sort, which is intimately related to matter, and certainly affected in some way by the motion of matter. For the present we must, for the sake of simplicity, be content to assume that the strains of this medium are, if it only bounds matter, continuous with those of matter, and if it permeates matter, are at places common to both matter and the medium identical with those of matter. (This may or may not be true. I only say that in the first development of the theory of this paper it must for simplicity be assumed.) This will not prevent us from regarding the slipping of the one medium over the other as the limit of a rapid shear. With this assumption the medium in question will appear in our equations merely as matter with zero density, but other physical quantities not zero.

Both for the medium referred to, and for matter, the statement would seem to remain true that the polarisation called electric displacement is a property that is carried about by the medium experiencing it.

3. In choosing the coordinates of any mechanical system it is, of course, only *necessary* to take them so that when given they completely specify the instantaneous position of the system. But as in all ordinary dynamical problems, so in the general electromagnetic problem, there may be all the difference in the world between one set and another in respect to the simplicity of the investigations in which they are employed, and the amount of light they throw on the interdependence of the parts of the system.

Now, I believe I am right in saying that all writers on the present subject take as the electric coordinates the three coordinates of the vector $\mathbf{D}d\varsigma$* for every element of volume $d\varsigma$ in space where $\mathbf{D}$ is the electric displacement at the point. According to the view just advanced that $\mathbf{D}$ is the measure of a property of the *matter* occupying the element $d\varsigma$, which is carried about with the matter, these are unsuitable coordinates. According to that view it is probable that the electric current is as intimately connected with the matter in which it resides, as is the electric displacement. It would seem to follow that the current components could not in general be considered as the rates of variation of the corresponding electric coordinates.

4. Suppose all space split up into a series of elementary parallelepipeda which *move with matter.* Let $\pm\, d\Sigma_a'$, $\pm\, d\Sigma_b'$, $\pm\, d\Sigma_c'$ be the six vector faces of one such parallelepiped. We shall take for our electric coordinates the three quantities $S\mathbf{D}'d\Sigma_a'$, $S\mathbf{D}'d\Sigma_b'$, $S\mathbf{D}'d\Sigma_c'$, where $\mathbf{D}'$ is the electric displacement at the point, for every element in space. [The reason for the dashes will appear immediately.]

Moreover, we assume that the same expressions, when $\mathbf{D}'$ is replaced by $\mathbf{C}'$, the current, are the rates of variation of the corresponding coordinates. In other words, the current $\mathbf{C}'$ at any point is defined by the equation

$$S\mathbf{C}'\,d\Sigma' = \frac{dS\mathbf{D}'d\Sigma'}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad . \quad . \quad . \quad (1),$$

where $d\Sigma'$ is any vector element of surface which *moves with matter*, and d/dt denote differentiation with regard to the time which follows the motion of matter. Thus the whole current through any surface which moves with matter = the rate of variation of the whole displacement through that surface.

B. *Mathematical Machinery.*

5. As might be expected, the mathematical machinery that appears to be most convenient for investigating as fully as possible the consequences of these assumptions, and others intimately connected with them, is novel. And I may remark in passing that what Professor TAIT persistently and with complete justice emphasizes as one of the greatest boons that Quaternions grant to ungrateful physicists, viz.,

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* Throughout this paper Mr. HEAVISIDE's practice of replacing MAXWELL's German letters by thick ordinary type is followed.

their *perfect naturalness*, seems to me to receive illustration in the methods about to be described.

The notation in the present paper will be mainly the same as that of my former paper on a "Proposed Extension of the Powers of Quaternion Differentiation."*

As in that paper (which will for the future be referred to as "the former paper"), a fixed position of all the matter in space will be taken as a standard of reference. Most of the following symbols have exactly the same meaning as before.

ρ is the coordinate vector of any particle of matter in the standard position; ρ' the coordinate of the same particle in the present position, so that ρ' may be regarded as a function of the independent variables, t (the time), and ρ. $d\varsigma$, $d\varsigma'$ denote elements of volume of the same particle in the standard and present positions; ds, ds' similar elements of surface; and $\mathrm{U}\nu$, $\mathrm{U}\nu'$ the unit normals at ds, ds'. In the present paper another notation will also be used, defined by

$$d\Sigma = \mathrm{U}\nu\, ds,\ d\Sigma' = \mathrm{U}\nu'\, ds' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

whence

$$\mathrm{U}\nu = \mathrm{U}\, d\Sigma,\ ds = \mathrm{T}\, d\Sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and similarly for $\mathrm{U}\, d\Sigma'$, $\mathrm{T}\, d\Sigma'$. This meaning of Σ is scarcely likely to clash with the usual summation meaning (which will also be freely used in the present paper), since in the present use the Σ will always be preceded by d, a combination that would be rare with the ordinary meaning.

With this notation equations (2) and (3) of the former paper take the somewhat briefer form

$$\int \phi\, d\rho = \iint \phi \mathrm{V}\, d\Sigma\, \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$\iint \phi\, d\Sigma = \iiint \phi \Delta\, d\varsigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

In connection with these equations it is well to call attention to the following usual conventions which will be strictly adhered to. The right-handed system of rotation is adopted. $\mathrm{U}\nu$, or $d\Sigma$, when regarded, as in the last equation, as the normal of the boundary of any region, is always drawn *from the region bounded.* Thus, if $\mathrm{U}\nu$ is regarded as the normal to the boundary of a dielectric at its junction with a conductor, it is drawn from the point of the bounding surface into the conductor. The positive direction, that of $d\rho$ in equation (3), round the boundary of a surface, is that of positive rotation round a proximate positive normal, $d\Sigma$ in equation (3). Thus the positive direction round the boundary of a magnetic shell whose positive normal is in the direction of magnetisation is that of the equivalent current.

∇ will have the usual meaning with regard to ρ, and ∇' the same meaning with

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* 'Proceedings of the Royal Society of Edinburgh,' 1890–91, p. 98.

regard to ρ'. Δ, a particular form of ∇, is used when we wish to imply that the differentiations of the ∇ are to refer to *all* the factors of a term. Thus

$$\mathrm{V}\,\mathbf{D}\Delta\mathbf{E} = \Sigma\,\partial\,(\mathrm{V}\mathbf{D}i\mathbf{E})/\partial x.$$

If σ be an independent variable vector, ${}_\sigma\nabla$, ${}_\sigma\Delta$, have the same meanings with regard to σ as ∇, Δ, with regard to ρ.

$\mathfrak{C}$ is a symbol of differentiation which is thus defined:—if ϖ be an independent variable self-conjugate linear vector function of a vector, given in terms of the scalars P and c by means of the equations

$$\begin{aligned} \varpi i &= \mathrm{P}i + \mathrm{N}j + \mathrm{M}k \\ \varpi j &= \mathrm{N}i + \mathrm{Q}j + \mathrm{L}k \\ \varpi k &= \mathrm{M}i + \mathrm{L}j + \mathrm{R}k, \end{aligned}$$

${}_\varpi\mathfrak{C}$ is a symbolic self-conjugate linear vector function of a vector given by

$$\begin{aligned} 2{}_\varpi\mathfrak{C}i &= 2i\frac{\partial}{\partial \mathrm{P}} + j\frac{\partial}{\partial \mathrm{N}} + k\frac{\partial}{\partial \mathrm{M}} \\ 2{}_\varpi\mathfrak{C}j &= i\frac{\partial}{\partial \mathrm{N}} + 2j\frac{\partial}{\partial \mathrm{Q}} + k\frac{\partial}{\partial \mathrm{L}} \\ 2{}_\varpi\mathfrak{C}k &= i\frac{\partial}{\partial \mathrm{M}} + j\frac{\partial}{\partial \mathrm{L}} + 2k\frac{\partial}{\partial \mathrm{R}}. \end{aligned}$$

Numerical suffixes are used exclusively to denote to what symbols the differentiations of a ∇ or $\mathfrak{C}$ refer, the operator and the operand having for this purpose the same suffix.

Let Q (α, β) be any function of two independent vectors α, β, which is linear in each Then ζ is defined by the equation

$$\mathrm{Q}\,(\zeta, \zeta) = \mathrm{Q}\,(\nabla_1, \rho_1) = \mathrm{Q}\,(i, i) + \mathrm{Q}(j, j) + \mathrm{Q}\,(k, k).$$

Similarly if R $(\alpha, \beta, \gamma, \delta)$ be linear in each of its constituents

$$\mathrm{R}\,(\zeta_1, \zeta_1, \zeta_2, \zeta_2) = \mathrm{R}\,(\nabla_1, \rho_1, \nabla_2, \rho_2),$$

and so to any number of pairs of ζ's.

At a given instant ρ' is a function of ρ only, and, therefore,

$$d\rho' = -\,\mathrm{S}\,d\rho\nabla\,.\,\rho' = \chi\,d\rho,$$

where χ is a linear vector function which is called the strain function. q, ψ, Ψ, m are all functions of χ given by the equations

$$\chi w = q\psi w q^{-1},$$

where q is a quaternion and ψ a *self-conjugate* linear vector function of a vector. χ' being the conjugate of χ,

$$\chi'\chi = \psi^2 = \Psi,$$

$$m = \frac{\mathrm{S}\,d\rho_a'\,d\rho_b'\,d\rho_c'}{\mathrm{S}\,d\rho_a\,d\rho_b\,d\rho_c} = \frac{d\varsigma'}{d\varsigma},$$

where $d\rho_a$, $d\rho_b$, $d\rho_c$ are three arbitrary independent increments of ρ, and $d\rho_a'$, $d\rho_b'$, $d\rho_c'$ the consequent increments of ρ'.

F and ϕ will have meanings closely connected but not identical with their meanings in the former paper. This will be explained later.

6. The displacement, current, magnetic force, &c., at the point ρ' will not be denoted by **D**, **C**, **H**, &c., but by **D**$'$, **C**$'$, **H**$'$, &c., with which the former symbols are connected in a way now to be described. In MAXWELL's 'Elect. and Mag.,' 2nd ed., § 12, he* remarks: "Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area. . . . In electrical science, electromotive and magnetic intensity belong to the first class, being defined with reference to lines. When we wish to indicate this fact we may refer to them as intensities. On the other hand, electric and magnetic induction, and electric currents, belong to the second class, being defined with reference to areas. When we wish to indicate this fact we shall refer to them as fluxes." Now in connecting dashed with undashed letters it is absolutely necessary to bear in mind whether the vectors indicated are intensities or fluxes. The connection between **D** and **D**$'$ will differ from that between **H** and **H**$'$.

7. Nearly all the physical vectors at a point will belong then to one of the following classes:—

Class I. Intensities.

(Examples: ∇, **A**, **E**, **H**, Θ, $d\Sigma/d\varsigma$, ${}_{\tau}\nabla l$.)

σ being a vector of this class, the three allied vectors, σ, σ', σ'', are connected by the equations

$$Sd\rho\sigma = Sd\rho'\sigma', \ \ \sigma' = \chi'^{-1}\sigma, \ \ \sigma'' = q^{-1}\sigma' q = \psi^{-1}\sigma \quad . \quad . \quad . \quad . \quad (5).$$

Class II. Fluxes.

(Examples: **B**, **C**, **D**, $d\rho/d\varsigma$, ${}_{\sigma}\nabla l$.)

τ being a vector of this class, the three allied vectors, τ, τ', τ'', are connected by the equations

$$Sd\Sigma\tau = Sd\Sigma'\tau', \ \ \tau' = m^{-1}\chi\tau, \ \ \tau'' = q^{-1}\tau' q = m^{-1}\psi\tau \quad . \quad . \quad . \quad . \quad (6).$$

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[*] This part of the present paper should be read in connection with MAXWELL's paper "On the Mathematical Classification of Physical Quantities," 'Collected Scientific Papers,' vol. 2, p. 257, or 'Proc. London Math. Soc.,' vol. 3, No. 34. In connection with the naturalness of the present methods, it may be of interest to note that the present paper was completed before I had seen this most suggestive paper of MAXWELL's.

I have not hesitated to put the symbolic vector, ∇, among the intensities since it obeys all the laws thereof. The *definitions* of the connection between σ, σ', and σ'', and between τ, τ', and τ'', may be taken as the first and third of equations (5) and (6) respectively. The second and fourth equation of each set are easily deduced from these by observing that $d\rho' = \chi d\rho$, $d\Sigma' = m\chi'^{-1}d\Sigma$ [equations (26) and (37) of former paper], and that both $d\rho$ and $d\Sigma$ are arbitrary vectors.

It should, perhaps, be noticed that these connections between σ, σ', and σ'' and between τ, τ', and τ'', although very useful and intimately connected with the physical nature of the vectors indicated are, after all, only definitions, and thus the phrase "where such and such a symbol is *defined* as a flux" will frequently occur below. This merely means that, having assigned the meaning of one of the three vectors, say τ', by a physical definition, the allied symbols, τ and τ'', are defined by saying that the symbol in question is a flux.

The connection between σ and σ' may be put in words, thus :—*If σ be an intensity, any line integral of σ' referred to the present position of matter is equal to the corresponding line integral of σ referred to the standard position of matter.* Of course, by the word "corresponding" it is implied that the two line integrals are to be taken through the same chains of matter. Similarly as to τ :—*If τ be a flux, any surface integral of τ' referred to the present position of matter is equal to the corresponding surface integral of τ referred to the standard position of matter.*

8. It is convenient to give here the following four simple but useful propositions.

Prop. I. If σ_a, σ_b be two intensities, $V\sigma_a\sigma_b$ is a flux.—By this is meant that $V\sigma_a'\sigma_b'$ bears the same relation to $V\sigma_a\sigma_b$ as does τ' to τ in equations (6). To prove

$$Sd\Sigma\sigma_a\sigma_b = m^{-1}S\chi'd\Sigma'\chi'\sigma_a'\chi'\sigma_b' \text{ [eq. (5) § 7]}$$
$$= Sd\Sigma'\sigma_a'\sigma_b' \text{ [Tait's 'Quaternions,' 3rd ed., § 158, eq. (3)].}$$

Prop. II. If σ, τ be an intensity and flux respectively, we have $S\sigma\tau d\varsigma = S\sigma'\tau'd\varsigma'$ $= S\sigma''\tau''d\varsigma'$.—For by equations (5) and (6) § 7, $S\sigma'\tau' = m^{-1}S\sigma\tau$, and $S\sigma''\tau'' = S\sigma'\tau'$. As particular cases we have

$$S\mathbf{B}\mathbf{H}d\varsigma = S\mathbf{B}'\mathbf{H}'d\varsigma',\ S\mathbf{C}\mathbf{A}d\varsigma = S\mathbf{C}'\mathbf{A}'d\varsigma',\ S\mathbf{D}\Theta d\varsigma = S\mathbf{D}'\Theta'd\varsigma' \quad . \quad . \quad . \quad (7).$$

Prop. III. If σ be an intensity $V\nabla\sigma$ is a flux.—By this is meant that $V\nabla'\sigma'$ bears the same relation to $V\nabla\sigma$ as does τ' to τ in equations (6). For any surface

$$\iint Sd\Sigma\nabla\sigma = \int Sd\rho\sigma \text{ [eq. (3) § 5 above]} = \int Sd\rho'\sigma' \text{ [eq. (5)]} = \iint Sd\Sigma'\nabla'\sigma' \text{ [eq. (3)].}$$

Hence, $Sd\Sigma\nabla\sigma = Sd\Sigma'\nabla'\sigma'$, or $V\nabla\sigma$ is a flux.

As particular cases, note that if, as we shall do directly, we assert that

$$4\pi\mathbf{C}' = V\nabla'\mathbf{H}', \quad \mathbf{B} = V\nabla\mathbf{A},$$

and that **B**, **C** are fluxes, and **H**, **A** intensities, it will follow that

$$4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}, \quad \mathbf{B}' = \mathrm{V}\nabla'\mathbf{A}'.$$

Prop. IV. If τ *be a flux* $\mathrm{S}\nabla\tau d\varsigma = \mathrm{S}\nabla'\tau' d\varsigma'$.—Proved by applying eq. (4), as we applied eq. (3) to prove Prop. III. As particular cases notice that

$$\mathrm{S}\nabla\mathbf{D}d\varsigma = \mathrm{S}\nabla'\mathbf{D}'d\varsigma', \quad \mathrm{S}\nabla\mathbf{C}d\varsigma = \mathrm{S}\nabla'\mathbf{C}'d\varsigma' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

9. Intimately connected with these two classes of vectors are two classes of linear vector functions of a vector.

In the following statements, as indeed throughout the paper, σ will denote an intensity, and τ a flux.

Class I. of Linear Vector Functions of a Vector.

(Examples :—The reciprocal of any function of Class II. ; ordinary stress, ϕ, Φ ; heat and electric conductivity, γ, R^{-1} ; specific inductive capacity, K ; magnetic permeability, μ).

Ω being of this class, the three allied symbols, Ω, Ω', Ω'', are connected by the equations

$$\left.\begin{array}{c} \mathrm{S}\sigma_a\Omega\sigma_b d\varsigma = \mathrm{S}\sigma_a'\Omega'\sigma_b' d\varsigma' = \mathrm{S}\sigma_a''\Omega''\sigma_b'' d\varsigma \\ \Omega' = m^{-1}\chi\Omega\chi', \quad \Omega'' = m^{-1}\psi\Omega\psi \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

σ_a and σ_b being any two intensities.

Class II. of Linear Vector Functions of a Vector.

(Example.—The reciprocal of any function of Class I., *e.g.*, electric resistance, R).

Υ being of this class, the three allied symbols Υ, Υ', Υ'' are connected by the equations

$$\left.\begin{array}{c} \mathrm{S}\tau_a\Upsilon\tau_b d\varsigma = \mathrm{S}\tau_a'\Upsilon'\tau_b' d\varsigma' = \mathrm{S}\tau_a''\Upsilon''\tau_b'' d\varsigma' \\ \Upsilon' = m\chi'^{-1}\Upsilon\chi^{-1}, \quad \Upsilon'' = m\psi^{-1}\Upsilon\psi^{-1} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

τ_a and τ_b being any two fluxes.

Of course, it is understood that Ω' and Υ' are not, as usual, the conjugates of Ω and Υ. Note, that if Ω or Υ is self-conjugate, then Ω' and Ω'' or Υ' and Υ'' are also self-conjugate. The first and second of each of the sets of equations (9) and (10) may be taken as the definitions of Ω', Ω'', Υ', Υ''. The third and fourth equations of each set can easily be proved by equations (5) and (6) to follow.

10. The following easily-proved propositions should be noticed :—

Prop. V. If Ω *be of Class I., then* Ω^{-1} *is of Class II., and if* Υ *be of Class II., then* Υ^{-1} *is of Class I.*

Prop. VI. $\Omega\sigma$ *is a flux, and* $\Upsilon\tau$ *is an intensity.*

Prop. VII. $S\nabla(\Omega\sigma)\,d\varsigma = S\nabla'(\Omega'\sigma')\,d\varsigma'$. (Props. IV. and VI.)

Prop. VIII. $\Omega' d\Sigma'$ *is the same function of* $\Omega d\Sigma$ *as* $d\rho'$ *is of* $d\rho$*, and* $\Upsilon' d\rho'$ *is the same function of* $\Upsilon d\rho$ *as* $d\Sigma'$ *is of* $d\Sigma$ *or* $\Omega' d\Sigma' = \chi\Omega d\Sigma$, $\Upsilon' d\rho' = m\chi'^{-1}\Upsilon d\rho$.

Prop. IX. $\Omega'\Delta' d\varsigma' = \chi\Omega\Delta d\varsigma$. [Prop. VIII. and eq. (4).]

11. Going, now, back to our definition of electric coordinates (§ 4), since for each element they may now be written $S\mathbf{D}d\Sigma_a$, $S\mathbf{D}d\Sigma_b$, $S\mathbf{D}d\Sigma_c$, and since $d\Sigma_a$, &c., are *constants*, we see that the choice of coordinates is equivalent to regarding $\mathbf{D}$ and not $\mathbf{D}'$ as the independent electric variable at any point. Further from eq. (1) § 4, and eq. (6) § 7, we have

$$S\mathbf{C}d\Sigma = dS\mathbf{D}d\Sigma/dt,$$

or, since $d\Sigma$ is an arbitrary *constant* vector,

$$\mathbf{C} = d\mathbf{D}/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

which is, of course, inconsistent with the equation $\mathbf{C}' = d\mathbf{D}'/dt$.

C.—*Preliminary Justification of the Foundations of the Present Theory.*

12. I have deliberately led up as quickly as possible to a description of the mathematical machinery to be used subsequently, as it has been necessary to notice incidentally some of the essential characteristics of the fundamental assumptions and the methods of investigating their consequences advocated in the present paper. As a preliminary justification of these assumptions, I cannot do better than indicate the line of thought which led up to them.

In studying MAXWELL's theory, and seeing how beautifully it was built up step by step from a mass of experimental facts, till the consistent whole stood revealed, it seemed to me that, notwithstanding the general harmony of its different parts, there was just here still something to be desired, some single plan that should govern *the whole.* This statement may not seem justifiable, so I instance two examples of the want of harmony. In one part of his treatise, the kinetic part, he works out the connections between the different parts of his theory by the general methods of Dynamics. But not so in the statical part. It would seem that the statical part of the subject, in such a plan as just mentioned, ought to appear as a particular case of the kinetic, whereas, in MAXWELL's treatise, the statical terms in the equations are merely *added on to those deduced from dynamical reasoning.* The same remark applies to the terms necessary to produce the mechanical effects of magnetism [compare for proof § 603 eq. (11) with the corresponding eq. (the last on p. 239) of § 619], and similar remarks, apply to the treatment in § 630 *et seq.* of the energy of the field. How to bring these various parts of the subject under the dynamical treatment I did not see, strictly on MAXWELL's own lines. Again, in considering the general equations of the electromagnetic field (§§ 608, 609) he speaks of a generalised force **E**. This generalised force, quite contrary apparently to dynamical analogies, has, not a single definite effect, partly kinetic and partly static, but two independent effects, one static and the other kinetic. On trying to trace out the reason of this, I could not arrive at any certain result strictly on MAXWELL's own lines. It *seemed* to me as if the double effect of **E** was simply assumed. [If it *merely* were analogous to an ordinary dynamical reaction, then it could not be associated with such external forces as result from electrolysis.]

13. Whether these and many other similar questions which occurred, some of which will appear below, can, strictly speaking, be denominated difficulties, is of no consequence. Suffice it that they led to the following considerations. MAXWELL has built up a theory whose axioms can* be put down in a definite form. Cannot, then, all his results (electrostatic, electrodynamic, magnetic, and electromagnetic) be developed as consequences of these axioms in one application of dynamical reasoning? Cannot we by such a single application obtain all MAXWELL's equations from (A) to (L) in §§ 591 to 614, as well as his stress results contained in other parts of the treatise, and by particular simplifying assumptions, shew that the ordinary electrostatic and magnetic theories are particular consequences of our general results?

This led me to attempt to apply in a perfectly rigorous and general manner the well-known equation

$$\delta\int \mathrm{L}\,dt + \Sigma\int \mathrm{Q}\,\delta q\,dt = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(where L is the Lagrangian function, "modified" if necessary, of any mechanical system of which q is a coordinate, and Q the external force of type q, and where the initial and final positions and times are not subject to variation) to the present case. The way I proposed to apply it was to assume all matter to be in any possible state as to strain and as to electric phenomena, then to vary all the geometrical coordinates by simply giving to each element of matter an arbitrary displacement, and also to vary all the electric coordinates, and trace the mathematical consequences. [Note that on MAXWELL's theory (at least as I understand it) these two variations are all that can be made, a variation in the magnetism being determined by the above variations.]

14. And it was here at the outset that the greatest difficulty of any met with in the investigation occurred. Consider a particular consequence of assuming that the electric coordinates are the three components of $\mathbf{D}'$ for every point of space. If by

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* It would be more correct to say "*some* of whose axioms." I wish to imply that I thought it advisable to fill in the remainder tentatively and seek the result.

the variation of geometrical coordinates an element ds' of surface where there is finite surface density of electricity be moved from P to P′, then in general the element of matter will *by the variation of the geometrical coordinates only* be entirely deprived of its charge, for this charge will be left behind at P. This result is, to say the least, an unfortunate one, and to be avoided, if by legitimate means it is possible. Still more disastrous results are arrived at if we assume that the components of $\mathbf{D}'$ for every element of *matter* are the electric coordinates, for then the charge in the whole of space is varied by a mere variation of the geometrical coordinates.

The legitimate way out of the difficulty seemed to be to assert that these electric coordinates, though theoretically permissible, were very unsuitable. To find suitable ones it was natural to use the principle that *the electric coordinates must be such that the variation in the geometrical coordinates does not alter the charge of any portion of matter.* This is, of course, ensured by assuming that $S\mathbf{D}'d\Sigma'$ is unaltered by variation of the geometrical coordinates, and from this it is but a step to the assertion that $S\mathbf{D}'d\Sigma'$ is itself a suitable electric coordinate.

Intimately connected with this question of the independent variation of geometrical and electrical coordinates is that of the correct expression for an electric current in (say) an arbitrarily moving fluid. It is not necessary to present all the reasons that occurred to me for the form already described (§ 4) as these are sufficiently indicated in the above considerations of variation of coordinates.

D. *An Analogy.*

15. The resemblances and differences between the present fundamental assumptions and what I take to be MAXWELL'S, are, perhaps, more clearly brought out by analogy.

I will first describe what I understand to be the analogy which MAXWELL allows himself throughout his theory, in order more closely to realise the interdependence of the various physical quantities considered, and as an aid to memory. The analogy contemplates the whole of space as being filled with an incompressible liquid. In dielectrics the liquid is, as it were, held in elastic meshes, in the form of closed cells, so that if it be displaced it tends to return to its original position. In the ideal conductor there are no such meshes, or rather there are meshes which do not form closed cells, so that the liquid can move through them, but is resisted while in motion. An actual body which admits some conduction, but behaves also like a dielectric will be typified by meshes which allow a slow leakage of the liquid. Now suppose into any space we introduce from some external source more liquid. This foreign liquid will be what is called the electric charge of that space, and it may be measured (since the liquid is incompressible) by the surface integral over the boundary of the space considered of the displacement of the original liquid outwards. Thus, "electric displacement" is represented in the analogy by a flux of the liquid.

The "conduction" current is measured by the current of *foreign* liquid, and the "displacement" current (indicated in the present paper by the term "dielectric" current) by that of the original liquid. In a simple conductor there is nothing to distinguish foreign from original liquid, and the conduction current in this case is represented by the whole liquid current.

A similar but not identical analogy will hold in the theory now advocated. For fixed matter the whole of the foregoing would be true, but not for moving matter. The liquid in the present analogy must not be incompressible, but must have a property in connection with *matter* which corresponds to the property of an incompressible liquid with reference to *space*. An incompressible liquid is one of which only one definite quantity can occupy an assigned space. In the present analogy we must say, instead, that the liquid is contained by matter, and that a given portion of matter always contains the same quantity of liquid. If by any means we pump foreign liquid into this portion, then an equal quantity of liquid must pass out of the boundary of that portion of matter into neighbouring matter, and thus in the present analogy as in the former, electric displacement will have for analogue the flux of the liquid, but not as in that case, across a surface fixed in space, but across a surface fixed relatively to matter.

Similar remarks apply to currents.

E. *Plan of the Paper.*

16. It will conduce to clearness to give some account here of the objects and aims of what is to follow. The part of the paper succeeding this introduction is in three main divisions: *The groundwork of the theory; The establishment of general results*, and *The detailed examination of these results.*

The groundwork of the theory, though not the longest of these, calls for most attention here. It is divided into two parts, *Fundamental assumptions* and *Preliminary dynamical and thermodynamical considerations.* I do not propose to give here a résumé of the different parts, but to call attention to certain prominent features.

The two most important of the fundamental assumptions are, perhaps, first, that in all cases $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, which I take to be one of the most characteristic features, if not the most characteristic, of MAXWELL's theory, and secondly, that the modified Lagrangian function per unit volume, though, of course, it contains $\mathbf{H}$, does not contain any term involving magnetic moment per unit volume or magnetic induction. Neither of these assumptions seems to be at variance with MAXWELL's, and, as hinted, the first is taken up mainly because it is a fundamental feature in his theory. From the first it follows that $\mathbf{C}$ must obey the laws of incompressibility, and this naturally leads to the assumption that $\mathbf{D}$ also invariably obeys those laws. The second leads to very important consequences, which, I believe, have not before been traced, and which I wish to call attention to here. Though not put quite in this form below they amount to this, that ${}_{\mathbf{H}}\nabla l$, where l is the modified Lagrangian function per unit volume of the standard position of matter, obeys the laws of incompressibility that round every circuit there is an electromotive force equal to the rate of decrease of the surface integral of $4\pi_{\mathrm{H}}\nabla l$ through the circuit, and that ${}_{\mathrm{H}}\nabla l - \mathbf{H}/4\pi$ appears in subsequent equations in such a manner as to compel us to identify it with the magnetic moment per unit volume.* It is clear, then, that $4\pi_{\mathrm{H}}\nabla l$ is, according to the present theory, the magnetic induction. As the theory is developed below it is convenient to define **B** as equal to $4\pi_{\mathrm{H}}\nabla l$ and call **B** the magnetic induction, leaving the justification till we examine the detailed consequences of the theory. It is well to insist on this result here, as it does not appear obvious in the work below, but only comes out when a general review of a great part of the paper is made. To put the matter in the form of a proposition :—

If the two fundamental assumptions are made—(1) *that* $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, *and* (2), *that* l, *the Lagrangian function per unit volume, can be expressed in terms involving* **H** *but independent of magnetic induction and of magnetic moment per unit volume, then the magnetic induction must be* $= 4\pi_{\mathrm{H}}\nabla l$.

17. The other most important features of the fundamental assumptions are first those already described with reference to the electric coordinates, and the expression for the current in terms of the displacement ; and secondly the manner in which are treated the two currents, conduction and dielectric (the latter being inappropriately, on the present theory, denominated the "displacement current"). If there are (and physicists seem agreed on the point) two independent currents whose *sum* appears in the equation $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, and whose *sum* obeys the laws of incompressibility, it seems to me of the nature of a truism that there must be also two independent electric displacements whose *sum* obeys the laws of incompressibility. I therefore, from the very beginning, recognise two displacements, **d** and **k**, which I call, for want of better names, the dielectric and conduction displacements.† This naturally leads to the contemplation of two independent kinds of electromotive force. This last, however, is subsequently satisfactorily disposed of.

18. Before leaving the fundamental assumptions, let me remark that though in some important respects the present theory may seem to differ from MAXWELL'S, it will be found, I think, that just where the difference seems to be most marked, is MAXWELL'S theory most vague. All the differences, if they really be such, have been forced on me unwillingly in the attempt to put into definite form what I take to be the essence

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[*] Strictly speaking, the last clause should be modified by the condition "if the present position be taken as the standard position." This, however, is only an accident of the particular form of enunciation, which, at the present stage, is unavoidable.

[†] Perhaps it would be better to call them the *elastic* and *frictional* displacements or the *reversible* and *irreversible* displacements. I wish to leave this point open for those better qualified to decide. Of the three sets of terms suggested above, the last seems to be the best. The only reason for adopting in the present paper the names given in the text is to imply the origin of the assumption that there are *two* such displacements. Of course, if we call the two displacements reversible and irreversible, we must also call the corresponding currents reversible and irreversible.

of MAXWELL'S theory. At any rate the results, though not in every respect identical with MAXWELL'S, are yet so nearly identical that the true matter for surprise is that they differ so little, and in such unimportant ways, from his.

It must be added, to prevent misconception of my own views, that I by no means consider proven what I regard as the key to MAXWELL'S theory, and what I have strictly adhered to in this paper, the assumption that under all circumstances $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$. My position rather is, that while this assumption may or may not be true, it is desirable to investigate as generally as possible what must be true, and what cannot be true if the assumption is made. In other words, I do not think that MAXWELL'S theory has yet had a fair trial, even at the hands of mathematicians, and the present paper is an attempt to provide more ways and means than hitherto have been available for such a trial. The methods adopted are equally applicable to other sets of fundamental assumptions.

19. Turning to the second part of the groundwork, the *preliminary dynamical and thermodynamical considerations*, it is necessary to remark that these considerations though not limited to an electric field, seemed absolutely necessary in order thoroughly to investigate the consequences of the assumptions. With regard to the first two sections of this part of the paper on *the modified kinetic energy and the free energy*, and on the *entropy* there is nothing which is likely to be questioned. In the third section on *frictional forces, conduction of heat and dissipation of energy*, I enunciate a principle which opens the way for much criticism. I would beg any readers to whom the form of enunciation is repugnant, to suspend their judgment as to the validity of the principle, not only until the first justification of it, but until they have seen it in action as it were, later in the paper. What was wanted was to bring this group of phenomena, which are undoubtedly closely connected, under the same sort of treatment as is accorded to the reversible phenomena of a system by means of its Lagrangian function, and the (dependent) entropy.

20. The way being thus paved, in the next principal division of the paper are deduced the general results of the theory, the most important of which are the equations of motion. These are considerably more general than the ordinary equations of the field, and thus we are led to the last division of the paper, the detailed examination of these results. The chief sub-divisions of this part are the comparison with MAXWELL'S results, a discussion from the point of view of the present theory of thermoelectric, thermomagnetic, and the HALL effects, and the transference of intrinsic energy through the field.

In comparing with MAXWELL'S results, wherever there is agreement, it is considered unnecessary to investigate further the detailed consequences. Where there is disagreement the physical consequences are traced with more detail, and in no case can it, I think, be said that the results of this part of the paper are condemnatory of the present theory. In this place, too, the bearing of the present theory on the question of convection currents is discussed.

Perhaps a clearer insight into the true bearings of the present theory is obtained by the attempt below to explain thermoelectric, thermomagnetic, and the HALL effects than by any other part of the paper. Especially clearly do some of the restrictions imposed by the condition $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$ come out.

21. In the last sub-division it will be found that I disagree entirely with Professor POYNTING's interpretation of his own results, and show how quite a different and, I think, simpler flux of energy may be made to account for the changes of intrinsic energy in different parts of the field. In particular, this interpretation would restore credence in what Professor POYNTING considers he has shown to be a false view, viz., that among other aspects of a current of electricity it may be looked upon as something conveying energy along the conductor. This part of the subject, although deduced from the present theory, is shown to be true on Professor POYNTING's own premisses.

22. It is well here to call attention to what might prove confusing otherwise. In what follows $\mathbf{E}$, $\mathbf{e}$, $\mathbf{F}$, Φ, and some allied symbols, stand for certain external forces. But there are three different meanings given in different parts of the paper to these symbols. They are originally defined as the whole external forces of the different types. But in treating of frictional forces, &c. (§§ 35 to 42) it is convenient to regard them as meaning only those parts of the forces which are *due to* friction and the like. Again from § 50 onwards it is convenient to regard them as meaning only those parts of the forces which are *independent of* friction and the like. This inconvenience is incurred to avoid the greater evil of a large additional array of symbols.

With this exception,* and one or two other trifling ones, which are noticed in their places, nowhere has the meaning of a symbol been changed throughout the paper.

II. GROUNDWORK OF THEORY.

A. *Fundamental Assumptions.*

23. We assume that the Lagrangian function, L, of all matter in space can be expressed in the form

$$\mathrm{L} = \iiint l d\varsigma + \iint l_s ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where l, l_s are functions of certain independent variables which determine the state of the body at the point. The volume integral extends throughout space, and the surface integral over certain specified surfaces. The entropy F of all matter in space will be assumed to be of the form

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* Since completing the paper I have discovered a notable exception which is *not* otherwise noted than in this footnote. It does not seem likely to lead to confusion; therefore I retain it. Most frequently in the present paper q stands for the typical *scalar* coordinate of a dynamical system, but it is not infrequently used, as in the former paper, for the *quaternion* of the rotation-operator $q\,(\quad)\,q^{-1}$.

$$F = \iiint f d\varsigma + \iint f_s ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and f and f_s will be determined from the values of l and l_s in a manner that will be described later on. All thermal phenomena not determined by F, and all forces of the nature of friction, will be supposed given by a third function X, given by

$$X = \iiint x d\varsigma + \iint x_s ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where x, x_s, unlike f, f_s, do not in any way depend upon L. The way in which these forces and the thermal phenomena depend upon X will be explained later. We shall call X the dissipation function. It is, in fact, a generalisation of Lord Rayleigh's dissipation function ('Theory of Sound,' 1st ed., § 81).

