PHILOSOPHICAL

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OF THE

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OF

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MDCCCLXIII.

CONTENTS

OF VOL. 152.

ERRATA.

Page 163, lines 2, 9, 13 (art. XLVII.), erase the word "non-polar."
— — line 7, *for* because *read* when.
— 588, line 10 from bottom, cancel p at the end of the formula, which should be

$$v\frac{Ct}{p}-\frac{1}{3}AJK\left(\frac{273\cdot7}{t}\right)^2.$$

— 755, 7 lines from bottom, *for* Plate A. *read* Plate XXVII.
— 864, 8 lines from bottom of text, *for* fig. 2 *read* fig. 3.
— — 6 lines from bottom of text, *for* fig. 3 *read* fig. 2.
Page 869, 13 lines from bottom of text, *for* more *read* less.

LIST OF ILLUSTRATIONS.

ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1862 by the PRESIDENT and COUNCIL.

The COPLEY MEDAL to THOMAS GRAHAM, Esq., F.R.S., for three Memoirs on the Diffusion of Liquids, published in the Philosophical Transactions for 1850 and 1851; for a Memoir on Osmotic Force in the Philosophical Transactions for 1854; and particularly for a Paper on Liquid Diffusion applied to Analysis, including a distinction of Compounds into Colloids and Crystalloids, published in the Philosophical Transactions for 1861.

The RUMFORD MEDAL to Professor KIRCHHOFF, of Heidelberg, for his Researches on the fixed Lines of the Solar Spectrum, and on the inversion of the bright lines in the Spectra of artificial light.

A ROYAL MEDAL to the Rev. Dr. T. R. ROBINSON, F.R.S., of Armagh, for the Armagh Catalogue of 5345 Stars, deduced from observations made at the Armagh Observatory, from the year 1826 up to 1854; for his Papers on the Construction of Astronomical Instruments, in the Memoirs of the Astronomical Society, and his Paper on Electro-Magnets, in the Transactions of the Royal Irish Academy.

A ROYAL MEDAL to Professor ALEXANDER W. WILLIAMSON, F.R.S., for his Researches on the Compound Ethers, and his subsequent communications in Organic Chemistry.

The BAKERIAN LECTURE was delivered by WARREN DE LA RUE, Esq., F.R.S.: it was entitled "On the Total Solar Eclipse of July 18th, 1860, observed at Rivabellosa, near Miranda de Ebro, in Spain."

The CROONIAN LECTURE was delivered by Professor A. KÖLLIKER, For. Memb. R.S.: it was entitled "On the Termination of Nerves in Muscles, as observed in the Frog; and on the Disposition of the Nerves in the Frog's Heart."

ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the Council-books and Journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions*; which was accordingly done upon the 26th of March 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgement of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

The Meteorological Journal hitherto kept by the Assistant Secretary at the Apartments of the Royal Society, by order of the President and Council, and published in the Philosophical Transactions, has been discontinued. The Government, on the recommendation of the President and Council, has established at the Royal Observatory at Greenwich, under the superintendence of the Astronomer Royal, a Magnetical and Meteorological Observatory, where observations are made on an extended scale, which are regularly published. These, which correspond with the grand scheme of observations now carrying out in different parts of the globe, supersede the necessity of a continuance of the observations made at the Apartments of the Royal Society, which could not be rendered so perfect as was desirable, on account of the imperfections of the locality and the multiplied duties of the observer.

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PHILOSOPHICAL TRANSACTIONS.

I. *On the Influence of Temperature on the Electric Conducting Power of Metals.*

By Augustus Matthiessen, *F.R.S.*, *and* Moritz von Bose.

Received December 5, 1861,—Read January 16, 1862.

The results obtained by different observers in their researches on the influence of temperature on the electric conducting power of metals do not agree at all together. The differences in their results may be partly owing to their not having tested pure metals, and partly to their not having taken into consideration the fact that, when a wire of a pure metal is heated for the first time to 100° C., an alteration in the conducting power of the wire is observed on its again being cooled; in fact, it is necessary to keep the wire for several days at 100° before its conducting power, on again being cooled, becomes constant.

In the experiments we are about to detail we have taken great care to employ only pure metals, as well as a method and a disposition of the apparatus with which great accuracy could be obtained.

The method employed for the determination of the resistances is fully described in the 'Philosophical Magazine' for February 1857. Fig. 1 shows the disposition of the apparatus. B is the trough in which the wires were heated: these were soldered to two thick copper wires F (4–5 millims. thick), bent as shown in the figure, and ending in the mercury-cups E, which were connected with the apparatus by two other copper wires, F′, of the same thickness. C is a piece of board placed in such a manner as to prevent the heat of the trough from radiating on the apparatus. The mercury-cups O are made of small blocks of wood, through which holes are bored just large enough to take the thick wires, and to the bottoms of which blocks amalgamated copper plates are fastened. Now it is clear that if the ends of the thick copper wires are filed flat, and well amalgamated, and the mercury-cups are filled with mercury, this method of connexion may be looked upon as a soldering of the copper plates to the wires, or, in other words, as a perfect connexion; for the wires may be removed as often as required, and on replacing them the same resistance is always observed. The wires F″, to which the normal wire (in the glass cylinder G) is soldered, are also 4–5 millims. thick. The reason why such thick wires were chosen was to make any difference in their resistance, caused by the change of temperature in the room or by the heating of the ends in the oil-bath, so small that no correction was necessary. This was proved to be the case by the following experiment:—After having soldered a wire in the trough to the ends of the thick copper wires, and determined its resistance with the normal wire generally used, the wire F′ at *e* was heated with the 6-Bunsen burner much above

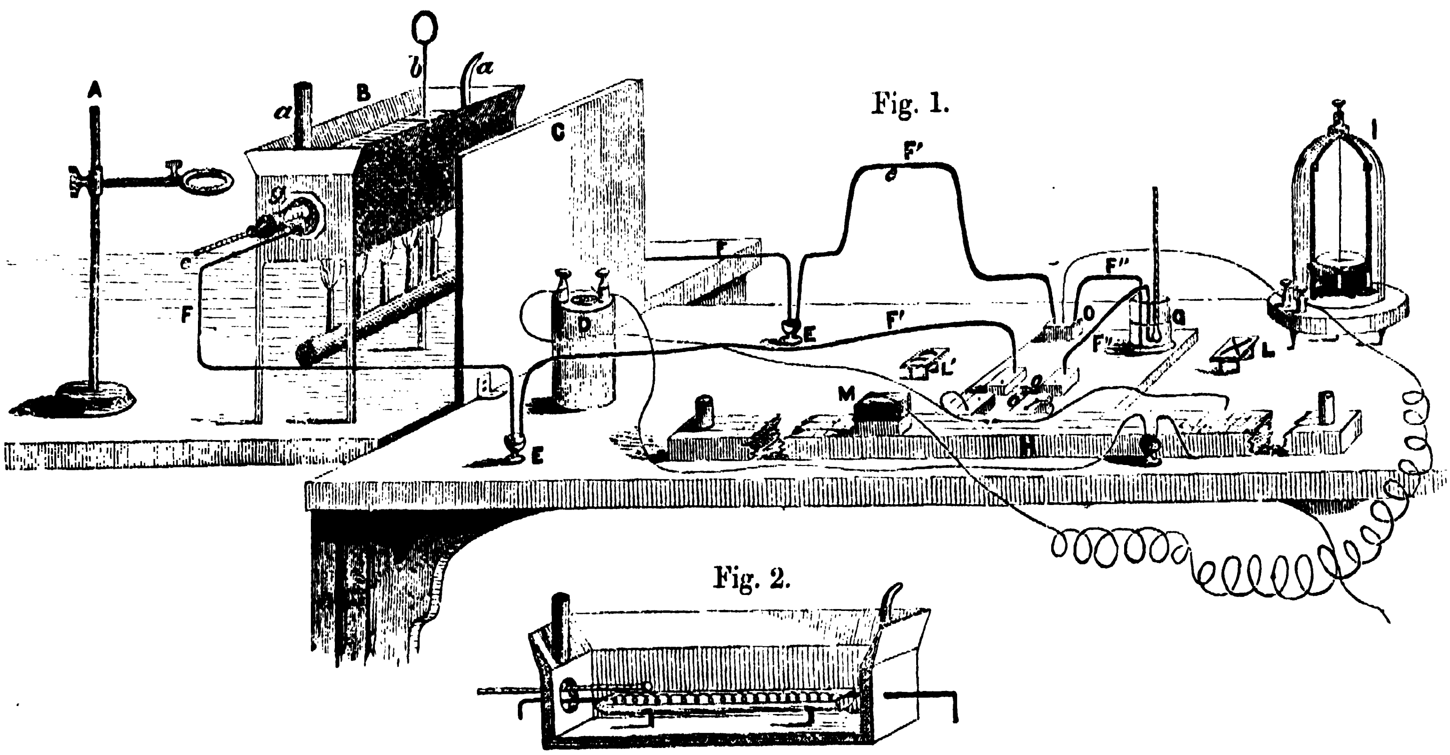

Fig. 1.

Fig. 2.

100° C., and the resistance of the circuit was again determined whilst the wire was at that temperature, when it was found to have increased only 0·08 per cent.; we did not, therefore, consider it necessary to make any correction for the increase of resistance caused by the heating of the ends of the thick wires in the trough. The resistance of the copper wires was determined at the ordinary temperature, and brought into calculation without further correction. Before the commencement of each series, all the ends of the wires dipping into the mercury-cups were carefully re-amalgamated. L, L′ are the two commutators fitting into four mercury-cups at *o*.

The wire stretched on the board H is of german silver instead of copper, as was formerly described; its half-length was 4550 millims. The length of the board is about 1500 millims.; the wire, therefore, was wound backwards and forwards several times on the one side; this is not visible in the figure. By using normal wires of different resistances, and by choosing proper lengths of the wire to be tested, it was always possible to begin the observations with the block M within 100 millims. of the middle of the wire. Great care was taken to lift the block M off the wire when it was moved, in order to prevent as much as possible its wearing. It may be mentioned that, although we generally worked with only one of the commutators, and therefore mostly used the one half of the wire, the zero-point of the wire only varied, during the whole of the experiments, which have taken almost a year to carry out, 3 millims. The zero-point was always determined before each series was begun. The distance the block M was moved when the resistance of a wire was determined, first at 0° and then at 100°, was, for pure metals in a solid state, about 800 millims., or about 8 millims. for 1°. As, however, the movement of the block M of 1 millim. caused a deflection of the needles of the galvanometer I of 20° to 30°, it is evident, with the apparatus employed, that the differences in the resistance of a wire to values less than those corresponding to 0°·1 C. can be accurately determined. Our results, moreover, prove this to be the case, as in many instances the difference between the observed and calculated conducting powers for the whole series do not amount to values equal to 0°·1 to 0°·2 C.

The trough B is a double one, the space between the inner and outer one being 20 millims. The dimensions of the inner trough were 400 millims. long, 80 millims. wide, and 80 millims. deep. Through the ends of both two holes of about 20 millims. wide were made, in which good corks were fitted, and through these passed the thick copper wires F; and also at one end a glass tube *d*, wide enough to allow the thermometer *c* to pass freely. A piece of india-rubber tubing, fitting over the glass tube *d*, and tightly round the thermometer, closed the tube, but allowed the thermometer to be moved either backward or forward with great ease. The tubes *a* are for filling the space between the inner and outer troughs with oil.

The wire to be tested lay in the trough, as shown in fig. 2, on a small glass tray, made by splitting a glass tube longitudinally, thereby preventing any possibility of its touching the trough, and also guarding it from being moved by the stirrer. A second trough, of somewhat smaller dimensions, was also used.

The use of an oil-bath for heating the wires has been objected to by a former observer*; it was therefore necessary to determine experimentally whether there was any real reason for the objection or not. He states that, as oil conducts electricity better on being heated than when cold, the differences between the conducting powers of cold and hot oil will materially affect the values obtained for the resistances of wire which had been determined at different temperatures in that liquid. In order to test the accuracy of this assertion, two copper plates of about 150 millims. diameter were connected, the one with the galvanometer, the other with a single Bunsen's cell; and to complete the circuit, this was connected with the galvanometer. A piece of filtering-paper, moistened with the olive-oil used, was placed between the copper plates, and these were pressed together with a weight. On completing the circuit not the slightest deflection of the needles was observed; the copper plates were then heated to above 100° C., and still no deflection was visible. To show that the connexions were good, a drop of water was put on the oiled paper; and immediately the needles of the galvanometer were sent with great violence to the stops. This proves that although oil may

* Arndsten, Poggendorff's 'Annalen,' vol civ. p. 1.

have a higher conducting power when hot than cold, yet in either case it is so infinitely small, that it cannot influence the results obtained in the manner just described.

Again, it was proved in a former research* that the formula for the correction of conducting power for temperature of a wire, deduced from the observations made in an oil- or air-bath, were exactly the same. Thus the formula obtained for an annealed wire of the gold-silver alloy heated in the oil-bath was

$$\lambda=15{\cdot}052-0{\cdot}01074t+0{\cdot}00000714t^2,$$

and that for the same wire heated in an air-bath was

$$\lambda=15{\cdot}059-0{\cdot}01077t+0{\cdot}00000722t^2.$$

As, however, more accurate results may be obtained by experimenting in an oil- than in an air-bath, on account of the wires taking more readily the temperature of the bath, and of their being more rapidly cooled if heated by the current passing through them, we have chosen this manner of heating the wires in preference to the other.

As oil, and more especially oil when hot, attacks most wires to a degree which would render the observations valueless, we were obliged to varnish them. The best varnish for the purpose is a solution of shell-lac in alcohol. For instance, a hard-drawn copper wire, not varnished, loses in conducting power after having been heated in an oil-bath to 100°, but if varnished, increases. To show that varnishing has no effect on the results, we give in Table I. the conducting power of a hard-drawn gold wire, first not varnished, and then varnished. Each result is the mean of two observations.

TABLE I.

Not varnished.				Varnished.			
T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.	
15·30	72·697	72·705	−0·008	13·85	73·120	73·085	+0·035
30·55	68·806	68·879	−0·073	30·95	68·756	68·782	−0·026
48·65	64·659	64·717	−0·058	49·55	64·523	64·520	+0·003
69·55	60·409	60·423	−0·014	68·40	60·636	60·645	−0·009
83·25	57·915	57·906	+0·009	84·55	57·704	57·680	+0·024
99·85	55·151	55·174	−0·023	98·70	55·346	55·352	−0·006
84·55	57·704	57·680	+0·024	84·90	57·645	57·620	+0·025
70·80	60·224	60·184	+0·040	70·25	60·318	60·289	+0·029
50·85	64·239	64·239	0·000	51·20	64·149	64·164	−0·015
30·95	68·746	68·782	−0·036	30·60	68·886	68·866	+0·020
16·80	72·343	72·316	+0·027	17·85	72·111	72·045	+0·066

The formula deduced from the observations, and from which the conducting powers were calculated, was

$$\lambda=76{\cdot}838-0{\cdot}27973t+0{\cdot}0006285t^2.$$

The thermometers used were:—1. One divided into degrees, each of which was 3·5

* Philosophical Magazine for February 1861.

millims. long. With very little practice the temperature could be read off to 0°·1 C. with accuracy. This thermometer was calibrated by ourselves, and afterwards compared with a normal thermometer from Kew Observatory, for which we were indebted to the kindness of Mr. BALFOUR STEWART. The corrected readings of our thermometer agreed perfectly with those of the Kew thermometer. 2. A normal thermometer from Messrs. NEGRETTI and ZAMBRA, divided into 0°·2 C. This was compared with the Kew thermometer and found to be correct. The boiling- and freezing-points of the thermometers were taken at intervals, and the necessary corrections made.

As the light in the room where the experiments were made came from above, and as the thermometers lay horizontally in the trough, by placing the eye in a position so that the division on the thermometer covered its reflexion on the column of mercury, all error of parallax was avoided. The thermometers were always read off with the help of the magnifying glass A through the oil in the glass tube *d*, so that the whole of the column of mercury had very nearly the temperature of the bath.

The normal wires were made of annealed german silver, and their resistances determined by comparing them with a hard-drawn wire of the gold-silver alloy *. They were soldered to two thick copper wires, varnished, and when used placed in the cylinder G, filled with oil, in which a thermometer hung. The temperature of the oil was taken immediately after each observation, and the conducting power of the normal wire corrected by the use of the formula

$$\lambda = 7{\cdot}803 - 0{\cdot}0034619t + 0{\cdot}0000003951t^2,$$

which was found by the determination of the conducting powers, at different temperatures, of a piece of wire from the same coil as that from which the normal wires were cut. In this paper we have taken as unit the conducting power of a hard-drawn silver wire at 0° C. $=100$ (that of the hard-drawn gold-silver alloy at 0° being $=15{\cdot}03$), in order to be able to compare at sight the present determinations with those made by one of us a short time ago †.

Before beginning a series, as already stated, all the ends of the wires dipping in the mercury-cups were re-amalgamated, and the zero-point of the scale redetermined. The current from the cell D was only allowed to pass through the apparatus for a second or two at a time, for fear of heating the wires, &c.

From 0° to 100° seven intervals were chosen at which observations were made, viz. 12°, 25°, 40°, 55°, 70°, 85°, 100°. With a little practice the flames of the 6-Bunsen burner could be regulated so as to come within a degree or two of the above temperatures. For about five minutes before, and whilst making the observations, the oil in the trough was stirred, one observer being at the trough whilst the other determined the resistances. Four observations at each interval were generally made on heating the wire to 100°, and again four at each interval on cooling (where this was not the case it will be mentioned with the series).

* Philosophical Magazine, February 1861.

† Philosophical Transactions, 1858 and 1860.

To save space, the mean only of the eight observations will be given, as otherwise the number of figures would be very great. Table I. may be taken as a fair example of the results obtained. The formulæ from which the conducting powers have been calculated is

$$\lambda = x + yt + zt^2,$$

where λ is the conducting power at t° C., x the conducting power at 0°, and y and z constants. The values for x, y, and z were deduced from the mean of the observations by the method of least squares.

We will now proceed to the experiments made with each metal, making at the same time a few remarks on their purification, &c., and then see what general laws and conclusions we may draw from the results obtained.

Silver.

Purified by precipitating nitrate of silver with hydrochloric acid, and reducing the washed chloride with pure carbonate of sodium. Wires 1, 2, and 3 were of different preparations. Table II. gives the results obtained with these wires.

TABLE II.

	First wire.		Second wire.		Third wire.	
	Hard drawn.	Annealed.	Hard drawn.	Annealed.	Hard drawn.	Annealed.
Length	1546 millims.	1535 millims.	1753 millims.	1741 millims.	1962 millims.	1953 millims.
Diameter	0·462 millim.	0·462 millim.	0·596 millim.	0·596 millim.	0·448 millim.	0·648 millim.
Conducting power found before heating the hard-drawn wires	97·645 at 15°·4	Reduced to 0°. 103·528	95·112 at 16°·0	Reduced to 0°. 101·149	94·053 at 16°·0	Reduced to 0°. 99·800
Conducting power after being kept at 100° for 1 day	98·138 at 16·2	104·364	96·618 at 15·6	102·585	95·241 at 15·4	100·839
Ditto, for 2 days	98·913 at 15·6	104·951	101·544 at 16·8	108·303*	96·337 at 16·0	102·223
Ditto, for 3 days	99·837 at 16·0	106·091	102·237 at 16·0	108·714	96·671 at 17·6	103·178
Ditto, for 4 days	99·212 at 18·4	106·377	101·427 at 19·2	109·162	97·917 at 15·6	103·747
Ditto, for 5 days	99·586 at 17·4	106·380	101·750 at 18·6	109·262	97·669 at 17·4	104·168
Ditto, for 6 days	....	....	....	....	97·322 at 18·2	104·100

The means of the conducting powers found for each of the following temperatures were—

* During the day the temperature of the oil increased, by mistake, to 130°.

First wire, hard drawn.				Second wire, hard drawn.				Third wire, hard drawn.			
T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
11·00	102·238	102·272	−0·034	12·20	103·927	103·927	0·000	9·60	100·534	100·546	−0·012
26·17	96·710	96·645	+0·065	23·70	99·520	99·523	−0·003	23·90	95·452	95·437	+0·015
38·25	92·490	92·505	−0·015	41·70	93·224	93·236	−0·012	38·95	90·476	90·507	−0·031
55·40	87·130	87·149	−0·019	56·20	88·703	88·708	−0·005	56·00	85·513	85·478	+0·035
68·85	83·389	83·374	+0·015	68·90	85·142	85·137	+0·005	68·15	82·244	82·252	−0·008
84·00	79·540	79·572	−0·032	85·45	81·078	81·036	+0·042	84·47	78·393	78·391	+0·002
101·30	75·831	75·813	+0·018	99·20	78·073	78·103	−0·030	98·60	75·477	75·484	−0·007
First wire, annealed.				Second wire, annealed.				Third wire, annealed.			
11·30	103·391	103·404	−0·013	8·00	106·447	106·426	+0·021	9·25	102·543	102·461	+0·082
24·25	98·589	98·576	+0·013	24·35	99·968	99·990	−0·022	25·55	96·371	96·495	−0·124
41·85	92·520	92·530	−0·010	38·05	95·051	95·077	−0·026	40·10	91·589	91·630	−0·041
56·45	88·006	87·965	+0·041	55·17	89·554	89·554	0·000	55·17	87·055	87·047	+0·008
67·75	84·670	84·714	−0·044	68·22	85·847	85·803	+0·044	68·55	83·483	83·367	+0·116
83·65	80·562	80·554	+0·008	83·62	81·882	81·888	−0·006	83·57	79·667	79·674	−0·007
98·80	77·046	77·042	+0·004	100·00	78·319	78·331	−0·012	100·00	76·124	76·163	−0·039

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire (hard drawn) . $\lambda=106{\cdot}651-0{\cdot}40948t+0{\cdot}0010370t^2$.

For first wire (annealed) . . $\lambda=107{\cdot}880-0{\cdot}40698t+0{\cdot}0009601t^2$.

For second wire (hard drawn) $\lambda=108{\cdot}928-0{\cdot}42389t+0{\cdot}0011407t^2$.

For second wire (annealed) . $\lambda=109{\cdot}802-0{\cdot}43138t+0{\cdot}0011667t^2$.

For third wire (hard drawn) . $\lambda=104{\cdot}209-0{\cdot}39124t+0{\cdot}0010133t^2$.

For third wire (annealed) . $\lambda=106{\cdot}088-0{\cdot}40160t+0{\cdot}0010235t^2$.

From the above Table it will be seen that, after heating a silver wire to 100° C. for some days, its conducting power is increased almost to the same extent as if it had been annealed, and that wires 1 and 2 were not completely hard drawn. On comparing the difference in the conducting powers produced by annealing the wires, we find for wire 3 it is only 6 per cent., whereas for wire 2 it is almost 10 per cent., taking the conducting power of the hard-drawn silver wire =100. In a former research* this difference was found to be—

			Reduced to 0°.
1.	Hard drawn	95·28 at 14°·0	100·47
	Annealed	103·98 at 14·8	109·98
2.	Hard drawn	95·36 at 14·6	100·78
	Annealed	103·33 at 14·6	109·20

* Philosophical Transactions, 1860.

These values have been reduced by using a formula which is the mean of the six deduced from the experiments; for although there is a difference in the formula obtained for the annealed and hard-drawn (or rather partially annealed) wires, yet it is so small that they may be considered the same, more especially as the difference between the one obtained for the different wires is far greater. Taking the mean of the above values, and assuming the influence of temperature on the conducting power of hard-drawn and annealed wires to be the same, we find the following formulæ:—

$$\text{For hard-drawn wires } \lambda = 100\cdot00 - 0\cdot38287t + 0\cdot0009848t^2.$$

$$\text{For annealed wires } \lambda = 108\cdot574 - 0\cdot41570t + 0\cdot0010624t^2.$$

Copper.

Wires 1 and 2 were of the same piece of electrotype copper prepared for us by Dr. H. Müller at Messrs. De la Rue and Co.'s. Wire 3 was cut off a piece of commercial electrotype copper from the same source. Table III. shows the results obtained with these wires.

Table III.

	First wire.		Second wire.		Third wire.	
	Hard drawn.	Annealed.	Hard drawn.	Annealed.	Hard drawn.	Annealed.
Length	2262 millims.	2245·5 millims.	1753 millims.	1738 millims.	1476 millims.	1461 millims.
Diameter	0·691 millim.	0·691 millim.	0·598 millim.	0·598 millim.	0·537 millim.	0·537 millim.
Conducting power found before heating the hard-drawn wires	95·672 at 10°·6	Reduced to 0°. 99·526	94·355 at 15°·0	Reduced to 0°. 100·021	92·568 at 20°·6	Reduced to 0°. 100·327
Conducting power after being kept at 100° for 1 day ...	96·324 at 9·9	99·943	94·965 at 13·2	99·971	93·263 at 19·0	100·461
Ditto, for 2 days...	96·750 at 11·8	101·097	94·880 at 14·2	100·268	93·720 at 18·0	100·563
Ditto, for 3 days...	96·914 at 12·2	101·418	94·501 at 15·9	100·524	93·434 at 19·0	100·645
Ditto, for 4 days...	97·950 at 9·8	101·671	94·153 at 17·2	100·656	93·278 at 19·6	100·708
Ditto, for 5 days...	98·437 at 8·7	101·682	95[illegible]70 at 14·4	101·074	92·865 at 20·6	100·649
Ditto, for 6 days...	....	....	94·327 at 18·2	101·230	92·738 at 21·1	100·705
Ditto, for 7 days...	....	....	96·575 at 12·7	101·469		

The means of the conducting powers found for each of the following temperatures were—

First wire, hard drawn.				Second wire, hard drawn.				Third wire, hard drawn.			
T.	Conducting power. Observed.	Conducting power. Calculated.	Difference.	T.	Conducting power. Observed.	Conducting power. Calculated.	Difference.	T.	Conducting power. Observed.	Conducting power. Calculated.	Difference.
16·86°	95·473	95·467	+0·006	19·17°	94·359	94·334	+0·025	12·65°	95·769	95·739	+0·030
29·88	91·063	91·002	+0·061	30·95	90·187	90·208	−0·021	25·61	91·061	91·076	−0·015
51·03	84·235	84·315	−0·080	48·53	84·518	84·544	−0·026	39·52	86·415	86·456	−0·041
69·52	78·997	79·044	−0·047	69·22	78·640	78·634	+0·006	53·92	82·069	82·090	−0·021
83·77	75·413	75·347	+0·066	83·77	75·015	74·968	+0·047	69·90	77·798	77·741	+0·057
98·60	71·829	71·838	−0·009	99·00	71·532	71·562	−0·030	84·87	74·172	74·142	+0·030
								99·92	70·951	70·987	−0·036
First wire, annealed.				Second wire, annealed.				Third wire, annealed.			
17·00°	95·535	95·567	−0·032	18·96°	94·987	94·959	+0·028	13·45°	96·954	96·934	+0·020
29·63	91·291	91·239	+0·052	31·86	90·424	90·449	−0·025	26·15	92·246	92·260	−0·014
50·22	84·687	84·726	−0·039	52·05	83·974	84·003	−0·029	39·35	87·727	87·753	−0·026
69·60	79·223	79·209	+0·014	70·27	78·836	78·829	+0·007	55·50	82·675	82·722	−0·047
83·42	75·636	75·638	−0·002	83·81	75·428	75·377	+0·051	69·90	78·742	78·686	+0·056
99·39	71·891	71·893	−0·002	99·57	71·757	71·784	−0·027	84·67	75·047	74·988	+0·059
								99·05	71·766	71·816	−0·050

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire (hard drawn) . . . $\lambda=101\cdot645-0\cdot37963t+0\cdot0007844t^2$.
For first wire (annealed) . . . . $\lambda=101\cdot791-0\cdot37959t+0\cdot0007921t^2$.
For second wire (hard drawn) . . $\lambda=101\cdot614-0\cdot39806t+0\cdot0009546t^2$.
For second wire (annealed) . . . $\lambda=102\cdot143-0\cdot39629t+0\cdot0009179t^2$.
For third wire (hard drawn) . . . $\lambda=100\cdot620-0\cdot39885t+0\cdot0010236t^2$.
For third wire (annealed). . . . $\lambda=102\cdot243-0\cdot40850t+0\cdot0010228t^2$.

The observations made with wires 1 and 2 were as follows: two at each interval on heating and two on cooling; again, two on heating and two on cooling, as shown in Table I.

On looking at the above, we observe that wire 1, after having been kept at 100° for several days, increased in conducting power almost to the same extent as if it had been annealed, wire 2 partially so, and wire 3 hardly at all. The annealing took place in a glass tube heated with a 4-Bunsen burner, whilst a current of hydrogen passed through it. Here, again, as in the case of the silver wire, we may assume that the formulæ of the hard-drawn and annealed copper wires are the same. In a former research* pure copper was found to conduct—

		Reduced to 0°.
1.	93·00 at 18·6°	99·877
2.	93·46 at 20·2	100·980
3.	92·02 at 18·4	99·824
4.	92·76 at 19·3	99·886
5.	92·99 at 17·5	99·453

* Philosophical Transactions, 1860.

The difference found between the conducting powers of hard-drawn and annealed wires was—

			Reduced to 0°.
6.	Hard drawn	95·31 at 11°·0	99·435
	Annealed	97·83 at 11·0	102·065
7.	Hard drawn	95·72 at 11·0	99·864
	Annealed	98·02 at 11·0	102·263

These values have been reduced to 0° as follows: take for instance the first, 93·00 at 18°·6. The mean of the six formulæ obtained for copper (see Table XV.) is

$$\lambda=100-0·38701t+0·0009009t^2;$$

and calculating the conducting power for 18°·6 by this formula, we find it equal to 93·114.

Now $$\frac{93·00}{93·114}=0·99877;$$

and if all the terms of the above formula be multiplied by this number, we deduce a formula by which the above value can be reduced. All the reductions given in this paper of former determinations were made in this manner, using the formulæ given in Table XV. The reductions to 0° in the Tables were made in a like manner, the only difference being that the formulæ found for the respective wires were used instead of the mean. Taking the mean of all the values found for copper, and using the mean for the formulæ given in Table XV., we find as the formula for correction of the conducting power for temperature of

A hard-drawn wire $\lambda=\ \ 99·947-0·38681t+0·0009004t^2$

An annealed wire $\lambda=102·213-0·39557t+0·0009208t^2$.

The values given as first term in the formulæ were found as follows: on referring to the paper* from which the conducting powers of copper were taken, it will be seen that each of them is the mean of three determinations. The reduced values therefore of 1 to 5, the mean of 6, hard drawn, and 7, hard drawn, and the mean of the first determinations of the three wires given in Table III., were added together, and the mean taken as the conducting power of a hard-drawn copper wire at 0° C. For the annealed, the per-centage differences of the values of 6, hard drawn and annealed, 7, ditto, and of the first determinations of the three wires in Table III. and the annealed ones, were added together, and the mean added to the value found for the hard-drawn wire (as a percentage amount). All the formulæ given as end-result with each metal have been constructed in this manner.

Gold.

Purified as described in the Philosophical Transactions, 1860, p. 175. Wires 1, 2, and 3 were of different preparations. The results obtained with these wires are given in Table IV.

* Philosophical Transactions, 1860.

Table IV.

	First wire.		Second wire.		Third wire.	
	Hard drawn.	Annealed.	Hard drawn.	Annealed.	Hard drawn.	Annealed.
Length...............	2214 millims.	2200 millims.	837 millims.		759·5 millims.	742·5 millims.
Diameter............	0·759 millim.	0·759 millim.	0·467 millim.		0·434 millim.	0·434 millim.
Conducting power found before heating the hard-drawn wires..............	73·239 at 13°·2	Reduced to 0°. 76·821	72·550 at 15°·1	Reduced to 0°. 76·561	67·530 at 36°·8	Reduced to 0°. 77·229
Conducting power after being kept at 100° for 1 day ...	72·746 at 15·2	76·854	73·359 at 12·6	76·733	71·868 at 19·4	77·223
Ditto, for 2 days...	72·751 at 15·1	76·832	....	....	71·854 at 20·1	77·405
Ditto, for 3 days...	....	....	....	....	72·191 at 19·0	77·457
Ditto, for 4 days...	....	....	....	....	72·396 at 18·0	77·394

The means of the conducting powers found for each of the following temperatures were—

First wire, hard drawn.				Second wire, hard drawn.				Third wire, hard drawn.			
T.	Conducting power. Observed.	Conducting power. Calculated.	Difference.	T.	Conducting power. Observed.	Conducting power. Calculated.	Difference.	T.	Conducting power. Observed.	Conducting power. Calculated.	Difference.
15°·95	72·567	72·536	+0·031	13°·36	73·222	73·212	+0·010	12°·44	73·854	73·841	+0·013
30·76	68·798	68·828	−0·030	24·79	70·329	70·325	+0·004	23·27	70·965	70·975	−0·010
50·06	64·392	64·410	−0·018	40·80	66·515	66·544	−0·029	39·42	67·002	67·013	−0·011
69·75	60·397	60·385	+0·012	55·65	63·306	63·312	−0·006	55·47	63·441	63·448	−0·007
84·31	57·742	57·722	+0·020	69·52	60·528	60·531	−0·003	70·56	60·455	60·435	+0·020
99·27	55·248	55·263	−0·015	84·12	57·905	57·854	+0·051	84·79	57·904	57·893	+0·011
				100·00	55·203	55·232	−0·029	99·00	55·635	55·647	−0·012
First wire, annealed.								Third wire, annealed.			
14°·92	74·020	73·992	+0·028	At the commencement of the determinations this wire was torn away from its place by the stirrer.				14°·10	74·327	74·293	+0·034
30·05	70·039	70·068	−0·029					26·31	71·067	71·095	−0·028
48·87	65·575	65·611	−0·036					40·51	67·582	67·621	−0·039
69·90	61·220	61·191	+0·029					53·72	64·645	64·628	+0·017
82·82	58·811	58·768	+0·043					70·17	61·229	61·220	+0·009
99·62	55·915	55·948	−0·033					85·36	58·422	58·388	+0·034
								99·30	56·029	56·056	−0·027

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire (hard drawn) . . . $\lambda=76{\cdot}838-0{\cdot}27973t+0{\cdot}0006285t^2$.

For first wire (annealed) . . . $\lambda=78{\cdot}161-0{\cdot}28935t+0{\cdot}0006664t^2$.

For second wire (hard drawn) . . $\lambda=76{\cdot}786-0{\cdot}27549t+0{\cdot}0005995t^2$.

For third wire (hard drawn . . $\lambda=77{\cdot}343-0{\cdot}29043t+0{\cdot}0007200t^2$.

For third wire (annealed) . . . $\lambda=78{\cdot}231-0{\cdot}28849t+0{\cdot}0006564t^2$.

The observations made with wire 1 (hard drawn) are given in Table I., those of the same wire (annealed) were made in the same manner.

Here we find no permanent change in conducting power with wire 1, after being kept at 100° for several days, and only a very slight increase with wires 2 and 3. The formulæ for the hard-drawn and annealed wires agree so closely that they may also, as with silver and copper, be considered the same.

In the paper just alluded to, the conducting power of pure gold was found—

		Reduced to 0°.
1.	72·68 at 19°·3	77·966
2.	73·08 at 23·3	79·524
3.	73·27 at 13·8	77·053
4.	73·99 at 15·1	78·178

The difference between hard-drawn and annealed wires was—

			Reduced to 0°.
5.	Hard drawn	74·20 at 14°·8	78·313
	Annealed	75·53 at 15·2	79·833
6.	Hard drawn	73·78 at 15·5	78·067
	Annealed	75·18 at 15·8	79·635

Taking the mean of the values as with copper, the following formulæ were deduced for the correction of conducting power for temperature:—

For hard-drawn wires $\lambda=77·964-0·28648t+0·0006582t^2$.

For annealed wires $\lambda=79·327-0·29149t+0·0006697t^2$.

Zinc.

Zinc free of arsenic was purified by distillation. All pressed wires. In Table V. the results obtained are given.

TABLE V.

	First wire.		Second wire.		Third wire.	
Length...............	502·2 millims.		394 millims.		372 millims.	
Diameter............	0·588 millim.		0·513 millim.		0·519 millim.	
		Reduced to 0°.		Reduced to 0°.		Reduced to 0°.
Conducting power found before heating the wires......	26·744 at 23°·1	29·093	26·903 at 18°·5	28·836	26·835 at 18°·0	28·639
Ditto, after being kept at 100° for 1 day............	26·695 at 23·7	29·103	27·081 at 17·5	28·919	26·784 at 18·5	28·636
Ditto, for 2 days...	....		26·980 at 18·5	28·919	26·885 at 17·4	28·632

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
11°·60	27·915	27·902	+0·013	11°·20	27·706	27·687	+0·019	11°·16	27·518	27·513	+0·005
24·24	26·639	26·653	−0·014	26·12	26·187	26·199	−0·012	25·96	26·088	26·090	−0·002
41·33	25·077	25·086	−0·009	39·55	24·951	24·959	−0·008	40·10	24·812	24·820	−0·008
55·08	23·925	23·926	−0·001	54·18	23·719	23·716	+0·003	56·85	23·423	23·428	−0·005
70·27	22·757	22·747	+0·010	72·32	22·330	22·330	0·000	71·73	22·306	22·295	+0·011
82·01	21·924	21·912	+0·012	85·77	21·407	21·414	−0·007	85·40	21·348	21·339	+0·009
98·07	20·865	20·875	−0·010	100·23	20·540	20·534	+0·006	98·95	20·462	20·472	−0·010

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire . . $\lambda=29{\cdot}114-0{\cdot}10727t+0{\cdot}0002372t^2$.

For second wire . $\lambda=28{\cdot}881-0{\cdot}10949t+0{\cdot}0002616t^2$.

For third wire . . $\lambda=28{\cdot}649-0{\cdot}10424t+0{\cdot}0002182t^2$.

No permanent alteration in the conducting power takes place after heating the wires for several days to 100°.

The value formerly found for the conducting power of zinc (precipitated galvanoplastically, fused and pressed) was—

	Reduced to 0°.
27·39 at 17°·6	29·220.

Treating these values as before, we find the formula for zinc to be

$$\lambda=29{\cdot}022-0{\cdot}10752t+0{\cdot}0002401t^2.$$

Cadmium.

The metal was purified as described in the Philosophical Transactions, 1860, p. 177. The wires were pressed. Table VI. shows the results.

Table VI.

	First wire.	Second wire.	Third wire.
Length...............	625 millims.	559 millims.	439 millims.
Diameter............	0·641 millim.	0·678 millim.	0·684 millim.

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
8°·87	23·327	23·329	−0·002	8°·89	23·374	23·400	−0·026	14°·60	21·849	21·859	−0·010
20·75	22·351	22·338	+0·013	21·59	22·280	22·270	+0·010	22·05	21·318	21·310	+0·008
34·47	21·241	21·255	−0·014	36·37	21·075	21·059	+0·016	39·65	20·072	20·061	+0·011
49·38	20·138	20·150	−0·012	48·52	20·157	20·146	+0·011	54·45	19·065	19·067	−0·002
63·39	19·188	19·186	+0·002	62·90	19·171	19·162	+0·009	68·10	18·179	18·194	−0·015
77·74	18·292	18·268	+0·024	80·00	18·109	18·131	−0·022	81·20	17·393	17·397	−0·004
93·55	17·325	17·339	−0·014					89·90	16·896	16·888	+0·008

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire . . $\lambda=24\cdot100-0\cdot088554t+0\cdot0001740t^2$.

For second wire . $\lambda=24\cdot240-0\cdot096753t+0\cdot0002548t^2$.

For third wire . $\lambda=24\cdot974-0\cdot078004t+0\cdot0001147t^2$.

The values obtained for the alteration in the conducting power of these wires after heating them for several days to 100°, have unfortunately been lost. It may, however, be stated that the differences were very small, and that there was a loss in conducting power.

The conducting power of cadmium was found in the paper already referred to—

	Reduced to 0°.
22·10 at 18°·8	23·678.

Deducing the formula for cadmium in the manner before described, we find

$$\lambda=23\cdot725-0\cdot087476t+0\cdot0001797t^2.$$

Pure cadmium, when heated to about 80°, becomes exceedingly brittle, in fact it may be powdered in a hot mortar with great ease. We should not have been able to carry out the determinations if the wires had not been varnished, as the movement of the oil by the stirrer would have caused them to fall to pieces. It is worthy of remark that this change in the molecular arrangement of the wires does not make itself apparent in the conducting power to any very marked extent.

Tin.

Purified by dissolving commercial tin in nitric acid, and reducing the washed oxide by heating it with lampblack. Pressed wires were used. Table VII. gives the results.

TABLE VII.

	First wire.		Second wire.		Third wire.	
Length	279 millims.		375 millims.		315 millims.	
Diameter	0·559 millim.		0·634 millim.		0·729 millim.	
		Reduced to 0°.		Reduced to 0°.		Reduced to 0°.
Conducting power found before heating the wires	10·970 at 18°·2	11·710	11·532 at 18°·1	12·324	12·285 at 18°·2	13·108
Ditto, after being kept at 100° for 1 day	11·124 at 19·4	11·926	11·442 at 19·1	12·273	12·291 at 18·4	13·124
Ditto, for 2 days	10·852 at 27·0	11·956	11·448 at 18·6	12·257	12·296 at 18·4	13·129
Ditto, for 8 days	10·835 at 28·0	11·980	11·444 at 18·4	12·264		

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
12°·9	11·4110	11·4202	−0·0092	11°·80	11·7144	11·7227	−0·0083	10°·00	12·649	12·660	−0·011
25·27	10·9320	10·9246	+0·0074	26·32	11·1287	11·1153	+0·0134	26·54	11·944	11·934	+0·010
40·55	10·3570	10·3436	+0·0134	40·04	10·5805	10·5732	+0·0073	39·52	11·408	11·391	+0·017
54·15	9·8498	9·8558	−0·0060	54·02	10·0451	10·0526	−0·0075	56·27	10·717	10·727	−0·010
70·53	9·2980	9·3046	−0·0066	70·02	9·4883	9·4961	−0·0078	70·30	10·189	10·202	−0·013
83·13	8·9033	8·9078	−0·0045	85·02	9·0102	9·0127	−0·0025	85·72	9·654	9·657	−0·003
100·90	8·3937	8·3881	+0·0056	98·50	8·6158	8·6096	+0·0062	97·30	9·279	9·270	+0·009

The formulæ deduced from the observations, and from which the conducting powers were calculated, were—

For first wire . $\lambda=11{\cdot}9613-0{\cdot}042902t+0{\cdot}00007422t^2$.

For second wire $\lambda=12{\cdot}2419-0{\cdot}044965t+0{\cdot}00008213t^2$.

For third wire . $\lambda=13{\cdot}1186-0{\cdot}046561t+0{\cdot}00007206t^2$.

We see from the results that wires 1 and 2 decrease to a small extent in conducting power, whereas wire 3 increases slightly after being heated to 100°.

The conducting power of tin was found—

	Reduced to 0°.
11·45 at 21°·0	12·351;

and calculating the formula of tin as before, we find

$$\lambda=12{\cdot}366-0{\cdot}044554t+0{\cdot}00007588t^2.$$

Lead.

Purified by reducing by heat the twice recrystallized acetate. Wires 1 and 2 were pressed; wire 3 drawn. No permanent alteration in the conducting power of the wires was observed after they had been kept at 100° for two days. Table VIII. shows the results.

TABLE VIII.

	First wire.	Second wire.	Third wire.
Length	416 millims.	453 millims.	389 millims.
Diameter	0·669 millim.	0·698 millim.	0·959 millim.

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
14°·55	7·9365	7·9336	+0·0029	14°·50	7·8685	7·8653	+0·0032	12°·40	7·9038	7·9022	+0·0016
25·40	7·6129	7·6152	−0·0023	27·50	7·5336	7·5392	−0·0056	26·20	7·4967	7·4968	−0·0001
40·30	7·2036	7·2071	−0·0035	40·37	7·1405	7·1397	+0·0008	39·60	7·1309	7·1324	−0·0015
54·80	6·8423	6·8420	+0·0003	54·80	6·7789	6·7775	+0·0014	54·60	6·7565	6·7585	−0·0020
70·33	6·4881	6·4863	+0·0018	69·63	6·4392	6·4370	+0·0022	69·70	6·4205	6·4187	+0·0018
84·52	6·1964	6·1929	+0·0035	84·80	6·1189	6·1218	−0·0029	84·40	6·1250	6·1229	+0·0021
99·35	5·9159	5·9189	−0·0030	100·10	5·8388	5·8381	+0·0007	98·85	5·8642	5·8658	−0·0016

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire . $\lambda=8\cdot3882-0\cdot032346t+0\cdot00007540t^2$.
For second wire $\lambda=8\cdot3147-0\cdot032055t+0\cdot00007307t^2$.
For third wire . $\lambda=8\cdot2925-0\cdot032468t+0\cdot00008011t^2$.

The value found for the conducting power of lead was

	Reduced to 0°.
7·77 at 17°·3	8·304.

Treating the mean of the values as above, the formula is

$$\lambda=8\cdot318-0\cdot032237t+0\cdot00007608t^2.$$

Arsenic.

Purified by sublimation. Small bars were cut from a comparatively solid piece and soldered to two copper wires; on account of the extreme brittleness of arsenic, the bars were placed in glass tubes closed at the ends with gypsum, through which the copper wires passed. As these were dried in a water-bath for several days, no permanent alteration of the conducting power of the bars was found after being heated in the oil-bath for two days. The values found for the conducting power of arsenic agree as well as could be expected, considering the bars were made by hand, and the metal somewhat porous. The difficulty of obtaining bars of metal of sufficient length is so great that we have been contented with two series. These are given in Table IX.

Table IX.

	First bar.	Second bar.
Length	50·4 millims.	55·5 millims.
Diameter	0·93 millim.	1·01 millim.

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.	
14°·20	5·0203	5·0180	+0·0023	13°·50	4·0051	4·0037	+0·0014
25·30	4·8007	4·8008	−0·0001	24·50	3·8371	3·8450	−0·0079
37·80	4·5710	4·5736	−0·0026	40·15	3·6367	3·6311	+0·0056
55·00	4·2854	4·2906	−0·0052	55·55	3·4447	3·4341	+0·0106
70·00	4·0767	4·0722	+0·0045	69·90	3·2559	3·2628	−0·0069
85·30	3·8810	3·8764	+0·0046	82·50	3·1144	3·1221	−0·0077
101·00	3·7005	3·7041	−0·0036	99·80	2·9485	2·9435	+0·0050

The formulæ deduced from the observations, and from which the conducting powers were calculated, were—

For first bar $\lambda=5\cdot3168-0\cdot021874t+0\cdot00005848t^2$.
For second bar $\lambda=4\cdot2078-0\cdot015506t+0\cdot00002843t^2$.

Taking the mean of the conducting powers at 0°, we deduce the formula for the correction of conducting power for temperature to be

$$\lambda=4\cdot7623-0\cdot018571t+0\cdot00004228t^2.$$

Antimony.

Purified by twice recrystallizing commercially pure tartrate of antimony and potassium, reducing by heat and re-fusing with antimonic acid. As antimony is so very brittle, it was not possible to manipulate with it in form of wire, it was therefore fused in the bowl of a tobacco-pipe, and when liquid allowed to run into the stem. After breaking off the bowl, the ends of the pipe were made so hot that the metal melted, and clean copper wires were pushed into the liquid metal, which on solidifying held them fast. The free ends of the copper wires were then soldered to the thick ones in the trough. Unfortunately in each case the copper wires in the pipe-stem became loose after heating for two or three days, and had to be therefore resoldered, so that no reliable determinations could be made as to the effect of heating to 100° for several days on the conducting power. It may be stated that the three wires lost in conducting power; but to what extent, we are of course not in a position to say. As the diameter of the pipe-stem could not be accurately determined, and as it could not be ascertained whether there were cavities in the wires (caused by contraction on cooling and crystallization) or not, the first observed conducting power was taken equal to 100. Table X. shows the results.

TABLE X.

The means of the conducting powers found for each of the following temperatures were—

First wire.				Second wire.				Third wire.			
T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
10·00°	100·000	100·052	−0·052	8·40°	100·000	99·999	+0·001	13·80°	100·000	99·901	+0·099
26·35	94·062	93·910	+0·152	25·60	93·947	93·850	+0·097	22·30	96·378	96·514	−0·136
40·40	88·982	89·089	−0·107	42·45	88·139	88·329	−0·190	38·65	90·552	90·527	+0·025
54·55	84·633	84·664	−0·031	57·80	83·707	83·731	−0·024	53·50	85·671	85·692	−0·021
70·65	80·126	80·152	−0·026	69·45	80·691	80·517	+0·174	69·65	81·118	81·082	+0·036
83·50	77·071	76·953	+0·118	86·85	76·138	76·159	−0·021	84·45	77·480	77·454	+0·026
99·40	73·430	73·484	−0·054	101·25	72·922	72·953	−0·031	98·80	74·448	74·480	−0·032

The formulæ deduced from the observations, from which the conducting powers were calculated, were—

For first wire . $\lambda=104\cdot095-0\cdot41487t+0\cdot0010755t^2$.

For second wire $\lambda=103\cdot190-0\cdot38721t+0\cdot00087\cdot48t^2$.

For third wire . $\lambda=105\cdot801-0\cdot44541t+0\cdot0012995t^2$.

The observed conducting powers in this and the foregoing Table do not agree so well with the calculated as the others, on account of the temperature of the bath never being exactly the same as that of the wire; for in the one case the heat had to traverse the glass tube filled with air, in the other the thickness of the pipe-stem, before reaching the metal.

The conducting power of antimony was found equal to

	Reduced to 0°.
4·29 at 18°·7	4·6172

Using this value as before described, we obtain a formula for antimony where

$$\lambda=4\cdot6172-0\cdot018389t+0\cdot00004788t^2.$$

Bismuth.

Purified by reducing the basic nitrate of bismuth with lampblack. Table XI. gives the results. The wires were pressed.

TABLE XI.

	First wire.		Second wire.		Third wire.	
Length...............	117 millims.		121·4 millims.		42·5 millims.	
Diameter............	0·596 millim.		0·596 millim.		0·217 millim.	
		Reduced to 0°.		Reduced to 0°.		Reduced to 0°.
Conducting power found before heating the wires......	1·1787 at 16°·6	1·2517	1·1036 at 18°·8	1·1773	1·2215 at 16°·6	1·2951
Conducting power after being kept at 100° for 1 day ...	1·3599 at 17·6	1·4494	1·3110 at 19·0	1·3995	1·3683 at 17·8	1·4569
Ditto, for 2 days...	1·3595 at 18·2	1·4521	1·3121 at 19·0	1·4006	1·3709 at 17·6	1·4587
Ditto, for 3 days...	1·3624 at 18·0	1·4541	1·3096 at 19·9	1·4023	1·3710 at 17·9	1·4603

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
9°·20	1·4059	1·4058	+0·0001	8°·60	1·3654	1·3641	+0·0013	9°·40	1·4129	1·4128	+0·0001
26·15	1·3226	1·3226	0·0000	24·00	1·2909	1·2935	−0·0026	25·65	1·3329	1·3339	−0·0010
39·50	1·2609	1·2614	−0·0005	38·75	1·2297	1·2287	+0·0010	43·05	1·2551	1·2538	+0·0013
57·25	1·1863	1·1858	+0·0005	55·30	1·1591	1·1593	−0·0002	57·45	1·1913	1·1912	+0·0001
68·95	1·1397	1·1397	0·0000	68·90	1·1058	1·1050	+0·0008	71·60	1·1315	1·1328	−0·0013
84·35	1·0833	1·0833	0·0000	84·00	1·0478	1·0474	+0·0004	88·60	1·0671	1·0666	+0·0005
96·35	1·0428	1·0429	−0·0001	95·90	1·0036	1·0042	−0·0006				

The formulæ deduced from the observations, by which the conducting powers were calculated, were—

For first wire . $\lambda=1\cdot4535-0\cdot0052883t+0\cdot00001060t^2$.

For second wire $\lambda=1\cdot4049-0\cdot0047972t+0\cdot000006453t^2$.

For third wire . $\lambda=1\cdot4603-0\cdot0051286t+0\cdot000007737t^2$.

From the above we see how bismuth increases in conducting power after being kept at 100° for one day. This increment is so rapid that it may be followed for the first two hours from five to five minutes. Wire 1 altered by one day's heating 16 per cent.; wire 2, 19 per cent.; and wire 3, 12 per cent. Wires 1 and 2 were cut from the same piece.

This behaviour explains why the conducting power of bismuth wires varies so much: for in the paper so often here alluded to, the maximum difference between twelve wires was found to be 22 per cent. In pressing the wires the heat applied to the press is never constant; so that, if pressed very warm, wires of high conducting power would probably be the result. The conducting power of bismuth was found equal to

	Reduced to 0°.
1·19 at 13°·8	1·2484

Taking the mean of the values as before, we find the formula for bismuth to be

$$\lambda=1\cdot2454-0\cdot0043858t+0\cdot000007134t^2.$$

Mercury.

Purified by allowing a solution of subnitrate of mercury to stand over the metal for several weeks, during which time it was often well shaken up with it. The determinations were made in a calibrated thermometer-tube, to the ends of which wide glass tubes (13 to 14 millims. wide) were fused and bent, as shown in fig. 3. Mercury prepared at different times was used for the determinations. For the experiments, the tube was filled with hot mercury, and its resistance was determined when cold. This was twice repeated; and the resistance being found the same each time, it was assumed that the tube filled in this manner did not contain air-bubbles; this is also proved by the close agreement of the formulæ found in the two cases for the variation of the conducting power at higher temperatures; for if in either case air-bubbles had been present, the formulæ must have differed to a much greater extent, as it can scarcely be assumed that in the two cases the bubbles were equal in bulk. The mercury was connected with the apparatus by amalgamated copper wires (4 to 5 millims. thick). Table XII. shows the results obtained.

Fig. 3.

TABLE XII.

Length ...=	269 millims.
Diameter =	1·424 millim.

The means of the conducting powers found for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.	
0°	1·6521	1·6530	−0·0009	0°	1·6529	1·6533	−0·0004
20·55	1·6276	1·6272	+0·0004	20·95	1·6272	1·6268	+0·0004
40·45	1·6003	1·6011	−0·0008	39·92	1·6010	1·6018	−0·0008
59·82	1·5750	1·5746	+0·0004	60·40	1·5741	1·5738	+0·0003
79·78	1·5465	1·5462	+0·0003	80·70	1·5454	1·5450	+0·0004
99·90	1·5162	1·5164	−0·0002	99·30	1·5174	1·5177	−0·0003

The formulæ deduced from the observations, by which the conducting powers were calculated, were—

For the first series . $\lambda=1·6530-0·0012240t-0·000001434t^2$.

For the second series $\lambda=1·6533-0·0012370t-0·000001297t^2$.

The value found for the conducting power of mercury was

	Reduced to 0°.
1·63 at 22°·8	1·6588

Taking the mean of the values as before, we find the formula for mercury to be

$$\lambda=1·656-0·0012326t-0·000001368t^2.$$

Tellurium.

Purified by dissolving the commercial metal in aqua regia, evaporating to dryness with excess of carbonate of sodium, fusing the residue, which was dissolved in water, and nitrate of barium added to precipitate any selenium present. The filtrate was evaporated to dryness with hydrochloric acid in excess, the residue dissolved in water, and precipitated by sulphurous acid.

On account of the low conducting power of tellurium, small bars of about 15 millims. in length and 3–5 millims. in diameter were used for the experiments. Bars I. and II. are of the same preparation. As the bars could not be accurately measured, we have called the first observed conducting power 100 in each case. Table XIII. gives the results.

Table XIII.

	Bar I.		Bar II.		Bar III.	
Conducting power found before heating the bars to 100°.	100 at 16°·4	......	100 at 15°·9	......	100 at 15°·6	
Ditto, after being kept at 100° for 1 day	79·145 at 15·4	......	86·50 at 13·0	......	83·16 at 12·6	
Ditto, for 2 days	45·449 at 16·0	......	76·51 at 13·6	......	69·23 at 14·1	
Ditto, for 3 days	22·378 at 16·0	......	70·43 at 16·4	......	61·25 at 16·9	
Ditto, for 4 days	16·129 at 15·0	......	65·68 at 16·6	......	54·92 at 17·2	
Ditto, for 5 days	8·068 at 15·2	......	61·68 at 16·8	......	50·69 at 17·8	
Ditto, for 6 days	6·989 at 15·0	......	56·85 at 17·2	......	46·11 at 16·6	
Ditto, for 7 days	5·781 at 14·2	......	54·88 at 16·6	......	42·35 at 16·4	
Ditto, for 8 days	4·830 at 15·5	......	51·33 at 16·1	......	38·64 at 15·8	
Ditto, for 9 days	4·621 at 16·8	......	46·27 at 15·6	......	35·31 at 16·2	
Ditto, for 10 days	4·302 at 15·3	......	45·26 at 16·2	......	33·50 at 16·4	
Ditto, for 11 days	4·181 at 15·0	Reduced to 0°.	42·10 at 16·6	......	30·97 at 16·8	
Ditto, for 12 days	4·1371 at 16·1	3·7662	41·31 at 17·4	......	29·98 at 18·2	
Ditto, for 13 days	4·0844 at 14·6	3·7646	39·28 at 16·0	......	28·21 at 15·6	
Ditto, for 14 days	....	......	37·72 at 17·1	......	26·73 at 16·8	
Ditto, for 15 days	....	......	35·35 at 15·4	......	23·68 at 15·4	
Ditto, for 16 days	....	......	32·23 at 15·6	......	19·43 at 16·0	
Ditto, for 17 days	....	......	29·92 at 17·0	......	16·65 at 17·6	
Ditto, for 18 days	....	......	28·11 at 17·6	......	14·43 at 17·0	
Ditto, for 19 days	....	......	26·25 at 16·2	......	12·59 at 16·4	
Ditto, for 20 days	....	......	25·54 at 13·0	......	11·68 at 14·4	
Ditto, for 21 days	....	......	24·12 at 13·4	......	10·34 at 13·6	
Ditto, for 22 days	....	......	23·29 at 12·8	......	9·32 at 13·6	
Ditto, for 23 days	....	......	22·00 at 13·6	......	8·64 at 14·1	
Ditto, for 24 days	....	......	21·45 at 14·1	......	7·92 at 13·8	
Ditto, for 25 days	....	......	20·86 at 14·6	......	7·36 at 14·6	
Ditto, for 26 days	....	......	20·17 at 15·8	......	6·97 at 14·2	
Ditto, for 27 days	....	......	19·74 at 16·0	......	6·66 at 14·8	
Ditto, for 28 days	....	......	19·68 at 13·0	......	6·52 at 15·8	
Ditto, for 29 days	....	......	19·65 at 12·2	Reduced to 0°.	6·35 at 15·8	
Ditto, for 30 days	....	......	19·633 at 12·0	20·145	6·12 at 12·6	
Ditto, for 31 days	....	......	19·633 at 11·9	20·137	6·04 at 12·0	Reduced to 0°.
Ditto, for 32 days	....	......	....	......	6·0330 at 11·8	5·6134
Ditto, for 33 days	....	......	....	......	6·0602 at 12·2	5·6191

The means of the conducting powers for each of the following temperatures were—

T.	Conducting power.		Difference.	T.	Conducting power.		Difference.	T.	Conducting power.		Difference.
	Observed.	Calculated.			Observed.	Calculated.			Observed.	Calculated.	
10°·40	3·9566	3·9575	−0·0009	3°·70	19·976	19·972	+0·004	4°·20	5·6646	5·6797	−0·0151
25·25	4·5212	4·5240	−0·0028	11·80	19·650	19·660	−0·010	22·[illegible]	6·7456	6·7392	+0·0064
38·85	5·3940	5·3846	+0·0094	22·40	19·477	19·466	+0·011	39·40	8·6703	8·5781	+0·0922
55·10	6·9089	6·9060	+0·0029	29·40	19·468	19·473	−0·005	53·90	10·8480	10·9426	−0·0946
70·45	8·8706	8·9019	−0·0313	29·40	19·468	19·496	−0·028	69·50	14·2472	14·3441	−0·0969
83·10	11·0316	11·0018	+0·0298	34·60	19·546	19·544	+0·002	83·60	18·4043	18·2406	+0·1637
99·40	14·3690	14·3764	−0·0074	39·70	19·689	19·653	+0·036	98·80	23·3209	23·3769	−0·0560
				55·30	20·513	20·464	+0·049				
				69·20	21·825	21·942	−0·117				
				86·40	25·096	25·019	+0·077				
				103·60	29·765	29·784	−0·019				

The formulæ deduced from the observations, and from which the conducting powers were calculated, were—

For first bar . . . . $\lambda=3{\cdot}7619+0{\cdot}011614t+0{\cdot}0006598t^2+0{\cdot}000002994t^3$.

For second bar to 29·4 . . $\lambda=20{\cdot}162-0{\cdot}055338t+0{\cdot}001085t^2$.

For second bar from 29·4 to 100 $\lambda=20{\cdot}014-0{\cdot}029569t+0{\cdot}00009390t^2+0{\cdot}000010635t^3$.

For third bar . . . . . . $\lambda=5{\cdot}5752+0{\cdot}019274t+0{\cdot}0013235t^2+0{\cdot}000003088t^3$.

From the above Table we learn that tellurium behaves in a very different manner from the other metals; for it will be seen how very much the conducting power decreases after it has been heated to 100° for some days, and how different is the time required before the conducting power of the different bars becomes constant, or, in other words, until the heating of the bars to 100° causes no further permanent alteration in the conducting power. Bar I. required 13 days; bar II. 32; bar III. 33. The first observed conducting power being taken equal to 100, bar I. is reduced to 4, bar II. to 19·6, and bar III. to 6. If we now look at the determinations of the conducting power at different temperatures of the three bars, we are struck at the great want of concordance in the results. With the first series we observe that the conducting power increases rapidly as the temperature rises; with the second it decreases with the rise of temperature to 29°·4, from which point it increases rapidly, as with bar I.; the third behaves as the first.

Bar I. showed no apparent difference in crystalline structure after being heated; it was thought very probable that the crystalline structure might have been altered by heating, and thus caused the enormous change in conducting power. The three bars, when first heated, behaved as metal to 70° or 80°, that is to say, they lost in conducting power up to that temperature, where it then began to increase. The temperature of this turning-point became lower after each day's heating, until, as in bars I. and II., it is below the lowest temperature at which observations were made.

The behaviour, therefore, of tellurium is intermediate between that of the metal and that of the metalloid; for, according to HITTORF*, selenium increases rapidly in conducting power with the temperature. Graphite and gas-coke† behave in the same manner; and BECQUEREL‡ found that gases when heated conduct better than when cold. From these facts we learn another marked difference in the physical properties of the metals and metalloids, viz. *that the metals lose in conducting power with an increase of temperature, whereas under the same circumstances the metalloids gain.*

In order to be better able to compare the results obtained with the pure metals, we give the following Tables. Table XIV. contains all the formulæ deduced from the observations by the method of least squares, with the conducting power of each metal taken =100 at 0°; Table XV. the mean of the formulæ found for each metal.

* POGGENDORFF's 'Annalen,' vol. lxxxvi. p. 214.

† Philosophical Transactions, 1858, p. 386.

‡ Ann. de Chim. et de Phys. (iii.) vol. xxxix. p. 388.

TABLE XIV.

Silver	I.	Hard drawn	$\lambda=100-0\cdot38394\,t+0\cdot0009723\,t^2$
		Annealed ..	$\lambda=100-0\cdot37725\,t+0\cdot0008900\,t^2$
	II.	Hard drawn	$\lambda=100-0\cdot38915\,t+0\cdot0010472\,t^2$
		Annealed ..	$\lambda=100-0\cdot39287\,t+0\cdot0010625\,t^2$
	III.	Hard drawn	$\lambda=100-0\cdot37544\,t+0\cdot0009724\,t^2$
		Annealed ..	$\lambda=100-0\cdot37855\,t+0\cdot0009647\,t^2$
Copper	I.	Hard drawn	$\lambda=100-0\cdot37351\,t+0\cdot0007716\,t^2$
		Annealed ..	$\lambda=100-0\cdot37291\,t+0\cdot0007781\,t^2$
	II.	Hard drawn	$\lambda=100-0\cdot39173\,t+0\cdot0009394\,t^2$
		Annealed ..	$\lambda=100-0\cdot38797\,t+0\cdot0008986\,t^2$
	III.	Hard drawn	$\lambda=100-0\cdot39639\,t+0\cdot0010173\,t^2$
		Annealed ...	$\lambda=100-0\cdot39954\,t+0\cdot0010003\,t^2$
Gold	I.	Hard drawn	$\lambda=100-0\cdot36405\,t+0\cdot0008181\,t^2$
		Annealed ...	$\lambda=100-0\cdot37017\,t+0\cdot0008526\,t^2$
	II.	Hard drawn	$\lambda=100-0\cdot35877\,t+0\cdot0007807\,t^2$
	III.	Hard drawn	$\lambda=100-0\cdot37551\,t+0\cdot0009309\,t^2$
		Annealed ...	$\lambda=100-0\cdot36877\,t+0\cdot0008390\,t^2$
Zinc	I.	..	$\lambda=100-0\cdot36845\,t+0\cdot0008147\,t^2$
	II.	..	$\lambda=100-0\cdot37911\,t+0\cdot0009058\,t^2$
	III.	..	$\lambda=100-0\cdot36385\,t+0\cdot0007618\,t^2$
Cadmium	I.	..	$\lambda=100-0\cdot36745\,t+0\cdot0007220\,t^2$
	II.	..	$\lambda=100-0\cdot39915\,t+0\cdot0010511\,t^2$
	III.	..	$\lambda=100-0\cdot33953\,t+0\cdot0004995\,t^2$
Tin	I.	..	$\lambda=100-0\cdot35867\,t+0\cdot0006205\,t^2$
	II.	..	$\lambda=100-0\cdot36730\,t+0\cdot0006709\,t^2$
	III.	..	$\lambda=100-0\cdot35492\,t+0\cdot0005493\,t^2$
Lead	I.	..	$\lambda=100-0\cdot38561\,t+0\cdot0008989\,t^2$
	II.	..	$\lambda=100-0\cdot38553\,t+0\cdot0008788\,t^2$
	III.	..	$\lambda=100-0\cdot39153\,t+0\cdot0009661\,t^2$
Arsenic	I.	..	$\lambda=100-0\cdot41141\,t+0\cdot0011000\,t^2$
	II.	..	$\lambda=100-0\cdot36851\,t+0\cdot0006757\,t^2$
Antimony	I.	..	$\lambda=100-0\cdot39855\,t+0\cdot0010332\,t^2$
	II.	..	$\lambda=100-0\cdot37524\,t+0\cdot0008477\,t^2$
	III.	..	$\lambda=100-0\cdot42099\,t+0\cdot0012283\,t^2$
Bismuth	I.	..	$\lambda=100-0\cdot36383\,t+0\cdot0007293\,t^2$
	II.	..	$\lambda=100-0\cdot34146\,t+0\cdot0004593\,t^2$
	III.	..	$\lambda=100-0\cdot35120\,t+0\cdot0005298\,t^2$
Mercury	I.	..	$\lambda=100-0\cdot074047t+0\cdot00008672t^2$
	II.	..	$\lambda=100-0\cdot074820t+0\cdot00007844t^2$

TABLE XV.

Silver	$\lambda=100-0\cdot38287t+0\cdot0009848t^2$
Copper	$\lambda=100-0\cdot38701t+0\cdot0009009t^2$
Gold	$\lambda=100-0\cdot36745t+0\cdot0008443t^2$
Zinc	$\lambda=100-0\cdot37047t+0\cdot0008274t^2$
Cadmium	$\lambda=100-0\cdot36871t+0\cdot0007575t^2$
Tin	$\lambda=100-0\cdot36029t+0\cdot0006136t^2$
Lead	$\lambda=100-0\cdot38756t+0\cdot0009146t^2$
Arsenic	$\lambda=100-0\cdot38996t+0\cdot0008879t^2$
Antimony	$\lambda=100-0\cdot39826t+0\cdot0010364t^2$
Bismuth	$\lambda=100-0\cdot35216t+0\cdot0005728t^2$
Mean of the above ...	$\lambda=100-0\cdot37647t+0\cdot0008340t^2$

From the last Table we see how closely the values found for the constants y or z agree together; and to show this more clearly, the conducting powers calculated from these formulæ for 0°, 20°, 40°, 60°, 80°, and 100° are given in Table XVI., together with values calculated from the mean of all the formulæ.

Table XVI.

T.	Silver.	Copper.	Gold.	Zinc.	Cadmium.	Tin.	Lead.	Arsenic.	Antimony.	Bismuth.	Calculated values from mean of formulæ.	Greatest difference from mean.
0°	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	100·00	0·00
20	92·74	92·62	92·99	92·92	92·93	93·04	92·62	92·56	92·45	93·18	92·80	0·38
40	86·26	85·96	86·65	86·50	86·46	86·51	85·96	85·82	85·73	86·83	86·27	0·56
60	80·57	80·01	80·98	80·75	80·60	80·59	80·04	79·80	79·84	80·93	80·41	0·61
80	75·67	74·80	76·01	75·66	75·35	75·10	74·85	74·50	74·77	75·49	75·23	0·78
100	71·56	70·31	71·70	71·23	70·70	70·11	70·39	69·88	70·54	70·51	70·69	1·01

Again, in Table XVII., we give the conducting power of the metals compared with hard-drawn silver wire at 0°=100, first at 0° and then at 100°, and, lastly, taking silver at 100°=100.

Table XVII.

	Conducting power		Taking silver =100 at 100°
	At 0°.	At 100°.	
Silver (hard drawn) ..	100·00	71·56	100·00
Copper (hard drawn)	99·95	70·27	98·20
Gold (hard drawn) ..	77·96	55·90	78·11
Zinc	29·02	20·67	28·89
Cadmium	23·72	16·77	23·44
Tin	12·36	8·67	12·12
Lead	8·32	5·86	8·18
Arsenic	4·76	3·33	4·65
Antimony	4·62	3·26	4·55
Bismuth	1·245	0·878	1·227

From these Tables we think we may deduce the law, that *all pure metals in a solid state vary in conducting power between* 0° *and* 100° *to the same extent*, more especially as we find that wires of one and the same metal show almost the same differences as were found between the mean results obtained for the different metals. In Table XVIII. two examples of this are given.

Table XVIII.

T.	Copper.		Cadmium.	
	I. annealed.	III. annealed.	II.	III.
0°	100·00	100·00	100·00	100·00
20	92·85	92·41	92·44	93·41
40	86·33	85·62	85·72	87·22
60	80·43	79·63	79·84	81·42
80	75·15	74·44	74·79	76·03
100	70·49	70·05	70·60	71·04

In Table XIX. the resistances of the copper wires 1, 2, and 3, and those calculated from the mean of all the formulæ, are given; we do this to show that the resistance of a wire does not increase in direct ratio to the temperature (as stated by some experimenters in this direction), but, on the contrary, the formula for correction of the resistance of a wire for temperature is

$$r=x+yt+zt^2,$$

and not

$$r=x+yt.$$

Table XIX.

First wire, hard drawn.			First wire, annealed.			Second wire, hard drawn.			Second wire, annealed.		
T.	Resistance.	Increase of resistance for 1°.	T.	Resistance.	Increase of resistance for 1°.	T.	Resistance.	Increase of resistance for 1°.	T.	Resistance.	Increase of resistance for 1°.
0°	98·382	......	0°	98·241	......	0°	98·412	......	0°	97·902	......
16·86	104·74	0·3771	17·0	104·67	0·3782	19·17	105·98	0·3948	18·96	105·28	0·3891
29·88	109·81	0·3825	29·63	109·54	0·3813	30·95	110·88	0·4028	31·86	110·59	0·3982
51·03	118·72	0·3985	50·22	118·08	0·3950	48·53	118·32	0·4102	52·05	119·08	0·4069
69·52	126·59	0·4057	69·60	126·23	0·4021	69·22	127·16	0·4153	70·27	126·85	0·4119
83·77	132·60	0·4085	83·42	132·21	0·4072	83·77	133·31	0·4166	83·81	132·58	0·4138
98·60	139·22	0·4142	99·37	139·10	0·4112	99·00	139·80	0·4181	99·57	139·36	0·4164
Third wire, hard drawn.			Third wire, annealed.			Resistance calculated from the mean of the six formulæ found for copper.			Resistance calculated from the mean of all the formulæ.		
0°	99·384	......	0°	97·806	......	0°	100	......	0°	100	......
12·65	104·42	0·3981	13·45	103·14	0·3966	20	107·97	0·3985	20	107·76	0·3880
25·61	109·82	0·4075	26·15	108·41	0·4055	40	116·33	0·4082	40	115·91	0·3977
39·52	115·72	0·4134	39·35	113·99	0·4113	60	124·98	0·4163	60	124·36	0·4060
53·92	121·85	0·4167	55·50	120·95	0·4170	80	133·69	0·4211	80	132·92	0·4115
69·90	128·54	0·4171	69·90	127·00	0·4176	100	142·22	0·4222	100	141·46	0·4146
84·87	134·82	0·4175	84·67	133·25	0·4186						
99·92	140·94	0·4159	99·05	139·32	0·4190						

The calculations from a formula of four or more terms, as

$$\lambda=x+yt+zt^2+at^3,$$

agree better with the observed values than that of three. An example of this is shown in Table XX., where the formulæ, deduced from observations made with a hard-drawn wire (of course previously heated to 100° for several days), of three and four terms, with the differences, are given.

TABLE XX.

T.	Conducting power.		Difference.	Conducting power, calculated from formula of four terms.	Difference.
	Observed.	Calculated from formula of three terms.			
10°·9	95·169	95·134	+0·035	95·166	+0·003
30·1	88·537	88·588	−0·051	88·534	+0·003
49·5	82·610	82·627	−0·017	82·605	+0·005
69·0	77·320	77·297	+0·023	77·304	−0·014
82·8	73·976	73·926	+0·050	73·966	+0·010
97·9	70·579	70·619	−0·040	70·580	−0·001

The formula of three terms, deduced from the observations, was

$$\lambda=99{\cdot}137-0{\cdot}37675t+0{\cdot}0008728t^2,$$

and that of four terms

$$\lambda=99{\cdot}307-0{\cdot}39301t+0{\cdot}0012318t^2-0{\cdot}000002193t^3.$$

From the above it will be seen how much better the observed values agree with the formula of four terms. We have, however, contented ourselves with a formula of three terms, as the conducting powers calculated from it agree with those observed to values corresponding to 0°·1 or 0°·2, and as the calculations for a formula of four terms would have increased the labour of the research to a very great extent. But it may be asked how it happens that the formulæ obtained for wires of one and the same metal vary so much, in fact, show differences almost equal to the mean of those deduced for the different metals?

That this is not due to errors of observation we have repeatedly satisfied ourselves; for compare only the formulæ of the hard-drawn (or rather partially annealed) and the annealed wires, and see how well they agree with each other. It appears, however, to be probably due to the molecular arrangement of the wires being different in each case. Take, for instance, the copper wires experimented with: wire 1 increased in conducting power by heating to 100° for several days, almost to the same extent as if it had been annealed, wire 2 partially so, and wire 3 hardly at all; and here it may be mentioned that silver and copper wires become softer and lose their elasticity, whereas gold does not seem to be annealed at all after having been kept at 100° for several days. Again, take cadmium, where we know that the wires become brittle and crystalline at 80°, and we find the formulæ vary more than those of any other metals; and, lastly, look at the results obtained with bismuth and tellurium, and there can be little doubt that the reason why the formulæ of the wires and bars of the same metal do not agree together is that the molecular arrangement is different in each; and that this is the cause of the differences in the formulæ, we may also assume from the fact that, when the wires on being heated do not at all or only to a very slight degree permanently alter in their conducting power, when cooled again, then the formulæ of wires of the same metal agree very closely with each other. Compare, for instance, those of lead, tin, mercury, &c.

The mean of the conducting powers given in the Tables agrees very well with the mean of the former determinations made with wires of metals of different preparation to that of those used for the experiments described in this paper.

The following questions have suggested themselves during the foregoing investigation, the answers to which we reserve for ourselves. It is intended to make them the subjects of short communications, which from time to time will be laid before the Royal Society:—

1. Will a hard-drawn wire become partially annealed by age? and, on the other hand, will an annealed wire become partially hard drawn?

2. Will bismuth or tellurium return to their original conducting power in time, or by exposure to intense cold?

3. Whether by heating tellurium or any of the metals to a higher temperature than 100° we should not arrive at the same result in a much shorter time.

4. What are the thermo-electric properties of bismuth, antimony, tellurium, &c. after being kept at 100° for several days? will they not have altered? It is remarkable that bismuth, which stands at one end of the thermo-electric series, should gain in conducting power after heating for some days, and that antimony and tellurium, at the other end of the series, should lose, the one slightly, the other, with a much higher thermo-electric number, to a very great extent.

5. Will tellurium conduct better in a melted state than the solid?

6. What law do the alloys follow as regards the influence of temperature on their conducting power?

II. *On the Aquiferous and Oviducal System in the Lamellibranchiate Mollusks.*

By George Rolleston, *Esq., M.D., F.L.S., Linacre Professor of Anatomy; and* C. Robertson, *Esq., Demonstrator of Anatomy, Oxford. Communicated by Dr.* Acland.

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Very different explanations have been offered of the means by which certain of the Lamellibranchiata are enabled to distend their muscular foot until the fluid with which it is swollen up causes it to appear all but transparent. These explanations, different as they are both in principle and in detail, admit yet of being reduced under one or other of three heads. Either they postulate the existence of a system of tubes homologous with the tracheæ of insects, and, like them, distinct from the animal's blood-vessels, as necessary for the explanation of the great changes of volume observed to take place in the mollusk's body; or they suppose these alterations of size to be effected by the agency of the blood-vascular system alone; or, thirdly, they hold the effect in question to be due to the joint working of these two systems of tubes.

Agassiz* refers the great distention observable in the foot of the *Natica heros*, of the *Pyrula carica* and *canaliculata*, and the Acephalous *Mactra solidissima*, to water inhaled by orifices more or less numerous, of less or greater calibre, in the muscular foot: these orifices, and the tubes in connexion with them, he speaks of as a water-vascular system, but he holds that they come into more or less direct and constant communication with the true blood-vascular system.

Theodor von Hessling†, who obtained the same result of injecting fully the blood-vascular system, by throwing in fluid from the glandular depression in the foot of the *Unio margaritifera*, as Agassiz did by a similar procedure with the similar depression in the foot of the Gasteropodous *Pyrulæ*, speaks of the system (which on these grounds he holds to be continuous) as but one system, and that a blood-vascular system, with certain orifices patent and communicating with the external medium in which the animal lives. Von Hessling holds also that the distention of the foot may be in part due to water inhaled through the organ of Bojanus, and mingled thus with the blood, as we shall presently describe.

M. Langer‡ holds that the organ of Bojanus is the route by which the water, upon which the change of volume in the animal's body depends, passes into it, and that this water passes into the blood-vessels, and not into any specialized water-vascular system.

* Zeitschrift für wiss. Zoologie. Pt. 7. p. 176, 1855.

† Perlmuscheln und ihre Perlen. Leipzic, 1859, p. 241.

‡ Denkschriften d. Kaiserlich. Akad. Wiss. xii. p. 55, 1856.

M. Lacaze Duthiers has discovered and described* yet another route than that of the organ of Bojanus, by which, in the *Dentalium* and *Pleurobranchus*, water from without can find its way into the interior of vessels carrying blood, and carrying it in these instances towards the heart, and not towards the gills.

Gegenbaur† differs from these authors merely in postulating the existence of orifices of exit as well as of entrance for the water; and these he holds to correspond with the puncta scattered over the foot-surface, and visible in great abundance occasionally along and near its free edge.

Von Rengarten‡ exactly reverses the functions thus supposed to belong to the punctated foot-pores, and the passage through the organ of Bojanus severally.

In a paper read by us§ before the Royal Society, February 3, 1859, we spoke of the water-vascular system as having its outlet in close approximation to the external orifice of the organ of Bojanus; and its inlet we suggested might be indicated by the position of the parasites which are not rarely to be seen studding the foot-surface and marking out the presence of its numerous pores. Gegenbaur, we observe‖, considers that the great liability of the foot to injury from the entrance of foreign bodies into these pores, is an argument for regarding them as exhalant rather than inhalant orifices.

Further investigations, carried on by us subsequently to the reading of that paper, showed us that our views as to the oviducal system in the Lamellibranchiata were founded in error. An exceedingly courteous notice of this mistake by M. Lacaze Duthiers¶ in the 'Proceedings of the Royal Society,' rendered an earlier retractation of this part of our paper unnecessary. Our views, on the other hand, as to the permeation of the bodies of the Lamellibranchiata by a system of vessels distinct from those in which the blood is contained, remain much what they were.

Before stating our views, and the arguments by which we would support them, we would say that the "perivisceral chamber" of the Brachiopoda, as described by Mr. Hancock** in a paper in the 'Philosophical Transactions,' which was published subsequently to the reading of our paper already referred to, holds much the same relation to the circulatory and reproductive and other viscera, as the system which we have called "aquiferous" in the Lamellibranchiata. As Mr. Hancock†† has himself pointed out the close correspondences of the two systems, we will but remark upon one point of discrepancy between them. In the Brachiopods the genitalia are packed into the main stems of the arborescent perivisceral system, in the direct course of the stream, if we may speak of it as a water-vascular system; in the Lamellibranchiata, or, at all events, in the family Unionidæ, the cæca of the generative gland are appended laterally to the

* Ann. des Sciences Naturelles, tom. xi. 1859, p. 255; tom. vii. Proceedings of the Royal Society, vol. x. p. 194.

† Grundzüge der vergleichenden Anatomie, p. 352, 1859.

‡ Diss. Inaug. Dorpat, 1853, cit. Von Hessling, *loc. cit.* p. 286.

§ Proceedings of the Royal Society, vol. ix. no. 34. p. 684. February 3, 1859.

‖ *Loc. cit.* p. 352.

¶ Proceedings of the Royal Society, vol. x. no. 37. p. 193.

** Philosophical Transactions for 1858. Read May 14, 1857.

†† *Loc. cit.* p. 844.

divergent twigs of the principal branches of the water-vascular tree; they do not lie in the direct course of the current of the aquiferous canals, and these canals, beyond and outside of them, break up into a very delicate minutely divided system of capillary tubes. What we shall attempt to prove is, that the orifices on either side of the foot in the Unionidæ lead not only to the generative gland, the products of which may be seen to issue forth from them at the spawning-season, but also to a system of tubes widely spread through the entire foot. We do not believe that any direct communication subsists either between the blood-vascular system and this system of tubes, or between either of these systems and the punctated depressions and inlets along the foot-edge. The blood-vessels seem to us to constitute a system of tubes closed, save at one point and at one lacuna. That point and that lacuna is the pericardial space—a cavity into which, besides the blood of the animal, the water in which it lives also finds its way. As the bivalve shell opens, it necessarily dilates this lacuna, and water is thus drawn into it through the compound sac known in the Acephala as the organ of BOJANUS. The water then gains access to the interior of the blood-vessels, as we shall proceed to show, and is carried onward within them. From the blood-vessels we suppose it to transude into the system of water-tubes everywhere in apposition with them, and, under normal conditions, to find its exit by these tubes, whilst under such abnormal circumstances as the sudden removal of the creature from the water, the sudden contraction of the muscular foot, causing jets of water to pour forth from the dilated semitransparent mass, may unload the infiltrated organ in a yet more expeditious manner. As to the way by which the water used by the mollusk for distending its foot comes into the body, we are at one with many other writers upon this subject; but we are not aware that our views, as to the method by which the animal disencumbers itself of the ingested fluid, are shared in by other authors.

Our arguments will be principally based upon the results of experiments made in the way of injection. The animals we operated upon were almost exclusively of the family Unionidæ; and, on account of the size of the specimens, as well as for other reasons, we employed chiefly the species *Anodon cygneus* and *Unio margaritifera*. In all our experiments we strove to reproduce, as nearly as possible, the conditions of the animal's natural life: our injections were always performed under water, by which and by other means as much support was given to the animal's body and its several parts as the water and the shell gave to it during life. Means were always adopted for securing that the animal died with its muscular system in a state of relaxation. We found the prussian blue injecting-fluid of Professor BEALE's* invention to possess many properties especially recommending it for use in our experiments, but we employed several other fluids as well.

Experiment 1.—If an *Anodon* or *Unio* (size is of little consequence in this experiment, though large size is a convenience in most) be removed from its shell without injuring the somewhat easily injured tissues which limit the secreting-structure of the

* How to work with the Microscope, p. 78, 1857.

organ of BOJANUS, and supported in water with its foot downwards in such a manner a to put its pericardial lacuna, and the parts in connexion with it, as nearly as possible into the condition in which they may be supposed to be in in the shell during life, and if an injection be then made into the pericardial lacuna, the following results will be seen to take place. The so-called "reddish-brown organ of KEBER" (a plexus of vessels rich in pigmentary deposit, continuous with other vessels not so coloured in the mantle and elsewhere, and bounding the pericardium on either side, and opening into it by several patent orifices at its anterior end) will become filled with the injecting-fluid first; next the gill-vessels, and sometimes together with them, yet not invariably, the systemic veins; and lastly the external orifice of the organ of BOJANUS will, on removing the animal from its prone position, be seen pouring out the injection on either side of the animal's foot.

Experiment 2.—A large *Anodon* was injected with a red stiffening-injection from the central branchial vein, a vessel readily injectible, lying as it does in the gill-cavity superiorly between the two innermost laminæ of the gills, in the angle where they become continuous with each other posteriorly to the posterior edge of the foot, with the following results:—The auricle and ventricle were filled to distention, the reddish-brown organ as well, and, besides the reddish-brown organ, the rest of the mantle, up to within a quarter of an inch of its free edge. No fluid, however, had penetrated into the pericardial space. The absence of penetration into the pericardium we have invariably had to record in our numerous injections from the branchial veins, even when the injection is noted as having been so entirely successful as to have passed through the aorta in such abundance as to inject in fine ramuscular divisions the edge of the muscular foot.

The former of these two experiments is so easy of performance, and yet proves so much, that we cannot but express our surprise at nowhere finding any record of its having been made by any of the different experimenters who have employed injections as a method for investigating the economy of mollusks. We have repeated it so frequently with the same results, as to have become quite convinced that the pericardial lacuna communicates, on the one hand, with blood going gillward, and on the other with the water in which the animal lies.

The uniformity with which our repetitions of Experiment 2 have led to the same negative result inclines us to doubt the existence of any direct communication between the aquiferous pericardial lacuna and the branchial veins properly so called. We are the more disposed to accept this conclusion, as in no mollusk whatever which is possessed of branchial vessels, except the *Pleurobranchus**, has the renal organ been shown to conduct the external water into the cavity of vessels homologous, not with the afferent, but with the efferent† branchial vessels of higher organisms.

Though Experiment 2 may seem to prove that the intravascular blood does not set in any very free current outwards into the pericardial space, especially when coupled with the observation that in multitudinous and varied injections of the different systems of blood-

* M. LACAZE DUTHIERS, Royal Society's Proceedings, *loc. cit.* † GEGENBAUR, Grundzüge, p. 367.

vessels we have* never succeeded in filling the pericardium from the blood-vessels, easy though it be, as in Experiment 1, to make the injected fluid take the reverse direction, more direct evidence is yet needed in support of our view of the organ of BOJANUS as the channel for an inwardly-setting current of water. The following considerations seem to us to show conclusively that; though Experiment 1 shows that it is possible for intrapericardial fluid to find its way outwards through the renal organ, such is not the direction usually taken by the fluid contained in the complex aquiferous system thus constituted.

1st. If we examine with the microscope the fluid contained in the pericardial space, we shall find it to contain, besides the morphological elements of blood, certain foreign bodies, such as the *Aspidogaster conchicola* and infusoria. Now these creatures must be supposed to have found their way inwards through the organ of BOJANUS.

2ndly. The external orifice of the organ of BOJANUS may be seen in a living *Anodon* (and, from its lying exposed in the gill-cavity, with yet greater ease in a *Unio margaritifera*) to execute movements of alternate opening and shutting, similar in character to those executed, as has been repeatedly noticed, by the analogous organ in the Pteropoda. These movements are repeated as frequently as once in every ten seconds (or oftener) in the *Unio margaritifera*; and they possess, there can be little doubt, in these as in other mollusks, the power of filling with water the cavities into which they lead.

3rdly. The glandular portion of the compound organ of BOJANUS has its opening into the pericardium guarded by a funnel-shaped projection which acts as a valve looking heartwards, and offers resistance consequently to fluid passing outwards from that lacuna.

4thly (Experiment 3). Fluid thrown in by the external orifice of the organ of BOJANUS, as it is either artificially, as in the *Anodon*, or naturally, as in the *Unio margaritifera*, exposed in the gill-cavity, finds it way even more easily into the pericardium than fluid thrown, as already described, into the pericardium finds its way into the gill-cavity by the reverse route. This experiment is but an imitation of what we may suppose to take place whensoever the animal by opening its valves dilates its pericardial space. As an immediate consequence of this dilatation, water is ingested into the blood-vascular system, and is forthwith applied to the purpose of distending the foot and protruding it through the opening valves.

Up to this point our views are in accordance with those adopted by several authors, though we are not aware that our method of proof has been employed by any other observers, so far as its detailed application is concerned.

We will now proceed to give our reasons for supposing that another system of tubes comes in aid of the blood-vascular system, and receives from it the fluid which that system has been the means of taking up in the manner described. Our arguments will go to show that water is transferred from the blood-vessels in the foot of the freshwater

* M. LANGER's language (Denkschriften d. Akad. Wiss. *loc. cit.* p. 43), in describing his success in such injections, is so qualified, "und sah *doch nicht immer*," as to allow one, without discourtesy, to give less weight to his views on this than on most other points.

mussel to another set of vessels, the main stem of which has the additional function of outlet to the generative gland. As, however, Von Hessling* holds that the system of pores in the foot plays no inconsiderable part in the work of supplying the distending foot with water, acting in aid of, and in alliance with, the system of the organ of Bojanus, and, with Agassiz and Von Rengarten, as already cited, holds this office to be exclusively discharged by this system of pores and inlets, we will begin by stating our reasons for demurring to these views, in which we ourselves at one time participated. It will be necessary to give the details of two sets of experiments, to show how we came to give up an opinion which can plead such high authorities as those we have cited for its defence.

Experiment 4.—A large *Anodon*, having died with its foot in a semidistended state, was injected from the venous sinus which receives the blood from the systemic veins and distributes it to the renal-portal system, with the prussian-blue injection already spoken of. The injection spread over the liver and over the whole of the generative gland, and the exclusively muscular part of the foot, spreading itself in especial richness along the free edge. No pressure which we subjected the foot to, when thus fully injected, caused any of the blue injection, easily and readily though it runs, to issue forth. Subsequently to this, a stiffening injection of red colour was thrown into the foot-mass from the oviducal outlets. This second injection spread itself very richly over the ovary, over the liver, and into the muscular foot, *along the free edge of which it issued in small jets without any pressure being applied.*

We will disregard, for the moment, the bearing which this experiment has upon the distinctness from the blood-vascular system of the system of tubes in the muscular foot, to which the stem opening under the name of oviduct into the mantle-cavity leads; and we will relate the details of another set of experiments, which led us to consider the phenomenon of the jets issuing from the foot-edge as due, in spite of the frequency with which we have seen it recur, to violence done, possibly unavoidably, to the delicate limitary tissues of these aquiferous tubes.

Experiment 5.—A *Unio margaritifera*, which had died with its foot quite relaxed, had the blue injecting-fluid introduced into its aorta, its venous system, and through the oviducal orifices, until the foot, from a state of perfect softness, became tense and swollen up. *On pressure, none of this triply-injected blue fluid could be made to issue forth from the foot-edge; but small hernia-like projections of transparent membrane rose out like bubbles all along the foot-edge.* They contained at first a transparent fluid, but after a little pressure they became filled with the blue injection. The thinness and transparency of these little sacs will account for the rarity of their appearance, and the comparative frequency with which jets of injected fluid have made themselves noticed in the region corresponding to the cæcal endings of tubes which these sacs must be held to represent. The depressions and pores which do exist in the foot of the Lamellibranchiate mollusk we believe to be glandular in character, and destitute of any direct

* Perlmuscheln und ihre Perlen, p. 286 *et seq.*

communication with the blood-vessels or other tubes in the animal's body. Now the *Unio margaritifera* stands in the same relation with reference to these foot-pores to the *Anodonta cygnea* as the *Pyrula carica* and *P. canaliculata* do, according to AGASSIZ, to the *Mactra*; that is to say, the foot of the *Unio* presents us with a gigantic pore, in the shape of a glandular depression of as much as an inch in length and two lines in diameter, whilst that of the *Anodon* is pierced but by microscopic inlets. VON HESSLING*, by whom this organ has been very accurately described, believes that injections can be made to pass, without rupture of any limitary membrane, from its cavity into the blood-vessels; and AGASSIZ holds a similar view with reference to the nearly similar structure in the *Pyrula*. But in the *Unio* just spoken of as so fully injected, as well as in several others similarly treated, though the sides and walls of this glandular depression were very richly injected, none of the injection could by pressure be made to issue out into the water in which the animal was lying. We should be inclined to consider this involution or glandular depression in the foot of the *Unio* as homologous with the foot-gland of the terrestrial Gasteropods; and the communication which has been held to exist between this Lamellibranchiate organ and its vascular system, we should not believe to be more direct than that which subsists between the muciparous foot-gland of the *Limax* and its venous system†.

It is not quite beside the purpose, to remark that the foot of one of the Unionidæ, when thoroughly distended, has a smooth bright appearance, so uniformly spread over the whole surface of its semigelatinous mass as to suggest the idea of the depressions having become everted and thus contributed to increase the size of the infiltrated organism. Though this appearance may not justify such an interpretation, yet it does seem quite inconsistent with the existence of patent pores communicating with the animal's blood-vessels.

We have repeatedly observed that, if a freshwater mussel die with its muscular foot in a state of contraction, no distention of the foot takes place, either by leaving the animal to soak in water till putrefaction sets in, or by artificial injection.

We will now proceed to state our reasons for holding the existence of a water-vascular system distinct from the blood-vessels of the Lamellibranchiata. SIEBOLD‡ states one of the objections urged against the existence of this system of vessels in the following words:—"The existence in these animals of a double system of lacunæ having this interpretation is attended with many difficulties. For then it must be admitted that one of these systems contains only water and the other blood, and it is difficult to understand how two kinds of wall-less canals can traverse the body without passing into each other." It is, however, demonstrable that in the Unionidæ, at all events, an all but perfectly closed system of blood-vessels exists. We have again and again, with various injecting fluids, found that they will pass from the aorta through a capillary system into a

* *Loc. cit.* p. 238. VON HESSLING, however, does not mention the occurrence of calcareous concretions impacted in this gland's duct. This we have observed.

† SIEBOLD, Anatomy of Invertebrata, p. 255, note 6, American edition. ‡ Ibid. p. 213.

systemic-venous system, from that into what may be called the renal-portal system of the organ of BOJANUS, and from that into the branchiæ, without any extravasation, or the formation of any lacuna anywhere. The pericardial space is, in the strictest sense of the phrase, a blood-lacuna; but, as already detailed, fluid cannot be made to pass into it from the blood-vessels, though such communication must take place to a certain extent during the life of the animal, and though the reverse direction of current is one easily demonstrable by artificial means, and is doubtless the ordinary one under normal conditions. There are two venous sinuses, however, in the Unionidæ, receiving, one after the other, the systemic-venous blood, and transmitting it into the organ of BOJANUS. The first of these* lies just within the muscular foot, along its superior and posterior edge; it subtends the second, the only one mentioned by authors, and opens into it by an orifice more or less perfectly guarded in different species of Unionidæ. This second sinus lies between the two opposed organs of BOJANUS; and from it the systemic-venous blood passes into the capillaries of the renal-portal system contained in those organs. But neither of these sinuses at all answers the character intended to be expressed by the term lacuna; they are homologous rather with the dilated great veins of certain vertebrata than with the lacunæ which do exist in certain molluscan families. There is the less occasion, however, to labour further at demonstrating the non-lacunar character of the blood-vascular system of the Unionidæ, as VON HESSLING†, in his recent book on the Pearl Mussel, confirms in this point the views previously enunciated by LANGER, adding to them a description of the histological characters of the vessels intervening between the arterial and venous systems in the *Unio*. It may be considered as beyond a doubt that a system of tubes all but entirely non-lacunar exists in these Lamellibranchiata, carrying their blood from the heart through a systemic, a renal, and a branchial system. No pressure that can in fairness be applied will cause any extravasation of fluid thus injected. Such pressure we have repeatedly applied to *Anodons* very fully distended by injection; and though it be not rare for fluid thrown in by the oviducal outlets to find its way out, as already described, by orifices along the foot, we have never found this to take place with the blood-vascular system.

In making use of the method of injections as a means for showing the independence of the several vascular trees in the Lamellibranchiate mollusks, we have sometimes injected the animal from the oviducal orifice alone, sometimes we have injected the same animal with a differently coloured fluid from its venous or from its arterial system, or from both; in a word, our injections have been either single, double, or triple.

* Into this sinus the cæca of the generative gland project somewhat freely from amongst the trabeculæ which run across what we call the roof of the muscular foot, from one side to the other; and it is here, we believe, that in injections from the oviducal outlets extravasation so often takes place into the blood-vessels.

† *Loc. cit.* p. 219. GEGENBAUR, in his 'Grundzüge,' p. 344, note, hints at some doubt still remaining in his mind as to the distinctness of these capillaries from the tissues they lie amongst. His work bears the same date (1859) as VON HESSLING's and we suppose both to have been published subsequently to the reading of our paper, February 3, 1859.

There is no difficulty in causing an injection to enter the body of any large individual of the family Unionidæ from its oviducal orifice; it is especially easy, however, to effect this in the *Unio margaritifera*, as the orifice is not in them, as in most species of the family, covered by the inner lamina of the inner gill, but, together with the orifice of the organ of BOJANUS, lies exposed and uncovered in the gill-cavity, and, besides this, is prolonged out in such a manner as to render the introduction of the syringe-pipe a very easy matter.

Experiment 6.—An injection thrown in by this orifice will spread itself over the whole of the viscera contained within the foot, not confining itself by any means to the ovary, but passing on beyond the area occupied by it or the male generative gland, into the exclusively muscular part of the foot, and distributing itself with especial richness along its free edge. That an injection thrown in by this orifice should thus spread itself would go some way towards showing that in the Lamellibranchiate, as in the Brachiopod mollusk, the ducts through which the generative products are extruded lead elsewhere as well as to the generative gland, were it possible to be sure that no transference of the injected fluid had taken place from tubes confessedly in connexion with the generative gland to another system of vessels—that, namely, which carries the blood. That such a transference does not rarely take place in one part of the blood-vascular system, we have already mentioned*; and hence arose the necessity for double injections, in which the blood-vascular system was (as has been and will again be described) injected and fully distended throughout the entirety of its own ramifications, before any fluid was thrown into the oviducal orifices, and by still mapping out a tree for itself, showed the independence of the system it led to. Single injections, however inferior to double ones, still furnish us with strong arguments for the view we are supporting. A freshwater mussel may have its whole visceral mass perfectly injected, either from the blood-vessels or from the oviducal system; but when thus injected, a practised eye has no difficulty in seeing into which of the two systems the injection has been thrown. The blood-vascular injection is seen to be contained in coarser tubes, and to form a less close network than the aquiferous, which, though confined within fine capillaries, gives, till closely inspected, an appearance almost of uniform diffusion, on account of the closeness of the network it forms.

Secondly, we will give the details of two double injections.

Experiment 7.—A double injection from the venous system and the oviducal in the same *Anodon*. A stiffening size injection of red colour was used for the oviducal or aquiferous system, and the prussian-blue injection, a more easily running fluid, for the venous system, with the following results. The red injection occupies the area corresponding to the generative gland, with coarse as well as with fine twigs, has imparted a faintish blush to the regions occupied by the liver and stomach, but has filled the interior of the exclusively muscular portion of the foot with so close and fine a network as to give it at a distance a uniform red appearance. The blue injection occupies much of the

* Note, p. 36.

foot-mass in common with and interposed between the red, its larger trunks holding the same position relatively to the larger red trunks as the larger systemic veins do to the larger generative ducts, but it has spread itself into the gills, which the red fluid has not.

Experiment 8.—A similar one to the preceding, but that the blood-vascular system was distended with the fluid used in Experiment 7 for the aquiferous, and *vice versâ*. The red fluid was thrown in by the aorta, it filled a large artery running parallel with the cap of the foot, it filled both labial tentacles, and it set, as it stiffened, in bossy masses along the edge of the foot, lastly it returned to the venous sinus and filled it and the organ of BOJANUS,—occupying thus the entire systemic and renal-portal vessels. The blue cold injection was thrown in by the orifices through which the generative products are extruded; and we shall see that it, when thus thrown in, disclosed the existence of a system of vessels distinct from those already so clearly marked out as coextensive with the systemic vessels. It spread itself chiefly over the ovary, but formed a fine plexus along the free edge of the foot beyond the artery described as running parallel with the edge of the foot, and figured as doing so by LANGER*.

This experiment must be thought to go a considerable way towards demonstrating the existence of a system of tubes distinct from, however closely apposed to, the blood-vascular system,—this system having been, in this experiment, filled with a rigid mass, and filled with it most thoroughly, as the injection of the organ of BOJANUS proves, and yet allowing the trees injectible from the aquiferous outlet to coexist side by side with it, even though the fluid they contained was so much more easily displaced than the stiffening size injection.

Thirdly, of triple injections.

The readiness with which injections pass from the arterial into the venous system make the triple injections which we have practised of less physiological value than at first sight might appear to be the case; and consequently we will content ourselves with giving the details of one such injection.

Experiment 9.—A large *Anodon* was injected from the venous sinus with a yellow stiffening injection; after this had been done, a blue-coloured fluid, also with size for its basis, was thrown into the aorta; and thirdly, a red injection of the same character was thrown in by the aquiferous opening. The blue fluid thrown into the arterial system drove the yellow fluid before it out of the systemic veins almost entirely, but it did not follow it into the renal-portal system of the organ of BOJANUS; this organ and the gills remained richly injected with yellow, to the exclusion of both the other colours; the red fluid, finally, which was thrown in by the aquiferous opening, spread itself in couples with the arterial blue over the entire visceral mass, filling alike the areas of digestive and of reproductive organs, and spreading itself with especial richness over the exclusively muscular part of the foot, which it will be recollected is the part of the animal most preeminently distended and distensible by both natural and artificial means.

Lastly, in a large individual of the freshwater-mussel family in which a stiffening

* Denkschriften der K. Akad. Wiss. Wien, viii. Bd. Taf. i. fig. 1.

or other injection has been thrown in by the orifice through which the generative products are extruded, a simple lens is sufficient to show that the tubes thus injected have the generative cæca affixed to them laterally, and pass on continuously into parts of the foot in which no generative cæca are lodged. It is most especially in that part of the muscular foot into which no viscera are packed, and which forms a belt of considerable width beyond and bounding the generative mass, and yet free from any admixture of its constituent elements (as the microscope will show), that we find the capillary network (shown to be in connexion with the oviducal outlet whilst clear of the terminal cæca) of the gland to attain its maximum development. Now this area is the area also of maximum distention in the distended foot. If in an *Unio* which has been injected from the blood-vascular system and from the oviducal, both with differently coloured injecting-fluids, a portion of the injected tissue be taken from this area and placed under one of the higher powers of the microscope, the fluid which has been thrown in by the oviducal orifice will be seen to be contained in tubes as well and sharply defined as those of the capillaries which the other injected fluid will show to be in connexion with the blood-vessels.

Whilst the analogy of the Echinodermata and many Annelids does away with any *à priori* improbability which may have seemed to attach to the possession by these mollusks of the system of tubes the existence of which we have been striving to demonstrate, the homology of the Brachiopoda furnishes us with a strong *à priori* presumption in favour of the correctness of our view. On the other hand, we cannot forbear pointing out the great improbability which must attach to a view which supposes a fluid of such morphological and such chemical characters as is the blood of the freshwater mussel to be diluted as it must be diluted on the hypothesis of the blood-vessels being the agents by which the animal voluntarily distends itself often to thrice its undistended bulk. How do the blood-corpuscles which we may take from the interior of the animal's heart behave when thus mixed with water under the microscope *? But it is not upon considerations such as these that we would lay most weight, but upon the evidence which injections of the several systems furnish to the unassisted eye, and upon the confirmation of that evidence which microscopic inspection furnishes.

* "Reagentien, wie ein *Ueberschuss von Wasser*, verdünnte Essigsäure, lösen bei der ersteren Art (Blutkörperchen) den scheinbar festen Inhalt auf, und lassen den Kern, wie die eingeschlossenen Körnchen, deutlich hervortreten. Ihre häufigen Formveränderungen, z. B. die "spiessigen hirschgeweihähnlichen Fortsätze," welche sie treiben, hängen *von unvermeidlichen Diffusionverhältnissen ab*, welchen sie bei der grossen Wassermenge gegenüber ihrer verhältnissmässig geringen Anzahl ausgesetzt sind. Während A. Ecker dieselben *durch eine Bildung von Vacuolen, in Folge deren Vergrösserung sie einreissen*, zu erklären sucht, halt Lieberkühn diese Zellenbildungen für Amöben mit selbständigen contractilen Bewegungen. In innigem Zusammenhang mit diesen Erscheinungen steht *das leichte Austreten des Zelleninhalts*, welche bisweilen in hellen und hyalinen Tropfen herumschwimmt, ja *nicht selten geht ein Zerfallen* desselben in zahlreiche kleine Tröpfchen noch innerhalb der Zellen vor sich, welche dadurch ein maulbeerartiges Ansehen bekommen, ebenso vereinigen die ausgetretenen Sarcodetropfen diese Körperchen zu den oben erwähnten Klümpchen und Flöckchen."—Von Hessling, *loc. cit.* pp. 219, 220.

III. *On the Contact of Curves.* *By* William Spottiswoode, *M.A.*, *F.R.S.*

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Let U=0 be the equation to the curve with which the curve V=0 has an "m-pointic contact;" in other words, let the curves U and V have m consecutive points in common. The degree of contact of which V is capable is equal to the number of independent constants contained in its equation; *i. e.* to the number of terms in its complete expression, less one. Thus if n be the degree of V, the degree of contact will be $=\frac{(n+1)(n+2)}{1.2}-1$.

If the curves U and V have only a single point in common, then the only conditions are

$$U=0,\ V=0. \qquad (1.)$$

If they have two consecutive points in common, then beside (1.) we have also

$$\left.\begin{aligned}\frac{\partial U}{dx}dx+\frac{\partial U}{dy}dy+\frac{\partial U}{dz}dz=0,\\ \frac{\partial V}{dx}dx+\frac{\partial V}{dy}dy+\frac{\partial V}{dz}dz=0,\end{aligned}\right\} \qquad (2.)$$

which, as is well known, lead to the conditions

$$\left.\begin{aligned}&\frac{\partial U}{dx}:\frac{\partial U}{dy}:\frac{\partial U}{dz}\\ &=\frac{\partial V}{dx}:\frac{\partial V}{dy}:\frac{\partial V}{dz};\end{aligned}\right\} \qquad (3.)$$

and if V be linear, $=lx+my+nz$, (3.) will suffice to determine the ratios $l:m:n$, and fix the position of the tangent V. The conditions (3.) may be expressed by a single equation thus: if α, β, γ be any arbitrary quantities, then the equation

$$\begin{vmatrix}\alpha & \beta & \gamma\\ \frac{\partial U}{dx} & \frac{\partial U}{dy} & \frac{\partial U}{dz}\\ \frac{\partial V}{dx} & \frac{\partial V}{dy} & \frac{\partial V}{dz}\end{vmatrix}=0, \qquad (4.)$$

considered as identical in α, β, γ, may be regarded as an expression for the required conditions.

The developed form of (4.) is

$$\left(\gamma\frac{\partial U}{dy}-\beta\frac{\partial U}{dz}\right)\frac{dV}{dx}+\left(\alpha\frac{\partial U}{dz}-\gamma\frac{\partial U}{dx}\right)\frac{\partial V}{dy}+\left(\beta\frac{\partial U}{dx}-\alpha\frac{\partial U}{dy}\right)\frac{\partial V}{dz}=0; \qquad (5.)$$

so that, writing

$$\left(\gamma\frac{\partial U}{dy}-\beta\frac{\partial U}{dz}\right)\frac{\partial}{dx}+\left(\alpha\frac{\partial U}{dz}-\gamma\frac{\partial U}{dx}\right)\frac{\partial V}{dy}+\left(\beta\frac{\partial U}{dx}-\alpha\frac{\partial U}{dy}\right)\frac{\partial}{dz}=\square, \quad . \quad . \quad (6.)$$

the effect of the operation $\square$ upon V is equivalent to the elimination of dx, dy, dz, from (1.) and their differentials (2.), and is expressed by the equation

$$\square V=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

If the curves have three consecutive points in common, we have, in addition to former conditions,

$$\left.\begin{aligned}\frac{\partial U}{dx}d^2x+\frac{\partial U}{dy}d^2y+\frac{\partial U}{dz}d^2z+\frac{\partial^2 U}{dx^2}dx^2+\frac{\partial^2 U}{dy^2}dy^2+\frac{\partial^2 U}{dz^2}dz^2+2\left(\frac{\partial^2 U}{dydz}dydz+\frac{\partial^2 U}{dzdx}dzdx+\frac{\partial^2 U}{dxdy}dxdy\right)=0,\\ \frac{\partial V}{dx}d^2x+\frac{\partial V}{dy}d^2y+\frac{\partial V}{dz}d^2z+\frac{\partial^2 V}{dx^2}dx^2+\frac{\partial^2 V}{dy^2}dy^2+\frac{\partial^2 V}{dz^2}dz^2+2\left(\frac{\partial^2 V}{dydz}dydz+\frac{\partial^2 V}{dzdx}dzdx+\frac{\partial^2 V}{dxdy}dxdy\right)=0,\end{aligned}\right\} \quad (8$$

from which dx, dy, dz, d^2x, d^2y, d^2z are to be eliminated by the help of (1.) and (2.), or (3.). Instead, however, of differentiating (2.), we may differentiate (3.), or, what is the same thing, (4.); and we shall then have an expression free from d^2x, d^2y, d^2z.

The results of the elimination may, moreover, in this case be grouped into a single expression. Writing for convenience

$$\frac{\partial U}{dy}\frac{\partial}{dz}-\frac{\partial U}{dz}\frac{\partial}{dy}=D_1,\quad \frac{\partial U}{dz}\frac{\partial}{dx}-\frac{\partial U}{dx}\frac{\partial}{dz}=D_2,\quad \frac{\partial U}{dx}\frac{\partial}{dy}-\frac{\partial U}{dy}\frac{\partial}{dx}=D_3, \quad . \quad . \quad . \quad (9.)$$

(3.) may be expressed by any two of the equations

$$D_1V=0,\quad D_2V=0,\quad D_3V=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

Differentiating these and combining their differentials with the first of (2.), we may form the quadratic system,

$$\left.\begin{array}{lll} D_1^2V\ =0 & D_1D_2V=0 & D_1D_3V=0 \\ D_2D_1V=0 & D_2^2V\ =0 & D_2D_3V=0 \\ D_3D_1V=0 & D_3D_2V=0 & D_3^2V\ =0, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.)$$

all of which are comprised in the one equation

$$\square^2V=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.)$$

And generally, by a similar train of reasoning, the constants of a curve V, having an m-pointic contact with U, will be determined by the equations

$$V=0,\quad \square V=0,\quad ..\ \square^m V=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.)$$

and if

$$\left.\begin{aligned} V&=(a, b, ..\ k\between x, y, z)^n \\ &=a\frac{\partial V}{da}+b\frac{\partial V}{db}+..\ k\frac{\partial V}{dk}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

then

$$a:b:\ldots k$$

$$=\left\|\begin{matrix}\frac{\partial V}{da} & \frac{\partial V}{db} & \cdots & \frac{\partial V}{dk}\\ \square\frac{\partial V}{da} & \square\frac{\partial V}{db} & \cdots & \square\frac{\partial V}{dk}\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \square^{m-1}\frac{\partial V}{da} & \square^{m-1}\frac{\partial V}{db} & - & \square^{m-1}\frac{\partial V}{dk}\end{matrix}\right\| \quad \ldots \ldots \quad (15.)$$

The equations (11.) are, however, not all independent. In the first place, as in (10.) any one of the three equations is a consequence of the other two, so in (12.) any one column or row is a consequence of the other two. Secondly, if any pair of binary combinations of the Ds, *e. g.* D_1D_2V and D_2D_1V, be developed, they will, with the help of (10.) or (3.), be found identical. This reduces the system (12.) to three independent conditions, of which

$$D_1^2V=0, \quad D_1D_2V=0, \quad D_2^2V=0 \quad \ldots \ldots \quad (16.)$$

is a type; as it should do. These three equations will suffice to determine the three independent constants of V, when it is capable of a 3-pointic contact, and no more, with U.

The actual calculations may be considerably simplified by using, instead of (13.), the following system,

$$V=0, \quad DV=0, \quad D^2V=0$$

(where, as before, D stands for any one of the symbols D_1, D_2, D_3). Of this system (16.) is a consequence.

By way of example, we have for the ordinary tangent

$$V=ax+by+cz=0,$$
$$DV=aDx+bDy+cDz=0,$$

whence

$$a:b:c$$

$$\left\|\begin{matrix}x & y & z\\ Dx & Dy & Dz\end{matrix}\right\| \quad \ldots \ldots \quad (17.)$$

$$=yDz-zDy : zDx-xDz : xDy-yDx$$

$$=y\frac{\partial U}{dy}+z\frac{\partial U}{dz} : -x\frac{\partial U}{dy} : -x\frac{\partial U}{dz}$$

$$=\frac{\partial U}{dx}:\frac{\partial U}{dy}:\frac{\partial U}{dz}. \quad \ldots \ldots \quad (18.)$$

The equation of the circle of curvature may be put in the form

$$hr^2+2(ayz+bzx+cxy)=0;$$

to which are to be added,

$$hD\,r^2+2(aD\,yz+bD\,zx+cD\,xy)=0,$$
$$hD^2r^2+2(aD^2yz+bD^2zx+cD^2xy)=0,$$

whence

$$h:a:b:c$$

$$\left\| \begin{matrix} r^2 & 2yz & 2zx & 2xy \\ Dr^2 & 2Dyz & 2Dzx & 2Dxy \\ D^2r^2 & 2D^2yz & 2D^2zx & 2D^2xy; \end{matrix} \right\| \quad \ldots \ldots \quad (19.)$$

also

$$Dr^2 = 2\left(z\frac{\partial U}{dy} - y\frac{\partial U}{dz}\right),$$

$$D^2r^2 = 2\left\{\left(\frac{\partial U}{dy}\right)^2 + \left(\frac{\partial U}{dz}\right)^2\right\} + 2\left(zD\frac{\partial U}{dy} - yD\frac{\partial U}{dz}\right),$$

$$Dyz = y\frac{\partial U}{dy} - z\frac{\partial U}{dz},$$

$$D^2yz = -2\frac{\partial U}{dy}\frac{\partial U}{dz} + yD\frac{\partial U}{dy} - zD\frac{\partial U}{dz},$$

$$Dzx = x\frac{\partial U}{dy} \quad D^2zx = xD\frac{\partial U}{dy},$$

$$Dxy = -x\frac{\partial U}{dz} \quad D^2xy = -xD\frac{\partial U}{dz}.$$

Moreover the minors formed from the first two rows of the matrix, and from the columns below written, have the following values:—

$$r^2\ ,\ 2yz = x\left\{-(y^2-z^2)\frac{\partial U}{dx} + xy\frac{\partial U}{dy} - xz\frac{\partial U}{dz}\right\},$$

$$r^2\ ,\ 2zx = x\left\{-yx\frac{\partial U}{dx} - (z^2-x^2)\frac{\partial U}{dy} + yz\frac{\partial U}{dz}\right\},$$

$$r^2\ ,\ 2xy = x\left\{zx\frac{\partial U}{dx} - zy\frac{\partial U}{dy} - (x^2-y^2)\frac{\partial U}{dz}\right\},$$

$$2zx,\ 2xy = 2x^3\frac{\partial U}{dx},$$

$$2xy,\ 2yz = 2xy^2\frac{\partial U}{dy},$$

$$2yz,\ 2zx = 2xz^2\frac{\partial U}{dz};$$

whence, finally,

$$h:a:b:c$$

$$= 4\frac{\partial U}{dx}\frac{\partial U}{dy}\frac{\partial U}{dz} + \frac{2xyz}{(n-1)^2}H$$

$$: 2\frac{\partial U}{dx}\left\{\left(\frac{\partial U}{dy}\right)^2 + \left(\frac{\partial U}{dz}\right)^2\right\} + (\ x^2-y^2-z^2)\frac{x}{(n-1)^2}H$$

$$: 2\frac{\partial U}{dy}\left\{\left(\frac{\partial U}{dz}\right)^2 + \left(\frac{\partial U}{dx}\right)^2\right\} + (-x^2+y^2-z^2)\frac{y}{(n-1)^2}H$$

$$: 2\frac{\partial U}{dz}\left\{\left(\frac{\partial U}{dx}\right)^2 + \left(\frac{\partial U}{dy}\right)^2\right\} + (-x^2-y^2+z^2)\frac{z}{(n-1)^2}H, \quad \ldots \quad (20.)$$

where H is the Hessian of U.

Proceeding by the same principles, we shall have merely to replace $\Box$ by D in (15.), and there will result a system of expressions for the constants of the curve V having an m-pointic contact with a given curve U.

The following method of reduction may sometimes be used with advantage. Let $\xi, \eta, \zeta, \ldots \phi, \chi, \psi$ stand for the powers and products of the variables forming the various terms of V. Then the coefficients of the curve V having an m-pointic contact with U will be proportional to the determinants of which the following is a type:

$$\begin{matrix} D^{m-1}\xi & D^{m-1}\eta & \ldots & D^{m-1}\psi \\ D^{m-2}\xi & D^{m-2}\eta & \ldots & D^{m-2}\psi \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \xi & \eta & \ldots & \psi. \end{matrix}$$

But this

$$=-\Sigma D^{m-2}\xi \begin{vmatrix} D^{m-1}\eta & D^{m-1}\zeta & \ldots & D^{m-1}\psi \\ D^{m-3}\eta & D^{m-3}\zeta & \ldots & D^{m-3}\psi \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot\,\cdot\,\cdot & & \cdot \\ \eta & \zeta & \ldots & \psi \end{vmatrix} = -\Sigma D^{m-2}\xi D \begin{vmatrix} D^{m-2}\eta & D^{m-2}\zeta & \ldots & D^{m-2}\phi \\ D^{m-3}\eta & D^{m-3}\zeta & \ldots & D^{m-3}\phi \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot\,\cdot\,\cdot & & \cdot \\ \eta & \zeta & \ldots & \phi \end{vmatrix}$$

$$=(-)^2\Sigma D^{m-2}\xi\ D\Sigma\ D^{m-3}\eta D \begin{vmatrix} D^{m-3}\zeta & \ldots & D^{m-3}\chi & D^{m-3}\psi \\ D^{m-4}\zeta & \ldots & D^{m-4}\chi & D^{m-4}\psi \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \zeta & \ldots & \chi & \psi \end{vmatrix}$$

$$=(-)^2\Sigma D^{m-2}\xi\ \Sigma\ D^{m-3}\eta\ D^2 \begin{vmatrix} D^{m-3}\zeta & \ldots & D^{m-3}\chi & D^{m-3}\psi \\ D^{m-4}\zeta & \ldots & D^{m-4}\chi & D^{m-4}\psi \\ \cdot\ \cdot & & \cdot & \cdot \\ \zeta & \ldots & \chi & \psi \end{vmatrix}$$

$$=(-)^{m-2}\ \Sigma D^{m-2}\xi\ \Sigma D^{m-3}\eta\ \ldots\ \Sigma D\phi\ D^{m-2} \begin{vmatrix} D\chi & D\psi \\ \chi & \psi \end{vmatrix} \quad \ldots\ldots\ldots \quad (21.)$$

In the case of the conic of 5-pointic contact this becomes

$$-\Sigma D^3x^2\Sigma D^2y^2\Sigma Dz^2D^3 \begin{vmatrix} 2Dzx & 2Dxy \\ zx & xy \end{vmatrix}$$

I now proceed to the calculation of the conic of 5-pointic contact. This may be effected by any of the three methods indicated above, viz. (1) by means of the symbol $\Box$, getting out the factors $(\alpha x+\beta y+\gamma z)^p$; (2) by means of the symbols D; (3) by the reduced formula (21.). As most of the steps have been calculated by two methods, I subjoin some of the leading formulæ and transformations which occur in the process.

Writing for convenience

$$\lambda=\gamma\frac{\partial U}{dy}-\beta\frac{\partial U}{dz},\quad \mu=\alpha\frac{\partial U}{dz}-\gamma\frac{\partial U}{dx},\quad \nu=\beta\frac{\partial U}{dx}-\alpha\frac{\partial U}{dy},$$

and putting

$$V=(abcfgh)(xyz)^2,$$

the general expressions (13.) become

$$\begin{aligned}
0&=(abcfgh)(xyz)^2,\\
0&=(abcfgh)(\lambda\mu\nu)(xyz),\\
0&=(abcfgh)(\lambda\mu\nu)^2+(abcfgh)(\square\lambda\,\square\mu\,\square\nu)(xyz),\\
0&=3(abcfgh)(\square\lambda\,\square\mu\,\square\nu)(\lambda\mu\nu)+(abcfgh)(\square^2\lambda\,\square^2\mu\,\square^2\nu)(xyz),\\
0&=3(abcfgh)(\square\lambda\,\square\mu\,\square\nu)^2+4(abcfgh)(\square^2\lambda\,\square^2\mu\,\square^2\nu)(\lambda\mu\nu)+(abcfgh)(\square^3\lambda\,\square^3\mu\,\square^3\nu)(xyz),
\end{aligned}$$

whence

$$a:b:c:f:g:h=$$

$$\left\|\begin{matrix}
3(\square\lambda)^2+4\lambda\square^2\lambda+x\square^3\lambda & \cdot & \cdot & 6\square\mu\square\nu+4(\mu\square^2\nu+\nu\square^2\mu)+y\square^3\nu+z\square^3\mu & \cdot & \cdot\\
3\lambda\square\lambda+x\square^2\lambda & \cdot & \cdot & 3(\mu\square\nu+\nu\square\mu)+y\square^2\nu+z\square^2\mu & \cdot & \cdot\\
\lambda^2+x\square\lambda & \cdot & \cdot & 2\mu\nu+y\square\nu+z\square\mu & \cdot & \cdot\\
x\lambda & \cdot & \cdot & y\mu+z\nu & \cdot & \cdot\\
x^2 & y^2 & z^2 & 2yz & 2zx & 2xy.
\end{matrix}\right\| \quad (22.)$$

The most convenient method for developing these formulæ is to begin with forming the 15 minors derived from the last two lines; and by means of them, the 20 minors derived from the last three lines; and so on. It may be noticed that in order to pass to the corresponding expressions in D, *e. g.* D_1, it is only necessary to make $\beta=0$, $\gamma=0$, and to divide out the αs. This gives

$$\lambda=0,\quad \mu=\frac{\partial U}{dz},\quad \nu=-\frac{\partial U}{dy}.$$

The results of the calculations are as follow: the third minors, formed from the two lower lines of (22.), and from the columns the lower constituents of which alone are written down, will be

$$y^2,\ z^2=-yz\frac{\partial U}{dx},\quad z^2,\ x^2=-zx\frac{\partial U}{dy},\quad x^2,\ y^2=-xy\frac{\partial U}{dz},$$

$$2zx,\ 2xy=2x^2\frac{\partial U}{dx},\quad 2xy,\ 2yz=2y^2\frac{\partial U}{dy},\quad 2yz,\ 2zx=2z^2\frac{\partial U}{dz},$$

$$x^2,\ 2yz=x\left(y\frac{\partial U}{dy}-z\frac{\partial U}{dz}\right),\quad x^2,\ 2zx=x^2\frac{\partial U}{dy}\quad x^2,\ 2xy=-x^2\frac{\partial U}{dz},$$

$$y^2,\ 2yz=-y^2\frac{\partial U}{dx},\quad y^2,\ 2zx=y\left(z\frac{\partial U}{dz}-x\frac{\partial U}{dx}\right)\quad y^2,\ 2xy=y^2\frac{\partial U}{dz},$$

$$z^2,\ 2yz=z^2\frac{\partial U}{dx},\quad z^2,\ 2zx=-z^2\frac{\partial U}{dy},\quad z^2,\ 2xy=z\left(x\frac{\partial U}{dx}-y\frac{\partial U}{dy}\right).$$

Similarly, for the second minors,

$$x^2,\ 2xy,\ 2xz = x^3(n-1)^{-2}\mathrm{H},$$

$$2xy,\ y^2\ ,\ 2yz = y^3(n-1)^{-2}\mathrm{H},$$

$$2xz,\ 2yz,\ \ z^2 = z^3(n-1)^{-2}\mathrm{H},$$

$$x^2,\ 2yz,\ 2zx = 2\left(\frac{\partial\mathrm{U}}{dy}\right)^2\frac{\partial\mathrm{U}}{dz} - zx^2(n-1)^{-2}\mathrm{H},$$

$$x^2,\ 2xy,\ 2yz = 2\frac{\partial\mathrm{U}}{dy}\left(\frac{\partial\mathrm{U}}{dz}\right)^2 - yx^2(n-1)^{-2}\mathrm{H},$$

$$2xy,\ \ y^2,\ 2zx = 2\left(\frac{\partial\mathrm{U}}{dz}\right)^2\frac{\partial\mathrm{U}}{dx} - xy^2(n-1)^{-2}\mathrm{H},$$

$$2xz,\ \ y^2,\ 2yz = 2\frac{\partial\mathrm{U}}{dz}\left(\frac{\partial\mathrm{U}}{dx}\right)^2 - zy^2(n-1)^{-2}\mathrm{H},$$

$$2xy,\ 2yz,\ \ z^2 = 2\left(\frac{\partial\mathrm{U}}{dx}\right)^2\frac{\partial\mathrm{U}}{dy} - yz^2(n-1)^{-2}\mathrm{H},$$

$$2zx,\ 2xy,\ \ z^2 = 2\frac{\partial\mathrm{U}}{dx}\left(\frac{\partial\mathrm{U}}{dy}\right)^2 - xz^2(n-1)^{-2}\mathrm{H},$$

$$y^2,\ \ z^2,\ 2yz = \left(\frac{\partial\mathrm{U}}{dx}\right)^3,$$

$$y^2,\ \ z^2,\ 2zx = -\left\{\left(\frac{\partial\mathrm{U}}{dx}\right)^2\frac{\partial\mathrm{U}}{dy} + yz^2(n-1)^{-2}\mathrm{H}\right\},$$

$$y^2,\ \ z^2,\ 2xy = -\left\{\left(\frac{\partial\mathrm{U}}{dx}\right)^2\frac{\partial\mathrm{U}}{dz} + y^2z(n-1)^{-2}\mathrm{H}\right\},$$

$$z^2,\ \ x^2,\ 2yz = -\left\{\left(\frac{\partial\mathrm{U}}{dy}\right)^2\frac{\partial\mathrm{U}}{dx} + z^2x(n-1)^{-2}\mathrm{H}\right\},$$

$$z^2,\ \ x^2,\ 2zx = \left(\frac{\partial\mathrm{U}}{dy}\right)^3,$$

$$z^2,\ \ x^2,\ 2xy = -\left\{\left(\frac{\partial\mathrm{U}}{dy}\right)^2\frac{\partial\mathrm{U}}{dz} + zx^2(n-1)^{-2}\mathrm{H}\right\},$$

$$x^2,\ \ y^2,\ 2yz = -\left\{\left(\frac{\partial\mathrm{U}}{dz}\right)^2\frac{\partial\mathrm{U}}{dx} + xy^2(n-1)^{-2}\mathrm{H}\right\},$$

$$x^2,\ \ y^2,\ 2zx = -\left\{\left(\frac{\partial\mathrm{U}}{dz}\right)^2\frac{\partial\mathrm{U}}{dy} + x^2y(n-1)^{-2}\mathrm{H}\right\},$$

$$x^2,\ \ y^2,\ 2xy = \left(\frac{\partial\mathrm{U}}{dz}\right)^3,$$

$$x^2,\ \ y^2,\ \ z^2 = \frac{\partial\mathrm{U}}{dx}\frac{\partial\mathrm{U}}{dy}\frac{\partial\mathrm{U}}{dz} - xyz(n-1)^{-2}\mathrm{H},$$

$$2yz,\ 2zx,\ 2xy = -2\frac{\partial\mathrm{U}}{dx}\frac{\partial\mathrm{U}}{dy}\frac{\partial\mathrm{U}}{dz} - 2xyz(n-1)^{-2}\mathrm{H}.$$

Similarly, for the first minors,

$$x^2,\ y^2,\ z^2,\ 2yz=\frac{\partial U}{dx}\Big\{\frac{\partial U}{dx}\Big(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\Big)-3H\Big(\frac{\partial U}{dy}\frac{\partial^2 U}{dxdz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dxdy}\Big)\Big\},$$

$$x^2,\ y^2,\ z^2,\ 2zx=\frac{\partial U}{dy}\Big\{\frac{\partial U}{dy}\Big(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\Big)-3H\Big(\frac{\partial U}{dz}\frac{\partial^2 U}{dydx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dydz}\Big)\Big\},$$

$$x^2,\ y^2,\ z^2,\ 2xy=\frac{\partial U}{dz}\Big\{\frac{\partial U}{dz}\Big(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\Big)-3H\Big(\frac{\partial U}{dx}\frac{\partial^2 U}{dzdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dzdx}\Big)\Big\},$$

$$\begin{aligned}y^2,\ z^2,\ 2zx,\ 2xy=\frac{\partial U}{dx}\Big\{&\frac{\partial U}{dy}\Big(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\Big)-3H\Big(\frac{\partial U}{dz}\frac{\partial^2 U}{dydx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dydz}\Big)\\ &-\frac{\partial U}{dz}\Big(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\Big)+3H\Big(\frac{\partial U}{dx}\frac{\partial^2 U}{dzdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dzdx}\Big)\Big\}-3yz\frac{H^2}{(n-1)^2},\end{aligned}$$

$$y^2,\ z^2,\ 2xy,\ 2yz=\frac{\partial U}{dx}\Big\{\frac{\partial U}{dx}\Big(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\Big)-3H\Big(\frac{\partial U}{dz}\frac{\partial^2 U}{dx^2}-\frac{\partial U}{dx}\frac{\partial^2 U}{dxdz}\Big)\Big\},$$

$$y^2,\ z^2,\ 2zx,\ 2yz=\frac{\partial U}{dx}\Big\{\frac{\partial U}{dx}\Big(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\Big)-3H\Big(\frac{\partial U}{dx}\frac{\partial^2 U}{dxdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dx^2}\Big)\Big\},$$

$$z^2,\ x^2,\ 2xy,\ 2zx=\frac{\partial U}{dy}\Big\{\frac{\partial U}{dy}\Big(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\Big)-3H\Big(\frac{\partial U}{dy}\frac{\partial^2 U}{dydz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dy^2}\Big)\Big\},$$

$$\begin{aligned}z^2,\ x^2,\ 2xy,\ 2yz=\frac{\partial U}{dy}\Big\{&\frac{\partial U}{dz}\Big(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\Big)-3H\Big(\frac{\partial U}{dx}\frac{\partial^2 U}{dzdy}-\frac{\partial U}{dy}\frac{\partial^2 U}{dzdx}\Big)\\ &-\frac{\partial U}{dx}\Big(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\Big)+3H\Big(\frac{\partial U}{dy}\frac{\partial^2 U}{dxdz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dxdy}\Big)\Big\}-3zx\frac{H^2}{(n-1)^2},\end{aligned}$$

$$z^2,\ x^2,\ 2yz,\ 2zx=\frac{\partial U}{dy}\Big\{\frac{\partial U}{dy}\Big(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dx}\Big)-3H\Big(\frac{\partial U}{dx}\frac{\partial^2 U}{dy^2}-\frac{\partial U}{dy}\frac{\partial^2 U}{dydx}\Big)\Big\},$$

$$x^2,\ y^2,\ 2zx,\ 2xy=\frac{\partial U}{dz}\Big\{\frac{\partial U}{dz}\Big(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\Big)-3H\Big(\frac{\partial U}{dy}\frac{\partial^2 U}{dz^2}-\frac{\partial U}{dz}\frac{\partial^2 U}{dzdy}\Big)\Big\},$$

$$x^2,\ y^2,\ 2yz,\ 2xy=\frac{\partial U}{dz}\Big\{\frac{\partial U}{dz}\Big(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\Big)-3H\Big(\frac{\partial U}{dz}\frac{\partial^2 U}{dzdx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dz^2}\Big)\Big\},$$

$$\begin{aligned}x^2,\ y^2,\ 2yz,\ 2zx=\frac{\partial U}{dz}\Big\{&\frac{\partial U}{dx}\Big(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\Big)-3H\Big(\frac{\partial U}{dy}\frac{\partial^2 U}{dxdz}-\frac{\partial U}{dz}\frac{\partial^2 U}{dxdy}\Big)\\ &-\frac{\partial U}{dy}\Big(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\Big)+3H\Big(\frac{\partial U}{dz}\frac{\partial^2 U}{dydx}-\frac{\partial U}{dx}\frac{\partial^2 U}{dydz}\Big)\Big\}-3xy\frac{H^2}{(n-1^2)},\end{aligned}$$

$$2yz,\ 2zx,\ 2xy,\ x^2=2\frac{\partial U}{dy}\frac{\partial U}{dz}\Big(\frac{\partial U}{dy}\frac{\partial H}{dz}-\frac{\partial U}{dz}\frac{\partial H}{dy}\Big)-3H\Big\{\Big(\frac{\partial U}{dy}\Big)^2\frac{\partial^2 U}{dz^2}-\Big(\frac{\partial U}{dz}\Big)^2\frac{\partial^2 U}{dy^2}\Big\},$$

$$2yz,\ 2zx,\ 2xy,\ y^2=2\frac{\partial U}{dz}\frac{\partial U}{dx}\Big(\frac{\partial U}{dz}\frac{\partial H}{dx}-\frac{\partial U}{dx}\frac{\partial H}{dz}\Big)-3H\Big\{\Big(\frac{\partial U}{dz}\Big)^2\frac{\partial^2 U}{dx^2}-\Big(\frac{\partial U}{dx}\Big)^2\frac{\partial^2 U}{dz^2}\Big\},$$

$$2yz,\ 2zx,\ 2xy,\ z^2=2\frac{\partial U}{dx}\frac{\partial U}{dy}\Big(\frac{\partial U}{dx}\frac{\partial H}{dy}-\frac{\partial U}{dy}\frac{\partial H}{dz}\Big)-3H\Big\{\Big(\frac{\partial U}{dx}\Big)^2\frac{\partial^2 U}{dy^2}-\Big(\frac{\partial U}{dy}\Big)^2\frac{\partial^2 U}{dx^2}\Big\}.$$

Proceeding to the calculation of the final expressions for the coefficients a, b, c, f, g, h of the conic of five-pointed contact, which is given rather more in detail, we have, writing $\frac{dU}{dx}=u,\ \frac{dU}{dy}=v,\ \frac{dU}{dz}=w,$

$$
\begin{aligned}
f=x^2,\ y^2,\ z^2,\ 2zx,\ 2xy = {} & D^4x^2(y^2,\ z^2,\ 2zx,\ 2xy) \\
& +D^4y^2(z^2,\ 2zx,\ 2xy,\ x^2) \\
& +D^4z^2(2zx,\ 2xy,\ x^2,\ y^2) \\
& +2D^4zx(2xy,\ x^2,\ y^2,\ z^2) \\
& +2D^4xy(x^2,\ y^2,\ z^2,\ 2zx)
\end{aligned}
$$

$$
\begin{aligned}
= & -(3(Dw)^2+4wD^2w-yD^3w)v\,(v\,D\,H-3HD\,v\,) \\
& +(3(Dv\,)^2+4v\,D^2v\,+zD^3v\,)w(wD\,H-3HD\,w) \\
& -(\qquad\qquad\qquad xD^3v\,)w(wD_2H-3HD_2w) \\
& -(\qquad\qquad\qquad xD^3w)v\,(v\,D_1H-3HD_1w) \\
= {} & 3(w^2(Dv)^2-v^2(Dw)^2)DH+4vw(wD^2v-vD^2w)DH \\
& +(w^2zD^3v+v^2yD^3w)DH-x(w^2D^3vD_2H+v^2D^3wD_1H) \\
& +3H\{3DvDw(vDw-wDv)+4vw(DvD^2w-DwD^2v) \\
& \qquad -yvDvD^3w-zwDwD^3v+xvD_1vD^3w+xwD_2wD^3v\} \\
= {} & 3(wDv-vDw)D(vw)DH+4vwD(wDv-vDw)DH \\
& -9(wDv-vDw)DvDwH+12vw(DvD^2w-DwD^2v)H \\
& +D^3v\{w^2(zDH-xD_2H)-3Hw(zDw-xD_2w)\} \\
& +D^3w\{v^2(yDH-xD_1H)-3Hv(yDv-xD_1v)\} \\
= {} & (n-1)^{-2}x^2\{3D(vw)HDH+4vw(DH)^2-9DvDwH^2\} \\
& +12vwH(DvD^2w-DwD^2v) \\
& +\ 3vwHD^2(vDw-wDv) \\
& -\ 3vwH(DvD^2w-DwD^2v),
\end{aligned}
$$

which, omitting the common factor $(n-1)^{-2}x^2$,

$$
\begin{aligned}
= {} & 3D(vw)HDH+4vw(DH)^2+9(n-1)^{-2}x^2H^3\frac{\partial^2U}{dydz}+9vw\mathfrak{A}H^2 \\
& +9\frac{3(n-2)}{n-1}vw\mathfrak{A}H^2-9\frac{x}{n-1}vwH\left(\mathfrak{A}\frac{\partial H}{dx}+\mathfrak{H}\frac{\partial H}{dy}+\mathfrak{G}\frac{\partial H}{dz}\right)-3vwHD^2H.
\end{aligned}
$$

But

$$3D(vw)HDH+6\frac{3(n-2)}{n-1}vw\mathfrak{A}H^2-6\frac{x}{n-1}vwH\left(\mathfrak{A}\frac{\partial H}{dx}+\mathfrak{B}\frac{\partial H}{dy}+\mathfrak{C}\frac{\partial H}{dz}\right)$$

$$=3(vDw+wDv)\left(v\frac{\partial H}{dz}-w\frac{\partial H}{dy}\right)H+6vw\left(Dw\frac{\partial H}{dy}-Dv\frac{\partial H}{dz}\right)H$$

$$=(3v^2Dw+3vwDv-6vwDv)H\frac{\partial H}{dz}-(3vwDw+3w^2Dv-6vwDw)H\frac{\partial H}{dy}$$

$$=3Hv(vDw-wDv)\frac{\partial H}{dz}-3Hw(wDv-vDw)\frac{\partial H}{dy}.$$

$$=-3(n-1)^{-2}x^2H^2\left(w\frac{\partial H}{dy}+v\frac{\partial H}{dz}\right).$$

Hence, excepting the common factor $(n-1)^{-2}x^2$, the total expression for f

$$=9(n-1)^{-2}x^2H^3\frac{\partial^2 U}{dydz}-3(n-1)^{-2}x^2H^2\left(\frac{\partial U}{dy}\frac{\partial H}{dz}+\frac{\partial U}{dz}\frac{\partial H}{dy}\right)$$

$$-vw\left\{3H\left(D^2H+\frac{x}{n-1}\left(\mathfrak{A}\frac{\partial H}{dx}+\mathfrak{B}\frac{\partial H}{dy}+\mathfrak{C}\frac{\partial H}{dz}\right)-\frac{3(n-2)}{n-1}H\mathfrak{A}\right)-4(DH)^2\right\}.$$

And comparing this with Mr. CAYLEY's expressions in arts. 14 and 15 of his memoir "On the Conic of Five-pointic Contact," in the Philosophical Transactions for 1859, pp. 376, 377; and bearing in mind that in his formulæ we must make $\lambda=1$, $\mu=0$, $\nu=0$, in order to institute a comparison; and lastly, dividing throughout by $(n-1)^{-2}x^2 9H^3$, and multiplying by 2, the expression above written becomes

$$2\frac{\partial^2 U}{dydz}-\frac{2}{3}\frac{1}{H}\left(\frac{\partial H}{dy}\frac{\partial U}{dz}+\frac{\partial H}{dz}\frac{\partial U}{dy}\right)-2\Lambda\frac{\partial U}{dy}\frac{\partial U}{dz},$$

which is in fact the coefficient of YZ in his general formula, viz.

$$\left(X\frac{\partial}{dx}+Y\frac{\partial}{dy}+Z\frac{\partial}{dz}\right)^2U-\left(\frac{2}{3}\frac{1}{H}\left(X\frac{\partial H}{dx}+Y\frac{\partial H}{dy}+Z\frac{\partial H}{dz}\right)+\Lambda\left(X\frac{\partial U}{dx}+Y\frac{\partial U}{dy}+Z\frac{\partial U}{dz}\right)\right)$$
$$\times\left(X\frac{\partial U}{dx}+Y\frac{\partial U}{dy}+Z\frac{\partial U}{dz}\right)=0.$$

The identity of the expressions for f, as deduced by the present method, with that deduced by Mr. CAYLEY having been thus demonstrated, it is unnecessary to pursue the calculations further.

The following are the principal subsidiary formulæ used in the foregoing calculations.

$$\delta=\alpha x+\beta y+\gamma z$$

$$y\nu-z\mu=\delta\frac{\partial U}{dx},\quad z\lambda-x\nu=\delta\frac{\partial U}{dy},\quad x\mu-y\lambda=\delta\frac{\partial U}{dz}$$

$$y\square\nu-z\square\mu=\delta\square\frac{\partial U}{dx},\quad z\square\lambda-x\square\nu=\delta\square\frac{\partial U}{dy},\quad x\square\mu-y\square\lambda=\delta\square\frac{\partial U}{dz}$$

$$x\square\frac{\partial U}{dx}+y\square\frac{\partial U}{dy}+z\square\frac{\partial U}{dz}=0.$$

Also writing

$$\square\lambda\frac{\partial U}{dx}+\square\mu\frac{\partial U}{dy}+\square\nu\frac{\partial U}{dz}=\Pi,$$

we have

$$\mu\square\nu-\nu\square\mu=\alpha\Pi,\ \nu\square\lambda-\lambda\square\nu=\beta\Pi,\ \lambda\square\mu-\mu\square\lambda=\gamma\Pi;$$

also

$$-\Pi=\begin{vmatrix} \frac{\partial^2 U}{dx} & \frac{\partial^2 U}{dxdy} & \frac{\partial^2 U}{dxdz} & \frac{\partial U}{dx} & \alpha \\ \frac{\partial^2 U}{dydx} & \frac{\partial^2 U}{dy^2} & \frac{\partial^2 U}{dydz} & \frac{\partial U}{dy} & \beta \\ \frac{\partial^2 U}{dzdx} & \frac{\partial^2 U}{dzdy} & \frac{\partial^2 U}{dz^2} & \frac{\partial U}{dz} & \gamma \\ \frac{\partial U}{dx} & \frac{\partial U}{dy} & \frac{\partial U}{dz} & 0 & 0 \\ \alpha & \beta & \gamma & 0 & 0 \end{vmatrix}=-\frac{\delta^2}{(n-1)^2}H,$$

where H is the Hessian of U. The corresponding reduced form of this expression, *i. e.* when $\alpha=1$, $\beta=0$, $\gamma=0$, is

$$\left(\frac{\partial U}{dz}\right)^2\frac{\partial^2 U}{dy^2}-2\frac{\partial U}{dz}\frac{\partial U}{dy}\frac{\partial^2 U}{dydz}+\left(\frac{\partial U}{dy}\right)^2\frac{\partial^2 U}{dz^2}=-\frac{x^2}{(n-1)^2}H.$$

And by means of these we may deduce the following:—

$$wDv-v\,Dw=(n-1)^{-2}x^2H,\ wD_1v-v\,D_1u=(n-1)^{-2}yxH,\ wD_2v-v\,D_2w=(n-1)^{-2}zxH,$$

$$uDw-wDu=(n-1)^{-2}xyH,\ uD_1w-wD_1u=(n-1)^{-2}y^2H,\ uD_2w-wD_2u=(n-1)^{-2}zyH,$$

$$vDu-uDv=(n-1)^{-2}xzH,\ vD_1u-uD_1v=(n-1)^{-2}yzH,\ vD_2u-uD_2v=(n-1)^{-2}z^2H.$$

Again,

$$yD_2H-zD_1H=3(n-2)\frac{\partial U}{dx}H,\ zDH-xD_2H=3(n-2)\frac{\partial U}{dy}H,\ xD_1H-yDH=3(n-2)\frac{\partial U}{dz}H$$

$$yD_2u-zD_1u=(n-1)\left(\frac{\partial U}{dx}\right)^2,\ zDw-xD_2w=xD_1v-yDv=(n-1)\frac{\partial U}{dy}\frac{\partial U}{dz}.$$

To which may be added,

$$\square\lambda D\,u+\square\mu D\,v+\square\nu D\,w=\delta H\lambda,$$

$$\square\lambda D_1u+\square\mu D_1v+\square\nu D_1w=\delta H\mu,$$

$$\square\lambda D_2u+\square\mu D_2v+\square\nu D_2w=\delta H\nu,$$

$$uDv+vD_1v+wD_2v=0,$$

$$D^2u=-(n-1)^{-2}\frac{\partial}{dx}(x^2H)-3Au+3xH,$$

$$D^2v=-(n-1)^{-2}\frac{\partial}{dy}(x^2H)-3Av,$$

$$D^2w=-(n-1)^{-2}\frac{\partial}{dz}(x^2H)-3Aw,$$

$$
\begin{aligned}
&u\mathrm{D}^2v \quad +v\mathrm{DD}_1v \ +w\mathrm{DD}_2v &&=-(n-1)^{-1}vx\mathrm{H},\\
&u\mathrm{D}^2w \quad +v\mathrm{DD}_1w+w\mathrm{DD}_2w &&=-(n-1)^{-1}wx\mathrm{H},\\
&x\mathrm{DD}_1u+y\mathrm{DD}_1v \ +z\mathrm{DD}_1w &&=\ (n-1)^{-2}xy\mathrm{H},\\
&y\mathrm{D}v \quad -x\mathrm{D}_1v \ =x\mathrm{D}_2w-z\mathrm{D}w=-(n-1)\ vw,\\
&v\mathrm{DD}_2w-w\mathrm{DD}_2u= \ \mathrm{D}\{(n-1)^{-2}yz\mathrm{H}\}-(n-1)^{-1}vy\mathrm{H},\\
&u\mathrm{DD}_1u-u\mathrm{DD}_1v= \ \mathrm{D}\{(n-1)^{-1}yz\mathrm{H}\}-(n-1)^{-1}wz\mathrm{H},\\
&\qquad z\mathrm{D}_1u \quad -y\mathrm{D}_2u \ =- \quad (n-1)^{-2}u^2,
\end{aligned}
$$

$$\mathrm{D}v\mathrm{D}w=-(n-1)^{-2}x^2\mathrm{H}\frac{\partial \mathrm{U}}{dydz}-vw\mathfrak{A},$$

$$\mathrm{D}v\mathrm{D}^2w-\mathrm{D}w\mathrm{D}^2v=(n-1)^{-2}x^2\left\{\mathrm{D}w\frac{\partial \mathrm{H}}{dy}-\mathrm{D}v\frac{\partial \mathrm{H}}{dz}-\mathfrak{A}\mathrm{H}\right\},$$

$$\mathrm{D}w\frac{\partial \mathrm{H}}{dy}-\mathrm{D}v\frac{\partial \mathrm{H}}{dz}=(n-1)^{-1}\left\{3(n-2)\mathfrak{A}\mathrm{H}-x\left(\mathfrak{A}\frac{\partial \mathrm{H}}{dx}+\mathfrak{H}\frac{\partial \mathrm{H}}{dy}+\mathfrak{G}\frac{\partial \mathrm{H}}{dz}\right)\right\},$$

whence

$$\mathrm{D}v\mathrm{D}^2w-\mathrm{D}w\mathrm{D}^2v=(n-1)^{-2}x^2\left\{\left(\frac{3(n-2)}{n-1}-1\right)\mathfrak{A}\mathrm{H}-\frac{x}{n-1}\left(\mathfrak{A}\frac{\partial \mathrm{H}}{dx}+\mathfrak{H}\frac{\partial \mathrm{H}}{dy}+\mathfrak{G}\frac{\partial \mathrm{H}}{dz}\right)\right\},$$

to which many others might be added.

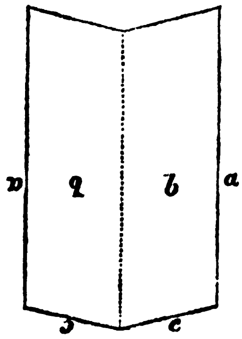
IV. *On Larixinic Acid, a crystallizable volatile principle found in the Bark of the Larch Tree* (Pinus Larix, *Linn.*). *By Dr.* John Stenhouse, *F.R.S.*

Received July 10,—Read November 21, 1861.

The most convenient way of preparing this somewhat singular substance consists in cutting the bark of the larch into small pieces, and then digesting it in water for twenty-four hours at a temperature of about 80° Cent. The solution, which has a deep reddish-brown colour, is then poured off on to a second portion of larch bark and digested as before. The concentrated infusion is then cautiously heated in an open porcelain dish, at the temperature of about 80° Cent., till it is converted into a syrup. A portion of this syrup is then distilled, either in glass or porcelain retorts, or, what is better than either, in a silver alembic. Iron retorts cannot be employed for this purpose, as the acetic acid which is always produced during distillation, by forming acetate of iron, instantly destroys the larixinic acid, by changing it into a deep-purple-coloured liquid. When a silver alembic cannot be procured, a very convenient way of distilling the extract of the larch is to pour it into a large Florence flask, the neck of which is passed obliquely through a cork or bung, which is inserted into a glass condenser. When the flask is cautiously heated on a sand-bath, the larixinic acid comes over with the first portions of the liquid, but becomes more abundant as the distillation proceeds, and usually forms large flat crystals which condense on the sides and neck of the receiver. The liquid which is distilled over, and which contains the greater portion of the larixinic acid, should be poured into small flat basins, and cautiously concentrated at about 60° Cent. When the greater portion of the water has been dissipated, it is advisable, especially in warm weather, to complete the operation by spontaneous evaporation; for unless the concentration of the aqueous solution of larixinic acid is conducted cautiously, the larixinic acid volatilizes along with the vapour of water, and is thereby lost. The highly concentrated solution of larixinic acid obtained in the way just described, on standing, deposits brownish-yellow crystals, which are impure larixinic acid. This is to be pressed between folds of blotting-paper, and again to be crystallized out of a small quantity of water. The larixinic acid may be rendered perfectly pure by subliming it once or twice. This is easily effected by placing the larixinic acid between two watch-glasses, or in any other suitable apparatus, and heating it cautiously on a sand-bath, or even on a water-bath, as the larixinic acid sublimes at the very low temperature of 93° Cent. The larixinic acid is a proximate principle, which exists ready formed in the larch. This is easily proved by distilling even a dilute infusion of the bark, when the liquid which passes over will be found to strike a deep purple colour with a persalt of iron, which is very persistent. The bark of old larch trees contains very little larixinic acid; but the bark of the small branches, and that of the stems of the larch when not more than from twenty to thirty years of age, contains very considerable quantities of this substance, the concentrated syrup from the portions of bark yielding more larixinic acid than an equal weight of catechu does of oxyphenic acid. Larixinic acid, after it has been purified by sublimation, forms beautifully white crystals, often more than an inch in length, of a brilliant silvery lustre, very much resembling benzoic acid in appearance. They sublime at 93° Cent., and melt at 153° Cent.; but its aqueous solutions volatilize at ordinary temperatures. I am indebted to the kindness of Professor W. H. Miller, of Cambridge, for the subjoined measurements of the crystals of larixinic acid.

"The crystals obtained by sublimation become rough so rapidly when exposed to the air, that very little confidence can be placed in the following results:—

Fig. 1.



"The crystals belong to the oblique system; they usually occur in twins, like the annexed figure. They are extremely thin in a direction perpendicular to *b*. Denoting by *l*, *m*, *n* faces in the zone *a b*, and by *r* a face in the zone *b c*, it appears that tan *b l*, tan *b m*, tan *b n* are nearly as the numbers 2, 3, 6, and that $ab=90°\ 0'$, $bc=90°\ 0'$, $ac=76°\ 0'$, $bl=49°\ 29'$, $bm=60°\ 20'$, $bn=74°\ 6'$, $br=75°\ 30'$. Cleavage *a* distinct, *c* imperfect.

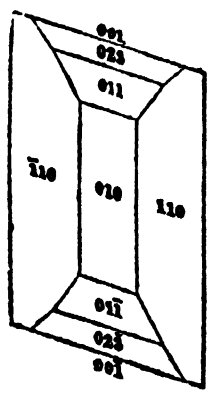
"The crystals of larixinic acid crystallized out of water are very imperfect; the angles must be regarded as very rough approximations. They are deduced, as well as I could deduce them, from a mean of a considerable number of observations by no means agreeing well with each other. The angle between the normals to two faces is taken as the measure of the angle between the faces.

"Oblique:—

100, 101 = 26° 22′

010, 111 = 74° 36′; 001, 101 = 44° 20′.

Fig. 2.



"Forms observed:—

010, 001, 110, 011, 023

Angles.		°	′
110, 001	=	71	58
010, 110	=	69	35
010, 011	=	59	37
010, 023	=	68	39
010, 001	=	90	0

"Cleavage 0 0 1 distinct, and very easily obtained."

The smell of the aqueous solution of larixinic acid is sweetish, like that of a syrup, but the smell of the sublimed acid is very peculiar and slightly empyreumatic. As larixinic acid emits a sensible odour at ordinary temperatures, in this respect it considerably resembles naphthaline and ordinary camphor. The taste of larixinic acid is slightly bitter and astringent. It reddens litmus paper very slightly, but a single drop of potash or ammonia, when added to a solution of a large quantity of larixinic acid, renders it alkaline. Larixinic acid is very soluble in boiling water, but is by no means very soluble in cold water, 87·88 parts of water at 15° C. dissolving 1 part of the acid only; but the solubility of larixinic acid in cold aqueous solutions is greatly increased by the addition of either acids or alkalies. Larixinic acid is deposited from its aqueous solutions in crystals which are very brittle, and often an inch or two in length. It likewise dissolves in cold alcohol, but to a much greater extent in hot alcohol. The crystals deposited from its alcoholic solutions are thicker and more distinctly formed than those from water. It also dissolves but sparingly in ether, and is deposited in crystals of very considerable lustre*. The following are the results of the analysis of the larixinic acid:—

I. 0·221 sublimed acid, dried *in vacuo*, gave 0·4633 carbonic acid and 0·1003 water.

II. 0·1993 sublimed acid, dried *in vacuo*, gave 0·417 carbonic acid and 0·0913 water.

III. 0·2272 larixinic acid, crystallized out of water, gave 0·4756 carbonic acid and 0·1030 water.

Calculated numbers.			Found.		
			I.	II.	III.
C_{20}	=	57·14	57·13	57·06	57·09
H_{10}	=	4·77	5·04	5·09	5·04
O_{10}	=	38·09	37·83	37·85	37·87

From these results it is evident that the carbon, hydrogen, and oxygen in larixinic acid are in the proportion of $C_2 H_1 O_1$, or some multiple of these numbers, $C_{20} H_{10} O_{10}$ being the numbers we have adopted as the more probable.

When a quantity of larixinic acid was dissolved in a great excess of liquid ammonia, a yellow-coloured solution was produced; when this was evaporated to dryness over sulphuric acid *in vacuo*, the larixinic acid was deposited in crystals which were nearly unaltered. It gave its characteristic reactions with salts of iron, and when boiled with milk of lime gave off no trace of ammonia. The combination which larixinic acid forms with ammonia is therefore so feeble that it is decomposed by the volatility of the ammonia. In this respect, therefore, and in its forming no hydrate, larixinic acid closely resembles both pyrogallic and oxyphenic acids.

When larixinic acid was treated with an excess of aqua potassæ it very readily dissolved, forming a yellowish solution. When dried over sulphuric acid *in vacuo*, the potash combination formed long flattish crystals having considerable lustre, but of a reddish-brown colour. These crystals, when pressed between folds of blotting-paper to free them from excess of potash and recrystallized *in vacuo*, yielded crystals which were more deeply coloured than the first. This potash combination is so very feeble that

* The crystals of larixinic acid catch fire readily, and burn with a bright flame, leaving no residue.

it is decomposed by carbonic acid. It contained a considerable quantity of potash; but I have not been able to obtain it of a constant composition.

A solution of larixinic acid gives no precipitate, either with lime-water or with saccharate of lime. The behaviour of larixinic acid with baryta is extremely singular and characteristic. When a solution of caustic baryta is added to a concentrated aqueous solution of larixinic acid, the latter being in excess, a bulky, semitransparent, gelatinous precipitate immediately falls, and if the solutions are concentrated fills the whole vessel. This precipitate, which considerably resembles hydrated alumina, is but slightly soluble in cold water; but it dissolves very readily in boiling water, from which it is again deposited on the cooling of the liquid. This baryta compound is readily decomposed by carbonic acid. When thrown on a filter and washed, the air being carefully excluded, it was dried *in vacuo* over sulphuric acid, and was then found to contain, as the mean of two experiments, 34·92 per cent. of baryta.

A solution of larixinic acid yields no precipitate with either basic or neutral acetate of lead, neither is it precipitated by nitrate or ammonio-nitrate of silver; but when its solution in the latter salt is boiled, the silver is reduced in a pulverulent state. Larixinic acid forms no precipitate with perchloride of platinum, even on the application of heat. It does not contain any nitrogen. It does not reduce oxide of copper when tried by TROMMER's test. It dissolves in concentrated sulphuric acid, but no conjugate combination is produced, as was ascertained by neutralizing with carbonate of baryta, larixinic acid being obtained unchanged. When larixinic acid is boiled with a mixture of hydrochloric acid and chlorate of potassa, it is decomposed, but without the formation of chloranile. When it is boiled with a solution of hypochlorite of lime, no coloration is produced. It is readily attacked by nitric acid, especially when assisted by heat; nitrous fumes are given off, and oxalic acid is the only fixed product. It is also readily attacked by bromine, especially when assisted by heat. Abundant vapours of hydrobromic acid are given off, the larixinic acid being entirely destroyed and converted into an uncrystallizable resin. The salts of copper produce an emerald-green colour in solutions of larixinic acid, but no precipitate. Chloride of manganese produces neither coloration nor precipitation. Protosulphate of iron strikes a brownish-red colour with solutions of larixinic acid, which acquire a brighter red colour on standing, resembling meconate of iron. Perchloride and persulphate of iron produce a beautiful purple dahlia-colour, which is very persistent, and stands dilution well. Its reactions with salts of iron are very characteristic of larixinic acid, which forms an excellent reagent for the detection of salts of iron, even in very minute quantity. In this way the presence of iron, in tolerably pure sulphate of copper, can readily be detected by the purple coloration produced. Larixinic acid does not affect neutral protonitrate of mercury in the cold; and on the application of heat no mercury is reduced.

Larixinic acid appears to be peculiar to the larch tree; at least I have not been able to find a trace of it in the bark of the spruce fir (*Abies excelsa*), or in that of the Scotch fir (*Pinus sylvestris*). Larixinic acid evidently belongs to that small group of substances of which pyrogallic acid and pyrocatechine, the oxyphenic acid of GERHARDT, are the only other members yet known. Larixinic acid is much less easily oxidizable than oxyphenic acid, which again is less easily oxidated than pyrogallic acid. Larixinic acid volatilizes at a much lower temperature than either of these two substances, from which it also differs in being a ready-formed proximate principle, and not an educt.

Addendum.—In consequence of the extremely feeble, if not somewhat doubtful, acid properties of the so-called larixinic acid, perhaps the name *Larixine* would be more appropriate; but in that case the name of pyrogallic acid should be altered to *pyrogalline*, and that of oxyphenic acid to *pyrocatechine*, the name originally given to it by ZWENGER.—J. S.

V. *On the Absorption and Radiation of Heat by Gaseous Matter.*—Second Memoir. *By*

JOHN TYNDALL, *F.R.S., Member of the Academies and Societies of Holland, Geneva, Göttingen, Zürich, Halle, Marburg, Breslau, Upsala, la Société Philomathique of Paris, &c.; Professor of Natural Philosophy in the Royal Institution.*

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§ 1. *Instruments.*

THE apparatus made use of in this inquiry is the same in principle as that employed in my last investigation*. It grew up in the following way:—A tube was first procured to receive the gases through which radiant heat was to be transmitted, but it was necessary to close the ends of this tube by a substance pervious to all kinds of heat, obscure as well as luminous. Rock-salt fulfils this condition, and accordingly plates of the substance an inch in thickness, so as to be able to endure considerable pressure, were resorted to. In the earliest experiments a cube of boiling water was placed before one end of this tube, and a thermo-electric pile connected with a galvanometer at the other; it was found that if the needle pointed to any particular degree when the tube was exhausted, it pointed to the same degree when the tube was filled with air. Thus tested, the presence of dry air, oxygen, nitrogen, or hydrogen had no sensible influence on the radiant heat passing through the tube.

In some of these trials the needle stood at 80°, in some at 20°, and in others at intermediate positions. I reasoned thus:—The quantity of heat which produces the deflection of 20° is exceedingly small, and hence a minute fraction of this quantity, even if absorbed, might well escape detection. On the other hand, the quantity of heat which produces the deflection of 80° is comparatively large, but then it would require a large absorption to move the needle even half a degree in this position. A deflection of 20° is represented by the number 20, but a deflection of 80° is represented by the number 710. While pointing to 80, therefore, an absorption capable of producing a deflection of 15 or 20 degrees on the lower part of the scale, would hardly produce a sensible motion of the needle. The problem then was, to work with a copious radiation, and at the same time to preserve the needle in a position where it would be sensitive to the slightest fluctuations in the absolute amount of heat falling upon the pile.

This problem was finally solved by converting the pile into a differential thermometer. Its second face was exposed, and a second source of heat was placed in front of that face. A moveable screen was interposed between the two, by the motion of which the same amount of heat could be caused to fall upon the posterior surface of the pile as

* Philosophical Transactions, 1861.

that received from the experimental tube by its anterior surface. When this was effected, no matter how high the previous deflection might be, it was completely neutralized, and the needle descended to zero.

Supposing this equality to have been established when the tube was exhausted, it is manifest that any gas, capable of absorbing even an extremely small proportion of radiant heat, would, if introduced into the tube, destroy the equilibrium of both sources. The second source of heat would now predominate, and a deflection of the galvanometer needle would be the consequence. The magnitude of this deflection would depend on the quantity of heat cut off by the gas, and properly reduced it became a strict measure of the absorption.

But in these experiments my first source of heat stood at some distance from the anterior end of the tube, and the heat, previously to entering the latter, had to cross a space of air which was not the subject of examination. This air-space I wished to abolish, so as to allow the calorific rays to enter the gas with all the qualities which they possessed at the moment of emission. I first thought of soldering the end of the experimental tube direct to the radiating surface, thus allowing the air to come into direct contact with the source. But it immediately occurred to me that the introduction of cool air into the tube would lower the temperature of the source, and that I could never know how far the indication of my galvanometer under such circumstances could be regarded as a true effect of absorption; hence I abandoned the idea of bringing the gases into contact with the radiating surface.

Instead of this arrangement an independent tube, 8 inches long, and of the same diameter as the experimental tube, was soldered on to the radiating plate. By means of a screw joint, the free end of this tube was connected air-tight with the experimental tube. Thus a chamber, from which the air could be removed, was introduced between the first plate of salt and the radiating surface. Two objects were thus secured; firstly, my source of heat was withdrawn from the action of irregular currents of air; and, secondly, the radiant heat entered the tube unchanged in quality save the infinitesimal change due to its passage through the diathermic salt.

To save the trouble and expense of a new Plate, I will ask permission to reprint in this memoir the Plate made use of in my last; a verbal reference will in most cases be sufficient to indicate the changes recently introduced. S S′ (Plate I.) it will be remembered represented the experimental tube, which was then made of brass polished within. Such a tube could not be used for any gases or vapours capable of attacking brass; and though I combated this difficulty, to some extent, by blackening the tube within, I could never feel at ease regarding the action of the gases upon the blackening substance. Many gases, moreover, present great difficulties on account of their affinity for atmospheric moisture. Hydrobromic and hydrochloric acid, for example, form dense fumes in the air, and however carefully they might have been dried, I should have been reluctant to base any inference on their deportment without actually having them under my eyes during experiment.