24. The absolute temperature of any element of matter will be denoted by θ. The vector Θ (assumed an intensity—§ 7 above) is defined by the equation

$$\Theta = \nabla\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Since both Θ and ∇ are intensities, we have

$$\Theta' = \nabla'\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

All electric and magnetic phenomena are supposed ultimately to depend upon the magnitudes and rates of variation of two fluxes (§ 7), $\mathbf{d}$, $\mathbf{k}$, called respectively the dielectric displacement and the conduction displacement. The whole displacement, $\mathbf{D}$, is defined as the sum of these two, so that

$$\mathbf{D} = \mathbf{d} + \mathbf{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

$\mathbf{D}$ must satisfy the two conditions of incompressibility for vectors, *i.e.*,

$$S\nabla\mathbf{D} = 0, \quad [Sd\Sigma\mathbf{D}]_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

the notation $[\quad]_{a+b}$ being as defined on p. 119 of former paper, *i.e.* the suffixes a and b denote the two regions bounded by a surface of discontinuity, and $[\quad]_{a+b}$ stands for $[\quad]_a + [\quad]_b$. Since $\mathbf{D}$ is a flux, it follows by Prop. IV., § 8, above, that

$$S\nabla'\mathbf{D}' = 0, \quad [Sd\Sigma'\mathbf{D}']_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The dielectric current, $\mathbf{c}$, the conduction current, $\mathbf{K}$, and the whole current, $\mathbf{C}$, all assumed to be fluxes, are given by the equations

$$\mathbf{c} = \dot{\mathbf{d}}, \quad \mathbf{K} = \dot{\mathbf{k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

$$\mathbf{C} = \dot{\mathbf{D}} = \mathbf{c} + \mathbf{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

25. The differentiations with regard to time implied by these dots are differentiations for a fixed element of the standard position of matter, *i.e.*, they are differentiations that follow the motion of matter. It is clear then that they are commutative with ∇, but not with ∇'. Hence, from equations (7) and (10),

$$\mathrm{S}\nabla\mathbf{C} = 0, \qquad [\mathrm{S}d\Sigma\mathbf{C}]_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

and, therefore, [equation (8) § 8 above]

$$\mathrm{S}\nabla'\mathbf{C}' = 0, \qquad [\mathrm{S}d\Sigma'\mathbf{C}']_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Since $\mathbf{C}$ satisfies the conditions of incompressibility, its surface integral over any surface only depends on the boundary of the surface, and may be expressed as the line integral of a vector $\mathbf{H}/4\pi$ round it. Thus, by equation (3), § 5,

$$4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

$\mathbf{H}$ is called the magnetic force, and is assumed to be an intensity, so that (Prop. III., § 8)

$$4\pi\mathbf{C}' = \mathrm{V}\nabla'\mathbf{H}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

All the vectors, including $\mathbf{H}$, hitherto mentioned, may be discontinuous. But they are assumed to be finite, so that $\iiint\mathbf{C}d\varsigma = 0$ for any infinitely small volume. Suppose this volume is a disc enclosing a part of a surface of discontinuity in $\mathbf{H}$. Then we have

$$0 = \iiint\mathrm{V}\nabla\mathbf{H}d\varsigma = \iint\mathrm{V}d\Sigma\mathbf{H}$$

by equation (4), §5 above. Hence

$$[\mathrm{V}d\Sigma\mathbf{H}]_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

so that the discontinuity in $\mathbf{H}$ is entirely normal to the surface. Similarly

$$[\mathrm{V}d\Sigma'\mathbf{H}']_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

26. From what has been said it follows that if **d**, **k** and their rates of variation are given for every point of space, $\mathbf{H}$ is not yet completely determined. It is, however, so determined by one more condition which is proved in § 48 below, and which is given here as we shall want to use it before proving it. $\mathbf{H}$ is one of the independent variables of which l is supposed an explicit function. The condition mentioned is that ${}_{\mathrm{H}}\nabla l$ satisfies the conditions of incompressibility. In other words, putting

$$4\pi_{\mathbf{H}}\nabla l = \mathbf{B} \quad \ldots \quad (17),$$

$$\mathrm{S}\nabla\mathbf{B} = 0, \qquad [\mathrm{S}d\Sigma\mathbf{B}]_{a+b} = 0 \quad \ldots \quad (18),$$

and **B** is called the magnetic induction.

These are proved by previously proving that

$$\mathbf{B} = \mathrm{V}\nabla\mathbf{A} \quad \ldots \quad (19),$$

where **A** is a vector which satisfies the condition

$$[\mathrm{V}d\Sigma\mathbf{A}]_{a+b} = 0 \quad \ldots \quad (20).$$

A is assumed to be an intensity, and **B** a flux, so that (§ 8 above),

$$\mathrm{S}\nabla'\mathbf{B}' = 0 \quad , \quad [\mathrm{S}d\Sigma'\mathbf{B}']_{a+b} = 0 \quad \ldots \quad (21),$$

$$\mathbf{B}' = \mathrm{V}\nabla'\mathbf{A}' \quad , \quad [\mathrm{V}d\Sigma'\mathbf{A}']_{a+b} = 0 \quad \ldots \quad (22).$$

This relation between **B** and **H** is not the usually accepted one, but it is certainly true on the present theory. It will appear later on that the value thus arrived at of **B**′, the magnetic induction at the point ρ', is independent of the particular position which is chosen as a standard of reference.

In the present theory **I**—assumed a flux—called the magnetic moment per unit volume is defined by the equation

$$\mathbf{B}' - \mathbf{H}' = 4\pi\mathbf{I}' \quad \ldots \quad (23),$$

from which it does not follow that $\mathbf{B} - \mathbf{H} = 4\pi\mathbf{I}$, since **B** and **I** are fluxes and **H** is an intensity. It does follow, however, that

$$\mathbf{B}'' - \mathbf{H}'' = 4\pi\mathbf{I}'' \quad \ldots \quad (24).$$

27. The equations of last article, it will be observed, do not represent fundamental assumptions. They are given here merely to indicate how the familiar symbols involved appear in the present theory. We now return to the fundamental assumptions.

The independent* variables, of which l is supposed a given explicit function, are

$$\theta,\ \Theta\ ;\ \rho',\ \dot{\rho}',\ \Psi\ ;\ \mathrm{d},\ \mathbf{D},\ \mathbf{C},\ \mathbf{H} \quad \ldots \quad (25);$$

x is supposed a given explicit function of

$$\theta,\ \Theta\ ;\ \Psi,\ \dot{\Psi}\ ;\ \mathbf{K},\ \mathbf{H} \quad \ldots \quad (26).$$

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* See § 31 below.

Nothing is here said about l_s and x_s, as it has been thought advisable to see how much can be explained without their aid. When we come to consider electrostatic contact potential difference—which for brevity we will in the future call contact-force—it will be found necessary to suppose l_s not zero. For ordinary friction, also, x_s must not be zero. The above assumptions will enable us to take account of (1) all MAXWELL's results, or results corresponding thereto; (2) the stresses, &c., resulting from variation of specific inductive capacity and magnetic permeability with strain and temperature; (3) thermoelectric, thermomagnetic, and the HALL effects; (4) many purely mechanical results whose details will be reserved for future treatment. They do not enable us to take account of (1) sliding friction; (2) electrolysis; (3) hysteresis and similar phenomena; (4) contact-force. All these, however, except (3), can be taken account of by slight additions to our present assumptions, as in the case of (4) will appear later.

The object of limiting as above the number of the independent variables entering into l and x is to free the mind from unnecessary vagueness. Moreover, the above assumptions are in one sense necessarily simpler than those made by Professor J. J. THOMSON ('Applications of Dynamics to Physics and Chemistry,' 1st ed., chap. vii.) to explain thermoelectric and thermomagnetic effects, in that the only quantity whose space-variations appear above in l or x is θ, a statement not true of Professor THOMSON's assumptions. With regard to the forms of l and x as functions of their independent variables, it is simplest at present to make no restrictions.

28. I am a little doubtful whether writers on the present subject recognize two semi-independent electric displacements at every point, but, as already remarked, it seems to me to follow, as a matter of course, from the assumption of two independent currents. The independent variables which have [§ 27 (25)] above been chosen to take account of these are $\mathbf{d}$ and $\mathbf{D}$, though, of course, $\mathbf{d}$ and $\mathbf{k}$ or $\mathbf{k}$ and $\mathbf{D}$ might have been chosen instead. MAXWELL generally, but not quite without exception, seems to use the symbol $\mathbf{D}$ for what I have called $\mathbf{d}$. I thought, however, that I should be following the usual custom of subsequent writers by using $\mathbf{D}$ for the whole displacement.

If there be two independent electrical displacements, it would seem as though we must assume, at any rate provisionally, the existence of two independent external electromotive forces of type $\mathbf{D}$ and $\mathbf{d}$. These we shall denote by $-\mathbf{E}$ and $-\mathbf{e}$ respectively. This, of course, means that the work done by the said external forces at the element $d\varsigma$, while $\mathbf{D}$, $\mathbf{d}$ change to $\mathbf{D}+d\mathbf{D}$ and $\mathbf{d}+d\mathbf{d}$ respectively, is $(\mathrm{S}\mathbf{E}\,d\mathbf{D}+\mathrm{S}\mathbf{e}\,d\mathbf{d})\,d\varsigma$. We shall also assume external surface forces of these types $-\mathbf{E}_s$, $-\mathbf{e}_s$, external ordinary forces $\mathbf{F}$ and $\mathbf{F}_s$ per unit volume and surface of the present position of matter,[*] and an external stress Φ, Φ being a self-conjugate linear vector function of Class I of § 9 above. This last statement means that the real

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[*] In the former paper $\mathfrak{F}$ meant the force per unit volume of the *standard* position of matter. The change has been made, since the equations of this paper are thereby simplified.

stress-function is not Φ, but Φ', *i.e.*, that the force exerted on a region at the element $d\Sigma'$ of its boundary is $\Phi' d\Sigma' = m^{-1}\chi\Phi\chi' d\Sigma'$.

29. The meaning of "external" may be defined as "not included in the form of L." Thus, the external forces include (1) all frictional forces given by X; (2) forces that, though not now included in the form of L, can be so included by generalising the meaning of l and l_s, so as to explain electrolysis, contact-force, and chemical phenomena; (3) forces that, through present ignorance we cannot include in X or L, though they should be so included. Thus, for instance, the external stress Φ may be supposed to be due entirely to viscosity and elastic fatigue, and the first of these will be accounted for by X.

30. $\delta\mathbf{D}'$ is due partly to variation of strain and partly to $\delta\mathbf{D}$; let $\delta'\mathbf{D}'$ be the latter part. $\mathbf{E}$ and e are assumed to be intensities. Hence (§ 8, Prop. II.)

$$(\mathrm{S}\mathbf{E}\,\delta\mathbf{D} + \mathrm{S}\mathsf{e}\,\delta\mathsf{d})\,d\varsigma = (\mathrm{S}\mathbf{E}'\,\delta'\mathbf{D}' + \mathrm{S}\mathsf{e}'\,\delta'\mathsf{d}')\,d\varsigma' \quad . \quad . \qquad . \quad . \quad (27).$$

A similar theorem is supposed to hold with regard to $\mathbf{E}_s$, e_s, viz.:

$$(\mathrm{S}\mathbf{E}_s\,\delta\mathbf{D} + \mathrm{S}\mathsf{e}_s\,\delta\mathsf{d})\,ds = (\mathrm{S}\mathbf{E}'_s\,\delta'\mathbf{D}' + \mathrm{S}\mathsf{e}'_s\,\delta'\mathsf{d}')\,ds' \quad . \quad . \qquad . \quad (28),$$

from which, since [§ 7, eq. (6)] $\delta'\mathbf{D}' = m^{-1}\chi\,\delta\mathbf{D}$, and

$$ds'/ds = \mathrm{T}\,d\Sigma'/\mathrm{T}\,d\Sigma = m\mathrm{T}\chi'^{-1}\mathrm{U}\nu = m\mathrm{T}^{-1}\chi'\mathrm{U}\nu'$$

$$\mathbf{E}'_s = \chi'^{-1}\mathbf{E}_s/\mathrm{T}\chi'^{-1}\mathrm{U}\nu, \quad \mathbf{E}_s = \chi'\mathbf{E}'_s/\mathrm{T}\chi'\mathrm{U}\nu' \quad . \quad . \quad . \quad . \quad . \quad (29),$$

and similarly for e_s, e'_s.

31. We must distinguish carefully between the independent variables of an element of matter which are given in the two lists (25) and (26) of § 27 and the independent variables of the system in general. These last consist only of

$$\theta,\ \rho',\ \mathsf{d},\ \mathbf{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

for every element of matter, for when these last and their time-rates of variation are assigned for all space, all the other quantities are determined. [It is not quite correct to talk of $\mathbf{D}$ as an *independent* variable on account of the equations of condition (7) of § 24.]

To enable us to develop the consequences of these fundamental assumptions, a digression on dynamics and thermodynamics must be made.

B. *Preliminary Dynamical and Thermodynamical Considerations.*

Ba. *The "Modified Kinetic" Energy and the "Free" Energy.*

32. It is not to be supposed that the coordinates we have assumed are sufficient to fix the position of all matter in space. The mathematical machinery we use cannot be supposed sufficiently fine to trace the motion of molecules. Such coordinates as would be required for that purpose are "ignored." Now (LARMOR, 'Proc. London Math. Soc.,' vol. 15, 1884, p. 173) in order that the principle expressed in eq. (1) of § 13, above, may be true under these circumstances, L must be, not the true Lagrangian function, but what ROUTH ('Elem. Rig. Dyn.,' 4th ed., § 420) has called a modified Lagrangian function. And that our principle may be true the particular type of modification is assigned, *i.e.*, the ignored coordinates are those whose momenta appear explicitly. And a further restriction is necessary (LARMOR, as above), viz., that the ignored coordinates must *only* appear through their momenta. That is, the ignored coordinates must be what Professor J. J. THOMSON ('Applications,' 1st ed., § 7) has called kinosthenic or speed coordinates. This last restriction, however, is not absolutely necessary if we take L to be the average value of the modified Lagrangian function for a small time, sufficiently large to allow the molecules to go through all their types of motion many times.

33. Whether these restrictions be imposed or not we have the following relation :—

$$\Lambda = \Sigma \dot{q}\, \partial \mathrm{L}/\partial \dot{q} - \mathrm{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where Λ is the whole energy of the motion due to a modified function L, and q is a coordinate whose velocity appears explicity. (Notice that *if* Λ were supposed expressed, not as a function of the $\dot{q}$'s, but as a function of the $\partial \mathrm{L}/\partial \dot{q}$'s, it would be the reciprocal function of L with regard to the $\dot{q}$'s [ROUTH'S 'Elem. Rig. Dyn.,' 4th ed., § 410.] It is not this reciprocal function only, because, for our purposes, it is more convenient to assume it an explicit function of the same quantities as L). To prove this, let* ϕ, Φ be a coordinate, whose momentum appears, and its momentum respectively, and let L_0 be the Lagrangian function of which L is the modified form. Thus

$$\Sigma \dot{q}\, \partial \mathrm{L}/\partial \dot{q} - \mathrm{L} = \Sigma \dot{q}\, \partial \mathrm{L}_0/\partial \dot{q} - (\mathrm{L}_0 - \Sigma \dot{\phi}\Phi).$$

[ROUTH'S 'El. Rig. Dyn.,' 4th ed., §§ 410, 420].

$$= \Sigma\, (\dot{q}\, \partial \mathrm{L}_0/\partial \dot{q} + \dot{\phi}\, \partial \mathrm{L}_0/\partial \dot{\phi}) - \mathrm{L}_0. \quad [\textit{ibid.}]$$

$$= 2\mathfrak{T} - (\mathfrak{T} - \mathfrak{V}). \quad [\mathfrak{T} = \text{kinetic energy}, \mathfrak{V} = \text{potential energy.}]$$

$$= \mathfrak{T} + \mathfrak{V} = \Lambda.$$

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[*] There is no danger of confusion of these meanings of ϕ, Φ with the stress meanings these symbols bear through the rest of this paper.

Equation (1) can be put in a more convenient form for our purpose. We have

$$\delta \mathrm{L} = \frac{d}{dt} \Sigma \frac{\partial \mathrm{L}}{\partial \dot{q}} \delta q + \Sigma \left(\frac{\partial \mathrm{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathrm{L}}{\partial \dot{q}} \right) \delta q + \Sigma \left(\frac{\partial \mathrm{L}}{\partial \phi} \delta \phi + \frac{\partial \mathrm{L}}{d \Phi} \delta \Phi . \right)$$

With the restrictions just mentioned, we have $\partial \mathrm{L} / \partial \phi = 0$ and $\delta \Phi = 0$. Hence

$$\left. \begin{aligned} \delta \mathrm{L} &= \frac{d}{dt} \Sigma \frac{\partial \mathrm{L}}{\partial \dot{q}} \delta q + \Sigma \left(\frac{\partial \mathrm{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathrm{L}}{d \dot{q}} \right) \delta q \\ &= 2 \frac{d}{dt} \mathfrak{T} \left(\delta q_a, \delta q_b, \ldots \right) + \Sigma \left(\frac{\partial \mathrm{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathrm{L}}{\partial \dot{q}} \right) \delta q \end{aligned} \right\} \quad . \; . \; . \; . \; . \quad (2),$$

where $2\mathfrak{T} \left(\delta q_a, \delta q_b \ldots \right)$ is to be defined as *the function which appears under the operator d/dt when* $\delta \mathrm{L}$ *is expressed as the sum of two quantities, one of which is a linear function of the variations of the retained coordinates, and the other is the rate of variation of a similar function.* We now have

$$\Lambda = 2 \mathfrak{T} \left(\dot{q}_a, \dot{q}_b, . \quad \right) - \mathrm{L} \quad . \; . \quad . \; . \; . \; . \; . \quad (3).$$

We shall show how, for our particular system, $\mathfrak{T} \left(\dot{q}_a \ldots \right)$ can be expressed in the form

$$\mathfrak{T} \left(q_a, q_b, \ldots \right) = \iiint \mathfrak{t} \, d\varsigma + \iint \dot{\mathfrak{t}} \, ds \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4),$$

where $\mathfrak{t}$, $\mathfrak{t}_s$ *are functions of the same independent variables as* l, l_s.

It is convenient to call $\mathfrak{T} \left(\dot{q}_a \ldots \right)$ the whole modified kinetic energy, and $\mathfrak{t}$, $\mathfrak{t}_s$ the modified kinetic energies per unit volume and surface respectively. And, similarly putting

$$\lambda = 2\mathfrak{t} - l, \quad \lambda_s = 2\mathfrak{t}_s - l_s \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5),$$

we shall call λ and λ_s the *free* energy[*] per unit volume and surface respectively. We thus have

$$\Lambda = \iiint \lambda d\varsigma + \iint \lambda_s ds \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6).$$

We shall then *assume* that the energy in any *finite region* is the integral on the right of this equation for that region. The surface integral in this case, of course, only applies to surfaces of discontinuity (as to physical quantities) in this region, and not to the true boundary of the region.

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[*] This term is adopted as a translation of HELMHOLTZ's '*freie Energie*' ('Wiss. Abh.,' II., 959). It is not, of course, the same as the intrinsic energy which we are about to determine by a method analogous to HELMHOLTZ's.

Bb. *The Entropy.*

34. If Q be the external force of type q, we have

$$d\mathrm{A} = \Sigma\,\mathrm{Q}dq,$$

where dq is the actual increment in q during the element of time. This can, of course, be proved directly from eq. (2). It must now be remembered that all the above variations are only true if we suppose the temperature of every element of matter kept constant. In other words, the last equation must, when we do not make this restriction, be replaced by

$$d\mathrm{A} - d_\theta\mathrm{A} = \Sigma\,\mathrm{Q}dq \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where $d_\theta\mathrm{A}$ stands for that part of the increment in A which is due to increment in temperature in all elements of matter during the element of time. Let now E be the intrinsic energy (including under this term the ordinary kinetic energy of matter as well as all other forms of energy) of all the matter in space. Thus by the fundamental property of entropy (TAIT'S 'Heat,' §§ 377, 378),

$$d\mathrm{E} = \Sigma\,\mathrm{Q}dq + \iiint \theta df d\varsigma + \iint \theta df_s ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

whence, from the last equation,

$$d\left(-\mathrm{E} + \mathrm{A} + \iiint \theta f d\varsigma + \iint \theta f_s ds\right) = d_\theta\mathrm{A} + \iiint f d\theta d\varsigma + \iint f_s d\theta ds \quad . \quad (9).$$

λ_s will be seen later on to be independent of Θ. Hence

$$\begin{aligned} d_\theta\mathrm{A} &= \iiint \left(\frac{\partial\lambda}{\partial\theta} d\theta - \mathrm{S}d\Theta_\Theta\nabla\lambda\right) d\varsigma + \iint \frac{\partial\lambda_s}{\partial\theta} d\theta ds \\ &= \iiint \left(\frac{\partial\lambda}{\partial\theta} + \mathrm{S}\nabla_\Theta\nabla\lambda\right) d\theta d\varsigma + \iint \left(\frac{\partial\lambda_s}{\partial\theta} - [\mathrm{SU}\nu_\Theta\nabla\lambda]_{a+b}\right) d\theta ds, \end{aligned}$$

[by putting $d\Theta = \nabla d\theta$, and applying § 5, eq. (4)]. Thus eq. (9) becomes

$$\left.\begin{aligned} & d\left(-\mathrm{E} + \mathrm{A} + \iiint \theta f d\varsigma + \iint \theta f_s ds\right) \\ & \quad = \iiint \left(f + \frac{\partial\lambda}{\partial\theta} + \mathrm{S}\nabla_\Theta\nabla\lambda\right) d\theta d\varsigma + \iint \left(f_s + \frac{\partial\lambda_s}{\partial\theta} - [\mathrm{SU}\nu_\Theta\nabla\lambda]_{a+b}\right) d\theta ds \end{aligned}\right\} (10).$$

Since the left of this equation is a perfect differential, so is the right. Hence we see that

$$f + \partial\lambda/\partial\theta + \mathrm{S}\nabla_\Theta\nabla\lambda \quad \text{and} \quad f_s + \partial\lambda_s/\partial\theta - [\mathrm{SU}\nu_\Theta\nabla\lambda]_{a+b}$$

must be functions of θ only. And further, by including these functions in $-\partial l/\partial\theta$ and $-\partial l_s/\partial\theta$ respectively—a proceeding that will not affect the equations of motion deduced from the form of L—we see that each of these quantities may be put equal to zero. With this extended meaning of L, then

$$f = -\partial\lambda/\partial\theta - \mathrm{S}\nabla_{\Theta}\nabla\lambda \qquad \dots\dots\dots\dots (11),$$

$$f_s = -\partial\lambda_s/\partial\theta + [\mathrm{SU}\nu_{\Theta}\nabla\lambda]_{a+b} \qquad \dots\dots\dots\dots (12).$$

We now see also from eq. (10) that

$$\mathrm{E} = \Lambda + \iiint \theta f d\varsigma + \iint \theta f_s ds \qquad \dots\dots\dots\dots (13),$$

or

$$\mathrm{E} = \iiint e d\varsigma + \iint e_s ds \qquad \dots\dots\dots\dots (14),$$

where

$$e = \lambda + \theta f = \lambda - \theta\partial\lambda/\partial\theta - \theta\mathrm{S}\nabla_{\Theta}\nabla\lambda \qquad \dots\dots\dots\dots (15),$$

$$e_s = \lambda_s + \theta f_s = \lambda_s - \theta\partial\lambda_s/\partial\theta + \theta[\mathrm{SU}\nu_{\Theta}\nabla\lambda]_{a+b} \qquad \dots\dots (16),$$

so that e, e_s may be called the intrinsic energy per unit volume and surface respectively.

Bc. *Frictional Forces, Conduction of Heat, and Dissipation of Energy.*

35. It has already [§ 27 (26)] been mentioned that x is a function of

$$\theta, \Theta\,;\ \Psi, \dot{\Psi}\,;\ \mathbf{K}, \mathbf{H} \qquad \dots\dots\dots\dots (17).$$

Of these Ψ and $\mathbf{K}$ are of the nature of velocities, and from the equation $4\pi\mathbf{G} = \mathrm{V}\nabla\mathbf{H}$ the same may be said of $\mathbf{H}$. Let us, then, briefly speak of them as "the velocities" involved in x. Similarly in the general theory where x_s is not assumed zero, it also will involve certain variables for like reasons called velocities. Let ξ, ξ_s be the functions which are reciprocal (ROUTH's 'El. Rig. Dyn.,' 4th ed., § 410) with regard to Θ and the velocities, to the functions x and x_s. Thus

$$x + \xi = -\mathrm{S}\Theta_{\Theta}\nabla x - \mathrm{S}\mathbf{K}_{\mathbf{K}}\nabla x - \mathrm{S}\mathbf{H}_{\mathbf{H}}\nabla x - \mathrm{S}\dot{\Psi}\zeta_{\dot{\Psi}}\text{Ɑ}x\zeta^{*} \qquad \dots\dots (18),$$

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[*] This seems a good opportunity to place on record a suggestion. There are some obvious objections to the method used in the present and former papers of indicating the independent variable of differentiation of a ∇ or Ɑ by an affix. It is somewhat hard to distinguish between **C** and **c** in the

and a similar equation would hold with regard to $x_s + \xi_s$, which, however, requires definite information as to the velocities involved in x_s. ξ is supposed (ROUTH, *ibid.*) expressed as a function, not of the variables (17), but of

$$\theta,\ {}_{\Theta}\nabla x\ ;\ \Psi,\ {}_{\dot\Psi}\text{Ɑ}x\ ;\ {}_{\mathrm{K}}\nabla x,\ {}_{\mathrm{H}}\nabla x \quad . \ . \ . \ . \ . \ . \ . \quad (19).$$

It is best at first to regard x as a function, not of **K**, but of **C** and **c** [§ 24, eq. (10)]. When later we make the assumption that, so far as it depends on these last two, it is a function of their difference (**K**) only, it will only have to be noticed that

$${}_{\mathbf{C}}\nabla x = {}_{\mathbf{K}}\nabla x = -\,{}_{\mathbf{c}}\nabla x \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (20).$$

Note that this gives

$$\mathrm{S}\mathbf{K}_{\mathbf{K}}\nabla x = \mathrm{S}\mathbf{C}_{\mathbf{C}}\nabla x + \mathrm{S}\mathbf{c}_{\mathbf{c}}\nabla x \quad . \ . \ . \ . \ . \ . \ . \ . \quad (21),$$

which shows [eq. (18)] that the statement that ξ is the reciprocal of x with regard to Θ and the velocities is still true.

Similarly, if it were assumed, as on a future occasion it will be assumed, that x_s was a function of

$$\theta\ ;\ [\dot\rho']_{a-b},\ \Psi_a,\ \Psi_b\ ;\ \mathrm{TSU}\nu\mathrm{K} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (22),$$

it would be best first to regard it as a function of

$$\theta\ ;\ \dot\rho_a',\ \dot\rho_b',\ \Psi_a,\ \Psi_b\ ;\ \mathbf{C}_a,\ \mathbf{C}_b,\ \mathbf{c}_a,\ \mathbf{c}_b \quad . \ . \ . \ . \ . \ . \ . \quad (23),$$

and later make the necessary restrictions.

36. We shall now suppose that the symbols Q, **F**, **F**$_s$, Φ, **E**, **E**$_s$, **e**, **e**$_s$ stand for those parts only of the external forces of the various types, which are owing to friction and the like. To determine their values we shall use the principle[*]

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present paper and Ψ and ψ in the former paper [paragraph following eq. 40)] when used as affixes. There are objections from the printer's and proof-reader's point of view when the affix is anything other than a mere letter. For instance, ${}_{\dot\Psi}\text{Ɑ}$ in the present case, and, still more, ${}_{\dot\rho'}\nabla$ in eq. (2), § 44, below, are objectionable on these grounds. Is it not, then, desirable to have, at any rate, an alternative notation? As an alternative to ${}_{\sigma}\nabla$, let me here suggest any one of the following: $\nabla|\sigma|$ $|\nabla\sigma|$ $[\nabla\sigma]$ $\nabla|\sigma$ $\nabla;\sigma$ $\nabla\,\cdot\!|\!\cdot\,\sigma$ $\nabla\sigma|$ $\nabla\sigma;$. Of these I should personally be inclined to favour $[\nabla\sigma]$ or $\nabla\sigma;$, the latter rather than the former. For instance, in this notation, eq. (18) would become

$$x + \xi = -\,\mathrm{S}\Theta\nabla\Theta\,;x - \mathrm{S}\mathbf{K}\nabla\mathbf{K}\,;x - \mathrm{S}\mathbf{H}\nabla\mathbf{H}\,;x - \mathrm{S}\dot\Psi\zeta\text{Ɑ}\Psi\,;x\zeta,$$

which, I think, shows that the notation is sufficiently striking, while it has the advantage of great simplicity.

[*] I have not been able to reduce this to simpler form or to substitute a simpler principle leading to the same results. I merely wished to make all the phenomena of the kind now being considered depend on some single scalar function X, much as the reversible phenomena depend on the single scalar

$$\iiint\left\{\theta\left(\delta\frac{x}{\theta}+\frac{\partial}{\partial\theta}\frac{\xi}{\theta}\cdot\delta\theta\right)+\dot{f}\,\delta\theta\right\}d\varsigma$$
$$+\iint\left\{\theta\left(\delta\frac{x_s}{\theta}+\frac{\partial}{\partial\theta}\frac{\xi_s}{\theta}\cdot\delta\theta\right)+\dot{f_s}\,\delta\theta\right\}ds+\Sigma\mathrm{Q}\,\delta\dot{q}=0\quad.\quad.\quad.\quad.\quad(24),$$

where now δ only implies such variations as are the consequences of varying *the velocities of the dynamical system and the temperature*, and where of course $\Sigma\mathrm{Q}\,\delta\dot{q}$ is given in our case by

$$\Sigma\mathrm{Q}\,\delta\dot{q}=\iiint\{-\,\mathrm{S}\mathbf{F}\,\delta\dot{\rho}'\,d\varsigma'+(\mathrm{S}\mathbf{E}\,\delta\mathbf{C}+\mathrm{S}\mathsf{e}\,\delta\mathsf{c})\,d\varsigma\}$$
$$+\iint\{-\,\mathrm{S}\mathbf{F}_s\,d\dot{\rho}'\,ds'+(\mathrm{S}\mathbf{E}_s\,\delta\mathbf{C}+\mathrm{S}\mathsf{e}_s\,\delta\mathsf{c})\,ds\}\quad.\quad.\quad.\quad.\quad.\quad(25).$$

This last equation would have to be modified if we contemplated finite sliding of one surface over another. In this paper, as already stated, we simplify by supposing this never to take place (except in § 64 below).

Equation (24) is more general than in this paper is required. Throughout this paper x_s, and therefore ξ_s, will be assumed zero.

37. The truth of the principle can be verified (as, admitting the restrictions just mentioned, will be shown directly) by proving that its consequences are in complete harmony with three recognised principles :—(1) that frictional forces can be explained by what Lord Rayleigh ('Sound,' 1st ed., vol. I., § 81) calls a dissipation function; (2) that the heat which is created by the destruction of energy in other forms, appears, in the first instance, at the elements of matter where the destruction takes place; (3) the fundamental principle of conduction of heat, that the rate of flow of heat out of any region across the element $d\Sigma'$ of its boundary $=\mathrm{S}\,d\Sigma'\gamma'\Theta'$ where γ' is a self-conjugate linear vector function, which is itself a function of the state of the medium at the point.

38. To show the truth of these statements in the limited circumstances mentioned, viz., when x_s is zero and there is no slipping, notice first what the effects of varying $\mathbf{C}$ and c only are. A variation in $\mathbf{C}$ will cause a variation in $\mathbf{H}$, since $4\pi\mathbf{C}=\mathrm{V}\nabla\mathbf{H}$ and $[\mathrm{VU}\nu\mathbf{H}]_{a+b}=0$. The device used in the calculus of variations to take account

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function L. When there are heat sources not included in our system (L and X) we ought to put $\dot{f}-h$ and $\dot{f_s}-h_s$ instead of $\dot{f}$ and $\dot{f_s}$ in eq. (24), $h\theta$ and $h_s\theta$ being the rate of supply of external heat per unit volume and surface respectively. The form of eq. (24) would perhaps be made more instructive by grouping together the terms

$$\iiint\dot{f}\,\delta\theta\,d\varsigma+\iint\dot{f_s}\,\delta\theta\,ds+\Sigma\mathrm{Q}\,\delta\dot{q}.$$

If $\dot{\mathrm{H}}$ be the rate of "absorption of heat" by a body ($d\varsigma$ or ds) of the system, this expression transforms into $\Sigma\,(\dot{\mathrm{H}}\,\delta\theta/\theta+\mathrm{Q}\delta\dot{q})$.

of such equations of condition is well known. In the present case it takes the following form: to the left of equation (24) add

$$-\iiint \mathrm{Sa}\,(\delta\mathbf{C} - \mathrm{V}\nabla\,\delta\mathbf{H}/4\pi)\,d\varsigma + (4\pi)^{-1}\iint \mathrm{Sa}_s\,d\Sigma\,\delta\mathbf{H}$$

where $\mathbf{a}$, $\mathbf{a}_s$ are vectors; $\delta\,\mathbf{C}$ and $\delta\,\mathbf{H}$ may *then* be regarded as independent. It is to be noted that there is but one $\mathbf{a}_s$ for an element of the bounding surface, *i.e.*, there is not one for each region bounded. In our notation this may be expressed by saying that $[\mathbf{a}_s]_a = [\mathbf{a}_s]_b$.

Now, by eq. (4), § 5, above,

$$\iiint \mathrm{Sa}\nabla\,\delta\mathbf{H}\,d\varsigma + \iint \mathrm{Sa}_s\,d\Sigma\,\delta\mathbf{H} = \iiint \mathrm{S}\,\delta\mathbf{H}\nabla\mathbf{a}\,d\varsigma + \iint \mathrm{S}\,(\mathbf{a}_s + \mathbf{a})\,d\Sigma\,\delta\mathbf{H}.$$

Hence, since the part contributed to the left of eq. (24) by $\delta\mathbf{H}$ is $-\iiint \mathrm{S}\,\delta\mathbf{H}_{\mathrm{H}}\nabla x\,d\varsigma$, we get, by equating to zero the coefficient of the arbitrary vector $\delta\mathbf{H}$,

$$4\pi_{\mathrm{H}}\nabla x = \mathrm{V}\nabla\mathbf{a} = \mathbf{b} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (26)$$

$$[\mathrm{VU}\nu\mathbf{a}]_{a+b} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (27),$$

$\mathbf{a}_s$ disappearing, since $[\mathbf{a}_s]_a = [\mathbf{a}_s]_b$.

Again, before considering what is contributed to the left of eq. (24) by $\delta\mathbf{C}$, it must be remembered that $\delta\mathbf{C}$ is not quite arbitrary, by reason of the equations of condition $\mathrm{S}\nabla\mathbf{C} = 0$, $[\mathrm{SU}\nu\mathbf{C}]_{a+b} = 0$. This is taken account of by adding to the left of[*] eq. (24)

$$\iiint \mathrm{YS}\nabla\,\delta\mathbf{C}\,d\varsigma + \iint \mathrm{Y}_s\mathrm{S}\,d\Sigma\,\delta\mathbf{C} = -\iiint \mathrm{S}\,\delta\mathbf{C}\nabla\mathrm{Y}\,d\varsigma + \iint (\mathrm{Y} + \mathrm{Y}_s)\,\mathrm{S}\,d\Sigma\,\delta\mathbf{C}$$

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[*] It may be objected that these equations of condition have already been taken account of in the treatment accorded to the more general equations of $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, $[\mathrm{VU}\nu\mathbf{H}]_{a+b} = 0$, and, therefore, it is *erroneous to take account of them again.* The answer to this is that it is not *necessary* to do this, but, on the other hand, it is not *erroneous*. We must expect as the result that the Y's will be, in a mathematical sense, redundant. That this actually is the case will appear in § 65 below. The reason for introducing them is to obtain the equations of the field in as familiar a form as possible, and to show the mathematical dependence of the existence of a potential on the equations $\mathrm{S}\nabla\mathbf{C} = 0$, $[\mathrm{SU}\nu\mathbf{C}]_{a+b} = 0$. The process may be paralleled in the subject of the Calculus of Variations. U, V, W being three functions of $x, y, \ldots, \partial x, \partial y, \ldots$, linear in the latter group, let it be required to satisfy the equation $\mathrm{U} = 0$ subject to the equations of condition $\mathrm{V} = 0$, $\mathrm{W} = 0$. The recognised method is to use the single equation $\mathrm{U} + \mathrm{AV} + \mathrm{BW} = 0$ instead of the three, A and B being functions of $x, y, \ldots$ determinable by the problem in hand. It would not be erroneous to add to the left of the last equation CW, where C was a function of the same kind as A and B. One of the two, B or C, would be mathematically redundant, but it might be convenient to introduce both and give arbitrarily, later on, some method of assigning a definite meaning to each.