The brass tube, then, which stretched from S to S′ in the figure is now replaced by one of glass, 2 feet 9 inches long, and 2·4 inches in diameter. The source of heat in my last-published inquiry was the cube of hot water C; but glass being far inferior to brass in reflecting power, I was unable with this source to bring out with due force the vast differences existing between various kinds of gaseous matter. I therefore had a copper hood constructed, and united by brazing with a tube 8 inches long, which was destined to form the vacuous chamber in front of the first plate of rock-salt. To heat the copper plate, a lamp formed on the principle of Bunsen's burner was made use of. The gas passed upwards by four hollow columns, each perforated for the admission of air. The mixture of air and gas escaped from these columns into a chamber shaped like the frustum of a cone, and over this chamber was placed a shade of thin sheet-iron, the top of which was narrowed to a slit one-eighth of an inch wide and 2 inches long. From this slit the mixture of gas and air issued, and formed upon ignition a sheet of flame. This was caused to glide along the back of the copper plate before referred to, which was thereby raised to a temperature of about 270° C. To preserve this source constant was one of the greatest difficulties of the investigation; for the slightest agitation of the surrounding air, or the slightest flickering of the flame itself, was sufficient to disturb the steadiness of the galvanometer and to render experiments in delicate cases impossible. The flame was surrounded by screens of pasteboard, these being again encompassed by towels, through the meshes of which the flame was fed; a suitable chimney produced a gentle draught and carried off the products of combustion; the rhythmic jumping of the flame itself was destroyed by screens of wire-gauze; in short, six weeks' practice was required to master all the difficulties of this portion of the apparatus. The "compensating cube" C′, the double screen H, and the thermo-electric pile P remain as before. They are exposed in the figure, but during the experiments they were surrounded by a close hoarding, all the chinks of which were stuffed with tow, so as to protect the cube and pile from the disturbing action of the air. To protect the anterior plate of rock-salt from the heat which might have been conducted to it from the source, the front chamber passed as before through a vessel V in which a current of cold water, constantly renewed, was caused to circulate.

§ 2. *Experiments.*

On two points I wished to set my mind at rest previous to starting on my vacation tour this year. These were the absorption of chlorine gas and of ozone. On the 16th, 17th, and 18th of June, I experimented on these two substances, and satisfied myself that chlorine was far outstripped by many colourless gases, and that ozone had a power of absorption very much greater than common oxygen.

The work was resumed on the 12th of September, and my first care was to examine whether my published experiments on moist and dry air stood the test of repetition. Professor MAGNUS had experimented on dry air and on air saturated with moisture, and found that the presence of the moisture had no influence on the absorption. I, on the contrary, had previously found, and stated, that dry air had only a small fraction of the absorptive energy of the same air when even partially saturated. I commenced my researches in September with a few experiments on this subject.

Half an atmosphere of the undried air of the laboratory admitted directly into the tube cut off an amount of heat which produced a deflection of 30 degrees.

My drying apparatus at this time consisted of a U-tube filled with fragments of pumice-stone wetted with sulphuric acid. Associated with this was a similar tube filled with like fragments, but moistened with caustic potash solution, to remove the carbonic acid of the air.

The air of the laboratory passed through both these tubes in succession, till a tension of 15 inches was attained, gave a deflection of 26 degrees.

This result surprised me, showing, as it seemed to do, a very close agreement between dry and moist air. On examining the drying-tubes, however, I found that by a mistake of arrangement the air had entered the sulphuric acid tube first, and passed straight from the potash into the experimental tube; thus partially reloading itself with moisture after it had been dried.

The air was now sent through both tubes, commencing with the potash—the deflection fell instantly to less than 5 degrees. Hence this experiment showed the absorption due to the moisture and carbonic acid of the air to be more than six times greater than that of the atmosphere itself. It will presently be seen that the difference here stated falls far short of the truth.

The potash and sulphuric acid were now abandoned, and the air was dried by passing it through a U-tube filled with fragments of chloride of calcium, which had lain in the tube for some months. The deflection produced by air thus dried was 40 degrees; that is to say, 10 degrees more than that produced by the undried air.

This result, and many others of a similar nature, were due to the imperfection of the chloride of calcium. I think chemists ought to be very cautious in the use of this substance as a drying agent. When pure and newly fused it may answer for this purpose, but when old it yields an impalpable powder, which proved in the highest degree perplexing to me in my first experiments. It is generally found, I believe, that a drying-tube of sulphuric acid gains more in weight than one of chloride of calcium, and from this it has been inferred that the quantity of moisture taken up by the former is greater than that taken up by the latter. The difference, however, may really be due to the mechanical carrying away of a portion of the chloride by the current of air.

On the 13th of September these experiments were resumed. The dry air then gave a deflection of less than 2 degrees; the air direct from the laboratory caused, in one experiment, the needle to move from 20 degrees on one side of zero to 28 on the other. In a second experiment the undried air caused the needle to move from 18° on one side of zero to 32° on the other.

Experiments made on the 17th entirely corroborated this result. Three successive trials made the action of the undried air of the laboratory 29°, 31°, and 30° respectively;

the deflection produced by the dried air being less than a single degree. On this day, therefore, the action of the aqueous vapour of the air was at least thirty times that of the air itself.

Almost every week-day during the last four months experiments similar to the above have been executed, and in no case have I observed a deviation from the result that the absorptive action of the aqueous vapour of the air is quite enormous in comparison with that of the air itself. Further on I will give an array of figures illustrating this point.

As I became more and more master of my apparatus, and more acquainted with the precautions necessary in delicate cases, the absorption of air and the elementary gases diminished more and more. I was induced to abandon the use of pumice-stone as well as of chloride of calcium, and to construct my drying apparatus in the following way. The internal portion of a massive block of pure glass was pounded to small fragments in a mortar; these were boiled in pure nitric acid, and afterwards washed several times with distilled water so as to remove all trace of the acid. They were then dried, afterwards moistened with pure sulphuric acid, and introduced by means of a funnel into a U-tube. The funnel was necessary to preserve the neck of the tube from all contact with the acid, the least action of which upon the corks used to close the tube was sufficient to entirely vitiate the experiments. At the top of each arm of the U-tube a quantity of dry fragments of glass was placed, so that if any dust or particles from the cork or sealing-wax chanced to reach the interior they fell upon the dry glass.

Similar precautions were taken with the caustic potash tube. Pure white marble was pounded to fragments and subjected to the action of a dilute acid, which removed the outer surface of the fragments. These were afterwards washed in distilled water and dried, then moistened with pure caustic potash, and introduced into the U-tube in the manner already described. It was sometimes necessary to perform this operation daily, and never on any occasion have I used tubes to dry a feeble gas which had been previously used to dry a powerful one.

In the present communication I shall have to touch upon many subjects which for want of time I have been unable to develope. The following is an example of these. Choosing a day of suitable temperature and moisture—a day on which the human breath shows no signs of precipitation—the action of the substances expired from the lungs may be most strictly determined by our apparatus. By breathing directly into the experimental tube, the action produced by the sum of the products of respiration might be accurately measured; by breathing through the sulphuric acid tube, the moisture of the breath would be withdrawn, and the difference between the action then observed and the former action would give that of the carbonic acid. In this way the products of respiration might be estimated singly, and the influence of various kinds of food and drink, or of physical exertion, on the respiration might be investigated in a manner hitherto unthought of.

I have to record the following experiments only in connexion with this subject. Placing a suitable tube between my lips, I filled my lungs with air; a stopcock which was interposed between me and the experimental tube being partially opened, I breathed through it slowly into the latter until the mercury gauge of the pump was depressed 15 inches. I had, at the time, two assistants, C. A. and R. C., and they subsequently breathed into the experimental tube the same quantity as myself. In the following Table the absorption produced by the breath of each is stated.

Action of the Products of Respiration on Radiant Heat.

Initials of person's name.	Absorption.
J. T.	62
J. T.	62
R. C.	66
R. C.	68
J. T. again	59
J. T.	59
R. C.	63
C. A.	62
J. T.	60·5

The absorption of dry air on the day that these results were obtained was found to be 1. *The same dry air inhaled, underwent a chemical change which augmented its absorptive energy at least* 60 *times.* I give this as a minor limit, and will not say how much I regard it as falling short of the truth.

The day afterwards the following results were obtained, the same amount as before being exhaled:—

Initials.	Absorption
J. T.	56
R. C.	62
J. T.	56
R. C.	59

In all cases R. C., who is the smallest and least robust man of the three, appeared to have the advantage. I will only add a few results obtained on the 6th of October, the quantity of air expired on the occasion depressing the mercurial column 5 inches.

Initials.		Absorption.
J. T.		33·5
R. C.		35
R. C.	After half a glass of Trinity Audit Ale	41
	Again	35
	After a teaspoonful of brandy	35
	After chewing and swallowing a small quantity of onion	40

After taking the ale and brandy my assistant washed his mouth and gargled his throat several times with cold water. I give these results merely as illustrative of one of the numerous applications of the apparatus. In all the experiments the tube remained perfectly transparent throughout, and, on pumping, the needle in each case returned accurately to zero.

§ 3.

In my last paper I brought the fact somewhat prominently forward that the elementary bodies which I had then examined were far less hostile to the passage of the longer undulations than the compound ones; and I founded at the time certain theoretic considerations on this fact*. I was desirous this year to extend the experiments to one or two of the coloured gases and vapours, and on the 20th of September resumed my experiments on chlorine. This gas is itself highly coloured, and of a specific gravity of 2·45; one of its compounds, hydrochloric acid, is quite transparent, and of specific gravity of only 1·26. Does the act of combination with hydrogen which renders chlorine gas more transparent to light render it also more transparent to heat? Chlorine prepared from hydrochloric acid and peroxide of manganese, and dried by passing it through sulphuric acid, was admitted into the tube till it depressed the mercury gauge 21 inches; the absorption of the gas was expressed by the number 44.

Hydrochloric acid was admitted till it depressed the gauge 19 inches; the absorption was 68. This experiment indicates that transparency to light and opacity to heat accompany the same act of chemical union.

The following results were afterwards obtained. I may remark that a subsidiary gauge was used, so as to prevent the destructive gases from entering the air-pump.

	Absorption.
Chlorine 15 inches	32
Chlorine 14 inches.	30
Chlorine 14 inches. . . .	30
Hydrochloric acid 14 inches .	47
Chlorine again	30
Hydrochloric acid	56

In all cases the effect of the compound gas was found to exceed that of the elementary one; so that *the chemical change which renders chlorine more transparent to light renders it more opake to obscure heat.*

Great care is required in experiments on hydrochloric acid, and great care was bestowed on the above. Previously to the introduction of the gas the experimental tube was filled with perfectly dry air, so as to leave a perfectly dry residue on exhaustion. The gas was allowed to stream through the drying-tube until all traces of air were expelled; then a joint was suddenly broken, and the retort was connected with the experimental tube. The gas thus passed directly from the retort through the drying apparatus into the vacuum. It was difficult to avoid sending in with the gas a few particles of moisture; but these, if such existed, appeared to be dissipated by the dynamic

* Philosophical Transactions, 1861.

heating of the gas on entering the tube, and kept dissipated by the flux of heat passing through it. At all events the closest scrutiny could detect no trace of mist or turbidity within the tube; it was perfectly transparent throughout. The chlorine, on the contrary, was intensely coloured.

I made many experiments with chlorine which had been collected over water, but something (what I know not yet) appeared to be in all cases carried along with the gas from the water into the tube, which materially augmented its absorption.

The above experiments were made in the early part of this inquiry, and before I had become aware of all the peculiarities of my apparatus. Subsequent experiments reduced in some degree the absorption both of chlorine and hydrochloric acid. Very careful experiments made on the 29th of October gave the following absorption for these two gases, at a tension of 30 inches:—

Chlorine	39
Hydrochloric acid	53

The chlorine and hydrochloric acid were removed from the experimental tube in the following manner:—A cock and connecting piece were attached to one end of the experimental tube, and from them a length of india-rubber tubing led to the flue of the laboratory stove. A gas-holder of air was put in connexion with the other end of the experimental tube, a system of drying-tubes intervening between the latter and the holder. By a slight water-pressure a stream of dry air was carried gently through the tube to the flue, and in this way the gases, which if pumped out would have injured the pistons, were speedily removed. As the dry air replaced the gases, the needle gradually descended to zero, its arrival there being indicative of the complete displacement of the gas. The perfect dryness of the air thus made use of was beautifully proved. Had the air contained moisture, it would instantly on its mixture with hydrochloric acid have rendered the medium within the tube turbid. However slight this turbidity might be, and however invisible to the eye, the galvanometer would have revealed it. But there was no movement in an upward direction; the needle gradually sunk from the moment the air entered.

As regards the influence of chemical union in the absorption of radiant heat, a still more severe test than that furnished by the substances last examined is presented by bromine and hydrobromic acid; for the opacity of the former to light is far greater than that of chlorine, while the two compounds are equally transparent. The density of bromine, moreover, is 5·54, whereas that of hydrobromic acid is only 2·75. The difficulty of operating with the acid compound of bromine is at least equal to that attendant on hydrochloric acid, and several successive days were spent in endeavouring to arrive at safe conclusions in connexion with this subject. Bromine dried with phosphoric acid was introduced into a flask furnished with a screw cap, which enabled it to be attached to the experimental tube. By turning a stopcock, the pure vapour was allowed slowly to enter until the mercury column was depressed two inches. From more than twenty experiments made with this substance, I should infer that the absorption of the quantity mentioned does not exceed

11,

while the absorption of hydrobromic acid of the same tension amounts to

30.

The hydrobromic acid was prepared by the action of glacial phosphoric acid (for a pure specimen of which I have to thank my friend Dr. FRANKLAND) on bromide of potassium. If the above figures represent the truth (and I have spared no pains to arrive at a right conclusion), we have here a most striking instance of *transparency to light and opacity to obscure heat being promoted by the self-same chemical act* *.

§ 4.

In the following Table is given the absorption of a number of gases at a common tension of one atmosphere.

TABLE I.

Name.	Absorption.
Air	1
Oxygen	1
Nitrogen	1
Hydrogen	1
Chlorine	39
Hydrochloric acid	62
Carbonic oxide	90
Carbonic acid	90
Nitrous oxide	355
Sulphuretted hydrogen	390
Marsh-gas	403
Sulphurous acid	710
Olefiant gas	970
Ammonia	1195

Air, oxygen, nitrogen, and hydrogen are all set down as equal to unity in the above Table. I do not mean thereby to affirm that there are no differences between these gases as regards their powers of absorption, but that the most powerful and delicate tests which I have hitherto applied have failed to establish a difference in a satisfactory manner. It is not improbable that the action of these gases may turn out to be less even than I have found it. For who can say that the best-constructed drying apparatus is really perfect? Besides, stopcocks must be greased, and hence may contribute an infinitesimal impurity to the air passing through them. I cannot even say that sulphuric acid, however pure, may not deliver a modicum of vapour to the current of air passing through it. At all

* A layer of liquid bromine, sufficiently opake to intercept the entire luminous rays of a gas-flame, is highly diathermanous to its obscure rays. An opake solution of iodine in bisulphide of carbon behaves similarly.—The details of these experiments shall be published in due time: they were publicly shown in my lectures many months ago.—June 18th, 1862.

events, if any further advance should be made in the purification of the gases, it will certainly only tend to augment the enormous differences exhibited in the above Table.

Ammonia, of the tension mentioned, stands highest in the above list as regards absorptive energy. I believe that a length of less than 3 feet of this gas, which to the vision is as transparent within the tube as the vacuum itself, is *perfectly black* to the rays emanating from the source here made use of. When the gas was in the tube, the interposition of a double metallic screen between the pile and source augmented the deflection very slightly. But I shall show further on that the ammonia in this experiment could not exhibit the full energy of its absorption, and that the length indicated is in all probability absolutely impervious to the heat issuing from our source.

It would be a mere affectation of accuracy to try to deal with smaller quantities of the first four substances mentioned in the Table than those with which I have here operated. Still, if such small quantities could be directly measured, the action of air, oxygen, hydrogen, and nitrogen, in comparison with that of the other substances at the same tension, would doubtless be greatly reduced. With the energetic gases the rays are most copiously struck down by the quantities which first enter the tube, the quantities which enter last producing in many cases an infinitesimal effect. Now I have shown in my last paper that for very small absorptions the effect is sensibly proportional to the quantity of gas present, and this would seem to justify us in assuming that for 1 inch of tension the absorption of air, oxygen, nitrogen, and hydrogen would be $\frac{1}{30}$th of the absorption at 30 inches. With all the other gases I have measured directly the absorption of a quantity possessing in each case a single inch of tension. Assuming the proportionality just referred to, and again calling the effect of air unity (the unit, however, being only $\frac{1}{30}$th of that in the last Table), the following are the relative absorptions:—

TABLE II.

Air	1
Oxygen	1
Nitrogen	1
Hydrogen	1
Chlorine	60
Bromine	160
Hydrobromic acid	1005
Carbonic oxide	750
Nitric oxide	1590
Nitrous oxide	1860
Sulphide of hydrogen	2100
Ammonia	7260
Olefiant gas	7950
Sulphurous acid	8800

Here we have the extraordinary result that for tensions of 1 inch of mercury *the* *absorption of ammonia is over seven thousand times, the absorption of olefiant gas seven thousand nine hundred and fifty times, while the action of sulphurous acid is eight thousand eight hundred times that of air.*

It is impossible not to be struck by the position of chlorine and bromine in this Table. They are elements, and notwithstanding their colour and density, they take rank after the transparent elementary gases. The perfectly transparent olefiant gas absorbs more than one hundred and thirty times the amount absorbed by the untransparent chlorine, and nearly fifty times the quantity absorbed by the intensely brown vapour of bromine. I cannot think this fact insignificant. Hitherto chemists have spoken to us of elements, and we have helped ourselves to conceptions regarding them and their compounds in the only way possible to our mental constitution. But our conceptions remained purely subjective, nor were we acquainted with any physical trait which would in any degree justify these conceptions. Here, however, we seem to touch the ultimate particles of matter. Starting from the idea that a gas absorbs such vibrations as are isochronic with its own, in all cases the compound gas reveals itself to the mind's eye with its molecules on the whole swinging more slowly than the uncombined atoms of which it is composed. Their absorption of the longer undulations proves their general coincidence in period with those undulations. We load the atom by the act of chemical union, and thereby render its vibrations more sluggish, that is to say, more fit to synchronise with the slowly recurrent waves of obscure heat.

In the foregoing Table I have given the absorption of nitric oxide as 1590, which is less than that of nitrous oxide, though the molecule of the former contains a greater number of atoms than that of the latter. It will be noticed *that those gases which on combining suffer no condensation are less energetic absorbers than those which suffer a reduction of volume.* Whether this rule is universal I am as yet unable to say.

It is very difficult to operate with nitric oxide; the affinity of the gas for oxygen is so enormous that the slightest trace of this substance gives rise to the brown fumes of nitrous acid. On first sending this gas into the experimental tube, 1 inch of it gave an absorption of 2040; but the needle slowly went up afterwards, until it finally indicated an absorption of 5100. On looking across the tube at this time, the brown hue of nitrous acid was discernible.

In a second experiment I made the vacuum as perfect as possible; on allowing nitric oxide to enter, the absorption was 1860, but the needle soon afterwards declared an absorption of 3060, the brown fumes appearing as before.

On filling the experimental tube with nitrogen, then exhausting, and allowing nitric oxide to enter, the absorption of 1 inch of the gas was 1680. On filling the experimental tube previously with hydrogen the absorption was 1590, which is that given in the Table. On letting in a mixture of air and nitric oxide till the tube was filled, the action last mentioned was augmented nearly twentyfold. Nitrous acid is therefore an extremely energetic gas. The difference between it and bromine is enormous when the colours of both are equally dense.

A close inspection of MELLONI'S Table* reveals, I think, the tendency of solid bodies also to become more transparent to heat as their composition becomes more simple. After rock-salt itself comes the element sulphur, and after it fluor-spar. But the case of lampblack will here occur to many, who regard it as the most powerful absorber and radiator yet discovered. No doubt the grouping of the atoms of an elementary substance may make it tantamount to a compound, and no doubt this is actually the case with lampblack; another eminent example of this kind is referred to in this paper in the section on ozone. LESLIE, however, found water to be a better radiator than lampblack, and WELLS found several substances which were more capable of being chilled by nocturnal radiation. On reflection, moreover, the following considerations arise. The lampblack of commerce and the soot of a lamp or candle, that is to say, the substances which have been hitherto employed in experiments on radiant heat, are copiously mixed with hydrocarbons, which are the most powerful absorbers and radiators in Nature. It might fairly be questioned whether the reputed experiments with lampblack really dealt with lampblack at all. But even the impure substance is to some extent transparent to radiant heat.

I have plates of black glass, rendered so by the solution of carbon in the glass while in a state of fusion, and which, though they are impervious to the rays of the most intense electric light, allow of a copious transmission of the rays of obscure heat. MELLONI'S beautiful experiments on glass of this character are well known; indeed mine are but repetitions of his. Another of MELLONI'S experiments which I have recently verified is the following. A plate of transparent rock-salt was placed over a smoky camphine lamp, and soot was deposited on its surface until it intercepted every ray of a brilliant jet of gas. The plate was placed between a source of heat possessing a temperature of 100° C., and a thermo-electric pile, a polished screen being placed between the salt and the source. As long as the screen remained, the needle of the galvanometer connected with the pile stood at zero; but the moment the screen was removed the needle promptly advanced, showing the instantaneous transmission across the layer of soot of a portion of the heat incident upon the salt. The actual numbers obtained in this experiment are these:—The deflection produced by the heat transmitted through the soot was 52°; which is equal to 90 units. The deflection produced when the layer of soot had been carefully removed, so as to leave both surfaces of the salt smooth and transparent, was 71°, which is equal to 300 units. The quantity transmitted through the soot is therefore to the total quantity as

$$90 : 300,$$

or as

$$30 : 100;$$

that is to say, the lampblack, which was perfectly opake to the light of a gas-jet, was transparent to fully 30 per cent. of the incident heat. On consulting MELLONI'S Table, I was gratified to find that he made the transmission by a plate similarly prepared

* 'La Thermochrôse,' p. 164.

27 per cent.; while a layer so opake that it cut off the beams of the sun itself transmitted 23 per cent. of the rays emitted by a source heated to 100° C.

At page 93 of 'La Thermochrôse,' MELLONI examines the absorption of this substance for all sorts of rays, and by a series of ingenious experiments, and reasonings remarkable for their clearness and precision, he arrives at the conclusion that lampblack absorbs with the same intensity all descriptions of radiant heat*. At page 284, however, he cites and discusses with the same precision a series of experiments made with smoked rock-salt, in which he shows that the same layer of lampblack transmits 8 per cent. of the rays from a lamp of Locatelli, 10 per cent. of those of incandescent platinum, 18 per cent. of those from copper heated to 400° C., and fully 23 per cent. of those emitted by a source of 100° C. Now a transmission of 8 per cent. implies an absorption of 92; while transmissions of 10, 18, and 23 per cent. imply absorptions of 90, 82, and 77. But that the self-same layer of lampblack absorbs 77 per cent. of the rays from one source and 92 per cent. of the rays from another, is at variance with the statement that lampblack absorbs heat from all sources with the same intensity. Suppose the surface of a thermo-electric pile to be coated by a layer of lampblack of the same thickness as that which coated MELLONI's plate of salt; 23 per cent. of the rays from a source of 100° C. would go right through such a layer and impinge upon the metal face of the pile; and if the latter were a good reflector, the heat incident upon it would be in great part retransmitted through the lampblack and lost to the instrument. For a source of 100° C., this loss would be many times greater than for a Locatelli lamp. Possibly, however, MELLONI meant simply to assert that for practical purposes the absorption by the face of his pile might be considered to be the same for all kinds of heat†.

§ 5.

I have now to record some new experiments on the action of *vapours* upon radiant heat. A number of glass flasks were prepared, of the shape and size of common test-tubes, each of which was furnished with a brass cap carefully cemented on to it. By means of this it could be attached to a stopcock, and thus connected with the experimental tube. The mode of operation was this. The liquid was introduced into the flask by means of a small glass funnel; the stopcock (S) was then attached to the flask and connected with a second air-pump, which was always kept at hand. The air above the liquid was removed, and the air dissolved in it was allowed to bubble away, until nothing remained but the pure liquid below and the pure vapour above it. The stopcock S was now shut off, and the flask united to the experimental tube. The exhaustion of the tube and stopcocks being complete, and the needle of the galvanometer at zero, the cock S was turned on and the mercury-gauge carefully observed at the same time.

* "Donc, le noir de fumée absorbe avec la même intensité toute sorte de rayonnements calorifiques" (p. 101).

† The sun, through the floating carbon of the London atmosphere, sometimes presents a most instructive appearance. Entirely shorn of his rays and of perfectly uniform brightness, his colour at times is as red as blood. This is doubtless mainly due to the comparative transparency of the floating carbon for the longer undulations.

No bubbling of the liquid was in any case permitted. The vapour entered silently and without the slightest commotion, and when the mercurial column was depressed to the extent required, the cock S was promptly turned off.

The energy with which the needle moves the moment a strong vapour enters is so extraordinary, that I was compelled to remove the stops which arrested the swing of the needle at 90°, lest the shock against them should derange the equilibrated magnetism of the astatic pair. The needle often swung far beyond a quadrant; and after it had come finally and permanently to rest, its position was observed in the following manner:— The dial of the galvanometer being well illuminated, a looking-glass was placed behind the instrument, at such an angle that when looked at horizontally the image of the dial was clearly seen. This image was observed by an excellent telescope, fixed at a distance of 11 feet from the galvanometer. Attached to the needle, and in continuation of it, was a bit of glass fibre of extreme fineness, which ranged over the graduated circle, and by means of it a very small fraction of a degree could be easily read off. I resorted to the expedient of observing from a distance, because I found that the approach of my person, perhaps through the diamagnetic action of my own body, had a sensible effect upon the needle of my instrument, which, I believe, surpasses in delicacy any hitherto constructed.

The *permanent* deflection of the needle was noted in all these experiments, and the value of the deflection, expressed in terms of one of the lower degrees of the galvanometer, was obtained from a table of calibration. To spare unnecessary labour, I omit the deflections in the following Table, and give the absorptions only produced by the vapours mentioned, at 0·1, 0·5, and 1·0 inch of tension.

TABLE III.

Name of substance.	Tensions. 0·1 inch.	0·5 inch.	1·0 inch.
Bisulphide of carbon	15	47	62
Iodide of methyl	35	147	242
Benzol	66	182	267
Chloroform	85	182	236
Methylic alcohol	109	390	590
Iodide of ethyl	158	290	390
Amylene	182	535	823
Sulphuric ether	300	710	870
Alcohol	325	622	
Formic ether	480	870	1075
Acetic ether	590	980	1195
Propionate of ethyl	596	970	
Boracic ether	620		

Let us compare some of the results of this Table of transparent vapours with the action of the highly coloured vapour of bromine. The absorption of bromine vapour at 1 inch tension is about 6, and at 0·1 of an inch tension would probably not exceed 1; hence, at 0·1 of an inch tension, bisulphide of carbon has probably 15 times the absorbent power of bromine; but bisulphide of carbon is the feeblest of the compound vapours that I have yet discovered. The strongest of these, boracic ether, has, according to the above estimate, and at the tension stated, *more than* 600 *times the absorbing energy of the strongly coloured bromine.*

The whole of the numbers in the above Table are referred to atmospheric air as unity; 0·1 of an inch of bisulphide of carbon vapour, for example, absorbs 15 times as much as a whole atmosphere of air. Let us compare, for an instant, the action of boracic ether with that of air. We arrive at an approximate comparison in this way. The absorption of the tenth of an inch of boracic ether is something more than that of a whole inch of methylic alcohol; by diminishing the quantity of methylic alcohol to one-tenth, we reduce its absorption from 590 to 109. The absorption of one-tenth of an inch of boracic ether is 620°; suppose it to diminish in the proportion above found for methylic alcohol, we should have for 0·01 of an inch of boracic ether an absorption of 111; that is to say, for $\frac{1}{3000}$th of an atmosphere of boracic ether, we should have an absorption 111 times that of a whole atmosphere of oxygen, nitrogen, hydrogen, or atmospheric air.

With the transparent elementary gases it is impossible to measure directly the absorption of 0·1 of an inch; but assuming, as before, that up to an absorption of 1 the effect is proportional to the quantity of gas present, the absorption of each of the elementary gases, at a tension of 0·1 of an inch, would be about 0·0033; hence the absorption of boracic ether of 0·1 of an inch tension is to that of air at the same tension as

$$620 : 0·0033,$$

which would give to the ether an energy 186,000 *times that of air.*

I have already spoken of the blackness of ammonia at 30 inches tension. Referring to Table I., its absorption is found to be 1195. In the last Table the vapour of acetic ether, possessing only one-thirtieth of the tension of the ammonia, produces apparently the same effect; its absorption is also 1195. Such facts give one entirely new ideas of the capabilities of matter, and our wonder will not be diminished by the results to be recorded further on.

With both gases and vapours we find that it does not follow that a gas which produces a larger effect than another at one tension should surpass that other at all tensions. Some gases start from a lower level than others, but finally attain an equal, or even a greater elevation. If their absorptions were represented by curves plotted from the same datum-line, these curves would in some cases approach, and in some cases cross each other. At a tension of 1 inch, for example, carbonic acid has more than double the absorptive power of carbonic oxide, whereas at a tension of 30 inches they are equal; indeed some of my experiments show carbonic oxide to have the advantage. On the 22nd of October, for example, I found the deflection produced by 2 inches of carbonic oxide to be 15°, while that of 2 inches of carbonic acid was 38°. The two gases at a tension of 30 inches gave these results:—

Carbonic oxide	52°
Carbonic acid	51·5

And again, on the 4th of November I obtained the following relative effects:—

	Tensions.	
	1·2 inches.	24 inches.
Carbonic oxide	12°	57°
Carbonic acid	37	54

The same remarks apply to vapours. Methylic alcohol, for example, starts at a lower level than the iodide of ethyl, but ascends more quickly, and finally reaches a much higher elevation. The same observation may be made of chloroform in comparison with benzol and the iodide of ethyl.

§ 6.

I have now to refer to a class of facts which surprised and perplexed me when I first observed them. As an illustration, I will first take the case of alcohol vapour. A quantity of this substance, sufficient to depress the mercury gauge 0·5 of an inch, produced an absorption which caused a deflection of 72° of the galvanometer needle.

While the needle pointed to this high figure, and previously to pumping out the vapour, I allowed dry air to stream into the tube, and happened while it entered to observe the effect upon the galvanometer. The needle, to my astonishment, sank speedily to zero, and went to 25° at the opposite side. The entry of the almost neutral air here not only abolished the absorption previously observed, but left a considerable balance in favour of the face of the pile turned towards the source. A repetition of the experiment brought the needle down to zero, and sent it to 38° on the opposite side. In like manner a very small quantity of the vapour of sulphuric ether produced a deflection of 30°; on allowing dry air to fill the tube the needle descended speedily to zero, and swung to 60° at the opposite side.

These results both perplexed and distressed me, for I imagined, on first observing them, that I had been throughout dealing with an effect totally different from absorption. I thought, at first, that my vapours had deposited themselves in opake films on my plates of rock-salt, and that the dry air on entering had cleared these films away, and allowed the heat from the source free transmission.

But a moment's reflection dissipated this supposition. The clearing away of such a film could at best but restore the state of things existing prior to its formation. It might be conceived of as bringing the needle again to 0°; but it could not possibly produce the negative deflection, which, in the case of ether vapour, amounted to the vast amplitude of 60°. Nevertheless I dismounted the tube, and subjected the plates of salt to a searching examination. I satisfied myself thus that no such deposition as that above surmised took place. The salt remained perfectly transparent while in contact with the vapour.

Some of the experiments recorded in the Bakerian Lecture for this year (1860) had taught me that the dynamic heating of the air when it entered the exhausted tube was sufficient to produce a very sensible radiation on the part of any powerful vapour contained within the tube, but I was slow to believe that the enormous effect above described could be thus accounted for. My first care was to determine the difference of temperature between a thermometer placed within the tube at the end furthest from the source, and one placed without it. I then examined, by an extremely sensitive thermometer, the increase of temperature produced by the admission of dry air into the tube, and the decrease consequent on pumping out; and found the former to be a considerable fraction of the total heat transmitted from the source. Could it be that the heat thus imparted to the alcohol and ether vapours, and radiated by them against the adjacent face of the pile, was more than sufficient to make good the loss by absorption? The *experimentum crucis* at once suggested itself here. If the effects observed be due to the dynamic heating of the air, we ought to obtain them even when the sources of heat made use of in the experiments are entirely abolished; and we should thus arrive at the solution of the novel, and at first sight utterly paradoxical problem, *To determine the radiation and absorption of gases and vapours without any source of heat external to the gaseous body itself.*

For the sake of brevity, I will call the heating of gas by its admission into a vacuum, the *dynamic heating* of the gas; and the chilling accompanying its pumping out, *dynamic chilling*. It would also contribute to brevity if I were allowed to call the radiation and absorption of the gaseous body, consequent on such heating and chilling, *dynamic radiation* and *dynamic absorption*, though I fear the terms are not unobjectionable.

§ 7. *On Dynamic Radiation and Absorption.*

Both the source of heat and the compensating cube were dispensed with, and the thermo-electric pile was presented to the end of the cold experimental tube. By a little management, the slight inequality of radiation against both faces of the pile, arising from differences in the various parts of the laboratory, was obliterated, and the needle of the galvanometer thus brought to 0°.

The vapours were admitted in the manner already described, until a tension of 0·5 of an inch was obtained. The air was then allowed to enter through a drying apparatus by an orifice of a constant magnitude. Two stopcocks, in fact, were introduced between the drying-tube and the experimental tube; one of these was kept partially turned on, and formed the gauge for the admission of the air. When the tube was to be exhausted, the second stopcock was turned quite off. When the tube was to be filled, this stopcock was turned full on; but the *gauge-cock* was never touched during the entire series of experiments.

Before, however, the mode of experiment was thus strictly arranged, a few preliminary trials gave me the following results:—

Nitrous oxide on entering caused the needle to swing in a direction which indicated the heating of the gas; the limit of its excursion was 28°, after which it slowly sunk to 0°.

The pump was now worked; the propulsion of the first portions of the gas from the tube was so much work done by the residue. That residue became consequently chilled; into it the face of the pile adjacent poured its heat, and a swing of the needle on the negative side of 0° was the consequence. The limit of the excursion was 20°.

Olefiant gas, operated on in the same manner, produced on entering the tube a swing of 67°, showing radiation; and on pumping out, a swing of 41°, showing absorption. After the pumping out of the gas, and without introducing a fresh quantity, *dry air* was again admitted; the swing produced by the dynamic radiation of the residue of the gas (0·2 of an inch intension) was 59°. On pumping out *very quickly*, the dynamic absorption produced a deflection of nearly 40°.

A little of the vapour of sulphuric ether was admitted into the tube; on the admission of dry air afterwards the needle swung from 0° to 61°; on pumping out, the needle ran up to 40° on the opposite side.

These and other experiments, which I confess gratified me exceedingly, showed that, without resorting to any source of heat external to the gaseous body itself, its radiation and absorption might be determined with extreme accuracy, and the reciprocity of both phenomena rendered strikingly clear. In fact, at this very time I had been devising an elaborate apparatus for the purpose of examining the radiation of gases and vapours, with a view to comparing it with their absorption; but no such apparatus would have given me results equal in accuracy to those placed within reach by the discovery of dynamic radiation and absorption.

The following Table is the record of a series of experiments in connexion with this subject. The vapour in each case was admitted till the mercury column fell half an inch, and dry air was admitted afterwards.

Table IV.—Dynamic Radiation and Absorption of Vapours.

	Deflections.	
	Radiation.	Absorption.
Bisulphide of carbon	14°	6°
Iodide of methyl	19·5	8
Benzol	30	14
Iodide of ethyl	34	15·5
Methylic alcohol	36	
Chloride of amyl	41	23
Amylene	48	
Alcohol	50	27·5
Sulphuric ether	64	34
Formic ether	68·5	38
Acetic ether	70	43

The paradox already referred to is here solved, and the explanation given of the extraordinary effect observed in the case of the alcohol and ether vapours when dry air entered the experimental tube. Dynamic radiation, moreover, and dynamic absorption go hand in hand; and if we compare both with Table III., we shall find the order of the substances precisely the same, although one set of results are obtained with a source of heat external to the gaseous body, and the other with a source of heat and cold within the body itself. Had I sufficient time at my disposal, I could develope this subject with advantage. The results just recorded constitute my first regular series of experiments, and no doubt augmented experience will enable me to attain more perfect results.

I could not well obtain half an inch of my most energetically acting vapour, namely, boracic ether; but one-tenth of an inch admitted into the tube and dynamically heated and chilled, gave—

Radiation.	Absorption.
56°	28°

Seeing the astonishing energy with which some of these vapours absorb and radiate heat, it may be asked how far the quantity of vapour may be reduced before its action becomes insensible. At present I will not venture to answer this question fully; certainly we should be dealing at least with millionths of our smallest weights. But I will here lay before the Society an account of one experiment, the result of which can hardly fail to excite astonishment. The experimental tube being exhausted, one-tenth of an inch of boracic ether vapour was admitted into it; the barometer stood at 30 inches at the time, hence the tension of the vapour within the tube was $\frac{1}{300}$th of an atmosphere.

Dynamically heated by dry air, the radiation of this vapour produced a deflection of 56°.

The tube was then exhausted to 0·2 of an inch, and the quantity of vapour reduced thereby to $\frac{1}{150}$th part of its first amount; the needle was allowed to come to zero, and the residue of vapour was dynamically heated as before: its radiation produced a deflection of 42°.

The pump was again worked till a vacuum of 0·2 of an inch was obtained, this residue containing of course $\frac{1}{150}$th of the quantity of ether present in the last. On dynamically heating this residue, its radiation produced a deflection of 20°*.

Two additional exhaustions, succeeded by dynamic heating, gave the deflections 14° and 10° respectively.

Tabulating the results so as to place each deflection beside the vapour-tension which produces it, we have the following view of the experiment:—

* This is less than the truth; my assistant having executed three or four strokes of the pump inadvertently while the dry air was not shut off, removing thereby a considerable proportion of the vapour which ought to be present at this stage of the experiment.

TABLE V.—Dynamic Radiation of Boracic Ether.

	Tension in parts of an atmosphere.	Deflection.
	$\frac{1}{300}$th	56°
$\frac{1}{150}\times\frac{1}{300}=$	$\frac{1}{45000}$th	42
$\frac{1}{150}\times\frac{1}{150}\times\frac{1}{300}=$	$\frac{1}{6750000}$th	20
$\frac{1}{150}\times\frac{1}{150}\times\frac{1}{150}\times\frac{1}{300}=$	$\frac{1}{1012500000}$th	14

The air itself, slightly warming the apparatus near the pile, produces a feeble radiation, amounting to 6° or 7°. I have purposely excluded the deflection 10°, in order to show that the effect was still diminishing when the experiment ended, the constant effect due to the air itself being not yet attained. I thus exclude two 0s from the denominator of my fraction which might fairly have appeared in it. The above result is, however, sufficiently extraordinary, showing as it does that the radiation of an amount of vapour possessing in our tube a tension of less than the thousand millionth of an atmosphere is perfectly measurable. It will also be borne in mind that the temperature imparted to this infinitesimal quantity of matter did not exceed 0·75 of a Centigrade degree.

These experiments, which I intend to develope on a future occasion, seem to give us new ideas as to the nature and capabilities of matter. A platinum wire raised to whiteness in a vacuum by an electric current, becomes comparatively cold in a second after the current has been interrupted; yet that wire, while ignited, was the repository of an immense amount of mechanical force. What has become of this? It has been conveyed away by a substance so attenuated that its very existence must for ever remain an hypothesis. But here is matter that we can weigh, measure, taste, and smell; that we can reduce to a tenuity, which, though expressible by numbers, defeats the imagination to conceive of it. Still we see it competent to arrest and originate quantities of force, which on comparison with its own mass are almost infinite, a small fraction of this force causing the double needle of the galvanometer to swing through considerable arcs. When we find common ponderable matter producing these effects, we have less difficulty in investing the luminiferous ether with those mechanical properties which have long excited the interest and wonder of all who have reflected upon the circumstances involved in the undulatory theory of light.

In the foregoing experiments dry air was used to warm the vapours, but similar differences ought to be exhibited by gases when heated by their own dynamic action. This is the case, as the following experiments show:—

TABLE VI.—Dynamic Radiation of Gases.

Name.	Radiation.
Air	7°
Oxygen	7
Hydrogen	7
Carbonic oxide	19
Carbonic acid	21
Nitrous oxide	31
Olefiant gas	63

I also satisfied myself of the energetic radiation of the two following gases, which, however, were used in irregular quantities. They were admitted into the tube from a large bolthead, until a common tension was established between the gas in the tube and the gas in the bolthead.

	Radiation.	Absorption.
Ammonia 15 in. tension	$56^{\circ}{\cdot}5$	$33^{\circ}{\cdot}5$
Sulphurous acid 16 in. tension . .	45	24

Let us reflect for an instant on the condition of our tube with its ½ inch of vapour at the moment when the latter has been heated by the entrance of the air. The gaseous column is heated throughout to the same temperature; the elastic condition of the luminiferous ether is the same for all the particles, and consequently their periods of vibration are all the same. Hence each molecule is in that precise condition which enables it to absorb most effectually the undulations emanating from its neighbours. The rays from the particles at the end of the tube most distant from the pile have to cross a space of nearly 3 feet before they reach the latter, this space being partially filled with molecules circumstanced as just described. Hence absorption to a comparatively greater extent must occur; and indeed we can imagine the tube so long that its frontal portion should furnish a vapour screen absolutely opake to the radiation of its hinder portion. Comparing ether vapour with olefiant gas, it is, I think, evident that the radiant points of the attenuated vapour which depresses the mercury column only 0·5 of an inch, are further apart than those of the gas which depresses the column 30 inches. Consequently there is a wider door open for the radiation of the distant ether particles towards the pile than for the distant particles of olefiant gas. The length of the whole column, in fact, might be available for the radiation of the vapour, and a part of it only available for the gas. Cut off this unavailable portion from the gas column, and we do not injure its efficacy; but cut off a similar length from the vapour column, and we may materially diminish its effect. Speaking generally, by reducing the column of ether and that of gas by the same amount, the diminution of radiation will be most sensibly felt where the radiant points are furthest asunder. Reasoning thus, it becomes evident that in a long tube the vapour may excel the gas in its amount of radiation, while with a short tube the gas may excel the vapour. Let us now test this reasoning by experiment.

The dynamic radiation of the following four substances has been tabulated thus:—

Sulphuric ether	64°
Formic ether	68·5
Acetic ether	70
Olefiant gas	63

The action of olefiant gas is therefore smallest when the length of the radiating column is 2 feet 9 inches.

Experiments of the same character were made with a tube 3 inches long, or of the former length, and the following results were obtained:—

Sulphuric ether	11°
Formic ether	12
Acetic ether	15
Olefiant gas	39

The verification of the above theoretic reasoning is here complete. It is proved that *in a long tube the dynamic radiation of the vapour exceeds that of olefiant gas, while in a short tube the dynamic radiation of the gas far exceeds that of the vapour.*

§ 8.

The apparatus with which these experiments were made is capable of very diverse uses. Attached to a compression pump, with it the relation between the mechanical force expended in compressing a gas and the heat developed might be accurately determined. If oxygen, hydrogen, nitrogen, or air were the body compressed, the conversion of *vis viva* into heat might be declared by a modicum of vapour always kept in the tube, while a compound gas would tell its own tale.

Another interesting point might be, and indeed has been settled by the apparatus. Some years ago a discussion was carried on between Professors CHALLIS and STOKES on LAPLACE's correction for the velocity of sound, Professor CHALLIS contending that LAPLACE had no right to his correction, inasmuch as the heat developed by the local compression of a mass of air of indefinite extension would be instantly wasted by radiation. Experiments, he argued, conducted in confined vessels furnish no ground for drawing conclusions regarding what occurs in the atmosphere, where the heat developed has an indefinite space to lose itself. In our experimental tube, though it is mechanically closed, indefinite extension, as regards the radiation of heat, is secured in one direction, and the means also exist of measuring the flux of heat in this direction. What is true for one direction is of course true for all, so that the apparatus will inform us of what must occur in the open atmosphere. Now with the most powerful radiating gases which I have examined the radiation continues a very sensible time, while the heat acquired by air on entering the tube is often a source of inconvenience on account of the inability of the air to disperse its heat by radiation. The question is therefore experimentally decided in favour of LAPLACE and his supporter.

I would here dwell for a moment on this comparative absence of radiating power on the part of air, and of the elementary gases generally. The air is the sole source of the heat which has warmed the vapours in our experiments on dynamic radiation; it is related to them precisely as a hot polished metal plate is to the coat of varnish which makes it a radiator. The air and the metal, both elements or mixtures of elements, are incompetent to impart motion to the luminiferous ether without the intermediation of a second body. They possess the motion, but they are so related to the ether that they cannot communicate this motion to it *directly*, or only in an extremely feeble degree. The atoms of air oscillate, but the ether does not swell. We have here a definite picture before the mind's eye, which, if the theory of an ether be true, is as certain as any conclusion of mathematics, and would hardly be rendered more certain if the physical vision were so sharpened as to be able to see the oscillating atom and the fluid in which it swings. I write thus strongly and definitely lest it should be imagined that I am dealing in vague conjectures in connexion with this subject. If I am vague, the mechanical theory of an ether must in reality bear the reproach. So far, however, from having a reproach to bear, the whole body of facts is in complete harmony with this theory.

Further, if, as all the facts declare, radiation and absorption are complementary acts, the one consisting in communication, the other in reception, and the one being strictly proportional to the other, no coincidence in period between the vibrations of a radiating body and those of oxygen, hydrogen, or air, could make any one of these substances a good absorber. The form of the atom, or some other attribute than its period of oscillation, must enter into the question of absorption. It is physically incapacitated from communicating motion, and hence in an equal degree incapacitated from accepting motion. The neutrality of elementary gases in the experiments on absorption already recorded does not arise from my accidentally choosing a source of heat whose periods do not synchronise with those of the gas; for however they might synchronise, the gas would still be a bad absorber. Even when the motion which their own absorbent power does not enable them to take up is mechanically imparted, or is communicated to them by contact, they are still incompetent to expend it upon the ether, which accepts all vibrations alike*.

§ 9.

Scents and effluvia generally have long excited the attention of observant men, and they have formed favourite illustrations regarding the divisibility of matter. Several chapters in the works of the celebrated ROBERT BOYLE are devoted to this subject, and eminent men in all countries have speculated more or less upon the extraordinary tenuity of the matter which is competent to produce sensible effects upon our organs of smell. Such a subject would of course in itself form a wide inquiry, which it is quite out of my power to develope at present. I think, however, that the apparatus which we have thus far made use of enables us to deal with the question in a manner hitherto unattainable.

A number of dry aromatic plants† were obtained from Covent Garden, the leaves and flowers of which were stuffed into glass tubes 18 inches long and a quarter of an inch in diameter. By means of my second air-pump, a current of dry air was first passed

* I can hardly imagine the bands in the spectra of metallic compounds to be produced by the vibration of the compound atom. All my experiments show the vast influence of chemical union on the rate of oscillation; the metal itself and the compound of that metal could hardly, in my opinion, oscillate alike. Hence I infer that decomposition has occurred when the bright and constant spectral bands are seen.—June 20th.

† I mean "dry" in the common acceptation of the term. They were not green, but withered; doubtless, strictly speaking, they contained aqueous vapour.

over them for some minutes. They were then connected with the experimental tube, which had its sources of heat arranged as already described. The tube was first exhausted and the needle brought to 0°, and dry air was then passed over the scented herbs until the tube was filled. The consequent deflection was noted, and from it the absorbent action of the odorous substance was deduced.

Thyme thus treated exercised thirty-three times the absorption of the air in which it was diffused.

Peppermint exercised thirty-four times the action of the air.

Spearmint exercised thirty-eight times the same action.

Lavender produced thirty-two times the action of the air.

Wormwood forty-one times the action of the air.

The following perfumes were obtained from Mr. Atkinson of Bond Street, and examined in this manner. Small squares of dried bibulous paper, all of the same size, were rolled into cylinders about 2 inches in length; each of these was moistened by an aromatic oil, and introduced into a glass tube between the drying apparatus and the experimental tube. The latter being first exhausted, was afterwards filled by a current of dry air which had passed over the scented paper. Calling the action of the air which formed the vehicle of the perfumes 1, the following absorptions were observed in the respective cases:—

Table VII.

Name of perfume.	Absorption.
Pachouli	30
Sandal Wood	32
Geranium	33
Oil of Cloves	33·5
Otto of Roses	36·5
Bergamot	44
Neroli	47
Lavender	60
Lemon	65
Portugal	67
Thyme	68
Rosemary	74
Oil of Laurel	80
Cassia	109

It would be interesting to examine the absolute weights of the substances which produced these effects; but this I suppose is a task which chemistry is unable to accomplish. In comparison with the air which carried the odours into the tube, their weight must be almost infinitely small. Still we find that the least energetic in the list has thirty times the effect of the air, while the most energetic produces 109 times the same effect. As regards the absorption of radiant heat, the perfume of a flower-bed may be more efficacious than the entire oxygen and nitrogen of the atmosphere above it.

After each scent had been introduced a stream of dry air was admitted at one end of the tube, while the pump was worked in connexion with the other. The perfume was thus cleared out until the needle returned to 0°. This was often a long operation, the odours clung with such tenacity to the apparatus. After the zero had been attained in the case of a strong perfume, a few minutes' rest of the pump sufficed to bring the scent from its hiding-places in the crevices and cocks of the apparatus, and almost to restore the original deflection. The quantity of those residues must be left to the imagination to conceive. If they were multiplied by billions they probably would not reach the density of the air.

Fearing that the more active perfumes might possibly prejudice the action of the more feeble ones which succeeded them, I made a series of experiments with the following essences, and obtained the results recorded:—

Camomile flowers	87
Spikenard	355
Aniseed	372

After this enormous effect I repeated the experiment with bergamot, and found its action to be exactly the same as that recorded in the Table.

I made a few experiments on musk, but obtained different results with it at different times. On the 16th of October I obtained some fresh musk from the perfumer's, placed it in a small glass tube, and carried dry air over it into the experimental tube. The first experiment gave me an absorption of

74,

the air which carried the perfume being unity. A second experiment, in which the air was admitted more quickly, the absorption was

72.

It would be idle to speculate upon the quantity of matter which produced this result. The stories regarding the unwasting character of this substance are well known; suffice it to say, that a quantity of its odour carried into the tube by a current of air of a minute's duration, produced an effect seventy-two times that of the air which carried it. Long-continued pumping failed to cleanse the tube and passages of the musk. It cannot be volatile, for an amount of ether vapour which produces a far greater action is speedily cleared away, while the cocks and connecting pieces of the air-pump had to be boiled in a solution of soda before they were fit for use after the experiments with this substance.

Two perfectly concurrent experiments with ordinary cinnamon, in which fragments of the substance were placed in a tube and had dry air passed over them, gave an absorption of

53.

Several kinds of tea, treated in the same manner, produced absorptions which varied between 20 and 28.

In the teas, cinnamon, musk, and the odorous plants already referred to, dry air had been passed over them for some time before they were examined. Still a small amount of aqueous vapour may have entered with the odours, and thus rendered the results to some extent of a mixed character.

§ 10. *Ozone.*

In my last memoir I alluded briefly to the action of ozone; but the experiments then made having been executed with a brass tube, I was very desirous of repeating them with a tube which could not be attacked by this extraordinary substance. Experiments with the glass tube, performed on the 16th, 17th, and 18th of last July, satisfied me that I had not over-estimated its power as an absorber of radiant heat.

In my first experiments I made use of large electrodes for the purpose of lessening the resistance to the passage of the current through the decomposing liquid. The oxygen thus obtained differed but little from ordinary oxygen.

This year I had three decomposing vessels constructed: in the first (No. 1) the platinum plates had about four square inches of surface, being rolled up to economise space; the plates of the second (No. 2) had two square inches of surface, while those of the third (No. 3) had only a square inch of surface each. Numerous experiments with these gave me the following constant results. Calling the absorption of ordinary oxygen 1,—

Electrolytic Oxygen.

From plates	Absorption.
No. 1	20
No. 2	34
No. 3	47

A series of experiments made on the following day gave these results:—

No. I.	21
No. II.	36
No. III.	47

I now cut away a portion of the plates of No. II. so as to make them smaller than those of No. III. The oxygen obtained with these plates gave an absorption of

65,

thus exceeding No. III. considerably. The plates of No. III. were now reduced so as to make them smallest of all; the oxygen which they delivered gave an absorption of

85.

I feared the development of heat with these smallest plates, and knowing heat to be very destructive of ozone, I surrounded the apparatus by a mixture of pounded ice and salt. The absorption of the oxygen thus obtained with the smallest plates amounted to

136.

By the results already recorded we have been prepared for the effect of minute quantities of matter; otherwise we could not fail to be struck with astonishment on finding a quantity of this substance, which would elude all attempts on the part of the chemist to determine its amount, producing an effect so stupendous in comparison with common oxygen. I have, moreover, strong reason to believe that I understate considerably the effect of the ozone. The experiments exhibit in an extremely striking manner the great influence of the density of the current at the place where the oxygen is liberated on the amount of ozone developed.

§ 11.

All the results here recorded had been obtained, when, turning to De la Rive's excellent treatise on Electricity, I found an allusion to the experiments of M. Meidinger on ozone. I had never previously heard any allusion made to this investigation, and was gratified to find in it the record of a very interesting piece of work.

M. Meidinger commences by showing the absence of agreement between theory and experiment in the decomposition of water, the difference showing itself very decidedly in a deficiency of oxygen *when the current was strong*. On heating his electrolyte, he found that this difference disappeared, the proper quantity of oxygen being liberated. He at once surmised that the defect of oxygen might be due to the formation of ozone; but in what way was still to be determined. If it were due to the great density of ozone in the tube which received the oxygen, the destruction of this substance by heat would restore the oxygen to its true volume. Strong heating, however, which destroyed the ozone, showed in repeated measurements no alteration of volume, hence M. Meidinger concluded that the defect which he had observed was not due to the ozone mixed with the oxygen itself. He finally concluded, and justified his conclusion by satisfactory experiments, that the loss of oxygen was caused by the formation of peroxide of hydrogen which was dissolved in the liquid, and thus withdrawn from the electrolytic gas. He was further led to experiment with electrodes of different sizes, and found the loss of oxygen much more considerable when a small electrode was used than with a large one; whence he inferred that the formation of ozone was facilitated by *augmenting the density of the current at the place where electrode and electrolyte meet.* Nothing could be more different from the method pursued by M. Meidinger than that by which I arrived at the same conclusion; and though I had no doubt of the accuracy of my experiments, it was pleasant to find them corroborated in such a remarkable and unexpected way. I may add, that since the perusal of M. Meidinger's paper I have repeated his experiments with my decomposition cells, and find that those which gave me the greatest absorption also show the greatest deficiency in the amount of oxygen liberated *.

* I have recently learned that M. De la Rive was the first to observe the influence of the size of the electrodes on the development of ozone.

The quantities of ozone with which I have operated must be perfectly unmeasurable by ordinary means. The action of the substance is like that of olefiant gas, or boracic ether—bulk for bulk it might transcend either. No elementary gas that I have examined behaves at all like ozone. In its swing through the ether it must powerfully disturb the medium. If it be oxygen, it must be oxygen packed into groups of atoms, which encounter vast resistance in moving through the ether. I sought to decide the question whether it is oxygen or a compound of hydrogen in the following way. Heat destroys ozone. If it were oxygen only, heat would convert it into the common gas; if it were the hydrogen compound which some chemists consider it to be, heat would convert it into oxygen plus aqueous vapour. The gas alone admitted into my tube would give the neutral action of oxygen, but the gas plus the aqueous vapour I hoped might give a sensibly greater action. I caused the dry electrolytic gas to pass through a glass tube heated to redness direct into the experimental tube. I afterwards introduced a drying-tube between the place where the gas was heated and the experimental tube. Hitherto I have not been able to establish with certainty a difference between the dried and undried gas. Previously to heating, the electrolytic oxygen had been scrupulously dried; if the act of heating developed aqueous vapour, I can only say that the experimental means which I have employed are unable to detect it. For the present, therefore, I hold the belief that ozone is produced by the packing of the atoms of elementary oxygen into oscillating groups; and that heating dissolves the bond of union and allows the atoms to swing singly, thus disqualifying them for either intercepting or generating the motion, which as systems they were competent to intercept and generate.

§ 12.

Since these researches were commenced, an eminent experimenter has been led by his own inquiries in another field to enter upon the investigation of gaseous diathermancy. On the 7th of February of the present year (1861), Professor MAGNUS communicated to the Academy of Sciences in Berlin a memoir on the Transmission of Heat through Gases*. The published notices of my experiments, commencing in May 1859, had escaped his attention, and his work is therefore to be regarded as independent of mine. Considering the very different methods which we have pursued, the general agreement between us must be regarded as remarkable.

The starting-point of Professor MAGNUS's investigation was the interesting experiment of Mr. GROVE, in which a platinum wire raised to whiteness by an electric current is suddenly cooled by an atmosphere of hydrogen. This action, which we have hitherto been disposed to attribute to the mobility of hydrogen, and its consequent high convective power, Professor MAGNUS was led to regard as an effect of conduction; and the thought induced him to examine the conductibility of gases generally. The mode of experiment adopted led him, not I think to the establishment of gaseous conductivity, but to results substantially the same as those that I had previously obtained. In fact

* POGGENDORFF's 'Annalen,' reprinted in Philosophical Magazine, S. 4. vol. xxii. p. 85.

the very experiments devised to show conductivity showed in a very striking manner the existence of athermancy, or opacity to radiant heat, in the case of a considerable number of gases.

The experiments on radiation, where obscure heat was made use of, were thus conducted. Two glass vessels, one much larger than the other, had their bottoms fused together; the larger one being turned upside down, the smaller one stood upright on the top of it. The mouth of the larger vessel was ground down, so that it could be placed like an ordinary receiver on the plate of an air-pump and exhausted, while through proper openings different gases could be afterwards admitted into it.

To the plate of the air-pump on which the above vessel was placed, was attached a thermo-electric pile with wires leading from it, through the plate, to a galvanometer; the axis of the pile was vertical, one face of it being turned downwards towards the plate, and the opposite face turned upwards towards the common surface of the two vessels which had been fused together.

Water was placed in the uppermost vessel, and caused to boil by conducting hot steam through it. Its under surface became thus heated to a temperature of 100° C. But this under surface constituted the upper surface of the vessel underneath. This latter, therefore, possessed a temperature of 100° C.; and it formed the source of heat made use of in the experiments.

Here Professor MAGNUS had a radiating surface of glass—a good radiator—kept at a constant temperature by the hot water above it; at a distance from this surface and turned towards it was the thermo-electric pile, defended from the radiation of the surface, or exposed to it, at pleasure, by the action of a moveable screen. The entire space between the pile and the radiating surface could either be rendered a vacuum, offering no resistance to the passage of the calorific rays, or else be filled by a gas the diathermancy of which was to be examined.

The concurrence of the experiments made with this apparatus and those made with mine is, as I have stated, remarkable. Some differences, however, exist between my friend and myself, a few remarks on which will not be without their use to those who may afterwards enter upon this extensive field of inquiry.

Experimenting in the ordinary way with his thermo-electric pile—using one of its faces only—Professor MAGNUS finds that air and oxygen cut off each more than 11 per cent. of the heat emanating from his source, while hydrogen cuts off more than 14 per cent.* I, on the contrary, with the most delicate means I could apply, failed to establish the absorption of these gases by experiments made in the ordinary manner†. In fact it was their neutrality that drove me to devise the principle of compensation, briefly referred to at the commencement of this memoir. I was so particular in the experiments which led me to the above negative result, that if the absorption amounted to one-tenth of that found by Professor MAGNUS I do not think it could have escaped me. Nor do I think that if such an action existed MELLONI could have concluded that the absorption

* Page 30. † Philosophical Transactions, 1861.

of a column of air fifteen times the length of that employed by Professor MAGNUS was absolutely insensible.

In the account of the experiments already published, where my source of heat was also 100° C., I have set down the absorption of air, oxygen, and hydrogen at about 0·33 per cent.; which is for air and oxygen thirty times, and for hydrogen over forty times less than that found by Professor MAGNUS.

In fixing the above figure for the absorption of these gases, I protected myself by assigning what I knew to be the superior limit of the effect, but I was morally certain at the time, that as soon as I could combine sufficient power and delicacy I should make the effect less. This I have done in my present inquiry, and find the absorption of the above gases to be under 0·1 per cent., which in the case of oxygen is less than $\frac{1}{100}$th, and in the case of hydrogen less than $\frac{1}{140}$th of the effect obtained by Professor MAGNUS with a tube less than half the length of mine. Making every allowance for the difference between our two sources of heat, the discrepancy between us is still enormous. In fact my conclusion is that these gases are practical vacua to radiant heat, and that the mixture of oxygen and nitrogen which constitutes the body of our atmosphere is the same.

While, however, in the case of the elementary gases the discrepancy between Professor MAGNUS and myself consists in a defect on my part, or an excess on his, with the powerful gases I obtained a considerably stronger action than he does. Thus with olefiant gas his absorption amounts to less than 54 per cent., whereas in mine it amounts to more than 72. This last result is what might only be expected, inasmuch as the length of gas traversed by the radiant heat was in the one case a little under 15 inches, and in the other 33.

Professor MAGNUS has further published an account of experiments in which a powerful gas-flame surrounded by a glass cylinder furnished the source of heat; the latter being augmented by a parabolic mirror of polished metal, placed behind the lamp. In this case the gases were enclosed in a glass tube 1 metre long and 35 millims. in diameter, the two ends of which were stopped with plates of glass 4 millimetres thick.

Two series of experiments were executed with this tube, in one of which the interior surface was covered with black paper, while in the other the glass was uncovered within. The former method is that pursued by Dr. FRANZ, and the result obtained by Professor MAGNUS in the case of atmospheric air and oxygen closely agrees with that obtained for the same gases by Dr. FRANZ. Professor MAGNUS makes the absorption in the case of the blackened tube about $2\frac{1}{2}$, and Dr. FRANZ about 3 per cent. for air and oxygen.

In the case of the unblackened tube, however, the absorption was found to be much more considerable. Here the absorption by air and oxygen amounted to 14·75 per cent., and with hydrogen it reached 16·23. This great difference between the unblackened and the blackened tube is ascribed by Professor MAGNUS to a change of quality which the heat undergoes by its reflexion from the interior glass surface.

One of my motives in introducing a glass tube into the present inquiry was, that I might be enabled to investigate the interesting question raised by this surmise of Professor MAGNUS. I have failed, however, to obtain his result. My naked glass tube, which is nearly of the same length as his, gives me a result which is more than 140 times less than his in the case of air and oxygen, and more than 160 times less than what he has obtained with hydrogen. Our sources of heat are, it is true, different, but the disadvantage is on my side; for assuredly the rays from a gas-jet are, if anything, less affected by the transparent elementary gases than those from my source. Had I time, I would repeat the experiments with a flame; but this, I regret to say, is out of my power at present.

Another difference between Professor MAGNUS and myself has reference to the influence of aqueous vapour. With both the gas-flame and the boiling water as sources of heat, he finds the effect of dry air to be precisely the same as that of air which he has allowed to pass in minute bubbles through water, and thus saturated with aqueous vapour.

I was engaged in experiments on this substance when my other duties compelled me to close this inquiry for a time. I believe, however, I may safely say, that not only is the action of aqueous vapour on radiant heat measurable, but *this action may be made use of as a measure of atmospheric moisture, the tube used in my experiments being thus converted into a hygrometer of surpassing delicacy.* Unhappily, as in other cases touched upon in this memoir, I have been unable to give this subject the development I could wish; but the results which I am in a position to record are nevertheless interesting.

On a great number of occasions I compared the air sent in directly from the laboratory into the experimental tube with the same air after it had been passed through the drying apparatus. Calling the action of the dry air unity, or supposing it rather to oscillate about unity (for the temperature of my source varied a little from day to day), on the following days the annexed absorptions were observed with the undried air of the laboratory:—

Absorptions by undried air.

October 23rd	63
October 24th	62
October 29th	65
October 31st	56
November 1st	50
November 4th	58
November 8th	49
November 12th	62

Nearly $\frac{9}{10}$ths of the above effects are due to aqueous vapour; which, therefore, in some instances *exerted nearly sixty times the action of the air in which it was diffused.*

The experiments which I have made on aqueous vapour have been very numerous and varied. Differing as I did from so cautious and able an experimenter, I deemed it due to Professor MAGNUS and myself to spare no pains in securing myself against error. I have experimented with air moistened in various ways, sometimes by allowing small bubbles of it to ascend through water, sometimes dividing it by sending it through the pores of common cane immersed in water. Between the drying apparatus and the experimental tube I have introduced tubes containing fragments of glass moistened with water, and allowed the air to pass over them; large effects were in all such cases obtained, the absorption being usually *more than eighty times that of dried air.* Fragments of unwetted glass, which had been merely exposed to the air of the laboratory, had dry air led over them into the experimental tube; the absorption was fifteen times that of dried air. A roll of bibulous paper, taken from one of the drawers of the laboratory, and to all appearance perfectly dry, was enclosed in a glass tube, and dry air carried between its leaves. The experiment was made five times in succession with the same paper, and the following absorptions were observed:—

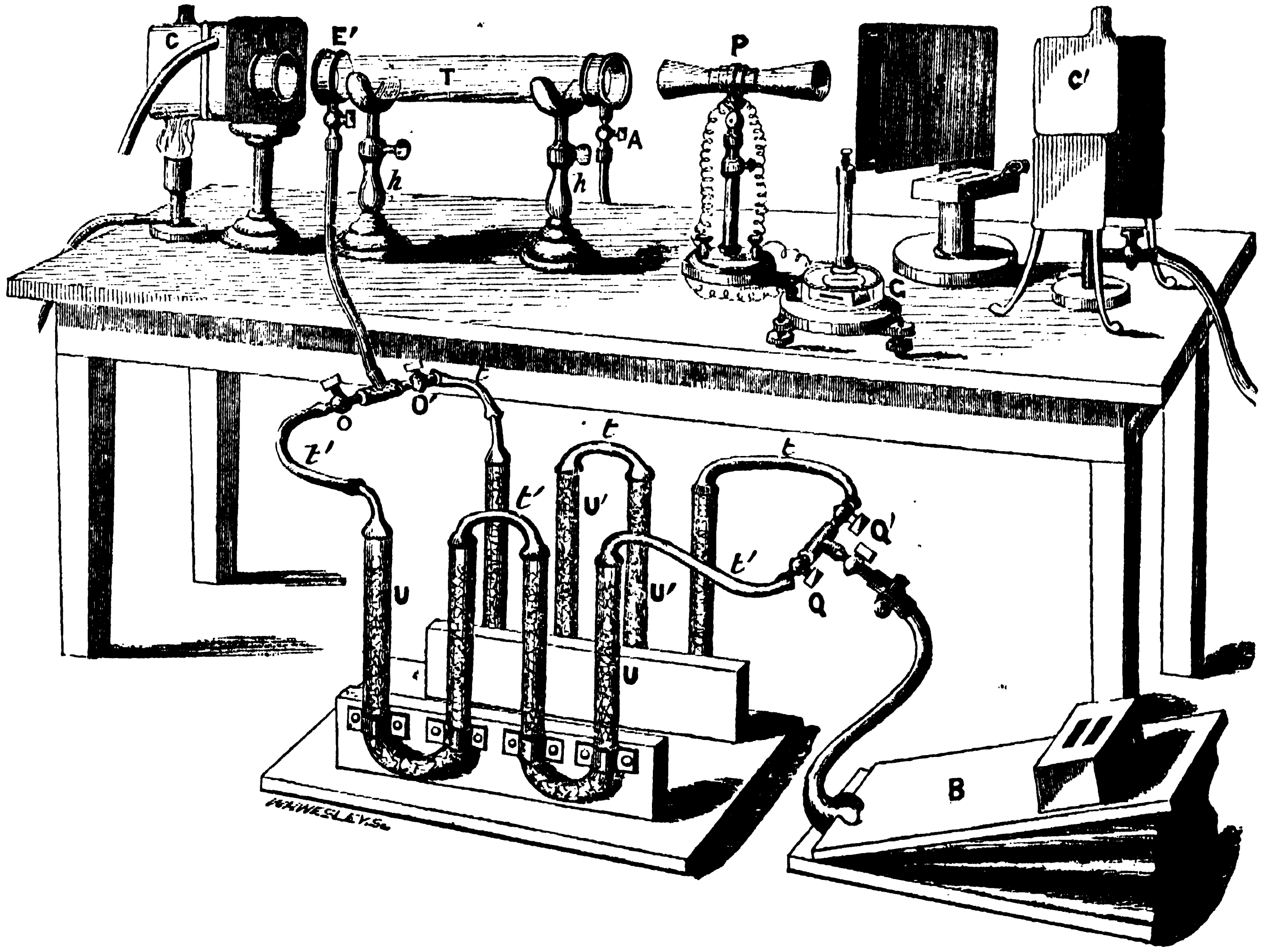
	Absorption.
No. 1	72
No. 2	62
No. 3	62
No. 4	47
No. 5	47

In fact, the action of aqueous vapour is exactly such as might be expected from the vapour of a liquid which MELLONI found to be the most powerful absorber of radiant heat of all he had examined.

Every morning, on commencing my experiments, I had an interesting example of the power of glass to gather a film of aqueous vapour on its surface. Suppose the tube mounted, and the air of the laboratory removed as far as the air-pump was capable of removing it. On allowing dry air to enter for the first time, the needle would move from 0° to 50°. On pumping out it would return to 0°, and on letting in dry air a second time it would swing almost to 40°. Repeated exhaustions would cause this action to sink almost to nothing. These results were entirely due to the vapour collected during the night in an invisible film on the inner surface of the tube, and which was removed by the air on entering, and diffused through the tube. If the dry air entered at the end of the tube nearest to the source of heat, on the first and second admissions, and sometimes even on a third, the vapour carried from the warm end to the cold end of the tube was precipitated as a mist upon the latter, for a distance sometimes of nearly a foot. The mist always disappeared on pumping out. It is needless to remark that facts of this character, of which I could cite many, were not calculated to promote incautiousness or rashness on my part. I saw very clearly how easy it was to fall into the gravest errors, and I took due precautions to prevent myself from doing so.

Knowing that a solution of salt was almost as opake to radiant heat as water itself, I was careful to examine whether the effects which I had observed with aqueous vapour might not be due to the precipitation of the vapour on the surfaces of the plates of salt used to stop my tube. The substance is well known to be very hygroscopic, and during the last three years the knowledge of this fact has rendered me careful to remove my polished plates every evening from the apparatus, and to keep them in perfectly dry air. Still, when it is remembered that the air on entering the tube is raised in temperature, and thus enabled to maintain a greater amount of vapour, and that the tube and plates of rock-salt form the channel for a flux of heat from the radiating source, the likelihood of precipitation occurring will seem but small. On examining the plates after the undried air of the laboratory was experimented with, no trace of precipitated moisture was observed upon their surfaces.

But to place the matter beyond all doubt, I abolished the plates of rock-salt altogether, and operated thus:—An india-rubber bag (B) was filled with air, and to its nozzle a T-piece, with the cocks Q Q′, was attached. The cock Q′ was connected with two tubes, U′ U′, each of which was filled with fragments of glass moistened with distilled water. The cock Q was connected with the tubes U U, each of which was filled with fragments of glass moistened by sulphuric acid. The other ends of these two series of tubes were connected with the cocks O O′, and from the T-piece between these cocks



a tube led to the end E′ of the open experimental tube T. The cock A at the other end of the experimental tube was placed in connexion with an air-pump. The pile P, the screen S, and the compensating cube C′ were used as in the other experiments. E is the end of the front chamber, and C the source of heat. In some experiments I had the end E closed by a plate of rock-salt, in others it was allowed to remain open; a distance of about 12 inches intervening between the radiating surface and the open end E′ of the experimental tube.

Closing the cocks Q and O, and opening Q′ and O′, gentle pressure being applied to the bag B, a current of moist air was slowly discharged at the end E′ of the experimental tube. The pump in connexion with A was then worked, and thus by degrees when the air was sucked into the tube T. The deflection of the galvanometer was 30°, the moist moist air filled the tube as completely as the arrangement permitted*; this deflection being due to the predominance of the compensating cube over the radiating source C.

The cocks Q′ and O′ were now closed, and Q and O opened; proceeding as before, a current of *dry* air was discharged at E′, and this air was drawn into the tube T in the manner just described. The moist air was thus displaced by dry; and, while the displacement was going on, the galvanometer was observed through the distant telescope. The needle soon commenced to sink, and slowly went down to zero; proving that a greater quantity of heat passed through the dry than through the moist air. The wet air was substituted for the dry, and the dry for the wet twenty times in succession, with the same constant result; the entrance of the humid air caused the needle to move from 0° to 30°, while the entrance of dry air caused it to fall from 30° to 0°. The air-pump was resorted to, because I found that when I attempted to displace the air by the direct force of the current from B, the temperature of the pile, or of the source, was so affected by the fresh air as to confuse the result. I may remark, that not only have I operated thus for days with aqueous vapour, but every result which I have obtained with vapours generally has been thus confirmed, so that all doubt as to the applicability of the rock-salt plates to researches of this nature may, I think, be abandoned†.

§ 13.

Whence, then, arise those differences between Professor MAGNUS and myself? I have no doubt that every one of his published results is the record of an experiment made with the utmost care which it is possible to bestow upon scientific work. The differences between us are, I imagine, to be referred to a radical defect in his apparatus. His desire was to do away with plates of all kinds between his source of heat and his pile, and hence he brought his gas *into direct contact with his source of heat.* The same thought had occurred to myself, and I was on the point of falling into the same error; but a series of experiments executed with reference to this point so early as the 26th of July, 1859, showed me that the accuracy of the results was entirely compromised by bringing

* Still, of course, only partially.

† It is sheer want of time that prevents me from describing more particularly the numerous experiments executed with open tubes.

the gas to be examined into contact with the source. I obtained thus an action forty times what I knew it ought to be, and was confirmed in the view which caused me to interpose a vacuous chamber in front of the experimental tube. Let me here record a few experiments made on the 4th of last November in connexion with this subject.

I first satisfied myself that the drying apparatus was in perfect condition, the air of the laboratory producing, when sent through it, an absorption of 1. This same air was sent into the front chamber, that is, into direct contact with the source. The galvanometer needle moved as it does in the case of absorbent gases, and at the end of two minutes declared a loss of heat equivalent to an absorption of 50. The front chamber is 8 inches in length; the experimental tube is 33 inches long; hence a column of 8 inches, in contact with the radiating surface, produced at least fifty times the effect of a column more than four times as long when the air was separated from the radiating surface.

I made the foregoing experiment three times in succession, and after two minutes found the needle pointing to precisely the same degree; the lowering of the source was perfectly constant and regular, and in all cases showed a loss equivalent to an absorption of 50.

It will be remembered that Professor MAGNUS obtained a greater absorption with hydrogen than with either oxygen or air. This result is perfectly explained by reference to the quicker convection of this gas. I operated with hydrogen as I did with air, first satisfying myself that a column of it 33 inches long exercised an absorption less than unity. In fact it could not be measured. The same hydrogen introduced into the first chamber, and allowed to remain there for two minutes, caused a withdrawal of heat from the source equivalent to an absorption of

65.

Now the absorption of air in Professor MAGNUS's experiments is to that of hydrogen as

11·12 : 14·21,

or as 50 : 64,

while my results of convection are as 50 : 65.

The coincidence is so perfect that I am disposed to regard it as in part accidental.

Substantially the same remarks apply to the experiments with the glass tube stopped with plates of glass 4 millimetres thick. According to MELLONI, 61 per cent. of the rays of a Locatelli lamp are absorbed by a plate of glass only 2·6 millimetres thick. Professor MAGNUS surrounded his flame by a glass cylinder, and this, it may be urged, partially sifted the heat of the lamp before it reached the end of the tube. But in so doing the glass cylinder itself must become intensely heated, and to the heat of the cylinder the glass ends of the tube would be *opake*. They would absorb it all. Cold air admitted into such a tube is exactly similar to cold air let into my front chamber, it chills what is in part the source of heat, and maintains that chill by convection. The heat applied may, in fact, be thus analysed. 1. We have a portion, almost wholly luminous, which went through the tube direct to the pile; 2, a portion *arrested* by the first glass plate; 3, a smaller portion *arrested* by the second glass plate; 4, we have the heat *radiated* by the first glass plate towards the second, and wholly absorbed by the latter; 5, we have the heat radiated by this latter against the pile. This analysis will probably enable us to understand how Professor MAGNUS obtained an absorption of only 2½ per cent. with the blackened tube, and as much as 14·75 per cent. with the unblackened one. With the latter, the source, and the plate of glass nearest the source, send a copious flux down the tube to the plate at the opposite end; the oblique rays are in great part reflected by the interior surface, and thus reach the plate adjacent to the pile. With the blackened tube this oblique radiation is entirely cut off, the rays incident on the interior surface being absorbed. Thus the plate of glass adjacent to the pile must be much more intensely heated with the unblackened tube than with the blackened one. The difference in the amount of heat received by the pile-end plate in the respective cases is rendered very manifest by the experiments of Professor MAGNUS himself, for he finds that with the same source, twenty-six times the amount of heat transmitted by the coated tube is transmitted by the uncoated one. What, therefore, Professor MAGNUS ascribes to a change of quality by reflexion, would, if I am correct, be due to the higher heating in the case of the naked tube, and consequent *greater chilling by the cold air*, of the plate of glass close to the pile. To this must be added the effect produced by cooling the distant end of the tube itself, to which heat has been communicated from the first glass plate by the process of conduction, and the cooling of which comes most into play when the tube is uncovered.

The difference between Professor MAGNUS and myself as regards the action of aqueous vapour admits now of easy explanation. His effect being one of convection, and not of absorption, the quantity of vapour present in his experiments—probably not more than 1 per cent. of the volume of the gas, certainly not 2 per cent.—vanished as a convecting agent, in comparison with the air.

It is hardly necessary to repeat these reflections with reference to the experiments of Dr. FRANZ. The taking of the chilling of his plates for absorption, has caused him to find no difference of effect when he doubled the length of his tube. With a tube 450 millimetres long, he finds precisely the same absorption as with a tube of 900. He finds the action of carbonic acid to be the same as that of air, although at atmospheric tensions the action of the former is 90 times that of the latter *. He finds the vapour of bromine more destructive to radiant heat than nitrous acid gas, whereas the latter is beyond comparison the most destructive. The heat rendered latent by the evaporation of the

* The sensible equality of all the transparent gases and air was regarded as evident by Dr. FRANZ. "It might be seen," he writes, "from the outset that no decided difference would be observed between them" (p. 342). Similarly, Professor MAGNUS, speaking of aqueous vapour, writes, "Although it might be foreseen with certainty that the small amount of aqueous vapour in the air could have no influence on the radiation," &c. (p. 43).

bromine of course augmented the chill, and thus magnified the effect which in reality he was measuring. In reference to heating the glass plates by the flame, made use of in his experiments, I will cite a single passage from the memoir of Dr. FRANZ. It refers to the vapour of iodine produced by throwing the substance on a heated surface in a vessel closed with glass plates. "The mirror," he writes, "showed a deflection of only 178. But as the glass plates through which the heat radiated had not yet assumed a temperature high enough to reduce the iodine, which had been precipitated upon them in crystals, to a state of vapour . . . . . it was necessary to wait, and allow the radiation of the lamp to continue till all the iodine was driven from the bottle *." This shows how much the glass plates could be heated by the radiation of the lamp; this heat on a particular occasion being sufficient to dissipate the solid iodine which had coated the glass plates.

§ 14.

As a dam built across a river causes a local deepening of the stream, so our atmosphere, thrown as a dam across the terrestrial rays, produces a local heightening of the temperature at the earth's surface. This, of course, does not imply indefinite accumulation any more than the river dam does, the quantity lost by terrestrial radiation being, finally, equal to the quantity received from the sun. The chief intercepting substance is the aqueous vapour of the atmosphere †, the oxygen and nitrogen of which the great mass of the atmosphere is composed being sensibly transparent to the calorific rays. Were the atmosphere cleansed of its vapour, the temperature of space would be directly open to us; and could we under present circumstances reach an elevation where the amount of that vapour is insensible, we might determine the temperature of space by direct experiment. Colonel STRACHEY has written an admirable paper on the aqueous vapour of the atmosphere ‡, in which he shows that the amount of vapour diminishes much more speedily with the elevation than might be inferred from the law of DALTON.

It might be possible to reach a height where, by preserving one face of a thermo-electric pile at the temperature of the locality, the other, protected from all terrestrial radiation, turned to the zenith, would assume the temperature of space in that direction §, while the consequent galvanometric deflection would give us the means of determining the difference in temperature between the two faces of the pile. Knowing one, we should therefore be able to determine the other; knowing the temperature of the locality, we could infer from it the temperature of stellar space. Many eminent writers,

* "Es musste bei fortdauernder Strahlung der Lampe der Zeitpunct abgewartet werden."

† The mildness of an island climate must be in part due to this cause. The direct tendency of the vapour is to check sudden fluctuations of temperature. Where it is absent, as at the surface of the moon, such fluctuations must be enormous. The face turned towards the sun drinks in the solar rays without let or hindrance, while the radiation of the face turned from the sun pours unchecked into space.

‡ Proceedings of the Royal Society, vol. xi. p. 182.

§ A *well* of cold air would be formed within the reflector, the lowest stratum of the well sharing the temperature of the face of the pile.

it is true, have supposed the upper atmospheric regions to be colder than space, the depression of temperature being due to the radiation of the aërial particles, just as a grass-blade is lowered, by its radiation, below the air which surrounds it. This notion must, I think, be abandoned; for, as far as experiment goes, it leads us to conclude that air, and particularly air in the higher atmospheric regions, behaves as a vacuum both as regards radiation and absorption.

§ 15.

In his paper on the conduction of heat by gases, Professor MAGNUS examines the question of convection, and has adduced some striking experiments to show that the cooling of an incandescent wire in hydrogen is not due to the convection of the gas. He finds that when the wire is enclosed in a narrow tube, with only a thin film of the gas surrounding it, and where therefore currents, in the ordinary sense, are hard to be conceived of, the gas still exercises its cooling power. It had often occurred to me to make this experiment; and when I first heard of its successful performance by Professor MAGNUS I adopted his conclusion, that the cooling was due to conduction.

Reflection, however, caused me to change my opinion. Suppose the wire to be stretched along the axis of a wide cylinder containing hydrogen, we should have convection, in the ordinary sense, on heating the wire. Where does the heat thus dispersed ultimately go? It is manifestly given up to the sides of the cylinder. The transfer by convection is a transfer ultimately to the sides of the cylinder, and if we narrow our cylinder we simply hasten the transfer. The process of narrowing may continue till a tube like that used by Professor MAGNUS is the result; the convection between centre and sides will still continue, and produce the same cooling effect as before. Whether we assume conduction or convection, the tube surrounding the wire must be supposed to possess sufficient conducting power to carry the heat off, otherwise it would become incandescent itself by the accumulation of the heat.

The reasoning of Professor MAGNUS in connexion with this subject is of extreme ingenuity. He contends that there is no reason why stronger currents should establish themselves in hydrogen than in other gases. Currents are due to differences of density produced by the expansion of a portion of the gas by heat. Now hydrogen actually expands *less* than other gases, and hence the differential action on which the currents depend is less in this gas than in the others. Professor MAGNUS alludes to the friction of the particles against each other, but considers this ineffective.

This reasoning leads us to the threshold of a question which might form the subject of a long and profitable investigation. For a given difference of density, is not the mobility of hydrogen greater than that of the other gases? The experiments above recorded, where different gases were brought into direct contact with the source of heat, seem to answer this question in the affirmative. I have had no time to pursue the question regarding hydrogen; but I have made a few experiments, which show the influence of density on the mobility of a gas in a very striking manner.

Having first so purified atmospheric air as to render it sensibly neutral to radiant heat, I allowed 15 inches of it to enter the front chamber F, and there to come into contact with the source of heat. Convection of course immediately set in, and its amount was accurately measured by the quantity of heat withdrawn from the radiating surface; this quantity, expressed in the units adopted throughout this memoir, was

62.

The quantity of gas in the front chamber was now doubled, that is, it now had an atmosphere of tension; the withdrawal of heat then was expressed by the number

68.

In the last experiment we had double the number of atoms loading themselves with heat and carrying it away; if their motion had been as quick as that of the atoms when half an atmosphere was used, they would have withdrawn sensibly double the amount of heat; but the fact is that half an atmosphere carried off 64, while a whole atmosphere carried off 68; hence the absolute swiftness of the atoms in the case of the denser air must be very much less than in the case of the rarer. In fact, the amount of heat withdrawn will be proportional on the one hand to the number of carrying particles, and on the other to the velocity with which they move; hence if v and v' be these velocities, we have

$$\frac{62}{68}=\frac{v}{2v'}, \text{ or } \frac{v}{v'}=\frac{62}{34}.$$

Thus, while the atoms of the rarer gas travel 62 units in a second, those of the denser gas travel only 34.

This retardation can, I think, arise from nothing else than the resistance offered by the particles of the air to the motion of their fellows. It must be borne in mind that the smallness of the increment observed on doubling the amount of gas was not due to the partial exhaustion of the source by the first quantity of gas. The heat of the source was such that the withdrawal of 64 of our units could not sensibly affect the subsequent convection.

Here, then, we see what a powerful effect density, or the internal resistance which accompanies density, has on the mobility of a gas; and there is every reason to suppose that the mobility of hydrogen is due to the comparative absence, in its case, of internal resistance. However this may be, the foregoing experiment enables us to draw some important inferences.

Storms at great heights must be greatly facilitated by the mobility of the particles of the air. In fact storms are cases of convection on a large scale, and in our front chamber we had one in miniature. With the same difference of temperature on the summit of Mont Blanc, the motion of convection would be very nearly twice as great as at the sea-level.

In the summer of 1859 I was fortunate enough to induce my friend Professor Frankland to accompany me to the summit of Mont Blanc, and to determine the comparative rates of combustion there and in the valley of Chamouni. Six candles were purchased, burnt for an hour at Chamouni, and the loss of weight determined. The same candles were lighted for the same time on the summit of the mountain, and the consumption determined. Within the limits of error, the consumption above was equal to that below. The *light* below was immensely greater than that above, still the amount of stearine consumed in the two cases was sensibly the same. Professor FRANKLAND surmised this to be due to the greater mobility of the rarefied air, which allowed a freer interpenetration of the flame by the oxygen *, and the foregoing experiments show that the augmentation of mobility is just such as would account for the observed effect.

* The influence of interpenetration is well seen in the exposed gas-jets of London, particularly in the butchers' shops on a Saturday night. A gust of wind, which carries oxygen to the centre of a flame, suddenly deprives it of light. A simple and beautiful experiment consists of passing a lighted candle swiftly to and fro through the air; the white light reduces itself to a pale-blue band. BUNSEN's burner is an illustration in the same line.

VI. *On the Calculus of Symbols.*

By WILLIAM SPOTTISWOODE, *M.A.*, *F.R.S.*

Made up of two Memoirs, one received November 4, Read November 21, 1861, the other received January 21, read January 30, 1862.

IN a paper published in the Philosophical Transactions for 1861, p. 79, Mr. W. H. L. RUSSELL has constructed systems of multiplication and division for functions of non-commutative symbols, subject to the same laws of combination as those in Professor BOOLE'S memoir "On a General Method in Analysis," Philosophical Transactions, 1844. In this calculus there are four systems of multiplication and division, viz. internal and external, both (1) when the functions are arranged in powers of ϱ, (2) when in powers of π, or, as they may be termed, the ϱ-arrangement, and the π-arrangement. In the paper in question the author has confined himself, so far as division is concerned, to the case most useful in practice, in which the divisor is linear. And of this he has discussed in full only the ϱ-arrangement.

§ 1. *Internal division of the π-arrangement by a linear factor.*

Adopting the same notation as Mr. RUSSELL, I propose here to investigate, in the first place, the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an *internal* factor of

$$\varphi_n(\varrho)\pi^n+\varphi_{n-1}(\varrho)\pi^{n-1}+\ldots\varphi_0(\varrho),$$

and to determine the quotient. This is partially discussed in pp. 73–75.

Let
$$\varrho\frac{d}{d\varrho}\psi=\psi',$$

then performing the actual divisions, for brevity writing ψ for $\psi(\varrho)$, and φ for $\varphi(\varrho)$,

$$\psi_1\pi+\psi_0\Big)\,\varphi_1\pi+\varphi_0\left(\frac{\varphi_1}{\psi_1}\right.$$

$$\varphi_1\pi+\varphi_1\frac{\psi_0}{\psi_1}$$

$$\overline{\varphi_0-\varphi_1\frac{\psi_0}{\psi_1}.}$$

Hence the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ will be

$$\varphi_0(\varrho)-\varphi_1(\varrho)\frac{\psi_0}{\psi_1}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Again,

$$
\begin{array}{l}
\psi_1\pi+\psi_0\,\big)\,\varphi_2\pi^2+\varphi_1\pi+\varphi_0\,\Big(\dfrac{\varphi_2}{\psi_1}\pi+\Big(\varphi_1-\varphi_2\dfrac{\psi_1'-\psi_0}{\psi_1}\Big)\dfrac{1}{\psi_1}\\
\qquad\varphi_2\pi^2+\varphi_2\dfrac{\psi_1'}{\psi_1}\pi+\varphi_2\dfrac{\psi_0}{\psi_1}\pi+\varphi_2\dfrac{\psi_0'}{\psi_1}\\
\qquad\Big(\varphi_1-\varphi_2\dfrac{\psi_1'+\psi_0}{\psi_1}\Big)\pi+\varphi_0-\varphi_2\dfrac{\psi_0'}{\psi_1}\\
\qquad\Big(\varphi_1-\varphi_2\dfrac{\psi_1'+\psi_0}{\psi_1}\Big)\pi+\Big(\varphi_1-\varphi_2\dfrac{\psi_1'+\psi_0}{\psi_1}\Big)\dfrac{\psi_0}{\psi_1}\\
\qquad\qquad\varphi_0-\varphi_1\dfrac{\psi_0}{\psi_1}+\varphi_2\Big\{\Big(\dfrac{\psi_0}{\psi_1}\Big)^2-\dfrac{\psi_0'\psi_1-\psi_0\psi_1'}{\psi_1^2}\Big\}.
\end{array}
$$

Hence the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_2(\varrho)\pi^2+\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ will be

$$
\varphi_0(\varrho)-\varphi_1(\varrho)\frac{\psi_0(\varrho)}{\psi_1(\varrho)}+\varphi_2\Big\{\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)^2-\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)'\Big\}=0. \quad \ldots\ldots \quad (2.)
$$

Before proceeding further, we may remark that the remainder, after internally dividing $\varphi_3(\varrho)\pi^3+\varphi_2(\varrho)\pi^2+\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ by $\psi_1(\varrho)\pi+\psi_0(\varrho)$, can differ from that last above found only in respect of the remainder arising from the division of $\varphi_3(\varrho)\pi^3$ by the factor in question; hence we have now only to divide the term $\varphi_3(\varrho)\pi^3$ by $\psi_1(\varrho)\pi+\psi_0(\varrho)$, and add the remainder so found to (2.), in order to have the condition required for the third degree. Proceeding to the division, writing $\chi=\frac{\psi_0}{\psi_1}$, and omitting for the present φ_3, which, since the division is internal, can be replaced as an external factor in the remainder, we have

$$
\begin{array}{l}
\pi+\chi\,\big)\,\pi^3\,\big(\pi^2-\chi\pi+\chi^2-2\chi'\\
\qquad\pi^3+\chi\pi^2+2\chi'\pi+\chi''\\
\qquad\quad-\chi\pi^2-2\chi'\pi-\chi''\\
\qquad\quad-\chi\pi^2-\chi^2\pi-\chi\chi'\\
\qquad\qquad(\chi^2-2\chi')\pi+\chi\chi'-\chi''\\
\qquad\qquad(\chi^2-2\chi')\pi+\chi^3-2\chi\chi'\\
\qquad\qquad\qquad-\chi^3+3\chi\chi'-\chi''.
\end{array}
$$

Hence the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_3(\varrho)\pi^3+\varphi_2(\varrho)\pi^2+\varphi_1(\varrho)\pi+\varphi_0(\varrho)$ will be

$$
\varphi_0(\varrho)-\varphi_1(\varrho)\frac{\psi_0(\varrho)}{\psi_1(\varrho)}+\varphi_2(\varrho)\Big\{\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)^2-\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)'\Big\}-\varphi_3(\varrho)\Big\{\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)^3-3\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)'+\Big(\frac{\psi_0(\varrho)}{\psi_1(\varrho)}\Big)''\Big\}=0, \quad (3.)
$$

the identity of which with Mr. RUSSELL's condition, given in p. 75 of his paper, I have verified.

For the fourth degree,

$$\begin{array}{l}
\pi+\chi)\pi^4(\pi^3-\chi\pi^2+(\chi^2-3\chi')\pi-(\chi^3-5\chi\chi'+3\chi'') \\
\underline{\pi^4+\chi\pi^3+3\chi'\pi^2+3\chi''\pi+\chi'''} \\
-\chi\pi^3-3\chi'\pi^2-3\chi''\pi-\chi''' \\
\underline{-\chi\pi^3-\chi^2\pi^2-2\chi\chi'\pi-\chi\chi''} \\
(\chi^2-3\chi')\pi^2+(2\chi\chi'-3\chi'')\pi+(\chi\chi''-\chi''') \\
\underline{(\chi^2-3\chi')\pi^2+(\chi^3-3\chi\chi')\pi+(\chi^2\chi'-3\chi'^2)} \\
-(\chi^3-5\chi\chi'+3\chi'')\pi-\chi^2\chi'+3\chi'^2+\chi\chi''-\chi''' \\
\underline{-(\chi^3-5\chi\chi'+3\chi'')\pi-\chi^4+5\chi^2\chi'-3\chi\chi''} \\
\underline{\underline{\chi^4-6\chi^2\chi'-4\chi\chi''+3\chi\chi'^2-\chi'''}}
\end{array}$$

and

$$\chi^4-6\chi^2\chi'-4\chi\chi''+3\chi'^2-\chi'''=-\chi(-\chi^3+3\chi\chi'-\chi'')+(-\chi^3+3\chi\chi'-\chi'')';$$

or if R_1, R_2, .. be the remainders of $\pi(\pi+\chi)^{-1}$, $\pi^2(\pi+\chi)^{-1}$, .., we have

$$R_2=-\chi R_1+R'_1,$$

$$R_3=-\chi R_2+R'_2,$$

$$R_4=-\chi R_3+R'_3.$$

And generally, if

$$\pi^n\ (\pi+\chi)^{-1}=\ Q_n+\ R_n(\pi+\chi)^{-1},$$

then

$$\pi^{n+1}(\pi+\chi)^{-1}=\pi Q_n+\pi R_n(\pi+\chi)^{-1},$$

the remainder of which must be contained in the last term. Performing the actual division, and remembering that $\pi R_n=R_n\pi+R'_n$,

$$\begin{array}{l}
\pi+\chi)R_n\pi+R'_n(R_n \\
\underline{R_n\pi+R_n\chi} \\
\underline{\underline{-\chi R_n+R'_n.}}
\end{array}$$

Hence we have generally,

$$R_{n+1}=-\chi R_n+R'_n,$$

and consequently, remembering that $R_0=1$, we have the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an internal factor of $\varphi_n(\varrho)\pi^n+\varphi_{n-1}(\varrho)\pi^{n-1}+\ldots\varphi_0(\varrho)$,

$$\varphi_0(\varrho)R_0+\varphi_1(\varrho)R_1+\ldots\varphi_n(\varrho)R_n=0, \quad \ldots\ldots\ldots\ldots \quad (4.)$$

where

$$R_{i+1}=-\frac{\psi_0(\varrho)}{\psi_1(\varrho)}R_i+\varrho\frac{d}{d\varrho}R_i=\left(-\frac{\psi_0(\varrho)}{\psi_1(\varrho)}+\varrho\frac{d}{d\varrho}\right)R_i.$$

The law of the quotients is best seen by actual division. In case of $\varphi_2\pi^2+\varphi_1\pi+\varphi_0$, given above, the quotient may be written

$\frac{1}{\psi_1}\times$

φ_1	φ_2	
0	1	π
1	$-\frac{1}{\psi_1}(\psi_1'+\psi_0)$	1

. . . . (5.)

For the case of a cubic function of π,

$$\psi_1\pi+\psi_0)\varphi_3\pi^3+\varphi_2\pi^2+\varphi_1\pi+\varphi_0\Big(\varphi_3\frac{1}{\psi_1}\pi^2+\Big(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\Big)\frac{1}{\psi_1}\pi+\Big(\varphi_1-\varphi_2\frac{\psi_1'+\psi_0}{\psi_1}+\varphi_3\frac{2\psi_1'^2+3\psi_0\psi_1'+\psi_0^2-\psi_1\psi''-2\psi_1\psi'}{\psi_1^2}$$

$$\frac{\varphi_3\pi^3+2\varphi_3\frac{\psi_1'}{\psi_1}\pi^2+\varphi_3\frac{\psi_1''}{\psi_1}\pi+\varphi_3\frac{\psi_0}{\psi_1}\pi^2+2\varphi_3\frac{\psi_0'}{\psi_1}\pi+\varphi_3\frac{\psi_0''}{\psi_1}}{\left(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\right)\pi^2+\left(\varphi_1-\varphi_3\frac{\psi_1''+2\psi_0'}{\psi_1}\right)\pi+\left(\varphi_0-\varphi_3\frac{\psi_0''}{\psi_1}\right)}$$

$$\frac{\left(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\right)\pi^2+\left(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\right)\frac{\psi_1'}{\psi_1}\pi+\left(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\right)\frac{\psi_0}{\psi_1}\pi+\left(\varphi_2-\varphi_3\frac{2\psi_1'+\psi_0}{\psi_1}\right)\frac{\psi_0'}{\psi_1}}{\left(\varphi_1-\varphi_2\frac{\psi_1'+\psi_0}{\psi_1}+\varphi_3\frac{2\psi_1'^2+3\psi_0\psi_1'+\psi_0^2-\psi_1\psi_1''-2\psi_1\psi_0'}{\psi_1^2}\right)\pi+\left(\varphi_0-\varphi_2\frac{\psi_0'}{\psi_1}+\varphi_3\frac{2\psi_1'\psi_0'+\psi_0\psi_0'-\psi_1\psi_0''}{\psi_1^2}\right)}.$$

The quotient of which may be written

$\frac{1}{\psi_1}\times$

φ_1	φ_2	φ_3	
0	0	1	π^2
0	1	$-\frac{1}{\psi_1}(2\psi_1'+\psi_0)$	π
1	$-\frac{1}{\psi_1}(\psi_1'+\psi_0)$	$\frac{1}{\psi_1^2}\begin{vmatrix}2\psi_1'+\psi_0 & \psi_1\\ \psi_1''+2\psi_0' & 2\psi_1'+\psi_0\end{vmatrix}$	1

. (6.)

Similarly, if the division be performed in the case of the quartic function, we shall find for the quotient of $(\varphi_4\pi^4+\varphi_3\pi^3+\varphi_2\pi^2+\varphi_1\pi+\varphi_0)(\psi_1\pi+\psi_0)^{-1}$,

$\frac{1}{\psi_1}\times$

φ_1	φ_2	φ_3	φ_4	
0	0	0	1	π^3
0	0	1	$-\frac{1}{\psi_1}(3\psi_1'+\psi_0)$	π^2
0	1	$-\frac{1}{\psi_1}(2\psi_1'+\psi_0)$	$\frac{1}{\psi_1^2}\begin{vmatrix}3\psi_1'+\psi_0 & \psi_1\\ 3\psi_1''+3\psi_0' & 2\psi_1'+\psi_0\end{vmatrix}$	π
1	$-\frac{1}{\psi_1}(\psi_1'+\psi_0)$	$\frac{1}{\psi_1^2}\begin{vmatrix}2\psi_1'+\psi_0 & \psi_1\\ \psi_1''+2\psi_0' & \psi_1'+\psi_0\end{vmatrix}$	$-\frac{1}{\psi_1^3}\begin{vmatrix}3\psi_1'+\psi_0 & \psi_1 & 0\\ 3\psi_1''+3\psi_0' & 2\psi_1'+\psi_0 & \psi_1\\ \psi_1'''+3\psi_0'' & \psi_1''+\psi_0' & \psi_1'+\psi_0\end{vmatrix}$	1

And likewise in general the quotient of $(\varphi_n\pi^n+\varphi_{n-1}\pi^{n-1}+\ldots\varphi_0)(\psi_1\pi+\psi_0)^{-1}$ will be represented by a square table giving for the coefficient of φ_n

$$(-)^{n-1}\frac{1}{\psi_1^{n-1}}\times\left|\begin{array}{cccc} \frac{i-1}{1}\psi_1'+\psi_0 & \psi_1 & \ldots & 0 \\ \frac{(i-1)(i-2)}{1.2}\psi_1''+\frac{i-1}{1}\psi_0' & \frac{i-2}{1}\psi_1'+\psi_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \psi_1^{(i-1)}+\frac{i-1}{1}\psi_0^{(i-2)} & \psi_1^{(i-2)}+\frac{i-2}{1}\psi_0^{(i-3)} & \ldots & \psi_1'+\psi_0 \end{array}\right| \quad (8.)$$

§ 2. *External division of the π-arrangement by a linear factor.*

I next investigate the condition that $\psi_1(\varrho)\pi+\psi_0(\varrho)$ may be an *external* factor of

$$\varphi_n(\varrho)\pi^n+\varphi_{n-1}(\varrho)\pi^{n-1}+\ldots\varphi_0(\varrho).$$

Performing the actual divisions, we have in the case of $n=1$,

$$\begin{array}{r} \psi_1\pi+\psi_0\big)\,\varphi_1\pi+\varphi_0\,\Big(\frac{\varphi_1}{\psi_1} \\ \varphi_1\pi+\psi_1\left(\frac{\varphi_1}{\psi_1}\right)'+\psi_0\frac{\varphi_1}{\psi_1} \\ \hline \varphi_0-\psi_1\left(\frac{\varphi_1}{\psi_1}\right)'-\psi_0\frac{\varphi_1}{\psi_1}; \\ \hline \end{array}$$

or, as the remainder may be more conveniently written,

$$\varphi_0-\varphi_1'+\varphi_1\frac{\psi_1'-\psi_0}{\psi_1}\ldots\ldots\ldots\ldots \quad (1.)$$

Again, in the case of $n=2$,

$$\begin{array}{r} \psi_1\pi+\psi_0\big)\,\varphi_2\pi^2+\varphi_1\pi+\varphi_0\,\Big(\frac{\varphi_2}{\psi_1}\pi+\frac{1}{\psi_1}\left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\} \\ \varphi_2\pi^2+\psi_1\left(\frac{\varphi_2}{\psi_1}\right)'\pi+\psi_0\frac{\varphi_2}{\psi_1}\pi \\ \hline \left\{\varphi_1-\psi_1\left(\frac{\varphi_2}{\psi_1}\right)'-\psi_0\frac{\varphi_2}{\psi_1}\right\}\pi+\varphi_0; \end{array}$$

or, transforming the remainder as in the former case, and continuing the division,

$$\begin{array}{l} \left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\}\pi+\varphi_0 \\ \left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\}\pi+\psi_1\left[\frac{1}{\psi_1}\left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\}\right]'+\frac{\psi_0}{\psi_1}\left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\} \\ \hline \qquad\qquad \varphi_0-\psi_1\left[\frac{1}{\psi_1}\left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\}\right]'-\frac{\psi_0}{\psi_1}\left\{\varphi_1-\varphi_2'+\varphi_2\frac{\psi_1'-\psi_0}{\psi_1}\right\}, \end{array}$$

which also may be transformed as follows:—

$$\phi_0-\phi'_1+\phi''_2+(\phi_1-2\phi'_2)\frac{\psi'_1-\psi_0}{\psi_1}+\phi_2\left\{\left(\frac{\psi'_1-\psi_0}{\psi_1}\right)^2-\left(\frac{\psi'_1-\psi_0}{\psi_1}\right)'\right\}. \quad . \quad . \quad . \quad (2.)$$

A similar process of division will be found, in the case of $n=3$, to lead to the following remainder:—

$$\left.\begin{array}{l}\phi_0-\phi'_1+\phi''_2-\phi'''_3+(\phi_1-2\phi'_2+3\phi''_3)\frac{\psi'_1-\psi_0}{\psi_1}+(\phi_2-3\phi'_3)\left\{\left(\frac{\psi'_1-\psi_0}{\psi_1}\right)^2-\left(\frac{\psi'_1-\psi_0}{\psi_1}\right)'\right\} \\ \quad +\phi_3\left\{\left(\frac{\psi'_1-\psi_0}{\psi_1}\right)^3-3\left(\frac{\psi'_1-\psi_0}{\psi_1}\right)\left(\frac{\psi'_1-\psi_0}{\psi_1}\right)'+\left(\frac{\psi'_1-\psi_0}{\psi_1}\right)''\right\}.\end{array}\right\} \quad (3.)$$

If Ψ_1, Ψ_2, .. represent the ψ-functions, coefficients of the ϕs in this expression, the law of their formation will be found to be as follows:—

$$\Psi_1 \;=\frac{\psi'_1-\psi_0}{\psi_1},$$
$$\Psi_2 \;=\Psi_1^2-\Psi'_1,$$
$$\Psi_3 \;=\Psi_1\Psi_2-\Psi'_2,$$
$$. \quad . \quad . \quad . \quad . \quad .$$

And generally we may write

$$\Psi_{i+1}=\Psi_1\Psi_i-\Psi'_i.$$

And if R_1, R_2, .. represent the remainders in the cases of $n=1$, $n=2$, .. respectively, we have

$$R_1=\phi_0-\phi'_1+\phi_1\Psi_1,$$
$$R_2=\phi_0-\phi'_1+\phi''_2+(\phi_1-2\phi'_2)\Psi_1+\phi_2\Psi_2,$$
$$R_3=\phi_0-\phi'_1+\phi''_2-\phi'''_3+(\phi_1-2\phi'_2+3\phi''_3)\Psi_1+(\phi_2-3\phi'_3)\Psi_2+\phi_3\Psi_3;$$

whence

$$R_2=R_1+\phi''_2-2\phi'_2\Psi_1+\phi_2\Psi_2,$$
$$R_3=R_2-\phi'''_3+3\phi''_3\Psi_1-3\phi'_3\Psi_2+\phi_3\Psi_3.$$

With a view to forming the expression for R_n, let the symbol $\binom{1}{0}$ affixed to R_i signify that in the expression for R_n the suffixes of the ϕs have all been increased by unity. Then, by the principle of division,

$$\begin{aligned}R_{n+} &=\phi_0-\psi_1\left[\frac{1}{\psi_1}R_n\binom{1}{0}\right]'-\frac{\psi_0}{\psi_1}R_n\binom{1}{0} \\ &=\phi_0-R'_n\binom{1}{0}+\Psi_1R_n\binom{1}{0} \\ &=\phi_0-\left(R_{n-1}-\phi_n^{(n)}+\frac{n}{1}\phi_n^{(n-1)}\Psi_1-\frac{n(n-1)}{1.2}\phi_n^{(n-2)}\Psi_2+..\right)'\binom{1}{0} \\ &\qquad +\Psi_1\left(R_{n-1}-\phi_n^{(n)}+\frac{n}{1}\phi_n^{(n-1)}\Psi_1-\frac{n(n-1)}{1.2}\phi_n^{(n-2)}\Psi_2+..\right)\binom{1}{0} \\ &=\phi_0-\left(R'_{n-1}-\Psi_1R_{n-1}\right)\binom{1}{0}+\phi_{n+1}^{(n+1)}-\frac{n+1}{1}\phi_{n+1}^{(n)}+..;\end{aligned}$$

and in this expression

$$\phi_0-\left(R'_{n-1}-\Psi_1 R_{n-1}\right)\binom{1}{0}=R_n,$$

while the general term of the ϕ-series, *i. e.* the coefficient of $\phi_{n+1}^{(n-r)}$, will be

$$(-)^{r-1}\left\{\frac{n(n-1)..(n-r+1)}{1.2..r}\Psi'_r-\frac{n(n-1)..(n-r)}{1.2..(r+1)}\Psi_{r+1}-\frac{n(n-1)..(n-r+1)}{1.2..r}\Psi_1\Psi_r\right\}$$

$$=(-)^{r-1}\frac{n(n-1)..(n-r+1)}{1.2..r}\left\{\Psi'_r-\Psi_1\Psi_r-\frac{n-r}{r+1}\Psi_{r+1}\right\}$$

$$=(-)^{r}\frac{(n+1)n(n-1)..(n-r+1)}{1.2..(r+1)}\Psi_{r+1},$$

which proves the general case; so that generally

$$R_{n+1}=R_n\pm\phi_{n+1}^{(n+1)}\mp\frac{n+1}{1}\phi_{n+1}^{(n)}\Psi_1\pm\frac{(n+1)n}{1.2}\phi_{n+1}^{(n)}\Psi_2\mp..,\quad\ldots\ldots\quad(4.)$$

the upper or lower sign being taken according as $(n+1)$ is even or odd; where R_{n+1} is the remainder after external division $\phi_{n+1}(\varrho)\pi^{n+1}+\phi_n(\varrho)\pi^n+..\phi_0$ by $\psi_1(\varrho)\pi+\psi_0(\varrho)$.

For the quotients Q_1, Q_2, .. we have immediately

$$Q_1=\frac{1}{\psi_1}\phi_1,$$

$$Q_2=\frac{1}{\psi_1}\left\{\phi_2\pi+R_1\binom{1}{0}\right\},$$

$$Q_3=\frac{1}{\psi_1}\left\{\phi_3\pi^2+R_1\binom{2}{0}\pi+R_2\binom{1}{0}\right\}.$$

.. ...

$$Q_n=\frac{1}{\psi_1}\left\{\phi_n\pi^{n-1}+R_1\binom{n-1}{0}\pi^{n-2}+R_2\binom{n-2}{0}\pi^{n-3}+..R_{n-1}\binom{1}{0}\right\}.$$

This completes the solution of the problem of division by a linear factor, both internal and external.

§ 3. *To divide* $\Sigma_{n=0}^{n=N}\phi_n\pi^n$ *internally by* $\Sigma_{m=0}^{m=M}\psi_m\pi^m$.

The first term in the quotient will obviously be

$$\frac{\phi_N}{\psi_M}\pi^{N-M},\quad\ldots\ldots\quad(1.)$$

and the product of this into the divisor may, by means of LEIBNITZ's theorem, be written thus:

$$\frac{\phi_N}{\psi_m}\Sigma_{p=0}^{p=N-M}[N-M,p]\ \Sigma_{m=0}^{m=M}\psi_m^{(N-M-p)}\pi^{m+p},\quad\ldots\ldots\quad(2.)$$

where $\psi_m^{(N-M-p)}$ means the result of the operation π^{N-M-p} or ψ_m alone, and

$$[N-M,p]=\frac{(N-M)(N-M-1)..(N-M-p+1)}{1.2..p}.$$

Then the remainder after subtraction from the dividend may be written thus:

$$\Sigma_{m+p=0}^{m+p=N-1}\left\{\phi_{m+p}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\phi_N[N-M,p]\psi_m^{(N-M-p)}\right\}\pi^{m+p}, \quad . \quad . \quad . \quad (3.)$$

since the coefficient of π^N vanishes. With a view to the second term in the dividend, the first term of the remainder (3.) is

$$\left\{\phi_{N-1}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\phi_N[N-M,p]\psi_m^{(N-M-p)}\right\}\pi^{N-1}, \quad . \quad . \quad . \quad . \quad (4.)$$

in which the limits of p and m are subject to the further condition $p+m=N-1$. The terms under the sign of summation will be evaluated hereafter. Putting the expression

$$(4)=\Phi_1\pi^{N-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

and, for the sake of symmetry,

$$\phi_N=\Phi_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

the first and second terms in the quotient will be $\frac{\Phi_0}{\psi_M}\pi^{N-M}$, and $\frac{\Phi_1}{\psi_M}\pi^{N-M-1}$, respectively; and, in the same manner as (2.), the product of the second term of the quotient into the divisor may be written thus,

$$\frac{\Phi_1}{\psi_M}\Sigma_{p_1=0}^{p_1=N-M-1}\Sigma_{m=0}^{m=M}[N-M-1,p]\psi_m^{(N-M-p-1)}\pi^{m+p_1}, \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

and the remainder thus:

$$\left.\begin{array}{l}\Sigma_{m+p=0}^{m+p=N-2}\left\{\phi_{m+p}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\Phi_0[N-M,p]\psi_m^{(N-M-p)}\right\}\pi^{m+p}\\ -\Sigma_{m+p_1=0}^{m+p_1=N-2}\left\{\frac{1}{\psi_M}\Sigma_{p_1=0}^{p_1=N-M-1}\Sigma_{m=0}^{m=M}\Phi_1[N-M-1,p_1]\psi_m^{(N-M-p_1-1)}\right\}\pi^{m+p_1}.\end{array}\right\} \quad . \quad (8.)$$

But since, when $p_1=N-M$, $[N-M-1,p_1]=0$, we may, without altering the value of (8.), change the superior limit of p_1 from $N-M-1$, to $N-M$; and by this means we may write the remainder (8.) in the following form:

$$\Sigma_{m+p=0}^{m+p=N-2}\left\{\phi_{m+p}-\frac{1}{\psi_m}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}(\Phi_0[N-M,p]\psi_m^{(N-M-p)}+\Phi_1[N-M-1,p]\psi_m^{(N-M-p-1)})\right\}\pi^{m+p}. \quad (9.)$$

Similarly, calling the first term of (9.) $\Phi_2\pi^{N-2}$, the third term in the quotient will be $\frac{\Phi_2}{\psi_M}\pi^{N-M-2}$, and the corresponding remainder

$$\left.\begin{array}{l}\Sigma_{m+p=0}^{m+p=N-3}\left\{\phi_{m+p}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}(\Phi_0[N-M,p]\psi_m^{(N-M-p)}+\Phi_1[N-M-1,p]\psi_m^{(N-M-p-1)}\right.\\ \qquad\left.+\Phi_2[N-M-2,p]\psi_m^{(N-M-p-2)})\right\}\pi^{m+p};\end{array}\right\}(10.)$$

and so generally the $(r+1)$th term in the quotient will be $\frac{\Phi_r}{\psi_M}\pi^{N-M-r}$, where

$$\Phi_r=\phi_{N-r}-\frac{1}{\psi_M}\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\Sigma_{q=0}^{q=r-1}\Phi_q[N-M-q,p]\psi_m^{(M-N-p-q)}, \quad . \quad . \quad . \quad . \quad (11.)$$

and the $(r+1)$th remainder

$$\Sigma_{m+p=0}^{m+p=N-r-1}\left\{\varphi_{m+p}-\Sigma_{p=0}^{p=N-M}\Sigma_{m=0}^{m=M}\Sigma_{q=0}^{q=r}\Phi_q[N-M-q,p]\psi_m^{(N-M-p-q)}\right\}\pi^{m+p}. \quad . \quad . \quad (12.)$$

The final remainder is the $(N-M+1)$th; and the expression will be derived from (10.) by replacing r by $N-M$.

It remains to develope the terms under the sign of summation in the expressions for the Φs. In the first place $\Phi_0=\varphi_N$ simply. In the case of Φ_1, the limiting values of p and m are

$$p=0,\ 1,\ ..\ N-M,$$
$$m=0,\ 1,\ ..\ M,$$
$$p+m=N-1.$$

These give as the only admissible values

$$\begin{array}{lll} p=N-M & , N-M-1, \\ m= \quad M-1, & M \quad , \end{array}$$

and consequently

$$\Phi_1=\varphi_{N-1}-\frac{\Phi_0}{\psi_M}\{\psi_{M-1}+(N-M)\psi'_M\}.$$

In the case of Φ_2, the only admissible values are

$$\begin{array}{llll} p=N-M & , N-M-1, & N-M-2, \\ m= \quad M-2, & M-1, & M \quad , \end{array}$$

giving

$$\Phi_2=\varphi_{N-2}-\frac{1}{\psi_M}\left\{\Phi_0\left(\psi_{M-2}+(N-M)\psi'_{M-1}+\frac{(N-M)(N-M-1)}{1.2}\psi''_M\right)\right.$$
$$\left.+\Phi_1(\psi_{M-1}+(N-M-1)\psi'_M)\right\}.$$

Before proceeding further, it may be well to illustrate these formulæ by an example.

Taking the case of $N=4$, $M=3$, we may determine the quotient and last remainder of internal division of

$$\varphi_4(\varrho)\pi^4+\varphi_3(\varrho)\pi^3+\varphi_2(\varrho)\pi^2+\varphi_1(\varrho)\pi+\varphi_0(\varrho)$$

by

$$\psi_3(\varrho)\pi^3+\psi_2(\varrho)\pi^2+\psi_1(\varrho)\pi+\psi_0(\varrho),$$

and thence the conditions that the latter may be an internal factor of the former.

By the formulæ given above, we have

$$\Phi_0=\varphi_4,$$
$$\Phi_1=\varphi_3-\frac{\varphi_4}{\psi_2}\{\psi_1+2\psi'_2\},$$
$$\Phi_2=\varphi_2-\frac{1}{\psi_2}\left\{\varphi_4(\psi_0+2\psi'_1+\psi''_2)+\left(\varphi_3-\frac{\varphi_4}{\psi_2}(\psi_1+2\psi'_2)\right)(\psi_1+\psi'_2)\right\},$$

which will determine the quotient

$$\frac{1}{\psi_2}(\Phi_0\pi^2+\Phi_1\pi+\Phi_2).$$

The last remainder is

$$\Sigma_{m+p=0}^{m+p=1}\left\{\varphi_{m+p}-\frac{1}{\psi_2}\Sigma_{p=0}^{p=2}\Sigma_{m=0}^{m=2}\Sigma_{q=0}^{q=2}\Phi_q[N-M-q,p]\psi_m^{(N-M-p-q)}\right\}\pi^{m+p};$$

whence for

$$m+p=1, \text{ i. e. } p=0,\ m=1, \text{ or } p=1,\ m=0,$$

we have

	$q=0$	$q=1$	$q=2$
$p=0$,	$[2, 0]=1$,	$[1, 0]=1$,	$[0, 0]=1$,
$p=1$,	$[2, 1]=2$,	$[1, 1]=1$,	$[0, 1]=0$.

Hence for $m=1$ the above expression gives

$$\Phi_0\psi_1''+\Phi_1\psi_1'+\Phi_2\psi_1,$$

and for $m=0$ it gives

$$2\Phi_0\psi_0'+\Phi_1\psi_0.$$

Again, for $m+p=0$, *i. e.* $p=0$, $m=0$, we have

$$\Phi_0\psi_0''+\Phi_1\psi_0'+\Phi_2\psi_0.$$

Hence the total remainder is

$$\left\{\varphi_1-\frac{1}{\psi_2}(\Phi_0\psi_1''+\Phi_1\psi_1'+\Phi_2\psi_1+2\Phi_0\psi_0'+\Phi_1\psi_0)\right\}\pi+\left\{\varphi_0-\frac{1}{\psi_2}(\Phi_0\psi_0''+\Phi_1\psi_0'+\Phi_2\psi_0)\right\}.$$

It may be useful to compare these results with the actual division, in the above example.

$$\psi_2\pi^2+\psi_1\pi+\psi_0\Big)\,\varphi_4\pi^4+\varphi_3\pi^3+\varphi_2\pi^2+\varphi_1\pi+\varphi_0\Big(\frac{\Phi_0}{\psi_2}\pi^2+\frac{\Phi_1}{\psi_2}\pi+\frac{\Phi_2}{\psi_2}$$

$$\left.\begin{array}{l}\varphi_4\pi^4+2\varphi_4\frac{\psi_2'}{\psi_2}\pi^3+\varphi_4\frac{\psi_2''}{\psi_2}\pi^2\\ \quad+\varphi_4\frac{\psi_1}{\psi_2}\pi^3+2\varphi_4\frac{\psi_1'}{\psi_2}\pi^2+\varphi_4\frac{\psi_1''}{\psi_2}\pi\\ \qquad+\varphi_4\frac{\psi_0}{\psi_2}\pi^2+2\varphi_4\frac{\psi_0'}{\psi_2}\pi+\varphi_4\frac{\psi_0''}{\psi_2}\end{array}\right\}$$

$$\Phi_1\pi^3+\left(\varphi_2-\frac{\varphi_4}{\psi_2}(\psi_2''+2\psi_1'+\psi_0)\right)\pi^2+\left(\varphi_1-\frac{\varphi_4}{\psi_2}(\psi_1''+2\psi_0')\right)\pi+\left(\varphi_0-\frac{\varphi_4}{\psi_2}\psi_0''\right)$$

$$\left.\begin{array}{l}\Phi_1\pi^3+\Phi_1\frac{\psi_2'}{\psi_2}\pi^2\\ \quad+\Phi_1\frac{\psi_1}{\psi_2}\pi^2+\Phi_1\frac{\psi_1'}{\psi_2}\pi\\ \qquad+\Phi_1\frac{\psi_0}{\psi_2}\pi+\Phi_1\frac{\psi_0'}{\psi_2}\end{array}\right\}$$

$$\Phi_2\pi^2+\left\{\varphi_1-\frac{1}{\psi_2}\Phi_0(\psi_1''+2\psi_0')-\frac{1}{\psi_2}\Phi_1(\psi_1'+\psi_0)\right\}\pi+\left\{\varphi_0-\frac{1}{\psi_2}\Phi_0\psi_0''-\frac{1}{\psi_2}\Phi_1\psi_0'\right\}$$

$$\Phi_2\pi^2+\frac{1}{\psi_2}\Phi_2\psi_1\pi \qquad\qquad +\frac{1}{\psi_2}\Phi_2\psi_0$$

$$\left\{\varphi_1-\frac{1}{\psi_2}\Phi_0(\psi_1''+2\psi_0')-\frac{1}{\psi_2}\Phi_1(\psi_1'+\psi_0)-\frac{1}{\psi_2}\Phi_2\psi_1\right\}\pi+\left\{\varphi_0-\frac{1}{\psi_2}\Phi_0\psi_0''-\frac{1}{\psi_2}\Phi_1\psi_0-\frac{1}{\psi_2}\Phi_2\psi_0\right\},$$

which agrees with the result before found.

Returning to the Φ functions, and writing for convenience the symbolical expression

$$\Sigma_{p=0}^{p=N-M-r}\Sigma_{m=0}^{m=M}[N-M-r,p]\psi_m^{(N-M-p-r)}=S_s[N-M-r,p]\psi_m^{(N-M-p-r)},$$
$$p+m=N-s,$$

where the suffix s indicates the number of units whereby the sum $p+m$ is less than N, we have

$$\Phi_0=\quad \phi_N,$$
$$\psi_M\Phi_1=\psi_M\phi_{N-1}-S_1\Phi_0[N-M,p]\psi_m^{(N-M-p)}$$
$$=\begin{vmatrix} \phi_{N-1} & S_1[N-M,p]\psi_m^{(N-M-p)} \\ \phi_N & \psi_M \end{vmatrix}, \quad \ldots\ldots\ldots\ldots \quad (13.)$$

$$\psi_M^2\Phi_2=\psi_M^2\phi_{N-2}-S_2(\Phi_0[N-M,p]\psi_m^{(N-M-p)}+\Phi_1\ [N-M-1,p]\psi_m^{(N-M-p-1)})$$
$$=\psi_M^2\phi_{N-2}-\Phi_0S_2\ [N-M,p]\psi_m^{(N-M-p)}-\Phi_1S_2[N-M-1,p]\psi_m^{(N-M-p-1)}$$
$$=\begin{vmatrix} \phi_{N-2} & S_2[N-M-1,p]\psi_m^{(N-M-p-1)} & S_2[N-M,p]\psi_m^{(N-M-p)} \\ \phi_{N-1} & S_1[N-M-1,p]\psi_m^{(N-M-p-1)} & S_1[N-M,p]\psi_m^{(N-M-p)} \\ \phi_N & 0 & S_0[N-M,p]\psi_m^{(N-M-p)} \end{vmatrix}; \quad \ldots \quad (14.)$$

and generally,

$$\Phi_r=\begin{vmatrix} \phi_{N-r} & S_r\ [N-M-r+1,p]\psi_m^{(N-M-p-r+1)} & \ldots & S_r\ [N-M-1,p]\psi_m^{(N-M-p-1)} & S_r[N-M,p]\psi_m^{(N-M-p)} \\ \phi_{N-r+1} & S_{r-1}[N-M-r+1,p]\psi_m^{(N-M-p-r+1)} & \ldots & S_{r-1}[N-M-1,p]\psi_m^{(N-M-p-1)} & S_r[N-M,p]\psi_m^{(N-M-p)} \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \phi_N & 0 & \ldots & 0 & S_0[N-M,p]\psi_m^{(N-M-p)} \end{vmatrix}. \quad (15.)$$

These formulæ give, for the example discussed above,

$$\Phi_0=\phi_4$$

$$\psi_2\Phi_1=\begin{vmatrix} \phi_3 & \psi_1+2\psi_2' \\ \phi_4 & \psi_2 \end{vmatrix}$$

$$\psi_2^2\Phi_2=\begin{vmatrix} \phi_2 & \psi_1+\psi_2' & \psi_0+2\psi_1'+\psi_2'' \\ \phi_3 & \psi_2 & \psi_1+2\psi_2' \\ \phi_4 & 0 & \psi_2 \end{vmatrix}.$$

And for the final remainder, the coefficient of π,

$$\psi_2^3\left\{\phi_1-\frac{1}{\psi_2}\Phi_0(\psi_1''+2\psi_0')-\frac{1}{\psi_2}\Phi_1(\psi_1'+\psi_0)-\frac{1}{\psi_2}\Phi_2\psi_1\right\}=\begin{vmatrix} \phi_1 & \psi_1 & \psi_0+\psi_1' & 2\psi_0'+\psi_1'' \\ \phi_2 & \psi_2 & \psi_1+\psi_2' & \psi_0+2\psi_1'+\psi_2'' \\ \phi_3 & 0 & \psi_2 & \psi_1+2\psi_2' \\ \phi_4 & 0 & 0 & \psi_2 \end{vmatrix},$$

and the terms independent of $\pi =$

$$\psi_2^3\left\{\phi_0-\frac{1}{\psi_2}(\Phi_0\psi_3''+\Phi_1\psi_0'+\Phi_2\psi_0)\right\} = \left|\begin{matrix} \phi_0 & \psi_0 & \psi_0' & \psi_0'' \\ \phi_2 & \psi_2 & \psi_1+\psi_2' & \psi_0+2\psi_1'+\psi_2'' \\ \phi_3 & 0 & \psi_1 & \psi_1+2\psi_2' \\ \phi_4 & 0 & 0 & \psi_2 \end{matrix}\right|,$$

both of which may be comprised under the single formula

$$\left|\begin{matrix} \pi & \phi_0 & \psi_0 & \psi_0' & \psi_0'' \\ 1 & \phi_1 & \psi_1 & \psi_0+\psi_1' & 2\psi_0'+\psi_1'' \\ 0 & \phi_2 & \psi_2 & \psi_1+\psi_2' & \psi_0+2\psi_1'+\psi_2'' \\ 0 & \phi_3 & 0 & \psi_2 & \psi_1+2\psi_2' \\ 0 & \phi_4 & 0 & 0 & \psi_2 \end{matrix}\right|$$

§ 4. *To divide $\Sigma_{n=0}^{n=N}\phi_n\pi^n$ externally by $\Sigma_{m=0}^{m=M}\psi_m\pi^m$.*

The first term in the quotient will be

$$\frac{\phi_N}{\psi_M}\pi^{N-M}, \quad \ldots \ldots \quad (1.)$$

the product of this into the divisor

$$\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\left(\frac{\phi_N}{\psi_M}\right)^{(m-p)}\pi^{N-M+p}, \quad \ldots \ldots \quad (2.)$$

and the remainder

$$\Sigma_{n=0}^{n=N}\phi_n\pi^n-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\left(\frac{\phi_N}{\psi_M}\right)^{(m-p)}\pi^{N-M+p}; \quad \ldots \ldots \quad (3.)$$

the first term of which is

$$\left.\begin{aligned} &\left\{\phi_{N-1}-\Sigma_{m=0}^{m=M}\psi_m[m,M-1]\left(\frac{\phi_N}{\psi_M}\right)^{(m-M+1)}\right\}\pi^{N-1} \\ =&\left\{\phi_{N-1}-\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)-M\psi_M\left(\frac{\phi_N}{\psi_M}\right)'\right\}\pi^{N-1}. \end{aligned}\right\} \quad \ldots \ldots \quad (4.)$$

Let $\phi_N=\Phi_0$, and let the coefficient of π^{N-1} above written $=\Phi_1$; then the next term in the quotient will be $\frac{\Phi_1}{\psi_M}\pi^{N-M-1}$, and the product of this into the divisor will take the same form as (2.), writing only Φ_1 for Φ_0, or ϕ_N. The remainder will be the same as (3.), with the addition of the term

$$-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\left(\frac{\Phi_1}{\psi_M}\right)^{m-p}\pi^{N-M+p-1}, \quad \ldots \ldots \quad (5.)$$

and the first term of the entire remainder will be

$$\left.\begin{aligned} &\left\{\phi_{N-2}-\Sigma_{m=0}^{m=M}\psi_m\left[[m,M-2]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+2)}+[m,M-1]\left(\frac{\Phi_1}{\psi_M}\right)^{(m-M+1)}\right]\right\}\pi^{N-2} \\ =&\left\{\phi_{N-2}-\psi_{M-2}\left(\frac{\Phi_0}{\psi_M}\right)-(M-1)\psi_{M-1}\left(\frac{\Phi_0}{\psi_M}\right)'-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Phi_0}{\psi_M}\right)''\right. \\ &\left.\quad -\psi_{M-1}\left(\frac{\Phi_1}{\psi_M}\right)-M\psi_M\left(\frac{\Phi_1}{\psi_M}\right)'\right\}\pi^{N-2}; \end{aligned}\right\} \quad \ldots \quad (6.)$$

and if we make this expression $=\Phi_2\pi^{N-2}$, the next remainder will be

$$\Sigma_{n=0}^{n=N}\phi_n\pi^n-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\left\{\left(\frac{\Phi_0}{\psi_m}\right)^{(m-p)}\pi^2+\left(\frac{\Phi_1}{\psi_m}\right)^{(m-p)}\pi+\left(\frac{\Phi_0}{\psi_m}\right)^{(m-p)}\right\}\pi^{N-M+p-3}, \quad (7.)$$

the first term of which is

$$\left.\begin{aligned}&\left\{\phi_{N-3}-\Sigma_{m=0}^{m=M}\psi_m\left[[m,M-3]\left(\frac{\Phi_0}{\psi_M}\right)^{m-M+3}+[m,M-2]\left(\frac{\Phi_1}{\psi_M}\right)^{m-M+2}+[m,M-1]\left(\frac{\Phi_0}{\psi_M}\right)^{m-M+1}\right]\right\}\pi^{N-2}\\&=\left\{\phi_{N-3}-\psi_{M-3}\left(\frac{\Phi_0}{\psi_m}\right)-(M-2)\psi_{M-2}\left(\frac{\Phi_0}{\psi_m}\right)'-\frac{(M-1)(M-2)}{1.2}\psi_{M-1}\left(\frac{\Phi_0}{\psi_m}\right)''-\frac{M(M-)(M-2)}{1.2.3}\psi_M\left(\frac{\Phi_0}{\psi_M}\right)'''\right.\\&\qquad-\psi_{M-2}\left(\frac{\Phi_1}{\psi_M}\right)-(M-1)\psi_{M-1}\left(\frac{\Phi_1}{\psi_M}\right)'-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Phi_1}{\psi_M}\right)''\\&\qquad\left.-\psi_{M-1}\left(\frac{\Phi_2}{\psi_M}\right)-M\psi_M\left(\frac{\Phi_2}{\psi_M}\right)'\right\}\pi^{N-3};\end{aligned}\right\}(8.)$$

and generally the $(r+1)$th term in the quotient will be

$$\frac{\Phi_r}{\psi_M}\pi^{N-M-r},$$

and the $(r+1)$th remainder

$$\Sigma_{n=0}^{n=N}\phi_n\pi^n-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\Sigma_{q=0}^{q=r}\left(\frac{\Phi_q}{\psi_M}\right)^{(m-p)}\pi^{N-M+p-q}. \quad . \quad . \quad . \quad . \quad (9.)$$

The formation of the Φs is as follows:—

$$\left.\begin{aligned}&\Phi_0=\phi_N,\\&\Phi_1=\phi_{N-1}-\Sigma_{m=0}^{m=M}\psi_m\,[m,M-1]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+1)},\\&\Phi_2=\phi_{N-2}-\Sigma_{m=0}^{m=M}\psi_m\left\{[m,M-1]\left(\frac{\Phi_1}{\psi_M}\right)^{(m-M+1)}+[m,M-2]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+2)}\right\},\\&\Phi_3=\phi_{N-3}-\Sigma_{m=0}^{m=M}\psi_m\left\{[m,M-1]\left(\frac{\Phi_2}{\psi_M}\right)^{(m-M+1)}+[m,M-2]\left(\frac{\Phi_1}{\psi_M}\right)^{(m-M+2)}\right.\\&\qquad\left.+[m,M-3]\left(\frac{\Phi_0}{\psi_M}\right)^{(m-M+3)}\right\},\\&\cdots\cdots\cdots\cdots\\&\Phi_s=\phi_{N-s}-\Sigma_{m=0}^{m=M}\psi_m\left\{[m,M-1]\left(\frac{\Phi_{s-1}}{\psi_M}\right)^{(m-M+1)}+[m,M-2]\left(\frac{\Phi_{s-2}}{\psi_M}\right)^{(m-M+2)}\right.\\&\qquad\left.+-[m,1]\left(\frac{\Phi_{s-M+1}}{\psi_M}\right)^{(M-1)}+[m,0]\left(\frac{\Phi_{s-M}}{\psi_M}\right)^{(M)}\right\}.\end{aligned}\right\} \quad . \quad (10.)$$

The final remainder is given by the formula

$$\Sigma_{n=0}^{n=N}\phi_n\pi^n-\Sigma_{m=0}^{m=M}\psi_m\Sigma_{p=0}^{p=m}[m,p]\Sigma_{q=0}^{q=N-M}\left(\frac{\Phi_q}{\psi_M}\right)^{(m-p)}\pi^{N-M+p-q}; \quad . \quad . \quad . \quad (11.)$$

and the general term of this π^{N-s} is to be found as follows:

$$N-M+p-q=N-s,$$

i. e.

$$p-q=M-s.$$

Then we have for

$$
\left.\begin{aligned}
&p=\mathrm{M}, \qquad q=s,\\
&\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[m,\mathrm{M}\quad]\left(\frac{\Phi_s}{\psi_\mathrm{M}}\right)^{(m-\mathrm{M})}=\Phi_s;\\
&p=\mathrm{M}-1,\ q=s-1,\\
&\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[m,\mathrm{M}-1]\left(\frac{\Phi_{s-1}}{\psi_\mathrm{M}}\right)^{(m-\mathrm{M}+1)}=\mathrm{M}\psi_\mathrm{M}\left(\frac{\Phi_{s-1}}{\psi_\mathrm{M}}\right)'+\psi_{\mathrm{M}-1}\left(\frac{\Phi_{s-1}}{\psi_\mathrm{M}}\right);\\
&p=\mathrm{M}-2,\ q=s-2,\\
&\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[m,\mathrm{M}-2]\left(\frac{\Phi_{s-2}}{\psi_\mathrm{M}}\right)^{(m-\mathrm{M}+2)}=\frac{\mathrm{M}(\mathrm{M}-1)}{1.2}\psi_\mathrm{M}\left(\frac{\Phi_{s-2}}{\psi_\mathrm{M}}\right)''+(\mathrm{M}-1)\psi_{\mathrm{M}-1}\left(\frac{\Phi_{s-2}}{\psi_\mathrm{M}}\right)'+\psi_{\mathrm{M}-2}\left(\frac{\Phi_{s-2}}{\psi_\mathrm{M}}\right);\\
&p=1, \qquad q=s-\mathrm{M}+1,\\
&\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[\ m,1\ ]\left(\frac{\Phi_{s-\mathrm{M}+1}}{\psi_\mathrm{M}}\right)^{(m-1)}=\mathrm{M}\psi_\mathrm{M}\left(\frac{\Phi_{s-\mathrm{M}+1}}{\psi_\mathrm{M}}\right)^{(\mathrm{M}-1)}+(\mathrm{M}-1)\psi_{\mathrm{M}-1}\left(\frac{\Phi_{s-\mathrm{M}+1}}{\psi_\mathrm{M}}\right)^{(\mathrm{M}-2)}+\ldots\psi_1\left(\frac{\Phi_{s-\mathrm{M}-1}}{\psi_\mathrm{M}}\right);\\
&p=0, \qquad q=s-\mathrm{M},\\
&\Sigma_{m=0}^{m=\mathrm{M}}\psi_m[\ m,0\ ]\left(\frac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right)^{(m)}=\psi_\mathrm{M}\left(\frac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right)^{(\mathrm{M})}+\psi_{\mathrm{M}-1}\left(\frac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right)^{(\mathrm{M}-1)}+\ldots\psi_0\left(\frac{\Phi_{s-\mathrm{M}}}{\psi_\mathrm{M}}\right);
\end{aligned}\right\}\quad(12.)
$$

the sum of all which will be found, on reference to the expressions for the formation of the Φs, to be equal to the first term of Φ_s, viz. $\phi_{\mathrm{N}-s}$; and consequently the coefficient of $\pi^{\mathrm{N}-s}$ vanishes for all values of s not exceeding the greatest value of q, viz. $\mathrm{N}-\mathrm{M}$. If, however, s is greater than $\mathrm{N}-\mathrm{M}$, by any number t, so that $s=\mathrm{N}-\mathrm{M}+t$, then the pairs of values

$$
\begin{aligned}
&p=\mathrm{M}, && q=s,\\
&p=\mathrm{M}-1, && q=s-1,\\
&\quad\cdot\ \cdot && \quad\cdot\ \cdot\\
&p=\mathrm{M}-t+1, && q=s-t+1
\end{aligned}
$$

are inadmissible, and the pairs

$$
\begin{aligned}
&p=\mathrm{M}-t, && q=s-t,\\
&p=\mathrm{M}-t-1, && q=s-t-1,\\
&\quad\cdot\ \cdot && \quad\cdot\ \cdot\\
&p=0, && q=s-\mathrm{M}
\end{aligned}
$$

alone remain; and consequently the coefficients of the powers of π, for $s>\mathrm{N}-\mathrm{M}$, do not vanish, and the remainder consists of a series of terms, the index of the highest power of π being

$$\mathrm{N}-\mathrm{M}+p-q=\mathrm{N}-s=\mathrm{N}-\mathrm{N}+\mathrm{M}-1=\mathrm{M}-1,$$

as it should be.

As an example, we may calculate by means of the formulæ given above, the final remainder in the external division of

$$\phi_4(\rho)\pi^4+\phi_3(\rho)\pi^3+\phi_2(\rho)\pi^2+\phi_1(\rho)\pi+\phi_0(\rho)$$

by

$$\psi_3(\varrho)\pi^3+\psi_2(\varrho)\pi^2+\psi_1(\varrho)\pi+\psi_0(\varrho),$$

viz.

$$\Sigma_{n=0}^{n=4}\varphi_n\pi^n-\Sigma_{m=0}^{m=2}\Sigma_{p=0}^{p=m}\Sigma_{q=0}^{q=2}\psi_m[m,p]\left(\frac{\Phi_q}{\psi_M}\right)^{(m-p)}\pi^{m+p-q}.$$

The conditions

$p=2,\ q=0$ give $\Phi_0\pi^4$

$p=2,\ q=1$ —— $\Phi_1\pi^3$

$p=2,\ q=2$ —— $\Phi_2\pi^2$

$p=1,\ q=0$ —— $\left\{2\psi_2\left(\frac{\Phi_0}{\psi_2}\right)'+\psi_1\left(\frac{\Phi_0}{\psi_2}\right)\right\}\pi^3$

$p=1,\ q=1$ —— $\left\{2\psi_2\left(\frac{\Phi}{\psi_2}\right)'+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)\right\}\pi^2$

$p=1,\ q=2$ —— $\left\{2\psi_2\left(\frac{\Phi_2}{\psi_2}\right)'+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)\right\}\pi$

$p=0,\ q=0$ —— $\left\{\psi_2\left(\frac{\Phi_0}{\psi_2}\right)''+\psi_1\left(\frac{\Phi_0}{\psi_2}\right)'+\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\right\}\pi^2$

$p=0,\ q=1$ —— $\left\{\psi_2\left(\frac{\Phi_1}{\psi_2}\right)''+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'+\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\right\}\pi$

$p=0,\ q=2$ —— $\left\{\psi_2\left(\frac{\Phi_2}{\psi_2}\right)''+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'+\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\right\}.$

Hence taking the sum of all the terms, the coefficients of π^4, π^3, π^2 vanish, and the final remainder is

$$\left\{\varphi_1-2\psi_2\left(\frac{\Phi_2}{\psi_2}\right)'-\psi_1\left(\frac{\Phi_2}{\psi_2}\right)-\psi_2\left(\frac{\Phi_1}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_1}{\psi_2}\right)\right\}\pi+\varphi_0-\psi_2\left(\frac{\Phi_2}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_2}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_2}{\psi_2}\right).$$

These results may be compared with the actual division,

$$\psi_2\pi^2+\psi_1\pi+\psi_0\Big)\varphi_4\pi^4+\varphi_3\pi^3+\varphi_2\pi^2+\varphi_1\pi+\varphi_0\Big(\frac{\Phi_0}{\psi_2}\pi^2+\frac{\Phi_1}{\psi_2}\pi+\frac{\Phi_2}{\psi_2}$$

$$\left.\begin{aligned}\varphi_4\pi^4+2\psi_2\left(\frac{\Phi_0}{\psi_2}\right)'\pi^3+\psi_2\left(\frac{\Phi_0}{\psi_2}\right)''\pi^2\\ +\ \psi_1\left(\frac{\Phi_0}{\psi_2}\right)\pi^3+\psi_1\left(\frac{\Phi_0}{\psi_2}\right)'\pi^2\\ +\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\pi^2\end{aligned}\right\}$$

$$\Phi_1\pi^3+\left\{\varphi_2-\psi_2\left(\frac{\Phi_0}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_0}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_0}{\psi_2}\right)\right\}\pi^2+\varphi_1\pi+\varphi_0$$

$$\Phi_1\pi^3+\qquad\left.\begin{aligned}2\psi^2\left(\frac{\Phi_1}{\psi_2}\right)'\pi^2+\psi_2\left(\frac{\Phi_1}{\psi_2}\right)''\pi\\ +\psi_1\left(\frac{\Phi_1}{\psi_2}\right)\pi^2+\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'\pi\\ +\psi_0\left(\frac{\Phi_1}{\psi_2}\right)\pi\end{aligned}\right\}$$

$$\Phi_2\pi^2+\left\{\phi_1-\psi_2\left(\frac{\Phi_1}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_1}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_1}{\psi_2}\right)\right\}\pi+\phi_0$$

$$\begin{array}{ll}\Phi_2\pi^2 & \left.\begin{array}{r}+2\psi_2\left(\frac{\Phi_2}{\psi_2}\right)'\pi+\psi_2\left(\frac{\Phi_2}{\psi_2}\right)\\ +\ \psi_1\left(\frac{\Phi_2}{\psi_2}\right)\pi+\psi_1\left(\frac{\Phi_2}{\psi_2}\right)'\\ +\psi_0\left(\frac{\Phi_2}{\psi_2}\right)\end{array}\right\}\end{array}$$

$$\overline{\Phi_3\pi+\left\{\phi_0-\psi_2\left(\frac{\Phi_2}{\psi_2}\right)''-\psi_1\left(\frac{\Phi_2}{\psi_2}\right)'-\psi_0\left(\frac{\Phi_2}{\psi_2}\right)\right\},}$$

which agrees with the results found above.

§ 5. *To divide $\Sigma_{n=0}^{n=N}\varrho^n\phi_n(\pi)$ internally by $\Sigma_{m=0}^{m=M}\varrho^m\psi_m(\pi)$.*

The first term of the quotient will be

$$\varrho^{N-M}\frac{\phi_N(\pi-M)}{\psi_M(\pi-M)}, \qquad (1.)$$

and the product of this into the divisor,

$$\Sigma_{m=0}^{m=M}\varrho^{N-M+m}\frac{\psi_m(\pi)}{\psi_M(\pi-M+m)}\phi_N(\pi-M+m). \qquad (2.)$$

The first term of the remainder will then be

$$\varrho^{N-1}\left\{\phi_{N-1}(\pi)-\frac{\psi_{M-1}(\pi)}{\psi_M(\pi-1)}\phi_N(\pi-1)\right\}=\varrho_{N-1}\frac{1}{\psi_M(\pi-1)}\begin{vmatrix}\phi_{N-1}(\pi) & \psi_{M-1}(\pi)\\ \phi_N(\pi-1) & \psi_M(\pi-1),\end{vmatrix} \qquad (3.)$$

and consequently the second term in the quotient will be

$$\left.\begin{array}{l}\varrho^{N-M-1}\left\{\phi_{N-1}(\pi-M)-\frac{\psi_{M-1}(\pi-M)}{\psi_M(\pi-M-1)}\phi_N(\pi-M-1)\right\}\frac{1}{\psi_M(\pi-M)}\\ =\varrho^{N-M-1}\frac{1}{\psi_M(\pi-M)\psi_M(\pi-M-1)}\begin{vmatrix}\phi_{N-1}(\pi-M) & \psi_{M-1}(\pi-M)\\ \phi_N(\pi-M-1) & \psi_M(\pi-M-1)\end{vmatrix}\end{array}\right\}. \qquad (4.)$$

The first term of the second remainder will then be

$$\left.\begin{array}{l}\varrho^{N-2}\left\{\phi_{N-2}(\pi)-\frac{\psi_{M-1}(\pi)}{\psi_M(\pi-1)\psi_M(\pi-2)}\begin{vmatrix}\phi_{N-1}(\pi-1) & \psi_{M-1}(\pi-1)\\ \phi_N(\pi-2) & \psi_M\ (\pi-2)\end{vmatrix}\right\}\\ =\varrho^{N-2}\frac{1}{\psi_M(\pi-1)\psi_M(\pi-2)}\begin{vmatrix}\phi_{N-2}(\pi) & \psi_{M-1}(\pi) & 0\\ \phi_{N-1}(\pi-1) & \psi_M(\pi-1) & \psi_{M-1}(\pi-1)\\ \phi_N(\pi-2) & 0 & \psi_M(\pi-2)\end{vmatrix}\end{array}\right\}. \qquad (5.)$$

And it is not difficult to see that the first term of the rth remainder will be

$$\varrho^{N-r}\frac{1}{\psi_M(\pi-1)\psi_M(\pi-2)..\psi_M(\pi-r)}\begin{vmatrix}\phi_{N-r}(\pi) & \psi_{M-1}(\pi) & .. & 0\\ \phi_{N-r+1}(\pi-1) & \psi_M\ (\pi-1) & .. & 0\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \phi_N(\pi-r) & 0 & .. & \psi_M(\pi-r),\end{vmatrix}\quad . \quad . \quad (6.)$$

in which determinant every column after the first consists of only two terms, viz. $\psi_{M-1}(\pi-s)$ and $\psi_M(\pi-s-1)$. Hence also the $(r+1)$th term of the quotient will be

$$\varrho^{N-M-r}\frac{1}{\psi_M(\pi-M)\psi_M(\pi-M-1)..\psi_M(\pi-M-r)}\begin{vmatrix}\phi_{N-r}(\pi-M) & \psi_{M-1}(\pi) & .. & 0\\ \phi_{N-r+1}(\pi-M-1) & \psi_M(\pi-1) & .. & 0\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \phi_N(\pi-r) & 0 & .. & \psi_M(\pi-r).\end{vmatrix}\quad (7.)$$

As to the other terms, than the first, of the various remainders. In the first remainder, the first term of which is given by (3.), the $(s+1)$th term will be found by making $n=N-s$, $m=M-s$, in the expression

$$\Sigma_{n=0}^{n=N}\varrho^n\phi_n(\pi)-\Sigma_{m=0}^{m=M}\varrho^{N-M+m}\frac{\psi_m(\pi)}{\psi_M(\pi-M+m)}\phi_N(\pi-M+m),$$

which gives

$$\varrho^{N-s}\left\{\phi_{N-s}(\pi)-\frac{\psi_{M-s}(\pi)}{\psi_M(\pi-s)}\phi_N(\pi-s)\right\}$$

$$=\varrho^{N-s}\frac{1}{\psi_M(\pi-s)}\begin{vmatrix}\phi_{N-s}(\pi) & \psi_{M-s}(\pi)\\ \phi_N(\pi-s) & \psi_M(\pi-s).\end{vmatrix}$$

Hence the entire first remainder may be expressed thus:

$$\Sigma_{s=0}^{s=M}\frac{\varrho^{N-s}}{\psi_M(\pi-s)}\begin{vmatrix}\phi_{N-s}(\pi) & \psi_{M-s}(\pi)\\ \phi_N(\pi-s) & \psi_M(\pi-s).\end{vmatrix}\quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

Similarly, the general expression for the second remainder is

$$\Sigma_{n=0}^{n=N-1}\varrho^n\phi_n(\pi)-\Sigma_{m=0}^{m=M}\varrho^{N-M+m-1}\frac{\psi_m(\pi)}{\psi_M(\pi-M+m)\psi_M(\pi-M+m-1)}\begin{vmatrix}\phi_{N-1}(\pi-M+m) & \psi_{M-1}(\pi-M+m)\\ \phi_N(\pi-M+m-1) & \psi_M\ (\pi-M+m-1),\end{vmatrix}$$

which may be transformed thus:

$$n=N-s,\quad m=M-s,$$

$$\Sigma_{s=0}^{s=M}\varrho^{N-s-1}\left\{\phi_{N-s-1}(\pi)-\frac{\psi_{M-s}(\pi)}{\psi_M(\pi-s)\psi_M(\pi-s-1)}\begin{vmatrix}\phi_{N-1}(\pi-s) & \psi_{M-1}(\pi-s)\\ \phi_N(\pi-s-1) & \psi_M(\pi-s-1)\end{vmatrix}\right\}$$

$$=\Sigma_{s=0}^{s=M}\varrho^{N-s-1}\frac{1}{\psi_M(\pi-s)\psi_M(\pi-s-1)}\begin{vmatrix}\phi_{N-s-1}(\pi\quad) & \psi_{M-s}(\pi\quad) & 0\\ \phi_{N-1}\ (\pi-s\quad) & \psi_M\ (\pi-s) & \psi_{M-1}(\pi-s\quad)\\ \phi_N\quad(\pi-s-1) & 0 & \psi_M\ (\pi-s-1).\end{vmatrix}\quad (9.)$$

And generally the expression for the tth remainder may be written

$$\Sigma_{s=0}^{s=M} \varrho^{N-s-t+1} \frac{1}{\psi_M(\pi-s)\psi_M(\pi-s-1)\,..\,\psi_M(\pi-s-t+1)} \begin{vmatrix} \phi_{N-s+t}(\pi) & \psi_{M-s}(\pi) & .. & 0 \\ \phi_{N-t+1}(\pi-s) & \psi_M(\pi-s) & .. & \psi_{M-t+1}(\pi-s) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \phi_N(\pi-s-t+1) & 0 & .. & \psi_M(\pi-s-t+1). \end{vmatrix} \quad (10.)$$

The last remainder is the $(N-M+1)$th. Then

$$\begin{aligned} t&=N-M+1, \\ N-s-t+1&=N-s-N+M-1+1=M-s \\ N-t+1&=N \quad -N+M-1+1=M \\ M-t+1&=M \quad -N+M-1+1=2M-N, \end{aligned}$$

and the remainder in question

$$=\Sigma_{s=0}^{s=M} \varrho^{M-s} \frac{1}{\psi_M(\pi-s)\psi_M(\pi-s-1)\,..\,\psi_M(\pi-s-N+M)} \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (11.)$$

$$\times \begin{vmatrix} \phi_{M-s}(\pi) & \psi_{M-s}(\pi) & .. & 0 \\ \phi_M(\pi-s) & \psi_M(\pi-s) & .. & \psi_{2M-N}(\pi-s) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \phi_N(\pi-s-N+M) & 0 & .. & \psi_M(\pi-s-N+M), \end{vmatrix}$$

in which the coefficient of ϱ^M vanishes, as it should. The last term, viz. that independent of ϱ,

$$=\frac{1}{\psi_M(\pi-M)\psi_M(\pi-M-1)\,..\,\psi_M(\pi-N)} \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (12.)$$

$$\times \begin{vmatrix} \phi_0(\pi) & \psi_0(\pi) & .. & 0 \\ \phi_M(\pi-M) & \psi_M(\pi-M) & .. & \psi_{2M-N}(\pi-M) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \phi_N(\pi-N) & 0 & .. & \psi_M(\pi-N); \end{vmatrix}$$

and if $N=1$, the result agrees with that given by Mr. RUSSELL*.

§ 6. *To divide* $\Sigma_{n=0}^{n=N}\varrho^n\phi_n(\pi)$ *externally by* $\Sigma_{m=0}^{m=M}\varrho^m\psi_m(\pi)$.

The first term of the quotient will be

$$\varrho^{N-M}\frac{\phi_N(\pi)}{\psi_M(\pi+N-M)}. \quad \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (1.)$$

The first remainder

$$\left.\begin{aligned} &\Sigma_{n=0}^{n=N}\varrho^n\phi_n(\pi)-\Sigma_{m=0}^{m=M}\varrho^{N-M+m}\frac{\psi_m(\pi+N-M)}{\psi_M(\pi+N-M)}\phi_N(\pi) \\ &=\Sigma_{s=0}^{s=M}\varrho^{N-s}\left\{\phi_{N-s}(\pi)-\frac{\psi_{M-s}(\pi+N-M)}{\psi_M(\pi+N-M)}\phi_N(\pi)\right\}, \end{aligned}\right\} \quad \cdot \cdot \cdot \cdot \cdot \cdot \quad (2.)$$

* Philosophical Transactions, vol. cli. p. 72.

the first term of which is

$$\varrho^{N-1}\left\{\phi_{N-1}(\pi)-\frac{\psi_{M-1}(\pi+N-M)}{\psi_M(\pi+N-M)}\phi_N(\pi)\right\},$$

whence the second term of the quotient will be

$$\left.\begin{aligned}&\varrho^{N-M-1}\frac{1}{\psi_M(\pi+N-M-1)}\left\{\phi_{N-1}(\pi)-\frac{\psi_{M-1}(\pi+N-M)}{\psi_M(\pi+N-M)}\phi_N(\pi)\right\}\\&=\varrho^{N-M-1}\frac{1}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\phi_{N-1}(\pi) & \psi_{M-1}(\pi+N-M)\\ \phi_N(\pi) & \psi_M(\pi+N-M).\end{vmatrix}\end{aligned}\right\}\quad . \; . \; (3.)$$

Similarly, the second remainder will be

$$\left.\begin{aligned}&\Sigma_{n=0}^{n=N-1}\varrho^n\phi_n(\pi)-\Sigma_{m=0}^{m=M}\varrho^{N-M+m-1}\frac{\psi_m(\pi+N-M-1)}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\phi_{N-1}(\pi) & \psi_{M-1}(\pi+N-M)\\ \phi_N(\pi) & \psi_M(\pi+N-M)\end{vmatrix}\\&=\Sigma_{s=0}^{s=M}\varrho^{N-s-1}\left\{\phi_{N-s-1}(\pi)-\frac{\psi_{M-s}(\pi+N-M-1)}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\phi_{N-1}(\pi) & \psi_{M-1}(\pi+N-M)\\ \phi_N(\pi) & \psi_M(\pi+N-M)\end{vmatrix}\right\}\\&=\Sigma_{s=0}^{s=m}\varrho^{N-s-1}\frac{1}{\psi_M(\pi+N-M-1)\psi_M(\pi+N-M)}\begin{vmatrix}\phi_{N-s-1}(\pi) & \psi_{M-s}(\pi+N-M-1) & 0\\ \phi_{N-1}(\pi) & \psi_M(\pi+N-M-1) & \psi_{M-1}(\pi+N-M)\\ \phi_N(\pi) & 0 & \psi_M(\pi+N-M)\end{vmatrix}\end{aligned}\right\}(4.)$$

The tth remainder

$$\left.\begin{aligned}&=\Sigma_{s=0}^{s=M}\varrho^{N-s-t+1}\frac{1}{\psi_M(\pi+N-M-t+1)\psi_M(\pi+N-M-t+2)\,..\,\psi_M(\pi+N-M)}\\&\times\begin{vmatrix}\phi_{N-s-t+1}(\pi) & \psi_{M-s}(\pi+N-M-t+1) & .. & 0\\ \phi_{N-t+1}(\pi) & \psi_M(\pi+N-M-t+1) & .. & \psi_{M-t+1}(\pi+N-M)\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \phi_N(\pi) & 0 & . & \psi_M(\pi+N-M);\end{vmatrix}\end{aligned}\right\}\quad . \; . \; . \; (5.)$$

and the last remainder, viz. the $(N-M+1)$th,

$$=\Sigma_{s=0}^{s=M}\varrho^{M-s}\frac{1}{\psi_M(\pi)\psi_M(\pi+1)\,..\,\psi_M(\pi+N-M)}\quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (6.)$$

$$\times\begin{vmatrix}\phi_{M-s}(\pi) & \psi_{M-s}(\pi) & .. & 0\\ \phi_M(\pi) & \psi_M(\pi) & .. & \psi_{2M-N}(\pi+N-M)\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ \phi_N(\pi) & 0 & .. & \psi_M(\pi+N-M).\end{vmatrix}$$

In the case considered by Mr. RUSSELL, viz. $M=1$, (6.) gives only the single term

$$\{\psi_1(\pi)\psi_1(\pi+1)\,..\,\psi_1(\pi+N-1)\}^{-1}\begin{vmatrix}\phi_0(\pi) & \psi_0(\pi) & .. & 0\\ \phi_1(\pi) & \psi_1(\pi) & .. & 0\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \phi_{N-1}(\pi) & 0 & .. & \psi_0(\pi+N-1)\\ \phi_N(\pi) & 0 & .. & \psi_1(\pi+N-1),\end{vmatrix}$$

and in this the coefficient of $\phi_i(\pi)$ is $(-)^i\times$

$$\begin{vmatrix}
\psi_0(\pi) & 0 & .. & 0 & 0 & .. & 0 & 0 \\
\psi_1(\pi) & \psi_0(\pi+1) & .. & 0 & 0 & .. & 0 & 0 \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
0 & 0 & .. & \psi_0(\pi+i-1) & 0 & .. & 0 & 0 \\
0 & 0 & .. & 0 & \psi_1(\pi+i) & .. & 0 & 0 \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
0 & 0 & .. & 0 & 0 & .. & \psi_1(\pi+N-2) & \psi_0(\pi+N-1) \\
0 & 0 & .. & 0 & 0 & .. & 0 & \psi_1(\pi+N-1)
\end{vmatrix}$$

$$=\psi_0(\pi)\psi_0(\pi+1)\,..\,\psi_0(\pi+i-1)\psi_1(\pi+i)\,..\,\psi_1(\pi+N-2)\psi_1(\pi+N-1);$$

whence the whole expression

$$=\Sigma_{i=0}^{i=N}(-)^i\phi_i(\pi)\frac{\psi_0(\pi)\psi_0(\pi+1)\,..\,\psi_0(\pi+i-1)}{\psi_1(\pi)\psi_1(\pi+1)\,..\,\psi_1(\pi+i-1)},$$

which agrees with the result given in the Philosophical Transactions, vol. cli. p. 73.

In the particular case of N=4, M=2, the final remainder in internal division is

$$\varrho\frac{1}{\psi_2(\pi-1)\psi_2(\pi-2)\psi_2(\pi-3)}\begin{vmatrix}
\phi_1(\pi) & \psi_1(\pi) & 0 & 0 \\
\phi_2(\pi-1) & \psi_2(\pi-1) & \psi_1(\pi-1) & \psi_0(\pi-1) \\
\phi_3(\pi-2) & 0 & \psi_2(\pi-2) & \psi_1(\pi-2) \\
\phi_4(\pi-3) & 0 & 0 & \psi_2(\pi-3)
\end{vmatrix}$$

$$+\frac{1}{\psi_2(\pi-2)\psi_2(\pi-3)\psi_2(\pi-4)}\begin{vmatrix}
\phi_0(\pi) & \psi_0(\pi) & 0 & 0 \\
\phi_2(\pi-2) & \psi_2(\pi-2) & \psi_1(\pi-2) & \psi_0(\pi-2) \\
\phi_3(\pi-3) & 0 & \psi_2(\pi-3) & \psi_1(\pi-3) \\
\phi_4(\pi-4) & 0 & 0 & \psi_2(\pi-4);
\end{vmatrix}$$

and in external division it is

$$\varrho\frac{1}{\psi_2(\pi)\psi_2(\pi+1)\psi_2(\pi+2)}\begin{vmatrix}
\phi_1(\pi) & \psi_1(\pi) & 0 & 0 \\
\phi_2(\pi) & \psi_2(\pi) & \psi_1(\pi+1) & \psi_0(\pi+2) \\
\phi_3(\pi) & 0 & \psi_2(\pi+1) & \psi_1(\pi+2) \\
\phi_4(\pi) & 0 & 0 & \psi_2(\pi+2)
\end{vmatrix}$$

$$+\frac{1}{\psi_2(\pi)\psi_2(\pi+1)\psi_2(\pi+2)}\begin{vmatrix}
\phi_0(\pi) & \psi_0(\pi) & 0 & 0 \\
\phi_2(\pi) & \psi_2(\pi) & \psi_1(\pi+1) & \psi_0(\pi+2) \\
\phi_3(\pi) & 0 & \psi_2(\pi+1) & \psi_1(\pi+1) \\
\phi_4(\pi) & 0 & 0 & \psi_2(\pi).
\end{vmatrix}$$

The expressions (10.) for the formation of the Φs admit of further development thus:

$$\Phi_0=\phi_N$$

$$\Phi_1=\phi_{N-1}-M\psi_M\left(\frac{\phi_N}{\psi_M}\right)'-\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)$$

$$\Phi_2=\phi_{N-2}-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\phi_N}{\psi_M}\right)''-(M-1)\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)'-\psi_{M-2}\left(\frac{\phi_N}{\psi_M}\right)$$

$$+\ M^2\psi_M\left(\frac{\phi_N}{\psi_M}\right)''+M\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)'+M\psi_M\left(\frac{\psi_{N-1}}{\psi_M}\right)'\left(\frac{\phi_N}{\psi_M}\right)-M\psi_M\left(\frac{\phi_{N-1}}{\psi_M}\right)'$$

$$+M\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)'+\psi_{M-1}\left(\frac{\psi_{N-1}}{\psi_M}\right)\left(\frac{\phi_N}{\psi_M}\right)-\psi_{M-1}\left(\frac{\phi_{N-1}}{\psi_M}\right)$$

$$=\frac{(M+1)M}{1.2}\psi_M\left(\frac{\phi_N}{\psi_M}\right)''+(M+1)\psi_{M-1}\left(\frac{\phi_N}{\psi_M}\right)'+\left\{-\psi_{M-2}+M\psi_M\left(\frac{\psi_{M-1}}{\psi_M}\right)'+\psi_{M-1}\left(\frac{\psi_{N-1}}{\psi_M}\right)\right\}\left(\frac{\phi_N}{\psi_M}\right)$$

$$-M\psi_M\left(\frac{\phi_{N-1}}{\psi_M}\right)'-\qquad\psi_{M-1}\left(\frac{\phi_{N-1}}{\psi_M}\right)$$

$$+\qquad\phi_{N-2};$$

or writing, by analogy to the Φs,

$$\Psi_0=\psi_{M-1}$$

$$-\Psi_1=\psi_{M-2}-M\psi_M\left(\frac{\Psi_0}{\psi_M}\right)'-\psi_{M-1}\left(\frac{\Psi_0}{\psi_M}\right),$$

the expression for Φ_2 becomes

$$\Phi_2=\frac{(M+1)M}{1.2}\psi_M\left(\frac{\phi_N}{\psi_M}\right)''+(M+1)\Psi_0\left(\frac{\phi_N}{\psi_M}\right)'+\Psi_1\left(\frac{\phi_N}{\psi_M}\right)$$

$$-M\psi_M\left(\frac{\phi_{N-1}}{\psi_M}\right)'-\qquad\Psi_0\left(\frac{\phi_{N-1}}{\psi_M}\right)$$

$$+\ \psi_M\left(\frac{\phi_{N-2}}{\psi_M}\right).$$

And so likewise writing

$$\Psi_2=\psi_{M-3}-\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Psi_0}{\psi_M}\right)''-(M-1)\psi_{M-1}\left(\frac{\Psi_0}{\psi_M}\right)'-\psi_{M-2}\left(\frac{\Psi_0}{\psi_M}\right)$$

$$-M\psi_M\left(\frac{\Psi_1}{\psi_M}\right)'-\qquad\psi_{M-1}\left(\frac{\Psi_1}{\psi_M}\right),$$

it will be found that

$$\Phi_3=_{N-3}-\frac{M(M-1)(M-2)}{1.2.3}\psi_M\left(\frac{\Phi_0}{\psi_M}\right)'''-\frac{(M-1)(M-2)}{1.2}\psi_{M-1}\left(\frac{\Phi_0}{\psi_M}\right)''-(M-2)\psi_{M-2}\left(\frac{\Phi_0}{\psi_M}\right)'-\psi_{M-3}\left(\frac{\Phi_0}{\psi_M}\right)$$

$$-\qquad\frac{M(M-1)}{1.2}\psi_M\left(\frac{\Phi_1}{\psi_M}\right)''-\qquad(M-1)\psi_{M-1}\left(\frac{\Phi_1}{\psi_M}\right)'-\qquad\psi_{M-2}\left(\frac{\Phi_1}{\psi_M}\right)$$

$$-\qquad M\psi_M\left(\frac{\Phi_2}{\psi_M}\right)'-\qquad\psi_{M-1}\left(\frac{\Phi_2}{\psi_M}\right)$$

$$
\begin{aligned}
= & -\frac{(M+2)(M+1)M}{1.2.3}\psi_M\left(\frac{\phi_N}{\psi_M}\right)''' - \frac{(M+2)(M+1)}{1.2}\Psi_0\left(\frac{\phi_N}{\psi_M}\right)'' - (M+2)\Psi_1\left(\frac{\phi_N}{\psi_M}\right)' - \Psi_2\left(\frac{\phi_N}{\psi_M}\right) \\
+ & \quad \frac{(M+1)M}{1.2}\psi_M\left(\frac{\phi_{N-1}}{\psi_M}\right)'' + (M+1)\Psi_0\left(\frac{\phi_{N-1}}{\psi_M}\right)' + \Psi_1\left(\frac{\phi_{N-1}}{\psi_M}\right) \\
- & \quad M\psi_M\left(\frac{\phi_{N-2}}{\psi_M}\right)'' - \Psi_0\left(\frac{\phi_{N-2}}{\psi_M}\right) \\
+ & \quad \psi_M\left(\frac{\phi_{N-3}}{\psi_M}\right).
\end{aligned}
$$

But the law of the expressions in the first form having been established above, it is unnecessary to pursue these latter formulæ further.

VII. *On the Theory of the Polyedra.* *By the Rev.* THOMAS P. KIRKMAN, *M.A.*, *F.R.S.*, *Honorary Member of the Literary and Philosophical Societies of Manchester and Liverpool.*

Received January 3,—Read January 23, 1862.

THE following memoir contains a complete solution of the problem of the *classification and enumeration* of the P-edra Q-acra. The actual *construction* of the solids is a task impracticable from its magnitude; but it is here shown, that we can enumerate them with an accurate account of their symmetry, to any values of P and Q.

Section first discusses fully the *symmetry of polyedra*, and gives a sketch of the Tables which it is required to form for the solution of our problem, viz. Tables of the classified P-edra Q-acra and Q-edra P-acra, and of their faces, summits, and edges, symmetrical and unsymmetrical.

Section second proves that all these Tables are given, if *certain data* are first obtained. The rest of the memoir is occupied by the investigation and construction of these assigned *data.*

Section third contains the analysis of a *polar or monozone summit* of a P-edron Q-acron: the *deltotomous effaceables* of the summit are defined and restored: the *reticulation* which is laid bare by the removal of the rays of the summit is analysed and reduced.

Section fourth is devoted to the *construction* of polar and monozone *reticulations*, and to their *registration* with their *signatures* of form and symmetry, in groups which suffice for our problem.

Section fifth gives formulæ for the zoned and zoneless *coronation* of a polar or monozone *perfect reticulation*, whereby it becomes a polyedron; and Tables of the *perfect summits* thus obtained are sketched out.

Section sixth enumerates and registers the results of *deltotomous effacements* about *perfect summits*, that is, summits about which all deltotomous effaceables have been restored, or which have no effaced effaceables, whereby the summit analysed in section third is obtained and registered.

Section seventh analyses the *polar summits of a janal axis*; the *rhombotomous effaceables* of the opposite summits are defined, and, as well as their deltotomous ones, restored, about either pole; the *janal reticulation* laid bare by the removal of the summits is analysed and reduced.

Section eighth constructs, enumerates, and registers, with their signatures of symmetry, the *fundamental* and *primitive* janal reticulations.

Section ninth contains the construction, enumeration, and registration of *janal subnucleus reticulations.*

Section tenth gives the like concerning *janal nucleus reticulations.*

Section eleventh gives a similar account of *perfect janal mixed reticulations.*

Sections twelfth and thirteenth are devoted to the *janal coronation* of janal reticulations, and to the enumeration and registration of *perfect janal summits.*

Sections fourteenth and fifteenth enumerate and register the results of deltotomous and rhombotomous effacements about perfect janal summits.

Section sixteenth analyses a *polyarchipolar summit*; effaceables are restored about all like archipoles; the *polyarchaxine reticulation* laid bare by the removal of these polar summits is reduced, and afterwards constructed with enumeration and registration of results: the formulæ for *polyarchaxine coronation* are given, and the results of effacement about the principal axes are enumerated and registered.

Section seventeenth gives the analysis, construction, and enumeration of *contrajanal anaxine pairs of edges*, which have neither polar nor zoned symmetry, but of which one edge is the reflected image of the other, and diametrically opposed to it.

The rest of the memoir is devoted to the enumeration of *plane reticulations*, i. e. partitioned polygons, the knowledge of which is taken for granted in all that precedes.

Section eighteenth enumerates and registers the symmetrical and asymmetric *plane penesolids*, i. e. plane reticulations laid bare by the removal of an edge of a polyedron.

Section nineteenth enumerates and registers the *primary plane reticulations*, symmetric and asymmetric, with their signatures of symmetry.

Section twentieth constructs, enumerates, and registers with their signatures, the *zoned plane reticulations.*

Section twenty-first gives the like account of the *zoneless plane reticulations.*

It is unfortunate that my previous labours on the partitions of the R-gon, that is, on plane reticulations, are of little utility for this problem of the polyedra, by reason of their too great generality, and of their not giving the number of *marginal triangles* in each partition. Yet the fundamental theorem on the *k*-divisions of the R-gon* has been the key to the *greatest difficulty* in this theory, which is to find the number of the asymmetric plane reticulations which have a given *marginal signature.*

Of the two sections here presented to the public, the first (arts. I.... XXXV.) is in itself a complete treatise on the symmetry and classification of Polyedra; and the second, together with the preceding introduction, puts the reader clearly in possession of the main outline of the argument. Much remains of the entire work, which is, however, completely written, and in the possession of the Royal Society, as well as extensive applications of the method to the enumeration of polyedra. These will be intelligible only when the general methods and formulæ of this memoir have been investigated and exhibited.

SECTION 1.—*On the Symmetry of Polyedra.*

I. The symmetry of polyedra is—

1. Zoned symmetry;
2. Zoneless axial symmetry;

* Philosophical Transactions, 1857, p. 225.

3. Mixed symmetry, both zoned and zoneless axial;
4. Neuter symmetry, neither zoned nor zoneless axial;
5. No symmetry.

The greater part of the polyedra are entirely asymmetric.

In considering this symmetry, we have no regard to mere lengths or inclinations of edges. The symmetry is *descriptive*, not *metrical*. For example, if the cutting edge of a wedge be removed by any quadrilateral section, the solid acquires, for our purposes, all the symmetry of the cube; and this remains, however the figure may be distorted.

1. *Zoned Symmetry.*

II. The polyedra which have a zoned symmetry, and only such, are

a. Monozone polyedra;

b. *m*-zoned monaxine heteroids;

c. Zoned triaxines;

d. *m*-zoned monarchaxines, having one principal and *m* secondary axes;

e. Zoned polyarchaxines, having the axial systems of the regular polyedra.

The terms will be explained below.

Def. *A zone is any closed line drawn or drawable on a polyedron, which divides it into halves, either of which is the reflected image of the other, no regard being had to mere lengths of edges.*

The closed line may or may not be all in one plane, and the halves may or may not be *metrically* equal. We have a right to conceive, when it is convenient, that the solid is constructed with the greatest possible symmetry, in which case the halves will be exactly equal, and the zone will be a plane, having to them the geometrical relation which a mirror has to an object touching it and to its image.

This relation we shall assume as always existing, however the polyedron may be distorted; that is, we assume that any zoneless edge will meet, on any zonal plane, if it be produced, the edge which is its reflected image in respect of that zonal plane.

For example, any section of a cube which passes through two opposite faces, and contains either two edges or none of the solid, is a zone.

Every zone has a *zonal signature*, Z, which describes it by the *number* of its zoned *features*, i. e. its *zonal faces*, and its *zonal summits*, through which it passes, and its *zonal* and *epizonal edges*, but gives no account of the *number of edges* in the zoned faces or summits, nor of the *order* of the zoned features.

Any edge contained by the zone Z, *is a zonal edge of* Z. *Any edge cut by the zone, is an epizonal edge of* Z.

It will create no confusion if we denote both the zone and its signature by the same name Z.

We represent zonal and epizonal edges by the symbols **0** and 0 (zero faces), and we write the number of such edges as an index over the proper symbol. All such indices in a zonal signature are *coefficients*.

III. a. *Monozone polyedra.*—A monozone polyedron has no symmetry but that of a single zone. It has the zonal signature

$$Z=\{g, G, \mathbf{0}^a, 0^b\},$$

which records that Z has g zoned summits, G zoned faces, a zonal edges, and b epizonal edges.

No two AA′ *of the zoned features of a monozone polyedron have the same configuration, or are one the reflected image of the other.*

For if A were identical in configuration with A′, there would be a *symmetry* of *repetition*, not essential to the zone Z. And if A were the reflected image of A′, there would be a zone different from Z, passing between A and A′.

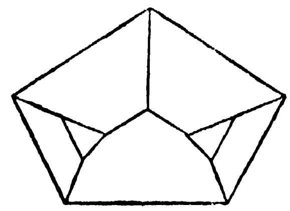
Every zoneless feature, B, *is twice read on a monozone polyedron, viz.* B *and its reflected image.*

The *undrawn* lines of Z are the *zonal traces* of the zoned faces and summits.

The *trace* of a zone is *agonal*, *diagonal*, or *monogonal*, according as it passes through one angle, two angles, or one angle only, of the face or summit.

Monogonal traces are seen only in *odd-angled* faces or summits.

A monozone 8-edron 12-acron is . . . .



whose zonal signature is

$$Z=\{2, \mathbf{2}, \mathbf{0}^1, 0^1\}.$$

IV. *Zoned axis.*—Any number of zonal planes may have a common line or axis, which is *a zoned axis.*

Def. *An axis is janal, whether it be zoned or zoneless,* if the configuration C, or if the reflected image of C, which is read at one extremity of the axis, can be read by an opposite eye at the other extremity, by turning the axis through any angle.

In particular, *a janal zoned axis* is said to be *objanal*, when the configuration C read at one *pole or extremity* is the configuration C′, read at the other pole (by an opposite eye in the axis), turned through two right angles, C′ being C inverted.

Also a *janal zoneless axis* is said to be *contrajanal*, if the configuration C read at one pole is the reflected image of the configuration C′ which can be read by an opposite eye at the other.

Def. *An axis, whether zoned or zoneless, is heteroid,* if the configuration C read at one pole cannot be read at the other, nor read inverted, nor reflected.

V. According to the character of the *polar features terminating an axis*, zoned or zoneless, the axis is *amphiedral*, *amphigonal*, *amphigrammic*, *gonogrammic*, *edrogrammic*, or *gonoedral*, terms which explain themselves.

An axis is *m*-zoned, when it is the intersection of *m* zones.

A zoned amphigrammic, edrogrammic, or gonogrammic axis is of necessity two-zoned; for a polar edge may be zonal in one zone, and epizonal in another, but cannot belong to a third.

VI. Theorem. *There are two different hemizonal sequences of configuration, and only two, read alternately upon the hemizones about a zoned axis.*

For, 1°, there cannot be fewer than two; because if two contiguous hemizones had exactly the same configuration, they would be each the reflected image of the other, and therefore not contiguous, but on opposite sides of a different zone.

And 2°, there cannot be more than two; for every hemizone A has on either hand, by definition of a zone, the same hemizone B, and B has on either hand the same hemizone A.

The propositions of the following article are easy deductions from the above theorem. We use the term perpendicular in a wide sense, in consideration of art. II.; that is, under the assumption of the greatest possible symmetry.

VII. When there is an even number of zones about a zoned axis, each is perpendicular to another, and has two identical hemizones.

If a $2m$-zoned axis is the only one of the solid, there are two different *entire zonal configurations* about the axis, viz. m zones Z alternating with the m zones Z′. The signatures of Z and Z′ may or may not be different; the configurations cannot be the same.

When there are $4m$ zones, each is perpendicular to one of the same configuration. When the number of zones is $4m+2$, each is perpendicular to one of a different configuration.

When the number of zones is odd, none is perpendicular to another.

When the axis is $2m$-zoned, there are, in each polar feature, m traces t alternating with m traces t'.

When the axis is $(2m+1)$-zoned, the polar face or summit has $2m+1$ identical traces.

About a $(2m+1)$-zoned axis there is *but one entire zonal configuration*, and consequently but one signature for all the zones. These zones have *not* each two identical hemizones.

Whether the r zones be odd or even, the 2r *semi-traces* present alternate configurations in the circuit of the pole.

Each trace has two like terminations, if r be even, and two different terminations, if r be odd.

VIII. Theorem. *If a zoned axis is the only axis of the polyedron, that axis is heteroid.*

For if not, the opposite poles will be either identical, or one the reflected image of the other.

If they are identical, there is an axis of *even repetition* perpendicular to the zoned axis, *i. e.* an axis about which in revolution of the solid, the same configuration is $2m$ times repeated to the eye; which is contrary to hypothesis.

If they are one the reflected image of the other, there is, at right angles to the zoned axis, a zone, whose intersections with those of the zoned axis are other axes of the solid; which is contrary to hypothesis. Wherefore the theorem is proved.

IX. b. *Zoned monaxine heteroids.*—A polyedron whose only axis is an m-zoned axis, is an m-*zoned monaxine heteroid polyedron.*

When m is odd, the zonal signature is

$$Z=\{(\sigma_p+g),\ (f_p+G)\ 0^a\ 0^b\},$$

where

$$\sigma_p+f_p=2$$

shows the number σ_p of polar summits, and f_p of polar faces, beside g *different* non-polar zonal summits, G different non-polar zonal faces, a different non-polar zonal edges, and b different non-polar epizonal edges.

The axis may be amphiedral, amphigonal, or gonoedral.

When m is even, the two zonal signatures are

$$Z=\{(\sigma_p+2g),\ (f_p+2G),\ 0_p^\alpha\ 0_p^\beta\ 0^{2a}\ 0^{2b}\},$$
$$Z_{\prime}=\{(\sigma_p+2g_{\prime}),\ (f_p+2G_{\prime}),\ 0_p^\alpha\ 0_p^\beta\ 0^{2a_{\prime}}\ 0^{2b_{\prime}}\},$$

where

$$\sigma_p+f_p+\alpha+\beta=2,$$
$$(\alpha=0=\beta \text{ if } m>2 \text{ (art. V.)})$$

describes the poles, which may give six different characters to the axis (V.), one for every solution of the equation $\sigma_p+f_p+\alpha+\beta=2$, where $\alpha=2$ differs not from $\beta=2$ in form.

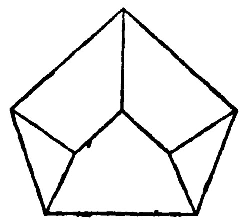
We consider two zoned features, of which one is the reflected image of the other, to be the same in configuration, and enumerate the two as one in our Tables of zoned features.

The above signatures show g, G, a, b, and $g_{\prime}$, $G_{\prime}$, $a_{\prime}$, $b_{\prime}$ for the number of their *different* zoned non-polar features. For each zone has two identical hemizones, when m is even, but not when m is odd (VII.).

The polar edge which $(\alpha+\beta>0)$ is zonal or epizonal in Z, is epizonal or zonal in $Z_{\prime}$. There is nothing to prevent the two zones Z $Z_{\prime}$ from having the same *signature*; but they cannot have the same *configuration*. And we always consider two zonal signatures which differ only in p subscribed, and in the ways of writing the factors of the same number of features, as *numerically the same zonal signature.* This is important to be remembered, when we inspect our Tables in considering the zone Z.

Whether m *be odd or even, every different non-polar feature is read* 2m *times on the solid,* namely, once in each of 2m *interzonal regions.*

A 2-zoned monaxine heteroid 6-edron 8-acron is . . . . . .

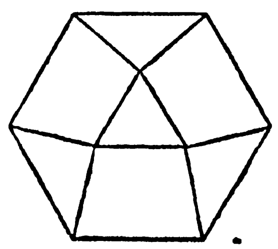


The zonal signatures are,

$$Z=\{(2.1)\ (2.\mathbf{1})\ 0_p^1\ 0_p^1\};$$
$$\sigma_p=0,\ g=1,\ G=\mathbf{1},\ \alpha=1=\beta,\ a=0=b;$$
$$Z'=\{(2.1)\ (2.\mathbf{2}),\ 0_p^1\ 0_p^1\};$$
$$\sigma_p=0,\ g_{\prime}=1,\ G_{\prime}=\mathbf{2},\ \beta=1=\alpha,\ a_{\prime}=0=b_{\prime}.$$

The axis is amphigrammic.

A 3-zoned monaxine heteroid 7-edron 9-acron is . . . . . .



whose signature is

$$Z=\{(1),\ (\mathbf{2}_p+\mathbf{2}),\ 0^3\};$$
$$\sigma_p=0,\ g=1,\ f_p=\mathbf{2},\ G=\mathbf{2},\ a=0,\ b=3.$$

The polar triangle has three monogonal traces; the polar hexagon has three agonal traces.

X. *Principal zoned axis. Secondary axes.*—A *principal axis* has more zones or not fewer zones than any other. The zones which intersect in it are *principal zones.*

Theorem. *If there be but one principal zoned axis,* A, *it is janal; the solid has one, and but one, zone besides those of the axis, and this zone is perpendicular to the axis* A.

For Z″, a zone not containing the principal axis, cannot divide the solid into halves of which one is the reflected image of the other, unless the axis is janal and at right angles to Z″. As the axis can be perpendicular to but one zone, the truth of the theorem is evident.

This Z″ is called, when $m>2$, the *secondary zone of the polyedron,* and its intersections with the zones of the principal axis are the *secondary zoned axes of the polyedron.*

XI. Theorem. *A* 2m*-zoned principal axis has* 2m *2-zoned secondary axes, all janal; the* 4m *secondary poles of these axes are in the secondary zone, of two alternate configurations, and the alternate secondary axes are different.*

The $2m$ axes are 2-zoned, being each the intersection of the secondary with a principal zone. The rest is evident, if we make a section of the solid in its secondary zone; for the 2m *traces* of this section are the 2m *axes* (VII.).

Theorem. *A* (2m+1)*-zoned principal axis has* 2m+1 *secondary axes. Their* 4m+2 *secondary poles are in the secondary zone, in the circuit of which they present two alternate configurations. The secondary axes are all 2-zoned, all alike, and all heteroid.*

XII. d. *Zoned monarchaxines.*—A polyedron having only one principal m-zoned axis is an m-*zoned monarchaxine janal polyedron,* which is also sufficiently described as an m-*zoned monarchaxine* (X.), where the number m does not include the secondary zone.

When $m=2$ no axis is principal, and there are three 2-zoned axes, any one of which is perpendicular to the other two. These three axes are of three different configurations; for if two semi-axes at right angles to each other were identical, they would reflect each other, and a zone would pass between them; whence it would follow that the third axis were no 2-zoned axis.

c. The solid in this case is called a *zoned triaxine polyedron.* It has three zones of different configurations.

The zonal signatures of a $2r$-zoned monarchaxine ($r>1$) are thus written, ZZ′ being principal zones, and Z″ being secondary:

$$Z=\{2(\sigma_p+\varsigma_p+2g),\quad 2(f_p+\varphi_p+2G),\quad \mathbf{0}_p^{2\alpha},\quad 0_p^{2\beta},\quad \mathbf{0}^{4a},\ 0^{4b}\},$$
$$Z'=\{2(\sigma_p+\varsigma'_p+2g'),\quad 2(f_p+\varphi'_p+2G'),\quad \mathbf{0}_p^{2\alpha'},\quad 0_p^{2\beta'},\quad \mathbf{0}^{4a'},\ 0^{4b'}\},$$
$$Z''=2r\{(\varsigma_p+\varsigma'_p+2g''),\quad (\varphi_p+\varphi'_p+2G''),\quad \mathbf{0}_p^{\beta+\beta'},\ 0_p^{\alpha+\alpha'},\ \mathbf{0}^{2c},\ 0^{2d}\},$$

where

$$\sigma_p+f_p=1$$

is the principal pole, summit or face, in the janal axis; and

$$\varsigma_p+\varphi_p+\alpha+\beta=1,\quad \varsigma'_p+\varphi'_p+\alpha'+\beta'=1,$$

ς_p denoting a polar summit, and φ_p a polar face, show the secondary poles, viz. r of either configuration (XI.). The numbers $gGab$, $g'G'a'b'$ record all the *different* zoned non-polar features of Z and Z', as $g''G''cd$ record those of Z''. The secondary polar edges are zonal in one zone and epizonal in another.

Every non-polar zoned feature of the r *zones* Z, *or of the* r *zones* Z', *occurs four times in each zone* Z *or* Z'. *Every non-polar zoned feature of* Z'' *occurs in it* 4r *times. Thus all such features are read* 4r *times.*

Every zoneless feature of the solid is read in it 8r *times,* in as many interzonal regions.

The principal axis is either amphiedral or amphigonal. The characters of the secondary axes, all janal, vary with the solutions of the equations enumerating their poles.

When $r=1$, the solid is a *zoned triaxine*, whose three signatures have one form, and may or may not be identical signatures. But the three configurations of the zones are always different.

The zonal signatures of a $(2r+1)$-zoned monarchaxine are

$$Z=\{(2\sigma_p+\varsigma_p+2g),\quad (2f_p+\varphi_p+2G),\ 0_p^\alpha\, 0_p^\beta\, \mathbf{0}^{2a}\, 0^{2b}\},$$

$$Z''=(2r+1)\{\varsigma_p+2g''),\ (\varphi_p+2G''),\quad 0_p^\beta\, 0_p^\alpha\, \mathbf{0}^{2c}\, 0^{2d}\},$$

where

$$\sigma_p+f_p=1$$

describes the principal pole, and

$$\varsigma_p+\varphi_p+\alpha+\beta=2$$

describes the secondary poles, of different configurations. Thus $\varsigma_p=2$ gives an amphigonal, $\varsigma_p=\alpha=1$ gives a gonogrammic axis, &c.

Every non-polar feature in the principal zone Z *is read twice in the solid in each of the* $2r+1$ *zones. Every non-polar feature in the secondary* Z'' *is read* $4r+2$ *times in* Z''.

Every zoneless feature of the solid is read $8r+4$ *times,* once in each of $8r+4$ interzonal regions.

A $(2r=)$4-zoned monarchaxine janal 22-edron 36-acron is here figured, of which the zonal signatures are

$$Z=\{(\,..\,),\ 2(\mathbf{1}_p+\mathbf{1}_p),\ 0^{4.1}\},$$

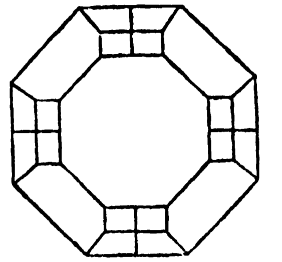
$$\sigma_p=\varsigma_p=g=0,\ b=f_p=\varphi_p=1,\ G=\alpha=\beta=a=0,$$

$$Z'=\{(2.1_p+4.1),\ (2.\mathbf{1}_p),\ \mathbf{0}^{4.1}\},$$

$$\sigma_p=\varphi'_p=0,\ \varsigma'_p=g'=f'_p=1=a',\ G'=\alpha'=\beta'=b'=0,$$

$$Z''=4\{(1_p+2.1),\ (\mathbf{1}_p),\ \mathbf{0}^{2.1}\},$$

$$\varsigma_p=1=g''=\varphi_p=c.$$



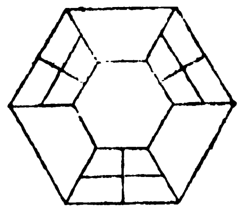
We see, on inspection of these signatures, that the principal axis is *amphiedral*, because Z has four polar faces, of which only two can be secondary poles. We see that Z'' has four polar summits in the two zones Z', and four polar faces in the two zones Z. Hence the secondary axes are alternately amphigonal and amphiedral.

The two zones Z have each four non-polar epizonal, and no zonal edges. The two zones Z' have each four non-polar zonal edges, and no epizonals.

The secondary zone has eight non-polar zonal edges and no epizonals, and it has eight non-polar summits.

Hence we can subtract, on inspection of the signatures, all the zoned faces, summits, and edges from the entire number in the solid, and thus determine the number of *different zoneless features* in each of the eight interzonal spaces.

A 3-zoned monarchaxine 17-edron 27-acron is . . . .

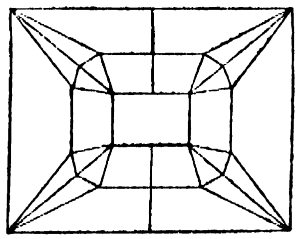


of which the signatures are

$$Z = \{(1_p+2.1),\ (2.\mathbf{1}_p+\mathbf{1}_p),\ \mathbf{0}^{2.1}\ 0^{2.1}\},$$
$$Z''=3\{(1_p+2.1),\ (\mathbf{1}_p),\ \mathbf{0}^{2.1}\}.$$

We read in these signatures that the principal axis is amphiedral, because Z has four polar features of which two faces must be principal poles, because the principal axis is janal. Either zone shows that the secondary axis is *gonoedral.*

A zoned triaxine 30-edron 26-acron is . . . .



of which the signatures are

$$Z = \{(2.1_p+4.1)\ (2.\mathbf{1}_p)\ \mathbf{0}^{4.1}\},$$
$$Z' = \{(\ .\ .\ )\ (2.\mathbf{1}_p+4.\mathbf{1})\ 0_p^{2.1}\ 0^{4.1}\},$$
$$Z''=\{(2.1_p+4.3)\ (\ .\ .\ )\ \mathbf{0}_p^{2.1}\ \mathbf{0}^{4.3}\}.$$

Z and Z′ have each two polar faces, wherefore their common axis is amphiedral. ZZ″ have each two polar summits, and have an amphigonal axis. Z′Z″ have an amphigrammic axis.

The non-polar zoned features are four summits, all alike in Z, and four zonal edges all alike; four epizonal edges alike, and four faces alike in Z′; twelve summits, as also twelve zonal edges, of three configurations in Z″.

There is nothing to prevent the three axes of a zoned triaxine having all one signature, and axes of *any* janal character. In every case the zonal signatures record accurately the configurations.

XIII. *System of principal poles. Zoned polyarchaxines.*—Let $A_1A_2A_3$ ... be a system of poles, zoned or zoneless, of one configuration, not all in one plane, in a polyedron P. If each pole be joined to those nearest it by lines drawn, if necessary, beneath the surface of P, it is evident that these lines will form a polyedron, Q, having edges all of one configuration, *i. e.* each being the intersection of an F-gon and an F′-gon. If F=F′, the polyedron Q is regular. If F<F′, let lines be drawn from the centres *f* of all the F-gons to all their angles. These lines produced will all pass through centres *f*′ of other faces F. This is all inevitable, by reason of our hypothesis that the poles $A_1A_2A_3$ ... have the same configuration. Wherefore the system of lines *ff*′ will form a regular polyedron, having as many edges as there are poles $A_1A_2A_3$ ....

Hence we have the

Theorem. *The number of like poles, zoned or zoneless, of any polyedron, which are not all in one plane, is equal to that either of the summits or of the edges of a regular polyedron.*

XIV. If there be a system of collateral polar summits $A_1A_2A_3\dots$ of the same configuration in any polyedron P, it can be reduced by sections passing only through edges A_1A_2, A_2A_3, A_3A_4, ... to a regular polyedron having only the edges A_1A_2, A_2A_3, A_3A_4, ...

If a system of principal polar summits be not collateral, sections of the solid can be made, removing all the edges of those summits, and laying bare a system of as many principal faces of a solid having fewer edges. The reciprocal of this has a system of principal polar summits, which can be treated like the preceding one; and thus we shall inevitably arrive finally at a solid having collateral principal summits, which reduces by a set of simple sections, as above shown, to a regular polyedron.

It is thus proved that the only systems of principal summits, zoned or zoneless, are those of the regular polyedra.

The following description of zoned polyarchaxines is easily verified by considering the regular solids.

XV. *A zoned triarchaxine polyedron* has three 4-zoned janal principal axes, four 3-zoned objanal secondary axes, and six janal 2-zoned tertiary axes. The axes may have various characters (V.).*

The zonal signatures of the zoned triarchaxine are

$$Z=\{(4s_p+4s_p''+8g)(4f_p+4f_p''+8G)\mathbf{0}_p^{4\alpha''}0_p^{4\beta''}\mathbf{0}^{8a}0^{8b}\},$$

$$Z_{\prime}=\{(2s_p+4s_p'+2s_p''+4g_{\prime})(2f_p+4f_p'+2f_p''+4G_{\prime})0_p^{2\alpha''}\mathbf{0}_p^{2\beta''}0^{4a_{\prime}}\mathbf{0}^{4b_{\prime}}\}.$$

There are six zones $Z_{\prime}$, and three zones Z.

The principal poles are $6(s+f)=6.1.$

The secondary poles are $8(s'+f')=8.1.$

The tertiary poles are $12(s_p''+f_p''+\alpha''+\beta'')=12.1.$

The secondary poles, as well as the tertiary, are of one name only.

The numbers g, G, a, b, $g_{\prime}$, $G_{\prime}$, $a_{\prime}$, $b_{\prime}$, $g_{\prime}$, &c., enumerate the non-polar zoned features which have all different configurations. The sum of these numbers $\geqq 0$.

Every non-polar zoned feature is read 24 *times on the solid. Every zoneless feature is read* 48 *times on the solid, in as many interzonal regions.*

XVI. Def. *A janal zoned axis is homozonal*, when, the axis being horizontal, two opposite eyes can see in the poles at the same time, one the trace t vertical between $t't'$, and the other t' vertical between tt. When there are but two traces in the pole, one eye will see the trace t vertical and t' horizontal, while the opposite eye sees t' vertical and t horizontal; and this is the configuration seen by opposite eyes in the secondary axis of a zoned tetrarchaxine.

A zoned tetrarchaxine polyedron has four heteroid principal 3-zoned axes, and three secondary homozonal 2-zoned axes. It has six identical zones, whose signature is

$$Z=\{2(s_p+s_p+g),\ 2(f_p+f_p'+G)\ \mathbf{0}_p^{\alpha'}\ 0_p^{\alpha'}\ \mathbf{0}^{2a}\ 0^{2b}\}.$$

The principal poles are

$$4(s_p+f_p)=4.2.$$

The secondary poles are

$$6(s'_p+f''_p+\alpha')=6.1.$$

Each of the six zones has $g+\mathrm{G}+a+b \gtreqless 0$ non-polar features all of different configuration.

Every zoned non-polar feature occurs 12 *times on the solid. Every zoneless feature is found* 24 *times, in as many interzonal regions.*

XVII. *A zoned hexarchaxine polyedron* has six objanal (IV.) principal 5-zoned axes, ten objanal secondary 3-zoned axes, and fifteen janal tertiary 2-zoned axes, which may have any terminations.

There are fifteen identical zones, whose signature is

$$\mathrm{Z}=\{2(s_p+s'_p+s''_p+2g),\quad 2(f_p+f'_p+f'''_p+2\mathrm{G}),\quad \mathbf{0}_p^{2\alpha''}0_p^{2\alpha''}\mathbf{0}^{4a}0^{4b}\}.$$

The principal poles of the solid are

$$6(2s_p+2f_p)=6.2,\quad (s_p+f_p=1);$$

the secondary poles are

$$10(2s'_p+2f''_p)=10.2,\quad (s'+f'=1);$$

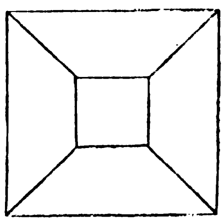
the tertiary poles are

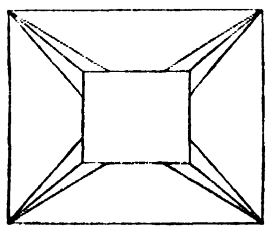
$$15(2s''_p+2f''_p+2\alpha'')=15.2,\quad (s''+f''+\alpha''=1).$$

Every zoned non-polar feature is read four times in each zone, i. e. *sixty times on the solid. Every zoneless feature is read* 120 *times, in as many interzonal spaces.*

Observe that two zoned signatures are numerically identical, if they differ only in p subscribed, and in the mode of exhibiting the factors of a number. There is nothing to prevent any two of the signatures having the same number of summits, face, and edges, of art. (II.... XVII.) from being spoken of as the same signature Z; but the polarity and repetition of the features differ with the symmetry.

For example, the two solids





have, the former the zones

$$\mathbf{Z}=\{(\,..\,)\,(4.\mathbf{1}_p),\ 0_p^{4.1}\},$$

$$\mathbf{Z}'=\{(4.\dot{1}_p)\,(2.\mathbf{1}_p),\ \mathbf{0}_p^{2.1}\},$$

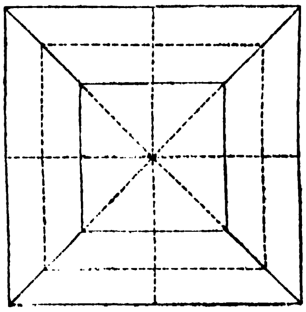
and the latter the zones

$$\mathbf{Z}_{,}=\{(\,..\,)\,(\mathbf{2}_p+2.\mathbf{1})\ 0^{2.2}\}$$

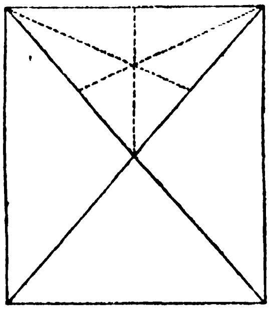
$$\mathbf{Z}'_{,}=\{(2.2)\,(\mathbf{2}_p)\ \mathbf{0}^{2.1}\}.$$

We see that, neglecting p subscribed, Z and $\mathrm{Z}_{,}$ are the same signature, as are Z′ and $\mathrm{Z}'_{,}$. The second solid is a 4-zoned monaxine heteroid, having only two polar zoned features. In the former every feature is polar.

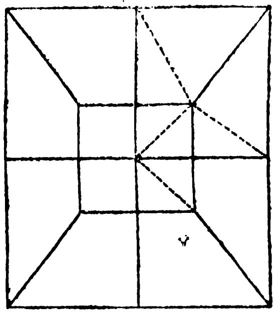
The following will suffice for examples of zoned polyarchaxines. Here are five zoned triarchaxines, in all of which, except the first, only half the solid is seen, the rest being conceived as below the page, which is a zonal plane. The dotted lines are undrawn zonal traces, which are, however, not all exhibited.



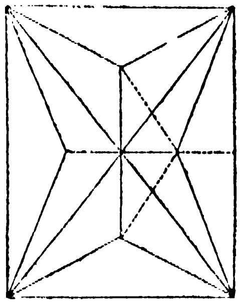
A



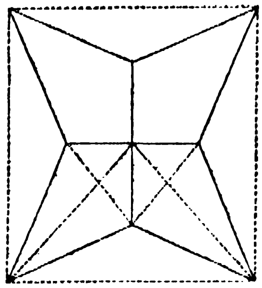
B



C



D



E

The zonal signatures are (art. XV.),—

A. $Z=\{(\ .\ .\ )(4.\mathbf{1}_p)0_p^{4.1}\}$, $Z_{/}=\{(4.1'_p)(2.\mathbf{1}_p)\mathbf{0}_p^{2.1''}\}$;

B. $Z=\{(4.1_p)(\ .\ .\ )\mathbf{0}_p^{4.1}\}$, $Z_{/}=\{(2.1_p)(4.\mathbf{1}'_p)0_p^{2.1''}\}$.

These two are reciprocals, both regular polyedra.

C. $Z=\{(4.1_p+4.1''_p)(\ .\ .)\mathbf{0}^{8.1}\}$, $Z_{/}=\{(2.1_p+4.1'_p+2.1''_p)(4.\mathbf{1}_{/})\mathbf{0}^{4.1_{/}}\}$;

D. $Z=\{(4.1_p)(\ .\ .)\mathbf{0}_p^{4.1''}\}$, $Z_{/}=\{(2.1_p+4.1'_p)(4.\mathbf{1})0_p^{2.1''}\mathbf{0}^{4.1_{/}}\}$;

E. $Z=\{(4.1_p)(4.\mathbf{1}''_p)\}$, $Z_{/}=\{(2.1_p+4.1'_p)(2.\mathbf{1}''_p)\mathbf{0}^{4.1_{/}}\}$.

The solid C has only amphigonal axes, and has eight non-polar edges in Z, all alike, and four non-polar faces, and four non-polar edges, in $Z_{/}$, either all alike. D has principal and secondary amphigonal, and tertiary amphigrammic, axes. The 12-edron E has principal and secondary amphigonal and tertiary amphiedral axes. Four of the six principal polar summits are in the plane of the page. The principal poles of E are tessaraces; the eight secondary poles are triaces; the twelve tertiary poles are quadrilaterals.

None of the solids drawn have zoneless faces, but they may have any number, if we load each of the forty-eight interzonal regions with the same polyedron R, zoned or zoneless. If R is zoneless, we shall impose R twenty-four times and its reflected image twenty-four times, so that the traces may be all preserved. The polyedron R may be polar or not, and symmetrical or not.

What precedes is sufficient to show that our signatures accurately record the zonal configuration. What more is required for placing the polyedron upon record, we shall see when we treat of registration.

2. *Zoneless Symmetry.*

XVIII. The polyedra which have a zoneless symmetry are—

a. r-ple monaxine heteroid polyedra;
b. r-ple monaxine contrajanal polyedra;
c. r-ple zoneless monarchaxine polyedra;
d. zoneless triaxine polyedra;
e. zoneless polyarchaxine polyedra, which have the axial system of the regular polyedra.

An *axis* zoned or zoneless is said to be of m*-ple repetition*, if the same feature presents itself m times in the same posture in a revolution of the solid about that axis.

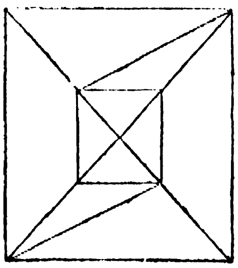
Every r*-zoned axis is an axis of* r*-ple repetition*; for though it has $2r$ like interzonal spaces about it, reflecting each other, no zoneless feature presents itself more than r times in the same posture in a revolution about the zoned axis.

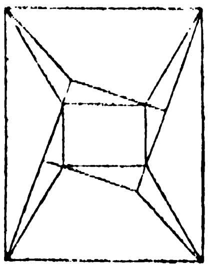
Every amphigrammic, edrogrammic, or gonogrammic axis, zoned or zoneless, is of necessity an axis of 2-ple repetition.

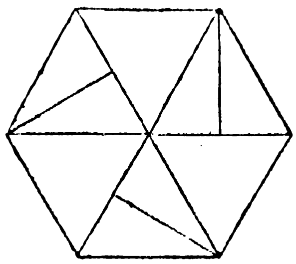
By an r*-ple axis*, when *zoned* is not added, we understand a zoneless axis of r-ple repetition.

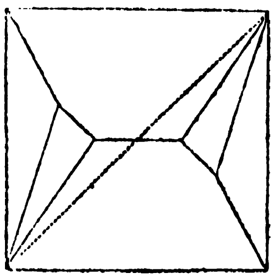
Heteroid r*-ple monaxine polyedron.*—A polyedron which has no zone, and but one zoneless r-ple heteroid axis, is an r*-ple monaxine heteroid polyedron.* The axis may be of any character. In this solid *every non-polar feature is read* r *times, and no more*, namely, once in each repeated sequence which presents itself in revolution about the axis.

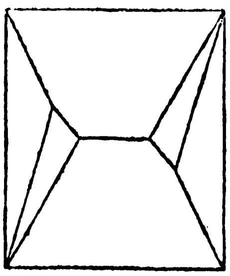
Such solids are the following:—


A


B


C


D


E

A has a 2-ple gonoedral axis; B has a 4-ple amphiedral axis; C has a triple gonoedral, D has a 2-ple amphigrammic, and E a 2-ple edrogrammic axis.

XIX. *Monaxine contrajanal polyedron.*—If we place any r-ple heteroid monaxine P,

($r > 1$), by a $2rm$-gonal face upon a mirror, and then turn P through an angle of m summits, while the image of P remains unmoved, that image, together with P so turned, forms an *r-ple monaxine contrajanal polyedron.* If, for example, $2rm = 4r$, there will be in the mirror a repeated sequence of four edges ABCD, and the configurations read by opposite eyes in the axis supposed parallel to the page and perpendicular to the lines will be ($m=2$),

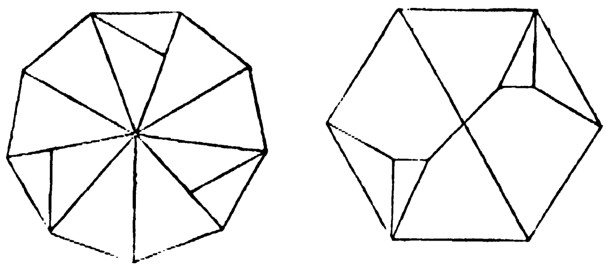
A B C D A B C D A B...
C D A B C D A B C D...

One eye sees A beyond C, and to the right of that B beyond D; the opposite eye sees A beyond C, and to the left of that B beyond D. The configurations are *contrajanal* (IV.).

We may take the last-drawn solid for the polyedron P, and conceive it laid by its square polar face in a mirror, and then turned through $m=1$ summit, while the image remains unmoved. The solid with the image will form a 2-ple monaxine contrajanal.

But it will be found impossible to produce this contrajanal configuration unless by employing a $2rm$-gonal polar face, and by turning the solid through m summits, $r > 1$ being the index of repetition.

The monaxine heteroids,



which have each a repeated sequence ABC, will give by this process attempted, no zoneless figures but monaxine heteroids. This could be easily demonstrated, but such demonstration would be of no future use to us in our problem. And our object here is simply to prove the existence of this class of solids.

Further, the configurations so obtained are not only *contrajanal* but *monaxine.* For there is no zoneless axis in the plane of the mirror, because P and its image in our construction do not form a repeated sequence in revolution about such an axis; and no point of P out of the mirror except the given pole of P can be a pole of repetition.

And as there is evidently still about either of the poles (of P and its image) an r-ple repetition, the indicated construction is an *r-ple monaxine contrajanal polyedron.*

But it is not to be supposed that all these solids have, like the one constructed, a closed circle of *zonoid edges*, viz. the edges in the mirror. But it is evident that such a circle is either drawn or drawable, in any janal polyedron, which shall present in the faces above and below it, the same repeated sequence of faces, either janal or contrajanal, to the two poles. Our object here is not to discuss the form, but to establish the existence, of these polyedra.

XX. *One principal zoneless axis. Zoneless r-ple monarchaxines.*—A principal axis of repetition, zoned or zoneless, has a higher repetition than another.

Let the r-ple heteroid monaxine P be placed by its polar mr-gonal face A in any way on A′ the same face of P′ identical with P, so that the summits of A and A′ may coincide.

As the two opposite poles $\alpha\alpha'$ of the solid PP′ are identical, there is an axis β of double repetition, zoned or zoneless, at right angles to the axis $\alpha\alpha'$, and, as this axis $\alpha\alpha'$ is r-ple, there must be r axes β, having r identical poles p and r identical poles $p_{\prime}$, one of each kind in each repeated sequence about $\alpha\alpha'$.

Let $r>2$, and let $\alpha\alpha'$ be the only principal zoneless axis of PP′. Then β is not zoned; for if it were, it would have at least two zones, neither of which would contain the zoneless poles $\alpha\alpha'$. There would then be at least four poles α, one between each pair of hemizones about β, and $\alpha\alpha'$ would not be the only principal axis. Wherefore β is zoneless, and there are r zoneless axes β.

Let pp' be two like poles of axes β most nearly contiguous. They form with the principal poles $\alpha\alpha'$ a repeated sequence; wherefore there is a zoneless pole $p_{\prime}$ of double repetition between p and p', of a different configuration from p, and there must be r zoneless poles p alternate with r zoneless poles $p_{\prime}$, around the axis $\alpha\alpha'$.

The solid PP′ $(r>2)$ is an r-*ple monarchaxine polyedron. It has* r *secondary axes, of double repetition, which are, according as* r *is even or odd, all janal and alternately different, or all heteroid and alike; and, in either case, the* 2r *poles present alternate configurations.*

In our construction there is a sequence of *zonoid edges* (in the faces A) in general effaceable; but we shall find that there are r-ple monarchaxines which have no such edges effaced, or effaceable, by effacements which shall preserve all summits.

Zonoid signature.—Every r-ple monarchaxine has a *zonoid signature*, which gives an exact enumeration of the *secondary poles*, whether they be faces, poles, or edges, but no account of the number of edges in polar faces or summits. This signature has the form

$$\zeta=r(\sigma_p+f_p+0^{\alpha}_p),$$

where

$$(\sigma+f+\alpha)=2,$$

and where every solution of the latter equation gives a different system of secondary poles. The distinction between the symbols **0** 0 (II.) in the account of zoneless polar edges vanishes. *The poles in* ζ *may be of two names, or of one name only.*

When $r=2$, there is no principal axis, the demonstration that β is zoneless fails, and β *may be* a zoned axis. In such case there are as many axes $\alpha\alpha'$ as there are zones about β perpendicular to $\alpha\alpha'$, and there cannot be more, because $\alpha\alpha'$ being an axis of repetition, must be central in the interzonal space in which it appears; and the limiting zones of that space must evidently be both of the same configuration, for otherwise $\alpha\alpha'$ would be no axis of repetition. In this case the symmetry is mixed, and will presently be discussed.

When $r=2$ and β is a zoneless axis, it follows that β is janal, otherwise $\alpha\alpha'$ would be no axis of repetition; and as the poles of $\alpha\alpha'$ and β form a repetition, there must be a third janal axis γ of even repetition at right angles to $\alpha\alpha'$ and to β.

One of these two axes $\beta\gamma$ may be an m-ple zoneless principal axis, of which the other two axes are secondaries.

Zoneless triaxines.—When $m=2$, the three zoneless axes $\alpha\alpha'$, β, and γ are all double janal axes, and the solid is a *zoneless triaxine polyedron.*

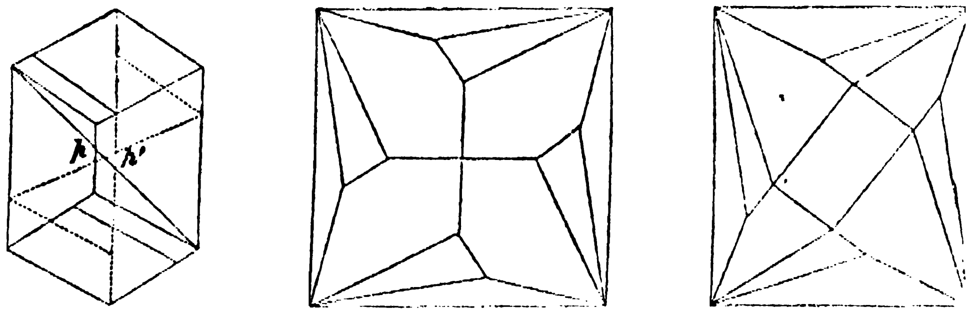
The zonoid signature of a zoneless triaxine, when it is recorded, has the form

$$\zeta=2\{\sigma_p+f_p+0_p^\alpha\},\quad (\sigma_p+f_p+\alpha=3),$$

showing six poles of three, of two names, or of one only. But we shall see that the registration of these signatures in zoneless triaxines is of no use for our purpose.

Every non-polar feature of an r*-ple monarchaxine or* (r=2) *triaxine is read* 2r *times upon the solid, namely,* r *times about either extremity of the* r*-ple janal axis.*

The following are such solids, in the two last of which only half the solid is seen, the other half, identical with that seen, being supposed below the paper.



A B C

The solid A, in which the dotted lines are below the page, is a zoneless triaxine whose zonoid signature is $(r=2)$,

$$\zeta=2\{1_p+1_p+0_p^1\},$$

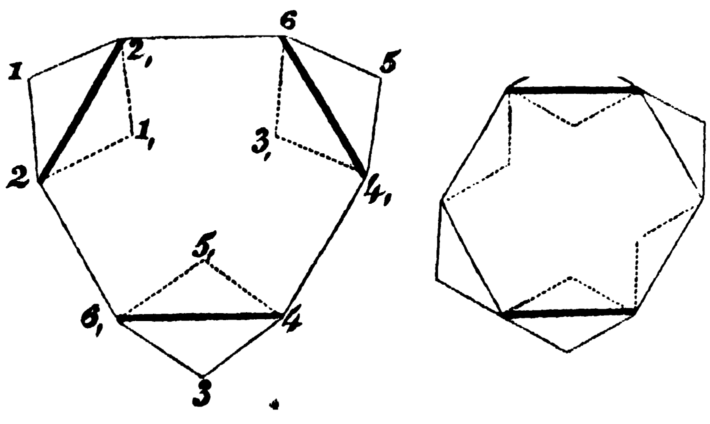
the three axes being amphigonal (pp'), amphiedral, and amphigrammic.

The solids B and C are 4-ple monarchaxines, half seen, whose zonoid signatures are

$$\zeta=4\{1_p+0_p^1\} \text{ for B and for C.}$$

The former has an amphigonal, the latter an amphiedral principal axis.

A simple mode of constructing zoned monarchaxines is to draw such *reticulations* as these, where the dark edges are effaceables:



The first is a 3-zoned monarchaxine, the second a zoneless triaxine, reticulation, having in the page an amphigonal and an amphigrammic axis. If we crown the former in one

polar face with the hexace 123456, and in the lower polar face with the hexace $1_,2_,3_,4_,5_,6_,$, the solid completed will be a triple monarchaxine. If we crown the opposite polar faces of the latter with tessaraces upon the marginal triangles, we form a zoneless triaxine polyedron.

XXI. *System of principal zoneless axes. Zoneless polyarchaxines.*—The following propositions have been sufficiently established by the proof in arts. XIII., XIV., that every system of principal axes is that of a regular polyedron.

We give an account in the zonoid signature of a polyarchaxine, of its *secondary and tertiary poles.*

A zoneless triarchaxine has three principal 4-ple janal axes, four secondary triple janal axes, and six double janal tertiary axes. Its zonoid signature is

$$\zeta=\{8(\sigma_p'+f_p')+12(\sigma_p''+f_p'')+0_p^{12\alpha''}\},$$

where

$$\sigma_p'+f_p'=1$$

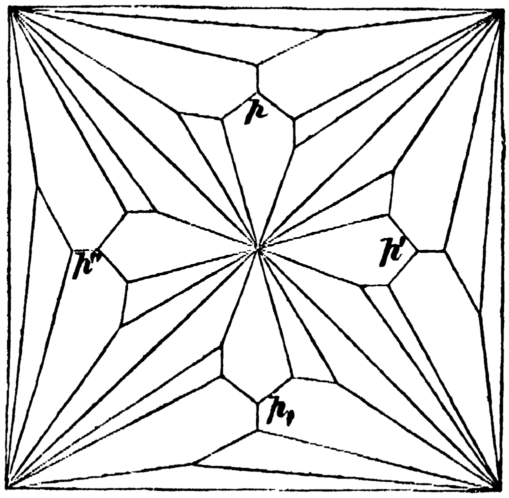
describes the secondary, and

$$\sigma_p''+f_p''+\alpha''=1$$

the tertiary poles.

The poles of the same rank are of one name only. Every non-polar feature is read twenty-four times in the solid, viz. four times about each pole of every principal axis.

Such a solid is here half-drawn, the portion unseen being identical with that seen, and below the page.



This triarchaxine 72-edron 62-acron has amphigonal principal and ($pp_,p'p''$) secondary axes, and amphigrammic tertiary axes. If we efface the twelve polar edges of the tertiary axes, we obtain a 60-edron 62-acron having amphiedral tertiary axes.

A *zoneless tetrarchaxine polyedron* has four triple heteroid principal axes, and three double janal secondary axes. Its zonoid signature is

$$\zeta=\{+6(\sigma_p'+f_p')+0_p^{6\alpha'}\},$$

where

$$\sigma_p'+f_p'+\alpha')=1$$

describes the secondary poles.

The principal poles may be of one name, or of two names.

Every non-polar feature is read twelve times on the solid, viz. thrice about the pole p of every principal axis.

A *zoneless hexarchaxine polyedron* has six quintuple janal principal axes, ten triple janal secondary axes, and fifteen double janal tertiary axes. Its zonoid signature is

$$\zeta=\{20(\sigma'_p+f''_p),\quad 30(\sigma''_p+f''_p)+0_p^{30\alpha''}\},$$

where

$$\sigma'_p+f'_p=1$$

gives the secondary, and

$$\sigma''_p+f''_p+\alpha''=1,$$

the tertiary poles.

Every non-polar feature is read sixty times on the solid, viz. five times about each extremity of every principal axis.

Zoneless polyarchaxines are easily constructed on zoned ones, by drawing lines in every principal, or secondary, or tertiary face, by which the zones are destroyed, and the repetition preserved; or by crowning like polar faces with polyedra which have a zoneless repetition of equal rank.

We shall see, by our processes of construction, that we always know the *name* of the pole opposite to any zoned or zoneless pole that we may be handling; for we always know the character (V.) of the axis; but the number of edges in that opposite pole is not thereby given.

We know of course the edges of every pole that we construct, but when the axis is heteroid, we do not always know the exact edges of the pole opposed.

Hence it may happen, in high values of P and Q, that we do not always know the exact feature opposite to a given principal pole of a tetrarchaxine. But this is not of the least consequence in our problem; and if we should wish to know what is the exact feature so opposite, a question that never arises in our argument, we can easily determine the point by other considerations.

3. *Mixed symmetry.*

XXII. The polyedra which have a mixed symmetry are—

a. r-zoned homozone polyedra.

b. r-ple monozone monaxine polyedra.

Homozone axes.—Let M M′ be two identical r-zoned polyedra having a $2rm$-gonal polar face F. There are m edges of F between two contiguous traces (VII.), wherefore the traces of F are either all agonal, or all diagonal. Let the identical polar faces FF′ of M and M′ be so united that the trace t of F shall cover the trace t of F′ of different configuration from t (VII.). Let pp' be two contiguous terminations of traces in the united faces FF′; and let PP′ be the poles of the r-zoned janal axis of the solid MM′. The sequence PpP′p' is a repetition; there is therefore a *pole of even repetition* between p and p', which is a zoneless pole, because no zone by hypothesis intervenes between p and p';

and there will be $2r$ of these zoneless poles; one central between every two hemizones about the r-zoned axis. And as the same sequence $PpP'p'$ is read about them all, the $2r$-zoneless poles have all one configuration, or configurations of which one is the reflected image of another.

Let PP′ be the only principal axis of the solid; then, as the pole P can recur only twice in a revolution about the zoneless poles, they are *all of double repetition.* Wherefore there are r-janal and similar axes of zoneless double repetition in a plane perpendicular to the r-zoned axis PP′.

1. Let r be odd; then every zone of the axis PP′ is perpendicular to one of the r zoneless axes, or symmetry would be impossible; and it is consequently *a repeating zone*, of which every non-polar feature is an *objanal monozone feature*, i. e. a feature f diametrically opposite to another which is to an opposite eye the inverted image of f (art. IV.).

Further, when r is odd, the poles $\alpha\alpha'$ of any one of the r zoneless axes are *contrajanal poles*: for, if not, the axis $\alpha\alpha'$ will be strictly janal, such that two opposite eyes in the axis will read exactly the same configurations from left to right: therefore the axis PP′ perpendicular to $\alpha\alpha'$ will be an axis of even repetition, since the same pole α recurs exactly in half a revolution about PP′; which is absurd, because, r being odd, PP′ is an axis of odd repetition (XVIII.).

Therefore $\alpha\alpha'$ is, when r is odd, a contrajanal axis.

2. Let r be even; then because the axis PP′ is of even repetition, the configurations read by two opposite eyes in the axis $\alpha\alpha'$ are strictly identical, and $\alpha\alpha'$ is a zoneless 2-ple strictly janal axis.

When r is even, either zone of M conspires to form the zone of the solid (MM′), as is evident from the position of the trace t upon the trace t'. For this reason the solid is called *homozone*, the two zones of M being confounded together. And this name is conveniently used to designate the solid (MM′), whether r be odd or even.

The axis PP′ is an r-*zoned homozone axis.*

When $r>2$, the axis PP′ is a principal axis, and the solid is a *homozone monarchaxine polyedron.* When $r=2$, there is no principal axis (XX.), as the zoned axis has, like the two zoneless ones, but a 2-ple repetition. The 2-zoned homozone is a *triaxine homozone polyedron.*

The zonal and zonoid signatures of the r-zoned homozone polyedron, for r odd or even, are

$$Z=\{(2\sigma_p+2g)(2f_p+2G)\ 0_p^\alpha\ 0_p^\alpha\ 0^{2a}\ 0^{2b}\},$$

where

$$\sigma_p+f_p+\alpha=1,\text{ and }\alpha=0,\text{ if }r>2,$$

$$\zeta=2r\{\varsigma_p+\varphi_p+0_p^\theta\},$$

where

$$\varsigma_p+\varphi_p+\theta=1$$

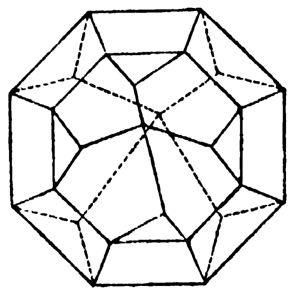
describes the zoneless pole.

Every non-polar zoned feature is read on the solid 2r *times*, namely, once in each hemizone about the zoned axis.

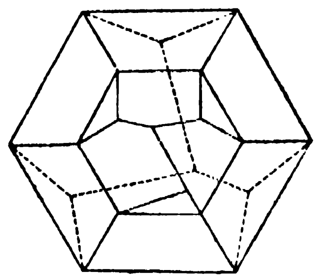
Every non-polar zoneless feature is read 4r *times*, viz. twice about each of the $2r$ zoneless poles.

In our construction the edges of the united faces FF' form a circuit of zonoid edges, generally effaceable, so that the zoneless axes may become amphiedral. But we shall learn that there are homozones which have no such lines effaced or effaceable.

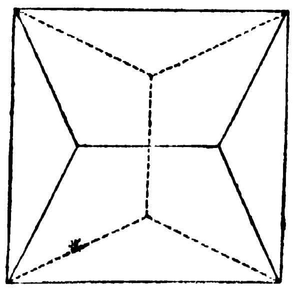
Homozone polyedra are the following:—



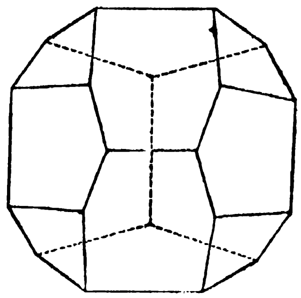
A



B



C



D

In the first the 4-zoned axis has polar tessaraces, and the four 2-ple zoneless axes are amphigrammic. The signatures of A are—

$$Z=\{(2.1_p+2.1)(2.\mathbf{3})\ \mathbf{0}^{2.1}\ 0^{2.2}\}(f_p=\alpha=0),$$
$$\zeta=\ 8\{0^1_p\},\quad (\sigma_p=\phi_p=0).$$

The signatures of the 3-zoned homozone B are—

$$Z=\{(2.1_p+2.1)\ (2.\mathbf{3})\ (\mathbf{0}^{2.1}\ 0^{2.2}\},$$
$$\zeta=\ 6\{0^1_p\}.$$

It has polar triaces, and three amphigrammic 2-ple zoneless axes.

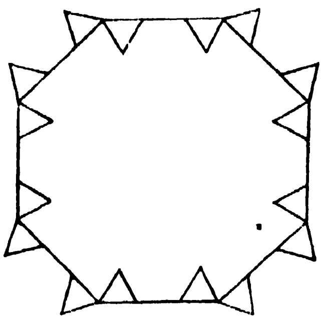
C is a 2-zoned homozone, *i. e.* a triaxine homozone, whose zoned axis is amphigrammic, the two zoneless ones being amphigonal. The signatures are—

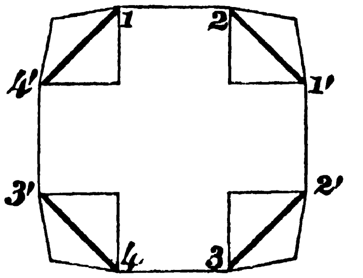
$$Z=\{(2.1)(2.\mathbf{2})\ \mathbf{0}^1_p\ 0^1_p\ 0^{2.1}\}(\sigma_p=f_p=g=a=0),$$
$$\zeta=\{4.1_p\}(\phi_p=\theta=0).$$

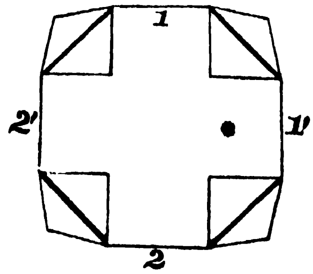
The fourth, D, has an amphigrammic zoned axis, and two amphiedral zoneless axes. Its signatures are—

$$Z=\{(2.1)(2.\mathbf{2})\ \mathbf{0}'_p\ 0'_p\ 0^{2.1}\},$$
$$\zeta=\{4.\mathbf{1}_p\},\ (\varsigma_p=\theta=0).$$

Homozones may be easily constructed by drawing janal reticulations like the following:







The interior marginal triangles are supposed to present in the inferior face of the reticulation precisely the configuration formed by the exterior ones in the upper polar face.

The first is a 4-zoned homozone reticulation, which becomes a 4-zoned homozone polyedron, if crowned in both the upper and lower faces by octaces whose rays pass to the eight marginal triangles.

The second is a 4-zoned monarchaxine reticulation, and becomes a 2-zoned homozone polyedron, if crowned in the upper face by an octace upon the four upper marginal triangles and upon the four summits 1 2 3 4, and if crowned below by an octace upon the four lower triangles and upon the summits 1′ 2′ 3′ 4′. The same reticulation in the third figure becomes a 2-zoned homozone polyedron, if crowned above by a hexace upon the four upper triangles and on the points 1 2, and by a hexace below on the lower marginal triangles and on the points 1′2′, whereby the points 1 2 1′ 2′ become four zoned triaces.

XXIII. *Monaxine monozone polyedra.*—Let F be a polar face of any $\overline{3+r}$-ple zoneless axis α of a polyedron P, and let P be placed on a mirror by the face F. The solid (PP′) formed by P and its image P′ is a $\overline{3+r}$-ple monozone monaxine polyedron.

The zoneless axis $(\alpha\alpha')$ of the solid (PP′) is *contrajanal* (IV.), and in the plane of the mirror there is a $\overline{3+r}$ times *repeating zone*, where r may be odd or even.

The zonal signature of the solid is, putting $r_{,}=3+r$,

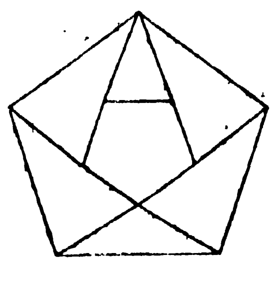
$$Z=r_{,}\{g,\ G,\ 0^{a},\ 0^{b}\},$$

which has $gr_{,}$ zoned summits of g configurations, $Gr_{,}$ zoned faces of G configurations, &c.

Every zoneless feature except the two zoneless poles is read $2r_{,}$ *times on the* $r_{,}$-ple *monozone monaxine.*

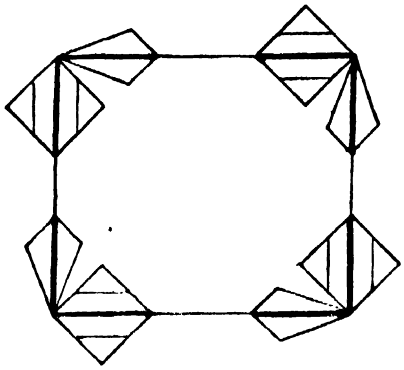
The designation of monaxine contrajanal belongs to these solids in strictness, as well as to those of art. XIX. But there can never be any confusion in our terms, if the zoneless polyedra of XIX. be called by that name, which is to be understood as zoneless, if the term monozone be wanting.*

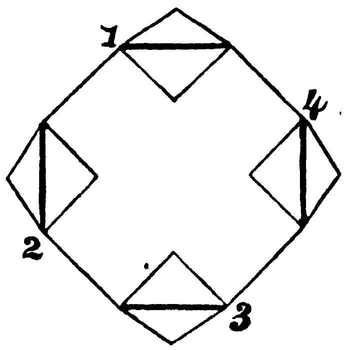
A 2-ple monaxine monozone is . . . .
whose zonal signature



$$Z=2\{1,\ 2,\ 0^{1}\}\ (a=0).$$

The zoneless axis is amphigrammic, 2-ple and contrajanal. Monaxine monozones are readily constructed on simple reticulations like these:





The former is a 4-ple monaxine monozone reticulation, which becomes a 4-ple monaxine monozone polyedron, if crowned in both the opposite polar faces by octaces upon the eight marginal triangles.

The latter is a 4-zoned monarchaxine reticulation, which becomes a 4-ple monaxine monozone polyedron, if crowned both above and below by octaces through the same four points 1 2 3 4 and through the marginal triangles. By such coronation the principal zones of the reticulation are destroyed, but the 4-ple repetition is preserved, and the secondary zone of the reticulation in the plane of the page is preserved also.

XXIV. Theorem. *There cannot be more than one principal axis in a mixed symmetry.*

For we have proved (XIII., XIV.) that the only systems of principal axes are those of the regular polyedra.

In a zoned polyarchaxine there can be no zoneless poles; for if there were one, there would be one at least in every interzonal region, *i. e.* there would be 24, 48, or 120; and such a number of similar poles has been proved impossible in XIII.

For a like reason there can be no zoned pole about any of the zoneless poles of a polyarchaxine, and consequently no zone; for if there were one zone there would be many, and therefore zoned axes and poles. Hence the theorem is proved.

From this theorem it follows that there cannot be in the solid PP′ of the preceding article any other axis than $\alpha\alpha'$. For if there were another, either (1) $\alpha\alpha'$ would be a principal axis, or (2) there would be one principal axis of more than $(3+r)$-ple repetition, or (3) there would be no principal axis.

1. If $\alpha\alpha'$ be a principal axis, every secondary axis β will be at right angles to $\alpha\alpha'$, otherwise there would be more than one axis ($\alpha\alpha'$) in the sequence repeated about β, which is impossible by the preceding theorem; and β will be a $2m$-ple axis, because the pole α occurs in half a revolution about β; but there is no axis of even repetition at right angles to $\alpha\alpha'$ by the reasoning of XXII. because $\alpha\alpha'$ is contrajanal. Therefore $\alpha\alpha'$ is no principal axis.

2. If there be a principal axis A different from $\alpha\alpha'$, since it is not at right angles to the zone FF′, there will be, by the definition of a zone (II.), more than one such axis A, which is impossible by the preceding theorem.

3. If there be no principal axis, there will be at least about the pole of $\alpha\alpha'$, $3+r$

identical poles of $(3+r)$-ple repetition, and these will form by the reasoning of XIII. a polyarchipolar system, which is absurd, if there be no principal axis.

It is therefore impossible that there can be any axis in PP′ except $\alpha\alpha'$, unless $r_{\prime}=2=3+r$ in the preceding article.

If we had taken α a double axis, we might or might not have completed in that article a monaxine monozone. Nothing prevents the existence of other double axes α, each perpendicular to a different zone (F).

The solid (PP′) constructed might thus have been a $(2m+1)$-zoned homozone, having $2m+1$ contrajanal double axes (XXII.).

4. *Neuter Symmetry.*

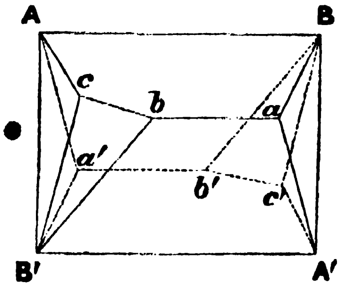
XXV. The polyedra which have a neuter symmetry are of one species only, viz.,

Contrajanal anaxine polyedra.—Let F be any $2m$-gonal zoneless non-polar face of a polyedron P, and let P be placed on a mirror by the face F, and turned through two right angles, while the image P′ remains unmoved. The solid PP′ constructed by P so turned, and by the image P′, *may be a contrajanal anaxine polyedron*. It will be such, if it has neither pole nor zone.

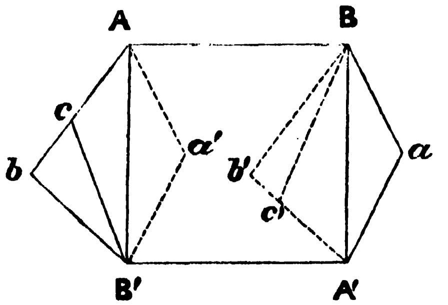
It is evident that to any edge *ab*, on one side of the mirror, there corresponds another diametrically opposite edge $a_{\prime}b_{\prime}$, on the other side, in the image, such that the configuration seen along *ab* to the right is that read along $a_{\prime}b_{\prime}$ to the left. The edges of the solid (PP′) in all cases form *janal anaxine pairs* (ab, $a_{\prime}b_{\prime}$) of edges diametrically opposite, unless P be such that by our construction we have completed a zoned or polar symmetry, in which case certain of the pairs will be zoned or polar.

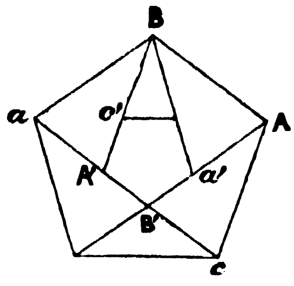
Every feature f *is twice read on a contrajanal anaxine polyedron, namely,* f *and its reflected image.*

Here is a contrajanal anaxine 10-edron 10-acron, where the dotted lines are supposed below the page. The configuration read along *ab* to the left is that read by an opposite eye along *a′b′* to the right, and the same is true of any other pair *bc*, *b′c′*. This solid is formed by crowning with edges *ab*, *a′b′* the *contrajanal penesolid* below on the left.



The same solid is constructed by drawing the *janal anaxine pair*,





AB′, A′B, in the quadrilaterals A′*c′*B*a*, A*a′*B′*c*, of the 2-ple monaxine monozone polyedron on the right, whereby both the zone and the axis are destroyed.

Observe that the term *janal* along with *anaxine* always means *contrajanal.*

XXVI. Def. *A janal anaxine pair on any polyedron are any two edges* ab, a_ib_i *diametrically opposite, non-polar, and zoneless, such that the configuration read along* ab *to the right is exactly that read by an opposite eye along* a_ib_i *to the left.*

A *janal anaxine* A-gon (*or* A-ace) *is any zoneless non-polar face* (*or summit*) *whose* A *edges form with those of an opposite* A-gon (*or* A-ace) A *janal anaxine pairs.*

For example, A′ and A are janal anaxine triaces in the last-drawn polyedron.

Thus we see that there are janal anaxine edges in solids which are not janal anaxine polyedra.

XXVII. It is important that we should here determine what kinds of polyedra have janal anaxine pairs.

1. Let P be a polyedron of zoned or mixed symmetry which has janal anaxine pairs. The janal anaxine pair ab, a_ib_i will either meet on a zone Z, or ab will meet $a'b'$ and a_ib_i will meet $a_i'b_i'$ on a zone Z_i (II.). The pair ab, a_ib_i being diametrically opposite may be supposed parallels. Then the zoneless ab meets its reflected image $a'b'$, and a_ib_i meets $a_i'b_i'$ on Z_i and the angle $(ab, a'b')$ has exactly the configuration of $(a_i'b_i', a_ib_i)$, and is diametrically opposite thereto.

Hence Z must be *a repeating zone*, to which an axis of even repetition is perpendicular, for the same configuration recurs about that axis in half a revolution.

It is then requisite and sufficient, in order that a zoned polyedron have janal anaxine pairs, *that it have a zone perpendicular to an axis, zoned or zoneless, of even repetition.*

XXVIII. The $(2m+1)$-zoned monarchaxine (XII.) has none of its $2m+1$ secondary axes perpendicular to a zone; for each of these axes is in one of the $2m+1$ zones.

The (2m+1)-*zoned monarchaxine has no janal anaxine edge.*

The $2m$-zoned monarchaxine polyedron has every secondary axis in one zone and perpendicular to another (XI., VII.).

The 2m-*zoned monarchaxine has janal anaxine pairs.*

The $(2m+1)$-*ple monaxine monozone has them not*, the axis being of odd repetition.

The 2m-*ple monaxine monozone has such edges.*

The (2m+1)-*zoned homozone has them* (XXII.).

The 2m-*zoned homozone has them not.*

The zoned triarchaxine polyedron has each of its six tertiary axes in two of its nine zones, and perpendicular to one of them, otherwise symmetry would be impossible.

The zoned triarchaxine has janal anaxine edges.

No secondary axis of a zoned tetrarchaxine is perpendicular to any of its six zones, for this axis is in two of them.

The zoned tetrarchaxine has no janal anaxine edges.

Each of the fifteen tertiary axes of a zoned hexarchaxine is in two of the fifteen zones, wherefore it is perpendicular to one of them, or symmetry would be impossible.

The zoned hexarchaxine has janal anaxine edges.

XXIX. 2. Let P be a polyedron of zoneless symmetry which has janal anaxine edges.

The edge ab makes with the zoneless pole α of P a configuration which is the reflected image of that made by the opposite edge a_ib_i with the opposite pole α of a janal axis. The only zōneless polyedron whose opposite janal poles can have such a configuration is the r-ple monaxine contrajanal polyedron (XIX.).

And we see, by considering the repeated sequence of faces above and below any closed line drawn or drawable on the polyedron equidistant from both poles, as, for example, (XIX.),

A B C D A B C D
C D A B C D A B, for r even $(=2)$,

and

A B C D A B C D A B C D
C D A B C D A B C D A B, for r odd $(=3)$,

that the edge $\frac{A}{C}$ diametrically opposite to $\frac{A}{C}$ in the former does not, but that the edge $\frac{C}{A}$ diametrically opposite to $\frac{A}{C}$ in the latter, does present, to two eyes either in the plane of the sequence or at the poles, a contrajanal configuration as compared with the first edge $\frac{A}{C}$.

The same thing is easily proved by taking any sequence of $2m$ faces, and comparing the cases of r odd and r even.

The same thing is also proved thus. If ab, a_ib_i are a janal anaxine pair about a zoneless axis of r-ple repetition, the line bisecting the two edges is a diameter (XXV.), and the plane containing it and the axis is a diametral plane. There are about the axis $r-1$ other similar janal anaxine diameters symmetrically disposed; and such symmetry is evidently impossible unless $r-1$ be even, that is, unless r be odd.

The r*-ple monaxine contrajanal polyedron has janal anaxine pairs if* r *be odd, but not if* r *be even.*

In all other zoneless polyedra, the opposite poles of a janal axis have configurations which are exact repetitions of each other to opposite eyes, wherefore the configuration read by one eye along an edge ab, which configuration includes the pole, cannot be the reflected image of that read along a_ib_i by an opposite eye.

No polyedron of zoneless symmetry, except the monaxine contrajanal, has janal anaxine edges.

XXX. We have proved that janal anaxine edges are found in the polyedra following:—

1. The zoned triaxines (XI.);
2. The $2m$-zoned monarchaxines (XII.);
3. The $(2m+3)$-zoned homozones (XXII.);
4. The zoned triarchaxines (XV.);
5. The zoned hexarchaxines (XVII.);
6. The $2m$-ple monaxine monozones (XXIII.);
7. The $2m+1$-ple monaxine contrajanals (XIX.).

Every zoneless and non-polar edge on these solids is a janal anaxine edge.

The signatures, zonal and zonoid, give the number of polar, zonal, and epizonal edges of the polyedron, whereby that of the zoneless and non-polar is accurately known; and as the number of repetitions of every feature is known by what precedes, *the exact number of different janal anaxine pairs upon all these solids is given by our signatures.*

Registration of P-*edra* Q-*acra and* Q-*edra* P-*acra.*

XXXI. It has been shown that all possible symmetrical P-edra Q-acra are comprised in the preceding classes. They are thus registered.

TABLES A.

1. *Zoned Symmetry.*

1. Monozone P-edra Q-acra (III.),

$$(PQ)\{Z\}=A,$$

where A is the number of monozones which have the zonal signature Z.

2. *Zoned monaxine heteroids* (VIII., IX.),

$$(PQ)Y^{2r}_{het}\ \{ZZ'\}=B,$$

$$(PQ)Y^{2r+3}_{het}\{\ Z\}=C,$$

where B is the number of these solids having the zonal signatures ZZ′ and a heteroid $2r$-zoned axis, of which Y expresses the character and nothing more (V.). In the same way, we record that there are C $(2r+3)$-zoned heteroids, having a given character of axis, and a given zonal signature Z.

We are content to know about these solids what is here registered, without asking what are the exact polar features; for we have a separate table of polar summits and faces, in which every polar A-ace and A-gon is recorded, with its zones, and with the character of its axis. If the question should arise, which, however, never does arise in our problem, what is the exact feature opposite to a given heteroid pole, it can easily be determined by reference to our processes of construction.

3. *Zoned triaxines* (XII.),

$$(PQ)(Y^2_{ja}Y'^2_{ja}Y''^2_{ja})\{ZZ'Z''\}=D,$$

where the zonal signature and characters of the janal axes of the D polyedra are recorded.

4. *Zoned monarchaxines* (XII.),

$$(PQ)X^{2r}_{ja}Y^2_{ja}Y'^2_{ja}\{ZZ'Z''\}=E,$$

$$(PQ)X^{2r+3}_{ja}Y^2_{het}\{\ ZZ''\ \}=F,$$

where X denotes an axis whose polar features are exactly registered, the principal poles being always known on these solids, and recorded with their zonal traces.

5. *Zoned triarchaxines* (XV.),

$$^3(PQ)X^4_{ja}Y^3_{obj}Y'^2_{ja}\{ZZ_{\prime\prime}\}=G.$$

The principal janal poles are always registered with their traces. The secondary and tertiary poles may not be exactly given.

6. *Zoned tetrarchaxines* (XVI.),

$$^{4}(PQ)Y^{3}_{het}Y'^{2}_{ja}\{Z\}=H.$$

The number of edges in the two heteroid principal poles, and in the janal secondary poles, may not always be known in this Table.

7. *Zoned hexarchaxines* (XVII.),

$$^{6}(PQ)X^{5}_{obj}Y^{3}_{obj}Y'^{2}_{ja}\{Z\}=I.$$

2. *Zoneless Symmetry.*

8. *Zoneless* r-*ple monaxine heteroids* (XVIII.),

$$(PQ)Y^{r}_{het}=J\ (r>1),$$

where the absence of zonal and zonoid signature, and the heteroid axis, are characteristic of the class.

9. *Zoneless* r-*ple monaxine contrajanals* (XIX.),

$$(PQ)X^{r}_{coja}=K\ (r>1),$$

where the polar feature is exactly registered.

10. *Zoneless triaxines* (XX.),

$$^{r}(PQ)y^{2}_{ja}y'^{2}_{ja}y''^{2}_{ja}=L.$$

Here are symbols y of double janal zoneless axes, of which not even the characters (V.) are registered. We shall see that it suffices for our problem to know the number L of *all* zoneless triaxine P-edra Q-acra. All the poles of them will be found in the following Table, which enumerates with exact description all janal poles. All that we care to know more of these L solids is how many amphigrammic axes they contain, and this we shall readily determine in the proper place (XLI.).

11. *Zoneless* 2r-*ple*, &c. *monarchaxines* (XX.),

$$(PQ)X^{2r}_{ja}\ Y^{2}_{ja}\ Y'^{2}_{ja}\ \{\zeta\}=M,\ (r>1),$$
$$(PQ)X^{2r+5}_{ja}Y^{2}_{het}\ \{\zeta\}=N,$$
$$(PQ)X^{3}_{ja}\ Y^{2}_{het}\ \{\zeta\}=N',$$
$$(PQ)Y^{3}_{ja}\ Y'^{2}_{(amphi..)}\{\zeta\}=N''.$$

In all these ζ denotes the zonoid signature common, as well as the other characters specified, to all the solids registered in the number M, N, &c.

The only zoneless monarchaxines, in which there can be any doubt about the number of edges in their principal poles, are those which have triple janal axes perpendicular to amphiedral, amphigonal, or amphigrammic secondary axes (XXXVIII.).

12. *Zoneless triarchaxines* (XXI.),

$$^{3}(PQ)X^{4}_{ja}Y^{3}_{ja}Y'^{2}_{ja}\{\zeta\}=P,$$

in which the principal poles are exactly specified, and where the zonoid signature gives the characters of the secondary and tertiary poles.

13. *Zoneless tetrarchaxines* (XXI.),

$$^{4}(\mathbf{P}Q)Y^{3}_{het}Y'^{2}_{ja}\{\zeta\}=Q.$$

We may not know always the edges in the pole opposed to a given tetrarchipole.

14. *Zoneless hexarchaxines* (XXI.),

$$^{6}(\mathbf{P}Q)X^{5}_{ja}Y^{3}_{ja}Y'^{2}_{ja}\{\zeta\}=R.$$

There are but few values of P and Q for which polyarchaxine polyedra are possible; and the ambiguities above denoted with respect to their principal poles can only exist for high values of P and Q, such as will not be calculated for a millennium or two.

3. *Mixed Symmetry.*

15. (r+3)-*ple monaxine monozones* (XXIII.),

$$(\mathbf{P}Q)_{mo.mo}X^{r+3}_{coja}\{Z\}=S,$$
$$(\mathbf{P}Q)_{mo.mo}Y^{2}_{coja}\{Z\}\mp S',$$

where the exact polar features are always known, except for 2-ple axes. This is, however, sufficient for our purpose, as we shall see (XXXIX.).

16. *Homozones* 2r-*zoned and* (2r+1)-*zoned* (XXII.).

$$(\mathbf{P}Q)_{hom}X^{2r}_{ja}\ Y^{2}_{ja}\ \{\ Z\zeta\}=T,\ (r\overline{\overline{>}}1),$$
$$(\mathbf{P}Q)_{hom}X^{2r+1}_{obja}Y^{2}_{coja}\{\ Z\zeta\}=T'\ (r>1),$$
$$(\mathbf{P}Q)_{hom}Y^{3}_{obja}\ Y'^{2}_{coja}\{^{3,6}Z\zeta\}=U,$$
$$(\mathbf{P}Q)_{hom}Y^{2}_{ja}\ Y'^{2}_{ja}\ \{\ ^{4}Z\zeta\}=W.$$

In all these entries the first axis is zoned, the second zoneless. In the third, U, $^{3,6}Z$ denotes a zonal signature of the triarchaxine form, or hexarchaxine, or possibly of both forms. When such a zone occurs in homozones along with a zonoid signature ζ, showing poles of a name not excluded from Z, we may not always know the exact edges of the zoned pole (vide XXXVII., XXXIX.).

When this ambiguity does not exist, the numbers U and W (where ^{4}Z denotes a signature of the tetrarchaxine form) will not be found in our Table; and they can only present themselves for high values of P and Q, which will not be calculated for the next thousand years. But it is enough for us that the numbers U W are exactly known, as this suffices for our problem.

4. *Neuter Symmetry.*

17. *Janal anaxine polyedra* (XXV.),

$$(\mathbf{P}Q)_{ja.an}=\Xi,$$

in which there is neither zone nor pole, Ξ being the entire number of the solids.

5. *Asymmetric Polyedra.*

18. $$(\mathbf{P}Q)_{as}=\Theta,$$

giving the entire number Θ of the solids.

This completes the Tables A of P-edra Q-acra, in which it is understood that every zonal and zonoid signature, and every zoneless repetition (r) will be found entered.

Another such Table is made of the Q-edra P-acra, which will have the form

$$(\mathbf{Q}P)\quad \{Z\}=A,$$
$$(\mathbf{Q}P)Y^{2r}\{Z\}=B, \&c.,$$

where the signatures will differ from the preceding only in the exchange of faces for summits, and of zonal for epizonal edges.

Registration of janal poles of P-*edra* Q-*acra and* Q-*edra* P-*acra.*

Tables B.

XXXII. A janal zoned polar A-gon, which is always the termination of a janal zoned axis, is described by its zonal traces in one of the following manners:—

$$A_{ja}^{m.di},\quad A_{ja}^{m.ag},\quad A_{ja}^{2h.agdi},\quad A_{obj}^{n.di},\quad A_{obj}^{n.ag},\quad A_{ja}^{n.mo},\quad A_{obj}^{n.mo},$$

where m is an odd or even number of traces diagonal or agonal, as the case may be, h is any number of diagonal alternate with as many agonal traces, and n is any odd number of traces, agonal, diagonal or monogonal, as the case may be, in the pole of a janal or objanal axis.

A janal polar zoneless A-gon is described thus by its repetition,

$$A_{ja}^{r},\quad A_{coja}^{r},$$

where r may be odd or even, the axis being janal or contrajanal. The summits of the A-gon form a sequence of configuration r times repeated in the circuit of the A-gon.

Def. *A janal zoned axis is heterozone, if the polyedron has zones of more than one configuration.*

Thus all janal zoned axes are conveniently divided into *heterozone* and *homozone*. A homozone axis or pole has but one zonal signature, as the solid has zones only of one configuration (XXII.). A heterozone axis or pole has always two or three zonal signatures. The $2r$-zoned heterozone pole has two different zones about its axis; the $(2r+1)$-zoned heterozone has only one zonal configuration about the principal axis; but it has also a secondary zone of different configuration (VII.). Hence the term heterozone applies usefully whether the number of principal zones be odd or even.

Heterozone janal polar faces.

$$A_{ja}^{2r.ag}\ sF\{ZZ'Z''\}=a,$$
$$A_{ja}^{2r.di}\ sF\{ZZ'Z''\}=b,\quad \{r\overline{\gtrless}1\},$$
$$A_{ja}^{2r.agdi}sF\{ZZ'Z''\}=c,$$

where Z'' always denotes the zone perpendicular to the axis of the registered janal pole.

The number s is that of the summits of the solid not in the A-gon, *i. e.* inferior to the A-gon; and F is the number of faces different from it, that is, inferior to it.

We have here

$$P=F+1, \quad Q=A+s.$$

We read that there are a A-gonal janal poles having r agonal traces of the zone Z, and r agonal traces of Z'; and a zone Z'' perpendicular to the axis of the A-gons. We read also c A-gonal janal poles having r agonal traces of Z and r diagonal traces of Z', with the secondary zone Z''.

When $2r=4$, the numbers a, b, c, when $ZZ'Z$ are the zoned signatures, will comprise the archipolar A-gons of triarchaxines having the zones ZZ'.

When $2r=2$, the numbers a, b, c enumerate the polar A-gons secondary in zoned monarchaxines, those of zoned triaxines, the tertiary polar A-gons of zoned triarchaxines and hexarchaxines, for the signatures may be $ZZ'Z$, or ZZZ, the case of the hexarchaxines. In this latter case only the term heterozone is improperly applied to any polar face above registered.

All $(2r+3)$-zoned heterozone A-gons are registered thus:

$$A_{ja}^{(2r+3)ag} sF\{ZZ''\}=d,$$
$$A_{ja}^{(2r+3)di} sF\{ZZ''\}=e,$$
$$A_{ja}^{(2r+3)mo} sF\{ZZ''\}=f.$$

Principal janal polar faces of zoned Polyarchaxines.

$$^3A_{ja}^{4y} sF\{ZZ''\}=\theta_1,$$
$$^6A_{obj}^{5y} sF\{\ Z\ \}=\theta_2,$$

where $4y$ means $4ag$ or $4di$, and $5y$ means $5ag$, $5di$ or $5mo$, as the case may be.

Radical janal polar faces of zoned Polyarchaxines.

$$^3A_{rad.ja}^{y} sF\{ZZ''\}=\theta'_1,$$
$$^6A_{rad.obj}^{y} sF\{\ Z\ \}=\theta'_2.$$

It is necessary, as we shall hereafter learn (§ 16.), to have a separate Table of all the *radical* archipoles of the polyarchaxines, which are, however, supposed to be included in the numbers θ_1 and θ_2 above written.

A radical polyarchipole janal or heteroid, whether face or summit, is a principal pole of a polyarchaxine collateral with other principal poles of the solid.

The triarchipoles will be found also under the numbers a, b, c $(r=2)$; for nothing prevents them being constructed and handled as monarchaxine poles.

The hexarchipoles may be constructed and handled as homozone poles, and they will consequently be found also in the Table following. This will, however, create no confusion in our results. There are no *janal* tretrarchipoles ^{4}A.

Homozone janal polar faces (XXII.).

$$A_{ja}^{2ry}\ sF\{Z\zeta\}=g,$$
$$A_{obj}^{(2r+3)y}sF\{Z\zeta\}=h,$$

where y stands for the *traces ag, di, mo, agdi*, as the case may be. Z is the zonal, and ζ the zonoid signature.

When $r=1$, the g homozones recorded are *homozone triaxines* (XXII.).

The number h will contain every secondary polar A-gon of all triarchaxines, and every primary and secondary polar A-gon of all hexarchaxines; for all these faces are in repeating zones to which an axis of 2-ple (zoned) repetition is perpendicular.

The number $g(r=1)$ will contain every secondary polar A-gon of a tetrarchaxine, for this solid has homozone secondary axes (XVI.).

Zoneless janal polar faces.

XXXIII.

$$A_{ja}^{r+3}sF\{\ \zeta\ \}=i,$$
$$A_{co.ja}^{r+3}sF\{Z''_{mo.mo}\}=j,$$
$$A_{co.ja}^{r+2}sF_{mo.co}\quad=k.$$

The first are monarchaxine janals (XX.), with zonoid signature; the second are monaxine monozones, with the signature of the zone perpendicular to the zoneless axis (XXIII.); the third are monaxine contrajanals of $(r+2)$-ple repetition (XIX.).

Zoneless janal polyarchipolar faces.

$$^3A_{ja}sF\{\zeta\}=\varphi_1,$$
$$^6A_{ja}sF\{\zeta\}=\varphi_2.$$

Radical zoneless janal polyarchipoles (§ 16.).

$$^3A_{rad.ja}sF\{\zeta\}=\varphi'_1,$$
$$^6A_{rad.ja}sF\{\zeta\}=\varphi'_2.$$

The register of janal polar faces is completed by the *2-ple janal and contrajanal polar faces*, the contrajanal axis being perpendicular to a zone. The janal are entered thus,

$$A_{ja}^2sF=l,$$

without zonoid signature; for since we can construct no janal symmetry on such a pair of opposite A-gons (as we shall see), but that of a zoneless triaxine, we require no account of the axes perpendicular to α that terminated by the A-gon.

This axis α being janal, must be perpendicular to two other $2m$-ple janal axes, β and γ (XX.); and the polyedron may be either a $2r$-ple monarchaxine, of which α is a secondary axis, or it may be a zoneless triaxine, of which α, β, γ (XX.) are three axes; or it may be, when β or γ is zoned, a $2r$-zoned homozone (XXII.) ($r\overline{\gtrless}1$).

Whatever the polyedron may be, any constructions on the opposite 2-ple A-gons will degrade the symmetry about any higher axis β, zoned or zoneless, to which α may be secondary, and the janal constructions will be all zoneless triaxines. For the entire symmetry about a $2r$-ple axis β cannot be preserved, if $r>1$, by janal constructions on the poles of one only of the r axes α perpendicular to β; that is, β, by constructions on α only, is degraded to a 2-ple repetition.

When β is a $2r$-zoned homozone axis, γ and α are identical in configuration; and any janal construction on the poles of α destroys the zones, and β becomes a zoneless 2-ple axis.

When the 2-ple axis is contrajanal, and perpendicular to a zone, it is entered thus,

$$A^2_{coja}sF\{Z''\}=m,$$

with the zone to which it is perpendicular. The solid on which A is found may be either a $(2r+3)$-ple homozone (XXII.), or a 2-ple monaxine monozone. In either case the A-gon is the reciprocal of a 2-ple A-ace constructed by our processes as a monaxine monozone, having a 2-ple contrajanal axis perpendicular to a given zone, and the entry is made as above. The only possible polar janal constructions on the two contrajanal polar faces are 2-ple monaxine monozones.

Janal amphigrammic poles.

The register of janal poles is completed by the janal polar edges of amphigrammic axes, zoned and zoneless.

The polar edge is registered as the intersection of two A-gons; and we write

$$s'=Q-2A+2$$

for the number of summits of the P-edron Q-acron not in the A-gons, and

$$F'=P-2$$

for that of the faces distinct from them.

$$(AA)^{2agdi}_{ja}s'F'\{ZZ'Z''\}=n,$$
$$(AA)^{2agdi}_{ja}s'F'\{\ Z\zeta\ \}=p,$$
$$(AA)^2_{ja}\ \ s'F'\qquad =q.$$

We read that there are n different polar edges of janal A-gons epizonal in Z and zonal in Z', Z'' being perpendicular to the amphigrammic axis.

We read that there are p polar edges of A-gons, both zonal and epizonal in Z, and perpendicular to zoneless axes of given zonoid signature.

And there are q amphigrammic zoneless axes, whose polar edges are intersections of A-gons. We have no occasion here to register any zonoid signature of other axes.

The n edges may be tertiary poles of zoned triarchaxines or hexarchaxines, or secondaries in zoned monarchaxines, or in zoned triaxines. In the first of these four cases we shall read $Z=Z'$, in the second, $Z=Z'=Z''$.

The p edges are either zoned poles of homozone triaxines, or possibly secondaries of zoned tetrarchaxines (XL.), if Z is a tetrarchaxine zonal signature.

The *q* zoneless polar edges may be tertiaries of zoneless hexarchaxines or triarchaxines, or secondaries of zoneless tetrarchaxines or monarchaxines, or poles of zoneless triaxines or homozone triaxines, or secondaries in *m*-zoned homozones, or axes of 2-ple monaxine monozones, or of 2-ple monaxine contrajanals. In any case we require no account of their symmetry, janal or contrajanal, as these polar edges can never be, like polar faces, the subjects of constructions. It suffices for our purpose that we know exactly the number of all janal zoned and zoneless amphigrammic axes.

Our object here is to state clearly what our Tables are supposed to contain. The mode of obtaining the Tables will be discussed in the sequel.

Registration of polar and non-polar faces of P-*edra* Q-*acra and* Q-*acra* P-*edra.*

TABLES C.

XXXIV. In these Tables will be found all the janal poles of the preceding Tables, as well as all heteroid polar faces. No symmetry is registered here but that of the face, therefore there is no account of poles secondary to the one considered. But the character of the axis (V.) is always suffixed, by one of the abbreviations,

am.go, *am.ed*, *am.gr*, *go.gr*, *go.ed*, *ed.go*.

We shall write x as the symbol of such abbreviation.

Zoned polar faces.

$$A_{x}^{2rdi} \quad sF\{ZZ'\}=a,$$
$$A_{x}^{2rag} \quad sF\{ZZ'\}=b,$$
$$A_{x}^{2ragdi} \quad sF\{ZZ'\}=c,$$
$$A_{x}^{(2r+1)di} sF\{\ Z\ \}=d,$$
$$A_{x}^{(2r+1)ag} sF\{\ Z\ \}=e,$$
$$A_{x}^{(2r+1)mo} s\dot{F}\{\ Z\ \}=f.$$

Here *s* and F are the summits and faces inferior to the A-gon.

We read that there are c A-gons which have *r* agonal traces of Z and *r* diagonal traces of Z″, of which A-gons some will be janal and others heteroid poles.

Zoned Tetrarchipoles.

$${}^{4}A_{x}^{3y} \quad sF\{Z\}=\theta_{3},$$
$${}^{4}A_{radx}^{3y} sF\{Z\}=\theta'_{3},$$

which, being heteroid poles, could not appear in the preceding Tables of polyarchipoles. The *y* denotes the traces.

Zoneless polar faces.

$$A_{x}^{r} sF=g \ (r>1),$$

showing that g A-gons have an *r*-ple zoneless repetition and the character x of axis, of which some will be janal and others heteroid poles.

Zoneless Tetrarchipoles.

$${}^4A^3_x \; sF\{\zeta\} = \varphi_3,$$

$${}^4A^3_{radx} sF\{\zeta\} = \varphi'_3.$$

Zoned non-polar faces.

$$A^{di}\, sF\{Z\} = d',$$

$$A^{ag}\, sF\{Z\} = e',$$

$$A^{mo} sF\{Z\} = f'.$$

Here there is no axis to be characterized.

Objanal monozone faces.

$$A^{di}_{obj} sF\{Z\} = j,$$

$$A^{ag}_{obj} sF\{Z\} = k,$$

$$A^{mo}_{obj} sF\{Z\} = l.$$

These are certain of the above *zoned non-polar faces*, which have an objanal symmetry by reason of their being faces in a *repeating zone*, to which an axis (α) of even repetition, zoned or zoneless, is perpendicular. This axis (α) appears not in this Table, but appears in general in the Tables which we shall learn to construct of *perfect objanal monozone summits*, which are the reciprocals of the faces here registered. Whatever α may be, the only janal symmetrical constructions possible on these faces are objanal monozone summits or reticulations, or simply janal anaxine pairs of edges, of whose construction we shall treat hereafter (§ 17.). The enumeration of our results is not dependent on our knowledge of the zoned or zoneless axis (α).

Zoneless non-polar faces.

Janal anaxine faces (XXVI.).

$$A_{ja.an} sF = h.$$

Asymmetric faces.

$$A_{as} sF = i.$$

This number i includes the h A-gons of the preceding entry of janal anaxine faces.

No face is enumerated under the numbers ghi, which is the reflected image of another.

Similar Tables, B and C, are supposed completed for the Q-edra P-acra. And they are all obtained from those of the reciprocal summits which we shall learn to construct, *and which are conceived as included in these Tables.*

The janal polar summits (Table B) and the polar summits (Table C) can be constructed for both Q-edra P-acra and P-edra Q-acra, by writing summits for faces and zonal for epizonal edges in all the signatures above given.

Registration of edges of P-edra Q-acra and Q-acra P-edra.

TABLES D.

Polar edges.

XXXV.
$$(AA)_x s' F' \{ZZ_i\} = a',$$
$$(AA)_x^2 s' F'_{az} = b',$$

where x is *am.gr*, *go.gr*, or *ed.gr* (V.). Here *az.* means *azone* or zoneless.

$$P = F' + 2,$$
$$Q = s' + 2.A - 2.$$

The amphigrammic polar edges may be either janal or heteroid. When they are janal, they will of course be found both in this Table and in the Table B of janal poles (XXXII., XXXIII.).

Non-polar zoned edges.

$$(AB)_{ep} s' F' \{Z\} = c' \ (B \gtreqless A),$$
$$(AA)_{zo} s' F' \{Z\} = d',$$

which record c' edges, each the intersection of a B-gon and A-gon, and each epizonal in the zone Z, and d' edges of A-gons zonal in Z.

The number d', and also the number c', when B=A, will not include polar edges of A-gons above entered in the zone Z, a signature which may belong equally to polar and to monozone polyedra (IX.).

Janal anaxine edges.

$$(AB)_{ja.on} s' F' = e' \ (B \gtreqless A),$$

showing that there are e' different janal anaxine edges of intersection of an A-gon and a B-gon, on all the P-edra Q-acra.

Asymmetric edges.

$$(AB)_{as} s' F' = f' \quad (B \gtreqless A),$$

which records the entire number f', including e' last registered, of asymmetric intersections of an A-gon with a B-gon on all the P-edra Q-acra.

One at least of the faces about each of the f' edges here registered is zoneless and non-polar, otherwise the edge would not be zoneless and non-polar. One of the faces may be polar or zonal non-polar. Thus we see that asymmetric or janal anaxine edges are found in faces which are not *asymmetric nor janal anaxine faces* (XXVI.).

Our Tables D of P-acra Q-edra would give us a Table D of the above edges in terms of the summits of the edges; but such a Table is of no use to us.

When the Tables A, B, C, D (XXXI. to XXXV.) are completed for P-edra Q-acra and Q-edra P-acra, so that no two features are recorded of which one is the reflected image of another, our problem of enumeration and classification of P-edra Q-acra is perfectly solved; and we shall see that we have the power of continuing such Tables to higher values of P and Q.

It is necessary to determine what *data* are required, and how they are to be employed, for the completion of these Tables; and next to find the means of obtaining these *data*. This will all be discussed, and the requisite general formulæ will be recorded, in the following sections.

SECTION 2.—*Problem of Classification and Enumeration of the* P-*edra* Q-*acra.*

XXXVI. (a) Let us suppose given for all polyedra of fewer than P+Q−2 edges, as far as we require them, the Tables A, B, C, D (XXXI. . . XXXV.).

(b) Let us suppose given for all the P-edra Q-acra, and for all the Q-edra P-acra, P and Q being definite numbers, all the polyarchaxine polyedra, with their poles and signatures, as entered in XXXI., XXXII., XXXIV.

(c) We suppose known also for P-edra Q-acra and for Q-edra P-acra all *janal poles*, with their signatures, as entered in Tables B (XXXII., XXXIII.).

(d) Let us conceive that all polar faces and summits of the same solids are known, as entered in Tables C (XXXIV.), and all polar edges, as entered in Tables D (XXXV.).

(e) Likewise all objanal monozone faces of the same solids, as entered in Tables C (XXXIV.).

(f) And also all monozone faces which have a diagonal trace of a single zone, *i. e.* the number d′ of Tables C (XXXIV.), for all signatures A and Z.

(g) And, finally, let us suppose that all the edges are given of Table D (XXXV.), for P-edra Q-acra, and for Q-edra P-acra.

We suppose given in the data (a), (b), (c), (d), (e), (f), (g), the numbers GHIPQR of Tables A (XXXI.), and all the Tables B, C, D, except only the numbers e′, f′, h, i of Tables C.

All the remaining numbers entered in Table A, and the numbers e′, f′, h, i *of Table C, can be determined by the data* (a), (b), (c), (d), (e), (f), (g).

When this has been proved, and when we have shown that we can obtain the data (a) . . . (g), our problem will be solved.

XXXVII. We have first to show that the data (a), (b) . . . (g) suffice for the determination of the sought numbers in Tables A and C. These numbers we shall consider in the order following:—EFTT′UKMNN′N″S′SDWLBCJAΞΘ in Table A (XXXI.), and e′, f′, h, i in Tables C (XXXIV.).

One difficulty of our problem lies in the danger incurred of enumerating the same solid more than once. Thus, all janal and objanal principal secondary or tertiary poles of zoned or zoneless polyarchaxines will be constructed by our processes simply as janal poles of a single axis: and we cannot be certain, without consideration, when we construct a 5-ple, 4-ple, 3-ple, or 2-ple axis, zoned or zoneless, that we do not thereby complete a polyarchaxine.

For example, if we load all the principal faces but two opposite ones of a regular 12-edron with 5-gonal pyramids (pentaces, ἄκη), we have in the two void faces an amphiedral axis of a 5-zoned homozone polyedron. If in our processes we were to charge this on its two pentagon poles with pentaces, we should certainly thereby construct an amphigonal 5-zoned homozone axis. But it would be an error to enumerate the construction among the homozones, for we have evidently here only completed a hexarchaxine.

In like manner, in constructing a 2-zoned janal axis, we cannot be sure that we have not been completing, by secondary or tertiary poles, a monarchaxine or a polyarchaxine.

E. F.

E. 2r-*zoned monarchaxines* (XXXI.).—When $r>2$, every $2r$-zoned heterozone polar face of Table B (XXXII.) gives a distinct $2r$-zoned monarchaxine of the number E; for no polyarchaxine has a $(6+2m)$-ple pole. Wherefore E is given for $r>2$, by (a, b, c) Table B (XXXII.), and the exact poles of the $2r$-zoned axis are known and can be registered for either polar faces or for their reciprocal summits.

Let $r=2$, $2r=4$; and let $J_{\prime}$ be the entire number of 4-zoned A-gonal poles having given traces (XXXII.) and the zonal signatures $\{ZZ'Z''\}$. Some of these $J_{\prime}$ poles may be principal poles of triarchaxines. When Z'' is not identical with one of the zones ZZ', this cannot happen, because no triarchaxine has three zonal signatures.

In the case then of $ZZ'Z''$ all different, every one of the $J_{\prime}$ poles gives a different 4-zoned monarchaxine, and they can be registered thus in Table A,

$$(\mathbf{P}Q)A^{4y}_{ja}Y^{2}_{ja}Y'^{2}_{ja}\{ZZ'Z''\}=J_{\prime}=E,$$

an entry which gives all that is indicated under the number E; for the principal pole is seen, and the signatures $ZZ'Z''$ give exactly the characters of the other poles and axes. Here y denotes the traces of the A-gonal pole.

Let next $J_{\prime\prime}$ be the number of 4-zoned A-gonal polar faces in Table B which have $Z''=Z'$ and definite traces and signatures; and let j be the number of triarchaxines in 5, Table A, which have a principal A-gonal face with the same traces and signatures $\{ZZ'\}$ (a) XXXVI. We have the number E in this case thus:—

$$J_{\prime\prime}-j=(\mathbf{P}Q)A^{4y}_{ja}Y^{2}_{ja}Y'^{2}_{ja}\{ZZ'Z'\}=E.$$

F. $(2r+1)$-*zoned monarchaxines*.—No zoned polyarchaxine (XV., XVI., XVII.) has $(2r+1)$-gonal poles properly janal. Therefore every janal $(2r+1)$-zoned pole of Table B gives one of the F polyedra required, with its signatures.

TT′. 2r-*zoned and* (2r+1)-*zoned Homozones* (XXXI.).—No polyarchaxine has a $2r$-zoned homozone axis, if $r>1$. Therefore all the poles registered under the number g, Table B, give polyedra here required with all their signatures, of the number T, for $r>1$.

No polyarchaxine has a $(2r+1)$-zoned homozone axis, if $r>2$; therefore all the poles entered under the number h in Table B, give polyedra to be entered under T′ in Table A, with their signatures, for $2r+1>5$. And the numbers TT′ are thus completed for the above values of r.

Let

$$A^{5y}_{0\zeta j}sF\{{}^{6}Z\zeta\}=c$$

be the number of 5-zoned janal poles entered under the number h in Table B, having the zone ${}^6Z$ of hexarchaxine signature (XVII.), y denoting the traces; and let ζ found in the c poles with ${}^6Z$ have a zonoid pole of name not excluded from ${}^6Z$. Then every principal polar A-gon of an hexarchaxine having those traces and the zone ${}^6Z$ is included in the number c. The Tables of XXXI. give us the number c' of the hexarchaxines which have this pole and zone by our hypothesis (a) XXXVI. Wherefore for this value $r=2$,

$$c-c'=T'=(PQ)A^{5y}_{obj}Y^2_{ja}\{{}^6Z\zeta\}$$

is the number T′ required for these signatures.

U. (16. XXXI.). Let m_1 be the entire number of 3-zoned homozone polar faces $A_1A_2A_3\ldots$ under the number h (XXXII.), which have the zonal signature Z_{ii} of the triarchaxine form (XV.) or of the hexarchaxine form (XVII.), or of both, a thing quite possible (XVII.); and let ζ in these homozone poles have a pole of name not excluded from Z_{ii}.

Let m_6 be the entire number of hexarchaxines having any secondary polar faces and this signature Z_{ii}, and let m_3 be the entire number of triarchaxines having any secondary polar faces and this signature Z_{ii}; then

$$m_1-m_3-m_6=U;$$

for all of the m_1 3-zoned homozone poles which are not secondary polar faces of hexarchaxines or triarchaxines, are to be enumerated as poles of 3-zoned homozone polyedra.

K. r-*ple monaxine contrajanals* (9, XXXI.).

XXXVIII. There can be no ambiguity in the enumeration of the r-ple monaxine contrajanals; for by its name and definition the solid can have no pole out of its single axis (XIX.). Wherefore every pole registered under the number k, Table B (XXIII.), gives a distinct polyedron of the number K under consideration.

MNN′N″ (11, XXXI.).

Zoneless monarchaxines.

No polyarchaxine has more than a 5-ple repetition. Hence for $(r+3)>5$, every pole registered under the number i in Table B (XXXIII.) gives one of the M solids with its signature when $r+3$ (Table B) is even, and one of the N solids if $r+3$ is odd.

M. For $r=4$, every janal pole registered in Table B under i, which is not archipolar on a zoneless triarchaxine, gives a 4-ple zoneless monarchaxine of the number M.

The only principal triarchaxine poles in the number i (XXXIII.) are among those 4-ple poles in which ζ has poles of one name only (XXI.). Let n_i be the entire number of 4-ple janal A-gons (i, XXXIII.) having poles of one given name only; and let n_{ii} be the number of zoneless A-gonal triarchipoles in Table A (5, XXXI.) which have tertiary poles of that name. Then n_i-n_{ii} is the number of 4-ple zoneless monarchaxine A-gons having this zonoid signature ζ, and this gives the entry

$$n_i-n_{ii}=(PQ)A^4_{ja}Y^2_{ja}Y'^2_{ja}\{\zeta\}=M \text{ (XXXI.)}.$$

N. For $r=5$ (i, XXXIII.), every janal pole not archipolar on a zoneless hexarchaxine (XXXI.), or not having poles in ζ of one name only, gives a 5-ple zoneless monarchaxine of the number N (XXXI.). When 5-ple zoneless A-gons are constructed which have poles of one name only in ζ, and which are found also in hexarchaxines having tertiary poles of that name, we obtain N in the same way as above by subtraction for $2r+5=5$.

N′. For $r=3$, the Table of triple zoneless poles (XXXIII.) which have not amphiedral, amphigonal, or amphigrammic secondary poles described in ζ, and which axes therefore cannot be tertiary axes in any triarchaxine or hexarchaxine (for these have all janal tertiary axes), gives exactly the number of triple zoneless monarchaxines of the signature.

N″. Let p be the number of triple janal zoneless poles, of all terminating features, of an axis of given character, perpendicular to amphiedral secondary axes, *i. e.* showing in ζ only polar faces; and let p' be the number of zoneless hexarchaxines, and p'' that of zoneless triarchaxines whose secondary and tertiary axes have the same characters (given by the signatures under the numbers P and R (XXXI.)) with those p axes and these secondaries: we have

$$p-p'-p''=(\mathbf{P}Q)Y^3_{ja}Y'^2_{amed}\{\zeta\}=N''_{,}$$

where X^3_{ja} can be written for Y^3_{ja}, if $p'=p''=0$.

In the same manner we can obtain N''_2 for amphigonal and N''_3 for amphigrammic secondary axes. Hence

$$N''=N''_1+N''_2+N''_3 \text{ (11, XXXI.)}$$

is given, and all *zoneless monarchaxines* can be enumerated both for P-edra Q-acra, and for Q-edra and P-acra.

SS′, *monaxine monozones.*

XXXIX. S. Every $(3+r)$-ple pole (j, XXXIII.) of a monaxine monozone gives a polyedron of the number S (XXXI.), with the proper signature Z.

S′ (XXXI.). Let s be the number of 2-ple contrajanal poles (α) of given *character* (V.), under the number m (XXXIII.), and having the zonal signature Z″; and let s' be the entire number of the $(2r+1)$-zoned homozones (XXXII. h) which have the zone Z″, and in their zonoid signature ζ a pole of the name α. Then

$$s-s'=(\mathbf{P}Q)_{mo.mo}Y^2_{coja}\{Z''\}=S' \text{ (XXXI.)};$$

which is thus known for every zone Z″ and for every character of axis.

D. *Zoned triaxines* (3, XXXI.).

XL. The Table B gives (a, b, c, XXXII.; n, XXXIII.) l for the number of 2-zoned heterozone poles of all names which are in zonal signatures ZZ′Z″, Z′Z″Z, or Z″ZZ′, which are all the same, the last-written zone in Table B being perpendicular to the axis carrying the janal pole recorded. If this be identical with one of the first two written zones, the pole *may be* tertiary on a zoned triarchaxine (XV.); if all the three signatures are alike, the pole *may be* tertiary on a hexarchaxine (XVII.); or the pole *may be* secondary on a $2r$-zoned monarchaxine (XII.), whatever be the signatures.

The number l comprises d tertiary poles of hexarchaxines, $d_{\prime}$ tertiary poles of triarchaxines, and $2d_{\prime\prime}$ secondary poles (XI.) of monarchaxines, which all have these zonal signatures. All other poles must be in zoned triaxines having these signatures; whence, as each (XII.) has three poles, we obtain, since d, $d_{\prime}$, and $d_{\prime\prime}$ are given,

$$D=\tfrac{1}{3}(l-d-d_{\prime}-2d_{\prime\prime}) \text{ (XXXI.)}.$$

W. *Homozone triaxines* (16, XXXI.).

There is never any ambiguity about the zoned poles of homozone triaxines, except when the zone, save in p subscribed, is of the form of a tetrarchaxine signature (^{4}Z) (XVI.), and when at the same time the pole named in ζ has the name of the secondary pole of the tetrarchaxine, named in (^{4}Z).

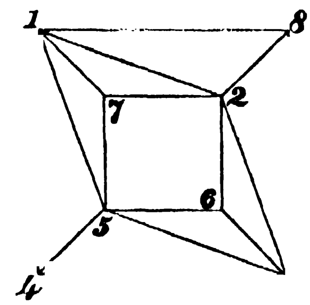
There are under the numbers g, p (XXXII., XXXIII.), $k_{\prime}$ 2-zoned homozone poles of all terminating features of a given name, which have the zone (^{4}Z) containing, though of course without p subscribed (XVII.), the feature named as secondary pole in ζ; and there are (6, XXXI.) $k_{\prime\prime}$ tetrarchaxines having this zonal signature, and the secondary pole named in ζ.

Wherefore $$k_{\prime}-k_{\prime\prime}=W$$

is the number of homozone triaxines which have these signatures $\{^4Z\zeta\}$.

It is worth while to show how the ambiguity spoken of can arise.

If we charge all the faces of a tetraedron P with tetraedra, and then efface two opposite edges of P, we obtain the solid here figured, where 1 2 3 5 are the summits of P, and 52 and 13 are the effaced edges. This is a homozone triaxine, in which the edges 32, 21, 15, 53 of P have become zoneless polar edges.



This homozone will be registered in our Table; and if, as will inevitably happen in our processes, we draw the polar edges 52, 13, we construct a homozone amphigrammic axis to which two 2-ple axes are perpendicular. But it would be an error to register the construction as a homozone triaxine; for it has a tetrarchaxine signature, and the same zonoid poles which the figured homozone has, and which can be made to appear by p subscribed in the zonal signature. We have simply completed a zoned tetrarchaxine. And this ambiguity may occur in far more complex constructions, in which the secondary poles may be faces or summits.

If, however, when we have completed a 2-zoned homozone by coronation either with lines as above, or with a pair of summits, we find that we have completed a zonal signature of tetrarchaxine form, which has no zonal faces, while our zonoid signature ζ shows a polar face, we have completed no tetrarchaxine.

L. *Zoneless triaxines* (10, XXXI.).

XLI. The Table B (XXXIII.) of janal poles (lmq) gives the entire number of 2-ple janal and contrajanal zoneless poles. Let these be

$$\mu=\mu'+q,$$

μ' being the number of amphiedral and amphigonal together, and q that of the amphigrammic.

Every zoneless polyarchaxine has one double zoneless pole, secondary or tertiary; every homozone polyedron has one; every $2r$-ple monarchaxine has two (XX.), and every 2-ple monaxine contrajanal has one; and these, by what precedes, are all enumerated. The zonoid signatures show in all cases how many of these secondary and tertiary or sole axes are amphigrammic. Let the number be

m_1 of amphigonal and amphiedral together;

m_2 of amphigrammic poles.

All the μ poles, except m_1+m_2, are poles of zoneless triaxines; for they can be nothing else, these being the only janal polyedra not already disposed of.

As each triaxine (XX.) has three of these poles, then are

$$\tfrac{1}{3}(\mu-m_1-m_2)=\text{L (10, XXXI.)}$$

zoneless triaxines, on which are $q-m_2$ amphigrammic axes. Thus *the entire number of polar edges on these* L *solids is known, and consequently the number of their different non-polar edges.*

We have thus demonstrated that we can enumerate by the data of art. XXXVI. all the janal polar P-edra Q-acra and Q-edra P-acra.

B, C. *r-zoned monaxine heteroids* (2. XXXI.).

XLII. Let δ be the number of different poles, in Table C (XXXIV.), terminating an r-zoned axis of given character and of given zonal signature {Z} or {ZZ″}. Let δ' be the number of these δ poles found on polyarchaxine monarchaxine and triaxine, heterozone or homozone, polyedra (XXXII.), in the same signature. All the rest must be heteroid poles of which every monaxine heteroid has two. Hence

$$\tfrac{1}{2}(\delta-\delta')=\text{B or }=\text{C (XXXI.)},$$

according as r is even or odd, is the number of r-zoned monaxine heteroids having this signature.

J. *r-ple zoneless monaxine heteroids* (8, XXXI.).

XLIII. Let ϵ be the number of zoneless poles terminating on r-ple axis of given character Y, in the Table C (XXXIV.). Let $\epsilon_{\prime}$ be the number of poles terminating r-ple zoneless axes of the character Y in the polyarchaxine monarchaxine triaxine and monaxine contrajanal, zoneless polyedra. This $\epsilon_{\prime}$ is known by what precedes, and $\epsilon_{\prime}=0$ of course, if $r>3$, and if the axis be gonoedral, edrogrammic, or gonogrammic. All the rest of the ϵ poles are on r-ple monaxine heteroids, which contain each two of them. Hence

$$\tfrac{1}{2}(\epsilon-\epsilon_{\prime})=\text{J},$$

and J is given for all values of $r>1$, and for every character of axis.

Thus all polar polyedra, zoned or zoneless, can be exactly registered, and *as the signatures give the number of polar and zoned edges on them all, we know upon them all the number of different zoneless and non-polar edges.*

A. *Monozone* P-*edra* Q-*acra.*

XLIV. Let Z be any zonal signature having b epizonal (or zonal) edges, without p subscribed. Let E be the number of non-polar epizonal (or zonal) edges which are found in the zone under the number c' (or d') in Table D (XXXV.); and let E′ be the number of *different* non-polar epizonal (or zonal) edges read in the zone Z, which will of course here show p subscribed (IX.) upon all the zoned polar P-edra Q-acra. This E′ is given by inspection of signatures, and the instalment of it contributed by different polar polyedra will vary according to the manner in which Z is written in their signatures (IX.) (XVII.).

The number of epizonals (or zonals) in Z found upon monozones will be E—E′, and as each solid has (III.) b of these, we obtain

$$\frac{1}{b}(\mathrm{E}-\mathrm{E}')=\mathrm{A}\ (1,\ \mathrm{XXXI.})$$

for the number of monozone P-edra Q-acra having the signature Z.

But it may be that $b=0$, or that Z has neither zonal nor epizonal edges. In that case Z must have only diagonally traced summits. Let s be the number of these summits, which are by hypothesis non-polar. We have in the datum (f) (XXXVI.) the number d' of diagonally traced summits non-polar in the zone Z, that are found on the P-edra Q-acra which have or have not zoned axes. And inspection of the signatures of all zoned axes previously enumerated tells us how many different non-polar summits in Z these axial solids contain. Let this be d''. Then

$$\frac{d'-d''}{s}=\mathrm{A}$$

is the number of monozone P-edra Q-acra which have the zone Z.

Thus all monozone P-edra Q-acra and Q-edra P-acra of all zonal signatures can be registered, and *the number of their zoned edges being given by their signatures, that of their different zoneless edges is known.*

Ξ. *Janal anaxine* P-*edra* Q-*acra* (17, XXXI.).

XLV. Let G be the entire of janal anaxine pairs entered in Table D (XXXV.) of the P-edra Q-acra; and let G′ be the number of janal anaxine pairs on all the solids enumerated in art. XXX. G′ is known by inspection of signatures of polyedra already constructed. Then, as every janal anaxine polyedron has $\frac{1}{2}(\mathrm{P}+\mathrm{Q}-2)$ of these pairs,

$$\frac{2}{\mathrm{P}+\mathrm{Q}-2}(\mathrm{G}-\mathrm{G}')=\Xi\ \ (17,\ \mathrm{XXXI.})$$

is the number of those polyedra.

Θ. *Asymmetric polyedra* (18, XXXI.).

XLVI. Let J be the entire number of asymmetric edges on the P-edra Q-acra, which are found in Table D (XXXV.). We have shown that the number of repetitions of every zoneless non-polar feature is given for all the symmetrical polyedra; and as the signatures give the number of polar and zoned edges on them all, we know how many different zoneless and non-polar edges are on them all.

Let this number be J'; then

$$\frac{1}{P+Q-2}(J-J')=\Theta$$

is the number of the asymmetric P-edra Q-acra.

XLVII. e', f'. *Monozone faces of* P-*edra* Q-*acra* (XXXIV.).

The number d' (XXXIV.) is supposed already known (f) (XXXVI.).

Let h_{AB} be the number of non-polar edges epizonal in the zone Z (XXXV., *c'*) (g) (XXXIV.), which are the intersection of an A-gon and a B-gon.

Let

$$2h_{AA}+h_{AB}+h_{AC}+\ldots=h_{A(Z)}$$

be the entire number of different edges of A-gons, epizonal in Z, where we write $2h_{AA}$, because these edges are each in two different A-gons.

The Table of polar faces (XXXII.) gives us the entire number of different edges of A-gons that are epizonal non-polar in Z in polar faces. A polar face may have either one trace or two different traces, whose signature is Z; for the two zones, though of different configuration, may have the same signature. In all cases, the number of *different* epizonals in Z of that polar face is given by inspection.

Let $p_{A(Z)}$ be the whole number of non-polar epizonals in Z of polar A-gons. There remain

$$h_{A(Z)}-p_{A(Z)}$$

edges epizonal in Z of A-gons, which edges are in non-polar A-gons. When A is even, each A-gon has two of them; when A is odd, each A-gon has only one of them. Wherefore for A even,

$$\tfrac{1}{2}(h_{A(Z)}-p_{A(Z)})=A^{ag}sF\{Z\}=e' \text{ (XXXIV.)},$$

and for A odd,

$$h_{A(Z)}-p_{A(Z)}=A^{mo}sF\{Z\}=f' \text{ (XXXIV.)}.$$

We can therefore enumerate all the monozone faces of the P-edra Q-acra, and of the Q-edra P-acra, and register each with its trace and zonal signature.

XLVIII. *Objanal monozone faces.*—When Z of the preceding article is the signature of a repeating zone, certain of the A-gons just enumerated will be the objanal monozone faces of (e), XXXVI. It is important that the objanal monozone faces should be separately registered, and it will be necessary to subtract these A-gons when they exist from the number of A-gons just found, in order that the *monozone* A-*gons*, which have a trace of Z, may be understood as not being *objanal monozone*. We shall find that objanal monozone summits are obtained by construction, and by these we know the reciprocal faces.

h, XXXIV. *Janal anaxine* A-*gons of* the P-*edra* Q-*acra.*

XLIX. By (g) XXXVI. and *e'*, XXXV., we know the number

$$2(AA_{ja.an})+(AB_{ja.an})+(AC_{ja.an})+\ldots=(A_{j.a})$$

of janal anaxine edges of A-gons, where we write $2(AA_{ja.an})$, because these edges are each in two different A-gons.

Some of these A-gons are polar and some zoned non-polar, the rest being janal anaxine A-gons.

The *polar zoned* A-*gons* are exactly those found on the first five of the seven classes of polyedra named in art. XXX.; and they are all either heterozone $2r$-zoned A-gons ($r\overline{\gtrless}1$), or homozone $(2r+3)$-zoned A-gons ($r\overline{\gtrless}0$), or hexarchaxine 2-zoned A-gons, which can only improperly be called heterozone, since their (tertiary) axis is perpendicular to a zone identical with its own two zones (XXXII). Now these are precisely the polar A-gons enumerated under the numbers *a*, *b*, *c*, XXXII. ($r>0$), and under the number *h* (XXXII.) for $r\overline{\gtrless}0$.

Wherefore we know by inspection of signatures the number of *different* zoneless edges in these zoned polar A-gons.

The *polar zoneless* A-*gons* of the third and last but one of the seven classes of art. XXX. are all either principal poles of $(2r+4)$-ple monaxine monozones, which are found under the number *j* (XXXIII.), for all odd values of r; or they are 2-ple poles of monaxine monozones, or 2-ple poles of $(2r+3)$-zoned homozones, which 2-ple A-gons are all found entered under the number *m* (XXXIII.).

The polar zoneless A-gons of the last of the seven classes of XXX. are all entered under the number *k* (XXXIII.), for $r=2n+1$.

Inspection of the signatures of repetition in all these polar A-gons, zoned or zoneless, gives us the number of different zoneless non-polar edges, that is of janal anaxine edges, in them all. Let this number be $(A'_{j.a})$.

The *zoned non-polar* A-*gons* are all given by (e) XXXVI., under the numbers (j, k, l) XXXIV.; whence that of their *different* zoneless edges is given also. Let this be (A''_{ja}). Then all the remaining

$$(A_{j.a})-(A'_{j.a})-(A''_{j.a})$$

janal anaxine edges of A-gons must be found in janal anaxine A-gons (XXVI.); and if we consider, as we may without confusion, the $2(AA)_{ja.an}$ above written as double edges, we can say that all these remaining edges are found in no A-gons which are not janal anaxine. Each A-gon has A of them; wherefore

$$\frac{(A_{j.a})-(A'_{j.a})-(A''_{j.a})}{A}=\text{h (XXXIV.)}$$

is the number of the janal anaxine A-gons on all the P-edra Q-acra, or of janal anaxine A-aces in the Q-edra P-acra.

i. *Asymmetric* A-*gons of* P-*edra* Q-*acra* (XXXIV.).

L. We know by (g), XXXVI., the number of asymmetric edges of A-gons of P-edra Q-acra, which are all entered under the number f' (XXXV.). We have

$$2(AA_{as})+(AB_{as})+(AC_{as})+\ldots=A_{as}$$

for this entire number. By inspection of our Table C (XXXIV.), we see that there are in all the polar and zoned A-gons entered under a, b, c, d, e, f, d′, e′, f′ (XXXIV.), which comprise all the symmetric A-gons of Table C, the number A'_{as} of different zoneless non-polar edges in these A-gons. The remaining $A_{as}-A'_{as}$ must be all edges in asymmetric A-gons; wherefore

$$\frac{A_{as}-A'_{as}}{A}\doteq \text{i (XXXIV.)}$$

is the number of asymmetric A-gons of the P-edra Q-acra. If we would know the number i″ of these which are not janal anaxine A-gons, we have

$$\text{i}''=\frac{A_{as}-A'_{as}-A_{j.a}+A'_{j.a}+A''_{j.a}}{A}.$$

As the Tables A, B, C, D are supposed to contain the same account of the features both of P-edra Q-acra and of Q-edra P-acra, we can obtain all the results of XXXVII.... L. alike for both P-edra Q-acra and for Q-edra P-acra; and with every face enumerated, we have also its reciprocal summit.

Thus we have demonstrated, in this second section, that the data of art. XXXVI. are sufficient for the entire completion of the Tables A, B, C, D (XXXI.... XXXV.), for faces and for summits.

All that remains for the complete solution of our problem of classification and enumeration of the P-edra Q-acra and Q-edra P-acra, is that we show how these data (XXXVI.) can be obtained and registered without ambiguity or repetition. We shall consider first the reciprocals of the faces (d) (f) (XXXVI.), and the edges (g) (XXXVI.).

NOTE ON ARTICLES XLIX. AND L.—The number A_{as} in art. L. is intended to comprise *all* edges of zoneless non-polar A-gons, and should have been defined as including the numbers b' and d' of XXXV.; for these, although symmetric edges, are often edges of asymmetric A-gons. The reader will therefore conceive that the numbers $(AA)_{ax.po.}=b'$ and $(AA)_{zo}=d'$, for every zonal signature, are included in the third line of L. And by the first-named edges in the seventh line of L. are meant edges not epizonal in those A-gons. In like manner the number $(A_{j.a.})$ in XLIX. is intended to comprise all zoneless polar edges, and all zonal edges in all zones, of A-gons on the solids of XXX.; for these are often edges of janal anaxine A-gons. The polars are in our signatures; the zonals ought to have been registered in a separate table in XXXV. as *objanal zonal edges, i. e.* edges $(AA)_{ob.zo.}$ (of A-gons) in a repeating zone (XXVII.). These are either diagonal traces drawn in objanal monozone (2A−2)-gons, or crowning edges of (2A−2)-gonal *objanal monozone penesolids*, of which we shall obtain in the sequel an accurate account. This will be all made clear in our applications.

VIII. *On a New Series of Organic Compounds containing Boron.*

By Dr. E. Frankland, *F.R.S.*

Received May 15,—Read May 22, 1862.

The substitution of a compound organic radical for an elementary constituent in inorganic compounds has proved itself to be one of the most important and fertile fields of modern chemical investigation. The application of this species of substitution to the inorganic compounds of metals has called into existence an entirely new and extensive family of organic substances—the organo-metallic compounds, bodies never met with in nature, distinguished by their well-marked affinities, and capable in some instances of effecting, in their turn, numerous substitutions of a like character. The realization of a similar substitution in the case of certain inorganic compounds of nitrogen and phosphorus has, in the hands of Hofmann, not only enriched the science with a host of new and interesting compounds, but has also brought our knowledge of the organic bases to a degree of completeness, which cannot be rivalled in any other class of organic compounds. Lastly, attempts have not been wanting to extend these reactions to the oxygen compounds of the metalloids; and although this portion of the field presents difficulties of a somewhat more formidable character, yet these attempts have not unfrequently been attended with success. Thus nitric oxide has been transformed into dinitroethylic and dinitromethylic acids*; sulphurous anhydride into ethylodithionic and methylotrithionic acids†; and carbonic anhydride into propionic and acetic acids‡.

The last-named reaction, confirming, as it did, the view previously expressed by Kolbe and myself§, that organic compounds in general are nothing more than substitutions of this nature effected in carbonic oxide, in carbonic acid, and possibly in other inorganic compounds of carbon, naturally awakened a desire to extend this inquiry to the oxygen compounds of boron and silicon, which are usually regarded as possessing certain important analogies with carbonic anhydride. With this end in view, boracic ether was submitted to the action of zincethyl by Mr. Duppa and myself. We found that the whole of the oxygen in boracic acid became replaced by ethyl, and in a short communication to the Royal Society ||, we described some of the properties of the remarkable body, boric ethide, thus formed. In the further study of this substance, and the exten-

* Philosophical Transactions for 1857, p. 59.

† Journal of Chemical Society, vol. x. p. 55, and p. 243.

‡ Ibid. vol. xi. p. 103; and Proceedings of the Royal Society, vol. x. p. 4.

§ Ann. der Chem. und Pharm. Bd. ci. s. 257. Proceedings of the Royal Institution of Great Britain for 1858.

|| Proceedings of the Royal Society, vol. x. p. 568.

sion of the research to the homologous methyl compound, I much regret having been deprived of the cooperation of my friend and fellow-labourer who had rendered me such valuable assistance at the commencement of the investigation, but who was reluctantly compelled to abandon its further prosecution.

The first attempt to replace oxygen by ethyl in boracic anhydride was made by exposing the latter in a finely pulverized condition to the action of zincethyl at various temperatures, but it was found that the zincethyl was utterly powerless to effect the desired substitution; neither did the anhydrous acid yield in the slightest degree to WANKLYN'S compound of sodiumethyl and zincethyl, although it was digested and heated with it for several days. There could scarcely be a doubt that the intractability of the anhydride was due in great measure to its total insolubility in the surrounding liquid, and therefore, in order to place it under conditions more favourable for the action of the organo-metallic body, it was converted into boracic ether.

The ether was prepared by ROSE'S process*, which consists in distilling an intimate mixture of sulphovinate of potash and dried borax. The best proportions were found to be two parts by weight of borax, and three parts of the sulphovinate; but the yield of ether was very small, the greater part of the product consisting of alcohol. The removal of the latter by rectification, as recommended by ROSE, involved the loss of much ether; recourse was therefore had to chloride of calcium for its abstraction, a method which gave very satisfactory results, the product of pure ether being more than doubled. The following is a sketch of the process finally adopted. About 3 lbs. of the mixed borax and sulphovinate of potash were put into an ordinary PAPIN'S digester, which was placed in a sand-bath and exposed to a very gradually increasing heat so long as volatile products came over. The crude distillate obtained from several such operations was then treated with about one-fourth of its weight of fused chloride of calcium, and agitated until the latter was dissolved. The liquid now separated into two layers, a lower one consisting of an alcoholic solution of chloride of calcium, and an upper one containing nearly all the boracic ether, which retained only a small proportion of alcohol in solution. The upper layer was decanted and submitted to distillation. It began to boil at about 85° C., but the thermometer soon rose to 118° C., between which temperature and 125° C. the greater part of the remaining liquid passed over and was reserved for the purposes of the investigation. A thick oily liquid remained in the retort, and appeared to consist of boracic acid united with a smaller proportion of oxide of ethyl.

On adding zincethyl to the boracic ether thus prepared, a considerable elevation of temperature gradually occurred, whilst at the same time a most penetrating and peculiar odour was developed, due apparently to the vapour of some volatile body, that not unfrequently burst into flame, when the cork was removed from the flask in which the reaction took place. Some preliminary experiments showed that this volatile body could be distilled unchanged from the mixture, and that it was neither miscible with, nor apparently decomposed by, water. It was also spontaneously inflammable, and the

* POGGENDORFF'S 'Annalen,' Bd. xcviii. s. 245.

beautiful green flame with which it burnt demonstrated the presence of boron as one of its constituents.

In order to prepare this body in sufficient quantity, several ounces of boracic ether were placed in a capacious flask closed by a doubly perforated cork. Through one of these perforations passed a thermometer, and through the other a short glass tube, one-fourth of an inch in diameter, and open at both ends: the bulb of the thermometer dipped into the boracic ether. Successive quantities of pure zincethyl were introduced through the short glass tube by means of a pipette, the elevation of temperature after each addition being allowed to subside before the next portion was added. The failure of a further addition of zincethyl to produce any rise of temperature was regarded as evidence of the completion of the reaction, which was not attained until a comparatively very large amount of zincethyl had been added.

The liquid in the flask was now submitted to distillation in an oil-bath. It began to boil at 94° C., and between this temperature and 140° C. a considerable quantity of a colourless liquid distilled over. The distillation then suddenly stopped, and, to avoid secondary products of decomposition by the application of a greater heat, the operation was interrupted. On cooling, the materials remaining in the flask solidified to a mass of large crystals of ethylate of zinc and zincethyl. On rectification, the distillate began to boil at 70° C., but the thermometer rapidly rose to 95°, at which temperature the last two-thirds of the liquid passed over and were received apart. The product thus collected exhibited a constant boiling-point on redistillation.

The combustion with oxide of copper of this liquid and the remaining boron compounds described in this paper presented some difficulties; owing partly to the volatility of boracic acid in aqueous vapour, and partly to the tendency of that acid when fused to encase particles of carbon and prevent their oxidation. Fortunately the errors thus introduced were not so considerable as to throw any doubts upon the analytical results, although in many cases the excess in the percentage of hydrogen and the deficiency in that of carbon are somewhat greater than usual. To estimate the boron in the liquid obtained as above described, advantage was taken of the complete decomposition of the compound when heated to 100° with concentrated nitric acid in sealed tubes. The whole of the boron was in this way converted into boracic acid, but the latter could not be determined by the direct evaporation of the nitric acid solution, the loss of boracic acid amounting in such an operation to 15 or 20 per cent. of the whole amount. None of the known processes for estimating this acid appeared to be eligible in the present instance, and it therefore became necessary to seek for a new one. After the trial of various methods with but indifferent success, the following experiments showed that the evaporation of the acid solution of boracic acid with a known weight of magnesia in excess, the residue being then ignited, presented a process which, although far from rigidly accurate, could not, in the case of the boron compound to be analysed, diminish the amount of boron to a greater extent than about 0·2 per cent.

I. 1·4310 grm. of boracic acid, treated as above described, suffered a loss equal to 2·55 per cent. of boracic acid, or ·73 per cent. of boron.

II. ·5957 grm. boracic acid, similarly treated, lost ·56 per cent. of boron.

III. 2·2477 grm. boracic acid, similarly treated, lost ·35 per cent. of boron.

IV. 1·1845 grm. boracic acid lost ·51 per cent. of boron.

V. ·4125 grm. boracic acid experienced neither loss nor gain.

VI. ·8398 grm. boracic acid lost ·48 per cent. of boron.

VII. 1·1637 grm. boracic acid lost ·58 per cent. of boron.

VIII. 1·4601 grm. boracic acid lost ·57 per cent. of boron.

IX. 1·6307 grm. boracic acid lost ·47 per cent. of boron.

Submitted to analysis, the new boron compound yielded the following results:—

I. ·1307 grm. gave ·3506 grm. carbonic acid and ·1818 grm. water.

II. ·1144 grm. gave ·3068 grm. carbonic acid and ·1625 grm. water.

III. ·1071 grm. gave ·0380 grm. boracic acid.

These numbers agree with the formula

$$B\begin{cases}C_4\,H_5\\C_4\,H_5\\C_4\,H_5.\end{cases}$$

	Calculated.		Found.			
			I.	II.	III.	Mean.
C_{12} . . .	72	73·55	73·16	73·14	——	73·15
H_{15} . . .	15	15·42	15·45	15·78	——	15·61
B . . .	10·9	11·03	——	——	11·08	11·08
	97·9	100·00				99·84

The new body may be conveniently termed *boric ethide*. It is evidently formed by the replacement of the three atoms of oxygen in boracic acid by three atoms of ethyl, according to the following equation:—

$$\underbrace{2B\begin{cases}C_4\,H_5\,O_2\\C_4\,H_5\,O_2\\C_4\,H_5\,O_2\end{cases}}_{\text{Boracic ether.}} + \underbrace{3\,Zn_2\begin{cases}C_4\,H_5\\C_4\,H_5\end{cases}}_{\text{Zincethyl.}} = \underbrace{2B\begin{cases}C_4\,H_5\\C_4\,H_5\\C_4\,H_5\end{cases}}_{\text{Boric ethide.}} + 6\underbrace{\left.\begin{matrix}C_4H_5\\Zn\end{matrix}\right\}O_2}_{\text{Ethylate of zinc.}}$$

The ethylate of zinc thus produced combines with zincethyl to form the crystalline compound above alluded to. Hence the very large amount of zincethyl which was found necessary to complete the reaction.

Boric ethide possesses the following properties:—It is a colourless mobile liquid of a pungent odour; its vapour is very irritating to the mucous membrane, and provokes a copious flow of tears. The specific gravity of boric ethide at 23° C. is ·6961; it boils at 95° C. A determination of the specific gravity of its vapour by GAY-LUSSAC's method gave the following numbers:—

Weight of boric ethide	·2839 grm.
Observed volume of vapour	96·68 cub. centims.
Temperature of oil-bath	149° C.
Height of barometer	760·5 millims.
Height of mercury inside of tube above that outside .	1·5 millim.
Height of column of oil	328·0 millims.

From these data the specific gravity of the vapour was calculated to be 3·4006. This number agrees very closely with that calculated upon the supposition that boric ethide is volumetrically composed like terchloride of boron, as is seen from the following calculation:—

1 vol. Boron vapour	·75319
3 vols. Ethyl	6·0117
The 4 vols. condensed to 2 vols. . . .	2)6·76489
	3·38244

The density of boric ethide vapour increases considerably as the temperature approaches the boiling-point; thus a determination made at 132° gave the number 3·5979, whilst a second showed the specific gravity of the vapour at 101°·6 to be no less than 3·757.

Boric ethide is insoluble in water, and is very slowly decomposed by prolonged contact with it. Iodine has scarcely any action upon it even at 100° C. It floats upon concentrated nitric acid for several minutes without change; but suddenly a violent reaction takes place, and crystals of boracic acid separate. When boric ethide vapour comes in contact with air, it produces slight bluish-white fumes, which in the dark are seen to proceed from a lambent blue flame. The liquid is spontaneously inflammable in air, burning with a beautiful green and somewhat fuliginous flame. In contact with pure oxygen it explodes. Excluded from the air, boric ethide is quite a stable body; a quantity of it kept in a sealed tube for two years exhibited, on examination, no evidence of any alteration.

When boric ethide is heated to 99° C. with strong hydrochloric acid over mercury, a considerable quantity of hydride of ethyl is slowly evolved; ·0517 grm. of boric ethide, thus treated as long as gas was evolved, gave 35·33 cubic centimetres at 11°·1 C. and 248·4 millimetres mercurial pressure, corresponding to 11·11 cub. centims. at 0° C. and 760 millims. pressure. The reaction

$$B\begin{cases}C_4H_5\\C_4H_5\\C_4H_5\end{cases} + HCl = B\begin{cases}C_4H_5\\C_4H_5\\Cl\end{cases} + \left.\begin{matrix}C_4H_5\\H\end{matrix}\right\}$$

requires that 11·31 cub. centims. of hydride of ethyl at 0° C. and 760 millims. pressure should be liberated. That the gas thus evolved is hydride of ethyl is established by the following data:—

I.

	Millims.	Temp.
Pressure of gas used (dry)	225·3	at 13°·9 C.
Pressure after action of anhydrous sulphuric acid (dry)	223·9	at 13°·9 C.

II.

Pressure of gas used (dry)	23·7	at 14°·7 C.
Pressure after addition of oxygen (dry)	253·9	at 14°·7 C.
Pressure after explosion (dry)	195·6	at 14°·7 C.
Pressure after absorption of carbonic acid (dry)	147·7	at 14°·7 C.

No. I. proves the absence of members of the olefiant gas family. No. II. proves that the gas has the composition and condensation of hydride of ethyl, one volume of which consumes on combustion 3·5 volumes of oxygen, and generates twice its volume of carbonic acid; the following being the experimental numbers:—

Vol. of combustible gas.		Vol. of oxygen consumed.		Vol. of carbonic acid generated.
23·7	:	82·5	:	47·9
1	:	3·48	:	2·01

If boric ethide be heated with water to 99° C. for several hours, it also appears to suffer an analogous decomposition, although with extreme slowness; even with hydrochloric acid, the action is so tedious that I have not been able to prepare a sufficient quantity of boric chlorodiethide ($B(C_4H_5)_2Cl$) to examine its properties. In the cold, a strong solution of hydrofluoric acid has no action upon boric ethide, which also suffers scarcely any change by being heated to 99° C. for four hours with concentrated sulphuric acid. Gently heated for fourteen days with sodium in a sealed tube, boric ethide underwent no visible change.

Ammonia-Boric Ethide.

If a few drops of boric ethide be passed up into a dry eudiometer filled with mercury, and dry ammoniacal gas be then admitted into the same tube, each bubble of gas collapses with a shock, like that produced by a bubble of steam projected into cold water. A large quantity of ammonia is thus absorbed by boric ethide with extreme energy. To prepare the compound thus formed in larger quantity, several grammes of boric ethide were placed in a small flask filled with nitrogen and surrounded with ice: a current of dry ammoniacal gas was now passed into the flask so long as it was absorbed; finally, the product thus obtained was warmed to expel excess of ammonia, and then exposed *in vacuo* over sulphuric acid for twenty-four hours. It did not crystallize, and could not be distilled, except *in vacuo*, without decomposition. Submitted to analysis, it yielded 61·43 per cent. of carbon and 15·43 per cent. of hydrogen. The formula

$$NH_3+B(C_4H_5)_3$$

requires 62·66 per cent. of carbon and 15·66 per cent. of hydrogen. The unavoidable slight oxidation of the boric ethide during the necessary manipulations affords a sufficient explanation of the deficiency in the amounts of carbon and hydrogen exhibited by the analysis. I should, however, have made renewed attempts to obtain this body in a state of greater purity, had not the investigation of the corresponding crystalline methyl compound described below, left no doubt that the formula above given expresses the composition of ammonia-boric ethide.

Ammonia-boric ethide is a somewhat oily liquid, possessing an aromatic odour and an alkaline reaction. Carbonic acid has no action upon it, even in the presence of water, but other acids decompose it instantly and liberate boric ethide. When it is exposed to a measured quantity of atmospheric air, there is scarcely any perceptible absorption of oxygen even after the lapse of several hours.

Boric Dioxyethide.

When boric ethide is placed in a flask and allowed to oxidize gradually, first in dry air and finally in dry oxygen, it forms a colourless liquid, which boils at 125° C., but cannot be distilled under atmospheric pressure without partial decomposition. At the ordinary temperature, this product of oxidation evaporates without residue in a stream of dry carbonic acid. It can be distilled *in vacuo* without decomposition, and a portion so rectified yielded on analysis the following results:—

I. ·2681 grm., burnt with oxide of copper and oxygen, gave ·5359 grm. carbonic acid and ·2720 grm. water.

II. ·2246 grm., similarly treated, gave ·4376 grm. carbonic acid and ·2303 grm. water.

Owing to the causes already mentioned, the complete combustion of this body was very difficult, nevertheless the above numbers agree sufficiently well with the formula

$$B(C_4H_5)_3O_4,$$

as is evident from the following comparison:—

	Calculated.		Found.	
			I.	II.
C_{12} . . .	72·0	55·42	54·52	55·10
H_{15} . . .	15·0	11·54	11·27	11·92
B . . .	10·9	8·39	——	——
O_4 . . .	32·0	24·65	——	——
	129·0	100·00		

I regard this liquid as a compound of vinic ether, with a body having the formula

$$B\begin{cases}C_4H_5\\O\\O\end{cases}$$

and derived from boracic acid by the substitution of one equivalent of ethyl for one of oxygen. For this body the name *boric dioxyethide* is appropriate, whilst its ethereal compound may be conveniently termed diethylate of boric dioxyethide. The formula of the latter will therefore be

$$B\begin{cases}C_4H_5\\O\\O\end{cases} + \left.\begin{matrix}C_4H_5\\C_4H_5\end{matrix}\right\}O_2, \text{ or } B\begin{cases}C_4H_5\\C_4H_5O_2\\C_4H_5O_2.\end{cases}$$

The formation of diethylate of boric dioxyethide from boric ethide may be thus represented:—

$$\begin{cases}C_4H_5\\C_4H_5\\C_4H_5\end{cases} + O_4 = B\begin{cases}C_4H_5\\C_4H_5O_2\\C_4H_5O_2.\end{cases}$$

This view of the constitution and mode of formation of the oxidized product is supported by its behaviour with water; for when diethylate of boric dioxyethide is placed in contact with water it is instantly decomposed, alcohol and *dihydrate of boric dioxyethide* being formed, according to the following equation:—

$$\underbrace{B\begin{cases}C_4H_5\\C_4H_5O_2\\C_4H_5O_2\end{cases}}_{\text{Diethylate of boric dioxyethide.}} + 2\left.\begin{matrix}H\\H\end{matrix}\right\}O_2 = \underbrace{B\begin{cases}C_4H_5\\HO_2\\HO_2\end{cases}}_{\text{Dihydrate of boric dioxyethide.}} + 2\left.\begin{matrix}C_4H_5\\H\end{matrix}\right\}O_2.$$

Dihydrate of boric dioxyethide may be conveniently prepared in a state of purity by agitating its aqueous solution with ether, which dissolves the boric compound. The ethereal solution must then be decanted, and on evaporation at common temperatures in a stream of dry carbonic acid, the new compound is left behind as a white and very volatile crystalline mass. The latter was sublimed at a gentle heat in a current of dry carbonic acid, and was made to condense in weighed tubes for analysis.

I. ·3870 grm., burnt with oxide of copper in a stream of air, gave ·4652 grm. carbonic acid and ·3285 grm. water.

II. ·5087 grm., oxidized with nitric acid in a sealed tube, the liquid supersaturated with a known weight of magnesia and then evaporated to dryness, gave ·2387 grm. boracic acid.

These numbers agree closely with the formula of dihydrate of boric dioxyethide:—

	Calculated.		Found.	
			I.	II.
C_4 . . .	24	32·47	32·78	——
H_7 . . .	7	9·47	9·43	——
B . . .	10·9	14·75	——	14·66
O_4 . . .	32	43·31	——	——
	73·9	100·00		

Dihydrate of boric dioxyethide is a colourless, volatile, crystalline body, very soluble in water, alcohol, and ether. It possesses an agreeable ethereal odour, and a most intensely sweet taste. Exposed to the air, it evaporates at ordinary temperatures, undergoing at the same time partial decomposition, and invariably leaving a slight residue of boracic acid. It may be sublimed without change at about 40° C. in a current of dry carbonic acid, and it then condenses in magnificent crystalline plates resembling naphthaline. It fuses at a gentle heat, and at a higher temperature boils with partial decomposition. Its vapour tastes intensely sweet.

Boric dioxyethide might be regarded as the anhydride of a bibasic acid: the diethylate of boric dioxyethide would then be the ether of this acid, whilst the volatile crystalline body just described would be the hydrated acid itself. The latter does in fact redden litmus paper, but in other respects its acid qualities are very obscure, and I have not been able to form definite salts with it. It therefore scarcely possesses a valid claim to a place amongst the acids.

Considering boric ethide to be formed by the substitution of the ethyl in zincethyl for the oxygen in boracic acid, Mr. DUPPA and myself expressed the reaction as follows:—

$$\underbrace{2B\begin{cases}C_4H_5O_2\\C_4H_5O_2\\C_4H_5O_2\end{cases}}_{\text{Boracic ether.}} + \underbrace{3Zn_2\begin{cases}C_4H_5\\C_4H_5\end{cases}}_{\text{Zincethyl.}} = \underbrace{2B\begin{cases}C_4H_5\\C_4H_5\\C_4H_5\end{cases}}_{\text{Boric ethide.}} + \underbrace{6\left.\begin{matrix}C_4H_5\\Zn\end{matrix}\right\}O_2}_{\text{Ethylate of zinc.}}$$

Another but less probable view of the change presents itself in the supposition that the three atoms of ethyl in boric ethide were already present in the boracic ether, the action of the zincethyl being simply to remove the whole of the oxygen from the boracic ether. KEKULÉ† has in fact adopted this latter view of the reaction.

So long as the organic radical of the zinc compound and that of the boracic ether are identical, it is impossible to prove whether the three individual atoms of ethyl in boric ethide were originally present in the boracic ether, or have been derived from the zincethyl. Indicating by an asterisk the atoms of ethyl which finally become part of the boric ethide, it is impossible to prove conclusively whether the reaction takes place according to the first or second of the following equations:—

$$(1)\quad 2B\begin{cases}C_4H_5O_2\\C_4H_5O_2\\C_4H_5O_2\end{cases} + 3Zn_2\begin{cases}C_4H_5{*}\\C_4H_5{*}\end{cases} = 2B\begin{cases}C_4H_5{*}\\C_4H_5{*}\\C_4H_5{*}\end{cases} + 6\left.\begin{matrix}C_4H_5\\Zn\end{matrix}\right\}O_2.$$

$$(2)\quad 2B\begin{cases}C_4H_5{*}O_2\\C_4H_5{*}O_2\\C_4H_5{*}O_2\end{cases} + 3Zn_2\begin{cases}C_4H_5\\C_4H_5\end{cases} = 2B\begin{cases}C_4H_5{*}\\C_4H_5{*}\\C_4H_5{*}\end{cases} + 6\left.\begin{matrix}C_4H_5\\Zn\end{matrix}\right\}O_2.$$

Although we cannot thus label, as it were, the atoms taking part in the reaction, we can unerringly trace the movements of the alcohol radicals, if we secure their identification by varying their composition in the two compounds used in the process. The

† Lehrbuch der org. Chemie, p. 489.

study of the action of zincmethyl upon boracic ether would obviously decide between these views. If boric ethide were produced from these materials, KEKULÉ's hypothesis would be established; but if, on the other hand, boric methide were the result of the reaction, then the correctness of the view originally taken by Mr. DUPPA and myself would be proved to be correct. The following are the results obtained in pursuing this inquiry.

Boric Methide.

When a strong ethereal solution of zincmethyl is added to boracic ether, an elevation of temperature to the extent of 8° or 10° C. is observed, whilst at the same time a most intensely pungent odour is developed; this odour, although it resembles that of boric ethide, is far more powerful, and more persistently irritating to the mucous membrane. A slow evolution of a spontaneously inflammable gas, burning with a splendid green flame, was also noticed; and this evolution of gas became more rapid when the warmth of the hand was applied to the flask containing the ingredients. Preliminary experiments proved that this gas was nearly insoluble in water, but almost completely soluble in alcohol, the residue remaining undissolved being marsh-gas derived from the action of the alcohol upon traces of zincmethyl vapour with which the gas was contaminated. The gas was not condensed by a freezing mixture of ice and salt. It was, with the exception of a small percentage of marsh-gas, instantaneously dissolved by solution of ammonia, which yielded the gas again unchanged when neutralized by an acid. Concentrated sulphuric acid was without action upon the gas.

These data led to the following plan for collecting the gas in a state of purity. About two ounces of boracic ether were mixed in a small flask with rather more than their own bulk of an ethereal solution of zincmethyl, of such strength as to be spontaneously inflammable in a high degree. The flask, loosely corked, was placed in ice-cold water, and allowed to stand for a couple of hours until the reaction was complete: it was then furnished with a bent tube passing through a cork, and designed to conduct the gas into a second flask placed in a freezing mixture of ice and salt; from this flask the gas passed into a third containing about half an ounce of strong solution of ammonia. The air in the whole of the apparatus was now displaced by nitrogen, and the flask containing the boracic ether and zincmethyl removed from the ice-cold water. A slow evolution of gas immediately commenced, and was kept up at a convenient speed by plunging the generating flask into cold water, to which heat was very slowly applied. The gas, in passing through the freezing mixture, deposited nearly the whole of the ether and zincmethyl vapour with which it was contaminated; and on reaching the solution of ammonia, the boron compound was instantaneously absorbed, whilst other gases, if present, passed through the ammonia unacted upon, and escaped into the atmosphere. The solution of ammonia soon became covered with a stratum of a lighter liquid, which increased in quantity until the stream of gas ceased to pass through. The ammonia-flask was now disconnected with the rest of the apparatus, and reserved for the next operation. The residue in the generating flask solidified to a crystalline mass on cooling.

It now only remained to disengage the gaseous boron compound from its combination with ammonia. For this purpose the ammonia-flask was fitted with a funnel-tube terminating beneath the surface of the liquid, and a gas-delivery tube, the latter leading to a LIEBIG'S potash apparatus charged with concentrated sulphuric acid; finally, the opposite extremity of the latter apparatus was connected with a mercurial gas-holder. To prevent dangerous explosions, on the elimination of the spontaneously inflammable gas from its ammonia compound, the whole of the air-spaces of the apparatus were filled with nitrogen. Everything being thus prepared, dilute sulphuric acid was gradually poured into the ammonia-flask through the funnel-tube, the contents of the flask being frequently agitated. No gas was evolved until the excess of ammonia was saturated; then, however, it was given off abundantly, and the addition of a few drops of sulphuric acid, from time to time, through the funnel-tube served to keep up a convenient current. The gas was allowed to pass freely through the depressed mercurial gas-holder until a sample of it proved, by its perfect solution in ammonia, that all nitrogen had been swept from the apparatus. The exit-tube of the gas-holder was now closed, and the gas collected in sufficient quantity for subsequent experiments.

The following determinations, together with the analysis of its ammonia compound, prove that this gas is *boric methide*, and that its formula is

$$B\begin{cases}C_2H_3\\C_2H_3\\C_2H_3.\end{cases}$$

I. An indefinite quantity of the gas was cautiously led over ignited oxide of copper, the carbonic acid and water produced being collected and weighed in the ordinary manner. 0·5875 grm. of carbonic acid and 0·3664 grm. of water were obtained. These numbers show that the atomic relation of carbon to hydrogen is as 2 : 3.

$$C : H = 2 : 3{\cdot}04.$$

II. A determination of the specific gravity of the gas gave the following results:—

Temperature of room at the moment of closing the flask	12°·2 C.
Height of barometer	759·4 millims.
Height of column of mercury in neck of flask above outer level of metal	4·5 millims.
Weight of flask and gas	28·7500 grms.
Temperature in balance-case	16°·1 C.
Weight of flask filled with dry air*	28·5882 grms.
Temperature in balance-case	15°·0 C.
Capacity of flask	141·7 cub. centims.

* The gas in the flask was replaced with dry air, by plunging the flask into a large beaker filled with carbonic acid, and then directing a rapid stream of the dry acid gas into the flask, so as to remove the spontaneously inflammable gas, which, rushing out of the neck of the flask with considerable velocity, reached the surface of the carbonic acid, and there formed beautiful wreaths of green flame. The flask was now several times exhausted and filled with dry air.

From these data the specific gravity of the gas was calculated to be 1·9108—a number which closely coincides with the calculated specific gravity of boric methide, which contains 1 volume of boron vapour and 3 volumes of methyl, the four volumes being condensed to two.

1 vol. Boron vapour . . .	·75319
3 vols. Methyl	3·10956
	2)3·86275
	1·93137

Boric methide is produced from boracic ether and zincmethyl by the following reaction:—

$$\underbrace{2B\left\{\begin{matrix}C_4H_5O_2\\C_4H_5O_2\\C_4H_5O_2\end{matrix}\right.}_{\text{Boracic ether.}}+\underbrace{3Zn_2\left\{\begin{matrix}C_2H_3\\C_2H_3\end{matrix}\right.}_{\text{Zincmethyl.}}=\underbrace{2B\left\{\begin{matrix}C_2H_3\\C_2H_3\\C_2H_3\end{matrix}\right.}_{\text{Boric methide.}}+\underbrace{6\left.\begin{matrix}C_4H_5\\Zn\end{matrix}\right\}O_2}_{\text{Ethylate of zinc.}}$$

The formation of boric methide under these circumstances proves conclusively that the corresponding ethyl compound is formed, not by the removal of the whole of the oxygen from boracic ether, but by the actual substitution of the three atoms of oxygen in boracic acid by three atoms of ethyl, whilst boric methide is in like manner produced by the similar substitution of methyl for oxygen,—a kind of substitution which is quite in harmony with the mode of formation of very numerous compounds in the organo-metallic family.

Boric methide exists at ordinary temperatures as a colourless and transparent gas, possessing a peculiar and intolerably pungent odour, irritating the mucous membrane, and provoking a copious flow of tears. Its specific gravity is 1·93137. It retains its gaseous condition when exposed to a cold of −16° C.; but at 10° C., and under a pressure of three atmospheres, it condenses to a colourless, transparent, and very mobile liquid. It is very sparingly soluble in water, but very soluble in alcohol and in ether. In contact with atmospheric air it takes fire spontaneously, burning with a bright green flame, which is very fuliginous if the volume of the flame be considerable. If the gas issue into the air through a tube $\frac{1}{10}$th of an inch in diameter, the amount of smoke is surprisingly great, two or three cubic inches of gas, when consumed in this way, filling the atmosphere of a capacious room with large comet-like flocks of carbonaceous matter. This curious phenomenon is probably due, in part at least, to the formation of a superficial coating of boracic acid, which envelopes the particles of carbon and prevents their combustion. Suddenly mixed with atmospheric air or oxygen, boric methide explodes with great violence. In contact with air, both boric methide and the vapour of boric ethide exhibit two distinct kinds of spontaneous combustion. Thus, when these bodies issue very slowly from a glass tube into the air, they burn with a lambent blue flame invisible in daylight, and the temperature of which is so low that a finger may be held in it for some time without much inconvenience. Under these circumstances partial oxidation only takes place, and it is to the products thus formed that the peculiar pungent odour of boric ethide and boric methide is due. When, on the other hand, these bodies issue into the air more rapidly, the lambent blue and nearly cold flame changes to the green and hot flame above mentioned. I have not examined the spectra of the two differently coloured flames from the same compound, but they doubtless present a widely different appearance, thus affording another instance of the dependence of the spectra of bodies upon temperature,—a phenomenon to which Dr. TYNDALL and myself recently called attention in the case of lithium *.

Boric methide is not acted upon by binoxide of nitrogen or by iodine. Solution of bichromate of potash scarcely affects it, but the addition of concentrated sulphuric acid at once determines the reduction of the chromic acid. When boric methide is allowed to bubble through water into chlorine, each bubble burns explosively with a bright flash of light and the separation of carbon. It has no tendency to unite with acids. Concentrated sulphuric acid has no action upon it; when mixed with hydriodic acid gas, it suffers no change; but, on the other hand, it is freely absorbed by solutions of the fixed alkalies, and by ammonia. If a very rapid current of the gas, mixed with half its volume of marsh-gas, be passed through a stratum of strong solution of ammonia only half an inch deep, not a trace of boric methide escapes absorption.

Ammonia-Boric Methide.

When dry ammoniacal gas is mixed with an equal volume of dry boric methide, both gases instantly disappear with the evolution of a considerable amount of heat, and the production of a white, volatile, crystalline compound. The latter is also formed when boric methide is passed into solution of ammonia. The colourless liquid stratum which forms upon the surface soon solidifies when it is placed over sulphuric acid *in vacuo*. A quantity of the compound obtained by this latter process was purified by solution in ether and subsequent recrystallization. On being submitted to analysis, it yielded the following results:—

I. ·2652 grm., burnt with oxide of copper, gave ·4862 grm. carbonic acid and ·401 grm. water.

II. ·2809 grm. gave ·5105 grm. carbonic acid and ·4217 grm. water.

III. ·2124 grm., decomposed by dilute hydrochloric acid in a stream of carbonic acid, gave a solution of chloride of ammonium, which, treated in the usual manner, yielded ·6450 grm. of chloride of platinum and ammonium.

These numbers lead to the formula

$$NH_3+B(C_2H_3)_3,$$

as is seen from the following comparison of calculated numbers with experimental results:—

* Philosophical Magazine, S. 4. vol. xxii. p. 472.

		Calculated.		Found.			
				I.	II.	III.	Mean.
C_6 .	36	49·39		49·99	49·56	——	49·77
H_{12} .	12	16·46		16·80	16·68	——	16·74
N. .	14	19·20		——	——	19·07	19·07
B. .	10·9	14·95		——	——	——	14·42
	72·9	100·00					100·00

The excess of carbon, and to some extent also that of hydrogen, in these analyses is doubtless due to the retention of a small amount of ether by the crystals, the great volatility of the latter preventing any protracted exposure over sulphuric acid; but the substance used in No. II. was exposed for a much longer time than that employed in No. I.

Ammonia-boric methide is deposited from its ethereal solution in magnificent arborescent crystals, which rapidly volatilize without residue when exposed to the air. They possess a caustic and bitter taste, and a very peculiar odour, in which both the smell of ammonia and of boric methide can be recognized. Ammonia-boric methide fuses at 56° C. and boils at about 110° C. In a current of air, or better, of carbonic acid, it sublimes at a very gentle heat, and condenses in magnificent arborescent crystals. Determinations of the specific gravity of its vapour, at three different temperatures, gave the following results:—

Weight of ammonia-boric methide used . . . . ·0580 grm.

	I.	II.	III.
Observed volume of vapour . .	57·9 cub. cent.	58·6 cub. cent.	59·9 cub. cent.
Temperature of oil-bath . . .	119°·0 C.	130°·0 C.	139° C.
Height of mercury inside of tube above that outside	99·9 millims.	93·1 millims.	88·5 millims.

Height of barometer . . . . 751·8 millims.

Height of column of cold oil . . . . 307 millims.