[eq. (4), § 5], where Y, Y_s are scalars, and, as with $\mathbf{a}_s$, $[\mathrm{Y}_s]_a = [\mathrm{Y}_s]_b$. Equating now to zero the coefficients of $\delta\mathbf{C}$, $\delta\mathbf{c}$, we get

$$\mathbf{E} = {}_{\mathrm{C}}\nabla x + \mathbf{a} + \nabla\mathrm{Y}, \quad \mathbf{e} = {}_{\mathrm{c}}\nabla x \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

$$\mathbf{E}_s = -[\mathrm{YU}\nu]_{a+b}, \quad \mathbf{e}_s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

It should be noticed that $\mathbf{b}$, defined by eq. (26), satisfies both the conditions of incompressibility

$$\mathrm{S}\nabla\mathbf{b} = 0, \quad [\mathrm{SU}\nu\mathbf{b}]_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

The first condition is obvious from the equation $\mathbf{b} = \mathrm{V}\nabla\mathbf{a}$. The second is easily deduced from the equation $[\mathrm{VU}\nu\mathbf{a}]_{a+b} = 0$. For this last asserts that the component of $\mathbf{a}$ parallel to the surface is the same for both regions bounded. Thus the line integral $\int \mathrm{S}\, d\rho\mathbf{a}$, which, by eq. (3), § 5, $= \iint \mathrm{S}\mathbf{b}\, d\Sigma$, taken over any closed curve on the surface, is the same for both regions. It follows that $[\mathrm{S}\, d\Sigma\mathbf{b}]_{a+b} = 0$. We naturally assume that $\mathbf{a}$ is an intensity and $\mathbf{b}$ a flux. Hence, by § 8,

$$\mathrm{V}\nabla'\mathbf{a}' = \mathbf{b}' \;, \quad [\mathrm{VU}\nu'\mathbf{a}']_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$\mathrm{S}\nabla'\mathbf{b}' = 0 \;, \quad [\mathrm{SU}\nu'\mathbf{b}']_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

39. Next suppose that the only variation implied in equation (24) is in $\dot{\rho}'$, and, therefore, in $\dot{\Psi}$. Thus

$$\iiint \theta\delta\,(x/\theta)\, d\varsigma = -\iiint \mathrm{S}\delta\dot{\Psi}\zeta_{\dot{\Psi}}\text{Ɑ}x\zeta d\varsigma \text{ [eq. (13) of former paper]}$$

$$= -\tfrac{1}{2}\iiint \mathrm{S}\delta\dot{\Psi}\zeta\chi^{-1}\Phi'\chi'^{-1}\zeta d\varsigma',$$

where Φ' is defined by saying that

$$\Phi = 2_{\dot{\Psi}}\text{Ɑ}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

and that Φ is a function of Class I. of § 9 above.* Now since [former paper, eq. (39)] $\Psi = \chi'\chi$, we have

$$\delta\dot{\Psi} = \delta\dot{\chi}'.\chi + \chi'\delta\dot{\chi}$$

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* What immediately follows is a particular case of a theorem required more than once below. Let Ω, χ and σ be as usual in this paper and let $\mathrm{Q}\omega = -\Sigma\beta\mathrm{S}\omega\sigma$. Then

$$\mathrm{S}\chi'\mathrm{Q}\zeta\Omega\zeta d\varsigma = \Sigma\mathrm{S}\beta\Omega'\sigma' d\varsigma'.$$

More generally, if (ω, ω') be any function of two vectors ω, ω' linear in each

$$(\mathrm{Q}\zeta, \chi\Omega\zeta)\, d\varsigma = \Sigma\,(\beta, \Omega'\sigma')\, .\, d\varsigma'.$$

and, therefore,

$$\mathrm{S}\delta\dot{\Psi}\zeta\chi^{-1}\Phi'\chi'^{-1}\zeta = \mathrm{S}\delta\dot{\chi}'\chi\zeta\chi^{-1}\Phi'\chi'^{-1}\zeta + \mathrm{S}\chi'\delta\dot{\chi}\zeta\chi^{-1}\Phi'\chi'^{-1}\zeta$$
$$= 2\mathrm{S}\delta\dot{\chi}\zeta\Phi'\chi'^{-1}\zeta \text{ [\textit{ibid.}, eq. (6)]}$$

and $\delta\dot{\chi}\omega = -\mathrm{S}\omega\nabla\,.\,\delta\dot{\rho}'$ [*ibid.*, eq. (25)]. Hence [*ibid.*, eq. (7)]

$$\iiint\theta\delta\,(x/\theta)d\varsigma = -\iiint\mathrm{S}\delta\dot{\rho}'_1\Phi'\chi'^{-1}\nabla_1 d\varsigma'$$
$$= -\iiint\mathrm{S}\delta\dot{\rho}_1'\Phi'\nabla_1'd\varsigma' \text{ [\textit{ibid.}, eq. (27)]}$$
$$= -\iint\mathrm{S}\delta\dot{\rho}'\Phi'd\Sigma' + \iiint\mathrm{S}\delta\dot{\rho}'\Phi'_1\nabla_1'd\varsigma' \text{ [eq. (4), § 5, above].}$$

Hence, from equations (24) (25), above,

$$\mathbf{F} = \Phi'\Delta'\ ,\ \mathbf{F}_s = -[\Phi'\mathrm{U}\nu']_{a+b} \quad \ldots\ldots\ldots \quad (34),$$

showing [*ibid.*, p. 107] that the presence in x of $\dot{\Psi}$ leads to a stress Φ.

40. Now, suppose the only variation of eq. (24) is that of temperature. In this case

$$\theta\left(\delta\frac{x}{\theta} + \frac{\partial\,(\xi/\theta)}{\partial\theta}\,\delta\theta\right) = \theta\frac{\partial}{\partial\theta}\frac{x+\xi}{\theta}\,\delta\theta - \mathrm{S}\,\delta\Theta_{\scriptscriptstyle\Theta}\nabla x$$
$$= -\frac{x+\xi}{\theta}\,\delta\theta - \mathrm{S}\nabla\,\delta\theta_{\scriptscriptstyle\Theta}\nabla x,$$

since [Routh's 'El. Rig. Dyn.,' 4th ed., § 410], $\partial\,(x+\xi)/\partial\theta = 0$. Also [§ 5, eq. (4) above]

$$-\iiint\mathrm{S}\nabla\,\delta\theta_{\scriptscriptstyle\Theta}\nabla x\,d\varsigma = -\iint\delta\theta\mathrm{S}\,d\Sigma_{\scriptscriptstyle\Theta}\nabla x + \iiint\delta\theta\mathrm{S}\nabla_{\scriptscriptstyle\Theta}\nabla x\,d\varsigma$$

Hence, the variation of θ leads to

$$j = (x+\xi)/\theta - \mathrm{S}\nabla_{\scriptscriptstyle\Theta}\nabla x \quad \ldots\ldots\ldots \quad (35),$$

$$\dot{f}_s = [\mathrm{SU}\nu_{\scriptscriptstyle\Theta}\nabla x]_{a+b} \quad \ldots\ldots\ldots\ldots \quad (36).$$

41. The first of the three statements in § 37 is now obvious, as far as **c** is concerned. With regard to **C** it must be remembered that **C** cannot be made to vary without varying **H**. Now [Rayleigh's 'Sound,' 1st ed., I., § 81] in order that frictional forces may be explained by a dissipation function X, in Lord Rayleigh's sense, the frictional force Q corresponding to an independent coordinate q should be $= -\partial\mathrm{X}/\partial\dot{q}$. For our purposes this is put more conveniently by saying that $\Sigma\mathrm{Q}\,\delta\dot{q} = -\Sigma\,\delta_{\dot{q}}\mathrm{X}$,

where the Σ implies that *any assigned group of independent velocities*, and no others, are varied, and where $\delta_{\dot{q}}X$ is the increment in X due to the particular variation $\delta\dot{q}$. Now, on account of the conditions,

$$4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H},\ [\mathrm{V}d\Sigma\mathbf{H}]_{a+b} = 0,\ \mathrm{S}\nabla\mathbf{C} = 0,\ [\mathrm{S}d\Sigma\mathbf{C}]_{a+b} = 0,$$

it is necessary that we consider the whole group of velocities, $\delta\mathbf{C}$, throughout space together. It is, then, as far as **C** is concerned, only necessary to prove that

$$\iiint \mathrm{S}\mathbf{E}\,\delta\mathbf{C}\,d\varsigma + \iint \mathrm{S}\mathbf{E}_s\,\delta\mathbf{C}\,ds = \iiint (\mathrm{S}\delta\mathbf{C}_{\mathbf{C}}\nabla x + \mathrm{S}\delta\mathbf{H}_{\mathbf{H}}\nabla x)\,d\varsigma,$$

the integrals extending throughout space. (As to the sign of these terms, it must be remembered that the force corresponding to **C** is not **E**, but $-$ **E**). This is proved quite easily* by means of eq. (4) § 5.

Similarly, with regard to $\dot{\Psi}$, it is only necessary to prove that

$$-\iiint \mathrm{S}\mathbf{F}\,\delta\rho'\,d\varsigma' - \iint \mathrm{S}\mathbf{F}_s\,\delta\dot{\rho}'\,ds' = \iiint \mathrm{S}\,\delta\dot{\Psi}\zeta_{\dot{\Psi}}\mathfrak{C}x\zeta\,d\varsigma,$$

and this is obvious from the mode in which equations (33) (34) were established.

42. To prove the second and third statements, let for any finite region $\iint_b$ denote an integration taken over the *true boundary* of that region, and $\iint_d$ an integral taken over both sides of any surface of discontinuity, as to physical quantities in the region, so that

$$\iint = \iint_b + \iint_d \quad \ldots\ldots\ldots\ldots \quad (37).$$

Then, if we can prove that for *any* finite region,

$$\left.\begin{aligned}&(\text{Rate of increase of heat} + \text{rate of doing work of frictional forces})\\ &\quad = -\iint_b \{\mathrm{S}d\Sigma\,(\theta_\Theta\nabla x + \mathrm{V}\mathbf{a}\mathbf{H}/4\pi - \mathrm{Y}\mathbf{C}) + \mathrm{S}\dot{\rho}'\Phi' d\Sigma'\}\end{aligned}\right\} \quad \ldots \quad (38),$$

it will follow that the energy supply required to account for (1) the increment of heat, (2) the work (negative) done by the frictional forces, consists of three parts, (1)

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* It should, perhaps, be noticed that $\delta\mathbf{C}$ and $\delta\mathbf{H}$ are now not perfectly arbitrary. We may assume that

$$\mathrm{S}\nabla\,\delta\mathbf{C} = 0\ ,\ 4\pi\,\delta\mathbf{C} = \mathrm{V}\nabla\,\delta\mathbf{H},$$

and from the equations $[\mathrm{VU}\nu\mathbf{a}]_{a+b} = 0$, $[\mathrm{VU}\nu\mathbf{H}]_{a+b} = 0$

$$[\mathrm{S}\mathbf{a}\mathrm{U}\nu\,\delta\mathbf{H}]_{a+b} = 0.$$

the work done on the boundary by the viscosity stress, (2) the work done on the boundary by the frictional electric forces, (3) a flux $-\theta_{\Theta}\nabla x$ of energy at every point of space. This may be put in, perhaps, the more familiar form :—the increment of heat in the region consists of three parts, (1) the work done *against* the frictional forces throughout the region, (2) the work done *by* the frictional forces (viscosity and electric) on the boundary, and (3) the surface integral taken inwards at the boundary of a flux $-\theta_{\Theta}\nabla x$. Stated in this way we see that equation (38) is equivalent to saying that the conduction of heat is due to a flux of heat $-\theta_{\Theta}\nabla x$ at every point of space, and that the frictional forces are sources of heat.* These are statements (2) and (3) of § 37 (except that here we have $-\theta_{\Theta}\nabla x$, and there we have a more definite form for flux of heat due to conduction.)

To prove eq. (38), note that the expression on the left

$$=\iiint\{(\theta\dot{f}+\mathrm{S}\mathbf{E}\mathbf{C}+\mathrm{S}\mathbf{e}\mathbf{c})\,d\varsigma-\mathrm{S}\mathbf{F}\dot{\rho}'\,d\varsigma'\}+\iint_{d}\{(\theta\dot{f}_{s}+\mathrm{S}\mathbf{E}_{s}\mathbf{C}+\mathrm{S}\mathbf{e}_{s}\mathbf{c})\,d\varsigma-\mathrm{S}\mathbf{F}_{s}\dot{\rho}'d\varsigma'\}$$

$$=\iiint\{[x+\xi-\theta\mathrm{S}\nabla_{\Theta}\nabla x+\mathrm{S}\mathbf{C}\,({}_{\mathbf{C}}\nabla x+\mathbf{a}+\nabla\mathrm{Y})+\mathrm{S}\mathbf{c}_{\mathbf{c}}\nabla x]\,d\varsigma-\mathrm{S}\dot{\rho}'\Phi'_{1}\nabla'_{1}\,d\varsigma'\}$$

$$+\iint_{d}\{\mathrm{S}d\Sigma\,(\theta_{\Theta}\nabla x+\mathrm{V}\mathbf{a}\mathbf{H}/4\pi-\mathrm{Y}\mathbf{C})+\mathrm{S}\dot{\rho}'\Phi'\,d\Sigma'\}.\dagger$$

Now put $\iint_{d}=\iint-\iint_{b}$ [equation (37)], and transform the integral $\iint$ by means of equation (4) § 5 above into a volume integral. In doing this note that by reversing the process of § 39 we get

$$-\iiint\mathrm{S}\dot{\rho}'\Phi'_{1}\nabla'_{1}\,d\varsigma'+\iint\mathrm{S}\dot{\rho}'\Phi'\,d\Sigma'=\iiint\mathrm{S}\dot{\Psi}\zeta_{\dot{\Psi}}\text{ᗡ}x\zeta\,d\varsigma.$$

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* It is possible at this stage that two objections may be taken to this reasoning. First it may be said that there ought to be no terms in the surface integral leading to the result that the frictional electric forces do work on the boundary. That this is not a sound objection will come out more clearly below, when the effect of $\mathbf{YC}$ will be found to in no way alter the ordinary views of the transference of electric energy through the field, and the effect of $\mathrm{V}\mathbf{aH}$ will be only to modify them in a way which would naturally be anticipated from the new hypothesis that $\mathbf{H}$ has some influence on the frictional forces of the field. Secondly, it may be said that besides the three terms mentioned in the text as contributing to rate of increase of heat, there should be a fourth due to such causes as the THOMSON and PELTIER effects. This statement is, however, undoubtedly wrong, as will appear more clearly when we come to the consideration of these effects. The explanation is that these effects are explained by terms in f. Hence, in equation (38) they are *included* on the left. If this is not considered convincing, let me call attention to equation (25), § 49 below, which asserts that the rate of increase of *intrinsic energy* (including that of the THOMSON effect, &c.), in any space = rate of doing work throughout the region of the external forces which are *not* due to friction + the rate of heat supply from external sources situated in the region + such a surface integral as now is under consideration (*i.e.*, confined to the true boundary).

† We have here for the sake of the next transformation added the term $\iint_{d}\mathrm{S}d\Sigma\mathbf{a}\mathbf{H}/4\pi$, since from the equations $[\mathrm{VU}\nu\mathbf{H}]_{a+b}=0$, $[\mathrm{VU}\nu\mathbf{a}]_{a+b}=0$, it follows that $[\mathrm{SU}\nu\mathbf{a}\mathbf{H}]_{a+b}=0$.

Thus we get for the expression on the left of eq. (38)

$$\iiint\{x + \xi + \mathrm{S}\Theta_{\Theta}\nabla x + \mathrm{S}\mathbf{C}_{\mathbf{C}}\nabla x + \mathrm{S}\mathbf{c}_{\mathbf{c}}\nabla x + \mathrm{S}\mathbf{H}_{\mathbf{H}}\nabla x + \mathrm{S}\dot{\Psi}\zeta_{\dot{\Psi}}\daleth x\zeta\}\, d\varsigma$$
$$- \iint_b \{\mathrm{S}d\Sigma\,(\theta_{\Theta}\nabla x + \mathrm{V}\mathbf{a}\mathbf{H}/4\pi - \mathrm{Y}\mathbf{C}) + \mathrm{S}\dot{\rho}'\Phi'\, d\Sigma'\}$$

of which the volume integral is zero [equations (18), (21)], and the surface integral is the expression on the right of equation (38).

To get the ordinary expression for the flux of heat due to conduction we have merely to suppose x to contain the term $-\mathrm{S}\Theta\gamma\Theta/2\theta$, where γ is a self-conjugate linear vector function of Class I., of § 9 above. The heat flux referred to the standard position of matter due to this term

$$= \theta_{\Theta}\nabla\,(\mathrm{S}\Theta\gamma\Theta/2\theta) = -\gamma\Theta,$$

and, therefore, by Prop. VI., § 10, the actual flux of heat is $-\gamma'\Theta'$.

43. It is known (Tait's 'Heat,' 1st ed., § 412) that if θ_0 be the lowest available temperature, $\theta_0\,\dot{\mathrm{F}}$ is the rate of dissipation or degradation of energy in Sir William Thomson's sense. Now by equations (35), (36),

$$\dot{\mathrm{F}} = \iiint(x + \xi)/\theta\, d\varsigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39),$$

so that $(x + \xi)\,\theta_0/\theta$ may be called the rate of dissipation of energy per unit volume. There seems very good reason then to call X the dissipation function. It only differs from Lord Rayleigh's function in the terms that lead to the conduction of heat.

If, as will usually be the case, x is quadratic in Θ and the velocities, $\xi = x$, and the rate of dissipation per unit of volume will be $2x\theta_0/\theta$. For instance, the rate of dissipation per unit volume of the standard position due to conduction $= -\mathrm{S}\Theta\gamma\Theta\theta_0/\theta^2$, and, therefore, per unit volume of the present position it is $-\mathrm{S}\Theta'\gamma'\Theta'\theta_0/\theta^2$.*

III. Establishment of General Results.

A. *Value of* δL *for a Finite portion of Matter.*

44. As already remarked (§ 34) the δ in equation (1) § 13 implies variation in everything but the temperature. This will be assumed for the present. Thus δl depends [§ 27 (25)] on the variations of

$$\rho',\ \dot{\rho}',\ \Psi\ ;\ \mathbf{d},\ \mathbf{D},\ \mathbf{C},\ \mathbf{H}.$$

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* I suppose this result has been noticed before, though I do not know by whom.

So far as it depends on ρ', l will be supposed only to contain the term $-\mathrm{D}_m\mathrm{W}$, where D_m is the density of matter in the standard position, and W an ordinary potential (quite independent, however, of electromagnetic phenomena). Of course, so far as it depends on $\dot\rho'$, l is supposed only to contain the term $-\mathrm{D}_m\dot\rho'^2/2$. Thus δl consists of the following parts :—

$$-\mathrm{S}\,\delta\rho'\nabla' l = \mathrm{D}_m\mathrm{S}\,\delta\rho'\nabla'\mathrm{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$-\mathrm{S}\,\delta\dot\rho'_{\dot\rho'}\nabla l = -\mathrm{D}_m\mathrm{S}\dot\rho'\,\delta\dot\rho' = -d\,(\mathrm{D}_m\mathrm{S}\dot\rho'\,\delta\rho')/dt + \mathrm{D}_m\mathrm{S}\ddot\rho'\,\delta\rho' \quad . \quad . \quad (2),$$

$$-\mathrm{S}\,\delta\mathbf{D}_{\mathbf{D}}\nabla l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$-\mathrm{S}\,\delta\mathbf{d}_{\mathbf{d}}\nabla l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$-\mathrm{S}\,\delta\mathbf{C}_{\mathbf{C}}\nabla l = -\mathrm{S}\,\delta\dot{\mathbf{D}}_{\mathbf{C}}\nabla l = -d\mathrm{S}_{\mathbf{C}}\nabla l\,\delta\mathbf{D}/dt + \mathrm{S}\delta\,\mathbf{D}d_{\mathbf{C}}\nabla l/dt \quad . \quad . \quad . \quad (5),$$

$$-\mathrm{S}\,\delta\mathbf{H}_{\mathbf{H}}\nabla l = -\mathrm{S}\mathbf{B}\,\delta\mathbf{H}/4\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

$$-\mathrm{S}\,\delta\Psi\zeta\text{Ɑ}l\zeta = m\mathrm{S}\,\delta\rho'_1\phi'\nabla'_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where Ɑ stands, as throughout the present paper it will stand, for ${}_{\Psi}\text{Ɑ}$, and where ϕ' is defined by saying that

$$\phi = -2\text{Ɑ}l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

and that ϕ is of Class I. in § 9 above. The proof of equation (7) is exactly parallel to the treatment of Φ in § 39 above, and, therefore, need not be given here.

45. The part of $\delta\mathrm{L}$ due to (7) is

$$\iiint \mathrm{S}\,\delta\rho'_1\phi'\nabla'_1\,d\varsigma' = -\iiint \mathrm{S}\,\delta\rho'\phi'_1\nabla'_1\,d\varsigma' + \iint \mathrm{S}\,\delta\rho'\phi'\,\delta\Sigma'$$

by eq. (4), § 5, above. The part* due to (6) is [§ 26, eq. (19)]

$$-(4\pi)^{-1}\iiint \mathrm{S}\nabla\mathbf{A}\,\delta\mathbf{H}\,d\varsigma = -(4\pi)^{-1}\iiint \mathrm{S}\mathbf{A}\nabla\,\delta\mathbf{H}\,d\varsigma - (4\pi)^{-1}\iint \mathrm{S}\mathbf{A}\,\delta\mathbf{H}\,d\Sigma.$$

When considering the whole of space this surface integral can be neglected, since by eq. (15), § 25, $[\mathrm{V}\,d\Sigma\mathbf{H}]_{a+b} = 0$, and by eq. (20), § 26, $[\mathrm{V}\,d\Sigma\mathbf{A}]_{a+b} = 0$. If, as for

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* This transformation which assumes a fact still to be proved (viz., that $\mathbf{B} = \mathrm{V}\nabla\mathbf{A}$, $[\mathrm{V}\,d\Sigma\mathbf{A}]_{a+b} = 0$) is given, not with the object of determining the equations of motion, in which process this fact will not be assumed, but to find the rate of change of energy in an assigned space.

the present we assume, we are considering that part of L contributed by a finite portion of matter we must retain the part of the surface integral due to the true boundary of the portion ($\iint_b$ of § 42). Thus the part of δL due to (6) is

$$-\iiint \mathrm{S}\mathbf{A}\,\delta\mathbf{C}\,d\varsigma - (4\pi)^{-1}\iint_b \mathrm{S}\mathbf{A}\,\delta\mathbf{H}\,d\Sigma$$

$$= -\frac{d}{dt}\iiint \mathrm{S}\mathbf{A}\,\delta\mathbf{D}\,d\varsigma + \iiint \mathrm{S}\dot{\mathbf{A}}\,\delta\mathbf{D}\,d\varsigma - (4\pi)^{-1}\iint_b \mathrm{S}\mathbf{A}\,\delta\mathbf{H}\,d\Sigma.$$

Collecting terms we have for any finite portion of matter

$$\begin{aligned}\delta \mathrm{L} = &-\frac{d}{dt}\iiint\{\mathrm{D}_m\mathrm{S}\dot{\rho}'\,\delta\rho' + \mathrm{S}\,\delta\mathbf{D}\,({}_{\mathrm{C}}\nabla l + \mathbf{A})\}\,d\varsigma \\ &+\iiint\{\mathrm{S}\,\delta\rho'\,[\mathrm{D}_m(\ddot{\rho}' + \nabla'\mathrm{W}) - m\phi'_1\nabla'_1] - \mathrm{S}\,\delta\mathbf{d}\,{}_{\mathrm{d}}\nabla l \\ &\qquad\qquad + \mathrm{S}\,\delta\mathbf{D}\,(d_{\mathrm{C}}\nabla l/dt + \dot{\mathbf{A}} - {}_{\mathrm{D}}\nabla l)\}\,d\varsigma \\ &+\iint \mathrm{S}\,\delta\rho'\phi'\,d\Sigma' - (4\pi)^{-1}\iint_b \mathrm{S}\mathbf{A}\,\delta\mathbf{H}\,d\Sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \end{aligned} \qquad (9).$$

B. *The Free Energy and Rate of Increase of Intrinsic Energy for any Finite portion of Matter.*

46. We now see from the principle enunciated in § 33, above, that the modified kinetic energy for all space $\mathfrak{T}_\infty$ is given by

$$2\mathfrak{T}_\infty = -\iiint\{\mathrm{D}_m\dot{\rho}'^2 + \mathrm{S}\mathbf{C}\,({}_{\mathrm{C}}\nabla l + \mathbf{A})\}\,d\varsigma.$$

Now

$$\begin{aligned} 4\pi\iiint \mathrm{S}\mathbf{C}\mathbf{A}\,d\varsigma &= \iiint \mathrm{S}\mathbf{A}\nabla\mathbf{H}\,d\varsigma \ [\text{§ 25, eq. (13)}] \\ &= \iiint \mathrm{S}\mathbf{H}\nabla\mathbf{A}\,d\varsigma + \iint \mathrm{S}\mathbf{H}\mathbf{A}\,d\Sigma \ [\text{§ 5, eq. (4)}] \\ &= \iiint \mathrm{S}\mathbf{B}\mathbf{H}\,d\varsigma \ [\text{§ 26, eq. (19)}], \end{aligned}$$

the surface integral vanishing by § 25, eq. (15) and § 26, eq. (20). Thus

$$2\mathfrak{T}_\infty = -\iiint\{\mathrm{D}_m\dot{\rho}'^2 + \mathrm{S}\mathbf{C}_{\mathrm{C}}\nabla l + \mathrm{S}\mathbf{B}\mathbf{H}/4\pi\}\,d\varsigma = \iiint(l + \lambda)\,d\varsigma \quad . \quad . \quad (10),$$

where λ is in value, but not in form (since we suppose it expressed in terms of the same independent variables as l) equal to the function which is reciprocal to l with regard to $\dot{\rho}'$, **C** and **H**. These three vectors may be [§ 35] called velocities, and thus λ is in value the reciprocal of l with regard to all the velocities involved in the latter. Adopting now the notation of § 33 and its *assumption* (end of § 33), we have

$$l + \lambda = 2\mathfrak{t} = -(\mathrm{D}_m\dot{\rho}'^2 + \mathrm{S}\mathbf{C}_{\mathrm{C}}\nabla l + \mathrm{S}\mathbf{BH}/4\pi) \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\lambda_s = \mathfrak{t}_s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

That l, the Lagrangian function (per unit volume), and λ, the free energy, should be reciprocal functions (in value only) with regard to the velocities they contain, is in accord with the fact (but not deducible from it) that a similar statement is true for an ordinary dynamical system [§ 33, eq. (1) above].

Let now Λ stand for the part of the free energy due to a finite portion of matter. Required $\dot{\Lambda}$. To find this, first obtain the rate of increase of Λ that would occur if all the circumstances were such as actually occur, except that the temperature of each element of matter is kept constant, and then add the part due to the rate of variation of temperature. To get the first of these we have at first to find the corresponding part of $\dot{\mathrm{L}}$ by changing all the δ's of eq. (9) into differentiations with regard to the time. Then we have to subtract the result from $\dot{\Lambda} + \dot{\mathrm{L}}$, which is given by eq. (11). Thus we get

$$\left.\begin{aligned}\dot{\Lambda} = \iiint (\dot{\theta}\,\partial\lambda/\partial\theta - \mathrm{S}\dot{\Theta}_{\Theta}\nabla\lambda)\,d\varsigma \\ - \iiint \{\mathrm{S}\dot{\rho}'\,[\mathrm{D}_m(\ddot{\rho}' + \nabla'\mathrm{W}) - m\phi_1'\nabla_1'] - \mathrm{S}\mathbf{c}_{\mathrm{d}}\nabla l + \mathrm{S}\mathbf{C}\,(d_{\mathrm{C}}\nabla l/dt + \dot{\mathbf{A}} - {}_{\mathrm{D}}\nabla l)\}\,d\varsigma \\ - \iint \mathrm{S}\dot{\rho}'\phi'\,d\Sigma' - (4\pi)^{-1}\iint_b \mathrm{S}\dot{\mathbf{A}}\mathbf{H}\,d\Sigma\end{aligned}\right\} (13).$$

It should be shown perhaps how the last integral appears. It comes from the term $-\mathrm{S}\mathbf{BH}/4\pi$ in $l + \lambda$ and from the two terms

$$-\frac{d}{dt}\iiint \mathrm{S}\mathbf{A}\,\delta\mathbf{D}\,d\varsigma - \frac{1}{4\pi}\iint_b \mathrm{S}\mathbf{A}\,\delta\mathbf{H}\,d\Sigma$$

in $\delta\mathrm{L}$. These three terms contribute to $\dot{\Lambda}$

$$\frac{d}{dt}\iiint (\mathrm{S}\mathbf{AC} - \mathrm{S}\mathbf{BH}/4\pi)\,d\varsigma + \frac{1}{4\pi}\iint_b \mathrm{S}\mathbf{A}\dot{\mathbf{H}}\,d\Sigma.$$

But, since, $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, $4\pi\mathrm{S}\mathbf{AC} - \mathrm{S}\mathbf{BH} = \mathrm{S}\mathbf{A}\Delta\mathbf{H}$, so that the volume integral can be transformed into a surface integral. Transforming and noticing that the part of the surface integral $\iint_d$ is zero, the last term in equation (13) is obtained.

We may now obtain $\dot{\mathrm{E}}$. To do this, combine the first integral of equation (13) with $d\,(\iiint \theta f\,d\varsigma + \iint \theta f_s\,ds)/dt$. We thus get by equations (11), (12) of § 34, and equation (4), § 5

$$\iiint \theta\dot{f}\,d\varsigma + \iint \theta\dot{f}_s\,ds - \iint_b \dot{\theta}\mathrm{S}\,d\Sigma_\Theta\nabla\lambda.$$

Thus, from eq. (13), § 34,

$$\left.\begin{aligned}\dot{\mathrm{E}} = \iiint \theta\dot{f}\,d\varsigma + \iint \theta\dot{f}_s\,ds - \iiint \{ &\mathrm{S}\dot{\rho}'\,[\mathrm{D}_m(\ddot{\rho}' + \nabla'\mathrm{W}) - m\phi_1'\nabla_1'] \\ &- \mathrm{S}\mathbf{c}_{\mathbf{d}}\nabla l + \mathrm{S}\mathbf{C}\,(d_{\mathbf{C}}\nabla l/dt + \dot{\mathbf{A}} - {}_{\mathbf{D}}\nabla l)\}\,d\varsigma \\ - \iint \mathrm{S}\dot{\rho}'\ddot{\phi}'\,d\Sigma' - \iint {}_b\mathrm{S}d\Sigma\,(\mathbf{V}\dot{\mathbf{A}}\mathbf{H}/4\pi + \dot{\theta}_\Theta\nabla\lambda)&\end{aligned}\right\} \quad . \quad (14).$$

C. *The Equations of Motion.*

47. The symbols **E**, **F**, Φ, &c., will now again be supposed to stand for the whole external forces including those due to friction. The parts contributed by all of these except Φ to $\Sigma\mathrm{Q}\,\delta q$ can be written down at once. By the former paper p. 107 the force per unit volume (of present position of matter) due to Φ is $\Phi'\Delta'$. Φ is assumed to be self-conjugate* and of Class I of § 9 above. Thus Φ' is also self-conjugate, and therefore there is, due to it, no couple per unit volume. The force per unit surface at a surface of discontinuity is $-[\Phi'\mathrm{U}\nu']_{a+b}$. Thus the part contributed to $\Sigma\mathrm{Q}\,\delta q$ by Φ is

$$\iint \mathrm{S}\,\delta\rho'\Phi'\,d\Sigma' - \iiint \mathrm{S}\,\delta\rho'\Phi_1'\nabla_1'\,d\varsigma'.$$

Hence collecting all the terms

$$\left.\begin{aligned}\Sigma\mathrm{Q}\,\delta q = - \iiint \mathrm{S}\,\delta\rho'\,(\mathbf{F} + \Phi_1'\nabla_1')\,d\varsigma' - \iint \mathrm{S}\,\delta\rho'\,(\mathbf{F}_s\,ds' - \Phi'\,d\Sigma') \\ + \iiint (\mathrm{S}\mathbf{e}\,\delta\mathbf{d} + \mathrm{S}\mathbf{E}\,\delta\mathbf{D})\,d\varsigma + \iint (\mathrm{S}\mathbf{e}_s\,\delta\mathbf{d} + \mathrm{S}\mathbf{E}_s\,\delta\mathbf{D})\,ds\end{aligned}\right\} \quad . \quad (15).$$

48. To obtain the equations of motion from these results, it must be remembered (§ 38) that while $\delta\rho'$ and $\delta\mathbf{d}$ are quite arbitrary, this is not the case with $\delta\mathbf{D}$ and $\delta\mathbf{H}$. We adopt the same method here as in § 38, *i.e.*, we add to the $\delta\mathrm{L}$ for all space

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* It is clear by the work in the former paper (pp. 106 to 108) that there is no *necessity* to make this simplification. On the other hand nothing seems gained by not making it.

$$-\iiint y\mathrm{S}\nabla\,\delta\mathbf{D}\,d\varsigma - \iint y_s\mathrm{S}\,d\Sigma\,\delta\mathbf{D} - \iiint \mathrm{S}\mathbf{A}\,(\delta\mathbf{C} - \nabla\delta\mathbf{H}/4\pi)\,d\varsigma + (4\pi)^{-1}\iint \mathrm{S}\mathbf{A}_s\,d\Sigma\,\delta\mathbf{H},$$

where

$$[y_s]_a = [y_s]_b, \qquad [\mathbf{A}_s]_a = [\mathbf{A}_s]_b,$$

and where y, y_s are scalars and $\mathbf{A}$, $\mathbf{A}_s$ vectors; $\delta\mathbf{D}$ and $\delta\mathbf{H}$ may then both be regarded as arbitrary. The expression to be added to δL may, by equation (4) § 5, be written :—

$$-\frac{d}{dt}\iiint \mathrm{S}\mathbf{A}\,\delta\mathbf{D}\,\delta\varsigma + \iiint\{\mathrm{S}\,\delta\mathbf{D}\,(\dot{\mathbf{A}} + \nabla y) + \mathrm{S}\,\delta\mathbf{H}\nabla\mathbf{A}/4\pi\}\,d\varsigma$$
$$-\iint\{(y + y_s)\,\mathrm{S}\,\delta\mathbf{D}\,\delta\Sigma - \mathrm{S}\,\delta\mathbf{H}\,(\mathbf{A} + \mathbf{A}_s)\,d\Sigma/4\pi\} \quad . \quad . \quad . \quad (16).$$

Equating now to zero, the coefficient of $\delta\mathbf{H}$ in the extended δL, we get

$$\mathbf{B} = \mathrm{V}\nabla\mathbf{A}, \qquad [\mathrm{V}\,d\Sigma\mathbf{A}]_{a+b} = 0,$$

the $\mathbf{A}_s$ disappearing on account of the relation $[\mathbf{A}_s]_a = [\mathbf{A}_s]_b$. This is the promised proof of equations (19) (20) of § 26, and, therefore, also of equation (18) of the same article.

49. δL and ΣQ δq are, [(9), (15), (16)], now in such a form that the consequences of equation (1), § 13, are seen by inspection. They give (writing D'_m for D_m/m, so that D'_m is the density of matter in the present position)

$$\mathrm{D}'_m\ddot{\rho}' = -\,\mathrm{D}'_m\nabla'\mathrm{W} + (\phi' + \Phi')_1\nabla'_1 + \mathbf{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

$$0 = -\,[(\phi' + \Phi')\,\mathrm{U}\nu']_{a+b} + \mathbf{F}_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

$$\mathrm{e} = {}_{\mathrm{d}}\nabla l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

$$\mathbf{E} = {}_{\mathrm{D}}\nabla l - d_{\mathrm{C}}\nabla l/dt - \dot{\mathbf{A}} - \nabla y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

$$\mathrm{e}_s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

$$\mathbf{E}_s = [y\mathrm{U}\nu]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Let, for a *finite* region

$$\left.\begin{array}{l}\mathrm{P} = \text{rate of doing work of external forces} \\ \qquad + \text{rate of supply of heat from external sources}\end{array}\right\} \quad . \quad . \quad (23),$$

so that P may be called the "power" of the external forces and heat sources. P may be divided into P_f the part due to the frictional forces, *i.e.*, the expression on the left of eq. (38) § 42, and P_e the power of the really external forces and sources, *i.e.*, forces and sources included neither in l nor x. From the equations of motion just obtained, and from eq. (14) § 46 above,

$$\dot{\mathrm{E}} = \mathrm{P} + \iiint \mathrm{S}\mathbf{C}\nabla y d\varsigma - \iint_d y\mathrm{S}\mathbf{C}d\Sigma - \iint_b \{ \mathrm{S}\dot{\rho}'\phi' d\Sigma' + \mathrm{S}d\Sigma\,(\mathrm{V}\dot{\mathbf{A}}\mathbf{H}/4\pi + \dot{\theta}_{\Theta}\nabla\lambda)\}.$$

Now, $\mathrm{S}\mathbf{C}\nabla y = \mathrm{S}\nabla(\mathbf{C}y)$, since $\mathrm{S}\nabla\mathbf{C} = 0$. Hence [eq. (4), § 5]

$$\iiint \mathrm{S}\mathbf{C}\nabla y d\mathrm{S} - \iint_d y\mathrm{S}\mathbf{C}d\Sigma = \iint y\mathrm{S}\mathbf{C}d\Sigma - \iint_d y\mathrm{S}\mathbf{C}d\Sigma = \iint_b y\mathrm{S}\mathbf{C}d\Sigma.$$

Hence

$$\dot{\mathrm{E}} = \mathrm{P} - \iint_b \{\mathrm{S}\dot{\rho}'\phi' d\Sigma' + \mathrm{S}d\Sigma\,(- y\mathbf{C} + \mathrm{V}\dot{\mathbf{A}}\mathbf{H}/4\pi + \dot{\theta}_{\Theta}\nabla\lambda)\} \quad . \quad . \quad (24).$$

Putting now $\mathrm{P} = \mathrm{P}_e + \mathrm{P}_f$ and substituting the expression on the right of eq. (38), § 42 for P_f, we get

$$\dot{\mathrm{E}} = \mathrm{P}_e - \iint_b \{\mathrm{S}\dot{\rho}'(\phi' + \Phi'_f)\, d\Sigma' + \mathrm{S}d\Sigma\,[- (y + \mathrm{Y})\,\mathbf{C} + \mathrm{V}\,(\dot{\mathbf{A}} + \mathbf{a})\,\mathbf{H}/4\pi$$
$$+ (\dot{\theta}_{\Theta}\nabla\lambda + \theta_{\Theta}\nabla x)]\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),$$

where now Φ_f' has been put for the Φ' of eq. (38), § 42, to distinguish it from the Φ' of equations (26), (27) below.