From these data the specific gravity of the vapour was calculated to be

I.	II.	III.
1·251	1·258	1·250

These numbers indicate that the vapour of ammonia-boric methide consists of equal volumes of boric methide and ammonia united without condensation:—

1 vol. Boric methide . . .	1·93137
1 vol. Ammonia	·5873
	2)2·51867
	1·25933

Thus the formula of ammonia-boric methide is a four-volume formula*, a state of

* H_2O_2 = 2 vols.

condensation which is usually considered to be abnormal, and which, where it occurs, is generally explained by the assumption of a decomposition of the body at the moment of conversion into vapour. The proof of the disunion or integrity of the vaporous molecule of ammonia-boric methide would be interesting in connexion with these so-called anomalous vapour-densities, but I have to regret my inability to offer any sufficiently decisive solution of this problem. The difficulty to be overcome is the finding of a reagent that will not decompose ammonia-boric methide at elevated temperatures, but which would absorb ammonia only, out of a mixture of this gas with boric methide, at a temperature above the boiling-point of ammonia-boric methide. Chloride of calcium does not decompose ammonia-boric methide; but although it readily absorbs ammonia at ordinary temperatures, yet it allows the whole of it to escape again at 110° C. Chloride of zinc decomposes ammonia-boric methide before the latter volatilizes. The same effect is produced by all the strong acids, which are therefore also inadmissible, whilst dry boracic acid does not absorb ammonia even at ordinary temperatures. The substance which appeared to be best adapted for this reaction was dry and recently fused chloride of copper. This salt does not decompose ammonia-boric methide below the boiling-point of the latter, whilst it readily absorbs ammonia, and retains it at a temperature of 160° C. I will now describe the mode in which an experiment with this substance was conducted, and the results which were obtained. A quantity of ammonia-boric methide was introduced into a graduated tube filled with mercury, and inverted in a vessel containing the same metal. The whole was now immersed in an oil-bath, and heat applied until the boron compound was converted into vapour, the volume of which, at a known temperature and pressure, was then observed. After the apparatus had been allowed to cool, a fragment of chloride of copper was passed up into the tube, and heat again applied. The boron compound soon melted and enveloped the fragment of chloride of copper: as the temperature approached the boiling-point of ammonia-boric methide, the latter slowly boiled off from the chloride of copper, and the vapour then occupied the same volume as that read off before the introduction of the chloride of copper. The mercury in the tube remained steady for two or three minutes; it then gradually ascended, and the contraction of the vapour-volume continued until it was reduced to exactly one-half, as indicated by the following numbers:—

Corrected volume of vapour before treatment with chloride of copper	35·67 cub. centims.
Ditto after treatment with chloride of copper	17·85 cub. centims.

By treatment with chloride of copper, 100 volumes of vapour were therefore reduced to 50·04 vols., the residue consisting of pure boric methide gas. It is obvious that this absorption may be due either to decomposition of the vapour of ammonia-boric methide *by chloride of copper* at an elevated temperature, or to the decomposition *by heat* of the boric compound into equal volumes of boric methide and ammonia, the latter being then absorbed by the chloride of copper. Unfortunately, the result of the experiment is not sufficiently decisive to compel the adoption of either of these hypotheses, although the formation of the vapour and its existence for a few minutes in contact with chloride of copper favour the first more than the second; thus indicating that the vapour of ammonia-boric methide consists of equal volumes of ammonia and boric methide united without condensation, a result which would harmonize with the very generally observed rule, that when two gases or vapours unite in equal volumes, the volume of the compound is equal to that of its constituents.

Ammonia-boric methide scarcely absorbs a perceptible amount of oxygen at ordinary temperatures, even after several days' exposure to the gas; but it takes fire below 100° C. when heated in contact with the air. Its vapour is also very inflammable; thus, when ammonia-boric methide is placed under the receiver of an air-pump, and the air is being withdrawn, the explosion of the mixture of air and vapour in the cylinders of the pump is frequently determined by the rise of temperature consequent upon the depression of the pistons when the rarefaction has become considerable.

Boric methide is also absorbed by aniline with great avidity. Acids expel the gas from this compound unchanged.

Terhydride of phosphorus has no action upon boric methide. A mixture of equal volumes of the two gases is spontaneously inflammable, burning with a yellowish-white flame, in which the characteristic green tinge attending the combustion of boric methide is no longer perceptible.

Compounds of Boric Methide with Potash, Soda, Lime, and Baryta.

Solution of caustic potash absorbs boric methide with great energy. The saturated solution, exposed over sulphuric acid *in vacuo*, dries down to a gummy mass, which scarcely exhibits signs of crystallization. The same body may be more conveniently formed by decomposing ammonia-boric methide with alcoholic solution of potash, taking care to employ an excess of the former. On evaporation over sulphuric acid *in vacuo*, the excess of the ammonia compound volatilizes, and is decomposed by the sulphuric acid with the elimination of boric methide: thus the potash compound evaporates in an atmosphere of boric methide. Nevertheless even by this method I did not succeed in obtaining the potash compound in a state of purity; potash-boric methide thus prepared yielding on analysis 47·93 per cent. of potash, and 42·86 per cent of boric methide, numbers only very remotely indicating the formula

$$KO\,B(C_2H_3)_3,$$

which requires 45·67 per cent. of potash and 54·33 per cent. of boric methide. The appearance of the compound, even after exposure to gentle heat *in vacuo*, suggested the presence of water, which could not, however, be expelled at a temperature below that at which potash-boric methide itself is decomposed.

Boric methide is also readily absorbed by solution of neutral carbonate of potash, bicarbonate of potash and potash-boric methide being apparently formed. Although boric methide and potash unite with remarkable energy, yet they are separated by acids with the greatest readiness; even carbonic acid in the presence of water can expel boric methide from its potash compound; thus, if an aqueous solution of potash-boric methide be passed into carbonic acid standing over mercury, the acid gas soon becomes replaced by pure boric methide.

Soda-boric methide, baryta-boric methide, and lime-boric methide are similar bodies, produced by the absorption of boric methide gas by caustic solutions of soda, baryta, and lime; they are all readily soluble in water and react alkaline.

Boric methide in combination with the alkalies and alkaline earths has almost entirely lost its powerful affinity for oxygen; nevertheless, when these bodies are placed in contact with a known quantity of oxygen over mercury for several days, the volume of the gas perceptibly diminishes.

The great difficulty, not to say danger, attending the gradual oxidation of considerable quantities of a gaseous and spontaneously inflammable body like boric methide, has prevented me from following this compound into its products of oxidation, as was done in the case of boric ethide. With a graduated supply of oxygen, however, boric methide appears to comport itself like boric ethide, and the compounds formed are probably homologous with diethylate and dihydrate of boric dioxyethide.

In conclusion, it can scarcely be doubted that the action upon boracic ether of the zinc compounds of the remaining alcohol radicals would produce the homologues of the bodies described in the foregoing pages. It may also be remarked, that the existence of bodies like boric dioxyethide, in which one-third of the oxygen in boracic anhydride is replaced by ethyl, altogether abolishes any supposed analogy between carbonic and boracic acids, whilst it proves that the composition of the latter acid is expressed by the formula BO_3, or some multiple of that formula. I am at present engaged in studying the action of zincethyl and sodiumethyl upon the ethers of silicic, carbonic, oxalic, and acetic acids.

IX. *On the Posterior Lobes of the Cerebrum of the Quadrumana.* By WILLIAM HENRY FLOWER, *F.R.C.S., Assistant-Surgeon to, and Demonstrator of Anatomy at, the Middlesex Hospital. Communicated by Dr.* SHARPEY, *Sec. R.S.*

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TIEDEMANN states that the hippocampus minor is not found in the brain of Monkeys, or of any other animals which he had examined, but is peculiar to Man*. In his figure of a horizontal section of the brain of *Simia nemestrina*, a part is described as "scrobiculus parvus loco cornu posterioris," and the drawing corresponds with the description. Many writers on human anatomy have followed TIEDEMANN'S statement; thus CRUVEILHIER observes, "Du reste, l'ergot [hippocampus minor] de même que la cavité digitale [posterior cornu of the lateral ventricle] n'existe guères que chez l'homme, sans doute parce que l'homme seul présente un grand développement de la partie occipitale du cerveau†."

More recently, the presence of a "posterior lobe" of the cerebrum, a "posterior horn of the lateral ventricle," and a "hippocampus minor" have been affirmed by an eminent authority in this country, to be the distinguishing characteristics of the human brain‡.

On the other hand, according to CUVIER, "les ventricules antérieurs ou latéraux n'ont de cavité digitale que dans l'homme et *dans les singes*. Cette partie n'existe dans aucun autre mammifère. Sa presence dépend de celle des lobes postérieurs§." M. SERRES, in his well-known work on the comparative anatomy of the brain, has the following passage:—"Le petit pied d'hippocampe, ou le relief de l'anse d'une anfractuosité dans la corne postérieure du grand ventricule, n'a encore été aperçu que dans l'homme; on l'efface en le déplissant par le procédé que l'on met en usage pour développer le grand ventricule latéral; je l'avais moi-même méconnu en procédant à sa recherche de cette manière. Je l'ai découvert au contraire chez les singes et les phoques, en pratiquant sur lè lobe postérieur une section verticale au niveau du genou postérieur du corps calleux: on enlève de cette manière toute la cavité anciroïde, et en l'entr'ouvrant, en comprimant légèrement le lobe postérieur, on voit le petit pied d'hippocampe. Dans les cerveaux qui ont été durcis par l'alcohol, on le met à découvert par cette section et

* The passage in full is, "Pedes hippocampi minores vel ungues, vel calcaria avis, quæ a posteriore corporis callosi margine tanquam processus duo medullares proficiscuntur, inque fundo cornu posterioris plicas graciles et retroflexas formant, in cerebro Simiarum desunt, nec in cerebro aliorum a me examinatorum mammalium occurrunt; homini ergo proprii sunt."—Icones Cerebri Simiarum et quorundam Mammalium rariorum. Heidelberg, 1821, p. 51.

† Anatomie Descriptive. Paris, 1836, tome iv. p. 697.

‡ Professor OWEN, Proc. Linn. Soc. 1858, and Annals and Mag. of Nat. Hist. June 1861.

§ Leçons d'Anatomie Comparée (3rd edit.), tome iii. p. 103.

par le procédé ordinaire. Je l'ai maintenant sous les yeux, chez le papion (*Simia sphynx*), chez le rhésus, chez une autre espèce de macaque, et chez le mandrill (*S. maimon*) *."

In LEURET'S 'Anatomie Comparée du Système Nerveux†' is an observation to the same effect; and in the continuation of the work by M. GRATIOLET, a passage occurs, which is so important in connexion with what has been written upon the subject by other authors, that I shall quote it in full‡:—

"Les ventricules latéraux s'enroulent, disons-nous sur les deux corps striés interventriculaires. Celle de leurs extrémités, qui répond à la massue antérieure du corps strié, est la *corne antérieure* ou *frontale* du ventricule latéral. L'autre extrémité, enroulée comme le corps strié lui-même, répond à sa massue inférieure. Nous la nommons *corne inférieure* ou *sphénoidale* du ventricule latéral. Tel est le ventricule dans la plupart des animaux mammifères. Mais dans l'*homme* et dans les *singes*, de la partie postérieure de l'arc du ventricule latéral se detache un prolongement un peu recourbé en dedans comme la corne d'un rhinoceros, ou comme une griffe. Ce prolongement est la *corne postérieure* ou *occipitale* du ventricule latéral. On le désigne encore sous le nom de *cavité ancyroïde*.

"Ce prolongement est fort remarquable; dans les *singes*, il a une grandeur énorme en égard à l'ensemble du ventricule latéral dont l'arc est fort petit. Dans l'homme, la prédominance passe à celui-ci. Cette observation est importante, parce qu'elle coïncide avec des observations faites sur la périphérie des hémisphères."

In a foot-note he adds, "Ce prolongement occipital du ventricule est particulier aux primates (singes) et à l'homme, et par conséquent, il caractérise fort bien le type d'organisation de ces êtres. Toutefois, il ne peut-être considéré comme un signe d'élévation, car il est beaucoup plus grand en égard à la partie enroulée du ventricule dans les singes, où son développement est énorme, que dans l'homme, où la partie enroulée l'emporte évidemment sur lui. Cette remarque est d'une haute importance, et fait voir que des dispositions, qui caractérisent un groupe élevé, ne peuvent toujours être choisies comme *critérium* des dispositions sériales à l'aide desquelles l'ensemble de ce groupe est zoologiquement conçu. Si l'on attachait à la considération de prolongement occipital une importance absolue, l'homme serait inférieur au singe. C'est une preuve entre mille que la faute dont les zoologistes doivent le mieux se garantir, c'est de prendre dans leurs raisonnements la partie pour le tout.

"On pourrait supposer, en considérant la grandeur des ventricules latéraux dans le fœtus, que cette grandeur de la cavité ancyroïde chez les singes résulte d'un arrêt de développement. Mais cette conclusion serait loin d'être exacte; en effet, aux lobes antérieurs qui, chez les singes, sont extrêmement réduits, correspond un ventricule tres-réduit dans toutes ses parties, tandis que le lobe postérieur, malgré la grandeur de son ventricule, a un développement relatif énorme."

It was not until the greater number of the observations related in the following paper

* Anatomie Comparée du Cerveau. Paris, 1826, tome ii. p. 470.

† Tome i. 1839, p. 402.

‡ Tome ii. 1857, p. 74.

were made, and the principal conclusions arrived at, that I met with this passage, which affords from an independent source a confirmation of their general accuracy, but at the same time leaves untouched the necessity for detailed descriptions of the condition of this portion of the brain in particular species. I also think it not improbable that renewed investigations into the development of the human brain may necessitate some modification in the conclusions contained in the last paragraph.

Some of the special monographs upon the brains of Quadrumana exhibit discrepancies almost equal to those existing in the more general statements upon the subject. Thus SCHROEDER VAN DER KOLK and VROLIK found in the Chimpanzee the "lateral ventricle distinguished from that of Man by the very defective proportions of the posterior cornu, wherein only a stripe is visible as an indication of a hippocampus minor*," while in a memoir on the brain of the same animal by Mr. MARSHALL, these parts are shown to be as fully developed as in an average example in the human brain†. Dr. ROLLESTON has recently described and figured a well-marked posterior cornu and hippocampus minor in the Orang‡; and Mr. HUXLEY, in an admirable memoir to which I shall have occasion hereafter to refer, has given a full account of the structure of the posterior lobes in *Ateles*, a monkey of the Platyrrhine or New World group§.

Such being the present state of the literature of a subject which is evidently within the reach of ordinary observation, it will be seen that its full elucidation must be based upon an appeal to nature, and not to authority. Therefore, with a desire to contribute some reliable data to an interesting branch of anatomical science, I have taken the opportunity recently afforded by the examination of the brains of numerous examples belonging to three families of Quadrumana of making the following observations. The greater number of the specimens described are from animals which have died during the past summer in the Gardens of the Zoological Society. For their transmission into my hands while in that perfectly fresh condition so essential to the success of cerebral investigations, I am indebted to the attention of Mr. BARTLETT, the able Superintendent of the Society's Collection.

As other considerations besides those of a purely anatomical character have been supposed to be involved in inquiries of this nature, I must state at the outset that these have been undertaken without reference to any theory as to the transmutation of species, or origin of the human race; whatever inferences others may draw from the facts related, for my own part I see no reason to assign any special importance, in determining the value of such a theory, to the condition of the particular portion of the cerebral organization now under consideration, especially as the general close resemblance between the physical structure of Man and the Quadrumana has long been a matter of common observation.

As it is to the brain of Man that the comparisons instituted in this paper chiefly refer, it will be necessary in the first instance to call attention to certain points in the struc-

* Nieuwe Verhandlingen der erste Klasse van het Koningl. Nederlandsche Instituut. Amsterdam, 1849.

† Nat. Hist. Review, July 1861. ‡ Ibid. April 1861. § Proc. Zool. Soc., June 11, 1861.

ture of that organ; and I must premise that in all the ensuing descriptions the nomenclature used is the one proposed by Mr. HUXLEY*, founded on that of M. GRATIOLET, to whom the merit of reducing to an intelligible and harmonious system the apparently confused and intricate surface-markings of the brain of the Primates is chiefly due†.

Plate II. fig. 1 is a sketch of the internal surface of the right hemisphere of the brain of an adult European. It has been carefully drawn from a specimen which had been prepared for the purpose, by injecting the carotid arteries with strong spirit, and hardening within the cranium, so that the general form and relative situation of the different parts have been accurately preserved. Five principal sulci are seen upon this face of the cerebral hemisphere. 1. The *calloso-marginal* (*i*, *i*), running lengthwise, parallel to, and between, the upper border of the corpus callosum and the superior margin of the hemisphere. 2. A nearly vertical fissure (*k*, *k*) placed about midway between the posterior end of the corpus callosum and the apex of the hemisphere. As it forms the line of demarcation between the parietal and occipital lobes, it has received the name of *occipito-parietal* ("scissure perpendiculaire interne" of GRATIOLET). 3. A deep and distinctly marked sulcus (*l*, *l*) commencing just below the posterior extremity of the corpus callosum, and running backwards and slightly upwards almost to the apex of the hemisphere, where it divides into an ascending and descending branch. This is the *calcarine* sulcus, so named from its relation to the part called, when seen from the interior of the ventricle, "calcar avis," or "hippocampus minor" ("anfractuosité de la cavité digitale," CRUVEILHIER; posterior portion of the "scissure des hippocampes," GRATIOLET). 4. The *dentate* sulcus, in which the small gyrus called "corpus dentatum" (corps godronné) is situated. This is the anterior portion of the "scissure des hippocampes" of GRATIOLET, and holds the same relation to the hippocampus major as the calcarine fissure does to the hippocampus minor. 5. A fissure (*n*, *n*) running more or less parallel to the last, but at a lower level, named from its connexion with the eminentia collateralis, the *collateral* sulcus. The relations of the last three sulci have been pointed out in detail by Mr. HUXLEY in the paper referred to.

The gyri on the inner surface have their boundaries defined by the above-named sulci, and are of a very simple character. They will be better understood by a reference to the figure, than by a verbal description. 17 is the marginal gyrus; 18 the callosal; 18′ the quadrate lobule; 19 the uncinate gyrus; 20 the dentate; and 25 the internal occipital lobule.

In the hemisphere of the hardened brain (Plate II. fig. 1) the space between the posterior edge of the corpus callosum and the extremity of the lobe was divided into four equal parts, and the sections indicated by the lines A, B, and C were made at the points thus determined. For accuracy of comparison, the same rule has been observed in all the sections of the Simian brains which follow. The figures A, B, and C represent the surface of such sections, and exhibit the distance to which the different sulci penetrate

* On the Brain of *Ateles paniscus*, Proc. Zool. Soc. June 11, 1861.

† Mémoire sur les Plis Cérébraux de l'Homme et des Primatès. Paris, 1854.

into the substance of the hemisphere. It must be observed that in the human brain there is some variation in this respect, but the average depth of the calcarine fissure is well illustrated in this example, as also the usual size of the posterior cornu. This cavity appears in section as a mere fissure curved round a delicate stratum of white substance, which covers internally the grey matter of the calcarine sulcus, and which constitutes the ventricular surface of the hippocampus minor. In section C no trace of the opening can be discovered.

Such is the appearance presented by this portion of the brain, in the highest state of perfection which that organ attains; a condition, however, always the least favourable for studying the morphological significance and true relations of its several parts. The original design, now obscured by special adaptive modifications, can only be traced either by observing the gradual evolution of the same parts from their most rudimentary condition, or by a comparison of similar structures in other animals of simpler organization.

By the first method we learn that while the hemispheres of the brain are mere sacs, and perfectly devoid of convolutions, the calcarine is one of the first of the sulci which appears on the surface (about the fifteenth week, according to TIEDEMANN), and coincident with it, an elevation is seen upon the interior of the ventricle, the future hippocampus minor. The eminence is, in fact, formed simply by an involution of the wall of the original ventricular cavity, and such portion of this cavity, situated in the posterior lobe, as has escaped being closed in by the growth of the surrounding cerebral substance, constitutes the posterior cornu. The variability of its extent in the human subject, and consequent apparent variability of the hippocampus minor, is well known*. Little physiological importance can, however, be attached to the size of the latter as commonly estimated by the projection into the ventricle. The real amount of cortical or ganglionic neurine surrounding the calcarine sulcus can only be ascertained by an examination of the length, depth, and complexity of that sulcus, and remains unaltered whatever may be the extent to which the cavity of the ventricle is closed: just as the size and form of the corpus striatum would be unaffected by the absence or closure of the portion of the lateral ventricle which lies in contact with it. Such an examination, both in different individuals of the human race, and in the brains of various animals, may supply important data in future investigations concerning the functions of special portions of the grey matter on the surface of the cerebrum.

* Many of the variations in the condition of these parts must be ascribed rather to pathological changes than to original conformation. Thus, in aged and debilitated subjects, the posterior cornu is often enlarged, and somewhat funnel-shaped, the calcarine projection being also more or less obliterated, a circumstance arising apparently from gravitation of the intra-ventricular fluid during long continuance of the recumbent posture. In atrophy of the cerebral substance, from whatever cause, attended by an increased size of the ventricular cavities, the change is usually most strikingly seen in the posterior cornu.

Order QUADRUMANA.

Family 1. *Catarrhina.*

Orang-Utang (*Pithecus satyrus*).—The brain of a young female of this species, which died in the Gardens of the Zoological Society, May 1850, is preserved in the Museum of the Middlesex Hospital. Although this specimen has retained its form and general characters unusually well, I do not propose to give any detailed account of it, being unwilling to multiply the too numerous imperfect descriptions which already exist of the cerebral anatomy of this animal—imperfect inasmuch as they are mostly taken from specimens the form of which has been more or less altered by preservation in spirit. A general flattening of the cerebral mass, contraction of the hemispheres, with loss of characteristic outline, obliteration of distinction between white and grey substance, and adherence of contiguous walls of sulci and cavities, render such brains ill adapted for the successful study either of the external characters or internal structure.

The following points relating to the posterior lobes of the cerebrum are, however, to be noted in this specimen. On looking directly down upon the centre of the upper surface of the brain, no part of the cerebellum is visible, either laterally or posteriorly. When viewed from one side, the posterior lobes are seen to project exactly as far backwards as completely to cover the cerebellum, but not to extend beyond it. But an examination of the interior of the cranium from which the brain was taken, shows that the shrinking of the hemispheres has reduced their dimensions in this direction. It is therefore perfectly evident that the posterior lobes of the cerebrum, according to any definition taken from external characters, exist in a very well-developed condition, although not prolonged backwards to quite so great an extent as they usually are in the human brain.

To examine the interior, the upper portions of the hemispheres were removed to the level of the inferior surface of the corpus callosum, and then further portions were carefully dissected away so as to expose the lateral ventricles with their three cornua. The general form of the cavity presents almost the exact counterpart of that in the human subject. The posterior cornu extends as far backwards as an average example in Man, being $\frac{5}{8}$ths of an inch long, and its apex being but $\frac{3}{8}$ths of an inch (= $\frac{1}{8}$th of the entire length of the hemisphere) from the surface of the posterior lobe. The projection of the hippocampus minor bears comparison with a very well-developed specimen of this structure as met with in the human brain. Its length is $\frac{5}{8}$ths of an inch, its breadth at the base $\frac{3}{16}$ths of an inch. The eminentia collateralis is more prominent than in many human brains. The hippocampus major has no distinct digital marks, but the convex border of its expanded termination has a slightly nodulated appearance. There is a complete correspondence of form in the ventricles of the two hemispheres, the posterior cornu extending backwards to a similar extent in both.

As it seemed desirable to possess an exact means of estimating the length of the posterior lobes in different animals by a criterion derived from internal structure, I have taken the most prominent part of the convex border of the hippocampus major as the limit between the antero-median and the posterior portions of the cerebrum. The former includes the anterior and middle cornua of the ventricle, the corpus striatum, thalamus opticus, and the hippocampus major; the latter, the posterior cornu and hippocampus minor, where these exist. In Man and the Quadrumana, the angle formed at the junction of the hippocampus major and minor readily indicates the exact spot on which to place the compasses (see Plate III. fig. 7). Such measurements should, if possible, be taken before the brain is removed from the cranial cavity. Let the length of the first part be called A, and that of the second B. In Man the average proportion in several examples of A to B is as 100 to 53. In this Orang's brain A measures exactly 2 inches, and B 1 inch, or as 100 to 50; so that the posterior lobe, as defined by internal structure, exhibits the same slight diminution upon that of the human brain, as was already estimated by the amount of covering of the cerebellum. It must be remembered, however, that this observation is taken from a spirit preparation.

Presbytes leucoprymnus (OTTO).—An adult female died in the Zoological Society's Gardens, August 20, 1861, and the brain was examined while in a perfectly fresh condition. The posterior lobes completely covered, and projected beyond, the cerebellum. The sulci on the internal face of the cerebral hemisphere (Plate II. fig. 2) bear a very close resemblance in general arrangement (though of course far less complex) to those of the human brain. The occipito-parietal (*k*, *k*) runs downwards and forwards, and almost meets the calcarine (*l*, *l*), the second internal annectent gyrus scarcely appearing on the surface. The calcarine sulcus differs from that, not only of Man, but of nearly all Apes, in extending to the extremity of the lobe, and even turning round to the outer surface, without dividing into branches. The three sections show that this sulcus penetrates more deeply into the substance of the hemisphere than in Man, but it is tolerably simple in its course, passing nearly directly inwards. The posterior cornu of the ventricle is distinctly open to within a very short distance of the hinder extremity of the lobe, and is of considerable vertical depth, being curled round the very prominent calcarine involution, or hippocampus minor.

In the left hemisphere a horizontal section was made, so as to expose the lateral ventricle (Plate III. fig. 5, *a*). The posterior cornu was seen, when opened from above, to take the same general course as in the brain of Man, viz. outwards, backwards, and finally somewhat inwards, but, owing to the depth of the calcarine fissure, and consequent great projection of the hippocampus minor, it is placed nearer to the external wall of the hemisphere than it is in Man. The measurements of the antero-median (A) and posterior (B) portions of the hemisphere are 1·5 inch and ·7 inch—A to B as 100 to 47. The latter is therefore proportionally less than in the human brain.

Cercopithecus.—Several examples of this genus have come under examination, including *C. sabæus*, *C. ruber*, and *C. mona*, but a description of *C. pygerythrus* (F. CUV.), the common Vervet Monkey, will suffice. In order to obtain a side view of the brain *in situ*, the right half of the cranium of an adult animal of this species was carefully removed with the saw and bone forceps, then the dura mater was taken away, and the contents of the cranial cavity exposed. While still in an undisturbed condition, the extent to which the posterior lobes of the cerebrum projected beyond the cerebellum was ascertained to be fully ⅛th of an inch. On comparing the form of the brain with that of the human subject, very great similarity is seen in the contour of the posterior half of the cerebrum, but the anterior lobes in the Monkey are much reduced, being narrowed almost to a point, flattened, and largely excavated in the orbital regions. Fig. 6 (Plate III.) is a profile view of this brain, and shows accurately the relative form and situation of the different parts while *in situ*, and the arrangement of the sulci upon the outer face of the hemisphere.

Fig. 3 (Plate II.) is a sketch of the convolutions of the inner face of the right hemisphere. The calcarine sulcus is very strongly marked, and describes a curve having the concavity upwards; it bifurcates as usual at the posterior end. The occipito-parietal sulcus, instead of running downwards and forwards as in Man and *Presbytes*, is directed somewhat backwards, and does not join the calcarine sulcus, but terminates at the upper margin of a prominent gyrus (the second internal annectent) which borders the last-named fissure superiorly. The sections show that the calcarine fissure extends to a greater depth than in any other genus (yet examined), and has a singular complexity of form, as there is concealed within it, and attached to its floor, a small but distinct gyrus (Plate II. fig. 3, and Plate III. fig. 7, 26). This convolution, which may be called from its position "calcarine," commences anteriorly by a slight elevation of the floor of the calcarine fissure, increases as it proceeds backwards, and comes to the surface where the sulcus turns up towards its termination; then, bounded superiorly by the lower branch of the sulcus, sweeps round the inferior border of the lobe, and becomes continuous with the infero-occipital gyrus of the outer side. It appears to be always present in *Cercopithecus*, *Macacus*, and *Cebus*, and probably in all the allied genera, but is absent in the highest and lowest members of the order. The consequence of the depth and complexity of the calcarine sulcus is that the involution of grey matter forming its walls is much increased, and bears a very large proportion to the mass of the lobe, and the cornu of the ventricle is thrown quite to the outer side of the hemisphere, being at its termination only separated by a thin stratum of white matter from the cortical layer of its external face. The walls of the cornu are in such close apposition that I have not been able to satisfy myself that it is in the adult *Cercopithecus* an actual cavity in the same sense as the remaining portion of the ventricle, especially as the staining of the lining membrane found in other parts rarely extends more than a quarter of an inch upon the surface of the hippocampus minor. But as fine sections of hardened brains show a line in which the cerebral substance is absent, having always the same definite extent, form, and direction,—as the slightest touch with the handle of the scalpel will separate the walls,—as in some genera it is undoubtedly as distinctly open as any other part of the cavity, and as it has been so considered by GRATIOLET, HUXLEY, and other competent observers, I have no hesitation in looking upon it as homologous to the posterior cornu in Man. Whether it is an actual or potential cavity is, however, of very little consequence, as there can be no question that the portion of the brain answering to that which in Man is called hippocampus minor, attains in *Cercopithecus* a really prodigious development in comparison with the size of the cerebral hemisphere.

The proportionate length of the posterior lobe, as measured upon a section made while the base of the brain still remained within the cranium, slightly exceeds that of Man, being to the antero-median portion as 54 to 100.

Macacus.—The brains of *M. silenus*, *M. nemestrinus*, *M. cynomolgus*, *M. sinicus*, *M. radiatus*, and *M. erythræus* have all been examined. They resemble one another so closely that the latter (the Rhesus Monkey) alone need be described. The length of the posterior lobes, both as to the extent to which the cerebellum is covered, and as ascertained by internal measurement, is slightly inferior to that of *Cercopithecus*. The convolutions of the inner face of the hemisphere (Plate II. fig. 4) have the same general arrangement as in that genus. The calcarine sulcus does not extend to quite so great a depth, but it conceals within it, though on rather a smaller scale, a similar gyrus (Calcarine, No. 26 B, Plate II. fig. 4). In adult examples the walls of the posterior cornu adhere very closely, but in a new-born Rhesus they were distinctly separate almost to the very end of the lobe. In this specimen the hemispheres were so elongated backwards as to project by nearly one-fourth of their length beyond the cerebellum.

Cynocephalus.—GRATIOLET has demonstrated that the principal cerebral characteristic of this genus is the great development of the occipital lobes. In a nearly full-grown example of *C. porcarius*, I find that they project $\frac{4}{10}$ths of an inch beyond the cerebellum, or rather more than $\frac{1}{6}$th of the entire length of the hemisphere; proportionally more, therefore, than in Man. Measured internally, the proportion of the posterior to the antero-median portion of the cerebrum is as 57 to 100—greater than in any other of the Catarrhine Apes. The calcarine sulcus is very deep and complex, as in the last two genera, but the specimen did not reach my hands in time to give any drawings.

Family 2. *Platyrrhina.*

Cebus apella.—The brain of this species presents a different form from that of *Cercopithecus*, or any of the Old World Apes. The hemispheres are much elongated and compressed laterally, so as to give a regularly oval outline to the entire cerebrum when seen from above. The cerebellum, though large, is entirely covered, and the posterior lobes are of great proportional length, being to the antero-median as 59 to 100.

The sulci upon the inner face of the cerebral hemisphere are shown in Plate III. fig. 5. The calcarine fissure at its anterior extremity joins the dentate, so that the callosal gyrus is not actually continuous with the uncinate as in Man and most of the Quadrumana; but at the point of union the sulcus is extremely shallow, or, in other words, the band which connects the two above-named gyri does not quite reach the surface. At first the sulcus runs somewhat in a downward direction, but ultimately takes a considerable sweep upwards. It penetrates deeply into the lobe, not quite to the same extent as in *Cercopithecus*, but is so disposed as to contain a similar convolution.

Hapale jacchus, the Common Marmoset.—In an adult specimen which died at the Zoological Society's Gardens, September 25, 1861, the brain was exposed *in situ* by clipping away different portions of the cranial bones, and the drawings (Plate III. figs. 8 & 9) made before its outline had been altered by removal from its bed in the skull. Seen from above, the two hemispheres form an elongated oval, slightly narrowed anteriorly. The olfactory lobes extend forwards beyond the cerebrum, but no part of the cerebellum is seen. The side view shows the elongation and flattening of the whole hemisphere, and the extent to which the posterior lobes project beyond the cerebellum. The orbital region of the anterior lobe is greatly excavated. The fissure of SYLVIUS is well marked, but on separating its lips no distinct median lobe is seen, the only indication of it being a very slight elevation of the floor of the middle third of the fissure. The outer surface of the hemisphere is perfectly smooth and free from sulci, a faint depression only occupying the situation of the antero-temporal, the most persistent of all the sulci of the outer face in the Quadrumana.

The inner surface of the hemisphere (Plate III. fig. 10), quite smooth in its anterior and superior portions, presents, nevertheless, three distinctly marked sulci, the dentate (*m*), the collateral (*n*), and, occupying exactly the same situation as in the higher Primates, the calcarine sulcus (*l*). This is quite simple, not bifurcated at the end, or joined by any other fissure, and describes a curve with the convexity upwards. There is no trace either of the occipito-parietal or the calloso-marginal sulcus. A section made at the middle of the posterior lobe (corresponding to the section B in the larger brains) shows that the calcarine fissure is of great depth and has a downward curve. The grey matter surrounding it occupies nearly the whole of the interior space of the lobe, forming a hippocampus minor of simple construction, but very great relative size. The very narrow stripe of medullary white matter between this involution and the external surface contains a distinctly marked crescentic opening, the section of the posterior cornu.

A horizontal section through the left hemisphere (Plate III. fig. 9) exhibits the great length of the posterior lobe as compared with the antero-median, viz. as 62 to 100. The posterior cornu in this view describes a regular curve, with the concavity inwards, and is seen to extend to within $\frac{1}{10}$th of an inch of the apex of the hemisphere. Its walls fell apart directly the section was made, and there appeared to be a distinct lining membrane, on which fine blood-vessels were seen to ramify.

The above description would apply almost equally well to the brain of *H. œdipus*, two examples of which have been dissected.

Thus, in the brain of these diminutive creatures and in that of Man, placed at opposite ends of an extensive series, and in many respects so widely removed from each other, are found certain well-marked common characters in the posterior lobes; and the principal distinction that we can draw between them, with respect to this portion of the brain, is, that in the Marmoset the whole lobe is more elongated, the calcarine fissure more deeply cut, the hippocampus minor more prominent, and the posterior cornu patent to a greater extent.

As has been already pointed out, from an examination of vertical sections of skulls, it is among members of this family that the occipital region of the cerebrum attains its greatest, and also, as far as the Apes properly so called are concerned, its least development; the first in *Chrysothrix*, one of the lowest, and the last in *Mycetes*, the genus usually placed at the head of the family. It is remarkable, also, that among the Catarrhina, as estimated by the same means, the backward development of the cerebrum, in relation to the cerebellum, appears to coincide with the order in which these animals are zoologically arranged, being least among the anthropoid Apes, and attaining its maximum in the *Cynocephali*.

Family 3. *Strepsirrhina.*

Most of the descriptions and figures of the brains of members of this family hitherto published are unsatisfactory. For example, that given by TIEDEMANN* of *Lemur mongoz*, though evidently drawn with great accuracy and care, represents a brain the form of which has been considerably altered by hardening in spirit. The same objection applies, but even more strongly, to the delineations of the brain of *Stenops* given by VROLIK† and by SCHROEDER VAN DER KOLK‡. That of *Tarsius*, in the admirable memoir of BURMEISTER §, may perhaps be excepted.

Having, therefore, lately had an opportunity of dissecting a Lemur in a fresh condition, I have thought it desirable to give a new figure of the external characters of the brain. The view of the upper surface (Plate III. fig. 11) was drawn after the removal of the skull-cap, while the brain was still in the head; the other two (Plate III. figs. 12 & 13) immediately after it was taken out, and with the assistance of a cast of the interior of the cranium. After the brain had been a fortnight in spirit, the hemispheres had lost one-fifth of their length, together with their characteristic outline, and had left about half of the cerebellum uncovered.

Lemur nigrifrons, GEOFF. (*L. mongoz*, LINN.?).—After removing the upper portion of the skull by a horizontal incision, and then taking away the dura mater, the surface of the encephalon was exposed. The part brought into view consisted of the cerebral hemispheres, with a small portion of the olfactory lobes projecting in front, and of the cerebellum behind. The general outline of the two cerebral hemispheres presented an oval figure, very narrow in front and broad behind, where it was deeply indented in the middle line. On looking directly down upon the centre of this oval, the portion of the cerebellum visible was part of the upper surface of the superior vermis, chiefly exposed by the divergence of the posterior apices of the cerebral hemispheres, and a very narrow

* *Op. cit.* tab. iv. figs. 1, 2, 3, 4.

† Bijdrage tot de Anatomie van den *Stenops Kukang*. Leiden, 1841.

‡ Nieuwe Verhand. der 1e Klasse v. h. Kon. Nederlandsche Inst. 1843.

§ Beiträge zur näheren Kenntniss der Gattung *Tarsius*. Berlin, 1846.

border of each lateral lobe. The extreme projection was ·15 of an inch behind the cerebrum. In front the olfactory lobes extended $\frac{1}{10}$th of an inch beyond the cerebrum. The hemispheres were 1·65 inch in length, and 1·3 inch across their broadest part. On gently separating the edges of the longitudinal fissure, the corpus callosum was seen to cover completely the corpora quadrigemina; its length was ·85 of an inch.

The brain was now removed. The general surface of the cerebrum is smooth, but marked with strongly defined, deeply cut, regular, and almost symmetrical sulci. The anterior or frontal lobes are attenuated, being flattened above, compressed laterally, and excavated below for the orbital plates. They are distinctly marked off from the temporal lobe by the fissure of SYLVIUS, which runs upwards and backwards to the parietal region, and has an abrupt and slightly bifurcated termination. The average depth of this fissure is nearly one-fourth of an inch, and on separating its lips, a small, smooth, but distinctly defined *insula* or median lobe of oval form was disclosed. This observation is important, as GRATIOLET says, "Le lobe central [insula] paraît particulier à l'homme et aux singes; peut-être voit-on quelque chose d'analogue dans les makis, mais on ne voit rien de semblable chez les autres mammifères." The temporal lobe is full and deep, and terminates posteriorly without any definite boundary in the posterior or occipital lobe. This last is shallow, and excavated on its under and inner surface for the cerebellum.

The sulci on the outer face of the hemisphere are—1. The fissure of SYLVIUS (*e*). 2. A well-marked longitudinal sulcus on the upper surface of the frontal lobe, inclining outwards posteriorly, probably corresponding with the infero-frontal (*a*). 3. A slight longitudinal indentation on the orbital surface of the same lobe. 4. A very distinct sulcus on the temporal lobe, parallel to, but extending rather higher than, the fissure of SYLVIUS, and curving forwards at its upper end (*f*): this is the antero-temporal (scissure parallèle). 5. A well-marked longitudinal sulcus on the upper surface of the parietal, and extending into the occipital lobe, marking off the upper limit of the angular gyrus. 6. A slight longitudinal indentation on the outer side of the occipital lobe. There is no trace of the temporo-occipital sulcus (scissure perpendiculaire externe), so well marked in the higher Apes, or of either of the parietal fissures; indeed the region on which they should be placed is very greatly reduced. In number, extent, and situation the sulci above described nearly correspond to those of *Callithrix moloch* (as figured by GRATIOLET), a Platyrrhine Ape about the same size as the Lemur.

Upon the internal face of the hemisphere (Plate III. fig. 14) are seen—1. The calloso-marginal (*i*), distinct only in the middle third of the hemisphere. 2. A very deeply marked calcarine sulcus (*l*, *l*), extending from below the posterior end of the corpus callosum, backwards and slightly upwards, to near the extremity of the hemisphere, where it ends abruptly without bifurcation. 3. Joining this, almost at a right angle, is the occipito-parietal (*k*), which does not quite reach the upper margin of the hemisphere. 4. A slight indication of the collateral sulcus (*n*). 5. A well-marked dentate sulcus (*m*).

The olfactory bulbs are in size intermediate between those of the lower Apes and those of the Carnivora. The corpora albicantia are represented by a single mass, which however is cleft posteriorly, indicating its separation into two portions. The corpora geniculata form distinct white nodules on the sides of the crura, but not visible until the edge of the temporal lobe is slightly lifted up. The pons is but little elevated. The medulla oblongata is very wide, and the tracts called corpora trapezoidea clearly marked out. The corpora quadrigemina resemble those of other Quadrumana, the anterior being larger and of more rounded form than the others. The cerebellum shows a marked inferiority to that of the true Monkeys. The median vermis, especially the inferior portion, is very large. The lateral vermis (flocculus) is also greatly developed, and forms the principal part of the lateral mass of the cerebellum. The body of the lobe is, however, not so much reduced as in the Carnivora.

To return to the cerebral hemispheres. A section was made through the right posterior lobe at the point B (Plate III. fig. 14). The calcarine sulcus is now seen to extend to about the middle of the section, but to be of the simplest form. The cortical layer which it carries with it (hippocampus minor) is bordered by a thin stratum of white substance, which is separated from the contiguous medullary cerebral matter, as in the other Quadrumana, by a fine crescentic line, indicating the presence of a posterior cornu of the lateral ventricle. In a horizontal section of the left hemisphere this cornu appeared as a mere fissure, with walls in close apposition, but traceable nearly to the termination of the hemisphere. In this view, the most marked difference between the parts displayed, and those of the ordinary Quadrumana, consisted in the comparative shortness of the posterior lobe, this being, as compared with the antero-median portion, only as 35 to 100.

None of the authors who have written upon the brains of the Lemuridæ, whose works I have been able to consult, describe a hippocampus minor. VROLIK expressly states that it is absent in *Stenops* ("L'éminence digitale, l'éminence collatérale de MECKEL manquent," *op. cit.* p. 79), and BURMEISTER alone mentions a posterior cornu to the ventricle in *Tarsius*, the only observation upon it being, that it is "very long." There can be no doubt, however, of the strict homology of the calcarine fissure, and its surrounding grey matter (hippocampus minor), in the Lemur, to that of the parts so described in Man and all the intermediate forms; and that in this low and almost aberrant member of the order, although of reduced length, corresponding with that of the hemisphere, it extends more deeply into, and bears a greater ratio to the surrounding mass of the lobe than it does either in Man or in the anthropoid Quadrumana.

The presence of the same parts is shown even more distinctly in the brain of a Galago (*Otolicnus*) preserved in the Museum of the Middlesex Hospital. The animal to which it belonged died in the Zoological Society's Gardens in 1852 *. While alive it was referred to *O. Garnettii* (OGILBY), but its dimensions did not agree with those of the type specimen of that species in the British Museum; its generic determination is, however, sufficient for the present purpose. For reasons given in the case of the Orang, I do not

* See Proc. Zool. Soc., March 23, 1852, p. 73.

purpose to give a description of the external characters of this brain, but only an account of such parts of the internal structure as have special reference to the subject of this paper. A horizontal section of both hemispheres has been made at the level of the corpus callosum, and the lateral ventricles are laid open. Fig. 15 (Plate III.) represents this dissection. A broad and very distinct posterior cornu extends backwards almost to the extremity of the hemisphere. Its floor and inner wall are raised into a prominence, having distinctly the characters of the hippocampus minor as found in Man and the higher Quadrumana, and corresponding with the deeply marked calcarine sulcus on the inner face of the lobe. The form of the eminence is somewhat triangular, the apex being directed backwards; but its surface is convex, both from above downwards and in the antero-posterior direction, so that the axis of the cavity into which it projects, though directed generally backwards, has first an outward inclination, and finally turns somewhat inwards. The anterior or broad end of the eminence is concave, being adapted to the curved posterior margin of the hippocampus major, from which it is separated by a deep groove. The length of the hippocampus minor is one-fourth of an inch; its breadth at the base almost as much. The part of the outer wall of the ventricle which projects into the angle between the hippocampi may be compared with the "eminentia collateralis" of the human brain*.

On comparing the posterior lobe in *Galago* with the same part in the true Apes, it is seen that there is, as in *Lemur*, a very marked reduction in length. This abbreviation is the more remarkable as there is no approach to it in the lowest of the Platyrrhine Monkeys. In the possession of a well-defined Sylvian fissure, a median lobe, and a calcarine sulcus, and in the general characters of the convolutions of the hemispheres, the brain of the Lemuridæ follows precisely the same type as that upon which the brain of Man and the other Quadrumana is formed, and differs essentially from that of the Carnivora and all other orders of Mammalia. But while the gradations of the brains of this type are tolerably regular and unbroken between the largest and the smallest of the series (i. e. *Homo* and *Hapale*), the Lemurs do not continue in precisely the same line of degradation, but rather should be placed as a small subseries parallel to the lower part of the large series, and distinguished from it by the shortness of the posterior lobes, the large size of the olfactory bulbs, and the inferior condition of the cerebellum.

With regard to the general characters of the posterior lobes throughout the series, although the examination of all the forms is not yet complete, the facts which have already been brought together are sufficient to justify the following conclusions:—

* A further examination of this specimen, and of the brains of some allied genera, leads me to doubt whether the above-described 'cavity' in the posterior lobe existed before dissection, the length of time that it had been in spirit having greatly facilitated this process. If it did not, it will justify the statement of the absence of the hippocampus minor by anatomists who have looked at this structure only in its relation to the posterior cornu, but at the same time will afford a further illustration of what I have endeavoured to show throughout this paper, viz. that the part of the brain to which this term has been applied can exist independently of the ventricular cavity.

1. That the posterior lobes, whether we understand by the term that portion of the cerebrum which lies over the cerebellum, or, taking our definition from internal structure, that part which is situated behind the hippocampus major, exist in all the Quadrumana, and are characterized in all by the presence of a deep longitudinal sulcus (calcarine) on their inner surface.

2. That the length of this part of the brain, in relation to that of the antero-median portion, varies in different members of the series, but is greater in many of the Apes than it is in Man, and attains its maximum in the smaller members of the family Platyrrhina.

3. That the depth and complexity of the characteristic involution of the cortical grey matter surrounding the calcarine sulcus (or, in other words, that part which, according to its homology with the structure so named in the human brain, must be called "hippocampus minor") is one of the most striking characteristics of the typical Simian brain, as it is greatest in *Cercopithecus*, *Macacus*, *Cynocephalus*, and *Cebus*, less in the anthropoid Apes, and least of all, in proportion to the mass of cerebral substance contained in the lobe, in Man.

Appendix.

TABLE showing the comparative length of the posterior lobes of the cerebrum in certain Quadrumana, and other Mammalia, measured upon the plan described at p. 191.

	Actual length in inches.		Proportion.	
	Antero-median portion.	Posterior portion.	Antero-median portion.	Posterior portion.
Homo (average)	4·40	2·35	100	53
Pithecus satyrus	2·00	1·00	100	50
Presbytes leucoprymnus	1·50	·70	100	47
Cercopithecus pygerythrus	1·65	·90	100	54
C. sabæus	1·65	·90	100	54
C. mona	1·20	·65	100	54
Macacus silenus	1·75	·95	100	54
M. erythræus	1·75	·90	100	52
Cynocephalus porcarius	2·20	1·25	100	57
Cebus apella	1·60	·95	100	59
Nyctipithecus felinus	1·20	·70	100	58
Hapale œdipus	·85	·50	100	59
H. jacchus	·80	·50	100	62
Lemur nigrifrons	1·00	·35	100	35
Stenops Javanicus	·90	·40	100	44
Otolicnus ——?	·85	·35	100	41
Pteropus Edwardsii	·79	·18	100	24
Erinaceus Europæus	·60	·07	100	11
Cercoleptes caudivolvulus	1·20	·40	100	33
Felis domesticus	1·20	·25	100	21
Canis familiaris	1·55	·55	100	35
Equus caballus	3·50	1·25	100	35
Sus scrofa	2·15	·55	100	25
Dicotyles torquatus	2·00	·40	100	20
Lepus cuniculus	1·00	·20	100	20

Explanation of the Plates.

PLATE II.

All the figures are of the natural size, except the first.

Fig. 1. Inner face of the right cerebral hemisphere of human brain, reduced one-half in linear dimensions. A, B, C. Sections of the posterior lobe at the points indicated by the lines so lettered.

Fig. 2. The same part of *Presbytes leucoprymnus.*

Fig. 3. The same part of *Cercopithecus pygerythrus.*

Fig. 4. The same part of *Macacus erythræus.*

PLATE III.

Fig. 5. The same part of *Cebus apella.*

Fig. 5 *a*. Horizontal section of left hemisphere of *Presbytes leucoprymnus.* The transverse lines show the mode of estimating the length of the posterior portion of the brain (B) as compared with the antero-median (A).

Fig. 6. Side view of the brain of *Cercopithecus pygerythrus,* showing the exact form when *in situ.* The outline is drawn from a cast of the interior of the cranium.

Fig. 7. Horizontal section of the brain of the same animal. On the right side the middle and posterior cornua are completely opened, so as to exhibit the relative size and situation of the two hippocampi. In exposing the hippocampus minor to this extent, the limits of the cornu (as seen in the sections A, B, C, fig. 3) have not been exceeded; but as the walls are more or less adherent, this must be regarded partly as a dissection. On the left side the walls of the cornu remain undisturbed, part of the brain only having been cut away to expose the commencement of the hippocampus major.

Fig. 8. Side view of the brain of *Hapale jacchus.*

Fig. 9. Upper surface of the same brain, the left hemisphere in section.

Fig. 10. Inner face of the right hemisphere of the same brain. B. Section of the posterior lobe.

Fig. 11. Upper surface of the brain of *Lemur nigrifrons.*

Fig. 12. Base of the same brain.

Fig. 13. Side view of the same.

Fig. 14. Inner face of the same. B. Section of the posterior lobe.

Fig. 15. Horizontal section of the brain of *Galago.* On the right side the section is carried rather low, and the hippocampus minor cut through; on the left the whole surface of this structure is exposed by opening the posterior cornu from above. The form of the brain is somewhat altered by keeping in spirit.

Nomenclature and lettering of all the Figures.

Gyri of the outer face:—

1. Infero-frontal.
2. Medio-frontal.
3. Supero-frontal.
1′. Supra-orbital.
4. Antero-parietal.
5. Postero-parietal.
5′. Postero-parietal lobule.
6. Angular.
7. Antero-temporal.
8. Medio-temporal.
9. Postero-temporal.
10. Supero-occipital.
11. Medio-occipital.
12. Infero-occipital.
13. First external annectent.
14. Second external annectent.
15. Third external annectent.
16. Fourth external annectent.

Gyri of the inner face:—

17. Marginal.
18. Callosal.
18′. Quadrate lobule.
19. Uncinate.
20. Dentate.
21–24. Internal annectent.
25. Internal occipital lobule.
26. Calcarine.

Sulci of the outer face:—

a. Infero-frontal.
b. Supero-frontal.
c. Antero-parietal.
d. Postero-parietal.
e. Sylvian.
f. Antero-temporal.
g. Postero-temporal.
h. Temporo-occipital.

Sulci of the inner face:—

i. Calloso-marginal.
k. Occipito-parietal.
l. Calcarine.
m. Dentate.
n. Collateral.
** Hippocampus major.
* Hippocampus minor.

X. *On Magnetic Calms and Earth-Currents.*

By CHARLES V. WALKER, *Esq.*, *F.R.S.*, *F.R.A.S.*

Received February 3,—Read February 13, 1862.

I USE the word "calm" in a purely negative sense—not storm. For many months the earth has shown few marked signs of activity. Very few notable earth-currents have attracted attention since those of 1861, January 22 to 26, referred to in the last paragraph but one of my paper "On Magnetic Storms, &c.," read February 14 *.

I had not proceeded far in the discussion of the results which form the subject of that communication, without discovering that I possessed close at hand, and under my immediate control, the means of verifying the conclusions to which I had arrived as to the general direction of earth-currents; of extending the observations to periods when the earth is free of signs from extraordinary activity; and of further pursuing my inquiries.

In sections 3 and 4 of Table XII.† I grouped together a series of observations; and in fig. 5 of Plate III. gave a graphic illustration of the same; and from those data deduced the approximate Direction of Earth-Currents; and determined the azimuth of the drift to be in turn about N.E. and S.W.

I was unable to include many cases in these sections of the Table. Simultaneous observations including both limiting lines were few; I was therefore glad to have so ready at hand the means of multiplying and modifying them. The groups of observations in the Table referred to were made on eight or nine different lines of telegraph, making various angles with the magnetic meridian; and were bounded on the one hand by the London-Tonbridge line, making an angle with the magnetic meridian of 13°W., and on the other hand by the Dover-London line, making an angle of 136° E., the two lines making with each other an angle of 149°; from which it could be deduced, and was shown, that the direction in which the currents moved was included within an arc of (180°−149°=)31°; and that this arc was situated midway between the lines in question (which are those set off on fig. 5, Plate III., and numbered there, as well as in Tables XI. and XII., 23 and 26 respectively), extending from 46° to 77° E. of magnetic north, or W. of magnetic south.

The Dover-London telegraph wires pass Tonbridge, where they enter my private office, being attached to a telegraph instrument placed there. This gives me the immediate command of the Dover-London line, numbered 26—25 in Table XI. p. 130, and in figs. 1 and 5, Plate III., and which is one of the limiting or boundary lines. By cutting off the communication with Dover, that is to say, by connecting the wire with

* Philosophical Transactions, 1861, p. 113.

† Ibid. 1861, p. 131.

the earth on the Dover side of Tonbridge, I obtain the other limiting line, London-Tonbridge, Nos. 23—24; or if the wires are connected with the earth on the London side of Tonbridge, I obtain the Dover-Tonbridge line. This is not given in my Table XI., but is very nearly identical with the Ashford-Tonbridge line there given, and numbered 32—31. It makes an angle of 118° E. with the magnetic meridian, and therefore falls, as shown in fig. 1, Plate III., intermediate between the other two, and incidentally is very useful in this investigation.

The earth-currents which form the subject of the present communication are not detected by the ordinary telegraph galvanometer. I have therefore prepared a small horizontal galvanometer. The coil is $2\frac{1}{4}$ inches long, $\frac{3}{4}$ of an inch wide, and $\frac{3}{4}$ of an inch high, and is filled with silk-covered copper wire, one yard of which weighs 5 grains; it is No. 35 of the Birmingham iron-wire gauge, corresponding to a diameter of $\frac{5}{640}$ inch. It is placed in the magnetic meridian. The needle is 1 inch in length, and carries a light index projecting from the E. side of the coil. The range of the index is about 55° on either side of 0°. The whole is covered by a glass shade. Earth-currents that attract no attention on the telegraph needles, produce on this instrument a deflection of 40° or 50°. It is placed in proper connexion with one of the Dover-London wires, and can be at any instant placed in circuit by merely pressing a spring, and thrown out of circuit by removing the pressure. Possession is obtained with equal facility of the London-Tonbridge or the Dover-Tonbridge section, by inserting a brass plug in one or other of two holes made in a divided brass disc. This is all the apparatus required. It is fixed on a slab within arm's reach from my chair. At any moment, when I see by the telegraph needles that the wires are unoccupied by telegrams, I can take the three complete observations in a few seconds. The word "up" is engraved in face of the N.E. quadrant, and the word "down" in face of the S.E.

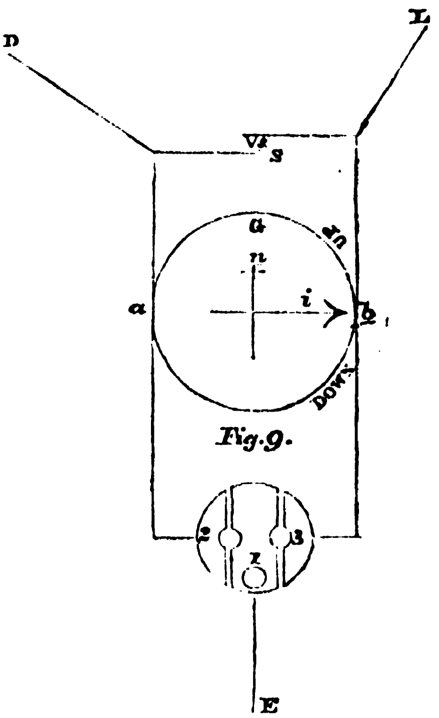


Fig. 9.

The arrangement of this miniature observatory will be readily understood from the following diagram (fig. 9).

L is the telegraph wire, entering the office from London; D the wire from Dover; S the contact spring, and *s* the stud on which it rests. When the spring and stud are in contact, the ordinary telegraph signals pass along them between D and L without being visible on the galvanometer G. The needle *n*, with its index *i*, and the circle of the galvanometer G, are given in one-third size; *a* and *b* are the terminals of the galvanometer wire. When the spring S is depressed or removed from contact with the stud *s*, as shown in the diagram, no current can pass between D and L without passing through the galvanometer G. 1, 2, and 3 show in one-third size a brass disc divided into three parts, and fixed upon a block of mahogany, insulated each from the others. The middle piece of the disc is connected by a wire with the gas- and water-pipes, and is therefore in connexion with the earth E. A brass plug is provided, which fits into the respective holes 1, 2, and 3. When in the centre hole 1, this instrument is not in action; when in the hole 2, it connects 1 and 2 together, so that the wire on the *a* or Dover side of the galvanometer is in direct communication with the earth E, and the galvanometer is in circuit between London and Tonbridge; when the plug is transferred to the hole 3, for like reasons the galvanometer is in circuit between Tonbridge and Dover. Observations on the whole line, or on either of the two sections, are readily and rapidly obtained by this arrangement.

The coil of the galvanometer is for convenience so wound, that when it is traversed by a positive current of electricity travelling *towards* London, or, to use railway language, "up" the line, the index moves in the right quadrant, or toward the word "up" engraved there; when, on the other hand, it is travelling *from* London, or "down" the line, the index moves to the left quadrant, or toward the word "down."

The following is a Table of Observations made during October 1861. They are taken in the order of the columns 1, 2, and 3. The letters *u* or *d* are entered against each observation, according as the index moves to the word "up" or "down." The time is taken at the end of the third observation. It is given in "Greenwich Mean Time," fractions of minutes being rejected; the local clock error being known by means of the "Time Signals" that come two or three times a day from Greenwich.

Table XIII.

Directions and Values of Earth-currents, collected at Tonbridge, 1861, October; from the London-Dover;—London-Tonbridge;—and Tonbridge-Dover lines.

Date.	Time.	Column 1.	Column 2.	Column 3.	Date.	Time.	Column 1.	Column 2.	Column 3.
		Dover-London ....*u*. London-Dover ....*d*.	Tonbridge-London ...*u*. London-Tonbridge ...*d*.	Dover-Tonbridge ......*u*. Tonbridge-Dover ......*d*.			Dover-London ....*u*. London-Dover ....*d*.	Tonbridge-London ...*u*. London-Tonbridge ...*d*.	Dover-Tonbridge ......*u*. Tonbridge-Dover ...
1861.	h m				1861.	h m			
Oct. 1.	9.31 A.M.	16° *u*	6° *d*	14° *u*	Oct. 4.	12.26 P.M.	40° *u*	55° *u*	50° *u*
	11.42 A.M.	15 *u*	35 *d*	30 *u*		4.50 P.M.	10 *d*	20 *d*	2 *d*
	2.20 P.M.	4 *u*	36 *d*		Oct. 5.	7.44 A.M.	35 *d*	50 *d*	45 *d*
	2.35 P.M.	3 *u*	28 *d*	7 *u*	Oct. 8.	11 A.M.	20 *d*	10 *d*	31 *d*
Oct. 2.	9.34 A.M.	9 *u*	4 *d*	11 *u*		12.56 P.M.	20 *u*	15 *u*	15 *u*
	12.24 P.M.	23 *u*	26 *d*	42 *u*		1.47 P.M.	45 *u*	10 *d*	50 *u*
	2.15 P.M.	17 *u*	35 *d*	20 *u*		3.12 P.M.	48 *u*	50 *d*	55 *u*
	2.47 P.M.	0	0	0	Oct. 9.	6.20 A.M.	12 *u*	20 *d*	25 *u*
	3.20 P.M.	6 *u*	28 *d*	18 *u*		9.20 A.M.	35 *d*	25 *u*	35 *d*
	3.35 P.M.	0	5 *d*	0	Oct. 10.	6.58 A.M.	10 *u*	15 *d*	20 *u*
Oct. 3.	7.2 A.M.	35 *d*	5 *d*	20 *d*		3.14 P.M.	30 *u*	50 *d*	50 *u*
	11.44 A.M.	32 *u*	14 *d*	38 *u*		10.0 P.M.	55 *d*	5 *u*	10 *d*
	5.20 P.M.	19 *d*	3 *d*	20 *d*	Oct. 11.	6.25 A.M.	55 *u*	50 *d*	55 *u*
	10.12 P.M.	20 *u*	0	20 *u*		9.6 A.M.	55 *d*	35 *u*	50 *d*
Oct. 4.	6.25 A.M.	33 *d*	3 *u*	45 *d*		10.25 A.M.	0	30 *u*	0
	12.24 P.M.	35 *u*	29 *d*	50 *u*	Oct. 12.	7.8 A.M.	5 *u*	40 *d*	40 *u*

TABLE XIII. (continued).

Date.	Time.	Column 1.	Column 2.	Column 3.	Date.	Time.	Column 1.	Column 2.	Column 3.
		Dover-London ....u. London-Dover ....d.	Tonbridge-London ...u. London-Tonbridge ...d.	Dover-Tonbridge ......u. Tonbridge-Dover ......d.			Dover-London ....u. London-Dover ....d.	Tonbridge-London ...u. London-Tonbridge ...d.	Dover-Tonbridge ......u. Tonbridge-Dover ......d.
1861.	h m				1861.	h m			
Oct. 12.	9.10 A.M.	20° d	20° u	40° d	Oct. 24.	7.41 A.M.	20° d	20° u	20° u
Oct. 14.	11.59 A.M.	10 d	10 d	10 d		7.47 A.M.	10 d	20 d	5 u
	12.52 P.M.	25 u	30 d	35 u		7.54 A.M.	15 u	22 d	35 u
	2.16 P.M.	10 u	12 d	11 d		9.20 A.M.	45 d	25 u	50 d
	2.45 P.M.	5 u	25 d	15 u		9.37 A.M.	55 d	45 u	55 d
Oct. 15.	11.34 A.M.	20 u	40 d	32 u		10.25 A.M.	30 d	15 u	43 d
	12.19 P.M.	18 u	16 d	25 u		11.24 A.M.	0	12 d	8 u
	1.23 P.M.	8 u	30 d	20 u		11.47 A.M.	13 d	25 u	38 d
	2.30 P.M.	2 u	15 d	7 u		12.29 P.M.	0	15 d	0
	3.25 P.M.	10 d	4 u	15 d		2.40 P.M.	22 u	55 u	15 u
	7.16 P.M.	20 u & d	5 u	40 to 60 d		2.44 P.M.	55 u	50 u	50 u
Oct. 16.	6.37 A.M.	10 d	5 u	35 d		3.32 P.M.	55 u	13 d	55 u
	7.44 A.M.	5 d	35 u	35 d		4.23 P.M.	7 u	35 u	0
	12.18 P.M.	40 u	50 d	45 u		5.3 P.M.	3 d	32 u	40 d
Oct. 17.	10.52 A.M.	23 u	10 d	30 u	Oct. 25.	6.14 A.M.	15 d	5 u	30 d
	11.25 A.M.	20 u	25 d	38 u		6.42 A.M.	50 d	20 u	55 d
	11.43 A.M.	26 u	26 d	40 u		10.21 A.M.	15 d	15 u	20 d
	2.13 P.M.	20 u	18 d	35 u		10.29 A.M.	40 d	10 u	33 d
	3.41 P.M.	10 u	20 d	15 u		11.10 A.M.	16 d	13 u	30 d
	5.6 P.M.	10 d	15 d	0		4.50 P.M.	20 u	20 d	30 u
Oct. 18.	6.35 A.M.	30 d	0	35 d		9.25 P.M.	25 d	15 u	40 d
	7.17 A.M.	35 d	25 u	45 d	Oct. 26.	6.10 A.M.	20 u	25 d	30 u
	8.36 A.M.	35 d	30 u	32 d		7.32 A.M.	20 d	35 u	20 d
	10.58 A.M.	14 u	10 d	28 u		7.44 A.M.	25 d	40 u	40 d
	11.37 A.M.	24 u	10 d	34 u		8.50 A.M.	0	5 d	0
	12.30 P.M.	40 u	30 d	48 u		10.54 A.M.	21 d	15 u	40 d
	12.57 P.M.	22 u	15 d	30 u		12.19 P.M.	0	10 u	0
	1.36 P.M.	20 u	25 d	40 u		12.57 P.M.	0	0	0
	2.44 P.M.	22 u	18 d	34 u		1.10 P.M.	20 u	20 d	26 u
	3.39 P.M.	15 u	12 d	16 u		1.26 P.M.	0	8 u	0
Oct. 19.	7.12 A.M.	25 d	10 u	38 d		4.1 P.M.	0	5 d	0
	12.1 P.M.	12 u	40 d	45 u		9.45 P.M.	0	10 d	0
Oct. 21.	7.7 A.M.	5 d	0	35 d		10.45 P.M.	10 d	10 d	10 d
	10.10 A.M.	28 d	15 u	32 d	Oct. 28.	7.15 A.M.	0	15 u	0
	1.24 P.M.	50 u	22 d			10.19 A.M.	0	20 d	8 u
	1.42 P.M.	50 u	10 d	50 u		10.35 A.M.	23 d	15 u	32 d
	2.21 P.M.	55 u	23 d	52 u		11.2 A.M.	5 d	24 d	0
	2.56 P.M.	43 u	15 d	47 u		11.18 A.M.	16 d	18 d	12 d
	3.35 P.M.	45 u	22 d	50 u		11.37 A.M.	20 d	10 u	35 d
Oct. 22.	6.15 A.M.	0	5 d	2 u		12.13 P.M.	20 d	4 u	27 d
	7.15 A.M.	5 d	0	25 d		12.57 P.M.	23 d	0	26 d
	7.57 P.M.	5 d	0	5 d		1.9 P.M.	30 d	14 u	40 d
Oct. 23.	6.13 A.M.	0	5 u	0		2.50 P.M.	10 d	3 u	14 d
	7.0 A.M.	5 d	0	0		3.21 P.M.	8 u	40 d	28 u
	7.31 A.M.	5 d	10 u	30 d					
	10.13 A.M.	18 d	35 u	32 d		10.10 P.M.	20 u	15 d	15 u
	1.1 P.M.	16 u	16 d	38 u		10.20 P.M.	5 d	10 d	10 d
	1.45 P.M.	40 u	0	45 u		10.30 P.M.	5 d	10 d	5 d
	3.9 P.M.	45 u	0	50 u		19.40 P.M.	5 u	25 d	20 u
Oct. 24.	6.20 A.M.	0	7 d	7 u		10.50 P.M.	15 d	5 u	5 to 20 d
	7.5 A.M.	0	0	2 u		11.0 P.M.	20 d	5 to 15 u	20 d

TABLE XIII. (continued).

Date.	Time.	Column 1.	Column 2.	Column 3.	Date.	Time.	Column 1.	Column 2.	Column 3.
		Dover-London ....*u*. London-Dover ....*d*.	Tonbridge-London ...*u*. London-Tonbridge ...*d*.	Dover-Tonbridge ......*u*. Tonbridge-Dover ......*d*.			Dover-London ....*u*. London-Dover ....*d*.	Tonbridge-London ...*u*. London-Tonbridge ...*d*.	Dover-Tonbridge ......*u*. Tonbridge-Dover ......*d*.
1861.	h m				1861.	h m			
Oct. 28.	11.10 P.M.	0°	10° *d*	3° *u*	Oct. 29.	4.43 A.M.	10° *d*	0° to 5° *u*	12° *d*
	11.20 P.M.	15 *u*	18 *d*	20 *u*		4.53 A.M.	10 *d*	0 to 5 *u*	15 *d*
	11.30 P.M.	5 *u*		5 *u*		5.0 A.M.	8 *d*	0 to 10 *u*	14 *d*
	11.40 P.M.	36 *d*	42 *u*	52 *d*		5.10 A.M.	8 *d*	8 *u*	18 *d*
	11.50 P.M.	25 *d*	22 *u*	35 *d*		5.20 A.M.	4 *d*	0 to 10 *u*	5 *d*
	12.0 P.M.	18 *d*	0	20 *d*		5.30 A.M.	0	0 to 15 *u*	0
Oct. 29.	12.10 A.M.	22 *d*	22 *d*	22 *d*		5.40 A.M.	4 *d*	0 to 4 *u*	5 *d*
	12.20 A.M.	10 *d*	0	15 *d*		5.50 A.M.	0	10 *u*	2 *d*
	12.30 A.M.	18 *d*	8 *d*	20 *d*		6.0 A.M.	0	5 *u* to 5 *d*	0
	12.40 A.M.	22 *d*	8 *u*	36 *d*		6.10 A.M.	0	0 to 5 *u*	0
	12.50 A.M.	10 *d*	7 *d*	13 *d*		6.20 A.M.	0	5 *d*	0
	1.0 A.M.	10 *d*	5 *d*	12 *d*		6.30 A.M.	0	0	0
	1.10 A.M.	15 *d*	6 *d*	20 *d*		6.40 A.M.	5 *u*	0	8 *u*
	1.20 A.M.	12 *d*	8 *d*	20 *d*		6.50 A.M.	5 *u*	10 *d*	25 *u*
	1.30 A.M.	18 *d*	6 *d*	20 *d*		7.7 A.M.	10 *u*	7 *u*	10 *u*
	1.40 A.M.	12 *d*	5 *d*	19 *d*					
	1.50 A.M.	15 *d*	5 *d*	21 *d*		7.30 A.M.	15 *u*	0	20 *u*
	2.0 A.M.	18 *d*	10 *u*	13 *d*		7.47 A.M.	15 *u*	0	15 *u*
	2.10 A.M.	18 *d*	0	20 *d*		10.5 A.M.	10 *u*	0 to 20 *d*	16 *u*
	2.20 A.M.	23 *d*	19 *u*	13 *d*		12.2 P.M.	8 *u*	40 *d*	28 *u*
	2.30 A.M.	18 *d*		20 *d*		12.25 P.M.	0	10 *d*	0
	2.40 A.M.	10 *d*	15 *u*	12 *d*		3.13 P.M.	10 *d*	12 *u*	16 *d*
	2.50 A.M.	5 *d*	0	10 *d*		3.29 P.M.	10 *d*	12 *u*	18 *d*
	3.0 A.M.	15 *d*	10 *d*	15 *d*	Oct. 30.	6.10 A.M.	0	10 *u*	10 *d*
	3.10 A.M.	13 *d*	5 *d*	19 *d*		6.30 A.M.	0	10 *u*	5 *d*
	3.20 A.M.	13 *d*	0	16 *d*		6.41 A.M.	0	20 *u*	8 *d*
	3.30 A.M.	15 *d*	0	20 *d*		7.42 A.M.	0	15 *u*	5 *d*
	3.40 A.M.	12 *d*	4 *d*	5 *d*		9.20 A.M.	17 *u*	1 to 10 *d*	20 *u*
	3.50 A.M.	10 *d*	0	12 *d*	Oct. 31.	6.12 A.M.	10 *d*	10 *u*	20 *d*
	4.0 A.M.	12 *d*	2 *d*	19 *d*		6.30 A.M.	0	10 *u*	0
	4.10 A.M.	14 *d*	10 *u*	20 *d*		9.43 A.M.	18 *u*	8 *u* to 12 *d*	21 *u*
	4.20 A.M.	10 *d*	5 *u*	20 *d*		11.59 A.M.	14 *u*	42 *d*	38 *u*
	4.30 A.M.	9 *d*	0 to 5 *u*	11 *d*		6.54 P.M.	0	0 to 15 *u*	4 *d*

Column 1, *u* is the direction 26—25 of Table XI.* and Plate III. figs. 1 & 5.
,, *d* ,, 25—26 ,, ,, ,,
Column 2, *u* ,, 24—23 ,, ,, ,,
,, *d* ,, 23—24 ,, ,, ,,
Column 3, *u* ,, 32—31 ,, ,, fig. 1.
,, *d* ,, 31—32 ,, ,, ,,

When column 1 contains an entry of an "up" current *u*, and column 2 of a "down"

* Philosophical Transactions, 1861, p. 130.

current *d*, we have the result of which a few cases were collected and given in Table XII. section 3, and shown graphically in Plate III. fig. 5. The result is further confirmed when the entry in column 3 corresponds in direction with that in column 1.

When the directions recorded in columns 1, 2, and 3 are *d*, *u*, and *d* respectively, we have the results given in Table XII. section 4.

As I shall have occasion to refer incidentally to the possible influence of heat or cold, &c. over the relative values of the currents registered in the three columns of Table XIII., I have given in Table XIV. the Meteorological Register taken at Tonbridge, and kindly furnished to me by Dr. Fielding. The barometer readings are corrected and reduced to sea-level at 32°.

Table XIV.—Meteorological Register taken at Tonbridge, October 1861.

	Barometer, 9 A.M.	Degree of moisture, 9 A.M.	Shade, maximum.	Shade, minimum.	Mean temperature.	Wind at 9 A.M.	Pluviometer, 9 A.M.
	in.	°	°	°	°		
October 1.	29·827	94	70·5	55·3	62·90	S.E.	0·015
2.	29·967	88	65	48	56·50	N.	0·080
3.	30·217	93	66	52	59	N.	0·010
4.	30·223	79	65·8	48	56·75	S.E.	0·005
5.	30·089	96	66·8	55·5	61·15	S.	0·000
6.	30·227	91	61	57	59	E.	0·010
7.	30·153	97	69·8	53·5	61·55	N.E.	0·080
8.	29·975	97	71·6	57	64·30	N.	0·010
9.	29·973	94	65	48	56·50	S.W.	0·040
10.	30·045	96	67·5	51·5	59·50	N.E.	0·030
11.	29·559	88	69	49·8	59·40	S.S.W.	0·070
12.	30·039	79	62	55	58·50	S.W.	0·320
13.	30·105	79	65·8	55·6	60·70	S.W.	0·005
14.	30·133	92	70·4	47·2	58·80	E.	0·000
15.	30·181	81	68·4	48·2	58·30	E.	0·000
16.	30·265	90	57	43·2	50·10	N.	0·000
17.	30·351	74	58·6	49·2	53·90	E.	0·000
18.	30·221	80	59	42	50·50	S.E.	0·000
19.	30·065	93	60	43	51·50	E.	0·000
20.	29·933	96	62	48	55	E.	0·000
21.	29·863	80	59	50·5	54·75	E.	0·020
22.	29·881	93	60·3	48·5	54·40	S.E.	0·030
23.	30·103	89	57·4	47	52·20	S.W.	0·310
24.	30·135	84	61·3	56	58·65	S.	0·010
25.	30·167	91	62	50·5	56·25	W.S.W.	0·030
26.	30·217	84	57	43	50	N.E.	0·000
27.	30·143	80	56	46·5	51·25	E.	0·000
28.	30·143	72	53·4	43	48·20	E.	0·000
29.	30·065	74	52·2	43	47·60	E.	0·000
30.	30·983	76	53·3	43	48·15	N.	0·015
31.	30·921	89	53	44	48·50	N.	0·000

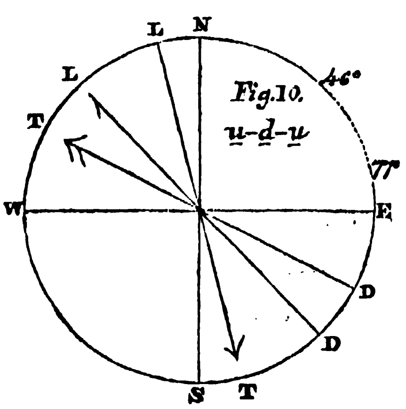
In Table XV. is given an analysis of the contents of Table XIII. Observations of a like character (as well as others to be hereafter referred to) were taken during the month of November. I have not thought it necessary to give them in detail, but have included a summary of them in this Table together with those of October.

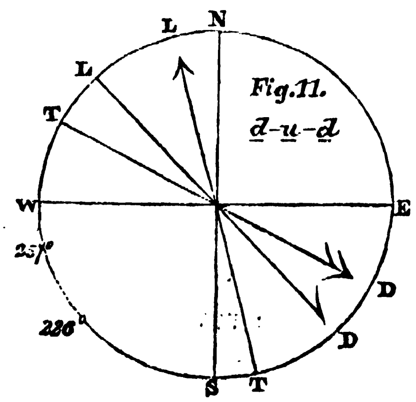
TABLE XV.—Analysis of Observations of Earth-Currents collected 1861, October and November.

	Col. 1.	Col. 2.	Col. 3.	October.	November.		
	0	0	0	3	2	5	
	u	*d*	*u*	78	37	115	
	d	*u*	*d*	83	27	110	
Normal	....	....	....	164	66		230
	u	*u*	*u*	1	10	11	
	d	*d*	*d*	25	6	31	
Abnormal	....	....	....	26	16		42
	u	*d*	*d*	1		1	
	d	*u*	*u*	1		1	
	d	*d*	*u*	1		1	
	u	*u*	*d*		1	1	
Exceptional	....	....	....	3	1		4
							276

In October there are three cases (000), and in November two, in which no deflection of the needle occurred in either of the three observations; and many individual cases are recorded. No particular stress is laid on these cases of no action. They merely indicate that the current, if any, was too small to affect the particular instrument used; a more delicate instrument might doubtless have given signs.

Of the 276 complete observations, 230, or five-sixths, are in accordance with the conclusions already arrived at—that the general direction of the drift of earth-currents is approximately N.E. or S.W. And the numbers of each kind come out nearly the same: 115 N.E. *u*, *d*, *u*;—and 110 S.W. *d*, *u*, *d*. The conditions of this group of results, which for convenience may be called *normal*, given graphically in Plate III. fig. 5, may be gathered in detail more readily from figs. 10 and 11.





N is the magnetic north; L—D, London-Dover; L—T, London-Tonbridge; T—D, Tonbridge-Dover. The respective lines of direction are further shown by the arrowheads,—one semi-barb indicating column 1; two, column 2; three, column 3. The arrow-heads pointing upwards all apply to "up" currents *u*; those pointing downwards, to "down" currents *d*. The dotted portion of the circumference of the circle is the arc of the horizon, within which the resultant is to be found. The degrees are given reckoned from the north, eastward round the circle. Fig. 10 represents the N.E. *normal*, or *u*, *d*, *u*, in Tables XIII. and XV.; fig. 11 the S.W. *normal*, or *d*, *u*, *d*.

From these Tables it appears that the *prevailing* currents, or those of most frequency, are from the N.E. or S.W.; this as well in calm periods as in periods of magnetic storms. In the absence of long-continued and consecutive observations, it is not easy to form an opinion as to whether the N.E. or S.W. currents prevail more or less at one part of the day than at another; or to what extent, if at all, the directions or alternations are influenced by local meteorological changes or conditions. In 'Les Archives des Sciences Physiques,' vol. xi. pp. 110–136, is a memoir by Father SECCHI, "On the Connexion of Meteorological Phenomena and Variations of the Intensity of Terrestrial Magnetism," in which he expresses very strongly his opinion that every rupture of meteorological equilibrium produces a rupture of electrical equilibrium, which can only be re-established by means of a current which discharges itself from place to place, which current cannot fail to act upon the magnetometers. Here is a wide field for research. Although I am not engaged in investigating the *origin* of the currents, I cannot avoid expressing my opinion that the value of existing currents, if not their direction (I speak locally), may be more or less influenced by meteorological changes, especially cloud, sunshine, or temperature. The currents at calm periods are at best but feeble. The resistances of the various parts of the telegraph wire through which they pass vary with the varying temperatures, so that it is quite reasonable to expect that, even when no change is taking place in the absolute value of the current travelling in the earth, the needle of the galvanometer may move forward or backward according as sunshine or cloud, heat or cold prevail here or there in the district under examination. This opinion is sanctioned in some degree by the result of some night observations made on October 28—29. My original observations were almost wholly made by day. On the night in question observations were made every ten minutes from 10.10 P.M. to 6.50 A.M. During this period there was evidently an excess of S.W. currents; the proportion of *u*, *d*, *u* to *d*, *u*, *d* currents was 1 : 2·7, whereas the day proportion for the month was 3·7 : 2·7. Also fifteen out of the twenty-five *d*, *d*, *d* currents were collected during the night; in fact there were only ten *u*, *d*, *u* currents in the whole fifty-six night observations. From 11.40 P.M. to 5.40 A.M. the London-Dover wire collected a continuous down *d* current, varying more or less in intensity; from 5.40 to 6.30 A.M., the current was too feeble to be appreciated; at 6.40 it was found in the reverse direction *u*, and was so when the observations were interrupted at 7.47.—These remarks in passing.

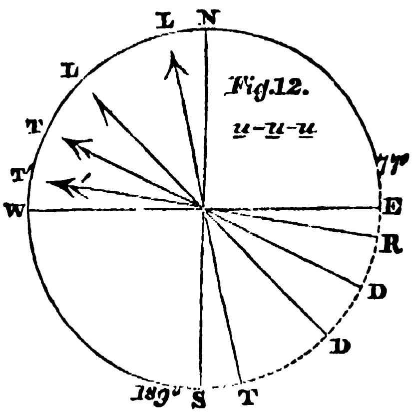
There is no consistency in the relations between two derived currents collected at the same time from the same earth-drift of electricity: this may be in great measure due, as I have hinted, to local meteorological interference, if not to the absolute differences of value in the different sections of the drift itself. I have taken at random, from the October observations, a few cases for illustration :—

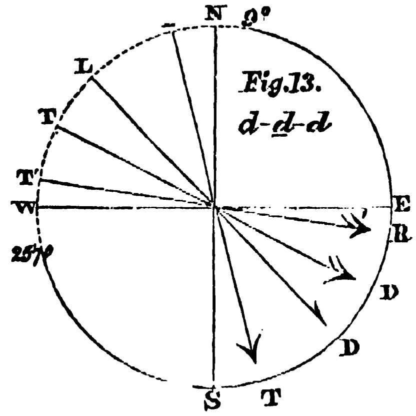
13 *u*	38 *u*
14 *u*	28 *u*
,, ...	31 *u*
15 *u*	15 *u*
,, ...	16 *u*
,, ...	20 *u*
,, ...	30 *u*
,, ...	35 *u*
16 *u*	14 *u*
,, ...	38 *u*
17 *u*	20 *u*
18 *u*	21 *u*
,, ...	25 *u*

The first column are Dover-London currents; the second, Dover-Tonbridge. Being read off on the same instrument, and under circumstances so favourable, they are strictly comparable. There are no relative values to be traced. For instance, 13° corresponds with 38°; 18° with 21° and 25°; 15° is coupled with various values—15°, 16°, 20°, 30°, and 35°, and so on. These illustrations may be extended at pleasure.

Since my original communication to the Royal Society, an "eighth article," by Professor LOOMIS, has appeared in the 'American Journal,' vol. xxxii., November 1861. On discussing the results accumulated in America, he infers that all the facts are consistent with the supposition of electric currents moving to and fro on the earth's surface, the average direction of which, on that continent, is from about N. 45° E. to S. 45° W., a result remarkably in accordance with the conclusions to which we have arrived by a somewhat different process for the S.E. part of England. He has also discussed, in quite another way, the magnetic disturbances in Europe; and he obtains a direction for the electric wave, connected with those disturbances, from N. 28° E. to S. 28° W. over the surface of Europe. These approximations are noteworthy.

It is plain, however, from Tables XIII. and XV., that currents from time to time flow from some point in the S.E. and in the N.W. quadrants. These directions may for the present be called *abnormal*. In the existing state of our knowledge, it is impossible to say whether they indicate the state of *transition* between the two *normal* quadrants. The cases are few in number. In October, 26 were recorded; in November, 16; or $\frac{2}{15}$ of the whole number. In December also I noticed a few. Like the *normal*, they are subdivided into two classes: *u*, *u*, *u*, 11; *d*, *d*, *d*, 31. The current was constant in the *d*, *d*, *d* direction, with a single interruption, from midnight October 28 to 1.50 A.M. October 29. Currents of this kind, as may be seen in Table XIII., are as definitely marked as the others; their values are equally large. They are expressed graphically in figs. 12 and 13.





The references and letters are the same as in figs. 10 and 11. The lines of direction are not favourable for arriving at an approximate place for the resultant. We have to seek it within the large arc of 131°, which includes respectively the whole of the S.E. or N.W. quadrant. I have the command at Tonbridge of the RamsgateHarbour-Tonbridge line, to which I shall have occasion hereafter to refer. It corresponds nearly with the Shalford-Red Hill, 34—33 of Table XI. and Plate III. fig. 1; it makes an angle with the magnetic meridian of 99°, reckoned eastward. Many observations on this line are given in Table XVI. column 3, and in all cases will be found to coincide in direction with those on the Tonbridge-Dover line. I have therefore been able to lay down this line in figs. 12 and 13. It is indicated by the letters R T′, and by three semibarbs. It reduces the arc to 112°, which, however, is still large, and which I cannot further reduce by the means at my command.

Since I wrote the note (1861, July) which appears in the 'Philosophical Transactions' for that year, Part I. p. 96, the Astronomer Royal has laid his proposition for erecting earth-current wires before the Board of Directors of the South-Eastern Railway, which has been referred to me; and it is needless to say has had my favourable report. The Directors have entertained the proposition most cordially; and have approved of the erection of the wires at cost price, and conceded the right of way and maintenance on the payment for each of a nominal sum of a few shillings annually.

The Greenwich-Dartford wire will make an angle of about 60° W. of N. (magnetic), nearly coincident with my Tonbridge-Ashford, or with the Tonbridge-Dover line of the present communication. The Greenwich-Croydon wire will make an angle of about 47° E. of N., or not far from the direction of my Ashford-Hastings line. By combining the wires at Greenwich, the Dartford-Croydon line may be obtained, which is 84° E. of N., or nearly my Ramsgate-Ashford direction. Treating these lines in the usual way, we shall have an arc of 36° or 107°. The former will be between N. and N.E., and out of the range. The latter, which however is very large, will include the range. I look forward with great interest to the completion of these wires, in order that we may see the result of uninterrupted observations, which cannot be obtained from our highly occupied telegraph wires.

I purpose erecting these wires as soon as the insulators are made; which will be constructed with great care, of a double concentric cup of ebonite, with an outer cup of French porcelain. The ebonite cups will be turned in a lathe before fitting, so as to present a perfectly smooth surface. They will be placed on the apex of our telegraph poles*.

Four *exceptional cases* presented themselves among the 276 observations. Their character is given in Table XV. They are beyond the reach of the system of analysis we have adopted. Whether the general direction of the current-drift was changing at the moment of observation, and was complete in one part of the district and not so in the other, I cannot, with these very rare and isolated data, pretend to say. The results are given precisely as read off. There was no reason to suspect any interferences from artificial currents; a few more cases may throw more light on this exceptional class.

A few entries will be found in the Table XIII. as thus: 5° to 15°; 0° to 10°; 5° *u* to 5° *d*, &c. The derived currents in these cases were unstable. A few of the earlier cases that occurred were questioned, and I was disposed to reject them, the impression being that they were due to interference from strong telegraph currents entering into the observing-wire. But instances occurred in which there was no reason to suspect interferences, they are therefore placed on record.

We speak of electric *currents* in this inquiry; the word conveys the idea of length without width. The currents in question necessarily and evidently cover large areas, presenting as it were an *electric plane*. Passing on from the determination of mere direction, I was able to survey the two sides of the same plane. By reference to the map (vol. 159, Plate II.), the RamsgateHarbour-London and the Dover-Tonbridge lines are not many degrees from being parallel. They are about 20 miles apart; the former is 67 miles, and the latter 45 miles in length. I have at Ashford junction a turnplate or switch. When desiring to make the observations on the Ramsgate-London line, I call Ashford and give the word "branch;" the reply is "yes" or "no," according as it is at liberty or not. If at liberty, the switch is turned, and I have the command of the wire from Ramsgate to London, the telegraph length of which is $97\frac{1}{2}$ miles; and then, by placing the plug in the hole 3 of fig. 9, for reasons already explained, the command is obtained of the Ramsgate-Tonbridge line.

Observations of this kind have been made from time to time; the results are given in Table XVI. The letters in this and subsequent Tables have the same references as those in Table XIII.

* I completed the two wires, and handed them over to the Astronomer Royal, June 30, 1862.

TABLE XVI.—Directions and Values of Earth-Currents collected at Tonbridge, 1861, November and December; from the Tonbridge-Dover, London-Ramsgate, and Tonbridge-Ramsgate lines.

Date.	Time.	Column 1.	Column 2.	Column 3.	Date.	Time.	Column 1.	Column 2.	Column 3.
		Dover-Tonbridge .... u. Tonbridge-Dover .... d.	Ramsgate-London .... u. London-Ramsgate .... d.	Ramsgate-Tonbridge, u. Tonbridge-Ramsgate, d.			Dover-Tonbridge .... u. Tonbridge-Dover .... d.	Ramsgate-London .... u. London-Ramsgate .... d.	Ramsgate-Tonbridge, u. Tonbridge-Ramsgate, d.
1861.	h m	°	°	°	1861.	h m	°	°	°
Nov. 17.	12.35 P.M.	28 *u*	10 *u*	14 *u*	Dec. 3.	6.13 A.M.	18 *d*	9 *d*	9 *d*
Nov. 20.	7.34 A.M.	25 *d*	15 *d*	20 *d*	Dec. 4.	6.20 A.M.	0	5 *d*	0
Nov. 21.	7.7 A.M.	0	0	0		1.25 P.M.	40 *u*	27 *u*	32 *u*
Nov. 22.	7.1 A.M.	15 *u*	12 *u*	0	Dec. 6.	6.39 A.M.	5 *u*	0	0
Nov. 25.	3.25 P.M.	14 *u*	0	7 *u*	Dec. 7.	6.24 A.M.	25 *u*	35 *u*	42 *u*
Nov. 26.	6.35 A.M.	35 *u*	20 *u*	15 *u*	Dec. 9.	7.18 A.M.	20 *u*	0	0
	3.24 P.M.	10 *u*	15 *u*	15 *u*	Dec. 13.	6.49 A.M.	10 *u*	20 *u*	20 *u*
Nov. 27.	6.19 A.M.	10 *u*	10 *u*	15 *u*	Dec. 14.	6.35 A.M.	15 *d*	10 *d*	20 *d*
	11.47 A.M.	15 *d*	11 *d*	10 *d*	Dec. 17.	6.30 A.M.	10 *d*	5 *d*	5 *d*
	12.32 P.M.	16 *d*	0	0	Dec. 19.	6.23 A.M.	0	15 *u*	20 *u*
	2.18 P.M.	28 *d*	8 *d*	20 *d*		8.18 P.M.	55 *d*	55 *d*	55 *d*
	2.38 P.M.	26 *d*	24 *d*	29 *d*	Dec. 20.	6.30 A.M.	5 *u*(?)	15 *d*	10 *d*
Nov. 28.	6.20 A.M.	0	0	2 *u*	Dec. 21.	6.24 A.M.	10 *u*	20 *u*	30 *u*
Nov. 29.	6.33 A.M.	5 *d*	2 *d*	0	Dec. 26.	6.34 A.M.	25 *u*	20 *u*	25 *u*
	1.13 P.M.	17 *d*	23 *d*	35 *d*	Dec. 27.	6.35 A.M.	15 *u*	15 *u*	20 *u*
Nov. 30.	6.22 A.M.	10 *d*	0	5 *d*	Dec. 28.	6.23 A.M.	0	2 *u*	2 *u*
Dec. 2.	2.27 P.M.	22 *u*	7 *d*	5 *d*	Dec. 31.	6.26 A.M.	30 *d*	25 *d*	30 *d*

Column 2 gives the Ramsgate-London results, or the survey of the north side of the parallelogram; and column 1 the Dover-Tonbridge results, or survey of the south side of the parallelogram. In every instance, with a solitary exception during the two months of observation, the directions coincide; the current or drift or electric plane is at least 20 miles wide, and the behaviour of its two limits is consistent. The proportion between the values of the currents on the two sides of the plane is not constant, but is a little better maintained than that of Table XIII. before referred to. The Ramsgate-London, or 67-mile line collected by $97\frac{1}{2}$ miles of telegraph wire, gives in the majority of cases a less value than the Dover-Tonbridge, or 45-mile line, collected in $46\frac{3}{4}$ miles of wire.

Column 3 of Table XVI., already noticed, is the Ramsgate-Tonbridge line. It makes a diagonal across the plane. The directions in all cases coincide with those of the other two lines, and so give a further evidence of consistency.

Tonbridge is almost in a direct line between London and Hastings, and very nearly equidistant. Lines 21—22, and 23—24, Plate III. fig. 1, give the true readings. I have a switch or turn-plate in the Telegraph Office at the Tonbridge junction, by means of which the Tonbridge-Hastings wire can be placed at my request in connexion with the

Tonbridge-London wire. The direct line between London and Hastings is 53 miles. I have thus an opportunity of making observations on the whole of this line, or on either half, the direction of all three being the same. The results of these observations are given in Table XVII. Column 1 contains the value of currents collected on the whole line of 53 miles; column 2, those on the London half of 27 miles; and column 3, those on the Hastings half of 26 miles.

TABLE XVII.—Directions and Values of Earth-Currents collected at Tonbridge, 1861, November and December; from the London-Hastings, London-Tonbridge, and Tonbridge-Hastings lines.

Date.	Time.	Column 1.	Column 2.	Column 3.	Date.	Time.	Column 1.	Column 2.	Column 3.
		Hastings-London .... *u*. London-Hastings .... *d*.	Tonbridge-London ... *u*. London-Tonbridge ... *d*.	Hastings-Tonbridge... *u*. Tonbridge-Hastings... *d*.			Hastings-London .... *u*. London-Hastings .... *d*.	Tonbridge-London ... *u*. London-Tonbridge ... *d*.	Hastings-Tonbridge .. *u*. Tonbridge-Hastings .. *d*.
1861.	h m	°	°	°	1861.	h m	°	°	°
Nov. 15.	12.34 P.M.	8 *d*	18 *d*	1 *d*	Nov. 29.	6.24 A.M.	2 *u*	5 *u*	0
	1.56 P.M.	8 *d*	24 *d*		Nov. 30.	6.17 A.M.	5 *u*	10 *u*	0
Nov. 16.	10.19 A.M.	4 *u*	44 *u*	6 *u*	Dec. 2.	11.48 A.M.	0	8 *d*	0
	10.53 A.M.	20 *d*	50 *d*	0	Dec. 3.	6.13 A.M.	5 *d*	0	5 *d*
Nov. 17.	12.35 P.M.	5 *u*	10 *u*	4 *u*	Dec. 4.	6.17 A.M.	10 *d*	15 *d*	5 *d*
Nov. 20.	7.37 A.M.	5 *d*	10 *d*	0		1.29 P.M.	0	16 *d*	1 *d*
	10.50 A.M.	7 *u*	8 *u*	4 *u*	Dec. 5.	6.24 A.M.	5 *d*	0	5 *d*
Nov. 21.	7.10 A.M.	0	5 *d*	0	Dec. 6.	6.37 A.M.	0		0
	9.35 A.M.	12 *u*	18 *u*	0	Dec. 7.	6.22 A.M.	0	5 *d*	0
	11.46 A.M.	4 *u*	6 *u*	3 *u*	Dec. 9.	7.16 A.M.	5 *d*	0	5 *d*
	12.49 P.M.	0	0	5 *u*	Dec. 13.	6.43 A.M.	0	0	0
Nov. 23.	6.22 A.M.	5 *d*	10 *d*	0	Dec. 14.	6.33 A.M.	5 *u*	10 *u*	0
	9.1 A.M.	0	10 *u*	0	Dec. 17.	6.29 A.M.	0	5 *d*	0
Nov. 25.	1.34 P.M.	6 *u*	12 *u*	4 *u*	Dec. 18.	6.20 A.M.	5 *d*	10 *d*	5 *d*
	2.47 P.M.	8 *u*	5 *u*	9 *u*	Dec. 19.	6.21 A.M.	5 *d*	10 *d*	5 *d*
	3.24 P.M.	8 *u*	23 *u*	7 *u*	Dec. 20.	6.20 A.M.	10 *d*	25 *d*	5 *d*
Nov. 26.	6.34 A.M.	0	10 *d*	0	Dec. 21.	6.22 A.M.	15 *d*	28 *d*	10 *d*
Nov. 27.	6.17 A.M.	0	0	0	Dec. 26.	6.30 A.M.	0	10 *d*	0
	12.32 P.M.	7 *d*	5 *d*	5 *d*	Dec. 27.	6.34 A.M.	0	0	0
	2.18 P.M.	0	3 *u*	0	Dec. 28.	6.21 A.M.	0	0	0
Nov. 28.	6.17 A.M.	0	5 *u*	2 *d*(?)	Dec. 31.	6.23 A.M.	0	0	0

If the value of these derived currents depended simply on the mere distance between the earth-plates or observing-stations, and their bearing each on the other, it is obvious that the values in columns 2 and 3 would be identical, or in this case nearly so. The London-Tonbridge wire-length is 41 miles; the Tonbridge-Hastings 33; so that the value on the latter length should be a little higher if anything, the resistance being less. But, with very rare exceptions, the Tonbridge-Hastings values are seen to be greatly below the Tonbridge-London. The contrast is remarkable. In some cases the differences are very conspicuous. I have made a sufficient number of observations, extended over two months, to satisfy myself that the one section is under all circumstances less active in derived currents than the other. This difference can only be attributed, as already suggested in my former communication (p. 109), to the different geological conditions of the two sections of country, a difference which may operate in two ways: either the resistance of the section may be relatively great, so that the earth-plates penetrate into a portion of the electric plane that is traversed by a current of low value, and hence the derived current is comparatively low; or the resistance of the section may be relatively small, so that, although the earth-plate may penetrate into a portion of the electric plane that is traversed by a current of higher value, yet the wire resistance, in contrast with the high conducting power of the earth section, may cause the derived current to have a relatively low value.

The London-Hastings line makes an angle of possibly 70° with the N.E. or S.W. resultant, that is with the *u*, *d*, *u* or the *d*, *u*, *d* currents; so that even with a good geological section and with a perfect knowledge of the relative resistances of its various parts, it would be no easy matter even to hint at the precise relation between the value of the current and the structure and arrangement of the strata. Mr. ROBERT HUNT has kindly furnished me with a geological section of the country between London and Hastings drawn by Mr. F. DREW. Between London and Tonbridge are included, London clay; Woolwich beds (sand); chalk; gault (clay); lower greensand, sand and a little limestone; clay; sand and sandstone; clay. Between Tonbridge and Hastings, sand and sandstone; clay and a little limestone. There are many faults also between Tonbridge and Hastings.

Column 1 of Table XVII. contains the values given by the whole length, from London to Hastings. They differ but little, save in one or two instances, from the Tonbridge-Hastings values; and are consequently very low in comparison with the Tonbridge-London values. These facts all indicate the very notable influence of local conditions, other than the meteorological variation already noted, over the relative value of the current in different parts of the plane.

It was a matter of considerable importance to determine with certainty whether the currents with which I was dealing were in whole, or only in part, earth-currents; whether, that is, any portion of the observed deflections of the galvanometer needle were due to electricity collected from the atmosphere by the suspended telegraph wires that were employed in these observations. To determine this I availed myself of the assistance of Ashford. After taking observations in the usual way between London and Dover, and between Tonbridge and Dover, I desired the Ashford clerk to detach the wire from his instrument and leave the end insulated. I thus had a wire of 67 miles connected with the earth at London, or one of 25 miles connected with the earth at Tonbridge, the end in either case being insulated at Ashford. It was desirable to make a considerable number of observations, at various hours of the day and under all conditions of weather, in order to test this question rigidly. The results are given in Table XVIII.

TABLE XVIII.—Observations showing, by detaching one end of the Telegraph wire from the earth, that currents collected at Tonbridge, 1861, October, November, and December, from the London-Dover and Tonbridge-Dover lines, were true and proper Earth-currents.

Date.	Time.	Column 1.	Column 2.	Column 3.	Date.	Time.	Column 1.	Column 2.	Column 3.
		Dover-London......... u. London-Dover......... d.	Dover-Tonbridge....... u. Tonbridge-Dover....... d.	Columns 1 and 2 with wire off earth at Ashford.			Dover-London .... u. London-Dover .... d.	Dover-Tonbridge ...... u. Tonbridge-Dover ...... d.	Columns 1 and 2 with wire off earth at Ashford.
1861.	h m				1861.	h m			
Oct. 18.	11.37 A.M.	24° *u*	34° *u*	0°	Nov. 27.	11.47 A.M.	10° *d*	15° *d*	0°
	1.36 P.M.	20 *u*	40 *u*	0		2.18 P.M.	17 *d*	28 *d*	0
	2.44 P.M.	22 *u*	34 *u*	0		2.38 P.M.	18 *d*	26 *d*	0
	3.39 P.M.	15 *u*	16 *u*	0	Nov. 28.	6.15 A.M.	0	0	0
Oct. 21.	10.10 A.M.	28 *d*	32 *d*	0	Nov. 29.	6.20 A.M.	0	5 *d*	0
Oct. 24.	11.47 A.M.	13 *d*	38 *d*	0	Nov. 30.	6.21 A.M.	5 *d*	10 *d*	0
	2.44 P.M.	55 *u*	50 *u*	0	Dec. 2.	11.49 A.M.	23 *u*	34 *u*	0
	4.23 P.M.	7 *u*	0	0		2.24 P.M.	15 *u*	22 *u*	0
Oct. 28.	11.18 A.M.	16 *d*	12 *d*	0	Dec. 3.	6.24 A.M.	15 *d*	18 *d*	0
	10.30 P.M.	5 *d*	5 *d*	0	Dec. 4.	6.15 A.M.	5 *d*	0	0
	11.0 P.M.	20 *d*	20 *d*	0		12.9 P.M.	20 *u*	13 *u*	0
	11.30 P.M.	5 *u*	5 *u*	0		1.24 P.M.	34 *u*	40 *u*	0
	12.0 P.M.	18 *d*	20 *d*	0		2.40 P.M.	20 *u*	16 *u*	0
Oct. 29.	0.30 A.M.	18 *d*	20 *d*	0	Dec. 5.	6.20 A.M.	0	0	0
	1.0 A.M.	10 *d*	12 *d*	0		12.34 P.M.	17 *u*	40 *u*	0
	1.30 A.M.	18 *d*	20 *d*	0	Dec. 6.	6.20 A.M.	0	0	0
	2.0 A.M.	18 *d*	13 *d*	0	Dec. 7.	6.20 A.M.	16 *u*	25 *u*	0
	2.30 A.M.	18 *d*	20 *d*	0	Dec. 9.	7.12 A.M.	15 *u*	20 *u*	0
	3.0 A.M.	15 *d*	15 *d*	0	Dec. 13.	6.40 A.M.	10 *u*	10 *u*	0
	3.30 A.M.	15 *d*	20 *d*	0	Dec. 14.	6.30 A.M.	5 *d*	15 *d*	0
	4.0 A.M.	12 *d*	19 *d*	0	Dec. 17.	6.17 A.M.	10 *d*	10 *d*	0
	4.30 A.M.	9 *d*	11 *d*	0	Dec. 18.	6.18 A.M.	15 *d*	10 *d*	0
	5.0 A.M.	8 *d*	14 *d*	0	Dec. 19.	6.20 A.M.	0	0	0
	5.30 A.M.	0	0	0	Dec. 20.	6.20 A.M.	0	5 *u*	0
	6.0 A.M.	0	0	0	Dec. 21.	6.18 A.M.	0	10 *u*	0
	6.30 A.M.	0	0	0	Dec. 26.	6.25 A.M.	20 *u*	25 *u*	0
Nov. 21.	11.46 A.M.	10 *d*	20 *d*	0	Dec. 27.	6.30 A.M.	10 *u*	15 *u*	0
Nov. 25.	1.31 P.M.	0	0	0	Dec. 28.	6.17 A.M.	0	0	0
	3.23 P.M.	16 *u*	14 *u*	0	Dec. 31.	6.19 A.M.	25 *d*	30 *d*	0
Nov. 26.	3.24 P.M.	5 *u*	10 *u*	0					

It will be seen that in no single instance was any deflection of the needle obtained when the wire was off at Ashford; so that we are right in regarding all the currents with which we have been dealing, as far as atmospheric electricity is concerned, as true and proper earth-currents.

Columns 1 and 2 are the deflections obtained on the London-Dover and Tonbridge-Dover lines in the usual way. Column 3 are the 0° resulting from repeating each of the two previous observations with the Ashford wire detached.

I have also made repeated observations on the effect of polarizing the earth-plates by the passage of a powerful current; and all with the same negative result. An observation is made, say on the Tonbridge-London line, and the deflection noted. A powerful current from the ordinary telegraph battery (a current which it would be imprudent to receive on the observing galvanometer) is transmitted for a given time, say half a minute, through the Tonbridge-London circuit, in the direction which would have produced the deflection previously observed. After this the observation is repeated. No appreciable difference has in any instance been found in the deflection. The direction remains the same; and the value unaltered. So that the currents, as far as the polarizing of the plates is concerned, may be regarded as true and proper earth-currents.

The alterations in direction of the currents and the varying values in either direction indicate that they are in nowise due to the mere electromotive force of the earth-plates employed.

I have mentioned in my former communication (pp. 94, 95) that the connexion with the earth is very frequently made by means of the fish-jointed rails. Some misgivings might arise on a first glance as to the influence of this arrangement over the results. The current collected under such circumstances, however, would be true in direction, but reduced in value. We need not enter into the discussion of this question from the fact that no such earth-connexion has been required for our present purposes.

To prevent misapprehension, I have thought it better to give a list of the earth-connexion employed at all the stations concerned in this investigation.

London	Gas-pipes; and a lightning conductor terminating in a wet pit of coke.
Tonbridge	Gas- and water-pipes.
Hastings	Gas- and water-pipes.
Ramsgate Harbour . . .	Water-pipes.
Dover	Gas-pipes.

I still continue taking observations; but they are all of the same character as those now placed on record, and merely afford further illustrations of the points that have been discussed. It would be premature to say that the subject is tolerably exhausted, as far as the means at my command are concerned; but I do not at this moment see any salient point that is within my reach. The steady daily and hourly march of these phenomena, and their relation, if any, to the like march of magnetometers, will soon be within the reach of Mr. AIRY; and we may be well assured that these and the other collateral questions will be ably discussed by him.

The results arrived at in these two communications may be briefly summed up as follows:—

1st. That currents of electricity are at all times moving in definite directions in the earth.

2nd. That their direction is not determined by local causes.

3rd. That there is no apparent difference, except in degree, between the currents collected in times of great magnetic disturbance, and those collected during the ordinary calm periods.

4th. That the prevailing directions of earth-currents, or the currents of most frequent occurrence, are approximately N.E. and S.W. respectively.

5th. That there is no marked difference in frequency, duration, or value, between the N.E. and the S.W. currents.

6th. That (at least during calm periods) there are definite currents of less frequency from some place in the S.E. and N.W. quadrants respectively.

7th. That the direction of the current in one part of a plane on the earth's surface (at least as far as the S.E. district of England is concerned) coincides with the direction in another part of the plane; and if the direction changes in one part, it changes in all parts of the plane.

8th. That the relation in value between currents in a given part of the plane and currents in another given part is not constant, but is influenced by local meteorological conditions, and varies from time to time.

9th. That the value of the current of a given length, moving in a given line of direction, is not necessarily the same as that of a current of the same length on the same line of direction produced, and that their relative value depends on the physical character of the earth interposed between the respective points of observation, and is tolerably constant.

10th. That the currents which have formed the bases of these investigations are derived currents from true and proper earth-currents, and neither in whole nor in any appreciable part have been collected from the atmosphere, nor are due either in whole or in any appreciable part to polarization imparted to earth-plates by the previous passage of earth-currents or of powerful telegraphic currents; nor are they due to any electromotive force in the earth-plates themselves.

11th. That the earth-currents in question (at least the powerful currents present at all times of great magnetic disturbance) exercise a *direct* action upon magnetometers, just as artificial currents confined to a wire exercise a direct action upon a magnet.

XI. *On the Spectrum of Carbon. By* JOHN ATTFIELD, *Esq., F.C.S., Director of the Laboratory of the Pharmaceutical Society; lately Demonstrator of Chemistry at St. Bartholomew's Hospital. Communicated by Dr.* FRANKLAND, *F.R.S.*

Received June 19,—Read June 19, 1862.

IT is well known that a mixture of coal-gas and air burns with a flame of slight luminosity. When such a flame is prismatically examined under favourable circumstances, as by the ordinary spectroscope, the light it emits is found to consist of four groups of rays of different refrangibility. These rays appear in the field of the instrument as faint yellow, light green, bright blue, and rich violet bands of light.

In 1856 SWAN* found that the spectrum thus obtained was common to all hydrocarbon flames. He showed that they were best seen in an olefiant gas-flame fed with air by a blowpipe jet, measured and recorded their distances from each other, searched for, but did not find, corresponding dark bands in the solar spectrum, and gave no theory in explanation of their origin.

On recently reading SWAN's paper by the light that Professors BUNSEN and KIRCHHOFF have thrown on the subject, I came to the conclusion that these bands must be due to incandescent carbon vapour; that, if so, they must be absent from flames in which carbon is absent, and present in flames in which carbon is present; that they must be observable equally in the flames of the oxide, sulphide, and nitride as in that of the hydride of carbon; and, finally, that they must be present whether the incandescence be produced by the chemical force, as in burning jets of the gases in the open air, or by the electric force, as when hermetically sealed tubes of the gases are exposed to the discharge from a powerful induction coil.

Experiment has fully confirmed the truth of this theory, and the following are the details of the investigation:—

To obtain intimate acquaintance with the spectrum in question SWAN's experiments were repeated. On feeding the flames with undiluted oxygen, instead of with air, still brighter spectra than he describes were obtained. The heat thus produced, however, volatilized potassium, sodium, lead, &c. from glass jets, and zinc and copper from brass jets; and this result ensued whether the oxygen was directed into the centre of, or made to surround the hydrocarbon flame. Finally, a mixture of olefiant gas and oxygen, or of coal-gas, saturated with benzole, and oxygen, was burned at an ordinary platinum oxyhydrogen safety-jet. In this way a cylindrical flame, half an inch in length and one-

* Edinb. Phil. Trans., vol. xxi. p. 411.

tenth of an inch in diameter, was obtained, and gave, on examination by the spectroscope, a brilliant well-defined spectrum.

The spectrum thus produced corresponds in appearance with the description of that observed by SWAN, excepting in the number of fine lines in each band of light. The yellow-green band, composed, according to the drawing accompanying SWAN's paper, of four lines, I find to contain six; the green band to have five instead of two; the blue five, that is one more than SWAN noticed; and the violet, beside being distinctly double, to have a faint hair-line between its two halves. Indeed, in this as in other spectra, the reduction of bands into groups of lines seems simply dependent on the refractive power of the spectroscope, an increased number of prisms causing greater dispersion of the spectrum, and, consequently, a division in a line or band that otherwise would appear to be single.

Having thus reproduced a satisfactory spectrum of the flame of a hydrocarbon, I next turned my attention to that of a nitrocarbon. Rejecting prussic acid vapour, on account of its containing hydrogen, I chose cyanogen. Cyanide of mercury was heated in a retort, and the cyanogen thus produced cooled and dried by passing over fragments of fused chloride of calcium contained in the neck of the retort. Ignited and examined by the spectroscope, this cyanogen flame gave a splendid series of bands, and these became still more distinct and brilliant on feeding the flame with oxygen by the platinum safety-tube already mentioned. Familiarity with the spectrum of hydrocarbon flames enabled me to detect it in this nitrocarbon light, other lines present being afterwards proved to be due to incandescent nitrogen.

But to establish the absolute identity of the hydro- and nitro-carbon spectra, excluding of course the lines due to nitrogen, they were simultaneously brought into the field of the spectroscope, one occupying the upper, the other the lower half of the field. This was readily effected after fixing the small prism, usually supplied with spectroscopes, over half of the narrow slit at the further end of the object-tube of the instrument. The light from the oxy-hydrocarbon flame was now directed up the axis of the tube by reflexion from the little prism, while that from the oxy-nitrocarbon flame passed directly through the uncovered half of the slit. A glance through the eye-tube was sufficient to show that the characteristic lines of the hydrocarbon spectrum were perfectly continued in the nitrocarbon spectrum. A similar arrangement of apparatus, in which the hydrocarbon light was replaced by that of pure nitrogen, showed that the remaining lines of the nitrocarbon spectrum were identical with those of the nitrogen spectrum. In this last experiment the source of the pure nitrogen light was the electric discharge through the rarefied gas.

The above experiments certainly seemed to go far towards proving the spectrum in question to be that of the element carbon. Nevertheless, the ignition of the gases having been effected in air, it was conceivable that hydrogen, nitrogen, or oxygen had influenced the phenomena. To eliminate this possible source of error, the experiments were repeated out of contact with air. A thin glass tube, 1 inch in diameter and

3 inches long, with platinum wires fused into its sides, and its ends prolonged by glass quills having a capillary bore, was filled with pure dry cyanogen, and the greater portion of this gas then removed by a good air-pump. Another tube was similarly prepared with olefiant gas. The platinum wires in these tubes were then so connected with each other that the electric discharge from a powerful induction coil could pass through both at the same time. On now observing the spectra of these two lights, in the simultaneous manner previously described, the characteristic lines of the hydrocarbon spectrum were found to be rigidly continued in that of the nitrocarbon. Moreover, by the same method of simultaneous observation, the spectrum of each of these electric flames, as they may be termed, was compared with the corresponding chemical flames, that is, with the spectra of the oxyhydro- and oxynitro-carbon jets of gas burning in air. The characteristic lines were present in every case. Lastly, by similar interobservation, a few other lines in the electric spectrum of the hydrocarbon were proved to be due to the presence of hydrogen, and several others in the electric spectrum of the nitrocarbon to be caused by the presence of nitrogen.

The electric discharge through cyanogen rapidly causes decomposition. The characteristic spectrum soon disappears, a black deposit is formed on the sides of the tube, and the spectrum of nitrogen alone remains. Olefiant gas is similarly affected, but not to the same extent. The glass tube is much blackened, but the spectrum is constant. Berthelot has, in fact, already shown that olefiant gas is decomposed by the electric current, acetylene being at the same time produced. Indeed, as acetylene may, according to Berthelot, be formed from its elements under the influence of the electric discharge, it is inconceivable that a hydrocarbon gas could be perfectly decomposed in such a tube as I have described.

The spectrum under investigation having then been obtained in one case when only carbon and hydrogen were present, and in another when all elements but carbon and nitrogen were absent, furnishes, to my mind, sufficient evidence that the spectrum is that of carbon.

But an interesting confirmation of the conclusion just stated is found in the fact, that the same spectrum is obtained when no other elements but carbon and oxygen are present, and also when carbon and sulphur are the only elements under examination. And first with regard to carbon and oxygen. Carbonic oxide burned in air gives a flame possessing a continuous spectrum. A mixture of carbonic oxide and oxygen burned from a platinum-tipped safety-jet also gives a more or less continuous spectrum, but the light of the spectrum has a tendency to group itself in ill-defined ridges. Carbonic oxide, however, ignited by the electric discharge in a semivacuous tube, gives a bright sharp spectrum. This spectrum was proved, by the simultaneous method of observation, to be that of carbon plus the spectrum of oxygen. With regard to carbon and sulphur almost the same remarks may be made. Bisulphide-of-carbon vapour burns in air with a bluish flame. Its spectrum is continuous. Mixed with oxygen and burned at the safety-jet, its flame still gives a continuous spectrum, though more distinctly furrowed than in the case of carbonic oxide; but when ignited by the electric current, its spectrum is well defined, and is that of carbon plus that of sulphur. That is to say, it is the spectrum of carbon plus the spectrum that is obtained from vapour of sulphur when ignited by the electric discharge in an otherwise vacuous tube.

Having thus demonstrated that dissimilar compounds containing carbon emit, when sufficiently ignited, similar rays of light, I come to the conclusion that those rays are characteristic of ignited carbon vapour, and that the phenomenon they give rise to on being refracted by a prism is the spectrum of carbon.

The spectrum of carbon is a very beautiful one. The lines composing each band of light regularly diminish in brightness in the direction of greatest refraction, and appear to retreat from the observer like pillars of a portico seen in perspective. It differs greatly from that of every other element that I am acquainted with; and though, in each of the experiments described, it was of course accompanied by the spectrum of either nitrogen, hydrogen, sulphur, or oxygen, its diagnosis was not thereby interfered with; it is, in fact, most widely different from, and cannot possibly be confounded with, either of them.

The brightest band of the carbon spectrum being blue, and its other constituents being on the one hand light green and on the other violet, the associated rays of ignited carbon vapour, as indeed seen by the naked eye in carbon flames, I conceive to be of a light-blue colour. The tint may be observed in the flame of a spirit-lamp, in a burning jet of carbonic oxide, in the blowpipe flame of any hydrocarbon, and at the base of a common candle flame. I have no hesitation in saying that should a source of heat be found of sufficient intensity to volatilize the diamond and ignite its vapour, blue will be the colour of the light emitted.

The subject of the *emission* of carbon light by carbon vapour naturally leads to the consideration of the *absorption* of carbon light by carbon vapour. This latter research I am compelled to defer for a time.

The investigation also suggests the important question, *Is the spectrum of a compound simply the sum of the spectra of its constituents?* I have made several experiments tending to confirm such a law, but must perform many more before coming to a decided conclusion. Some observations made by Professor Plücker in the course of an examination of the effects of the electric discharge on rarefied gases* seem to indicate that a compound has a spectrum different from that of the superposed spectra of its constituents.

Finally, I beg to offer my best thanks to Dr. Frankland for allowing me the use of his laboratory and apparatus in making this research, and to J. P. Gassiot, Esq., and Captain G. W. Puget (34th Regt.) for the loan of induction coils.

* Poggendorff's 'Annalen,' Bd. cv. S. 77, and Bd. cvii. S. 533.

XII. *On the Theory of Probabilities.*

By George Boole, *F.R.S., Professor of Mathematics in Queen's College, Cork.*

Received June 10,—Read June 19, 1862.

This paper has for its object the investigation of the general analytical conditions of a Method for the solution of Questions in the Theory of Probabilities, which was proposed by me in a work entitled "An Investigation of the Laws of Thought" (London, Walton and Maberly, 1854).

The application of this method to particular problems has been illustrated in the work referred to, and yet more fully in a 'Memoir on the Combination of Testimonies and of Judgments' published in the Transactions of the Royal Society of Edinburgh (vol. xxi. Part 4). Some observations, too, on the general character of the solutions to which the method leads, founded upon induction from particular cases, were contained in the original treatise, and the outlines, still in some measure conjectural, of their general theory were given in an Appendix to the Memoir. But the complete development of that theory was attended with analytical difficulties which I have only lately succeeded in overcoming. It involves discussions relating to the properties of a certain functional determinant, and to the possible solutions of a system of algebraic equations of peculiar form—discussions which will, I trust, be thought to possess a value, as contributions to Mathematical Analysis, independent of their present application.

As concerns the nature of the problems to which the method is applicable, it may be stated that they are such that the numerical elements, both given and sought, are the probabilities of events or states of things the definitions of which, and the connexions of which, are capable of expression by logical propositions. There is ground for believing that all questions whatever involving probability are ultimately reducible to this general form. This point, however, I do not purpose to discuss here. It has been already in some degree considered in the Memoir referred to.

In order to explain more fully the necessity for the present investigation, it will be requisite to state the fundamental principles upon which the method in question rests. There are only two of them which can possibly afford matter for discussion.

1st. The expression in language of the data of a problem in the Theory of Probabilities is to a certain extent arbitrary, because it depends upon the extent of meaning of the primary simple terms employed to express the events the conceptions of which it involves. But the choice of simple terms is, if we consider it with respect to our absolute power of choice, arbitrary. Any complex combination of events can be contemplated as a single whole in thought, and expressed by a single term. The invention of new simple terms to express what was before expressed by a combination of terms is a normal phenomenon in the growth of language.

Now the first principle upon which the method rests is the following:—

Principle I.—The different forms which a problem may be made to assume by different elections with respect to the simple terms of its expression are mutually equivalent.

For instance, if the following data were given,

The probability of rain is p,
The probability of rain with snow is q,

the form which the problem would assume in a language in which there was no word for snow, but in which the combination of snow with rain was called sleet, would be

The probability of rain is p,
The probability of sleet is q,

with the added condition, expressed as a logical proposition, that sleet always implies rain. And this as a statement of the data would, it is affirmed, be equivalent to the former statement. If these were the data of an actual problem, the event of which the probability is sought would require similar translation.

I desire to guard here against a possible misapprehension. I have said that the choice of simple terms, if considered with respect to our power of choice, is arbitrary. I do not mean by this to affirm that the actual growth of language is arbitrary. We know that it is far otherwise. Unity of sensuous impression in the early stages of its growth, unity of thought in the latter, seems to govern the invention and introduction of simple terms. It has indeed been said that there is a λόγος in the constitution of things of which language in its varied forms is the human reflexion, but never without the inseparable human element of choice and voluntary power.

It is then affirmed that whatever the grounds of fitness or propriety (and the existence of such grounds is fully conceded) may be, which have governed the actual choice of the simple terms of language, those grounds have nothing whatever to do with the calculation of probability. This depends upon the *information* contained in the data, information supposed to be derived from actual experience, or at least to be of such a nature that experience might have furnished it.

The different forms in which a problem is capable of being expressed, though differing in consequence of the different arbitrary elections which are possible with respect to its simple terms, are not independent of each other. They are connected together by the Laws of Thought, and pass one into the other by the processes of the Calculus of Logic, which is an organized expression of those Laws.

Among these forms there is one which presents exclusive advantages. It is that in which those events, however originally expressed, the probabilities of which constitute the data, are assumed as the simple events of the problem, and expressed by logical symbols corresponding to the simple terms of ordinary language; the event of which the probability is sought being also expressed logically by means of the same symbols. The Calculus of Logic enables us to do this, determining at the same time in an *explicit* form, *i. e.* in a form capable of expression in ordinary language by definite logical propositions, the connexion which exists among all the events in question—a connexion which in the original form of the data was only *implied.*

This leads us to the statement of the second Principle of the Method.

Principle II.—When the data have been translated into probabilities of events connected by conditions logical in form and explicitly known, the problem may be constructed from a scheme of corresponding ideal events which are free, and of which the probabilities are such that when they (the ideal events) are restricted by the same conditions as the events in the data, their calculated probabilities will become the same as the given probabilities of the events in the data.

To take a material illustration: the problem, in the form to which it is reduced by the Calculus of Logic in accordance with Principle I., might be represented by the supposition of an urn containing balls distinguished by certain properties, *e. g.* by colour, as white or not white, by form, as round or not round, by material, as ivory or not ivory, and by the supposition that, while these properties enter into every conceivable combination, all the balls in which certain combinations are found are attached by strings to the sides of the urn, so that only the balls in which the remaining combinations are realized can be drawn. Suppose, further, that the probabilities of drawing under the actual conditions a white ball, a round ball, an ivory ball, &c. are given, and the probability of drawing a free ball fully defined with respect to the above elements of distinction is required. The principle affirmed is that we must proceed as if the balls were all free, and with probabilities such that the calculated probability of drawing any one of the balls which under the previous supposition are free, would be the same as under that supposition it is given to be.

Confining ourselves to the above material case, I remark, that the supposed mode of solution represents, 1st, a *possible* order of things; 2ndly, an order of things in which no preference is given to any one combination over any other which falls under the same category, or mode of thought. All the procedure of the theory of probabilities is founded upon the mental construction of the problem from some hypothesis, either, 1st, of events known to be independent; or, 2ndly, of events of the connexion of which we are totally ignorant; so that, upon the ground of this ignorance, we can again construct a scheme of alternatives all equally probable, and distinguished merely as favouring or not favouring the event of which the probability is sought. In doing this we are not at liberty to proceed arbitrarily. We are subject, first, to the formal Laws of Thought, which determine the possible conceivable combinations; secondly, to that principle, more easily conceived than explained, which has been differently expressed as the "principle of sufficient reason," the "principle of the equal distribution of knowledge or ignorance*," and the "principle of order." We do not know that the distribution of

* Knowledge and ignorance being in the theory of probabilities supplementary to each other, the equal distribution of the one implies that of the other.

I take this opportunity of explaining a passage in the 'Laws of Thought,' p. 370, relating to certain applications of the principle. Valid objection lies not against the principle itself, but against its applicaproperties in the actual urn is the same as it is conceived to be in the ideal urn of free balls, but the hypothesis that it is so, involves an equal distribution of our actual knowledge, and enables us to construct the problem from ultimate hypotheses which reduce it to a calculation of combinations.

I pass from the particular and material to the general problem. In the form to which this is brought by the Calculus of Logic, the probabilities are those of events of which certain combinations are, as a logical consequence of the original definitions of those events, impossible. It might, at first sight, appear that this establishes a fundamental difference between this problem and that of the urn, in which certain combinations are prevented from issuing by a material hindrance. In the one case the restriction appears as logically necessary, in the other as only actual.

Upon this I remark, that the data of the problem in its ultimate reduced form *might* result from the same kind of dependence as in the actual data; that they, in fact, *would* thus result if the mind of the observer were capable of contemplating, and were in a position to contemplate, each of the events in this ultimate translated form simply as a whole, and of recording, through an approximately infinite series of observations, what combinations of those wholes come into being, and what do not, in the actual universe. What appears as necessary in the translated data would now appear as actual—as a result of observation; what is impossible would be received as non-existent. The question is, then, whether the difference between the conception of what is impossible from involving a logical contradiction, and the conception of what in the actual constitution of things never exists, is of a kind to affect expectation. I do not hesitate to say that it is not. We are concerned with events in so far as they are capable of happening or not happening, of combining or not combining; but we are not concerned with the reasons in virtue of which they happen or do not happen, combine or do not combine. If we went beyond this, we should enter upon a metaphysical question to which I presume that no answer can, upon rational grounds, be given, viz. upon the question whether, when two things or events are in the actual constitution of things incapable of happening together, it would, if our knowledge were sufficiently extended, be found that the resulting conceptions of them were logically inconsistent.

I have but one further observation on Principle II. to make. It is that in the general problem we are not called upon to interpret the ideal events. The whole procedure is, like every other procedure of abstract thought, formal. We do not say that the ideal events exist, but that the events in the translated form of the actual problem are to be considered to have such relations with respect to happening or not happening as a certain system of ideal events would have if conceived first as free, and then subjected, without their freedom being otherwise affected, to relations formally agreeing with those to which the events in the translated problem are subject.

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tion through arbitrary hypotheses, coupled with the assumption that any result thus obtained is necessarily *the* true one. The application of the principle employed in the text, and founded upon the general theorem of development in Logic, I hold to be *not* arbitrary.

We are now able to explain more clearly the nature of the analytical investigation which will follow. Let $p_1, p_2, \ldots p_n$ represent the probabilities given in the data. As these will in general not be the probabilities of unconnected events, they will be subject to other conditions than that of being positive proper fractions, viz. to other conditions beside

$$\left.\begin{array}{l} p_1 \overline{\gtrless} 0,\ p_2 \overline{\gtrless} 0 \ldots p_n \overline{\gtrless} 0 \\ p_1 \overline{\lessgtr} 1,\ p_2 \overline{\lessgtr} 1 \ldots p_n \overline{\lessgtr} 1. \end{array}\right\} \quad \ldots \ldots \ldots \quad (1.)$$

Those other conditions will, as will hereafter be shown, be capable of expression by equations or inequations reducible to the general form

$$a_1p_1 + a_2p_2 \ldots + a_np_n + a \overline{\gtrless} 0,$$

$a_1, a_2, \ldots a_n, a$ being numerical constants which differ for the different conditions in question. These, together with the former, may be termed the conditions of possible experience. When satisfied they indicate that the data *may* have, when not satisfied they indicate that the data *cannot* have, resulted from actual observation. On the other hand, the ideal events are regarded as independent, and their probabilities, which enter as auxiliary quantities into the process of solution, are subject to no other condition than that of being positive proper fractions. It is the general object of the analytical investigation to establish the two following conclusions, viz.,—

1st. The probabilities of the ideal independent events, as involved in the method under consideration, will in the process be determinable, without ambiguity, as positive proper fractions whenever the data satisfy the conditions of possible experience, and not otherwise.

And, as a consequence of the above,

2ndly. The probability determined by the method will have such a value as it consistently might have had if, instead of being calculated from the data, it had been determined by observation under the same experience as the data.

These conclusions rest upon the ground of certain analytical theorems relating to functional determinants, and to the possible solutions of simultaneous algebraic equations, which will be demonstrated in this paper. But, in order to explain the application of those theorems, it will be necessary to show, first, how the "conditions of possible experience" in problems in the Theory of Probabilities may be determined; secondly, what the analytical method in question for the solution of such problems is.

Determination of the Conditions of possible Experience.

The method for determining the conditions of possible experience given in the 'Laws of Thought,' chap. xix., may be advantageously replaced by the following one, which is taken from the 'Memoir on the Combination of Testimonies and of Judgments,' already referred to.

Let the events in the data be resolved into the ultimate possible alternatives which they involve, and let the unknown probabilities of these alternatives be represented by λ, μ, ν, &c.; then, as the probability of each event in the data is equal to the sum of the probabilities of the alternatives which it involves, we shall have a system of equations connecting λ, μ, ν, &c. with $p_1, p_2, \ldots p_n$, the probabilities supposed given. Again, $\lambda, \mu, \nu \ldots$, as probabilities, are subject to the conditions

$$\lambda \geqq 0,\ \mu \geqq 0,\ \nu \geqq 0, \ldots \text{\&c.},$$

and, as alternatives mutually excluding each other, to the condition

$$\lambda+\mu+\nu+\ldots=1,$$

or the condition

$$\lambda+\mu+\nu+\ldots \leqq 1,$$

according as the alternatives in question together make up certainty or not.

Thus we have a system consisting of equations and inequations from which λ, μ, ν, &c. must be eliminated. To effect this elimination we must determine as many of the quantities $\lambda, \mu, \nu \ldots$ as possible from the equations, substitute their values in the inequations, and then eliminate the remainder of the quantities $\lambda, \mu, \nu \ldots$ by means of the theorem that if we have simultaneously

$$\lambda \leqq a_1,\ \lambda \leqq a_2, \ldots \lambda \leqq a_m,$$
$$\lambda \geqq b_1,\ \lambda \geqq b_2, \ldots \lambda \geqq b_n,$$

then we have the system of conditions of which the type is

$$a_i \geqq b_j,$$

a_i representing any one of the set $a_1, a_2, \ldots a_m$, and b_j any one of the set $b_1, b_2, \ldots b_n$. Thus there are mn conditions in all.

This method is illustrated in the following problem, in the expression and solution of which it is to be noticed, that when in the Calculus of Logic an event is represented by x, the event which consists in its not happening is denoted by $1-x$, or for brevity by $\bar{x}$; that when two events are represented by x and y, their concurrence is denoted by xy, the happening of the first without the second by $x\bar{y}$, and so on.

Problem. Given that the probability of the concurrence of the events x and y is p, of the events y and z, q, and of the events z and x, r. Required the conditions to which p, q, and r must be subject in order that the above data may be consistent with a possible experience.

Resolving the events xy, yz, xz into the possible alternations out of which they are formed, let us write

$$\text{Prob. } xyz=\lambda,\ \text{Prob. } xy\bar{z}=\mu,\ \text{Prob. } x\bar{y}z=\nu,\ \text{Prob. } \bar{x}yz=\varrho.$$

Then we have the equations

$$\lambda+\mu=p,\ \lambda+\varrho=q,\ \lambda+\nu=r,$$

together with the inequations

$$\lambda \geqq 0,\ \mu \geqq 0,\ \nu \geqq 0,\ \varrho \geqq 0,$$
$$\lambda+\mu+\nu+\varrho \leqq 1.$$

From the equations we find

$$\mu=p-\lambda,\ \varrho=q-\lambda,\ \nu=r-\lambda,$$

which, substituted in the inequations, give

$$\lambda \geqq 0,\ p-\lambda \geqq 0,\ q-\lambda \geqq 0,\ r-\lambda \geqq 0,$$
$$p+q+r-2\lambda \leqq 1;$$

and it only remains to eliminate λ. Now from the above,

$$\lambda \overline{\overline{<}} p,\ \lambda \overline{\overline{<}} q,\ \lambda \overline{\overline{<}} r,\ \lambda \overline{\overline{>}} 0,\ \lambda \overline{\overline{>}} \frac{p+q+r-1}{2},$$

therefore

$$p \overline{\overline{>}} 0,\ q \overline{\overline{>}} 0,\ r \overline{\overline{>}} 0,$$

$$p \overline{\overline{>}} \frac{p+q+r-1}{2},\ q \overline{\overline{>}} \frac{p+q+r-1}{2},\ r \overline{\overline{>}} \frac{p+q+r-1}{2}.$$

The last three conditions are reducible to the simpler form,

$$p \overline{\overline{>}} q+r-1,\ q \overline{\overline{>}} r+p-1,\ r \overline{\overline{>}} p+q-1.$$

Such are the conditions of possible experience in the data.

Suppose, for instance, it was affirmed as a result of medical statistics that, in two-fifths of a number of cases of disease of a certain character, two symptoms x and y were observed; in two-thirds of the cases the symptoms y and z were observed; and in four-fifths of the cases the symptoms z and x were observed; so that, the number of cases observed being large, we might on a future outbreak of the disease consider the fractions $\frac{2}{5}$, $\frac{2}{3}$, and $\frac{4}{5}$ as the probabilities of recurrence of the particular combinations of the symptoms x, y, and z observed. The above formulæ would show that the evidence was contradictory. For, representing the respective fractions by p, q, and r, the condition $p \overline{\overline{>}} q+r-1$ is not satisfied. (*Edinburgh Memoir.*)

In applying the above method to the *à priori* limitation of questions in the theory of probabilities, it will be necessary to represent the probability sought by a single letter u, and treat this as if it were one of the numerical data. The resolution of the event of which the probability is sought into alternatives belonging to the same scheme as those of the events in the data gives us a new equation, which must be combined with the equations involving p, q, r, &c. The elimination of λ, μ, ν, &c. then determines not only the conditions of possible experience limiting p, q, r, but also the conditions which u must satisfy *à priori*, whatever method for its actual determination may be employed.

Thus, if from the foregoing data it were required to determine the *à priori* limits of Prob. xyz, i. e. of the probability of the conjunction of the events x, y, z, we should have as the additional equation

$$u=\lambda,$$

and therefore, after elimination of λ, μ, ν,

$$u \overline{\overline{<}} p,\ u \overline{\overline{<}} q,\ u \overline{\overline{<}} r,$$

$$u \overline{\overline{>}} 0,\ u \overline{\overline{>}} \frac{p+q+r-1}{2},$$

the conditions required.

It will, however, in most of the following investigations suffice to consider the conditions of possible experience in the data alone, because it will be shown that when these are satisfied the corresponding conditions for the probability sought, when its value is determined by the method of the following section, will also be satisfied.

Statement of the Method for the Solution of Questions in the Theory of Probabilities.

For the general demonstration of this method the reader is referred to the 'Laws of Thought,' chap. xvii. For the purpose of the analytical investigation the statement of the method will suffice.

Let s, t, v, &c. represent the events of which the probabilities are given, p, q, r, &c. those probabilities, and w the event of which the probability is sought; then, whatever the definitions of $s, t \ldots$ and w may be, and whatever connecting relations may exist, it is always possible by the Calculus of Logic to determine the logical dependence of w upon s, t, &c. in the following most general form, viz.

$$w=\mathrm{A}+0\mathrm{B}+\tfrac{0}{0}\mathrm{C}+\tfrac{1}{0}\mathrm{D}.$$

Here A, B, C, D are logical combinations of the events s, t, &c., and the connexion in which these stand to the event w and to each other is the following: A expresses those combinations of s, t, &c. which are entirely included in w, i. e. which cannot happen without our being permitted to say that w happens. B represents those combinations which may happen but are not included under w; so that when they happen we may say that w does not happen. C represents those combinations the happening of which leaves us in doubt whether w happens or not. D those combinations the happening of which would involve logical contradiction.

It follows from the above that the *translated* form of the problem is

Given Prob. $s=p$, Prob. $t=q$, Prob. $v=r$, &c., s, t, $v \ldots$ being regarded as events subject to the explicit logical condition

$$\mathrm{A}+\mathrm{B}+\mathrm{C}=1.$$

Required the probability u of the event of which the logical expression is

$$w=\mathrm{A}+\tfrac{0}{0}\mathrm{C};$$

and it is shown (Laws of Thought, p. 265), upon grounds essentially the same as those expressed in Principles I. and II. of this paper, that the solution of the problem is involved in the following *algebraic* equations, viz.

$$\frac{\mathrm{V}_s}{p}=\frac{\mathrm{V}_t}{q}\ldots=\frac{\mathrm{A}+c\mathrm{C}}{u}=\mathrm{V}, \qquad \text{(I.)}$$

in which the functions V, V_s, $\mathrm{V}_t \ldots$ are formed in the following manner, viz.,—

1st. V is derived from A+B+C without change of form by interpreting s, t, &c. no longer as logical symbols, but as symbols of quantity. They represent the probabilities of the ideal events of Principle II.

2ndly. V_s is the sum of those terms in V which contain s as a factor, V_t the sum of those which contain t as a factor, &c.

The quantity c is an arbitrary constant, admitting of any value between 0 and 1.

To effect the solution, the quantities s, t, &c. are to be eliminated from the system (I.), and u then determined as a function of p, q, $r \ldots$ and c. The arbitrary constant c may not appear in the final result, because the developed form of w may not contain any terms affected with the symbol $\frac{0}{0}$. When such terms do appear, the constant c admits of an interpretation indicating what new data are required to make the solution definite*.

It is proper here to observe that the conditions of possible experience can be determined as well from the 'translated' as from the original form of the problem. That the results will agree is evident *à priori*, but it may be desirable to point out the analytical connexion of the two processes. I will take the example just considered, and then offer some general remarks on the subject.

Representing the events xy, yz, zx by s, t, v, the translated data would be found to be

$$\text{Prob. } s=p, \quad \text{Prob. } t=q, \quad \text{Prob. } v=r,$$

s, t, and v being connected by the explicit logical condition

$$stv+s\bar{t}\bar{v}+\bar{s}t\bar{v}+\bar{s}\bar{t}v+\bar{s}\bar{t}\bar{v}=1.$$

It is easily shown that the first member of this equation represents the sum of those combinations of the events s, t, v, with respect to happening or failing, which involve no logical contradiction.

If, then, we represent under the above condition

$$\text{Prob. } stv=\lambda', \quad \text{Prob. } s\bar{t}\bar{v}=\mu', \quad \text{Prob. } \bar{s}t\bar{v}=\nu', \quad \text{Prob. } \bar{s}\bar{t}v=\varrho',$$

we shall have

$$\lambda'+\mu'=p, \ \lambda'+\nu'=q, \ \lambda'+\varrho'=r,$$
$$\lambda'\overline{>}0, \ \mu'\overline{>}0, \ \nu'\overline{>}0, \ \varrho'\overline{>}0,$$
$$\lambda'+\mu'+\nu'+\varrho'\overline{<}1.$$

This system of equations and inequations agrees with that employed in the previous solution, if we only make

$$\lambda'=\lambda, \ \mu'=\mu, \ \nu'=\varrho, \ \varrho'=\nu,$$

so that the elimination of λ', μ', ν', ϱ' will lead to the same results as before.

In general it may be observed that each combination of s, t, v which is possible without logical contradiction, gives, on substituting for s, t, v... their expressions in the simple terms of the original problem, either a single combination of those simple terms, or a sum of such combinations; but the same combination of those simple terms will not arise from two different combinations of s, t... It is clear from this that the systems of united equations and inequations arising in the two forms of the problem will be related in the following manner, viz.—

For each positive quantity λ' in the one set, there will exist either a single positive quantity λ, or a sum of such quantities $\lambda_1+\lambda_2+$&c. in the other; but each such sum is inseparable, and the elements it is composed of are distinct from those of any other sum arising from any other of the quantities λ'... It is evident, then, that the final results of elimination will be the same. The same formal processes which eliminate single quan-

* Laws of Thought, p. 267.

tities in the one case, will eliminate the corresponding single quantities, or sums of single quantities, in the other.

Simplification of the General Equations for the Solution of Questions in the Theory of Probabilities.

Let us express the system (I.) in the form

$$\frac{V_s}{V}=p, \quad \frac{V_t}{V}=q, \text{ \&c.,}$$

$$u=\frac{A+cC}{V},$$

and let us suppose the quantities p, q... (and therefore s, t...) to be n in number. Then all the terms in V will be composed of products of s, t... $\bar{s}$, $\bar{t}$..., each term involving either s or $\bar{s}$, either t or $\bar{t}$, &c., but not the combinations $s\bar{s}$, $t\bar{t}$, &c. Each term is therefore homogeneous and of the nth degree.

It follows, therefore, that if we divide the numerator and denominator of each of the first members of the above system by $\bar{s}\,\bar{t}\,\bar{v}$..., and then make

$$\frac{s}{\bar{s}}=x_1, \quad \frac{t}{\bar{t}}=x_2, \quad \frac{v}{\bar{v}}=x_3, \text{ \&c.,}$$

and if at the same time we, for symmetry, change p, q, r... into p_1, p_2, ..p_n, the system will assume the following form,—

$$\frac{V_1}{V}=p_1, \quad \frac{V_2}{V}=p_2 \ldots \frac{V_n}{V}=p_n,$$

$$u=\frac{A+cC}{V},$$

in which V, A, C are formed from their former values by suppressing $\bar{s}$, $\bar{t}$, $\bar{v}$, &c., or, which is the same thing, changing each of them into unity, and then changing s, t, v... into x_1, x_2, x_3..., while V_1 consists of those terms of V which contain x_1, V_2 of those which contain x_2, and so on.

In its new form V is a rational and entire function of x_1, x_2, ...x_n not involving powers of those quantities, and with all its coefficients equal to unity. Again, as s, t, &c. are from the theory of their origin required to be positive proper fractions, x_1, x_2, ... x_n are, from the nature of their connexion with s, t..., required to be positive quantities. And it is sufficient that x_1, x_2, ... x_n be determinable as positive quantities in order that s, t... may be determinable as positive fractions.

Now we shall proceed to show that x_1, x_2, ... x_n are determinable as positive quantities precisely when p_1, p_2, ...p_n satisfy the conditions of possible experience. We shall further show, as a consequence of this, that the value of the probability sought, when determined by the General Rule, will, under the same conditions, lie within such limits as if it were itself given by the same experience. In the order of this proof, we shall first demonstrate the theorems of pure Analysis upon which the conclusions depend, then in a distinct section make the particular application.

Analytical Theorems relating to Functional Determinants and Systems of Algebraic Equations.

A symmetrical determinant may be conveniently expressed in the form

$$\begin{vmatrix} A_1 & A_{12} & \dots & A_{1n} \\ A_{21} & A_2 & \dots & A_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1} & A_{n2} & \dots & A_n \end{vmatrix} \quad \dots\dots\dots\dots \quad (I.)$$

the conditions of symmetry being

$$A_{ij}=A_{ji},\ A_{ii}=A_i.$$

It is desirable to employ fixed language in referring to this. We shall therefore call the quantities A_1, $A_2 \dots A_n$ the 'principal elements,' and the diagonal series of terms which they form the 'principal diagonal.' The elements A_{ij}, when i and j differ, we shall call 'subordinate elements.' The element A_i, together with all the subordinate elements which occur upon the same horizontal or vertical line of the determinant, we shall designate the 'i-system of elements.' Lastly, in comparing two rows or two columns of elements together, those elements will be said to correspond which occupy the same numerical place in their respective rows or columns.

The following Lemma will next be established.

Lemma.—A symmetrical determinant expressed in the form (I.) will be unaltered in value, if from each subordinate element of its i-system we subtract the corresponding element of its j-system multiplied by a quantity λ, which is invariable for the same system,—and for the principal element A_i substitute $A_i-2\lambda A_{ij}+\lambda^2 A_j$.

It is known that a determinant vanishes if two of its lines or columns are identical, and it is known as a consequence of this that if from a particular line or column of a determinant the corresponding elements of another line or column, multiplied each by the same constant, are subtracted, the determinant is unaltered in value. From the ith line of the above symmetrical determinant subtract, term by term, λ times the jth line, and then from the ith column of the resulting determinant subtract λ times the jth column. As respects any subordinate element, the result will obviously accord with the statement in the Lemma. But the element A_i will be successively converted into

$$A_i-\lambda A_{ji}$$

$$(A_i-\lambda A_{ji})-\lambda(A_{ij}-\lambda A_j).$$

The last expression, since $A_{ji}=A_{ij}$, is reducible to

$$A_i-2\lambda A_{ij}+\lambda^2 A_j.$$

Upon this property the demonstration of the following general proposition will be founded.

PROPOSITION I.

Let the symmetrical determinant (I.) *possess the following properties, viz.*:—

1st. *That all its elements are linear homogeneous rational functions of certain quantities a, b, c, &c., unlimited in number.*

2ndly. *That if the coefficients of any one of these quantities a in the elements of any particular line or column taken in order are $\alpha_1, \alpha_2, \dots \alpha_n$, and in any other line or column $\beta_1, \beta_2, \dots \beta_n$, then these two series of quantities are respectively proportional.*

3rdly. *That the principal terms $A_1, A_2, \dots A_n$ are positive,* i. e. *that the coefficients of all the quantities a, b, c, &c. which appear in these terms are positive.*

Then the developed determinant will be itself positive, and will consist of products of the quantities a, b, c, &c. without powers, each product affected by a positive sign.

First, it may be observed that any letter a of the set $a, b, c \dots$ which appears in the subordinate term A_{ij} will appear in both the principal terms A_i, A_j.

For let m be the coefficient of a in A_{ij}, and therefore also in A_{ji}; let l be the coefficient of a in A_i, and n its coefficient in A_j. Thus to the elements A_i, A_{ji} in the i-column correspond A_{ji}, A_j in the j-column. Hence, by the definition of the determinant,

$$l:m::m:n,$$

$$\therefore \quad m^2=ln,$$

which implies that neither l nor n vanishes, so that a appears in A_i and A_j.

Secondly, we shall show that the determinant can, without alteration of its final developed value, be reduced to a form in which any letter a of the system $a, b, c \dots$ shall appear in only one system of elements, and therefore only in the principal term of that system, since every subordinate term is common to *two* systems.

Let us suppose a to be contained in two at least of the systems of elements, and for convenience of expression, let these be the 1-system and the n-system. Let, then, $\alpha_1, \alpha_2, \dots \alpha_n$ be the successive coefficients of a in $A_1, A_{21}, \dots A_{n1}$, and therefore, by definition of the determinant, $\lambda\alpha_1, \lambda\alpha_2, \dots \lambda\alpha_n$, its coefficients in $A_{n1}, A_{n2}, \dots A_n$. Any of the quantities $\alpha_1, \alpha_2, \dots \alpha_n$ may be 0. But by the Lemma above demonstrated the determinant may, without alteration of value, be reduced to the following form, viz.:—

$$\left|\begin{matrix} A_1, & A_{12}, \dots & A_{1n}-\lambda A_1 \\ A_{21}, & A_2, \dots & A_{2n}-\lambda A_{21} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot \\ A_{n1}-\lambda A_1, & A_{n2}-\lambda A_{12} \dots & A_n-2\lambda A_{n1}+\lambda^2 A_1 \end{matrix}\right| \quad \dots\dots\dots \quad \text{(B.)}$$

Now in the determinant thus transformed the quantity a will no longer occur in the n-system.

This is obvious with respect to the subordinate elements of that system. With respect to the principal element, we observe that the coefficient of a is

in A_1, equal to α_1,

in A_{n1}, equal to $\lambda\alpha_1$,

. . . .

in A_n, equal to $\lambda\times\lambda\alpha_1$ or $\lambda^2\alpha_1$,

whence the coefficient of a in $A_n-2\lambda_{n1}+\lambda^2A_1$ is equal to 0.

Thus a has been eliminated from the n-system, and as the process has not affected any elements but those which belong to the n-system, it will not affect the relations under which a enters into the other systems.

Consider then any other quantity b in the set a, b, c, then by hypothesis the coefficients of b in any line or column of elements

$$A_{i1}, A_{i2}, .. A_{in}, \text{ or } A_{1i}, A_{2i}, .. A_{ni}$$

may be represented by

$$\mu_i\beta_1, \mu_i\beta_2, .. \mu_i\beta_n,$$

$\beta_1, \beta_2, .. \beta_n$ being an arbitrary set of quantities which are the same for all lines or columns, while μ_i differs for different lines or columns, and vanishes for those in which b does not enter.

It is to be noted that as $A_{ij}=A_{ji}$, we have in general

$$\mu_i\beta_j=\mu_j\beta_i,$$

while as the principal elements of the determinant (I.) are positive, we have always $\mu_i\beta_i=$a positive quantity.

Now reverting to the derived determinant (B.), we see that its ith line or column of elements will be

$$A_{i1}, A_{i2}, \ldots A_{in}-\lambda A_{i1},$$

and its jth line or column

$$A_{j1}, A_{j2}, \ldots A_{jn}-\lambda A_{j1},$$

supposing i and j to be both less than n.

In these lines or columns the successive coefficients of b will therefore be

$$\mu_i\beta_1, \mu_i\beta_2, \ldots \mu_i\beta_n-\lambda\mu_i\beta_1,$$
$$\mu_j\beta_1, \mu_j\beta_2, \ldots \mu_j\beta_n-\lambda\mu_j\beta_1,$$

which stand to each other in the constant ratio $\mu_i : \mu_j$.

Now let $j=n$. The coefficients of b in the nth line or column of (B.) are obviously

$$\mu_n\beta_1-\lambda\mu_1\beta_1, \mu_n\beta_2-\lambda\mu_1\beta_2, \ldots \mu_n\beta_n-2\lambda\mu_1\beta_n+\lambda^2\mu_1\beta_1,$$

of which the last term may be reduced as follows,

$$\mu_n\beta_n-2\lambda\mu_1\beta_n+\lambda^2\mu_1\beta_1=\mu_n\beta_n-\lambda\mu_1\beta_n-\lambda\mu_n\beta_1+\lambda^2\mu_1\beta_1=(\mu_n-\lambda\mu_1)(\beta_n-\lambda\beta_1);$$

so that the series of coefficients of b becomes

$$(\mu_n-\lambda\mu_1)\beta_1, (\mu_n-\lambda\mu_1)\beta_2 \ldots (\mu_n-\lambda\mu_1)(\beta_n-\lambda\beta_1),$$

and they are now seen to stand to those of b in the i-line and column in the constant ratio $\mu_n-\lambda\mu_1 : \mu_i$.

We have, lastly, to prove that the new principal element $A_n-2\lambda A_{1n}+\lambda^2 A_1$ is positive.

Let N be the coefficient of any one of the quantities $a, b, c \ldots$ in the above element, L its coefficient in the principal element A_i, and M its coefficient in each of the subordinate elements common to the two systems of which the above are the respective principal elements, viz. in $A_{in}-\lambda A_{i1}$ and $A_{ni}-\lambda A_{1i}$. Then, by what has already been proved,

$$L : M :: M : N,$$

$$\therefore\ M^2 = LN;$$

but L is positive; therefore N is so, and the principal element in question consists wholly of positive terms.

The above demonstration shows that the elimination of a from the n-system produces a new determinant equivalent to the original one, and in which the characters noted in the original one still remain. Should a occur in any other system or systems of elements of the new determinant beside the 1-system, it can, by repetitions of the same process, be eliminated thence. Ultimately, then, it will only remain in the 1-system, and therefore only in the principal term of that system. Again, as it enters that term in the first degree, it follows that the developed determinant will involve only the first power of a. Hence, as a may represent any of the quantities $a, b, c, \ldots$, it is seen that no powers, but only products of these quantities, will appear in the developed determinant.

Let us represent the determinant, after the elimination of a from all the elements but A_1, in the form

$$\begin{vmatrix} A_1 & B_{12} & \ldots & B_{1n} \\ B_{21} & C_2 & \ldots & C_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ B_{n1} & C_{n2} & \ldots & C_n \end{vmatrix}.$$

Now let ah_1 represent that term in A_1 which involves a. Then the portion of the determinant which involves a will be

$$ah_1 \begin{vmatrix} C_2 & \ldots & C_{2n} \\ \cdot & \cdot & \cdot \\ C_{n2} & \ldots & C_n \end{vmatrix}.$$

And here it is to be observed that ah_1 is positive, while the new determinant to which it is attached as a coefficient possesses all the characters of the old one. This determinant we can therefore transform in the same way, so as to eliminate any other letter b from all but a single principal element, which we shall suppose to contain it in a term bh_2. That portion of the original determinant which involves ab will therefore assume the form

$$abh_1h_2 \begin{vmatrix} D_3 & \ldots & D_{3n} \\ \cdot & \cdot & \cdot \\ D_{n3} & \ldots & D_n \end{vmatrix}.$$

Ultimately, then, as the result of such processes continued, the portion of the original determinant which involves any particular combination of n letters selected from $a, b, c \ldots$ will consist of the product of a series of positive terms, each of which has appeared in some residual principal element. Every such combination being positive, it follows that the determinant itself consists solely of positive terms.

Proposition II.

If V *be any rational entire function of the n variables* $x_1, x_2, \ldots x_n$, *but involving no powers of those variables above the first, and if, further, all the different terms of* V *have positive signs, then the determinant*

$$\begin{vmatrix} V & V_1 & V_2 & \ldots & V_n \\ V_1 & V_1 & V_{12} & \ldots & V_{1n} \\ V_2 & V_{21} & V_2 & \ldots & V_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ V_n & V_{n1} & V_{n2} & \ldots & V_n \end{vmatrix}$$

in which V_i *denotes the sum of the terms in* V *which contain* x_i, *and* V_{ij} *the sum of the terms in* V *which contain* x_i, x_j, *will, when developed as a rational and entire function of* $x_1, x_2, \ldots x_n$, *consist wholly of terms with positive coefficients.*

From the definition it is plain that in general

$$V_{ij}=V_{ji}, \quad V_{ii}=V_i,$$

whence the above determinant is symmetrical.

Again, all its elements are homogeneous linear functions of the terms in V.

Again, if $\alpha, \alpha_1, \alpha_2, \ldots \alpha_n$ represent the successive coefficients of any one of the terms of V in any row or column of the determinant, and $\beta, \beta_1, \beta_2 \ldots \beta_n$ the successive corresponding coefficients of the same term in any other row or column of the determinant, the one series of coefficients shall be proportional to the other.

Let us compare the first column and the i-column headed with the element V_i. Selecting any term in V, suppose it to contain x_i, then in whatever element of the first column that term is found, it will be found in a corresponding element of the i-column, and in each case with unity for its coefficient, since all the elements are mere collections of terms from V. But when it is not found in a particular element of the first column, it will not be found in the corresponding element of the i-column. The entire series of coefficients in the one being then the same as that in the other, the common ratio of the corresponding terms is unity.

Suppose, secondly, that the proposed term is found in V and not in V_i; then in all the elements of the i-column its coefficient is 0, so that the series of coefficients in the i-column might be formed from those in the first column by multiplying the latter successively by 0. This again represents a common ratio.

The same reasoning may be applied to the comparison of any two columns of the determinant. Thus in comparing the i-column and the j-column:—terms of V which contain both x_i and x_j will be found in corresponding elements of both columns—terms which contain x_i but not x_j will be wholly absent from the j-column. Thus in all cases if $\alpha, \alpha_1, \alpha_2, \ldots \alpha_n$ represent the coefficients of a term of V in one column, its coefficients in any other column, taken in the same order, will be of the form $\lambda\alpha, \lambda\alpha_1, \lambda\alpha_2 \ldots \lambda\alpha_n$, the coefficient λ being either 1 or 0.

Lastly, the principal elements consist, as do all the elements, of positive terms.

Therefore by the last proposition the developed determinant will consist of products (without powers higher than the first) of different terms of V, and the coefficients of all such products will be positive.

Therefore the determinant will be expressible as a rational entire function of $x_1, x_2, \ldots x_n$ with positive coefficients.

The rapidity with which the complexity of the determinant increases as the number of variables increases is remarkable. For example, if $n=2$ and $V=axy+bx+cy+d$, the determinant is

$$\begin{vmatrix} axy+bx+cy+d & axy+bx & axy+cy \\ axy+bx & axy+bx & axy \\ axy+cy & axy & axy+cy \end{vmatrix};$$

and its calculated value will be found to be

$$abcx^2y^2+abdx^2y+acdxy^2+bcdxy,$$

consisting of four positive terms.

But if $n=3$ and

$$V=axyz+byz+cxz+dxy+ex+fy+gz+h,$$

the developed determinant will consist of fifty-eight positive terms. Its calculated value will be found in the Memoir on Testimonies and Judgments.

Proposition III.

The functions $V, V_1, V_2, \ldots V_n$ *being defined as above, if* V *be complete in form,* i. e. *if it consist of all the terms which according to definition it can contain, each with a positive coefficient, then the system of equations*

$$\frac{V_1}{V}=p_1, \quad \frac{V_2}{V}=p_2 \ldots \frac{V_n}{V}=p_n \quad \ldots \ldots \ldots \quad (1.)$$

will, when $p_1, p_2, \ldots p_n$ *are proper fractions, admit of one solution, and only one, in positive values of* $x_1, x_2, \ldots x_n$.

We shall show, first, that the above proposition is true when $n=1$, secondly, that on the hypothesis that it is true for $n-1$ variables, it is true for n variables. Hence it will follow that it is true generally.

Suppose $n=1$. Then $V=ax_1+b$, whence the system (1.) reduces to the single equation

$$\frac{ax_1}{ax_1+b}=p_1,$$

$$x_1=\frac{bp_1}{a(1-p_1)},$$

whence, since a and b are positive, and p is a positive fraction, x_1 is positive.

Thus the proposition is true when $n=1$.

Now, let $x_1=0$, and let $x_2, x_3 \ldots x_n$ be determined to satisfy the last $n-1$ equations of the system (1.). These $n-1$ equations will, when $x_1=0$, form a system of the same nature with respect to the $n-1$ variables $x_2, x_3 \ldots x_n$, as (1.) is with respect to the n variables $x_1, x_2, \ldots x_n$. This will be at once seen by taking any particular example. Hence by hypothesis $x_2, x_3 \ldots x_n$ will be determinable as positive quantities, and their values substituted in the first member of the first equation of (1.) will reduce it to the form

$$\frac{Ax_1}{Ax_1+B},$$

A and B being finite and positive. Hence the function $\frac{V_1}{V}$ will become 0.

Secondly, let any finite positive value be assigned to x_1. The last $n-1$ equations of the system (1.) will again form a system of the same nature as before, and will by hypothesis determine a set of finite positive values for $x_2, x_3, \ldots x_n$. These values again substituted in $\frac{V_1}{V}$, will give to it again the form

$$\frac{Ax_1}{Ax_1+B},$$

A and B being finite and positive. Hence as x_1 is finite and positive, $\frac{V_1}{V}$ will be a positive fraction.

Lastly, let x_1 be infinite. Still the last $n-1$ equations of the system (1.) will assume the same form as before. Determining thence $x_2, x_3 \ldots x_n$, and substituting in $\frac{V_1}{V}$, we have

$$\frac{V_1}{V}=\frac{Ax_1}{Ax_1+B},$$

in which A and B are finite and positive and x_1 is infinite. Hence $\frac{V_1}{V}=1$. It is seen then that as x_1 varies from 0 to infinity, $x_2, x_3, \ldots x_n$ being at the same time always by hypothesis determined to satisfy the last $n-1$ equations of the system (1.), the function $\frac{V_1}{V}$ will vary from 0 through positive fractional values to unity. It is manifest, too, that it varies continuously. If then it vary by continuous *increase*, it will once, and only once in its change, become equal to p_1, and the whole system of equations thus be satisfied together. I shall show that it does vary by continuous *increase*.

If it vary continuously from 0 to 1 and not by continuous increase, it must in the course of its variation assume at least once a maximum or minimum value. Let us then seek the condition of possibility of

$$\frac{V_1}{V} = \text{a maximum or minimum},$$

the variables being subject to the relations

$$\frac{V_2}{V}=p_2, \quad \frac{V_3}{V}=p_3 \ldots \frac{V_n}{V}=p_n.$$

Here, proceeding in the usual way by differentiation, we have

$$\frac{VdV_1-V_1dV}{V^2}=0, \quad \frac{VdV_2-V_2dV}{V^2}=0, \ldots \frac{VdV_n-V_ndV}{V^2}=0,$$

or

$$\frac{dV}{V}=\frac{dV_1}{V_1}=\frac{dV_2}{V_2}\dots=\frac{dV_n}{V_n}.$$

Let the common value of these fractions be represented by $-dt$, then we have a system of $n+1$ equations of which the first is

$$Vdt+dV=0,$$

while the n others are of the type

$$V_i dt+dV_i=0.$$

The complete system, therefore, on effecting the total differentiations, becomes

$$Vdt+\frac{dV}{dx_1}dx_1 \dots +\frac{dV}{dx_n}dx_n=0,$$

$$V_1 dt+\frac{dV_1}{dx_1}dx_1 \dots +\frac{dV_1}{dx_n}dx_n=0,$$

$$\dots\dots\dots\dots$$

$$V_n dt+\frac{dV_n}{dx_1}dx_1 \dots +\frac{dV_n}{dx_n}dx_n=0.$$

Now from the nature of the function V we have

$$\frac{dV}{dx_i}=\frac{V_i}{x_i},\quad \frac{dV_i}{dx_i}=\frac{V_i}{x_i},\quad \frac{dV_i}{dx_j}=\frac{V_{ij}}{x_j},$$

so that the above equations become

$$Vdt+V_1\frac{dx_1}{x_1}+V_2\frac{dx_2}{x_2}\dots+V_n\frac{dx_n}{x_n}=0,$$

$$V_1 dt+V_1\frac{dx_1}{x_1}+V_{12}\frac{dx_2}{x_2}\dots+V_{1n}\frac{dx_n}{x_n}=0,$$

$$V_2 dt+V_{21}\frac{dx_1}{x_1}+V_2\frac{dx_2}{x_2}\dots+V_{2n}\frac{dx_n}{x_n}=0,$$

$$\dots\dots\dots\dots$$

$$V_n dt+V_{n1}\frac{dx_1}{x_1}+V_{n2}\frac{dx_2}{x_2}\dots+V_n\frac{dx_n}{x_n}=0,$$

and the elimination of $dt, \frac{dx_1}{x_1}, \frac{dx_2}{x_2}\dots\frac{dx_n}{x_n}$ from these equations gives the sought condition of possibility of a maximum value of $\frac{V_1}{V}$, consistently with the satisfaction of the last $n-1$ equations of the system (1.).

This condition is therefore expressed by the equation

$$\begin{vmatrix} V & V_1 & V_2 \dots V_n \\ V_1 & V_1 & V_{12} \dots V_{1n} \\ V_2 & V_{21} & V_2 \dots V_{2n} \\ \dots & \dots & \dots\ \ \dots \\ V_n & V_{n1} & V_{n2} \dots V_n \end{vmatrix}=0.$$

But we have already seen (Prop. II.) that the first member of this equation is essentially positive for positive values of $x_1, x_2 .. x_n$. Hence the function $\frac{V_1}{V}$ varies by continuous increase, and on the hypothesis that the proposition to be proved is true for $n-1$ variables, it is true for n variables.

Therefore, connecting this with the former result, the proposition is true universally.

Proposition IV.

If V *be an incomplete function, some of the terms belonging to the complete form being wanting, but the terms present having their coefficients positive, it will in general be necessary not only that the quantities $p_1, p_2, \dots p_n$ should be positive fractions, but also that they should satisfy certain inequations of the form*

$$a_1p_1+a_2p_2\dots+a_np_n+b \overline{\gtrless} 0,$$

in order that the system

$$\frac{V_1}{V}=p_1, \quad \frac{V_2}{V}=p_2 \dots \frac{V_n}{V}=p_n \quad \dots \dots \dots \quad (1.)$$

may admit of a solution in positive values of $x_1, x_2 \dots x_n$.