50. In §§ 38, 39, it will be remembered that **E**, **F**, &c., stood for those parts only of the external forces which were *due to* X. Let, now, these symbols stand for those parts only of the external forces which are *not involved in* X. Thus in equation (20) we must change $\mathbf{E}$ into $\mathbf{E} + {}_{\mathrm{C}}\nabla x + \mathbf{a} + \nabla \mathrm{Y}$ [eq. (28), § 38], and similarly for the rest of the equations of motion. We thus get

$$\mathrm{D}_m'\ddot{\rho}' = - \mathrm{D}_m'\nabla'\mathrm{W} + (\phi' + \Phi_f' + \Phi')_1\nabla_1' + \mathbf{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

$$0 = - [(\phi' + \Phi_f' + \Phi')\,\mathrm{U}\nu']_{a+b} + \mathbf{F}_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

$$\mathbf{e} = {}_{\mathrm{d}}\nabla l - {}_{\mathrm{c}}\nabla x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

$$\mathbf{E} = {}_{\mathrm{D}}\nabla l - (d_{\mathrm{C}}\nabla l/dt + {}_{\mathrm{C}}\nabla x) - (d\mathbf{A}/dt + \mathbf{a}) - \nabla\,(y + \mathrm{Y}) \quad . \quad . \quad (29).$$

$$\mathbf{e}_s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

$$\mathbf{E}_s = [(y + \mathrm{Y})\,\mathrm{U}\nu]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

We collect here, partly for reference, partly to show more clearly the actual stage we have now reached, the other chief equations of the field.

$$\phi = -2\mathfrak{A}l \quad \ldots \ldots \quad (32).$$

$$\Phi_f = \quad 2_{\dot{\Psi}}\mathfrak{A}x \quad \ldots \ldots \quad (33).$$

$$\left.\begin{array}{ll} 4\pi_{\mathrm{H}}\nabla l = \mathbf{B} = \mathrm{V}\nabla\mathbf{A}, & [\mathrm{VU}\nu\mathbf{A}]_{a+b} = 0 \\ \mathrm{S}\nabla\mathbf{B} = 0, & [\mathrm{SU}\nu\mathbf{B}]_{a+b} = 0 \end{array}\right\} \quad \ldots \ldots \quad (34).$$

$$\left.\begin{array}{ll} 4\pi_{\mathrm{H}}\nabla x = \mathrm{b} = \mathrm{V}\nabla\mathrm{a}, & [\mathrm{VU}\nu\mathrm{a}]_{a+b} = 0 \\ \mathrm{S}\nabla\mathrm{b} = 0, & [\mathrm{SU}\nu\mathrm{b}]_{a+b} = 0 \end{array}\right\} \quad \ldots \ldots \quad (35).$$

$$\left.\begin{array}{ll} 4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}, & [\mathrm{VU}\nu\mathbf{H}]_{a+b} = 0 \\ \mathrm{S}\nabla\mathbf{C} = 0, & [\mathrm{SU}\nu\mathbf{C}]_{a+b} = 0 \\ \mathrm{S}\nabla\mathbf{D} = 0, & [\mathrm{SU}\nu\mathbf{D}]_{a+b} = 0 \end{array}\right\} \quad \ldots \ldots \quad (36).$$

$$\left.\begin{array}{l} \mathbf{D} = \mathrm{d} + \mathrm{k}, \quad \mathbf{C} = \mathrm{c} + \mathbf{K} \\ \mathrm{c} = \dot{\mathrm{d}}, \quad \mathbf{K} = \dot{\mathrm{k}}, \quad \mathbf{C} = \dot{\mathbf{D}} \end{array}\right\} \quad \ldots \ldots \quad (37).$$

$$\left.\begin{array}{l} \mathrm{c}' = \dot{\mathrm{d}}' + \mathrm{V}\nabla'_1\mathrm{V}\mathrm{d}'\dot{\rho}'_1 = \partial\mathrm{d}'/\partial t + \mathrm{V}\nabla'\mathrm{V}\mathrm{d}'\dot{\rho}' - \dot{\rho}'\mathrm{S}\nabla'\mathrm{d}' \\ \mathbf{C}' = \dot{\mathbf{D}}' + \mathrm{V}\nabla'_1\mathrm{V}\mathbf{D}'\dot{\rho}'_1 = \partial\mathbf{D}'/\partial t + \mathrm{V}\nabla'\mathrm{V}\mathbf{D}'\dot{\rho}' \end{array}\right\} \quad \ldots \quad (38).$$

[The last set has not yet been proved, as it is more convenient to discuss it along with the detailed results, though clearly itself a general result.] Roughly speaking, of these equations [(25) to (38)], it may be said that (36) and (37) contain the assumptions of the present theory, and the rest the consequences of those assumptions.

Two remarks may be made here. It is clear that, since in the equations (25) to (38), y and Y occur only under the form $y + \mathrm{Y}$, there is nothing by means of which we could experimentally distinguish them. Putting, then,

$$y + \mathrm{Y} = v \quad \ldots \ldots \quad (39),$$

we shall generally in the future speak only of v. It may be conveniently called the potential, though, as we shall see later, this is not in accordance with MAXWELL'S usage of the term; and, what is perhaps of more importance, there is something arbitrary about it apart from the arbitrary additive constant which every potential involves.

The second thing to notice is that the **E** of equation (29) is not what is usually known as the electromotive force. The physical fact that is usually stated by saying that $\mathbf{E} = \mathrm{RK}$, must with the present notation be stated by saying that $\mathrm{E} = 0$, since $-\mathrm{RK}$ appears on the right of equation (29) as a part of the term $-{}_{\mathrm{C}}\nabla x$. This, of course, is due to the fact that **E** of equation (29) is physically defined as the part of the electromotive force not depending on friction.

D. *Change of Variables in l, λ, and x.*

51. In what follows with reference to change of variables, we shall always speak as if the change had reference only to l. Exactly similar reasoning applies to similar changes of variables in any other function such as λ, x, or a part only of any one of these. There is, indeed, no reason why the function should be a scalar.

So far, l has been assumed an explicit function of the list of variables (25), § 27. These are by far the most convenient variables for most mathematical operations, and we shall continue as often as otherwise so to regard l. For many physical interpretations, however, it is necessary to regard l, or a part of it, expressed in terms of other variables. Consider, for instance, air as a dielectric. This will be taken account of by supposing l to contain a term quadratic in d. Suppose, now, we compress the air till its density is (say) doubled. We know as a matter of experimental fact, that the specific inductive capacity will not thereby be largely altered. This will mean, *not* that the quadratic expression in d is but slightly altered in form, *but* that the equal expression in d' is thus slightly altered. Moreover, to express simply the fact of electric and magnetic isotropy of fluids requires that the independent variables should be the dashed letters. Let then

$$l d\varsigma = l' d\varsigma' = l'' d\varsigma' \text{ or } l = ml' = ml'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where

$$\left.\begin{array}{llllllll} l \text{ is an explicit function of } & \theta, \Theta & ; \rho', \dot{\rho}', \Psi; & \mathrm{d}, & \mathbf{D}, & \mathbf{C}, & \mathbf{H} & \\ l' \qquad ,, \qquad\qquad ,, & \theta, \Theta' & ; \rho', \dot{\rho}', \Psi; & \mathrm{d}', & \mathbf{D}', & \mathbf{C}', & \mathbf{H}'; & q \\ l'' \qquad ,, \qquad\qquad ,, & \theta, \Theta'' & ; \rho', \dot{\rho}', \Psi; & \mathrm{d}'', & \mathbf{D}'', & \mathbf{C}'', & \mathbf{H}'' & \end{array}\right\} \quad . \quad (2).$$

Defining λ', λ'' similarly, it may be said here what will appear incidentally later, that λ' and l', and again, λ'' and l'', are related to one another exactly as are λ and l; *i.e.* [§ 46, eq. (11)]

$$\left.\begin{array}{l} l' + \lambda' = -\mathrm{D}'_{m}\dot{\rho}'^{2} - \mathrm{S}\mathbf{C}'\,{}_{\mathrm{C}}\nabla' l' - \mathrm{S}\mathbf{H}'\,{}_{\mathrm{H}}\nabla' l' \\ l'' + \lambda'' = -\mathrm{D}'_{m}\dot{\rho}'^{2} - \mathrm{S}\mathbf{C}''\,{}_{\mathrm{C}}\nabla'' l'' - \mathrm{S}\mathbf{H}''\,{}_{\mathrm{H}}\nabla'' l'' \end{array}\right\} \quad . \quad . \quad . \quad . \quad (3),$$

where ${}_{\mathrm{C}}\nabla'$ is put for ${}_{\mathrm{C}'}\nabla$, &c.

52. In (2) it is to be noticed that one more variable, viz., q, occurs in l' than in l or l''. The reason is obvious, but on account of the fact, it is easiest to arrive at formulæ transforming differentiations of l into the corresponding ones of l' by first considering the similar relations between l and l''.

Let σ and τ be taken as a typical independent variable intensity and flux respectively. l' is obtained from l'' merely by changing every σ'' and τ'' into $q^{-1}\sigma' q$ and $q^{-1}\tau' q$ respectively. (§ 7.)

By considering the increment in l' and l'' due to an increment in a σ'' or τ'', we at once obtain

$$ {}_\sigma\nabla' l' = q_\sigma\nabla'' l'' q^{-1}, \quad {}_\tau\nabla' l' = q_\tau\nabla'' l'' q^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4). $$

By a similar process it is easy to see that

$$ \left.\begin{aligned} \frac{\partial l}{\partial\theta}\,d\varsigma = \frac{\partial l'}{\partial\theta}\,d\varsigma' &= \frac{\partial l''}{\partial\theta}\,d\varsigma' \\ {}^{;}_{\rho'}\nabla l d\varsigma = {}^{;}_{\rho'}\nabla l' d\varsigma' &= {}^{;}_{\rho'}\nabla l'' d\varsigma' \\ \nabla' l = \nabla'(ml') &= \nabla'(ml'') \\ \text{Ɑ} l' &= \text{Ɑ} l'' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5). $$

53. We proceed to find the corresponding relations for the other variables. Let us in l and l'' vary Ψ and every σ and τ, and σ'' and τ''. Thus,

$$ -\Sigma S\delta\sigma_\sigma\nabla l - \Sigma S\delta\tau_\tau\nabla l - S\delta\Psi\zeta\text{Ɑ} l\zeta = \delta l = l''\delta m + m\delta l'' $$
$$ = l''\delta m + m\{-\Sigma S\delta\sigma''{}_\sigma\nabla'' l'' - \Sigma S\delta\tau''{}_\tau\nabla'' l'' - S\delta\Psi\zeta\text{Ɑ} l''\zeta\}. $$

Now (§ 7)

$$ \tau'' = m^{-1}\psi\tau, \quad \sigma'' = \psi^{-1}\sigma. $$

Hence

$$ \delta\tau'' = m^{-1}(\delta\psi - m^{-1}\delta m . \psi)\tau + m^{-1}\psi\delta\tau. $$
$$ \delta\sigma'' = -\psi^{-1}\delta\psi\psi^{-1}\sigma + \psi^{-1}\delta\sigma. $$

Substituting these values and equating the vector coefficients of the arbitrary vectors $\delta\sigma$ and $\delta\tau$, we obtain

$$ {}_\sigma\nabla l = m\psi^{-1}{}_\sigma\nabla'' l'' = m\chi^{-1}{}_\sigma\nabla' l' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6), $$

$$ {}_\tau\nabla l = \psi_\tau\nabla'' l'' = \chi'{}_\tau\nabla' l' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7), $$

the last result in each of these being given by equation (4). These equations show that ${}_\sigma\nabla l$, ${}_\tau\nabla' l'$, ${}_\tau\nabla'' l''$ bear to one another exactly the same relations as τ, τ', τ'', which may be expressed by saying that they are fluxes.* Similarly, ${}_{\tau}\nabla l$ is an intensity. This particular result can, of course, be proved by a simpler process than the above. We now see that the meaning of $\mathbf{B}'$, obtained by defining $\mathbf{B}$ as a flux, $= 4\pi_{\mathrm{H}}\nabla l$, and likewise the meaning of $\mathfrak{b}'$ is independent of the particular position of matter we take as the standard. We also see similarly that the various terms in $\mathbf{E}'$, $\mathfrak{e}'$, resulting from regarding these vectors as intensities, and utilising equations (28), (29), § 50, will be independent of the particular standard position chosen. And again, by Prop. II., § 8, we now see that equations (3) of last section must be true.

54. Putting now $\delta\sigma = 0$, $\delta\tau = 0$, the equation $\delta l = l''\,\delta m + m\,\delta l''$ gives

$$- \mathrm{S}\,\delta\Psi\zeta\text{Ɑ}l\zeta = - m\mathrm{S}\,\delta\Psi\zeta\text{Ɑ}l''\zeta + (l'' + m^{-1}\Sigma\mathrm{S}\psi\tau_{\tau}\nabla''l'')\,\delta m$$
$$+ (m\Sigma\mathrm{S}\psi^{-1}\,\delta\psi\psi^{-1}\sigma_{\sigma}\nabla''l'' - \Sigma\mathrm{S}\,\delta\psi\tau_{\tau}\nabla''l'').$$

Now, by former paper, eq. (18),

$$6m = \mathrm{S}\zeta_1\zeta_2\zeta_3\mathrm{S}\psi\zeta_1\psi\zeta_2\psi\zeta_3.$$

Hence

$$2\,\delta m = \mathrm{S}\,\delta\psi\zeta\mathrm{V}\psi\zeta_1\psi\zeta_2\mathrm{S}\zeta\zeta_1\zeta_2,$$

or, by eq. (10) of former paper,

$$\delta m = - m\mathrm{S}\,\delta\psi\zeta\psi^{-1}\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Similarly, since m^2 is the same function of Ψ as m is of ψ,

$$\delta m = - \frac{m}{2}\mathrm{S}\,\delta\Psi\zeta\Psi^{-1}\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Also, for future use, note that since $\delta m = - \mathrm{S}\,\delta\psi\zeta_{\psi}\text{Ɑ}m\zeta = - \mathrm{S}\,\delta\Psi\zeta\text{Ɑ}m\zeta$ these equations give (former paper, p. 105),

$${}_{\psi}\text{Ɑ}m = m\psi^{-1}, \quad \text{Ɑ}m = \tfrac{1}{2}m\Psi^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

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* It is interesting to notice a particular result of this. Since Θ is an intensity, ${}_{\Theta}\nabla l$ is a flux. Hence [Prop. IV., § 8] $\mathrm{S}\nabla_{\Theta}\nabla l = m\mathrm{S}\nabla'_{\Theta}\nabla'l'$. Dismissing the particular notation of this paper for the moment, and putting x, y, z for the coordinates of ρ and λ, μ, ν for those of ρ', this may be written

$$m^{-1}\left\{\frac{\partial}{\partial x}\left(\frac{\partial l}{\partial\frac{\partial\theta}{\partial x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial l}{\partial\frac{\partial\theta}{\partial y}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial l}{\partial\frac{\partial\theta}{\partial z}}\right)\right\} = \frac{\partial}{\partial\lambda}\left(\frac{\partial l'}{\partial\frac{\partial\theta}{\partial\lambda}}\right) + \frac{\partial}{\partial\mu}\left(\frac{\partial l'}{\partial\frac{\partial\theta}{\partial\mu}}\right) + \frac{\partial}{\partial\nu}\left(\frac{\partial l'}{\partial\frac{\partial\theta}{\partial\nu}}\right).$$

If we add [equation (5), § 52, above] $- m^{-1}\partial l/\partial\theta$ to the left of this equation and $- \partial l'/\partial\theta$ to the right, we get a well-known theorem of JACOBI'S. Comparing with the form of this theorem given in TODHUNTER'S 'History of the Calculus of Variations,' § 323, equation (2), his G, Γ, v, ϕ, Π are our l, ml', θ (regarded as a function of ρ), θ (regarded as a function of ρ'), and m^{-1} respectively. See also TODHUNTER'S 'Functions of LAPLACE, LAMÉ, and BESSEL,' § 298, equation (17), and the supplementary volume of BOOLE'S 'Differential Equations,' p. 216.

In the equation for δl put $\delta\psi\psi^{-1}\sigma = -\delta\psi\zeta S\zeta\psi^{-1}\sigma$ and $\delta\psi_{\tau}\nabla'' l'' = -\delta\psi\zeta S\zeta_{\tau}\nabla'' l''$; for δm substitute from eq. (9); and for l'', ${}_{\tau}\nabla'' l''$ and ${}_{\sigma}\nabla'' l''$, substitute in terms of l, ${}_{\tau}\nabla l$ and ${}_{\sigma}\nabla l$. Thus

$$\begin{aligned} -S\,\delta\Psi\zeta\complement l\zeta = & -S\,\delta\Psi\zeta\{m\complement l''\zeta + \tfrac{1}{2}(l + \Sigma S\tau_{\tau}\nabla l)\,\Psi^{-1}\zeta\}, \\ & -S\,\delta\psi\zeta\{\Sigma_{\sigma}\nabla l S\zeta\psi^{-1}\sigma - \Sigma\tau S\zeta\psi^{-1}{}_{\tau}\nabla l\}. \end{aligned}$$

Now, let Ω, υ be two functions of Class I. of § 9, the first given by

$$\Omega = -2\complement l + 2m\complement l'' + (l + \Sigma S\tau_{\tau}\nabla l)\,\Psi^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

from which [eq. (9) § 9, eq. (32) § 50, and Prop. II. § 8]

$$\Omega' = \phi' + 2\chi\complement l''\chi' + l' + \Sigma S\tau'_{\tau}\nabla' l' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Let υ be given by

$$\upsilon\omega = \Sigma\,(\tau S_{\tau}\nabla l\psi^{-2}\omega - {}_{\sigma}\nabla l S\sigma\psi^{-2}\omega) \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

from which are easily deduced

$$\upsilon'\omega = \Sigma\,(\tau' S_{\tau}\nabla' l'\omega - {}_{\sigma}\nabla' l' S\sigma'\omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

$$\upsilon''\omega = q^{-1}\upsilon'(q\omega q^{-1})\,q = \Sigma\,(\tau'' S_{\tau}\nabla'' l''\omega - {}_{\sigma}\nabla'' l'' S\sigma''\omega) \, . \quad . \quad . \quad . \quad (15).$$

From the last value for δl we now have

$$S\,\delta\Psi\zeta\Omega\zeta = 2S\,\delta\psi\zeta\upsilon\psi\zeta,$$

or

$$S\,\delta\psi\zeta\Omega\psi\zeta = S\,\delta\psi\zeta\upsilon\psi\zeta.$$

Hence (former paper, p. 105) the pure part of $\Omega\psi =$ ditto $\upsilon\psi$, *i.e.*,

$$\Omega\psi = \upsilon\psi + mV\eta\;(\;),$$

where η is a vector to be determined. Hence

$$\Omega\omega = \upsilon\omega + mV\eta\psi^{-1}\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Therefore

$$\Omega'\omega = \upsilon'\omega + \chi V\eta q^{-1}\omega q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

and

$$\Omega''\omega = \upsilon''\omega + \psi V\eta\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

From the last equation and the fact that $V\zeta\Omega''\zeta = 0$, it is easy to deduce that

$$\eta = (\psi + S\zeta\psi\zeta)^{-1}\,\Sigma V\,(\sigma''{}_{\sigma}\nabla'' l'' + \tau''{}_{\tau}\nabla'' l'') \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

and by taking the pure part of both sides of eq. (17) we get

$$2\Omega'\omega = \Sigma \mathrm{V}(\tau'\omega_\tau \nabla' l' - \sigma'\omega_\sigma \nabla' l') + \omega \Sigma \mathrm{S}(\tau'_\tau \nabla' l' - \sigma'_\sigma \nabla' l') + \chi \mathrm{V} \eta q^{-1} \omega q + q \mathrm{V} \chi' \omega \eta \cdot q^{-1},$$

whence, putting

$$[\phi'] = -2\chi \text{Ɑ} l'' \chi' = -2\chi \text{Ɑ} l' \chi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

we have from eq. (12)

$$\left.\begin{aligned} 2\phi' = 2[\phi'] - \{2l' + \Sigma \mathrm{S}(\tau'_\tau \nabla' l' + \sigma'_\sigma \nabla' l')\} + \Sigma \mathrm{V}\{\tau'(\quad)_\tau \nabla' l' - \sigma'(\quad)_\sigma \nabla' l'\} \\ + \{\chi \mathrm{V} \eta q^{-1}(\quad) q + q \mathrm{V} \chi'(\quad) \eta \cdot q^{-1}\} \end{aligned}\right\} (21).$$

Note that the terms here depending on η may be put in the form

$$\chi \mathrm{V} \eta q^{-1}(\quad) q + q \mathrm{V} \chi'(\quad) \eta \cdot q^{-1} = q \varpi q^{-1}(\quad) q \cdot q^{-1} \quad . \quad . \quad . \quad (22),$$

where ϖ is the self-conjugate linear vector function given by

$$\varpi = \psi \mathrm{V} \eta(\quad) + \mathrm{V} \psi(\quad) \eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

For purposes of physical interpretation it is often legitimate to assume the present and standard positions to coincide. In this case $q = 1$, $\chi = \chi' = 1$, so that

$$2\phi' = 2[\phi'] - \{2l + \Sigma \mathrm{S}(\tau_\tau \nabla l + \sigma_\sigma \nabla l)\} + \Sigma \mathrm{V}\{\tau(\quad)_\tau \nabla l - \sigma(\quad)_\sigma \nabla l\} \quad (24),$$

and if, further, l is a homogeneous quadratic function of the σ's and τ's,

$$2\phi' = 2[\phi'] + \Sigma \mathrm{V}\{\tau(\quad)_\tau \nabla l - \sigma(\quad)_\sigma \nabla l\} \quad . \quad . \quad . \quad . \quad . \quad (25).$$

55. For future use we will make two deductions from these results. First suppose that

$$l = l_0 + m(2\pi \mathrm{K}_0^{-1} \mathrm{d}'^2 - \mu_0 \mathbf{H}'^2/8\pi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

where K_0, μ_0 are absolute constant scalars—the specific inductive capacity and magnetic permeability of a vacuum, and where l_0 is expressed in terms of the undashed letters. Thus it is only in the part of l independent of l_0 that the change of variables is made. In this part there is one τ', viz., d', and one σ', viz., $\mathbf{H}'$; and $\eta = 0$. Hence

$$\phi'\omega = -2m^{-1}\chi \text{Ɑ} l_0 \chi' \omega - 2\pi \mathrm{K}_0^{-1} \mathrm{d}'\omega \mathrm{d}' - \mu_0 \mathbf{H}'\omega \mathbf{H}'/8\pi \quad . \quad . \quad . \quad (27).$$

Next let l_0' be what l_0 becomes when expressed in terms of the dashed letters.

Note that l_0' does not stand towards l_0 in the same way as l' towards l, as appears by the equations

$$ml' = l, \qquad l_0' = l_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

In utilising equation (21), then, the analogue of l' will be $m^{-1}l'_0$. Thus the part contributed by l'_0 to the terms $[\phi'] - l'$ on the right of equation (21) will be

$$\begin{aligned} -2\chi\text{Ɑ}(m^{-1}l_0')\chi' - m^{-1}l_0' &= -2m^{-1}\chi\text{Ɑ}l_0'\chi' + m^{-1}l_0'\chi\Psi^{-1}\chi' - m^{-1}l_0', \text{ [equation (10)]}, \\ &= -2m^{-1}\chi\text{Ɑ}l_0'\chi', \end{aligned}$$

since [former paper, equation (39)] $\Psi^{-1} = \chi^{-1}\chi'^{-1}$. Assuming, then, ϕ_0 to be of the first class of § 9, and defined by

$$\phi_0 = -2\text{Ɑ}l_0' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29),$$

equation (21) gives

$$\begin{aligned} 2\phi' = 2\phi'_0 &+ (\mu_0\mathbf{H}'^2/4\pi - 4\pi\mathrm{K}_0{}^{-1}\mathbf{d}'^2) \\ &- \Sigma\mathrm{S}(\tau'_\tau\nabla'l' + \sigma'_\sigma\nabla'l') + \Sigma\mathrm{V}\{\tau'(\quad)_\tau\nabla'l' - \sigma'(\quad)_\sigma\nabla'l'\} \\ &+ \{\chi\mathrm{V}\eta q^{-1}(\quad)q + q\mathrm{V}\chi'(\quad)\eta\,.\,q^{-1}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30), \end{aligned}$$

where l', σ', τ', and η have exactly the same meanings as before, so that, indeed,

$$l' \equiv l'_0/m + 2\pi\mathrm{K}_0{}^{-1}\mathbf{d}'^2 - \mu_0\mathbf{H}'^2/8\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

E. *Connection between* **E** *and* **e**.

56. So far it has been assumed that there are two independent kinds of external force denoted by **E** and **e**, and by $\mathbf{E}_s$ and $\mathbf{e}_s$. This is contrary to the usual custom, but seems to me to be a necessary consequence of assumptions always made as to the difference in nature between what is ordinarily called the displacement current and the conduction current.

The independent variables required to fix the electric state at a point have for mathematical convenience been taken as **D** and **d**. These are, perhaps, not the most natural. It would seem from the ordinary views as to the two kinds of current as if the dielectric displacement **d**, and the conduction displacement **k** are the most natural. Moreover, I believe it is generally held that **d** has exclusively to do with the potential energy of electrification. It seems, then, likely to lead to correct results to assume that if **d** and **k** were taken as the independent coordinates, there would never be any external force of type **d**.

As this conclusion may seem open to question let us put the matter in a different way. If (regarding **d** and **k** as the independent electric coordinates) we could be certain that we had found the full expressions for l, l_s, x, x_s, *both* types of external electromotive force would be zero. But we can with considerable certainty say that we have not found these completely, so far as they depend upon **k** and **K** (electrolysis, &c.). On the other hand, it is by no means so obvious that we have not found them completely so far as they depend on **d** and **c**. Let us then assume that the external force (exclusive of frictional forces, of course) of the latter type is zero. If we can point to no experimental facts contradicted by this assumption, we may consider that the simplification is warranted.

57. Now (§ 28) the work done per unit volume by the external forces **E**, **e** of equations (28), (29), § 50 above, while **D** and **d** suffer the increments $d\mathbf{D}$ and $d\mathbf{d}$ respectively, is

$$\begin{aligned} \mathrm{S}\mathbf{E}\,d\mathbf{D} + \mathrm{S}\mathbf{e}\,d\mathbf{d} &= \mathrm{S}\mathbf{E}\,(d\mathbf{d} + d\mathbf{k}) + \mathrm{S}\mathbf{e}\,d\mathbf{d} \\ &= \mathrm{S}\,(\mathbf{E} + \mathbf{e})\,d\mathbf{d} + \mathrm{S}\mathbf{E}d\mathbf{k}. \end{aligned}$$

Hence, if **d** and **k** be taken as the coordinates, the forces of those types would be $\mathbf{E} + \mathbf{e}$ and **E** respectively. The assumption just made then leads to

$$\mathbf{E} + \mathbf{e} = 0, \quad \mathbf{E}_s + \mathbf{e}_s = 0 \quad \ldots\ldots\ldots \quad (1),$$

where it must be remembered that the exact meaning of these four symbols is that given to them in § 50 above, not the meaning they had previous to that section. If we assumed that x was independent of **H**, equation (1) would be equally true of the previous meanings of the symbols.

We shall now always suppose **e** to be replaced by $-\mathbf{E}$. With regard to $\mathbf{e}_s$ and $\mathbf{E}_s$, note that by means of equation (1) and equations (30), (31), and (39) of § 50

$$[v\mathrm{U}\nu]_{a+b} = 0 \quad \ldots\ldots\ldots\ldots \quad (2),$$

which shows that what we have called the potential is continuous throughout space. This will be found to lead to the result that contact-force cannot be explained without a slight extension of the independent variables of l, or the assumption that l_s is not zero. It does not, however, prevent on present assumptions an explanation of the PELTIER effect.

IV. DETAILED EXAMINATION OF THESE RESULTS.

A. *Maxwell's Results.*

58. The justification of the present theory, where it differs from accepted theory, must be based on an examination of its results in detail. First, then, let us compare

with MAXWELL'S results. With the exception of (1) the expression for current in terms of displacement for a moving body, and (2) certain of his mechanical results which I hold to be inconsistent with certain others of his own, it will be found that his results flow from the equations now established.

We put down, then, simple forms of l and x, the first involving as independent variables Ψ, $\mathbf{d}$ and $\mathbf{H}$, and the second Ψ and $\mathbf{K}$ only, and compare the results with MAXWELL'S. Besides MAXWELL'S results we shall find that this form of l is sufficient to take account of the interdependence of magnetisation and strain, and of specific inductive capacity and strain. After that we add certain terms to, and otherwise generalise l and x, still, however, regarding them as involving no independent variables except such as occur in the lists (25), (26), of § 27. Thermoelectric, thermomagnetic, and the HALL phenomena are thereby accounted for and discussed in detail. Finally, to account for electrostatic contact-force (and incidentally capillary phenomena), we shall assume l to contain certain independent variables not in the list (25) of § 27, and shall adopt a certain form for l_s.

59. For MAXWELL'S results it is only necessary to assume

$$l = 2\pi \mathrm{S}\,\mathbf{d}\mathrm{K}^{-1}\mathbf{d} - \mathrm{S}\mathbf{H}\mu\mathbf{H}/8\pi - \mathrm{S}\mathbf{I}_0\mathbf{H} \quad \text{. . . . . . . (1)},$$

$$x = -\mathrm{S}\mathbf{K}\mathrm{R}\mathbf{K}/2 \quad \text{. . . . . (2)},$$

where $\mathbf{I}_0$ is a flux, μ and K* are self-conjugate functions of Class I. of § 9, and R is one of Class II, all four being functions of strain and temperature. From these statements, and § 9, it follows that

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* I did not notice when first [former paper, p. 119] using K in this signification that it already had a special quaternion meaning (conjugate of a quaternion). As this meaning is never required in the present paper, and very rarely in physical applications, I have nevertheless retained the present meaning for K.

I take this opportunity of apologising for the apparent want of system in my notation. It has been brought about by an attempt to compromise between accepted notation and a system of notation more suitable for quaternion methods. May I suggest the following system? First, let the Greek alphabet be left as a happy hunting ground for symbols of every denomination (vectors, scalars, linear vector functions, &c.); secondly, let the ordinary alphabets, A, B . . ., a, b . . ., be used for scalars and linear vector functions of a vector (which so often in important cases reduce to scalars) only; thirdly, let bold type be used for vectors only and write $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ instead of i, j, k; fourthly, let HAMILTON'S K, S, T, U, V be transferred to the German alphabet; fifthly, let the rest of the two German alphabets be retained for mathematicians who are hard pressed for suitable symbols; sixthly, let the symbols of differentiation be quite independent of the above restrictions. The following somewhat chaotic but classified list of some of the chief symbols used in the present paper may serve to convince the sceptic that *some such* system is necessary. (1.) *Linear vector functions of a vector* (20), A B, C K, R, a, b, c, r, Υ, Φ, Ψ, Ω, γ, μ, ϖ, υ, ϕ, χ, ψ. (2.) *Vectors* (31), i, j, k, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, $\mathbf{H}$, $\mathbf{I}$, $\mathbf{K}$, $\mathbf{L}$, $\mathbf{N}$, $\mathbf{P}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{h}$, Θ, $d\Sigma$, α, ϵ, η, $\mathrm{U}\nu$, ρ, σ, τ, ω. (3.) *Scalars* (32), D, E, F, H, P, Q, W, X, Y, Z f, g, h, l, m, n, q, s, t, u, v, x, y, z, θ, λ, ξ, $\mathfrak{D}$, $\mathfrak{T}$, $\mathfrak{B}$, t. (4.) *Symbols of differentiation and variation*, Ɑ, d, ∂, Δ, ∇, δ, ζ. (5.) *Symbols of peculiar quaternion meaning*, S, T, U, V.

$$\left.\begin{aligned} l' &= 2\pi S\mathrm{d}'\mathrm{K}'^{-1}\mathrm{d}' - S\mathbf{H}'\mu'\mathbf{H}'/8\pi - S\mathbf{I}'_0\mathbf{H}' \\ l'' &= 2\pi S\mathrm{d}''\mathrm{K}''^{-1}\mathrm{d}'' - S\mathbf{H}''\mu''\mathbf{H}''/8\pi - S\mathbf{I}''_0\mathbf{H}'' \end{aligned}\right\} \dots\dots \quad (3),$$

$$x' = -S\mathrm{K}'\mathrm{R}'\mathrm{K}'/2, \quad x'' = -S\mathrm{K}''\mathrm{R}''\mathrm{K}''/2 \dots\dots \quad (4).$$

60. Most of MAXWELL's results are collected together in § 619 of his 'Electricity and Magnetism,' 2nd edition. In our notation they are

$$\mathbf{B}' = V\nabla'\mathbf{A}' \dots\dots \quad (A)\ (5),$$

$$\mathbf{E}'_0 = V\dot{\rho}'\mathbf{B}' - \partial\mathbf{A}'/\partial t - \nabla' z \dots\dots \quad (B)\ (6),$$

where $\partial/\partial t$ denotes differentiation with regard to time at a fixed point of space, and where z is some scalar put for MAXWELL's Ψ. Equation (C) we omit for the present, as it requires more detailed discussion than the others.

$$\mathbf{B}' = \mathbf{H}' + 4\pi\mathbf{I}' \dots\dots \quad (D)\ (7).$$

$$4\pi\mathbf{C}' = V\nabla'\mathbf{H}' \dots\dots \quad (E)\ (8).$$

$$\mathrm{d}' = \mathrm{K}'\mathbf{E}'_0/4\pi \dots\dots \quad (F)\ (9).$$

$$\mathbf{K}' = \mathrm{R}'^{-1}\mathbf{E}'_0 \dots\dots \quad (G)\ (10).$$

Equation (H) we also omit as in this, the present theory certainly gives a result different from MAXWELL's.

$$\mathbf{B}' = \mu'\mathbf{H}' \dots\dots \quad (L)\ (11),$$

"when the magnetisation arises from the magnetic induction," MAXWELL adds. The equations omitted are

$$\text{Mechanical force due to field} = V\mathbf{C}'\mathbf{B}' - D'\nabla' z - n'\nabla'\Omega \quad . \quad (C)\ (12).$$

$$\mathbf{C}' = \mathbf{K}' + \partial\mathrm{d}'/\partial t \dots\dots \quad (H)\ (13).$$

$$D' = -S\nabla'\mathrm{d}'^{*} \dots\dots \quad (J)\ (14).$$

$$n' = S\nabla'\mathbf{I}' \dots\dots \quad (15).$$

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[*] The omission of the *minus* sign in MAXWELL's equation $e = S\nabla\mathfrak{D}$ is obviously a misprint. [See equation (J) § 612.]

In these, D', n' have been substituted for MAXWELL's e, m, as the latter symbols already have, in the present paper, a different meaning. "When the magnetic force can be derived from a potential"

$$\mathbf{H}' = -\nabla'\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

[There is no risk of this scalar Ω being confused with the Ω of §§ 9, 10, 54 of the present paper.] In addition to these, he gives in § 613 the surface equation corresponding to equation (14), viz.,

$$D_s' = [SU\nu'\mathbf{d}']_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (K)\ (17),$$

where D_s' has been put for his σ.

61. Equation (5) is the same as equation (22), § 26, above, (7) as (23) § 26, (8) as (14) § 25. We can now show that equations (6), (9), (10), (11) all follow if we assume that there is no external force other than that due to friction.

By the last paragraph of § 50 above, we see that what MAXWELL calls $\mathbf{E}$ is not likely to be what on the present theory we call $\mathbf{E}'$. To compare with ordinary theories, then, it is convenient to introduce a new intensity $\mathbf{E}_0$ defined by

$$\mathbf{E}_0 = R\mathbf{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Since $\mathbf{E}_0$ and $R\mathbf{K}$ (§ 10, Prop. VI, above) are both intensities, equation (10) follows. To prove (9), note that

$$_{\mathbf{c}}\nabla x = -{}_{\mathbf{K}}\nabla x = -R\mathbf{K} = -\mathbf{E}_0$$
$$_{\mathbf{d}}\nabla l = -4\pi K^{-1}\mathbf{d},$$

so that putting $\mathbf{e}$ of equation (28), § 50, equal to zero,

$$\mathbf{d} = K\mathbf{E}_0/4\pi,$$

from which equation (9) follows by Prop. VI, § 10. Again,

$$\mathbf{B} = 4\pi_{\mathbf{H}}\nabla l = \mu\mathbf{H} + 4\pi\mathbf{I}_0$$

and therefore

$$\mathbf{B}' = \mu'\mathbf{H}' + 4\pi\mathbf{I}_0' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

which implies that the part of $\mathbf{B}'$ "induced" by magnetic force, is $\mu'\mathbf{H}'$. This is equation (11).

62. To prove equation (6), note first that putting $\mathbf{E} = 0$, equation (29) of § 50 [modified by equation (39) of § 50] gives

$$\mathbf{E}_0 = -\dot{\mathbf{A}} - \nabla\upsilon \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Next note that by d/dt, or a dot, is denoted differentiation with regard to time, which follows the motion of matter, and by $\partial/\partial t$, a differentiation at a fixed point of space, so that d/dt is commutative with ∇, but not with ∇', and $\partial/\partial t$ with ∇', but not with ∇. Hence, as is well known,

$$d/dt = -\mathrm{S}\dot{\rho}'\nabla' . + \partial/\partial t \quad . \; . \; . \; . \; . \; . \; . \; . \quad (21).$$

Now, by equation (20),

$$\begin{aligned} \mathbf{E}'_0 &= \chi'^{-1}\mathbf{E}_0 = -\chi'^{-1}d(\chi'\mathbf{A}')/dt - \nabla' v \\ &= -\dot{\mathbf{A}}' - \chi'^{-1}\dot{\chi}'\mathbf{A}' - \nabla' v. \end{aligned}$$

Now [former paper, equation (25)],

$$\chi'\omega = -\nabla_1 \mathrm{S}\omega\rho'_1,$$

so that

$$-\chi'^{-1}\dot{\chi}'\mathbf{A}' = \chi'^{-1}\nabla_1\mathrm{S}\mathbf{A}'\dot{\rho}'_1 = \nabla'_1\mathrm{S}\mathbf{A}'\dot{\rho}'_1,$$

and, by equation (21),

$$-\dot{\mathbf{A}}' = -\partial\mathbf{A}'/\partial t + \mathrm{S}\dot{\rho}'\nabla' . \mathbf{A}',$$

therefore

$$\begin{aligned} \mathbf{E}'_0 &= -\partial\mathbf{A}'/\partial t + \mathrm{S}\dot{\rho}'\nabla' . \mathbf{A}' + \nabla'_1\mathrm{S}\mathbf{A}'\dot{\rho}'_1 - \nabla' v \\ &= -\partial\mathbf{A}'/\partial t + \mathrm{S}\dot{\rho}'\nabla' . \mathbf{A}' - \nabla'_1\mathrm{S}\mathbf{A}'_1\dot{\rho}' - \nabla'(v - \mathrm{S}\mathbf{A}'\dot{\rho}') \\ &= -\partial\mathbf{A}'/\partial t + \mathrm{V}\dot{\rho}'\mathrm{V}\nabla'\mathbf{A}' - \nabla' z \\ &= -\partial\mathbf{A}'/\partial t + \mathrm{V}\dot{\rho}'\mathbf{B}' - \nabla z, \end{aligned}$$

where

$$z = v - \mathrm{S}\mathbf{A}'\rho' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (22).$$

This proves equation (6). Of course, this more complicated form of equation (20) is necessary for some purposes, but the simpler form is more useful in discussing the general theory. From the simpler form, indeed, we may see at once that MAXWELL'S result must follow, since it implies the truth of the principle from which he deduces his result. That principle is ('Elect. and Mag.,' 2nd ed., § 598) that the line integral of $\mathbf{E}'_0$ round any closed curve moving with matter equals the rate of decrease of the line integral of $\mathbf{A}'$ round the same curve. Since both $\mathbf{E}_0$ and $\mathbf{A}$ are intensities, this may in our notation be expressed by saying that the line integral of $\mathbf{E}_0$ round the corresponding *fixed* curve equals the rate of decrease of the line integral of $\mathbf{A}$ round the fixed curve. This last is clearly insured by the equation $\mathbf{E}_0 = -\dot{\mathbf{A}} - \nabla v$.