For let $Ax_rx_sx_t\dots$ be any term in V, A being a constant which is positive in all the terms, but which may be different in the different terms. Suppose that in V_i there exist e terms like the above, and let the several ratios of these terms to V be denoted by $\lambda_1, \lambda_2 \dots \lambda_e$. Then the ith equation of the system (1.) will become

$$\lambda_1+\lambda_2\dots+\lambda_e=p_i, \quad \dots \dots \dots \quad (2.)$$

and the system (1.) will be converted into a system of n equations of this nature. We will suppose that there exist m distinct quantities of the nature of $\lambda_1, \lambda_2 \dots \lambda_e$ in the first members of this transformed system, and we will represent these by $\lambda_1, \lambda_2 \dots \lambda_m$. Then, if these constitute all the ratios of the separate terms of V to V itself, we have a new equation,

$$\lambda_1+\lambda_2\dots+\lambda_m=1. \quad \dots \dots \dots \quad (3.)$$

If they do not constitute all those separate ratios, we have, on the contrary, an inequation,

$$\lambda_1+\lambda_2\dots+\lambda_m \overline{\lessgtr} 1. \quad \dots \dots \dots \quad (4.)$$

Lastly, the condition that $\lambda_1, \lambda_2 \dots \lambda_m$ are positive fractions, gives the inequations

$$\lambda_1 \overline{\gtrless} 0, \ \lambda_2 \overline{\gtrless} 0 \dots \lambda_m \overline{\gtrless} 0. \quad \dots \dots \dots \quad (5.)$$

The conditions $\lambda_1 \overline{\lessgtr} 1$, &c. are already implied in (3.) or (4.).

The λ quantities are thus subject to a system of united *equations* and *inequations*, from which they must be eliminated by the method already explained.

The result of such elimination will be a final system of inequations connecting $p_1, p_2, \dots p_n$. Equations connecting these quantities can only present themselves when the equations of the original system are not independent, or, which really falls under the same hypothesis, when one or more of the variables $x_1, x_2 \dots x_n$ is wholly absent from that system. Thus if x_1 were a common factor of all the terms of V, it would divide out from the numerators and denominators of the system, which would thus become a system of n simultaneous equations connecting the $n-1$ variables $x_2, x_3 \dots x_n$. Considered with reference to these variables, therefore, the equations of the system would not be independent.

All resulting inequations will be capable of expression under the one general form,

$$a_1p_1+a_2p_2\dots+a_np_n+b \overline{\gtrless} 0,$$

the coefficients $a_1, a_2, \dots a_n$ and b being positive, negative, or vanishing, numerical constants. For any inequation which presents itself in the form

$$a'_1p_1+a'_2p_2\dots+a'_np_n+b \overline{\lessgtr} 1$$

may be transformed into

$$-a'_1p_1\dots-a'_np_n+1-b \overline{\gtrless} 0.$$

Again, the general inequation

$$a_1p_1+a_2p_2\dots+a_np_n+b \overline{\gtrless} 0$$

determines an inferior limit of p_1 when a_1 is positive, and a superior limit of p_1 when a_1 is negative.

For in the former case we have

$$p_1 \overline{\gtrless} -\left(\frac{a_2}{a_1}p_1\dots+\frac{a_n}{a_1}p_n+\frac{b}{a_1}\right),$$

the second member of which is an inferior limit of p_1; and it will be observed that the calculated value of this member may be positive, as there is no general restriction on the signs of $a_2, \dots a_n, b$.

In the latter case, changing a_1 into $-a'_1$, and observing that a_1 is positive, we have

$$p_1 \overline{\overline{<}} \frac{a_2}{a'_1}p_2+\frac{a_3}{a'_1}p_3\dots+\frac{a_n}{a'_1}p_n+\frac{b}{a'_1},$$

the second member of which is a superior limit of p_1.

Lastly, the final system of inequations is totally independent of the numerical value of the coefficients of V. The only restriction is that these coefficients are positive.

Proposition V.

Let V *be incomplete in form; then, provided that the equations*

$$\frac{V_1}{V}=p_1,\ \frac{V_2}{V}=p_2\dots\frac{V_n}{V}=p_n \quad \dots\dots\dots \quad (1.)$$

are independent with respect to the quantities $x_1, x_2, \dots x_n$, and that the inequations of condition deducible by the last proposition are satisfied, the equations will admit of one solution, and only one, in positive finite values of $x_1, x_2, \dots x_n$.

The proof of this proposition will, in its general character, resemble the proof of

Proposition III. It will be shown that when we assign to x any value *between* the limits 0 and infinity, the quantities $x_2, x_3 \dots x_n$ will admit of determination from the last $n-1$ equations of the system as positive finite quantities, and the function $\frac{V_1}{V}$ will receive a value falling within the limits assigned by Proposition IV. to the quantity p_1; that when x_1 is equal to 0 or infinity, $x_2, x_3 \dots x_n$ will admit of determination either as positive finite quantities, or as limits (0 and ∞) of such quantities; and that these values together will give to $\frac{V_1}{V}$ a value coinciding with the highest of the inferior, or lowest of the superior limits of p_1, as determined by Proposition IV.; that when x_1 varies from 0 to ∞, $x_2, x_3 \dots x_n$ being determined as above, $\frac{V_1}{V}$ will vary by continuous increase from the highest of the inferior to the lowest of the superior limits of p_1, and once in its variation become equal to p_1. Thus the truth of the proposition for n variables will flow necessarily from its assumed truth for $n-1$ variables. And on this ground it will be shown that it may ultimately be reduced to a direct dependence upon Proposition III.

In the system (1.) let x_1 receive any finite positive value, and let V by the substitution of this value become U; the last $n-1$ equations of (1.) will thus assume the form

$$\frac{U_2}{U}=p_2,\ \frac{U_3}{U}=p_3 \dots \frac{U_n}{U}=p_n, \quad \dots \dots \quad (2.)$$

in which the quantities $p_2, p_3 \dots p_n$ satisfy the conditions to which the direct application of Proposition IV. to this reduced system of equations would lead.

For what is important to notice in the change from V to U is, that any two terms in V which differ only in that one contains x_1 and the other does not, reduce to a single term in U. The effect of the change upon the primary system of equations and inequations formed in the application of Proposition IV. to the system (1.) is the following:—

1st. The equation between $\lambda_1, \lambda_2 \dots$ derived from the first equation of (1.) will be annulled.

2ndly. In the remaining equations connecting $\lambda_1, \lambda_2 \dots$ some *pairs* of those quantities may be replaced by single quantities, with corresponding changes in the inequations. Thus if $\lambda_1+\lambda_2$ be replaced by μ, the inequations

$$\lambda_1 \geqq 0,\ \lambda_2 \geqq 0$$

will be replaced by what they before *implied*, viz.

$$\mu \geqq 0.$$

But these changes do not affect the truth of the relations, or introduce any new relations. They cannot, therefore, lead to any new final conditions. The conditions connecting $p_2, p_3 \dots p_n$, in accordance with Proposition IV. in the system (2.), must have been already involved in the equations connecting $p_1, p_2 \dots p_n$ in the system (1.).

Hence by hypothesis the system (2.) gives one set of positive finite values of

$x_2, x_3 \dots x_n$ corresponding to the assumed positive finite value of x_1. And these values together make $\frac{V_1}{V}$ a positive proper fraction. We may notice that, representing $\frac{V_1}{V}$ under the form

$$\frac{Ax_1}{Ax_1+B},$$

it cannot be that either A or B is wanting so as to reduce $\frac{V_1}{V}$ to the value 0 or 1. For if A were wanting, V would not contain x_1 at all, as by hypothesis it does; and if B were wanting, V would contain x_1 in every term. Thus x_1 would divide out from the system (1.), which would thus become a system of n equations between $n-1$ variables, and would cease to be independent, as by hypothesis it is.

But when $x_1=0$, or $x_1=$ infinity, the form of V, considered as a function of $x_2, x_3 \dots x_n$, will not generally be the same as in the case last considered; and the conditions connecting $p_2, p_3, \dots p_n$ will no longer be such that we can affirm the possibility of deducing from the last $n-1$ equations of the system (1.), as transformed, positive finite values of $x_2, x_3, \dots x_n$.

The theory of this case depends upon a remarkable transformation.

The most general form of the inequations of condition connecting $p_1, p_2, \dots p_n$, as determined by Proposition IV., is

$$a_1p_1+a_2p_2\dots+a_np_n+b \overline{\gtrless} 0. \qquad (3.)$$

Hence, from the nature of the system (1.), it follows that the function

$$a_1V_1+a_2V_2\dots+a_nV_n+bV \qquad (4.)$$

must consist wholly of positive terms. Therefore V must consist of terms which would either appear in the development of the above function with positive signs, or not appear in it at all. Let $Ax_rx_sx_t\dots$ be any term of V. Then, as the coefficient of this term in (4.) would be

$$a_rA+a_sA+a_tA\dots+bA,$$

and as A is positive, we have

$$a_r+a_s+a_t\dots+b \overline{\gtrless} 0,$$

a general condition which determines not what terms have actually entered, but what could alone possibly have entered into the constitution of V.

From the system (1.) we have

$$\frac{a_1V_1+a_2V_2\dots+a_nV_n+bV}{V}=a_1p_1+a_2p_2\dots+a_np_n+b.$$

Hence if we write

$$a_1V_1+a_2V_2\dots+a_nV_n+bV=H,$$

we have

$$\frac{H}{V}=a_1p_1+a_2p_2\dots+a_np_n+b, \qquad (5.)$$

an equation by which we may replace any one of the equations of the system (1.), and which has the peculiarity that for every term $Ax_r x_s x_t \dots$ which appears in the numerator H the particular condition

$$a_r + a_s + a_t \dots + b > 0$$

is satisfied.

Let K be the aggregate of those terms in V for which the remaining particular condition

$$a_r + a_s + a_t \dots + b = 0$$

is satisfied. Then $V = H + K$. If we now substitute (5.) in place of the first equation of the system (1.) and then write $H+K$ for V, $H_1 + K_1$ for V_1, &c., the system becomes converted into the following one, viz.

$$\frac{H}{H+K} = a_1 p_1 + a_2 p_2 \dots + a_n p_n + b, \quad \frac{H_2+K_2}{H+K} = p_2, \quad \frac{H_3+K_3}{H+K} = p_3, \dots \frac{H_n+K^n}{H+K} = p_n. \quad . \quad . \quad (6.)$$

Now let us transform the above equations by assuming

$$x_2 = x_1^{\frac{a_2}{a_1}} y_2, \quad x_3 = x_1^{\frac{a_3}{a_1}} y_3 \dots x_n = x_1^{\frac{a_n}{a_1}} y_n.$$

The general type of these equations is

$$x_i = x_1^{\frac{a_i}{a_1}} y_i,$$

and it includes the particular case of $i=1$, provided that we suppose, as we shall do, $y_1 = 1$.

Then representing, as before, any term of V by $Ax_r x_s x_t \dots$, we have

$$Ax_r x_s x_t \dots = Ax_1^{\frac{a_r + a_s + a_t \dots}{a_1}} y_r y_s y_t \dots$$

Let this substitution be made in the different terms both of the numerators and denominators of the fractions which form the first members of the above system, and then let each numerator and denominator be multiplied by $x_1^{\frac{b}{a_1}}$. The result will be the same as if for each term $Ax_r x_s x_t \dots$ in numerator or denominator we substituted the term

$$Ax_1^{\frac{a_r + a_s + a_t \dots + b}{a_1}} y_r y_s y_t \dots$$

In considering the effect of this transformation we will first suppose a_1 positive, and afterwards suppose it negative.

Case 1; the coefficient a_1 positive. Here, since for all the terms in H and in $H_2, H_3 \dots H_n$ we have

$$\frac{a_r + a + a_t \dots + b}{a_1} > 0,$$

all such terms in the transformed equations will be affected with positive powers of x_1.

And since for all terms in K, $K_2, \dots K_n$ we have

$$\frac{a_r + a_s + a_t \dots + b}{a_1} = 0,$$

all such terms in the transformed equations will be free from x_1.

Now let $$a_1p_1+a_2p_2\ldots+a_np_n+b=0.$$

This, as a_1 is positive, is to suppose that p_1 coincides with one of its own inferior limits. We must suppose this to be the highest of those limits, as otherwise some of the other limiting conditions would be violated. Now, since all the terms in H are affected with positive powers of x_1, while those in K do not contain x_1, the first equation of the system (6.) will be satisfied by $x_1=0$, provided that the remaining $n-1$ equations give finite positive values for $y_2\ldots y_n$. But the vanishing of x_1 reduces these equations to the form

$$\frac{K_2}{K}=p_2,\ \frac{K_3}{K}=p_3,\ldots\ \frac{K_n}{K}=p_n. \quad \ldots \ldots \quad (7.)$$

It is therefore necessary to show that p_2, $p_3\ldots p_n$ in this system are actually subject to the conditions to which the application of the method of Proposition IV. to the system itself would lead.

The n quantities p_1, $p_2\ldots p_n$ are by hypothesis subject to the conditions furnished by the application of the method of Proposition IV. to the original system (1.). In applying this method each of the original equations yields an equation of the form

$$\lambda_1+\lambda_2\ldots+\lambda_e=p_i; \quad \ldots \ldots \quad (8.)$$

and to the equations thus formed are added the inequations

$$\lambda_1+\lambda_2\ldots+\lambda_m\leqq 1,$$
$$\lambda_1\geqq 0,\ \lambda_2\geqq 0,\ \ldots\lambda_m\geqq 0;$$

λ_1, $\lambda_2\ldots\lambda_m$ having reference to the whole system of original equations.

Now the satisfaction of the equation

$$\frac{H}{H+K}=0$$

by the value $x_1=0$, involves the vanishing of all those quantities of the system λ_1, $\lambda_2\ldots\lambda_m$, which are derived from terms in V that are also found in H. Hence the λ quantities that do not vanish are those derived from terms in V which appear in K.

Again, the condition

$$a_1p_1+a_2p_2\ldots+a_np_n+b=0$$

shows that the system of equations of which (8.) is the type are not independent. They must, under the particular circumstances of the case, be such that the above equation shall be derivable from them. Hence one of these equations may be rejected. If we reject the first, viz. the one which contains p_1, and then reduce the others by making the λ quantities which are not derived from K to vanish, the system typified by (8.) evidently reduces to the system which we should have to employ if we applied the method of Proposition IV. directly to the system of $n-1$ equations (7.). Hence the quantities p_2, p_3, $\ldots p_n$ satisfy the final conditions to which that application would lead, and therefore by hypothesis the equations (7.) admit of solution by a single system of finite positive values of y_2, y_3, $\ldots y_n$.

Now in general

$$x_i=x_1^{\frac{a_i}{a_1}}y_i.$$

Hence since $x_1=0$ and y_i is finite and positive for all values of i from 2 to n, we see that x_i will be 0 for all values of i for which a_i is positive, finite and positive for all values of i for which a_i is 0, and infinite for all values of i for which a_i is negative.

Case 2; the coefficient a_i negative. Here the inequation of condition (3.) must be supposed to determine the lowest of the superior limits of p_1, and therefore when p_1 coincides with that limit we have

$$a_1p_1+a_2p_2\ldots+a_np_n+b=0.$$

The transformations remaining formally the same as before, the following results will present themselves.

The terms in H and in $H_2, H_3\ldots H_n$ will be affected with negative instead of positive powers of x_1. Hence the same determination of $y_2, y_3\ldots y_n$ from the last $n-1$ equations of (6.), which before followed from the assumption $x_1=0$, will now follow from the assumption $x_1=\infty$, which at the same time satisfies the first equation of (6.).

The equation

$$x_i=x_1^{\frac{a_i}{a_1}}y_i$$

shows, since a_1 is here negative and x_1 infinite, that x_i will be infinite for those values of i for which a_i is negative, finite for those values of i for which a_i is 0, nothing for those values of i for which a_i is positive.

In all these cases the values 0 and ∞ appear as limits of finite positive values. This results from the connexion of the second member of the first equation of the system (6.) with the condition (3.).

Lastly, as the incompleteness of form of V only causes certain terms of the developed determinant of Proposition II. to vanish, but leaves the signs of the terms which remain positive, it follows that as x_1 varies from 0 to infinity, $x_2, x_3, \ldots x_n$ being always determined by the last $n-1$ equations of (1.), the function $\frac{V_1}{V}$ will vary by continuous increase between the limits above investigated, viz. from the highest inferior to the lowest superior limit of p_1. Once, therefore, in its progress it becomes equal to p_1, and all the equations are satisfied together.

The above reasoning establishes rigorously that if the proposition is true for $n-1$ variables, it is true for n variables. It remains then to consider the limiting case of $n=1$.

Here, however, only the complete form of V, viz. $V=ax+b$, leads to a definite value of x, and this, as has been seen, is finite and positive. If we give to V the particular form ax, the equation $\frac{V_x}{V}=p$ becomes

$$\frac{ax}{ax}=p, \text{ or } p=1,$$

which determines p, but leaves x indefinite. If we employ the other particular form $V=b$, we obtain no equation whatever, and here again x is indefinite. But as the reducing transformations are all definite, the above indefinite forms cannot present themselves in the last stage of the problem when the original equations are independent and admit of definite solution.

The proposition is therefore established.

Application.

The general system of algebraic equations upon which the solution of questions in the theory of probabilities depends, is a particular case of that discussed in Proposition V. Its peculiarity is, that all the coefficients which appear in the function V are equal to unity.

The conditions of possible experience, as determined by the method, agree with the conditions shown in Proposition IV. to be necessary, and in Proposition V. to be sufficient, in order that $x_1, x_2 \ldots x_n$ may be determinable as positive finite quantities. For in both cases the quantities λ_1, λ_2, &c. correspond to the different terms in V, and in both cases the equations among those quantities depend simply on the forms of the functions $V_1, V_2 \ldots V_n$, and therefore ultimately on the form of V, irrespectively of the values of the positive coefficients of V. In both cases the systems of inequations are the same.

It follows, therefore, that precisely when the data represent a possible experience, the probabilities of the ideal events from which in the process of solution the problem is mentally constructed admit of determination as positive proper fractions.

Again, as the process for determining the *à priori* limits of the probability sought rests ultimately upon the assumption that the ratio of any term or partial aggregate of terms in V to V itself is a positive fraction, and as this assumption is satisfied when $x_1, x_2 \ldots x_n$ are positive quantities, it follows that the calculated value of the probability sought will always lie within the limits which it would have had if determined by observation from the same experience as the data.

But though the test last mentioned is one which must necessarily be satisfied by a true method, it is of infinitely less theoretical importance than that from which it is derived, viz. the test which consists in the absolute connexion between possibility in the data and formal consistency in the method.

As the conclusions of Propositions IV. and V. depend upon the form of the function V and the fact that its coefficients are positive, it follows that if in the application of the method to questions of probability we substituted any other positive values for unity in the coefficients of V, leaving the rest of the process as before, we should still be able to determine $x_1, x_2, \ldots x_n$ as positive quantities, or as limits of such, and the altered value of the probability sought would still be consistent with the experience from which the data are supposed to be derived. It would, however, properly speaking, be a value of interpolation, not a probability.

I will close with a few remarks upon the general nature of the method, and of the solutions to which it leads.

1st. The probability determined is not precisely of the same nature as the probabilities given.

For the data are supposed to be derived from experience; and therefore, on the supposition that the future will resemble the past, the events of which the probabilities are given will in the long run recur with a frequency proportioned to their probability.

But the probability determined is always an intellectual rather than a material probability. We cannot affirm that in the long run an event will occur with a frequency proportional to its calculated probability; but we can affirm that it is more likely to occur with this than with any other precise degree of frequency; that if it do not occur with this degree of frequency, the data are in some measure *one-sided.*

At the same time the limits of possible deviation are determined.

2ndly. General solutions obtained by the method do sometimes, but not always, admit* of being verified by other methods. I believe that this is solely because it is not often possible to solve the problem by other methods without introducing hypotheses which are of the nature of additional data, and, in effect, limit the problem. Every general solution, however, admits of a number of particular verifications by necessary consequence from the theorems established in this paper.

3rdly. It has been seen that a calculated probability is not necessarily a definite numerical value. It may be of the form $A+cC$, in which c is an arbitrary positive fraction. Here it is implied that the probability admits of any value between A and $A+C$. If, further, $A=0$ and $C=1$, it is implied that the probability may have any value between 0 and 1,—is therefore quite indefinite. This would really arise if we applied the method to a case in which the event of which the probability is sought had absolutely no connexion with those of which the probabilities are given.

Hence in the present theory the numerical expression for the probability of an event about which we are totally ignorant is not $\frac{1}{2}$, but c†. Hence, also, when all the probabilities given are measured by $\frac{1}{2}$, it is not to be concluded (upon the ground of *e nihilo nihil*) that the probability sought will also be $\frac{1}{2}$.

4thly. While extending the real power of the theory of probabilities, the method tends in some cases to diminish the apparent value of its results. For all problems in which the data admit of logical expression can be solved by it; but the resulting solutions, founded upon the bare data, may be of an indeterminate character, in place of the determinate results to which ordinary methods, aided by hypotheses not really involved in the data, lead.

This is the case with the problem of the combination of different grounds of belief or opinion. The general solution is indefinite. In two limiting cases, however, it assumes a definite form; one of these, which agrees with the formula generally accepted, representing the extreme cumulative force of testimonies, the other the mean weight of

* Professor DONKIN has verified a general solution (Laws of Thought, p. 362).

† See on this subject a paper by Bishop TERROT, Edinburgh Transactions, vol. xxi. part 3.

judgments. Both these, however, occur as limiting cases, and they can only be applied with confidence under extreme circumstances, such as probably never occur in human affairs. (*Edinburgh Memoir*, pp. 630–645.)

5thly. I have, in effect, remarked that there is reason to suppose that all questions in the theory of probabilities can ultimately be reduced to questions in which the immediate subjects of probability are *logical*, i. e. involve no other essential relations than those of genus and species, whole and part. This is a question of theoretical rather than of practical interest. For instance, whether the formula of the arithmetical mean, which is the basis of the theory of astronomical observations, is self-evident, or whether it rests upon an ultimate logical basis, or whether, as I am inclined to believe, it may lawfully be regarded in either of these distinct but not conflicting lights, the superstructure remains the same.

XIII. *On the Calculus of Symbols.*—Second Memoir. *By* W. H. L. Russell, *Esq.*, *A.B.* *Communicated by* Arthur Cayley, *Esq.*, *F.R.S.*

Made up of two Memoirs: one received January 7,—Read January 30, 1862; the other received June 18,—Read June 19, 1862.

This Memoir is a continuation of one on the Calculus of Symbols which I had the honour to lay before the Society in December 1860, and which has since been published in the Philosophical Transactions. I commence this paper with some extensions of the method given in the former memoir for resolving functions of non-commutative symbols into binomial factors. I then explain a method, analogous to the process for extracting the square root in ordinary algebra, for resolving such functions into equal factors. I next investigate a process for finding the highest common internal divisor of two functions of non-commutative symbols, or, in other words, of finding if two linear differential equations admit of a common solution. After this, I give a rule for multiplying linear factors of non-commutative symbols, analogous to the ordinary algebraical rule for linear algebraical factors. I then resume the consideration of the binomial theorem explained in the former memoir. Two new forms of this binomial theorem are here given; and the method by which these forms are proved identical will, I hope, be considered an interesting portion of symbolical algebra, and as exhibiting in a remarkable manner its peculiar nature.

In the next place, I proceed to calculate the general values of the coefficients which occur in the form of the binomial theorem given in the first memoir; I then obtain an expression for the symbolical coefficient of the general term of the multinomial theorem as previously explained; and also a theorem for the multiplication of symbolical factors emanating from each other after a given law; lastly, I investigate a binomial theorem reciprocal to the binomial theorem already considered.

In the former memoir I explained a process by which the symbolical function

$$\rho^n\phi^n(\pi)+\rho^{n-1}\phi_{n-1}(\pi)+\rho^{n-2}\phi_{n-2}(\pi)+\ldots+\phi_0(\pi)$$

could be resolved in all possible cases into factors of the form

$$(\rho\psi_1^{(n)}\pi+\psi_0^{(n)}\pi)(\rho\psi_1^{(n-1)}\pi+\psi_0^{(n-1)}\pi)\ldots(\rho\psi_1\pi+\psi_0\pi).$$

I shall now give a method by which the same symbolical function may be resolved into factors of the form

$$(\rho^\alpha\psi_\alpha^{(n)}\pi+\psi_0^{(n)}\pi)(\rho^\beta\psi_\beta^{(n-1)}\pi+\psi_0^{(n-1)}\pi)\ldots(\rho^\mu\psi_\mu(\pi)+\psi_0\pi).$$

By pursuing methods similar to those employed in the preceding paper, we find the following equations as the condition that $\rho^2\psi_2(\pi)+\psi_0(\pi)$ may be an internal factor of the given symbolical function,

$$\phi_0(\pi)-\frac{\psi_0(\pi)}{\psi_2(\pi-2)}\phi_2(\pi-2)+\frac{\psi_0\pi\psi_0(\pi-2)}{\psi_2(\pi-2)\psi_2(\pi-4)}\phi_4(\pi-4)-\ldots=0,$$

$$\phi_1\pi-\frac{\psi_0(\pi)}{\psi_1(\pi-2)}\phi_3(\pi+2)+\frac{\psi_0\pi\psi_0(\pi-2)}{\psi_2(\pi-2)\psi_2(\pi-4)}\phi_5(\pi-4)-\ldots=0.$$

Again, the conditions that $\varrho^2\psi_2(\pi)+\psi_0(\pi)$ may be an external factor of the same symbolical function will be given by the equations

$$\phi_0(\pi)-\frac{\psi_0\pi}{\psi_2\pi}\phi_2(\pi)+\frac{\psi_0\pi\psi_0(\pi+2)}{\psi_2\pi\psi_2(\pi+2)}\phi_4(\pi)-\ldots=0,$$

$$\phi_1(\pi)-\frac{\psi_0(\pi+1)}{\psi_2(\pi+2)}\phi_3(\pi)+\frac{\psi_0(\pi+1)\psi_0(\pi+3)}{\psi_2(\pi+1)\psi_2(\pi+3)}\phi_5(\pi)-\ldots=0.$$

We may also find in like manner the conditions that $\varrho^3\psi_3(\pi)+\psi_0(\pi)$, $\varrho^4\psi_4(\pi)+\psi_0(\pi)$ may be internal or external factors of the given symbolical function: in every case the number of equations of condition will be equal to the degree of (ϱ) in the given factor.

By applying the method of divisors, as explained in the former paper, we may ascertain the forms of $\psi_2(\pi)$, $\psi_0(\pi)$ in order that $\varrho^2\psi_2(\pi)+\psi_0(\pi)$ may be an internal factor of the given symbolical function. In the present case, however, $\psi_2(\pi)$ must be a divisor both of $\phi_n(\pi)$ and $\phi_{n-1}(\pi)$, $\psi_0(\pi)$ a divisor both of $\phi_1(\pi)$ and $\phi_0(\pi)$,—a consideration which will greatly simplify the process. We proceed, in like manner, should there be no internal factors of the form $\varrho^2\psi_2\pi+\psi_0\pi$, $\varrho^2\psi_2(\pi)+\psi_0(\pi)$, to ascertain if there are any of the form $\varrho^3\psi_2(\pi)+\psi_0(\pi)$, &c.; and continuing the investigation as before, we are able, in all possible cases, to resolve the given symbolical function

$$\varrho^n\phi_n(\pi)+\varrho^{n-1}\phi_{n-1}(\pi)+\ldots+\phi_0(\pi)$$

into binomial factors.

Hence, in any linear differential equation,

$$X_r\frac{d^ru}{dx^r}+X_{r-1}\frac{d^{r-1}u}{dx^{r-1}}+X_{r-2}\frac{d^{r-2}u}{dx^{r-2}}+\ldots=X,$$

if we put

$$\varrho=x,\ \pi=x\frac{d}{dx},$$

we shall be able in all possible cases to reduce it to a series of equations:—

$$\varrho^\alpha\psi_\alpha^{(n-1)}(\pi)u_{n-1}+\psi_0^{(n-1)}(\pi)u_{n-1}=X,$$

$$\varrho^\beta\psi_\beta^{(n-2)}(\pi)u_{n-2}+\psi_0^{(n-2)}(\pi)u_{n-2}=u_{n-1},$$

$$\&c.\qquad=\&c.,$$

$$\varrho^\mu\psi_\mu\quad(\pi)u\quad+\psi_0\quad(\pi)u\quad=u_1,$$

a series of binomial equations, each of which may be treated by the methods due to Professor Boole. I shall now explain a process analogous to that denominated 'evolution' in ordinary algebra. To resolve the symbolical function

$$\varrho^{2n}\phi_{2n}(\pi)+\varrho^{2n-1}\phi_{2n-1}(\pi)+\ldots+\phi_0(\pi)$$

into two equal factors.

For this purpose, let us assume

$$\varrho^{2n}\varphi_{2n}(\pi)+\varrho^{2n-1}\varphi_{2n-1}(\pi)+\varrho^{2n-2}\varphi_{2n-2}(\pi)+\ldots+\varphi_0(\pi)$$
$$=(\varrho^n\theta_n(\pi)+\varrho^{n-1}\theta_{n-1}(\pi)+\varrho^{n-2}\theta_{n-2}(\pi)+\ldots+\theta_0(\pi))^2.$$

From this we find, equating the coefficients of ϱ^{2n},

$$\theta_n(\pi)\theta_n(\pi+n)=\varphi_{2n}(\pi),$$

from whence

$$\theta_n(\pi+n)\theta_n(\pi+2n)=\varphi_{2n}(\pi+n);$$

or

$$\theta_n(\pi+2n)=\frac{\varphi_{2n}(\pi+n)}{\varphi_{2n}(\pi)}\theta_n(\pi),$$

whence

$$\theta_n(\pi)=\frac{\varphi_{2n}(\pi-n)\ \varphi_{2n}(\pi-3n)\varphi_{2n}(\pi-5n)\ldots}{\varphi_{2n}(\pi-2n)\varphi_{2n}(\pi-4n)\varphi_{2n}(\pi-6n)\ldots}.$$

Again, equating the coefficients of ϱ^{2n-1}, we shall have

$$\theta_{n-1}(\pi+n)\theta_n\pi+\theta_n(\pi+n-1)\theta_{n-1}(\pi)=\varphi_{2n-1}(\pi)\ldots; \quad \ldots\ldots \quad \text{(A.)}$$

$\theta_n(\pi)$, $\varphi_{2n-1}(\pi)$ are known rational functions of (π), wherefore assume

$$\theta_n(\pi)=a+b\pi+c\pi^2+\ldots+k\pi^n,$$
$$\varphi_{2n-1}(\pi)=\alpha+\beta\pi+\gamma\pi^2+\ldots+\varkappa\pi^r,$$

where a, b, c, ... α, β, γ, ... are known, and

$$\theta_{n-1}(\pi)=\mathrm{A}+\mathrm{B}\pi+\mathrm{C}\pi^2+\ldots+\mathrm{K}\pi^{r-s},$$

where A, B, C are known. They may be easily found by equating the coefficients of (π) in equation (A.) and $\theta_{n-1}(\pi)$ thus determined.

If we equate the coefficients of ϱ^{2n-2}, we shall have

$$\theta_{n-2}(\pi+n)\theta_n\pi+\theta_{n-2}(\pi)\theta_n(\pi+n-2)+\theta_{n-1}\pi\theta_{n-1}(\pi+n-1)=\varphi_{2n-2}(\pi),$$

from which $\theta_{n-2}(\pi)$ may be determined in like manner. By this method we may in all possible cases reduce a proposed differential equation,

$$\mathrm{X}_{2r}\frac{d^{2r}u}{dx^{2r}}+\mathrm{X}_{2r-1}\frac{d^{2r-1}u}{dx^{2r-1}}+\ldots+\mathrm{X}_1\frac{du}{dx}+\mathrm{X}_0u=\mathrm{X},$$

to the form

$$\left\{\Xi_r\frac{d^r}{dx^r}+\Xi_{r-1}\frac{d^{r-1}}{dx^{r-1}}+\ldots+\Xi_0\right\}^2u=\mathrm{X},$$

when Ξ_r, Ξ_{r-1}, &c. are rational functions of (x).

The methods for finding the highest common divisor in ordinary algebra apply equally to the present Calculus, as will be seen by the following examples:—

To find the highest common internal divisor of the symbolical functions $\varrho^2+\varrho-\pi^2$ and $\varrho^2+\varrho(\pi^2+\pi+1)-\pi^3$.

Take $\rho^2+\rho-\pi^2$ for a divisor, and proceed as follows:—

$$\begin{array}{l} \rho^2+\rho-\pi^2)\rho^2+\rho(\pi^2+\pi+1)-\pi^3 \\ \qquad\qquad\underline{\rho^2+\rho \qquad\qquad -\pi^2} \\ \qquad\qquad\quad \rho(\pi^2+\pi)-\pi^3+\pi^2 \end{array}$$

It is easily seen that the remainder may be written $\pi(\pi-1)(\rho-\pi)$.

Hence, taking $\rho-\pi$ as divisor,

$$\begin{array}{l} \rho-\pi)\rho^2+\rho-\pi^2(\rho+\pi \\ \qquad\quad\underline{\rho^2-\rho\pi} \\ \qquad\qquad \rho(\pi+1)-\pi^2 \\ \qquad\qquad \underline{\rho(\pi+1)-\pi^2} \end{array}$$

Hence $\rho-\pi$ is the highest common divisor of $\rho^2+\rho-\pi^2$ and $\rho^2+\rho(\pi^2+\pi+1)-\pi^3$.

Again, to find the highest common internal divisor of $\rho^2\pi(\pi+1)+\rho(\pi^3+\pi^2)+\pi^2(\pi-1)$ and $\rho^3\pi-2\rho^2\pi+\rho(\pi^2+\pi)+\pi^4$.

The first of these quantities is equivalent to $(\pi-1)(\rho^2\pi+\rho(\pi^2+\pi)+\pi^2)$.

We take $\rho^2\pi+\rho(\pi^2+\pi)+\pi^2$ for divisor, and proceed as follows:—

$$\begin{array}{l} \rho^2\pi+\rho(\pi^2+\pi)+\pi^2)\rho^3\pi-2\rho^2\pi+\rho(\pi^2+\pi)+\pi^4(\rho-(\pi+1) \\ \qquad\qquad\qquad\quad \underline{\rho^3\pi+\rho^2(\pi^2+\pi)+\rho\pi^2} \\ \qquad\qquad\qquad\quad -\rho^2(\pi^2+3\pi)+\rho\pi+\pi^4 \\ \qquad\qquad\qquad\quad \underline{-\rho^2(\pi^2+3\pi)-\rho(\pi^3+3\pi^2+2\pi)-\pi^2(\pi+1)} \\ \qquad\qquad\qquad\qquad\quad \rho(\pi^3+3\pi^2+3\pi)+\pi^4+\pi^3+\pi^2 \end{array}$$

The remainder is equivalent to $(\pi^2+\pi+1)(\rho\pi+\pi^2)$.

Hence, taking $\rho\pi+\pi^2$ for a new divisor,

$$\begin{array}{l} \rho\pi+\pi^2)\rho^2\pi+\rho(\pi^2+\pi)+\pi^2(\rho+1 \\ \qquad\quad \underline{\rho^2\pi+\rho\pi^2} \\ \qquad\qquad\quad \rho\pi+\pi^2 \\ \qquad\qquad\quad \underline{\rho\pi+\pi^2} \end{array}$$

Hence $\rho\pi+\pi^2$ is the highest common internal divisor of

$$\rho^2\pi(\pi+1)+\rho(\pi^3+\pi^2)+\pi^2(\pi-1)$$

and

$$\rho^3\pi-2\rho^2\pi+\rho(\pi^2+\pi)+\pi^4.$$

It is evident, as was mentioned in the introduction to this memoir, that this process is equivalent to finding the conditions that two linear differential equations may have a common solution.

I shall next proceed to find the general term of the continued product

$$(\rho+\theta_1\pi)(\rho+\theta_2\pi)(\rho+\theta_3\pi)\ldots(\rho+\theta_n\pi).$$

This product, when developed, will be of the form

$$\varrho^n\phi_n(\pi)+\varrho^{n-1}\phi_{n-1}(\pi)+\ldots+\varrho^m\phi_m(\pi)+\ldots+\varrho\phi_1\pi+\phi_0(\pi).$$

Then $\phi_m(\pi)$ is given by the following rule:—

Write down the following symbolical product:

$$\theta_1(\pi+m)\theta_2(\pi+m)\theta_3(\pi+m)\ldots\ldots\theta_{n-m}(\pi+m);$$

take every possible combination of the quantities 1, 2, 3,...n taken $(n-m)$ at a time, and substitute them as the weight* of θ in this continued product, diminishing (m) in each factor by the increment of the weight of the factor; add all the results together, and we obtain the value of $\phi_m(\pi)$. The truth of this rule will be manifest to every one who will consider the following result obtained by actual multiplication:—

$$(\varrho+\theta_1\pi)(\varrho+\theta_2\pi)(\varrho+\theta_3\pi)(\varrho+\theta_4\pi)(\varrho+\theta_5\pi)=\varrho^5$$
$$+\varrho^4\{\theta_1(\pi+4)+\theta_2(\pi+3)+\theta_3(\pi+2)+\theta_4(\pi+1)+\theta_5(\pi)\}$$
$$+\varrho^3\{\theta_1(\pi+3)\theta_2(\pi+3)+\theta_1(\pi+3)\theta_3(\pi+2)+\theta_2(\pi+2)\theta_3(\pi+2)+\theta_1(\pi+3)\theta_4(\pi+1)$$
$$+\theta_2(\pi+2)\theta_4(\pi+1)+\theta_3(\pi+1)\theta_4(\pi+1)+\theta_1(\pi+3)\theta_5\pi+\theta_2(\pi+2)\theta_5(\pi)+\theta_3(\pi+1)\theta_5\pi+\theta_4\pi\theta_5\pi\}$$
$$+\varrho^2\{\theta_1(\pi+2)\theta_2(\pi+2)\theta_3(\pi+2)+\theta_1(\pi+2)\theta_2(\pi+2)\theta_4(\pi+1)+\theta_1(\pi+2)\theta_3(\pi+1)\theta_4(\pi+1)$$
$$+\theta_2(\pi+1)\theta_3(\pi+1)\theta_4(\pi+1)+\theta_1(\pi+2)\theta_2(\pi+2)\theta_5\pi+\theta_1(\pi+2)\theta_3(\pi+1)\theta_5\pi$$
$$+\theta_2(\pi+1)\theta_3(\pi+1)\theta_5\pi+\theta_1(\pi+2)\theta_4\pi\theta_5\pi+\theta_2(\pi+1)\theta_4\pi\theta_5\pi+\theta_3\pi\theta_4\pi\theta_5\pi\}$$
$$+\varrho\{\theta_1(\pi+1)\theta_2(\pi+1)\theta_3(\pi+1)\theta_4(\pi+1)+\theta_1(\pi+1)\theta_2(\pi+1)\theta_3(\pi+1)\theta_5\pi$$
$$+\theta_1(\pi+1)\theta_2(\pi+1)\theta_4\pi\theta_5\pi+\theta_1(\pi+1)\theta_3\pi\theta_4\pi\theta_5\pi+\theta_2\pi\theta_3\pi\theta_4\pi\theta_5\pi\}$$
$$+\theta_1\pi\theta_2\pi\theta_3\pi\theta_4\pi\theta_5\pi.$$

I now come to the investigation of the two new forms of the binomial theorem as explained in the former memoir.

It is evident, in the first place, that in multiplying any binomial $(\varrho^2+\varrho\theta(\pi))^n$ by $\varrho^2+\varrho\theta(\pi)$, the result in this case will be the same whether we employ internal or external multiplication.

Let

$$(\varrho^2+\varrho\theta(\pi))^n=\varrho^{2n}+\varrho^{2n-1}\phi_{2n-1}(\pi)+\varrho^{2n-2}\phi_{2n-2}\pi+\ldots,$$

where $\phi_{2n-1}(\pi)$, $\phi_{2n-2}(\pi)$, $\phi_{2n-3}(\pi)$ are unknown functions of (π) which we seek to determine.

Then multiplying externally and internally by $(\varrho^2+\varrho\theta(\pi))$

$$\begin{aligned}(\varrho^2+\varrho\theta(\pi))^{n+1}&=\varrho^{2n+2}+\varrho^{2n+1}\phi_{2n-1}(\pi)\quad+\varrho^{2n}\phi_{2n-2}(\pi)\\&\qquad+\varrho^{2n+1}\theta(\pi+2n)\quad+\varrho^{2n}\theta(\pi+2n-1)\phi_{2n-1}(\pi)+\ldots\\&=\varrho^{2n+2}+\varrho^{2n+1}\phi_{2n-1}(\pi+2)+\varrho^{2n}\phi_{2n-2}(\pi+2)+\ldots\\&\qquad+\varrho^{2n+1}\theta(\pi)\qquad+\varrho^{2n}\theta\pi\phi_{2n-1}(\pi+1)+\ldots\end{aligned}$$

* The use I have here made of the term 'weight' will be familiar to every one who is conversant with the modern Higher Algebra.

From whence, by equating the coefficients of (ϱ),

$$\varphi_{2n-1}(\pi)+\theta(\pi+2n)=\varphi_{2n-1}(\pi+2)+\theta(\pi),$$

$$\therefore \quad \left(\varepsilon^{2\frac{d}{d\pi}}-1\right)\varphi_{2n-1}(\pi)=\left(\varepsilon^{2n\frac{d}{d\pi}}-1\right)\theta(\pi),$$

whence

$$\varphi_{2n-1}(\pi)=\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi).$$

Again,

$$\varphi_{2n-2}(\pi)+\theta(\pi+2n-1)\varphi_{2n-1}(\pi)=\varphi_{2n-2}(\pi+2)+\varphi_{2n-1}(\pi+1)\theta\pi,$$

$$\therefore \quad \varphi_{2n-2}(\pi)=\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left\{\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\,\theta(\pi)\right\}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta\pi\right\}-\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left\{\frac{\varepsilon^{(2n+1)\frac{d}{d\pi}}-\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta\pi\right\}\theta\pi.$$

We shall now investigate another form of this expansion, by which we shall be able to obtain a remarkable expression for the general term. We shall express the unknown functions by a notation slightly differing from that which we have just employed. The reason for so doing will be easily seen by the reader.

Let

$$(\varrho^2+\varrho\theta\pi)^n=\varrho^{2n}+\varrho^{2n-1}\varphi_1^{(n)}\pi+\varrho^{2n-2}\varphi_2^{(n)}(\pi)+\dots$$

Then

$$(\varrho^2+\varrho\theta(\pi))^{n+1}=\varrho^{2n+2}+\varrho^{2n+1}\varphi_1^{(n)}(\pi+2)+\varrho^{2n}\varphi_2^{(n)}(\pi+2)+\dots$$

$$+\varrho^{2n+1}\theta(\pi)+\varrho^{2n}\theta(\pi)\varphi_1^{(n)}(\pi+1)+\dots$$

$$=\varrho^{2n+2}+\varrho^{2n+1}\varphi_1^{(n+1)}(\pi)+\varrho^{2n}\varphi_2^{(n+1)}(\pi)+\dots;$$

$$\therefore \quad \varphi_1^{(n+1)}(\pi)=\varphi_1^{(n)}(\pi+2)+\theta(\pi),$$

or

$$\varphi_1^{(n+1)}(\pi)-\varepsilon^{2\frac{d}{d\pi}}\varphi_1^{(n)}\pi=\theta(\pi).$$

Wherefore, solving this equation in finite differences, we have

$$\varphi_1^{(n)}(\pi)=\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi).$$

Again,

$$\varphi_2^{(n+1)}\pi=\varphi_2^{(n)}(\pi+2)+\varphi_1^{(n)}(\pi+1)\theta\pi,$$

whence

$$\varphi_2^{(n+1)}\pi-\varepsilon^{2\frac{d}{d\pi}}\varphi_2^{(n)}\pi=\theta\pi\varepsilon^{\frac{d}{d\pi}}\varphi_1^{(n)}\pi.$$

Hence

$$\varphi_2^{(n)}(\pi)=\left\{\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\left(\theta(\pi)\varepsilon^{\frac{d}{d\pi}}\right)\left\{\varepsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\theta(\pi)$$

$$=\varepsilon^{-\frac{d}{d\pi}}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\theta(\pi)\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\right\}\theta(\pi)$$

$$=\varepsilon^{-\frac{d}{d\pi}}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right\}^2;$$

and similarly,

$$\varphi_r^{(n)}\pi=\varepsilon^{-\frac{d}{d\pi}}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\varepsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right\}^r,$$

where, however, a proper correction must be added after each performance of the symbol Σ.

As there are some peculiarities connected with this form, I shall calculate its value at length for the values $r=1$ and $r=2$.

First, let $r=1$. Then

$$\phi_1^{(n)}(\pi)=\epsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\epsilon^{-2n\frac{d}{d\pi}}\theta(\pi)$$

$$=\epsilon^{2(n-1)\frac{d}{d\pi}}\Pi_1-\frac{\theta(\pi)}{\epsilon^{2\frac{d}{d\pi}}-1}.$$

Let $n=1$. Then

$$\phi_1^{(1)}\pi=\theta\pi,\quad\text{and}\quad \Pi_1=\frac{\epsilon^{2\frac{d}{d\pi}}\theta(\pi)}{\epsilon^{2\frac{d}{d\pi}}-1}.$$

Consequently

$$\phi_1^{(n)}(\pi)=\frac{\epsilon^{2n\frac{d}{d\pi}}-1}{\epsilon^{2\frac{d}{d\pi}}-1}\theta(\pi),$$

which coincides with the value of the coefficient of ϱ^{2n-1} obtained by the former process.

Again,

$$\phi_2^{(n)}(\pi)=\epsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\epsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\epsilon^{(2n-1)\frac{d}{d\pi}}\Sigma\epsilon^{-2n\frac{d}{d\pi}}\theta(\pi)$$

$$=\epsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\epsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\epsilon^{\frac{d}{d\pi}}\frac{\epsilon^{2n\frac{d}{d\pi}}-1}{\epsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)$$

$$=\epsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\theta(\pi-2n)\epsilon^{\frac{d}{d\pi}}.\frac{\epsilon^{2n\frac{d}{d\pi}}-1}{\epsilon^{2\frac{d}{d\pi}}-1}\theta(\pi-2n)$$

$$=\epsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\left(\epsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right)\epsilon^{\frac{d}{d\pi}}\frac{\epsilon^{2n\frac{d}{d\pi}}-1}{\epsilon^{2\frac{d}{d\pi}}-1}\left(\epsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right)$$

$$=\epsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\left\{\left(\epsilon^{-2n\frac{d}{d\pi}}\theta(\pi)\right)\left(\frac{\epsilon^{\frac{d}{d\pi}}}{\epsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)\right\}$$

$$-\epsilon^{2(n-1)\frac{d}{d\pi}}\Sigma\epsilon^{-2n\frac{d}{d\pi}}\left\{\theta(\pi).\frac{\epsilon^{\frac{d}{d\pi}}}{\epsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right\}$$

$$=\epsilon^{2(n-1)\frac{d}{d\pi}}\Pi_2+\epsilon^{2(n-1)\frac{d}{d\pi}}\left(\frac{\epsilon^{-2n\frac{d}{d\pi}}}{\epsilon^{-2\frac{d}{d\pi}}-1}\theta(\pi)\right)\left(\frac{\epsilon^{\frac{d}{d\pi}}}{\epsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)$$

$$-\epsilon^{2(n-1)\frac{d}{d\pi}}\frac{\epsilon^{-2n\frac{d}{d\pi}}}{\epsilon^{-2\frac{d}{d\pi}}-1}.\left(\theta(\pi)\frac{\epsilon^{\frac{d}{d\pi}}}{\epsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)$$

$$=\epsilon^{2(n-1)\frac{d}{d\pi}}\Pi_2-\left(\frac{\theta(\pi)}{\epsilon^{2\frac{d}{d\pi}}-1}\right)\left(\frac{\epsilon^{(2n-1)\frac{d}{d\pi}}}{\epsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)$$

$$+\frac{1}{\epsilon^{2\frac{d}{d\pi}}-1}\left(\theta(\pi)\frac{\epsilon^{\frac{d}{d\pi}}}{\epsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right),$$

where Π_2 is a function of (π) to be determined.

For this purpose put $n=1$, then $\varphi_2^{(n)}(\pi)=0$, and

$$\Pi_2=\left(\frac{\theta(\pi)}{\varepsilon^{2\frac{d}{d\pi}}-1}\right)\left(\frac{\varepsilon^{\frac{d}{d\pi}}\theta(\pi)}{\varepsilon^{2\frac{d}{d\pi}}-1}\right)-\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right);$$

$$\therefore\ \varphi_1^{(n)}(\pi)=\left(\frac{\varepsilon^{2(n-1)\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)\left(\frac{\varepsilon^{(2n-1)\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)-\frac{\varepsilon^{2(n-1)\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right).$$

We next proceed to show the identity of this value of the coefficient of ϱ^{2n-2} with that formerly obtained.

The truth of the following theorem is easily seen:—

$$\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)f_1(\pi)f_2(\pi)=\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)f_1(\pi)\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)f_2(\pi)+f_2(\pi)\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)f_1(\pi)+f_1(\pi)\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)f_2(\pi).$$

Hence

$$\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)\varphi_1^n\pi=\left\{\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}-1\right)\theta(\pi)\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta(\pi)\right\}+\left\{\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}-1\right)\theta(\pi)\right\}\left\{\frac{\varepsilon^{(2n-1)\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right\}$$

$$+\left\{\left(\varepsilon^{(2n-1)\frac{d}{d\pi}}\right)\theta\pi\right\}\left\{\left(\frac{\varepsilon^{2(n-1)\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\right)\theta(\pi)\right\}-\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}-1\right)\left(\theta(\pi)\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right).$$

But

$$\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}-1\right)\left(\theta\pi.\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)=\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}\theta(\pi)\right)\left(\frac{\varepsilon^{(2n-1)\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)-\theta\pi\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi);$$

$$\therefore\quad\left(\varepsilon^{2\frac{d}{d\pi}}-1\right)\varphi_1^{(n)}\pi=\left(\varepsilon^{2(n-1)\frac{d}{d\pi}}\theta(\pi)\right)\left(\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta(\pi)\right)+\left(\frac{\varepsilon^{2(n-1)\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right)\left(\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta(\pi)\right)$$

$$-\theta(\pi)\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta\pi-\theta(\pi).\frac{\varepsilon^{(2n-1)\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta\pi+\theta(\pi).\frac{\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta\pi$$

$$=\left\{\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta\pi\right\}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta(\pi)\right\}-\left\{\frac{\varepsilon^{(2n+1)\frac{d}{d\pi}}-\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right\}\theta(\pi),$$

or

$$\varphi_1^{(n)}\pi=\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left\{\frac{\varepsilon^{2n\frac{d}{d\pi}}-1}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta\pi\right\}\left\{\varepsilon^{(2n-1)\frac{d}{d\pi}}\theta\pi\right\}-\frac{1}{\varepsilon^{2\frac{d}{d\pi}}-1}\left\{\frac{\varepsilon^{(2n+1)\frac{d}{d\pi}}-\varepsilon^{\frac{d}{d\pi}}}{\varepsilon^{2\frac{d}{d\pi}}-1}\theta(\pi)\right\}\theta\pi,$$

which agrees with the value of the symbolical coefficient of ϱ^{2n-2} as before obtained.

It is proper to add that the same method of investigation applies to all binomials of the form $(\varrho^{\alpha}+\varrho^{\beta}\theta(\pi))^n$, of which I have, for the sake of simplicity, selected the case $(\varrho^2+\varrho\theta\pi)^n$.

I now come to the calculation of the coefficients of the general term of the form of the binomial theorem as given in the first memoir.

Let us assume

$$\begin{aligned}(\rho^2+\rho\theta(\pi))^n = {} & \rho^{2n} + A_1^{(0)}\rho^{2n-1}+A_2^{(0)}\rho^{2n-2}+\&c.\\ & +(A_1^{(1)}\rho^{2n-1}+A_2^{(1)}\rho^{2n-2}+A_3^{(1)}\rho^{2n-3}+\&c.)\pi\\ & +(A_1^{(2)}\rho^{2n-1}+A_2^{(2)}\rho^{2n-2}+A_3^{(2)}\rho^{2n-3}+\&c.)\pi^2+\&c.\\ & +(A_1^{(r)}\rho^{2n-1}+A_2^{(r)}\rho^{2n-2}+A_3^{(r)}\rho^{2n-3}+\&c.)\pi^r+\&c.\end{aligned}$$

Then we have, writing $\theta_m(\pi)$ for $\frac{\theta^{(m)}(\pi)}{1.2.3\ldots.m}$,

$$A_1^{(0)}=\Sigma\theta(2n)$$

$$A_2^{(0)}=\Sigma\theta(2n-1)\Sigma\theta(2n)$$

$$A_3^{(0)}=\Sigma\theta(2n-2)\Sigma\theta(2n-1)\Sigma\theta 2n$$

$$\&c.=\&c.$$

$$A_s^{(0)}=\Sigma\theta(2n-s+1)\Sigma\theta(2n-s+2)\ldots.\Sigma\theta 2n$$

$$A_1^{(1)}=\Sigma\theta_1(2n)$$

$$A_2^{(1)}=\Sigma\theta(2n-1)\Sigma\theta_1(2n)+\Sigma\theta_1(2n-1)\Sigma\theta(2n)$$

$$\begin{aligned}A_3^{(1)}=&\Sigma\theta_1(2n-2)\Sigma\theta(2n-1)\Sigma\theta(2n)\\ &+\Sigma\theta(2n-2)\Sigma\theta_1(2n-1)\Sigma\theta(2n)\\ &+\Sigma\theta(2n-2)\Sigma\theta(2n-1)\Sigma\theta_1(2n).\end{aligned}$$

$$\&c.=\&c.$$

$$\begin{aligned}A_s^{(1)}=&\Sigma\theta_1(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta(2n)\\ &+\Sigma\theta(2n-s+1)\Sigma\theta_1(2n-s+2)\ldots\Sigma\theta(2n)\\ &+\ldots\\ &+\Sigma\theta(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta_1(2n)\end{aligned}$$

$$A_1^{(2)}=\Sigma\theta_2(2n)$$

$$\begin{aligned}A_2^{(2)}=&\Sigma\theta_2(2n-1)\Sigma\theta 2n+\Sigma\theta_1(2n-1)\Sigma\theta_1(2n)\\ &+\Sigma\theta(2n-1)\Sigma\theta_2(2n)\end{aligned}$$

$$\begin{aligned}A_s^{(2)}=&\Sigma\theta_2(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta(2n)\\ &+\Sigma\theta(2n-s+1)\Sigma\theta_2(2n-s+2)\ldots\Sigma\theta(2n)\\ &+\ldots\\ &+\Sigma\theta(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta_2(2n)\\ &+\Sigma\theta_1(2n-s+1)\Sigma\theta_1(2n-s+2)\ldots\Sigma\theta(2n)\\ &+\Sigma\theta_1(2n-s+1)\Sigma\theta(2n-s+2)\ldots\Sigma\theta_1(2n)\\ &+\ldots\end{aligned}$$

Where there are (s) terms in the first part of this expression, and $s\cdot\frac{s-1}{2}$ in the second:

$$\begin{aligned}A_s^{(r)}=&\Sigma\theta_{\alpha_1}(2n-s+1)\Sigma\theta_{\beta_1}(2n-s+2)\Sigma\theta_{\gamma_1}(2n-s+3)\ldots\Sigma\theta_{\nu_1}(2n)\\ &+\Sigma\theta_{\alpha_2}(2n-s+1)\Sigma\theta_{\beta_2}(2n-s+2)\Sigma\theta_{\gamma_2}(2n-s+3)\ldots\Sigma\theta_{\nu_2}(2n)\\ &+\Sigma\theta_{\alpha_3}(2n-s+1)\Sigma\theta_{\beta_3}(2n-s+2)\Sigma\theta_{\gamma_3}(2n-s+3)\ldots\Sigma\theta_{\nu_3}(2n)\\ &+\ldots\end{aligned}$$

where $\alpha_1, \beta_1, \gamma_1, \ldots \nu, \alpha_2, \beta_2, \gamma_2, \ldots \nu_2$, &c. are all the whole numbers which satisfy the equation

$$\alpha+\beta+\gamma+\ldots+\nu=r.$$

In the preceding investigation we have used $\theta_m(\pi)$ as an abbreviation for $\frac{\theta^{(m)}\pi}{1.2.3\ldots m}$, where $\theta^{(m)}\pi$ is the mth function derived from $\theta(\pi)$. In the following investigation, it is proper to remark that $\theta_1(\pi)$, $\theta_2(\pi)$, $\theta_3(\pi)\ldots$ are any rational and entire functions of (π) whatever.

To find an expression for the general term of the multinomial theorem, by which

$$(\varrho^{\alpha}+\varrho^{\alpha-1}\theta_1(\pi)+\varrho^{\alpha-2}\theta_2(\pi)+\varrho^{\alpha-3}\theta_3(\pi)+\ldots)^n$$

is expanded in powers of (ϱ).

Let us assume

$$(\varrho^{\alpha}+\varrho^{\alpha-1}\theta_1(\pi)+\varrho^{\alpha-2}\theta_2(\pi)+\ldots)^n=\varrho^{\alpha n}+\varrho^{\alpha n-1}\phi_1^{(n)}(\pi)+\varrho^{\alpha n-2}\phi_2^{(n)}(\pi)+\varrho^{\alpha n-3}\phi_3^{(n)}(\pi)+\ldots$$

Then multiplying internally by the factor

$$\varrho^{\alpha}+\varrho^{\alpha-1}\theta_1(\pi)+\varrho^{\alpha-2}\theta_2(\pi)+\varrho^{\alpha-3}\theta_3\pi+\ldots,$$

and equating coefficients of like powers of (ϱ), we have the following series of equations:—

$$\phi_1^{(n+1)}(\pi)-\phi_1^{(n)}(\pi+\alpha)=\theta_1(\pi),$$

$$\phi_2^{(n+1)}(\pi)-\phi_2^{(n)}(\pi+\alpha)=\theta_1(\pi)\phi_1^{(n)}(\pi+\alpha-1)+\theta_2(\pi),$$

$$\phi_3^{(n+1)}(\pi)-\phi_3^{(n)}(\pi+\alpha)=\theta_1(\pi)\phi_2^{(n)}(\pi+\alpha-1)+\theta_2(\pi)\phi_1^{(n)}(\pi+\alpha-2)+\theta_3(\pi),$$

and thus we proceed: hence we have, putting the symbol $\varepsilon^{(n+1)\alpha\frac{d}{d\pi}}\Sigma\varepsilon^{-(n+1)\alpha\frac{d}{d\pi}}=\Pi$,

$$\phi_1^{(n)}(\pi)=\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1\pi,$$

$$\phi_2^{(n)}\pi=\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_2(\pi)$$

$$\phi_3^{(n)}\pi=\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1\pi+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_2(\pi)+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_2(\pi)\varepsilon^{-2\frac{d}{d\pi}}\Pi\theta_1\pi+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_3\pi$$

$$\phi_4^{(n)}(\pi)=\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)$$

$$+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_2\pi+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1\pi\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_2(\pi)\varepsilon^{-2\frac{d}{d\pi}}\Pi\theta_1\pi+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_2(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1\pi\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_1\pi$$

$$+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_1(\pi)\varepsilon^{-\frac{d}{d\pi}}\Pi\theta_3(\pi)+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_3(\pi)\varepsilon^{-3\frac{d}{d\pi}}\Pi\theta_1(\pi)+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_2\pi\varepsilon^{-2\frac{d}{d\pi}}\Pi\theta_2\pi+\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_4(\pi).$$

We easily see that the general term may be expressed thus: construct the formula

$$\varepsilon^{-\alpha\frac{d}{d\pi}}\Pi\theta_a(\pi)\varepsilon^{-a\frac{d}{d\pi}}\Pi\theta_b(\pi)\varepsilon^{-b\frac{d}{d\pi}}\Pi\theta_c(\pi)\varepsilon^{-c\frac{d}{d\pi}}\ldots\Pi\theta_e(\pi),$$

and give to $\alpha, \beta, \gamma, \ldots \nu$ all the values which satisfy the equation

$$a+b+c+\ldots+e=r,$$

where $\alpha n-r$ is the index of (ϱ). Then the sum of all the terms so formed will be the required result.

To determine the product of the factors

$$(\varrho+\chi_n(\pi))(\varrho+\chi_{n-1}(\pi))\ldots(\varrho+\chi(\pi))\ldots(\varrho+\chi_{n-1}(\pi))(\varrho+\chi_n\pi),$$

whence $\chi_1(\pi)$, $\chi_2(\pi)$, $\chi_3(\pi)$ emanate from each other after a given law.

Let

$$(\varrho+\chi_n(\pi))\ldots(\varrho+\chi\pi)\ldots\varrho+\chi_n(\pi)=\varrho^{2n+1}+\varrho^{2n}\varphi_1^{(n)}\pi+\varrho^{2n-1}\varphi_2^{(n)}\pi+\ldots$$

Then multiplying internally and externally by $\varrho+\chi_{n+1}(\pi)$,

$$\begin{aligned}&\varrho^{2n+3}+\varrho^{2n+2}\varphi_1^{(n+1)}\pi+\varrho^{2n+1}\varphi_2^{(n+1)}\pi+\&c.\\&=\varrho^{2n+3}+\varrho^{2n+2}\{\varphi_1^{(n)}(\pi+1)+\chi_{n+1}(\pi+2n+2)+\chi_{n+1}(\pi)\}\\&\quad+\varrho^{2n+1}\{\varphi_2^{(n)}(\pi+1)+\varphi_1^{(n)}(\pi+1)\chi_{n+1}(\pi+2n+1)\\&\quad+\varphi_1^{(n)}(\pi)\chi_{n+1}(\pi)+\chi_{n+1}(\pi+2n+1)\chi_{n+1}(\pi)\}.\end{aligned}$$

Hence

$$\varphi_1^{(n+1)}\pi-\varphi_1^{(n)}(\pi+1)=\chi_{n+1}(\pi+2n+2)+\chi_{n+1}\pi,$$

whence

$$\varphi_1^{(n)}(\pi)=\varepsilon^{n\frac{d}{d\pi}}\Sigma\left(\varepsilon^{(n+1)\frac{d}{d\pi}}+\varepsilon^{-(n+1)\frac{d}{d\pi}}\right)\chi_{n+1}(\pi),$$

and also

$$\begin{aligned}\varphi_2^{(n+1)}(\pi)-\varphi_2^{(n)}(\pi+1)&=\varphi_1^{(n)}(\pi+1)\chi_{n+1}(\pi+2n+1)\\&\quad+\varphi_1^{(n)}(\pi)\chi_{n+1}(\pi)+\chi_{n+1}(\pi+2n+1)\chi_{n+1}(\pi);\end{aligned}$$

$$\begin{aligned}\therefore\quad\varphi_2^{(n)}(\pi)&=\varepsilon^{n\frac{d}{d\pi}}\Sigma\varepsilon^{-(n+1)\frac{d}{d\pi}}\left\{\varepsilon^{(2n+1)\frac{d}{d\pi}}\chi_{n+1}(\pi)\right\}\left\{\varepsilon^{n\frac{d}{d\pi}}\Sigma\left(\varepsilon^{(n+1)\frac{d}{d\pi}}-\varepsilon^{-(n+1)\frac{d}{d\pi}}\right)\varepsilon^{\frac{d}{d\pi}}\chi_{n+1}(\pi)\right\}\\&\quad+\varepsilon^{n\frac{d}{d\pi}}\Sigma\varepsilon^{-(n+1)\frac{d}{d\pi}}\chi_{n+1}(\pi)\varepsilon^{n\frac{d}{d\pi}}\Sigma\left(\varepsilon^{(n+1)\frac{d}{d\pi}}-\varepsilon^{-(n+1)\frac{d}{d\pi}}\right)\chi_{n+1}(\pi)\\&\quad+\varepsilon^{n\frac{d}{d\pi}}\Sigma\varepsilon^{-(n+1)\frac{d}{d\pi}}\chi_{n+1}(\pi)\left\{\varepsilon^{(2n+1)\frac{d}{d\pi}}\chi_{n+1}(\pi)\right\};\end{aligned}$$

and in like manner we find the values of the succeeding symbolical coefficients.

I now come to the form of the binomial theorem which is reciprocal to that previously investigated.

To expand $(\pi^2+\theta(\varrho).\pi)^n$ in powers of (π).

Let us assume

$$(\pi^2+\theta(\varrho).\pi)^n=\pi^{2n}+\varphi_{2n-1}(\varrho).\pi^{2n-1}+\varphi_{2n-2}(\varrho).\pi^{2n-2}+\ldots,$$

we know that

$$\pi^r\theta(\varrho)=\theta(\varrho).\pi^r+r\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho).\pi^{r-1}+r.\frac{r-1}{2}\left(\varrho\frac{d}{d\varrho}\right)^2\theta\varrho\pi^{r-2}+\ldots$$

Hence, multiplying internally and externally by $\pi^2+\theta(\varrho)\pi$, we shall have

$$\begin{aligned}&\pi^{2n+2}+\varphi_{2n-1}(\varrho)\pi^{2n+1}+\varphi_{2n-2}(\varrho).\pi^{2n}+\varphi_{2n-3}(\varrho).\pi^{2n-1}+\ldots\\&\quad+\left\{\theta(\varrho)\pi^{2n}+2n\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)\pi^{2n-1}+2n.\frac{2n-1}{2}\left(\varrho\frac{d}{d\varrho}\right)^2\theta(\varrho)\pi^{2n-2}+\ldots\right\}\pi\\&\quad+\varphi_{2n-1}(\varrho)\left\{\theta(\varrho).\pi^{2n-1}+(2n-1)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)\pi^{2n-2}+\ldots\right\}\pi\\&\quad+\varphi_{2n-2}(\varrho)\left\{\theta(\varrho).\pi^{2n-2}+(2n-2)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho).\pi^{2n-3}+\ldots\right\}\pi\\&\quad+\&c.\\&=\pi^{2n+2}+\left\{\varphi_{2n-1}(\varrho).\pi^2+2\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-1}(\varrho).\pi+\left(\varrho\frac{d}{d\varrho}\right)^2\varphi_{2n-1}(\varrho)\right\}\pi^{2n-1}\\&\quad+\left\{\varphi_{2n-2}(\varrho).\pi^2+2\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-2}(\varrho).\pi+\left(\varrho\frac{d}{d\varrho}\right)^2\varphi_{2n-2}(\varrho)\right\}\pi^{2n-2}\end{aligned}$$

$$+\ \&\text{c.}\ +\theta\varrho\pi^{2n+1}+\theta(\varrho).\left(\varphi_{2n-1}(\varrho).\pi+\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-1}(\varrho)\right)\pi^{2n-1}$$

$$+\theta(\varrho)\left(\varphi_{2n-2}(\varrho).\pi+\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-2}(\varrho)\right)\pi^{2n-2}+\ \&\text{c.}$$

Hence equating coefficients of the powers of (ϱ),

$$\varphi_{2n-2}(\varrho)+2n\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho).+\varphi_{2n-1}(\varrho).\theta(\varrho)=2\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-1}(\varrho)+\varphi_{2n-2}(\varrho)+\theta(\varrho)\varphi_{2n-1}(\varrho),$$

whence

$$\varphi_{2n-1}(\varrho)=n\theta(\varrho),$$

$$\varphi_{2n-3}(\varrho)+2n.\frac{2n-1}{2}\left(\varrho\frac{d}{d\varrho}\right)^2\theta(\varrho)+(2n-1)\varphi_{2n-1}(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)+\varphi_{2n-2}(\varrho)\theta\varrho$$

$$=\left(\varrho\frac{d}{d\varrho}\right)^2\varphi_{2n-1}(\varrho)+2\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-2}(\varrho)+\theta(\varrho)\left(\varrho\frac{d}{d\varrho}\right)\varphi_{2n-1}(\varrho)+\theta(\varrho)\varphi_{2n-2}(\varrho)+\varphi_{2n-3}(\varrho),$$

whence

$$\varphi_{2n-2}(\varrho)=n(n-1)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)+\frac{n(n-1)}{2}(\theta(\varrho))^2;$$

$$\therefore\quad (\pi^2+\theta(\varrho).\pi)^n=\pi^{2n}+n\theta\varrho.\pi^{2n-1}+\left\{n(n-1)\left(\varrho\frac{d}{d\varrho}\right)\theta(\varrho)+\frac{n(n-1)}{2}(\theta(\varrho))^2\right\}\pi^{2n-2}+\dots,$$

the required expansion.

XIV. *On the Calculus of Functions.* *By* W. H. L. Russell, *Esq.*, *A.B.* *Communicated by* Arthur Cayley, *Esq.*, *F.R.S.*

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One of the first efforts towards the formation of the Calculus of Functions is due to Laplace, whose solution of the functional equation of the first order, by means of two equations in finite differences, is well known. Functional equations were afterwards treated systematically by Mr. Babbage; his memoirs were published in the Transactions of this Society, and there is some account of them in Professor Boole's Treatise on the Calculus of Finite Differences. A very important functional equation was solved by Poisson in his Memoirs on Electricity; which suggested to me the investigations I have now the honour to lay before the Society.

I have commenced by discussing the linear functional equation of the first order with constant coefficients, when the subjects of the unknown functions are rational functions of the independent variable, and have shown how the solution of such equations may in a variety of cases be effected by series, and by definite integrals. I have then considered functional equations with constant coefficients of the higher orders, and have proved that they may be solved by methods similar to those used for equations of the first order. I have next proceeded with the solution of functional equations with variable coefficients.

In connexion with functional equations, I have considered equations involving definite integrals, and containing an unknown function under the integral sign; the methods employed for their resolution depend chiefly upon the solution of functional equations, as effected in this paper. The Calculus of Functions has now for a long time engaged the attention of analysts; and I hope that the following investigations will be found to have extended its power and resources.

Let the functional equation be

$$\varphi\left\{\frac{n^2+2n-8}{4r}-nx+rx^2\right\}-\alpha\varphi(x)=\mathrm{F}(x),$$

where φ is an unknown, and F a known function.

Let

$$x=\frac{n}{2r}+\frac{2}{r}\cos z,$$

and the equation becomes

$$\varphi\left\{\frac{n}{2r}+\frac{2}{r}\cos 2z\right\}-\alpha\varphi\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\}=\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\}$$

or if

$$\varphi\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\}=\varphi_1(z),$$

we shall have

$$\phi_1(2z)-\alpha\phi_1(z)=\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\}.$$

Let

$$z=\frac{1}{2^{v+1}},$$

then

$$\phi_1\left(\frac{1}{2^v}\right)-\alpha\phi_1\left(\frac{1}{2^{v+1}}\right)=\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos\frac{1}{2^{v+1}}\right\}.$$

Let

$$\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos\frac{1}{2^{v+1}}\right\}=\chi\left(\frac{1}{2^{v+1}}\right),$$

then

$$\phi_1\left(\frac{1}{2^v}\right)-\alpha\phi_1\left(\frac{1}{2^{v+1}}\right)=\chi\left(\frac{1}{2^{v+1}}\right),$$

and

$$\phi_1\left(\frac{1}{2^v}\right)=\chi\left(\frac{1}{2^{v+1}}\right)+\alpha\chi\left(\frac{1}{2^{v+2}}\right)+\alpha^2\chi\left(\frac{1}{2^{v+3}}\right)+\ldots+\frac{\mathrm{C}}{\alpha^{v+1}}.$$

Now

$$\int_0^\pi\frac{(\chi(\varepsilon^{i\theta})+\chi(\varepsilon^{-i\theta}))d\theta}{1-2\alpha\cos\theta+\alpha^2}=\frac{2\pi}{1-\alpha^2}\chi(\alpha),$$

whence (α) is less than unity: then if

$$\varepsilon^{\omega}=2^{v+n+1},$$

and

$$\omega=(v+1)\log_\varepsilon 2+n\log_\varepsilon 2,$$

also

$$f(\theta)=\chi(\varepsilon^{i\theta})+\chi(\varepsilon^{-i\theta}),$$

we shall have

$$\chi\left(\frac{1}{2^{v+n+1}}\right)=\frac{1}{2\pi}\int_0^\pi\frac{\varepsilon^{\omega}-\varepsilon^{-\omega}}{\varepsilon^{\omega}-2\cos\theta+\varepsilon^{-\omega}}f(\theta)d\theta$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\pi d\theta d\varrho f(\theta)\cdot\frac{\varepsilon^{(\pi-\theta)\varrho}+\varepsilon^{-(\pi-\theta)\varrho}}{\varepsilon^{\pi\varrho}-\varepsilon^{-\pi\varrho}}\sin\{\varrho\log_\varepsilon 2^{v+1}+n\varrho\log_\varepsilon 2\};$$

$$\therefore\quad\chi\left(\frac{1}{2^{v+1}}\right)+\alpha\chi\left(\frac{1}{2^{v+2}}\right)+\alpha^2\chi\left(\frac{1}{2^{v+3}}\right)+\alpha^3\chi\left(\frac{1}{2^{v+4}}\right)+\ldots$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\pi d\theta d\varrho f(\theta)\cdot\frac{\varepsilon^{(\pi-\theta)\varrho}+\varepsilon^{-(\pi-\theta)\varrho}}{\varepsilon^{\pi\varrho}-\varepsilon^{-\pi\varrho}}$$

$$\{\sin(\varrho\log_\varepsilon 2^{v+1})+\alpha\sin(\varrho\log_\varepsilon 2^{v+1}+\varrho\log_\varepsilon 2)+\alpha^2\sin(\varrho\log_\varepsilon 2^{v+1}+2\varrho\log_\varepsilon 2)+\&c.\}$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\pi d\theta d\varrho f(\theta)\cdot\frac{\varepsilon^{(\pi-\theta)\varrho}+\varepsilon^{-(\pi-\theta)\varrho}}{\varepsilon^{\pi\varrho}-\varepsilon^{-\pi\varrho}}\cdot\frac{\sin(\varrho\log_\varepsilon 2^{v+1})-\alpha\sin(\varrho\log_\varepsilon 2^{v})}{1-2\alpha\cos(\varrho\log_\varepsilon 2^{v+1})+\alpha^2}.$$

Hence

$$\phi(x)=\frac{1}{\pi}\int_0^\infty\int_0^\pi d\theta d\varrho f(\theta)\cdot\frac{\varepsilon^{(\pi-\theta)\varrho}+\varepsilon^{-(\pi-\theta)\varrho}}{\varepsilon^{\pi\varrho}-\varepsilon^{-\pi\varrho}}\cdot\frac{\sin\left\{\varrho\log_\varepsilon\dfrac{2}{\cos^{-1}\frac{2rx-n}{4}}\right\}-\alpha\sin\left\{\varrho\log_\varepsilon\dfrac{1}{\cos^{-1}\frac{2rx-n}{4}}\right\}}{1-2\alpha\cos\left\{\varrho\log_\varepsilon\dfrac{1}{\cos^{-1}\frac{2rx-n}{4}}\right\}+\alpha^2}$$

$$+\frac{\mathrm{C}}{\left(\cos^{-1}\frac{2rx-n}{4}\right)^{\frac{\log_\varepsilon\alpha}{\log_\varepsilon\frac{1}{2}}}},$$

where

$$f(\theta)=\mathrm{F}\left\{\frac{n}{2r}+\frac{1}{r}\cos\cos\theta(\varepsilon^{\theta}+\varepsilon^{-\theta})+\frac{i}{r}\sin\cos\theta(\varepsilon^{\theta}-\varepsilon^{-\theta})\right\}$$
$$+\mathrm{F}\left\{\frac{n}{2r}+\frac{1}{r}\cos\cos\theta(\varepsilon^{\theta}+\varepsilon^{-\theta})-\frac{i}{r}\sin\cos\theta(\varepsilon^{\theta}-\varepsilon^{-\theta})\right\}.$$

Next let the functional equation be

$$\varphi\left\{\left(\frac{n^3}{27r^2}-\frac{4n}{3r}\right)+\left(\frac{n^2}{3r}-3\right)x+nx^2+rx^3\right\}-\alpha\varphi(x)=\mathrm{F}(x).$$

Let

$$x=-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos z,$$

and the equation will be transformed into

$$\varphi\left\{-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos 3z\right\}-\alpha\varphi\left\{-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos z\right\}=\mathrm{F}\left\{-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos z\right\};$$

this may be written

$$\varphi_1(3z)-\alpha\varphi_1(z)=\chi(z).$$

Let

$$z=\frac{1}{3^{v+1}}$$

$$\varphi_1\left(\frac{1}{3^v}\right)-\alpha\varphi_1\left(\frac{1}{3^{v+1}}\right)=\chi\left(\frac{1}{3^{v+1}}\right),$$

whence

$$\varphi_1\left(\frac{1}{3^v}\right)=\chi\left(\frac{1}{3^{v+1}}\right)+\alpha\chi\left(\frac{1}{3^{v+2}}\right)+\alpha^2\chi\left(\frac{1}{3^{v+3}}\right)+\dots$$

The same method evidently applies, and we thus obtain the value of $\varphi(x)$.

In the same way we may treat the equation

$$\varphi(a_0+a_1x+a_2x^2+a_3x^3+\dots+a_mx^m)-\alpha\varphi(x)=\mathrm{F}(x),$$

where $a_0, a_1, a_2, \dots a_{m-2}$ must be supposed given in terms of a_{m-1}, a_m.

We will now consider the equation

$$\varphi\left\{\frac{(n-r)r'^2+2rn'^2+2r(rr'-nr'-2nn')x+((n-r)r'^2+2rn'^2)x^2}{(n'-r')r^2+2r'n^2+2r'(rr'-n'r-2nn')x+((n'-r')r'^2+2r'n'^2)x^2}\right\}-\alpha\varphi(x)=\mathrm{F}(x).$$

Let

$$x=\frac{n+r\cos z}{n'+r'\cos z}.$$

Then the equation becomes

$$\varphi\left\{\frac{n+r\cos 2z}{n'+r'\cos 2z}\right\}-\alpha\varphi\left\{\frac{n+r\cos z}{n'+r'\cos z}\right\}=\mathrm{F}\left\{\frac{n+r\cos z}{n'+r'\cos z}\right\},$$

which may be treated as before: as we may write it

$$\varphi_1(2z)-\alpha\varphi_1(z)=\chi(z).$$

Similar methods will evidently apply to the equation

$$\varphi\left\{\frac{a_0+a_1x+a_2x^2+\dots+a_mx^m}{b_0+b_1x+b_2x^2+\dots+b_mx^m}\right\}-\alpha\varphi(x)=\mathrm{F}(x).$$

The general linear functional equation of the nth order with constant coefficients is

$$\phi(\psi^n x)+a\phi(\psi^{n-1}x)+b\phi(\psi^{n-2}x)+\dots+h\phi\psi(x)+k\phi(x)=F(x),$$

where the subject ψx is supposed to be the same as in the preceding equations.

Let $x=\chi(z)$, and $\psi(x)=\psi\chi(z)=\chi(mz)$ suppose; then $\psi^2(x)=\chi m^2 z$, $\psi^3 x=\chi m^3 z$, &c., and the equation becomes

$$\phi\chi(m^n z)+a\phi\chi(m^{n-1}z)+b\phi\chi(m^{n-2}z)+\dots+h\phi\chi(mz)+k\phi\chi(z)=F\chi(z).$$

Let $z=\frac{1}{m^{v+n}}$, and the equation becomes

$$\phi\chi\left(\frac{1}{m^v}\right)+a\phi\chi\left(\frac{1}{m^{v+1}}\right)+b\phi\chi\left(\frac{1}{m^{v+2}}\right)+h\phi\chi\left(\frac{1}{m^{v+n-1}}\right)+k\phi\chi\left(\frac{1}{m^{v+n}}\right)=F\chi\left(\frac{1}{m^{v+n}}\right).$$

This equation may be written

$$\left\{1+a\varepsilon^{\frac{d}{dv}}+b\varepsilon^{2\frac{d}{dv}}+\dots h\varepsilon^{(n-1)\frac{d}{dv}}+k\varepsilon^{n\frac{d}{dv}}\right\}\phi\chi\left(\frac{1}{m^v}\right)=F\chi\left(\frac{1}{m^{v+n}}\right),$$

or

$$\left\{\left(1+\alpha_1\varepsilon^{\frac{d}{dv}}\right)\left(1+\alpha_2\varepsilon^{\frac{d}{dv}}\right)\left(1+\alpha_3\varepsilon^{\frac{d}{dv}}\right)\dots\left(1+\alpha_n\varepsilon^{\frac{d}{dv}}\right)\right\}\phi\chi\left(\frac{1}{m^v}\right)=F\chi\left(\frac{1}{m^{v+n}}\right),$$

$$\therefore\quad \phi\chi\left(\frac{1}{m^v}\right)$$

$$=A_1\left\{F\chi\left(\frac{1}{m^{v+n}}\right)-\alpha_1 F\chi\left(\frac{1}{m^{v+n+1}}\right)+\alpha_1^2 F\chi\left(\frac{1}{m^{v+n+2}}\right)+\&c.\right\}$$

$$+A_2\left\{F\chi\left(\frac{1}{m^{v+n}}\right)-\alpha_2 F\chi\left(\frac{1}{m^{v+n+1}}\right)+\alpha_2^2 F\chi\left(\frac{1}{m^{v+n+2}}\right)+\&c.\right\}$$

$$+A_3\left\{F\chi\left(\frac{1}{m^{v+n}}\right)-\alpha_3 F\chi\left(\frac{1}{m^{v+n+1}}\right)+\alpha_3^2 F\chi\left(\frac{1}{m^{v+n+2}}\right)+\dots\right\}+\&c.$$

$$+\frac{C_1}{(-\alpha_1)^v}+\frac{C_2}{(-\alpha_2)^v}+\frac{C_3}{(-\alpha_3)^v}+\&c.,$$

where A_1, A_2, A_3, &c. are certain functions of α_1, α_2, α_3, &c., and C_1, C_2, C_3, &c. are arbitrary constants, from whence we at once obtain the value of $\phi(x)$.

We now proceed to consider functional equations with variable coefficients. And first let the equation be

$$\phi(x)-\chi(x)\phi\left\{\frac{a+bx}{c+ex}\right\}=F(x),$$

where $\chi(x)$ and $F(x)$ are known rational functions of (a).

Let

$$x=u_z,\quad \frac{a+bu_z}{c+eu_z}=u_{z+1},$$

then

$$u_{z+1}=\frac{a+bu_z}{c+eu_z}.$$

Suppose a solution of this equation of finite differences to be

$$u_z=\frac{A+Bz}{C+Ez},$$

the equation becomes

$$\phi u_z - \chi(u_z)\phi(u_{z+1}) = \mathrm{F}u_z.$$

Let

$$\chi u_z = \alpha \cdot \frac{(\alpha_1+z)(\alpha_2+z)(\alpha_3+z)\dots}{(\beta_1+z)(\beta_2+z)(\beta_3+z)\dots},$$

and the equation may then be written

$$\alpha^z \frac{\Gamma(\alpha_1+z)\Gamma(\alpha_2+z)\Gamma(\alpha_3+z)\dots}{\Gamma(\beta_1+z)\Gamma(\beta_2+z)\Gamma(\beta_3+z)\dots}\phi(u_z) - \alpha^{z+1}\frac{\Gamma(\alpha_1+z+1)\Gamma(\alpha_2+z+1)\Gamma(\alpha_3+z+1)\dots}{\Gamma(\beta_1+z+1)\Gamma(\beta_2+z+1)\Gamma(\beta_3+z+1)\dots}\phi(u_{z+1})$$

$$= \alpha^z \frac{\Gamma(\alpha_1+z)\Gamma(\alpha_2+z)\Gamma(\alpha_3+z)\dots}{\Gamma(\beta_1+z)\Gamma(\beta_2+z)\Gamma(\beta_3+z)\dots}\mathrm{F}u_z.$$

Hence

$$\phi(u_z) = \mathrm{F}u_z + \alpha \cdot \frac{(\alpha_1+z)(\alpha_2+z)(\alpha_3+z)\dots}{(\beta_1+z)(\beta_2+z)(\beta_3+z)\dots}\mathrm{F}(u_{z+1}) + \alpha^2 \frac{(\alpha_1+z)(\alpha_1+z+1)(\alpha_2+z)(\alpha_2+z+1)\dots}{(\beta_1+z)(\beta_1+z+1)(\beta_2+z)(\beta_2+z+1)\dots}\mathrm{F}(u_{z+2})$$

$$+ \dots + \frac{\mathrm{C}}{\alpha^z} \cdot \frac{\Gamma(\beta_1+z)\Gamma(\beta_2+z)\dots}{\Gamma(\alpha_1+z)\Gamma(\alpha_2+z)\dots}.$$

$\mathrm{F}u_z$ is a rational function of (z), and may therefore be decomposed into a series of terms of the form

$$\mathrm{F}u_z = \frac{1}{h_1+k_1z} + \frac{1}{h_2+k_2z} + \frac{1}{h_3+k_3z} + \dots$$

Hence

$$\phi u_z = \frac{1}{h_1+k_1z} + \alpha \frac{(\alpha_1+z)(\alpha_2+z)(\alpha_3+z)\dots}{(\beta_1+z)(\beta_2+z)(\beta_3+z)\dots} \cdot \frac{1}{h_1+k_1(z+1)} + \&c.$$

$$+ \frac{1}{h_2+k_2z} + \alpha \frac{(\alpha_1+z)(\alpha_2+z)(\alpha_3+z)\dots}{(\beta_1+z)(\beta_2+z)(\beta_3+z)\dots} \cdot \frac{1}{h_2+k_2(z+1)} + \&c.$$

$$+ \frac{1}{h_3+k_3z} + \alpha \frac{(\alpha_1+z)(\alpha_2+z)(\alpha_3+z)\dots}{(\beta_1+z)(\beta_2+z)(\beta_3+z)\dots} \cdot \frac{1}{h_3+k_3(z+1)} + \&c.$$

$$+ \frac{\mathrm{C}}{\alpha^z} \cdot \frac{\Gamma(\beta_1+z)\Gamma(\beta_2+z)\Gamma(\beta_3+z)\dots}{\Gamma(\alpha_1+z)\Gamma(\alpha_2+z)\Gamma(\alpha_3+z)\dots}.$$

We may obtain a multiple integral which shall be equivalent to any of the above series, by remembering that

$$\frac{\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\dots(\beta+n-1)} = \frac{\Gamma\beta}{\Gamma\alpha\Gamma(\beta-\alpha)}\int_0^1 v^{n+\alpha-1}(1-v)^{\beta-\alpha-1}dv$$

$$\frac{1}{\beta(\beta+1)(\beta+2)\dots(\beta+n-1)} = \frac{\Gamma\beta \,.\, \varepsilon}{2\pi}\int_{-\infty}^{\infty} \frac{\varepsilon^{iv}dv}{(1+iv)^{\beta+n}},$$

also

$$\frac{1}{h+kn} = \int_0^{\infty} \varepsilon^{-v(h+kn)}dv,$$

and summing accordingly. We may hence immediately deduce the value of $\phi(x)$.

It is evident that the functional equation

$$\phi(a+bx+cx^2) - \alpha \frac{\psi(x)}{\psi(a+cx+cx^2)}\phi(x) = \mathrm{F}(x),$$

when ψ_1 and F_1 are known functions of (x), may be reduced to the form

$$\phi_1(a+bx+cx^2)-\alpha\phi_1(x)=F(x)\psi(a+bx+cx^2)$$

by making

$$\phi_1(x)=\psi(x)\phi(x).$$

The second equation has already been considered; it becomes, therefore, interesting to ascertain what equations are included under the form of the first equation, in other words, to consider what forms the algebraical expression $\frac{\psi(x)}{\psi(a+bx+cx^2)}$ can possibly take. The following are a few of them:—

$$\frac{\sqrt{(x+2\sqrt{ac}-b)}}{\sqrt{a}+\sqrt{c}.x},\quad \frac{x-a}{x(b+cx)},\quad \frac{x^2-a^2}{(2a+bx+cx^2)x(b+cx)},\quad \frac{\sqrt{\{x^2-2bx+b^2-4ac}}{a-cx^2}.$$

Many others may, in like manner, be imagined; and the same methods, *mutatis mutandis*, apply to functional equations of the higher orders with variable coefficients.

I now come to the consideration of equations involving definite integrals, when the equation contains an unknown function under the sign of definite integration.

Let us take the equation

$$\int_0^1 \frac{F_1(\alpha)+F_2(\alpha).x^2}{F_3(\alpha)+F_4(\alpha)x^2+F_5(\alpha).x^4}dx\phi(x)=F(\alpha),$$

where $\phi(x)$ is an unknown function of (x) not containing (α), $F_1(\alpha)$, $F_2(\alpha)\ldots$, $F_5(\alpha)$ rational functions of (α) which is supposed to vary independently of (x), to determine $\phi(x)$.

Suppose the equation can be written in the form

$$\int_0^1 \frac{\lambda(x)dx}{\sqrt{1-x^2}}\left\{\frac{1-(\chi(\alpha))^2}{1-2\chi\alpha(1-2x^2)+(\chi\alpha)^2}+\frac{\mu(1-\alpha^2)}{1-2\alpha(1-2x^2)+\alpha^2}\right\}=F(\alpha),$$

where

$$\lambda(x)=\phi(x)\sqrt{1-x^2}.$$

Let $x=\sin\frac{\theta}{2}$, and the equation reduces to

$$\int_0^\pi d\theta\lambda\left(\sin\frac{\theta}{2}\right)\left\{\frac{1-(\chi\alpha)^2}{1-2\chi\alpha\cos\theta+(\chi\alpha)^2}+\frac{\mu(1-\alpha^2)}{1-2\alpha\cos\theta+\alpha^2}\right\}=2F(\alpha);$$

or if

$$\psi(\alpha)=\int_0^\pi d\theta.\frac{(1-\alpha^2)\lambda\left(\sin\frac{\theta}{2}\right)}{1-2\alpha\cos\theta+\alpha^2},$$

we find

$$\psi\chi(\alpha)+\mu\psi(\alpha)=2F(\alpha).$$

Suppose the solution of this equation be determined by the former investigations to be

$$\psi(\alpha) = f(\alpha),$$

then

$$\int_0^\pi \frac{(1-\alpha^2)d\theta\lambda\left(\sin\frac{\theta}{2}\right)}{1-2\alpha\cos\theta+\alpha^2}=f(\alpha).$$

Assume

$$\lambda\sin\frac{\theta}{2}=a_0+a_1\cos\theta+a_2\cos 2\theta+a_3\cos 3\theta+\ldots,$$

then since

$$\frac{1-\alpha^2}{1-2\alpha\cos\theta+\alpha^2}=1+2\alpha\cos\theta+2\alpha^2\cos 2\theta+\ldots,$$

we have

$$\pi(a_0+a_1\alpha+a_2\alpha^2+\ldots)=f(\alpha);$$

$$\therefore \qquad a_0+a_1\varepsilon^{i\theta}+a_2\varepsilon^{2i\theta}+\ldots=\frac{1}{\pi}f(\varepsilon^{i\theta})$$

$$a_0+a_1\varepsilon^{-i\theta}+a_2\varepsilon^{-2i\theta}+\ldots=\frac{1}{\pi}f(\varepsilon^{-i\theta}).$$

Hence

$$\lambda\left(\sin\frac{\theta}{2}\right)=\frac{1}{2\pi}\{f\varepsilon^{i\theta}+f\varepsilon^{-i\theta}\},$$

whence $\phi(x)$ can be determined.

Similar treatment will of course apply to the equation

$$\int_0^1\frac{F_1\alpha+F_2\alpha x^2+F_3\alpha\,.\,x^4+\ldots}{F_n(\alpha)+F_{n+1}(\alpha)x^2+\ldots}\phi(x)=F(\alpha),$$

but the functional equation employed for its solution (when possible by this method) will be of a higher order.

Let us, lastly, consider the equation

$$\int_0^1 F_1(x)\phi(x\psi(\alpha))=F(\alpha),$$

to find ϕ, where F_1x is a known function of (x) not containing (α), and (α) varies independently of (x).

Let $\psi(\alpha)=\beta$, then $\alpha=\psi^{-1}\beta$, and the equation becomes

$$\int_0^1 F_1(x)\phi(x\beta)=F\psi^{-1}(\beta).$$

Let

$$\phi(x\beta)=A_0+A_1x\beta+A_2x^2\beta^2+\ldots,$$

then we shall have

$$A_0\int_0^1 dxF_1x+A_1\int_0^1 dx\,.\,xF(x)\,.\,\beta+A_2\int_0^1 dxx^2F(x)\,.\,\beta^2+\ldots$$

$$=F\psi^{-1}0+F'\psi^{-1}0\,.\,\beta+F''\psi^{-1}0\,.\,\frac{\beta^2}{1.2}+\ldots;$$

then

$$A_0=\frac{F\psi^{-1}(0)}{\int_0^1 dxF_1x},\quad A_1=\frac{F'\psi^{-1}0}{\int_0^1 dx\,.\,xF(x)}\ldots$$

Hence

$$\phi(\xi)=\frac{F\psi^{-1}0}{\int_0^1 dxF_1x}+\frac{F'\psi^{-1}0}{\int_0^1 dx\,.\,xF_1x}\,.\,\xi+\frac{F''\psi^{-1}0}{\int_0^1 dx\,.\,x^2F_1x}\,.\,\frac{\xi^2}{1.2}+\frac{F'''\psi^{-1}0}{\int_0^1 dx\,.\,x^3\,.\,F_1\,.\,x}\,.\,\frac{\xi^3}{1.2.3}+\ldots,$$

ξ being any variable.

Now suppose

$$\frac{1}{\int_0^1 dx\,.\,x^nF_1x}=\chi(n),$$

and that we are able to express $\chi(n)$ by a definite integral, so that

$$\chi(n)=\int f(v)(\lambda(v))^n dv,$$

the integral being supposed to have certain finite limits, we shall have

$$\phi(\xi)=\int f(v)\mathrm{F}\psi^{-1}(\xi\lambda(v))\,.\,dv.$$

Thus if

$$\int_0^1(x+1)\phi(x\psi(\alpha))=\mathrm{F}(\alpha),$$

we have

$$\phi(\xi)=\frac{1}{2}\xi\frac{d}{d\xi}\mathrm{F}\psi^{-1}(\xi)+\frac{3}{4}\mathrm{F}\psi^{-1}\xi-\frac{1}{8}\int_0^\infty \varepsilon^{-\frac{3v}{4}}dv\mathrm{F}\psi^{-1}(\xi\varepsilon^{-v}).$$

XV. *On the Difference in the Magnetic Properties of Hot-Rolled and Cold-Rolled Malleable Iron, as regards the power of receiving and retaining Induced Magnetism of Subpermanent Character. By* GEORGE BIDDELL AIRY, *Astronomer Royal.*

Received April 22,—Read May 15, 1862.

IN reflecting on the differences exhibited by different iron-built ships in the change of their subpermanent magnetism, it has often occurred to me as a subject worthy of experimental investigation, whether a portion of this difference might not depend on the temperature at which the plates of iron are passed through the rollers in the last stage of their manufacture. No favourable opportunity of making these experiments presented itself until, in the course of the last winter, I became aware that Mr. FAIRBAIRN had been engaged in experiments on the difference of the strength of plates of malleable iron, according as they had been rolled at a high or at a low temperature. I immediately requested Mr. FAIRBAIRN'S kind offices for procuring for me bars adapted to magnetic experiment, divided into the four classes of—1. Hot-Rolled, with the length of the bars parallel to the direction in which the rolling had lengthened the iron, or parallel to the direction of fibre; 2. Hot-Rolled, with the length of the bars transverse to the direction of fibre; 3. Cold-Rolled, with the length of the bars parallel to the direction of fibre; 4. Cold-Rolled, with the length of the bars transverse to the direction of fibre (which classes will hereafter be described by the words, 1. Hot-Rolled Longitudinal; 2. Hot-Rolled Transversal; 3. Cold-Rolled Longitudinal; 4. Cold-Rolled Transversal). Upon Mr. FAIRBAIRN'S application, the bars which I requested were promptly and gratuitously furnished by RICHARD SMITH, Esq., Superintendent of LORD DUDLEY'S Iron Works at the Round Oak Works near Dudley.

The number of bars was 24, namely, 6 in each of the four classes above described. Each bar was 16 inches long, 4 inches broad, and about $\frac{1}{4}$ inch thick: the aggregate weight of the bars in each class was,—1st, 28 lbs. 8 oz.; 2nd, 28 lbs. 10 oz.; 3rd, 27 lbs. 10 oz.; 4th, 27 lbs. 8 oz. The manufacture of the bars is described to me in substance as follows:—The hot-rolled and cold-rolled bars were all manufactured in the same way up to the stage of producing sheets of iron of the desired thickness; the last rollings having commenced with large bars at a welding heat, and having terminated with the bars (now converted into sheets) at a dull red heat. Then the sheets to be cold-rolled were allowed to cool to a perfectly cold state, and in that state were rolled afresh between other rollers. After this, the experimental bars were cut out of the sheets. Each set of six bars was packed in one box, with the maker's inscription on every bar reading forward in the same direction in all.

The bars when received by me, after resting some days in a room, were all placed upright in the same direction relatively to the direction of the maker's inscription, and the distinctive number of the bar was painted on the upper end of each; the end on which it was painted being that which is called "the Lettered End." The bars rested thus for several days, with the lettered end upwards.

The following apparatus was prepared for the experiments:—A wooden frame was constructed about 11½ feet in length; and this length, in the use of the frame, was placed very approximately in the direction of magnetic E. and W. Its ridge was 21 feet N. of the old front, or 13 feet N. of the new front, of the anteroom of the Magnetic Observatory. The upper and essential part of the frame consisted of two planes, each about 2 feet broad; of which one was very approximately in the position transverse to the direction of dip at Greenwich; and the other, at right angles to the former, included in its plane the direction of dip. These are called "Equatorial Plane" and "Dip Plane" respectively. Ledges of wood were attached to the planes, for the support of flagstones resting on the planes and lying parallel to them. And whether the wooden surface or the stone surface was employed, a frame of laths was placed upon it, which retained the length of each bar in the position nearest to the vertical, and prevented one bar from touching another. In this position the bars were struck with a hammer. If they were upon the equatorial plane, any induced magnetism was instantly struck out of them; if upon the dip plane, they became powerfully magnetized, the lower end having the same properties as the marked end of a compass-needle, or being charged with austral magnetism.

For testing the magnetism, a vertical wooden rod was provided, carrying two horizontal planes or stages, also of wood. The upper stage supported a prismatic Kater's compass, with which a well-defined mark (the ball-mast) was viewed, and its apparent azimuth was read. The lower stage supported the bar under experiment, which was placed in a horizontal position below the compass, at the distance of 5 inches, as nearly as could be conveniently measured, between the centre of the compass-needle and the centre of the bar. The reading of the compass-card under view increases as the card turns in the direction opposite to that of the sun's diurnal motion. From this it will appear that when the end of a bar, which has been downwards on the dip surface, is placed eastward on the lower plane, the austral magnetism of that end will attract the needle's south end towards the east, and will cause the card-reading to increase.

The bars were numerated as follows:—

(Hot-Rolled Longitudinal) Nos. 1, 2, 3, 4, 5, 6 (painted in white);
(Hot-Rolled Transversal) Nos. 7, 8, 9, 10, 11, 12 (painted in black);
(Cold-Rolled Longitudinal) Nos. 13, 14, 15, 16, 17, 18 (painted in red);
(Cold-Rolled Transversal) Nos. 19, 20, 21, 22, 23, 24 (painted in blue).

In the conduct of the experiments, the following rules were uniformly followed:—

The bars of the different classes were systematically intermixed, the same order being preserved in the whole series of experiments. Thus the order in which they were placed upon either plane (equatorial or dip), and the order in which their effects in disturbing the compass were observed, was always the following:—

Order of position, or experiment.	No. of bar.	Order of position, or experiment.	No. of bar.	Order of position, or experiment.	No. of bar.
1	1	9	3	17	5
2	7	10	9	18	11
3	13	11	15	19	17
4	19	12	21	20	23
5	2	13	4	21	6
6	8	14	10	22	12
7	14	15	16	23	18
8	20	16	22	24	24

The bars, when on the equatorial plane, were struck in the order 1 to 24 of the "Order of Position, or Experiment;" when on the dip plane, they were struck in the opposite order. They were always placed on the experimental stage under the compass in the order 1 to 24 of the "Order of Position, or Experiment." A printed skeleton form had been prepared, and the entries in it of the compass-readings were made without the slightest risk of confusion. The lettered end of the bar was always in the first instance placed towards the east; and, as soon as the compass-card was read, the bar was reversed in length, and its lettered end placed towards the west, and the compass-card was again read. The difference between these two readings (which has been carefully verified by examination, and by collateral formation and difference of sums of readings) is the number recorded in this paper. The zero, or card-reading when no bar was near, was usually taken at the beginning and at the end of each experiment, merely to enable me to detect erroneous readings, but no further use was made of it.

In two instances in Experiment 13, the presumption of error of 5° (undetected at the proper moment) was so strong, and the uncertainty as to the side on which the error was made was so great, that I judged it best to omit them, and to multiply the result for four bars by $\frac{3}{2}$, in order to produce the result for six bars.

The "Diff. readings" in the following Tables denotes the excess, with its proper algebraical sign, of the compass-reading when the lettered end of the bar is East above the compass-reading when the lettered end is West, and is taken as measure of the intensity of magnetism of the bar. When the bars have been struck with the lettered end downwards, the "Diff. readings" is positive.

Experiment 1, 1862, February 6, 0^h.

The bars were brought from the painting-room, and were immediately tried.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
	° ′		° ′		° ′		° ′
1	+18 5	7	+ 5 15	13	+ 1 0	19	+25 20
2	+13 30	8	+ 7 40	14	+ 8 0	20	+14 5
3	+14 30	9	+ 7 45	15	+ 7 20	21	+18 20
4	+ 7 0	10	+ 7 30	16	+ 1 10	22	+12 50
5	+11 20	11	+ 6 50	17	+20 30	23	+ 6 20
6	+12 10	12	+ 3 40	18	+ 7 0	24	+17 30
Sum...	+76 35	Sum...	+38 40	Sum...	+45 0	Sum...	+94 25

Total sum . . . +254° 40′.

The sign of the magnetism in this experiment is opposite to that which would have been given by terrestrial induction upon the bars in the position in which they had been standing for many days. It appears therefore that the magnetism had been produced by some circumstance in the manufacture, and that the terrestrial action upon the bars in a quiescent state had not reversed or destroyed it.

The bars were deposited on the flagstones of the equatorial plane, lettered end upwards.

Experiment 2, February 7, 0^h.

The bars, without being struck or subjected to any other violence, were tried.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
	° ′		° ′		° ′		° ′
1	+18 0	7	+ 3 0	13	− 2 10	19	+25 0
2	+13 45	8	+ 6 45	14	+ 6 30	20	+13 20
3	+12 20	9	+ 6 0	15	+ 5 25	21	+18 10
4	+ 5 0	10	+ 4 10	16	+ 0 20	22	+11 50
5	+11 35	11	+ 4 25	17	+19 40	23	+ 5 30
6	+10 30	12	+ 3 10	18	+ 7 15	24	+17 40
Sum...	+71 10	Sum...	+27 30	Sum...	+37 0	Sum...	+91 30

Total sum . . . +227° 10′.

The magnetism, it appears, had very slightly diminished. The bars were returned to the equatorial plane.

Experiment 3, February 7, 1^h.

Each bar as it lay on the flagstones of the equatorial plane, lettered end upwards, was struck with three rather heavy blows, from the unlettered to the lettered end. The hammer used was a geological hammer, weighing (with handle) about $2\frac{3}{4}$ lbs., and it fell in the blows about 2 feet. The bars were immediately examined.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
	° ′		° ′		° ′		° ′
1	−2 10	7	− 2 20	13	−4 35	19	+2 40
2	+0 55	8	− 2 40	14	+0 40	20	−1 30
3	+0 50	9	− 2 0	15	−7 45	21	+3 50
4	−4 40	10	− 1 20	16	−2 15	22	+0 50
5	−1 20	11	− 7 20	17	+0 10	23	−3 50
6	−2 10	12	− 6 0	18	+6 25	24	+7 0
Sum...	−8 35	Sum...	−21 40	Sum...	−7 20	Sum...	+9 0

Total sum . . . −28° 35′.

Upon the whole, the antecedent magnetism is destroyed, and a small amount of magnetism of the opposite kind is left, but it is distributed very capriciously among the bars. It seems to have no relation to the former magnetism.

The bars were immediately placed on the flagstones of the dip plane, with lettered end upwards, and were immediately struck with the same hammer and in the same manner as above, but were not examined on this day.

Experiment 4, February 7, 22^h.

The bars had been resting on the flagstones of the dip plane since Feb. 7, 1^h, with lettered end upwards, and without further disturbance were taken from it for examination.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	− 22° 10′	7	− 21° 0′	13	− 27° 0′	19	− 18° 40′
2	− 19 50	8	− 19 10	14	− 23 40	20	− 25 40
3	− 18 0	9	− 17 30	15	− 30 30	21	− 14 30
4	− 21 20	10	− 23 20	16	− 29 40	22	− 20 50
5	− 21 20	11	− 20 40	17	− 25 15	23	− 26 10
6	− 19 40	12	− 20 30	18	− 25 20	24	− 30 0
Sum...	−122 20	Sum...	−122 10	Sum...	−161 25	Sum...	−135 50

Total sum . . . −541° 45′.

The bars were immediately returned to the flagstones of the dip plane, in the same position as before, with lettered end upwards, and were immediately struck for the next experiment.

Experiment 5, February 7, 23^h.

The bars on the flagstones of the dip plane, with lettered end upwards, were struck each with three blows, with the same hammer and with the same force as in Experiment 5.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	− 21° 20′	7	− 19° 10′	13	− 25° 5′	19	− 20° 0′
2	− 20 30	8	− 18 40	14	− 23 40	20	− 25 10
3	− 16 40	9	− 18 10	15	− 28 40	21	− 17 25
4	− 19 50	10	− 17 10	16	− 29 0	22	− 20 27
5	− 16 40	11	− 15 10	17	− 23 10	23	− 27 40
6	− 20 20	12	− 20 40	18	− 25 20	24	− 31 0
Sum...	−115 20	Sum...	−109 0	Sum...	−154 55	Sum...	−141 42

Total sum . . . −520° 57′.

The magnetism of the bars is, upon the whole, slightly diminished. The bars were placed on the flagstones of the dip plane, with lettered end downwards, and were left undisturbed in that state.

Experiment 6, February 10, 23^h.

The bars, which had been resting undisturbed on the dip plane with lettered end downwards for three days, were taken out for examination without striking or disturbance.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	−16° 10′	7	−12° 35′	13	− 17° 50′	19	− 17° 40′
2	−14 20	8	−14 10	14	− 21 10	20	− 23 10
3	−15 0	9	−14 40	15	− 24 20	21	− 14 30
4	− 8 40	10	−13 40	16	− 23 10	22	− 17 30
5	−15 30	11	−12 30	17	− 19 20	23	− 23 35
6	−14 20	12	−16 0	18	− 20 30	24	− 23 0
Sum...	−84 0	Sum...	−83 35	Sum...	−126 20	Sum...	−119 25

Total sum . . . −413° 20′.

About $\frac{1}{5}$th part of the magnetism has been destroyed by three days' exposure to antagonistic terrestrial magnetism.

The bars were returned to the dip-plane flagstones in the same position, with lettered end downwards.

Experiment 7, February 11, 0^h.

The bars, immediately after the last examination, were returned to the flagstone dip plane with lettered end downwards, and each bar was struck once lightly at its centre with a joiner's hammer, weighing with its handle about $\frac{1}{2}$ lb., and were immediately examined.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+12° 10′	7	+16° 10′	13	+ 9° 0′	19	+17° 0′
2	+11 40	8	+15 30	14	+15 0	20	+11 20
3	+15 20	9	+12 0	15	+13 10	21	+16 30
4	+ 8 0	10	+11 0	16	+ 8 20	22	+ 9 40
5	+12 10	11	+10 15	17	+16 10	23	+ 8 40
6	+12 40	12	+10 40	18	+15 30	24	+ 5 40
Sum...	+72 0	Sum...	+75 35	Sum...	+77 10	Sum...	+68 50

Total sum . . . +293° 35′.

The light blow has destroyed the former magnetism, and has given an opposite magnetism of nearly three-fourths its amount.

The bars were returned to the flagstones of the dip plane in the same position, with lettered end downwards.

Experiment 8, February 12, 0^h.

The bars had not been disturbed in any way since the last examination. They had been resting on the flagstones of the dip plane, with lettered end downwards.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+14° 30′	7	+17° 10′	13	+ 7° 55′	19	+18° 30′
2	+11 25	8	+10 20	14	+16 0	20	+ 9 45
3	+11 55	9	+ 8 50	15	+12 10	21	+16 15
4	+ 8 20	10	+ 8 50	16	+ 9 40	22	+10 10
5	+11 0	11	+10 30	17	+13 30	23	+ 8 30
6	+11 10	12	+ 9 5	18	+13 15	24	+ 5 30
Sum...	+68 20	Sum...	+64 45	Sum...	+72 30	Sum...	+66 40

Total sum . . . +272° 15′.

It is remarkable that the magnetism, though favoured by the inductive force of terrestrial magnetism, has in this day's rest somewhat diminished.

The bars were returned to the flagstones of the dip plane, with lettered end downwards.

Experiment 9, February 12, 1^h.

The bars on the flagstones of the dip plane, with lettered end downwards, were struck each three times (from top to bottom) with the light hammer and with light blows, and were immediately examined.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+15° 55′	7	+16° 0′	13	+14° 10′	19	+18° 15′
2	+11 5	8	+14 10	14	+17 0	20	+14 30
3	+12 55	9	+12 20	15	+14 10	21	+18 50
4	+11 15	10	+13 15	16	+14 15	22	+18 5
5	+13 5	11	+12 55	17	+15 50	23	+10 25
6	+12 30	12	+12 40	18	+16 10	24	+11 0
Sum...	+76 45	Sum...	+81 20	Sum...	+91 35	Sum...	+91 5

Total sum . . . +340° 45′.

The magnetism is not much increased.

The bars were placed on the equatorial plane flagstones.

Experiment 10, February 15, 1^h.

The bars had been lying undisturbed on the equatorial plane, lettered end downwards.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+11° 20′	7	+12° 0′	13	+11° 30′	19	+15° 45′
2	+11 25	8	+ 9 55	14	+13 5	20	+12 50
3	+10 20	9	+ 9 45	15	+ 7 55	21	+17 5
4	+ 6 40	10	+10 30	16	+ 7 5	22	+16 55
5	+ 9 40	11	+10 35	17	+10 25	23	+ 6 55
6	+ 7 30	12	+ 9 50	18	+ 9 20	24	+ 9 5
Sum...	+56 55	Sum...	+62 35	Sum...	+59 20	Sum...	+78 35

Total sum . . . +257° 25′.

No specific cause is known for this decided decrease of magnetism.

The bars were returned to the flagstone equatorial plane.

Experiment 11, February 15, 2^h.

The bars, lying on the flagstones of equatorial plane, with lettered end downwards, were struck each in the middle of its length, with a single blow of the light hammer. The hammer was raised about 18 inches.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+1° 20′	7	+0° 35′	13	− 4° 20′	19	+3° 25′
2	−0 20	8	−0 25	14	− 1 10	20	−1 40
3	−0 35	9	−0 55	15	− 5 55	21	+3 50
4	−3 0	10	−0 55	16	− 5 5	22	+0 15
5	+0 30	11	−0 20	17	− 2 35	23	−5 30
6	−1 20	12	−1 35	18	− 2 25	24	−5 30
Sum...	−3 25	Sum...	−3 35	Sum...	−21 30	Sum...	−5 10

Total sum . . . −33° 40′.

The bars were returned to the flagstones of equatorial plane, lettered end downwards.

Experiment 12, February 20, 23^h.

The flagstones had been removed, and the bars were lying on the wooden boards of the equatorial plane, with the lettered end upwards. Each bar was slightly rapped three times with a wooden mallet, weighing about $1\frac{7}{8}$ lb.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+1° 0′	7	−1° 20′	13	−4° 50′	19	+0° 10′
2	−1 50	8	−2 30	14	−1 30	20	−1 55
3	−1 5	9	−2 20	15	−7 10	21	+3 5
4	−4 5	10	−2 40	16	−6 20	22	−0 35
5	−1 30	11	−1 50	17	−3 30	23	−6 10
6	−3 20	12	−2 40	18	−4 25	24	−3 25
Sum...	−10 50	Sum...	−13 20	Sum...	−27 45	Sum...	−8 50

Total sum . . . −60° 45′.

It would appear that, after a certain diminution, the shocks on the equatorial plane have no tendency to reduce the magnetism further.

The bars were placed on the wood of the dip plane, with lettered end upwards.

Experiment 13, February 21, 0^h.

The bars upon the boards of the dip plane, with lettered end upwards, were fairly struck each three times with the mallet.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	−18° 20′	7	....	13	−19° 25′	19	−15° 5′
2	−15 30	8	−14 20	14	−19 10	20	−20 5
3	−17 50	9	−15 40	15	−23 40	21	−10 25
4	−17 55	10	....	16	−24 20	22	−14 0
5	−15 50	11	−14 0	17	−19 55	23	−20 5
6	−14 25	12	−14 10	18	−20 0	24	−21 20
Sum...	−99 50	Sum...	−87 15	Sum...	−126 30	Sum...	−101 0

Total sum . . . −414° 35′.

The readings for bars 7 and 10 appeared suspicious, and have been omitted, the "sum" for the second series being formed by multiplying the sum for bars 8, 9, 11, 12 by $\frac{6}{4}$. It scarcely differs from that which would have been given by the use of the recorded numbers.

The bars were returned to the wood of the dip plane, with lettered end downwards.

Experiment 14, February 21, 1h.

The bars upon the wood surface of the dip plane, with lettered end downwards, were struck each with one light blow of the wooden mallet.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+ 8° 5′	7	+ 7° 15′	13	+ 1° 20′	19	+ 4° 10′
2	+ 6 50	8	+ 3 35	14	+ 6 55	20	— 0 10
3	+ 5 20	9	+ 5 10	15	+ 3 45	21	+ 6 55
4	+ 5 20	10	+ 5 20	16	— 0 50	22	— 1 50
5	+ 8 20	11	+ 7 55	17	+ 8 50	23	+ 0 40
6	+ 8 25	12	+ 6 35	18	+ 9 55	24	+ 3 15
Sum...	+42 20	Sum...	+35 50	Sum...	+29 55	Sum...	+13 0

Total sum . . . +121° 5′.

The bars were placed on the wood of the dip plane, with lettered end upwards, and were struck each three times with the wooden mallet, and so left. The mallet was raised about a foot for each blow.

Experiment 15, February 22, 0h.

The bars were not disturbed before examination.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	— 19° 10′	7	— 16° 40′	13	— 23° 15′	19	— 18° 30′
2	— 18 50	8	— 18 50	14	— 24 5	20	— 21 10
3	— 17 50	9	— 18 15	15	— 26 55	21	— 13 55
4	— 18 30	10	— 19 35	16	— 24 10	22	— 17 55
5	— 17 35	11	— 15 50	17	— 22 40	23	— 24 45
6	— 15 45	12	— 18 25	18	— 23 35	24	— 26 45
Sum...	—107 40	Sum...	—107 35	Sum...	—144 40	Sum...	—123 0

Total sum . . . —482° 55′.

The bars were placed on the wood of the equatorial plane, lettered end downwards.

Experiment 16, March 24, 0h.

The bars had been lying 30 days undisturbed on the equatorial plane.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	—16° 50′	7	—14° 50′	13	— 20° 15′	19	— 15° 5′
2	—14 10	8	—14 40	14	— 19 30	20	— 18 55
3	—14 20	9	—12 45	15	— 20 50	21	— 10 45
4	—14 15	10	—13 30	16	— 18 20	22	— 13 55
5	—13 40	11	—11 50	17	— 18 55	23	— 21 10
6	—12 15	12	—13 45	18	— 19 20	24	— 22 35
Sum...	—85 30	Sum...	—81 20	Sum...	—117 10	Sum...	—102 25

Total sum . . . —386° 25′.

Experiment 17, March 24, 1^h.

Immediately after the last examination, the bars were placed on the wood of the equatorial plane, lettered end downwards, and were struck with the wooden mallet very lightly, the mallet being lifted about 2 inches, each bar three times. As soon as the series of blows was finished, the bars were struck again in the same manner. They were then immediately examined.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
	° ′		° ′		° ′		° ′
1	−11 5	7	−11 45	13	−17 40	19	−11 20
2	−10 15	8	−11 10	14	−15 5	20	−14 15
3	− 7 5	9	−10 15	15	−13 30	21	− 7 55
4	−10 30	10	− 9 45	16	−11 55	22	−11 45
5	−11 5	11	− 9 50	17	−12 50	23	−19 0
6	− 9 10	12	−11 30	18	−15 50	24	−20 50
Sum...	−59 10	Sum...	−64 15	Sum...	−86 50	Sum...	−85 5

Total sum . . . −295° 20′.

The bars were placed on the wood of the equatorial plane, lettered end upwards.

Experiment 18, March 24, 2^h.

The bars on the wood of the equatorial plane, lettered end upwards, were struck with the mallet, each bar three times, the mallet being raised more than 1 foot; and the series of blows was then repeated. The bars were examined immediately.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
	° ′		° ′		° ′		° ′
1	− 1 45	7	− 3 30	13	− 1 5	19	+ 1 30
2	− 2 0	8	− 1 5	14	+ 0 50	20	− 1 5
3	− 2 5	9	− 0 35	15	− 3 10	21	+ 3 45
4	− 1 25	10	− 0 5	16	− 2 15	22	+ 1 20
5	− 1 10	11	− 0 5	17	− 0 55	23	− 3 35
6	− 0 15	12	− 0 40	18	+ 0 10	24	− 0 15
Sum...	− 8 40	Sum...	− 6 0	Sum...	− 6 25	Sum...	+ 1 40

Total sum . . . −19° 25′.

The bars were returned to the wood of equatorial plane, lettered end upwards.

Experiment 19, March 25, 22^h.

The bars were shifted to the wood of dip plane, with lettered end downwards, and were struck heavily with the geological hammer swung out horizontally to arm's length, each bar three times. They were examined immediately.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
	° ′		° ′		° ′		° ′
1	+ 21 45	7	+ 19 40	13	+ 24 55	19	+ 28 10
2	+ 18 25	8	+ 19 0	14	+ 26 15	20	+ 24 55
3	+ 17 25	9	+ 20 25	15	+ 21 30	21	+ 27 5
4	+ 15 20	10	+ 15 30	16	+ 23 25	22	+ 26 25
5	+ 18 15	11	+ 16 35	17	+ 22 10	23	+ 20 15
6	+ 15 55	12	+ 15 50	18	+ 24 45	24	+ 29 10
Sum...	+107 5	Sum...	+107 0	Sum...	+143 0	Sum...	+156 0

Total sum . . . +513° 5′.

The bars were returned to the wood of dip plane, lettered end upwards.

Experiment 20, March 25, 23^h.

The bars, on wood of dip plane, lettered end upwards, were struck heavily with the geological hammer at arm's length, each three times. They were then immediately placed with lettered end downwards, and struck with blows of similar force, each three times. They were then examined immediately.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
	° ′		° ′		° ′		° ′
1	+ 21 20	7	+ 18 50	13	+ 25 55	19	+ 27 25
2	+ 17 30	8	+ 18 30	14	+ 24 35	20	+ 19 35
3	+ 16 15	9	+ 18 10	15	+ 24 10	21	+ 27 0
4	+ 17 0	10	+ 19 25	16	+ 24 55	22	+ 26 55
5	+ 20 0	11	+ 16 40	17	+ 25 0	23	+ 23 0
6	+ 17 35	12	+ 16 30	18	+ 26 30	24	+ 35 10
Sum...	+109 40	Sum...	+108 5	Sum...	+151 5	Sum...	+159 5

Total sum . . . +527° 55′.

It is worthy of remark that the diff. reading for the bar No. 20 (8th in the order of examination) at first appeared to be 24° 25′. After examining bar No. 5, bar No. 20 was again examined, and the diff. readings was found to be 19° 35′ as is given above. If the first reading was correct (which there is no special reason for doubting), it would appear that the bar lost one-fifth part of its magnetism in a few minutes.

The bars were returned to the dip plane, lettered end upwards. On March 27, 1h, they were moved to the equatorial plane, lettered end upwards.

Experiment 21, March 31, 1h.

The bars had been resting undisturbed on the equatorial plane since March 27.

Hot-rolled longitudinal.		Hot-rolled transversal.		Cold-rolled longitudinal.		Cold-rolled transversal.	
No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.	No.	Diff. readings.
1	+12° 40′	7	+12° 25′	13	+ 16° 5′	19	+ 23° 35′
2	+13 40	8	+15 10	14	+ 21 30	20	+ 19 30
3	+12 55	9	+16 15	15	+ 17 40	21	+ 21 40
4	+15 5	10	+14 15	16	+ 19 0	22	+ 22 50
5	+15 55	11	+15 35	17	+ 22 30	23	+ 22 10
6	+15 50	12	+15 5	18	+ 24 35	24	+ 32 5
Sum...	+86 5	Sum...	+88 45	Sum...	+121 20	Sum...	+141 50

Total sum . . . +438° 0′

The bar No. 20 does not appear to have undergone any further change in its magnetism.

After this, the apparatus was dismounted, and the bars were packed up in the same order as at first.

I shall now proceed to exhibit the principal results deducible from the details above.

For abbreviation, I shall sometimes use the expression "tendency 0" to denote that the operation on the bars has been performed while they were lying on the equatorial plane; "tendency +" to denote that the bars were lying on the dip plane, with lettered end downwards; "tendency —" to denote that they were on the dip plane with lettered end upwards.

I. Aggregate of magnetism in the twenty-four bars.

Exp. 1.	After standing several days, tendency —	+254° 40′
2.	After resting one day, tendency 0.	+227 10
3.	Struck heavily, on flagstones, large iron hammer, tendency 0 . .	— 28 35
4.	Struck heavily, tendency —; examined after a day's rest, with tendency —	—541 45
5.	(Immediately after Exp. 4), struck heavily, tendency — . . .	—520 57
6.	Not struck, but under + tendency for three days	—413 20
7.	(Immediately), struck lightly with light iron hammer, tendency +	+293 35
8.	After resting quietly one day, under + tendency.	+272 15
9.	(Immediately), struck fairly with light iron hammer, tendency +	+340 45
10.	After three days' rest, tendency 0	+257 25
11.	(Immediately), struck fairly with light iron hammer, tendency 0	— 33 40

From this time the bearing is on wood, and the blows are given with a wooden mallet, unless otherwise mentioned.

Exp. 12.	Lightly struck, tendency 0	$-\ 60^{\circ}\ 45'$
13.	Fairly struck, tendency —	$-414\ 35$
14.	Very lightly struck, tendency + ...	$+121\ \ 5$
15.	Fairly struck, tendency —, rested one day	$-482\ 55$
16.	After thirty days' rest, tendency 0	$-386\ 25$
17.	Struck very lightly, tendency 0	$-295\ 20$
18.	Struck fairly, tendency 0	$-\ 19\ 25$
19.	Struck heavily with geological hammer, tendency + ...	$+513\ \ 5$
20.	(Immediately), first struck heavily, tendency —; then struck heavily (immediately), tendency + ...	$+527\ 55$
21.	After one day's rest, tendency —, and four days' rest, tendency 0	$+438\ \ 0$

It appears that the greatest amount of magnetism is about $\pm 530^{\circ}$, and this requires a heavy blow; but it is nearly or quite indifferent whether the bars are supported on stone or on wood, and whether the blow is given by an iron hammer or a wooden mallet. This amount of magnetism, the bars lying with tendency 0, is diminished in one or two days by about one-fifth part, and is scarcely diminished further in thirty days. The loss of magnetism is not greater when the bars, instead of lying with tendency 0, are placed in a tendency opposite to the magnetism with which they are charged. When the charge of magnetism is smaller than the maximum, the diminution in a day or two is nearly in the same proportion.

It appears also that the effect of violence on the bars while lying with tendency 0 is not completely to destroy the magnetism; and that sometimes the magnetism is actually increased by the violence.

II. Proportions of the Aggregate of Magnetism carried by the different classes of Bars; excluding those experiments in which the Bars had been struck while lying with tendency 0.

		Hot-rolled longitudinal.	Hot-rolled transversal.	Cold-rolled longitudinal.	Cold-rolled transversal.
Experiment 4,	tendency —	·226	·225	·297	·250
5,	—	·222	·209	·298	·272
6,	—	·204	·203	·306	·289
7,	+	·245	·258	·263	·235
8,	+	·251	·238	·266	·245
9,	+	·225	·239	·269	·267
10,	+	·221	·244	·231	·306
13,	—	·241	·211	·306	·244
14,	+	·349	·295	·247	·107
15,	—	·223	·222	·299	·255
16,	—	·221	·211	·303	·266
17,	—	·201	·218	·293	·288
19,	+	·209	·208	·279	·305
20,	+	·207	·205	·287	·302
21,	+	·196	·202	·277	·323
Mean		·229	·226	·281	·264

These results appear to leave no doubt that, as a general rule, the cold-rolled iron will, under similar circumstances, take up a heavier charge of magnetism than the hot-rolled iron, in the proportion of 545 : 455, or 1·2 : 1·0, very nearly. It also appears that the cold-rolled longitudinal takes a heavier charge than the cold-rolled transversal.

The departures from this result, in Experiments 8, 9, 10, 14, are confined to small charges of magnetism, in which accident might have greater play. They are also confined to magnetism of + character; of this, perhaps, the following is the explanation.

The bars when received from the manufacturer had + magnetism (see Experiments 1 and 2). It would seem therefore that the hot-rolled bars differ from the cold-rolled bars in this particular, that the hot-rolled bars retain rather more of their primitive magnetism, under all changes, than the cold-rolled bars retain.

III. Proportions of the Aggregate of Magnetism, without regard of sign, carried by the different classes of Bars, when they have been struck while lying with tendency 0.

	Hot-rolled longitudinal.	Hot-rolled transversal.	Cold-rolled longitudinal.	Cold-rolled transversal.
Experiment 3.	·184	·464	·157	·193
11.	·102	·106	·638	·154
12.	·178	·220	·457	·145
18.	·380	·264	·282	·073
Mean	·211	·264	·384	·141

I do not attach great importance to the apparent tendency of the cold-rolled longitudinal bars to retain much magnetism and of the cold-rolled transversal bars to retain little. The observed deviations are small, and a trifling accidental error may materially disturb the results.

IV. Spontaneous Losses of Magnetism in the Bars of different Classes, when they have rested undisturbed.

	Hot-rolled longitudinal.	Hot-rolled transversal.	Cold-rolled longitudinal.	Cold-rolled transversal.	Sum.
Exp. 1 to 2, 1 day, tendency 0	5° 25′	11° 10′	8° 0′	2° 55′	27° 30′
5 to 6, 3 days, tendency +, opposing	31 20	25 25	28 35	22 17	107 37
7 to 8, 1 day, +, favouring	3 40	10 50	4 40	2 10	21 20
9 to 10, 3 days, 0	19 50	18 45	32 15	12 30	83 20
15 to 16, 30 days, 0	22 10	26 15	27 30	20 35	96 30
20 to 21, 1 day, −, opposing; 4 days, 0 ...	23 35	19 20	29 45	17 15	89 55
Sum...........................	106 0	111 45	130 45	77 42	426 12

V. Proportions of the Losses of Magnetism in the Bars of different Classes to the Aggregate of Losses.

	Hot-rolled longitudinal.	Hot-rolled transversal.	Cold-rolled longitudinal.	Cold-rolled transversal.
Experiment 1 to 2.	·197	·407	·291	·106
5 to 6.	·293	·236	·266	·207
7 to 8.	·172	·509	·218	·101
9 to 10.	·238	·225	·387	·150
15 to 16.	·230	·272	·285	·214
20 to 21.	·263	·215	·331	·192
Means	·232	·311	·296	·162

It appears, as I think (though the last line of Table IV. would lead to a different conclusion), that there is a real difference in the retentive powers for magnetism among the different classes of bars; and that the cold-rolled bars, under the circumstances of the experiments, lose less magnetism, spontaneously, than the hot-rolled bars lose. If, instead of comparing the absolute losses of magnetism, we had compared the proportions for loss of magnetism in each class to entire magnetism in that class, the difference would have been still more remarkable.

XVI. *On the Relations of the Vomer, Ethmoid, and Intermaxillary Bones.*

By JOHN CLELAND, *M.D., Demonstrator of Anatomy in the University of Edinburgh.*
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IN a paper read before the Royal Physical Society of Edinburgh, in March 1859*, I have drawn attention to the fact, not before noticed, that in Mammalia the lateral masses of the ethmoid always lie edge to edge with the dilated part of the vomer, and are usually united with it to form one bone. I have shown that in the human subject the sphenoidal spongy bones represent, in a disrupted and altered form, the plates by which in other mammals this union is effected; and these plates I have ventured to call the ethmovomerine laminæ.

A further inquiry into the variations of these bones and their development has not only confirmed my former statements, but led to other conclusions of most important morphological bearing.

It will be convenient to consider, in the first place, the typical connexions of the vomer in the mammal. We shall then examine the peculiar modifications which the vomer, ethmoid, and intermaxillaries undergo in the human subject. Afterwards we shall review in detail the varieties of their connexions in a number of mammalian families, and endeavour to deduce their morphological relations. Lastly, we shall glance at the arrangement of these bones in the other classes of Vertebrata, and draw such additional morphological conclusions as our observations shall seem to warrant.

Typical Connexions of the Mammalian Vomer.

In all young mammals the septal cartilage of the nose extends backwards to the pre-sphenoid bone, and the vomer embraces its inferior edge. The vomer is known to be developed from two centres of ossification, one on each side; it always consists therefore fundamentally of two alæ; and between these a groove runs along its whole length, and is occupied by the septal cartilage.

Anteriorly it articulates generally with the mesial-palatine processes of the intermaxillaries. Sometimes this articulation does not take place, to wit, where the intermaxillaries have either no mesial-palatine processes or very short ones; but, when it exists, it is always of such a character that the vomerine groove is continued forwards by a groove between the intermaxillaries, while also the inferior aspect of the mesial processes of the intermaxillaries lies in a continuous line with the inferior edge of the vomer.

* Edinburgh New Philosophical Journal, October 1859.

Since the vomerine groove extends from the presphenoid to the intermaxillaries, it is obvious that the primary laminæ of the vomer (its alæ) can have no contact with the bones forming the back part of the palate. The articulation which the vomer forms usually with the maxillaries, and often with the palatals, is brought about by the production, downwards and backwards from the line of junction of the alæ, of a mesial plate, which is an after-development, and of varying extent in different species.

Between the vomer and the central plate of the ethmoid the only connexion that ever occurs is of a very secondary description. No doubt the appearances in human skulls are often such as would incline one to a different conclusion; but in the skulls of other animals there is, even in the adult condition, in the majority of cases, no contact between the two bones at all; and when there is, it is a contact of contiguity, not of continuity. The central plate of the ethmoid grows from above downwards at the expense of the septal cartilage, and is an ossification of that structure; while, on the other hand, the vomer only embraces the cartilage, never invades it.

But the most intimate and constant connexion of the vomer is with the lateral masses of the ethmoid; although no one who had confined his attention to the arrangement in adult human skulls would think that these bones were at all related. This connexion can be very well seen in the Dog, the Cat, the Sheep, or the Pig. In all these animals, and indeed, as far as I am aware, in all mammalia except Man, the Orang, the Horse, the Elephant, and the Giraffe (keeping the Cetacea out of view at present), the vomer and lateral masses of the ethmoid form one continuous bone. The free superior edge of each ala of the vomer can be traced all the way back in contact with the mesial structures embraced; and, forming a sharp angle with this edge when examined from above, but appearing to be gradually continuous with the ala when looked at from beneath, a very thin but often broad plate of bone passes outwards underneath the ethmoidal convolutions to the external inferior angle of the lateral mass of the ethmoid, and is continuous with the principal lamina which forms the framework of that bone. This plate is what I have called the ethmovomerine lamina. The vomer frequently extends some distance backwards beyond its junction with the ethmovomerine laminæ; but the position of the latter is so far constant that their posterior edges are always in contact with the sphenoidal processes of the palate-bones. The place of junction of the ethmovomerine lamina and ethmoid always corresponds to the upper margin of the internal aspect of the foramen called in human anatomy sphenopalatine, but which has only a seeming relation to the sphenoid bone in the human subject, and none whatever in other mammals, and which it will be better therefore in this paper to call the nasal foramen of the palate-bone.

The Vomer and Sphenoidal Spongy Bones in Man.

If we now proceed to examine the development of the vomer and the bones connected with it in the human subject, we shall see how beautifully the adult appearances are derived from conditions exactly corresponding to those just described. The key to these appearances is to be found principally in the sphenoidal spongy bones. These curious little bones reach their maximum of development at a very early age. Of four specimens beside me illustrating this condition, No. 1 (Plate IV. fig. 1) is from an infant possibly about a year old; No. 2 (fig. 2) from a child two or three years old; No. 3 (fig. 3) from a child which, judging from the appearances of the rest of the skull, may have been four years old; and No. 4 from an infant a few months old. In all these cases the body of the presphenoid is distinguishable from the postsphenoid by a mark at the line of junction, and projects forwards in continuation of it, in a somewhat rounded form, presenting little of that compression which has taken place to so great an extent in the adult. The presphenoid at this date is quite unconnected with the sphenoidal sinuses, and consists simply of the body just described, and the wings of INGRASSIAS. The body dips between the sphenoidal spongy bones, but is separated from them by some thickness of tissue. In Nos. 1 and 2, the sphenoidal spongy bones are already adherent to the ethmoid; in 3 and 4 they are unattached either to the sphenoid or ethmoid. They are of the shape of hollow pyramids with their apices directed backwards, and their inner aspects parallel to their fellows. Their cavities (the commencing sphenoidal sinuses) open at their bases in front into the nasal fossæ.

There are, I believe, three distinct ossifications which go to the formation, normally, of each sphenoidal spongy bone: certainly there are three portions which require description. A superior plate, very distinctly marked off in the specimens alluded to, limits the sphenoidal sinus more or less perfectly on the superior and internal aspects; it forms a convexity directed upwards and inwards, and lies in contact with the fibrous tissue which separates it from the body of the sphenoid. An inferior plate forms the whole or the greater part of the floor of the sphenoidal sinus, and may be said to be the only recognizable part in the adult. Internally this plate comes in contact with the lower margin of the superior plate, beneath which it is prolonged downwards so as to lie edge to edge with the corresponding lamina of the vomer, immediately in front of the thick dilated part. Posteriorly it lies above the sphenoidal process of the palate-bone. Besides these two plates, there enters properly into the formation of the sphenoidal spongy bone a third element (an orbital plate) connected with the anterior and external part of the inferior plate, and whose characteristic property is that by an aspect which articulates behind with the sphenoid, in front with the ethmoid, inferiorly with the palatals, and sometimes above with the frontal, it takes a small part in the formation of the orbit. It may be attached to the ethmoid, palatal, or sphenoid bone, instead of to the remainder of the sphenoidal spongy bone. In specimen No. 3 it is certainly unusually large and distinct, but it can be easily distinguished in Nos. 1 and 2 also. It is absent in No. 4; but there it exists incorporated with the orbital process of the palate-bone, at least that process projects upwards between the sphenoid and ethmoid. This enlarged form of the orbital process of the palate-bone is not at all uncommon; and that it arises from union with it of the orbital element of the sphenoidal spongy bone I am satisfied, from its occupying the position of that element. Moreover, in an adult specimen, in which I carefully disarticulated a palate-bone with an orbital process of that description, I found that that process had the whole sphenoidal spongy bone in union with it*.

It will be seen from this description that the sphenoidal spongy bone forms an arch of communication between the ethmoid and exactly that part of the vomer from which in other mammals the ethmovomerine lamina springs. Also it is it, and not the sphenoid, which completes with the palate-bone the sphenopalatine foramen; thus it agrees entirely with the ethmovomerine lamina in its relations. If there be yet any link wanting to prove its identity with that lamina, we shall find it supplied when we examine the skull of the young Orang.

In a fœtus of the third or fourth month, the vomer consists merely of two alæ which meet beneath the septal cartilage and form one bone. The inferior edge of the scooped bone so formed presents, in a fœtus of the fifth or sixth month, a broad surface marked by a raphe in the middle line for articulation with the maxillaries strictly so called, *i. e.* the part behind the anterior palatine foramen (fig. 4, B). This surface narrows behind into a mere edge to articulate with the palatals; while in front it ceases abruptly, and only the lamina bounding the groove is prolonged on the intermaxillary part of the palate. Already the intermaxillaries have begun to be elevated in the middle line above the level of the maxillaries, so as to form the crista incisiva, consisting of a process on each bone ("semicrista incisiva" of Henle), and a groove between the two on such a level as to continue the groove of the vomer, which it does throughout life.

In the skulls of young subjects (for example in the specimen referred to as No. 1 (fig. 1, B), the vomer is seen in its most characteristic form. The alæ have reached their maximum of development. At their posterior extremity is seen the thick dilated part which passes back beneath the sphenoid, and lies between the upper extremities of the pterygoids, which are sometimes called *vaginal processes*, but improperly, since they only lie edge to edge with the vomer and never sheath it. In front of the dilatation the margins of the alæ continue for some distance to ascend as they pass forwards, and it is at this part, as far forwards as the points at which they begin to slope downwards, that the sphenoidal spongy bones are in contact with them. Also, by the elongation of the face, the vomerine groove has become sloped, while the extremity which rests on the crista incisiva is, by the increased elevation of that process, raised above the level of the maxillaries; and the space thus left between the groove and the hard palate is spanned by a mesial plate, about one line deep in front and three behind. As the face developes further this mesial plate becomes deeper, and sometimes there is a process continuous with the anterior perpendicular edge of the mesial plate, sent downwards between the incisive foramina, behind the crista incisiva (fig. 4, A). This downward process is occasionally a separate bone, and is so in a specimen of mine taken from a subject

* Henle mentions that the orbital process of the palate-bone "aids in closing the sphenoidal sinus when its wall is imperfect." The condition which he refers to is no doubt that mentioned in the text. See Henle, Handbuch der System. Anat. des Menschen, i. 174.

possibly about ten or twelve years old. But apparently the most frequent arrangement in the adult is that the crista incisiva is prolonged backwards so as to articulate with the maxillaries behind the anterior palatine canal; and in that case, as the maxillary margin of the vomer falls short of the canal, the process in question does not exist.

At an early age the vomer begins to lose its characteristic form and its individuality; and the sphenoidal spongy bones lose theirs even sooner. In consequence of this, the ordinary descriptions, however faithfully they may give an account of the appearances in adult skulls, are imperfect as descriptions of these separate bones. The fact is that the maturity of a skull does not correspond with the period of most perfect development of the individual bones of which it is composed. This is recognized by us when we resort to young skulls for specimens of disarticulated bones, and would be acknowledged still more readily in describing an elephant's skull or a whale's; but, with regard to the bones in question, the period of this most characteristic condition is so early that it has escaped due attention, and the natural result has followed, that the adult appearances have not been properly understood.

The alterations which the sphenoidal spongy bones undergo take place in connexion with the increased hollowing out of the sphenoidal sinuses. As these sinuses dilate, the superior plates of the spongy bones are pressed against the body of the presphenoid, which also becomes narrowed and deepened, until, instead of a thick bone in the middle line and a lamina supporting it on each side, we meet with only one central lamina between two large sinuses, and that deviating often a great distance to one side. The processes which projected downwards to the vomer disappear by the dilatation of the sinuses, until there is nothing left of them but a slight ridge corresponding to the edge of the vomer; inside which, what was once a perpendicular lamina forms part of the convex floor of the sinus, and on reaching the middle line is incorporated with the body of the presphenoid to form the septum sphenoidale. Thus the sphenoidal spongy bones become blended with the sphenoid, while at the same time their appearance of continuity with the vomer is gradually effaced; so much so, that the latter is habitually and correctly described as resting, in the adult, against their under surface. In the Anatomical Museum of this University there is a skull, seemingly from a subject about twelve years of age, in which the descending processes of the sphenoidal spongy bones have at the fore part not yet begun to be rounded away, but articulate accurately edge to edge with the vomer.

The sphenoidal and other sinuses no doubt have the important function of serving as caves for the reverberation of the voice, but they are likewise useful in giving lightness to the bones they occupy; and though in old persons they cease to communicate with the nose, they nevertheless grow larger and larger as age advances and the bones grow denser, until, as in a specimen beside me, the antrum of HIGHMORE invades the malar bones, and the sphenoidal sinuses so hollow-out the sphenoid that the walls of the Vidian canals are laid bare and traverse them like tubes. In these aged subjects the septum sphenoidale is a very thin partition indeed, and is usually very much driven to one side.

The alterations which take place with growth in the form of the vomer are results of the peculiar conformation of the human face, which is developed with a view to speech and expression, and not, as in the lower animals, for the mere prehension of food. The more the maxilla projects the wider is the gape allowed; but in Man, especially in the civilized races, instead of there being any such projection, the palate is short and the incisors perpendicular; and, as adult life approaches, the nasal fossæ and antra of HIGHMORE are enlarged to such an extent by downward growth of the maxillæ as to render the perpendicular height of the posterior nares quite a distinctive characteristic of the human skull. In consequence of this elongation of the face, the distance between the cribriform plate and the palate is rapidly increased. This distance the central plate of the ethmoid and the vomer must span; and as they stretch to accomplish this object, they become attenuated and most frequently lose their symmetry. Often indeed the first degeneration from its typical form which the vomer undergoes is a deviation of its laminæ from the middle line. This is found even in children.

As the central plate of the ethmoid grows downwards, its edge, which is advancing at the expense of the septal cartilage, corresponds with that structure in thickness. But only the edge is so thick; the rest of the plate gets thinner and thinner as it becomes elongated, and may even present irregular perforations. The vomer, on the other hand, embraces the cartilage from below. But when the central plate of the ethmoid, growing down from above, comes in contact with the alæ as they stretch upwards in the perichondrium, the ossifications from above and below unite, and if the middle line has been perfectly preserved, the cartilage between them is surrounded above, below, and on each side by bone. Very rarely this vestige of cartilage disappears entirely, and there is then a thin lamina of bone extending from the cribriform plate to the palate, and in it a diagonal of stronger bone stretching from the crista incisiva to the sphenoid. Generally, however, the deviation from the middle line is considerable, and the central plate of the ethmoid, though anchylosed with both alæ at the back part, is in front united more with one lamina than the other (usually most with that on the concave side of the septum), while the other tends to undergo atrophy. The cartilage between them is thus left more exposed on one side than the other, or sometimes is exposed on one side at one part, and on the other side further back. Occasionally, even in old subjects, a thread of cartilage can be traced back into one of the sphenoidal sinuses.

When the vomer is described as only grooved at its upper and back part, and exhibiting a cul-de-sac for the reception of the rostrum, the description is that of an irregular piece of bone consisting of the vomer and part of the ethmoid. The specimens chosen for study by the describers have been taken from skulls in which the vomer has already become thoroughly united to the central plate; and when it has been sought to separate it, the central plate has given way at its weakest part. In other instances the fracture is effected through the alæ of the vomer. It is utterly impossible to separate a vomer accurately from the central plate after the two bones have come into contact. The ridge which one so frequently meets with on one side of the vomer in specimens purchased with disarticulated skulls, is formed by an atrophied ala of the vomer, or by the edge of the central plate of the ethmoid projecting to one side, which it may do in an infinite variety of ways.

The same cause accounts for the imperfect description of the inferior margin of the vomer. In particular, the projection which lies on the crista incisiva is very liable to be destroyed by anchylosis; and even when it is not anchylosed it often requires very careful disarticulation to exhibit it. Sometimes the crista itself undergoes changes. It may have the margins of its groove worn away, and present a flat surface having the fore part of the vomer imbedded between its halves; or it may be completely bent over to one side, so that one semicrista shall lie above the other, while the fore part of the vomer, lying between them, is necessarily distorted, but may yet admit of disarticulation. At an early age the united margins of the maxillaries behind the anterior palatine canal become elevated into a sharp ridge, and the margins of the originally flat surface of the vomer in contact with them bend down to grasp it, and become irregular, thin away, and disappear more or less completely. Vestiges of them, as they die away, may sometimes be seen still very distinct on careful disarticulation, even in adult subjects. The edges of the palate-bones often form a more prominent ridge than the maxillaries, rising into a spine a couple of lines or more in height; and the portion of vomer in contact with this is a thin edge.

These remarks, I believe, are sufficient to show that if the human vomer is to be described with accuracy in its character as a separate bone, it must be studied as it shows itself in early life*.

If it be asked why, even in the early condition, when its form and connexions most resemble the typical arrangement, the human vomer should yet differ from the arrangement in other mammals by having the arch of communication between it and the ethmoid a separate bone, some explanation will be arrived at by considering that in Man the ethmoid is very feebly developed, and that the distance between the ethmoid and vomer is increased,—circumstances which depend on the feeble development of the sense of smell, and on the rapid curvature of the arch of the cranium. To the cursory glance the human ethmoid presents a rather well-developed appearance, since it forms a large part of the inner wall of the orbit, and the orbital plate is a peculiarity confined to Man and the Quadrumana. Yet in Man the ethmoid is rudimentary, and in Monkeys still more so; and the existence of the orbital plate is a mere consequence of the great curvature of the cranial arch. In the generality of mammals the ethmoid is shielded by the frontal and palatals; but in Man the palatals are removed from it by the downward development of the face; while by the great size of the anterior lobes of the brain it is pushed from under the shelter of the frontal, and the cribriform plate is depressed into the horizontal plane, instead of occupying, as in the Cat, the Sheep, &c., an almost vertical position. But whereas in most mammalia the lateral masses of the ethmoid

* Of this the first MONRO was well aware. See the Works of ALEXANDER MONRO, M.D., published by his son in 1781, p. 120.

are composed of large turbinations imbedded in a dense mass of small leaflets, and both leaflets and turbinations lie at right angles to the cribriform plate, in Man there are no such leaflets and only two stunted turbinated processes, which, instead of descending to fill the nasal cavity, immediately curve backwards so as to occupy as little room as possible. And not only is the ethmoid so rudimentary, but by the curvature of the cranial arch, the vomer is moved away from it; its anterior extremity sloping rapidly to a lower level, and its posterior being pushed backwards under the sphenoid. Of this latter fact we may be convinced by comparing the position of the human vomer with that of the same bone in the Cat. The Cat has got sphenoidal sinuses as well as Man, but they are floored-in entirely by projections downwards from the wings of INGRASSIAS; and the part of the vomer which is continued into the ethmovomerine laminæ lies, not underneath the sinuses, but underneath the ethmoid, immediately in front of the sinuses. Moreover, it is very important to keep in mind that the margin of the ethmoid which is placed posteriorly in the human skull is that which is morphologically inferior, and which corresponds to that to which the ethmovomerine lamina is habitually attached, and to which therefore it behoved that the human vomer should be connected. This will be best illustrated by using the vomer and ethmoid of the Sheep, Dog, or Rabbit for comparison. It will then be easily understood, on referring to Plate IV. fig. 5, how there comes to be a space left between the ethmoid and vomer, to bridge over which the large and peculiar development of ethmovomerine laminæ as sphenoidal spongy bones becomes necessary (fig. 5).

Varieties presented by the Vomer and Intermaxillaries in different Classes of Mammalia.

In the *Quadrumana* the vomer comes to a point anteriorly, which is so directed that the vomerine groove is continuous, as in other animals, with the upper surface of the intermaxillaries; but it never comes further forwards than just to touch them, and sometimes falls short of them by a slight interval. This is owing to a want of development of the mesial-palatine processes of the intermaxillaries, these bones having in the Quadrumana begun to suffer that atrophy which they undergo in the human subject; and as their bulk is occupied almost entirely by the sockets of the large incisor teeth, while, on the other hand, the cartilages of the nose are little developed, there is no appearance of the crista incisiva found in Man. The ethmoid and vomer are continuous, according to the normal plan, save only in the Orang; and we may therefore confine our attention to the arrangement in it and in the Chimpanzee. In two skulls of Chimpanzees in the University Museum, the posterior part of the vomer arches out on each side into a lamina which passes up in front of the sphenoid to be continuous with the orbital plate of the ethmoid, as in other Quadrumana; but both are skulls of animals of such an age that they present anchylosis of many sutures. In the Orang, however, there are certainly separate sphenoidal spongy bones. The examination of three young Orang skulls gives the following results. In the largest skull, the sphenoidal spongy bone is quite distinct on the right side, both from the orbital plate of the ethmoid and from the vomer; but on the left side their suture of connexion with the ethmoid has begun to disappear, and it scarcely comes into contact with the vomer. In the smallest skull, the sphenoidal spongy bones are perfectly free from the ethmoid on both sides, and also free from the vomer, though in contact with it; but while that on the left side is perfectly free from the sphenoid, the right one is anchylosed with the anterior inferior margin of that bone. In the remaining skull, the sphenoidal spongy bones are separate from the orbital plates of the ethmoid, but are anchylosed with the sphenoid both in front of the foramen opticum and below, also with the vomer, and with a turbination of the ethmoid. In all three cases the sphenoidal spongy bones take part in the formation of the orbit, while the palatal has no orbital plate.

The shape of the posterior extremity of the vomer affords us a very distinctive difference between the skulls of the Orang and Chimpanzee, which I believe has not hitherto been noticed. In the Orang it is expanded, flat, and with irregular edges. In the Chimpanzee it is thick, comparatively narrow, with straight edges, bifurcated, and fitted into a groove so as to leave a canal between it and the sphenoid. These characters are well marked even in the extremely young skull. In the Gorilla the posterior extremity of the vomer has expanded irregular edges as in the Orang, while it is bifurcated and a canal is left above it as in the Chimpanzee* (fig. 6).

In the *Bat*, the intermaxillaries are extremely small; they do not come together in the middle line, and have no mesial-palatine processes; they therefore have no connexion with the vomer.

In the *Carnivora*, the vomer and ethmoid are quite typical in their arrangement. In the Cats, the Dogs, and the Bears, the mesial plate of the vomer reaches a short way back upon the palatals. In the new-born Kitten the sphenoid, vomer, and intermaxillaries are seen in a very instructive condition. The body of the presphenoid is not yet compressed into a septum, and the two projections from the wings of INGRASSIAS, which afterwards meet beneath so as to enclose the sinuses, are as yet but little processes; the vomer passes forwards in a straight line from the front of the body of the presphenoid; and beyond the vomer this line is continued by the intermaxillaries, while the surface for articulation of the vomer with the maxillaries is as yet a mere knob. This arrangement is calculated strongly to suggest the idea of a series of centra (fig. 7, D).

* In the engraving of the base of the skull of a Chimpanzee in the Trans. Zool. Soc. vol. iii. pl. 60, illustrating Professor OWEN's paper, "Osteological Contributions to the Natural History of the Chimpanzee," the canal along the middle line of the sphenoid is represented as completed behind the vomer by junction of the roots of the pterygoid bones, or in other words, by the vaginal processes meeting together in the middle line. This completely conceals from view any parallel-bordered part of the vomer which may be supposed to rest between these and the body of the sphenoid. The skull represented is that of a male. The representation of the vomer of the Gorilla in pl. 63, illustrating the same paper, agrees with the description given above. My sketch was taken from a Gorilla's skull in the Museum of the Jardin des Plantes, Paris.

In the skull of a young male Chimpanzee in Dr. ALLEN THOMSON's possession the pterygoids have already almost met in the middle line, as represented in Professor OWEN's drawing, while the vomer is compressed between them. Most probably this is a sexual characteristic.

The posterior extremity of the feline vomer, as I have observed it in the Lion, Tiger, Leopard, and Cat (fig. 7, A), is of a well-marked constant form, viz. the posterior margin of the ethmovomerine laminæ is a straight line across, and behind it the vomer sends back a narrow process between the palatals. In the canine family the ethmovomerine laminæ are very broad, and their posterior margins slope backwards and inwards to the posterior extremity of the vomer (fig. 8).

In the Bears and their allies the mesial-palatine processes of the intermaxillaries come in contact with the lateral plates of the same bones behind the incisive foramina, and in the middle line between them there is also a characteristic foramen (fig. 9). This foramen varies very much in different species, and also in individuals. The *Ursus ornatus*, in three specimens in the Museum at the Jardin des Plantes at Paris, has it larger than any other bear's skull in that collection. The *Ursus maritimus* has it always of considerable size. In *Ursus arctos* and *Ursus americanus* it is small and variable. The Coati Mondi has this foramen very large; and I observe it, but small, in the Weasel, the Marten, the Glutton, and the Hyæna, very small in the Badger, and not at all in the Civet and the Otter. The inferior margin of the vomer of the Weasel, notwithstanding the length of the hard palate, is very short; it articulates with the intermaxillaries, and only for a very short distance with the maxillaries.

In the Seal the ethmoid is so compressed that the elements of which it is composed are brought close together; so close that beneath the nasal bones the central plate, which is slightly flattened out above, comes in contact with and is soldered to the outer plates of the lateral masses (fig. 10). There is no need in this instance of a special ethmovomerine lamina to unite the vomer and lateral mass; they are in contact, and are anchylosed directly with one another. At an early age the central plate of the ethmoid of the Seal becomes closely connected with the vomer; yet even after it has extended down to the vomerine groove, one can mark this distinction, viz. that the central plate replaces the septal cartilage, while the vomer surrounds it. The vomer merely touches the intermaxillaries, and does not reach back to the palate bones.

The arrangement in the Hedgehog is interesting in this respect,—that, owing, I presume, to the unarched form of the head, the inferior margin of the central plate of the ethmoid intervenes for about a third of an inch between the vomer and presphenoid; so that if the maxillary and palate bones be removed from a hedgehog's skull, there are seen extending in a straight line forwards, the bodies of the occipital, postsphenoid, and presphenoid bones, the central plate of the ethmoid with a slightly flattened edge lying between the ethmovomerine laminæ, then the vomer, and lastly, the mesial-palatine processes of the intermaxillaries; the whole presenting the appearance of a series of homologous elements (fig. 11).

Ruminantia.—In the Ox, the Sheep (fig. 12), the Deer, the Camel, and the Alpaca, the vomer has the typical arrangement. The inferior margin is not prolonged back sufficiently far to articulate with the palate bones, and its anterior extremity is scarcely prolonged upon the intermaxillaries, but fits on to them in such a way as to make their mesial processes continuous with it. In the Giraffe, however, there is an exception to the usual connexions; the superior parts of the palatals floor-in the greater part of the ethmoidal turbinations, and the alæ of the vomer only come in contact for a very short distance in front of the palatals with prolongations inwards from the framework of the lateral masses of the ethmoid, but are not anchylosed to them. The arrangement is extremely similar to that in the Horse, even in the shape of the back part of the vomer.

There are some interesting varieties in the intermaxillaries of ruminants. In the animals which we have hitherto examined, the angle of junction of the mesial-palatine and lateral plates of the intermaxillary has been the most anterior part of that bone, and the groove for the septal cartilage has been open in front, so that, as in the human subject, the cartilage could be prolonged forwards beyond the intermaxillaries. But in the Sheep there is a slight, a very slight inclination of the lateral plates to prolong themselves forwards beyond the points at which the mesial palatine processes come off from them (fig. 13). These points are at a little distance from the middle line, and the prolongations forwards of the lateral plates are inclined inwards, so that in front of the mesial plates there is a little space left in the middle line. Into this space the anterior extremity of the septal cartilage, slightly dilated, dips down, and the tendency of the tips of the intermaxillaries is to embrace it. This arrangement is so faint in the Sheep that it would appear unworthy of attention, were it not that in the Camel and the Alpaca it is carried out to a most distinct and unmistakeable extent. In them the mesial-palatine processes, which are but slender in comparison with the lateral plates, arise at such a distance from the middle line that at their bases they are first directed inwards to meet one another before they are directed backwards. The space for the extremity of the septal cartilage is large, and almost converted into a foramen by the prolonged tips of the lateral plates approaching the middle line in front of it; and as they do so, they turn their inner aspects downwards, and embrace the cartilage on its upper border. Thus there can remain no doubt that the portion of the cartilage which projects through the space into the palate is really its anterior extremity (Plate V. fig. 14).

In the Giraffe this relationship of the intermaxillaries to the septal cartilage is very distinct. The mesial-palatine processes are very large, while the lateral plates are comparatively slender; but the continuation forwards of the latter to embrace the septal cartilage is well marked (Plate IV. fig. 15).

Pachydermata.—In the Horse, as already mentioned, the connexion of the vomer with the lateral masses of the ethmoid is of an exceptionally slight description. The leaflets of the ethmoid, which in most instances lie in contact for a considerable extent with the ethmovomerine laminæ, are in this case floored-in completely by the superior parts of the palate bones. But even in these circumstances a slender lamina, immediately in front of the palate bone, and in contact with its nasal foramen, passes downwards and inwards on each side from the ethmoidal turbinations, and articulates with the ala of the vomer, though it is not anchylosed to it (fig. 16). These peculiar relations in the skull of the Horse seem principally to depend on the palate bones being pushed further forwards than usual; a circumstance which is probably connected with the habitual nasal respiration of the Horse*, as, by the posterior nares being placed well forwards, the most direct and free connexion between the nasal passages and respiratory organs is obtained. The lateral plates of the intermaxillaries of the Horse meet for some distance in the middle line in front of the septal cartilage, in the same way as we have seen them tending to do in the Camel; but the mesial-palatine processes arise quite from the fore part, and are in contact with each other at their origins, so that the extremity of the cartilage is enclosed completely in a cul-de-sac, and does not project into the palate (Plate V. fig. 17).

In the Pigs, the vomer articulates with the intermaxillaries, maxillaries, and palatals below, and forms one bone with the lateral masses of the ethmoid, according to the typical mode. The posterior extremity of the vomer is always very narrow, and the intermaxillaries are as in the Carnivora. We may notice also the little bone developed in the anterior extremity of the septal cartilage, in connexion with the snout.

In the skull of a sucking-pig, I have observed to advantage some of the more important morphological relations of the vomer (Plate IV. fig. 18). The central and cribriform plates of the ethmoid have not yet begun to ossify; there are separate centres of ossification in a number of the leaflets of the lateral masses, and larger ones in the great turbinated processes; and on each side the vomer is continuous (with only a slight trace of the junction) with a mass of bone consisting of the ethmovomerine lamina and part of the lateral mass of the ethmoid. It is but fair to add, that already the vomer is anchylosed to the body of the sphenoid. The articular surface on the inferior edge of the vomer for the maxillaries is flat, with a raphe in the middle line, and abruptly ceases where the intermaxillaries fit on; and the inferior surface of the mesial processes of the intermaxillaries is continuous with the maxillary surface of the vomer; a state of matters exactly similar to what we have noticed in the young human subject, and which may also be seen in the skull of a new-born puppy.

The Hippopotamus has the vomer and intermaxillaries arranged like those of the Pig. The ethmovomerine laminæ are broad, but in the specimen which I examined, although it was a well-grown one, there were suture markings between them and the vomer.

In the young Elephant (Plate V. fig. 19) the intermaxillaries come in contact with one another by means of large triangular surfaces, which reach to a considerable height above the level of the floor of the nares behind. They have no mesial-palatine processes. There is a small anterior palatine canal between them and the maxillaries in the middle line. The vomer is but a slightly developed bone: in the specimen before me it presents in the greater part of its extent superiorly a mere edge to articulate with the septal cartilage, and no vestiges of alæ. At the posterior part, however, it is bifurcated, and comes in contact with the lateral masses of the ethmoid, but is not anchylosed to them. At its

* Pointed out by Sir CHARLES BELL in his 'Anatomy and Philosophy of Expression,' 3rd edit. pp. 126 and 134.

anterior extremity it presents two processes; one passing into the anterior palatine canal, the other inclined somewhat upwards behind the surfaces of contact of the intermaxillaries. The superior and inferior margins of the vomer are thus rendered continuous with those of the intermaxillaries.

In the two-horned species of Rhinoceros the intermaxillaries are very small, and consist merely of lateral plates, without vestige of mesial-palatine processes; while the anterior extremity of the vomer stops short on the palate plates of the maxillaries, some distance behind their anterior margin. But in *Rhinoceros indicus* (which unfortunately I have not examined) a process of considerable size is described as arising from behind the superior margin of the intermaxillary; and in *R. tichorhinus*, not only is the osseous septum of the nose rendered complete by this process reaching up to the nasal bone, but the thick nasals are prolonged downwards to the level of the palate, and articulate with the anterior extremities of the intermaxillaries, so as to form a complete arch of bone in front of the nostrils, the only instance of this among Mammals *.

In the Tapir we have another instance of intermaxillaries without any mesial-palatine processes, and the anterior extremity of the vomer stopping short upon the maxillary bones; the intermaxillaries, however, are of considerable size, and come in contact with one another for a considerable distance in front of and above the septal cartilage, exactly as in the case of the Horse, only much more extensively (fig. 20).

Rodentia.—In the Rodentia the vomer is remarkable for the very little tendency it evinces to articulate with the maxillaries. Its superior connexions are typical, *i. e.* it is continuous with the lateral masses of the ethmoid, and the margin of junction of the ethmoid and ethmovomerine lamina enters into the formation of the nasal foramen of the palate bone, as seen from within. In all those that I have examined the vomer is bifid in front, and fits on edge to edge with those parts of the mesial processes of the intermaxillaries which bound the groove for the septal cartilage. The mesial processes of the intermaxillaries are well developed, and are expanded laterally in connexion with Jacobson's organs in a very characteristic fashion. The edges for articulation with the vomer are sometimes, as for instance in the Hare, very minute: but in other species with strong incisors they are well developed; thus they are of considerable size in the Paca (fig. 21); while in the Porcupine they are remarkably elevated, and the vomer sends down a process from its mesial plate between them.

In the Rat, the Beaver, the Porcupine, and the Paca, the vomer comes in contact with the anterior extremity of the maxillary part of the palate plate by a little, slightly dilated point; in the Squirrel it scarcely comes in contact. But the most interesting condition is seen in the Hare and Rabbit: they have only one single large foramen incisivum; for although the mesial-palatine processes of the intermaxillaries project well backwards, the palate plates of the maxillaries do not come far enough forwards to meet them: the vomer does not even approach the maxillaries, but its posterior margin terminates inferiorly in a thickened angle, which articulates with the intermaxillaries in such a manner as to be

* Cuvier, Ossemens Fossiles, tom. ii. Meckel, Anatomie Comparée (traduit), tom. iv. p. 273.

continuous with their inferior margin; and between this angle and the point where the vomerine groove is continued on to the intermaxillaries, there intervenes a considerable space in which the vomer comes in contact with the intermaxillaries by the edge of a triangular development of mesial plate (fig. 22).

In the *Edentata* the vomer has its superior connexions typical. In *Orycteropus* its inferior margin reaches back to the palatals, in *Manis* and *Myrmecophaga* it does not; and in *Bradypus* it only articulates with the anterior part of the maxillaries. The intermaxillaries are very small. They are smallest in *Bradypus*. In *B. didactylis* and *B. torquatus* they are separate; in *B. tridactylis* they are fused into one little plate, which is in contact with the maxillaries by its posterior and lateral angles, and anteriorly turns upwards upon the septal cartilage. In *Myrmecophaga* the intermaxillaries are somewhat better developed, but the incisive foramina being large, the mesial-palatine processes do not quite come in contact with the vomer, even though the latter projects some distance forwards beyond the maxillaries.

The intermaxillaries of the Armadillo are very characteristic. Their lateral plates are broad, and meet together in the middle line of the palate behind the incisive foramina. These foramina are small; and the mesial-palatine processes which separate them are slender, but are prolonged back upon the superior aspect of the line of junction of the lateral plates, so that thus the vomer rests upon the mesial-palatine processes, according to the general rule, and does not come in contact with the lateral plates.

In the *Marsupiata* the relations of the vomer are normal. Sometimes, as in the Koala, the inferior margin articulates with the palatals; sometimes, as in the Kangaroo, it does not. The groove for the septal cartilage is open in front, as in the Carnivora. In the Kangaroo (fig. 23) the ethmovomerine laminæ are very broad: the vomer exhibits on each side a peculiar lateral ridge, which extends forwards from the point where the ethmo vomerine lamina comes off, and articulates in front with a very long prolongation backwards of the mesial process of the intermaxillary, which extends between it and the maxillary. This ridge exists also in the Wombat, the Phalangers, and the Opossum; probably it is a constant marsupiate characteristic.

Cetacea.—In the Manati and Dugong we again meet with the arrangement of the vomer and intermaxillaries which we found in the Tapir, viz. the vomer is quite unconnected with the intermaxillaries, and the latter have no traces of mesial-palatine processes. The lateral plates of the intermaxillaries are very well developed, and meet each other above and in front of the septal cartilage, as they do in the Horse and the Tapir. In the Manati the anterior extremity of the vomer projects beyond the anterior margin of the maxillaries; in the Dugong it falls considerably short of it. In the Dugong the turbinations of the lateral masses of the ethmoid are very slight; but their framework is strong, and united by ethmovomerine laminæ to the vomer (fig. 24). In the Manati the lateral masses of the ethmoid are considerably more developed; they also are united to the vomer.