Thus in the results contained in equations (5) to (11) the present theory is in complete agreement with MAXWELL'S. Equations (14), (15), (17) may be taken as definitions. Equations (12), (13), (16) remain. Of these the last implies several other equations involving Ω and **I**. It may be left for the present. *On the present theory equations* (12), (13) *are not true.*

63. It remains then to investigate the physical bearing of the points of difference. Equation (13), of course, could not be expected to represent the results of the present theory, from the definition of a current adopted in § 4 above. Equation (13) asserts that the dielectric current is $\partial\mathbf{d}'/\partial t$. The question is by what on the present theory this statement must be replaced. Since **c** a flux $= \dot{\mathbf{d}}$,

$$\mathbf{c}' = m^{-1}\chi\mathbf{c} = m^{-1}\chi d\,(m\chi^{-1}\mathbf{d}')/dt = \dot{\mathbf{d}}' + m^{-1}\chi\,\frac{d}{dt}(m\chi^{-1})\,\mathbf{d}'.$$

Now by former paper equations (9), (11),

$$m\chi^{-1}\omega = -\tfrac{1}{2}\mathrm{V}\nabla_1\nabla_2\mathrm{S}\omega\rho'_1\rho'_2.$$

Hence

$$\frac{d}{dt}(m\chi^{-1})\,\omega = -\mathrm{V}\nabla_1\nabla_2\mathrm{S}\omega\dot{\rho}'_1\rho'_2 = \mathrm{V}\nabla_1\chi'\mathrm{V}\omega\dot{\rho}'_1\ [\textit{ibid.}, \text{equation (25)}]$$
$$= \mathrm{V}\chi'\nabla'_1\chi'\mathrm{V}\omega\dot{\rho}'_1 = m\chi^{-1}\mathrm{V}\nabla'_1\mathrm{V}\omega\dot{\rho}'_1,$$

by TAIT'S 'Quaternions,' 3rd. ed., § 157, equation (2). Hence

$$\left.\begin{aligned}\mathbf{c}' &= \dot{\mathbf{d}}' + \mathrm{V}\nabla'_1\mathrm{V}\mathbf{d}'\dot{\rho}'_1 = \partial\mathbf{d}'/\partial t + \mathrm{V}\nabla'\mathrm{V}\mathbf{d}'\dot{\rho}' - \dot{\rho}'\mathrm{S}\nabla'\mathbf{d}'\\ \mathbf{C}' &= \dot{\mathbf{D}}' + \mathrm{V}\nabla'_1\mathrm{V}\mathbf{D}'\dot{\rho}'_1 = \partial\mathbf{D}'/\partial t + \mathrm{V}\nabla'\mathrm{V}\mathbf{D}'\dot{\rho}'\end{aligned}\right\}\quad . \quad . \quad . \quad (23),$$

which equations have already been given in anticipation, in equation (38), § 50. In the case of an incompressible substance (solid or fluid) $\mathrm{S}\nabla'\dot{\rho}' = 0$, and, therefore,

$$\mathbf{c}' = \dot{\mathbf{d}}' + \mathrm{S}\mathbf{d}'\nabla'\,.\,\dot{\rho}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24),$$

and for a rigid body whose angular velocity (vector) is η this simplifies further to

$$\mathbf{c}' = \dot{\mathbf{d}}' - \mathrm{V}\eta\mathbf{d}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

Thus, on the present theory neither $\dot{\mathbf{d}}'$ nor $\partial\mathbf{d}'/\partial t$ is the dielectric current.

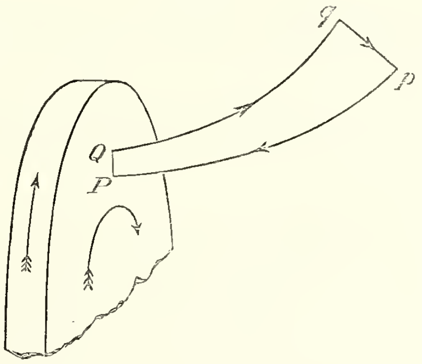
The effect of the difference between the theories will be very slight in most experimental work, though it will, of course, lead to different results in the solution of certain problems which involve currents in moving bodies.

64. There is one experimental result, however, in connection with which equation (23) has considerable interest. In the 'Phil. Mag.,' V., vol. xxvii [1889], p. 445, Professor

Rowland and Mr. Hutchinson describe the experiments by which they have proved that a moving charged body acts on surrounding bodies as it should on the convection current theory. Now, this can be shown *accurately* to follow from equation (23) if we make the double assumption (1) that the medium in contact with the moving conductor is at rest, and (2) that the slipping which thus takes place may be regarded as the limit of a rapid shear; and *approximately* to follow without the assumption.

First, then, assume there is no slipping. Let the motion be steady. If the moving dielectric be itself charged, we see by the term $-\dot{\rho}'\mathrm{S}\nabla'\mathfrak{d}'$ in $\mathfrak{c}'$ that the effect of its motion is to cause the current due on the convection current theory to its charge. Since the motion is steady, $\partial\mathfrak{d}'/\partial t = 0$. To take account of the remaining term $\mathrm{V}\nabla'\mathrm{V}\mathfrak{d}'\dot{\rho}'$ of $\mathfrak{c}'$, consider the current through a strip of surface constructed thus:—Take an elementary line PQ in the surface of the conductor. Through all points of PQ draw the lines of electrostatic induction (lines at every point of which the tangent is parallel to $\mathfrak{d}'$). Bound the strip of surface thus obtained at any distance from PQ by another element *pq*. In fig. 1 the arrows indicate (1) the direction of motion of the conductor, (2) the positive direction (PQ*qp*) *round* the strip when the positive direction *through* it is that of the motion at PQ. The current through PQ$qp = -\iint \mathrm{S}\mathfrak{c}'d\Sigma'$ taken over the strip. The part contributed to this by the term $\mathrm{V}\nabla'\mathrm{V}\mathfrak{d}'\dot{\rho}'$ of $\mathfrak{c}'$ is

Fig. 1.



$$-\iint \mathrm{S}d\,\Sigma'\nabla'\mathrm{V}\mathfrak{d}'\dot{\rho}' = -\int \mathrm{S}d\,\rho'\mathfrak{d}'\dot{\rho}'$$

by equation (3), § 5. The parts contributed to the line integral by the lines of induction *p*P, Q*q* are zero. Hence the current through the strip

$$= -\mathrm{S}\overline{\mathrm{PQ}}\mathfrak{d}'\dot{\rho}' + \mathrm{S}\overline{pq}\mathfrak{d}'\dot{\rho}'$$

where $\overline{\mathrm{PQ}}$, $\overline{pq}$ stand as usual for the *vectors* PQ, *pq*. The first of these terms is the rate of flow in the direction of motion of electrostatic charge through the element

PQ. Hence, if at *pq* $\dot{\rho}'$ is small enough to be neglected in the above expression, the whole current which on the present theory would be flowing through the strip PQ*qp* is the same as the current due to surface charge, which on the convection current theory would be flowing in the same direction across the element PQ. The extension to the case when slipping is allowed and the dielectric is at rest is obvious.

With regard to the plausibility of this explanation, it must be remembered that in this paper we admittedly do not take account of the independent motion in the very same space of two mediums such as air and ether. Now, probably,[*] the ether is at rest relative to the conductor, and it is reasonable to suppose that the relative motion of the conductor and the ether is of more importance in connection with the part $V\nabla' V\,\mathrm{d}'\dot{\rho}'$ of c' than the relative motion of the air and the conductor. On the other hand, as the air carries about with it any charge it possesses, it is the motion of the air we must consider in interpreting the term $-\dot{\rho}' S\nabla'\mathrm{d}'$. Indeed, if we suppose the ether only to *bound* the conductor and the molecules of air, and that the ether is mainly at rest (*i.e.*, acts to the conductor and the molecules of air much as an ocean of perfect fluid, which could *slide* over surfaces, and was originally at rest, would act to the conductor and molecules supposed immersed in it) the explanation is *complete.* [I do not wish to imply that I endorse this theory of the relative behaviour of the ether and matter].

On the whole, I think it may be said that this test of the correctness of eq. (23) is fairly well met.

65. Before comparing eq. (12) with the corresponding results of the present theory, it is necessary to make one or two remarks on passages from MAXWELL'S 'Electricity and Magnetism.' In the quotations I am about to make I have in every case changed MAXWELL'S notation to the notation used above, as leading to a clearer comparison of results. Consistently with this, I have always substituted Quaternion language for the corresponding Cartesian.

In the first place[†] I wish to discuss MAXWELL'S views concerning the scalar he calls Ψ, and which has been above denoted by z [equations (6) (12) (22)]. In his second volume he seems to intend the symbol always to have the same meaning. The first place in which it occurs in this volume is in § 598, where he is investigating the expression for $\mathbf{E}'_0$. After proving that

$$\mathbf{E}'_0 = V\dot{\rho}'\mathbf{B}' - \partial\mathbf{A}'/\partial t - \nabla' z,$$

he proceeds: "The terms involving the new quantity z are introduced for the sake of giving generality to the expression for $\mathbf{E}'_0$. They disappear from the integral when

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[*] According to a report in 'Nature,' September, 1891, p. 454, Professor LODGE described to the British Association experiments which go to prove this. I have not yet seen details of the experiments.

[†] Before going further, attention may be recalled to the footnote of § 38 above.

extended round the closed circuit. The quantity z is, therefore, indeterminate as far as regards the problem now before us, in which the total electromotive force round the circuit is to be determined. We shall find, however, that when we know all the circumstances of the problem, we can assign a definite value to z, and that it represents, according to a certain definition, the *electric potential* at the point ρ'." Now, I have looked in vain through the subsequent part of his treatise to find the promised definition of electric potential, and I have tried hard on MAXWELL'S own assumptions to see how the *definite value* he here speaks of is to be assigned, and I have totally failed. He nowhere shows how to assign a definite value to $\mathbf{A}'$; whereas he certainly assigns a definite value to $\mathbf{B}'$, and also from equations (9) (10) above, he also clearly assigns a definite value to $\mathbf{E}_0'$. From the equation just given, then, it follows that $\nabla' z$ must be indefinite in order to counterbalance the arbitrary part of $\partial\mathbf{A}'/\partial t$, which is necessarily of the form ∇' (*some scalar*).* Leaping over this difficulty of MAXWELL'S assertions, however, *i.e.*, supposing $\partial\mathbf{A}'/\partial t$ definite, the question still remains what is the definite value of z? Light *seems* to be thrown on the question by the assertion above that it is the "electric potential," and the following, taken from § 630 of his treatise :—

"The energy of the system may be divided into the Potential Energy and the Kinetic Energy.

"The potential energy, due to electrification, has already been considered in § 85. It may be written

$$W = \tfrac{1}{2}\Sigma\mathfrak{D}z,$$

where $\mathfrak{D}$ is the charge of electricity at a place where the electric potential is z, and the summation is to be extended to every place where there is electrification.

"If d' is the electric displacement, the quantity of electricity in the element of volume $d\varsigma'$ is

$$\mathfrak{D} = -\,S\nabla'\mathrm{d}'d\varsigma',$$

and

$$W = -\tfrac{1}{2}\iiint zS\nabla'\mathrm{d}'d\varsigma'$$

where the integration is to be extended throughout all space." He then shows that it follows that

$$W = \tfrac{1}{2}\iiint S\mathrm{d}'\nabla' z d\varsigma',$$

and proceeds :

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* For $V\nabla'\,\mathbf{A}'$ is assigned for every point of space and $[Vd\Sigma'\mathbf{A}']_{a+b} = 0$. It is well known that when this much and no more of a vector is known, it contains an arbitrary term ∇' (*a scalar*), and that this is the full extent of its arbitrariness.

"If we now write $\mathbf{E}_0'$ the electromotive force instead of $-\nabla' z$, we find

$$W = -\tfrac{1}{2}\iiint \mathrm{S}d'\mathbf{E}_0' d\varsigma'.$$

"Hence, the electrostatic energy of the whole field will be the same if we suppose that it resides in every part of the field where electrical force and electrical displacement occur, instead of being confined to the places where free electricity is found." Were it not for this last statement, the interpretation I should put on the whole of the above passage would be expressed thus:—In the *particular case* of electrostatics $\mathbf{E}_0' = -\nabla' z$ and $W = -\frac{1}{2}\iiint \mathrm{S}d'\mathbf{E}_0' d\varsigma'$. In the *general case*, where the electricity is not stationary, $\mathbf{E}_0'$ cannot be put in the form $-\nabla' z$; but *we shall nevertheless assume* that the equation $2W = -\iiint \mathrm{S}d'\mathbf{E}_0' d\varsigma'$ is still true. This seems to me the interpretation that presents least difficulty, but it seems hard to reconcile it with the last sentence quoted, which implies that the equation $2W = \iiint \mathrm{S}d'\nabla' z d\varsigma'$ is *exactly the same* as the equation $2W = -\iiint \mathrm{S}d'\mathbf{E}_0' d\varsigma'$. There seems only one other possible interpretation of the passage, but that lands us in hopeless difficulties. This explanation is that the $\mathbf{E}_0'$ which occurs in § 598, where it *cannot be put* in the form $-\nabla' z$, has a different meaning from the $\mathbf{E}_0'$ which occurs in §§ 630, 631, where it is $= -\nabla' z$. If he has changed the meaning of $\mathbf{E}_0'$, we may presume that matters have not been further complicated by a change in the meaning of z. In this case §§ 630, 631 may be put thus:—

(1) It is assumed that the energy of the field can be divided into two parts, electrostatic and electromagnetic.

(2) The former of these, in the absence of electric currents, can be put in the form $\frac{1}{2}\iiint \mathrm{S}d'\nabla' z d\varsigma'$ where z is a scalar. It is assumed that this statement is also true when there are electric currents present.

(3) It is assumed that the z appearing in this expression is the same as the z which occurs in the general equation $\mathbf{E}_0' = \mathrm{V}\dot{\rho}'\mathbf{B}' - \partial\mathbf{A}'/\partial t - \nabla' z$; and it is convenient to give it the name electric potential.

It will be acknowledged that these assumptions are more unwarrantable than the one required for the first interpretation, and therefore I shall understand the passage to be thus, as at first, correctly interpreted. But if this be so, we are as far off as ever from the conclusion that z has a definite value which can appropriately be called the electric potential.

66. This is no mere question of terms, for [equation (12), above] MAXWELL asserts that in the expression for the force due to the field occurs a term $-\mathrm{D}'\nabla' z$, and here the indefiniteness is not counterbalanced by the corresponding indefiniteness of $\partial\mathbf{A}'/\partial t$. There are more ways than one of compromising to get out of the difficulty.* The

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* For instance, we may (arbitrarily) render $\mathbf{A}'$ definite by the equations $\mathrm{S}\nabla'\mathbf{A}' = 0$ $[\mathrm{S}d\Sigma'\mathbf{A}']_{a+b} = 0$, and thus render $\nabla' z$ definite; and we may then *assert* that equation (12) is correct.

course followed here is, of course, to abide by what the present theory leads to, and then to choose that particular interpretation of the above passages which appears least at variance with our results. [It will be seen from the above that the statements at the end of § 50, above, are true, viz., that the potential v of the present theory is certainly not [equation (22), § 62] the same as Maxwell's potential z, and that without some such assumption as $\mathrm{S}\nabla'\mathbf{A}' = 0$, $[\mathrm{S}d\Sigma'\mathbf{A}']_{a+b} = 0$ our potential, like Maxwell's, is indefinite apart from an arbitrary additive constant. *This* question of the arbitrariness of the potential *is* one merely of terms.]

It is, perhaps, unnecessary now to say that the position I wish to maintain is that Maxwell has not investigated in a perfectly general manner the consequences of his own theory, and that, consequently, some of his *general* equations may prove inconsistent with that theory. Equation (12) I hold to be such an equation. So little right, indeed, has he to put this down as one of his general results that it is, I hold, inconsistent with other parts of his treatise. For instance, if the equation were consistent with equation (4), § 640 ('Elect. and Mag.,' 2nd edition), we should have $\nabla_1'\mathrm{S}\mathbf{H}_1'\mathbf{I}' = \nabla'\Omega\mathrm{S}\nabla'\mathbf{I}'$, which is certainly* not the case in general on Maxwell's theory. I shall not, then, compare the mechanical results of the present theory with equation (12) at all, but shall adopt the simpler process of comparing the stress which results from the present theory with that which Maxwell obtains in Chapter V. of Part I., and Chapter XI. of Part IV.

67. Before this comparison another matter must be considered. Maxwell, in accordance with, I think, universal custom, supposes that a molecular couple exists due to magnetism. In the first place this extraordinary exception to our ordinary

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* As might be expected, the relation is true in very many important problems whose details have been worked out, but it is not true in general, even when there are no currents. Dropping the special notation of this paper for the moment, let r, x have their usual Cartesian meanings. Denote differentiations with regard to r by dashes. Let F be any function of r. If there be no currents, and if

$$\Omega = x\mathrm{F},$$

then will

$$-\mathbf{H} = \nabla\Omega = i\mathrm{F} + \rho x\mathrm{F}'/r,$$

and [from the relation $\mathrm{S}\nabla(\mathbf{H} + 4\pi\mathbf{I}) = 0$, which is the only equation to be satisfied]

$$\mathbf{I} = i\,(r\mathrm{F}' + 3\mathrm{F})/4\pi.$$

In this case

$$4\pi\nabla_1\mathrm{S}\mathbf{I}\mathbf{H}_1 = -4\pi\mathrm{S}\mathbf{I}\nabla.\nabla\Omega = (r\mathrm{F}' + 3\mathrm{F})\,\{i2x\mathrm{F}'/r + \rho\,[\mathrm{F}' + x^2 d\,(\mathrm{F}'/r)/dr]/r\},$$

and

$$4\pi\nabla\Omega\mathrm{S}\nabla\mathbf{I} = -(x/r)\,d\,(r\mathrm{F}' + 3\mathrm{F})/dr.\{i\mathrm{F} + \rho x\mathrm{F}'/r\},$$

which are clearly not in general equal. The above expression for **I** is, of course, not the general one for this case, as we may add to it a term $\mathrm{V}\nabla\sigma$ where σ is any vector. Also, it is assumed that F is such that both **H** and **I** are everywhere continuous, *i.e.*, F and F′ are everywhere continuous. For instance, put $\mathrm{F} = (r - a)^2$ from $r = 0$ to $r = a$, and $\mathrm{F} = 0$ from $r = a$ to $r = \infty$.

conceptions of stress seems to me quite unnecessary on general grounds. It is well known that to every magnetic distribution there is an analogous conceivable distribution of ordinary statical electricity. In the ordinary action-at-a-distance theories the mutual mechanical effects of different parts of a magnetic system would be exactly the same as the corresponding effects in the analogue. Why, then, should it be considered unnecessary in the case of electrostatics, but necessary in the case of magnetics, to postulate a molecular couple? Why not, in other words, say that the stress which MAXWELL would suppose existent in the electric analogue is exactly the stress really existent in the magnetic system? In the second place, although the process seems viciously needless, we may, if we like, conceive any physical phenomena involving stress as causing a molecular couple which is exactly balanced by a stress-couple. [It must be so equilibrated in order to insure against infinite angular acceleration of an element of matter—supposing, of course, that the ultimate constitution of matter were not heterogeneous.] This latter stress-couple will be entirely of the nature of a reaction, since (former paper, p. 108) it is entirely independent of the potential energy of strain. In the present case, then, in which we suppose electromagnetic phenomena to produce stress, we shall have one stress exactly equilibrating another stress, neither of them having anything to do with the Lagrangian function. This is only another way of saying that no physical conception whatever is gained by the supposition that the particular physical phenomenon produces a stress-couple. We shall, then, consider it necessary to compare our results only with the pure part of the stress which MAXWELL supposes to exist.

Thus in § 641 MAXWELL arrives at the conclusion that the stress required to produce observed electromagnetic phenomena is υ where

$$8\pi\upsilon\omega = -2\mathbf{H}'\mathrm{S}\omega\mathbf{B}' + \omega\mathbf{H}'^2 = 8\pi\{\phi'\}\omega + \mathrm{V}(\mathrm{V}\mathbf{B}'\mathbf{H}'.\omega) \quad . \quad . \quad . \quad (26),$$

where $\{\phi'\}$ denotes the pure stress given by

$$8\pi\{\phi'\}\omega = -\mathbf{H}'\mathrm{S}\omega\mathbf{B}' - \mathbf{B}'\mathrm{S}\omega\mathbf{H}' + \omega\mathbf{H}'^2 = -\mathrm{V}\mathbf{B}'\omega\mathbf{H}' - 4\pi\omega\mathrm{S}\mathbf{I}'\mathbf{H}' \quad . \quad (27).$$

Now, what in the former paper (p. 108) was called the couple stress part of υ, namely, $\mathrm{V}(\mathrm{V}\mathbf{B}'\mathbf{H}'.\omega)/8\pi$, produces a couple per unit volume $\mathrm{V}\mathbf{B}'\mathbf{H}'/4\pi = \mathrm{V}\mathbf{I}'\mathbf{H}'$, and this must be equilibrated by some other couple per unit volume even when the body is not in equilibrium. This couple can only result from a couple stress, $\mathrm{V}\epsilon\omega$, which produces a couple, 2ϵ, per unit volume; and this is quite independent of the potential energy of strain, and therefore of the Lagrangian function. Thus, $2\epsilon + \mathrm{V}\mathbf{I}'\mathbf{H}' = 0$. If now, in addition to the stress $\upsilon\omega$ we take account of the *reactionary* stress $\mathrm{V}\epsilon\omega$, we simply get the pure stress $\{\phi'\}\omega$. We shall then merely compare the stress (pure) which flows from the present theory with $\{\phi'\}$.

68. It is well-known that MAXWELL's stress can only be looked on as a normal

type of stress. It by no means explains all the known facts. It does explain satisfactorily such known mechanical actions of real conductors—conveying currents and bearing charges—and magnets on one another as are of the nature of apparent actions at a distance. It does not at all explain the many known mechanical actions of one part of a conductor or magnet on another part which can be tested only by observing the (small) strains resulting. In other words, conductors and magnets are found to behave mechanically, as they would if MAXWELL'S supposed stress acted outside them, but not as if this stress existed internally. It need not therefore be matter for surprise if, on the present theory, what would appear as the most suitable stress to regard as normal should differ from MAXWELL'S. It is only necessary that *just outside* conductors and magnets it should be identical with MAXWELL'S.

To see what on the present theory should be regarded as a normal type we must discuss, from the physical point of view, the results of §§ 54, 55 above. As far as I can see (but this is, of course, largely a matter for personal judgment) on the present theory we should recognize two normal types of stress—one for fluids and one for solids. The reason is that we may assume fluids to be magnetically and electrically isotropic, and that fluids are subject to indefinitely large strains. On the other hand, solids, even if magnetically and electrically isotropic when unstrained, cannot be considered so when strained and, moreover, their strains cannot exceed a certain—usually very small—amount without the form of l being permanently altered.

For bodies which are electrically isotropic, however large their strain, it is needless to say that we must regard the Lagrangian function as given in terms of the dashed letters. For such bodies, η of equation (19) § 54 is zero. [For let α'', β'' . . . be the vectors of which l'' is an explicit function. Since the body is isotropic the value of l'' must remain unaltered if we rotate α'', β'' . . . all to the same extent round the same axis. In particular, if we increase α'', β'' . . . by $\mathrm{V}\epsilon\alpha''$, $\mathrm{V}\epsilon\beta''$, . . . where ϵ is an infinitely small vector, l'' must remain unaltered, *i.e.*, the increment $-\Sigma \mathrm{S}\epsilon\alpha''{}_{\alpha}\nabla'' l'' = 0$. Since ϵ is arbitrary, it follows that $\Sigma \mathrm{V}\alpha''{}_{\alpha}\nabla'' l'' = 0$]. By equation (20) § 54 above, we see that the assumption that $[\phi'] = 0$ amounts to assuming that *the Lagrangian function of unit volume of the body when strained, however largely, is the same as the Lagrangian function of unit volume of the body when unstrained.* By equation (29), § 55, we see that the assumption $\phi_0' = 0$ amounts to assuming that *that part of the Lagrangian function which causes the body to differ from vacuum, and which is contributed by a given mass of the body, is unaffected by strain.* Hence from equations (21), § 54, and (30), § 55, we have

$$\left.\begin{array}{l}\textit{For an isotropic body, of which the Lagrangian function per unit}\\ \quad\textit{volume is unaffected by strain}\\ \quad 2\phi' = -\left\{2l' + \Sigma \mathrm{S}\left(\tau'{}_{\tau}\nabla' l' + \sigma'{}_{\sigma}\nabla' l'\right)\right\}\\ \qquad\qquad\qquad + \Sigma \mathrm{V}\left\{\tau'(\;)_{\tau}\nabla' l' - \sigma'(\;)_{\sigma}\nabla' l'\right\}\end{array}\right\} \quad . \quad (28).$$

$$\left.\begin{array}{l}\textit{For an isotropic body, of which that part of the Lagrangian}\\ \quad\textit{function per unit mass, which causes the body to differ from}\\ \quad\textit{vacuum, is unaffected by strain}\\ \quad 2\phi' = (\mu_0 \mathrm{H}'^2/4\pi - 4\pi \mathrm{K}_0{}^{-1}\,\mathrm{d}'^2) - \Sigma \mathrm{S}\,(\tau'_\tau \nabla' l' + \sigma'_\sigma \nabla' l') \\ \qquad\qquad\qquad\qquad\qquad + \Sigma \mathrm{V}\{\tau'(\ )_\tau \nabla' l' - \sigma'(\ )_\sigma \nabla' l'\}\end{array}\right\} \quad . \quad (29).$$

It is scarcely necessary to say, that of course it is not meant to be here implied that there is any body whatsoever whose general Lagrangian function—whether per unit mass or per unit volume—is even approximately unaffected by strain. It is only for brevity that we verbally contemplate such a body. There seems little or no reason for choosing one rather than the other of these two stresses as the normal type of stress for fluids. Both of them would satisfy the condition that for a gas which, however large its strain, always behaved like a vacuum, the normal stress would be the vacuum stress which resulted from identical values of **H**′ and **d**′. As the stress of equation (29), however, agrees more closely with the pure part of MAXWELL'S stress than that of equation (28), we will call the stress of (29) the normal stress for fluids.

For solids it is harder to find a suitable normal stress, but as by far the greater number of them (non-magnetic bodies) behave magnetically approximately like a vacuum, it seems to me that the most suitable is obtained by supposing $\text{Ɑ}l_0$ of equation (27), § 55 to be zero. In this case, of course, we assume that for solids the normal type of stress is the stress that, with identical values of **H**′ and **d**′ would exist in a vacuum. Thus

$$\textit{For a vacuum, } \phi'\omega = -2\pi \mathrm{K}_0{}^{-1}\,\mathrm{d}'\omega \mathrm{d}' - \mu_0 \mathrm{H}'\omega \mathrm{H}'/8\pi \quad . \quad . \quad . \quad (30),$$

but it is needless to say that this is not perfectly satisfactory. The question may be asked why, in the present case, the stress of equation (29) should not be still retained as the normal one? The answer is, that the equation [(30), § 55] from which it is derived, and which actually must, in every exact discussion, be taken in its place, is a wholly unsuitable one for a solid, while equation (27), § 55, is a suitable one. Again it may be asked, Why not retain the stress equation in its original form $\phi' = -2m^{-1}\chi \text{Ɑ} l \chi'$, for solids? The answer to this is, that the important fact that the great majority of solids behave magnetically like a vacuum is not thereby readily taken account of.

69. To compare these stresses and their effects with MAXWELL'S, it must first be noted that MAXWELL has only investigated the electrostatic part of his stress for the case of a series of charged conductors surrounded by a dielectric that behaves electrostatically like a vacuum. I consider myself at liberty then to substitute anything for the electrostatic part of his stress which reduces to his for that particular case. The stress he obtains in Chapter V. of Part I. of his treatise is $-2\pi \mathrm{K}_0{}^{-1}\,\mathrm{d}'(\ )\,\mathrm{d}'$. For the particular case mentioned this may be written

$$\mathrm{Vd}'(\ )\,{}_{\mathrm{d}}\nabla' l'/2 - \mathrm{Sd}'({}_{\mathrm{d}}\nabla' l' + 4\pi \mathrm{K}_0^{-1}\,\mathrm{d}')/2,$$

since for that case ${}_{\mathrm{d}}\nabla' l' = -\,4\pi \mathrm{K}_0^{-1}\,\mathrm{d}'$. We shall assume that this is the correct expression in general, since thereby the stress of equation (29) is rendered identical with the pure part of MAXWELL's stress. The pure part of his electromagnetic stress is the $\{\phi'\}$ of equation (27) above. Let us then put

$$\phi_m'\omega = \mathrm{Vd}'\omega_{\mathrm{d}}\nabla' l'/2 - \omega\mathrm{Sd}'({}_{\mathrm{d}}\nabla' l' + 4\pi \mathrm{K}_0^{-1}\,\mathrm{d}')/2 - \mathrm{V}\mathbf{B}'\omega\mathbf{H}'/8\pi - \omega\mathrm{S}\mathbf{I}'\mathbf{H}'/2 \quad (31),$$

or, if we assume that the *complete* expression for x' is $-\,\mathrm{S}\mathbf{K}'\mathrm{R}'\mathbf{K}'/2$ [equations (28), § 50 and (20), § 35],

$$\phi_m'\omega = -\,\mathrm{Vd}'\omega\mathbf{E}'_0/2 + \omega\mathrm{Sd}'(\mathbf{E}'_0 - 4\pi \mathrm{K}_0^{-1}\,\mathrm{d}')/2 - \mathrm{V}\mathbf{B}'\omega\mathbf{H}'/8\pi - \omega\mathrm{S}\mathbf{I}'\mathbf{H}'/2 \quad (31a),$$

and call ϕ_m' MAXWELL's stress. [Of course I do not thereby mean to render MAXWELL responsible for this form.] If we regard $\mathrm{Vd}'(\ )\,{}_{\mathrm{d}}\nabla' l'/2$ as the correct generalisation of MAXWELL's electrostatic stress we may indicate it by calling $[\phi_m']$ the second form of MAXWELL's stress where

$$[\phi_m']\,\omega = \mathrm{Vd}'\omega_{\mathrm{d}}\nabla' l'/2 - \mathrm{V}\mathbf{B}'\omega\mathbf{H}'/8\pi - \omega\mathrm{S}\mathbf{I}'\mathbf{H}'/2 \quad . \ . \ . \ . \quad (32),$$

which gives, on the assumption that the complete expression for x' is $-\,\mathrm{S}\mathbf{K}'\mathrm{R}\,\mathbf{K}'/2$,

$$[\phi_m']\,\omega = -\,\mathrm{Vd}'\omega\mathbf{E}'_0/2 - \mathrm{V}\mathbf{B}'\omega\mathbf{H}'/8\pi - \omega\mathrm{S}\mathbf{I}'\mathbf{H}'/2 \quad . \ . \ . \quad (32a).$$

If we now assume that the only variables of l' are $\mathbf{H}'$ and d', equation (28) gives

$$2\phi' = -\,\{2l' + \mathrm{S}\,(\mathrm{d}'_{\mathrm{d}}\nabla' l' + \mathbf{B}'\mathbf{H}'/4\pi)\} + \mathrm{Vd}'(\quad)\,{}_{\mathrm{d}}\nabla' l' - \mathrm{V}\mathbf{B}'(\quad)\,\mathbf{H}'/4\pi \ . \quad (28a),$$

of which the following particular cases should be noted :—

If l' be quadratic in d' *and* $\mathbf{H}'$,

$$\left.\phi' = \mathrm{Vd}'(\quad)\,{}_{\mathrm{d}}\nabla' l'/2 - \mathrm{V}\mathbf{B}'(\quad)\,\mathbf{H}'/8\pi\right\} \quad . \ . \ . \ . \ . \ . \quad (28b),$$

If l' be given by (3), § 59, *and x' by* (4),

$$\left.\phi' = -\,\mathrm{Vd}'(\quad)\,\mathbf{E}'_0/2 - \mathrm{V}\mathbf{B}'(\quad)\,\mathbf{H}'/8\pi + \mathrm{S}\mathbf{I}_0'\mathbf{H}'/2\right\} \quad . \ . \ . \ . \quad (28c),$$

from which it follows from equation (32a) that in this case

$$\phi' = [\phi_m'] + \mathrm{S}\,(\mathbf{I}'_0 + \mathbf{I}')\,\mathbf{H}'/2 \quad . \ . \ . \ . \ . \ . \ . \quad (28d,\ 32b),$$

so that this stress differs from the second form of MAXWELL's stress by a hydrostatic pressure which is zero for non-magnetic bodies.

Under the same circumstances (l' a function of $\mathbf{H}'$ and $\mathbf{d}'$ only), equation (29) gives

$$\begin{aligned}\phi' = \mathrm{V}\mathbf{d}'(\quad)_{\mathbf{d}}\nabla' l'/2 - \mathrm{S}\mathbf{d}'({}_{\mathbf{d}}\nabla' l' + 4\pi \mathrm{K}_0{}^{-1}\,\mathbf{d}')/2 \\ - \mathrm{V}\mathbf{B}'(\quad)\mathbf{H}'/8\pi - \mathrm{S}\mathbf{I}'\mathbf{H}'/2 + (\mu_0 - 1)\,\mathbf{H}'^2/8\pi \quad . \quad . \quad . \quad . \quad (29a).\end{aligned}$$

Hence, with the electromagnetic system of units for which $\mu_0 = 1$,

$$\phi' = \phi_m{}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29b,\ 31b),$$

in which it should be noticed there is no necessity to assume that the complete form of x' is $-\mathrm{S}\mathbf{K}'\mathbf{R}'\mathbf{K}'/2$, nor is it assumed, as in equation (28*d*, 32*b*) that l' has the particular form given in equation (3), § 59.

To sum up, of the two equations (28) and (29), (28) agrees more closely with MAXWELL as to the electrostatic part, and (29) more closely as to the electromagnetic part. On the whole, equation (29) agrees more closely than (28).*

Of course, the normal stress [eq. (30)] we have adopted for solids is by no means the same as MAXWELL'S, except for non-magnetic bodies whose specific inductive capacities are the same as for a vacuum. But this does not prevent our normal stress explaining all that MAXWELL'S stress explains, and, indeed, from the remarks at the beginning of last section, it is now evident that for all useful purposes either the one stress or the other will serve equally well.

70. We have now compared the results of the present theory with all MAXWELL'S results contained in equations (5) to (17), § 60, above, except (16). Except for equations (12), (13), the agreement is exact, and I think it may now be claimed that what the present theory gives instead of equation (12), agrees, as well as (12), with known facts, and what it gives instead of (13) agrees better than (13).

Equation (16) itself is obvious enough since it merely asserts that $\mathbf{H}'$ has a potential when there are no currents in the field. But it suggests another question—does the present theory lead to the *ordinary* mathematical theory of electromagnetism? It can be easily shown to do so. The mechanical results when expressed in terms of $\mathbf{H}'$ and $\mathbf{I}'$ have just been shown to result in the same forces and moments on conductors and magnets regarded as wholes, as does MAXWELL'S stress. These are all the mechanical demands of the ordinary theory. Equation (19), § 61, shows that the relations between the whole magnetic moment per unit volume, the permanent magnetic moment per unit volume, and the magnetic force at the point, may on the present theory, be regarded as the same as in the ordinary theory. Only one other

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* Notwithstanding this, and the fact that I have in this paper called the stress of equation (29) the normal stress, I think equation (28) is to be preferred, partly because of the greater simplicity of the assumptions which lead to it, and partly because of the greater simplicity of the electrostatical results flowing from it.

demand is made by the ordinary theory. **H**$'$ and **A**$'$ on the one hand, must be determined in terms of **C**$'$ and **I**$'$, on the other, by means of particular relations.