In the carnivorous Cetacea the vomer is largely developed, and its inferior margin extends back the whole length of the palate. The intermaxillaries prolong the vomerine groove for a greater or less distance according to the genus. The distinctness of the central plate of the ethmoid from the vomer is very clearly seen, inasmuch as the former terminates abruptly where it becomes continuous with the cartilage in front, while the grooved surface of the latter is continued smoothly forwards. In vertical sections of skulls of the Dolphin and Grampus in the Museum of this University, although many sutures are obliterated, a straight line passing forwards from the inferior margin of the presphenoid indicates distinctly the place of contact of the still separate vomer and central plate of the ethmoid (fig. 25). The olfactory apparatus and lateral masses of the ethmoid are entirely absent; but the posterior parts of the alæ of the vomer are enormously expanded to take their place, and pass up on each side of the central plate of the ethmoid in front of the frontals, even as far as the nasals, forming the whole posterior wall of the nares, viz. the whole of that wall which in other Mammals is formed by the ethmoid. I notice also in *Deductor globiceps* that the expanded margin of the vomer, with the assistance of an angle of the maxillary, completes for the palate bone its nasal foramen (fig. 26).

Monotremata.—In the Ornithorhynchus the inferior margin of the vomer extends quite back to the posterior extremity of the hard palate, which is formed by the meeting of the pterygoids. In the Echidna it does not extend back so far. The condition of the intermaxillaries in the Ornithorhynchus is extremely interesting. The large bones which extend forwards from the maxillaries at a considerable distance from one another, and which give the form to the broad flat bill, are beyond all question intermaxillaries, and correspond exactly to the intermaxillaries of the Bat, or any other animal in which the mesial-palatine processes of these bones are not developed. Utterly unconnected with these, in front of the vomer, and continuing on its upper aspect the vomerine groove, is the little bone which has been recognized both by CUVIER and MECKEL as corresponding with the mesial-palatine processes of intermaxillaries, which it no doubt does; but it is very interesting to find these represented by a bone so distinctly separated from the lateral plates of the intermaxillaries. It leads us to the consideration of the remarkable arrangement which exists in cases of cleft palate in the human subject, contrasted with the natural arrangement.

In cases of *complete cleft palate in the human subject* the inferior margin of the vomer is free, and is seen in the middle line of the open roof of the mouth. The intermaxillaries, fused or separate, articulate with its anterior extremity, and continue forwards in the same straight line, and support the incisor teeth when they are developed. They are entirely disconnected from the maxillaries, and are developed on the under aspect of the septal cartilage, which is a mesial structure; while the maxillaries are developed in lateral laminæ which, in these instances, fail to reach the middle line. Yet in the normal condition, marvellous as it may appear, the intermaxillary grows from the same centre of ossification as the maxillary *, and therefore must be con-

* See on this subject, in the 'Comptes Rendus' (Dec. 1858 and Jan. 1859), various papers by M. Em.

sidered as at least commencing from what had originally been a lateral lamina. On the one hand, in the normal condition, the mesial processes of the intermaxillaries are obviously present; and in the cleft-palate condition, on the other hand, there can be no doubt that the lateral plates are represented as well as the mesial processes, since the incisor teeth are developed on the mesial bone. Thus it is certain that, according as the palate is completely closed or remains cleft, the whole intermaxillary—both mesial and lateral portion—comes from one or other of two parts which at an early period are always quite distinct. This is a fact of great importance.

Morphological Conclusions respecting the Vomer, Ethmoid, and Intermaxillaries in Mammals.

I had desired that this paper should be as strictly observational and as little theoretical or controversial in its character as might be; but I find it is impossible, now that I have arrived at this point, to refrain from indicating in what direction the observations just made appear to tend. I am conscious that by entering on this theme I render myself liable to the charge of presumption, in asking to be heard upon matters which have been discussed by the greatest authorities. My excuse is, that it is the observational part of my subject which compels me into the theoretical: and in venturing an opinion upon certain segments of the skull, I shall endeavour to limit myself as much as possible to what seem to me to be deductions to which the facts discussed inevitably lead; facts, some of them at least, not hitherto known, or not previously collated, and therefore not till now at the disposal of the theorist; but a knowledge of which, I cannot help thinking, is indispensable to the just conception of the segments to which they relate.

I gladly embrace this opportunity of acknowledging my obligations to Professor GOODSIR for the use of books and specimens placed at my disposal, and for valuable information bearing upon topics treated of in this paper; and if, for the reason just mentioned, I have taken it upon me to adopt conclusions differing from his in certain details (although not more than from those of others), I do not forget that to him entirely I owe my morphological training; nor am I the less sensible of the advantage which I have enjoyed in being frequently indoctrinated by him in those great principles of Morphology which he illustrated in his communications to the British Association in 1856 *.

1. The first proposition which I shall make is, that the whole septum of the nose is continued forwards from the line of centra formed by the basilar process of the occipital and the bodies of the postsphenoid and presphenoid †. This statement can be best verified in some animal in which the body of the presphenoid is more developed than in

ROUSSEAU, showing that the intermaxillaries are at no period normally separate from the maxillaries; and by M. LARCHER, showing that in cleft palate they are distinct.

* Edinburgh New Philosophical Journal, Jan. 1857, pp. 118–181.

† The embryological aspect of this proposition is considered below at page 315.

Man. Take, for example, the Cat (fig. 7, B & C). The posterior aspect of the body of the presphenoid in the Cat exactly corresponds to the anterior extremity of the postsphenoid, presenting an appearance exactly similar to what the body of a vertebra exhibits on removal of its epiphyses; viz. a part in the middle, broad below and narrow above, formed by the centrum, and the two superior angles formed by the alæ. If we look at the same bone from the front, we see plainly that the septum between the sphenoidal sinuses consists entirely of the anterior extremity of the centrum greatly compressed; and just as the posterior extremity of the presphenoidal centrum is continuous with the postsphenoidal, so is its anterior extremity continuous with the septal cartilage of the nose and the central plate of the ethmoid. This continuity is in some respects even better seen in the new-born Kitten, as above described. Now we know that, morphologically, it is of little importance whether cranial bones are developed in the primordial cartilage of the skull, of which the septal cartilage is the remains, or are developed round it. The basisphenoid in fishes, for example, is developed round it, while in mammals it is developed in it. Therefore the central plate of the ethmoid, the vomer, and the mesial processes of the intermaxillaries all claim from their position to be centra as much as the basioccipital and basisphenoid. With regard to the mesial processes of the intermaxillaries, that they play the part of a centrum is shown, not merely by their constant relation to the septal cartilage, but by their articulation with the vomer being of such a description as to make them continuous with that bone both on the upper and under aspect, and by their condition in cleft-palate.

2. The vomer, lateral masses of the ethmoid, and palate-bones belong to one segment. This seems to be an altogether unavoidable conclusion, from the constant connexion of these bones in so invariable a manner.

What meaning can be attached to the remarkable way in which, in the human subject, the vomer and lateral masses of the ethmoid, notwithstanding their altered forms and positions, preserve the relations to one another which they exhibit in other animals, but that they are members of one segment? And how else can we explain that the place of the flattened ethmoid and ethmovomerine laminæ of the Dugong is occupied in the Delphinidæ by expansions of the vomer, which both form the posterior walls of the nostrils and articulate with the palatals?

This conception of the construction of the ethmoidal segment, as well as the reasons just mentioned for considering that the mesial-palatine processes of the intermaxillaries play the part of a centrum, is at variance with the view of Professor GOODSIR, that the vomer and the intermaxillaries are members of one "sclerotome*."

Also if the connexions of the vomer and lateral masses of the ethmoid recorded above are considered sufficient to prove that these bones are parts of one segment, it will at once appear evident that the lateral masses of the ethmoid and the ethmovomerine laminæ form an incomplete neural arch. In that case we must differ from Professor

* *Op. cit.* p. 138.

OWEN, who considers that the central plate of the ethmoid represents in a coalesced condition the prefrontals of the fish or reptile*, and plays the part which we allot to the lateral masses in the neural arch of the vomerine segment. It cannot do so, for these reasons: 1st, that while the lateral masses of the ethmoid are continuous with the vomer, the central plate is never truly continuous, but only contiguous to the vomer; 2ndly, that the central plate must, for reasons above stated, play the part of a centrum; and 3rdly, that, as Professor GOODSIR has shown, Professor OWEN's view is inconsistent with the relation of the olfactory nerves to the central plate†.

3. The frontal and the central plate of the ethmoid belong to one segment. The fact of the central plate of the ethmoid having no early connexion with the vomer, as well as its tendency to remain distinct even after the vomer and it have come in contact, shows that it is not part of the same centrum as the vomer; therefore, inasmuch as we have already concluded that it is a centrum, it can form no part of the vomerine segment. On the other hand, the presphenoidal centrum is complete without it. In these circumstances it becomes apparent that the central plate of the ethmoid is the centrum of a segment intervening between the vomerine and presphenoidal. This opinion is strengthened by our remembering that in the Sheep the central plate of the ethmoid is anchylosed to the centrum of the presphenoid before uniting with the lateral masses, and that in the Hedgehog it appears on the base of the skull for a considerable distance between the presphenoid and vomer. The frontal forms the neural arch belonging to this centrum, and is in constant connexion with it. But if, as above proved, that margin of the os planum of the human subject which lies superiorly is morphologically posterior, then the margin of the frontal which articulates with it, viz. the inner edge of the orbital plate, is morphologically anterior; and therefore not only the foramen cæcum, as Professor GOODSIR believes‡, but also the space occupied by the cribriform lamina, lies within the arch formed by the frontal and the central plate of the ethmoid. Thus the cribriform lamina, which in point of development is a mere lateral expansion of the central plate, forms a screen across the entrance into the cavity of the ethmovomerine arch in the plane of segmentation, and has no further morphological importance than may be supposed to attach to the tentorium cerebelli. By the turning up of its anterior extremity to touch the frontals and nasals, the central plate of the ethmoid divides the neural arch of the segment to which it belongs, as well as those in front of it, into a

* OWEN 'On the Archetype and Homologies of the Vertebrate Skeleton,' pp. 131 & 135.

† *Op. cit.* p. 149. The strength of Professor GOODSIR's argument rests in this: that, according to Professor OWEN's theory, the olfactory nerves in the mammal are made to lie outside a neural arch, through which they pass in the reptile and fish; which involves the supposition that the points of egress of the olfactory nerves have been moved in the mammal one segment backwards. According to the theory advanced in this communication, although no doubt the bone corresponding to the mammalian central plate of the ethmoid passes upwards on the outside of the olfactory nerves, and does not rise up between them, yet in both fishes and mammals the olfactory nerves are contained within the neural arch of the segment.

‡ *Op. cit.* p. 142.

right and left portion, exactly as the vomer, by a prolongation downwards, divides the anterior hæmal arches.

4. The intermaxillaries, maxillaries, and nasals are the osseous elements of one segment. Of this we might find sufficient evidence in their relations in mammals; and indeed the proposition does not stand in need of much proof, if it be once considered certain that the vomer, lateral masses of the ethmoid, and palatals belong to one segment, and that the mesial-palatine processes of the intermaxillaries are the centrum of the segment following. But that we may fully understand the parts played by the intermaxillaries, maxillaries, and nasals respectively, we must defer our remarks on this subject till we have examined the arrangements in other classes.

We shall now glance for a single moment at the manner in which this explanation of the anterior segments of the skull affects the general view of the cranial segmentation.

That there is an occipital segment, and that the postsphenoid and parietals are portions of another segment, is generally admitted; and we have concluded that the frontal and central plate of the ethmoid are elements of a third segment. These then are the three segments which roof-in the cranial cavity. They are all complete above. Their centra are successively smaller in their order from behind; and the floor of the neural arches which they form gradually turns upwards as it passes forwards.

In front of the occipital segment lie the petrous bones*, and connected with them are the organs of hearing. In front of the parietal segment lies the presphenoid, connected with which are the organs of vision. In front of the frontal segment lies the ethmo-vomerine segment, and connected with it are the organs of smell. None of the three segments connected with the special senses has a complete neural arch. With the most posterior of them no centrum is connected; the second (the presphenoid) has a small centrum; and the most anterior has a larger centrum—the vomer: and the tendency of these centra, contrary to the tendency of the centra of the brain-protecting segments, is to curve downwards as they pass forwards. Thus there are two alternating sets of sclerotomes, which may be distinguished as the protective and the sensory; while foremost of all is an imperfect and peculiar seventh and terminal sclerotome—the facial, which may be considered as binding these two sets together. On this seven-segmented plan I believe the head to be formed in all vertebrata; and although it be true that in certain cases there are no sclerous elements of the cranium developed in connexion with either the ear or the eye, these organs are nevertheless themselves portions of segments lying between those in connexion with which the neighbouring protective sclerotomes are developed. The upward tendency of the protective, and the downward tendency of the sensory sclerotomes, is not seen at all in the fish, but becomes more and more observable as we ascend to reptiles, birds, and mammals; and is exhibited best of all in man, in whose structure the idea of vertebrate creation is completed (fig. 27).

These few and imperfect remarks have been necessitated by the consideration that no

* It is unnecessary for the purpose of this argument to discuss the positions of the interparietal, mastoid, or squamous.

segment of the skull can be properly viewed apart from the others. While on the other hand, were I to enter into further details, I should be led far beyond the subject of this paper, and such details are not required for the elucidation of the points under consideration.

Birds, Reptiles, and Fishes.

The Vomer and Intermaxillary in the Bird.—There is only one intermaxillary in the bird. It sends no mesial processes backwards on the palate. Its lateral plates articulate with the maxillaries exactly as the mammalian intermaxillaries do. Its distinctive peculiarity is, that from the place of union of its two halves in front it sends up long processes between the nostrils, which reach back to the roof of the skull. These processes close-in the septal cartilage above and in front, exactly as it is closed in by the intermaxillaries in the Horse, the Tapir, and the Dugong. If the parts of the intermaxillaries of the Dugong which meet in the middle line were prolonged back to the frontal, their arrangement would altogether resemble the intermaxillary of the bird, for they are situated as much in front of the nostril as it is. The arrangement also in *Rhinoceros tichorhinus* only differs from the beak of the bird, in that the bones roofing the nostrils are prolonged down to the intermaxillaries, instead of the latter passing up to the former. The nostrils of the bird, therefore, correspond to those of the mammal, and the bones which roof them (ethmoido-frontals of GOODSIR) are the nasals.

Fortunately, opinions are agreed as to which bones are the palatals of the bird. They approach each other behind and come in contact in the middle line, and form by their junction a grooved surface, which glides backwards and forwards on the basisphenoid. But very frequently a small bone (the vomer of CUVIER, OWEN, &c., the entopterygoid of GOODSIR) is intercalated between them, so situated as to furnish a continuation of this groove along the inferior edge of the septal cartilage; and with this little bone the palatals articulate edge to edge. It is often absent, as in the Gallinaceæ, and in different birds it presents different shapes: thus it is broad in the Crow, and a vertical plate of considerable size in the Duck (fig. 28); and in the Gull and Guillemot, as well as in the Albatros, which has it of very large size, it bears a most striking resemblance to the mammalian vomer. That it belongs to the same segment as the palatals is as obvious as that the vomer, lateral masses of the ethmoid, and palatals in the mammal belong to one segment; and if that proposition has been proved by the foregoing observations, there can be no further doubt that this bone in the bird's skull is the vomer.

The framework of the lateral masses of the ethmoid of the mammal is not represented in the bird; but the turbinations for the distribution of the olfactory nerves, which in the mammal are attached to that framework, are in the bird, when they are ossified, usually in close connexion with the palatals, which are members of the same segment. This can be seen in the Gull and in the Albatros. In the Parrot there is an exception to the rule.

If the nasals of the bird correspond with the nasals of the mammal, then the frontals of the bird (sphenoido-frontals of GOODSIR) must correspond with the frontals of the mammal; and indeed they have the same relation to the cranial cavity, and are met, like those of the mammal, by the anterior extremity of a bone belonging to the series of centra. But if they are the frontals, then the bone which meets them, viz. the interorbital plate (prefrontal of OWEN), corresponds to the mammalian central plate of the ethmoid, at least in the anterior part of its extent *.

The Vomer in Reptiles and Fishes.—Of the vomer in reptiles and fishes it is not necessary, for the purposes of this paper, to say much. If the vomer, palatals, and lateral masses of the ethmoid in the mammal form one segment, and if this segment is that to which the olfactory nerves belong, then it requires no argument to prove that these bones must be respectively represented in reptiles and fishes by the vomer and palatals of CUVIER and OWEN, and the prefrontals; for these comply with all the necessary conditions. The prefrontals and palatals are always in contact, and sometimes, as in the Crocodilia, in such a way as to indicate in the strongest manner that they are parts of one sclerotome; while the vomer articulates always with one or other of these two pairs of bones, and sometimes with both. Moreover the vomer in reptiles and fishes always lies along the inferior margin of the septal cartilage (or middle frontal process of the primordial cranium). In reptiles, however, it presents a series of variations. While in the Turtle it is a single bone, and disposed much as in the mammal, except that inferiorly it appears prominently in the palate †, in other reptilia it is in two parts—a right and a left. In the Crocodile these parts lie side by side, and inferiorly articulate with the palate plates of the palatals, while superiorly they curve outwards and come in contact edge to edge with the superior extremities of the same bones. On the other hand, in the Serpents it is the inferior edges of the vomerine bones which curve outwards. In the Lizards they merely come in contact in the middle line; and in the Batrachia they do not even meet.

If it be asked, what corresponds in reptiles and fishes to the central plate of the ethmoid, I reply that it is the interorbital septum. This structure is frequently completely ossified in fishes so as to form a single distinct bone; and, as Professor GOODSIR has pointed out ‡, it completes a neural ring with the great frontal. According to the theory now advanced, it differs from the central plate of the mammalian ethmoid only in that it stretches outwards and upwards to meet the frontal, instead of the frontal stretching downwards, spanning the whole arch to meet it; and in that it does not project upwards in the middle line so as to divide the arch into lateral halves.

According to this view the centrum of the frontal segment always lies between the

* The posterior part of the interorbital plate appears to belong to the presphenoid, as has been pointed out by Professor HUXLEY in his Lecture "On the Theory of the Vertebrate Skull," pp. 10 & 11. See Royal Society's Proceedings, Nov. 18, 1858.

† The vomer appears, however, in the palate of even some mammalia, viz. in certain Cetacea. In *Hyperoodon* it even appears at two different places.

‡ *Op. cit.* p. 158.

orbits; and the peculiarity of its position in the mammal consists merely in this,—that by the curving downwards of the frontal and vomerine neural arches, and the curving upwards of the frontal centrum, the latter is entirely concealed by the former.

The Intermaxillaries in Reptiles and Osseous Fishes.—We now approach a most interesting series of intermaxillaries, which afford the clue to the explanation of the segment to which these bones belong. In the Crocodiles the arrangement of the intermaxillaries is like that in the mammalia, but by the great elongation of the maxillaries they are far removed from the vomerine bones. The articulation of their anterior extremities is like that in the Horse; they pass up in front of the septal cartilage and close it in. In the Chelonians the nasals are absent; and the intermaxillaries are united into a single bone, which is placed between the anterior extremities of the maxillaries, and in the Tortoise articulates with the vomer. From the Chelonians we pass to the Lizards, and in them also we find the intermaxillary single and articulating with the anterior extremities of the maxillaries; but it differs from the chelonian intermaxillary by sending upwards and backwards in the middle line a process to articulate with the nasals, which corresponds to the processes projected upwards by the intermaxillary of the bird, and which is often the most developed part of the bone. In *Varanus* and others the nasals are represented by a single bone.

In the Serpents we again meet with a single intermaxillary which articulates in the middle line with the nasals; and as it is but loosely connected with the maxillaries, and forms the anterior extremity of the line of centra, it puts one strongly in mind of the arrangement in cases of cleft-palate in the human subject.

It greatly resembles also another bone, viz. the nasal in fishes (nasal of OWEN); and the resemblance consists in this, that, like the reptilian intermaxillary, the nasal of the fish, when present, forms the anterior member of the series of centra.

Sometimes this relation of the nasal in fishes is very much disguised by the thickening of the vomer which takes place in connexion with the development of vomerine teeth,—a thickening which gives to the vomer the appearance of projecting downwards, and which often looks as if it formed the termination of the line of centra, while the extremity of the nasal appears to be situated altogether above that line. This is the case, for example, in the Cod; but when, in the fresh state, a vertical section is made through the nasal and vomer, it is seen that, not the vomer, but the extremity of the nasal bounds anteriorly the cartilaginous bar of the base of the skull (fig. 29). There is, in addition, a nodule of cartilage * attached in front of the nasal, in the Cod and many other fishes, which seems to be a portion of that bar, separated from the main part by the interruption of the nasal. But that the nasal lies in the line of centra is best seen in some of the fishes which have no vomerine teeth. It is beautifully seen in *Malapterurus*, in which the whole shape of the nasal singularly resembles the intermaxillary of *Boa* or *Python* (fig. 30).

One might well believe from its relation to the line of centra that the nasal of the fish

* Recognized as the vomer by Professor GOODSIR, *op. cit.* p. 140.

was the true intermaxillary, were it not that the bones universally acknowledged as intermaxillaries in the fish have claims to the name which cannot be set aside, inasmuch as a complete chain of forms of intermaxillaries, of which they form one link, can be traced from reptiles to fishes through the Batrachia. The position therefore which I maintain is this, that in the sclerotome formed by the nasals, intermaxillaries, and maxillaries, when the centrum is represented by a bony formation, it is derived in mammals, birds, and reptiles from the intermaxillaries, and in fishes from the nasal: the Ornithorhynchus being the only animal, as far as I know, in which it is a separate bone.

If it be repugnant to any preconceived notion to believe that the same morphological element (the centrum) can be derived in one instance from one developmental element, and in another instance from a different one in the same segment, we have only to keep in memory the exactly analogous instance of the intermaxillaries in the human subject, which usually arise from the maxillaries, and yet in certain conditions of development take origin in connexion with the septal cartilage (see above, p. 303).

To return to the series of forms of intermaxillaries in reptiles and fishes: the intermaxillaries of the Frog are two small bones articulating with one another in the middle line, and externally with the anterior extremities of the elongated maxillaries; they give off processes towards the nasals, and very slight mesial-palatine processes. Exactly in the same way do the intermaxillaries of the Salmon articulate with one another, and with the maxillaries; the only difference being that the maxillaries hang downwards instead of being directed horizontally backwards. In most fishes the intermaxillaries have superior mesial processes, which pass backwards over the nasal, and evidently correspond with the superior mesial processes in frogs, lizards, and birds. Frequently, as in the Cod, the maxillaries and intermaxillaries are so loosely connected, that the idea is given of two separate arches; but, on the other hand, it is to be remembered that not only is this looseness of connexion (considered apart from the question of development, to be afterwards noticed) no reason why they may not be members of one segment, but also that the intermaxillaries are always in contact with one another in the middle line, while the maxillaries are not so; that the arrangement is often such as has just been mentioned as existing in the Salmon; and that sometimes the maxillary and intermaxillary are even more closely soldered into one arch, of which the intermaxillary forms the proximal, the maxillary the distal part.

Morphological Conclusions respecting the Construction of the Facial Segment throughout the Vertebrata.

Ere endeavouring to discover from the above data in what precise manner the maxillaries, intermaxillaries, and nasals unite to form the facial sclerotome, it will be well to state distinctly that they cannot be considered as forming in the mammal a continuous neural ring behind the nostrils, however tempting at first sight that idea may appear; for such an hypothesis would render inexplicable the union of the intermaxillaries above the anterior extremity of the septal cartilage, which occurs in the Dugong, Tapir, &c., besides that it could not be applied to other than mammalian forms.

The real constitution of the facial segment will become evident if we consider the series of appearances assumed by it, passing upwards from the fish. In the fish the intermaxillaries and maxillaries hanging downwards form a sufficiently evident incomplete hæmal arch; while the ideal cylinder formed by the series of neural arches is brought to a close in front by the nasal, which rests, like the nasals in the other classes, upon the cartilage in connexion with which the centra are developed; and when the centrum of this sclerotome is represented by a plate of bone, it is a process of the nasal which represents it. Obviously the nostril of the fish lies behind the facial segment, for both the intermaxillaries and maxillaries lie in front of the nostril (fig. 31).

The facial segment of the Frog differs from that of the fish in that the maxillaries are directed horizontally backwards, and come in contact with bones behind; and in that the centrum of the segment is represented by a process derived, not from the nasals, but from the intermaxillaries. Further, the nasals of the Frog differ from the nasal of the fish in being expanded to protect the nostrils, like those of most reptilia, birds, and mammals. The exact signification of this expansion we shall consider anon.

The relation of the nostril of the Frog to the facial segment is the same as in the fish, but inasmuch as it communicates with the mucous surface, and this communication lies between the maxillary and the palatal, we can now see distinctly that the nostril is a passage lying between the segments to which these bones belong,—what has been named by Professor Goodsir a "metasomatomic" opening*.

In lizards, serpents, and birds the facial sclerotome is constituted as in the Frog; and the hæmal arch has still, as in the Frog, its distal extremities directed backwards instead of downwards. According to this view the superior mesial processes of the intermaxillaries are to be considered morphologically as projecting not so much upwards as backwards, and as lying above the nostrils, while the hæmal arch lies in front of the nostrils.

Lastly, in the mammalia, by the union of the palate plates of the maxillaries the hæmal arch is completed, and the ring of bone which surrounds the incisive foramina is the anterior extremity of the ideal cylinder formed by the series of hæmal arches (fig. 27). This is best conceived of by looking at this ring in the Hare or the Rabbit. There we see the single incisive foramen bounded in the middle line in front by the mesial-palatine processes of the intermaxillaries, while extending backwards from them is the hæmal arch formed by intermaxillaries and maxillaries exactly as in other classes, except that it is complete.

The fact that in certain mammals the intermaxillaries limit the anterior extremity of the septal cartilage as much as they do in birds and lizards, embracing it at a point morphologically anterior to the nostrils, appears to me to prove that this is the true explanation of the facial sclerotome in the mammalia, and that the nostrils throughout the vertebrata are intersegmental openings lying between the two most anterior segments of the skull.

* To Professor Goodsir we are also indebted for the terms "sclerotome, myotome," &c. and other additions to explicit morphological nomenclature.

If these conclusions are correct, the connexion of the nasals with the intermaxillaries and maxillaries behind the nostrils, although so constant, is of altogether secondary morphological importance. But it is undeniable that by the great extent of this connexion in mammals, and by the non-development, in most of them, of any part of the intermaxillaries above the septal cartilage, and the separation of the nasals from the intermaxillaries in the middle line, where in birds and lizards they come in contact, there is a marked attempt, so to speak, on the part of Nature to convert the nasals, intermaxillaries, and maxillaries of the mammal into a neural arch behind the nostrils. This tendency reaches its maximum in the human skull, in which the hæmal ring of the facial sclerotome has almost, if not altogether, disappeared, and the whole bulk of the elements of the hæmal arch may be said to be devoted to the formation of what we may call the pseudo-arch behind the nostrils. It may be described as an effort to open up the closed extremity of the neural cylinder, and at the same time to close the open extremity of the hæmal cylinder*.

Still, however, the morphological interpretation of the mammalian nostrils is that they are intersegmental clefts. They are similar clefts to those in which the eye and ear are developed; and the alar cartilages pass round the olfactory clefts exactly as the tarsal cartilages pass round the optic clefts, and the pinna of the ear and the tympanic bone round the auditory clefts.

Thus these structures are morphologically as well as functionally comparable; they encircle openings which pass from the dermal to the mucous surface, and which primarily lie in the transverse plane; although in Mammalia the malar, which belongs to the facial segment†, passes backwards and sends up a process behind the orbit, and, in a similar manner, the intermaxillary and maxillary send up processes behind the nostril. That these processes, however, are not situated so entirely behind the nostril as might on first thoughts be supposed, will be perceived if we take into consideration that the arch formed by the intermaxillaries and maxillaries lies in the plane of the palate, and that processes pointing upwards at right angles to that plane are therefore directed ideally backwards. Thus the nasal process of the maxillary is ideally inferior to the nostril, in the same way as we have seen that the superior mesial process of the intermaxillary of the bird or lizard is ideally superior to the nostril.

Let it not be supposed that such a structure as the trunk of the Elephant presents any obstacle to the theory of the nostrils here offered. The fusion in the middle line of structures undoubtedly lateral in their ideal position is well exemplified in cases of sympodia; and in those monsters in which the head is only developed as far forwards as the ears, the pinnæ of the ears are united at their bases, forming a short tube.

* If we take into consideration that the facial segment is the segment especially engaged in expression, I think that I shall not be considered too fanciful in saying that the gradual closing of the hæmal cylinder and opening of the neural cylinder by the disposition of the bones of the facial segment is in harmony with the increasing development of innervation from fishes up to man.

† This statement, made in passing, is illustrated by the attachment to the maxillaries alone of the largely developed malars of the Sloth family and of the vestiges of malars in *Myrmecophaga*.

With regard to the nasals: to understand the exact morphological part which is played by them, we must have recourse to embryology. The only theory of the segmentation of the skull, as far as I know, in which the teachings of embryology have been taken into account and been sought to be explained, is that of Professor GOODSIR. He is of opinion that the maxillaries are developed, not, as has been supposed, in the maxillary lobes, but in the lateral frontal processes of REICHART*; and certainly, if the maxillary lobes are homotypic with the pair of visceral laminæ behind them, this hypothesis presents the only escape from a most serious difficulty; for, while the maxillary lobe arises from behind the eye, all theories agree in representing the maxillary bone as belonging to a segment in front of the eye. But although it is with the utmost diffidence that I would express an opinion differing from Professor GOODSIR'S, I take courage to do so in the present instance, inasmuch as his view does not agree with the conclusions arrived at above from the consideration of adult forms, and because, from observations made on embryo lambs, I am convinced that embryologists have been right in considering that the maxillary bones are developed from the maxillary lobes. I believe that the true solution, and that which will be found to explain all the phenomena, is this: that the cleft between the maxillary lobe of the embryo and the lateral frontal process is not transverse in position, but longitudinal; that it does not separate an anterior from a posterior segment, but that it divides the inferior elements of more than one segment from the corresponding superior elements. It will be observed that if the maxillary bone belongs to a segment in front of the eye, and if it is really developed from the maxillary lobe, it follows as a necessary consequence that the nature of the cleft between the maxillary lobe and lateral frontal process is as now stated. For on one side of the cleft is the blastema in which the palatal and maxillary afterwards appear, while on the other side is that in which appear the frontal and nasal, and doubtless also the lateral mass of the ethmoid. Moreover, this idea of the fusion of segments by longitudinal cleft is by no means an unwarranted assumption, as may possibly be alleged. In support of it we observe,—

1st. The maxillary lobe, as it grows, runs alongside of the lateral frontal and middle frontal processes, and does not strike out at right angles to them.

2nd. The permanent severance of the pterygoid of the fish from the base of the skull is an instance of longitudinal fission of at least one sclerotome, remaining throughout life.

3rd. The hypothesis of fission of segments by longitudinal clefts is necessary to explain the peculiar condition of the elements of the face in cartilaginous fishes, if the head is to be considered as segmented at all; for no one can suppose that the upper jaw of the cartilaginous fish, and the parts which unite it to the skull, are all parts of one segment. According to my hypothesis, the condition is merely an arrestment of development; for I judge from the maxillary lobe in mammals being originally separate from the superior parts of the face, and afterwards united to them, that longitudinal fission is a symptom of degeneration at the extremity of the series of segments, which

* *Op. cit.* p. 120.

tends to disappear as the segments involved become more fully developed*. I believe, therefore, that the palatals and maxillaries combine to form the upper jaw of the cartilaginous fish (as may be seen to advantage in the Sturgeon), and that the extremity of the snout represents the intermaxillary†; and that thus, for example, the weapon borne by the Sawfish is an intermaxillary bearing teeth. Among mammals, to a certain extent similar is the arrangement in Cyclopian monsters: the nose is represented in them by a proboscis above the single eye, and the maxillary forms no superior connexion in front of the eye.

4th. Tendency to longitudinal fission of the anterior cranial segments is exhibited in the permanent duplicity, in many animals, of the vomer and of the intermaxillaries; also in the compression of the septum of the nose, and separation of the segments into two alternating sets, as above described. Moreover, the flattening at the base of the skull of the substance surrounding the chorda dorsalis, and its division into two trabeculæ where the chorda ceases, are phenomena, as it appears to me, indicating the same thing; and in this last case also the divided parts unite as development proceeds‡.

In deference to the opinions of Professor HUXLEY§, I may here state that I cannot think that the cessation of the notochord is proof of the immediate cessation of segmentation. The keenest advocate of the segmentation of the cranium will admit that in the face the segment or segments are of a degenerated description. So are those at the caudal extremity. But while in the caudal segments the arches degenerate before the centra, in the cranium the centra degenerate very soon. It is in harmony with this that the chorda dorsalis is one of the first elements to disappear in the cranial segments; while the division of the basis cranii into trabeculæ is to be accounted for in the way above shown‖.

* In this respect longitudinal and intersegmental fissures resemble one another. The parts of the embryo which are developed in immediate contact with the cavity of the ovum, and in which the great systems are fully represented, viz. the thorax and abdomen, are those in which the various tissues, the osseous, muscular, nervous, and vascular, ultimately exhibit the most complete segmentation; but it is only in the head and neck—an extremity of the embryo in which the systems are not typically developed, (and there it is only in the visceral walls, which have in the head and neck the most rudimentary development,)—that segments are for a time partially separated by fissures of the blastema. These intersegmental fissures are directly connected with the rudimentary condition of the parts which they separate, and disappear as the latter become more fully developed; and I have wished in the text to indicate that that also is the case with longitudinal fissures.

† At least that part of the intermaxillary which, as above shown, is in osseous fishes combined with the nasal.

‡ The foramen in the basioccipital of *Phoca vitulina* is doubtless a phenomenon of the same description.

§ *Op. cit.* p. 52.

‖ In justification of the plan pursued in this communication, in which the main argument is drawn from comparative anatomy and then shown to be in accordance with development, I shall here state why I cannot go so far as to concur in Professor HUXLEY's doctrine (*op. cit.* p. 5), that "the study of the gradations of structure presented by a series of living beings may have the utmost value in suggesting homologies, but the study of development alone can finally demonstrate them." Were that doctrine true, it

Conceiving, then, the development of the anterior part of the cranium to take place in the manner above stated, I consider that the nasals have the same relation to the segment to which they belong as the frontal has to its segment, inasmuch as they are continuous in position with the frontal, except in those lower vertebrata in which the prefrontals complete a well-developed intervening arch. They are more closely connected with the corresponding elements behind them than with the remaining elements of their own segment, as a natural consequence of the fission of segments. And now it will be understood how it is that the nasals are often so far sundered from the intermaxillaries: it is, that the preceding segment having a very elongated centrum (the vomer) and a very imperfect neural arch (the lateral masses of the ethmoid), the intermaxillaries are projected forwards in front of the vomer, while the nasals cling behind to the frontals.

With respect to the expanded form of the nasals in many reptiles, in birds, and in mammals, I apprehend that they are not to be considered as spreading downwards to form an imperfect arch of the neural series, but as spreading outwards to protect two intersegmental passages; for it is the nostrils, and not, in any sense, a continuation of the cranial cavity, which they protect. This will be best understood by looking at the Frog's skull, in which the osseous cranial cavity is closed-in at the fore part of the prefrontals, and the neural arch of the segment in front is reduced to zero, the position where it exists in certain fishes being represented in the Frog by the line of junction of the nasals with one another and with the mesial cartilage below; while the expansions of the nasals over the nostrils are like the projections outwards of two transverse processes, and are strictly similar to those elongated projections of the frontals which roof-in

would be impossible finally to demonstrate the correspondence of any two structures in different animals, since we do not see one developed out of the other. But, on the contrary, there is a vast number of such correspondences so plainly obvious as to need no demonstration. That which makes these correspondences so evident is simply the comparison of the bones; and it is the business of the anatomist, in cases of correspondence less obvious, to submit the structures which he compares to a careful scrutiny, until, by minute examination of their relations, and determining what is constant and what is variable, he is able to give a certain judgment upon points which appear to the uninitiated eye obscure. But if the prosecution of such researches renders certain the correspondence of a number of elements in different animals, will not the light thrown by the varied relations of these elements upon the laws which regulate their arrangement furnish as certain and secure data on which to build as can be obtained from the use of any scientific instrument whatever? It has not been the instruments used, but the manner in which they have been handled, that has led to the discrepancies of morphological theories. I have an interest to insist on this matter, because it is impossible that the questions discussed in the present communication can be settled by embryological evidence, for these reasons: viz. the maximum segmentation of the cranium is found, not in the embryo, but at the period of most characteristic development of the bones composing it; and the whole history of the sclerotomes to whose elucidation this communication is devoted is not given in any single species; but, on the contrary, were we to search in the mammal for the stages illustrated in the structure of fishes, we should find that at the period at which these stages would fall due to be represented, there is as yet no differentiation of tissues, and the osseous system whose history we seek to trace has not begun to appear.

the orbits in Whales. In other animals the arrangement is perhaps not so distinct, and there may be a difficulty, especially in mammals, to determine if the nasals do not, in part, at least, form a continuation of the roof of the cranial cavity; for in them the neural arch of the ethmoidal sclerotome is, as we have seen, a continuation of that cavity, and is open in front. But, inasmuch as in the mammal the olfactory-sense capsules are withdrawn into the interior of the ethmoidal neural arch, the cranial cavity is closed in by soft parts at a point posterior to the nasals; and I therefore incline to think that the nasals, protecting in this, as in other cases, the nostrils, are, like those of the Frog, similar in nature to the orbital processes of the cetacean frontals.

NOTE.—Since writing the above, I have deemed that it would be advisable to add here some further details as to the morphological structure of the cranium in the mammal, even though it is impossible in this place to give the arguments on which these details are founded. The descending process of the occipital bone, the mastoid, the squamous, the external angular process of the frontal, and the lacrymal are serially corresponding structures, belonging respectively to the occipital, petrous, parietal, frontal, and ethmovomerine segments, and tending to project from their neural arches. There are no hæmal arches in connexion with the protective segments. Into the composition of the hæmal arch of the petrous segment enter the styloid processes, the stylohyoid ligaments, and the small cornua of the hyoid; while the great cornua of the hyoid belong not only to a segment behind, but to the true splanchnic skeleton; that is to say, to the series of sclerous structures internal to the primary vascular arches, and which are best developed in the branchial arches of fishes. The other hæmal arches are, as has been stated, the pterygoid, palatal, and maxillary, belonging respectively to the presphenoid, ethmovomerine, and facial segments. The maxillary arch, having assumed in the mammal as much as possible the aspect of a neural arch, as has been above explained, although in reality hæmal, has a bone radiating from it—the malar, which tends to form connexions with the bones radiating from the neural arches behind, viz. with the lacrymal, external angular process of the frontals, and the squamous. The lower jaw is not a hæmal arch at all; that is to say, it is not an arch belonging to a single segment and corresponding to the palate-bones, or to a pair of ribs, but is a limb-arch. For I hold that Professor GOODSIR has distinctly shown that the shoulder-girdle and pelvic girdle are not rib-arches, and that limbs do not belong merely to single segments; but seeing, on the other hand, that these girdles pursue an arched direction, I conceive that they are to be considered as arches external to the series of rib-arches, and supporting radiations, just as rib-arches sometimes do. Such a limb-arch is the lower jaw; but the visceral laminæ being very imperfect in the head, it is intimately connected with arches bounding the hæmal cavity. In the fish it supports extensive radiations, viz. the opercular apparatus, but in the mammal it bears no radiations. The quadrate jugal in the bird has every appearance of being a radiation homologous to the operculum; for not the quadrate jugal, but the bone called mastoid by Professor OWEN, corresponds to the mammalian squamous. The quadrate bone of the bird has been abundantly proved to correspond to the incus of the mammal. The elements of the limb-arch are the incus, malleus, and lower jaw; and its superior member, the incus, is, throughout the vertebrate series, connected with the mastoid and squamous bones. As for the stapes, it is a radiation of the petrous sclerotome, and corresponds to the structure which supports the eye of the Shark, and to the ethmoidal turbinations in the presphenoidal and ethmovomerine sclerotomes. While, on the one hand, the rib-arches belong to the individual segments of the body, on the other there are three limb-arches, corresponding to the three great regions of the body: one for the head—the region in which the highest development of organs of animal life takes place; one for the cervical region—the region in which is the highest development of the vascular organs; and one for the abdomen—the region in which the greatest development is found of organs of vegetable life.

EXPLANATION OF THE PLATES.

PLATES IV. & V.

Fig. 1. Bones from specimen referred to in the text as No. 1.

A. The vomer, ethmoid, sphenoidal spongy bones, and left palate and maxillary bones, from the skull of an infant, slightly enlarged, and viewed from behind:—*a*, orbital plate of the ethmoid; *b*, posterior extremity of the vomer; *c*, sphenoidal process of the palate-bone; *d*, orbital surface of the palate-bone, and, immediately above it, the orbital portion of the sphenoidal spongy bone: between the two processes of the palate-bone is the spheno-palatine foramen, completed above by the inferior portion of the sphenoidal spongy bone: *e*, the superior portion of the sphenoidal spongy bone; *f*, inferior portion.

B. Another view taken from the same specimen:—*a*, *b*, *c*, the parts of the inferior margin of the vomer which articulate with the palate, maxillary, and intermaxillary bones respectively; *d*, inferior aspect of the sphenoidal spongy bone; *e*, orbital plate of the ethmoid, seen in perspective; *f*, inferior turbinated process of the ethmoid.

Fig. 2. Bones from specimen referred to in the text as No. 2.

A. Ethmoid, with sphenoidal spongy bones attached:—*a*, cribriform plate; *b*, os planum; *c*, superior portions of the sphenoidal spongy bones, with space between them for the body of the presphenoid; *e*, *e*, orbital portions; *f*, *f*, inferior portions.

B. Sphenoid:—*a*, body of postsphenoid; *b*, body of presphenoid.

Fig. 3. Bones from specimen referred to in the text as No. 3.

A:—*a*, part of the orbital plate of the frontal; *b*, os planum; *c*, presphenoid;

d, external pterygoid plate; *e*, palate-bone; *f*, orbital surface of sphenoidal spongy bone; *g*, the space between the orbital and sphenoidal processes of the palate-bone, bridged over by the sphenoidal spongy bone.

B and C. Left sphenoidal spongy bone; viewed in B from its external, and in C from its superior aspect:—*a*, superior portion; *b*, orbital portion; *c*, inferior portion.

Fig. 4. A. Vomer, with the portions of the palate, maxillary, and intermaxillary bones with which it was articulated:—*a*, the process which rests upon the intermaxillaries; *b*, the process which descends behind the intermaxillaries.

B. Vomer from a fœtus:—*a*, *b*, *c*, the margins for articulation with the palate, maxillary, and intermaxillary bones respectively.

Fig. 5. Diagram of the vomer, ethmoid, and sphenoidal spongy bones of the human subject, with the vomer and ethmoid of the Sheep represented in dotted lines. The superior edge *a* of the human ethmoid corresponds to the edge *a'* of the Sheep's ethmoid; while *c* is the sphenoidal spongy bone rendered necessary in the human subject to fill up the space between *b'* and *b*.

Fig. 6. A. Posterior extremity of the vomer in the Orang.

B. The same in the Chimpanzee.

C. The same in the Gorilla.

Fig. 7. Bones from the Cat.

A. Vomer and ethmoid seen from below:—*a*, margin of vomer for articulation with the maxillaries; *b*, *b*, ethmovomerine laminæ.

B and C. Views of the presphenoid from the front and from behind.

D. Portion of base of skull of a new-born kitten:—*a*, intermaxillaries; *b*, vomer; *c*, presphenoidal centrum; *d*, postsphenoidal centrum; *e*, basioccipital; *f*, ethmoidal turbinations; *g*, pterygoid; *h*, tympanic.

Fig. 8. Vomer and ethmoid of the Fox:—*a*, ethmovomerine laminæ; *b*, margin of vomer for articulation with the maxillaries and palatals; *c*, margin for articulation with the intermaxillaries; *d*, groove corresponding to the upper part of the nasal foramen of the palate-bone.

Fig. 9. Intermaxillaries of *Ursus maritimus*.

A. Inferior aspect.

B. Superior aspect:—*a*, maxillary; *b*, mesial-palatine process of intermaxillary; *c*, incisive foramen; *d*, foramen peculiarly ursine; *e*, vomer.

Fig. 10. Vomer and ethmoid of *Phoca vitulina*:—*a*, vomerine groove; *b*, lateral mass of the ethmoid; *c*, central plate; *d*, expanded superior margin of central plate.

Fig. 11. Part of the skull of the Hedgehog:—*a*, frontal; *b*, nasals; *c*, left intermaxillary; *d*, vomer, with right ethmovomerine lamina removed; *e*, flat inferior margin of the central plate of the ethmoid; *f*, presphenoid; *g*, groove on the left ethmovomerine lamina corresponding to the nasal foramen of the palate-bone.

Fig. 12. The vomer and lateral masses of the ethmoid of the Lamb, seen from below:—*a*, the inferior margin of the vomer, rough posteriorly for articulation with the maxillaries, and smooth anteriorly where it comes in contact with the intermaxillaries; *b*, *b*, the grooves which complete the nasal foramina of the palate-bones. The spaces between these grooves and the margins of the vomer represent the ethmovomerine laminæ; and on the outer aspects of the grooves are the small orbital surfaces of the ethmoid.

Fig. 13. Intermaxillaries of the Sheep:—*a*, *a*, incisive foramina; *b*, *b*, portions of the maxillaries; *c*, space for the anterior extremity of the septal cartilage.

Fig. 14. Intermaxillaries of the Alpaca. Letters as in fig. 13.

Fig. 15. Intermaxillaries of the Giraffe. Letters as in fig. 13.

Fig. 16. Part of the skull of the Horse:—*a*, vomer (the anterior half removed); *b*, part of the palate-bone; *c*, part of the pterygoid; *d*, ethmovomerine lamina; *e*, nasal foramen of the palate-bone.

Fig. 17. Section of the fore part of the skull of the Horse:—*a*, fore part of the vomer, resting on the maxillaries, and articulating in front with the intermaxillaries; *b*, incisive foramen; *c*, cul-de-sac for the anterior extremity of the septal cartilage.

Fig. 18. Vomer, &c. from the skull of the sucking Pig:—*a*, presphenoid; *b*, portion of the vomer which rests on the intermaxillaries; *c*, ethmovomerine lamina and part of the ethmoid; *d*, various distinct ossifications belonging to the ethmoid.

Fig. 19. Vomer, &c. of the young Elephant:—*a*, *b*, surfaces by which the intermaxillary and maxillary come in contact with their fellows of the opposite side; *c*, incisive foramen; *d*, *e*, processes of the vomer, by which its inferior and superior margins are made continuous with the corresponding margins of the intermaxillaries.

Fig. 20. Anterior nares of the Tapir:—*a*, vomer; *b*, maxillary; *c*, palate-plate of maxillary; *d*, intermaxillary; *e*, foramen corresponding to both incisive foramina and the space for the point of the septal cartilage in the Camel &c.; *f*, lateral mass of the ethmoid; *g*, expanded superior margin of the central plate of the ethmoid; *h*, united nasals.

Fig. 21. Part of the skull of *Cœlogenys*:—*a*, *b*, mesial borders of the maxillary and intermaxillary; *c*, vomer.

Fig. 22. View of the articulations of the vomer in the Rabbit. Above are the vomer and ethmoid forming one bone. Beneath are the bones of the upper jaw of the left side, and a portion of the intermaxillary of the right side adherent:—*a*, anterior extremity of the vomer grooved for the cartilaginous septum of the nose; *b*, the part of the vomer which articulates with *c*, the extremity of the expanded mesial processes of the intermaxillaries forming turbinations in connexion with JACOBSON'S organ.

Fig. 23. Part of the skull of the Kangaroo:—*a*, presphenoid; *b*, frontal; *c*, ethmovome-

rine lamina; *d*, nasals; *e*, central plate of the ethmoid; *f*, lateral ridge of the vomer peculiar to marsupiate animals.

Fig. 24. Part of the skull of the Dugong:—*a*, frontal; *b*, intermaxillary; *c*, vomer; *d*, maxillary; *e*, *e*, palatals; *f*, *f*, ethmovomerine laminæ and lateral masses of the ethmoid.

Fig. 25. Section of skull of *Delphinus orca*:—*a*, intermaxillary; *b*, maxillary; *c*, *c*, vomer; *d*, palatal; *e*, central plate of the ethmoid.

Fig. 26. Skull of *Delphinus globiceps* from above:—*a*, intermaxillary; *b*, *b*, maxillary; *c*, palatal; *d*, pterygoid; *e*, nasal foramen of palate-bone; *f*, central plate of the ethmoid; *g*, expansion of the ala of the vomer; *h*, the part of the frontal which would have been concealed by the nasal, had not that bone been removed.

Fig. 27. Diagram of the two most anterior segments of the mammalian skull, and of the neural arches of the other segments. The dotted part represents the extent of the septal cartilage. The centra are darkened. Two dotted lines and arrowheads show the direction of the anterior hæmal arches:—*a*, occipital segment; *b*, auditory segment; *c*, sphenoparietal segment; *d*, optic segment; *e*, frontal bone; *f*, central plate of the ethmoid; *g*, lateral mass of the ethmoid; *h*, vomer; *i*, palatal; *k*, nasal; *m*, intermaxillary; *n*, maxillary.

Fig. 28. Vomer and palatals of the Duck, viewed from above.

Fig. 29. A. Section of the skull of the Cod. The cranial cavity is represented dark; the cut surfaces of bones are marked with oblique lines, and the cartilages are left white, while the interorbital septum is dotted:—*a*, intermaxillary; *b*, vomer; *c*, basisphenoid; *d*, basioccipital; *e*, nasal; *f*, frontal; *g*, supra-occipital.

B. A similar section through the nasal and vomer, after the cartilage has been removed:—*b*, vomer; *e*, nasal; *h*, prefrontal.

Fig. 30. A. Intermaxillary and vomer of *Python*:—*a*, intermaxillary; *b*, vomer.

B. Nasal and vomer of *Malapterurus*:—*a*, nasal; *b*, vomer.

Fig. 31. Diagram of the two most anterior segments of the skull of the fish:—*a*, prefrontal; *b*, vomer; *c*, palatal; *d*, nasal; *e*, intermaxillary; *f*, maxillary; *g*, a dotted circle indicating the position of the nostril. The centra are darkened. The dotted lines and arrowheads show the direction of the hæmal arches.

XVII. *On the Properties of Electro-deposited Antimony* (concluded). *By* G. GORE, *Esq.* *Communicated by Professor* STOKES, *Sec. R.S.*

Received May 24,—Read June 19, 1862.

Second variety of active Electro-deposited Antimony.

92. IN addition to the variety of electro-deposited antimony obtained from a solution of teroxide of antimony and hydrochloric acid, a second variety may be obtained from a solution of terbromide of antimony in the following manner.

93. Dissolve one part of teroxide of antimony in 10 parts of hydrobromic acid of sp. gr. about 1·3, filter the solution through a funnel loosely plugged with asbestus, and electrolyse it by means of three SMEE's elements and an anode of antimony in the usual manner, at a speed of deposition of about 3 to 5 grains per square inch per hour. The exact proportion of the ingredients of the solution is not a matter of great importance.

94. The electrolytic conduction is free, and a coherent metallic deposit is quickly obtained. The deposit, when thin, is occasionally quite bright and black, and very similar in appearance to that obtained in the chloride solution, but generally it is of a much lighter colour and quite dull in aspect; it is at first scaly, exhibiting the unequal cohesive action already described (17.), but in a less degree than the first variety, and emitting but slightly audible crackling sounds during its formation. During the process of deposition solitary bubbles of gas occasionally adhere to the receiving surface, and cause deep conical holes in the deposit, especially if the solution is not sufficiently dilute; in escaping, these bubbles sometimes emit a chirping sound; in some instances the holes were so numerous as to give to the substance the appearance of a metallic sponge.

95. The deposited substance is firm and moderately hard, but rather less so than that formed in the chloride solution; its fractured surface has also a bright metallic lustre similar to that of the first variety. The specific gravity of two specimens was 5·415 and 5·472 at 60° Fahr.

96. By contact with a red-hot wire, it exhibits a similar molecular and thermic change to that manifested by the chloride variety (19.). A piece $\frac{1}{8}$th of an inch thick, at 60° Fahr., touched by a red-hot wire, exhibited strong molecular action at the points touched, but the action did not spread throughout the mass. A second similar piece, heated to about 200° Fahr. in a porcelain capsule upon the surface of boiling water, when touched by the heated wire manifested stronger and more extended change, but did not discharge the whole of its heat except by numerous contacts of the wire. A third piece in a platinum dish upon the surface of a solution of chloride of calcium at 268° Fahr., evolved all its heat instantly with explosive violence and projection of pieces of the metal by a

single contact of the red-hot wire. Pieces $\frac{1}{10}$th of an inch thick in a platinum dish upon a solution of chloride of calcium at 295° Fahr., did not discharge their heat by repeated scratching with a sharp steel pointer, but instantly discharged it by contact of a heated wire, and shattered themselves to pieces.

97. On placing a small fragment upon melted fusible alloy, and gradually heating the latter, the fragment discharged all its heat suddenly and powerfully, and shattered all to pieces, when the alloy obtained a temperature of 318° Fahr. A similar piece upon the surface of mercury discharged itself similarly when the mercury attained a temperature of 322° Fahr. And another piece, $\frac{1}{8}$th of an inch thick, weighing 33 grains, immersed in a solution of chloride of calcium at 272° Fahr., suddenly discharged its heat and shattered to pieces at the end of four minutes, when the solution was at 275° Fahr. A fragment $\frac{1}{10}$th of an inch thick, formed three years and four months previously (viz. in December 1858), was put in a platinum dish upon a solution of chloride of calcium at 268° Fahr., and touched with red-hot wire; it instantly discharged its heat with violence, scattering fragments of the metal in all directions. The temperature of sudden discharge appears from these circumstances to lie between 270° and 300° Fahr.; but depends upon several conditions, especially upon the degree of rapidity with which the heat is carried away in relation to that with which it is generated. The fractured surface of pieces suddenly discharged is brighter than that of the unchanged substance.

98. Thick pieces of the active substance may be easily reduced to powder in a mortar without causing them to discharge their heat. A small quantity of the powder formed into a train upon the surface of glass or metal at 60° Fahr., and touched by a red-hot wire, only exhibited the change near where it was touched; but if the powder was previously placed in a platinum dish upon a solution of chloride of calcium at 265° Fahr., and then touched at one end by the heated wire, it discharged its heat gradually, and the change, attended by evolution of fumes, was slowly propagated to the distant extremity.

99. The active substance was *gradually* discharged of its heat in the following manner:—A piece barely $\frac{1}{8}$th of an inch thick, and weighing 40·05 grains, was suspended during thirty minutes in a solution of chloride of calcium at 252° Fahr.; when taken out, washed with water, dried at 240° Fahr., cooled and weighed, it had lost 2·45 grains =6·117 per cent. It was then reheated in a platinum dish upon a solution of chloride of calcium at 250° Fahr., and touched six or eight times with a red-hot wire; not the slightest sign of molecular change occurred, and its weight remained but slightly altered: on breaking, it was found to have become less brittle and of stronger cohesion; its fractured surface was quite dull and earthy, and exhibited the appearance of a dark inner core surrounded by a layer of a lighter colour, as if an action or change had proceeded from the outer surface towards the centre.

100. 20·3 grains of it, fused in an analysis-tube (82.), gave 17·20 grains in the form of a metallic button, =79·52 per cent. of metal, and 20·48 per cent. of volatile matter, in the original unchanged substance; and there distilled into the bent part of the tube a

quantity of colourless buttery substance, which was slightly semifluid at 60° Fahr., and doubtless consisted of terbromide of antimony and a little aqueous hydrobromic acid, but was not further examined. In another case, two portions of the unchanged substance lost respectively 18·42 and 20·40 per cent. of volatile matter by fusion in the analysis-tube; the two portions were parts of a single piece of the deposited substance, the one losing 18·42 per cent. being from the upper part, and the other from its lower part, as it hung in the electrolyte.

101. Another mode of *gradually* discharging its heat was as follows:—A piece $\frac{1}{8}$th of an inch thick was placed upon the surface of mercury at 260° Fahr. during $1\frac{1}{4}$ hours, at the end of which time it was found, by repeated contacts of a red-hot wire, to have entirely lost its heating power; its cohesion was greatly increased, and its fractured surface was dull and earthy in appearance; its odour and taste were also acid: a portion was reduced to powder; the powder also possessed an acid taste, and strongly reddened damp litmus paper, whilst the powder of the *unchanged* substance did not. Another piece $\frac{1}{10}$th of an inch thick, weighing 23·52 grains, was placed upon platinum foil upon the surface of mercury at 265° Fahr., and kept at that temperature forty-five minutes, then cooled and weighed; it had lost 1·38 grain =5·86 per cent.; it was then reheated on the foil and mercury to 270° Fahr., and touched several times with a red-hot wire, but no signs of the change occurred.

102. The electro-chemical equivalent of this substance was examined in the following manner:—Two similar depositing cells, one filled with the chloride solution (90.) and the other with the bromide liquid (93.), and containing anodes of antimony and previously weighed cathodes of sheet platinum of equal size, were connected in a single line with five SMEE's elements, and deposits simultaneously formed upon them during two days; they were then taken out, wiped dry, allowed to stand twelve hours to further dry the one from the bromide solution on account of its porosity, and then weighed; then replaced in the liquid to receive further deposits during two days, and again dried and weighed. In two determinations of this kind there were obtained respectively 50·07 and 50·11 parts of the bromide deposit for every 42·5 parts of the chloride deposit, or for every single equivalent, or 32·2 parts of zinc consumed; and in two other determinations 51·2 and 51·4 parts of bromide deposit were obtained. Each of these quantities of deposit contains the same amount of metallic antimony, viz. 40 parts, or $\frac{1}{3}$rd of an equivalent. The variation in the numbers obtained was probably caused by the porosity of the deposit formed in the bromide solution.

Third variety of active Electro-deposited Antimony.

103. A third variety of heat-giving electro-deposited antimony remains to be described, and was obtained in the following manner:—Dissolve 1 part by weight of teroxide of antimony in 15 parts of hydriodic acid of sp. gr. 1·25; filter the solution from any oxide that may remain undissolved, and electrolyse it by two or three SMEE's elements very feebly excited and an anode of antimony in the usual manner, and at a speed of deposition not

exceeding 1 grain per square inch per hour. The exact proportion or strength of the ingredients is not important, provided the acid is not too concentrated, sufficient oxide is dissolved in it, and the electric current is very feeble.

104. The deposit is generally at first scaly, especially if the voltaic current is rather strong; but afterwards it is dull in appearance and grey like that formed in the bromide solution, and the tendency to the evolution of gas is much greater than in that liquid. It is also liable to cohesive action and crackling (17. 94.), but in a less degree than the other kinds. Its cohesion is very feeble, and it is much more friable than the bromide variety; its fractured surface is dull and earthy in appearance, and it has a much less metallic character than either of the other kinds, unless it has been deposited with extreme slowness; it is then sometimes bright and black like a blackleaded surface of iron; its cohesion is then also greater, and its fractured surface black and bright. The specific gravity of one specimen which had been formed moderately slowly was 5·27 at 60° Fahr.

105. Repeatedly in depositing this variety in a coherent state at a moderate speed, I observed at the lower part of the deposit on each side of the cathode horizontal lines about $\frac{3}{8}$ths of an inch apart, of a lighter colour than the rest of the deposit. The contrast of colour was caused by a black powdery deposit adhering to the other portions and not upon the lines, and appeared to be more prominent at the lines; but what was the cause of the lines I have not ascertained.

106. On immersing dry pieces in water a hissing sound is produced, evidently by powerful absorptive action, and numerous bubbles of gas are evolved from the whole of its surface during about five or ten seconds, and a few small ones afterwards.

107. A small piece, $\frac{1}{9}$th of an inch thick, placed upon platinum foil upon mercury at 200° Fahr. and touched by a red-hot wire, exhibited no signs of evolving heat; but with the mercury at 338° Fahr., and touched as before, it discharged its heat feebly and slowly, and evolved iodide of antimony. A second similar piece, heated from 300° Fahr. upwards upon platinum foil upon mercury, discharged its heat with evolution of iodide of antimony when the mercury had attained a temperature of 350° Fahr.; but the thermic action was feeble, and occupied about 20 seconds. Its temperature of sudden discharge appears therefore to be about 340° Fahr., and the amount of heat evolved is apparently very much less than with the other varieties. If exposed to strong sunlight during two hours, it acquires a reddish-brown colour externally.

108. The dried substance, fused in an analysis-tube (82.), gave 77·76 per cent. of a metallic button, and a solid, red, easily fusible sublimate, together with a little moisture, manifestly teriodide of antimony, and a little aqueous hydriodic acid.

109. The electro-chemical equivalent of this variety was examined in the same manner as the preceding one (102.). In two determinations, first with slow action (0·5 gr. per square inch per hour), 50·39 parts of deposit were obtained; and second, with very slow action (0·2 gr. per square inch per hour), 48·07 parts were obtained for every single equivalent, or 42·5 parts in the chloride solution. The numbers were lessened in con-

sequence of a small quantity of the deposited substance, which was not firmly deposited, falling off in a state of powder to the bottom of the electrolyte.

Further particulars respecting the first variety.

110. Several experiments (of which the following is an example) were made to determine the influence of speed of deposition upon the state of aggregation, &c. of the deposited metal. Five parts of pure oxychloride of antimony were dissolved in 20 parts of pure hydrochloric acid of sp. gr. 1·15, and the solution (contained in one vessel) electrolysed simultaneously by two separate currents from two single SMEE'S elements, one of which contained plates 6·3 times the size of the other, each charged with the same exciting liquid, viz. 1 measure of acid to 18 of water: the antimony anode of the large battery possessed six times the amount of surface of that of the other, but the cathodes were of equal size. The speed of deposition by the small battery was 0·75 gr. per square inch per hour, and by the large one 3·22 grs., or 4·26 times the amount. The rapid deposit exhibited much scaly cohesive action (17.), but the slow one did not. The slow deposit consisted nearly wholly of grey crystalline metal, which did not possess the thermic power, whilst the rapid one was nearly wholly black and amorphous, and possessed the usual heating quality. Owing to the decline of the current, the deposit upon both sides of the slow one was at first black and amorphous, and then became grey; and that upon the back of the rapid one consisted of alternate layers, black, grey, and black, produced by variations of the electric current, by gradual decline, and additions of acid.

111. It is particularly worthy of notice, that whether the alteration of electric power was gradual, as by slow exhaustion of the battery liquid, or sudden, as upon the additions of acid, the change of aggregation and of chemical composition of the deposited metal was sudden in both cases, in accordance with that rigid distinction or want of gradation between the two varieties of deposit already noticed (3.). The cohesion between the alternate layers was much more feeble than between the parts of the layers themselves, and the layers could be readily separated by means of a knife; the lines of junction were quite distinct and definite, although the layers were in perfect contact, and no films of depositing liquid appeared to be enclosed between them. These sudden changes of aggregation are particularly interesting.

112. It appears from this and other similar experiments, that with a suitable liquid, &c., when the quantity of the electric current is small in relation to the amount of receiving surface, the deposited metal is grey and crystalline, and possesses no heating power; and that when it is relatively large the metal is dark-coloured and amorphous, and has the peculiar thermic properties; other circumstances, such as high temperature of the solution, or excess of acid (9. 10. 11.), will also produce grey metal.

113. In several instances, when the active variety had been *rapidly* deposited, it was less bright and smooth, its colour was lighter, its fractured surface was coarse, and on

discharging its heat by a heated wire, the thermic change was unusually rapid and strong, shattering the metal throughout into scaly particles with almost explosive violence. The shattered mass consisted of parallel scaly layers, possessing an angular position with regard to the receiving surface, their horizontal edges pointing upwards at an angle of about 40° towards the surface of the liquid.

114. Two deposits were also simultaneously formed, both of the black amorphous kind, but at *equal* and nearly uniform rates of deposition, viz. 0·55 gr. per square inch per hour, upon cathodes of equal size and at equal distances from the anodes, by two separate currents, in a similar solution to that of the previous experiment (110.) contained in a single vessel, one current being generated by one SMEE's element of large surface, and the other by twenty of small surface, the exciting liquid in each battery being the same, and the plates the same distance asunder. Both the deposits appeared alike, except that the one formed by the single element had a few small nodules of grey metal formed upon it during the latter part of the process. The active deposit formed by the twenty elements lost 7·4 per cent. by fusion in the analysis-tube (82.), and 2·7 per cent. by discharging it in the air at 60° Fahr. by a heated wire; and that formed by the single element lost respectively 7·1 and 1·86 per cent. under similar circumstances. These differences of results in the two cases may have been due to other circumstances than the difference of number of elements.

115. It has been already shown (66. 75. 83.) that the discharge of heat may take place gradually, or even in fragmentary portions, by careful management of the temperature in an air-bath, or easily by immersing the substance for a greater or less period of time in boiling water. By suitable treatment, different parts of a given piece of the active substance may also be either wholly or partly discharged of their heat. In one instance about $\frac{3}{4}$ths of an inch of the lower end of an active bar, 2 inches long, $\frac{1}{2}$ an inch wide, and $\frac{3}{16}$ths of an inch thick, was immersed in cold water, the water heated to boiling in 6 minutes, and kept boiling 20 minutes; the lower end was then found to have lost its heating power, whilst the upper end remained unaltered; and in several other similar experiments similar results were obtained. In these and all other cases where the active substance was slowly discharged, the substance became much harder and much more difficult to break.

116. The thermo-electric relation of the changed to the unchanged substance was examined as follows:—Two similar active bars were made of the shape of the annexed figure, containing small silver studs (A and B) in their extremities, enclosed by the antimony during the process of deposition; one bar was discharged of its heat gradually by immersing it in boiling water one hour;

it was then placed parallel to the other, but separated by bits of cork and india-rubber; the two studs at one end of the compound bar were connected together by silver wire, and that end immersed in a solution of chloride of calcium at 60° Fahr.; the other studs and ends, being previously connected by silver wires with a galvanometer, were wrapped in oil-silk and immersed in water at 60° Fahr.; the chloride-of-calcium solution was now heated to 212° Fahr. in 35 minutes; a gradually increasing deflection, amounting at its maximum to 42½ degrees, occurred, the discharged bar being thermo-electro-positive to the active one; the temperatures were maintained during 2 hours and 40 minutes, during which time the deflection decreased to 38½ degrees. A second similar experiment was made with a pair of bars like the last, one of them being wholly active, and the other previously made inactive at one end only, viz. the end most distant from the galvanometer, by gradual discharge in boiling water during 30 minutes. With one end of this compound bar in water at 50° Fahr., and the other in boiling water, a deflection of 28 degrees occurred, the discharged end being thermo-electro-positive to the active end of the adjoining bar. A similar experiment with a single active bar of the above angular shape, with its ends dipping into two vessels of water, one at 57° and the other at 212° Fahr., and connected with the galvanometer by silver wires, gave a deflection of 42½ degrees, which decreased to 40 degrees in 1 hour and 25 minutes.

117. In numerous experiments the suddenly and also the gradually discharged substance was found in every instance to be decidedly electro-positive to undischarged portions of the same pieces in the following liquids—pure sulphuric, hydrochloric, or nitric acids, freely diluted with water; also in aqueous solutions of ammonia, sesquicarbonate of ammonia, hydrate of potash, and carbonate of soda.

118. A bar of the unchanged substance placed axially in a copper wire helix, the ends of which were connected with a galvanometer, on having its heat suddenly discharged by contact of a heated wire with one of its extremities excited no appreciable electric current in the wire.

Comparison of the three varieties.

119. They each require a solution containing an excess of free acid in addition to the amount necessary to dissolve the teroxide, the proportion of free acid necessary being greatest with the chloride and least with the iodide.

120. The rate of deposition admissible to obtain a firm deposit, and also the range of speed of deposition, is greatest in the chloride and least in the iodide solution; being about 0·5 to 10·0 grains per square inch per hour in the chloride, 3 to 5 grains in the bromide, and 0·5 grain or less in the iodide. The tendency to the evolution of hydrogen gas at the cathode is least in the chloride and greatest in the iodide: this tendency frequently causes the iodide deposit to be completely disintegrated.

121. Each of the deposits exhibits the phenomenon of unequal cohesion and cracking (17.), but that formed in the chloride solution exhibits it the most strongly. The metal obtained from the iodide solution is the most friable, and that from the chloride the least so. The chloride compound is the most metallic in appearance, and the iodide one the least. The specific gravities of the three varieties are also in series; that of the chloride deposit is 5·8, the bromide one 5·44, and the iodide one 5·25.

122. They each possess the power of evolving heat, and this quality appears at

different temperatures in the three substances, being at about 200° Fahr. in the chloride deposit, about 280° in the bromide, and about 340° in the iodide. The manifestations of heat evolved also vary, being considerable in the chloride deposit (73.), apparently less in the bromide one, and but very small in the iodide compound. They each become much less metallic in appearance by a gradual discharge of their heat, and the gradual discharge is also attended by an increase of cohesive force, most in the chloride and least (if any) in the iodide deposit. The iodide compound is also the most affected by solar light.

123. The chloride deposit contains about 6·3 per cent. of saline matter, the bromide about 20, and the iodide about 22·2. The pseudo-electro-chemical equivalent (if I may apply this term to such cases) varies in each variety, and is dependent upon the amount of saline matter which unites with the true equivalent of metallic antimony deposited; it is about 42·5 with the chloride deposit, 50 with the bromide, and 51 with the iodide.

Conclusion.

124. A probably correct explanation of the formation and properties of these several metallic deposits is as follows:—The electric current in passing decomposes the salt of antimony, setting free 1 equivalent of chlorine, bromine, or iodine, and $\frac{1}{3}$rd of an equivalent of antimony for every single equivalent of zinc consumed (48. 102. 109.). The antimony in the act of depositing, being in what is termed the "nascent" state, unites chemically, in a comparatively feeble or unstable manner, with the elements of the electrolyte, combining with them in an indefinite and somewhat variable proportion. The $\frac{1}{3}$rd equivalent, or 40 parts of antimony, carry down from $2\frac{1}{2}$ to 3 parts from the chloride solution, about 10 parts from the bromide liquid, and about 11 parts from the iodide solution, and thus occasion about 42·75 parts of deposit in the first liquid, about 50 parts in the second, and 50·5 parts in the third solution, for each equivalent of zinc consumed.

125. Another explanation, which has nearly an equal weight of evidence in its favour, is that the antimony is deposited in the "*amorphous*" state, and the chloride or other salt is enclosed mechanically in it during the process of deposition, and that the change consists in the assumption by the metal of the crystalline state, whereby it is converted into an inconceivable number of crystals of insensible size, and the imprisoned salt is set free.

126. Antimony is not the only metal that manifests this property of uniting during electrolysis with portions of the electrolyte; several other metals also show it: it is well known that silver deposited by electrolysis from a solution of the double cyanide of silver and potassium containing a little bisulphide of carbon, is harder and very much brighter than that deposited from the same solution without that ingredient; this bright deposit is not wholly metallic silver, but contains a small proportion (about 1 per cent.) of some other ingredients of the electrolyte.

I have not examined the laws or conditions that regulate the proportions of the substances that unite by this species of combination.

127. The substance formed in either of these solutions of antimony may be viewed as a feeble chemical compound of metallic antimony with a salt of antimony: that it is not a purely mechanical mixture is rendered very probable by the fact of its powder not reddening moistened litmus paper, especially in the case of that obtained from the bromide solution, which contains as much as 20 per cent. of saline and acid matter, whereas immediately after the gradual change has occurred it has the power of reddening litmus strongly (101.), and the salt of antimony contained in it may be much more readily extracted by dilute hydrochloric acid (83.); these facts also indicate that the change is, at least in part, a case of chemical decomposition, the substances remaining together after the decomposition in a state of mechanical mixture.

128. The compound deposited is evidently not a *direct* result of electrolysis, otherwise it would be deposited in the proportion of its electro-chemical equivalent; nor is it a *definite* chemical compound, because there is no equivalent or atomic proportion between the quantity of metallic antimony and that of the salt with which it is combined, and because the proportions of these two ingredients are in each case somewhat variable.

129. The decomposition of these deposits, like that of peroxide of hydrogen, is attended by evolution of much heat; it cannot, however, like the decomposition of that substance, be viewed as a result of a tendency of the separated ingredients to assume the gaseous or even the liquid state, because the compound set free is in each case in a nearly solid condition, but must be referred to some other and at present unknown cause.

130. The union of the nascent antimony with the salt of antimony is evidently dependent upon the rate of deposition; for when the speed of deposition in the chloride solution fell below about 0·75 or 0·5 grain per square inch per hour, the antimony lost its power of combining with the antimonial salt, and was deposited alone in the grey or crystalline state (110.). It is also dependent upon the temperature of the liquid (11.), and upon the proportions of the ingredients of the solution (9.).

131. Numerous and varied electrolytic experiments were made with solutions of arsenic, and small portions of scaly deposit were obtained from the aqueous fluoride, which exhibited in a comparatively feeble and indistinct degree (like the third variety of active antimony) a similar thermic property to that manifested by electro-deposited antimony.

XVIII. The Bakerian Lecture.—*On the Total Solar Eclipse of July* 18*th*, 1860, *observed at Rivabellosa, near Miranda de Ebro, in Spain.*

By Warren De La Rue, *Esq.*, *PhD.*, *F.R.S.*, *Hon. Sec. Royal Astron. Soc.*, *Treasurer Chem. Soc.*, *&c.*

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My attention was first called to the Solar Eclipse of 1860, in the latter part of the year 1858, on the occasion of my visiting Russia, when Dr. Mædler placed in my hands a copy of his anticipative pamphlet, entitled "L'Eclipse Solaire du 18 Juillet, 1860."

This paper contained a Map of Spain, with certain lines indicating the position of the central path of the moon's shadow, the limits of totality, and its epoch at various localities; and it occurred to me, on perusing it, that, if circumstances should permit of my observing the eclipse, Santander would be very convenient for the disembarcation and erection of the instruments I should, in all probability, require for photographic observations, to the prosecution of which my successful researches in astronomical photography led me to think I ought to devote myself. On communicating my plans to Mr. Vignoles, he strongly recommended me to cross to the southern side of the Pyrenees in order to avoid the mists which are caused by the condensation of vapours from the ocean against the northern slopes of the mountains. Subsequently Mr. Vignoles published an eclipse-map of Spain on a very large scale, and I selected Miranda for my station; but he suggested that I should place my observatory at Rivabellosa, about two miles from that town.

It is fortunate that I changed my station from Santander to Rivabellosa, as many of those astronomers who selected the former place were prevented by the state of the atmosphere from observing the eclipse.

On my journey to Russia, I stopped at Königsberg and made the acquaintance of Dr. Luther, who showed me the Daguerreotype of the total eclipse of 1851, which had been taken by Dr. Busch with the Königsberg heliometer. Great credit is due to Dr. Busch for that successful pioneering experiment, more especially when due allowance is made for the uncertainty then existing as to the brilliancy of the prominences, and for the state of the photographic art at that epoch. In the interval of seven years, however, astronomical photography had made great progress; and I recollect being much struck with the very indifferent definition of the protuberances in the Daguerreotype, from which I inferred the impracticability of deriving any conclusive evidence respecting the nature of such appearances from photographs, unless more distinct ones could be obtained. The inspection of the Königsberg Daguerreotype subsequently exercised some influence on my plan of procedure. Discarding all thoughts of employing the Daguerreotype process, because the collodion process was far more sensitive and convenient, I chose the latter as best suited to my purpose, although I knew perfectly well, from experience, how frequently the collodion-film is rendered defective by specks, streaks, and even minute holes. It was open to me to employ an achromatic or a reflecting telescope of ordinary construction, and to place the sensitive plate in the principal focus; but I was aware that the largest telescope I could possibly take with me would only give an image of a very moderate size, and that any of the before-named defects in the collodion might fall over and obliterate, or so confuse the impression of any prominence in one photograph, as to render its identification with its impression in a subsequent photograph a matter of impossibility. These considerations led me to think that it would be very desirable to employ the Kew photo-heliograph, because in this instrument the primary focal image of the sun is enlarged from about half an inch in diameter to nearly 4 inches, which is a scale amply sufficient to counterbalance the disadvantages of the collodion process; but, on the other hand, the light is thus attenuated sixty-four times, besides being absorbed to some extent in passing through the two lenses composing the secondary magnifier, an ordinary Huyghenian eyepiece; and this question consequently presented itself, Would it be possible with such an enfeebled image to get even a single impression during the whole duration of the totality? This was an extremely doubtful matter. By employing the Kew heliograph one would evidently run the risk of returning without any pictures of the *totality*, however many might be procured of the other phases of the eclipse *.

At the meetings of the Astronomical Society, and on other occasions, I made inquiries of those astronomers who had witnessed the eclipse of 1851, respecting the intensity of the light of the corona and red flames, as compared with that of the moon, and the relative brightness of the one to the other; but their answers did not tend to increase my hopes in respect of the possibility of procuring photographs of the totality by means of the Kew instrument. The general impression I formed from the information thus derived was, that the light emitted by the corona and red flames, taken together, was about equal to that of a full moon—less rather than greater; but no one recollected precisely the brightness of the prominences as compared with that of the corona. With this imperfect information as a guide, an attempt was made at Kew to photograph the moon, but not the slightest impression could be procured of our satellite by an exposure of the sensitive plate, during one whole minute †, to its image in the heliograph. My expectation of success in getting pictures of the totality was not great after this trial; nevertheless I still thought it desirable to carry on the experiment to the end, on account of the value of the results if I should fortunately succeed. It occurred to me several times to fit up also a photographic apparatus to an achromatic telescope, but I finally concluded that to attempt too many things would be sure to result in complete failure. I endeavoured, however, to stimulate other astronomers to

* Report on Celestial Photography, by the author, in the Reports of British Association, 1859, p. 152.

† While this paper was passing through the press a very faint impression of the moon was procured with the Kew heliograph in three minutes, with chemicals which gave a very strong impression of it in four seconds in the focus of my reflector of 13 inches aperture and 10 feet focal length.—August 1862.

make photographic experiments in the manner which I have indicated, as offering a greater probability of at least a partial success, so that the chances of obtaining pictures might be multiplied. With this object, I circulated as rapidly as I could my Report to the British Association on "Celestial Photography," which passed through the press in May 1860, and of which copies were extensively sent both to English and foreign astronomers at the latter end of May and the beginning of June.

I have now in this narrative to go back some months earlier than the period just alluded to, in order to connect it with the Himalaya Expedition, an expedition originating solely with, and organized by, the Astronomer Royal. When the year 1859 was drawing to a close, and I was turning my thoughts to the preparation which would be required for July 1860, Mr. AIRY mentioned that, if I had any intention of observing the eclipse, he might possibly be in a position to afford me some facilities for so doing, as he had it in contemplation to make an application to the Admiralty for a ship to convey intending observers to Spain, in the event of a sufficient number of astronomers expressing their willingness to join the expedition which he intended to organize. I expressed the satisfaction I felt in learning that he, the official head of astronomy in England, was willing to take the matter in hand, because I felt persuaded that, under his general direction, the expedition would prove a successful one; and I at once volunteered to form one of his party.

It was intimated to me that, in the event of my taking charge of the Kew heliograph, I should not be expected to entail upon myself the expenses of fitting it up for the object contemplated, or the personal expenses of the assistants who might accompany it, it having been from the first intended, that a grant from the Government Fund should be asked for to defray these charges. When, therefore, I had finally decided on taking charge of the instrument, I was requested to propose such a sum as I thought fully adequate to the purpose, and I named £150, which was granted. The entire expenses of the photographic expedition amounted to more than three times that sum, the balance being defrayed by myself.

The actual preparations were commenced at the latter end of January 1860, first of all by Mr. BECKLEY, the mechanical assistant of the Kew Observatory, and were continued, so far as his other occupations would permit, until the month of June; but, in spite of every exertion on his part, so much remained to be done, that in June I engaged Mr. REYNOLDS (now my private photographic assistant) to aid in completing the arrangements. My party, besides myself, was, after a few changes, thus finally constituted:—Mr. BECKLEY, Mr. REYNOLDS, Mr. DOWNES, and Mr. E. BECK.

Among the preparations to be made was a stand for the telescope, the cast-iron pedestal of the Kew heliograph being too heavy for convenient transport. It was necessary, moreover, to make some contrivance for supporting the frame of the polar axis in a position adapted to certain limits of latitude, within which I might fix my station; and it was thought that this could be best arranged by making a new cast-iron pedestal composed of several pieces which took apart for the convenience of carriage*.

* This iron stand has been left *in situ*, and thus marks the precise locality of my observatory.

Originally, merely a temporary tent in which to develope the photographs was procured; but when it was known that H.M.S. 'Himalaya' would be placed at the disposal of the Astronomer Royal, I put this aside, and caused a complete photographic observatory to be constructed, part to contain the heliograph with a removable roof, and part divided off and fitted up as a photographic room, with a cistern, to be filled from the outside, a sink, and with tables and shelves to hold the apparatus and photographs. This observatory took to pieces, and every part was marked when in its place, so that no time need be lost in putting it together again in its destined position. Besides the ordinary roof, there was another covering, consisting of strong canvas, supported at the distance of about three feet from the walls and roof of the developing-room. The object of this was to prevent the overheating of the photographic room, a circumstance most detrimental to photography. This canvas was kept wetted with water, in order that the evaporation might lower the temperature of the stratum of air between it and the observatory, and it fulfilled the object perfectly. The canvas, when the observatory was not in use, was drawn over the room containing the heliograph, and protected the instrument from rain.

The print exhibits the arrangement of the observatory, when secured after the day's work. The position of the canvas, and the simple arrangement for maintaining it at the proper distance from the house, and also the outside cistern, are well shown in the picture, which is copied from a photograph taken on the occasion of Mr. Airy's visit to my station. When the observatory was at work, the canvas was removed from the front (the south side) and tied back as far as the upright which is seen on the western side, the top boards also being removed. The front boards were of a height which admitted of our observing the sun above them whenever it was desirable to do so.

Photographic chemicals were prepared in duplicate; part of the collodion intended to be used was mixed with the iodizing solution in London, and after subsidence was carefully decanted previous to packing, in order to avoid the defects before alluded to; but collodion and iodizing solution were also taken separately, so that some might be prepared on the spot, and used, if found free from defects, in that state of extreme sensitiveness which exists in collodion freshly iodized with the potassium iodizer. Nitrate-of-silver baths, prepared in the ordinary way with crystallized nitrate of silver, were taken, and were used in depicting the several phases of the eclipse, with the exception of those of totality. In taking the latter pictures the baths used were made with nitrate of silver which had been fused carefully in my own laboratory, and were so extremely sensitive that they would give photographs of the full moon in the focus of my reflector in less than a second of time, while with the usual bath five seconds were barely sufficient to give a picture of similar intensity.

As few astronomers perhaps are aware of the number of materials required for such an expedition, I here give the list of contents of one of the boxes of chemicals.

Packages.	Contents.	Packages.	Contents.
3	Half-pint bottles of Collodion.	1	½ oz. Iodide of Potassium.
1	Four-ounce Bottle of Collodion, iodized.	1	Ounce Measure.
		1	Gallon Distilled Water.
1	Half-ounce Bottle of Pyrogallic Acid.	1	Set of Scales and Weights.
		3	Plate-drainers.
1	Six-ounce Bottle of Acetic Acid.	1	4 oz. of Tripoli.
1	1½-pound Bottle of Hyposulphite of Soda.	1	Packet Cotton Wool.
		1	Glass Funnel.
1	Case containing Oxide of Silver and dilute Nitric Acid, in separate bottles, for correcting the bath, in case of need.	1	Retort Stand.
		1	Lantern.
		3	Bottles of Varnish.
			Test Papers.
2	24 oz. of Nitrate-of-silver Bath.		Filtering Paper.
1	2 oz. Crystals Nitrate of Silver.	4	8-oz. Mixing Glasses for Collodion.
1	4 oz. fused Nitrate of Silver.		

The apparatus, when completed, weighed 34 cwt., and was made up into thirty packages for convenience of transport. Among the miscellaneous requisites were included:—distilled water weighing 139 lbs.; engineers' and carpenters' tools weighing 113 lbs.; lanterns, lamp-oil, spirit-lamp, and spirits of wine, weighing together 73 lbs.; a small stove and kettle for boiling water, and, lastly, some preserved provisions, in case the party should be compelled to encamp for a few days. Owing, however, to the excellent arrangements most kindly made by Mr. VIGNOLES, the latter were quite unnecessary.

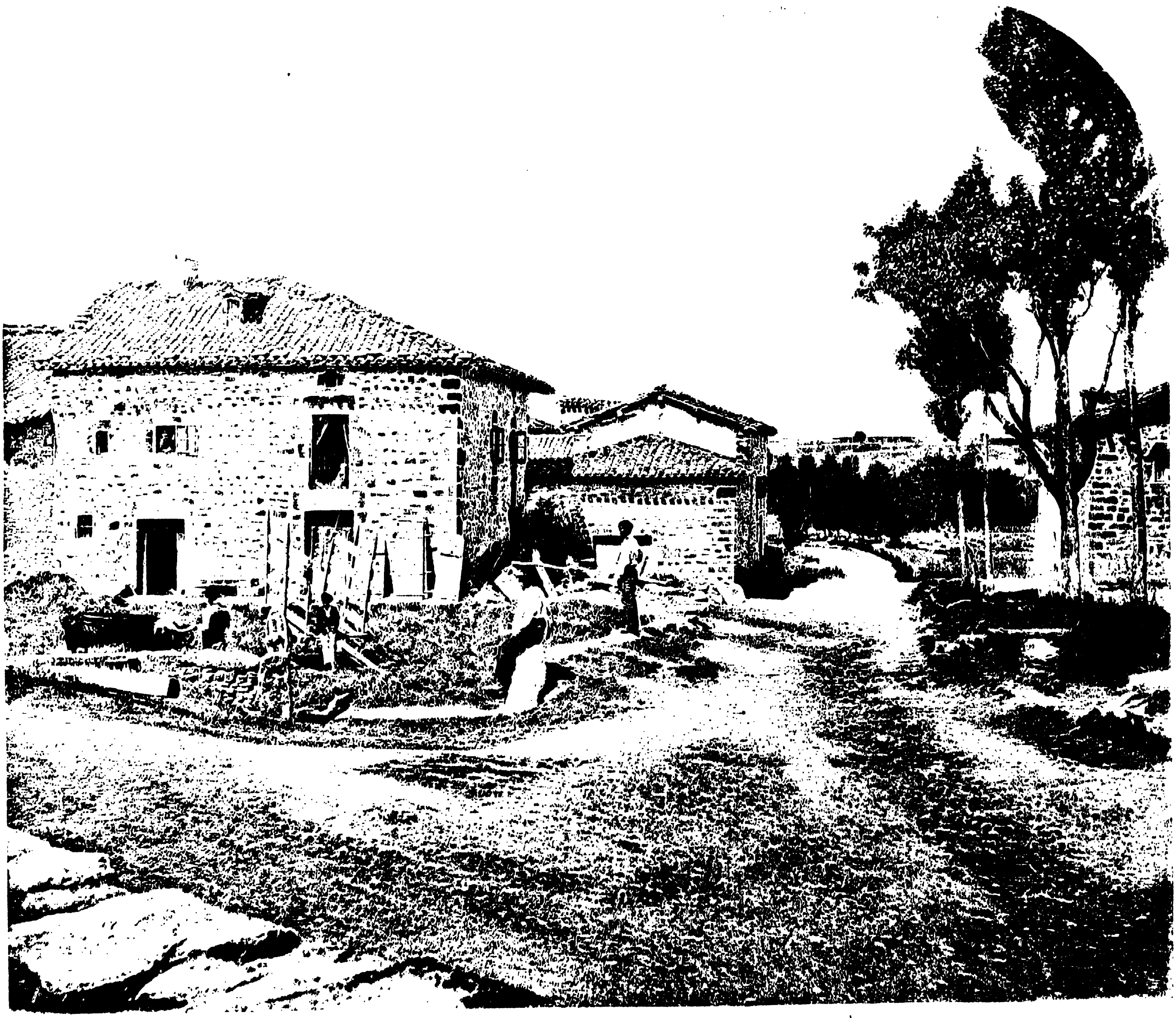
As my plans became matured, it occurred to me that it would be desirable to make determinations of geographical position; and I therefore borrowed from the Kew Observatory a small transit theodolite with 6-inch altitude and azimuth circles, both reading with the verniers only to one minute of arc. The optical part of the instrument was found to be very indifferent, and the readings of the altitude circle were, from some cause, not so accordant from time to time as they ought to have been. I took with me also three chronometers—a box chronometer, a pocket chronometer, both indicating Greenwich mean time, and my journeyman sidereal chronometer.

I was induced to make preparations for eye-observations (which I did not originally contemplate), partly in order that I might be in a position to interpret from my own sketches and recollections the results of the photographs, and partly because in case I should fail in making photographs, I might still be able to contribute something to the series of optical observations. I therefore took with me a beautiful achromatic of 3 inches' aperture, which Mr. DALLMEYER had kindly lent me for the occasion. This telescope was mounted by Messrs. TROUGHTON and SIMMS on a most convenient and steady altazimuth-stand, designed by the Astronomer Royal. The equatorial movement was effected by the joint action of two radius bars, which enabled me to keep the sun exactly within a tangential square, which I had had ruled upon glass and placed in the focus of an eyepiece to be described hereafter.

Lastly, through the kindness of Messrs. ELLIOTT and Mr. CASELLA, I obtained the loan of some meteorological instruments. Messrs. ELLIOTT lent me one of their excellent aneroid barometers. Mr. CASELLA lent me a marine barometer and a standard thermometer, both verified at Kew, the readings of which were used in the reductions of the astronomical observations.

The apparatus was sent to Plymouth on the 5th of July, whence we set sail on the 7th; on the 9th we reached our destination. Mr. VIGNOLES met the 'Himalaya' in a small steamer which he had chartered to convey ashore the astronomers who intended to land at Bilbao, with their apparatus and luggage; but I am placed under a further obligation to him, not only for his kind and liberal hospitality during my stay at Bilbao, but also for dispatching my apparatus, as soon as it was landed, to Rivabellosa, which is situated at a distance of seventy miles from the port of Bilbao, and is only accessible through a pass difficult for the transmission of heavy baggage.

On the evening of the 10th we left Bilbao in a diligence which I had engaged to convey my party to Rivabellosa, at which place we arrived on the next day, after a journey very trying to our chronometers.



[illegible]'S HOUSE AND THRASHING FLOOR. MR. DE LA RUE'S LODGINGS.

THE VILLAGE OF RIVABELLOSA.

Printed by the ordinary Letterpress, from a Block produced by means of Photography and Electrometallurgy.
Absolutely untouched by the graver.

The instruments reached Rivabellosa on the evening of the 11th; the previous part of the day had been occupied in taking a general survey of the country around the village, with the object of selecting a site whereon to erect our observatory, and I at length settled on one of the thrashing-floors* which are to be seen in great numbers in that country in the open fields. It was about 60 feet in diameter, and close to the road, which we found to be a great convenience, inasmuch as the water required for our use had to be brought from a distance. Moreover, it was level, and extremely hard and dry. I had hardly selected this site when I learned, with some concern, that the harvest had commenced, and that the proprietor intended to make use of the floor on the morrow for his thrashing operations, which it is the custom of the country to complete immediately after the reaping. However, Don SIMON, land-surveyor of the Bilbao and Tudela Railway, who explained this to me, kindly undertook to negotiate for the hire of the station; but the owner, when informed that his thrashing-floor was the best adapted for my purpose of any place I had seen, at once said that it was quite at my disposal, and, although he had to convey his grain to a distance, refused any remuneration.

The instruments were conveyed to the thrashing-floor, and the transit theodolite unpacked in time to make an observation of the sun soon after 10 o'clock on the morning of the 12th. By the evening the observatory was erected and in actual operation, and a photograph of the sun was obtained on the 14th. To my staff was at last added Mr. S. CLARK, who had acted as interpreter, and who kindly volunteered his assistance during the eclipse. And I am greatly indebted to the gentlemen composing my staff for their most efficient assistance. With a self-denial hardly to be expected under the circumstances, each carried out steadily his allotted task, and thus contributed to a result which is most gratifying, after so much preparation and trouble.

Upwards of forty photographs were taken during the eclipse, and a little before and after it,—two being taken during the totality, on which are depicted the luminous prominences, with a precision as to contour and position impossible of attainment by eye-observations.

Photographs of the sun were taken on the two days succeeding the eclipse, namely, the 19th and 20th, and the instruments were then taken down and packed.

On the 26th the Himalaya Expedition re-embarked on board the 'Himalaya,'—myself, staff, and baggage being reconveyed to the vessel from Bilbao through the kindness of Mr. VIGNOLES, who accompanied us to England. On the 28th we landed at Portsmouth. I must here take occasion to express my best thanks to Captain SECCOMBE and the officers of the 'Himalaya' for their great kindness during the outward and homeward voyage. My thanks are also due to Señor MONTESINO, Chairman of the Bilbao and Tudela Railway, for assistance rendered; also to Don SIMON, and to Mr. BENNISON and Mr. PRESTON, gentlemen belonging to Mr. VIGNOLES's staff.

* In the accompanying print a thrashing-floor, similar to the one I employed, is represented. The Plate is inserted partly with the object of calling attention to Mr. PAUL PRETSCH's phototype process, which I have recently employed to furnish representations of sun-spots. (See Monthly Notices Roy. Ast. Soc. vol. xii. p. 278.)

After having obtained the photographs and got them home safely, my work was only begun. I knew that they contained in themselves most valuable records; but it did not, in the first instance, appear so clearly how I could turn them to account. The two totality pictures presented the most interest, and to them I first turned my attention; and as it was evident that no measurements ought to be made on the originals, I then bethought me of the best means of multiplying them. In the first instance, I got some enlarged positive copies photographed by Mr. Downes, and, made some little progress in measuring them; but I soon found that I should require others, and, on attempting to have some made a little later in the year, I experienced an amount of difficulty I never could have anticipated. The original negatives proved to be so extremely intense, that nothing short of unobscured sunlight would penetrate them and reveal their details; so that it was only by working for many selected days with Mr. Downes, that I succeeded in getting a sufficient number of positive copies for my purpose. Those who remember how remarkably dull and wet the year 1860 was in England will readily understand that the selected days were few and far between, so that it was fast approaching winter before I had got on much with the work on the photographs. I also had recourse to the albumen process, and obtained a few copies of the size of the original by superposition, without the intervention of the camera. These were made in my presence, at Messrs. Negretti and Zambra's, and from these positives some negatives were taken. Although these copies did not aid me greatly, it was fortunate they were taken; for in course of time the original negative No. 26, the second of totality, gave indications of decay, and on attempting to save it by revarnishing it, the collodion expanded, and crinkled up so much that, except as a record of what was done, the original negative is spoiled. I have protected it from further injury by cementing it with Canada balsam to a second glass plate; but one of the albumen negatives must now supplement it, if more copies are ever taken by direct superposition. Enlarged negatives, however, exist; but, as something is always lost in copying, the damage to the original negative is unfortunate*.

Measurements of the enlarged positives on glass of the totality pictures soon proved to me that the most accurate results could be obtained by measuring the photographs of the other phases, and that these results would indicate the path of the moon, and the position of the centres at the epoch of totality, independently of any determinations of geographical position. For this object it became necessary to measure the original negatives, because the slightest deviation of the optical axis of the copying camera from a right angle to the plane of the sensitive plate, or the least eccentricity, would cause a distortion of the cusps. I had, therefore, to devise an instrument for measuring the photographs; and having considered how the object could be effected, I put one in hand with Messrs. Troughton and Simms, who constructed it for me with their usual skill and precision. After the instrument had been made, it was found to be convenient that some of the parts should be provided with a means of adjustment; so that although it was commenced in February of 1861, it was not until July 18th that it was

* The whole of the original negatives have been deposited with the Astronomer Royal.

completely finished and ready for use. Since that time, every spare moment has been devoted to the final accomplishment of this work; and, taking into account the interruptions I am subject to, I feel convinced that it could not have been done in less time, although I candidly confess that the delay in sending in this Report must appear scarcely warranted.

Determination of the Geographical Position.

Previous to starting for Spain, I made certain preparations for facilitating the after-operations which I might have to carry out; for I knew that the time allowed for getting the instruments into position would not be more than sufficient, even if each day permitted of observations being made. For this reason, I computed, with the formula

$$\frac{\sin(\alpha \mp \lambda)}{\sin \alpha} \quad (\alpha = \text{the polar distance, } \lambda = \text{the colatitude}),$$

a series of star constants for all the stars likely to be visible in my instrument. I also computed a table of corrections, to be applied to the times of equal altitudes of the sun for intervals of two, three, four, five, and six hours, for each day from July 12 to July 22; and similar corrections to the azimuths of equal altitudes, to enable me to lay off at once a meridian line and erect a mark. My star constants did not, however, prove of much service; for it was rarely that I could get a glimpse of the stars; so that on only two occasions was I enabled to make any observations at all, and then only with the greatest difficulty, although I watched patiently for opportunities through or between the clouds.

I have already stated that I took with me three chronometers. My journeyman sidereal chronometer, Leplastrier 2915, is a very old one. In consequence of the wear of the fusee, this chronometer, which is an eight-day one, varies in its daily rate from 8·6 seconds to 17 seconds, losing; but during long periods the rate is pretty uniform, as will be seen from the following observations:—

Leplastrier 2915.

	sec.
From June 30 to July 3, losing daily	10·36
From July 3 to July 30, losing daily	12·51
From July 30 to August 7, losing daily	12·06
From August 7 to August 17, losing daily	11·29
On July 3, 21^h 24^m, it was fast of Cranford sidereal time . . .	194·8
Cranford observatory is west of Greenwich in longitude . . .	97·5
Hence, on July 3, 21^h 24^m, Leplastrier No. 2915 was fast of Greenwich sidereal time	97·3
	194·8
On July 30, 6^h 30^m, it was slow of Cranford sidereal time . .	138·0

The pocket chronometer, Frodsham No. 9768, was found to have the following rates and errors:—

	sec.
From June 26 to June 29, its daily rate was losing	1·05
From June 29 to June 30, its daily rate was losing	2·84
From June 30 to July 3, its daily rate was losing	2·00
On June 26, 15h, it was slow of Greenwich mean time	3·0
On July 3, 15, it was slow of Greenwich mean time	15·0

From these data, the mean daily rate was 1·71 seconds losing. On the 17th or 18th of July this chronometer tripped, most likely in consequence of its having been touched by the inmates of my lodgings; and it was therefore useless to make any determination of its error on my return from Spain.

From a comparison with the box chronometer to be next mentioned, for the loan of which I was indebted to the kindness of Mr. FRODSHAM, I estimated the error on Greenwich mean time of Frodsham 9768 to be 3·2 sec. slow, for July 3, 15^h, instead of 3 sec., as found by observation. The comparison was made on June 25, 22^h: two hours later

	sec.	
Mr. ELLIS found the box chronometer, Frodsham No. 3094, to be on June 26, 0^h	1·9	slow of Greenwich mean time.
In two hours 3094 would have gained	0·08	
At the time of comparison 9768 was slow of 3094	0·50	
Between the comparison and the ascertaining of the error of 9768, namely 17 hours, the latter would lose	0·74	
Estimated error of Frodsham 9768, on June 26, 15^h	3·22	

This difference between 3·2 sec. and 3·0 sec. might have been occasioned by the journey of No. 3094 to Greenwich, but in any case was so small that, for the subsequent calculations, I retained without correction the error afterwards determined by observation on July 3, 15^h, when Frodsham 9768 was slow of Greenwich mean time 15·0 sec.; whence on July 5, 0^h, its error would have been 17·3 sec. slow of Greenwich mean time.

The box chronometer, Frodsham No. 3094, was (through the kindness of the Astronomer Royal) compared at Greenwich by Mr. ELLIS, whose determination of its errors are given below.

	h	m	sec.	
June 26 . . .	0	0	1·9	slow of Greenwich mean time.
June 27 . . .	0	0	1·0	slow of Greenwich mean time.
June 28 . . .	0	0	0·5	slow of Greenwich mean time.
June 29 . . .	0	0	1·3	fast of Greenwich mean time.
June 30 . . .	0	0	2·3	fast of Greenwich mean time.
July 1 . . .	0	0		fast of Greenwich mean time.
July 2 . . .	0	0	3·6	fast of Greenwich mean time.
July 3 . . .	0	0	4·5	fast of Greenwich mean time.

The average daily rate of this chronometer was therefore gaining 0·91 sec.; and assuming this rate to have continued, on July 5 0^h its error would have been fast of Greenwich mean solar time 6·3 seconds.

On the supposition that the pocket chronometer would continue to lose 1·71 sec. daily,
and that the box mean-time chronometer would continue to gain daily 0·91 sec.
box chronometer Frodsham 3094 would gain over Frodsham 9768 . . 2·62 secs. daily.

The subjoined Table contains the assumed errors of each chronometer, the estimated difference between the two chronometers, and the difference actually observed as nearly as possible at noon of each day—the observation being reduced to noon:—

Date.	Assumed error of Frodsham 3094.	Assumed error of Frodsham 9768.	Estimated difference between 9768 and 3094.	Observed difference between 9768 and 3094.
	sec.	sec.	sec.	sec.
July 5.	fast 6·3	slow 17·3	−23·6	−24·5
July 6.	fast 7·2	slow 19·0	−26·2	
July 7.	fast 8·1	slow 20·7	−28·8	
July 8.	fast 9·0	slow 22·4	−31·4	
July 9.	fast 9·9	slow 24·1	−34·0	−35·0
July 10.	fast 10·8	slow 25·8	−36·6	−37·0
July 11.	fast 11·7	slow 27·5	−39·2	−37·5
July 12.	fast 12·6	slow 29·2	−41·8	−37·5
July 13.	fast 13·6	slow 30·9	−44·5	−38·5
July 14.	fast 14·5	slow 32·6	−47·1	−40·0
July 15.	fast 15·4	slow 34·3	−49·7	−41·5
July 16.	fast 16·3	slow 36·1	−52·4	−42·5

On examining the two box chronometers immediately after our arrival at Rivabellosa, Leplastrier 2915 was found to be apparently uninjured, but I was chagrined to find that Frodsham No. 3094 had been most severely disturbed by the joltings of our vehicle, notwithstanding the protection of its outside padded case, and an extra precaution I had taken to press shavings into its own case, to keep it firm in its place. The cap of the glass had become unscrewed, the glass had shaken out, and the chronometer itself, shifting from its normal position, had risen out of its seat; fortunately, however, the glass could not move far, on account of the wadding, and the hands were consequently uninjured. I succeeded in replacing the chronometer, and in putting the glass into its frame; but it thenceforward took up an entirely new rate, as was evident on comparing the differences between its readings and those of the pocket chronometer. An inspection of the foregoing Table shows that up to the 10th the two chronometers maintained the average rate assigned to each; for example, the computed difference minus the observed difference on that day amounted to only −0·4 second. After my return to England, chronometer No. 3094 was, with the Astronomer Royal's kind permission, again compared by Mr. Ellis, who found the following errors from Greenwich mean time:—

For Frodsham 3094.

	h	m	secs.
August 10	0	0	2·8 slow.
August 11	0	0	3 slow.
August 13	0	0	2·7 slow.
August 14	0	0	3 slow.
August 15	0	0	2·2 slow.
August 16	0	0	2·6 slow.
August 17	0	0	2·8 slow.
August 18	0	0	2·7 slow.
August 20	0	0	2·5 slow.

its average daily rate being losing 0·03 second, whereas, before starting, it was gaining 0·91 second.

On the 16th of July, Mr. OTTO STRUVE, Dr. WINNECKE, and Lieut. OOM visited my station; and I took advantage of their doing so, to make a comparison of chronometers. Dr. WINNECKE estimated the error of Frodsham No. 3094 to be fast of Greenwich mean time 7·5 seconds; consequently as the pocket chronometer Frodsham No. 9768 was slow of No. 3094 42·5 seconds, it follows, from this comparison, that it was −42·5 seconds +7·5 seconds =35 seconds slow of Greenwich mean time, which differs by only −1·1 second from the error assigned to it, by applying its mean daily losing rate of 1·71 second.

It appears, therefore, that the pocket chronometer could be relied on up to the 16th of July. On correcting the assumed error from Greenwich mean time of No. 3094 by comparisons with the pocket chronometer, we obtain the following:—

Frodsham No. 3094.

	Computed error from Greenwich mean time, taking No. 9768 as the standard.	Assumed error from Greenwich mean time applying its average rate.
July 9	fast 10·9 seconds	fast 9·9 seconds
July 10	fast 11·2 seconds	fast 10·8 seconds
July 11	fast 10·0 seconds	
July 12	fast 8·3 seconds	
July 13	fast 7·6 seconds	
July 14	fast 7·4 seconds	
July 15	fast 7·2 seconds	
July 16	fast 6·4 seconds	

With the advantage of the comparison made on the occasion of Mr. STRUVE's visit, I have confidence in assuming the error of Frodsham 3094 to have been from +6·4 to +7·5 seconds, say +7 seconds, on July 16 at noon.

On the 19th the Astronomer Royal and his party honoured me with a visit, and I had the great satisfaction of hearing from our leader that he was well satisfied with the success of my photographic operations, and also with the arrangements of the observatory, and the many preparations which had been made to secure the result.

At 2^h 30^m, a comparison was made between Mr. AIRY's pocket chronometer Molyneux No. 1007 and the box chronometer Frodsham No. 3094; Molyneux was slow of Frodsham 3094 43 seconds.

Molyneux 1007 was slow of Greenwich mean time,

	h	sec.	
On July 16 . .	23	28·8	losing daily 7·2 seconds.
July 17 . .	22	35·7	gaining daily 2·7 seconds.
July 18 . .	22	33·0	

Applying the latter rate, Molyneux would appear to have been, on July 19, 2^h 30^m, 32·5 seconds slow, and consequently Frodsham 3094 10·5 seconds fast of Greenwich mean time. I believe that Molyneux 1007 could not be greatly depended on; but, the comparison of chronometers having been made, I place the result on record, although I am not able to make it accord with the other observations within several seconds.

Observations.

The following observations were made with the transit theodolite. During the day the instrument had to remain exposed to the sun; and this caused the several parts to expand very unequally, and kept the bubble in the level always in motion—a circumstance which proved very troublesome.

Estimation of Longitude.

July 12. Four pairs of reduced observations of equal altitudes of the sun showed that at local mean noon

Frodsham 3094 was fast of local mean time . . . 11 min. 51·9 sec.

July 14. Two pairs of reduced observations of equal altitudes of the sun showed that at local mean noon

Frodsham 3094 was fast of local mean time . . . 11 min. 51·3 sec.

With reduced observations of the azimuths of equal altitudes of the sun on the 12th and the 14th, northern and southern adjustable meridian marks were placed, the first against a building, the second against some trees; both sufficiently distant to give distinct vision of the mark, which was a cross × of wood, moveable in a top and a bottom groove in a wooden frame.

Attempts were made on the night of the 13th to obtain observations of stars; but the weather was too cloudy; by dint of perseverance, however, I did manage to get, through breaks in the clouds, the meridian altitude of α Lyræ, and an altitude of the pole star out of the meridian, presently to be referred to in the determinations of latitude.

On the night of the 14th I was more fortunate, and was able to obtain observations of a high and low star, and finally to adjust the meridian marks by means of an observation of δ Ursæ Minoris on the meridian. As soon as the observation of δ Ursæ Minoris was made, an assistant at each of the marks held a lantern near it, so that being illuminated I might see it with the theodolite and thus be enabled to make signals to them for the permanent adjustment of the marks, which was successfully accomplished. The following day being Sunday, no work was done; but on Monday Mr. PRESTON volunteered to aid me in projecting my meridian towards the station of the Astronomer Royal at Pobes. Using, as a signal flag, a bed sheet, which was not at all larger than was necessary in order to be well seen, I was able to direct Mr. PRESTON where to erect a staff with a similar appendage, in the line of my meridian, towards the north, and at a distance of about seven miles from Rivabellosa. Mr. O. STRUVE, as I before said, visited us on the 16th and undertook to work out the geodetic survey, which it was agreed should be made to connect the Pobes and Rivabellosa stations.

July 14. Observations of α Scorpii and ζ Herculis, when reduced, showed that at 16 hour 28 min.

Leplastrier 2915 was fast of local sidereal time . . . . 11 min. 1·7 sec.

The weather afforded no other opportunity for star observations for longitude determinations until the night of the 18th, when I was too much fatigued to avail myself of it.

July 16, 0 h. A transit of the sun showed that

Frodsham 3094 was fast of local mean time . . . . . 11 min. 49·9 sec.

July 20, 0 h. A transit of the sun showed that

Frodsham 3094 was fast of local mean time . . . . . 11 min. 41·5 sec.

July 20, 8 h. 0 min. sidereal time. The transit of the sun being observed simultaneously with the sidereal chronometer, showed that

Leplastrier 2915 was fast of local sidereal time . . . . 9 min. 55·8 sec.

From the foregoing observations are derived the following results.

Frodsham No. 3094.

	Fast of Rivabellosa, mean solar time.			Daily rate.	Error on Greenwich mean time as estimated by comparison with Frodsham 9768.
	h	m	sec.	sec.	sec.
July 12th . .	0	11	51·9		Fast 8·3
				Losing 0·3	
July 14th . .	0	11	51·3		Fast 7·4
				Losing 0·7	
July 16th . .	0	11	49·9		Fast 6·4
				Losing 2·1	
July 20th . .	0	11	41·5		

whence the Longitude was West of Greenwich

	m	sec.
July 12 . . .	11	43·6
July 14 . . .	11	43·9
July 16 . . .	11	43·5
Mean . . .	11	43·7

Taking Dr. WINNECKE's estimate of the error of Frodsham No. 3094 on the 16th, namely, 7·5 seconds fast of Greenwich, we get for the longitude

West 11 min. 42·4 seconds

Applying the average daily losing rate of 12·51 sec. since July 3rd, 21 h. 24 min. for Leplastrier, we derive the following results from the observations made with that chronometer:—

Leplastrier No. 2915

	h	m	Was fast of local sidereal time. m	sec.	Error on Greenwich sidereal time by applying mean daily rate.	m	sec.
July 14 . .	16	47	11	1·7	Slow	0	37·9
July 20 . .	8	0	9	55·8	Slow	1	48·4

whence the longitude was West of Greenwich

	m	sec.
July 14 . .	11	39·6
July 20 . .	11	44·2
Mean .	11	41·9

By combining all the foregoing results, and taking the arithmetical mean, we obtain for my Observatory at Rivabellosa

	m	sec.
	11	43·7
	11	42·4
	11	41·9
the longitude . . . . West	11	42·7

Observations for Latitude.

July 12. The reduced zenith distance of the sun's upper limb, when on the meridian, gave as the latitude of Rivabellosa

N. 42° 42′ 37″

July 13. An observation of α Lyræ on the meridian and of *Polaris* out of the meridian, when reduced, gave for the latitude freed from error of the level,

N. 42° 42′ 19″.

July 14. An observation of the zenith distance of the sun's upper limb previous to the meridian passage at the hour angle 24° 5′ 15″ gave, when reduced, the latitude

N. 42° 41′ 48″.

July 20. The reduced zenith distance of the sun's upper limb, when on the meridian, gave the latitude

N. 42° 41′ 17″.

Combining the foregoing determinations of latitude, we obtain

	°	′	″
July 12. Observations of sun	42	42	37
July 13. Observations of stars	42	42	19
July 14. Observations of sun	42	41	48
July 20. Observations of sun	42	41	17
as the mean . . . . Latitude N.	42	42	

The foregoing numbers are not so accordant as I could desire; but they are, I believe, as good as could have been obtained with the instrument, particularly under the circumstance of its continual exposure to the sun during its employment in the day-time.

Elevation of the Station.

Before leaving Bilbao on the 10th, the aneroid barometer was read off, when it stood at 30·019 in., temperature 71° Fahr. On arriving at Rivabellosa it indicated 28·473 in., the temperature being 65° Fahr. With these numbers I estimated the height to be 1481 feet above the ground floor of Mr. VIGNOLES' house at Bilbao, which is several feet above the mean sea-level.

Mr. PRESTON, however, was so kind as to connect my station by levelling with a normal point on the railway, and made its height to be 1572 feet 4 inches above the mean sea-level.

Recapitulation.

The geographical position of my observatory at Rivabellosa was, therefore,

Latitude N. 42° 42′, Longitude W. 11 min. 42·7 sec.,

and its height above the mean high-water mark 1572 feet 4 inches. Mr. STRUVE has communicated to me that the geodetic connexion of Rivabellosa and Pobes showed the geographical position of my observatory to be

Latitude N. 42° 43′ 24″, Longitude W. 11 min. 41·3 sec.

Lastly, I estimate the error of the mean-time chronometer, Frodsham No. 3094, July 18th, 0^h, to have been 4·6 seconds fast of Greenwich mean time, by assuming a progressive increase in its losing rate from July 16th to July 18th, and taking the mean between Dr. WINNECKE'S and my own estimate for its error on July 16th.

After the return of the expedition, Mr. CARRINGTON kindly made some extensive calculations to admit of a direct comparison of my observed results with the demands of theory. To Mr. FARLEY I am also much indebted for special computations of the moon's position in respect of the sun's, in a form the most convenient for comparison with measurements hereafter to be mentioned.

Abstract of the Results of Mr. CARRINGTON'S *Calculations for Rivabellosa.*

Assumed position of station:—

Geographical latitude	42° 42′
Longitude W. of Greenwich	$11^m\ 42^s{\cdot}7$.
Height above sea-level	1572 feet.

Whence the following elements:—

Geocentric latitude	=42° 30′·5.
Log. distance from earth's centre . . .	9·9993676.

The true positions of the sun and moon are those of LE VERRIER'S and HANSEN'S

Tables, as derived from the Special Circular issued by the Superintendent of the Nautical Almanac.

For the totality the apparent positions were calculated for 3 h. 0 m. and 3 h. 5 m. Greenwich mean time, with the following results:—

Zenith distance of sun's centre	40° 33′.
Angle at sun between pole and zenith . . .	47° 59′.
Moon's semidiameter	16′ 33″·0.
Sun's semidiameter	15′ 44″·8.

Totality began at 3 h. 0 m. 37·4 s. at 101° 38′ on the sun.
Totality ended at 3 h. 4 m. 1·6 s. at 312° 56′ „
Direction of motion from 297° 17′ to 117° 17′ „

Relative motion during totality	92″·8.
Nearest approach of centres	13″·0.

Duration 3 min. 24·2 secs.

For the first contact the apparent positions were calculated for 1 h. 40 m., 1 h. 50 m., and 2 h. 0 m. Greenwich mean time, with the following results:—

Zenith distance of sun's centre	28° 49′.
Angle at sun between pole and zenith	35° 50′.
Moon's semidiameter	16′ 34″·6.

First contact at 1 h. 47 m. 56·0 s. at 296° 54′ on the sun.

Rate of approach of centres per minute . . .	24″·87.

For the last contact the apparent positions were calculated for 4 h. 0 m., 4 h. 10 m., and 4 h. 20 m. Greenwich mean time, with the following results:—

Zenith distance of sun's centre	52° 45′.
Angle at sun between pole and zenith	51° 36′.
Moon's semidiameter	16′ 30″·7.

Last contact at 4 h. 10 m. 15·2 s. at 117° 20′ on the sun.

Rate of retreat of centres per minute	29″·85.

The formulæ used in computing the moon's parallax and apparent semidiameter were

$$\theta = \text{Æ of zen.} \qquad m = \frac{\varrho \cos\varphi' \sin p}{\cos\delta} \qquad n = \frac{\varrho \sin\varphi' \sin p}{\sin\gamma}$$

$$\alpha' - \alpha = -\frac{1}{\sin 1''}\left\{m \,.\, \sin\overline{\theta-\alpha} + \tfrac{1}{2}\, m^2 \sin 2\,.\,\overline{\theta-\alpha} + \tfrac{1}{3}\, m^3 \sin 3\,.\,\overline{\theta-\alpha}\right\}$$

$$\delta' - \delta = -\frac{1}{\sin 1''}\left\{n \,.\, \sin\overline{\gamma-\delta} + \tfrac{1}{2}\, n^2 \sin 2\,.\,\overline{\gamma-\delta} + \tfrac{1}{3}\, n^3 \sin 3\,.\,\overline{\gamma-\delta}\right\}$$

$$\tan\gamma = \tan\varphi' \,.\, \frac{\cos\frac{1}{2}\,.\,\overline{\alpha'-\alpha}}{\cos(\theta - \frac{1}{2}\alpha' + \alpha)}.$$

$$R' = R \,.\, \sin(\gamma - \delta') \,.\, \operatorname{cosec}(\gamma - \delta).$$

Mr. FARLEY'S *Elements of the Eclipse for Rivabellosa.*

The following calculations are based on the same latitude and longitude as those of Mr. CARRINGTON.

Greenwich mean time.	Apparent distance of ⊙ and ☾ centres.	Angle of line joining centres, N. towards E.	Aug^d. S. D. or Radius of ☾.	Ratio of Lunar to Solar radius.
d. h m	′ ″	° ′	′ ″	
July 18. 1 45	33 32·8	296 54	16 34·5	1·0526
1 55	29 22·0	296 54	16 34·4	1·0525
2 5	25 8·9	296 52	16 34·3	1·0524
2 15	20 53·0	296 47	16 34·1	1·0523
2 25	16 34·4	296 38	16 33·9	1·0521
2 35	12 12·2	296 23	16 33·7	1·0518
2 45	7 47·2	295 48	16 33·4	1·0515
2 55	3 19·3	293 40	16 33·1	1·0511
3 5	1 13·8	126 59	16·32·8	1·0508
3 15	5 48·2	119 20	16 32·5	1·0505
3 25	10 26·7	118 22	16 32·2	1·0502
3 35	15 9·1	117 57	16 31·9	1·0498
3 45	19 55·2	117 43	16 31·6	1·0495
3 55	24 45·0	117 32	16 31·3	1·0492
4 5	29 39·0	117 22	16 30·9	1·0488
4 15	34 37·2	117 14	16 30·5	1·0484

	h	m	sec.	
Time of first contact . . .	1	47	57	at 296° 54′ N. towards E.
middle . . .	3	2	20	duration of totality 3 min. 20 sec.
last contact . . .	4	10	15	at 117° 18′ N. towards E.

Nearest approach of centres 0′ 12″·7.

OBSERVATIONS OF THE ECLIPSE.

I. *Observations with the unassisted Eye, and with the Telescope.*

A splendid day on Sunday the 15th was succeeded by one of the grandest and most awful thunder-storms I have ever witnessed; and the 16th was cloudy, almost without intermission. The day previous to the eclipse had been completely overcast, with the exception of a short interval about noon; but even then the sun could only just be seen through a cloud somewhat thinner than those which obscured the rest of the heavens. The climate had therefore proved anything but propitious, and every interval of fine weather had to be diligently made use of for the adjustment of the instruments and the prosecution of observations. Fortunately an opportunity had presented itself on two days for practice in observing the sun with the Dallmeyer between 1 h. 30 min. and 4 P.M., and for special practice at about 3 o'clock. It was ascertained that during the progress of the eclipse the radius bars would have to be changed from one leg of the tripod-stand to the other, and arrangements were made to prevent the necessity for doing this during or near the period of totality.

To this instrument I had fitted an eyepiece of my own contrivance, which I described, verbally, at one of the meetings of the Astronomical Society, and which was in consequence adopted by Mr. PRITCHARD. No account having been published of this appendage, and experience having proved its value in eclipse observations, I think it desirable to describe it here. It will be remembered that Mr. HODGSON some time ago proposed that a piece of polished glass should be used as a diagonal reflector in observing the sun; and "Hodgson's solar eyepiece" has been generally adopted, and is a most convenient and efficient instrument. It occurred to me, that if the glass reflector were made in the form of a parallelogram, of such dimensions that a moiety of its surface would suffice for the field of the telescope, one-half of the upper reflecting surface might be silvered and the other left plain, and that the addition of a suitable contrivance would enable the observer to draw into position the unsilvered or the silvered surface, according as either partial or total reflexion might be required. The silver film is so extremely thin that it in no way affects the focus, yet it is susceptible of the highest possible polish. It was not a convenient plan to silver only half the mirror; so, when the whole had been silvered *, one-half of the silver was neatly removed by means of a cloth, wetted with cyanide of potassium, strained over the forefinger. The roughened back of the reflector was freed from silver, and the plate then washed thoroughly with distilled water and allowed to dry. A little pad of wash-leather, well charged with dry rouge by rubbing it on a second piece of leather on which some rouge-powder had been placed, very soon removed the peach-like bloom from the silver surface, and produced a perfect polish.

The construction of the eyepiece will be readily understood by reference to the accompanying wood-engravings, wherein the same letter refers always to the same part.

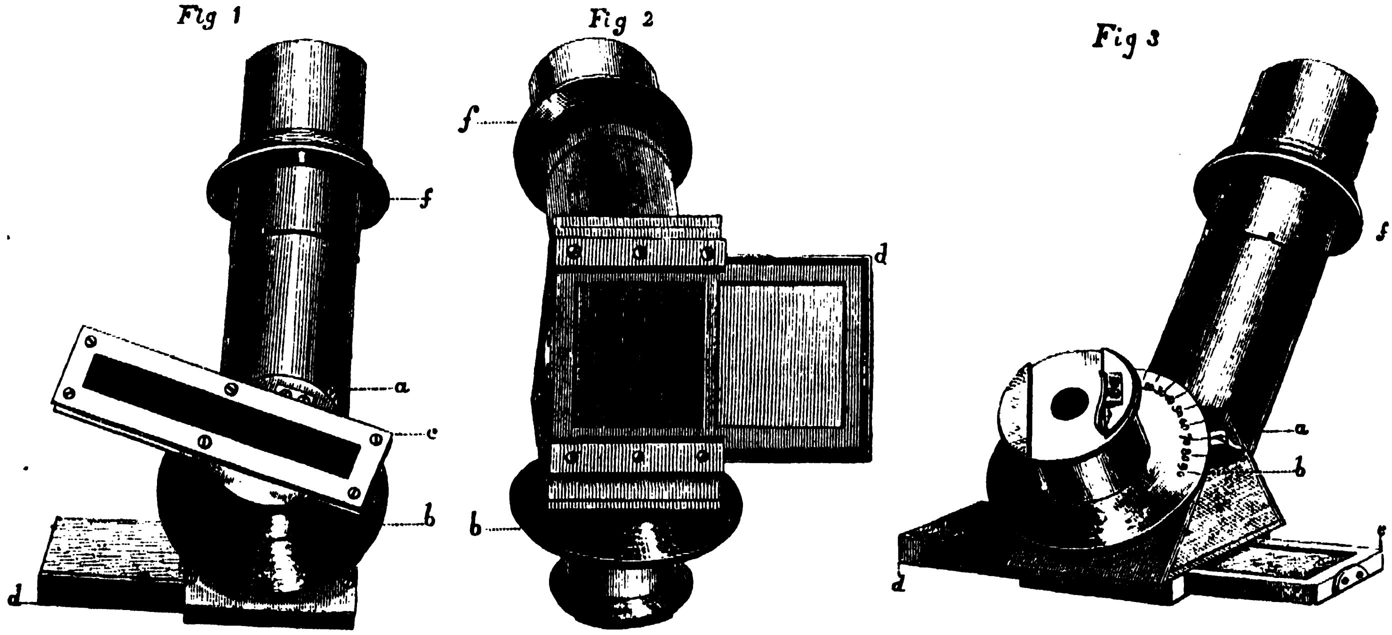
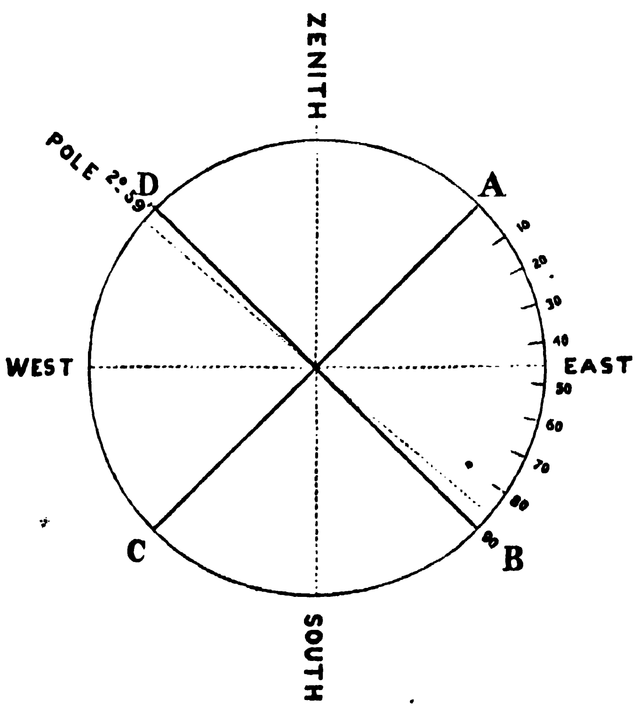


Fig. 1 is a front view, the plain glass reflector being in operation. Fig. 2 is a view of the under side, the plain glass being still in position for use. Fig. 3 shows the glass

* A method for silvering glass has been described by myself and Dr. MÜLLER in the 'Monthly Notices of the Astronomical Society,' vol. xix. p. 171.

reflector drawn out so as to place the silvered surface in the field. *f* is the socket-adapter, which screws into the telescope; it has a slit cut into it to receive a pin fixed on the sliding tube, the object of which is to keep the position-lines of the eyepiece, when once adjusted, in their proper position; *e* is the glass reflector, the half towards *d* being silvered, that towards *e* plain; *d* is a covering to protect the sliding reflector; *b* is a circle fixed to the body of the eyepiece, having one quadrant divided into nine spaces of 10° each; *a* is an index attached to a positive eyepiece, which can be moved by it through an arc of 90°; *c* is a graduated sun-shade, composed of a wedge of dark glass and another of white glass, reversed in position, so as to form, when combined, a parallel plate. This is held firmly in its place by means of a spring, shown in fig. 3, which, while it holds the shade firmly in any required position, also allows its instant removal at pleasure. The glass reflector *e*, as soon as the observer desires to use the silvered surface, can be drawn forward in a small fraction of a second, without disturbing any other part of the instrument.

In the focus of the positive eyepiece was fixed a piece of parallel glass on which were etched several lines; this micrometer-plate was carried round with the eyepiece whenever the index, *a* (figs. 1 & 3), was moved. A reference to Plate VI., which contains a fac-simile of my hand-drawings, and also a representation of the position-lines, will render clear the following explanation. Four principal lines on the glass plate formed a tangential square calculated to enclose exactly the moon's disk, which in fact it accomplished with great precision; four other lines surrounded the first square at the distance of exactly 1′ of arc; and a third series formed a third square at the same distance from the second. Joining the angles of the squares were two diagonal fainter lines, which served to measure angles of position, while the several squares served to measure distances. The angles of the tangential square may be designated A, B, C, D. As



soon as the axis of the telescope-stand had been adjusted in a vertical position by means of screws affixed to the feet of the tripod-stand, and of a level attached to the vertical axis of the telescope, a distant mountain peak was made to run along one of the lines D A or C B by causing the telescope to turn on the vertical axis when the index of the eyepiece stood at zero; and the position of the whole eyepiece was regulated by means of a slight axial adjustment of the sliding tube until the line in question, and the course of the object horizontally across the field, showed an absolute coincidence. The diagonal lines D and A were then each 45° from the zenith; and the angle between the pole and the zenith at the period of totality being 47° 59′, the wire D, to the west of the zenith, was 2° 59′ east of the pole. This will readily be seen by the annexed diagram, which shows the apparent positions of the zenith and of the east, west, and south points in the field of the eyepiece, the zenith being represented in its natural position, but the east and west points reversed, right for left; so that, in referring any measurement to D, 2° 59′ must be added to reduce them to positions from the north point towards the east.

At last the eventful 18th of July arrived, and appeared hopelessly cloudy. The sky was watched with the most intense anxiety by us all; and I am free to confess that *my* nerves were in the most feverish state of agitation. Not the slightest break in the clouds or mist was visible until about 10 o'clock, when a streak of clear sky gave us the first faint gleam of hope. At noon the sky began to clear very generally, not so much by the clouds being carried off by the wind, as by their melting away in the air. An attempt was made to get an observation of the sun; but the clouds were so dense that it was only just as he was passing away from the field of the telescope that his following limb could be made out. Soon after this the clouds, which had been dissolving and gradually becoming thinner, disappeared all at once; and we had a magnificent sky, absolutely cloudless, except near the horizon. The heights towards the north in the direction of Pobes were perfectly free from mist; moreover we could discern through the telescope Mr. Bonomi and Mr. J. Beck on a low hill a few miles to the west of our station, and it gave us pleasure to know that at least two of the several parties were as fortunate as ourselves. The mists surrounded the higher peaks of the Pyrenees so pertinaciously that we expressed to each other great fears for our good friend Mr. Vignoles; for we were aware that he intended to carry out his plan of observations from one of the highest peaks, and we afterwards heard with the greatest regret that our apprehensions had been realized.

About twenty minutes before the commencement of the eclipse, an occurrence took place which very nearly brought all our labours to a calamitous termination. Mr. Preston had placed at our disposal his excellent and handy servant Juan, whom we had always found obliging, and very ingenious in expedients whenever any temporary arrangements had to be made. In order that he might have an opportunity of looking at the eclipse, I smoked a piece of glass for him with a wax lucifer-match, and he then, on his own account, prepared several pieces for the bystanders in a similar way. The demand soon increased so much, that he was scarcely able to keep pace with it, and at length became so excited that he threw away the matches in all directions without extinguishing them, and some, falling in the standing corn, set it on fire. The corn was very thin, and fortunately the wind was blowing from the position of the fire towards the thrashing-floor; otherwise but a few minutes only could have elapsed before the conflagration would have assumed such dimensions as to be beyond the power of man to control. Happily, a few seconds after the occurrence, the crackling sound and the smell of burning straw drew my attention to the spot, and, water being at hand, the fire was got under before it had spread more than a few feet.

The Alcalde of Miranda had intimated to me, a few days before, that he was instructed to place at my disposal as many of the civic guard as I might think necessary to prevent interruption; but my experience of the consideration evinced by the Spaniards was such, that I replied that one or two would be quite sufficient. Shortly before the commencement of the eclipse, there arrived five mounted guards, who were of great use in preventing the crowd from encroaching on the thrashing-floor, which an excusable curiosity to watch our proceedings tempted them often to do. It is right to add that I could not persuade the guards to take any present whatever, their reply being that their orders on this head were imperative, and that, moreover, they had felt pleasure in being of service. When we were on the point of commencing our observations, about 200 persons had assembled round our observatory, and, although they conducted themselves perfectly well in other respects, their talking quite prevented my hearing the beats of the chronometer. They seemed to think that the eclipse could only be seen from my station; and it was with some difficulty that a number were persuaded to go to an adjoining height, whence the effects on the landscape and the progress of the shadow could really be better observed. I explained this, through the kindness of a gentleman from Miranda who spoke French, and who showed his faith in what I stated by leading the way. The Alcalde of Rivabellosa, CIVILO GUINEA, to whom I was indebted for facilitating my operations, explained to those who remained around the station the necessity for silence, and they thenceforth carried on their conversation in a tone which caused us no inconvenience. It is indeed impossible to speak too highly of the good feeling manifested throughout by the Spaniards of all grades, who endeavoured in every way to promote our objects.

Owing to my pocket chronometer having tripped, and become many minutes fast of Greenwich mean time, some confusion arose about the period of first contact, and a photograph, which it was intended to procure as close as possible to that event, was lost in consequence of the plates being prepared too soon, and none being ready when it actually took place. The error of the pocket chronometer was only discovered when it was too late, and it was then found to be faster than Frodsham 3094 by 8 min. 11 sec.

The first contact was observed at 1 h. 56 min. 55 beats.

	m	sec.	h	m	sec.
5 beats to 2 seconds			1	56	22
Error, fast of Frodsham	8	11			
Error, Frodsham fast of Greenwich mean time . . .		4·4		8	15·4
Making the observed Greenwich mean time of first contact to be			1	48	6·6

	h	m	sec.
The occultation of the largest solar spot, which I call *c*, occurred at 2 h. 10 m. 125 beats	=2	10	50
Deducting the difference between the pocket Frodsham and Frodsham 3094		8	11
It gives	2	2	39

as the time by Frodsham 3094. This agrees within 0·5 second of the time noted by Mr. Beckley when a photograph of the phenomenon was taken in accordance with my signal, namely, 2 h. 2 m. 39·5 secs. Attention is called to the very exact accordance of the times recorded by myself and by Mr. Beckley, because I shall hereafter have to draw attention to certain conclusions which I have deduced, the soundness of which is dependent in part upon the epochs of the photographs having been accurately registered by Mr. Beckley.

After the occultation of the spot *c*, nothing worthy of record occurred until about 2 h. 12 m., when a cluster of clouds formed very rapidly and unexpectedly in the immediate neighbourhood of the sun, and completely put a stop to both optical and photographic observations. The clouds melted away about six minutes after they had formed, and thenceforward until the end of the eclipse all went on without interruption.

I had never before witnessed so great an obscuration of the sun as that presented by this eclipse many minutes even before the totality occurred, and I was particularly struck by the change of colour in the sky, which had been gradually losing its azure blue and assuming an indigo tint, while at the same time I remarked that the surrounding landscape was becoming tinged with a bronze hue, which to my mind suggested the idea that the light of the sun near the periphery is not only less intense than, but possibly different in quality from, that of the centre*. Spectrum experiments at future great eclipses, when the sun's crescent is reduced to an extremely narrow line, would set this question at rest, and might also have an important bearing on the line of investigations so ably inaugurated by Kirchhoff and Bunsen, into the composition of the sun's atmosphere. Another phenomenon could not fail to attract attention. When the sun's visible disk was reduced to a very narrow crescent, the shadows of all near objects became extremely black and sharply defined, whilst the lights, by contrast, assumed a peculiarly vivid intensity, the aspect of nature strongly recalling to mind the effects produced by the illumination of the electric light. Several minutes before the totality, the whole contour of the brown-looking lunar disk could be distinctly seen in the heavens.

Only a few brief seconds, unfortunately, could be spared from the telescope after the totality had actually commenced; but when I had once turned my eyes on the moon encircled by the glorious corona, then on the novel and grand spectacle presented by the surrounding landscape, and had taken a hurried look at the wonderful appearance

* In connexion with this remark, compare Sir John Herschel on the Chemical Rays of the Spectrum. Philosophical Transactions, 1840, Art. 82.

of the heavens, so unlike anything I had ever before witnessed, I was so completely enthralled, that I had to exercise the utmost self-control to tear myself away from a scene at once so impressive and magnificent, and it was with a feeling of regret that I turned aside to resume my self-imposed duties. I well remember that I wished I had not encumbered myself with apparatus, and I mentally registered a vow, that, if a future opportunity ever presented itself for my observing a total eclipse, I would give up all idea of making astronomical observations, and devote myself to that full enjoyment of the spectacle which can only be obtained by the mere gazer.

Although, possibly, not more than twenty seconds were devoted to observations with the unassisted eye, the phenomena remain strongly impressed on my memory, and at the time of writing this account, sixteen months after the event, I have it now pictured before me mentally, as vividly as if it had but just occurred. The darkness was not nearly so great as I had been led to expect from the accounts which I had read of former total eclipses; and although I had a lantern at hand, I did not require it, either in making my drawings or for reading the divisions of the micrometer quadrant on the eyepiece. The illumination was markedly distinct from that which occurs in nature on any other occasion, and certainly was greater than on the brightest moonlight night; and yet, at the time, the light appeared to me less than what I remembered of bright moonlight. It was only by subsequent trials, in endeavouring to make out details of the drawings which I had made of the phenomena, and to distinguish between colours under various circumstances of moonlight and twilight, that I was able to form a proper appreciation of the amount of light; and the best account I can give of it is, that it most resembles that degree of illumination which exists in a clear sky soon after sunset, when after having made out a first-magnitude star, other stars of less brilliancy can be discerned one after another. The light was good enough and sufficiently polychromatic to enable me to distinguish the colours of near objects; but those in the distance appeared to be illumined by the most unearthly hues.

Immediately surrounding the corona, the sky had an indigo tint, which extended to within about thirty or twenty-five degrees of the horizon, while lower down it appeared to me to be modified by a tinge of sepia. It became red as it approached the horizon, close to which, and just above the mountains, it was of a brilliant orange. The mountains appeared, by contrast, of an intensely dark yet brilliant blue. I saw two stars to the east of the sun, which by the aid of Mr. HIND's diagram I have since identified as Jupiter and Venus; but I had not time to search for more, or, most probably, I should have seen others. These planets, and also Castor, were made out by Messrs. ROBERT SWANSON, HARRY EDMONDS, and MATTHEWS, in the employ of Messrs. BRASSEY and Co., and were identified by them in my copy of Mr. HIND's diagram.

The effect of totality upon the bystanders was most remarkable. Until the beginning of totality, the murmur of the conversation of many tongues had filled the air; but then in a moment every voice was hushed, and the stillness was so sudden as to be perfectly startling; then the ear caught the sound of the village bells, which had been tolling unheeded during the eclipse, and this circumstance added much to the solemn grandeur of the occasion.

The time I could spare was far too short for any exact observations of the corona; however, I knew that Mr. PRITCHARD, Mr. OOM, Mr. BONOMI, and other observers intended to make special delineations and measurements of that phenomenon, and I therefore confined my attention to its general characteristics. It appeared to me to glow with a silvery-white light, softening off into a very irregular outline, while from its general boundary shot out several long streamers. It extended generally to about 0·7 or 0·8 of the moon's diameter beyond her periphery. Close to the moon, and reaching not further than 2′, the light was very brilliant, and several zones of gradations of brightness appeared to exist, but the very bright zones would all be comprised in a circle about 0·25 of the moon's diameter.

The observations just recited were made in the brief interval I could afford between the telescopic observations, which I will now proceed to describe. In order to facilitate my operations, I had prepared two diagrams exactly representing the appearance of the micrometer lines in the telescope, and, by chance, I had made the tangential square of such dimensions as to include a circle precisely 4 inches in diameter, which had been coloured to render it more readily distinguishable, and which represented the moon. Four inches happened to be almost exactly the diameter of the moon's disk on the screen of the heliograph; so that, later on, the photographs and the two drawings made during the totality, were readily compared by the superposition of each upon each.

On the diagrams I had painted fifteen streaks of various tints, some of which I believed might resemble the colour of the prominences, and some I knew would be useful as a contrast, to enable the eye to form a more correct judgment. The chromatic scale I here insert contains a selection of the tints painted round my diagram.

SCALE OF COLOURS WITH WHICH THE PROMINENCES WERE COMPARED.

Several minutes (probably five) before totality, I entirely removed the dark glass, and found that the sun's image might be looked at without the slightest inconvenience after reflexion by the plain glass. I could then see in the telescope, as I had shortly before seen with the naked eye, the whole of the lunar disk, which appeared of a deep sepia brown, nearly, but not quite, black, and, to my great surprise, I perceived a luminous prominence, about 20° to the west of the zenith, shining with great brilliancy, although, on account of the plain-glass surface being then in use, the greater part of its light passed through the glass, and was therefore not reflected to the eye. I then, cautiously, but rather quickly, brought into action the silvered surface, and beheld with delight that the luminosity of the prominence, which I will call A*, was so great that there could be very little doubt of our obtaining the much wished-for photographic pictures.

I now watched carefully for the so-called Baily's Beads, but no such phenomena presented themselves,—at which, however, I felt no surprise, for I had always believed that they arose, in all probability, from the atmospheric disturbance of an image formed by a telescope wanting in perfect definition. The Dallmeyer I used was so perfect that I did not think I should see anything of the kind.

To the east of the zenith, about 20°, a floating cloud, quite detached from the moon's limb, and distant from it more than 0'·5, next attracted my attention. This cloud, which I will call C, appeared about 1'·5 long, and was inclined about 50° or 60° to the moon's limb. It had two curvatures, both convex on the edge most distant from the moon, and was decidedly of a rose tint, but of a much paler hue than the published accounts of previous eclipses had led me to expect. I compared the prominence carefully with my scale of tints, and found that it very nearly matched the colour marked *c*. It must therefore have been of a yellowish pink (approaching a salmon); for *c* on my chromatic scale was a mixture of carmine and cadmium yellow. This prominence (C) presented a great amount of detail, and reminded me of the aspect of a cirrus cloud glowing with the illumination of a setting sun. I should here remark that, in comparing my scale of colours with the luminous prominences, I depended on the general light of the heavens, and that I did not employ my lamp, which, I found, completely changed their appearance.

The prominence A was generally more brilliant, and did not seem to me to be so pink as the detached cloud; I could, moreover, detect a tinge of yellow in its brilliant light. It also showed considerable structure, appearing to consist of several streaks, curved inwards, while from the summit came two peach-coloured faint streamers, bending over in opposite directions downwards towards the moon's limb.

I paid most particular attention to the prominence A, because I knew from its position that it was critically placed for the observation of any change of position-angle in reference to the moon's centre; and I also remarked carefully the prominence C, and sketched all that I could make out by the most careful scrutiny. On comparing my drawings with the photographs, it will be perceived that a certain boomerang-like prominence in the photograph is wanting in my hand-drawings, and that there are also other prominences visible in the photograph which are not shown in the drawings. This is a curious circumstance, hereafter to be more particularly dwelt upon; but it is right to mention it here, because it affords me the opportunity of saying that, at all events, as regards the boomerang, I am certain that it was not visible in the telescope; for I observed so carefully in the neighbourhood of the floating cloud, that it is next to an impossibility that such an object could have escaped detection.

* See Index Map, Plate XV.

In the eastern quadrant (in reference to the zenith) a long line of prominences, extending over 70° on the moon's limb, was visible at the commencement of the totality, but before the end of totality it was covered by the lunar disk. This streak (which I call I) terminated in a hook, about 1'·5 high, bent upwards, and was *much indented on the concave side*, where it was in contact with the moon's edge. It was extremely brilliant, and, although it presented in parts a pink colour, was not uniformly so coloured, but to my eye had here and there a considerable admixture of yellowish white. In the first photograph of the totality is depicted a curious branching prominence, not unlike the fallen stump of a tree, which I did not observe, and therefore did not record in the sketches. I do not state so positively that this prominence was not visible, for this reason, that I did not pay such special attention to that part of the field, my eye being directed more particularly to the prominences A and C; but I have a strong impression that it was not visible. Just about the part where this would be, the corona appeared to me, in the telescope, to be particularly bright; but, besides a mere sheet of brilliant light, I saw nothing to delineate.

About half a minute after the commencement of totality, the progress of the moon uncovered, in the western quadrant (in reference to the zenith), a small peak, like that of a mountain, which I will call R. As the eclipse progressed, this prominence became more and more uncovered, and another smaller peak appeared, the whole contour reminding me somewhat of the hull and masts of a ship in full sail. Just before the reappearance of the sun this prominence reached apparently about 1'·5 beyond the moon's limb.

Extending from the southern base of this prominence, there came into view, about a minute before the end of totality, a long streak of prominences much *indented and irregular on the concave side.* This streak extended over fully 50° on the moon's limb, when it had been fully uncovered by her onward course. It was pretty generally of a decided rose tint. Just previously to the reappearance of the sun, I remarked a sort of carmine glow near that part of the moon's limb where the crescent of the sun was first re-formed.

Plate VI. is a most exact fac-simile of the two drawings, black representing white, which I made during the totality, and it is desirable that I should make a few remarks about them.

Figure 1 was begun, as nearly as I can recollect, about thirty seconds after the commencement of totality. As a preliminary step, the moon's disk was brought exactly within the tangential square, and the position of the prominence A, in respect of the line D, was noted first of all, and at once marked down on the left-hand diagram; the hook I was then referred to the line B, and the mountain R to the line D, the latter being registered with great care. The floating cloud and the other prominences were then filled in, possibly not quite so carefully. The details were next drawn in, black representing white, and the first diagram was completed as rapidly as possible, yet as faithfully as the short time at command would permit. I was aware, whilst so occupied, that by the addition of detail after detail in the several prominences I was exaggerating their dimensions; but there was too little time to spare to rub out and commence anew.

When the first drawing was completed, about a minute and a half after the commencement of the totality, I looked away from the telescope in order to make the eye observations which I have already described, and before I resumed my work at the telescope an interval of half a minute may have elapsed, but certainly not more. The next thing I did was to measure the angular position of the prominence A; and after bringing the moon well into the tangential square, I moved the wire D through the arc necessary to bring it into contact with the side of that prominence nearest to D, which brought the index to an exact coincidence with one of the divisions on the quadrant; I noted down 10° for the angle moved through; but this is an evident error, for the angle was as nearly as possible 20°, which, added to 2° 59′, makes the position-angle of the western boundary of that prominence 22° 59′, from the north towards east, which is not far from its true position at that time.

Whilst measuring this prominence, I asked Mr. REYNOLDS, whose allotted task it was to develope the photographs after their exposure in the heliograph, whether anything could be seen on the first plate of the totality; and learning, with a thrill of intense pleasure, that the operation had completely succeeded, I made no further measurements, knowing full well that I should get them far better in the photographs.

Immediately after this, I commenced my second drawing, given in Plate VI., and noted down the position of the prominences A and R very exactly, by referring them to position-line D; and I then filled in the other details. As very little time remained for the completion of the drawing, I devoted my attention chiefly to the prominence R and a faint hooked prominence about 45° to the west of the position-line D, which did not imprint its image on the second photograph to the extent I should have expected from its dimensions in my sketch.

Between the completion of the first sketch and the commencement of the second, I estimate that there was an interval of about one minute, and that the second sketch was therefore commenced as nearly as possible 2½ minutes after the beginning of totality.

Thus, before commencing sketch No. 1, there elapsed,

	min.	sec.
From the beginning of total obscuration	0	30
To complete No. 1 sketch it required	1	0
Time consumed by eye observations, away from the telescope.	0	30
The measurement of prominence A occupied	0	30
Interval elapsed from the beginning of totality to the commencement of sketch No. 2	2	30

By placing a horn protractor on the original sketches, the following measurements were made:—

Protuberance.	Synonym.	Part measured.	Distance from position line D towards east, in degrees and decimals of a degree. First drawing.	Second drawing.
A.	Cauliflower. Wheatsheaf.	First boundary, *a* * .	23°·0	20°·0
		Second boundary, *a′* .	28·0	24·5
		Middle	25·5	22·25
C.	Detached cloud.	First point, *c* . . .	58·7	49·0
		Last point, *c′* . . .	69·0	58·5
R.	Mountain peak.	First point, *r′* . . .	347·0	343·5

In order to reduce these measures to position angles from the North towards East, it is necessary to add to them 2° 59′, say 3°, which will give us—

	First drawing.	Second drawing.	Apparent angular motion in the interval.
A. Middle	28°·5	25°·25	3°·25
C. First point, *c* . .	61·7	52·0	9·7
R. First point, *r′* . .	350·0	346·5	3·5

As I before stated, the positions of A and R were laid down with great care; and it will be hereafter seen that their deviations from the positions given by measurements of the photograph are remarkably small. The measurement of all the details, however, do not agree so well, because the same care could not be devoted to the laying down of their positions.

These drawings show that there was a decided angular shifting of the luminous prominence A, and of others, in reference to the moon's centre; and taking into account the probable interval between the two drawings, namely two minutes, the amount of angular motion of A is a very near approximation to the angular change which must actually have occurred. As mentioned above, there is in the drawings an exaggeration of the dimensions of the prominences, which renders them unfit for the precise determination of the moon's actual progression in the line of motion during the period of totality; nevertheless they afford excellent evidence that there was, in fact, a covering and an uncovering of prominences, which, taken in connexion with the change in the position-angle of the protuberance A with reference to the moon's centre, can only be explained on the assumption that these extraordinary appendages belong to the sun, and not to the moon.

Furthermore, it would be quite possible to make out, with considerable although not with absolute accuracy, from these drawings, the direction of the moon's motion, and the extent to which the prominences first seen were obscured by the progress of the lunar disk, and others uncovered on the opposite side as the moon continued her course. For instance, it will be remarked on inspection, that the streak of prominences, almost 1′ in

* The letters refer to the index map, Plate XV.

height, depicted in the eastern quadrant of fig. 1, Plate VI., is almost entirely covered in fig. 2, and that the difference of position of the moon in the two pictures, when measured by a suitable scale, indicates a motion of about 50″ in the interval of two minutes which they include,—a result very near to the truth, for the actual progression in that period was 54″·5. The photographs, however, as will be hereafter seen, are so much better adapted for such determinations as these, that it is not worth while to dwell more upon the conclusions to be derived from the hand drawings.

In the two coloured drawings, Plates VII. and VIII., I have depicted the result of my telescopic observations; to facilitate my doing which at some future convenient time, I made a coloured sketch on the afternoon of the eclipse. This coloured sketch, the black-and-white drawings made at the telescope and shown in fac-simile in Plate VI., together with my photographs, which I have not hesitated to use to correct any errors of position or dimension in the sketches, have enabled me to give in these drawings what I believe to be a very truthful representation of the appearance of the prominences, immediately after the commencement and just before the end of totality. The corona I do not give as an absolutely true representation of that phenomenon, but as fairly resembling its general appearance. It has been derived from the photographs, so far as they show it.

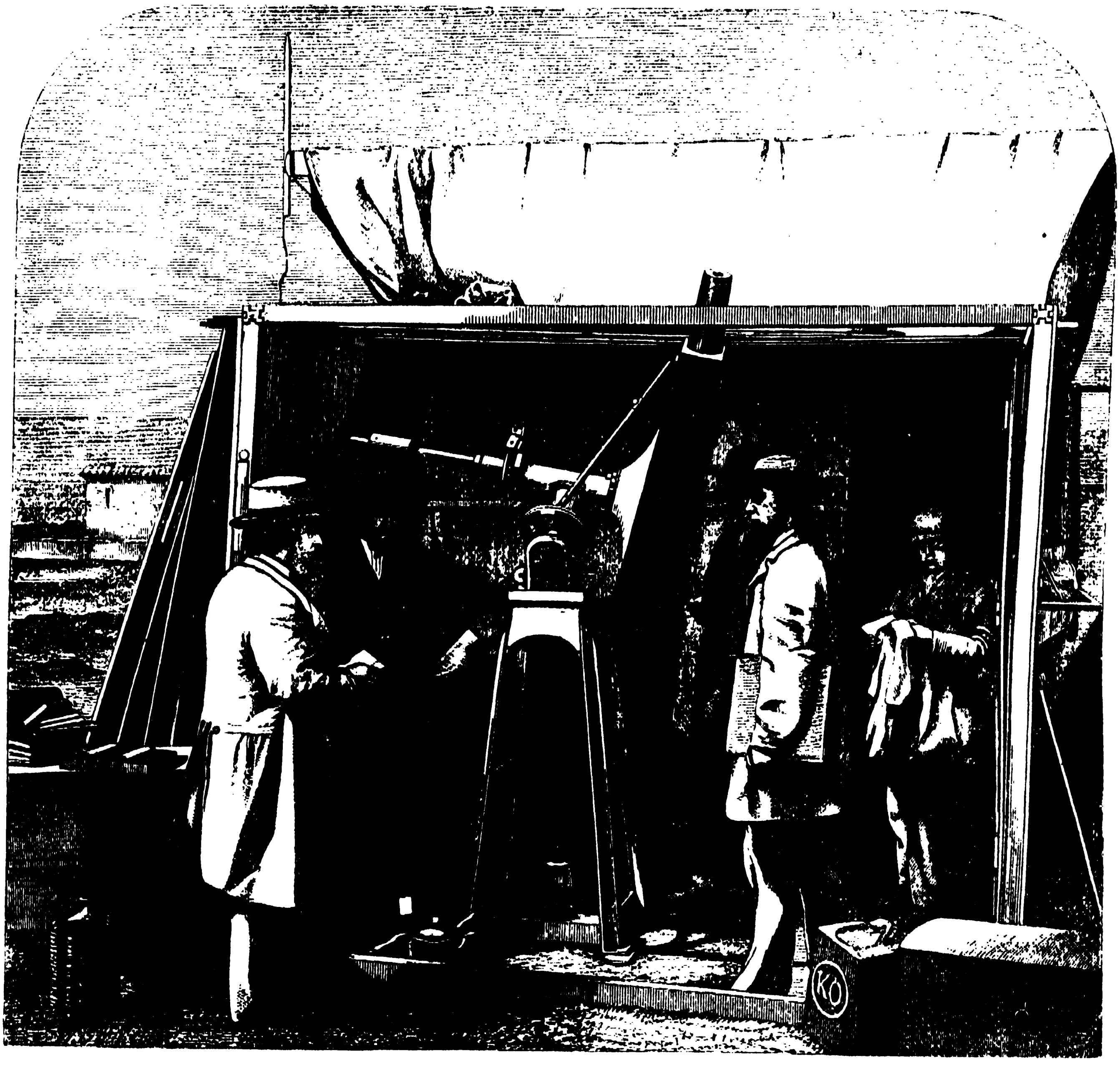
II. *Account of the Photographic Observations.*

The Kew heliograph, with which the photographs were obtained, is represented in the accompanying engraving *. It was devised by myself, for the special object of making photographs of the sun's disk, at the request of the Council of the Royal Society, in accordance with a recommendation to that effect by Sir JOHN HERSCHEL.

It has an equatorial mounting of the ordinary form, after the so-called German model, to which is attached a clock-work driver. The tube is square in section, and larger at the lower end than at the upper or object-glass end. The object-glass has 3·4 inches' clear aperture and 50 inches' focal length; the primary focal image of the sun at his mean distance is 0·466 inch in diameter; but before it is allowed to fall on the sensitive plate, it is enlarged to about 3·8 inches by means of an ordinary Huyghenian eyepiece. In the plane of the focus of the posterior lens of this eyepiece are attached two position-wires, which cross at right angles, and which were adjusted, approximately, into a position at an angle of 45° to a parallel of declination. The object-glass is so constructed as to ensure the coincidence of the chemical and visual foci; but this coincidence being somewhat disturbed by the Huyghenian secondary magnifier, the amount of adjustment required to effectuate the best chemical focus was ascertained very carefully by a series of experiments.

* The engraving was copied from a photograph taken at Rivabellosa. The front boards of the observatory were taken out in order that this might be done. Mr. DOWNES, who was charged with the preparation of the plates, is standing in the doorway leading to the developing room. Mr. REYNOLDS has a plate-holder ready to place in the heliograph, and Mr. BECKLEY is observing the time with the chronometer.

For sun-pictures, and the photographs of the several phases of the eclipse, the aperture of the object-glass was reduced to about 2 inches in diameter by means of a stop; but the light of the sun is so extremely powerful, that, even with this small aperture, combined with the enlargement of the primary image and its consequent reduction in intensity by 64 times, the shortest exposure possible with the ordinary means of uncovering and covering the object-glass would be far too long, and would give none but solarized pictures. For this reason the instrument has attached to it an instanta-



neous apparatus of a peculiar construction. It consists of a sliding plate with a square aperture sufficiently large to permit of the passage of all the rays; this aperture is fitted with a sliding piece, actuated by a screw which projects through and a few inches beyond the telescope-tube; by means of this screw the aperture may be completely opened, closed, or reduced to a slit of any required width; a divided scale being affixed to the screw for that purpose. The projecting screw connected with the slide is shown in the engraving, on the underside of the tube.

Previous to taking the picture, the sliding plate is drawn up just so high that the unperforated part of it completely shuts off the sun's image; it is held in this position by means of a small thread attached to it at one end and looped at the other, the loop being passed over a hook on the top of the tube; and the slide is pulled downwards, in opposition to the thread, by means of a spring of vulcanized caoutchouc attached to the inferior side of the tube. When the picture is about to be taken, the retaining thread is set on fire*, and the rectangular aperture, as soon as the sliding plate becomes released, flashes across the axis of the secondary object-glass—thus allowing the different parts of the sun's image to pass through it in succession, and to depict themselves one after another, after enlargement, on the collodion-plate. Although the time of exposure is so short as to be scarcely appreciable, yet it is necessary to regulate its duration; and it is therefore controlled by adjusting, 1st, the strength of the vulcanized caoutchouc spring; 2ndly, the width of the aperture. In practice, the opening is usually varied between one-twentieth and one-fortieth of an inch.

A number of plates, with ground rims and edges, were cleaned in London, so as to permit of their examination, and all defective ones were rejected; forty-eight selected plates were then numbered consecutively, and arranged in boxes marked very distinctly A, B, C, D, so as to ensure their being taken out in the proper order during the eclipse. The heliograph was furnished with three plate-holders, in order that no interruption might occur in the succession of the photographs; and as these were filled, they were placed in such a way that each plate was sure to be exposed in its numerical order. A few spare plates were also cleaned, and marked A, B, C, D, E, F, G, H, I, &c.

On the day previous to the eclipse the plates were again carefully cleaned, and replaced in their proper order in their respective boxes.

On the 18th the following plates were placed in the heliograph, and the time of taking each photograph noted by Mr. Beckley, with any requisite remarks. The time was observed with Frodsham No. 3094, whose error at Greenwich mean noon was, as already stated, fast of Greenwich mean time 4·6 seconds, and whose daily rate was losing 2·1 seconds. The exact time of depiction was ascertained by listening to the click which the instantaneous slide made in striking home upon a stop, when it had flashed across, in front of the secondary magnifier.

* Mr. Clark, who undertook this task, is represented in the engraving with a lighted taper in his hand.

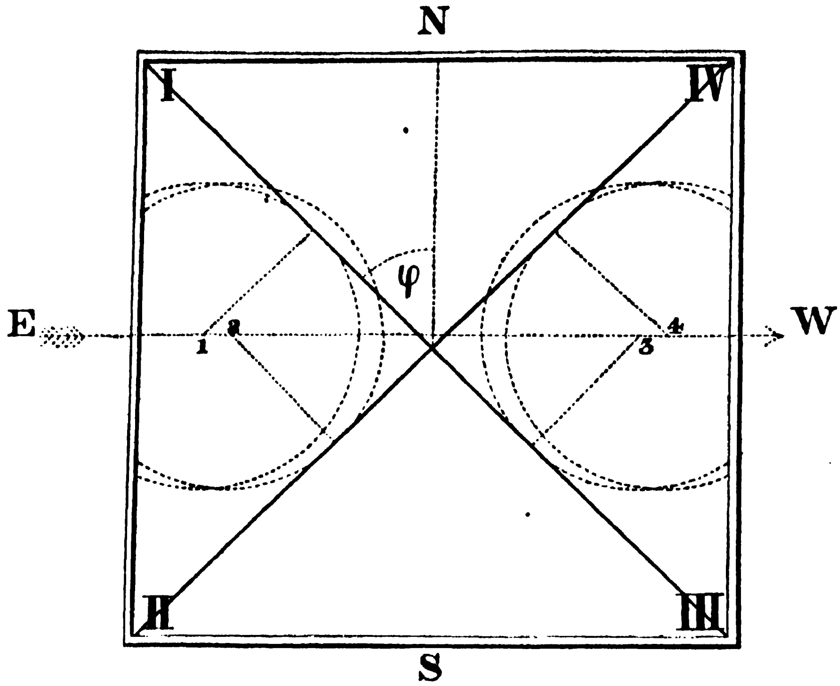
No. or letter.	h	m	sec./2	Remarks.
1.	23	38	1	Through a cloud.
2.	....			Spoiled.
3.	....			Spoiled.
4.	0	23	105	
A.	0	29	45	
B.	0	34	34	
C.	0	42	24	
D.	0	49	93	
E.	1	3	23	
F.	1	13	85	
G.	1	25	45	
H.	1	29	76	
I.	1	41	76	
5.	1	45	3	
6.	1	47	96	
7.	1	52	71	The first of the eclipse.
8.	1	56	15	
9.	2	2	79	Occultation of spot *c*.
10.	2	3	41	
11.	2	7	21	
12.	2	11	31	
13.	....			Clouds.
14.	2	20	24	
15.	2	22	97	
16.	2	27	103	
17.	2	33	36	
18.	....			Spoiled.
19.	2	36	43	
20.	2	41	55	
21.	2	46	46	

No. or letter.	h	m	sec./2	Remarks.
22.	2	48	60	Occultation of spot *b*.
23.	2	51	53	A solarized picture, in consequence of the full aperture being used and the instantaneous slide being detached.
24.	2	55	88	Id., the time uncertain.
25.	....			Totality; time not noted.
26.	....			Totality; time not noted.
27.	....			Shaken by wind; time not noted; the instantaneous slide had not yet been replaced.
28.	3	13	116	
29.	3	17	32	Reappearance of spot *c*.
30.	3	20	24	
31.	3	24	35	
32.	3	26	112	
33.	3	34	12	
34.	....			Spoiled.
35.	3	37	48	
36.	3	41	46	
37.	3	44	4	
38.	....			Forgot to uncover the plate.
39.	3	51	116	Spot *a* uncovered completely.
40.	....			Forgot to uncover the plate.
41.	3	59	5	Reappearance of spot *b*.
42.	4	2	96	
43.	4	5	98	
44.	4	10	75	
45.	4	16	40	

The diagram shows the appearance of the cross wires when projected on the glass screen. The image of the sun, being twice reversed, is finally depicted on the screen in its natural position, north being at the top, south at the bottom, east to the left hand, and west to the right hand. In the positive photographs of the eclipse, printed from the negatives, the pictures are likewise erect, and the points similarly situated. Calling the wires I., II., III., IV., I. would have approximately the position-angle of 45°, II. 135°, III. 225°, and IV. 315°. In the measurements, hereafter to be described, the several photographs were so placed on the measuring instrument as to cause its circle to read respectively one or other of these angles, according as either I., II., III., or IV. was employed in adjusting the photograph, the correction to the measured angles, necessitated by the deviation of that wire from its assumed position in reference to a parallel of declination, being subsequently applied.

The wires were found not to be absolutely at right angles.

	°	′
IV.— I. measured	89	59·3
I.— II.	89	52
II.—III.	90	8·7
III.—IV.	90	
	360	0



At $4^h\ 35^m$, when the heliograph was pointed to the west of the meridian, observations were made to determine the deviation of the position-wires, from an angle of 45° to a parallel of declination, by the method described by Mr. CARRINGTON in the 'Monthly Notices' of the Astronomical Society, vol. xiv. p. 153; and the observations were repeated on July 19 at $11^h\ 55^m$, when the heliograph was pointed east of the meridian.

July 18, $4^h\ 35^m$, by Frodsham 3094 uncorrected.

The sun's limb made contacts with wires I. and III.

	sec.
at an interval of	194
" "	195
" "	193
Sum	582

and with wires II. and IV.

at an interval of	188·5
" "	191
" "	188
	567·5

582 log = 3·7649

567·5 log = 3·7540

Angle of φ 45° 43′·5 log tan. = 0·0109

July 19, $11^h\ 55^m$.

The sun's limb made contacts with wires I. and III.

	sec.
at an interval of	189·5
" "	188
Sum	377·5

and with wires II. and IV.

at an interval of	. . . .	193·5
" "	. . . .	193
Sum	. . . .	386·5

whence angle $\phi=44°\ 19'\cdot5$.

	h	min.			°	′
At	4	35	P.M.	$\phi=$	45	43·5
At	11	55	A.M.	$\phi=$	44	19·5
Interval	4	40	diff.	+	1	24·0
Interval . . . . =		280		=+		84

$$\frac{+84}{280}=+0'\cdot3 \text{ the change of the angle } \phi \text{ in 1 minute.}$$

	h	m	sec.
No. 6 photograph was taken at	1	47	48
Deducting	11	55	0
	1	52	48=112·8 min.

$112\cdot8\times0'\cdot3=34'$, $44°\ 19'\cdot5+34'=\ 4°\ 53'\cdot5$, the position of wire I. in reference to a circle of R.A. at the epoch of No. 6 photograph; hence the correction to be applied to the assumed position of 45° is $-6'\cdot5$.

With the foregoing data has been calculated the following Table of corrections, to be applied to the assumed position of the wires at the epochs of the several photographs.

No.	Correction to the wires.	No.	Correction to the wires.	No.	Correction to the wires.
6.	−6′·5	20.	+ 9′·5	32.	+23′·2
7.	−5·1	21.	+11·0	33.	+25·4
8.	−4·0	22.	+11·7	35.	+26·4
9.	−2·0	23.	+12·6	36.	+27·6
10.	−1·8	24.	+14·0	37.	+28·4
11.	−0·7	25.	+15·5	39.	+29·8
12.	+0·5	26.	+16·5	41.	+31·9
14.	+3·2	27.	+18	42.	+33·0
15.	+4·0	28.	+19·4	43.	+33·9
16.	+5·5	29.	+20·3	44.	+35·3
17.	+7·1	30.	+21·2	45.	+37·0
19.	+8·0	31.	+22·5		

In these corrections I have not taken into account a small error in the computed angle of ϕ, which arises in consequence of the wires not being at right angles; for, on examining them, I found that the heat of the sun had caused a curvature in one, and I could not, without much trouble, have ascertained the correction with precision. It was computed that it would not, however, amount in any case to more than $+2'$. My measurements of position-angle, hereinafter given, are moreover liable, from the difficulty of adjusting the photographs, to discordances to the like extent, as will be easily conceived when it is stated that 2′ on the sun's limb do not occupy more than the space of $\frac{1}{1000}$th of an inch on the photographs obtained with the Kew heliograph.