Let us suppose—merely to get rid of the dashes—the standard position to coincide with the actual position. One difference between MAXWELL'S theory and the ordinary theory is that according to the latter it is assumed that each individual magnetic molecule and each elementary current has its own influence—independently of the rest—in producing terms in **A** and **H**. Thus, **H** consists of two parts, the first depending only on the magnetism and the second only on the currents. The first $= -\nabla\Omega$, where $\Omega = -\iiint \mathrm{S}\mathbf{I}\nabla u d\varsigma$ [MAXWELL'S 'Elect. and Mag.,' 2nd ed., § 383, equation (3)], where u^{-1} is the distance of the element $d\varsigma$ from the point under consideration, and where in the differentiations of ∇u the end of u^{-1} at the element $d\varsigma$ is supposed varied. The second part is obtained on the assumption that each closed current causes a term in **H** which the corresponding magnetic shell would cause. The second part is thus found to be $\nabla\iiint u\mathbf{C}d\varsigma$.[*] The first part of **A** is supposed to depend on **I** in the same way as the **A**, called the vector potential, of Part III. of MAXWELL'S treatise depends, *i.e.*, $= \iiint \mathrm{V}\mathbf{I}\nabla u d\varsigma$ [§ 405, equation (22)]. The second part, as with **H** is obtained by assuming that any closed current will cause a term in **A** equal to the term in **A** that would be caused by the corresponding magnetic shell. The second part is thus found to be $\iiint u\mathbf{C}d\varsigma$.[†] We will suppose that the ordinary theory also admits that **A** is arbitrary in containing a term ∇w, where w is a scalar. (This is only to render the comparison with the present theory simpler. Perhaps it ought to be said that the **A** thus obtained in terms of **I** and **C** on the ordinary theory is found to satisfy the conditions $\mathrm{S}\nabla\mathbf{A} = 0$, $[\mathrm{S}d\Sigma\mathbf{A}]_{a+b} = 0$, and that the present theory only agrees with the ordinary theory if we arbitrarily impose those relations.) All this may be expressed thus. Defining $\mathbf{A}_0$ and Ω by

$$\mathbf{A}_0 = \iiint u\mathbf{C}d\varsigma, \ \Omega = -\iiint \mathrm{S}\mathbf{I}\nabla u d\varsigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

we shall have

$$\mathbf{A} = \mathbf{A}_0 + \iiint \mathrm{V}\mathbf{I}\nabla u d\varsigma + \nabla w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

$$\mathbf{H} = -\nabla\Omega + \nabla\mathbf{A}_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

If q be any quaternion function of the position of a point which may be discontinuous at certain surfaces, we have

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[*] The magnetic force at an external point due to a shell of strength $c = c\nabla\iint \mathrm{S}d\Sigma\nabla u = -c\iint \mathrm{S}d\Sigma\nabla.\nabla u = c\iint \mathrm{V}d\Sigma\nabla.\nabla u$ [since $\nabla^2 u = 0$] $= c\int d\rho\nabla u = c\nabla\int u d\rho$. The reason for the change of sign in ∇ on crossing the integral sign is that when outside one end and when inside the other end of u^{-1} is naturally supposed in the differentiations of ∇u to vary.

[†] This is not inconsistent with §§ 616, 617 of 'Elect. and Mag.,' 2nd edit., for there MAXWELL is considering the two parts together.

$$4\pi q = \nabla^2 \iiint uq\,d\varsigma = -\nabla \iiint \nabla uq\,d\varsigma = \nabla\left(\iiint u\nabla q\,d\varsigma - \iint u\,d\Sigma q\right).$$

Now, on the present theory (by means of the equations $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, $[\mathrm{V}d\Sigma\mathbf{H}]_{a+b} = 0$, and by elimination of **B** from the equations $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I}$, $\mathrm{S}\nabla\mathbf{B} = 0$, $[\mathrm{S}\,d\Sigma\mathbf{B}]_{a+b} = 0$),

$$\nabla\mathbf{H} = 4\pi(\mathbf{C} - \mathrm{S}\nabla\mathbf{I}),\ [d\Sigma\mathbf{H}]_{a+b} = -\,4\pi\,[\mathrm{S}d\Sigma\mathbf{I}]_{a+b}.$$

Hence substituting **H** for q,

$$\mathbf{H} = \nabla\left(\iiint u\mathbf{C}\,d\varsigma - \iiint u\mathrm{S}\nabla\mathbf{I}\,d\varsigma + \iint u\mathrm{S}\,d\Sigma\mathbf{I}\right) = \nabla\left(\iiint u\mathbf{C}\,d\varsigma + \iiint \mathrm{S}\mathbf{I}\nabla u\,d\varsigma\right).$$

This is equation (35). Again substituting **A** for q, and putting

$$4\pi w = \iiint u\mathrm{S}\nabla\mathbf{A}\,d\varsigma - \iint u\mathrm{S}\,d\Sigma\mathbf{A},$$

we get [equations (19), (20), § 26]

$$4\pi(\mathbf{A} - \nabla w) = \nabla \iiint u\mathbf{B}\,d\varsigma = -\iiint \mathrm{V}\nabla u\mathbf{B}\,d\varsigma = 4\pi \iiint \mathrm{V}\mathbf{I}\nabla u\,d\varsigma + \iiint \mathrm{V}\mathbf{H}\nabla u\,d\varsigma$$

Also,

$$\iiint \mathrm{V}\mathbf{H}\nabla u\,d\varsigma = \iiint u\mathrm{V}\nabla\mathbf{H}\,d\varsigma - \iint u\mathrm{V}\,d\Sigma\mathbf{H} = 4\pi\iiint u\mathbf{C}\,d\varsigma.$$

This proves equation (34).

That the present theory (and MAXWELL'S), so far as **H** and **A** depend upon **I** and **C**, thus leads exactly to the ordinary theory is of some importance. One consequence is, that the mechanical action between bodies carrying currents and the induction of currents by the variation of position and magnitude of other currents and magnets, must necessarily be independent of the nature of the medium separating them, so long as that medium is non-magnetic. This is in direct contrast with the known large influence the medium separating two charges of electricity has on the mutual actions of the bodies bearing the charges. On the present, as on MAXWELL'S theory, this is simply owing to the fact that the ordinary theory of magnetism is in the points just mentioned, accurately true, whereas the ordinary theory (action-at-a-distance, with consequently no difference of specific inductive capacity for different media) of electrostatics is not even approximately true. [Whether or not the theory I have called the "ordinary" theory has actually ever been formulated is of little consequence. I have, in the above, accurately enough described what I mean by the term.]

B. *Modifications necessary on account of Hysteresis.*

71. This seems to be the place to consider what bearing the phenomena of hysteresis have upon such theories as the present. No theory of electromagnetism can be considered complete unless it takes this important group of facts into account. I do not here propose to give a theory of hysteresis—so that the present theory must be in this sense confessed incomplete—but it is necessary to notice what modifications ought strictly to be made in the assumptions hitherto adopted.

Professor EWING ('Phil. Mag.,'[*] V., vol. 30 [1890], p. 205), has given a theory which adapts itself to dynamical methods such as the present. In his theory the phenomena of hysteresis depend upon the fact that groups of molecules can have various stable configurations, different groups at any instant having very different degrees of stability. The stability of a group is liable by variation of $\mathbf{H}$ and Ψ to break down, so that the group takes up another configuration of greater or less stability, and the oscillations which necessarily ensue on the change result to our senses in the production of heat. On this view hysteresis is a phenomenon that prevents us, if we would take full account of the facts, from ignoring certain coordinates we have hitherto ignored. We can, however, go on ignoring these coordinates if we suppose l not to have a constant form in terms of the variables not ignored above, but a form which depends on the particular state as to these groups of molecules of an element of volume. We must, then, suppose certain variables—call them hysteresis-coordinates—which define the relative numbers of groups of different kinds. Of these l will be a function, but they are not of the nature of ordinary dynamical coordinates. Their value merely determines the instantaneous form of l as a function of ordinary coordinates, so that if one or more of the hysteresis-coordinates change, the form of l changes and a new dynamical era begins. In fact, they are very analogous to θ, and like θ they must not be varied when the dynamical coordinates are varied in order to obtain the equations of motion. A mathematical development of Professor EWING's theory may be supposed to furnish the *nature* of these variables, and experiment must then be appealed to at once to test the theory, and if the test be favourable, to find the exact form of l in terms of the variables. And from the mathematical development, or that combined with experiment, we must look to find the laws of variation of the hysteresis coordinates when $\mathbf{H}$ and Ψ vary.

72. This, of course, is only to be looked upon as an ideal procedure of events, which, perhaps, for many years cannot come about. Meanwhile, tentative hypotheses as to the nature of these variables might be made. For instance, it might be assumed that l is always correctly given by equation (1), § 59 above, and that the vector $\mathbf{I}_0$ is the sole hysteresis coordinate. In this case μ (and K ?) would, of course, be assumed a function of $\mathbf{I}_0$ as well as of Ψ. Though this is probably much too

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[*] Or 'Nature,' Oct., 1891, p. 566.

simple a theory for the explanation of all hysteresis phenomena, yet I believe it could be made to account for nearly all the known facts.* But, at present, even if this simple assumption were made, we are very much in the dark as to how $\mathbf{I}_0$ varies with $\mathbf{H}$ and Ψ, and are compelled to fall back on such pure conjectures as are illustrated in the foot-note. To mention only one thing—nearly all the detailed experiments on hysteresis deal only with variations of $\mathbf{H}$ *parallel to itself.*

73. Thus it is useless to attempt a satisfactory theory of hysteresis at present, though we can see vaguely how, perhaps, in the future it may be made to fit into the present theory.

But these considerations show that we must be very cautious in discussing results which depend upon the form of l in terms of $\mathbf{H}$, for they imply that we are very ignorant of this form, even when we know how $\mathbf{I}$ varies with $\mathbf{H}$ under assigned circumstances. Thus, for instance, in equation (30), § 55, we can learn little of the true meaning of ϕ'_0 so far as it depends upon $\mathbf{H}'$. The rest of ϕ' in this equation, however, being independent of the form of l, gives us information of no doubtful character.

C. *On the Strains accompanying these Stresses.*

74. The object of the present paper is to discuss the *general* theory of electromagnetism. It is not proposed, therefore, to deal more than is absolutely necessary in particular problems. A word, however, must be said as to a certain class of problems connected with the stresses just investigated.

After the question of hysteresis has been settled in some such way as just indicated, it will be possible to discuss in detail the exact form of l'_0 of equation (29), § 55 above. To do this, the data on which to argue will generally be the strains which accompany electromagnetic phenomena. This necessitates the consideration of such strains.

75. It must not be supposed that these strains will bear the same relation to the stresses as strains bear to the ordinary stresses considered in the mathematical theory of elasticity. From equations (26), (27), § 50 above, we see that in the case of equilibrium (no external stress)

$$\mathrm{D}_m{}'\nabla'\mathrm{W} - \mathbf{F} = \phi'\Delta', \quad \mathbf{F}_s = [\phi'\mathrm{U}\nu']_{a+b}.$$

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* By suitably choosing the form of μ in terms of $\mathbf{I}_0$, and the four functions now to be introduced. Let $|\omega|$, $[\omega]$, $\{\omega\}$ be three positive scalar functions of $\mathrm{T}\omega$, and let $\mathrm{Q}\omega$ be a vector function of the form $\mathrm{T}\omega$ *function* $(\mathrm{U}\omega)$—not in general linear—such that $\mathrm{S}\omega\mathrm{Q}\omega$ is always negative. The form of Q, like that of μ, is a function of $\mathbf{I}_0$. Let $\mathbf{H}$ be the present and $\mathbf{h}$ any previous value of $\mathbf{H}$. Then assume that

$$\dot{\mathbf{I}}_0 = \mathrm{Q}\mathbf{N} \text{ where } \mathbf{N} = |\mathbf{H}|\dot{\mathbf{H}}\int^{\mathbf{H}} [\mathbf{h}]\,\epsilon^{-\int_{\mathbf{h}}^{\mathbf{H}}\{\mathbf{h}\}\,\mathrm{T}d\mathbf{h}}\,\mathrm{T}\,(d\mathbf{h} + \mathbf{UHT}d\mathbf{h}),$$

the lower limit of the first integral sign being, strictly, the value of $\mathbf{H}$ at an indefinitely remote epoch, but practically at a time determined by the exponential. I give this merely to show in what *sort* of way we may suppose $\mathbf{I}_0$ to depend on the history of the body.

Taking for simplicity the case where there is no external force (F), or force potential (W), we have

$$\phi'\Delta' = 0, \quad [\phi' \mathrm{U}\nu']_{a+b} = 0.$$

Substituting now from equation (27), § 55 for ϕ', we see that this means that there is equilibrium owing to the simultaneous existence of three stresses: (1) the ordinary elasticity theory stress, owing to terms which only involve Ψ; (2) the stress which is independent of l_0 and, therefore, depends *only* on electromagnetic quantities; (3) a stress due to terms in l_0, which involve both Ψ and electromagnetic quantities. When these last are linear in Ψ, the resulting stress will depend, like the second, upon electromagnetic quantities only. If not linear, they will depend both upon Ψ and the electromagnetic quantities. It is quite possible that there should be no strain at all, and yet a very sensible stress due to electromagnetic actions.

In fact, in solving the elasticity problem—having given the distribution throughout the field, of dielectric displacements, of currents, and of magnetisation, required the strain at any point—the only way in which the electromagnetic data can be used is, by finding the force per unit volume and surface respectively due to them, and then treating these forces as external. That is, the knowledge of the *stress* which produces the mechanical effects of electromagnetism is of no use in discovering the strain actually resulting; all the knowledge we can thus utilise is that of the *forces* (per unit volume and surface) due to such stresses. This shows that, to find the true expression for l it is not sufficient to investigate experimentally what strain accompanies a given displacement, or current, or magnetisation at a point.[*] The problem is much more complicated. The shapes of all the bodies present must be assumed of quite as great importance as the electromagnetic quantities in deciding the form of l from such experiments.

76. These remarks may be illustrated by considering the effect of MAXWELL'S stress in two different cases. Choosing one shape of soft-iron body it will be found that the magnetisation will, according to MAXWELL'S stress, compress the body; choosing another shape, expansion results.

Suppose we have (1) an anchor ring of soft iron, (2) surrounding this a layer of air of uniform thickness, (3) surrounding this n coils of insulated uniformly distributed wire carrying a current c. Take columnar coordinates r, ϑ, z, the axis of z being the axis of the anchor ring, and let i, j, k be unit vectors (functions of the position of a point) in the directions of dr, $d\vartheta$, dz respectively. At any point inside the coil we have $\mathbf{H}' = 2ncj/r$. Assuming μ' to be a constant scalar $\mathbf{I}' = (\mu' - 1)\,\mathbf{H}'/4\pi$. Hence, from equation (27), § 67,

$$\{\phi'\}\,\Delta' = -\,\Delta'\mathrm{S}\mathbf{I}'\mathbf{H}'/2 = -\,(\mu' - 1)\,\nabla_1{}'\mathrm{S}\mathbf{H}'\mathbf{H}_1{}'/4\pi = -\,n^2c^2\,(\mu' - 1)\,i/\pi r^3.$$

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[*] This seems to be the meaning of the third sentence of the small print on p. 269 of vol. 15., 'Encyc. Brit.,' 9th ed.

At the surface of the soft iron $\mathbf{H}'$ is tangential, and therefore continuous. From this it easily follows that

$$[\{\phi'\}\mathrm{U}\nu']_{a+b} = 0.$$

Hence, due to MAXWELL'S stress, there is in this case no superficial force and no bodily force in the air, but there is a bodily force in the iron directed towards the axis. The iron will therefore be compressed.

77. For the other case, notice that the force per unit surface due to the electromagnetic part of MAXWELL'S stress is $-[\{\phi'\}\mathrm{U}\nu']_{a+b}$ and by equation (27), § 67,

$$-8\pi[\{\phi'\}\mathrm{U}\nu']_{a+b} = [\mathbf{B}'\mathrm{S}\mathbf{H}'\mathrm{U}\nu' + \mathbf{H}'\mathrm{S}\mathbf{B}'\mathrm{U}\nu' - \mathrm{U}\nu'\mathbf{H}'^2]_{a+b}.$$

This can be put in several different forms, of which, perhaps, the following are the most useful

$$\left.\begin{aligned} -8\pi[\{\phi'\}\mathrm{U}\nu']_{a+b} &= 4\pi[\mathbf{I}'\mathrm{S}\mathrm{U}\nu'\mathbf{H}']_{a+b} + (\mathbf{H}_t' + \mathbf{B}_n')[\mathrm{S}\mathrm{U}\nu'\mathbf{H}']_{a+b} \\ &= (4\pi\bar{\mathbf{I}}' + \mathbf{H}_t' + \mathbf{B}_n')[\mathrm{S}\mathrm{U}\nu'\mathbf{H}']_{a+b} + 4\pi[\mathbf{I}'\mathrm{S}\mathrm{U}\nu'\overline{\mathbf{H}}']_{a+b} \end{aligned}\right\} \quad . \quad (36),$$

where the bar indicates the mean value for the two regions bounded by the surface, and the suffixes n and t denote normal and tangential components respectively. Thus $\mathbf{B}_n'$ and $\mathbf{H}_t'$ have the same value on both sides of the surface. When $\mathbf{B}'$ is parallel to $\mathbf{H}'$, $\mathbf{B}' = \mu'\mathbf{H}'$ where μ' is a scalar, not necessarily constant. (But if not constant it has here a different meaning from what it has in the rest of this paper.) In this case the tangential component of $-8\pi[\{\phi'\}\mathrm{U}\nu']_{a+b}$ is zero. For the first expression of equation (36) gives for the component in question

$$[(4\pi\mathbf{I}_t' + \mathbf{H}_t')\,\mathrm{S}\mathrm{U}\nu'\mathbf{H}']_{a+b} = \mathbf{H}_t'[\mu'\mathrm{S}\mathrm{U}\nu'\mathbf{H}']_{a+b} = \mathbf{H}_t'[\mathrm{S}\mathrm{U}\nu'\mathbf{B}']_{a+b} = 0.$$

So long then as we deal with magnetically isotropic media this surface traction is normal.

Consider a magnetically isotropic body surrounded by a non-magnetic medium, and let the magnetic region be denoted by the suffix a, so that $\mathbf{I}_b' = 0$. In accordance with what has been just proved we consider only the normal part of the traction. Thus,

$$-8\pi[\{\phi'\}\mathrm{U}\nu']_{a+b} = 4\pi[\mathbf{I}'_n\mathrm{S}\mathrm{U}\nu'\mathbf{H}']_{a+b} + \mathbf{B}'_n[\mathrm{S}\mathrm{U}\nu'\mathbf{H}']_{a+b}.$$

Let now $[\mathbf{H}'_n]_a = \mathrm{H}\mathrm{U}\nu'_a$. Thus

$$4\pi[\mathbf{I}'_n]_a = (\mu' - 1)\,\mathrm{H}\mathrm{U}\nu'_a, \qquad \mathbf{B}'_n = \mu'\mathrm{H}\mathrm{U}\nu'_a,$$

$$[\mathrm{S}\mathrm{U}\nu'\mathbf{H}']_a = -\mathrm{H}, \qquad [\mathrm{S}\mathrm{U}\nu'\mathbf{H}']_{a+b} = (\mu' - 1)\,\mathrm{H},$$

the last coming from the fact that $[\mathbf{H}'_n]_b = \mathbf{B}'_n$. Thus

$$-8\pi\left[\{\phi'\}\mathrm{U}\nu'\right]_{a+b} = (\mu' - 1)^2\mathrm{H}^2\mathrm{U}\nu'_a\,;$$

or

$$-\left[\{\phi'\}\mathrm{U}\nu'\right]_{a+b} = -2\pi\left[\mathrm{I}'^{2}_{n}\mathrm{U}\nu'\right]_a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

Hence for both paramagnetic and diamagnetic isotropic bodies surrounded by non-magnetic media, MAXWELL's stress leads to a surface traction which is always a tension (except as in the anchor-ring when it is zero).

78. Consider now the well-known ordinary case of a soft-iron ellipsoid (μ' a constant scalar) brought into a uniform field. Inside the ellipsoid $\mathbf{B}'$, $\mathbf{H}'$, and $\mathbf{I}'$ are all constant, and therefore $\{\phi'\}\Delta' = 0$, so that there is no bodily force. Also since [equation (27), § 67]

$$\{\phi'\}\Delta' = \mathrm{V}\mathbf{C}'\mathbf{B}' - \nabla'_1\mathrm{S}\mathbf{I}'\mathbf{H}'_1 + \mathrm{V}\nabla'\mathrm{V}\mathbf{I}'\mathbf{H}'/_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (38),$$

there is no bodily force in the surrounding medium. Hence, in the present case, the only force is the tension at the surface. The ellipsoid will therefore be expanded by MAXWELL's stress.

D. *Thermoelectric, Thermomagnetic, and* HALL *Effects.*

79. It will be found convenient to discuss these various effects together.

The natures of the thermoelectric and HALL effects are well known and need no description here. The thermomagnetic effects are perhaps not so well-known. The original papers of VON ETTINGSHAUSEN and NERNST (the discoverers of these effects) are in 'Wied. Ann.,' xxxi. (1887), 737 and 760, xxxiii. (1888), 126, 129, 474. The effects are briefly described in Professor J. J. THOMSON's 'Applications of Dynamics to Physics and Chemistry,' 1st ed., § 57. The principal features of these effects are that the electromotive forces due to differences of temperature are modified in two ways by the presence of magnetic force. First, parallel to Θ there is an electromotive force that varies approximately as $\mathbf{H}^2\mathrm{T}\Theta$ (the "longitudinal" thermomagnetic effect); and, secondly, at right angles to both Θ and $\mathbf{H}$ there is an electromotive force $\mathrm{BV}\Theta\mathbf{H}$, where B is a scalar dependent on the temperature, but approximately independent of $\mathrm{T}\Theta$ and $\mathrm{T}\mathbf{H}$ (the "transversal" thermomagnetic effect). The latter effect is especially large in bismuth. There is evidence that these effects are closely connected with the HALL effect.

80. The natural way to discuss these results would appear to be to attempt to explain them by suitable terms in l. But on the present theory it is possible that they may be explained by terms in x. According to the first explanation they would be reversible phenomena, and according to the second irreversible phenomena involving dissipation of energy. Thermoelectric effects are certainly at present looked upon by physicists as reversible phenomena.

The two explanations—which will for the future be referred to as the theory of reversibility and the theory of irreversibility respectively—will be found in many respects very analogous, though, of course, we must expect some striking difference of results. On the theory of this paper the most striking would seem to be that, while thermoelectric effects must in the main be explained on the theory of reversibility, the explanation of the thermomagnetic effects by this theory is inadmissible by reason of certain collateral consequences.

81. Let ϖ—which has no connection with the ϖ of § 54 above—be a linear vector function of a vector, itself a function of θ, Ψ, and $\mathbf{H}$; and let

$$\varpi = [0] + [1] + [2] = \Sigma [n] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $[n]$ is a homogeneous function of degree n in the vector $\mathbf{H}$.

In particular, let ϖ be given by

$$\varpi\omega = \mathrm{A}\omega + \mathrm{BV}\omega\mathbf{H} - \omega\mathrm{S}\mathbf{H}\mathrm{C}\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where A, B, C are linear vector functions of a vector, themselves functions of Ψ and θ only. B and C, but not A, may for simplicity be assumed self-conjugate. Notice that $\mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla . [n] = - n [n]$, and therefore

$$(1 + \mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla.)\, \varpi = \Sigma (1 - n) [n] = \mathrm{A} + \mathrm{S}\mathbf{H}\mathrm{C}\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Similarly,

$$(\mathrm{S}\mathbf{K}_{\mathrm{K}}\nabla. + \mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla. + \mathrm{S}\Theta_{\Theta}\nabla.)\, \mathrm{S}\mathbf{K}\varpi\Theta = - \mathrm{S}\mathbf{K} (2 [0] + 3 [1] + 4 [2])\, \Theta \quad . \quad (4),$$

ϖ and A will be assumed to be of a class given by

$$\left.\begin{aligned} \mathrm{S}\tau\varpi\sigma d\varsigma = \mathrm{S}\tau'\varpi'\sigma' d\varsigma' = \mathrm{S}\tau''\varpi''\sigma'' d\varsigma' \\ \varpi' = \chi'^{-1}\varpi\chi', \qquad \varpi'' = \psi^{-1}\varpi\psi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and, of course, exactly similarly for A. It should be noticed that unlike the two classes of § 9 above, ϖ and A have not the property that if ϖ or A is self-conjugate, so also is ϖ' or A$'$ and ϖ'' or A$''$. But they have another simple property, namely, that if ϖ is a scalar,

$$\varpi = \varpi' = \varpi'' . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

It may also be noticed that $\varpi\sigma$ is an intensity, and $\varpi_c\tau$ a flux where ϖ_c stands for the conjugate of ϖ.

B is assumed to be of Class II. of § 9 above, and C of a class given by

$$\left.\begin{aligned} \mathrm{S}\sigma_a\mathrm{C}\sigma_b = \mathrm{S}\sigma_a'\mathrm{C}'\sigma_b' = \mathrm{S}\sigma_a''\mathrm{C}''\sigma_b'' \\ \mathrm{C}' = \chi\mathrm{C}\chi', \quad \mathrm{C}'' = \psi\mathrm{C}\psi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

so that it is very closely allied to Class I. of § 9 above.

It will now be seen that to obtain ϖ' or ϖ'' from ϖ it is only necessary to change A, B, C, and **H** into A$'$, B$'$, C$'$, and **H**$'$, or into A$''$, B$''$, C$''$, and **H**$''$ respectively. The statement is obvious so far as A, C, and **H** are concerned. With regard to B, we have, by §§ 7, 9,

$$\begin{aligned}\mathrm{B}'\mathrm{V}\omega\mathbf{H}' &= m\chi'^{-1}\mathrm{B}\chi^{-1}\mathrm{V}\omega\chi'^{-1}\mathbf{H}\\ &= \chi'^{-1}\mathrm{B}\mathrm{V}\chi'\omega\mathbf{H}\ [\text{Tait's 'Quaternions,' 3rd ed., § 157, eq. (2)}],\end{aligned}$$

which, with equation (5), proves the statement.

82. For the theory of reversibility, it is assumed that l contains a term g given by $g = -\mathrm{S}\mathbf{D}\varpi\Theta$. For the theory of irreversibility, it is assumed that x contains a term g given by $g = -\mathrm{S}\mathbf{K}\varpi\Theta$. Denote the various parts of **E**, f &c., depending upon g by the suffix g. It conduces to clearness to arrange the general results of these two assumptions in parallel columns thus:—

Theory of Reversibility.

l contains a term g given by

$$g = -\mathrm{S}\mathbf{D}\varpi\Theta \quad . \quad . \quad . \quad . \quad (8).$$

This contributes terms $\mathbf{E}_g$, $\mathbf{E}_{sg}$ to the right of equations (29), (31), § 50, given by

$$\mathbf{E}_g = \varpi\Theta\,, \quad \mathbf{E}_{sg} = 0 \quad . \quad . \quad . \quad (9).$$

In this is not included the part of $\dot{\mathbf{A}}$ due to g, but this is practically given by equations (15), (16), below. By equation (11), § 46,

$$\lambda_g = \mathrm{S}\mathbf{D}\,(1 + \mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla.)\,\varpi\Theta \quad . \quad . \quad (10).$$

By equations (11), (12), § 34 (putting ϖ_θ for $\partial\varpi/\partial\theta$)

$$\begin{aligned}f_g = &-\mathrm{S}\mathbf{D}\,(1 + \mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla.)\,\varpi_\theta\Theta\\ &+ \mathrm{S}\mathbf{D}\,(1 + \mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla.)\,\varpi\Delta \quad . \quad (11).\end{aligned}$$

$$f_{sg} = -\left[\mathrm{S}\mathbf{D}\,(1 + \mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla.)\,\varpi\mathrm{U}\nu\right]_{a+b} \quad . \quad . \quad . \quad (12).$$

Hence to the *left* of equations (35), (36), § 40, are contributed for a steady field

$$\begin{aligned}\theta\dot{f}_g = \theta\{&-\mathrm{S}\mathbf{C}\,(1 + \mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla.)\,\varpi_\theta\Theta\\ &+ \mathrm{S}\mathbf{C}_1\,(1 + \mathrm{S}\mathbf{H}_{1\mathrm{H}}\nabla.)\,\varpi_1\nabla_1\} \quad . \quad (13).\end{aligned}$$

$$\theta\dot{f}_{sg} = -\theta\left[\mathrm{S}\mathbf{C}\,(1 + \mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla.)\,\varpi\mathrm{U}\nu\right]_{a+b} \quad . \quad . \quad (14).$$

By equations (34), § 50

$$\begin{aligned}\mathrm{B}_g &= -4\pi_{\mathrm{H}}\nabla\mathrm{S}\mathbf{D}\varpi\Theta\\ &= 4\pi\,\{\mathrm{V}\mathbf{B}\mathbf{D}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{D}\Theta\} \quad . \quad . \quad (15).\end{aligned}$$

Theory of Irreversibility.

x contains a term g given by

$$g = -\mathrm{S}\mathbf{K}\varpi\Theta \quad . \quad . \quad . \quad . \quad (8\text{A}).$$

This contributes terms $\mathbf{E}_g$, $\mathbf{E}_{sg}$ to the right of equations (29), (31), § 50, given by

$$\mathbf{E}_g = -\varpi\Theta\,, \quad \mathbf{E}_{sg} = 0 \quad . \quad . \quad (9\text{A}).$$

In this is not included the part of **a**, due to g, but this is practically given by equation (16A) below.

Contributed to the *right* of equations (35), (36), § 40, are terms (for any field, steady or not) given by

$$\begin{aligned}\theta\dot{f}_g = &-\mathrm{S}\mathbf{K}\,(2\,[0] + 3\,[1]\\ &+ 4\,[2])\,\Theta - \theta\mathrm{S}\mathbf{K}_1\varpi_1\nabla_1 \quad . \quad . \quad (13\text{A}).\end{aligned}$$

$$\theta\dot{f}_{sg} = \theta\left[\mathrm{S}\mathbf{K}\varpi\mathrm{U}\nu\right]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14\text{A}).$$

Theory of Reversibility.

Hence for a steady field

$$\dot{\mathbf{B}}_g = 4\pi\,(\mathrm{V}\mathbf{B}\mathbf{C}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{C}\Theta) \quad . \quad (16).$$

and if this is the only part of $\dot{\mathbf{B}}$ for a steady field, we have

$$0 = \mathrm{S}\nabla\dot{\mathbf{B}}_g = 4\pi\mathrm{S}\nabla\,(\mathrm{V}\mathbf{B}\mathbf{C}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{C}\Theta) \quad . \quad . \quad (17).$$

$$0 = [\mathrm{SU}\nu\dot{\mathbf{B}}_g]_{a+b} = 4\pi\,[\mathrm{SU}\nu\,(\mathrm{V}\mathbf{B}\mathbf{C}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{C}\Theta)]_{a+b} \quad . \quad . \quad . \quad (18).$$

The following equations will be explained below :—

$$\sigma = \theta\,(\partial\mathrm{P}/\partial\theta - d\mathrm{P}/d\theta) \quad . \quad . \quad (19).$$

$$\Pi = \theta\,[\mathrm{P}]_{a-b} \quad . \quad . \quad . \quad . \quad . \quad (20).$$

$$\frac{d(\Pi/\theta)}{d\theta} + \frac{[\sigma - \theta\mathrm{P}_\theta]_{a-b}}{\theta} = 0 \quad . \quad . \quad (21).$$

$$\left.\begin{aligned} \mathrm{E} &= -\int \mathrm{S}\,d\rho\mathbf{E}_g = \int_{\theta_0}^{\theta} (\Pi/\theta)\,d\theta \\ &= \Pi - \Pi_0 + \int_{\theta_0}^{\theta} [\sigma - \theta\mathrm{P}_\theta]_{a-b}\,d\theta \end{aligned}\right\} \quad . \quad (22).$$

Contributed to equation (27) § 55 we have

$$\phi_g' = 2m^{-1}\mathrm{S}\mathbf{D}\varpi_1\Theta.\chi\mathfrak{C}_1\chi' \quad . \quad . \quad (23).$$

Hence, for a steady field,

$$\phi_g' = 2m^{-1}\mathrm{S}\mathbf{C}\varpi_1\Theta.\chi\mathfrak{C}_1\chi' \quad . \quad . \quad (24).$$

Theory of Irreversibility.

For any field, steady or otherwise,

$$\mathbf{b}_g = 4\pi\,(\mathrm{V}\mathbf{B}\mathbf{K}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{K}\Theta) \quad . \quad (16\mathrm{A})$$

and if g is the only term in x containing $\mathbf{H}$, we have

$$0 = \mathrm{S}\nabla\mathbf{b}_g = 4\pi\mathrm{S}\nabla\,(\mathrm{V}\mathbf{B}\mathbf{K}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{K}\Theta) \quad . \quad (17\mathrm{A}).$$

$$0 = [\mathrm{SU}\nu\mathbf{b}_g]_{a+b} = 4\pi\,[\mathrm{SU}\nu\,(\mathrm{V}\mathbf{B}\mathbf{K}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{K}\Theta)]_{a+b} \quad . \quad . \quad . \quad (18\mathrm{A}).$$

The following equations will be explained below :—

$$\sigma = -\,(2\mathrm{A} + \theta\,d\mathrm{A}/d\theta) \quad . \quad . \quad (19\mathrm{A}).$$

$$\Pi = \theta\,[\mathrm{A}]_{a-b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20\mathrm{A}).$$

$$\frac{d(\Pi\theta)}{d\theta} + \theta\,[\sigma]_{a-b} = 0 \quad . \quad . \quad . \quad (21\mathrm{A}).$$

$$\left.\begin{aligned} \mathrm{E} &= -\int \mathrm{S}\,d\rho\mathrm{E}_g = -\int_{\theta_0}^{\theta} (\Pi/\theta)\,d\theta \\ &= \Pi - \Pi_0 + \int_{\theta_0}^{\theta} [\sigma]_{a-b}\,d\theta \end{aligned}\right\} \quad (22\mathrm{A}).$$

83. Before discussing these equations in detail, it will be shown how in the theory of reversibility certain very important restrictions must be imposed on the generality of ϖ as a function of θ, Ψ, and $\mathbf{H}$, in order that known experimental facts shall not be contradicted. These restrictions seem, for the most part, to depend on the particular form of theory adopted in this paper; but as the particular features of the theory which are thus involved are held by many physicists, it is of interest to notice exactly what part of our fundamental assumptions causes the restrictions. The direct cause of the trouble is the equation $\mathbf{B} = 4\pi_{\mathbf{H}}\nabla l$. Now, it will be remembered (§ 16 above) that this equation flows from the assumptions (1) that l contains $\mathbf{H}$ and not $\mathbf{B}$ explicitly, and (2) that $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$. If, then, it can be proved experimentally that the consequences of the restrictions developed below (the chief of which is that thermomagnetic phenomena involve dissipation of energy, and that a part of the thermoelectric phenomena do the same), are contrary to fact; one or both of these assumptions must be relinquished. Thus, perhaps, in this unexpected quarter, will be found a practical test of the truth of MAXWELL's fundamental assumption $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$. The consequences of relinquishing either of the above assumptions would probably be much the same, since it would lead to the physicist being compelled to recognise with

Professor J. J. Thomson ['Applications,' § 17 (4)] magnetic coordinates independently of electric coordinates. It is interesting to note in this connection that Professor J. J. Thomson (*ibid.*, § 59) working on somewhat different lines from this paper, has also found that thermomagnetic phenomena have a distinct bearing on the equation $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$. His conclusion is that this equation must be given up, and that instead we shall have

$$\mathrm{V}\nabla\mathbf{H} = 4\pi\mathbf{C} + (4\pi/3)\,\mathrm{B}\,(\Theta\mathrm{S}\nabla\mathrm{d} - \mathrm{S}\Theta\nabla\,.\,\mathrm{d}),$$

B being here assumed to be a scalar. He assumes that thermomagnetic phenomena are reversible.

84. The first equation that challenges attention is (16). It might be thought that it was a truism that $\mathbf{B}$ should remain constant in a steady field. This, however, is not the case. If the steady increase of $\mathbf{B}$ implied by this equation does not produce a steady increase in some physical quantity which can be measured directly, the field will remain steady in the ordinary sense though $\mathbf{B}$ increases. Now the physically measurable phenomena depending on $\mathbf{B}$ can be conveniently divided into three groups, (1) the stress at the point — which leads to mechanical phenomena capable of measurement, (2) the effect it has in modifying the value of $\mathbf{H}$ at all points of the field — which again leads to mechanical phenomena capable of measurement, (3) its effect on the induction of currents. As to (1) it is certainly true that in many instances above $\mathbf{B}$ does occur in the expression for the stress at a point, but this is in the case of solids purely a mathematical result. By equation (27) § 55 we see that the rate of variation of $\mathbf{B}$ will in general have no effect on the stress at a point. With regard to (2) the question must be asked, how does the value of $\mathbf{B}$ affect the values of $\mathbf{H}$, $\mathbf{C}$, &c., at points other than that considered? The answer is — solely by reason of the equations $\mathrm{S}\nabla\mathbf{B} = 0$, $[\mathrm{SU}\nu\mathbf{B}]_{a+b} = 0$, where it must be remembered $\mathbf{B}$ is an explicit function of $\mathbf{H}$, $\mathbf{C}$, &c. If then the rates of variation $\mathrm{S}\nabla\mathbf{B}$ $[\mathrm{SU}\nu\mathbf{B}]_{a+b}$ due to g are zero, the steady increase of $\mathbf{B}_g$ will not produce a steady increase of $\mathbf{H}$, $\mathbf{C}$, &c., at any point of space. But these conditions are insured by equations (17), (18). Hence we see that $\dot{\mathbf{B}}_g$ need not cause time variation of any mechanical phenomena. As to (3) the only electric effect of $\mathbf{B}$ (due to $-\dot{\mathbf{A}}$ in $\mathbf{E}$) is one which remains constant so long as $\dot{\mathbf{B}}$ remains constant, and, therefore, does not affect the steadiness of the field. Hence equation (16) presents no difficulty.

Equations (15), (17), and (18), however, do present very formidable difficulties. It has been stated that the influence of $\mathbf{B}$ on the electromagnetic quantities of the field is due to the equations $\mathrm{S}\nabla\mathbf{B} = 0$, $[\mathrm{SU}\nu\mathbf{B}]_{a+b} = 0$. It must now be added that the manner in which it thus affects the field depends on the form of its expression in terms of $\mathbf{H}$, $\mathbf{C}$, &c. Now unless the principal term in $\mathbf{B}$ is one depending on $\mathbf{H}$ only, the behaviour of the body in question would not at all approximate to the magnetic behaviour we know that bodies exhibit. For instance, we know that bismuth always behaves very approximately as if it were non-magnetic. This requires that the principal term in $\mathbf{B}'$ should be $\mu_0\mathbf{H}'$. It is needless to say that this will not in general be the case if $\mathbf{B}$ involves $\mathbf{D}$. In fact, in a steady field, the only conditions we should be able to assert that $\mathbf{B}$ imposed upon the field would be those which resulted from equations (17), (18), which do not contain $\mu_0\mathbf{H}'$ at all.