At the moment of taking the photographs, the collodion was in a soft and moist condition, but subsequently, when the measurements were made, it had become dry. It became, therefore, not only a matter of interest, but of fundamental importance, to ascertain whether there had been any contraction of the collodion while drying, and, if so, whether the contraction had been uniform. Much care and attention were necessary in order to determine this point. By observing the positions of specks on the glass in respect of markings on a photograph while wet, it could be seen whether they retained their relative positions when the collodion had dried. The result, however, proved that there was no appreciable contraction, except in thickness, and that the collodion film did not become distorted, provided the rims of the glass plate had been well ground. I cannot show this more strikingly than by citing the measured radius of the sun on two photographs, namely, Nos. 6 and 45, and the measurement of the angles between the position-wires depicted on them. The radius of the sun No. 6 was found to be 1906·5, that of No. 45 1906·0 thousandths of an inch.

	Angle between IV. and I.	Angle between I. and II.	Angle between II. and III.	Angle between III. and IV.
No. 6	89° 59′·3	89° 52′	90° 8′·7	90° 0′
No. 45	90 2	89 53	90 6	89 58·5
Diff. 6—45 . .	−2·7	− 1	+ 2·7	+ 1·5

These differences, which are extremely small, do not exceed those obtained in measuring at different times the same photograph, and depend somewhat on the judgment exercised in causing the images of the position-wires to be bisected exactly by the wire of the microscope.

Photographs of the various phases of the partial eclipse, either previous to or after totality, exhibit a very curious phenomenon. The concave edge of the sun in immediate contiguity with the moon's limb, appears brighter than the other neighbouring parts of the crescent, while the convex limb of the sun bordered by the dark background of the sky, does not appear at all brighter than its proximate parts. This brightening of that part of the sun's disk which borders on the moon's limb, extends only for the space of a narrow line beyond the latter, but is remarkably conspicuous. As it cannot be accounted for by assuming the existence of a lunar atmosphere, it naturally excited a desire to trace out its cause. The Astronomer Royal, to whom I pointed out the fact, ascribed it to the effect of contrast, and I have subjected this hypothesis to the test of experiment in the following manner:—Having made some photographic prints of the sun's crescent on paper, which showed the appearance in a striking manner, I cut out about half of the crescent with sharp scissors, in such a way that the visible surface of the sun might be lifted up like a tongue, and replaced in its normal position within the background at pleasure; on smoothing the part so cut out, and causing it to occupy its original place, the bright line was apparent, but it disappeared when the crescent was lifted up, and a sheet of white paper was interposed between it and the dark ground of the photograph. These phenomena occurred when the photograph was examined with the naked eye, with the aid of spectacles, or, from a short distance, with a sharply defining telescope by Ross. Viewed in either of these ways, the brightening was found to begin immediately beyond the edge of the white paper as it was introduced more or less under the crescent.

For the purpose of illustrating this paper on the occasion of its being read before the Society, I prepared a representation of one of the photographs of partial phase, 3 feet in diameter, in which, bearing in mind the well-known fact that there is on the solar disk a gradual diminution of the intensity of the light from the centre to the periphery, I carefully reduced the brightness of the solar crescent in due gradation towards the convex boundary. In the first instance the background was not painted in, and I expected that when it was completed a brightening would immediately occur. Such, however, was not the case.

On calling Professor STOKES's attention to this failure in producing the phenomenon of brightening by artistic means, he suggested that I should renew the attempt by using a real photograph of the sun and a dark disk for the moon*. On this plan I succeeded in making eclipse-pictures artificially, which showed the brightening very distinctly. From these experiments I am inclined to believe that Mr. AIRY's explanation is the true one, although it is a curious subjective fact that the parts possessing superior illumination exhibit to our perception an extremely bright line, bordering immediately on the dark limb of the moon, while the less bright parts towards the circumference present no such appearance, although they also are contrasted with the dark background.

In order to study other points connected with the photographs, I had made, on glass, some enlarged copies, in which the moon's disk was increased in some cases to 9 inches, in others to 13 inches in diameter. It was found that measurements could be made on these with considerable accuracy, by means of a graduated beam compass reading to thousandths of an inch, and I had proceeded to some extent in this way, when it occurred to me to have an instrument constructed expressly for measuring the original negatives. The study of the enlarged copies led, however, to a method of producing charts of the prominences with complete fidelity; and the plan will, I think, hereafter prove applicable to the production not only of astronomical, but also of other graphic representations derived from photographs. In order to carry out this method, a table had to be constructed with a square hole cut in it somewhat smaller than the glass positive to be worked upon; a recess surrounding the hole was made in the top of the table, just the size of the glass, and of a depth corresponding to the average thickness of the plates. Four spring clips served to hold the glass firmly in its seat. Parallel with

* For the lunar disk I employed photographic paper darkened to the same tint as the background of the solar photograph. These disks were in some cases neatly inserted in a circular hole in the solar picture, and in other cases pasted on it. In either case the surface was polished and made uniform by passing the picture through a rolling-press.

one side of the table were inserted two brass plates with long slots through which two screws worked. These screws passed through a straight edge, which could be adjusted so as to cause a right-angled drawing-triangle resting against it to assume any required position with respect to the image of the position-wires depicted on the photograph. A long glass mirror was attached to the frame of the table underneath the top, in such a way as to be adjustable to the angle best suited to reflect light through the transparent positive. In front of and above the table was placed an inclined screen formed of tissue paper, to diminish the direct light, a certain amount of which was required to show the position of the etching-point. Without the aid of this screen, the direct light would have been too powerful, and would have prevented the details of the transparent photograph from being seen by the light transmitted from below after reflexion from the mirror.

In the first place, the centre of the picture was found, and marked with a diamond point. A drawing-triangle, with one angle of 90° and two of 45°, was now placed over the photograph, with one of its sides resting against the adjustable straight edge, when its hypothenuse would coincide approximately in direction with the images of the wires. By adjusting the straight edge, the hypothenuse of the drawing-triangle was brought to exact coincidence with either of the wires, and the straight edge, against which it rested, was then (by means of the screws passing through it) clamped in position. By sliding the triangle along in contact with the straight edge, a line parallel with the wire was next set off passing through the centre, and marked slightly on the periphery of the picture by scratching with a diamond point through the collodion film. On taking in the beam-compasses a chord corresponding to 45° plus the known + or − error of the wires, a circle of right ascension, or a parallel of declination, could be made to pass through the centre, and, the points of its intersection with the lunar disk having been marked, any angles of position could be ascertained, by taking the chord between any part of a protuberance to be measured and the normal points thus set off.

If it were desired to produce an etching of any photograph, the outline of the protuberances or of the sun's disk and spots, or of the crescent of the sun, as the case might be, was traced very carefully with an etching-point through the collodion, with the aid of a lens. When this had been done, the plate was warmed by holding it before a bright clear fire, and a piece of composition, consisting of a mixture of paraffin and white wax, rubbed over it; the heat of the plate caused the waxy mixture to melt, and thus a very even, thin, and translucent etching-ground was laid on the glass. The outline was now traced a second time, in this instance through the wax, and a camel-hair pencil, wetted with liquid hydrofluoric acid, was rapidly run over the parts traced. In about a minute the acid was removed with blotting-paper, and the plate rinsed with water, and again dried with bibulous paper. When quite dry, the wax was melted by holding the plate before the fire, and wiped off with a cloth. If the etching proved satisfactory, it was again covered with an etching-ground, then centered on the circular dividing-engine, and degrees and subdivisions set off, starting from a normal point previously marked on the plate. It was then put on the straight-line engine, and a scale of minutes and seconds of arc set off from the moon or the sun's periphery, in accordance with the previously calculated value in arc of subdivisions of an inch; both sets of division were then etched in the same way as the outline.

Sometimes, according to the position in which the photograph was taken*, the etching was performed at the back of the plate, to correspond with the previous tracing through the collodion on its face. In this case the collodion picture might be allowed to remain as "a witness" (as workmen call it) of the correctness of the etching. In other cases, if the original negative had been purposely turned over, so as to present the opposite face to the camera, then the etching was made through the collodion, which had to be removed before the subsequent operations about to be described were performed.

An etched glass plate, if filled with printing ink, could be made to give a print by placing india proof-paper over it, and, after superposing a sheet of glazed paper upon this, rubbing the latter carefully with a burnisher; but it would not be advisable to attempt to take many impressions in this way. However, by the well-known processes of electrotype, copper duplicates of the glass plate can be procured, which can be printed from in the ordinary copper-plate press; and as the glass plate is only used for furnishing the matrices, and is not injured thereby, the printing-plates may be procured without practical limit as to number. In this way Plates XIII., XIV., XV., XVI. and XVII. were obtained. The original glass plate of Plate XV. was, however, made in a somewhat different manner from the others. Originally, it was a photograph of the sun; after the outline of the sun and his spots had been etched, and the normal line marked thereon, the collodion was entirely removed, to permit of the plate being superposed, accurately, first over Plate XIII., and then over Plate XIV. Previously, however, Plate XV. was coated with the transparent etching-ground, and the luminous prominences depicted on Plate XIII. traced off, care being taken to ensure the parallelism of the normal line of one plate with that of the other, and internal contact between the peripheries of the sun and moon respectively. The same thing was done with Plate XIV., the prominences visible in the two pictures being placed in coincidence. In this way the pictures of the prominences could be made to assume their proper position around the sun's picture. In order to facilitate this operation, a positive picture had been previously taken with the enlarging-camera, from both the original totality negatives laid one over the other, and combined suitably together, so as to form in one picture a correct representation of the whole of the prominences. When the two totality pictures had been traced off on Plate XV., a line was drawn to join the two positions of the moon's centre, which had been set off from Plate XIII. and Plate XIV. respectively; this line was then prolonged to show the path of the moon's centre during the period of totality; lines were also drawn to join these positions of the moon's centre, and the sun's centre, and prolonged to the periphery of the sun, to indicate the points of disappearance and reappearance of the sun's limb. When etched, this plate was

* By placing the original negative in the copying-camera with the collodion film either turned towards the lens or away from it, the picture produced was either in its natural position or reversed right for left.

angularly divided concentrically with the sun, and a scale of minutes and seconds of arc etched, starting from the sun's limb, by which means the prominences were referred to the sun's centre, and their angles of position and heights above his periphery could be read off with a fair degree of accuracy.

In the three Plates, XIII., XIV. and XV., a wrong correction was, however, applied for the errors of the wires in determining the zero of the angular divisions, namely $+23'$ for both totality pictures, instead of $+15'{\cdot}5$ for the first totality picture and $+16'{\cdot}5$ for the second; so that in taking angles of position of the prominences, the readings on Plate XIII. must be corrected by applying the number $-7'{\cdot}5$; those on Plate XIV. by applying the correction $-6'{\cdot}5$, and those on Plate XV. by applying $-7'{\cdot}0$.

Moreover, a small error in determining the centre in Plate XIV. also interferes with the absolute correctness of the position-angles and of the heights of the prominences above the moon's periphery. Subsequently to this being etched, I discovered the fact that the centre should have been placed about $5''$ of linear space nearer 270°, in a direction from 90° to that point, and $4''$ nearer 360°. The angular positions of some of the principal prominences, determined by measurement of the original negatives, will be hereinafter given, so that no difficulty will be experienced in correcting the position of the other prominences as read off from the Plates. The prominences in these Plates are represented in their natural (erect) position, and this is also the case with the sun-spots in Plate XV.; the position-angles are laid down from North towards East. The North point (360°) is consequently at the top, the East point (90°) is on the left hand, the South point (180°) is at the bottom, and the West point (270°) on the right hand.

In order to facilitate reference to the prominences, I have designated them on Plate XV. by capital letters, commencing with the prominence situated at right angles to the path of the moon across the sun's disk, which I call A; and I then follow on towards the east with the other capital letters, the small letters being employed, either alone, or with one or more dashes, to mark the subordinate parts.

The three principal sun-spots are marked *a*, *b*, *c* in the order of increase of their several position-angles.

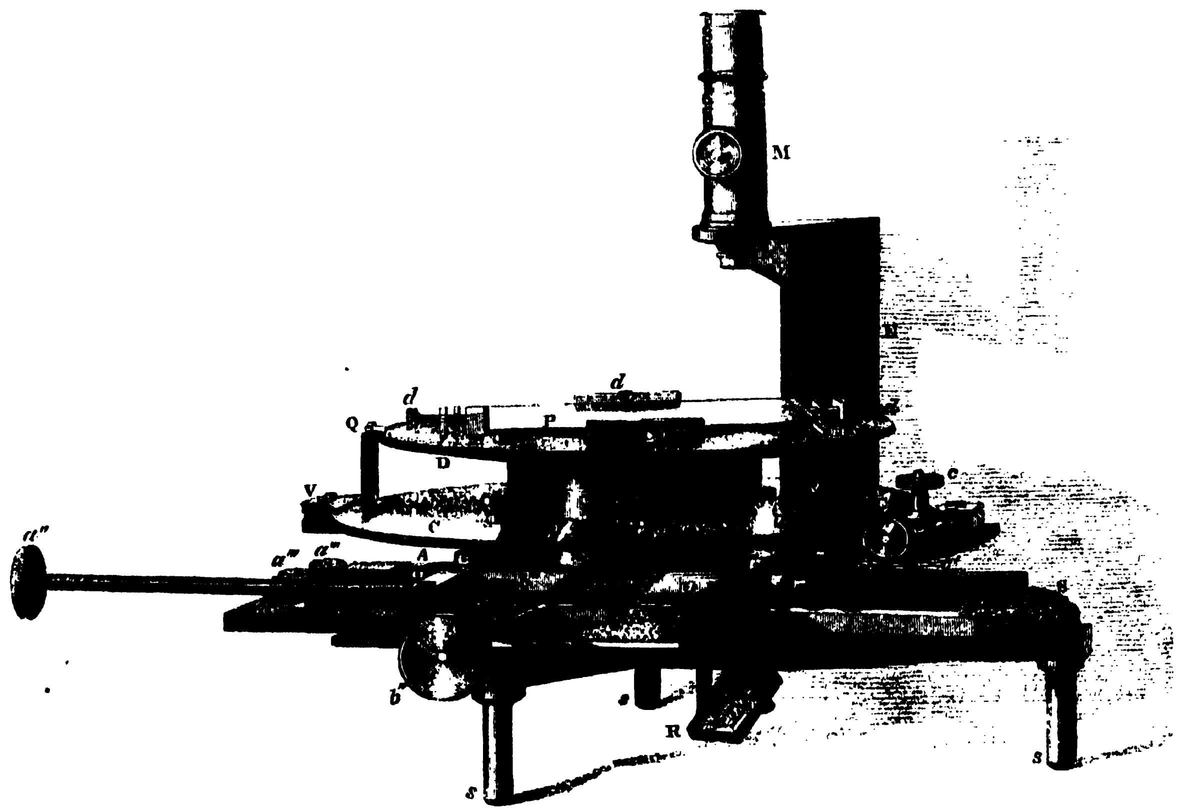
In Plates XIII. and XIV. the details were drawn in on the back of the glass plate, and the collodion pictures still remain intact; Plate XV. was drawn on the face of the enlarged positive, which had been taken intentionally in a reversed position, by reversing the original negative in the copying-camera. The correction in the position of the glass negative on account of its thickness was duly made; that is to say, the totality pictures having been copied with the collodion turned towards, and the sun-picture with the collodion turned from the lens, the collodion was in this way carried from the lens a quantity equal to the thickness of the glass plate. The holder supporting the original negative was therefore moved towards the lens a similar quantity, and the relative sizes of the pictures, as a matter of course, remained undisturbed.

Plate XV. will be found useful as an index map of the prominences, and will facilitate comparisons of the results obtained by the various astronomers who observed the eclipse. Moreover, the position of the sun's axis being given on it, an idea may be formed by mere inspection, of the general distribution in heliocentric latitude of these appendages. Plate XVI. was obtained by etching a positive copy of No. 22 photograph, and Plate XVII. by etching one of No. 28 photograph. These photographs were obtained with the same setting of the copying-camera as the sun-picture, the original of Plate XV. They show the contours of the sun and moon, and furnish a means for comparing the concave outline of the luminous prominences with the profile of the moon which was in juxtaposition with it.

Reserving, for the present, an account of the measurements actually made on the engraved-glass originals of Plates XIII., XIV., and XV., I will proceed to describe a new measuring-instrument, and the measurements made by its means, not only of the totality-pictures, but also of the original negatives of the sun taken before and after the eclipse, as well as of the different phases of the eclipse. These consist of the direct measurement of the sun's radius, the direct measurement of the moon's radius (where possible), the measurement of the chord joining the cusps, the measurement of the distance between the chord and the peripheries of the sun and the moon, and also the distances of these peripheries, the determination of the angles of position of the cusps, and their angular openings, and, lastly, the position-angles of the luminous prominences. The heights of the prominences could not be determined by means of the micrometer, in consequence of an inadvertence on the part of the makers, who by mistake made it somewhat too small; here, however, the engraved-glass originals of Plates XIII., XIV., and XV. supply the numbers with sufficient accuracy, so that this oversight is of no practical moment.

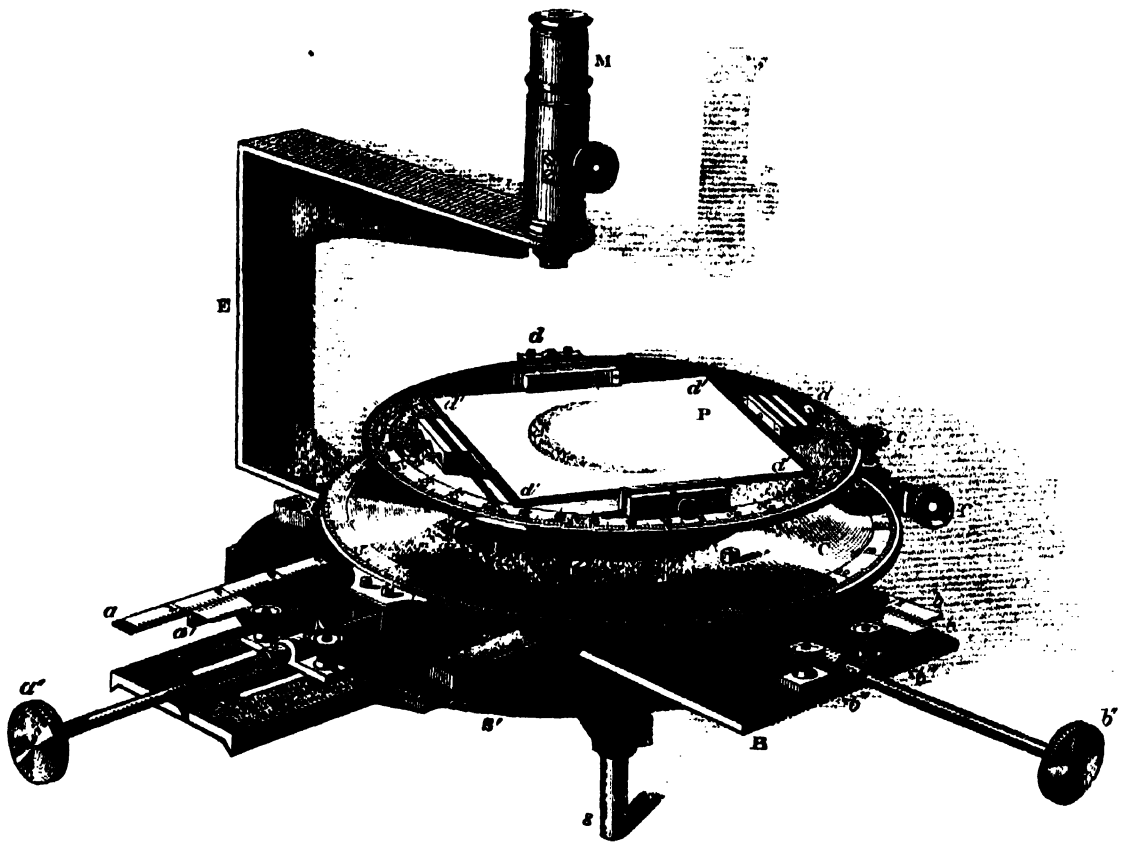
Figures 1 and 2 represent the micrometer in two different positions.

Fig. 1.



In both figures the same letter is employed to designate the same part, capital letters being used for the principal parts, and the same small letter, either alone or with one or more dashes, for the subordinate details attached to it. S is a tripod stand sup-

Fig. 2.



ported on the legs s; to this stand is firmly fixed the arm E, which supports the fixed microscope M in the centre of the instrument. This microscope can be adjusted to focus by means of the milled head shown in the engraving; and its positive eyepiece can also be adjusted by sliding it up or down, so as to bring to focus wires crossing at right angles and fixed in the centre of the field of view in such a way that their directions correspond respectively with the positions of the slides A and B, presently to be spoken of. R, fig. 1, is a plain mirror, which is adjustable so that it may reflect light through the photograph. A is a slide to which are attached all the other parts of the apparatus; it moves freely, and without vibration, between guide-bars fixed on the top plate of the tripod stand, and shown in fig. 1. The top plate of the tripod stand has a round hole in it, and the bottom plate of the slide A is perforated with a similar hole.

A steel rod a'', screwed for a certain distance at the point, works into a tapped hole in the slide A; by taking hold of the milled head a'', the slide A may be rapidly moved the whole length of the guide-bars, the rod carrying with it the clamping-piece with its two screws a''', a'''. These clamping-screws slide through two slotted holes in the top plate of the stand; when the nuts a''', a''' are screwed tight the slide is held fast, but by turning the milled head a'' the micrometer screw is also turned, and the slide can be moved for any short distance.

Attached to the slide, and moving with it, is a scale a, a little more than 4 inches in length, and divided into inches, tenths, and $\frac{1}{50}$ths, the inches being numbered 0 to 4. On the stand is fixed the vernier a' (fig. 2), which reads to $\frac{1}{1000}$th of an inch, and by estimation to $\frac{1}{2000}$th.

On the slide A are two guide-bars, between which the slide B works at right angles to A; the guide-pieces are adjustable by means of set screws a'''' (fig. 1), to ensure the rectangularity of the slide B. Slide B is moved by means of the steel rod b'', screwed at one end, and carrying its clamping-bar and screws b''' in the same way as the slide A. It has also an attached scale b, and a vernier b', fixed to the slide A, which reads with the scale b to the same quantities as the vernier a' of the scale a.

On the slide B, which is perforated, is a hollow axis, somewhat more than 4 inches in diameter, in which works the hollow axis of the divided circle C, which reads by means of its vernier V to 10″ of arc. The circle C is clamped by the clamp c, and can then be moved by means of the milled head c', attached to a tangent screw. The vernier V and clamp c are fixed to the slide B.

On the hollow axis of the circle C is a second divided circle D, which reads to minutes of arc, and is divided into four quadrants; the vernier Q (fig. 1) being attached to the lower circle C. The circle D is only used for axial adjustments, to bring the position-wires depicted on the photographs to parallelism with the wires in the microscope M.

The circle D has four dogs fastened on it, through which work the screws d, which carry along with them four pressure plates, with two projecting wires in each, to act against the photographic plate P, and make it central with the instrument. The photograph rests on the four ivory studs d'.

The photograph to be measured is in all cases placed with the collodion side downwards, in order to ensure a constant distance from the microscope; if placed upwards, any variations in the thickness of the various glass plates would necessitate a change in the position of the microscope at each operation.

In the first instance it is necessary to determine that particular position of the slides A and B in which the axis of the circle C corresponds with the centre of the cross wires of the microscope, which is very accurately and rapidly accomplished in the following manner:—A glass plate of suitable size has ruled upon it with a writing-diamond two lines which intersect each other as nearly as may be at right angles: this plate is placed face downwards on the ivory supports d', and the slides A and B are brought approximately to and clamped in the central position, which is at about the division 2 inches on their respective scales; by means of the screws d, the cross on the glass is made to coincide with the cross wires of the microscope. The circle C is now turned through half a revolution, when the cross on the glass plate will be found to have shifted. Half this deviation, according to its direction, is corrected by the screws a'' and b'' of the slides A and B, and half by the centering screws d of the circle D. After a few trials, the cross on the glass plate will not shift during the rotation of the circle C; when this is the case, the verniers of the slides A and B are observed, and the readings noted down.

The centre of the circle C (and consequently of D) was found to coincide with the cross of the microscope when A read 2·0025 inches and B 1·981 inch; and these positions were found not to alter materially for several months, during which they were from time to time tested.

The next thing to be ascertained was the rectangularity of the two slides, which was done in this way. By means of the upper circle D, one of the ruled lines on the glass plate was made to coincide with a wire of the microscope after the lower circle C had been clamped to read 360°. The slide B remaining central, slide A was unclamped and drawn out to the full extent; any deviation between the line on the glass plate and the centre of the cross in the microscope was then corrected by moving the tangent screw c', and, the slide being again pushed back to the centre, a few trials soon brought about an exact coincidence of the line on the plate and the wire of the microscope during the travelling of the slide A. The vernier V was next read off accurately, and the slide A brought to its normal position, 2·0025 inches. The circle C was then moved through exactly 90°, and clamped securely, and the slide B, having been unclamped, was pulled out to the extent of its path. If the same line on the glass plate maintained its coincidence with the cross of the microscope during the travelling of slide B, the slides were necessarily at right angles; if not, the error was corrected by moving the slide B, with respect to A, by means of the adjusting-screws a''''. If the deviation was considerable, it became necessary, after its rectification, to re-ascertain the normal central position of slides A and B. The greatest deviation from rectangularity amounted, after final adjustment, to 0° 0′ 10″, a quantity which could in no way affect the measurements.

The thickness of the wire of the microscope was ascertained to be 0·0003 inch, amounting to about 0″·15. The measurements could not generally be made nearer than 0·001 inch, equal to 0″·5, and in most cases the *centre* of the wire was brought as nearly as possible to coincidence with the point to be measured; this minute correction consequently has not been applied.

The measurement of the photographs was effected in the following manner:—In the first place, the slide B was set at its central point 1·981 inch, and the slide A pulled out so far as was judged necessary to bring the periphery of the sun (or moon, as the case might be) under the centre of the cross. The photograph to be measured was then laid on the supports, due regard being paid to the position of the wires depicted thereon, so that they might coincide with the corresponding divisions on the circle C, that is, I. with 45°, II. with 135°, III. with 225°, and IV. with 315°. The photograph was next centered by means of the four screws d', and the requisite movement of the slide A by means of its screw a'', the slide B remaining at rest. When the periphery of the sun, notwithstanding its somewhat irregular outline, maintained, during the rotation of the circle C, a general coincidence with the centre of the microscope, the vernier a' was read off; this reading, minus the reading of the central position of A, namely, 2·0025, gave the measurement of the radius of the sun. The slide A was then unclamped and traversed right across, so as to bring the limb of the sun, after passing to the other side of the centre of the instrument, into coincidence with the cross when the circle C was rotated, and the vernier a' again read off. The difference between the first and second readings gave the diameter of the sun-picture. Thus, let

	in.	
The first reading of photograph No. 20 be	3·907	
Deduct the position of the centre on the scale *a* . . .	2·0025	
Radius of the sun=	1·9045	

	in.	
The second reading of same photograph	0·094	
And the first reading having been	3·907	
Difference between first and second reading gives . .	2)3·813	sun's diameter.
	1·9065	radius.

These measurements, in the example cited, differ from each other 0·002 inch, which, as will be hereafter seen, is equal to about 1″ of arc. In Table I., columns 7 and 8, are given two series of measurements of the sun in these ways: in column 7 the numbers were obtained by the first, in column 8 by the second method.

The next operation was the rectification of the position of the photograph by means of the wires depicted on it. I must here call attention to the circumstance, that in the various phases of the eclipse the wires cannot be traced beyond the crescent of the sun; for the time occupied in taking the photograph is so extremely short, that the background of the sky in the immediate neighbourhood of the sun does not in the slightest degree depict itself on the collodion plate. The wires are visible on the sun's crescent, because they intercept his action, and produce a blank space corresponding to their shadows, but evidently the dark body of the moon and the adjacent sky could have no such effect. In photographs of the partial phases, therefore, the length of the wires depicted corresponds with the extent of the crescent unobscured, and is shorter the nearer the picture is to the totality. A wire which is visible in some cases is afterwards covered by the moon, so that the correct apposition of the picture could not be effected, in every instance, by means of the same wire. In the totality-pictures the whole four wires are visible, from the fact of one or other of them having intercepted either the light of the prominences or that of the corona, the latter even having depicted itself sufficiently to render the shadow of the wires quite perceptible. In the other phases, either wire IV. or wire II. was used for rectifying the picture; if wire II. were used, the circle was clamped to read 134° 54′ 10″, because this had been found, by measurement, to be the position which corresponded to 315° for wire IV.

Supposing wire IV. to have been visible, and the circle clamped so as to read exactly 315° after centering, it would generally happen that the picture of wire IV. was neither under the centre of the microscope, nor parallel with the wire corresponding to the slide A, because, in the first place, the picture may not have been taken exactly in the centre of the heliograph; and in the second place, in centering, the image of the wire may not have been brought exactly parallel with a normal diameter from 315° to 135° of the circle C. If such were the case, by means of the upper circle D the image of the wire was first made approximately parallel with a wire of the microscope, and then, by means of the screw *b*″ of the slide B, was brought exactly under this wire of the microscope, so that the latter bisected longitudinally the broad image of the wire in the photograph; the slide A was now unclamped, and drawn along: if the microscope-cross continued to bisect, in the direction of its length, the wire of the photograph, the operation had been successful; if not, by again turning the upper circle D through a small arc, and moving the slide B sufficiently to cause bisection, this coincidence was finally brought about. Slide B was then screwed back to its original central position, namely 1·981 inch, and slide A to that position in which the periphery of the sun came exactly under the centre of the cross of the microscope, which in the example cited would be 3·907 inches.

The instrument was then in a position for measuring the position-angles of the cusps, which was effected by rotating the circle C so as to bring first one and then the other cusp under the microscope. Table III., columns 8 and 9, contains a series of such measurements, corrected, however, for the ascertained error of the position-wires of the heliograph, which is given in column 7, for the epoch of each photograph: the original numbers, before the correction was applied, were the means of three measurements for each cusp.

The position-angles of the cusps gave the means of finding the position-angle of the line joining the centres of the sun and moon, which is at right angles to the chord joining the cusps. For example, in photograph No. 20, the measurement of which has been quoted by way of illustration,

	°	′	″
The position-angle of the northern cusp was found to be	13	49	30*
That of the southern cusp	218	19	
Adding	360		
And dividing by 2	2)592	8	30
We obtain for the position-angle of the line joining the sun and moon's centres	296	4	15

The circle C was now fixed, so as to read the angle thus found to be that of the line joining the centres of the sun and moon, and the slides A and B were both unclamped, and so placed as to bring one of the cusps exactly to coincide with the centre of the microscope-cross. Slide A was then clamped, and slide B drawn along so as to bring the other cusp under the microscope. If it coincided with the centre of the cross, the operation was so far completed; but if not, by causing the circle C to move through a small arc, and adjusting A a little, the coincidence of both cusps was brought about. In the photograph cited, the circle had to be brought to read 296° 0′ 30″; and this, and other numbers obtained in a like manner, are the lines of centres given in Table III., column 11, corrected for the errors of the wires of the heliograph.

The coincidence of the cusps with the microscope-cross having been effected, slide B was moved so as to bring one cusp exactly under the centre of the microscope; and its position having been read off on the vernier b', the other cusp was made central, and the vernier b' again read off, the difference between the two readings giving the length

* These numbers are not corrected for the errors of the position-wires as in Table III. columns 8 and 9.

of the imaginary chord joining the cusps. Thus, in the photograph under consideration,

The first reading of vernier b' was . .	3·841
The second reading	0·124
Length of the chord = difference . . .	3·717
Length of the semichord or sine . . .	1·8585

Slide B was now brought to its normal central position, 1·981 inch, and clamped, and vernier a' of the slide A being read off, gave the position of the imaginary chord. The slide A was now moved sufficiently to bring, first, the moon's periphery and then the sun's periphery into coincidence with the centre of the microscope-cross, and the readings were recorded in each case. These readings gave the means of finding the versed sine of the moon, the versed sine of the sun, and the distance of the moon's periphery from the sun's periphery.

Thus the first reading of the vernier a' gave the position of the chord . .	2·407
The second reading the position of the moon's limb	1·154
Their difference the versed sine of the moon	1·253
Again, the first reading was	2·407
The third reading giving the position of the sun's limb	0·094
And their difference the versed sine of the sun	2·313
Position of the moon's periphery as above	1·154
Position of the sun's periphery as above	0·094
Their difference gave the distance of the sun and moon's peripheries . .	1·060

Table II., columns 2 to 10 inclusive, gives a series of readings of the verniers a' and b', and the resulting determinations of the chords and the versed sines of the cusps of the sun and the moon.

By making the vernier of the circle C read 360°, plus the correction for the error of the wires of the heliograph, the position of the cusps could be read off in differences of right ascension and declination from the sun's (or the moon's) centre. Thus, in the photograph quoted, No. 20, the circle was set at 0° 9′ 30″, and the northern cusp made to coincide with the centre of the cross of the microscope by moving both the slides; the same thing was done with respect to the southern cusp. The readings were as follows:—

Northern Cusp.

Slide A	3·853	Slide B	2·429
Centre	2·0025	Centre	1·981
Difference of declination . .	+1·8505	Difference of R.A. . . .	+0·448

Southern Cusp.

Slide A	0·505	Slide B	0·803
Centre	2·0025	Centre	1·981
Difference of declination . .	−1·4975	Difference of R.A. . . .	−1·178

Although the differences of declination and right ascension of the cusps from the centres were taken out in every case, they will not be recorded in this report, because the results have been worked out by other measurements. I cite the above merely to show the applicability of the instrument to the measuring of two such coordinates as declination and right ascension.

Measurements of the moon's radius could, I found, be obtained with great accuracy by centering all the photographs which had been taken so near totality as to give a large proportion of the lunar periphery. Experience in centering for the sun had previously proved that the operation could be performed with ease, even in those cases where a large portion of the sun's disk was shut off by the moon. The totality-negatives were also measured, and I had two positive copies on albuminized glass made by superposing the negatives; and these also were measured. In the case of the totality-pictures, besides the radius, the whole diameter was measured, by causing the slide A to carry the picture from one side to the other of the centre of the reading-microscope. These measurements are given in Table I. column 9.

The readings, it will be observed, are in inches and decimals of an inch, the values of which in *arc* are based entirely on an assumed diameter for the sun, in consequence of no steps having been taken in Spain to obtain data for an absolute scale. This was not attempted, because at that time it was not contemplated that such an extensive use of the photographs would afterwards be made. It might be practicable to ascertain the value in arc of an inch on the screen of the heliograph; but this would have no retrospective application to the pictures obtained in Spain, because it would not be possible to place the object-glass, the secondary lens, and the screen in relative positions absolutely the same*. By assuming for the semidiameter of the sun its tabular value, the arbitrary measurements were translated into measures of arc; but it will be seen that any error in the tabular semidiameter of the sun necessarily affects all the numbers based upon it.

TABLE I.

In Table I. are given the following quantities:—Column 1 contains the progressive numbers of the photographs. Where a break occurs in the consecutive order, it arises from the circumstance that the omitted plate was spoiled after the picture had been taken, or that the photograph could not be taken, for the reasons already given.

Column 2 contains the times noted as the epochs of the photograph, to the nearest half-second. In observing the time, the half-second beats of the chronometer were counted, from 1 to 120, a method which, I think, is less likely to lead to error than the

* Since this paper was handed in, the Kew heliograph has been removed to Cranford, and attempts have been made to obtain an absolute value in arc for the scale of the micrometer by procuring pictures of objects situated at the distance of about a mile. Hitherto these attempts have been attended with only partial success, on account of the want of definition of the resulting pictures, consequent on the feebleness of the light and the movements of the intervening atmosphere. I have it in contemplation to continue the experiments and to erect a scale of equal parts about 80 feet long, at a known distance, with the view of ascertaining the radial distortion of the image, and the value of each increment from centre to circumference.

counting whole seconds by two steps, when using a chronometer which beats half-seconds.

Column 3 contains the computed errors of the chronometer for the several epochs, and Column 4 the epochs of the photographs, corrected by deducting the numbers in column 3 from those of column 2.

The time-intervals in columns 5 and 6, the nature of which is sufficiently explained by their superscriptions, will be found useful in checking the numbers given in this and the subsequent Tables, they having been employed in the calculations.

Columns 7 and 8 give the results of the measurements of the sun's radius by the two methods already explained. On examining these numbers certain discrepancies will be apparent; and they may, on the whole perhaps, appear at first sight not so accordant as might have been expected. The greatest difference in column 7 is between No. 29, in which the sun's radius measured 1909·5 thousandths of an inch, and No. 16, in which it measured 1900·5 thousandths—the difference being $\frac{9}{1000}$ths of an inch, or about 4″·5. Some of this difference is due to errors in centering; for on taking the means of columns 7 and 8 for the photographs Nos. 8 and 16, the difference of radius was reduced to $\frac{7}{1000}$ths, $=3''{\cdot}5$. Measures of the same photograph may vary in difficult cases, on account of the irregularity or faintness of the sun's border, from $\frac{1}{1000}$th to $\frac{3}{1000}$ths of an inch, but in most cases they are in complete agreement. There is, however, a real difference of photographic diameter in different pictures; for in disturbed states of the atmosphere the sun's diameter is enlarged by irradiation; and, on the other hand, when once the instantaneous apparatus has been so adjusted as to produce the best effect, any great diminution in the intensity of the light renders the picture more feeble, and the periphery of the sun consequently less distinctly defined; and it is just barely possible that the fainter portions of the limb are not depicted at all, whence would arise a diminution in the size of the picture.

The mean of the measurements given in column 7 is 1904·17, and of those in column 8, 1905·65,—the difference not being greater when converted into arc than 0″·7, while the mean of both sets of numbers is 1904·91. By assuming the radius of the sun to be 15′ 44″·8, as calculated by Mr. Hind from Leverrier's Tables, the value of $\frac{1}{1000}$th of an inch of my scale becomes 0″·4960, the logarithm of which is 9·6954654. This number has been employed in the reduction of the several measurements to their equivalents in arc.

Column 9 contains measurements of the moon, which are very accordant. The original negative of the second totality-picture presented greater difficulty in centering than the first totality-picture, in consequence of the triplication of the images of the protuberances; a positive albumen copy of it on glass was more easily centered. The greatest discordance in the measures is 4·5 thousandths, which are equivalent to 2″·2. The mean of all the measures gives 2002·25 thousandths =16′ 33″·1 as the radius of the moon, which agrees almost exactly with the number of Mr. Carrington, 16′ 33″, and that of Mr. Farley, 16′ 32″·9.

In column 10 are given the distances actually measured between the peripheries of the sun and moon in a direction at right angles to a line joining the cusps; these numbers differ in a few cases from those which were obtained for the same photographs by deducting the numbers in column 4 from those in column 3, Table II., but it has not been thought necessary to alter them in Table I. The cases in which discrepancies occur are the following:—

		inch.
No. 8.	in which the peripheral distances in Table II. differ from those in Table I. by	−0·003
No. 17.	,, ,, ,,	+0·001
No. 19.	,, ,, ,,	−0·001
No. 22.	,, ,, ,,	+0·003
No. 29.	,, ,, ,,	−0·001

The numbers for these particular photographs in Table II. result from a series of second measurements of several of the photographic plates, which it was found necessary to make again for Table II., in consequence of some minute errors in reading the vernier b' in the first series.

Column 11, Table I., gives the numbers of column 10 reduced to the adopted mean solar radius, namely, 1904·91 thousandths of an inch.

Column 12 gives the differences of the peripheral distances for two consecutive photographs, and, neglecting the augmentation of the moon's semidiameter, the approximate approach or retreat of the centres in the interval between their epochs. By dividing these numbers by those in column 6, were obtained the numbers in column 13, which are very nearly the rates of approach or retreat of the sun and moon's centres per minute.

Column 14 gives the approach and retreat per minute for the longer periods bracketed, and column 15 the same numbers reduced to seconds of arc. These rates of approach and retreat of the centres are affected by any errors in registering the time of the photographs, and also by all errors of measurement. The numbers do not run quite smoothly, and yet perhaps they are as good as could be expected. No account was taken of the augmentation of the moon's semidiameter, except for the middle of the eclipse; but the rates of approach or retreat for the beginning and end, even without this correction, admit of a comparison with the computed numbers, as the change of semidiameter was not great during the intervals.

	Approach of sun and moon's centres per minute at the commencement.	Relative motion of centres per minute at the middle.	Retreat of sun and moon's centres per minute at the end.
Measured . . .	25″·26	27″·84	30″·05
CARRINGTON . .	24·87	27·27	29·85
FARLEY	25·14	27·40	29·61

TABLE I.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.
						Measured radius of sun, in thousandths of an inch,								
No. of photograph.	Observed time, Frodsham 3094.	Error of Frodsham 3094 at noon, 4·6 seconds fast; daily losing rate 2·1 seconds; deduced error at epoch.	Epochs of photographs, corrected to Greenwich mean time.	Interval elapsed since the epoch of No. 6.	Interval elapsed between two consecutive photographs.	By measuring distance of periphery from centre.	By reversal and traversing slide A, to eliminate errors of centering.	Measured radius of the moon, in thousandths of an inch, by measuring distance of periphery from centre, &c.	Measured distance of sun and moon's peripheries at right angles to a line joining the cusps, in thousandths of an inch.	Distances reduced to the mean radius 1904·91, in thousandths of an inch.	Approach or retreat of peripheries in the interval, in thousandths of an inch.	Approach or retreat of the peripheries (or of the centres nearly) per minute, in thousandths of an inch.	Approach or retreat of centres per minute, calculated for the intervals bracketed, in thousandths of an inch.	Numbers in column 14 reduced to seconds of arc.
	h m sec./2	sec.	h m sec.	min.	min.									″
6	1 47 96	+4·4	1 47 43·6	0·000		1906·5	1907·2							
7	52 71	+4·4	52 31·1	4·792	4·792	1907·5	1909·5	....	3592	3585·2				
8	56 15	+4·4	56 3·1	8·325	3·533	1909·5	1906·5	....	3403	3397·5	187·7	53·12		
9	2 2 79	+4·4	2 2 35·1	14·858	6·533	1905·5	1906·0	....	3074	3072·6	324·9	49·73	50·92	25·26
10	3 41	+4·4	3 16·1	15·542	0·684	1905·5	1906·5	....	3040	3038·2	34·4	50·29		
11	7 21	+4·4	7 6·1	19·375	3·833	1902·5	1904·5	....	2840	2842·1	196·1	51·16	51·71	25·65
12	11 31	+4·4	11 11·1	23·458	4·083	1904·5	1905·5	....	2628	2627·9	214·2	52·46		
14	20 24	+4·4	20 7·6	32·400	8·942	1904·0	1905·7	....	2176	2176·1	451·8	50·53		
15	22 97	+4·4	22 44·1	35·008	2·608	1903·0	1904·7	....	2040	2041·1	135·0	51·77	50·80	25·20
16	27 103	+4·4	27 47·1	40·058	5·050	1900·5	1905·5	....	1776	1777·8	263·3	52·14		
17	33 36	+4·4	33 13·6	45·500	5·442	1906·5	1908·0	....	1489	1487·1	290·7	53·42	52·19	25·89
19	36 43	+4·4	36 17·1	48·558	3·058	1901·5	1903·0	2001·5	1332	1333·9	153·2	50·10		
20	41 55	+4·4	41 23·1	53·658	5·100	1904·5	1906·5	2004·5	1060	1059·7	274·2	53·76		
21	46 46	+4·4	46 18·6	58·583	4·925	1904·0	1904·7	2002·5	792	792·2	267·5	54·31	53·87	26·72
22	48 60	+4·4	48 25·6	60·700	2·117	1904·5	1906·5	2001·5	680	679·8	112·4	53·09		
23 a	51 53	+4·3	51 22·2	63·643	2·943	....	....	....	523	523·0	156·8	53·28	53·28	26·43
24 b	55 88	+4·3	55 39·7	67·935	4·292	....	....	....	298	298·0	225·0	52·42		
25	First totality.	....	....	....	......	....	....	2001·5 d 2000·0 e	......	....	......	......		
26	Second totality.	....	....	....	......	....	....	2004·5 f 2000·0 g	......	....	......	......	56·19	27·84 h
27 c	....	....	....	....	......	....	....		......	....	......	......		
28	3 13 116	+4·3	3 13 53·7	86·168		1902·5	1903·0	2001·5	20	20·0				
29	17 32	+4·3	17 11·7	89·468	3·300	1909·5	1911·0	2003·5	556	556·6				
30	20 24	+4·3	20 7·7	92·402	2·934	1900·5	1902·0	2003·5	740	738·0	181·4	54·97	56·71	28·13
31	24 35	+4·3	24 13·2	96·493	4·091	1904·5	1905·0	2002·5	908	909·8	171·8	58·55		
32	26 112	+4·3	26 51·7	99·135	2·642	1902·5	1904·0	....	1142	1142·1	232·3	56·78		
33	34 12	+4·3	34 1·7	106·302	7·167	1903·0	1904·7	....	1289	1290·1	148·0	56·02	56·15	27·85
35	37 48	+4·3	37 19·7	109·602	3·300	1902·5	1904·0	....	1692	1692·9	402·8	56·20		
36	41 46	+4·3	41 18·7	113·585	3·983	1904·5	1907·0	....	1886	1887·6	194·7	59·00	57·16	28·35
37	44 4	+4·3	43 57·7	116·235	2·650	1902·5	1904·5	....	2116	2115·1	227·5	57·12		
39	51 116	+4·3	51 53·7	124·168	7·933	1903·5	1905·5	....	2259	2260·7	145·6	54·94	57·97	28·75
41	59 5	+4·2	58 58·3	131·245	7·077	1902·5	1904·5	....	2720	2720·6	459·9	57·97	58·88	29·20
42	4 2 96	+4·2	4 2 43·8	135·003	3·758	1904·5	1906·0	....	3135	3137·3	416·7	58·88		
43	5 98	+4·2	5 44·8	138·020	3·017	1902·5	1906·5	....	3366	3366·4	229·1	60·96	60·59	30·05
44	10 75	+4·2	10 33·3	142·828	4·808	1904·0	1904·0	....	3547	3547·8	181·4	60·13		
45	16 40	+4·2	16 15·8	148·537	5·709	1906·0	1907·5	....						
						1904·17	1905·65	2002·25 Moon's mean radius =log 3·3015184						
						Sun's mean radius 1904·91 =log 3·2798745								

a 23 is a solarized picture, and the contour not perfectly defined.
b 24 also is a solarized picture; and in its epoch there is reason to believe that there is some error.
c 27 is solarized; many impressions exist on the plate, in consequence of the telescope having been agitated by a gust of wind. Its epoch was not noted.
d 25. These measurements were made on the original negative.
e 25. These measurements were made on an albumen positive copy, obtained by printing from the negative by superposition (*i. e.* not by means of a camera).
f 26. From the original negative.
g 26. From the albumen positive.
h The augmentation of the radius of the moon, in the cases of Nos. 22 and 28 was taken account of in the computation of this quantity.

$\frac{2002 \cdot 25}{1904 \cdot 91} = 1 \cdot 0511$ {ratio of lunar to solar radius.

97·3 = excess of moon's over sun's radius at the middle of eclipse.

Tabular radius of sun .... =944″·8 = log 2·9753399
Sun's measured radius in 1000ths of an inch... = log 3·2798745
∴ Value of $\frac{1}{1000}$th of an inch in sec. of arc 0″·4960 = log 9·6954654

Moon's radius in 1000ths of an inch .... = log 3·3015184
Value of $\frac{1}{1000}$th of an inch in sec. of arc ...... = log 9·6954654
∴ Moon's radius = 993″·1 = log 2·9969838
= 16′ 33″·1.

Column 11.

The numbers in this column were obtained as follows:—

Nos. in Col. 7 = r
" 8 = r'
" 10 = d
" 11 = d'

$$d' = \frac{d \cdot 1904 \cdot 91 \times 2}{r + r'}$$

Table II.

In Table II., columns 2, 3, and 4, is given a series of measurements from which are derived the versed sines of the sun and moon, and in columns 7 and 8 other measurements from which are obtained the lengths of the chord joining the cusps given in column 9. Columns 11 and 12 the resulting semidiameters of the moon and sun respectively, calculated upon these data. On account of the change in the apparent diameter of the moon during the eclipse, the measures in column 11 are not adapted for giving a mean result of the whole series; but the case is different for the sun, and hence the average of the measures of column 12 has been taken out for comparison with the mean semidiameter obtained by direct measurement: the mean semidiameter of the sun given by Table II. is 1903·1 thousandths of an inch, which differs by only −1·91 thousandths =−0″·9 from the value given in Table I. With respect to the moon's radius, it should be borne in mind that the photographs near the commencement and the end of the eclipse are not well adapted for such calculations, as a very minute error in measuring the chord or the versed sine introduces a great error in the resulting calculated semidiameter; and any rounding off or indistinctness of the cusps, especially near the epochs of commencement and end, militates greatly against exact determinations of the moon's radius by the method employed. For these reasons, it has been necessary to omit certain numbers of column 11, in deducing the averages for the moon's semidiameter, namely, Nos. 7, 10, 14, 39, 41, 42, and 43: by bringing together into three groups the remaining calculated semidiameters of the moon, we obtain for the mean epochs of these groups the following results, as compared with those deduced from Mr. Farley's numbers for the same epochs.

	Mean epoch, $2^h\ 17^m$.	Mean epoch, $3^h\ 1^m$.	Mean epoch, $3^h\ 34^m$.
Moon's radius, De La Rue	993″·8	993″·3	990″·9
Moon's radius, Farley	994·1	992·9	991·9
Difference	− 0·3	+ 0·4	− 1·0

These numbers are remarkably near the computed numbers, and render manifest that even so minute a change as the decrease in the moon's semidiameter during the eclipse is traceable in the photographs. Taking the differences of semidiameter at the first and last epochs, the augmentation of the moon's radius becomes more apparent, and not far from the true numbers; thus

	Moon's semidiameter. De La Rue.	Moon's semidiameter. Farley.
At . . $2^h\ 17^m$	993″·8	994″·1
At . . $3^h\ 34^m$	990·9	991·9
Difference . . .	2·9	2·2

Table II.—Calculations of the Radii of the Sun and Moon, from measurements of the Chords and Versed Sines.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	
	Slide A.					Slide B.						
No.	Position of a line joining the cusps.	Position of moon's periphery.	Position of sun's periphery.	Versed sine of the moon $=b$, in 1000ths of an inch.	Versed sine of the sun $=b'$, in 1000ths of an inch.	Position of the first cusp.	Position of the second cusp.	Length of chord, in 1000ths of an inch.	Sine, or semi-chord, $=a$, in 1000ths of an inch.	Calculated radius of the moon in 1000ths of an inch.	Calculated radius of the sun in 1000ths of an inch.	
7	3787	3683	91	104	3696	2643·0	1317·5	1325·5	662·7	2163·4	1907·4	A second measurement gave a similar result; the cusps are rounded off, and difficult to measure.
8	3693	3496	96	197	3597	2848·0	1114·0	1734·0	867·0	2006·3	1903·0	
9	3525	3168	94	357	3431	3117·0	835·5	2281·5	1140·7	2000·9	1905·1	
10	3504	3133	93	371	3411	3148·5	807·5	2341·0	1170·5	2032 0	1906·3	Same result on remeasurement; the northern cusp not well adapted for measurements.
11	3403	2936	96	467	3307	3267·0	694·0	2573·0	1286·5	2005·5	1903·7	
12	3294	2724	96	570	3198	3386·0	582·0	2804·0	1402·0	2009·2	1906·3	
14	3045	2271	95	774	2950	3575·0	393·0	3182·0	1591·0	2022·2	1904·0	This plate is fogged, and very difficult to measure.
15	2977	2136	96	841	2881	3615·0	353·0	3262·0	1631·0	2002·0	1902·2	
16	2825	1868	92	957	2733	3687·0	272·0	3415·0	1707·5	2001·8	1900·0	
17	2668	1583	93	1085	2575	3764·0	206·0	3558·0	1779·0	2000·9	1902·0	
19	2572	1428	97	1114	2475	3794·0	174·0	3620·0	1810·0	2003·9	1899·3	
20	2407	1154	94	1253	2313	3841·0	124·0	3717·0	1858·5	2004·8	1903·2	
21	2222	889	97	1333	2125	3870·0	93·0	3777·0	1888·5	2004·2	1901·6	
22	2133	774	91	1359	2042	3874·0	86·0	3788·0	1894·0	1999·3	1899·3	This plate difficult to measure, from the effects of atmospheric disturbance during exposure.
23	......	......	...	......	......	......	......	......	......	......	......	Solarized, and too indistinct at the cusps.
24	......	......	...	......	......	......	......	......	......	......	......	Ditto ditto.
27	......	......	...	......	......	......	......	......	......	......	......	Ditto ditto.
28	2031	655	99	1376	1932	3880·0	82·0	3798·0	1899·0	1998·5	1899·3	
29	2186	831	92	1355	2094	3880·0	87·0	3793·0	1896·5	2004·7	1905·8	
30	2308	1007	99	1301	2209	3859·0	105·0	3754·0	1877·0	2004·5	1902·0	
31	2460	1239	97	1221	2363	3829·0	133 0	3696·0	1848·0	2009·0	1904·1	
32	2547	1386	97	1161	2450	3802·0	159·0	3643·0	1821·5	2009·4	1902·1	
33	2786	1788	96	998	2690	3715·0	250·0	3465·0	1732·5	2002·8	1902·9	
35	2899	1983	97	916	2802	3658·0	308·0	3350·0	1675·0	1989·4	1901·6	
36	3022	2209	93	813	2929	3588·0	376·0	3212·0	1606·0	1992·7	1904·7	
37	3099	2355	96	744	3003	3529·0	432·0	3097·0	1548·5	1983·4	1900·7	
39	3348	2815	95	533	3253	3325·0	635·0	2690·0	1345·0	1963·5	1904·6	The southern cusp rounded, and impression faint.
41	3559	3231	96	328	3463	3057·0	901·0	2156·0	1078 0	1935·5	1899·3	An extremely faint impression, and the cusps measured with great difficulty.
42	3680	3461	95	219	3585	2880·0	1072·0	1808·0	904·0	1975·3	1906·5	A faint impression, and difficult to measure.
43	3773	3639	92	134	3681	2683·0	1285·0	1398·0	699·0	1890·3	1906·9	An extremely faint impression, and the cusps measured with great difficulty.
										Mean.........	1903·1	

No.	Moon's S.D.	Sun's S. D.	No.	Moon's S.D.	Sun's S. D.	No.	Moon's S.D.	Sun's S. D.		Augmentation of moon's radius.	Farley's augmentation of moon's radius.
8	2006·3	1903·0	20	2004·8	1903·2	31	2009·0	1904·1	h m		
9	2000·9	1905·1	21	2004·2	1901·6	32	2009·4	1902·1	2 17	+0″·5	+1″·2
11	2005·5	1903·7	22	1999·3	1899 3	33	2002·8	1902·9	3 1	0·0	0·0
12	2009·2	1906·3	28	1998·5	1899·3	35	1989·4	1901·6	3 34	−2·4	−1·0
15	2002·0	1902·2	29	2004·7	1905·8	36	1992·7	1904·7			
16	2001·8	1900·0	30	2004·5	1902·0	37	1983·4	1900·7			
17	2000·9	1902·0									
19	2003·9	1899·3									
Means......	2003·8	1902·7		2002·7	1901·9		1997·8	1902·7	Between 2h 17m and 3h 34m	−2·9	
Value in arc.........	993″·8	943″·7		993″·3	943″·3		990″·9	943″·7			
Calculated from Farley's numbers	994·1	944·8		992·9	944·8		991·9	944·8	Ditto Farley's	−2·2	
De La Rue—Farley	−0·3	−1·1		+0·4	−1·5		−1·0	−1·1	Difference..	−0·7	
Mean epoch	2h 17m			3h 1m			3h 34m				

Radius of the moon, column 11 $=r$.
Radius of the sun, column 12 $=r'$.
Versed sine of moon, column 5 $=b$.
Versed sine of sun, column 6 $=b'$.
$\frac{\text{Chord}}{2}$ of cusps, column 10 $=a$.

$$r=\frac{a^2}{2b}+\frac{b}{2} \qquad r'=\frac{a^2}{2b'}+\frac{b'}{2}.$$

TABLE III.

In Table III., column 2, the epoch of the photographs in Greenwich mean time is again given as in Table I.; in column 3 is given a series of numbers obtained by adding to the peripheral distances of the sun and moon (as shown in Table I., column 10) the number 97·3, which expresses in thousandths of an inch the excess of the mean measured radius of the moon over the mean measured radius of the sun, namely, 2002·2−1904·9=97·3.

Column 4 contains the numbers in column 3 reduced to seconds of arc, and column 6 the same numbers corrected by the quantities in column 5, which contains the corrections necessary on account of the augmentation of the moon's semidiameter from the mean diameter at the middle of the eclipse. The numbers in column 5 are derived by interpolation from Mr. FARLEY'S calculations. The corrected numbers in column 6 show the distances of the sun and moon's centres at the epochs given in column 2.

Column 7 contains the errors of the wires of the heliograph from the assumed position of 45° for wire I., for the epochs of the several photographs. These numbers have been applied to the numbers in columns 8, 9, and 10, in which, respectively, are given the corrected angles of position of the cusps, and the line joining the sun and moon's centres.

Column 12 contains the measures of half the angles between the cusps, taken from the sun's centre, the numbers being half the differences between the position-angles of each pair of cusps, which are given in columns 8 and 9. The angles in this column were employed in the computation of Table IV.

Table III.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
No.	Epoch of photograph, Greenwich mean time.	Distance of the sun and moon's peripheries increased by 97·3, the excess of the lunar radius at the middle of the eclipse in 1000ths of an inch.	The numbers in column 3 reduced to seconds of arc.	Correction to be applied for augmentation of moon's radius; interpolated from Mr. Farley's numbers.	Resulting distances of sun and moon's centres.	Error of position-wires of photoheliograph, and of their imprints on the photographs.	Position-angle of the cusps N. towards E. corrected by the quantities in column 7. Northern cusp.	Position-angle of the cusps N. towards E. corrected by the quantities in column 7. Southern cusp.	Line of centres derived from the measurement of position-angles of cusps.	Line of centres ascertained by measuring the angle at right angles to an imaginary line joining the cusps.	Half the angle included between the cusps.
	h m sec.		″	″	″	′	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
7	1 52 31·1	3682·5	1826·4	+1·4	1827·8	+ 5·1	317 17 41	278 24 41	296 51 11	a	20 26 30
8	56 3·1	3494·8	1733·4	+1·4	1734·8	+ 4·0	324 11 55	269 42 30	296 57 12	297 5 00	27 14 42
9	2 2 35·1	3169·9	1572·2	+1·3	1573·5	+ 2·0	333 46 25	259 56 55	296 51 40	296 58 40	36 54 45
10	3 16·1	3135·5	1555·2	+1·3	1556·5	+ 1·8	334 52 32	258 52 27	296 52 29	56 52	38 0 2
11	7 6·1	2939·4	1457·9	+1·3	1459·2	+ 0·7	339 18 18	254 2 48	296 40 33	42 48	42 37 45
12	11 11·1	2725·2	1351·6	+1·2	1352·8	− 0·5	344 1 45	249 16 45	296 39 15	37 30	47 22 30
14	20 7·6	2273·4	1127·6	+1·0	1128·6	− 3·2	353 22 2	239 42 57	296 32 29	28 12	56 49 32
15	22 44·1	2138·4	1060·6	+1·0	1061·6	− 4·0	355 55 00	237 31 00	296 43 00	38 00	59 12 00
16	27 47·1	1875·1	930·0	+0·9	930·9	− 5·5	0 50 45	232 19 15	296 35 00	37 00	64 15 45
17	33 13·6	1584·4	785·8	+0·7	786·5	− 7·1	6 0 21	227 6 21	296 33 21	33 6	69 27 00
19	36 17·1	1431·2	709·9	+0·7	710·6	− 8·0	8 51 31	223 42 51	296 17 11	13 30	72 34 20
20	41 23·1	1157·0	573·8	+0·5	574·3	− 9·5	13 59 00	218 28 30	296 13 45	10 00	77 45 15
21	46 18·6	889·5	441·2	+0·4	441·6	−11·0	19 50 46	212 52 21	296 21 33	26 00	83 29 12
22	48 25·6	777·1	385·4	+0·3	385·7	−11·7	21 22 37	209 34 12	295 28 24	26 52	85 54 12
23	51 22·2	620·3	307·7	+0·2	307·9	−12·6	28 24 59	200 42 42	294 33 50	b	f
24	55 39·7	395·3	196·1	+0·1	196·2	−14·0	33 53 50	190 56 47	292 25 18	c	g
27	....	117·3	58·2	−0·3	57·9	−18·0	359 41 00	260 47 00	130 14 00	d	h
28	3 13 53·7	653·9	324·3	−0·5	323·8	−19·4	30 1 19	208 45 24	119 23 21	119 24 24	89 22 2
29	17 11·7	835·3	414·3	−0·6	413·7	−20·3	34 6 18	203 18 53	118 42 35	118 41 28	84 36 17
30	20 7·7	1007·1	499·5	−0·7	498·8	−21·2	37 47 2	199 20 32	118 33 47	31 12	80 46 45
31	24 13·2	1239·4	614·7	−0·8	613·9	−22·5	42 3 30	194 17 00	118 10 15	9 50	76 6 45
32	26 51·7	1387·4	688·1	−0·9	687·2	−23·2	44 41 37	191 23 2	118 2 19	3 42	73 20 42
33	34 1·7	1790·2	887·9	−1·1	886·8	−25·4	52 14 14	183 30 49	117 52 31	117 49 24	65 38 17
35	37 19·7	1984·9	984·5	−1·2	983·3	−26·4	55 50 24	179 33 24	117 41 54	38 24	61 51 30
36	41 18·7	2212·4	1097·3	−1·3	1096·0	−27·6	60 4 36	175 16 21	117 40 28	37 36	57 35 52
37	43 57·7	2358·0	1169·5	−1·4	1168·1	−28·4	62 55 24	172 19 54	117 37 39	37 54	54 42 15
39	51 53·7	2817·9	1397·6	−1·6	1396·0	−29·8	72 30 18	162 30 3	117 30 10	32 48	44 59 52
41	58 58·3	3234·6	1604·3	−1·9	1602·4	−31·9	82 42 24	152 22 54	117 32 39	e	34 50 15
42	4 2 43·8	3463·7	1717·9	−2·0	1715·9	−33·0	88 59 00	145 33 25	117 16 12	23 30	28 17 12
43	5 44·8	3645·1	1807·9	−2·1	1805·8	−33·9	95 25 54	138 51 54	117 8 54	7 54	21 43 00

a Not measured.
b, c, d Not sufficiently distinct to ascertain the position of the line joining the cusps.
e Too faint to ascertain the position of the line joining the cusps with sufficient precision.
f, g, h The exact termination of the cusps could not be ascertained with precision.

No. 7. 296° 51′·2
No. 8. 296 57·2
Mean ... 296 54·2 Mean epoch 1h 54m·3

No. 9. 296 51·7
No. 10. 296 52·5
Mean ... 296 52·1 Mean epoch 2 2·9

Interval 8m·6 } Increase of position-angle per minute = 0′·24

No. 37. 117° 37′·6
No. 39. 117 30·2
Mean ... 117 33·9 Mean epoch 3h 47m·9

No. 41. 117 32·6
No. 42. 117 16·2
No. 43. 117 8·9
Mean ... 117 19·3 Mean epoch 4 2·5

Interval 14m·6 } Decrease of position-angle per minute = 1′·0.

From Table III. it is possible to derive several elements of the eclipse; for example, the epochs of first and last contacts, and the duration of the eclipse.

The distance of the sun and moon's centres at the epoch of

No. 7 was . . .	1827·8″	No. 41 was . . .	1602·4
No. 8 was . . .	1734·8	No. 42 was . . .	1715·9
No. 9 was . . .	1573·5	No. 43 was . . .	1805·8

whence is derived, as the mean motion per minute, in the interval between

7 and 9 . . . =25″·263
41 and 43 . . . =30·022

	At first contact.		Last contact.
The augmented moon's radius was	994″·5	. . . . . .	990″·7
The sun's radius	944·8	. . . .	944·8
	1939·3		1935·5

By deducting from these sums of the radii the corresponding distances of the centres in photographs 7, 8, and 9, and in 41, 42, and 43 respectively, and dividing the numbers so obtained by the corresponding rates of approach or retreat of the centres, the intervals are derived which have elapsed between the epochs of the first contact and the epochs of the photographs, on the one hand; and on the other, the intervals which must have elapsed between the epochs of the photographs and the last contact of the sun and moon. By subtracting from the epochs of Nos. 7, 8, and 9, and adding to those of Nos. 41, 42, and 43 their several intervals, the following periods result:—

	First contact.				Last contact.		
	h	min.	sec.		h	min.	sec.
No. 7 . .	1	48	6·4	No. 41 . .	4	10	4·0
No. 8 . .	1	47	57·4	No. 42 . .	4	10	2·7
No. 9 . .	1	48	6·4	No. 43 . .	4	10	4·0
Mean . .	1	48	3·4*	Mean . .	4	10	3·6

whence the duration of the eclipse is found to have been 2 h. 22 min. 0·2 sec. .

	°	′
The position-angle of the line joining the centres at 1 h. 54·3 min. was . .	296	54·2
and by adding the decrease in the position-angle since the period of first contact, as calculated from the numbers in Table III.		1·5
we obtain for period of first contact the position-angle	296	56

which agrees very nearly with the angle calculated by Mr. CARRINGTON and Mr. FARLEY, namely 296° 54′. In the same manner, the position-angle of the line joining the centres at the epochs of last contact was found to be 117° 12′

Mr. CARRINGTON'S number is . . .	117	20
Mr. FARLEY'S	117	18

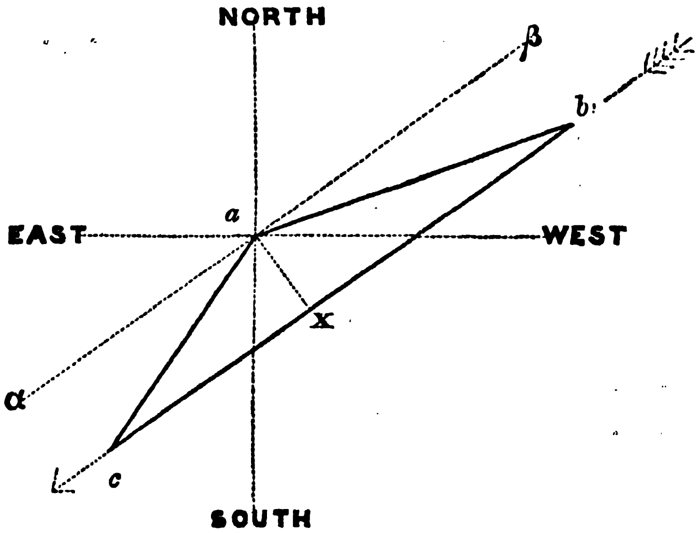
* The time of first contact observed with the achromatic was 1 h. 48 m. 6·6 sec. p. 354.

The beginning and end of the eclipse do not agree with the calculations; for example—

	First contact.			Last contact.			Duration.		
	h	min.	sec.	h	min.	sec.	h	min.	sec.
De La Ruè . . .	1	48	3·4	4	10	3·6	2	22	0·2
Carrington . . .	1	47	56	4	10	15·2	2	22	19·2
Farley	1	47	57	4	10	15	2	22	18

But it is possible that the discrepancy may partly arise in consequence of the assumption of a greater angular measure for the diameters of the sun and moon than they in reality subtend; and this view is supported by my measures of the distances of the sun and moon's centres, which, as a whole, come out greater than the computed distances. It will be seen that, as the reduction of my arbitrary measures to their equivalents in arc is dependent on the tabular value for the sun's semidiameter, if this be in excess of the truth, my distances of the centres must come out greater than the real values; and this is actually the case, as will be hereafter seen. If the diameter either of the sun or of the moon, or of both, be less than the tabular numbers, the first contact must happen later, and the last contact sooner than the computed times: the epochs of these phenomena, derived from the measure of the peripheral distances of the sun and moon given above, tend to show that some correction is necessary to the tabular diameters.

The epoch of the middle of the eclipse, the direction of motion of the moon's centre, and the nearest approach of the centres of the sun and moon may also be derived from the photographs, by means of the distances of the centres and the epochs of two photographs, one before and one after totality.



In the above diagram, let a represent the position of the sun's centre, b the position of the moon's centre previous to totality, at the epoch of No. 22 photograph, and c the position of the moon's centre in photograph No. 28, after totality; $b\,c$ will represent the motion of the moon's centre across the solar disk during the interval, and the line $\beta\,\alpha$, parallel to $b\,c$ and passing through the sun's centre, the direction of motion referred to the sun's centre; aX shows the nearest approach of the centres of the sun and moon; X the position of the centre of the moon at the middle of the eclipse.

$a\,b$, the distance of the centres at the epoch of No. 22, was . . . . 385″·7
$a\,c$, the distance of the centres at the epoch of No. 28, was . . . . 323″·8

the position-angle of the line joining the centres at the epoch of No. 22 was	295°	28′	24″
the position-angle of the line joining the centres at the epoch of No. 28 was	119	23	21
whence the angle $c\,a\,b$	=176	5	3

	h	min.	sec.
The epoch of No. 22 was . . .	2	48	25·6
" No. 28 was . . .	3	13	53·7

and the interval, in minutes and decimals of a minute, 25·468.

	°	′	″
From these data the angle X $c\,a$, equal to the angle $\alpha\,a\,c$, was found to be	2	7	43·7
And the angle X $b\,a=\beta\,a\,b$	1	47	13·3

	″
The side bX was computed to be	385·512
The side cX was computed to be	323·576
And the line $b\,c$, which represents the space travelled over during the interval, was consequently	709·088

	min.	sec.
The proportion of the interval of time occupied by the moon in travelling from b to X was computed to be	13	50·8
The proportion in travelling from X to c	11	37·3

		″
The mean rate of motion per minute	0	27·84
The nearest approach of centres, aX, was found to be. . .	0	12·03

	h m sec.		h m sec.
The epoch of No. 22 was . .	2 48 25·6	That of No. 28 was . . .	3 13 53·7
Add time-interval b to X . .	0 13 50·8	Deduct time-interval X to c .	0 11 37·3
Middle of the totality . . .	3 2 16·4		3 2 16·4

	° ′ ″		° ′ ″
The position-angle of $a\,b$ at the epoch of 22 was . .	295 28 24	The position-angle of $a\,c$ at the epoch of 28 was . .	119 23 21
Adding the angle $\beta\,a\,b$ . .	1 47 13·3	Deducting the angle $\alpha\,a\,c$.	2 7 43·7
We obtain as the direction of motion of the moon's centre during the totality from	297 15 37·3	to	117 15 37·3

By combining Nos. 22 and 29, and Nos. 23 and 28, similar numbers were computed, which, together with the preceding results, are given in the following summary:—

	Direction of motion of moon's centre.	Nearest approach of centres.	Relative motion of centres per minute.	Middle of totality.		
				h	m	sec.
Nos. 22 and 28 .	297° 15′ 37″·3 to 117° 15′ 37″·3	12″·03	27″·84	3	2	16·4
Nos. 22 and 29 .	297 8 53·6 to 117 8 53·6	11·27	27·78	3	2	18·4
Nos. 23 and 28 .	297 2 14·1 to 117 2 14·1	13·29	28·02	3	2	20·9
Mean . .	297 8 55 117 8 55	12·20	27·88	3	2	18·6

The following are the numbers computed for the same elements by

				h	m	sec.
Carrington . .	297° 17′ 0 to 117° 17′ 0	13″	27″·27	3	2	19·5
Farley . . .	. . . .	12·7	27·35	3	2	20·0

Table IV.

In Table IV. column 1, are given the computed sines of half the angles of the opening of the cusps, which are set forth in column 12, Table III.; the sun's mean measured radius, namely, 1904·91, being employed in the computations. The resulting numbers correspond to the semichords given in Table II. column 10; but in most instances the calculated is greater than the measured semichord or sine.

In column 2 are given the cosines of the same angles referred to the sun. These numbers correspond to the distance of the sun's centre from the imaginary chord joining the cusps.

In column 3 is set forth the augmented semidiameter of the moon for the epoch of each photograph, the increase or decrease from the mean measured diameter 2002·2 being derived from Mr. Farley's values.

Calling the sines in column 1$=a$, and the augmented lunar semidiameter $=b$, the cosine of the moon for the angle represented by the same sine was derived by the formula $\sqrt{(b-a)(b+a)}$; column 4 gives these cosines referred to the lunar radius, and represents the distances of the moon's centre from the chord joining the cusps.

The distances of the sun and moon's centres could evidently be derived by taking out the sums or differences of such numbers as those in columns 2 and 4: in the cases actually under consideration, the distances of the moon and sun's centres result from the addition of these quantities; they are given in thousandths of an inch in column 5, and reduced to seconds of arc in column 6.

In column 7 are given the like quantities, derived from measurements of the distance of the peripheries, which are merely a repetition of the values given in Table III. column 6.

In column 8 are set forth the mean distances of the centres of the sun and moon, ascertained by taking the arithmetical mean between the numbers in columns 6 and 7.

In column 9 are the same distances computed by interpolation of the values calculated by Mr. Farley.

Column 10 gives the differences between the mean distances in column 8, and those in column 9, or DE LA RUE—FARLEY. The mean of the differences will be seen to be $+4''\cdot1$; that is, the distances of the centres of the sun and moon come out greater than the computed distances by $4''\cdot1$. This tends to show that the semidiameters of the sun and moon jointly, are less in reality by $4''$ than their tabular values. It is not intended to urge this as an absolute proof, but merely as supporting that view, which is further corroborated by the fact that the first contact occurred later, and the last contact sooner, than the predicted times. The distances of the sun and moon's centres, obtained by calculation from the angular opening of the cusps, will be presently employed to furnish data respecting the commencement and end of the eclipse, &c.; and it will be seen that the times thus obtained differ from those derived from the peripheral distances, and that they approach more nearly to the predicted times. The optical distortion of the sun's image would occur in the direction of a radius, and would not affect the numbers derived from measurements of the angular opening of the cusps, provided the picture were concentric with the optical axis of the instrument; while it would affect the numbers based on the measures of the distances of the sun and moon's peripheries, so that the quantity $4''$ is probably, from that cause, in excess of the true correction. The measurements of the angular openings of the cusps, and the measurements of the distances of the peripheries, both present peculiar difficulties. The difficulty of determining the precise termination of the cusp, especially when blunted by a lunar mountain, leads one to make the angular opening greater than it ought to be, and, consequently, the cosines and the distance of the centres less than they really are. On the other hand, the optical distortion of the image, combined with the irregularities of the peripheries of the sun and moon, tends to make the measurements of the distances of the peripheries, and consequently the distance of the centres, greater than they are in reality. These liabilities to error have, therefore, in the two cases, an opposite effect on the final results; hence a mean of the numbers obtained by the two methods will probably approach very nearly to the correction to be applied to the semidiameters of the sun and moon taken conjointly.

TABLE IV.

No.	1. Sun. Sine of ½ ∠ of opening of the cusps in 1000ths of an inch, =a.	2. Sun. Cosine of ½ ∠ of the opening of the cusps in 1000ths of an inch.	3. Mean radius of the moon, corrected for augmentation of S. D. in 1000ths of an inch, =b.	4. Moon. Cosine of ½ ∠ of the opening of cusps $\sqrt{(b-a)(b+a)}$, in 1000ths of an inch.	5. Distance of the centres of the sun and moon; col. 2 + col. 4 in 1000ths of an inch.	6. Distance of centres of sun and moon, reduced to seconds of arc.	7. Distance of centres brought from column 6, Table III.	8. Distance of centres. Mean of columns 6 and 7. Seconds of arc.*	9. Distance of centres computed for epochs of photographs, by interpolating Mr. Farley's numbers.	10. Measurements minus Farley's numbers.	11. Angle of position of the line joining centres, derived from column 10, Table III.*	12. Angle of position of line of centres N. towards E., computed from Mr. Farley's numbers.	13. Measured angle of position minus Farley's angle of position of line of centres.
7	666·7	1784·9	2005·0	1890·9	3675·8	1823″·1	1827″·8	1825″·4	1824″·2	+1″·2	296° 51′·2	296° 54′·0	− 2′·8
8	872·1	1693·6	2005·0	1805·4	3499·0	1735·4	1734·8	1735·1	1735·4	−0·3	296 57·2	296 53·8	+ 3·4
9	1144·1	1523·1	2004 8	1646·3	3169·4	1572·0	1573·5	1572·7	1570·0	+2·7	296 51·7	296 52·5	− 0·8
10	1172·8	1501·1	2004 8	1626·0	3127·1	1551·0	1556·5	1553·7	1552·7	+1·0	296 52·5	296 52·3	+ 0·2
11	1290·1	1401·1	2004·8	1534·6	2935·7	1456·0	1459·2	1457·6	1455·1	+2·5	296 40·5	296 50·9	−10·4
12	1401·6	1290·0	2004·6	1433·2	2723·2	1350·6	1252·8	1351·7	1350·6	+1·1	296 39·2	296 48·9	− 9·7
14	1594·4	1042·3	2004·2	1214·4	2256·7	1119·3	1128·6	1123·9	1120·4	+3·5	296 32·5	296 42·4	− 9·9
15	1636·2	975·4	2004·2	1157·4	2132·8	1057·8	1061·6	1059·7	1053·0	+6·7	296 43·0	296 40·0	+ 3·0
16	1715·9	827·2	2004·0	1035·2	1862·4	923·7	930·9	927·3	921·4	+5·9	296 35 0	296 33·8	+ 1·2
17	1783·7	668·7	2003·6	912·6	1581·3	784·3	786·5	785·4	778·7	+6·7	296 33·3	296 25·7	+ 7·6
19	1817·5	570·5	2003·6	843·3	1413·8	701·2	710·6	705·9	698·2	+7·7	296 17·2	296 18·5	− 1·3
20	1861 6	404 0	2003·2	739·8	1143 8	567·3	574·3	570 8	563·0	+7·8	296 13·7	296 0·6	+13·1
21	1892·6	216·1	2003 0	655·8	871·9	432·4	441·6	437·0	432·1	+4·9	296 21·5	295 31·2	+10·3
22	1900·0	136·1	2002 8	633·4	769·5	381·7	385·7	383 7	375·4	+8·3	295 28·4	295 4·1	+24·3
23	....	....	....	....	....	....	307·9	307·9	296·6	......	294 33·8	294 26·5	+ 7·3
24	....	....	....	....	....	....	196·2	196·2	181·2	......	292 25·3	....	....
27	....	....	....	....	....	....	57·9	57 9	....	......	130 14·0	....	....
28	1904·8	21·0	2001·2	613·6	634·6	314·7	323 8	319·2	317·9	+1·3	119 23·3	120 10·7	−47·4
29	1896·5	179·1	2001·0	638·2	817·3	405·4	413·7	409·5	409·3	+0·2	118 42 6	119 7·3	−24·7
30	1880·3	305·2	2000·8	683·9	989·1	490·6	498·8	494·7	491·0	+3·7	118 33·8	118 50·3	−16·5
31	1849·2	457·2	2000·6	763·4	1220 6	605·4	613·9	609·6	605·0	+4·6	118 10·2	118 26·5	−16·3
32	1825·0	546·0	2000·4	819·1	1365·1	677·1	687·2	682·1	679·3	+2·8	118 2·3	118 17·3	−15·0
33	1735·3	785·8	2000·0	994·3	1780·1	882·9	886 8	884·8	881·6	+3·2	117 52·5	117 59·4	− 6·9
35	1679·7	898·5	1999·6	1084·9	1983·4	983·7	983·3	983·5	975·7	+7·8	117 41·9	117 53·7	−11·8
36	1608·3	1020·7	1999·4	1187·8	2208·5	1095·4	1096·0	1095·7	1089·7	+6·0	117 40·5	117 48·2	− 8·7
37	1544·7	1100·7	1999·2	1269·1	2369·8	1175·4	1168·1	1171·7	1165·5	+6·2	117 37·6	117 44 4	− 6·8
39	1346·9	1347·0	1999 0	1477·1	2824·1	1400·7	1396·0	1398·3	1395·0	+3·3	117 30·2	117 35·4	− 5·2
41	1088·2	1563·5	1998·4	1676·1	3239 6	1606·8	1602·4	1604·6	1601·8	+2·8	117 32·6	117 28·0	+ 4·6
42	902·7	1677·4	1998·2	1782·7	3460·1	1716·1	1715·9	1716·0	1712·3	+3·7	117 16·2	117 24·3	− 8·1
43	704·8	1769·7	1998·0	1869·6	3639·3	1805·0	1805·8	1805·4	1801·3	+4·1	117 8·9	117 21·4	−12 5
										Mean +4·1			

* Plate XVIII. exhibits a graphic representation of the path of the moon's centre, set off, in accordance with the numbers of columns 8 and 11, by means of an instrument constructed specially for such operations.

Proceeding as before, but with the distances of the centres given in Table IV., column 6, we obtain, as the rate of approach of the centres per minute between 7 and 9, 24″·945; and the rate of retreat of the centres per minute, between 41 and 43, 29″·255, whence we derive the following:

First contact.	h	m	sec.	Last contact.	h	m	sec.
No. 7 . .	1	47	51·6	No. 41 . .	4	10	12·4
No. 8 . .	1	47	52·7	No. 42 . .	4	10	13·8
No. 9 . .	1	47	51·6	No. 43 . .	4	10	12·4
Mean . .	1	47	52		4	10	12·9

Duration of the Eclipse . . . 2 h 22 m 20·9 sec.

These numbers agree fairly with the calculations; for example—

	First contact. h	m	sec.	Last contact. h	m	sec.	Duration. h	m	sec.
De La Rue	1	47	52	4	10	12·9	2	22	20·9
Carrington	1	47	56	4	10	15·2	2	22	19·2
Farley	1	47	57	4	10	15·0	2	22	18

	Direction of motion of the moon's centre during totality.	Nearest approach of centres.	Relative motion of the centres per minute.	Middle of eclipse. h m sec.
Nos. 22 and 29	297° 8′ 25″ to 117° 8′ 25″	11″·10	27″·34	3 2 22·6
Nos. 22 and 28	297 14 34·1 to 117 14 34·1	11·79	27·33	3 2 23·2
Mean . .	297 11 29 to 117 11 29	11·44	27·34	3 2 22·9

The following are numbers computed for the same elements by

				h m sec.
Carrington . .	297° 17′ 0″ to 117° 17′ 0″	13″·0	27″·27	3 2 19·5
Farley . . .	. . . .	12·7	27·35	3 2 20

By combining these results with the results of the peripheral measures already given, we obtain

	First contact. h	m	sec.	Last contact. h	m	sec.
By peripheries . .	1	48	3·4	4	10	3·6
By cusps	1	47	52·0	4	10	12·9
Mean . . .	1	47	57·7	4	10	8·2

Duration of the Eclipse . . . 2 h 22 m 10·5 sec.

	Direction of motion.	Nearest approach of centres.	Relative rate of motion of centres.	Middle of totality. h m sec.
Peripheries .	297° 8′·9 to 117° 8′·9	12″·20	27″·88	3 2 18·6
Cusps . . .	297 11·5 to 117 11·5	11·44	27·34	3 2 22·9
Mean .	297 10 to 117 10	11·8	27·61	3 2 20·7

These numbers agree very closely with the theoretical numbers, the chief difference being in the epoch of the end of the eclipse, which is earlier by 7 seconds than Mr. CARRINGTON'S computation, and by 6·8 seconds than that of Mr. FARLEY.

The following are the differences:—

	De La Rue—Carrington. sec.	De La Rue—Farley. sec.
First contact	+1·7	+0·7
Last contact	−7·0	−6·8
Duration	−8·7	−7·5
Middle of eclipse	+1·2	+0·7
Nearest approach of centres	−1″·2	−0″·9
Direction of motion of the moon's centre during the totality	−7′	
Relative motion of the centres per minute	+0″·34	+0″·26
Position-angle of the line joining the centres at the first contact	+2′	+2′
Ditto at the last contact	−8′	−6′

The periods of first contact, and the middle of the eclipse, are accordant, but not so that of the end of the eclipse, the duration being less than the computed duration by 8 seconds. The half of this, or 4 seconds, would correspond to a distance moved through of 1″·9, by which quantity the radii of the moon and of the sun jointly would be smaller than the computed values. Without any desire to attach more importance to the results of the photographic measurements than they merit, I believe that I have made out satisfactorily that astronomical photography is capable of furnishing data on which great reliance can be placed, and which it would be difficult to collect in any other way. It possesses the advantage, in the case of sun-pictures, of instantaneous registration, and permits of measurements being made calmly and at leisure, and of their being repeated as often as may be considered desirable. Its employment in connexion with means of measurement will undoubtedly suggest future improvements; and although it is impossible at present to predict the destiny of astronomical photography, it appears likely that it will take a high rank among the methods of observation.

Solar Spots.

With the view of ascertaining whether any connexion exists between the luminous prominences and the faculæ, or the spots on the solar disk, photographs of the sun were obtained as soon as the heliograph could be got to work, and others would have been taken on each day previous to the eclipse if the weather had proved favourable. Several photographs were secured on the 14th, but only one on the 16th, which was not measured, having been overlooked; spots *a* and *b* were then well on the solar disk. Afterwards, until the 18th, none could be procured, but on the 19th and 20th several were obtained. With the exception of a spot *d*, which became visible on the 20th, with a position-angle 86° 26′*, a group of spots, extending on the 18th from 114° 30′ to 121° 30′†, and a double spot X visible on the 14th in position-angle 243° 53′ to 245° 34′‡, there were none between which and the luminous prominences any connexion could be presumed to exist. The group 114° 30′ to 121° 30′ was surrounded by many faculæ; the spots in it underwent considerable changes on the 19th and 20th; the faculæ extended evidently beyond the visible portion of the sun's surface on the 18th; for a part which was not in sight on the 18th came into view on the 19th and 20th. Just in the neighbourhood of these faculæ there was visible in the telescope during the totality, a very brilliant sheet of light. On the 18th, besides the group of spots surrounded by faculæ just mentioned, and other small spots delineated in the index map, Plate XV., there were three conspicuous spots, which I have designated by the letters *a*, *b*, *c*. Spot *c* was visible on the 14th, but the two others had not yet come round; the three spots *a*, *b*, *c* continued to be visible on the 19th and 20th.

On the whole, however, no very intimate relation was discoverable between the prominences and the sun-spots; and recent photographic researches having convinced me that the formation of spots and their changes are among the least frequent of the great disturbances always occurring in the solar photosphere, I take this opportunity of stating my opinion that future investigations will rather tend to disprove any very close connexion between them.

On the 14th, at $4^h\ 12^m\ 6^s{\cdot}6$ Greenwich mean time, the spots visible on the sun's disk were the following:—

Spot.		Position-angles.	Distance from the centre in a decimal of the radius.
	c	45° 37′	·3825
Cluster	α	112 21	·5548
Cluster	β	113 6	·5169
Cluster	γ	114 25	·4984
Cluster	δ	114 35	·5811
Cluster	ε	114 51	·6075
X first nucleus . . .		243 53	·8899
X second " . . .		245 34	·9078

The spots α, β, γ, δ, and ε, somewhat changed, were still on the disk on the 18th, but I did not notice any spot which could have been brought by rotation into proximity with the western limb of the sun, with the exception of X, which was at some little distance in longitude on the hemisphere turned away from the earth.

Of the group of small spots surrounded by faculæ, visible on the eastern edge of the sun on the 18th, the following were selected and measured on photograph No. 6, whose epoch is $1^h\ 47^m\ 43^s{\cdot}6$.

* See in the index map, Plate XV., the prominences E and F.

† See in the index map, Plate XV., the prominences H and G.

‡ See in the index map, Plate XV., the prominence L, which, however, was at some distance from the position of X.

Spot.	Position-angle.	Distance from the centre in a decimal of the radius.
ζ	113° 33′	·9979
η	114 40	·9953
θ	116 31	·9643
ι	117 37	·9984

In consequence of the partial breaking up of the spots on the 19th and 20th, it was not easy to identify them, and the Greek letters may possibly not refer in all cases to the same spot. The following, selected from many other small spots surrounded by faculæ, were measured on a photograph taken on the 19th at $0^h\ 9^m\ 51^s$ Greenwich mean time:—

Spot.	Position-angle.	Distance from the centre in a decimal of the radius.
ζ	117° 26′	·9518
θ	119 45	·8872
ι	120 19	·9607
κ	124 48	·9166

On the 20th, civil reckoning, or astronomical reckoning 19th day $23^h\ 46^m\ 45^s$, the photograph K was taken, when a fresh spot *d* had made its appearance on the eastern limb; the following are the results of the measurements of this spot, and of some others in the group surrounded by faculæ. All these spots had altered greatly since the previous day.

Spot.	Position-angle.	Distance from the centre in a decimal of the radius.
d	86° 26′	·9961
ζ	121 3	·8713
ι	123 53	·8896
θ	126 10	·7733
κ	130 52	·8231

Table V.

Table V. contains the results of the measurements of the principal spots (*a*, *b*, *c*), the angles of position being corrected for the errors of the wires, and the distances given in a decimal of the radius of the sun. In the case of each spot, the particular edge measured is indicated on Plate XV. by a dotted line, and by means of Table V. my results may be reduced to those of other observers who may have measured a different part of the same spot.

Column 1 gives the number of the photograph; column 2 the date of the photograph; columns 3, 8, and 13 the distances of the spots *a*, *b*, *c* from the sun's centre in a decimal of the radius; columns 4, 9, 14 the averages of several measures; columns 5, 10, and 15 the position-angles; columns 6, 11, 16 the average position-angle for several photographs; and lastly, columns 7, 12, and 17 the mean epochs of the means of the measures.

Table V.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.
No.	Greenwich mean time.	Spot *a*. Distance from the sun's centre in decimal of radius.		Spot *a*. Position-angle corrected for error of the wire.	Spot *a*. Mean.	Spot *a*. Mean epoch.	Spot *b*. Distance from the sun's centre in decimal of radius.		Spot *b*. Position-angle corrected for error of the wire.	Spot *b*. Mean.	Spot *b*. Mean epoch.	Spot *c*. Distance from the sun's centre in decimal of radius.		Spot *c*. Position-angle corrected for error of the wire.	Spot *c*. Mean.	Spot *c*. Mean epoch.
	d h m sec.			° ′	° ′	h m			° ′	° ′	h m			° ′	° ′	h m
	14 4 12 6·6	Not on disk.	...	......	...	......	......	...	......	......	......	·3825	...	45 37		
6	18 1 47 43·6	·6872		74 35·8			·6772		129 1·5			·6064		301 5·5		
7	52 31·1	·6877		75 6·5			·6798		129 16·8			·6070		300 54·3		
8	56 3·1	Defective.		74 48·3			·6764		129 16·0			·6019	·6053	301 3·0	301 0	1 54·72
9	2 2 35·1	Defective.	·6867	......	75 00	1 57·33	·6759	·6780	129 5·0	129 12	1 58·21	·6061		300 58·0		
10	3 16·1	·6868		75 8·2			·6784		129 14·3							
11	7 6·1	·6853		75 21·3			·6801		129 18·3							
12	11 11·1	·6832		75 21·0			·6764		129 23·5							
14	20 7·6	·6837		75 23·2			·6748		129 25·2							
15	22 44·1	·6867	·6852	75 3·0	75 16	2 20·46	·6762	·6758	129 29·0	129 23	2 20·46					
16	27 47·1	·6871		75 17·5			·6759		129 14·5							
17	33 13·6	·6809		75 20·6			·6752		129 41·1							
19	36 17·1	·6817		75 16·6			·6748		129 42·6							
20	41 23·1	·6825	·6821	75 0·0	75 13	2 39·31	·6730	·6737	129 31·5							
21	46 18·6	·6833		75 16·6			·6723		129 33·6	129 34	2 41·13					
22	48 25·6	......	...	......	...	......	·6730		129 22·7							
29	3 17 11·7	......	...	......	...	......	......	...	......	......	......	·6172		300 48·3		
30	20 7·7	......	...	......	...	......	......	...	......	......	......	·6170		300 22·7		
31	24 13·2	......	...	......	...	......	......	...	......	......	......	·6179	·6175	300 28·6	300 34	3 22·10
32	26 51·7	......	...	......	...	......	......	...	......	......	......	·6179		300 35·8		
33	34 1·7	......	...	......	...	......	......	...	......	......	......	·6198		300 25·4		
35	37 19·7	......	...	......	...	......	......	...	......	......	......	·6163		300 28·4		
36	41 18·7	......	...	......	...	......	......	...	......	......	......	·6187	·6189	300 30·6	300 29	3 39·16
37	43 57·7	......	...	......	...	......	......	...	......	......	......	·6207		300 31·4		
39	51 53·7	......	...	......	...	......	......	...	......	......	......	·6188		300 23·8		
41	58 58·3	Too faint.	...	......	...	......	·6611		130 4·9	......	......	·6260	·6219	300 24·9		
42	4 2 43·8	·6755		75 6·0			·6676		130 2·0			·6209		300 39·0	300 29	3 57·87
43	5 44·8	·6734	·6729	74 56·0	74 53	4 8·82	·6676	·6635	130 10·9	130 10	4 6·85	·6261		300 33·9		
44	10 33·3	·6743		74 41·3			·6619		130 6·3			·6241	·6238	300 53·3	300 33	4 10·86
45	16 15·8	·6684		74 49·5			·6595		130 27·0			·6212		300 13·0		
46	19 0 9 51	·5312	...	70 48·8	...	......	·5454	...	140 18·0	......	......	·7483		296 53·0		
K	23 46 40	·3581	...	59 31·0	...	......	·4294	...	161 26·0	......	......	·8713		295 41·0		

July 18, $2^h\ 2^m\ 35^s$ Greenwich mean time, the moon had occulted spot *c*, at a distance of ·6061 of the sun's radius from the sun's centre, and between the position-angles 300° 58′ and 302° 33′, reckoning from the respective edges of the spot.

At $3^h\ 11^m\ 11^s{\cdot}7$ the spot *c* was partly uncovered by the moon's edge at a distance of ·6172, and between the angles 300° 48′ and 302° 45′.

At $3^h\ 58^m\ 58^s{\cdot}3$ the moon's limb had partially passed off spot *b*, at a distance of ·6611, between the angles 128° 3′ and 130° 5′.

I am not aware that any practical use has ever been made of the occultation of a sun-spot by the moon during an eclipse; but as it is probable that during the eclipse under discussion such phenomena were recorded by a great number of observers, as on the occasion of previous eclipses, I have thought it desirable to make the measurements given in Table V., which will, I believe, afford better means than have been before available for turning such observations to account.

Photographs of the Totality.

Copies of the two totality-pictures which accompanied this paper were produced in the following way: the original negatives were placed in the focus of an enlarging-camera, and positive collodion copies on glass procured, on which the lunar disk was enlarged to 9 inches in diameter. These positive copies were then placed in the focus of the camera, and a number of negatives made, to print any impressions that might be required, in which the lunar disk was reduced to the sizes of the several engravings accompanying the paper. The photographic copies therefore are two removes from the original, and, something being lost at each operation, they do not present all the details visible in prints taken direct from the original negatives. The corona, for example, which is depicted on the original negatives, is to a great extent lost in the copies, because in bringing clearly out the details of the prominences, the corona in most cases becomes over-printed.

A few positive 9-inch copies on glass have been presented to Observatories and public Societies; but it was not possible to do this very extensively, in consequence of the extreme difficulty of copying, occasioned by the density of the original negatives. They could only be procured on days when the sun was perfectly unobscured by haze or cloud, and ultimately the injury to the second original negative prevented my continuing the work sufficiently long to obtain a supply as great as I desired. Secondary copies can, however, still be procured.

In Plate IX. are given mezzotint fac-similes of the two totality-pictures the size of the originals; although they will serve to give a general idea of the photographs, and to illustrate what has to be said respecting them, they are, after all, but imperfect substitutes for the photographs themselves. Copies of the photographs would have been inserted in this memoir, had past experience of the permanence of such pictures warranted the Council in doing so*.

* In the Author's copies, Plate IX. *a*, photographic copies from the originals are given. They are two removes from the originals.

In order that the phenomena presented by the photographs may be clearly understood, I propose to give an explanation of certain appearances in them, which might otherwise occasion some difficulty. First, with regard to No. 25 photograph (the first totality-picture) (Plate IX. fig. 1). The sensitive plate was in the heliograph and the slide which covered it removed, a minute or so before totality, a temporary screen being first held just before the object-glass, so as to stop off all light. Thus every thing was in readiness, and it was only necessary to remove the temporary screen, to expose the plate at the proper moment. The very instant of the disappearance of the sun, I gave the signal for its removal, which was immediately done, and Mr. BECKLEY, who was watching the chronometer, gave the signal for covering the object-glass exactly one minute after it had been uncovered. I had given instructions that no attempt was to be made to note the precise epoch of total obscuration; for each operator had too much to occupy his attention to admit of any work being done which was not absolutely essential to the photographic operations. In order to regulate the time of exposure, the precise position of the second-hand of the chronometer was noticed when the plate was first exposed, and the signal was given for replacing the screen when the second-hand had completed a revolution.

The telescope followed the motion of the sun so well that the prominences retained a perfectly fixed position on the sensitive plate; and from the results on the second plate, presently to be spoken of, it is known that they must have depicted themselves to some extent, though very faintly, in a second. On the other hand, the comparatively feeble corona would have required even a longer period to thoroughly imprint itself than the whole time allotted for the first picture. Consequently, as the moon moved from the west to the east, she kept shutting off the prominences and corona on the east, thus stopping further action; while, on the west, she permitted fresh portions of the corona to commence a new action. The luminous prominences, when once they had produced their effect, could not be obliterated, although they might be subsequently covered by the moon; for it is well known in photography that latent images remain on the plate for a long time, and become apparent on applying the developing fluid. In the case of the corona, the full effect not having been produced even at the end of the operation, it will be evident that its picture was necessarily the most intense on the eastern side, just at that part where the moon's periphery had arrived at the close of the work; while it is clear that on the western side the action would continuously follow up the moon's progress, and that therefore an impression gradually becoming fainter towards the lunar disk, would indicate the point which the moon had reached when the image of the eclipse was shut off. With this explanation, the appearance of the moon's edge beyond the prominences on the eastern side in the first totality-picture can present no difficulty. If, during the exposure of the plate, fresh prominences had become uncovered on the western side, they would have imprinted themselves; and if the plate had remained in the heliograph during the entire period of totality, the whole of the prominences would necessarily have recorded themselves on a single plate, although only a part had been visible at one time, and the plate would have shown very nearly the position of the moon at the conclusion of the operation. Unless an instantaneous picture of the phenomena of totality could be procured*, as in the case of the other phases, no photograph would show the precise state of matters at any one moment; consequently, if it be desired to know what was the condition of things at any one instant of the period during which the plate was in the heliograph, for example, at the commencement of the totality, and a minute afterwards, recourse must be had to the expedient of completing the circle of the lunar disk for the position she occupied at these two epochs respectively. The photograph itself affords the necessary data for effecting this; for it will be found that the disk of the moon, as depicted on the photograph 25, represented in Plate IX. fig. 1, is not quite a complete circle, and that the longest diameter is in a direction at right angles to prominence A (see Plate XV.).

By measuring the diameter in the direction of prominence A with a divided beam compass, and taking the half of the quantity as a radius, it was a matter of no great difficulty to find the centre of the picture for the two epochs in question, namely, near the commencement and near the end of the first minute, as indicated by the photograph, and to draw in the lunar disk from either centre. In this way two photographs, α and β, 7 inches in diameter, were corrected, and served as originals for Plates X. and XI., which show the state of the phenomena as accurately as if two instantaneous pictures had been taken.

Plate X., which is the copy of α, represents the appearance of the phenomena of totality at the commencement, and Plate XI. the copy of β nearly at the end of the first minute. A line drawn to the centres, laid down for the two epochs, was found to correspond absolutely with the direction ascertained independently to be that of the motion of the moon's centre, and measured 23″. Allowing a period of five seconds for the production of a picture sufficiently intense to show itself clearly on the plate, both at the commencement and at the end of the exposure, the period traceable would be fifty seconds, and $\frac{23'' \times 60}{50} = 27''{\cdot}6$ would be the motion of the moon's centre during a minute, a result not differing by more than a few tenths of a second of arc from the mean derived from the measures of the other phases of the eclipse.

In Plate IX. fig. 1, representing the untouched photograph No. 25, a portion of the prominence R is visible; but it came into view after the commencement of totality, and was therefore painted out in completing the lunar disk in the touched photograph α, represented in Plate X.; at the epoch shown in photograph β, represented in Plate XI., however, the lunar disk had revealed so much of prominence R as is seen in the original picture.

The dark lines in the original photograph, represented in Plate IX. fig. 1, situated respectively above the floating cloud C, and across the broad part of G, are the images in shadow of the position-wires. In the original negatives and positive copies taken

* The possibility of doing this during future total eclipses will be presently pointed out.

carefully from them, the continuations of the images of these wires are depicted in positions diametrically opposite. The wires are not central with the picture of the lunar disk, which was adjusted by means of the finder to be as much on the right of the plate as possible, in order that the details on the eastern limb might not be lost. It will be perceived that the light was inflected sufficiently during the taking of the picture to complete the image of the protuberance G, notwithstanding the intervening wire.

About eighty seconds were required for covering and taking out photograph 25, placing plate No. 26 in the heliograph, drawing back the slide which covered it, allowing time for the vibrations imparted to the instrument to cease, and removing the temporary cover from the telescope; so that the exposure of plate No. 26 commenced about two minutes and twenty seconds after the commencement of total obscuration, and continued until within a second of the reappearance of the sun, having been, like plate No. 25, exposed as nearly as possible one minute to the actinic influence of the prominences. The action of the prominences was, however, on account of the accidental disturbance of the heliograph, not allowed to continue on one part of the plate during the whole time; and hence their impressions are not so strongly depicted as those of the prominences on photograph No. 25, represented in Plate IX. fig. 2.

On this account the appearance of photograph No. 26 was not easy at first to comprehend; and it gave me considerable difficulty for some time to make out with precision the true nature of the result obtained. A gust of wind had arisen close upon the time this picture was being taken, and I was induced to imagine that the telescope had so been caused to vibrate; but further examination of the picture showed that this could not have been the case, for three very distinct images were imprinted of the prominence E (the boomerang), proving that after each disturbance the instrument followed the sun's movement correctly. This afforded a clue to what had actually occurred; and I found on inquiry that two of my assistants had looked at the eclipse through the finder of the heliograph, and had, as it appeared, inadvertently disturbed it in right ascension, which the wear of the worm-wheel and tangent-screw permitted them to do. Fortunately a firm radius-bar, which I had had made previous to leaving England, held the telescope so firmly in declination that it could not be readily moved in that direction; if it had been so moved, the resulting picture might have defied interpretation, and have rendered the photograph useless for exact measurements.

The clue once obtained, it was easy to make out three impressions of the wheatsheaf A (Plate XV.), three of the floating cloud D, three of each of the three points h', h'', h''' of the fallen tree H, and less easily three of the mountain-peak R because they overlapped. There were produced only two impressions of prominence Q on the western side, because at the epoch of the first action, when the point I was as yet partly visible, the moon still covered Q. The next impression was formed just when a part of Q had become visible, and when also part of the spade L had been revealed. The prominence L appears to have been sufficiently brilliant to imprint itself continuously while its image was traversing the plate. In the last impression on the plate, the prominences and corona continued their action undisturbed during the remainder of the time of exposure: this picture is the one which includes those images of the several prominences depicted furthest to the right in each case, Plate IX. fig. 2. During the exposure of the plate in its third period there was a slight irregularity in the motion of the driving-apparatus, which to a very small extent enlarged the prominences in the direction of right ascension.

In making a representation of the state of the phenomena at the end of totality, it was only necessary to paint out the two impressions of each of the several prominences belonging to the two other periods depicted on the photograph, and to correct the slight exaggeration caused by the irregularity of the driving-apparatus. In this way was produced the touched photograph γ, represented in Plate XII., which faithfully shows the state of matters about a second, or less, before the reappearance of the sun. It has been possible to make out, from photograph 26, three corrected pictures, showing the appearance of the prominences at three different epochs of that period of totality during which it was in the heliograph; but only one of the resulting pictures has been engraved, namely, that shown in Plate XII.

The irregularity of many of the protuberances on the concave side adjacent to the lunar disk is very striking, and appeared to me, while observing the eclipse with the achromatic, to be greater than could be attributable to the indentation which would be caused by any amount of irregularity on the lunar periphery. The extent of this irregularity could be readily estimated during the other phases of the eclipse with the telescope, and is also depicted clearly on the several photographs, which afford a permanent record of the moon's profile. In Plate XVI., which was produced by etching an enlarged positive copy of photograph 22 and electrotyping from it, is shown the profile of the moon's limb between the position-angles 44°·5 and 191°; in Plate XVII., produced in a like manner from photograph 28, the moon's profile is depicted between the position-angles 228° and 9°; altogether the plates exhibit 287° of the moon's outline, with which the concave edge of the luminous prominences shown in Plates XIII. and XIV. may be compared. As the moon moved onwards, the great amount of indentation of the concave side of the protuberances appeared to me to become less on the eastern side and greater on the western side. Some of the irregularity on the concave side is undoubtedly due to the periphery of the lunar disk, but all of it cannot be so accounted for. We may assume that some prominences are not in absolute contact with the sun's photosphere, but, on the contrary, are supported at a distance from it, as in the case of the floating cloud D. Notably, in that part of the prominence G between the position-angles 112° and 124° the irregularity of the concave boundary cannot be accounted for by the form of the moon's limb; on the other hand, in support of the position that in certain cases the irregularity must be due to the profile of the moon's disk, we have good evidence in the second totality-photograph, Plate IX. fig. 2, where part of the luminous prominence Q is depicted at two epochs, namely, just as it became visible, and at the end of totality. It will be seen that the part q'—q'' has the same amount of indentation at both epochs. In most cases the irregularity of the contour of the prominences appears to be much greater than that portion of the moon's limb corresponding in position-angle with it.

On comparing the results of the expedition of 1860 with those obtained in 1851, one cannot fail to be impressed with the general similarity in the aspect of the prominences at the two epochs: on both occasions were seen luminous masses of vast extent, perfectly detached from the sun, and far beyond the lunar disk; the same irregularity of outline on the convex side, running out into points; the same apparent outpouring of faint vapours, falling as it were towards the sun (as in the faint portions of prominence A); lastly, although not seen with the eye in 1860, there is recorded by the photographic retina a similar hooked projection to that seen in 1851, and named by the Astronomer Royal "the boomerang," from its approximation in form to the Australian weapon bearing that name.

Reference to Plate XV. affords better means of judging of the dimensions of the protuberances depicted on the two photographs than the photographs themselves, on account of the latter merely showing their distance from the moon's periphery, whereas in the index map, Plate XV., the distances from the sun's disk are shown. We thus see that while the two photographs give a fair representation of the height above the sun's periphery of the prominences D, E, F, G, M, N, O, P, Q, R, they do not do so in respect of the remainder. Notably, the prominence A, if it extended inwards to the sun's periphery, and was not, like the floating cloud D, supported at some distance from it, must have been of twice the height revealed to us; and again, the prominence K must, as regards the brighter part of it, have been nearly four times the height seen above the moon's limb. The brighter part of the prominence K, at the epoch of the second totality-picture, was covered by the moon; and although the fainter hooked part projected beyond the lunar disk, it was too faint to imprint itself in the second picture, on account of the disturbance of the telescope, which did not allow it to remain sufficiently long on one part of the plate. In the first picture it depicted itself clearly, and, if it had done so in the second, would have admitted, in connexion with A, of a chord being drawn which would have afforded a capital basis for measurements. The actinic power of "the boomerang" is quite remarkable; for it imprinted itself, distinctly, three times on photograph 26, although it was invisible to the eye. Not only, therefore, is photography of value in recording phenomena visible to the human eye, but it is also able to render evident bodies which emit only those rays which belong to the invisible part of the spectrum.

The question whether the luminous prominences would appear as bright or dark markings on the sun's disk admits of a probable solution by means of photography, which furnishes data as to the degree of luminosity of the prominences relatively to that of the sun's photosphere. It will be recollected that I have stated that one of the prominences (A) was visible some minutes previous to totality, and that it continued visible to other observers several minutes after the reappearance of the sun. It so happened that photograph No. 28 was taken about twenty seconds after the reappearance of the sun, when the distance between the moon and the sun's peripheries was only $9''\cdot6$. This photograph was obtained with the full aperture (3·4 inches) of the heliograph, and the plate was exposed by removing a temporary cover which had been placed before the object-glass, and replacing it as quickly as possible. The time of exposure would certainly not exceed a second, yet the image is completely solarized (bleached) from over-exposure; moreover, the wind, which rose suddenly at that period, violently shook the heliograph in the direction of right ascension, by successive impulsions against the object-end, which projected beyond the walls of the observatory; and many impressions of the solar crescent are consequently depicted on the plate, on which, however, not the slightest trace of prominence A could be made out. With the aperture of the object-glass reduced to 2 inches in diameter, and using the instantaneous apparatus, a picture of partial phase could under similar circumstances have been procured in $\frac{1}{20}$th of a second, and therefore in less than $\frac{1}{58}$th of a second with the full aperture of the telescope. Moreover, as in the second totality-picture (No. 26) the prominences were depicted three times, in consequence of two disturbances of the telescope in right ascension, during the minute the plate was exposed in the heliograph, we know that on the average twenty seconds are about sufficient to bring out the picture of the luminous prominences strongly. These triplicate images are not, however, of equal intensity, one being very faint; and therefore, assigning to this latter (what its appearance warrants) an exposure of half the time of the other two, we have twelve seconds as the time required to depict the most luminous of the prominences fairly. It results, therefore, that the light of the luminous prominences is fully $58\times12=696$ times less bright than that of the photosphere of the sun.

On August 12, 1862, I succeeded, as I have already stated in the foot-note, page 334, in obtaining an *extremely* faint impression of the moon with the Kew heliograph in three minutes. The full aperture of the object-glass was employed, and the chemicals used were in the highest degree of sensitiveness. An impression of the luminous prominences of equal intensity would, according to data furnished by the second totality-picture No. 26, have been produced in a second. It may therefore be safely estimated that the image of the luminous prominences (for equal areas) is 180 times more brilliant than that of the moon. Assuming for the ratio of the light of the sun in comparison with the light of the moon 200,000 to 1, it would follow that the image of the luminous prominences is $\frac{200,000}{180}=1111$ times less brilliant than that of the sun; taking the mean of the two estimates it would be 900 times less brilliant than that of the sun.

Although in all probability the prominences are less bright than the dark nuclei of the solar spots, it does not follow that they would appear as very dark markings on the sun's disk, for to a great extent they may permit of the transmission of the light emitted by the photosphere; and, besides, it is by no means probable that there is any intimate connexion between the solar spots and the prominences, for the vast extent of the sun's limb which is surrounded by the prominences precludes such an idea, and leads to the conviction that they are far more generally distributed on the solar disk, and of proportions greatly exceeding any which the spots ever attain to.

Since the prominences would appear to be scattered so widely over the sun's surface, the question has arisen whether it would be possible to render these wonderful appendages apparent at other periods than those of total eclipses of the sun. For the purpose of solving this problem, Mr. JAMES NASMYTH devised an apparatus which, in part, consisted of a cylindrical box, blackened inside, having in one end an aperture of such dimensions that it exactly permitted of the passage of the sun's image when projected by a telescope, whilst the surface surrounding the aperture was sufficiently large to receive the images of all objects situated beyond the solar periphery.

The Astronomer Royal also has made experiments with the same view, using, in part, the Nasmyth apparatus; but the existence of the luminous prominences could not be detected by its means, in all probability on account of the great amount of illumination of that part of our atmosphere which is in apparent contiguity with the sun. On the occasion of Professor PIAZZI SMYTH's experimental visit to the Peak of Teneriffe he took out with him this apparatus, because it was thought that the more attenuated stratum of atmosphere at that elevation would interfere less with the success of the experiment. Only negative results were, however, obtained, and the problem remains to be solved.

Is it probable that photography may lead to a solution of the difficulty? I am inclined to think that it may possibly do so. It would, however, be quite futile to attempt to delineate the luminous prominences, when beyond the sun's periphery, by means of photography, after the experience afforded by the experiments before cited; for most unquestionably they would not produce an image so intense as that of our own atmosphere in apparent contiguity with the sun's disk and illuminated by his rays. My hope is that their forms may be depicted on the brighter solar disk itself, and their existence rendered evident by means of the stereoscope, which has already enabled me to make out the real nature of the radiating lines on the lunar surface.

During the year 1861, by means of my 13-inch equatorial reflector, I succeeded in procuring photographs of the sun's surface, on a scale of 3 feet for the sun's diameter. These colossal photographs were obtained by enlarging the focal image by means of a secondary magnifier, constructed especially to ensure a flat field and the coincidence of the visual and chemical foci. They show, in a remarkably striking manner, the mottling of the sun's photosphere, which appears to be entirely composed of an undulating mass of waves, like the surface of the sea agitated by wind.

Two pictures of the same sun-spot, taken at an interval sufficiently great to admit of the sun's rotation causing the necessary angular shift of its position, evidently possess the stereoscopic relation. By placing them in the stereoscope in such a way that the positions of the two pictures, relatively to each other, shall be reversed, that last taken being placed on the left, that first taken to the right (supposing the image to be erect), I have obtained a stereoscopic picture of a sun-spot, and some surrounding faculæ, which represented the various parts of the picture in their true relative positions in regard to altitude, and in other respects. I have ascertained in this way that the faculæ occupy the highest positions of the sun's photosphere, the spots appearing like holes in the penumbræ, which appeared lower than the brighter regions surrounding them; in one case parts of the faculæ were discovered to be sailing over a spot, apparently at some considerable height above it.

My hope of rendering evident the luminous prominences is dependent upon an extension of this experiment. I believe that, with a careful adjustment of the time of exposure of the sensitive plate, I shall succeed in obtaining the outline of the luminous prominences (the so-called red flames) as very delicate dark markings on the more brilliant mottled background of the photosphere. These delineations, except with the aid of the stereoscope, would be confounded with the other markings of the sun's surface, but they would assume their true aspect, and stand out from the rest, as soon as two suitable pictures were viewed by the aid of that instrument.

The difficulties in the way of doing this are, however, of a special kind, as will readily be seen from the following considerations: when the aperture of the instantaneous slide is at a maximum, and the rapidity of motion at a minimum, a picture of the sun will result which will be of one uniform maximum density, without the slightest trace of any marking even of a dark sun-spot. As the aperture is reduced and the velocity of the slide augmented the spots will become depicted, but no trace of the penumbræ will be seen; then we shall get the penumbræ, and subsequently traces of the faculæ and of the general mottling of the sun's disk; lastly, by still further reducing the aperture, the faculæ and the mottling will be well brought out, but especially the latter. The photographic process, it will be recollected, is one of progressive action, and even the faintest parts of the picture may, by a long exposure, produce as much intensity as the bright parts do by a shorter action; and evidently, with sufficient time, all distinctions of bright and less bright must cease to exist on the photographic plate, if all parts have produced the maximum density of effect which the plate is capable of affording. It rarely (it might be said never) happens that all parts of the picture are portrayed with the best effect; and in heliography the apparatus has to be variously adjusted, according as the spots or the mottlings of the sun's surface are required to be especially well shown.

Measurements of the Totality-Photographs.

The main object of the observations of the total eclipse of 1860 was to ascertain whether the luminous prominences are objective phenomena belonging to the sun, or whether they are merely subsidiary phenomena, produced by some action of the moon's edge on light emanating originally from the sun. If the luminous prominences are attached to the sun, it is evident that they would continually change their positions with respect to the moon's centre as the moon moved across the solar disk. For example, a prominence situated in the direction of the moon's path would maintain its position-angle in reference to the moon's centre unchanged, but it would be gradually and at length entirely covered by the moon. On the other hand, a prominence situated at right angles to the moon's path would change its position-angle in respect of the moon's centre, but would remain uncovered to almost the same extent at the end of totality as at the commencement. Moreover, prominences situated in positions intermediate between these two would change their respective position-angles, referred to the moon's centre, less, but would be covered more, in the proportion of their relative degrees of proximity to the line of the moon's path; and *vice versâ*, the change in position-angle of the several prominences would be greater in proportion as they were situated nearer to a line at right angles to the moon's path*. No stronger evidence that the prominences belong to the sun can be adduced, than that of a change in the angular position of a prominence in reference to the moon's centre, because it is not probable that *different* parts of the moon's periphery would produce precisely the same effect on light emanating from the sun. The relative changes of position may be calculated for any given locality; for which purpose it is necessary to know either its geographical position, or to determine the relative motions of the sun and moon during an eclipse by other means. In a former part of this paper I have stated that the geographical position was ascertained, also that the exact path of the moon's centre across the solar disk was made out by certain measurements of the photographs; and in Plate XVIII. I have given a graphic representation of the moon's path in reference to the sun's centre during the eclipse. Mr. Carrington and Mr. Farley's elements of the eclipse, founded on the geographical position, have already been cited, and will be presently made use of for computing the changes of position of the several luminous prominences which should occur if they belong to the sun.

If the prominences belong to the sun, photographic images of the same protuberances taken at any one locality, at different epochs of totality, ought to coincide exactly when the photographs are superposed one over the other†; and measurements of their positions with respect to the moon's centre ought to correspond with their computed positions. Moreover, photographs taken at different places (sufficiently near in longitude) ought to agree in their details,—although, for very distant localities, it is possible that they might not do so; for it is conceivable that during long intervals a change might occur in the luminous prominences, or fresh prominences might be brought into view by the sun's rotation.

Furthermore, on consideration it will be evident that to two observers differently situated the protuberances would not have precisely the same dimensions; that is, they would appear to project more or less beyond the moon's limb. Within the zone of

* An inspection of Plate XV. will render this evident, and it will be seen that the angle γ gradually diminishes as the prominence is more distant from A.

† That this was the case was shown, on the occasion of this paper being read, by sliding the first totality-picture over the second, and projecting their images on a screen by means of an electric lamp. It was thus seen that the pictures of the several prominences correspond exactly in form and position, each to each, in the two photographs.

totality, indeed, some prominences might be visible at one station and not at another, in consequence of the parallactic shift of the moon with respect to the sun.

Photographs offer great advantages over eye observations in determining changes of position in the prominences; but nevertheless there are some difficulties even in photographic measurements.

For example, with objects which are terminated by a softened outline, there is some difficulty in determining, absolutely, where the boundary ceases to exist: this was found to be especially the case with the luminous prominences depicted on the original negative, and also with the prominences as shown in positive photographic copies taken from it by superposition on an albumenized plate. Some doubt also existed in regard to the centering of No. 26 photograph; but this was found not to affect the results so much as the uncertainty of the precise termination of the prominences, in the photographs both negative and positive.

Table VI.

Table VI. gives the results of a series of angular measures of the luminous prominences, with reference to the moon's centre, both on the original negatives and on the albumen positive copies. Columns 4, 5, 6, 7 refer to the first totality-picture; columns 8, 9, 10, and 11 to the second totality-picture. Columns 7 and 11 are the measured positions, corrected for the errors of the wires, for the epochs of the two photographs; column 12 the difference in the position-angle of certain prominences at the commencement and at the end of totality. A mere inspection of column 12 renders it evident that the nearer a prominence is situated to 27° 10′, in reference to the sun's centre (which is the case with prominence A), the greater is its angular shift in reference to the moon's centre; and the nearer a prominence is to the line of motion of the moon's centre, the less is the angular change, as, for example, the bead h''' of the protuberance H.

TABLE VI.—Angular Position of the Prominences referred to the Moon's centre.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
			No. 25 photograph, 1st of totality, measured with the new micrometer.				No. 26 photograph, 2nd of totality, measured with the new micrometer.				
Letter designating protuberance on diagram.	Synonym.*	The part measured.	Original negative.	Albumen positive (copy).	Mean.	Mean corrected for error of wires, by adding 15′·5.	Original negative.	Albumen positive (copy).	Mean.	Mean corrected for error of wires, by adding 16′·5.	Angular motion between the epochs of the two photographs.
A.	Cauliflower, Wheatsheaf	*a*	28° 10′	27° 47′·5	27° 58′·7	28° 14′	22° 15′	22° 37′	22° 26′	22° 42′·5	5° 32′
		a′	32 1·5	31 48	31 54·7	32 10	....	26 8	26 8	26 24	
B.	Small spot	...	46 30	45 18	45 54	46 9					
C.	Detached cloud	*c*	58 14·5	58 2·5	58 8·5	58 24	53 23	53 45	53 34	53 50	4 34
D.	Base of Boomerang	*d*	72 16·5	72 2	72 9·2	72 25	67 57	68 16	68 6·5	68 23	4 2
E.	"	*e′*									
F.	....	*f*	85 17·5	85 18	85 17·7	85 33					
H.	Part of trunk of fallen tree	*h′*	....	....	....	....	105 16·5	105 19	105 17·7	105 34	
		h″	....	....	....	....	106 28	106 35·5	106 31·7	106 48	
		h‴	109 30	....	109 30	109 45	108 33	108 42	108 37·5	108 53	0 52
G.	Part of long eastern protuberance..	*g′*	134 9	134 12	134 10·5	134 26					
I.	Mitre	*i′*	155 1	155 3	155 2	155 17					
K.	Hook	*k″*	195 9	195 17	195 13	195 28					
L.	Spade	*l*	....	....	....	....	261 33	261 58	261 45·5	262 2	
		l′	....	....	....	....	263 24	263 18	263 21	263 37	
M.	Bead	*m*	....	....	....	....	275 58	275 52	275 55	276 11	
Q.	....	*q*	....	....	....	....	298 13	298 9	298 11	298 27	
		q″	....	....	....	....	309 35·5	309 26	309 30·7	309 47	
R.	Mountain, Ship	*r*	....	....	....	....	340 59·5	341 26	341 12·7	341 29	
		r′	348 26·5	348 12	348 19·2	348 35	344 26	344 51	344 38·5	344 55	3 45
	"	*r″*	....	....	....	....	351 3	351 22	354 12·5	351 29	

By MR. CARRINGTON's calculations the relative motion of the centres during totality was 92″·8 = $\beta\alpha$, Plate XV.

The moon's radius at totality by measurements of the photographs was 993″·1 = $a\beta$, Plate XV.

The position-angle of the moon's path $\beta\alpha$ prolonged in the direction of α was ascertained by measurements of photographs to be .... 117° 10′

The position-angle of prominence A at the point a was ascertained by measurements of photographs to be 28 14

Whence the angle $a\beta\alpha$, Plate XV. .... = 88 56

With these elements the angle γ = the angular shift of prominence A during totality was found to be 5° 21′.

In a like manner, taking into account the distance of each prominence from the moon's centre, the angular shift of the other prominences was calculated, whence

Angular shift of prominence A = 5° 21′
" " " " C = 4 35
" " " " E = 3 46
" " " " H = 0 41
" " " " R = 3 56

The edge *a* of the prominence A (the cauliflower or wheatsheaf) is not far from a line at right angles to the path of the moon's centre, and is well situated, therefore, for ascertaining whether the change of position-angle accords with the demands of that hypothesis which assumes that the luminous prominences belong to the sun.

Disregarding, for the moment, the errors in the assumed places of the wires, we have the following results.

Position of the prominence A, measured on the edge *a*:—

	First totality-picture.	Second totality-picture.	Angular shift.
Original negative . .	28° 10′	22° 15′	5° 55′
Albumen positive . .	27 47·5	22 37	5 10·5
Difference . . .	— 22·5	+ 22	

Both these values of the angular shift have to be diminished by 1′, in consequence of the alteration of the position-wires of the instrument in the interval between the two epochs, and they become respectively 5° 54′ and 5° 9′·5.

Considering the difficulties experienced, from the causes before mentioned, in making measurements of position-angle, it is quite justifiable to take the mean of the above results; for, besides the uncertainty in determining the exact boundary of a prominence in the photograph, there is superadded, also, the difficulty of placing the photograph, quite correctly, in its proper angular position on the measuring-instrument, on account of the wires being very faintly imprinted on the western side of the picture in the first, and on the eastern side in the second, totality-picture.

The mean of the two measures gives 5° 32′ as the angular shift in the position of prominence A. Assuming, in accordance with theory, the motion of the moon's centre to have been 92″·8 during totality, we have 5° 21′ as the theoretical change of position-angle, which, deducted from the mean 5° 32′, gives the difference of 0° 11′.

On taking an average of the measurements of all the prominences, the difference between the measured and the computed angular shift is only half this quantity, as will presently be seen.

Table VII.

Table VII. contains the results of measurements of positive photographs 9 inches in diameter. Columns 4, 5, 6, 7, and 8 relate to No. 1 totality-picture; and columns 9, 10, 11, 12, and 13, to No. 2 totality picture. Columns 7 and 12 show, respectively, the differences in the determinations of the position-angles of the several prominences in No. 1 and No. 2 totality-pictures, on the 9-inch photograph and the original negatives and positive-albumen copies by superposition. I give preference to the measures by means of the instrument of the original negatives and the direct albumen copies, so far as regards the position-angles; but for measuring the amount of motion of the moon's centre during the totality, I am, for the reason already assigned, dependent on the 9-inch photographs. The whole difficulty, with respect to the latter, consisted in exactly ascertaining the position of the centre of the moon in the two pictures, and in measuring correctly a picture of the sun, enlarged to the same scale, with a divided beam compass, in order to obtain a value in arc for the measurements taken in inches and decimals of an inch. This accomplished, it was only necessary to measure the distance from the moon's centre or periphery of certain parts of the prominences in both totality-pictures, and to compare the results of the measures obtained on one picture with those on the other, for the purpose of ascertaining the amount of motion of the lunar disk in reference to the several prominences, and to reduce the resulting numbers to their value in arc.

The amount of motion is given in column 14 for those prominences which are visible in both photographs.

Thus, in a direction nearly at right angles to the path of the moon's centre, the apparent motion of the periphery was 1″, while at 83° from that point, or 7° from the line of motion of the lunar centre, it was 93″.

The motion of the moon's periphery, in respect of the several prominences situated at an angle θ would be 92″·8*. sin (θ—27° 10′), 27° 10′ being the position of a line at right angles to the motion of the lunar centre, as deduced from the photographic measures already given. For the following calculations, the angle θ was obtained from Plate XV., in which the prominences are referred to the sun's centre.

	Part measured.	Angle θ.		Measured motion.	Computed motion.	Measure —computation.
		°	′	″	″	″
Prominence A	*a*	26	9	1	1·6	—0·6
C	*c*	56	34	44	45·6	—1·6
C	*c′*	61	39	51	52·5	—1·5
E	*e*	67	4	65	59·5	+5·5
H	*h‴*	110	9	93	92·1	+0·9
R	*r*	346	14	60	60·9	—0·9
						+0·3 Mean.

From Table VI. are derived the following numbers:—

	Part measured.	Angular shift in respect of the moon's centre during totality.				
		Measured.		Calculated.		Measure—computation.
		°	′	°	′	′
Prominence A	*a*	5	32	5	21	+11·0
C	*c*	4	34	4	35	— 1·0
E	*e′*	4	2	3	46·5	+15·5
H	*h‴*	0	52	0	41·5	+10·5
R	*r′*	3	45	3	56·5	—11·5
						+ 4·9 Mean.

=1″·4 motion of the moon's centre in excess of the computed quantity.

It would be extremely difficult to obtain more convincing proofs that the luminous prominences belong to the sun than the foregoing numbers offer; but I have still one more to bring forward.

* The relative motion of the centres during totality as calculated by Mr. Carrington.

Table VII.

Designating letter on Plate XV.	Name of protuberance.	Part measured.	Distances of Prominences from the Moon's Periphery and their Position-angles referred to the Moon's centre. No. 25 photograph. First totality-picture. Measures on the enlarged positive copy, the original of Plate XIII.	Numbers in column 4, corrected for error of the zero by subtracting 8′.	Mean of the measures of the original negative, and albumen print therefrom, Table VI. column 7.	Column 5 minus column 6.	Measured height of the prominences above the moon's periphery.	No. 26 photograph. Second totality-picture. Measures on the enlarged positive copy, the original of Plate XIV.	Numbers in column 9 corrected for error of zero by subtracting 7′.	Mean of the measures of the original negative and the albumen print therefrom, Table VI. column 11.	Column 10 minus column 11.	Measured height of the prominences above the moon's periphery, corrected for error of centering.	Motion of the moon in covering and uncovering prominences.	Distance and position-angles referred to the sun's centre. Both totality-pictures engraved round No. 6 photograph of sun. Plate XV. Position-angle on sun's centre by graphic process.	Height of prominences above the sun's periphery, corrected for error of centering in those cases in which the positions are derived from the copy of No. 26.
1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.
A.	Cauliflower ...	a.	28° 40′	28° 32′	28° 14′	+18′	0′ 40″	22° 30′	22° 23′	22° 42′·5	−19′·5	0′ 39″	0′ 1″	26° 9′	1′ 18″
		a′.	32 24	32 16	32 10	+ 6		26 35	26 28	26 24	+ 4			29 39	
C.	Detached cloud	c.	58 30	58 22	58 24	− 2	0 44	54 14	54 7	53 50	+17	0 00	0 44	56 34	0 54
		c′.	....	....	....	......	1 45	....	....	....	......	0 54	0 51	61 39	1 53
D.	....	d.	72 20	72 12	72 25	−13	0 40	68 5	67 58	68 23	−25	....	....	....	....
E.	Boomerang ...	e.	68 50	....	....	......	2 31	....	....	....	......	1 26	1 5	67 4	2 40
		e′.	72 20	72 12	72 25	−13	....	....	....	....	......	....	......	....	....
F.	....	f.	85 40	85 32	85 33	− 1	....	....	....	....	......	....	......	85 4	....
		f′.	....	....	....	......	0 38	....	....	....	......	....	......	....	0 39
H.	Fallen tree ...	h′.	....	....	....	......	1 50	105 30	105 23	105 34	−11	0 12	1 38	....	....
		h″.	....	....	....	......	1 38	107 5	106 58	106 48	+10	0 3	1 35	....	....
		h‴.	110 0	109 52	109 45	+ 7	1 33	108 55	108 48	108 53	− 5	0 00	1 33	110 9	....
G.	Long protuberance ...	g.	....	....	....	......	1 3	....	....	....	......	....	......	....	1 2
		g′.	134 30	134 22	134 26	− 4	1 5	....	....	....	......	....	......	136 14	1 14
I.	Mitre	i.	....	....	....	......	1 6	....	....	....	......	....	......	....	1 28
		i′.	155 20	155 12	155 17	− 5	....	....	....	....	......	....	......	157 29	....
K.	Hook	k.	....	....	....	......	0 44	....	....	....	......	....	......	....	1 38
		k″.	195 50	195 42	195 28	+14	....	....	....	....	......	....	......	198 19	0 54
L.	Spade	l.	....	....	....	......	....	262 20	262 13	262 2	+11	0 21	......	260 4	0 48
		l′.	....	....	....	......	....	264 5	263 58	263 37	+21	....	......	....	....
M.	Bead	m.	....	....	....	......	....	276 40	276 33	276 11	+22	0 16	......	274 44	0 24
Q.	....	q.	....	....	....	......	....	298 40	298 33	298 27	+ 6	0 37	......	297 54	0 37
		q″.	....	....	....	......	....	309 40	309 33	309 47	−14	0 31	......	309 49	0 31
R.	Mountain	r′.	348 50	348 42	348 35	+ 7	0 00	344 45	344 38	344 55	−17	1 00	1 00	346 14	1 16
						Mean +1′·3					Mean +0′·0				

It is known that Señor AGUILAR, Professor MONTSERRAT, and Father SECCHI obtained photographs of the eclipse at Desierto de las Palmas, south of the central line, whilst my own were taken on the north of it. The latitude of my position was 42° 42′ N., that of Señor AGUILAR 40° 4′ 4″ N.; my position, measured on a meridian, is about 18′ north of the central line of the eclipse, Señor AGUILAR's about 4′ to the south of the central line.

Señor AGUILAR was so good as to send me paper copies of the four photographs taken at Desierto de las Palmas; and I have been able, in return, to send him copies of mine on glass, 9 inches in diameter. I have enlarged the copies presented to me to exactly the same dimensions (9 inches for the moon's diameter). One amongst the number, namely, the first taken, is sufficiently good, when magnified, to admit of measures being made on it with a fair amount of accuracy, although there is evidence, in the woolliness of the photograph, of the telescope not having followed the sun very well. Notwithstanding a want of precision in the details, I was able to ascertain with certainty that the distances between the point r' of R and a of A, between a of A and c of C, between c of C and g of G, between g of G and i of I, and between i of I and k'' of K, in Señor AGUILAR's photographs, correspond exactly with the same points in mine. I have already mentioned that when my first totality negative was superposed over the second the several prominences exactly coincided in their relative positions, and that the distance between any given points of two prominences on my first totality-photograph is absolutely the same in the second totality-photograph. Here we have the evidence carried still a step further; for the distances between given points in two prominences in Señor AGUILAR's photographs accord entirely with the distances between the same points on my own.

I may mention that the prominence G in Señor AGUILAR's photograph, from the commencement of the broad part on towards h of the fallen tree H, is much confused; H is not seen, because it is mixed up with G, which in consequence is as broad from h to g' as in its broadest part. With this explanation, however, there will be experienced no difficulty in comparing the photographs. The boomerang is not depicted on Señor AGUILAR's photographs. As my position was north of the central line, and the moon's centre was, as we have seen, shifted, by parallax, about 12″ below the sun's, it follows that I ought to have seen more of the prominence A, and less of I and K, than could be seen at Desierto de las Palmas; this is fully borne out by the photographs taken at the respective localities. The height of A above the moon's periphery is 40″ in my first totality-picture; in the corresponding picture taken at Desierto de las Palmas it is 32″, the difference being 8″. Prominence K in my first totality-picture, measured at k', is 44″, in Señor AGUILAR's 60″, difference 16″. The mean of the two measures $\frac{8+16}{2}=12''$, the relative parallactic displacement of the moon's disk at the two stations of Rivabellosa and Desierto de las Palmas, ascertained as nearly as the want of definition in the photograph obtained at the latter station permitted. It is probably less by about 2″ than the true displacement.

In conclusion, the two totality-pictures No. 25 and No. 26, when reduced to a suitable size and placed in the stereoscope, No. 25 on the left, and No. 26 on the right, afford a very beautiful view of the phenomena of totality, and one which could not be enjoyed by mortal eyes in looking at the real eclipse. Not only does the stereoscope render evident the fact of the moon being an object intervening between the observer and the sun, but it also shows it as a sphere. The triplication of the prominences must be corrected in No. 26; but any attempt to complete the lunar disk by painting on a positive copy, as α, β, and γ photographs, the originals of Plates X., XI., and XII., is immediately detected, and the corrected lunar disk appears perfectly flat. In placing the photographs in the stereoscope, the prominence A must be placed upwards, and at right angles to the line joining the centres of the photographs.

APPENDIX.

Having brought to a successful issue the photographic record of a total eclipse, it may not be out of place to point out for the guidance of others what steps I would recommend should hereafter be adopted.

In the foot-note to p. 334, I have mentioned that with my 13-inch reflector intense photographs of the moon were obtained in four seconds, and that, under precisely similar atmospheric circumstances, it required three minutes to obtain a feeble impression of the moon with the Kew heliograph, which, for the present, is mounted on an outrigger attached to the declination axis of my reflector. It will be remembered that for the totality-pictures obtained at Rivabellosa, under exceptionally favourable conditions in respect of the sun's altitude and the state of the atmosphere, the sensitive plate was exposed exactly one minute, the resulting photograph being remarkably dense, even to a fault. A picture of the moon, of greater intensity than the feeble image given by the Kew heliograph, could be obtained with my reflector in a second, so that it would produce pictures of the prominences in $\frac{1}{180}$th part of the time required by the heliograph, or in $\frac{60}{180}=\frac{1}{3}$ of a second. Making sufficient allowance for the difficulties in determining the exact ratio of actinic intensity in the foci of the two instruments, and also for a condition of the atmosphere less favourable than that under which the 'Himalaya' photographers worked, it may be safely estimated that, with a 13-inch reflector, perfect pictures of the prominences could be procured in two seconds. A 13-inch reflector would, however, be a cumbrous instrument to transport and erect at a distance from home; but a 9-inch reflector—or its equivalent, a 6-inch refractor, specially corrected for the actinic rays, is within the compass of such an expedition. These telescopes might be mounted with clockwork drivers on rigid equatorial stands, which must, however, be so designed as to admit of an adjustment of the polar axis to suit various latitudes. For each instrument an observatory should be constructed to take to pieces. Each observatory would require not less than four plate-holders, and about six baths to contain nitrate of silver, which must have been carefully fused*. The collodion employed should be iodized a month before use with the cadmium iodizer, and, before starting, carefully decanted into clean vessels, tied over with bladder to prevent evaporation.

I would recommend that three instruments, having a focal length of about 10 feet, should be prepared and kept, with the portable observatory, in readiness to be placed at the disposal of any such expedition as that organized by the Astronomer Royal. These instruments would give pictures of the luminous prominences in four seconds at the outside; and under favourable atmospheric conditions, in less than a second.

Observers intending to take charge of an instrument should practise with it in taking lunar photographs, previous to starting, so as to familiarize themselves completely with it. Not fewer than four persons should accompany each telescope, and two of them ought to be accomplished photographers.

Pictures of the partial phases would be best obtained with such an instrument as the Kew heliograph.

Lastly, I have deemed it to be desirable that positive copies of the eclipse-pictures should be placed in some Institution readily accessible to the public, and I have therefore presented to the South Kensington Museum a series of enlarged positive copies on glass, 9 inches in diameter, which are at present exhibited in the International Exhibition, Class XIII.

* "Report on Celestial Photography," by the author, in the Report of the British Association for 1859.

XIX. *On a New Method of Approximation applicable to Elliptic and Ultra-elliptic Functions.*—Second Memoir*. *By* CHARLES W. MERRIFIELD. *Communicated by* W. SPOTTISWOODE, *Esq.*, *F.R.S.*

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SINCE my first memoir on this subject was read before the Society, Mr. SYLVESTER has published a method, more general than mine, of applying rational approximation to facilitate the computation of the integrals of irrational functions. This method, at which he had arrived independently, included, *à majori*, the one which was the subject of my memoir. Aided by his papers, my subsequent studies have enabled me to view the method with more generality, as well as with more precision and completeness of detail, and I am now able to present it in a sufficiently finished and practical form for the immediate use of the computer. I have also computed auxiliary Tables, to render its application easier in certain cases.

Any rational formula, which gives approximately the value of a function to be integrated, may be integrated in lieu of it, and the result will in general be an approximate value of the integral sought. But for such a process to be of any practical utility, the convergence of the formula must be excessive, for the complexity of the integral forms is so great that the labour would be enormous, unless the terms were very few in number. In the discovery of formulæ sufficiently convergent for the purpose, lies the success of the method.

We are by no means restricted to functions under a square root, or even to pure radical forms at all. The principle applies with equal generality to functions which are given implicitly as roots of equations, and thus to a class of differential equations; and Mr. SYLVESTER has well remarked that these formulæ not only afford facilities for computation, as by a method of quadratures, but also enable us to assign superior and inferior limits to an integral, without losing its generality of form.

I shall begin with the approximation to the square root, giving it in its general form, and explaining its exact analytical signification. I shall then show its application to Elliptic Functions, and how, in the ordinary cases, certain simple reductions can be effected, which greatly lessen the labour of computation; and I shall give these reductions for the cases more commonly occurring, with some examples and working formulæ. I shall then add a short account of the extension of the method.

The paragraphs in the first two sections of this paper bear a consecutive number for convenience of reference.

* For the First Memoir, see the Philosophical Transactions for 1860, p. 223.

SECTION I.—*Approximants to the Square Root.*

1. Mr. SYLVESTER gives, for the approximants to the square root, the following statement:—

"Let r be an approximate value of $\sqrt{N}$; then by that mode of application of NEWTON'S method of approximation to the equation $x^2=N$, which is equivalent to the use of continued fractions, we may easily establish the following theorem, viz., that

$$\frac{r^2+N}{2r},\quad \frac{r^3+3rN}{3r^2+N},\quad \frac{r^4+6r^2N+N^2}{4r^3+4rN},\quad \frac{r^5+10r^3N+5rN^2}{5r^4+10r^2N+N^2},$$

will be successive approximations to $\sqrt{N}$."

2. Their general form is

$$y=\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i}\sqrt{N}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

which is always rational. In this form the approximation to $\sqrt{N}$ as i increases is obvious. The method of my previous memoir is simply the particular case of $i=2^k$.

3. If we wish to approximate to $N^{-\frac{1}{2}}$, we may take the reciprocal of (1.), or, what is simpler, we may divide (1.) by N, thus obtaining

$$z=\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i}\,\frac{1}{\sqrt{N}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Before we can integrate these formulæ, we must reduce them by means of the method of rational fractions; the simplest and most general way is as follows:—

4. Let ϱ be an ith root of unity; then, obviously,

$$\log(1-x^i)=\log(1-\varrho x)+\log(1-\varrho^2x)+\ldots\ldots+\log(1-\varrho^ix).$$

Multiplying the differential coefficient of this by $(-x)$, we obtain

$$\frac{ix^i}{1-x^i}=\frac{\varrho x}{1-\varrho x}+\frac{\varrho^2x}{1-\varrho^2x}+\frac{\varrho^3x}{1-\varrho^3x}+\ldots\ldots+\frac{\varrho^ix}{1-\varrho^ix};$$

and since $\frac{1}{1-x^i}=1+\frac{x^i}{1-x^i}$ and $\frac{1+x^i}{1-x^i}=1+\frac{2x^i}{1-x^i}$,

$$\frac{i}{1-x^i}=\frac{1}{1-\varrho x}+\frac{1}{1-\varrho^2x}+\frac{1}{1-\varrho^3x}+\ldots\ldots+\frac{1}{1-\varrho^ix},$$

$$i\frac{1+x^i}{1-x^i}=\frac{1+\varrho x}{1-\varrho x}+\frac{1+\varrho^2x}{1-\varrho^2x}+\frac{1+\varrho^3x}{1-\varrho^3x}+\ldots\ldots+\frac{1+\varrho^ix}{1-\varrho^ix}.$$

Making $x=\frac{r-\sqrt{N}}{r+\sqrt{N}}$, we may thus divide $N^{\pm\frac{1}{2}}$ into i fractions, each of the form

$$\frac{1}{i}\,\frac{(r+\sqrt{N})+\varrho^k(r-\sqrt{N})}{(r+\sqrt{N})-\varrho^k(r-\sqrt{N})}\,N^{\pm\frac{1}{2}},$$

k being any integer not exceeding i.

5. If we add the pairs k and $i-k$, we obtain for the sum of the pair,

$$\frac{2(r+\sqrt{N})^2-2(r-\sqrt{N})^2}{(r+\sqrt{N})^2+(r-\sqrt{N})^2-(\varrho^k+\varrho^{i-k})(r^2-N)}\,N^{\pm\frac{1}{2}}$$

$$=\frac{8rN}{2(r^2+N)-(\varrho^k+\varrho^{i-k})(r^2-N)} \text{ or } \frac{8r}{\text{same denominator}},$$

according to whether the upper or lower sign be taken. Now, because ϱ is an ith root of unity, $\varrho^k+\varrho^{i-k}=2\cos\frac{2k\pi}{i}$, and the sum of the pair reduces itself, for $\sqrt{N}$, to

$$\left.\begin{aligned} y_k&=\frac{1}{i}\frac{4rN}{(r^2+N)-\cos\frac{2k\pi}{i}(r^2-N)}=\frac{1}{i}\frac{2rN}{r^2\sin^2\frac{k\pi}{i}+N\cos^2\frac{k\pi}{i}} \\ &=\frac{1}{i}\frac{2rN}{N+\sin^2\frac{k\pi}{i}(r^2-N)}=\frac{1}{i}\frac{2rN}{r^2-\cos^2\frac{k\pi}{i}(r^2-N)}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

For $N^{-\frac{1}{2}}$ we have the simpler forms,

$$\left.\begin{aligned} z_k&=\frac{1}{i}\frac{4r}{(r^2+N)-\cos\frac{2k\pi}{i}(r^2-N)}=\frac{1}{i}\frac{2r}{r^2\sin^2\frac{k\pi}{i}+N\cos^2\frac{k\pi}{i}} \\ &=\frac{1}{i}\frac{2r}{N+\sin^2\frac{k\pi}{i}(r^2-N)}=\frac{1}{i}\frac{2r}{r^2-\cos^2\frac{k\pi}{i}(r^2-N)}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

All that remains is to integrate these terms, and sum them.

6. Our grouping the terms in pairs has limited the value of k to range from i to $\frac{1}{2}(i-1)$ when i is odd. There is an odd term which, however, presents no difficulty, being simply $\frac{r}{i}$ in the case of $\sqrt{N}$, and $\frac{r}{iN}$ in the case of $N^{-\frac{1}{2}}$. When i is even, k is limited to range from 1 to $\frac{1}{2}i-1$, and the odd term becomes $\frac{r^2+N}{ir}$ in the case of $\sqrt{N}$, and $\frac{r^2+N}{iNr}$ in the case of $N^{-\frac{1}{2}}$. It is important to bear in mind that the term just mentioned is an odd term, and therefore not affected with the coefficient 2, which appears in the terms composed of pairs corresponding to imaginary roots.

7. The value of i, which I consider to be most useful for general purposes, is $i=8$: in this case the odd term becomes $\frac{r^2+N}{8r}$ or $\frac{r^2+N}{8Nr}$, and the other values of $\frac{k\pi}{i}$ are three in number, viz. 22° 30′, 45°, 67° 30′. With proper precautions $i=8$ will almost always give seven or more figures correct.

8. If we now give infinite values to k and i and pass from the summation to the definite integral, we have $\left(\text{putting } \lambda=\frac{k}{i}\right)$

$$z=N^{-\frac{1}{2}}=\int_0^{\frac{1}{2}\lambda}\frac{2r\,d\lambda}{r^2+(N-r^2)\cos^2\lambda\pi}=\frac{2}{\pi}\frac{r}{N}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{1+\frac{r^2-N}{N}\sin^2\phi};$$

and since

$$\int_0^{\frac{1}{2}\pi}\frac{d\phi}{1+p\,.\sin^2\phi}=\frac{\frac{1}{2}\pi}{\sqrt{1+p}},$$

this is an identical equation, as it ought to be.

9. This use of approximants, therefore, is simply the application of the method of

quadratures to a definite integral, which we substitute for the surd proposed for evaluation.

10. It would appear at first sight that a full application of the method of quadratures in the ordinary way, with the help of differences, would give better results than the mere summation of the ordinates. But this is not the case; for the differences diverge immediately. If we use differential coefficients for the quadrature, instead of differences, we have an opposite anomaly, namely that the correction of the summation appears to be absolutely *nil*, inasmuch as the differential coefficients which appear in the series are all of odd order, and the numerator of each of them contains the factor $\sin\phi\cos\phi$, which vanishes at both the limits 0 and $\frac{1}{2}\pi$. LEGENDRE has discussed this point. See the Appendix to the second volume of his 'Fonctions Elliptiques,' p. 578.

11. The application of the method to integrations, then, lies in the substitution for

$$\int_0^t \frac{M}{\sqrt{N}}\,dt \text{ of } \int_0^t\int_0^{\frac{1}{2}} \frac{2Mr\,.\,d\lambda\,.\,dt}{r^2+(N-r^2)\cos^2\lambda\pi},$$

in which, since λ and t are perfectly independent of each other, we may change the order of integration, thus obtaining

$$\int_0^{\frac{1}{2}}\left\{\int_0^t \frac{2Mr\,.\,dt}{r^2+(N-r^2)\cos^2\lambda\pi}\right\}d\lambda\ ;$$

and the rest of the operation depends upon our being able to perform the integration in } generally, and then to determine the integral in λ by quadratures. The great advantage of the method turns upon the easy application of the method of quadratures, in consequence of our not requiring to *difference* the ordinates.

12. One way of exhibiting generally the degree of convergence is as follows: $N^{\pm\frac{1}{2}}$ always lies between

$$N^{\pm\frac{1}{2}}\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i} \text{ and } N^{\pm\frac{1}{2}}\frac{(r+\sqrt{N})^i-(r-\sqrt{N})^i}{(r+\sqrt{N})^i+(r-\sqrt{N})^i},$$

and the error of either is therefore always less than their difference,

$$N^{\pm\frac{1}{2}}\frac{4(r^2-N)^i}{(r+\sqrt{N})^{2i}-(r-\sqrt{N})^{2i}}.$$

13. There is another mode, by which, in any given case, we may see how far it is necessary to carry our work in order to obtain a given number of decimals correctly in the result. Let θ_m be determined by the equation

$$\int_0^{\theta_m}\frac{d\theta_m}{\cos\theta_m}=m\int_0^{\theta_1}\frac{d\theta_1}{\cos\theta_1},$$

and let $\sin\theta_1=\frac{\sqrt{N}}{r}$ or $\frac{r}{\sqrt{N}}$, whichever may be less than unity; then the mth approximant will be $\frac{\sqrt{N}}{\sin\theta_m}$. This is easily seen from the general term of the approximant, since

$$\int\frac{d\theta}{\cos\theta}=\frac{1}{2}\log_e\left(\frac{1+\sin\theta}{1-\sin\theta}\right).$$

14. A table of meridional parts, such as is given in the books on Navigation, if carried far enough, would solve this equation. I have calculated an auxiliary Table for the purpose, as follows:—

Let $\operatorname{cosec}\phi-1=z$, $\log_e\tan\left(\frac{1}{4}\pi+\frac{1}{2}\phi\right)=y$, then

$$y=\frac{1}{2}\log_e\frac{2+z}{z}=\frac{1}{2}\log 2-\frac{1}{2}\log z+\frac{1}{2}\log\left(1+\frac{1}{2}z\right)$$

$$=\frac{1}{2}\log 2-\frac{1}{2}\log z+\frac{z}{4}-\frac{z^2}{16}+\frac{z^3}{48}-\frac{z^4}{128}+\ldots\ldots\ldots$$

To bring this formula to the same unit as the common Table of meridional parts, we must multiply it by the number of minutes in the arc equal to unity, or by $L=3437{\cdot}74677\ 07849\ 4$, whence we have $\frac{1}{2}L\log_e 2=1191{\cdot}43224\ 08243\ 2$, and $\frac{1}{2}L\log_e 10=3958{\cdot}85223\ 39129\ 100$. These data give the following Table, the argument being the common logarithm of z with its sign changed; that is, the number of places which are correct:

$-\log z$.	y.	$-\log z$.	y.	$-\log z$.	y.
1	5234·14859	6	24944·54650	11	44738·80681
2	9117·70966	7	28903·39796	12	48697·65905
3	13068·84816	8	32862·25012	13	52656·51128
4	17026·92712	9	36821·10235	14	56615·36352
5	20985·70200	10	40779·95458	15	60574·21575

15. As a simple example, let $N=3$, $r=2$; $\therefore\frac{\sqrt{N}}{r}=\sin 60°$: the meridional parts for $60°=4527$; and in order that the error may not exceed unity in the tenth place of figures, we must have m or $i=\frac{40780}{4527}=9$; so that we must make $i=9$ at least, for the 10th figure to be correct.

16. These methods of course only exhibit the degree of approximation on the surd itself. The proportionate approximation is generally greater on the integral than on the simple surd, because the first approximant is usually so chosen as to be identical with the surd at one of the limits, and it is only near the other limit that the discrepancy tells.

SECTION II.—*Details of Reduction and Computation.*

17. The chief assistance, which can be provided *à priori* for the computer, consists in the exhibition and discussion, for the ordinary forms, of the integral $\int_0^t\frac{2Mrdt}{r^2+(N-r^2)\cos^2\lambda\pi}$ and of the auxiliary functions which present themselves in its reduction.

18. In applying these methods to elliptic integrals, the radical and the first approximant r must both be of a simple form, and it is advisable that r^2-N or $N-r^2$ should be

of a square form. For the common form of the elliptic radical $\sqrt{(1-\sin^2\theta.\sin^2\phi)}$, our choice is practically limited to

(1) $r=1$, (2) $r=\cos\phi$, (3) $r=\sin\theta.\cos\phi$,
(4) $r=\cos\theta$, (5) $r=\cos\theta.\sin\phi$.

And on these suppositions I now proceed to the integration of the general form of the reduced approximant for $\int_0^\phi(1-\sin^2\theta.\sin^2\phi)^{-\frac{1}{2}}d\phi=\int z d\phi$. I omit mention of the constants of integration, because very slight changes in the function may alter them. The first of our three cases require, as they stand, no constant, and these are the most usefu' cases.

(1) $r=1,\ r^2-\mathrm{N}=\sin^2\theta.\sin^2\phi,$

$$z_k=\frac{2}{1-\sin^2\theta.\cos^2\frac{k\pi}{i}.\sin^2\phi},$$

$$\int z_k d\phi=2\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\tan^{-1}\left\{\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}\tan\phi\right\}.$$

(2) $r=\cos\phi,\ r^2-\mathrm{N}=-\cos^2\theta.\sin^2\phi,$

$$z_k=\frac{2\cos\phi}{\cos^2\phi+\cos^2\theta.\cos^2\frac{k\pi}{i}.\sin^2\phi}=\frac{2\cos\phi}{1-\left(1-\cos^2\theta.\cos^2\frac{k\pi}{i}\right).\sin^2\phi},$$

$$\int_k z d\phi=\left(1-\cos^2\theta.\cos^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\log_{\epsilon}\left\{\frac{1+\sin\phi\left(1-\cos^2\theta.\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}{1-\sin\phi\left(1-\cos^2\theta.\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}\right\}.$$

(3) $r=\sin\theta.\cos\phi,\ r^2-\mathrm{N}=-\cos^2\theta,$

$$z_k=\frac{2\sin\theta.\cos\phi}{\left(1-\cos^2\theta.\sin^2\frac{k\pi}{i}\right)-\sin^2\theta.\sin^2\phi},$$

$$\int z_k d\phi=\left(1-\cos^2\theta.\sin^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\log\left\{\frac{\left(1-\cos^2\theta.\sin^2\frac{k\pi}{i}\right)^{\frac{1}{2}}+\sin\theta.\sin\phi}{\left(1-\cos^2\theta.\sin^2\frac{k\pi}{i}\right)^{\frac{1}{2}}-\sin\theta.\sin\phi}\right\}.$$

(4) $r=\cos\theta,\ r^2-\mathrm{N}=-\sin^2\theta\cos^2\phi,$

$$z_k=\frac{2}{\cos\theta}\frac{1}{1+\tan^2\theta.\cos^2\frac{k\pi}{i}.\cos^2\phi},$$

$$\int z_k d\phi=2\left(1-\sin^2\theta.\sin^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\tan^{-1}\left\{\cos\theta.\tan\phi\left(1-\sin^2\theta.\sin^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\right\}.$$

(5) $r=\cos\theta\sin\phi,\ r^2-\mathrm{N}=-\cos^2\phi,$

$$z_k=\frac{2\cos\theta.\sin\phi}{\cos^2\theta.\sin^2\phi+\cos^2\frac{k\pi}{i}.\cos^2\phi},$$

$$\int z_k d\phi = 2\left(\cos^2\frac{k\pi}{i} - \cos^2\theta\right)^{-\frac{1}{2}} \tan^{-1}\left\{\frac{\cos\theta}{\cos\phi}\left(\cos^2\frac{k\pi}{i} - \cos^2\theta\right)^{-\frac{1}{2}}\right\}$$

$$= \left(\cos^2\theta - \cos^2\theta\frac{k\pi}{i}\right)^{-\frac{1}{2}} \log_e \left\{\frac{1 - \frac{\cos\phi}{\cos\theta}\left(\cos^2\theta - \cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}{1 + \frac{\cos\phi}{\cos\theta}\left(\cos^2\theta - \cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}}\right\}.$$

(6) If we make $t = \tan\frac{1}{2}\phi$, we obtain

$$(1 - \sin^2\theta . \sin^2\phi)^{-\frac{1}{2}} d\phi = 2(1 - 2\cos 2\theta . t^2 + t^4)^{-\frac{1}{2}} dt.$$

Taking $r = 1 - t^2$, the terms which we have to integrate are of the form

$$\int \frac{4(1-t^2)\,dt}{(1-t^2)^2 + 4\cos^2\theta . \cos^2\frac{k\pi}{i} . t^2}.$$

Putting $q^2 = 1 - \cos^2\theta . \cos^2\frac{k\pi}{i}$, we have

$$\int z_k d\phi = \frac{1}{q}\log_e\left(\frac{t^2 + 2qt + 1}{t^2 - 2qt + 1}\right).$$

The same expression serves for the integral

$$\int \frac{2dt}{\sqrt{(1 + 2\cos 2\theta . t^2 + t^4)}},$$

if we put $q^2 = 1 - \sin^2\theta . \cos^2\frac{k\pi}{i}$.

19. It will be observed that the first four cases, and the sixth, depend upon a radical of the form $\sqrt{(1 - \sin^2 A . \sin^2\omega)}$, where ω is restricted to the selected values of $\frac{k\pi}{i}$. Assuming the modulus sin A not to vary, it would therefore in general be better to begin by computing the radical for the selected values. I have computed, and I append to this paper, a Table of this radical, the selected values of $\frac{k\pi}{i}$ being 22° 30′, 45°, and 67° 30′, while A ranges by whole degrees from 1° to 90° inclusive. Every entry but the last in the 2nd, 3rd, and 4th columns of the Table was computed by myself in duplicate with VEGA's ten-figure logarithms, by the help of two or more of the following formulæ, some of which are from LEGENDRE.

20. Putting Δ for $\sqrt{(1 - \sin^2 A . \sin^2\omega)}$,

(1) Make $\sin A . \sin\omega = \sin M$; then $\Delta = \cos M$,

$\log\sin M = \log\sin A + \log\sin\omega$, $\log\Delta = \log\cos M$; or else

(2) Make $\tan A . \cos\omega = \tan M$; then $\Delta = \cos A . \sec M$,

$\log\tan M = \log\tan A + \log\cos\omega$, $\log\Delta = \log\cos A + \text{ar. co.} \log\cos M$.

Moreover, let L be the tabular angle nearest to the angle M: it is not necessary to obtain the value of M: so that we have simultaneously,

$\log\sin M = \log\sin L \pm s$,

$\log\tan M = \log\tan L \pm t$,

$\log\cos M = \log\cos L \mp c$;

then we shall also have, and with great approximation,

$$\begin{aligned}\log s&=\log(t\,.\cos^2 \mathrm{L})\mp(t-t\,.\cos^2 \mathrm{L})\\&=\log(c\,.\cot^2 \mathrm{L})\mp(c+c\,.\cot^2 \mathrm{L}),\\ \log c&=\log(s\,.\tan^2 \mathrm{L})\pm(s+s\,.\tan^2 \mathrm{L})\\&=\log(t\,.\sin^2 \mathrm{L})\pm(t-t\,.\sin^2 \mathrm{L}),\\ \log t&=\log(s\,.\sec^2 \mathrm{L})\mp(s-s\,.\sec^2 \mathrm{L})\\&=\log(c\,.\operatorname{cosec}^2 \mathrm{L})\mp(c-c\operatorname{cosec}^2 \mathrm{L}).\end{aligned}$$

I have given the whole set of six, but my Table was computed with the pair for log c. By way of example, I add a specimen copy of one of my working sheets. The use of so many as ten figures is not altogether unnecessary, because otherwise, when Δ is nearly equal to unity, the value of $\log(1-\Delta)$ or of $\log\frac{1-\Delta}{1+\Delta}$ cannot be had with exactness.

21. The following formulæ will also be found in many cases preferable, both for exactness and facility, to the ordinary use of logarithmic tables by means of differences. These formulæ, as well as those of the previous paragraph, are but applications of TAYLOR'S theorem, reduced to a shape fit for the computer. Even where only seven figures are required their application is frequently much easier, and gives more exact results, than interpolation by differences. In what follows, x is supposed to be the nearest tabular entry.

22. *To find* $\log y$ *from* $\log\tan y$.—Let us assume simultaneously

$$\log y=\log x\pm l,\quad \log\tan y=\log\tan x\pm t.$$

Putting $u=\log x$, $z=\log\tan x$, we have

$$\frac{du}{dz}=\frac{\sin 2x}{2x}\ \text{and}\ \frac{d^2u}{dz^2}=\mathrm{M}\frac{\sin 2x}{2x}\left(\cos 2x-\frac{\sin 2x}{2x}\right),$$

M being the modulus of the logarithms.

Hence, by TAYLOR'S theorem,

$$l=t\,\frac{\sin 2x}{2x}\left\{1\mp\mathrm{M}t\left(\frac{\sin 2x}{2x}-\cos 2x\right)\right\}\ \text{nearly.}$$

Taking the logarithm, this becomes

$$\begin{aligned}\log l&=\log\left(t\,\frac{\sin 2x}{2x}\right)\mp t\left(\frac{\sin 2x}{2x}-\cos 2x\right)\\&=\log\left(\frac{t\,.\sin x\,.\cos x}{x}\right)\mp\frac{t\,.\sin x\,.\cos x}{x}\pm t\mp 2t\sin^2 x.\end{aligned}$$

The latter is the better shape for a working formula, because $\log\sin x$ and $\log\cos x$ are found in the same page and line as $\log\tan x$, while $\log\sin 2x$ must be looked for elsewhere. The first term alone is sufficient when x is small; but when x much exceeds 45°, $\cos 2x$ changes its sign, and even the entire formula is insufficient. The maximum value of the coefficient of t in the second term is 1·0631, corresponding to $x=78°\ 33'\ 26''\cdot5$. In many cases, where the first term alone is insufficient, a rough interpolation, made at sight from the following Table, will answer the purpose; it is a Table of the value of $\left(\frac{\sin 2x}{2x}-\cos 2x\right)$ and of its logarithm, from $x=45°$ to $x=90°$.

45°	0·63662	9·80387	70°	1·02910	0·01246
50	0·73816	9·86815	75	1·05658	0·02390
55	0·83149	9·91986	80	1·06216	0·02619
60	0·91349	9·96070	85	1·04334	0·01843
65	0·98041	9·99141	90	1·	0·

The Table shows that, past 45°, the formula

$$\log l=\log\left(\frac{t\sin x\cos x}{x}\right)\mp t$$

is a better approximation than when the $\mp t$ is omitted. It is to be remarked that t is at its minimum for $x=45°$, and increases both towards $x=0$ and $x=90°$. Near the latter limit, where great accuracy is required, we must proceed as follows.

Find the correction for the logarithm of the complement of the arc by the above process, and then find $\log(\frac{1}{2}\pi-y)$ from $\log y$. For this purpose, I observe that $\log y=\log x\pm l$ is equivalent to $y=x.10^{\pm l}$, hence

$$\tfrac{1}{2}\pi-y=\tfrac{1}{2}\pi-x.10^{\pm l}=(\tfrac{1}{2}\pi-x)-x(10^{\pm l}-1).$$

Now, let $\pm\text{A}=10^{\pm l}-1$, whence

$$\log(\pm m\text{A})=\log(\pm l)\pm\tfrac{1}{2}l-\frac{1}{12}\text{M}l^2 \text{ nearly, and also}$$

$$\log(\tfrac{1}{2}\pi-y)=\log(\tfrac{1}{2}\pi-x)-\left(\frac{\pm m\text{A}x}{\frac{1}{2}\pi-x}\right)-\tfrac{1}{2}\text{M}\left(\frac{\pm m\text{A}x}{\frac{1}{2}\pi-x}\right)^2-\dots$$

It is not often that the third term of either formula will be required.

I have gone into all this detail, because the inverse tangent is continually presenting itself in all these integrations, and because no book that I know shows the proper way of handling it.

23. The following constants are needed for these and similar formulæ:—

$10+\log m=9·63778\ 43113\ 00537$, $\log\text{M}=0·36221\ 56886\ 99463$,

$10+\log 1°=8·24187\ 73675\ 90828$, $-\log 1°=1·75812\ 26324\ 09172$,

$10+\log 1'=6·46372\ 61172\ 07184$, $-\log 1'=3·53627\ 38827\ 92816$,

$10+\log 1''=4·68557\ 48668\ 23541$, $-\log 1''=5·31442\ 51331\ 76459$.