85. The conclusion we are bound to come to is that on the present theory there must be no term in l which will result in ${}_{\mathrm{H}}\nabla l$ containing $\mathbf{D}$. Thus the thermomagnetic effects must be left to the theory of irreversibility.

It might be thought that all difficulty would vanish if, in g, instead of $\mathbf{D}$ we substituted $\mathbf{d}$, since this would involve terms in $\mathbf{e}$ similar to those given above for $\mathbf{E}$ and since $\mathbf{E} = -\mathbf{e}$. This, however, is not the case. No term in $\mathbf{e}$ can affect steady currents. The equation $\mathbf{E} + \mathbf{e} = 0$ determines the value of $\mathbf{d}$ but does not affect the value of $\mathbf{C}$. Again, since in equations (13) (14) we should have $\mathbf{c}$ instead of $\mathbf{C}$, we see that no thermal effects would, owing to g, occur in a steady field.

86. None of these difficulties and restrictions are met with in the theory of irreversibility. We shall find, however, that the thermoelectric consequences of that theory are inconsistent with known facts. Both theories, therefore, are assumed as true in part.

In connection with thermoelectricity it is necessary first to establish equations (19) to (22) and (19A) to (22A). In these equations the notation adopted is that of Professor CHRYSTAL'S article in the 'Encyc. Brit.,' 9th ed., vol. viii., p. 97. He there considers a circuit of two unstressed isotropic metals a and b, one of the junctions being at temperature θ, and the other at temperature θ_0. The positive direction round the circuit is taken as that from the metal a to the metal b at the junction θ. Π, σ* are the PELTIER and THOMSON effects at the temperature θ, and Π_0 the PELTIER effect at the temperature θ_0. E is the electromotive force round the circuit due to thermoelectric effects.

To consider such a case as this, B and C are, of course, ignored, and A is supposed a scalar. For the future, in order to distinguish more clearly between the theories, P will be substituted for A on the theory of reversibility. It is necessary now to distinguish between $\partial/\partial\theta$ and $d/d\theta$. The former denotes differentiation when θ, Ψ are taken as independent variables. Now in all the commoner experiments on thermoelectricity, different parts of the circuit are not similarly strained, but similarly stressed. This may be taken account of by regarding Ψ as a function of θ. Regarding it as such $d/d\theta$ denotes *total* differentiation with regard to θ.

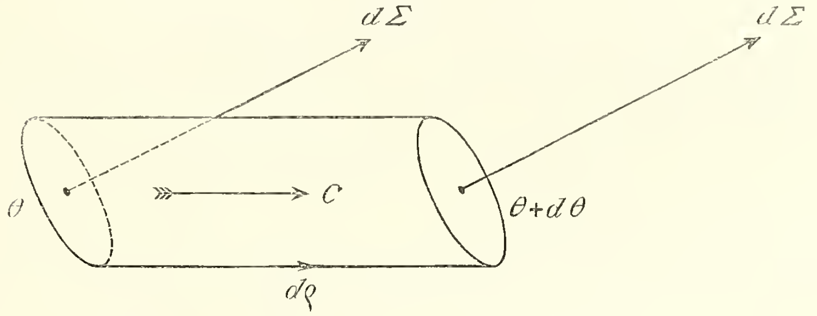
87. σ, the THOMSON effect, is defined by saying that the heat "absorbed" by the metal between two sections at temperature θ and $\theta + d\theta$, while a unit quantity of electricity passes in the direction from the first to the second section, is $\sigma d\theta$, or the rate at which heat is absorbed in this part equals the rate at which electricity is flowing

---

* There seems no danger in using σ here for this *scalar*, though in the rest of the present paper it is taken as the type of an intensity (*vector*), nor in using E here for the electromotive force round the circuit, though in the rest of the paper it stands for intrinsic energy.

in the assigned direction × $\sigma d\theta$. Consider an elementary (generally oblique) cylinder whose generating lines are parallel to $\mathbf{C}$ or $\mathbf{K}$ and whose faces are coincident with the isothermal surfaces θ and $\theta + d\theta$. Let $d\rho$ be the vector in the direction from the section θ to the section $\theta + d\theta$, representing a generating line, and $d\Sigma$ the vector area of either face, drawn inwards at the section θ and outwards at the section $\theta + d\theta$ (fig. 2). The rate of "absorption" of heat means the rate at which energy disappears as heat (positive when it causes *fall* of temperature) and appears as some other form of energy, in the present case that of "electrical separation." Taken per unit volume of the standard position of matter this $= \theta \times$ what is contributed to the *left* of equation (35), § 40. [The truth of this statement should be clearly recognised. The principal term on the left of equation (35), § 40, is $c\dot{\theta}/\theta$ where c is the capacity for heat per unit volume. Any other positive term contributed to this side will therefore tend to render $\dot{\theta}$ negative.] Hence the rate of absorption of heat per unit volume due to the terms now under consideration will be + the expression on the right of equation (13), § 82, and − the expression on the right of equation (13A). Putting $\varpi =$ the scalar P or A, and noting that $S\nabla\mathbf{C} = 0$, and that for a steady field $S\nabla\mathbf{K} = 0$, we see that on the theory of reversibility the rate of absorption of heat for our element $-Sd\Sigma d\rho$ of volume is

Fig. 2.



$$\theta(-P_\theta S\mathbf{C}\nabla\theta + S\mathbf{C}\nabla P)(-Sd\Sigma d\rho)$$

and on the theory of irreversibility

$$(2AS\mathbf{K}\nabla\theta + \theta S\mathbf{K}\nabla A)(-Sd\Sigma d\rho).$$

Now $\nabla P = \nabla\theta \,.\, dP/d\theta$ and $\nabla A = \nabla\theta \,.\, dA/d\theta$, and since $\mathbf{C}$ is parallel to $d\rho$, these two may be interchanged in the expressions just given. Thus the rates of absorption of heat on the theories of reversibility and irreversibility respectively, are

$$-\theta(-\partial P/\partial\theta + dP/d\theta)\, Sd\rho\nabla\theta S\mathbf{C}\, d\Sigma$$

and

$$-(2\mathrm{A}+\theta\, d\mathrm{A}/d\theta)\, \mathrm{S}d\rho\nabla\theta\mathrm{S}\mathbf{C}\, d\Sigma.$$

But the rate of absorption of heat also $= \sigma\, d\theta \times$ the rate of flow through the element from θ to $\theta + d\theta$

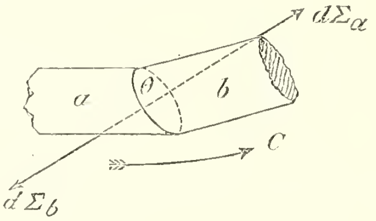
$$= \sigma\mathrm{S}\, d\rho\nabla\theta\mathrm{S}\mathbf{C}\, d\Sigma.$$

Equating these different expressions for the same quantity, we get equations (19) and (19A).

Equations (20) and (20A) are obtained in an exactly similar way. It need only now be said that from equation (14) the rate of absorption of heat at an element of the boundary of the metals a and b [fig. 3] is

$$\theta\,[\mathrm{PS}\mathbf{C}\, d\Sigma]_{a+b} = -\,\theta\,[\mathrm{P}]_{a-b}\,\mathrm{S}\mathbf{C}\, d\Sigma_a,$$

Fig. 3.



and also that by the definition of the PELTIER effect Π, it is $= \Pi \times$ the rate of flow from the metal a to the metal b

$$= -\,\Pi\mathrm{S}\mathbf{C}\, d\Sigma_a,$$

from which equation (20) follows. Similarly for equation (20A). Equations (21) and (21A) are easily deduced from those now established.

88. In connection with equations (22) and (22A) it is advisable to make what may be looked upon as a digression, to examine whether, on the present theory, we have a right to identify the line integral of any part of $\mathbf{E}$ round a circuit with what, in the laboratory, is known as the electromotive force of the particular kind round it. To test this, we must see whether the total line integral of R$\mathbf{K}$ round the circuit = what is called the whole resistance of the circuit × the whole current.

In equations (29), (31), § 50 occurs a scalar $y + \mathrm{Y}$ or v which, unlike the other terms in the equations, does not depend directly or indirectly (as is the case with $d\mathbf{A}/dt + \mathbf{a}$) upon the form of l or x. Consider now any closed curve which, if it anywhere crosses a surface of discontinuity, passes, we shall suppose, from the region a to the region b. Then, in the expression $-\int \mathrm{S}\mathbf{E}\, d\rho - \Sigma\mathrm{S}\mathbf{E}_s\mathrm{U}\nu_a$ this unknown scalar does not appear as can be easily seen by equations (29), (31). Before taking this line integral, remove R$\mathbf{K}$ to the left of equation (29), § 50, keeping all the other terms on the right.

It is the line integral (including the terms contributed by $\Sigma \mathrm{S}\mathbf{E}_s \mathrm{U}\nu_a$*) that thus appears on the right, which is ordinarily called the total electromotive force round the curve.*

Let us examine whether this statement is consistent with the one above about whole current and whole resistance. Suppose the motion steady so that **K** obeys the laws of incompressibility. Consider an infinitely small tube of flow, and let this be the line along which the integral is taken. Let c be the whole current flowing along the tube. Consider an elementary right section of the tube of length $\mathrm{T}d\rho$, and cross-section $\mathrm{T}d\Sigma$. Thus

$$\mathrm{K} = c\mathrm{UK}/\mathrm{T}d\Sigma, \quad d\rho = \mathrm{UKT}d\rho,$$

and, therefore, the part contributed to the integral $-\int \mathrm{S}d\rho \mathrm{RK}$ by the element in question is

$$-c\mathrm{SUKRUK}\,.\,\mathrm{T}d\rho/\mathrm{T}d\Sigma.$$

If then we choose to define as follows: (1) $-$ SUKRUK $=$ the specific resistance of the body at the point, (2) specific resistance × $\mathrm{T}d\rho/\mathrm{T}d\Sigma =$ the resistance of the element, (3) the sum of the resistances of all the elements $=$ the whole resistance of the tube, we shall have

$$-\int \mathrm{S}d\rho\mathrm{RK} = \textit{current flowing along tube} \times \textit{whole resistance of tube},$$

or

$$\textit{conductivity of tube} \times (-\int \mathrm{S}d\rho\mathrm{RK}) = \textit{current flowing along tube},$$

which defines the conductivity as the reciprocal of the whole resistance. Now split any finite tube of flow into an infinite number of such elementary tubes, call the sum of the conductivities of the elementary tubes the whole conductivity of the finite tube, and call the reciprocal of this last the whole resistance of the finite tube. We shall then have that the *mean* of the values of $-\int \mathrm{S}d\rho\mathrm{RK}$ for the elementary tubes $=$ the whole resistance of the finite tube × the whole current along it. All this may, I think, be said to be in complete agreement with the ordinary theory, but it serves to call attention to the fact—important in connection with the longitudinal effect mentioned in § 79 above—that anything which interferes with the ordinary lines of flow will alter the apparent resistance.

89. To return to our immediate purpose, we are now at liberty to say that the line integral of any term contributed to the right of equation (29), § 50, round a closed circuit implies an equal electro-motive force round the circuit in the ordinary sense. Equations (22) and (22A) are easily seen to follow.

Comparing, now, equations (20), (21), (22), (20A), (21A), and (22A), with equations (4), (5), (6), and (7) on p. 97, vol. 8, 'Encyc. Brit.,' 9th ed., we see the results of the theory of reversibility only differ from the ordinary theory in having $\sigma - \theta \mathrm{P}_\theta$ in place of σ, while those of the theory of irreversibility differ widely.

Thus the former theory explains thermoelectric effects satisfactorily. But we shall see that we cannot suppose A zero. Hence we must on the present theory suppose that the main thermoelectric effects are reversible, but that there are subsidiary irreversible ones that with the present means of experiment it would be practically impossible to disentangle from the former.

90. The detailed comparison between the two theories is most clearly made by means of the thermoelectric diagram. From equation (22A) we see that on the theory of irreversibility the thermoelectric power, instead of being Π/θ, is $-\Pi/\theta$. To any one who is acquainted with the ordinary thermoelectric diagram, the following statements will be sufficiently obvious from the accompanying figures:—

Theory of Reversibility.

Abscissa = θ.

Ordinate = − thermoelectric power with respect to lead

$$= -\frac{\Pi\,(\text{lead})}{\theta} = -\mathrm{P}.$$

E = the area marked in fig. 4.

$\sigma - \theta\mathrm{P}_\theta$ = PS of fig. 5.

Π = + area marked in fig. 6.

$\Pi - \Pi_0$ = + area marked in fig. 7.

$\int_{\theta_0}^{\theta}[\sigma - \theta\mathrm{P}_\theta]_{a-b}\,d\theta$ = − area marked in fig. 8.

Theory of Irreversibility.

Abscissa = θ.

Ordinate = − thermoelectric power with respect to lead

$$= +\frac{\Pi\,(\text{lead})}{\theta} = +\mathrm{A}.$$

E = the area marked in fig. 4.

σ = − 3PR of fig. 5 = − 2PQ − PS.

Π = − area marked in fig. 6.

$\Pi - \Pi_0$ = − area marked in fig. 7.

$\int_{\theta_0}^{\theta}[\sigma]_{a-b}\,d\theta$ = + (area marked in fig. 8).
+ 2 (area marked in fig. 4).
= + (area marked in fig. 7).
+ (area marked in fig. 4).

Fig. 4.

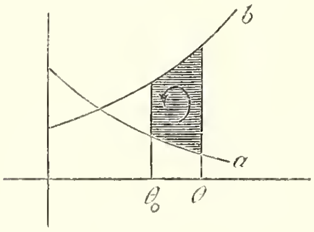
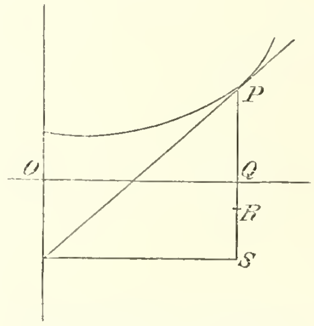


Fig. 5.



[In this figure OQ is the axis and QS = 3QR.]

Fig. 6.

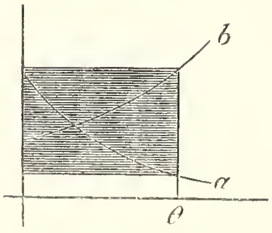


Fig. 7.

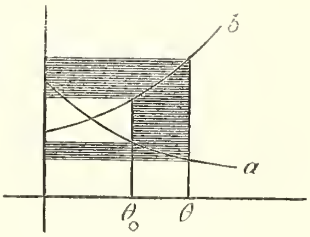
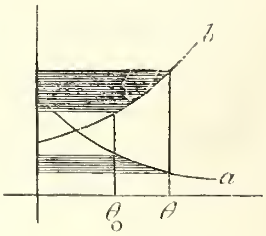


Fig. 8.



The following results may be noted :—

No series of experiments confined to determinations of the electromotive forces resulting from differences of temperature at the junctions of thermoelectric circuits can distinguish between the two theories.

On the theory of reversibility the following statement is true; on the theory of irreversibility the contrary is true. In a thermoelectric circuit of two metals, if a galvanic current be passed across the junction in the same direction as that of the current that would be produced by heating the junction, the effect is absorption, and vice versâ [TAIT's 'Heat,' 1st ed., § 192]. In many cases this statement has been verified experimentally, and no one, I think, has ever asserted that he has obtained the contrary. Hence the main thermoelectric effects *cannot* be explained on the theory of irreversibility.

On the theory of reversibility $\sigma - \theta P_\theta$ *takes the place of* σ *in the ordinary theory.*

91. This last statement is of importance in connection with a difficulty purposely passed over till now. By equation (23) § 82 it appears that **D** enters into the expression for the stress unless P be independent of the strain. Thus the stress depends on the electric history of the substance. This involves difficulties of two kinds. First it shows that in all bodies for which P is not thus independent the stress would probably be widely different from what it is ordinarily assumed to be. This, however, would not affect the apparent mechanical effect on a conductor as a whole since, as already noticed, that effect depends upon the stress just outside the conductor, *i.e.*, the stress in the surrounding *dielectric* in which **D** does not increase indefinitely. But one would think that its effects on the mutual behaviour of different parts of a conductor would have been observed. The second difficulty is connected with equation (24). It might be thought a truism that ϕ' should remain constant in a steady field. But as with $\mathbf{B}_g'$ (§ 84) this is not the case. We saw in § 75 that so long as any term which contributes to ϕ' contributes zero to $\phi'\Delta'$ and $[\phi'\mathrm{U}\nu']_{a+b}$, it produces no effect on the motion and strains of bodies, that is, no mechanical effect whatever. The conditions for a steady field, so far as stress is concerned, are, therefore, that $\dot{\phi}'\Delta' = 0$ and $[\dot{\phi}'\mathrm{U}\nu']_{a+b} = 0$. If then g is the only term in l containing **D**, equation (24) gives for a steady field

$$m^{-1}\mathrm{S}\mathbf{C}\varpi_1\Theta \,.\, \chi\mathrm{C}_1\chi'\Delta' = 0,\ [m^{-1}\mathrm{S}\mathbf{C}\varpi_1\Theta \,.\, \chi\mathrm{C}_1\chi'\mathrm{U}\nu']_{a+b} = 0 \quad . \quad . \quad . \quad (25).$$

To see what *may* be the approximate effect of this, let us assume ϖ to be a scalar (P) whatever be the strain, and let the scalar be a function of the temperature and the density of the body only. Then

$$\begin{aligned}\dot{\phi}_g' &= 2m^{-1}\mathrm{S}\mathbf{C}\Theta \,.\, \chi\mathrm{C}\mathrm{D}_m'\chi' \,.\, \partial\mathrm{P}/\partial\mathrm{D}_m' \\ &= 2\mathrm{D}_m\, m^{-1}\mathrm{S}\mathbf{C}\Theta \,.\, \chi\mathrm{C}\,(m^{-1})\,\chi' \,.\, \partial\mathrm{P}/\partial\mathrm{D}_m' \ [\S\ 49] \\ &= -\ \mathrm{D}_m'\mathrm{S}\mathbf{C}'\Theta' \,.\, \partial\mathrm{P}/\partial\mathrm{D}_m' \ [(10)\ \S\ 54 \text{ and II. } \S\ 8].\end{aligned}$$

From the first of equations (25) it follows that the scalar $SC'\Theta' D_m' \partial P/\partial D_m'$ must be constant throughout any single conductor; and from the second, that it must be constant throughout any number of conductors in contact, and if the conductors are anywhere bounded by a dielectric, *i.e.*, *invariably*, this constant value must be zero. Hence with present assumptions $SC'\Theta'$ must be zero everywhere in a steady field. [This is not quite accurate since if P, regarded as a function of D_m' is a maximum or minimum, $\partial P/\partial D_m' = 0$. It does not seem hopeful to pursue this supposition however.] This presents no difficulty in ordinary cases, since **C** and Θ—to drop the dashes as no longer necessary—would generally be very approximately at right angles in any case. If, however, we contemplate such a case as the attempt by ordinary means to force a galvanic current and a stream of heat in the same direction through a conductor, some very curious consequences are involved. The most obvious of them seem to be that both the heat and electric apparent conductivities would be *largely* altered. That no such *large* alterations in these physical quantities have been observed I believe to be the case. These difficulties may be wholly imaginary. If, which seems on other grounds most probable, P is not even approximately a scalar when the body is strained, we should not be able to deduce that $SC\Theta$ was even approximately zero. In this case, the adjustments brought about in a steady field of the kind just contemplated by reason of the equations (25), would probably be mainly strain adjustments that would not cause **C** and Θ to vary much, if at all, from parallelism. These strains, of course, might very well have hitherto escaped detection.

These difficulties are, however, sufficiently serious to make it necessary to consider the results of assuming P independent of the strain. The most important of these results are easily seen to be (1) that, although the connections between the PELTIER effect and the electromotive forces in a circuit of different metals whose junctions are at different temperatures would on the theory of reversibility be the same as is usually supposed, yet there would on that theory, taken alone, be no THOMSON effect [equation (19)], and (2) that there would be no thermoelectric effects in a circuit of a single metal whose various parts were variously stressed. These two, then, would have to be explained on the theory of irreversibility, and no quantitative connection need be expected between the THOMSON effect and the main thermoelectric effects.

92. These difficulties seem to me not to be confined to the particular form of theory developed in this paper. For instance, there seems as much reason to suppose Professor J. J. THOMSON'S σ_x, σ_y, σ_z ('Applications,' 1st ed., § 53) to be independent of the strain as the present P. And I may remark in passing that similar statements may be made with regard to the C′ (*ibid.*, § 43) introduced to explain the HALL effect. [By § 84 above, it is obvious that on the present theory it is impossible to explain the HALL effect by such a term owing to the results other than the HALL effect that would ensue from the term.]

93. Our chief conclusions, so far, may be thus summarised:—

(1.) *If* P *be assumed to be dependent upon* Ψ, *the theory of reversibility suffices to explain all the known experimental facts of thermoelectricity.*

(2.) *If, as there is some reason to believe,* P *be independent of* Ψ, *the main thermoelectric effects must be explained on the theory of reversibility, but the* THOMSON *effect and the thermoelectric effects observed in a circuit of a single metal differently stressed in different parts must be explained on the theory of irreversibility. In this case there is no such connection between the* THOMSON *effect and other thermoelectric effects as is usually supposed.*

(3.) *On the present theory it is impossible to explain thermomagnetic phenomena by the theory of reversibility.*

94. There is little to be said with regard to the thermomagnetic phenomena themselves, as our knowledge of them is almost confined to what is expressed by equations (2) and (9A). It is necessary to remark, however, that the C of equation (2) may be—probably is—not the main cause of what has been described in § 79 above as the longitudinal effect, since an apparent longitudinal effect would be caused (§ 88 above) by any interference with the lines of flow of electricity, and by the variation in the resistance due to any cause. More than one effect of these kinds will be noticed below.

But, of course, the existence of B and C may involve results of a kind other than thermomagnetic, which are practically measurable. Besides equations (2) and (9A), B and C occur in equations (13A) to (18A). Equations (13A) (14A) do not require notice, since in the present state of accuracy of experimental knowledge of thermoelectric quantities the influence of B and C in these equations is negligible. With regard to equations (16A) to (18A), we can trace approximately the effects of B and C in one important class of cases.

95. Suppose we have a plate of uniform thickness (small) in which a current is flowing placed in a strong uniform magnetic field. On account of the current, of course, the uniformity will be disturbed, but only slightly if the strength is great enough. Outside the plate (except in certain conducting wires) we assume that there is no current. Hence by equation (18A) we see that at every point of the boundary of the plate

$$\mathrm{SU}\nu\,(\mathrm{V}\mathbf{BK}\Theta - 2\mathrm{C}\mathbf{H}\mathrm{S}\mathbf{K}\Theta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

Equation (17A) gives by equation (4) § 5 for any portion of the plate

$$\iint \mathrm{S}d\Sigma\,(\mathrm{V}\mathbf{BK}\Theta - 2\mathrm{C}\mathbf{H}\mathrm{S}\mathbf{K}\Theta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

This is the form in which can be most easily discussed the effect of equation (17A). Let the region to which equation (27) refers be taken as a cylinder, one face of the cylinder being in one face of the plate, and the parallel face somewhere inside the plate. Since the plate is thin, we may suppose the faces of the cylinder large compared with the curved surface, and may, therefore, neglect the portion contributed to the integral of equation (27) by the curved surface compared with the rest of the integral. Now by equation (26) the part of the integral contributed by the face of the plate is zero. Hence putting i for $\mathrm{U}\nu$ we have, *approximately*, at any point of the plate

$$\mathrm{S}i\,(\mathrm{VB}\mathbf{K}\Theta - 2\mathrm{C}\mathbf{H}\mathrm{S}\mathbf{K}\Theta) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (28).$$

Assuming B and C to be scalars, this may be put in the form

$$\mathrm{S}i\mathbf{K}^{-1}\Theta = 2\mathrm{CB}^{-1}\,\mathrm{S}i\mathbf{H}\mathrm{S}\mathbf{K}^{-1}\Theta \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (29).$$

Now assuming, which will be approximately—*exactly* at the boundary—true, that i is perpendicular to $\mathbf{K}$, we have

$$\Theta = -\,i\mathrm{S}i\Theta + \mathbf{K}\mathrm{S}\mathbf{K}^{-1}\Theta + i\mathbf{K}\mathrm{S}i\mathbf{K}^{-1}\Theta.$$

Hence by equation (2), § 81,

$$\begin{aligned}\varpi\,(\Theta + i\mathrm{S}i\Theta) = (\mathrm{A} - \mathrm{C}\mathbf{H}^2)\,(\mathbf{K}\mathrm{S}\mathbf{K}^{-1}\Theta + i\mathbf{K}\mathrm{S}i\mathbf{K}^{-1}\Theta)&\\ + \mathrm{B}\,(\mathrm{V}\mathbf{K}\mathbf{H}\mathrm{S}\mathbf{K}^{-1}\Theta + \mathrm{V}i\mathbf{K}\mathbf{H}\mathrm{S}i\mathbf{K}^{-1}\Theta).&\end{aligned}$$

Let us split this vector up into its components parallel to the three vectors $\mathbf{K}$, i, and $\mathrm{V}\mathbf{K}\mathbf{H}$. For this purpose notice that since $\mathbf{K}$ is perpendicular to i,

$$\mathrm{V}i\mathbf{K}\mathbf{H} = -\,\mathbf{K}\mathrm{S}i\mathbf{H} + i\mathrm{S}\mathbf{K}\mathbf{H}$$

$$i\mathbf{K} = (\mathrm{V}\mathbf{K}\mathbf{H} + i\mathrm{S}i\mathbf{K}\mathbf{H})/\mathrm{S}i\mathbf{H}.$$

Thus

$$\begin{aligned}\varpi\,(\Theta + i\mathrm{S}i\Theta) = \mathbf{K}\{&(\mathrm{A} - \mathrm{C}\mathbf{H}^2)\,\mathrm{S}\mathbf{K}^{-1}\Theta - \mathrm{B}\mathrm{S}i\mathbf{K}^{-1}\Theta\mathrm{S}i\mathbf{H}\}\\ &+ i\mathrm{S}i\mathbf{K}^{-1}\Theta\{(\mathrm{A} - \mathrm{C}\mathbf{H}^2)\,\mathrm{S}i\mathbf{K}\mathbf{H}/\mathrm{S}i\mathbf{H} + \mathrm{B}\mathrm{S}\mathbf{K}\mathbf{H}\}\\ &+ \mathrm{V}\mathbf{K}\mathbf{H}\{(\mathrm{A} - \mathrm{C}\mathbf{H}^2)\,\mathrm{S}i\mathbf{K}^{-1}\Theta/\mathrm{S}i\mathbf{H} + \mathrm{B}\mathrm{S}\mathbf{K}^{-1}\Theta\} \quad . \; . \quad (30)\end{aligned}$$

from which, by substituting for $\mathrm{S}i\mathbf{K}^{-1}\Theta$ from equation (29), we obtain

$$\begin{aligned}\varpi\,(\Theta + i\mathrm{S}i\Theta) = \mathrm{B}^{-1}\mathrm{S}\mathbf{K}^{-1}\Theta\,[&\mathbf{K}\mathrm{B}\,\{\mathrm{A} - \mathrm{C}\,(\mathbf{H}^2 + 2\mathrm{S}^2 i\mathbf{H})\}\\ &+ i2\mathrm{C}\,\{(\mathrm{A} - \mathrm{C}\mathbf{H}^2)\,\mathrm{S}i\mathbf{K}\mathbf{H} + \mathrm{B}\mathrm{S}\mathbf{K}\mathbf{H}\mathrm{S}i\mathbf{H}\}\\ &+ \mathrm{V}\mathbf{K}\mathbf{H}\,\{2\mathrm{C}\,(\mathrm{A} - \mathrm{C}\mathbf{H}^2) + \mathrm{B}^2\}]\,. \; . \; . \; . \; . \; . \quad (31).\end{aligned}$$

96. This transformation is not likely to give us clearer ideas of what takes place when a stream of heat is made to flow in the plate which is large compared with the

streams due to ordinary electric resistance and thermoelectric phenomena. The original form of ϖ is more suitable for that purpose. We assume then that the only heat effects are due to purely electrical causes. In this case, if the faces of the plate are thermally similar, we may assume that $Si\Theta$ has opposite values at points situated symmetrically on opposite sides of the plane midway between the faces of the plate. The effect of $\varpi i Si\Theta$ will be then merely to make the current stronger or weaker in the middle of the plate than near the faces, and, therefore, to increase the apparent resistance of the plate. We have already seen that unless P (assumed invariably to be a scalar in this connection) be independent of the strain, $S\mathbf{K}^{-1}\Theta$ is zero, so that the whole expression on the right of equation (31) is zero. If, however, P be independent of the strain, the term in $V\mathbf{KH}$ in equation (31) would indicate a HALL effect. The presence of $S\mathbf{K}^{-1}\Theta$ in this term, however, serves to show that probably this is not the true explanation of the HALL effect.

The HALL effect may be explained by saying that there is an electromotive force $hV\mathbf{KH}$, where h may be called the coefficient of the HALL effect. It has been found experimentally that this coefficient is by no means independent of $T\mathbf{H}$—that, in fact, in certain cases it changes sign when a definite strength of magnetic field is reached. The above work indicates how, on the present theory, we may seek to explain such an effect. For this purpose it must be remembered that equations (30) and (31) are only true if g is the only term in x which involves $\mathbf{H}$.

97. Let us now assume that the electric resistance is a function of $\mathbf{H}$, and let us incorporate in g the term of x thus depending upon $\mathbf{H}$. We must add a term $-S\mathbf{K}r\mathbf{K}/2$ to the former value of g; r being a function (of Class II. of § 9) of θ, Ψ and $\mathbf{H}$. For the sake of definiteness give r the form $-bS\mathbf{H}c\mathbf{H}$, where b, c are functions of the same classes as B and C respectively. Thus, as can be easily seen, to get r' or r'' from r we have merely to change b, c and $\mathbf{H}$ into b', c', and $\mathbf{H}'$ or b'', c'', and $\mathbf{H}''$ respectively. Equations (8A), (9A), (13A), and (16A) must now be changed to

$$g = -S\mathbf{K}\varpi\Theta + S\mathbf{K}b\mathbf{K}S\mathbf{H}c\mathbf{H}/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

$$\mathbf{E}_g = -\varpi\Theta + b\mathbf{K}S\mathbf{H}c\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

$$\theta\dot{f}_g = -S\mathbf{K}\left(2\left[0\right] + 3\left[1\right] + 4\left[2\right]\right)\Theta + 2S\mathbf{K}b\mathbf{K}S\mathbf{H}c\mathbf{H} - \theta S\mathbf{K}_1\varpi_1\nabla_1 \quad . \quad (34),$$

$$\mathbf{b}_g/4\pi = VB\mathbf{K}\Theta - 2C\mathbf{H}S\mathbf{K}\Theta - c\mathbf{H}S\mathbf{K}b\mathbf{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

Hence, in place of equation (28) we now have

$$Si\left(VB\mathbf{K}\Theta - 2C\mathbf{H}S\mathbf{K}\Theta - c\mathbf{H}S\mathbf{K}b\mathbf{K}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).$$

Assuming B, C, b, c, to be all scalars, and putting, as is permissible in this case, $b = 1$, instead of equation (29) we have

$$\mathrm{S}i\mathbf{K}^{-1}\Theta = \mathrm{B}^{-1}\,\mathrm{S}i\mathbf{H}\,.\,(2\mathrm{CS}\mathbf{K}^{-1}\Theta + c) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (37),$$

and in place of (31),

$$\begin{aligned}\mathbf{E}_g - \varpi i\mathrm{S}i\Theta = -\,\mathrm{B}^{-1}[&\mathbf{K}\mathrm{B}\,\{\mathrm{S}\mathbf{K}^{-1}\Theta[\mathrm{A} - \mathrm{C}\,(\mathbf{H}^2 + 2\mathrm{S}^2 i\mathbf{H})] - c\,(\mathbf{H}^2 + \mathrm{S}^2 i\mathbf{H})\} \\ &+ i\,(2\mathrm{CS}\mathbf{K}^{-1}\Theta + c)\{(\mathrm{A} - \mathrm{C}\mathbf{H}^2)\,\mathrm{S}i\mathbf{K}\mathbf{H} + \mathrm{B}\mathrm{S}\mathbf{K}\mathbf{H}\mathrm{S}i\mathbf{H}\} \\ &+ \mathrm{V}\mathbf{K}\mathbf{H}\,\{(2\mathrm{CS}\mathbf{K}^{-1}\Theta + c)\,(\mathrm{A} - \mathrm{C}\mathbf{H}^2) + \mathrm{B}^2\mathrm{S}\mathbf{K}^{-1}\Theta\}] \quad . \quad . \quad . \quad . \qquad (38)\end{aligned}$$

which simplifies when $\mathrm{S}\mathbf{K}^{-1}\Theta$ is zero (which it certainly approximately is in the experiments made to determine the coefficient of the HALL effect, whether P be independent of Ψ or not) to

$$\begin{aligned}\mathbf{E}_g - \varpi i\mathrm{S}i\Theta = \mathrm{B}^{-1}c\,[&\mathbf{K}\mathrm{B}\,(\mathbf{H}^2 + \mathrm{S}^2 i\mathbf{H}) \\ &- i\,\{(\mathrm{A} - \mathrm{C}\mathbf{H}^2)\,\mathrm{S}i\mathbf{K}\mathbf{H} + \mathrm{B}\mathrm{S}\mathbf{K}\mathbf{H}\mathrm{S}i\mathbf{H}\} \\ &- \mathrm{V}\mathbf{K}\mathbf{H}\,(\mathrm{A} - \mathrm{C}\mathbf{H}^2)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (39).\end{aligned}$$

Owing to the term in i on the right as well as the term in i on the left, there may be an apparent increase of resistance. The term in $\mathbf{K}$ shows that there will also be an increase $-\,c\,(\mathbf{H}^2 + \mathrm{S}^2 i\mathbf{H})$ in the resistance. The term in $\mathrm{V}\mathbf{K}\mathbf{H}$ shows that the present assumptions lead to a HALL effect, whose coefficient $= -\,\mathrm{B}^{-1}\,c\,(\mathrm{A} - \mathrm{C}\mathbf{H}^2)$.

With regard to the new term $2\mathrm{S}\mathbf{K}b\mathbf{K}\mathrm{S}\mathbf{H}c\mathbf{H}$ in equation (34), it should be noticed that since it is quadratic in $\mathbf{K}$, it would have no influence on the apparent THOMSON effect, but only upon the apparent resistance as measured by heat effects.

That we can explain the HALL effect on the present theory is of some interest, because, as remarked in § 92 above, we cannot explain it on the present theory in the ordinary way. Nor can we hope to explain it by the term $-\,{}_{\mathbf{K}}\nabla x$ in $\mathbf{E}$ [equations (29), § 50, and (20), § 35] for $\mathrm{V}_{\mathbf{K}}\nabla\mathrm{V}\mathbf{K}\mathbf{H} = \mathrm{V}\zeta\mathrm{V}\zeta\mathbf{H} = -2\mathbf{H}$, whereas $\mathrm{V}_{\mathbf{K}}\nabla\ ({}_{\mathbf{K}}\nabla x) = 0$. It should be noticed that the difficulties in the way of explaining the HALL effect by a term $-\,\mathrm{C}'\mathrm{S}\mathbf{C}\mathbf{D}\mathbf{H}/2$ do not apply to explaining the magnetic rotation of the plane of polarised light, since this is equally well explained by substituting $\mathbf{d}$ for $\mathbf{D}$.

98. One effect of g still remains for consideration. It is necessary to consider it if only to show that it leads to no results large enough to be experimentally tested. It also helps to show how the various interferences with the lines of flow, several times mentioned above, are mainly brought about.

In writing down equation (9A) it was mentioned that $\mathbf{E}_g$ did not contain the part of $\mathbf{a}$ due to g. We have indirectly taken account of the effect of $\mathbf{a}_g$ in part, by the considerations just given in leading up to the explanation of the HALL effect. We have not, however, thereby taken full account of $\mathbf{a}_g$. To do this in the manner illustrated above, we should require to study the effect $\mathbf{b}_g$ had in modifying the whole electromotive force instead of the part $\mathbf{E}_g$. We proceed then, to a more general examination of the effect of the term $-\,\mathbf{a}_g$ in $\mathbf{E}$.

We will now drop the suffix g from $\mathbf{a}$ and $\mathbf{b}$, since we shall not suppose x to contain

H except by reason of the term g, so that $\mathbf{b}_g$ of equation (35) now stands for the full value of **b**. $S\nabla\mathbf{a}$ and $[SU\nu\mathbf{a}]_{a+b}$ are arbitrary. Let us, since it does not affect any physically measurable quantity, assume them both to be zero. Since $\mathbf{b} = V\nabla\mathbf{a}$, $[Vd\Sigma\mathbf{a}]_{a+b} = 0$, if we considered an analogue in which **b** stood for an electric current, $4\pi\mathbf{a}$ would be the magnetic force due to the currents in a space containing no magnets. This analogue will serve to give us a very fair general idea of the effect of $-\mathbf{a}$ in **E**. Since [equation (35)] every term of **b** contains **K**, **b** will be confined to conductors where there are conduction currents. Thus, in the same conductor we shall have the "real current" and the "current of the analogue." The current of the analogue may be disposed with reference to the real current in one of three ways. It may be mainly parallel to the real current, or it may circulate round the real current mainly at right angles to it, or it may circulate round it spirally. In the first and third cases we see by the analogy that there would be an electromotive force due to **a** in the general field approximately parallel everywhere to the part of the magnetic force due to the real current. In the conductor itself, then, the resulting electromotive force would cause the real current to move spirally, and would, therefore, apparently increase the resistance. In the second and third cases we see by the analogy that there would result effects due to the local state of affairs, so that where **b** was large there the effect of **a** on **E** would be large. In a case of this sort we should have to examine further before we could say what the local effect would be.

It is easy to see that in the experiments for determining the quantities connected with the HALL and thermomagnetic effects, the second of the above cases very approximately represents the state of affairs. For, by equation (35), it is only in the plate, where Θ or **H** is very large compared with the rest of the circuit, that **b** will have a sensible value. Hence there must be a strong local current of the analogue, that is, a current which does *not* go round the circuit parallel to the real current.

It follows that the main physical effects of **a** are those that were considered in dealing with the HALL effect.

E. *Contact Electromotive Force.*

99. Our knowledge of this is not very accurate, but, besides the fact that contact-force certainly exists, and that it has been in numerous individual cases measured with fair accuracy, the following seems to stand out with considerable certainty.

If the (apparent) electromotive force from one material, a, to another, b, when they are in contact be denoted by $a \mid b$, then the equation

$$a \mid b + b \mid c + c \mid a = 0 \qquad \text{. . . . . (1)}$$

is true if all three materials are conductors, but is not true if they are not all conductors.

This force cannot apparently be explained by any term in l of the kind we have hitherto supposed l to contain. Suppose, then, l to contain a term* $\mathrm{S}\mathbf{d}a\Delta$ where a is of the same class as A or ϖ [equations (5), § 81.] Thus $a\sigma$ is an intensity, and $a_c\tau$ a flux, a_c being the conjugate of a.

The portion of L contributed by this term is $\iiint \mathrm{S}\mathbf{d}a\Delta d\varsigma$, or [eq. (4), § 5], $\iint \mathrm{S}\mathbf{d}a d\Sigma$. Hence the effects of supposing l to contain a term $\mathrm{S}\mathbf{d}a\Delta$ are identical with those of supposing l_s to contain a term $[\mathrm{S}\mathbf{d}a\mathrm{U}\nu]_{a+b}$.

So far as electric phenomena are concerned it is quite easy to see the effect of this term. In place of equations (31), § 50, and (2), § 57, we shall have

$$\mathbf{E}_s = [v\mathrm{U}\nu]_{a+b} = [a\mathrm{U}\nu]_{a+b} \quad \ldots\ldots\ldots \quad (2),$$

or, if a be a scalar,

$$[v]_{a-b} = [a]_{a-b} \quad \ldots\ldots\ldots \quad (3).$$

100. Although this term involves a contact force it does not explain the known facts, since, as can be easily seen, the contact force here obtained is such that equation (1) would be invariably true. We seem then to be driven to the conclusion that to explain contact force, l_s cannot any longer be assumed to be zero.

Adopting the suggestion of Professor CHRYSTAL, 'Encyc. Brit.,' 9th ed., vol. 8, p. 85, we will assume that there is no real contact force between conductors. This simplification is not, of course, necessary on the present theory, but the simpler the assumptions—so long as they are not intrinsically improbable—the better. Professor CHRYSTAL shows that the assumption serves to explain all the known facts, the apparent contact force between conductors being explained by the difference of their contact forces with one and the same dielectric.

We now assume that l_s contains the term† $\mathrm{S}a\mathrm{U}\nu_a[\mathbf{d}]_{a+b}/2$ where a is of the same class as before, but now depends on Ψ_a, Ψ_b and θ, and where of course the suffix a has nothing to do with the linear vector function a. It is assumed that a is zero for a surface of separation of two conductors.

In place of equations (2), (3), we now have

$$\mathbf{E}_s = [v\mathrm{U}\nu]_{a+b} = a\mathrm{U}\nu_a. \quad \ldots\ldots\ldots \quad (4)$$

$$[v]_{a-b} = a \quad \ldots\ldots\ldots \quad (5).$$

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* It may be asked, Why make the differentiations act on a as well as $\mathbf{d}$, why not assume the term to be $\mathrm{S}\mathbf{d}_1 a\nabla_1$? The answer is that this leads to a more complicated result, namely (1), to the same contact force as the term chosen, (2), to a term $-a\Delta$ in $\mathbf{E}$, and (3), to a stress which involves the space derivatives of $\mathbf{d}$. It is best to assume, if possible, a term which involves what we know to be true and nothing more.

† Perhaps it would be better, as more general to suppose l_s to contain the term $[\mathrm{S}\alpha\mathrm{U}\nu\,\mathbf{d}]_{a+b}$ where α is of the same class as a, and α_b is not merely characteristic of the substance b, but depends on *both* the substances bounded, *i.e.*, where, in general, $\alpha_{a-b} + \alpha_{b-c} + \alpha_{c-a}$ is not zero. Equations (4), (5), (6) will still be true if we put $a = \alpha_{a-b}$.

It might be thought that equation (4) could not be true in general unless a was a scalar in general. This, however, is not the case. From equation (4) it certainly follows that $\mathrm{VU}\nu a\mathrm{U}\nu = 0$, but this, by virtue of the dependence of a on strain is merely an equation of condition satisfied by the strain. The equation $[v\mathrm{U}\nu]_{a+b} = a\mathrm{U}\nu_a$ may indeed be written

$$[v]_{a-b} = -\mathrm{U}\nu a\mathrm{U}\nu \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6),$$

which is, of course, more general than equation (5), since here a is not assumed a scalar. It may be noted that $\mathrm{V}\, d\Sigma' a'\, d\Sigma' = m\chi \mathrm{V}\, d\Sigma a\, d\Sigma$, so that the two conditions, $\mathrm{VU}\nu' a' \mathrm{U}\nu' = 0$ and $\mathrm{VU}\nu a\mathrm{U}\nu = 0$, are identical.

101. It should be remarked here that on the present theory the term just introduced would have no thermal effect in steady fields, and, therefore, no connection with the PELTIER effect. (See § 85 above.)

We have been obliged to suppose l_s no longer zero. Before discussing the modifications this entails in the general results above, the last application in the present paper of those results will be made.

This is the place where it would be proper to discuss electrolysis in connection with the present theory. This I do not propose to do, because the mathematical machinery of this paper would require some important modifications to enable us to deal with such subjects as diffusion, the motion of the ions, &c., and because the subject is a very large one, and would, perhaps, unduly extend the length of the present paper.

F. *The Transference of Energy through the Field.*

102. On the present theory, in which the principle enunciated in equations (24), (25), § 36, required strong confirmation, it was necessary to show that it agreed in every particular with the generally accepted views as to frictional forces being derivable from a dissipation function in Lord RAYLEIGH'S sense, and also with the much more certainly established truths treated of in the theory of conduction of heat. The only way to establish this last seemed to be to show that as a result of the principle there was a time flux of intrinsic energy, one term of which was what, in the theory of conduction of heat, is called the time flux of heat. This led to the necessity of finding the time flux of intrinsic energy in general. We are thus brought on to ground which has hitherto been regarded as belonging exclusively to Professor POYNTING—the transference of energy through the field.

103. Let us now examine how far the results of the present theory are consistent with those of Professor POYNTING. Let $\mathbf{L}$ be a flux such that

$$\dot{\mathrm{E}} - \mathrm{P}_e = \iint_b \mathrm{S}\mathbf{L}\, d\Sigma \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1),$$

so that $\mathbf{L}$ may be called the time flux of intrinsic energy. By Proposition VIII., § 10,

$$(\phi' + \Phi_f')\, d\Sigma' = \chi(\phi + \Phi_f)\, d\Sigma.$$

Hence we see by equation (25), § 49, that

$$\mathbf{L} = -(\phi + \Phi_f)\,\chi'\dot{\rho}' + v\mathbf{C} - \mathrm{V}(\dot{\mathbf{A}} + \mathbf{a})\,\mathbf{H}/4\pi - (\dot{\theta}_\Theta \nabla\lambda + \theta_\Theta \nabla x) \quad . \quad . \quad (2).$$

Now, ('Phil. Trans.,' 1884, Part II., pp. 343 to 349) Professor POYNTING's result expressed in similar notation would be (calling the time flux of energy $\mathbf{P}$)

$$\mathbf{P} = -(\phi + \Phi_f)\,\chi'\dot{\rho}' - \mathrm{V}\{\nabla v + (\dot{\mathbf{A}} + \mathbf{a})\}\,\mathbf{H}/4\pi - (\dot{\theta}_\Theta \nabla\lambda + \theta_\Theta \nabla x). \quad . \quad (3).$$

It is scarcely necessary to remark that we have here generalised his expression* by the insertion of the terms $-(\phi + \Phi_f)\,\chi'\dot{\rho}' - \mathrm{V}\mathbf{aH}/4\pi - (\dot{\theta}_\Theta \nabla\lambda + \theta_\Theta \nabla x)$ and have substituted for his $\mathbf{E}$ what he means by it, namely, $-(\dot{\mathbf{A}} + \nabla v)$. It might be thought at first that this is not quite what he means by $\mathbf{E}$ since he incorporates in it terms depending on the motion of the body with reference to the lines of magnetic induction. Remembering, however, that equation (6), § 60, and the equation $\mathbf{E}_0 = -(\dot{\mathbf{A}} + \nabla v)$ are identical, it will be seen that these terms have been here incorporated.

104. The *direct* interpretation of equations (2) and (3) is, of course, widely different. Let us see if they have the same physical significance, that is, whether they lead to the same rate of increase of intrinsic energy in any finite space.

For this purpose it must be asked whether or not $\iint_b \mathrm{S}(\mathbf{L} - \mathbf{P})\, d\Sigma$ is zero. Now

$$4\pi(\mathbf{L} - \mathbf{P}) = 4\pi v\mathbf{C} + \mathrm{V}\nabla v\mathbf{H} = \mathrm{V}\nabla(v\mathbf{H}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Hence $4\pi\iint_b \mathrm{S}(\mathbf{L} - \mathbf{P})\, d\Sigma = \iint_b \mathrm{S}\, d\Sigma \nabla(v\mathbf{H})$, or by equation (3), § 5,

$$4\pi\iint_b \mathrm{S}(\mathbf{L} - \mathbf{P})\, d\Sigma = \int_b v\mathrm{S}\, d\rho\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where the line integral† is to be taken over all lines of discontinuity on the true

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* I only say—generalised his *expression*—since *some such* terms as have been added in the text would, on Professor POYNTING's own theory, be included in the vector $\mathbf{L}$, defined by equation (1), as the time flux of *intrinsic* energy. The result of the present paper is, however, in all strictness much more general than his, since it has not among other things been assumed that all the bodies in space are isotropic with reference to specific inductive capacity, resistance, and magnetic permeability.

† It may be well to notice that by the conventions of § 5, above, if the closed curve be regarded as bounding, not the regions of the true boundary, but the part of the surface of discontinuity in the region of space under consideration, the sign of the line integral must be changed.

boundary of the region considered, that is, over the trace on that surface of surfaces of discontinuity. The element $d\rho$ is, of course, taken twice, namely, once for each of the two regions of the true boundary which it bounds. Since $d\rho$ is in the surface of discontinuity, and the component of $\mathbf{H}$ parallel to that surface is not discontinuous, we see that $[\mathrm{S}d\rho\mathbf{H}]_{a+b} = 0$. Hence the part contributed by $d\rho$ to the line integral may be written $[v]_{a-b}\,[\mathrm{S}d\rho\mathbf{H}]_a$ or

$$v_{a-b}\mathrm{S}d\rho_a\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

If then v is continuous, the line integral is zero. It has already appeared [equation (2), § 57] that if l_s is everywhere zero this is the case. Hence, unless l_s exist, the physical results of supposing $\mathbf{P}$ to be the time flux of intrinsic energy are identical with those of supposing $\mathbf{L}$.

If l_s exist, we have at present no right to say that on the present theory $\mathbf{L}$ may be taken as the time flux; but in § 111, below, this will be proved. The conclusion is then that, to explain the rate of variation of energy, Professor POYNTING's flux $\mathbf{P}$ must be supplemented by a finite flux $\mathbf{P}_s$ along surfaces of discontinuity in the potential, where

$$4\pi\mathbf{P}_s = [v\mathrm{VU}\nu\mathbf{H}]_{a+b} = v_{a-b}\mathrm{VU}\nu_a\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

[In verifying the sign of this expression attention must be paid to the caution in the last foot-note.] This of course is rather an unnatural, though by no means absurd, result, and therefore I think it better to regard $\mathbf{L}$ as, more probably than $\mathbf{P}$, representing the true time flux of intrinsic energy. Another reason for preferring $\mathbf{L}$ to $\mathbf{P}$ is that for a field at rest, *i.e.*, such that $\dot{\rho}'$, $\dot{\theta}$, Θ, $\mathbf{C}$ and $\mathbf{c}$ are everywhere zero, $\mathbf{L}$ is zero, whereas $\mathbf{P} = \mathrm{V}\mathbf{H}\nabla v/4\pi$.

In now contrasting Professor POYNTING's result with that of the present paper, we will suppose v continuous.

105. Very shortly after the first publication of Professor POYNTING's paper, Professor J. J. THOMSON in criticising it remarked ('B. A. Reports,' 1885, p. 151). "This [Professor POYNTING's] interpretation of the expression for the variation in the energy seems open to question. In the first place it would seem impossible, *à priori*, to determine the way in which the energy flows from one part of the field to another by merely differentiating a general expression for the energy in any region, with respect to the time, without having any knowledge of the mechanism which produces the phenomena which occur in the electromagnetic field; for although we can, by means of HAMILTON's or LAGRANGE's equations, deduce from the expression for the energy the forces present in any dynamical system, and therefore the way in which the energy will move, yet for this purpose we require the energy to be expressed in terms of the coordinates fixing the system, and it will not do to take any expression which happens to be equal to it. The problem of finding the way in which the energy is transmitted in a system whose mechanism is unknown, seems to be an

indeterminate one ; thus, for example, if the energy inside a closed surface remains constant we cannot, unless we know the mechanism of the system, tell whether this is because there is no flow of energy either into or out of the surface, or because as much flows in as flows out. The reason for this difference between what we should expect and the result obtained in this paper is not far to seek." He then goes on to point out* how, so far from **P** being necessarily the time flux of energy, $\mathbf{P} + \mathrm{V}\nabla\epsilon$ where ϵ is any vector, such that at surfaces of discontinuity $[\mathrm{VU}\nu\epsilon]_{a+b} = 0$, might equally well be taken as the time flux of energy. It so happens that (assuming v continuous) $\mathbf{L} - \mathbf{P}$ is such a vector, so that the difference between the result arrived at in this paper and Professor POYNTING's is just such a case as Professor THOMSON warned us to expect.

We cannot then say that either **L** or **P** is the time flux of energy, but only that if we assume either the one or the other (**P** being supplemented with $\mathbf{P}_s$) to be the flux, the real changes of intrinsic energy will be accounted for.

106. Notwithstanding Professor THOMSON's warning, many subsequent writers seem to have taken Professor POYNTING's theories for established facts. The following statement of Professor POYNTING especially seems to have grown to be accepted almost universally as a commonplace truth [' Phil. Trans.,' 1884, Part II., p. 361]:—"I think it is necessary that we should realise thoroughly, that if we accept MAXWELL's theory of energy residing in the medium, we must no longer consider a current as something conveying energy along the conductor. A current in a conductor is rather to be regarded as consisting essentially of a convergence of electric and magnetic energy from the medium upon the conductor, and its transformation there into other forms." Now, if we take **L** as the true time flux of energy, we see that one way in which we *must* regard a current is precisely the way Professor POYNTING denies us, namely, "as something conveying energy along the conductor." In fact, from the term $v\mathbf{C}$ in **L**, we see that in this respect, as in so many others, a current and the potential are the exact analogue of a liquid current and its pressure. Without doubt, the view that **L** is the true flux is simpler for steady fields than the view that **P** is. This statement is not so obvious—perhaps on the whole not true—for varying fields.

It is easy to contrast in detail the two views in all the particular cases Professor POYNTING considers. This may, therefore, be omitted here.

G. *The General Effects of the Existence of l_s.*

107. The general equations above established must now be modified on account of l_s. In considering electrolysis on the present theory it would be necessary to suppose l_s to contain **D** or **C** or both, as well as **d**. For the sake of simplicity we shall not

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[*] This is not put quite in the form Professor THOMSON puts the case.

make this supposition. l_s will be assumed a function, then, of d_a, d_b, θ, Ψ_a, Ψ_b. As throughout the rest of this paper, we assume that there is no slipping at the surface. This leads to a relation between Ψ_a and Ψ_b. Let i, j, k as usual stand for a set of mutually perpendicular unit vectors, which are, however, functions of the position of a point. Let

$$i = \mathrm{U}\nu_a, \quad i' = \mathrm{U}\nu_a' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

j and k are thus parallel to the surface.

We have

$$\Psi\omega = \Sigma i \mathrm{S} i \Psi i \mathrm{S} i\omega + \Sigma\,(j\mathrm{S}k\omega + k\mathrm{S}j\omega)\,\mathrm{S}j\Psi k$$

$$= \Sigma i \mathrm{S} i\omega\,(\chi i)^2 + \Sigma\,(j\mathrm{S}k\omega + k\mathrm{S}j\omega)\,\mathrm{S}\chi j \chi k,$$

where the summation sign implies that i, j, k are to be changed cyclically. Since there is no slipping the strains in the surface of each region bounded are the same or

$$\chi_a j = \chi_b j, \qquad \chi_a k = \chi_b k$$

Hence, putting

$$2\overline{\Psi} = \Psi_{a+b}, \qquad 2\Gamma = \Psi_{a-b} i = [\Psi \mathrm{U}\nu]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

it at once follows that

$$\Psi_{a-b}\omega = -\,2\Gamma \mathrm{S} i\omega + 2i\,(\mathrm{S}j\Gamma\mathrm{S}j\omega + \mathrm{S}k\Gamma\mathrm{S}k\omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Thus Ψ_a and Ψ_b can be expressed in terms of the independent variables $\overline{\Psi}$ and Γ. That these last are independent is seen thus. The deformation in the neighbourhood of a point on the bounding surface requires for complete specification a knowledge of the following three things : (1) the pure strain of an element of the surface, (2) the displacement of a point in the region a near to the element of surface relative to the latter when purely strained, (3) a similar displacement in the region b. These three are independent, and each requires three scalars to specify it—nine in all, the same number as is required to specify $\overline{\Psi}$ and Γ. Thus l_s is a function of the variables d_a, d_b, $\overline{\Psi}$ and Γ.

108. The part of δl_s depending on $\delta\Psi_a$ and $\delta\Psi_b$ is $-\,\mathrm{S}\delta\overline{\Psi}\zeta\overline{\text{Ɑ}}l_s\zeta - \mathrm{S}\delta\Gamma_\Gamma\nabla l_s$* where $\overline{\text{Ɑ}}$ stands for ${}_{\overline{\Psi}}\text{Ɑ}$. Put now

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* We may dispense with Γ altogether thus. Put $\overline{\Psi} + \mathrm{V}\Gamma(\quad) = \Pi$. Then with the meaning of ${}_\Pi\text{Ɑ}$ explained on page 103 of former paper,

$${}_\Gamma\nabla = \mathrm{V}\zeta_\Pi\text{Ɑ}\zeta, \; \overline{\text{Ɑ}} = {}_\Pi\text{Ɑ} - \mathrm{V}_\Gamma\nabla(\quad)/2,$$

i.e., $\overline{\text{Ɑ}}$ is the pure part, and ${}_\Gamma\nabla/2$ the rotation-vector of ${}_\Pi\text{Ɑ}$.

$$m_s = ds'/ds = \mathrm{T}d\Sigma'/\mathrm{T}d\Sigma = m\mathrm{T}\chi'^{-1}i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

$$\left.\begin{aligned} \{\phi_s'\} &= -\, m_s^{-1}\{\chi\overline{\daleth}l_s\chi' - \tfrac{1}{2}[\chi_{\Gamma}\nabla l_s\mathrm{S}(\quad)\chi\mathrm{U}\nu + \chi\mathrm{U}\nu\mathrm{S}(\quad)^{3}_{\varepsilon}\chi_{\Gamma}\nabla l_s]\} \\ \phi_s' &= [\{\phi_s'\}]_{a+b} \end{aligned}\right\} \quad . \quad . \quad (5).$$

Then it is easy to show, after the manner of § 39, that

$$\delta l_s = m_s[\mathrm{S}\delta\rho_1'\{\phi_s'\}\nabla_1']_{a+b} - \mathrm{S}\delta\mathrm{d}\,[{}_{\mathrm{d}}\nabla l_s]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

To see what modifications must be made in equation (9), § 45, consider first the first term on the right of equation (6). This contributes to $\delta\mathrm{L}$ for a finite portion of space $\iint \mathrm{S}\delta\rho_1'\,\{\phi_s'\}\,\nabla_1' ds'$. Put in this $\nabla_1' = -\,i'\mathrm{S}i'\nabla_1' - i'\mathrm{V}i'\nabla_1'$. Thus

$$\iint \mathrm{S}\delta\rho_1'\{\phi_s'\}\,\nabla_1' ds' = -\iint \mathrm{S}i'\nabla_1'\mathrm{S}\delta\rho_1'\,\{\phi_s'\}\,i'ds' - \iint \mathrm{S}\delta\rho_1'\,\{\phi_s'\}\,(i'\mathrm{V}i'\nabla_1')\,ds'.$$

109. We will anticipate somewhat here, as the effect is a considerable simplification. The first of the terms on the right involves the vector $-\mathrm{S}i'\nabla'.\delta\rho'$, and this is the only term involving space derivatives of $\delta\rho'$ that cannot be transformed into terms involving $\delta\rho'$ only. Now the vector coefficient of this vector, like that of $\delta\rho'$ in $\delta\mathrm{L} + \Sigma\mathrm{Q}\delta q$, must be zero. For $-\,\mathrm{S}i'\nabla'.\delta\rho'$ is the normal space rate of variation of $\delta\rho'$, and it is clear that we can *at the surface* change this arbitrarily without changing $\delta\rho'$ at any point of space. [This cannot be said of the tangential space rates of variation of $\delta\rho'$, for a change in these causes a change of the same order of magnitude in $\delta\rho'$ at all points of the surface.] Hence we obtain the equation

$$\phi_s' i' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

or by equation (5), since $i' = mm_s^{-1}\chi'^{-1}i$,

$$\left.\begin{aligned} &[m\chi\,\{\daleth l_s\mathrm{U}\nu + \tfrac{1}{2}\,({}_{\Gamma}\nabla l_s - \mathrm{U}\nu\mathrm{S}\mathrm{U}\nu_{\Gamma}\nabla l_s)\}]_{a-b} \\ &\qquad = [m\chi]_{a+b}\,\overline{\daleth}l_s i + \tfrac{1}{2}\,[m\chi]_{a-b}\,({}_{\Gamma}\nabla l_s - i\mathrm{S}i_{\Gamma}\nabla l_s) = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (8).$$

The geometrical meaning of equation (7) should be noticed. It reduces the six coordinates of the self-conjugate ϕ_s' to three. ϕ_s' operating on any vector reduces it to the tangent plane. It may be said to act only on vectors in the plane and to strain them in the plane.

110. We now have

$$\iint \mathrm{S}\delta\rho_1'\,\{\phi_s'\}\,\nabla_1'\,ds' = -\iint \mathrm{S}\delta\rho_1'\phi_s'\,(i'\mathrm{V}i'\nabla_1')\,ds',$$

ds' on the left being taken, as usual, twice, but on the right only once. That we may substitute $S\delta\rho_1'\phi_s'(i'Vi'\nabla_1')$, or $S\delta\rho_1'[\{\phi_s'\}]_{a+b}(i'Vi'\nabla_1')$ for $[S\delta\rho_1'\{\phi_s'\}(i'Vi'\nabla_1')]_{a+b}$ is obvious, from the fact that the space derivatives of $\delta\rho'$ involved are only the tangential ones, which are the same for both regions bounded, because there is no slipping. The boundary of the surface in question, like the boundary of any volume, must be supposed to involve not only the geometrical boundary, but also any lines of discontinuity on it. With this meaning for the boundary, we have, by equation (3), § 5,

$$\iint S d\rho_1'\{\phi_s'\}\nabla_1'\,ds' = -\int S\delta\rho'\phi_s'(i'd\rho') + \iint S\delta\rho'\phi_{s1}'(i_1'Vi'\nabla_1')\,ds' \quad . \quad . \quad (9).$$

The geometrical meaning of $i_1'Vi'\nabla_1'$ should be noticed. By equation (3), § 5, we have

$$\iint i_1'Vi'\nabla_1'\,ds' = \int i'd\rho' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

from which, by limiting the portion of surface considered to an element bounded by lines of curvature, it can easily be deduced that

$$i_1'Vi'\nabla_1' = i'(r^{-1} + r'^{-1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where r, r' are the principal radii of curvature, reckoned positive or negative, according as the centre of the corresponding curvature is in the region a or b $[i' = U\nu_a']$. Thus $-i_1'Vi'\nabla_1'$, or $Vi'\nabla'.i'$ may be called the vector curvature of the surface at the point.*

Since $\phi_s'i' = 0$, we see by equation (11) that equation (9) may be written

$$\iint S\delta\rho_1'\{\phi_s'\}\nabla_1'\,ds' = -\int S\delta\rho'\phi_s'(i'd\rho') + \iint S\delta\rho'\phi_{s1}'(i'Vi'\nabla_1')\,ds' \quad . \quad (12),$$

ough this last simplification is not needed for our purposes.

111. We are now in a position to see what alterations l_s occasions in the various equations given above. This may be done in the following semi-tabular form.†

Add to right of 45 (9)

$$-\iint S\delta d\,[_d\nabla l_s]_{a+b}\,ds + \iint S\delta\rho'\phi_{s1}'(i'Vi'\nabla_1')\,ds' - \int S\delta\rho'\phi_s'(i'd\rho') \quad . \quad . \quad (13).$$

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* Using, for the moment, the notation of TAIT's 'Quaternions,' 3rd edition, §§ 296, *et seq.*, for ρ and dashes, it seems to me that lucidity would be gained by calling ρ'' the vector curvature of the curve at the point considered. Thus the vector curvature of a curve is a vector whose tensor is the ordinary scalar curvature, and which is drawn from the point on the curve in question towards the centre of curvature. By analogy, I would call the vector, drawn from a point on a surface towards the concave side, and equal in magnitude to the sum of the principal curvatures, the vector curvature of the surface at the point.

† In what follows " 45 (9) " stands for " § 45, equation (9)."

Add to 3rd term of 46 (10)

$$\iint (l_s + \lambda_s)\, ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

46 (11) *Unaltered* . . . . (15).

In place of 46 (12)

$$\lambda_s = -\, l_s, \quad \mathrm{t}_s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Add to right of 46 (13)

$$\iint (-\,\dot{\theta}\partial l_s/\partial\theta + \mathrm{S}\dot{\mathrm{d}}_{\mathrm{d}}\nabla l_s)\, ds - \iint \mathrm{S}\dot{\rho}'\phi_{s1}'(i'\mathrm{V}i'\nabla_1')\, ds' + \int \mathrm{S}\dot{\rho}'\phi_s'(i'd\rho') \quad . \quad (17).$$

Add to right of 46 (14)

$$\iint \mathrm{S}\dot{\mathrm{d}}_{\mathrm{d}}\nabla l_s\, ds - \iint \mathrm{S}\dot{\rho}'\phi_{s1}'(i'\mathrm{V}i'\nabla_1')\, ds' + \int \mathrm{S}\dot{\rho}'\phi_s'(i'd\rho') \; . \quad . \quad . \quad . \quad (18).$$

47 (15), 49 (17), 50 (26) *Unaltered* . . . . (19).

Add to right of 49 (18) and 50 (27)

$$-\,\phi_{s1}'(i'\mathrm{V}i'\nabla_1') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Besides the equation $\phi_s'i' = 0$, we here have

$$[\phi_s'(i'd\rho')]_{e+f+} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

the suffix $e+f+$ indicating two or more superficial regions bounded by the curve.

49 (19), 49 (20), 50 (28), 50 (29) *Unaltered.* . . . . (22).

In place of 49 (21) and 50 (30)

$$\mathrm{e}_s = [{}_{\mathrm{d}}\nabla l_s]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

49 (22), 50 (31) *Unaltered* . . . . (24).

49 (23), (24), (25) *Unaltered* . . . . (25),

[since all the new terms added to 46 (14) are clearly accounted for by P_c. It will be observed that this statement is true of any term of L not involving a velocity, *i.e.*, of any term which merely contributes to the potential energy].

57 (1) *Unaltered* . . . . (26).

In place of 57 (2)

$$[v\mathrm{U}\nu + {}_{\mathrm{d}}\nabla l_s]_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).^*$$

112. It is well to point out what the exact physical significance of ϕ'_s is. It implies the existence of a membranous stress, *i.e.*, a stress such as a perfectly flexible membrane could exhibit.

To investigate the properties of such a stress in a membrane coincident with the actual surface, let i', j', k' be three mutually perpendicular unit vectors, so that [equation (1), § 107] the two latter are parallel to the surface. Consider an elementary triangle in the surface at the point under consideration, whose vector edges, taken in the positive direction round the triangle, are yj', $-yj' + zk' = d\rho'$ and $-zk'$. Let $y\mathbf{F}_y$, $\mathbf{F}$, and $z\mathbf{F}_z$ be the forces exerted by the rest of the membrane on the triangle across these three faces respectively. Since all other forces on the element are of a higher order of smallness than these three, we have as the equation of motion

$$\begin{aligned}
\mathbf{F} &= -y\mathbf{F}_y - z\mathbf{F}_z \\
&= -\mathbf{F}_y\mathrm{S}j'\,d\rho' + \mathbf{F}_z\mathrm{S}k'\,d\rho' \\
&= -\mathbf{F}_y\mathrm{S}k'\,(i'\,d\rho') - \mathbf{F}_z\mathrm{S}j'\,(i'\,d\rho') \\
&= -\Phi_s'\,(i'\,d\rho'),
\end{aligned}$$

where Φ_s' is a linear vector function of a vector. This equation does not completely determine Φ_s' since $i'd\rho'$ is not perfectly arbitrary, but confined to a plane. The arbitrary part of Φ_s' having no physical bearing on the problem in hand may be chosen at will. For present purposes it is convenient to define Φ_s' completely by the equation

$$\Phi_s'\omega = -\mathbf{F}_y\mathrm{S}k'\omega - \mathbf{F}_z\mathrm{S}j'\omega,$$

where ω is a perfectly arbitrary vector. This gives

$$\Phi_s'i' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

[This is not always the most convenient way of choosing the arbitrary part of Φ_s' as, for instance, in the study of surface tension.]

Since the membrane is perfectly flexible $\mathbf{F}_y$ and $\mathbf{F}_z$ are parallel to the tangent plane and, therefore, Φ_s' only operates on vectors in the plane, and strains them in the plane. Thus Φ_s' has four disposable scalars.

Calling the side of $d\rho'$ towards which $i'd\rho'$ points the negative side, the above amounts to saying that the force exerted across the element $d\rho'$ *by* the part of the membrane on the positive side *on* the part on the negative side is $-\Phi_s'(id\rho')$. [The direction round any closed curve on the membrane, which is that of positive rotation

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* As with equation (6), § 100, this may be put in the form $v_{a-b} = [\mathrm{U}\nu\mathrm{d}\nabla l_s]_{a-b}$

round i' is, of course, considered positive. Thus, for such a closed curve, $i'd\rho'$ points inwards. This is the reason for taking the positive and negative sides as just defined. It also accounts for the sign given to Φ_s', since the latter is thus brought into harmony with the sign of the linear vector function which represents an ordinary stress.] This stress will be called the stress Φ_s'.

113. We now seek the force and couple per unit surface due to the stress Φ_s'. For this purpose, first take a finite portion of the surface. The force exerted by the stress on any portion of the surface is

$$-\int \Phi_s' \, (i'd\rho') = -\iint \Phi_{s1}' \, (i_1'\mathrm{V}i'\nabla_1') \, ds'$$

by equation (3), § 5. Hence the force per unit surface due to the stress is

$$-\Phi_{s1}' \, (i_1'\mathrm{V}i'\nabla_1').$$

Again, the couple for a finite portion of the surface round any arbitrary origin is

$$-\int \mathrm{V}\rho'\Phi_s' \, (i'd\rho') = -\iint \mathrm{V}\rho'\Phi_{s1}' \, (i_1'\mathrm{V}i'\nabla_1') \, ds' - \iint \mathrm{V}\zeta\Phi_s' \, (i'\mathrm{V}i'\zeta) \, ds'.$$

Hence (by the force per unit surface just obtained) the couple per unit surface

$$= -\mathrm{V}\zeta\Phi_s' \, (i'\mathrm{V}i'\zeta) = \mathrm{V}\zeta\Phi_s'\zeta - \mathrm{V}i'\Phi_s'i' = \mathrm{V}\zeta\Phi_s'\zeta,$$

by equation (28). Assuming, which will be the case if there be no other couple per unit surface, as is certainly true in our case, that there is no such stress couple per unit surface, we see that $\mathrm{V}\zeta\Phi_s'\zeta = 0$ or Φ_s' is self-conjugate. Thus Φ_s' is of exactly the same type as ϕ_s' and has three disposable coordinates only. [It is not necessary to assume this couple zero since the problem may be treated in an exactly similar manner to that of general stress (former paper, p. 106, *et seq.*).]

114. Now suppose Φ_s' is an "external" stress in the actual surface under consideration. The part of $\Sigma \mathrm{Q}\delta q$ due to it will be $\iint \mathrm{S}\delta\rho'\Phi_{s1}' \, (i'\mathrm{V}i'\nabla_1') \, ds' - \int \mathrm{S}\delta\rho'\Phi_s' \, (i'd\rho')$, so that the only way in which the expression (13) is affected by these new terms is that ϕ_s' must be changed into $\phi_s' + \Phi_s'$. Similarly for all the subsequent expressions in which ϕ_s' occurs. This shows that ϕ_s' is a stress of the kind contemplated.

The bearing of this on capillary phenomena will not be discussed here, because this is foreign to the objects of the present paper. It was necessary in this paper to show the general results flowing from the existence of l_s.

It should, however, be remarked that this stress, though not affecting the mechanical action of the field on a body as a whole, would affect the strains of a body, and probably be sometimes comparable in this effect with the similar effects resulting from the dependence of l on strain.

115. In conclusion, let me remark that in several respects the above investigations might be generalised. It is not hard to take account of the slipping of surfaces over one another, both with regard to reversible and irreversible phenomena. It is harder, but not impossible, to take account of the existence and relative motion in identically the same portion of space of two media, such as the ether and air, or as two different kinds of matter, as in diffusion and chemical phenomena (though, of course, in the two last cases, the two media do not probably really exist in the same portion of space —a statement not proven). I have refrained from this in the present paper for two reasons: first, to keep the length of the paper within reasonable bounds, and secondly, not to render the subject more intricate than is absolutely necessary. My aim has been not so much to establish incontrovertible results as to develope a new method of treatment, more powerful, and in reality much simpler than those which are in use to-day. If I have succeeded in convincing my readers that this method is worthy of study, the main object of the present paper is attained. Meanwhile the matters that have been just indicated can be left over for future consideration.

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TO THE

PHILOSOPHICAL TRANSACTIONS (A)

FOR THE YEAR 1892.

HARRISON AND SONS, PRINTERS IN ORDINARY TO HER MAJESTY, ST. MARTIN'S LANE, LONDON, W.C.

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—— Part II...	1	2	6	1842. Part I...	0	16	0	1863. Part I...	1	14	0	—— Part III...	1	12	0
1818. Part I...	1	10	0	—— Part II...	1	2	0	—— Part II...	1	7	6	1884. Part I...	1	8	0
—— Part II...	1	0	0	1843. Part I...	0	10	0	1864. Part I...	0	11	0	—— Part II...	1	16	0
1819. Part III...	0	10	0	—— Part II...	1	10	0	—— Part II...	1	7	6	1885. Part I...	2	10	0
1820. Part I...	0	15	0	1844. Part I...	0	10	0	—— Part III...	1	10	0	—— Part II...	2	5	0
—— Part II...	1	15	0	—— Part II...	1	10	0	1865. Part I...	2	2	0	1886. Part I...	1	8	0
1821. Part I...	1	4	0	1845. Part I...	0	16	0	—— Part II...	1	5	0	—— Part II...	1	15	0
—— Part II...	1	1	0	—— Part II...	1	0	0	1866. Part I...	1	14	0	1887. (A)	1	3	0
1822. Part I...	2	0	0	1846. Part I...	0	7	6	—— Part II...	2	7	6	—— (B.)	1	16	0
—— Part II...	1	12	0	—— Part II...	1	12	0	1867. Part I...	1	3	0	1888. (A.)	1	10	0
1823. Part II...	1	8	0	—— Part III...	1	12	0	—— Part II...	1	15	0	—— (B.)	2	17	6
1824. Part I...	0	12	6	—— Part IV...	1	12	0	1868. Part I...	2	5	0	1889. (A.)	1	18	0
—— Part II...	1	0	0	1847. Part I...	0	14	0	—— Part II...	2	0	0	—— (B.)	1	14	0
—— Part III...	1	4	0	—— Part II...	0	16	0	1869. Part I...	2	10	0	1890. (A.)	1	16	6
1825. Part I...	1	4	0	1848. Part I...	1	0	0	—— Part II...	3	3	0	—— (B.)	1	5	0
1826. Part I...	1	2	6	—— Part II...	0	14	0	1870. Part I...	1	10	0	1891. (A.)	2	2	0
—— Part II...	0	12	6	1849. Part I...	1	0	0	—— Part II...	1	18	0	—— (B.)	3	3	0
—— Part III...	2	0	0	—— Part II...	2	5	0	1871. Part I...	1	10	0	1892. (A.)	2	1	0
—— Part IV...	1	2	6	1850. Part I...	1	10	0	—— Part II...	2	5	0	—— (B.)	2	2	0

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