24. As an example of finding the inverse tangent, let it be required to find $\log y$ and $\log(\frac{1}{2}\pi-y)$ from

$\log\tan y=9·02313\ 50437$. Here we must take

$\log\tan x=9·02303\ 57359$

$+t=\quad 9\ 93078$; $\therefore\ x=6°\,1'\,10''=21670''$

$\frac{1}{2}\pi-x=302330''$

$\log \sin x = 9{\cdot}0206346$	$\log \sin^2 x = 8{\cdot}0413$	$\log 21670 = 4{\cdot}33585\ 89113$
$\log \cos x = 9{\cdot}9975988$	$\log 2 = 0{\cdot}3010$	$\log 1'' = 4{\cdot}68557\ 48668$
$\log t = 5{\cdot}9969834$	$\log t = 5{\cdot}9970$	$\log x = 9{\cdot}02143\ 37781$
ar. co. $\log x = 0{\cdot}9785662$	$\log (2 \sin^2 x) = 4{\cdot}3393$	$l = \quad +9\ 85786$
$\log \frac{t \sin 2x}{2x} = 5{\cdot}9937830$	$2 \sin^2 x = \quad 21,800$	$\log y = 9{\cdot}02153\ 23567$
2nd correction $\quad -14$	$\frac{t \sin 2x}{2x} = \quad 985,8$	
$\log l = 5{\cdot}9937816$	$1007,6$	
$\frac{1}{2}l \quad +493$	$-t = -993,1$	$\log 302330 = 5{\cdot}48048\ 12441$
$\log mA = 5{\cdot}9938309$	2nd correction $= \quad 14,5$	$\log 1'' = 4{\cdot}68557\ 48668$
$\log x = 9{\cdot}0214338$	The comma cuts off the eighth decimal.	$\log (\frac{1}{2}\pi - x) = 0{\cdot}16605\ 61109$
ar. co. $\log (\frac{1}{2}\pi - x) = 9{\cdot}8339439$		correction $= \quad -70666$
log correction $= 4{\cdot}8492086$		$\log (\frac{1}{2}\pi - y) = 0{\cdot}16604\ 90443$

Verification.—The numbers corresponding to these logarithms of y and of $\frac{1}{2}\pi - y$ are 0·10508 29743 and 1·46571 33525, the sum of which, to the very last figure, is exactly $\frac{1}{2}\pi$.

25. *To find* $\log \frac{y+1}{y-1}$ *from* $\log y$.—Let $\log \frac{y+1}{y-1} = \log \frac{x+1}{x-1} \mp p$, and $\log y = \log x \pm q$; then $\log p = \log \left(\frac{2qx}{x^2-1}\right) \mp \frac{1}{2}q \mp \frac{q}{x^2-1}$, nearly. This formula obviously fails where y is near unity; in this case $\log \frac{y+1}{y-1}$ cannot be had with great accuracy, unless y itself be given absolutely. All the cases of $\int \frac{dy}{y^2 - a^2}$ may be included in the above formula by giving proper signs to p and q. It may save trouble to remark that x must not always be taken to the extreme limit of the Table, because $\log (x+1)$ and $\log (x-1)$ have also to be taken out. As an example, let

$\log y = 0{\cdot}36290\ 63835$		
$\log x = 0{\cdot}36285\ 93030$		$x = 2{\cdot}306,\ x+1 = 3{\cdot}306,\ x-1 = 1{\cdot}306$
$q = \quad +4\ 70805$		$\log (x+1) = 0{\cdot}51930\ 28492$
$\log q = 5{\cdot}6728411$		$\log (x-1) = 0{\cdot}11594\ 31769$
ar. co. $\log (x^2-1) = 9{\cdot}3647540$		sum $= 0{\cdot}63524\ 60261$
$5{\cdot}0375951$	$\frac{q}{x^2-1} = 0{\cdot}0001090$	difference $= 0{\cdot}40335\ 96723$
$\log x = 0{\cdot}3628593$	$\frac{1}{2}q = \quad 135$	$-p = \quad -5\ 02761$
$\log 2 = 0{\cdot}3010300$	2nd corr^n^ $= 0{\cdot}0001225$	$\log \frac{y+1}{y-1} = 0{\cdot}40330\ 93962$
$5{\cdot}7014844$		
2nd correction $= \quad -1225$		
$\log p = 5{\cdot}7013619$		

This example has been so chosen as to admit of easy verification. In fact $y=2\cdot30625$, and $\log\frac{y+1}{y-1}=\log\frac{529}{209}=0\cdot40330\ 93959\ 24$. The error is therefore only of three units in the tenth decimal place, where there was no reason to expect accuracy.

26. The only other formulæ which I shall give are the following, for finding the logarithm of a number, and *vice versâ*. They are indispensable where more than seven figures are required.

Let $\log(x\pm h)=\log x\pm k$, then

$$\log k=\log\left(\frac{mh}{x}\right)\mp\frac{1}{2}\frac{mh}{x}\text{ nearly,}$$

$$\log h=\log(\mathrm{M}xk)\pm\frac{1}{2}k\text{ nearly.}$$

The values of $\log m$ and $\log \mathrm{M}$ have been given in paragraph 23.

27. As an example of the application of the method to the evaluation of elliptic integrals of the third class, let us take the integral

$$\int_0^{\phi}\frac{d\phi}{(1-\sin^2\alpha.\sin^2\phi)(1-\sin^2\theta.\sin^2\phi)^{\frac{1}{2}}}$$

for the values $\alpha=45°$, $\theta=30°$, $\phi=60°$.

I have selected these values because they can be obtained without reduction or interpolation from the Table of $\Delta(\theta,\phi)$ which I have given, and also because $\sin^2\alpha=\sin\theta$, and therefore the integral can be reduced to one of the first class, *plus* an inverse tangent, thus admitting of easy verification. For this case

$$z_k=\frac{2}{(1-\sin^2\alpha.\sin^2\phi)\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}.\sin^2\phi\right)},$$

$$\int z_k\,d\phi=\frac{2\sin^2\alpha}{\sin^2\alpha-\sin^2\theta.\cos^2\frac{k\pi}{i}}\cdot\frac{1}{\cos\alpha}\tan^{-1}(\cos\alpha.\tan\phi)$$

$$-\frac{2\sin^2\theta\cos^2\frac{k\pi}{i}}{\sin^2\alpha-\sin^2\theta.\cos^2\frac{k\pi}{i}}\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}\right)^{-\frac{1}{2}}\tan^{-1}\left\{\left(1-\sin^2\theta.\cos^2\frac{k\pi}{i}\right)^{\frac{1}{2}}\tan\phi\right\}.$$

Making $\frac{k\pi}{i}$ successively 22° 30′, 45°, 67° 30′, and, for the odd term, 90°, we find, after a few obvious reductions, that eight times the value of the integral is

$$\left\{\frac{17}{3}+\frac{2}{\Delta^2(45°,67\tfrac{1}{2})}+\frac{2}{\Delta^2(45°,22\tfrac{1}{2})}\right\}\frac{1}{\cos 45°}\tan^{-1}\{\cos 45°.\tan 60°\}$$

$$-\frac{1}{\cos 30°}\tan^{-1}\{\cos 30°.\tan 60°\}-\frac{2}{3}\frac{1}{\Delta(30°,45°)}\tan^{-1}\{\Delta(30°,45°).\tan 60°\}$$

$$-\frac{\tan^{-1}\{\Delta(30°,67\tfrac{1}{2}).\tan 60°\}.\cos^2 22\tfrac{1}{2}}{\Delta^2(45°,67\tfrac{1}{2}).\Delta(30°,67\tfrac{1}{2})}-\frac{\tan^{-1}\{\Delta(30°,22\tfrac{1}{2}).\tan 60°\}.\cos^2 67\tfrac{1}{2}}{\Delta^2(45°,22\tfrac{1}{2}).\Delta(30°,22\tfrac{1}{2})}.$$

As these inverse tangents range generally from 45° to 60°, I computed them by the shortened formula of paragraph 22, namely $\log\left(\frac{t.\sin x.\cos x}{x}\right) \pm t$; this being sufficient to give eight figures of decimals accurately. I found

$$\log \tan^{-1}\{\cos 45° \tan 60°\} = 9{\cdot}94747\ 15296,$$
$$\log \tan^{-1}\{\cos 30° \tan 60°\} = 9{\cdot}99246\ 23739,$$
$$\log \tan^{-1}\{\Delta(30°, 45°) \tan 60°\} = 0{\cdot}00766\ 92607,$$
$$\log \tan^{-1}\{\Delta(30°, 67\tfrac{1}{2}).\tan 60°\} = 9{\cdot}99727\ 33807,$$
$$\log \tan^{-1}\{\Delta(30°, 22\tfrac{1}{2}).\tan 60°\} = 0{\cdot}01665\ 09657.$$

I hence obtained the following values:—

For the positive terms.	For the negative terms.
7·10091 3039	1·13483 2441
4·37212 6152	0·72539 4027
2·70421 6251	1·66839 9478
14·17725 5442	0·16728 4032
3·69590 9978	3·69590 9978
8)10·48134 5464	
1·31016 8183 value required	

A more exact value of the integral, otherwise obtained, is

$$\tfrac{1}{2}F(30°, 60°) + \tan^{-1}\left(\frac{4 \sin 60°}{\sqrt{13}}\right) = 1{\cdot}31016\ 8161,$$

which differs from the previous value by 2 units in the eighth decimal place.

28. In order to find how many places ought to have been accurately obtained, I observe that the method followed gives $N = \frac{13}{16}$, $r=1$, whence

$$\log\left(\frac{1}{r}\sqrt{N}\right) = 9{\cdot}95491 = \log \sin 64°\ 20'\ 30''.$$

The corresponding meridional parts are 5086·5, which must be multiplied by $i=8$, giving 40692·0. Referring to the Table in paragraph 14, I find that this nearly corresponds to *ten* places correct, and therefore that the integral ought to be correct to at least that extent. That it is not so, is due to my having curtailed the formula for finding the logarithms of the inverse tangents. But my object was only to give seven decimals correct, and my going beyond that was simply because, with a ten-figure Table, putting down the additional figures gave me less trouble (once I had to use more than seven) than abbreviation would have done. This remark may at first sight seem strange to any one who has not had some practice in using large Tables. But the logarithmic corrections are given in the shape of arithmetical complements: with reference to the 10th figure, therefore, considered as an integer, the index is right as it stands, and we need not bestow thought on the proper placing of the correction, as we must if we use any other number of figures.

29. If we had been content with five decimals, the calculation would have been very easy, for in that case we might have used six-figure logarithms, and have made $i=4$, thus omitting the terms containing $22^\circ\frac{1}{2}$ and $67^\circ\frac{1}{2}$. We should get

7·10091 3039	1·13483 2441
1·86022 6468	0·72539 4027
4)5·24068 6571	1·86022 6468
1·31017 1643 value required.	

30. It is worth while to notice a case which will sometimes occur, namely (using the notation of the last example), that the values may be so selected as to give, for one of the values of k, $\sin\alpha=\sin\theta.\cos\frac{k\pi}{i}$, and thus each of the terms into which $\int z_k d\phi$ was divided would become infinite. Of course the difficulty is only apparent; for in this case the proper value is $\int z_k d\phi=\int\frac{2d\phi}{(1-\sin^2\alpha.\sin^2\phi)^2}$, of which the integral may be at once found by differentiating the expression $\frac{\sin\phi.\cos\phi}{1-\sin^2\alpha.\sin^2\phi}$.

SECTION III.—*Extension of the Method.*

In respect of rapid approximation and precision of limit, the foregoing processes leave nothing to be desired, as far as concerns the radical of the square root; but they do not go beyond that. Mr. SYLVESTER has given an elegant extension of the method to radicals of a higher index, by means of symmetric functions*.

The more general problem before us is that of approximating to the integrals of irrational functions by means of rational substitutions.

Let ϕ and ψ be functional symbols, and y a function of z; then, that $\phi(z).y_m$ and $\phi(z):y_m$ should both be approximations to $\phi(z)$, depends upon y_m approaching unity as m increases. Assuming that y_m and y_1 are connected by the equation $y_m=\psi(m, y_1)$, our problem is to choose ψ so that, in the first place, the approximation shall be exceedingly rapid, and, in the next place, that $\phi(z).y_m$ and $\phi(z):y_m$ shall both (or at least one of them) be thoroughly manageable, and easily integrable. In the case of the approximants already given, the equation $y_m=\psi(m,y_1)$ has been made $\int\frac{dy_m}{1-y_m^2}=m\int\frac{dy_1}{1-y_1^2}$.

I am acquainted with three general methods which effect the object more or less. The first is the obvious one afforded by the Newtonian approximation to the roots of an equation; viz., let a be a first approximate solution, obtained by trial, of the equation $fx=0$, and call $f'x$ the differential coefficient of fx; then a second approximation is

* See the Philosophical Magazine for December 1860, Supplementary Number, vol. xx. p. 525, note A.

$a-\frac{fa}{f'a}=b$; a third approximation will evidently be $b-\frac{fb}{f'b}=c$, and so forth. If we apply this method to the pure equation $x^n=p$, the convergent terms which we obtain are as follows:—

$$b=\frac{(n-1)a^n+p}{na^{n-1}},$$

$$c=\frac{(n-1)\{(n-1)a^n+p\}^n+n^n.p.a^{n(n-1)}}{n^2a^{n-1}\{(n-1)a^n+p\}^{n-1}}, \text{ \&c.}$$

The second method is that of the reversion of series; it is sufficiently discussed by ARBOGAST*.

The third method was suggested to me by Mr. CAYLEY's remark that Mr. SYLVESTER's third approximation is a particular case, for $n=2$, of the common form (of the books on the binomial theorem) $\sqrt[n]{\mathrm{N}}=\frac{(n+1)\mathrm{N}+(n-1)a^n}{(n-1)\mathrm{N}+(n+1)a^n}\cdot a$, approximately, a being a first approximation. In order to gain generality, and thereby symmetry, I shall pass from the particular form $\sqrt[n]{\mathrm{N}}$ to the more general $\varphi^{-1}\mathrm{N}$ by the following Lemma:—

Let $$\mathrm{N}=a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots, \qquad (1.)$$

and let $x_1, x_2, x_3, x_4 \ldots$ be determined by the system of equations,

$$\left.\begin{aligned}\mathrm{N}&=a_0\left(1+\frac{a_1}{a_0}x_1\right)\\&=a_0\left(1+\frac{a_1}{a_0}x_2\left(1+\frac{a_2}{a_1}x_1\right)\right)\\&=a_0\left(1+\frac{a_1}{a_0}x_3\left(1+\frac{a_2}{a_1}x_2\left(1+\frac{a_3}{a_2}x_1\right)\right)\right),\end{aligned}\right\} \qquad (2.)$$

and so forth; also let $\mathrm{N}-a_0=\mu$, and

$$1=(\mu-a_1x-a_2x^2-a_3x^3-\ldots)(\lambda_0+\lambda_1x+\lambda_2x^2+\lambda_3x^3+\ldots), \qquad (3.)$$

then $$x_1=\frac{\lambda_0}{\lambda_1},\quad x_2=\frac{\lambda_1}{\lambda_2},\quad x_3=\frac{\lambda_2}{\lambda_3}\ldots\, x_n=\frac{\lambda_{n-1}}{\lambda_n}.$$

For, if we substitute these values in the equations (2.) after placing them in the following form,

$$\left.\begin{aligned}\mu=\mathrm{N}-a_0&=a_1x_1\\&=a_1x_2+a_2x_2x_1\\&=a_1x_3+a_2x_3x_2+a_3x_3x_2x_1,\end{aligned}\right\} \qquad (4.)$$

and so forth, we obtain

$$\mu=\frac{a_1\lambda_0}{\lambda_1}=\frac{a_1\lambda_1+a_2\lambda_0}{\lambda_2}=\frac{a_1\lambda_2+a_2\lambda_1+a_3\lambda_0}{\lambda_3}=\text{\&c.}, \qquad (5.)$$

which are the same equations as we should get by multiplying the two series in (3.) and equating to zero the coefficients of x and of its powers. The coefficient $\lambda_0=\frac{1}{\mu}$, obviously.

* Calcul des Dérivations, pp. 288–296.

The coefficients λ may now be found in a variety of ways; by solving equations (4.) or (5.), by simple division, or by ARBOGAST'S processes *. The object of the preceding lemma is to connect the quantities x_n with the coefficients of division and of recurring series. Our results in any way are,

$$x_1=\frac{\mu}{a_1},\ x_2=\frac{a_1\mu}{a_2\mu+a_1^2},$$

$$x_3=\frac{a_2\mu^2+a_1^2\mu}{a_3\mu^2+2a_2a_1\mu+a_1^3},$$

$$x_4=\frac{a_3\mu^3+2a_2a_1\mu^2+a_1^3\mu}{a_4\mu^3+(2a_3a_1+a_2^2)\mu^2+3a_2a_1^2\mu+a_1^4}.$$

If for a_0, a_1, &c. we substitute the coefficients of the binomial theorem, so as to make $N=(a+x)^n$, we obtain

$$a+x_1=\frac{(n-1)a^n+N}{na^n}\cdot a,$$

$$a+x_2=\frac{(n-1)a^n+(n+1)N}{(n+1)a^n+(n-1)N}\cdot a,$$

$$a+x_3=\frac{(n^2-1)a^{2n}+(4n^2+2)a^nN+(n^2-1)N^2}{(n+1)(n+2)a^{2n}+4(n^2-1)a^nN+(n-1)(n-2)N^2}\cdot a.$$

Making $n=2$, we obtain Mr. SYLVESTER'S approximants to the square root, and λ_n is then the coefficient of x^n in the development by ascending powers of

$$\frac{1}{(N-a^2)-2ax-x^2};$$

and so far the method agrees with the Newtonian approximation by continued fractions; but from this point the two methods diverge. For $n=3$, λ_n is the coefficient of x^n in the development of

$$\frac{1}{(N-a^3)-3a^2x-3ax^2-x^3};$$

and the successive approximants are

$$\frac{2a^3+N}{3a^3}\cdot a,\ \frac{a^3+2N}{2a^3+N}\cdot a,\ \frac{4a^6+19a^3N+4N^2}{10a^6+16a^3N+N^2}\cdot a,\ \frac{5a^9+45a^6N+30a^3N^2+N^3}{15a^9+51a^6N+15a^3N^2}\cdot a,\ \&c.;$$

while the second approximant obtained by successive substitution is

$$\frac{16a^9+51a^6N+12a^6N^2+2N^3}{36a^9+36a^6N+9a^3N^2}\cdot a.$$

What these methods all effect is simply a rational approximation to the value of y in the equation $\phi(y,z)=0$. Then, making $y=\frac{du}{dz}$, we have only to integrate in order to find the value of u. They thus constitute a means of approximately solving, in respect of u, differential equations of the form $\phi\left(z,\frac{du}{dz}\right)=0$; but they do not effect the solu-

* See his 'Calcul des Dérivations,' pp. 26, 29; or DE MORGAN, 'Diff. Calc.' p. 331.

tion of this equation in respect of z, and still less do they solve the more general form $\phi\left(u, z, \frac{du}{dz}\right)=0$.

It may suggest processes of reduction in some cases to remark, that there are many other functions of y_m and $\phi(z)$, which will approximate to $\phi(z)$ as m increases, besides the simple product or quotient of $\phi(z)$ by y_m.

There is one point about these higher approximants, of which a solution, even if accompanied with considerable restrictions, would be extremely desirable,—I mean the resolution of the denominators into factors. I do not suppose that the problem, in its perfectly general form, admits of a compact solution; but any class of cases, of even moderate generality, for which it could be elegantly solved, would probably have very useful applications. The criterion of convergence and the measure of approximation would also have their interest.

Specimen sheet of work for the Table. Arc of 60°.

log sin 22½ =	9·58283	96605	83*	log tan² L =	9·0913166	log cos L =	9·97473	27132
log sin A =	9·93753	06316	96	log s =	5·3160144	c =	+	25545
log sin M =	9·52037	02922	79		4·4073310	log Δ =	9·97473	52677
log sin L =	9·52039	09944		s + s tan² L =	−233			
s =	−2	07021						
s tan² L =		25546		log c =	4·4073077			

log sin 45° =	9·84948	50021	68*	log tan² L =	9·7781471	log cos L =	9·89794	07883
log sin A =	9·93753	06316	96	log s =	4·1137763	c =	−	7797
log sin M =	9·78701	56338	64		3·8919234	log Δ =	9·89794	00086
log sin L =	9·78701	43344		s + s tan² L =	+ 21			
s =	+	12995						
s tan² L =		7821		log c =	3·8919255			

log sin 67½ =	9·96561	53459	21*	log tan² L =	10·2501550	log cos L =	9·77806	24352
log sin A =	9·93753	06316	96	log s =	4·7816118	c =	− 1	07593
log sin M =	9·90314	59776	17		5·0317668	log Δ =	9·77805	16759
log sin L =	9·90313	99296		s + s tan² L =	+168			
s =	+	60480						
s tan² L =	1	07589		log c =	5·0317836			

log tan 22½ =	9·61722	43146	62*	log sin² L =	8·6140815	log cos 22½ =	9·96561	53459 21*
log cos A =	9·69897	00043	36	log t =	5·5451559	log sec L =	0·00911	84784
log tan M =	9·31619	43190	08		4·1591374		9·97473	38243
log tan L =	9·31615	92213		t − t sin² L =	+327	c =	+	14430
t =	+ 3	50877				log Δ =	9·97473	52673
t sin² L =		14018		log c =	4·1591701			

log tan 45 =	10·00000	00000	00*	log sin² L =	9·3009948	log cos 45 =	9·84948	50021 68*
log cos A =				log t =	5·3429218	log sec L =	0·04845	06017
log tan M =	9·69897	00043	36		4·6439166		9·89793	56039
log tan L =	9·69894	79790		t − t sin² L =	+176	c =	+	44049
t =	+2	20253				log Δ =	9·89794	00082
t sin² L =		44047		log c =	4·6439342			

log tan 67½ =	10·38277	56853	38*	log sin² L =	9·7730722	log cos 67½ =	9·58283	96605 83*
log cos A =	9·69897	00043	36	log t =	4·7719327	log sec L =	0·19521	55228
log tan M =	10·08174	56896	74		4·5450049		9·77805	51834
log tan L =	10·08175	16044		t − t sin² L =	− 24	c =	−	35075
t =	−	59147				log Δ =	9·77805	16759
t sin² L =		35075		log c =	4·5450025			

Note.—The entries marked *, and the whole of the letter-press, were printed on the sheets. The letter A on this page corresponds to θ in the Table.

Table of the value of the function log $\Delta(\theta, \omega)$ or log $\sqrt{(1-\sin^2\theta \, . \, \sin^2\omega)}$ for four values of ω, viz. 22° 30′, 45°, 67° 30′, and 90°.

θ.	log Δ (θ, 22° 30′).	log Δ (θ, 45°).	log Δ (θ, 67° 30′).	log cos θ.	θ.
1°	9·99999 03138	9·99996 69274	9·99994 35386	9·99993 38497	1°
2	9·99996 12643	9·99986 77197	9·99977 41349	9·99973 53589	2
3	9·99991 28792	9·99970 24074	9·99949 17311	9·99940 44063	3
4	9·99984 52048	9·99947 10407	9·99909 62308	9·99894 07898	4
5	9·99975 83051	9·99917 36910	9·99858 74990	9·99834 42260	5
6	9·99965 22633	9·99881 04507	9·99796 53618	9·99761 43489	6
7	9·99952 71805	9·99838 14325	9·99722 96070	9·99675 07098	7
8	9·99938 31764	9·99788 67716	9·99637 99831	9·99575 27754	8
9	9·99922 03891	9·99732 66254	9·99541 62004	9·99461 99270	9
10	9·99903 89748	9·99670 11740	9·99433 79300	9·99335 14589	10
11	9·99883 91084	9·99601 06211	9·99313 48042	9·99194 65764	11
12	9·99862 09825	9·99525 51957	9·99183 64168	9·99040 43940	12
13	9·99838 48090	9·99443 51515	9·99041 23213	9·98872 39328	13
14	9·99813 08170	9·99355 07689	9·98887 20337	9·98690 41185	14
15	9·99785 92542	9·99260 23560	9·98721 50300	9·98494 37781	15
16	9·99757 03868	9·99159 02501	9·98544 07479	9·98284 16370	16
17	9·99726 44986	9·99051 48183	9·98354 85857	9·98059 63156	17
18	9·99694 18915	9·98937 64594	9·98153 79027	9·97820 63255	18
19	9·99660 28857	9·98817 56060	9·97940 80200	9·97567 00654	19
20	9·99624 78189	9·98691 27246	9·97715 82198	9·97298 58164	20
21	9·99587 70469	9·98558 83197	9·97478 77462	9·97015 17377	21
22	9·99549 09429	9·98420 29331	9·97229 58056	9·96716 58605	22
23	9·99508 98979	9·98275 71478	9·96968 15661	9·96402 60827	23
24	9·99467 43204	9·98125 15899	9·96694 41603	9·96073 01625	24
25	9·99424 46358	9·97968 69297	9·96408 26837	9·95727 57115	25
26	9·99380 12870	9·97806 38852	9·96109 61968	9·95366 01869	26
27	9·99334 47337	9·97638 32245	9·95798 37258	9·94988 08840	27
28	9·99287 54524	9·97464 57677	9·95474 42643	9·94593 49269	28
29	9·99239 39363	9·97285 23905	9·95137 67741	9·94181 92587	29
30	9·99190 06948	9·97100 40265	9·94788 01877	9·93753 06317	30
31	9·99139 62526	9·96910 16705	9·94425 34100	9·93306 55951	31
32	9·99088 11517	9·96714 63813	9·94049 53211	9·92842 04835	32
33	9·99035 59484	9·96513 92852	9·93660 47788	9·92359 14023	33
34	9·98982 12144	9·96308 15797	9·93258 06231	9·91857 42135	34
35	9·98927 75363	9·96097 45359	9·92842 16784	9·91336 45194	35
36	9·98872 55150	9·95881 95031	9·92412 67607	9·90795 76446	36
37	9·98816 57654	9·95661 79121	9·91969 46797	9·90234 86165	37
38	9·98759 89157	9·95437 12781	9·91512 42488	9·89653 21441	38
39	9·98702 56072	9·95208 12065	9·91041 42888	9·89050 25944	39
40	9·98644 64943	9·94974 93945	9·90556 36388	9·88425 39665	40
41	9·98586 22425	9·94737 76364	9·90057 11640	9·87777 98629	41
42	9·98527 35294	9·94496 78273	9·89543 57674	9·87107 34581	42
43	9·98468 10429	9·94252 19663	9·89015 64024	9·86412 74638	43
44	9·98408 54812	9·94004 21611	9·88473 20869	9·85693 40901	44
45	9·98348 75524	9·93753 06317	9·87916 19193	9·84948 50022	45

TABLE

(*continued*).

θ.	log Δ (θ, 22° 30′).	log Δ (θ, 45°).	log Δ (θ, 67° 30′).	log cos θ.	θ.
46	9·98288 79722	9·93498 97136	9·87344 50980	9·84177 12731	46
47	9·98228 74657	9·93242 18620	9·86758 09423	9·83378 33303	47
48	9·98168 67640	9·92982 96543	9·86156 89166	9·82551 08951	48
49	9·98108 66053	9·92721 57944	9·85540 86587	9·81694 29168	49
50	9·98048 77326	9·92458 31150	9·84910 00105	9·80806 74967	50
51	9·97989 08943	9·92194 45794	9·84264 30543	9·79887 18039	51
52	9·97929 68415	9·91927 32846	9·83603 81532	9·78934 19787	52
53	9·97870 63285	9·91660 24627	9·82928 59976	9·77946 30249	53
54	9·97812 01111	9·91392 54820	9·82238 76555	9·76921 86852	54
55	9·97753 89457	9·91124 58470	9·81534 46333	9·75859 13013	55
56	9·97696 35878	9·90856 71996	9·80815 89399	9·74756 16513	56
57	9·97639 47918	9·90589 33160	9·80083 31625	9·73610 87645	57
58	9·97583 33090	9·90322 81075	9·79337 05490	9·72420 97077	58
59	9·97527 98869	9·90057 56154	9·78577 51006	9·71183 93361	59
60	9·97473 52675	9·89794 00084	9·77805 16759	9·69897 00043	60
61	9·97420 01874	9·89532 55788	9·77020 61055	9·68557 12291	61
62	9·97367 53742	9·89273 67330	9·76224 53190	9·67160 92909	62
63	9·97316 15478	9·89017 79885	9·75417 74828	9·65704 67649	63
64	9·97265 94174	9·88765 39622	9·74601 21530	9·64184 19615	64
65	9·97216 96810	9·88516 93618	9·73776 04387	9·62594 82593	65
66	9·97169 30239	9·88272 89739	9·72943 51756	9·60931 32999	66
67	9·97123 01175	9·88033 76506	9·72105 11125	9·59187 80116	67
68	9·97078 16179	9·87800 02961	9·71262 51046	9·57357 54170	68
69	9·97034 81644	9·87572 18497	9·70417 63081	9·55432 91617	69
70	9·96993 03790	9·87350 72689	9·69572 63771	9·53405 16846	70
71	9·96952 88643	9·87136 15099	9·68729 96519	9·51264 19176	71
72	9·96914 42028	9·86928 95088	9·67892 33280	9·48998 23640	72
73	9·96877 69551	9·86729 61579	9·67062 76041	9·46593 53400	73
74	9·96842 76594	9·86538 62846	9·66244 57824	9·44033 80750	74
75	9·96809 68303	9·86356 46269	9·65441 43168	9·41299 62305	75
76	9·96778 49569	9·86183 58088	9·64657 27859	9·38367 51767	76
77	9·96749 25025	9·86020 43157	9·63896 37732	9·35208 80330	77
78	9·96721 99032	9·85867 44678	9·63163 26324	9·31787 89102	78
79	9·96696 75670	9·85725 03958	9·62462 71226	9·28059 88450	79
80	9·96673 58731	9·85593 60134	9·61799 68925	9·23967 02300	80
81	9·96652 51705	9·85473 49940	9·61179 28070	9·19433 24413	81
82	9·96633 57778	9·85365 07456	9·60606 61106	9·14355 53039	82
83	9·96616 79819	9·85268 63874	9·60086 74357	9·08589 44712	83
84	9·96602 20377	9·85184 47281	9·59624 56795	9·01923 45656	84
85	9·96589 81672	9·85112 82461	9·59224 67793	8·94029 60083	85
86	9·96579 65594	9·85053 90708	9·58891 24439	8·84358 45184	86
87	9·96571 73697	9·85007 89667	9·58627 88989	8·71880 01636	87
88	9·96566 07187	9·84974 93212	9·58437 57166	8·54281 91639	88
89	9·96562 66934	9·84955 11322	9·58322 48116	8·24185 53184	89
90	9·96561 53459	9·84948 50022	9·58283 96696	—log. infin.	90

XXX. *On Simultaneous Differential Equations of the First Order in which the Number of the Variables exceeds by more than one the Number of the Equations.* By GEORGE BOOLE, *F.R.S., Professor of Mathematics in Queen's College, Cork.*

Received June 19,—Read June 19, 1862.

It is a fundamental proposition of analysis that a system of n differential equations of the first order containing $n+1$ variables admits of n integrals, each of which is expressed by a function of the variables equated to an arbitrary constant.

But when a system of n differential equations of the first order connects $n+r$ variables, r being greater than unity, no existing theory assigns in a general manner the number of theoretically possible integrals of the above species, or shows us how to discover them. Yet such cases are of great importance.

I wish to develope here the theory of a method for the solution of the above classes of equations, which was published by me in the 'Proceedings of the Royal Society' for March 6th of the present year, and which enables us to assign the number of theoretically possible integrals, and to reduce their discovery to the solution of a system of simultaneous differential equations equal in number to the number of integrals, and expressible as exact differential equations.

The solution of the problem as thus reduced may be effected by known methods, but I have thought it desirable to discuss this part of the subject also in direct sequence to the other, and in conformity with its method.

Of the Connexion between ordinary and partial Differential Equations.

It has been found convenient, in researches bearing upon the general theory of differential equations, to use the term 'integral' in two distinct senses, viz. to denote, as above, a relation satisfying the differential equation or system of equations, and expressed by the equating of a function of the variables to a constant, and to denote the function itself. The particular sense intended will always be shown by the connexion.

With this convention two systems of differential equations will be said to be equivalent when they have in either of the above senses (and the one implies the other) the same system of integrals. This will explain the meaning of the following proposition.

PROPOSITION I.—*A system of n ordinary differential equations of the first order connecting $n+r$ variables may be converted into an equivalent system of r linear partial differential equations of the first order.*

Let $x_1, x_2, \dots x_{n+r}$ be the variables, then the supposed given system of differential equations may, by algebraic solution with respect to $dx_1, dx_2, \ldots dx_n$, be reduced to the form

$$\left.\begin{aligned} dx_1 &= A_{11}dx_{n+1}+A_{12}dx_{n+2}\ldots+A_{1r}dx_{n+r}, \\ dx_2 &= A_{21}dx_{n+1}+A_{22}dx_{n+2}\ldots+A_{2r}dx_{n+r}, \\ &\ldots\ldots\ldots\ldots \\ dx_n &= A_{n1}dx_{n+1}+A_{n2}dx_{n+2}\ldots+A_{nr}dx_{n+r}, \end{aligned}\right\} \quad \ldots\ldots \quad \text{(I.)}$$

the coefficients A_{11}, A_{12}, &c. being functions of the variables.

Let $P=c$ be an integral of the system. Then

$$\frac{dP}{dx_1}dx_1+\frac{dP}{dx_2}dx_2\ldots+\frac{dP}{dx_{n+r}}dx_{n+r}=0.$$

Substituting in this equation the values of $dx_1, dx_2, \ldots dx_n$ given by (I.), we have

$$\begin{aligned} &\left(\frac{dP}{dx_{n+1}}+A_{11}\frac{dP}{dx_1}+A_{21}\frac{dP}{dx_2}\ldots+A_{n1}\frac{dP}{dx_n}\right)dx_{n+1} \\ +&\left(\frac{dP}{dx_{n+2}}+A_{12}\frac{dP}{dx_1}+A_{22}\frac{dP}{dx_2}\ldots+A_{n2}\frac{dP}{dx_n}\right)dx_{n+2} \\ &\ldots\ldots\ldots\ldots \\ +&\left(\frac{dP}{dx_{n+r}}+A_{1r}\frac{dP}{dx_1}+A_{2r}\frac{dP}{dx_2}\ldots+A_{nr}\frac{dP}{dx_n}\right)dx_{n+r}=0. \end{aligned}$$

As the differentials $dx_{n+1}, dx_{n+2}, \ldots dx_{n+r}$ are now independent, we have, on equating their coefficients separately to 0,

$$\left.\begin{aligned} &\frac{dP}{dx_{n+1}}+A_{11}\frac{dP}{dx_1}+A_{21}\frac{dP}{dx_2}\ldots+A_{n1}\frac{dP}{dx_n}=0, \\ &\frac{dP}{dx_{n+2}}+A_{12}\frac{dP}{dx_1}+A_{22}\frac{dP}{dx_2}\ldots+A_{n2}\frac{dP}{dx_n}=0, \\ &\ldots\ldots\ldots\ldots \\ &\frac{dP}{dx_{n+r}}+A_{1r}\frac{dP}{dx_1}+A_{2r}\frac{dP}{dx_2}\ldots+A_{nr}\frac{dP}{dx_n}=0, \end{aligned}\right\} \quad \ldots\ldots \quad \text{(II.)}$$

a system of r linear partial differential equations, the common integrals of which will be the integrals of the system (I.). We say 'the *common* integrals of which,' because in fact these equations express the conditions to all of which P must be subject in order that $P=c$ may be an integral of the system (I.).

The formal connexion between the systems (I.) and (II.) deserves to be carefully noticed. The several partial differential equations of the system (II.) may be formed by inspection from the columns in the right-hand member of the system (I.) by the following rule. For the differential dx_{n+i} in any column write the differential coefficient $\frac{dP}{dx_{n+i}}$, to this add the series of differential coefficients $\frac{dP}{dx_1}, \frac{dP}{dx_2}, \ldots \frac{dP}{dx_n}$ multiplied in succession by the descending coefficients of the column, and equate the final result to 0.

The symmetrical form of the equation

$$\frac{dP}{dx_1}dx_1+\frac{dP}{dx_2}dx_2\ldots+\frac{dP}{dx_{n+r}}dx_{n+r}=0$$

shows that by an exactly similar rule any system of n partial differential equations of the first order, the terms of which consist of the differential coefficients of P multiplied by functions of the independent variables $x_1, x_2, \dots x_{n+r}$, may be converted into an equivalent system of r common differential equations of the first order.

For the proposed system of partial differential equations is by algebraic reduction expressible in the form

$$\left.\begin{aligned} \frac{dP}{dx_1}&=A_{11}\frac{dP}{dx_{n+1}}+A_{12}\frac{dP}{dx_{n+2}}\dots+A_{1r}\frac{dP}{dx_{n+r}},\\ \frac{dP}{dx_2}&=A_{21}\frac{dP}{dx_{n+1}}+A_{22}\frac{dP}{dx_{n+2}}\dots+A_{2r}\frac{dP}{dx_{n+r}},\\ \frac{dP}{dx_n}&=A_{n1}\frac{dP}{dx_{n+1}}+A_{n2}\frac{dP}{dx_{n+2}}\dots+A_{nr}\frac{dP}{dx_{n+r}}.\end{aligned}\right\}\quad\dots\dots\dots\quad\text{(III.)}$$

If the values of $\frac{dP}{dx_1}, \frac{dP}{dx_2}, \dots \frac{dP}{dx_n}$ in this system be substituted in the previous general equation, and the coefficients of the differential coefficients $\frac{dP}{dx_{n+1}}, \frac{dP}{dx_{n+2}}, \dots \frac{dP}{dx_{n+r}}$ in the result be separately equated to 0, we shall have

$$\left.\begin{aligned} dx_{n+1}+A_{11}\,dx_1+A_{21}\,dx_2\dots+A_{n1}\,dx_n=0,\\ dx_{n+2}+A_{12}\,dx_1+A_{22}\,dx_2\dots+A_{n2}\,dx_n=0,\\ dx_{n+r}+A_{1r}\,dx_1+A_{2r}\,dx_2\dots+A_{nr}\,dx_n=0.\end{aligned}\right\}\quad\dots\dots\dots\quad\text{(IV.)}$$

These equations may in like manner be formed by inspection from the columns of the second member of (III.), by writing for $\frac{dP}{dx_{n+i}}$ in any column dx_{n+i}, adding to this $dx_1, dx_2, \dots dx_n$ multiplied by the descending coefficients of the column, and equating the final sum to 0. The rule for the one case differs from that for the other only in that differentials take the place of differential coefficients.

As an objection may be felt to the legitimacy of that step of the above process in which, the differential coefficients $\frac{dP}{dx_1}, \frac{dP}{dx_2}, \dots \frac{dP}{dx_n}$ being eliminated, the coefficients of the remaining ones are separately equated to 0, I will point out another mode of procedure which leads to the same result, and which is founded upon LAGRANGE's method of solution. Let the equations of the system (III.) be added together after having been multiplied respectively by $\lambda_1, \lambda_2, \dots \lambda_n$, which are to be regarded as indeterminate functions of the variables $x_1, x_2, \dots x_{n+r}$. The result will be a linear partial differential equation of the first order, of which the Lagrangian auxiliary system of ordinary differential equations will be

$$\frac{dx_1}{\lambda_1}=\frac{dx_2}{\lambda_2}\dots=\frac{dx_n}{\lambda_n}=\frac{-dx_{n+1}}{A_{11}\lambda_1\dots+A_{n1}\lambda_n}\dots=\frac{-dx_{n+r}}{A_{1r}\lambda_1\dots+A_{nr}\lambda_n}.$$

Hence, eliminating $\lambda_1, \lambda_2, \dots \lambda_n$, or more strictly speaking the ratios

$$\frac{\lambda_1}{\lambda_n}, \frac{\lambda_2}{\lambda_n}, \dots \frac{\lambda_{n-1}}{\lambda_n},$$

we have

$$A_{11}\ dx_1 \ldots + A_{n1}\ dx_n = -dx_{n+1},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$A_{1r}\ dx_1 \ldots + A_{nr}\ dx_n = -dx_{n+r},$$

which agrees with the system (IV.).

Of the Determination of the Number of Integrals of a system of Differential Equations of the First Order and Degree.

We still suppose the given system of differential equations to be expressed under the general form (I.), and reduced by Prop. I. to the equivalent partial differential system (II.).

Now if for the expression of that system we introduce a series of symbols $\Delta_1, \Delta_2, \ldots \Delta_r$ defined as follows, viz.

$$\Delta_i = \frac{d}{dx_{n+i}} + A_{1i}\frac{d}{dx_1} + A_{2i}\frac{d}{dx_2} \cdots + A_{ni}\frac{d}{dx_n}, \quad \ldots \ldots \quad (1.)$$

the system will assume the form

$$\Delta_1 P = 0,\ \Delta_2 P = 0, \ldots\ \Delta_r P = 0, \quad \ldots \ldots \quad (2.)$$

and we shall now establish the following proposition

PROPOSITION II.—*If* $\Delta_i P = 0$, $\Delta_j P = 0$ *represent any two linear partial differential equations of the system* (II.), *then will the equation of which the symbolical expression is*

$$(\Delta_i \Delta_j - \Delta_j \Delta_i) P = 0 \quad \ldots \ldots \quad (3.)$$

also be a linear partial differential equation of the first order, and it will be satisfied by all the common integrals of the two equations from which it is formed.

For, representing any one of the quantities $x_1, x_2, \ldots x_n$ by x, and any function of those quantities by X, Δ_i consists of a series of terms of the form $X\frac{d}{dx}$. Again, representing any one of the same series of quantities $x_1, x_2, \ldots x_n$ by y, and any function of them by Y, Δ_j will consist of terms of the form $Y\frac{d}{dy}$. Hence $(\Delta_1\Delta_2 - \Delta_2\Delta_1)P$ will consist of terms of the form

$$\left(X\frac{d}{dx}\,Y\frac{d}{dy} - Y\frac{d}{dy}\,X\frac{d}{dx}\right)P.$$

Effecting the differentiations, this term becomes

$$X\frac{dY}{dx}\,\frac{dP}{dy} + XY\frac{d^2P}{dxdy} - Y\frac{dX}{dy}\,\frac{dP}{dx} - YX\frac{d^2P}{dxdy},$$

or

$$X\frac{dY}{dx}\,\frac{dP}{dy} - Y\frac{dX}{dy}\,\frac{dP}{dx},$$

which involves only the first differential coefficients of P. Hence (3.) will be a linear partial differential equation of the first order.

Hence also the ultimate form of (3.) will be the same as if Δ_i, when operating on $\Delta_j P$, operated only on the coefficients $A_{1j} \ldots A_{nj}$ involved in Δ_j, and *vice versâ*. Thus the ultimate form of (3.) will be

$$(\Delta_i A_{1j}-\Delta_j A_{1i})\frac{dP}{dx_1}+(\Delta_i A_{2j}-\Delta_j A_{2i})\frac{dP}{dx_2}\ldots\ldots+(\Delta_i A_{nj}-\Delta_j A_{ni})\frac{dP}{dx_n}=0. \quad . \quad . \quad . \quad (4.)$$

Secondly, the equation $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$ will be satisfied by any common integral of $\Delta_iP=0$ and $\Delta_jP=0$.

For let $\phi=c$ be a common integral of the latter equations. Then, identically,

$$\Delta_i\phi=0, \quad \Delta_j\phi=0;$$

therefore, since Δ_j and Δ_i involve only operations of differentiation together with algebraic ones,

$$\Delta_j\Delta_i\phi=0, \quad \Delta_i\Delta_j\phi=0;$$

$$\therefore \quad \Delta_i\Delta_j\phi-\Delta_j\Delta_i\phi=0;$$

whence (3.) is also identically satisfied.

Proposition III.—*If, by the successive application of Prop. II., and by permitted processes of algebraic elimination, we derive from the system of partial differential equations*

$$\Delta_1P=0, \ \Delta_2P=0, \ldots\ldots \Delta_rP=0,$$

into which the given system of differential equations has been converted, a final system of partial differential equations which, while including the above system, shall be such that the application of Prop. II. to any pair of the equations contained shall lead only to an identity, then the number of integrals of the given system of differential equations will be equal to the number of variables they contain, diminished by the number of partial differential equations of the above final system.

The developed form of the system

$$\Delta_1P=0, \ \Delta_2P=0, \ \ldots \ \Delta_rP=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

is the following:—

$$\left.\begin{array}{l}\dfrac{dP}{dx_{n+1}}+A_{11}\dfrac{dP}{dx_1}\ldots+A_{n1}\dfrac{dP}{dx_n}=0,\\[2ex] \dfrac{dP}{dx_{n+2}}+A_{12}\dfrac{dP}{dx_1}\ldots+A_{n2}\dfrac{dP}{dx_n}=0,\\[2ex] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\\[2ex] \dfrac{dP}{dx_{n+r}}+A_{1r}\dfrac{dP}{dx_1}\ldots+A_{nr}\dfrac{dP}{dx_n}=0.\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Comparing these with (4.), Prop. II., which is the developed form of the equation $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$, we see that the latter equation is necessarily algebraically independent of the above system; for no equation derived from that system by algebraic processes could be free, as (4.) is, from all the differential coefficients $\frac{dP}{dx_{n+1}}\cdots\frac{dP}{dx_{n+r}}$.

Again, as $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$ is satisfied by all the common integrals of $\Delta_iP=0$ and $\Delta_jP=0$, it follows that the system of $r+1$ equations,

$$\Delta_1P=0, \ldots \Delta_rP=0, \ (\Delta_i\Delta_j-\Delta_j\Delta_i)P=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

will be satisfied by all the common integrals of the system (1.). To this system of $r+1$

equations we can also give a form analogous to the developed form of the system (2.). It will be noticed that the differential coefficients $\frac{dP}{dx_{n+1}} \ldots \frac{dP}{dx_{n+r}}$ appear there, each in only one equation, and each with the coefficient unity. Now let the last equation of (3.) in its developed form be divided by the coefficient of $\frac{dP}{dx_n}$, and let also the value of $\frac{dP}{dx_n}$ which it gives be substituted in the other equations of (3.); then we shall have in the whole a system of $n+1$ equations possessing the same general character as the system (2.). To this new system the same procedure may be applied, viz. the genesis of a new equation by means of Prop. II., and the transference of another differential coefficient $\frac{dP}{dx_{n-1}}$ to the list of those which form the respective *first* terms of the equations of the system. We will suppose this procedure to have been repeated until a system composed of m partial differential equations *such that the further application of Prop.* II. *leads only to identities* has been formed. If $n+r-m=p$, that system will be of the form

$$\left.\begin{aligned} &\frac{dP}{dx_{p+1}}+H_{11}\frac{dP}{dx_1}\ldots+H_{p1}\frac{dP}{dx_p}=0,\\ &\frac{dP}{dx_{p+2}}+H_{12}\frac{dP}{dx_1}\ldots+H_{p2}\frac{dP}{dx_p}=0,\\ &\ldots\ldots\ldots\ldots\\ &\frac{dP}{dx_{p+m}}+H_{1m}\frac{dP}{dx_1}\ldots+H_{pm}\frac{dP}{dx_p}=0.\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4.)$$

And if, in analogy with former notation, we write

$$\frac{d}{dx_{p+i}}+H_{1i}\frac{d}{dx_1}+H_{2i}\frac{d}{dx_2}\ldots+H_{pi}\frac{d}{dx_p}=\Delta_i,$$

it will take the symbolical form

$$\Delta_1P=0,\ \Delta_2P=0,\ldots\ \Delta_mP=0;$$

but it will differ from all former systems of equations in that all the conditions represented by

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0 \quad \ldots\ldots\ldots\ldots \quad (5.)$$

will be identically satisfied,—satisfied, in consequence not of any ascertained peculiarity of the integral P, but of the constitution of the system of symbols $\Delta_1, \Delta_2, \ldots\Delta_m$.

The course of argument has shown that the common integrals of the system (4.) will be identical with those of the parent system (2.). Now we shall show that the existence of the condition (5.) renders the integration theoretically possible; that the system of p ordinary differential equations into which, by Prop. II., the final system of partial differential equations (4.) is resolvable admits of exactly p integrals. As $p=n+r-m$, this is to say that the number of integrals is equal to the number of original variables diminished by the number of final partial differential equations.

The proof of this will consist of two parts:—1st. It will be shown that, if a system of p integrals exists, the conditions represented by (8.) will be identically satisfied.

2ndly. It will be shown that, when the conditions represented by (5.) are identically satisfied, the solution either of the final system of partial differential equations (4.), or of the corresponding system of ordinary differential equations, by a system of p integrals is theoretically possible.

It will follow from these conjoined, that the number of actually existing integrals is exactly p.

1st. The system of ordinary differential equations corresponding to (4.) may be expressed in the form

$$\left.\begin{array}{l} dx_1 = H_{11}dx_{p+1} + H_{12}dx_{p+2} \ldots + H_{1m}dx_{p+m}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ dx_p = H_{p1}dx_{p+1} + H_{p2}dx_{p+2} \ldots + H_{pm}dx_{p+m}. \end{array}\right\} \quad . \ . \ . \ . \ . \ . \quad (6.)$$

Now suppose this system to have p integrals. Then, by means of these, $x_1, x_2, \ldots x_p$ can be eliminated from the coefficients H_{11}, &c. in the second members, which will thus become exact differentials of those functions of the variables $x_{p+1} \ldots x_{p+m}$ which express the values of $x_1, \ldots x_p$. Hence we shall have the system of conditions

$$\frac{d}{dx_{p+i}}H_{kj} = \frac{d}{dx_{p+j}}H_{ki}, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (7.)$$

k representing any integer from 1 to p, and i, j any integers from 1 to m, and the bracketed symbols of differentiation referring to H_{kj}, H_{ki} *as transformed*. Hence, the unbracketed symbols referring to the prior state of the functions, we have

$$\begin{aligned}\left(\frac{d}{dx_{p+i}}\right) &= \frac{d}{dx_{p+i}} + \frac{dx_1}{dx_{p+i}}\frac{d}{dx_1} \ldots + \frac{dx_p}{dx_{p+i}}\frac{d}{dx_p} \\ &= \frac{d}{dx_{p+i}} + H_{1i}\frac{d}{dx_1} \ldots + H_{pi}\frac{d}{dx_p} \\ &= \Delta_i.\end{aligned}$$

In the same way

$$\left(\frac{d}{dx_{p+j}}\right) = \Delta_j;$$

so that (7.) becomes

$$\Delta_i H_{kj} - \Delta_j H_{ki} = 0. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8.)$$

Now if we construct, in analogy with (4.), Prop. II., the developed form of the conditional equation (5.) of the present section, we shall have

$$(\Delta_i H_{1j} - \Delta_j H_{1i})\frac{dP}{dx_1} \ldots + (\Delta_i H_{pj} - \Delta_g H_{pi})\frac{dP}{dx_p} = 0,$$

or, Σ denoting summation from $k=1$ to $k=p$,

$$\Sigma(\Delta_i H_{kj} - \Delta_j H_{ki})\frac{dp}{dx_k} = 0.$$

Now as by (8.) the coefficients vanish identically, the equation, and generally the system of conditional equations of which it is the type, will be identically satisfied.

2ndly. We proceed to show that the system of m linear partial differential equations (7.), represented under the form

$$\Delta_1 P=0,\ \Delta_2 P=0,\ \ldots \Delta_m P=0,$$

and satisfying identically the system of conditions represented by

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0,$$

will admit of p integrals expressing distinct values of P; and the system of ordinary differential equations (6.) corresponding to the above system of partial differential equations will be expressible as a system of exact differential equations, and will by integration give the above systems of integrals.

Beginning with the first partial differential equation of the system (4.), and forming the corresponding Lagrangian system of ordinary differential equations

$$dx_{p+1}=\frac{dx_1}{H_{11}}\ldots=\frac{dx_p}{H_{p1}},$$

$$dx_{p+2}=0,\quad dx_{p+m}=0,$$

we see that the integrals of this system will be of the form

$$u_1 \ =c_1 \quad \ldots u_p \ =c_p,$$

$$x_{p+2}=c_{p+2}\ldots x_{p+m}=c_{p+m},$$

$u_1\ldots u_p$ being functions of all the variables $x_1,\ldots x_{p+m}$, among which, by virtue of the integrals of the second line, $x_{p+2},\ldots x_{p+m}$ may be regarded as constant. The general integral will be

$$F(u_1,\ldots u_p,\ x_{p+2},\ldots x_{p+m})=0,$$

the form of F being perfectly arbitrary.

Now the general form of *any* equation of the system (4.) is

$$\frac{dP}{dx_{p+i}}+H_{1i}\frac{dP}{dx_1}\ldots+H_{pi}\frac{dP}{dx_p}=0\,. \quad \ldots \quad (9.)$$

Let us transform this by assuming

$$u_1\ldots u_p,\ x_{p+1}\ldots x_{p+m}$$

as independent variables. Then referring the right-hand members of the following equations to the new, the left-hand to the old system of variables, we have

$$\frac{dP}{dx_{p+i}}=\frac{dP}{dx_{p+i}}+\frac{dP}{du_1}\frac{du_1}{dx_{p+i}}\ldots+\frac{dP}{du_p}\frac{du_p}{dx_{p+i}},$$

$$\frac{dP}{dx_1}=\frac{dP}{du_1}\frac{du_1}{dx_1}\ldots+\frac{dP}{du_p}\frac{du_p}{dx_1},$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{dP}{dx_p}=\frac{dP}{du_1}\frac{du_1}{dx_p}\ldots+\frac{dP}{du_p}\frac{du_p}{dx_p};$$

whence, substituting in (9.),

$$\frac{dP}{dx_{p+i}}+\frac{dP}{du_1}\left(\frac{du_1}{dx_{p+i}}+H_{1i}\frac{du_1}{dx_1}\dots+H_{pi}\frac{du_1}{dx_p}\right)$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$+\frac{dP}{du_p}\left(\frac{du_p}{dx_{p+i}}+H_{1i}\frac{du_p}{dx_1}\dots+H_{pi}\frac{du_p}{dx_p}\right)=0,$$

or

$$\frac{dP}{dx_{p+i}}+(\Delta_i u_1)\frac{dP}{du_1}\dots+(\Delta_i u_p)\frac{dP}{du_p}=0; \qquad (10.)$$

and in this equation we may give to i the successive values $1, 2, \dots p$. If, then, we write

$$\frac{d}{dx_{p+i}}+(\Delta_i u_1)\frac{d}{du_1}\dots+(\Delta_i u_p)\frac{d}{du_p}=\Delta'_i,$$

we see that the proposed transformation will have the effect of converting

$$\Delta_1,\ \Delta_2, \dots \Delta_p$$

into

$$\Delta'_1,\ \Delta'_2, \dots \Delta'_p$$

respectively, and the system of partial differential equations into

$$\Delta'_1P=0,\ \Delta'_2P=0, \dots \Delta'_pP=0.$$

The developed form of the first of these equations is

$$\frac{dP}{dx_{p+1}}+(\Delta_1 u_1)\frac{dP}{du_1}\dots+(\Delta_1 u_p)\frac{dP}{du_p}=0.$$

But since $u_1, \dots u_p$ are integrals of $\Delta_1P=0$, we have

$$\Delta_1u_1=0, \dots \Delta_1u_p=0,$$

so that the equation $\Delta'_1P=0$ reduces to

$$\frac{dP}{dx_{p+1}}=0.$$

We learn from this that x_{p+1} will not explicitly appear in P after the transformation which introduces $u_1, \dots u_p$.

The developed form of the remaining $p-1$ equations represented by (10.) will be

$$\left.\begin{aligned}&\frac{dP}{dx_{p+2}}+(\Delta_2 u_1)\frac{dP}{du_1}\dots+(\Delta_2 u_p)\frac{dP}{du_p}=0,\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ &\frac{dP}{dx_{p+m}}+(\Delta_m u_1)\frac{dP}{du_1}\dots+(\Delta_m u_p)\frac{dP}{du_p}=0;\end{aligned}\right\} \qquad (11.)$$

and we shall next show that the variable x_{p+1} will not present itself in the coefficients (Δ_2u_1), &c.

The general form of such coefficients is Δ_iu_s, where i has any value from 2 to m, and s any value from 1 to p.

Now if in (5.), which is true independently of the nature of the function P, we make

$j=1$ and $P=u_s$, we have

$$(\Delta_i\Delta_1-\Delta_1\Delta_i)u_s=0.$$

But $\Delta_1 u_s=0$; therefore, by the above,

$$\Delta_1\Delta_i u_s=0.$$

Now $\Delta_i u_s$ is expressible at most as a function of $u_1, \ldots u_p, x_{p+1} \ldots x_{p+m}$. But this transformation converts, as has been seen, Δ_1 into $\frac{d}{dx_{p+1}}$. Thus we have

$$\frac{d}{dx_{p+1}}\Delta_i u_s=0,$$

so that $\Delta_i u_s$ is free from x_{p+1}. Thus the system (11.) is free from x_{p+1}.

Lastly, since by the above transformation $\Delta_2 \ldots \Delta_p$ are converted into $\Delta'_2 \ldots \Delta'_p$, the system of conditions $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$ is converted into $(\Delta'_i\Delta'_j-\Delta'_j\Delta'_i)P=0$.

It is thus seen that the system of p partial differential equations

$$\Delta_1 P=0,\ \Delta_2 P=0, \ldots \Delta_p P=0,$$

containing $p+m$ independent variables $x_1, x_2, \ldots x_{p+m}$, and satisfying the conditions

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0,$$

is convertible into a system of $p-1$ partial differential equations,

$$\Delta'_2 P=0,\quad \Delta'_3 P=0,\quad \Delta'_p P=0,$$

containing $p+m-1$ independent variables $u_1 \ldots u_p, x_{p+2} \ldots x_{p+m}$, and satisfying the condition

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0.$$

And as this system possesses the same character as that upon which the previous transformation depended, it will admit of transformation into a system of $p-2$ partial differential equations containing $p+m-2$ independent variables; and so on until we arrive at a single final partial differential equation containing $p+1$ independent variables, and having therefore p distinct integrals, which will be the common integrals of the primary system of partial differential equations as well as of the system of ordinary differential equations to which they correspond.

Cor. The property of the coefficients $\Delta_2 u_1$, &c. of the system (11.), of being free from the variable x_{p+1}, enables us, by properly determining the integrals of the partial differential equation $\Delta_1 P=0$, to reduce the system to a form of great simplicity.

Let $\Delta_i u_j$ be any one of those coefficients. Its developed form is

$$\left(\frac{d}{dx_{p+i}}+H_{1i}\frac{d}{dx_1}\ldots+H_{pi}\frac{d}{dx_p}\right)u_j. \quad \text{(12.)}$$

Now as this expression will, after the performance of the differentiations, be free from x_{p+1}, and as the differentiations are none of them with respect to x_{p+1}, we can give to x_{p+1} in it any particular value before differentiation without affecting the final result. Let us then suppose that in $H_{1i}, \ldots H_{pi}$, and in u_j, x_{p+1} is made equal to 0. Now it is possible so to determine the integrals u_j as functions of the variables $x_1, \ldots x_{p+r}$, that when $x_{p+1}=0$ each u_j shall reduce to x_j. For this purpose it is only necessary to choose as arbitrary constants a set of arbitrary values of $x_1, x_2, \dots x_p$ corresponding to $x_{p+1}=0$. Let $x'_1, x'_2, \dots x'_p$ be such arbitrary constants, and let the given system of integrals be reduced to the form

$$u_1=x'_1, \dots u_p=x'_p,$$

and the functions $u_1, \dots u_p$ will possess the required property. Changing, then, each u_j into x_j, the expression (12.) reduces to H_{ji}, and it only remains to express this in terms of $u_1, \dots u_p$; which, as $x_{p+1}=0$, is done by merely changing $x_1 \dots x_p$ into $u_1, \dots u_p$.

Thus the system (11.) is reduced to

$$\frac{dP}{dx_{p+2}}+(H_{12})\frac{dP}{dx_1}\dots+(H_{p2})\frac{dP}{dx_p}=0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{dP}{dx_{p+m}}+(H_{1m})\frac{dP}{dx_1}\dots+(H_{pm})\frac{dP}{dx_p}=0,$$

where the brackets denote that in the enclosed portion x_{p+1} is to be made 0, and $x_1, x_2, \dots x_p$ converted into $u_1, u_2, \dots u_p$.

Now this form is identical, the above conversion of letters excepted, with that of the system (4.), omitting that equation of the latter system by the integration of which the forms of $u_1, \dots u_p$ are determined.

It follows from the above that, obtaining the integrals of $\Delta_1 P=0$ in such a form that the arbitrary constants shall represent the arbitrary values of $x_1 \dots x_p$ when $x_{p+1}=0$, and representing the functions which are equal to those arbitrary constants by $x'_1, x'_2, \dots x'_p$, then if in the remaining equations $\Delta_2 P=0 \dots \Delta_m P=0$ we change $x_1, \dots x_p, \frac{d}{dx}, \dots \frac{d}{dx_p}$ to $x'_1, \dots x'_p, \frac{d}{dx'_1}, \dots \frac{d}{dx'_p}$, and x_{p+1} and $\frac{dP}{dx_{p+1}}$ to 0, the common integrals of the transformed system of $p-1$ will be the same as those of the previous system of p partial differential equations. In the same way a third system of $p-2$ partial differential equations may be formed, and so on, till we obtain a single final partial differential equation which will have the common integrals of the parent system. By this method, which is due to JACOBI and NUTANI, all the labour of the successive transformations is avoided. The successive integrals thus introduced are termed '*Haupt-integrale.*'

Instead of applying the foregoing methods, general or particular, to the final system of partial differential equations, we may apply it to the solution of the corresponding final system of ordinary differential equations. In this case they would really represent the method of solution known as the variation of parameters, and the conditions $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$ would secure the sufficiency of that method. If in the system of ordinary differential equations (6.) we regard $x_{p+2}, \dots x_{p+m}$ as constant, we get

$$\left.\begin{array}{l} dx_1-H_{11}dx_{p+1}=0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ dx_p-H_{p1}dx_{p+1}=0. \end{array}\right\} \quad \dots \dots \dots \quad (13.)$$

Integrate this in the form

$$u_1=c_1, \dots u_p=c_p;$$

then, treating $c_1, \dots c_p$ as functions of the variables before regarded as constant, and endeavouring to satisfy the unreduced equations (6.), we obtain, in virtue of the above conditions, a system of differential equations equal in number to the system given, but containing one variable fewer. The system (13.) by which the forms of $u_1, \dots u_p$ are here determined, is the Lagrangian auxiliary of the first partial differential equation $\Delta_1 P=0$ integrated in the other method; and so in each subsequent stage. And with respect to the other parts of the process, it obviously makes no difference whether we take as the new variables $u_1, \dots u_p$, or $c_1, \dots c_p$ under the condition (necessarily involved in the method of the variation of parameters) that they shall after integration be replaced by $u_1, \dots u_p$. But it would not have sufficed simply to refer the solution of the final system of ordinary differential equations to the method of the variation of parameters, first, because the necessity and sufficiency of the conditions which form the ground of that method and are the warrant of its success were to be shown; secondly, because the connexion of the systems of ordinary differential equations which arise in the method of parameters with the successive partial differential equations forms an essential part of the demonstration.

General Rule.

The results of the foregoing inquiry may be collected into the following General Rule:—

To find the number p of possible integrals of a system of n differential equations of the first order connecting $n+r$ variables $x_1, x_2, \dots x_{n+r}$, and to determine those integrals.

Rule.—Suppose P an integral of the given system. Determine from the given system $dx_1, dx_2, \dots dx_n$ as functions of the other differentials. Substitute these values in the equation

$$\frac{dP}{dx_1}dx_1+\frac{dP}{dx_2}dx_2 \dots +\frac{dP}{dx_{n+r}}dx_{n+r}=0,$$

and equate to 0 the coefficients of the remaining differentials. This will give a system of r partial differential equations of the form

$$\frac{dP}{dx_{n+1}}+A_{11}\frac{dP}{dx_1}\dots+A_{1n}\frac{dP}{dx_n}=0,$$

$$\dots\dots\dots\dots$$

$$\frac{dP}{dx_{n+r}}+A_{r1}\frac{dP}{dx_1}\dots+A_{rn}\frac{dP}{dx_n}=0,$$

r of the differential coefficients appearing each only in one equation and with coefficient unity. Representing these equations in the symbolical form

$$\Delta_1 P=0, \ \Delta_2 P=0, \dots \Delta_r P=0,$$

deduce any equation or equations of the form

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0.$$

If *all* such prove to be identities, the given system of differential equations admits of n integrals, and is reducible to a system of exact differential equations. But if *any* such equation is not an identity, it will constitute a new partial differential equation of the form

$$B_1\frac{dP}{dx_1}+B_2\frac{dP}{dx_2}\dots+B_n\frac{dP}{dx_n}=0.$$

And this, combined with the previous ones, will enable us to form a system of $r+1$ partial differential equations, in which $\frac{dP}{dx_n}, \frac{dP}{dx_{n+1}}, \dots \frac{dP}{dx_{n+r}}$ appear each in only one equation and with coefficient unity. Upon this system let the same process be repeated as upon the previous system of r partial differential equations, and so continually repeated until we arrive at a final system of partial differential equations such that, if that system be represented in the form

$$\Delta_1 P=0\dots \Delta_m P=0,$$

the condition

$$(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$$

shall be identically satisfied for every pair.

Then, the number of such partial differential equations being m, the number of integrals of the original system of partial differential equations will be $n+r-m$, i. e. it will be equal to the number of the original variables diminished by the number of final partial differential equations.

And if by that final system we eliminate m of the differential coefficients from

$$\frac{dP}{dx_1}dx_1+\frac{dP}{dx_2}dx_2\dots+\frac{dP}{dx_{n+r}}dx_{n+r}=0,$$

and equate to 0 the coefficients of the remaining differential coefficients, we shall have a system of $n+r-m$ differential equations expressible as exact differential equations for the determination of the integrals.

Actually to determine these, we should endeavour in the first instance to reduce the final system of differential equations, as such reduction is theoretically possible, to a system of exact differential equations. If the means of doing this are not obvious, the method of the variation of parameters or the equivalent methods of Prop. III. must be applied.

Lastly, if the process which consists in the application of the theorem $(\Delta_i\Delta_j-\Delta_j\Delta_i)P=0$ do not stop with the formation of the final system of partial differential equations, but lead to algebraic relations among the variables, the given system of differential equations will have no integrals properly so called, but it may admit of *solutions* analogous to those the theory of which has been developed by Pfaff, Jacobi, and others for the differential equation

$$X_1dx_1+X_2dx_2\dots+X_ndx_n=0.$$

Applications.

1st. Suppose it required to find the number of integrals of the form $P=c$, which the system of differential equations

$$dz=(t+xy+xz)dx+(xzt+y-xy)dy,$$
$$dt=(y+z-3x)dx+(zt-y)dy$$

admits, and to determine such integrals.

Eliminating dz and dt from the equation

$$\frac{dP}{dx}dx+\frac{dP}{dy}dy+\frac{dP}{dz}dz+\frac{dP}{dt}dt=0,$$

and equating to 0 the coefficients of dz and dt in the result, we have

$$\frac{dP}{dx}+(t+xy+xz)\frac{dP}{dz}+(y+z-3x)\frac{dP}{dt}=0, \quad \ldots \quad (1.)$$

$$\frac{dP}{dy}+(xzt+y-xy)\frac{dP}{dz}+(zt-y)\frac{dP}{dt}=0. \quad \ldots \quad (2.)$$

Hence writing

$$\Delta=\frac{d}{dx}+(t+xy+xz)\frac{d}{dz}+(y+z-3x)\frac{d}{dt},$$

$$\Delta'+\frac{d}{dy}+(xzt+y-xy)\frac{d}{dz}+(zt-y)\frac{d}{dt},$$

and forming the equation

$$(\Delta\Delta'-\Delta'\Delta)P=0,$$

we have, on rejecting a common algebraic factor,

$$x\frac{dP}{dz}+\frac{dP}{dt}=0. \quad \ldots \quad (3.)$$

By substituting in (1.) and (2.) the value of $\frac{dP}{dt}$ hence obtained, we have the system of three equations,

$$\frac{dP}{dx}+(3x^2+t)\frac{dP}{dz}=0,$$

$$\frac{dP}{dy}+y\frac{dP}{dz}=0,$$

$$\frac{dP}{dt}+x\frac{dP}{dz}=0.$$

Now if upon any two of these we repeat the same process as upon (1.) and (2.), we obtain as the result $0=0$. Thus the system of partial differential equations is complete.

As then there are three equations in this final system, while the number of original variables was four, the primary system will admit of one integral of the form $P=c$.

To obtain this integral, eliminate $\frac{dP}{dx}$, $\frac{dP}{dy}$, $\frac{dP}{dt}$ from the above equations and

$$\frac{dP}{dx}dx+\frac{dP}{dy}dy+\frac{dP}{dz}dz+\frac{dP}{dt}dt=0,$$

and equate to 0 the coefficient of $\frac{dP}{dz}$ in the result. We find

$$dz-(t+3x^2)dx-ydy-xdt=0,$$

the integral of which is

$$z-xt-x^3-y^2=c.$$

2nd. The solution of the partial differential equation

$$Rr+Ss+Tt+U(s^2-rt)=V,$$

as well as of the special equations

$$Rr+Ss+Tt=V,$$

$$Rr+Ss+Tt+U(s^2-rt)=0,$$

the theory of which constitutes an exception to that of the more general form, depends in general upon the integration of three simultaneous differential equations between five variables. To this integration the method of the foregoing sections is applicable.

The only cases for which the theory of the ultimate solution can be said to be complete, are those in which the auxiliary system of common differential equations admits either three integrals of the form $P=c$, or two integrals of that form.

We may apply the method of the foregoing sections, not only to the determination of the integrals, but also to the discovery of the *à priori* conditions connecting the coefficients R, S, T, U, V in order that each of these species of integration may be possible.

For example, the solution of the equation

$$Rr+Ss+Tt+(s^2-rt)=V$$

depending upon the integration of the system

$$dq=-m_1dx+Rdy,$$

$$dp=-m_2dy+Tdx,$$

$$dz=\ pdz\ +qdy,$$

in which m_1 and m_2 are roots of the equation

$$m^2-Sm+RT-V=0,$$

let it be required to determine the conditions under which the system admits three integrals.

Eliminating dq, dp, dz between the above equations and

$$\frac{dP}{dx}dx+\frac{dP}{dy}dy+\frac{dP}{dz}dz+\frac{dP}{dp}dp+\frac{dP}{dq}dq=0,$$

and equating to 0 the coefficients of dx and dy in the result, we obtain two partial differential equations which may be thus represented, viz.

$$\Delta P=0, \qquad \Delta' P=0,$$

in which

$$\Delta=\frac{d}{dx}-m_1\frac{d}{dq}+T\frac{d}{dp}+p\frac{d}{dz},$$

$$\Delta'=\frac{d}{dy}+R\frac{d}{dq}-m_2\frac{d}{dp}+q\frac{d}{dz}.$$

That there may be three integrals, it is here necessary that there should be but two partial differential equations in the completed system. Hence the equation

$$(\Delta\Delta'-\Delta'\Delta)P=0$$

must vanish identically.

Developing this, we have the conditions

$$\Delta R+\Delta' m_1=0, \quad \Delta m_2+\Delta' T=0, \quad \Delta q-\Delta' p=0.$$

Now on performing the operations denoted by Δ and Δ', the last equation gives

$$m_2-m_1=0.$$

Hence referring to the quadratic, we see that

$$S^2-4RT+4V=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

To this must be added the two other reduced conditions,

$$\Delta R+\Delta' m=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (II.)$$

$$\Delta m+\Delta' T=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (III.)$$

m representing one of the equal roots of the reduced quadratic.

The first of the above conditions was given by AMPÈRE*. The others are probably new. Satisfied, they enable us to predict that the partial differential equation under consideration admits a complete primitive involving three constants, and a general primitive arising from the variation of those constants in subjection to any two arbitrary conditions.

3rd. We have supposed each linear partial differential equation employed in the processes of this paper to be of the form

$$A_1\frac{dP}{dx_1}+A_2\frac{dP}{dx_2}\dots+A_n\frac{dP}{dx_n}=0,$$

and we have supposed each system of partial differential equations which arises, to be so reduced that each equation shall have some one of the partial differential coefficients of P entering into it alone and with a coefficient equal to unity.

The first of these conditions is virtually sufficiently general, because any linear partial differential equation can be deprived of its second member. The advantage of the second condition is that each newly-formed equation will be really new, and not an algebraic combination of the old ones.

But neither of these conditions is necessary. From two linear partial differential equations of the form

$$\Delta_1 P=H, \quad \Delta_2 P=K,$$

in which H and K are functions of the independent variables, arises a new equation,

$$(\Delta_1\Delta_2-\Delta_2\Delta_1)P=\Delta_1 K-\Delta_2 H, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

which will be satisfied by all the simultaneous integrals of the equations from which it is derived.

It may be rigorously proved that, in applying this process, the generated system

* Journal de l'Ecole Polytechnique, Cahier 18^e^.

(including the original equations) will be complete when no new equation arises from the combination of any one of the equations with any one of the equations of the original system.

I will illustrate this by investigating the conditions of integrability of the expression

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right) dx.$$

If this expression admit of an integral V, it is easy to see that V will satisfy the two partial differential equations

$$\frac{dV}{dx}+y_1\frac{dV}{dy}+y_2\frac{dV}{dy_1}\dots+y_n\frac{dV}{dy_{n-1}}=\phi, \quad \dots\dots\dots \quad (2.)$$

$$\frac{dV}{dy_n} \qquad =0, \quad \dots\dots\dots \quad (3.)$$

in which

$$y_1=\frac{dy}{dx}, \; y_2=\frac{d^2y}{dx^2} \dots\dots,$$

and ϕ stands for

$$\phi(x, y, y_1, \dots y_n).$$

The above are, in fact, the partial differential equations which we should obtain by Prop. I. as the equivalents of the system of ordinary differential equations,

$$dV=\phi dx,$$

$$dy=y_1dx, \; dy_1=y_2dx, \dots dy_{n-1}=y_ndx.$$

If we write

$$\left(\frac{d}{dx}\right)=\frac{d}{dx}+y_1\frac{d}{dy}+y_2\frac{d}{dy_1}\dots+y_n\frac{d}{dy_{n-1}},$$

the above partial differential equations become

$$\left(\frac{d}{dx}\right)V=\phi\dots(\text{I.}), \qquad \frac{dV}{dy_n}=0\dots(\text{II.})$$

The combination of (I.) with (II.) (by the theorem (1.)), then of (I.) with the result, and so on, gives a series of equations which may be thus expressed:—

$$\frac{dV}{dy_{n-1}}=\Delta_n\phi, \quad \dots\dots\dots \quad (\text{III.})$$

$$\frac{dV}{dy_{n-2}}=\Delta_{n-1}\phi, \quad \dots\dots\dots \quad (\text{IV.})$$

. . . . .

$$\frac{dV}{dy_0} =\Delta_1\phi, \quad \dots\dots\dots \quad (\text{V.})$$

$$0 =\Delta_0\phi, \quad \dots\dots\dots \quad (\text{VI.})$$

in which

$$\Delta_r=\frac{d}{dy_r}-\left(\frac{d}{dx}\right)\frac{d}{dy_{r+1}}+\left(\frac{d}{dx}\right)^2\frac{d}{dy_{r+2}}-\&c.,$$

$$\Delta_0=\frac{d}{dy}-\left(\frac{d}{dx}\right)\frac{d}{dy_1}+\left(\frac{d}{dx}\right)^2\frac{d}{dy_2}-\&c.$$

The combination of (II.) with (III.) ... (V.) gives the series of conditions

$$\frac{d}{dy_n}\Delta_n\phi=0,\ \frac{d}{dy_n}\Delta_{n-1}\phi=0,\ \dots\ \frac{d}{dy_n}\Delta_1\phi=0. \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(VII.)}$$

The conditions of integrability are expressed by (VI.) and (VII.). These satisfied, the equations (III.), (IV.), ... (V.) show that ϕdx can be expressed as an exact differential with respect to x, y, y_1, y_{n-1}, in the form

$$\phi dx=(\phi-y_1\Delta_1\phi-y_2\Delta_2\phi\ \dots\ -y_n\Delta_n\phi)dx$$
$$+\Delta_1\phi dy+\Delta_2\phi dy_1\ \dots\ +\Delta_n\phi dy_{n-1},$$

a result first established by M. SARRUS.

XXXI. (I.) *On the Dicynodont Reptilia, with a Description of some Fossil Remains brought by* H.R.H. Prince Alfred *from South Africa, November* 1860. *By Professor* Owen, *F.R.S. &c.*

Received January 23,—Read February 20, 1862.

On the return of His Royal Highness Prince Alfred from the Cape of Good Hope, in 1860, he honoured me by transmitting some fossil remains, with the following Note:—

"Dear Professor Owen,

"In the course of my journey in South Africa I met with two very interesting Fossil Remains; one, the larger, being the head of a *Dicynodon*: and I hope you will accept them from me as being the best specimens I obtained, upon the Prince Consort's suggestion, on the occasion of your last Lecture, of which I retain the most agreeable recollection.

"Yours truly,

"Alfred."

"Windsor Castle, November 15th."

I feel it a duty, at the present season of grief at a National bereavement, which becomes daily more appreciated, to communicate to the Society this evidence of the lamented Prince Consort's unfailing interest in our sciences, as manifested by his desire that Lectures on Natural History should be delivered before the Royal offspring; and also as showing that, amidst the manifold occupations of the Prince Consort, he allowed no opportunity to escape in which his influence could be, in any way, exerted to aid in the direct advancement of science.

In the present instance, this influence, through the zealous fulfilment by Prince Alfred of his father's suggestion, has procured for us valuable additional evidence of the peculiar structure and characters of some rare extinct animals, which, under any circumstances, I should have felt bound to submit to the Royal Society.

Some delay has been occasioned by the necessity of removing the very hard matrix from the varied and intricate surfaces of the imbedded petrified bones,—a labour requiring to be performed under my immediate supervision, or by my own hands.

The smaller of the two fossils referred to by H.R.H. Prince Alfred, has thus proved to be one of the most perfect specimens of the petrified skull of a dicynodont Reptile which, hitherto, has come under my observation. It is figured, of the natural size, in Plates XIX. & XX. figs. 1, 2, 3, 4 & 5.

Ptychognathus Alfredi, OWEN.

By the angular contour of the profile (figs. 1 & 3), in which the almost straight line of the vertex (*v o*) meets the equally straight occipital line (*o b*) at a right angle, and the straight facial or naso-premaxillary line (*v n*) meets that of the vertex at an obtuse angle, the specimen belongs to the subgeneric form of the dicynodont family called *Ptychognathus*.

The occiput (fig. 4) shows the same crocodilian extent of ossification as in other members of this singular family.

The articular tubercle is formed, as in *Ptychognathus declivis**, by the basioccipital (fig. 4, $_1$) at the lower and middle part, and, in less proportions, by the exoccipitals (*ib.* $_2$) which just meet at the upper part of the tubercle. The foramen magnum is a rather narrow oval, with the small end upwards, exclusively bounded, externally, by the exoccipitals; the basioccipital appearing at the interspace of the exoccipitals immediately within the cranial cavity. The basioccipital in advance of the tubercle, has its lower surface at first concave, and then convex, lengthwise; and is more deeply concave transversely (fig. 5, $_1$) through the production of its sides into the short and thick hypapophyses, figs. 4 & 5, *hy*. Its junction with the basisphenoid (fig. 5, $_5$) is by an almost straight transverse harmonia. The hypapophyses (*hy*) are divided by a short notch from the paroccipitals (fig. 4, $_4$), which abut, as in *Crocodilia* and *Chelonia*, against the inner side of the tympanics ($_{28}$), but here against their lower half, fixing these bones more firmly in their position. The paroccipitals, which, as in the Crocodile, are exogenous growths of the exoccipitals (*ib.* $_2$), are divided by a wider and more shallow notch from the exoccipital and mastoid, $_8$. The superoccipital (fig. 4, $_3$) forms the upper part of the occipital surface, is of a vertically oblong or subtriangular form, with the apex downward, and terminating some distance above the foramen magnum. The sides of the superoccipital concavity are formed by the backwardly projecting and slightly diverging hinder end of the parietals (fig. 4, $_7$); between which and the exoccipitals and tympanics the broad and angular mastoid (*ib.* $_8$) are wedged, forming the upper, outer, and posterior outstanding backwardly bent ridge, for the attachment of the cervical muscles, and presenting the sutural surface to the upper end of the tympanic, $_{28}$. The upper surface of the parietal (fig. 2, $_7$) presents a roughish non-articular tract of about 8 lines in breadth, at the fore part of which is the venous foramen. This median tract is made slightly concave across by the elevation of the ridges (*t*, *t*) bounding the smooth, longitudinally concave, tracts affording origin to the temporal muscle.

The frontals ($_{11}$, $_{11}$) form the chief part of the flattened vertex or upper platform of the cranium (figs. 2 & 4, *f*, *t*). They extend longitudinally from the foramen parietale forward to about one-third of an inch from the prefronto-nasal ridge (fig. 2, *f*); and transversely to the superorbital ridges (*s*), of which they form about the middle third part. The median or frontal suture may be traced along a great part of their extent: the surface is slightly raised at this part, and is depressed laterally, between the median

* Quarterly Journal of the Geological Society, vol. xvi. (April 1859) p. 49, pl. 1. figs. 3–5.

rising and the slightly elevated orbital borders. There is no trace of the pair of tubercles which, in *Ptychognathus declivis* *, stand transversely on the frontal platform, in the line bisecting the middle of the orbits: they have not been chipped away in clearing off the matrix, either in the present skull or in that of *Ptychognathus latirostris* †. The median rising at the frontal suture is continued backward over the part where it divides to go to each tubercle, leaving an intermediate shallow depression, in *Ptychognathus declivis*. The frontal tubercles may be, therefore, reckoned among the cranial characters distinguishing *Ptychognathus declivis* from *Ptychognathus latirostris*, and from the skull of the species under description. In this, as in the previously defined species of *Ptychognathus*, the vertex or upper surface of the skull is bounded anteriorly by a low ridge (fig. 2, *f*, *f*), extending transversely with a slight curve, convex forward, from the fore part of one superorbital border to the other. The ridge is formed by the nasals (fig. 2, $_{15}$) at the middle, and by the prefrontals (*ib.* $_{14}$) at the sides. The latter are strong, large, angular bones (figs. 1, 2 & 3, $_{14}$), dividing by a prominent tuberosity the fore from the upper borders of the orbit: the frontal surface of the prefrontal is divided by the before-described ridge from the facial surface; both are slightly concave, the latter of greater extent, reaching almost to the hinder angle of the external nostril (figs. 1 & 3, *n*), and articulating with the nasal (*ib.* $_{15}$) above, and with the lacrymal (*ib.* $_{73}$) below. The nasals ($_{15}$), divided by a median suture, bend abruptly, at the prefronto-nasal ridge, from the upper to the fore or facial part of the skull, at an open angle. They are united for about an inch as they slope upon the face (fig. 2), and then diverge to form the upper border of the nostrils, receiving at the angles, so formed, the upper and hinder ends of the coalesced premaxillaries, *ib.* $_{22}$. The premaxillaries continue the facial line, begun by the nasals, straight to the upper margin of the mouth, which is directed forward. There is a slightly elevated line along the place of the obliterated medial suture (fig. 2, *m*), parallel with which is a pair of similar linear ridges (fig. 2, *p*, *p*), dividing the median from the lateral surfaces of the premaxillary: these surfaces have almost the same breadth, except that the lateral surface (fig. 1, $_{22}$) increases near the oral border; it is narrowest where it forms the fore part of the nostril, *n*.

The maxillary (figs. 1 & 3, $_{21}$) is chiefly remarkable, as in the rest of the genus, for the strong ridge-like prominence (*r r*) of the socket of the long and large canine tusk, *c*. Between this and the outer premaxillary ridge (fig. 1, *p*) the sides of the face are slightly sinuous, the convexity above gradually changing to a deeper concavity below, in the transverse or vertical direction; lengthwise the sides of the face are slightly concave as they converge towards the mouth; except at the canine alveoli, which are a little convex lengthwise at their fore part. Above, the maxillary bounds the lower part of the nostril, and there unites with the premaxillary ($_{22}$) and lacrymal ($_{73}$); below, it is continued from the canine-socket inward and downward, forming the edentulous sectorial border which meets the corresponding part of that border of the mandible ($_{32}$), the fore part of which, in advance of the maxillary, is strongly bent upward ($_{32'}$) to meet the eden-

* Quarterly Journal of the Geological Society, vol. xvi. p. 50, pl. 1. fig. 3. † Ibid. p. 51.

tulous trenchant border of the premaxillaries, *n*. The back part of the maxillary is continued beneath the orbit in a pointed form, articulating by a strong, oblique and extensive suture with the malar (fig. 1, 26), by which and the lacrymal the maxillary is removed from the orbital border itself.

The lacrymal (73), forming the middle of the fore part of the orbit, extends upon the face apparently to the nostril. The malar curving back along the lower part of the orbit, the border of which it there forms, bifurcates behind to connect itself with the postfrontal (figs. 2 & 4, 12) above and with the squamosal (fig. 1, 27) behind; but their limitary sutures I cannot satisfactorily make out. There is no other bony bridge overspanning the temporal fossa except the normal malo-squamosal zygomatic arch. The squamosal combines with the mastoid in affording the articular sutural surface to the upper end of the tympanic.

The squamosal (fig. 1, 27) is compressed, of 8 lines in vertical extent, by 2 in transverse; and increases vertically as it passes backward. The tympanic pedicle is of great length, broad and compressed from before backward at its upper part, becoming narrower as it descends, but gaining in thickness to where it receives the broad abutment of the paroccipital; below which it slightly expands in every direction to form the convex articular surface for the mandible. The tympanic pedicle is fixed immoveably, and by its size and connexions forms an unusually strong 'point d'appui' for the vigorous actions of the lower jaw. It consists of two bones, united by a broad overlapping squamous suture. The upper portion includes the mastoid and squamosal elements; the latter extends to near the articular condyle, along the fore part of the pedicle; the lower portion, or tympanic proper (28), forms the lower half of the back part of the pedicle, and expands below to form the joint.

The mandible of *Ptychognathus* (fig. 1, 30–32) resembles that of the *Chelonia* in its edentulous condition, general proportions, and comparative simplicity of structure, and that of the *Crocodilia* in the vacuity left between the dentary (32), angular (31), and surangular (30) elements; but it is peculiar and ptychognathic in the sudden vertical expanse and upward curve of its symphysial end (32′), the vertical diameter here being three times that of the articular end. I cannot distinguish an articular from a surangular element: the suture between this and the angular extends near and runs parallel with the upper border of the surangular to the vacuity, towards which it bends. No coronoid process is developed, nor is there any coronoid or complementary ossicle as in modern Lizards, Chelonians, and Crocodiles. The splenial element (fig. 5, 29) extends far back, as in *Chelonia*. The dentary elements, confluent at the symphysis, are deeply notched at their narrow hinder part, the upper projection being the longest. A horizontal ridge extends from the upper part of the notch forward as far as the depression, lodging the end of the canine tusk when the mouth is shut. The vacuity is situated halfway between the two ends of the mandibular ramus, not in the posterior third as in the Crocodile. The symphysis is broad as well as high, and the rami meet there so as to form, below, an arch or curve, concave backward, fig. 5, *s*. The fore part of the symphysis is convex both vertically and transversely. The longitudinal ridge on the outside of the dentary, parallels that formed by the canine alveolus of the maxillary; and the alveolar borders in both incline inwards, to meet at the oral margin behind the great tusks. The mandibular ridge resembles the maxillary one in *Oudenodon*; and one is led to speculate whether in that genus the maxillary ridge may have related to a rudimentary upper canine, and the mandibular ridge to that of a lower one, in the embryo. In both *Ptychognathus* and *Oudenodon*, however, these parallel ridges in the upper and lower jaws, with the inward convergence of the alveolar plates, recall the structure of those parts in *Scelidosaurus* *, and which is indicated by the mandibles to have existed, in a certain degree, in *Hylæosaurus* and *Iguanodon*.

The fossil skull above described was obtained from a greenish sandstone of the Rhenosterberg. As it manifests specific distinctive characters, I propose to refer it to a species of *Ptychognathus*, under the name of the young Prince by whom it has been made available in advancing our knowledge of South African fossils.

The *Ptychognathus Alfredi*, like *Ptychognathus latirostris*, differs from *Pt. declivis* in the more circular orbits, the absence of the frontal tubercles, the right angle at which the occiput and vertex meet, and the greater depth and thickness of the facial part in proportion to its length.

But on comparing *Ptychognathus Alfredi* with *Ptychognathus latirostris*, the facial part of the skull presents less breadth in proportion to its length and depth, and the lower jaw is narrower in proportion to its length; the paroccipital is also relatively narrower.

The occipital region in *Ptychognathus declivis* (Plate XXI. fig. 2) shows nearly the whole extent of the masto-tympanic pedicle, and well exemplifies the singular breadth of the hinder surface of the cranium in these dicynodont reptiles. It also shows the precondyloid foramina, and the small vacuity between the exoccipital and masto-tympanic bones. The side view of the skull of *Ptychognathus latirostris* (Plate XXI. fig. 1) well exemplifies the ridged structure of the jaws and the composition of the mandible.

In the corresponding view of the skull of *Ptychognathus Alfredi* (Plate XX. fig. 3) is shown the extent of the pulp-cavity of the canine (*c*), closely conforming to that in the instances of ever-growing tusks in the mammalian class. As in former specimens of true *Dicynodon*, the section made to show the base of the tusk exhibits no trace of any germ of a successional tooth. The tusk curves forward, downward, and slightly inward. The points of both tusks have been broken away.

Dicynodon tigriceps, OWEN.

The larger fossil, obtained by H.R.H. PRINCE ALFRED from the Karoo-beds, in the district of Graaf Reinet, is a skull of a true *Dicynodon*, equalling in size that of the *Dicynodon tigriceps* †. It has been obliquely crushed, but under circumstances of sur-

* "Monograph on British Fossil Reptiles," Palæontological Society's volume for 1860, p. 12, pls. 4 & 5.

† Transactions of the Geological Society, 2nd Series, vol. vii. p. 233.

rounding support, which have kept the lower jaw in its natural connexions with the tympanic pedicle.

In the breaking up of the hard rock in which this fossil has been imbedded, the right maxillary, tympanic, and zygomatic arch have been removed; the fore part of the upper jaw, from the beginning of the sockets of the canine tusks, and the corresponding end of the lower jaw, are also broken away. The length of the remaining portion of the skull now exposed is 1 foot 6 inches; but this is somewhat more than it would naturally be; owing to the left half of the broad occipital region (Plate XXII. 4, 28) having been bent backward from the transverse to almost the longitudinal position, in the line of the skull's axis, and this with so little disturbance of the connexions of its elementary bones as exemplifies, with other similar condition of the skull, the gradual operation of the disturbing force, and the condition of surrounding support that has made the pressure act upon the brittle fossil as if it had been a plastic material. For, after close observation and reflection upon all the appearances presented by this fossil, I infer that the cosmic movements affecting the matrix have operated after the sediment in which the dead body of the old reptile was buried had become, with its contents, hardened into stone.

After careful removal of the matrix from the remaining petrified bones of the right side of the skull, the occipital tubercle (1) was worked out in its whole length: it projects from the lower plane of the basioccipital (*h*) to the extent of 1 inch 8 lines, and from the foramen magnum (*f*) about 1 inch 4 lines; the vertical diameter of the base of the tubercle is 1 inch 6 lines; it slightly expands to its articular convexity. The right side of the tubercle having been broken away, the compact or close granuloid texture of the bone is here displayed.

The occipital hypapophyses (*h*) are 2 inches in length; the left paroccipital (4) expands to a breadth of 5 inches, where it abuts against the broad masto-tympanic pedicle (28). An extent of 6 inches by 4 inches of the smooth posterior surface of this singularly expanded lamelliform bone is here preserved. The small vacuity (*a*) between the par- and exoccipitals has been converted into a foramen by the meeting of extensions of ossification from 2 to 4. The foramen magnum (*f*), naturally of a vertically oval form, is here made narrower by the slow lateral squeezing of the occiput; its long diameter is 1 inch 4 lines. The lower angle of the superoccipital (3) is preserved, making an extent of the well-ossified occipital plane reaching to 4 inches above the occipital tubercle.

The removal of the outer part of the maxillary and the zygomatic arch has brought into view part of the interorbital septum, and the upper and outer part of the bony palate; structures that have not been shown by previous specimens of *Dicynodon*.

The descending cranial plate of the frontal, where it forms the inner wall of the orbit and the rhinencephalic continuation of the cranial cavity, is shown at 11, Plate XXII.

The basisphenoid (*ib.* 5) is short and deep, and sends out a process uniting with one from the pterygoid, to abut against the tympanic pedicle.

The presphenoid (*ib.* 9) projects forward as a compressed plate 10 lines in vertical diameter; in its length or extent of ossification it exceeds that in the *Chelonia*, and more resembles that in the *Crocodilia*; at its base there arise some thin rays of ossification, which ascend to unite with a similarly attenuated lamelliform base of the orbitosphenoid. I infer that there was persistent cartilage in this part of the skull. The orbitosphenoid (*ib.* 10) becomes thicker as it ascends to unite with the under part of the hinder third of the frontal, and with part of the parietal. It is perforated by a foramen opticum of elliptical form, 6 lines in long diameter.

Below and anterior to the presphenoid is seen a small part of the vomer (*ib.* 13), where it expands laterally to join the palatine (*ib.* 20) and pterygoid, *ib.* 24. The pterygoid, about 5 inches in length, contracts as it extends backward, bounds above or mesially the outer part of a long elliptical foramen (posterior nostril, *ib. n*), and then bends downward and outward to join the basisphenoid and abut against the lower end of the tympanic.

There appears to have been a part of the interorbital space unossified, about 3 inches in length, 1½ inch broad below, but suddenly contracting to a width of ½ an inch at the upper part.

The exposed bottom of the right socket of the canine tusk (*ib.* fig. 1, *c*) shows the similarly exposed pulp-cavity of the beginning of that tusk, which measures 1½ inch across. The walls of the tusk increase in thickness to about 3 lines in an extent of 1½ inch. A greater extent of the left socket (Plate XXII. fig. 2) is preserved, showing the concentrically lamelliform structure of the base of that tusk.

Sufficient of the mandible is preserved to demonstrate its characteristic dicynodontal composition. The alveolar border of the dentary element is toothless. The ramus rapidly augments in depth towards and at the symphysis, where a portion is broken away. The posterior part of the dentary (*ib.* fig. 1, 32) is notched or bifurcates to form an oblong vacuity at the middle of the mandibular ramus. Traces of a long surangular (*ib.* fig. 1, 30), which developes no coronoid process, of a broader or deeper angular (*ib.* 31), and also of a splenial (*ib.* 29), are discernible amongst the elements of the lower jaw.

As a whole, the present instructive specimen exemplifies the near equality in size of some of the strange extinct two-tusked reptiles of South Africa to the existing Walrus; it also shows that in the structure of the bony palate the *Dicynodon* combines, as in other parts of the skull, crocodilian with chelonian and lacertian characters.

Explanation of the Plates.

PLATE XIX.

Fig. 1. Side view of skull of *Ptychognathus Alfredi*: nat. size.
Fig. 2. Upper view of the same skull: nat. size.

PLATE XX.

Fig. 3. Side view, with section of tusk, of the same skull: nat. size.
Fig. 4. Back view of the same skull.
Fig. 5. Under view of ditto.

PLATE XXI.

Fig. 1. Side view of skull of *Ptychognathus latirostris*: nat. size.
Fig. 2. Back view of *Ptychognathus declivis*.

PLATE XXII.

Dicynodon tigriceps.

Fig. 1. Side view of skull: $\frac{1}{3}$ nat. size.
Fig. 2. Socket and base of canine tusk: nat. size.

(II.) *On the Pelvis of the* DICYNODON. *By Professor* OWEN, *F.R.S. &c.*

The following description is taken from a part of the petrified skeleton of a *Dicynodon* equalling in size the species, *D. tigriceps*, Ow., to which the cranium described in the preceding paper belongs, and exemplifying the structure of the pelvic part of the trunk of that extinct animal.

This instructive fossil is from the same locality in the Graaf Reinet district as that from which the specimen of the skull of the *Dicynodon tigriceps* was obtained which is described in the seventh volume of the 2nd Series of the Geological Transactions *. It formed the nucleus of a huge nodule of greyish-blue argillo-ferruginous limestone, transmitted from that locality by A. G. BAIN, Esq., F.G.S., and is now in the British Museum.

In a front view of this specimen (Plate XXIII. fig. 1) seven successive vertebræ are seen; but there is an appearance of slight dislocation at the expanded and co-adapted articular ends of the two anterior of these (D & S_1), which indicates that they were not anchylosed together; the rest seem to have coalesced, although traces of the intervertebral articulations remain. One-half of the last vertebra (Plate XXIV. S_6) is broken away. The length of the six entire centrums is 1 foot 2 inches.

The first vertebra (Plate XXIII. fig. 1, D) supports a pair of long, comparatively slender curved ribs (*pl*), articulated to strong outstanding transverse processes (*d*). These were the last or hindmost pair of free ribs, and indicate great breadth in that part of the trunk.

The second vertebra (*ib.* S_1), which is the first of the sacral series, sends out a pair of broad and thick parapophyses (*ib. p*), to each of which is attached a longer and broader pleurapophysis (*ib. pl*). This rapidly expands as it extends outward, and underlaps, or

* 4to. 1855, p. 233.

passes anterior to, the iliac bone (*ib.* $_{62}$). It resembles in shape the human scapula, but is much thicker. The hinder and inner border of this sacral rib is thick, and smoothly rounded; the front border is thinner, and is slightly concave; the outer border appears to have been straight, but is somewhat mutilated. This expanded termination of the rib rests on the ventral side of the ilium, concealing much of that bone in a front view of the pelvis. It has not been anchylosed therewith; decomposition of the ligamentous uniting matter, and subsequent partial dislocation, have allowed the matrix to insinuate itself between the sacral rib and the ilium, as seen at *a*, fig. 2, Plate XXIII.

Of the short, thick, transversely extended ribs of the five succeeding sacral vertebræ, the left of the penultimate (*pl*) is the best preserved towards the pelvic cavity; it presents a smooth convex or rounded surface about an inch in breadth, is slightly bent with the concavity towards the outlet of the pelvis, and abuts against the ischium.

The bodies of the sacral vertebræ ($s_{1 \ldots 5}$) are contracted at the middle, and slender there in proportion to the pelvis, but are rapidly and much expanded at their articular ends; consequently they are very concave lengthwise, both below and at the sides; but are smooth and convex transversely, yielding a semicircular transverse section. The parapophyses go off near the base of the neural arch. The first centrum (s_1) is nearly 3 inches in length, and as much in breadth at the articular end, but is only 1 inch 3 lines across the middle; the rest ($_{2 \ldots 5}$) slightly diminish in length as they approach the tail.

The ilium (Plates XXIII. and XXIV. $_{62}$) is a strong straight triangular bone, at least 10 inches in length from the upper border of the acetabulum (Plate XXIII. fig. 2, *c*); above which it is contracted to a breadth of $3\frac{1}{2}$ inches and a thickness of 2 inches, and then expands to a breadth of $8\frac{1}{2}$ inches, measured along its oblique anterior border or labrum, Plate XXIV. fig. 4, *l l'*. The front part of the anterior two-thirds of the ilium expands into a rough flattened surface, 6 inches in length, and 3 inches in breadth anteriorly (*ib.* fig. 4, *r*), to which the back surface of the expanded first sacral rib is ligamentously attached. The inner surface of the ilium extends 6 inches behind this articulation, and is almost flat, but rather sinuous: coarse bony ridges or rays of ossification appear on this surface, near the labrum (*i'*), diverging thereto. The outer surface of the ilium is moderately concave transversely. This anterior expanded part of the ilium passes behind and in advance of the last pair of free ribs; the relations of which, and of the first sacral ribs, remind one of those of the answerable vertebræ in the pelvis of birds. The ribs of the second and third sacral vertebræ also abut against the ilium.

The ischium (Plates XXIII. and XXIV. $_{63}$), behind or beyond the acetabulum, forms a short and very thick prominence (*ib.* fig. 2, *t*), which receives the abutment of the ribs of the fifth sacral (Plate XXIV. pl_5): below or behind this it receives a similar abutment from the ribs of the last sacral vertebra (pl_6). The space, corresponding to the 'great ischiadic foramen' in Edentate mammals, is thus divided into two vacuities. The ischium becomes thinner, but is of great breadth where it forms the lower wall of the pelvis and converges towards its fellow to form, with the pubis, the long symphysis (Plate XXIII. fig. 2, *y*); here it again increases in thickness. A fracture, with a slight dislocation on the left side, has taken place where the suture with the pubis may originally have existed; but on the right side no trace of such suture is visible; and the perfect state of the surface of the pelvis at this part demonstrates as complete a confluence of the two bones, with each other and the ilium, to form a large 'os innominatum,' as in mammals.

The pubis (Plates XXIII. and XXIV. 63) is remarkable for the bold and broad anterior convexity (*b*) of its iliac half, the inner part of which is perforated by an elliptical aperture (figs. 1 & 4, *f*), answering to that in the pubis of the Monitor*. The outer part of the bone is produced into a short obtuse process (*r*), less developed proportionately than that in the Monitor's pubis. Beyond this process the inferior border of the pubis bends downward and forward, rendering the antero-inferior surface of the half of the bone next the symphysis concave. The symphysis makes a slight angle near its beginning (Plate XXIII. fig. 1, *y*), projecting towards the pelvic cavity: the suture, there, is obliterated. The inferior or symphysial wall of the pelvis measures 14 inches across in a straight line from the border of one acetabulum to that of the other. The length of the symphysis (*ib.* fig. 2, 63, *y*, 64) in a straight line cannot have been less than 8 inches; but the upper or anterior margin is wanting (*ib.* fig. 1, 63). Its external contour is first concave, then convex, lengthwise. The broad, subquadrate ischio-pubic walls of the pelvis on each side the symphysis are slightly concave, outwardly, both vertically and transversely; the hinder three-fourths of these appear to have been formed by the ischium (64): there is no 'obturator' space or vacuity, merely the outlet of the pubic perforation appears externally. Both ischia and pubis combine to form a continuous tract of bone at the symphysis, which presents a thick protuberance at its lower or outer part near its termination (Plate XXIII. 64). The thick posterior border of the ischium is concave below the acetabulum, lengthwise, expanding into an angular tuberosity.

The outlet of the pelvis (Plate XXIV. fig. 3) is of a semielliptic form, 9 inches in transverse and 4 inches in the fore-and-aft diameters. If the inlet or brim of the pelvis (Plate XXIII. fig. 1) be defined by the smooth thick convex border of the first sacral ribs (*pl*), it presents an oval form, and measures 11 inches in transverse and 10 inches in fore-and-aft diameters, the latter being taken from the middle of the first sacral vertebra to the symphysis.

From the study of the above-described most interesting portion of the dicynodont skeleton, we learn—

1st. That there were no lumbar vertebræ, *i. e.* none bearing the technical anatomical characters of such†; but that free ribs continued to be developed to the pelvic or sacral series.

2nd. That the sacral series includes six vertebræ.

3rd. That the ilium, ischium, and pubis coalesce into an 'os innominatum.'

* Cuvier, 'Ossemens Fossiles,' pl. 17. fig. 39, *b*.

† This negative character is open to the same kind of objection as that relative to the 'hippocampus minor' in animals below man.

4th. That the junction of the ossa innominata with the vertebral column is effected in two ways—by an overlapping or squamous syndesmosis, and by the usual abutments: thus the anterior bony wall or surface of the pelvis, analogous to that formed by the expansion of the iliac bones in mammals, is here formed by the expanded ribs of the first sacral vertebra.

5th. That the ischium of the right side joins that of the left, and the right pubis joins the left pubis; and that both pairs of pelvic hæmapophyses are coextended and confluent, not only along a continuous ischio-pubic symphysis, as in mammals, but so as to obliterate the intervening vacuities called '*foramina ovalia* seu *obturatoria*,' thereby repeating the character of the connate abdominal hæmapophyses in the chelonian plastron.

In the comparison of this new and, at present, unique type of pelvic structure, it is interesting to observe, in connexion with the mammalian tusks in the skull, a mammalian condition of the ischio-pubic symphysis. I lay less stress on the degree of coalescence expressed by the term 'innominate bone,' because in some lizards I have observed a like confluence of iliac, ischial, and pubic bones, yet never with that amount of expansion of the iliac element which the *Dicynodon* shows, and in which, again, may be discerned a mammalian characteristic. If the remains of the huge reptiles of the extinct Dinosaurian order had not revealed to us an extent of sacrum so much surpassing that of all living Saurians, one would have laid more stress on the character of the six sacral vertebræ in the *Dicynodon*, as repeating that in some mammalia. But in this modification we may not be justified in inferring more than that, like the *Megalosaurus* and *Iguanodon*, a heavy trunk was in part supported on a pair of huge hind limbs, and the weight thereupon transferred by a larger proportion of the vertebral column in the *Dicynodon* than in the prone, crawling Crocodiles and Lizards of the present day.

As the lacertian characters prevail in the skull of the *Dicynodon*, so likewise do they in the pelvis: the backward production of the iliac bones, their confluence with the ischium and pubis, never met with in Crocodiles or Chelonians, and, above all, the oblique perforation of the pubis near its acetabular expansion, are all repetitions of structures known only, among existing Reptilia, in the lizard tribes. But the massive and entire anterior or ventral bony walls of the pelvis, the thick tumid acetabular halves of the pubic bones, and the great expanse of the over- or rather under-lapping pleurapophyses of the first sacral vertebra, are dicynodontal specialities, and suggest immense strength in this part of the massive framework of these strange extinct Reptilia.

Explanation of the Plates.

PLATE XXIII.

Dicynodon tigriceps.

Fig. 1. Front view of pelvis: $\frac{1}{4}$ nat. size.

Fig. 2. Side view of pelvis: $\frac{1}{4}$ nat. size.

PLATE XXIV.

Fig. 3. **Back view of sacrum and pelvis:** $\frac{1}{4}$ **nat. size.**

Fig. 4. **Os innominatum, inner surface:** $\frac{1}{4}$ **nat. size.**

The outline restored by dots, where abraded.

(III.) *Notice of a Skull and parts of the Skeleton of* RHYNCHOSAURUS ARTICEPS. *By Professor* OWEN, *F.R.S. &c.*

The rocks in South Africa containing the Dicynodont Reptilia, appear, by other fossils, especially of some plants, to belong to the Triassic period. It is in the New Red Sandstones of Europe and of our own island, that reptilian remains have been discovered which offer the nearest approach, though the gap is wide, to the dicynodont type.

A lizard which, in biting, with trenchant edentulous jaws, may also have pierced its prey by a pair of produced weapons analogous to the tusks of *Dicynodon*, has left its remains in the New Red Sandstone of the Grinsill quarries near Shrewsbury. In this singular species (*Rhynchosaurus articeps**) the premaxillary bones, by their shape and structure—the bony tissue of the produced tips acquiring the hardness and almost the texture of dentine—and by the production of their sharp end (Plate XXV. fig. 2, 22′) beyond the mandible (*ib.* 32), may have inflicted wounds and served a purpose, like those of the upper curved tusks of the *Dicynodon*.

I take the present opportunity of briefly noting a second example of this exceedingly rare fossil reptile, found, like the first, in the Grinsill sandstone, and liberally transmitted to me for examination and description by the Directors of the Museum of Natural History at Shrewsbury. This fossil consists of the skull almost entire (Plate XXV. figs. 1 and 2), with about fifteen of the anterior vertebræ (*ib.* figs. 3 and 4) more or less mutilated; and in the same slab of stone are parts or impressions of several ribs, a humerus (*ib.* fig. 3, 53), antibrachial (*ib.* 54, 55) and metapodial (*ib.* 57) bones. All these parts agree sufficiently in size and other characters with the first-discovered specimen to be referable to the same species.

The apparent greater breadth of the cranium is due to its having been crushed almost flat (fig. 1), which has correspondingly expanded the mandibular rami (fig. 3, 31, 31); but the force has been applied either so gradually, or, whether gradually or suddenly, with such equable support from the surrounding matrix, that there has occurred little fracture or dislocation.

The temporal muscles have encroached upon the sides of the parietal (fig. 1, 7), so as to develope a straight median crest along that bone. The outer and back part of the temporal fossa is bounded by a strong triangular mastoid (*ib.* 8), which unites by an oblique suture with the long postfrontal, 12. The prefrontals are seen at 14 in fig. 1. A portion of the lower or true zygomatic arch, not preserved in the first specimen, is

* Transactions of the Cambridge Philosophical Society, vol. vii.

seen at $_{24}$, in fig. 2. The long, slender, curved and dense premaxillaries ($_{22}$) are in the same good state of preservation. The lower jaw is likewise preserved *in situ*, as when the mouth is shut. There is the same compound structure and entire or imperforate outer surface of the ramus as in the original specimen. The symphysis is very short, fig. 3, *y*. A long and slender bone, on the inner side of the articular end of each ramus, appears to be a part or 'cornu' of the hyoid apparatus, fig. 3, $_{40}$, $_{40}$. The vertebræ are biconcave, with broad horizontally flattened zygapophyses and a moderately high, subquadrate compressed, fore-and-aft extended spine. The ribs acquire thoracic length at about the eighth vertebra from the head, and are longitudinally grooved as in the Ichthyosaur. There is an impression of a broad scapula (*ib.* $_{51}$) and broader coracoid (*ib.* $_{52}$) near the remains of the proximal half of a humerus, *ib.* $_{53}$. The humerus has an expanded proximal end, and a concave outline behind; it has also a large medullary cavity with a compact wall. The radius (*ib.* $_{54}$) and the ulna (*ib.* $_{55}$) are distinct. The remains of apparently a metatarsus show two of the larger and two of the rather more slender of these bones. All bespeak a reptile capable of progression on dry land, as well of swimming in the sea; of one that might leave impressions of its foot-prints on a tidal shore.

The skull of a rare New Zealand Lizard is figured in Plate XXV. fig. 5, as coming nearest to the Rhynchosaur in the proportions of the divided premaxillaries (*ib.* $_{22}$), each with a large and long tooth, which, were it completely confluent with the bone, would add still more to its resemblance to the New Red Sandstone fossil.

Explanation of the Plate.

PLATE XXV. *Rhynchosaurus articeps.*

Fig. 1. Upper view of skull.
Fig. 2. Side view of skull.
Fig. 3. Under view of the mandible and fore part of skeleton.
Fig. 4. Portion of the vertebral column.
Fig. 5. Side view of the skull of *Rhynchocephalus* (recent).

All the figures are of the natural size.

XXXII. *On the Differential Coefficients and Determinants of Lines, and their Application to Analytical Mechanics.* *By* A. Cohen, *Esq.* *Communicated by Professor* Stokes, *Sec. R.S.*

Received May 8,—Read June 19, 1862.

Chapter I.

1. I propose in these pages to prove the principal theorems of dynamics in a manner which appears to me both simpler and more methodical than that in which they are generally proved; and I believe that I shall be able, by applying a few conceptions which spring naturally from the principles of higher algebra and statics, to give a clear interpretation to most of the more complicated formulæ in dynamics, as well as to the several analytical steps which lead to those formulæ.

2. There are many reasons why the diagonal AD, which is constructed on the straight lines AB, AC, should be considered as the *sum* of those two lines. Those reasons may be found developed in De Morgan's 'Double Algebra,' in Warren 'On Imaginary Quantities,' and in the Tract of Benjamin Gompertz 'On Imaginary Quantities.'

I shall therefore call AD (AD being the diagonal of the parallelogram constructed on AB and AC) the *complete sum* of AB and AC, and the two lines AB and AC will be called the *components* of AD. Moreover, denoting AB, AC, AD by P, Q, R respectively, I shall express their relation to one another by the equation

$$R=(P)+(Q).$$

I shall likewise denote by $(-Q)$ a line equal and opposite to Q, and define $(P)-(Q)$ to be the same as $(P)+(-Q)$, calling it *the complete difference* of P and Q.

All lines which have the same length and direction will be considered as equal to one another, so that any line is equivalent to a line through the origin having the same length and direction.

3. It evidently follows from the above definitions, that the complete sum of AB and BC is AC, and the complete difference of AB and AC is BC.

4. Suppose now Q to represent a line which varies with the time t both in length and direction. The *complete difference* of the two consecutive values of Q after an increment of time Δt may be called the *complete increment* of Q, and may be denoted by $\Delta(Q)$. Moreover, if we divide the length of $\Delta(Q)$ by Δt, and take the limit of that ratio, then the line which has that limit for its length, and which has for its direction the direction of $\Delta(Q)$, when Δt diminishes without limit, will be called the complete differential coefficient of Q, and will be denoted by $D_t(Q)$.

5. It will be sometimes found convenient to denote a line of length r, which is parallel or perpendicular to a line P, by ($r \parallel$ to P) or ($r \perp^{r}$ to P). Moreover, if n represent a

numerical quantity, then nP may be used to denote a line which is in the direction of P, and whose length bears to that of P the ratio of n to 1.

6. The following Lemmas, which will be of constant use, are all but self-evident:—

I. If $R=(P)-(Q)$, then $(R)-(P)+(Q)=0$. In short, the ordinary rule of signs holds good.

II. $(nP)\pm(nQ)=n\{(P)\pm(Q)\}$.

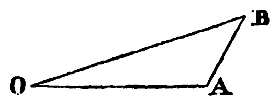
III. The projection on any line or plane of the complete sum or difference of two lines is equal to the sum or difference of their respective projections on the line or plane.

7. Whenever there is no risk of any mistake, the brackets may be omitted in the above and similar formulæ, and the *complete* sum or difference of lines may be spoken of simply as their sum or difference.

What has been hitherto said may be of course extended to all magnitudes whatsoever which can be adequately represented by straight lines, such as forces, velocities, axes of couples, axes of rotation, and accelerations, &c.

8. The application which can be made in dynamics of this conception of the complete differential coefficient of a line, will become at once apparent from the following considerations.

Suppose a particle to be moving from A to B. Let O be any fixed point. Then the particle's velocity is, according to its very definition, represented in magnitude and direction by the limit of $\frac{AB}{\Delta t}$. But AB is the complete difference of O B and O A, or the complete increment of the radius vector O A, and therefore the velocity, being the limit of $\frac{AB}{\Delta t}$, is the complete differential coefficient of the radius vector.



In the next place let O A and O B in the last figure represent in magnitude and direction the successive velocities of a particle at times t and $t+\Delta t$ respectively. Then, since $OB=(OA)+(AB)$, it follows that, if with the velocity OA we compound the velocity AB, we shall obtain the velocity OB, and therefore the particle's acceleration is represented by the limit of $\frac{AB}{\Delta t}$, or by the complete differential coefficient of the velocity OA. Hence we have the following proposition:—

If a particle's velocity and acceleration be represented by straight lines, the velocity will be represented by the complete differential coefficient of the radius vector drawn from a fixed point to the particle, and the acceleration will be represented by the complete differential coefficient of the velocity. Or more briefly, *the velocity is the complete differential coefficient of the radius vector, and the acceleration is the complete differential coefficient of the velocity, and is therefore the second complete differential coefficient of the radius vector.* So that if R denote the radius vector, and V and F denote respectively the velocity and acceleration, we have

$$V=D_t(R), \quad F=D_t(V)=D_t^2(R).$$

9. Such then being the connexion which exists between the differential coefficient of the radius vector and the velocity and acceleration of a particle, I will proceed to prove some of the principal propositions concerning the differential coefficients of lines, and to apply them to the dynamics, or rather the kinematics of a particle.

The first proposition is the following:—

If P, Q, R represent straight lines, and if we have

$$(P)\pm(Q)=R,$$

then

$$D_t(P)\pm D_t(Q)=D_t(R).$$

For let P, Q, R after an interval of time Δt become P′, Q′, R′ respectively, then we have

$$(P')\pm(Q')=R',$$

and therefore, by Lemma I. of section 6, it follows that

$$\{(P')-(P)\}\pm\{(Q')-(Q)\}=(R')-(R),$$

or

$$\Delta(P)\pm\Delta(Q)=\Delta(R).$$

Therefore by Lemma II. of section 6, we have

$$\left(\frac{\Delta(P)}{\Delta t}\right)\pm\left(\frac{\Delta(Q)}{\Delta t}\right)=\frac{\Delta(R)}{\Delta t};$$

and taking the limit of both sides of this equation, we obtain

$$D_t(P)\pm D_t(Q)=D_t(R).$$

Similarly it may be shown that

$$D_t\{(P)\pm(Q)\pm(R)\}=D_t(P)\pm D_t(Q)\pm D_t(R).$$

Moreover, denoting the second complete differential coefficient by D^2_t, it follows that

$$D^2_t\{(P)\pm(Q)\pm(R)\}=D_t\{D_t(P)\pm D_t(Q)\pm D_t(R)\}$$
$$=D^2_t(P)\pm D^2_t(Q)\pm D^2_t(R).$$

10. Suppose now a line Q to have Q_x, Q_y, Q_z for its components parallel to the axes of coordinates Ox, Oy, Oz; it is evident from Lemma III. of section 6, that

$$Q=(Q_x)+(Q_y)+(Q_z).$$

It follows, therefore, from the preceding section, that

$$D_t(Q)=D_t(Q_x)+D_t(Q_y)+D_t(Q_z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

and

$$D^2_t(Q)=D^2_t(Q_x)+D^2_t(Q_y)+D^2_t(Q_z). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II.)}$$

These equations are true whether the axes of coordinates are fixed or move. But supposing the axes to be *fixed* axes, let q_x, q_y, q_z be the respective lengths of Q_x, Q_y, Q_z. Then it is evident that, as the direction of Q_x does not vary, $D_t(Q_x)$ is a line whose direction is that of Q_x or Ox, and whose length is $\frac{dq_x}{dt}$; and similarly, $D^2_t(Q_x)$ is a line whose direction is that of Ox, and whose magnitude is $\frac{d^2q_x}{dt^2}$. Similar results hold good for

$D_t(Q_y)$, $D_t^2(Q_y)$, &c. Therefore the two equations (I.) and (II.) evidently show that the components of $D_t(Q)$ and $D_t^2(Q)$ parallel to Ox are respectively equal to $\frac{dq_x}{dt}$ and $\frac{d^2q_x}{dt^2}$.

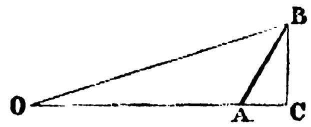
11. It is easy to apply the above results to the velocity and acceleration of a particle. For let Q in the last section stand for the radius vector of a moving particle, then the components of the radius vector are respectively equal to x, y, and z; and since the velocity is the complete differential coefficient, and the acceleration is the second complete differential coefficient of the radius vector, it follows from the last section that the components parallel to Ox of the velocity and of the acceleration are respectively equal to $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$. So that if v_x, v_y, v_z be the components of the velocity, and f_x, f_y, f_z be the components of the acceleration, we have the elementary formulæ

$$v_x=\frac{dx}{dt}, \text{ and similarly } v_y=\frac{dy}{dt}, \quad v_z=\frac{dz}{dt};$$

$$f_x=\frac{d^2x}{dt^2}, \text{ and similarly } f_y=\frac{d^2y}{dt^2}, f_z=\frac{d^2z}{dt^2}.$$

12. Our next proposition will arise from investigating the complete differential coefficient of a line Q, which varies both in magnitude and direction with the time t.

Let Q at time t be the line O A, and let it become O B at time $t+\Delta t$, so that we have



$$\Delta(Q)=(OB)-(OA)=AB.$$

Produce OA to C, making OC=OB, and draw BC. Let OA and OB have for their respective lengths q and $q+\Delta q$, and let angle B O A$=\alpha$.

Then

$$\Delta(Q)=AB=(AC)+(CB).$$

Therefore, by Lemma II. of section 6,

$$\frac{\Delta(Q)}{\Delta t}=\left(\frac{AC}{\Delta t}\right)+\left(\frac{CB}{\Delta t}\right). \quad \ldots \ldots \ldots \ldots \quad \text{(I.)}$$

Now diminish Δt indefinitely and take the limit of the last equation. The limit of $\frac{\Delta(Q)}{\Delta t}$ is $D_t(Q)$, the complete differential coefficient of Q. The limit of $\frac{AC}{\Delta t}$ is evidently a line whose length is the limit of $\frac{AC}{\Delta t}$, or $\frac{dq}{dt}$, and whose direction is that of OA or of Q. Finally, since COB is an isosceles triangle, $\frac{CB}{\Delta t}$ has for its limit a line whose length is OA limit of $\frac{\alpha}{\Delta t}$, and whose direction is perpendicular to OA or Q, and in the plane in which Q is moving at time t; and if ϖ be the rate at which the direction of Q is varying at time t, ϖ=limit of $\frac{\alpha}{\Delta t}$. Therefore, taking the limit of equation (I.), we obtain

$$D_t(Q)=\left(\frac{dq}{dt} \parallel \text{ to } Q\right)+(q\varpi \perp^{r} \text{ to } Q),$$

the latter line ($q\varpi \perp^r$ to Q) being in the plane in which Q is moving at time *t*. This is the fundamental proposition concerning the differential coefficient of a line, and may be stated in the following form:—

The complete differential coefficient of a line Q, *whose length is* Q *and whose direction is at time* t *varying with an angular velocity* ϖ, *is the complete sum or is compounded of two lines, one being* $\frac{dq}{dt}$ *in the direction of* Q, *and the other being* $q\varpi$ *in a direction perpendicular to* Q *and in the plane in which* Q *is moving at time* t. The former of these two lines would evidently be the complete differential coefficient of Q, if the *length* of Q *only* varied, and the latter would be its complete differential coefficient if the *direction* of Q *only* varied; and in this sense it may therefore be said that the complete differential coefficient of a line is the *complete sum of the two partial differential coefficients* obtained by varying separately the length and the direction of Q. One of these partial differential coefficients may be called the *length-differential coefficient*, and the other the *direction-differential coefficient* of Q, and the complete sum of these two constitutes the complete differential coefficient of Q.

13. Let Q in the preceding section stand for the velocity of a moving particle. Then $D_t(Q)$ will be the particle's acceleration, q will be the velocity v, and the direction of Q will be that of the tangent to the particle's path. Finally, ϖdt will be the angle between two consecutive tangents, so that $\varpi dt = \frac{ds}{\varrho}$, ds being an element of the particle's path, and ϱ the absolute radius of curvature. Therefore $\varpi = \frac{1}{\varrho}\frac{ds}{dt} = \frac{v}{\varrho}$. It follows then at once from the last section, that $D_t(Q)$, the particle's acceleration, is compounded of $\frac{dv}{dt}$ along the tangent, and $v\varpi$ or $\frac{v^2}{\varrho}$ perpendicular to the tangent and in the plane in which the radius vector is moving at time t. In other words, the resolved part of the acceleration along the tangent is $\frac{dv}{dt} = \frac{d^2s}{dt^2}$, and the resolved part along the absolute radius of curvature is $\frac{v^2}{\varrho}$.

14. The same fundamental proposition of section 12 enables us to investigate $D_t^2(Q)$, the second complete differential coefficient of a line Q, if we suppose that line to move always in the same plane. We have, namely,

$$D_t(Q) = \left(\frac{dq}{dt} \parallel \text{to } Q\right) + (\varpi q \perp^r \text{to } Q).$$

Now, in order to find $D_t^2(Q)$, we must ascertain the complete differential coefficients of $\left(\frac{dq}{dt} \parallel \text{to } Q\right)$ and of $(\varpi q \perp^r \text{to } Q)$. The complete differential coefficient of the former line $\left(\frac{dq}{dt} \parallel \text{to } Q\right)$ is, by section 12,

$$\left(\frac{d^2q}{dt^2} \parallel \text{to } Q\right) + \varpi\left(\frac{dq}{dt} \perp^r \text{to } Q\right).$$

Again, the complete differential coefficient of the line ($q\varpi \perp^r$ to Q) is similarly the complete sum of the line $\left(\frac{d}{dt}(q\varpi)\perp^r \text{to Q}\right)$ and of a line whose length is $q\varpi^2$ and whose direction is perpendicular to the line ($q\varpi \perp^r$ to Q), and whose direction is therefore evidently opposite to that of Q. Hence $D_t^2(Q)$ is the complete sum of

$$\left(\frac{d^2q}{dt^2} \parallel \text{to Q}\right) \text{ and } \left(\varpi\frac{dq}{dt}\perp^r \text{to Q}\right) \text{ and } \left(\frac{d}{dt}(q\varpi)\perp^r \text{to Q}\right) \text{ and } (-q\varpi^2 \parallel \text{to Q}).$$

Therefore

$$D_t^2(Q)=\left\{\left(\frac{d^2q}{dt^2}-q\varpi^2\right) \parallel \text{to Q}\right\}+\left\{\left(q\varpi+\frac{d}{dt}(q\varpi)\right)\perp^r \text{to Q}\right\}.$$

In other words, the components of $D_t^2(Q)$ parallel and perpendicular to Q are respectively

$$\frac{d^2q}{dt^2}-q^2\varpi \text{ and } q\varpi+\frac{d}{dt}(q\varpi).$$

15. This last result may be easily applied to the dynamics of a particle. For, let Q stand for the radius vector of a particle moving in a given plane. Let that radius vector have r for its length and ϖ for its angular velocity; then, since the acceleration equals the second differential coefficient of the radius vector, it follows at once from the last section, that the components of the acceleration parallel and perpendicular to the radius vector are respectively

$$\frac{d^2r}{dt^2}-r\varpi^2 \text{ and } r\varpi+\frac{d}{dt}(r\varpi), \text{ or } \frac{1}{r}\frac{d}{dt}(r^2\varpi).$$

16. This last result is, however, but a particular instance of the connexion which exists between the actual motion of a particle, and its motion relatively to axes which move in the same plane as the particle moves. It will be found that that connexion may be easily deduced from the solution of the following problem:—

"Supposing the axes of coordinates Ox and Oy to move about O in the plane of xy with an angular velocity ϖ at time t, it is required to find the complete differential coefficient of a line Q which moves in that plane, the lengths of Q's components along the moving axes being given."

Let Q have for its components Q_x and Q_y, and let the respective lengths of these be q_x and q_y. Then, by Lemma III. of section 6, we have

$$Q=(Q_x)+(Q_y).$$

Whence it follows that

$$D_t(Q)=D_t(Q_x)+D_t(Q_y).$$

Now since Q_x and Q_y vary in direction as well as magnitude, and since the angular velocity of their change of direction is ϖ, we have, by the fundamental proposition in section 12,

$$\left.\begin{aligned} D_t(Q_x)&=\left(\frac{dq_x}{dt} \parallel \text{to O}x\right)+(\varpi q_x \perp^r \text{to O}x),\\ D_t(Q_y)&=\left(\frac{dq_y}{dt} \parallel \text{to O}y\right)+(\varpi q_y \perp^r \text{to O}y).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I.)}$$

But since the lines ($\varpi q_x \perp^r$ to Ox) and ($\varpi q_y \perp^r$ to Oy) are respectively proportional and perpendicular to Q_x and Q_y, and since the latter lines have Q for their complete sum, it evidently follows that the former two lines have for their complete sum a line which is perpendicular to Q, and whose length is ϖq.

Hence

$$D_t(Q)=\left(\frac{dq_x}{dt} \parallel \text{ to } Ox\right)+\left(\frac{dq_y}{dt} \parallel \text{ to } Oy\right)+(\varpi q \perp^r \text{ to } Q). \quad . \quad . \quad . \quad . \quad \text{(II.)}$$

17. This last formula is true whether the axes be rectangular or oblique, and may be made the basis of all the formulæ of relative motion in one plane.

It may be observed that the line

$$\left(\frac{dq_x}{dt} \parallel \text{ to } Ox\right)+\left(\frac{dq_y}{dt} \parallel \text{ to } Oy\right)$$

is what would be the complete differential coefficient of Q if the coordinate axes were fixed; and it may therefore be called the complete differential coefficient relative to the moving axes, or, more briefly, the *relative differential coefficient* of Q. So that the above formula shows that *the complete differential coefficient of* Q is *its relative differential coefficient together with a line* ($\varpi q \perp^r$ to Q), the latter line being drawn towards the direction in which the axes are revolving.

18. If the axes of coordinates be rectangular, then the line ($\varpi q_x \perp^r$ to Ox) is evidently the same as ($\varpi q_x \parallel$ to Oy), and the line ($\varpi q_y \perp^r$ to Oy) is the same as ($-\varpi q_y \parallel$ to Ox); and therefore, looking at the equations (I.) in section 16, we see that

$$\begin{aligned} D_t(Q)&=D_t(Q_x)+D_t(Q_y)\\ &=\left(\left(\frac{dq_x}{dt}-\varpi q_y\right) \parallel \text{ to } Ox\right)+\left(\left(\frac{dq_y}{dt}+\varpi q_x\right) \parallel \text{ to } Oy\right). \end{aligned}$$

In other words, the components of $D_t(Q)$ parallel to Ox and Oy are respectively

$$\frac{dq_x}{dt}-\varpi q_y \text{ and } \frac{dq_y}{dt}+\varpi q_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(III.)}$$

The same result may be also deduced from formula (II.) in the same section, if we resolve the line ($\varpi q \perp^r$ to Q) along the rectangular axes of x and y.

19. Let us apply the above formulæ first to the velocity of a particle.

Suppose, then, a particle to move in a given plane, and that the rectangular axes of coordinates in that plane revolve about the origin with an angular velocity ϖ at time t. Let v_x and v_y be the components of the particle's velocity along the moving axes. Then, since the velocity is the complete differential coefficient of the radius vector, and since x and y are the components of that radius vector, it follows from the formulæ (III.) of the last section, that

$$v_x=\frac{dx}{dt}-y\varpi,$$

$$v_y=\frac{dy}{dt}+x\varpi.$$

If the radius vector be chosen as axis of x, then $x=r$, $y=0$; therefore $v_x=\frac{dx}{dt}$ along the radius vector, $v_y=r\dot{\varpi}$ perpendicular to the radius vector, where ϖ is the angular velocity of the radius vector.

20. Let us now apply the same formulæ (III.) of section 18 to the acceleration of a particle. Let v_x and v_y, as before, denote the components of the velocity, and let f_x and f_y denote the components of the acceleration of the particle. Then, since the acceleration is the complete differential coefficient of the velocity which has v_x, v_y for its components, it follows at once from the formulæ (III.), that

$$f_x=\frac{dv_x}{dt}-v_y\varpi,$$

$$f_y=\frac{dv_y}{dt}+v_x\varpi.$$

Suppose now the axis of x to be the radius vector, then we have already shown that $v_x=\frac{dv}{dt}$, $v_y=r\varpi$. Therefore by substituting these values in the last formulæ, we see that f_x, the acceleration along the radius vector, is $\frac{d^2r}{dt^2}-r\varpi^2$, and f_y, the acceleration perpendicular to the radius vector, is $\frac{dt}{d}(r\varpi)+\varpi\frac{dr}{dt}=\frac{1}{r}\frac{d}{dt}(\varpi r^2)$, which is the same result as was obtained in section 15.

21. Returning to the more general case, we have, as before,

$$f_x=\frac{dv_x}{dt}-v_y\varpi,$$

$$f_y=\frac{dv_y}{dt}+v_x\varpi;$$

and substituting in these the values already obtained for v_x and v_y, namely, $\frac{dx}{dt}-y\varpi$, $\frac{dy}{dt}+x\varpi$, we find

$$\left.\begin{aligned} f_x&=\frac{d^2x}{dt^2}-2\frac{dy}{dt}\varpi-\varpi^2x-y\frac{d\varpi}{dt},\\ f_y&=\frac{d^2y}{dt^2}+2\frac{dx}{dt}\varpi-\varpi^2y+x\frac{d\varpi}{dt},\end{aligned}\right\} \quad \ldots\ldots \quad \text{(IV.)}$$

which are the formulæ for the components of the acceleration in terms of the coordinates of the particle.

22. The last formulæ (IV.) may also be obtained in the following manner.

Let Q_x and Q_y represent respectively the components of the radius vector Q along the axes of x and y. Then

$$Q=(Q_x)+(Q_y).$$

Therefore

$$D_t^2(Q)=D_t^2(Q_x)+D_t^2(Q_y).$$

But since the axes of x and y revolve at time t with an angular velocity ϖ, and since Q_x and Q_y have x and y for their respective lengths, it follows from section 14, that

$$D_t^2(Q_x)=\left\{\left(\frac{d^2x}{dt^2}-x\varpi^2\right) \parallel \text{ to } Ox\right\}+\left\{\frac{1}{x}\frac{d}{dt}(x^2\varpi) \perp^r \text{ to } Ox\right\},$$

and that

$$D_t^2(Q_y)=\left\{\left(\frac{d^2y}{dt^2}-y\varpi^2\right) \parallel \text{ to } Oy\right\}+\left\{\frac{1}{y}\frac{d}{dt}(y^2\varpi) \perp^r \text{ to } Oy\right\}.$$

Whence it is easy to see that the components of $D_t^2(Q)$, or of the particle's acceleration, are

$$\left.\begin{aligned} f_x&=\frac{d^2x}{dt^2}-x\varpi^2-\frac{1}{y}\frac{d}{dt}(\varpi y^2),\\ f_y&=\frac{d^2y}{dt^2}-y\varpi^2+\frac{1}{x}\frac{d}{dt}(\varpi x^2),\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (V.)$$

which equations are clearly the same as those obtained in the preceding section.

We shall soon prove similar formulæ for the more general case of a particle and axes of coordinates moving in any manner whatsoever in space of three dimensions, and therefore, in order to prevent needless repetition, we shall postpone the further discussion and complete interpretation of the equations (IV.) or (V.).

23. It is, however, interesting here to observe that all the results already obtained may be readily deduced from the principles of what Professor De Morgan has called "Double Algebra." According to those principles, namely, the radius vector R whose length is r and whose inclination to a fixed line is θ, is symbolically represented by $r\varepsilon^{\theta\sqrt{-1}}$, so that we have

$$R=r\varepsilon^{\theta\sqrt{-1}}.$$

Therefore

$$D_t(R)=\varepsilon^{\theta\sqrt{-1}}\left(\frac{dr}{dt}+r\frac{d\theta}{dt}\sqrt{-1}\right);$$

and the last expression represents the complete sum of

$$\left(\frac{dr}{dt} \parallel \text{ to R}\right) \text{ and } \left(r\frac{d\theta}{dt} \perp^r \text{ to R}\right).$$

This result is the same as that arrived at in section 12.

Again, $D_t^2(R)=D_t(D_t(R))=$

$$\varepsilon^{\theta\sqrt{-1}}\left(\frac{d^2r}{dt^2}+\sqrt{-1}\,\frac{d}{dt}\left(r\frac{d\theta}{dt}\right)+\frac{d\theta}{dt}\sqrt{-1}\left(\frac{dr}{dt}+r\frac{d\theta}{dt}\sqrt{-1}\right)\right)$$

$$=\varepsilon^{\theta\sqrt{-1}}\left(\frac{d^2r}{dt^2}-r\overline{\frac{d\theta}{dt}}^2+\sqrt{-1}\left(\frac{d}{dt}\left(r\frac{d\theta}{dt}\right)+\frac{dr}{dt}\frac{d\theta}{dt}\right)\right);$$

and this expression represents the complete sum of a line

$$\frac{d^2r}{dt^2}-r\overline{\frac{d\theta}{dt}}^2 \parallel \text{ to R and a line } \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)+\frac{dr}{dt}\frac{d\theta}{dt} \perp^r \text{ to R}.$$

This result is the same as that arrived at in section 14.

Finally, in order to obtain the formulæ for relative motion, we have merely to put

$$R = r\varepsilon^{(\theta+\alpha)\sqrt{-1}},$$

where θ is the angle made by R with the moving axis of x, and α is the angle made by that moving axis with a fixed line. It follows then that

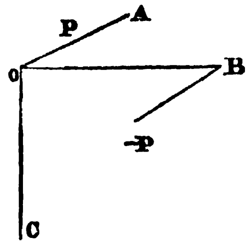
$$D_t(R) = \varepsilon^{\alpha\sqrt{-1}}\frac{d}{dt}\left(r\varepsilon^{\sqrt{-1}}\right) + r\frac{d\alpha}{dt}\sqrt{-1}\,\varepsilon^{(\theta+\alpha)\sqrt{-1}}.$$

Now it is evident that $\varepsilon^{\alpha\sqrt{-1}}\frac{d}{dt}\left(r\varepsilon^{\theta\sqrt{-1}}\right)$ represents *the relative differential coefficient* of R, and $r\frac{d\alpha}{dt}\sqrt{-1}\,\varepsilon^{(\theta+\alpha)\sqrt{-1}}$ represents a line $r\frac{d\alpha}{dt}$ ⊥r to R. We thus obtain the same result as in section 16.

By differentiating again it would also be easy to deduce the result of section 20, if we observe that $\varepsilon^{\alpha\sqrt{-1}}\frac{d^2}{dt^2}\left(r\varepsilon^{\theta\sqrt{-1}}\right)$ represents the particle's relative acceleration whose components are $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$.

Chapter II.

24. In order to extend the formulæ which we have proved for the motion of a particle in one plane to the motion of a particle in space, it will be found very convenient to make use of a conception which presents itself in statics, as soon as the equilibrium of a solid body is treated of in that science.



Let O A and O B be any two straight lines drawn from the origin O. If then O A represent a force P, and if we apply at B a force $-$P, we shall obtain *a couple*. Let O C be the *axis* of that couple. We know then from statics that, if O A and O B have for their projections on the axes of coordinates X, Y, Z and x, y, z, then O C has for its projections

$$zY - yZ, \quad xZ - zX, \quad yX - xY. \qquad \text{(I.)}$$

Now the relation which the line O C bears to the lines O A and O B is one which not only presents itself in statics, but which also plays a very important part in the differentiation of lines, and in the dynamics both of a particle and of a body. For this reason it will be proper to treat of the relation in question quite independently of statical considerations; and since the expressions (I.), which are the projections of O C, are evidently what are called *determinants*, I shall call the line O C *the determinant of* O B *to* O A.

Hence we have the following definition:—

"The determinant of a line Q to a line P is a line which is equal to twice the area of the triangle of which the lines P and Q drawn from the origin are sides, and which is perpendicular to that area, and the line is moreover drawn in such a direction that, to an eye looking along it towards the origin, the revolution of Q towards P appears to be a revolution in the positive direction."

The determinant of Q to P may be briefly denoted by

$$\det (Q, P).$$

It is evident from the above definition, that the determinant of Q to P is a line equal and opposite to the determinant of P to Q.

Moreover, if the projections of P on the axes of coordinates be p_x, p_y, p_z, and those of Q be q_x, q_y, q_z, then it follows from the formulæ (I.), that the determinant of Q to P or det (Q, P) has for its projections

$$q_y p_z - q_z p_y, \quad q_z p_x - q_x p_z, \quad q_x p_y - q_y p_x. \quad \ldots \ldots \quad \text{(II.)}$$

25. The connexion which exists between the notion of a determinant of lines and the elementary conceptions of dynamics may be easily made apparent. For suppose a particle at the extremity B of O B to be revolving about the line O A with an angular velocity represented in magnitude by O A, then if O C be drawn perpendicular to the plane A O B, and equal to twice the area A O B, it is evident that O C will represent the linear velocity of the particle. But O C is then by definition the same thing as the *determinant of* O A *to* O B. Whence it follows *that the determinant of* O A *to* O B *represents the velocity of the point* B, *due to a rotation whose axis and angular velocity are represented by* O A.

This result, together with the result of the preceding section, may then be recapitulated in the following manner. If V represent the velocity of a particle at the extremity of the radius vector R, and the particle rotates about an axis which is represented by the line Ω, then, if the angular velocity is represented by the length of Ω, we have

$$V = \det (\Omega, R).$$

Secondly, if P represent a force at the origin and R represent the radius vector at the extremity of which a force −P acts, then the axis of the couple (P, −P) is det (P, R) or det (R, −P); so that det (R, P) is what French writers call "the moment-axis of a force P with respect to the origin."

26. Such, then, being the connexion between determinants and statical and dynamical conceptions, I will proceed to prove some of the more important propositions concerning the determinants of lines.

The most important theorem concerning the determinants of lines is the following:—
"If P, P′, and Q be three straight lines drawn from the origin, then

$$\det (P, Q) + \det (P', Q) = \det \{(P) + (P'), Q\}."$$

This proposition might be easily proved by geometry, but it is at once deducible from statics. For, consider two couples having a common arm Q, and having forces P and P′ respectively acting at the extremity of Q at the origin O. The resultant of those two couples will be a couple having the same arm Q, and having for its force acting at O the resultant of P and P′, or (P)+(P′). Now it is proved in statics that the *axis* of this resultant couple is the complete sum of the axes of two component couples. Therefore, substituting for those axes the equivalent determinants, we see that the determinant of (P)+(P′) to Q is the complete sum of the determinant of P to Q, and of the determinant of P′ to Q. Thus we have

$$\det(P, Q)+\det(P', Q)=\det\{(P)+(P'), Q\} \quad \text{(I.)}$$

And it may similarly be shown that

$$\det(P, Q)-\det(P' Q)=\det\{(P)-(P'), Q\} \quad \text{(II.)}$$

The same proposition follows also easily from a consideration of the linear expressions (II.) in section 24 for the projections of a determinant, and is in fact equivalent to a fundamental theorem concerning algebraic determinants, which theorem may be found in SALMON'S 'Lessons on Higher Algebra,' section 19, page 9.

27. The proposition proved in the last section will be found to be of constant use in explaining and shortening analytical processes in mechanics. One useful application can be made of it in proving "the parallelogram of angular velocities." For taking Q to be the radius vector of a particle, and P and P′ to represent two axes and angular velocities of rotation then the formula (I.) of the last section translates itself at once by means of section 25 into the following proposition:—"The linear velocity of a particle due to a rotation whose axis and angular velocity are represented by the line P, compounded with the linear velocity due to a rotation similarly represented by the line P′, is equivalent to the linear velocity due to a rotation represented by the complete sum of P and P′." And this is evidently the same as the proposition called "the parallelogram of angular velocities of rotation."

28. Let us next investigate the complete differential coefficient of det (P, Q).

We will first premise that, if m be any numerical quantity, it follows evidently from the definition of a determinant, that

$$\det(mP, Q)=m\det(P, Q)=\det(P, mQ). \quad \text{(I.)}$$

Suppose now that P and Q after an interval of time Δt become respectively (P)+(ΔP), (Q)+(ΔQ), the sign + here denoting the *complete* sum. Then the complete increment of det (P, Q) is

$$\left.\begin{aligned}&\det(P+\Delta P, Q+\Delta Q)-\det(P, Q)=\\&\quad\det(P+\Delta P, Q+\Delta Q)-\det(P+\Delta P, Q)+\det(P+\Delta P, Q)-\det(P, Q).\end{aligned}\right\} \quad \text{(II.)}$$

But it follows from formula (II.) of section 26, that

$$\det(P+\Delta P, Q+\Delta Q)-\det(P+\Delta P, Q)=\det(P+\Delta P, \Delta Q);$$

and similarly,

$$\det(P+\Delta P, Q)-\det(P, Q)=\det(\Delta P, Q).$$

Substitute, then, these values in the above equation (II.), and divide both sides by Δt by means of the above formula (I.), and finally let Δt diminish without limit. We thus obtain for the complete differential coefficient of det (P, Q)

$$\det(P, D_t(Q))+\det(D_t(P), Q).$$

We have therefore the equation

$$D_t\{\det(P, Q)\} = \det\{P, D_t(Q)\} + \det\{D_t(P), Q\}.$$

The same equation may also be proved by considering the algebraical determinants which represent the projections of det (P, Q); and it may in fact be easily deduced from the following identical equation,

$$\frac{d}{dt}(q_z p_y - q_y p_z) = \left(\frac{dq_z}{dt} p_y - \frac{dq_y}{dt} p_z\right) + \left(q_z \frac{dp_y}{dt} - q_y \frac{dp_z}{dt}\right).$$

29. It may be here observed that the formulæ (I.) and (II.) in section 26, and the formulæ (I.) and (II.) in section 28, show that there exists an intimate *symbolical* connexion between det (P, Q) and the product P, Q. In fact the only difference between their symbolical properties consists in P and Q *not* being commutative in the expression det (P, Q), and being so in the expression for the product.

30. There is one more proposition which is often very useful in analytical dynamics. Let it be required to find det (R, Q′), where Q′ itself equals det (P, Q). Let the required line det (R, Q′) be denoted by U. Then, by the definition of a determinant, U is perpendicular on R and on Q′, which last line is itself perpendicular on the plane containing P and Q. Hence it follows that U is perpendicular on R and in the plane containing P and Q.

We have still to find the magnitude of U. For this purpose let the angle which R makes with the plane containing P and Q be ψ, so that ψ is the complement of the angle between R and Q′.

Moreover, let θ be the angle between P and Q, and let the magnitudes of P, Q, Q′, R be denoted by p, q, q', r respectively. Then, since U is det (R, Q′), it follows from the definition of a determinant, that the length of U equals $rq' \sin\left(\frac{\pi}{2} - \psi\right)$ or $rq' \cos\psi$. Similarly, $q' = pq \sin\theta$. Hence the length of U equals $pqr \sin\theta \cos\psi$.

There are two cases especially which frequently occur in dynamics, first, when R is identical with Q, and secondly, when R is perpendicular on Q.

Let us first take the case of R being identical with Q; then $\psi = 0$ and $r = q$. Therefore the required determinant is a line in the plane containing P and Q, and perpendicular on R or Q, and its length equals $pq^2 \sin\theta$.

If, moreover, Q is perpendicular on P, then the required determinant is in the direction of P, and its length equals pq^2, since $\theta = \frac{\pi}{2}$. So that, if P is perpendicular on Q, we see that det {Q, det (P, Q)} is a line pq^2 in the direction of P; and therefore evidently det {Q, det (Q, P)} is a line pq^2 opposite to P.

Let us now take the case of R being perpendicular on Q. Then it might be easily proved by spherical trigonometry, that $\sin\theta \cos\psi$ equals the cosine of the angle between R and P. But we will prove this by analysis, because in doing so we shall meet with formulæ which will be of use in the sequel.

Let, then, the components of P, Q, Q′, R parallel to any three axes of coordinates be denoted by p_x, p_y, p_z, q_x, &c., q'_x, &c., r_x, &c. Then, if we denote the components of U=det (R, Q′) by u_x, u_y, u_z, we have, by section 24,

$$u_x = q'_z r_y - q'_y r_z; \qquad \text{(I.)}$$

and since Q′=det (P, Q), we have

$$q'_x = q_z p_y - q_y p_z,$$
$$q'_y = q_x p_z - q_z p_x,$$
$$q'_z = q_y p_x - q_x p_y.$$

Hence, substituting the values of q'_z and q'_y in (I.), we get

$$u_x = p_x(q_y r_y + q_z r_z) - q_x(p_y r_y + p_z r_z).$$

Now by hypothesis R is perpendicular on Q: hence

$$q_x r_x + q_y r_y + q_z r_z = 0;$$

therefore

$$q_y r_y + q_z r_z = -q_x r_x.$$

Therefore

$$u_x = -q_x(p_x r_x + p_y r_y + p_z r_z).$$

But as q and r denote the magnitudes of P and R, it is evident that

$$p_x r_x + p_y r_y + p_z r_z = pr \cos \phi,$$

where ϕ denotes the angle between P and R. Therefore

$$u_x = -prq_x \cos \phi.$$

Similarly

$$u_y = -prq_y \cos \phi,$$
$$u_z = -prq_z \cos \phi.$$

Therefore the line U of which u_x, u_y, u_z are the components is a line in direction opposite to Q, and whose length equals $prq \cos \phi$, q being the length of Q. Hence if R be perpendicular on Q, then det (R, det (P, Q)) equals $-pqr \cos \phi$ in the direction of Q.

It follows from the above proposition, that, if Q′ or det (P, Q) represent a force or acceleration which acts at the extremity of the radius vector R, and if Q be perpendicular on R, then the moment-axis of that force or acceleration about the origin is $-pqr \cos \phi$ in the direction of Q, and the moments of such force or acceleration about the coordinate axes are respectively $-pr \cos \phi\, q_x$, $-pr \cos \phi\, q_y$, $-pr \cos \phi\, q_z$; and as ϕ is the angle between P and the radius vector R, $pr \cos \phi = xp_x + yp_y + zp_z$, if x, y, z be the coordinates of the extremity of the radius vector.

Chapter III.

31. We are now in a condition to treat fully of the motion of a particle in space of three dimensions; and it will be found that the propositions which have just been proved concerning the determinants of lines, will enable us to show how all the results

arrived at as to a particle's motion in one plane may be extended to motion in space generally.

32. Suppose Q to represent a line drawn from the origin, varying both in direction and magnitude in any manner whatsoever, and let it be required to investigate $D_t(Q)$ the complete differential coefficient of Q.

Let the length of Q be q at time t, and let the direction of Q be revolving at time t about a line whose direction is that of the line represented by Ω, and let the length of Ω be the angular velocity ϖ, with which Q's direction is revolving at time t.

It has been already shown in section 12 that $D_t(Q)$ is in all cases the complete sum of the two partial differential coefficients which are obtained by varying separately the length and direction of Q. Now the former partial differential coefficient is evidently $\left(\frac{dq}{dt} \parallel \text{to } Q\right)$, and the other partial differential coefficient is, by section 25, equal to det (Ω, Q). Hence we have the following fundamental equation,

$$D_t(Q)=\left(\frac{dq}{dt} \parallel \text{to } Q\right)+\det(\Omega, Q). \quad \text{(I.)}$$

33. It is not difficult to deduce from the last equation the expression for $D_t^2(Q)$, the second complete differential coefficient of Q. In order to find that expression we must take the complete differential coefficient of each of the expressions of which the right-hand member of equation (I.) is composed. For this purpose represent for a moment the line $\left(\frac{dq}{dt} \parallel \text{to } Q\right)$ by Q_1. Then it follows from the fundamental formula of the preceding section, that

$$D_t^2(Q)=D_t(Q_1)+D_t(\det(\Omega, Q)).$$

Now the formula (I.) of the last section gives evidently

$$D_t(Q_1)=\left(\frac{d^2q}{dt^2} \parallel \text{to } Q_1\right)+\det(\Omega, Q_1),$$

or

$$\left(\frac{d^2q}{dt^2} \parallel \text{to } Q\right)+\det(\Omega, Q_1).$$

Moreover we have, according to section 28 of the preceding Chapter,

$$D_t\{\det(\Omega, Q)\}=\det\{D_t(\Omega), Q\}+\det\{\Omega, D_t(Q)\}.$$

But since

$$D_t(Q)=(Q_1)+\det(\Omega, Q),$$

it follows from section 26 of the preceding Chapter, that

$$\det(\Omega, D_t(Q))=\det(\Omega, Q_1)+\det\{\Omega, \det(\Omega, Q)\}.$$

Therefore, collecting the above results, we obtain

$$D_t^2(Q)=D_t(Q_1)+D_t\{\det(\Omega, Q)\}=$$

$$\left(\frac{d^2q}{dt^2} \parallel \text{to } Q\right)+2\det(\Omega, Q_1)+\det(D_t(\Omega), Q)+\det\{\Omega, \det(\Omega, Q)\}.$$

The two last terms of this expression are evidently what would be $D_t^2(Q)$ if $\frac{dq}{dt}$ were zero, that is to say, if Q did not vary in *magnitude*; and $\left(\frac{d^2q}{dt^2} \parallel \text{to } Q\right)$ is evidently what $D_t^2(Q)$ would be if Q did not vary in *direction*; so that we have the following proposition:—

"$D_t^2(Q)$ is the complete sum of the two partial second differential coefficients obtained by varying separately the length and the direction of Q, together with 2 det (Ω, Q_1), where Q_1 is the line $\left(\frac{dq}{dt} \parallel \text{to } Q\right)$."

34. Suppose Q to be R the radius vector of a moving particle, the length of which radius vector is r, then $D_t^2(Q)$ is the particle's acceleration; Q_1 is $\left(\frac{dr}{dt} \parallel \text{to } R\right)$, and is therefore the velocity along the radius vector. If, then, we denote this by R_1, the equation arrived at in the last section shows that the acceleration is compounded of

$$\left(\frac{d^2r}{dt^2} \parallel \text{to } R\right) + 2 \det(\Omega, R_1),$$

and of what would be the particle's acceleration if R did not vary in magnitude, that is to say, if the particle simply revolved about the origin. And this latter acceleration is again compounded of det $(D_t(\Omega), R)$ and det $\{\Omega, \det(\Omega, R)\}$. The last line is, by section 30, a line drawn from the extremity of R, or from the particle, perpendicular to and towards Ω, and whose magnitude is $\varpi^2 p$, p being the length of that perpendicular.

35. The above result is, however, but a particular instance of the theory of the motion of a particle relatively to axes which revolve about the origin, a subject which we are now in a condition to treat of very simply in its utmost generality. That theory will be found to depend upon the solution of the following problem:—

"Supposing the axes of coordinates Ox, Oy, Oz to revolve round the origin O about an axis Ω at time t with angular velocity ϖ (which is the length of Ω), it is required to find the complete differential coefficient of a line Q, the components of Q along the coordinate axes being given."

Let Q have for its components Q_x, Q_y, Q_z, and let the respective lengths of these be q_x, q_y, q_z. Then evidently

$$Q=(Q_x)+(Q_y)+(Q_z).$$

Therefore

$$D_t(Q)=D_t(Q_x)+D_t(Q_y)+D_t(Q_z).$$

But, by the fundamental formula of section 32, we have

$$D_t(Q_x)=\left(\frac{dq_x}{dt} \parallel \text{to } Ox\right)+\det(\Omega, Q_x),$$

$$D_t(Q_y)=\left(\frac{dq_y}{dt} \parallel \text{to } Oy\right)+\det(\Omega, Q_y),$$

$$D_t(Q_z)=\left(\frac{dq_z}{dt} \parallel \text{to } Oz\right)+\det(\Omega, Q_z).$$

Now we know, from section 26 of Chapter II., that

$$\begin{aligned}&\det(\Omega, Q_x)+\det(\Omega, Q_y)+\det(\Omega, Q_z)\\&=\det\{\Omega, (Q_x)+(Q_y)+(Q_z)\}\\&=\det(\Omega, Q).\end{aligned}$$

Therefore

$$D_t(Q)=\left(\frac{dq_x}{dt}\parallel \text{to } Ox\right)+\left(\frac{dq_y}{dt}\parallel \text{to } Oy\right)+\left(\frac{dq_z}{dt}\parallel \text{to } Oz\right)+\det(\Omega, Q).$$

This formula is true whether the axes be rectangular or oblique, and may be made the basis of all the formulæ of relative motion.

It may be observed that, if the coordinate axes did not move, $D_t(Q)$ would be equivalent to

$$\left(\frac{dq_x}{dt}\parallel \text{to} Ox\right)+\left(\frac{dq_y}{dt}\parallel \text{to} Oy\right)+\left(\frac{dq_z}{dt}\parallel \text{to} Oz\right).$$

So that the line represented by the last expression may be called the differential coefficient of Q *relatively to the moving axis*, or, more briefly, *the relative differential coefficient of* Q. The above formula of the last section therefore shows that *the complete differential coefficient of* Q *is the relative differential coefficient of* Q *together with det* (Ω, Q).

This proposition exactly corresponds with the proposition in section 17 of Chapter I.

36. Assuming now the axes of coordinates to be rectangular, we know, from formulæ (II.) in section 24 of Chapter II., that $\det(\Omega, Q)$ has for its components

$$\begin{aligned}&q_z\varpi_y-q_y\varpi_z \text{ parallel to } Ox,\\&q_x\varpi_z-q_z\varpi_x \text{ parallel to } Oy,\\&q_y\varpi_x-q_x\varpi_y \text{ parallel to } Oz.\end{aligned}$$

Therefore it follows from the preceding section, that the components of $D_t(Q)$ are

$$\begin{aligned}&\frac{dq_x}{dt}+q_z\varpi_y-q_y\varpi_z,\\&\frac{dq_y}{dt}+q_x\varpi_z-q_z\varpi_x,\\&\frac{dq_z}{dt}+q_y\varpi_x-q_x\varpi_y.\end{aligned}$$

These formulæ are in fact simply the analytical expression of the fundamental proposition in the preceding section, and correspond exactly to the formulæ in section 18 of Chapter I.

37. Let us now apply the above formulæ to dynamics, and first to the velocity of a particle.

Suppose, then, a particle to move in space in any manner whatsoever, and suppose that the rectangular axes of coordinates revolve about a line Ω at time t with angular velocity ϖ, ϖ being the length of Ω. Let v_x, v_y, v_z be the components of the particle's velocity, and ϖ_x, ϖ_y, ϖ_z the components of Ω. Then, since the velocity is the complete differential coefficient of the radius vector R of the particle, and since x, y, z are the components of

R, it follows at once from the formulæ of the preceding section, that

$$v_x=\frac{dx}{dt}+z\varpi_y-y\varpi_z,$$

$$v_y=\frac{dy}{dt}+x\varpi_z-z\varpi_x,$$

$$v_z=\frac{dz}{dt}+y\varpi_x-x\varpi_y.$$

These formulæ simply express the fact, that the absolute velocity is equivalent to the relative velocity together with det (Ω, R).

38. Let us next apply the formulæ to the acceleration of a particle. Let, as before, v_x, v_y, v_z be the components of the particle's velocity V, and let f_x, f_y, f_z be the components of the particle's acceleration. Then, since the acceleration is the complete differential coefficient of the velocity, of which v_x, v_y, v_z are the components, it follows at once from section 36, that

$$f_x=\frac{dv_x}{dt}+v_z\varpi_y-v_y\varpi_z,$$

$$f_y=\frac{dv_y}{dt}+v_x\varpi_z-v_z\varpi_x,$$

$$f_z=\frac{dv_z}{dt}+v_y\varpi_x-v_x\varpi_y.$$

If we substitute in the last equations the values obtained for v_x, v_y, v_z in the preceding section, we obtain

$$f_x=\frac{d^2x}{dt^2}+z\frac{d\varpi_y}{dt}-y\frac{d\varpi_z}{dt}+2\left(\frac{dz}{dt}\varpi_y-\frac{dy}{dt}\varpi_z\right)+(y\varpi_z-z\varpi_y)\varpi_y-(x\varpi_z-z\varpi_x)\,\varpi_z,$$

and similar formulæ for f_y and f_z.

These are the ordinary formulæ. It would not be difficult to deduce their real meaning from their analytical form; but it will be better first to prove the result of such interpretation in a different and more direct manner.

39. We have already seen that, if V denote the particle's absolute velocity, and R the radius vector, and if V_1 denote the relative velocity which has $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ for its components, then

$$V=V_1+\det(\Omega, R).$$

Let then F denote the particle's absolute acceleration, and let F_1 denote the particle's relative acceleration which has $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ for its components; then

$$F=D_t(V)=D_t(V_1)+D_t\det(\Omega, R). \quad \ldots\ldots \quad (I.)$$

Now it follows from the fundamental proposition in section 35, that

$$D_t(V_1)=F_1+\det(\Omega, V_1).$$

Moreover we know from section 28, that

$$D_t \det(\Omega, R) = \det(D_t(\Omega), R) + \det(\Omega, D_t(R)).$$

But since $D_t(R) = V = V_1 + \det(\Omega, R)$, therefore

$$\det(\Omega, D_t(R)) = \det(\Omega, V_1) + \det\{\Omega, \det(\Omega, R)\}.$$

Hence, collecting the above results, and substituting them in the equation (I.), we find

$$F = F_1 + 2\det(\Omega, V_1) + \det(D_t(\Omega), R) + \det\{\Omega, \det(\Omega, R)\}.$$

It may be observed as to this formula, that if $V_1 = 0$, that is to say, if the particle had no *relative* motion, and moved as if rigidly connected with the axes of coordinates, then the two first terms of the last equation would vanish; and therefore its other two terms are what the acceleration would be if the particle had no relative motion, and they represent what may therefore be conveniently termed the particle's *system-acceleration.* French writers have given to this acceleration the name of "accélération d'entraînement;" it is the acceleration of a point which is in the position of the moving particle, and which is supposed to be rigidly connected with the system of moving axes, and I therefore propose to call it "system-acceleration." Using then this expression, we have the following proposition:—"The acceleration of a particle is equivalent to its acceleration relatively to a system of axes revolving about a fixed point, together with the system-acceleration corresponding to the particle and together with an acceleration equal to $2\det(\Omega, V_1)$, V_1 being the particle's relative velocity, and Ω the axis about which the system is revolving at time t." Or, more briefly, a particle's absolute acceleration equals the complete sum of its relative acceleration and of its system-acceleration together with $2\det(\Omega, V_1)$. Such is the brief expression of Corioli's beautiful and very useful proposition concerning relative motion.

40. We have just seen that the particle's system-acceleration is compounded of $\det(D_t(\Omega), R)$ and $\det\{\Omega, \det(\Omega, R)\}$. As regards the latter line, it is clear from section 30 that it is in the direction of the line drawn from the particle perpendicular to and towards the axis Ω, and that its magnitude is $\varpi^2 p$, p being the length of that perpendicular. It is therefore equal and opposite to what is usually called "the centrifugal force."

As regards the other line $\det(D_t(\Omega), R)$, it is of course at once determined as soon as $D_t(\Omega)$ is known. Now if $\varpi_x, \varpi_y, \varpi_z$ be the components of Ω, it follows from the fundamental proposition in section 35 that $D_t(\Omega)$ is equivalent to

$$\left(\frac{d\varpi_x}{dt} \parallel \text{to } Ox\right) + \left(\frac{d\varpi_y}{dt} \parallel \text{to } Oy\right) + \left(\frac{d\varpi_z}{dt} \parallel \text{to } Oz\right) + \det(\Omega, \Omega).$$

But it is clear, from the very definition of a determinant, that $\det(\Omega, \Omega)$ is zero. Hence we see that the components of $D_t(\Omega)$ are $\frac{d\varpi_x}{dt}, \frac{d\varpi_y}{dt}, \frac{d\varpi_z}{dt}$. This is an important proposition often used in the dynamics of a rigid body, and generally proved by means of a good deal of analytical work. It is usually expressed in the following manner. If

$\varpi_1, \varpi_2, \varpi_3$ be the components of ϖ along fixed axes, and $\varpi_x, \varpi_y, \varpi_z$ be its components along moving axes which coincide with the former at time t, then $\frac{d\varpi_1}{dt}=\frac{d\varpi_x}{dt}, \frac{d\varpi_2}{dt}=\frac{d\varpi_y}{dt}, \frac{d\varpi_3}{dt}=\frac{d\varpi_z}{dt}$. It is evident that this amounts to saying that $D_t(\Omega)$ has $\frac{d\varpi_x}{dt}, \frac{d\varpi_y}{dt}, \frac{d\varpi_z}{dt}$ for its components; and we have just seen how that proposition follows at once from the fundamental theorem in section 35, and from the self-evident fact that $D_t(\Omega, \Omega)=0$.

41. Recapitulating then the results of the two last sections, we see that a particle's system-acceleration is equivalent to det $(D_t(\Omega), R)$ minus the centrifugal force, and that the absolute acceleration of the particle is compounded of the relative acceleration of the particle, its system-acceleration, and 2 det (Ω, V_1), V_1 being the particle's relative velocity.

If we now look back on the analytical expressions obtained in section 38 for the components of the absolute acceleration, it will be easy to see their full meaning. The expression $\frac{d^2x}{dt^2}$ is the component of the relative acceleration. The expression $z\frac{d\varpi_y}{dt}-y\frac{d\varpi_z}{dt}$ is the component of det $(D_t(\Omega), R)$, since, as we have seen, $D_t(\Omega)$ has for its components $\frac{d\varpi_x}{dt}, \frac{d\varpi_y}{dt}, \frac{d\varpi_z}{dt}$. The expression $2\left(\frac{dz}{dt}\varpi_y-\frac{dy}{dt}\varpi_z\right)$ is the component of 2 det (Ω, V_1), since V_1 has for its components $\frac{d\varpi_x}{dt}, \frac{d\varpi_y}{dt}, \frac{d\varpi_z}{dt}$. Finally, it may be easily shown by analytical geometry, that the expression

$$(y\varpi_z-z\varpi_y)\varpi_y-(x\varpi_z-z\varpi_x)\varpi_z$$

is the component of the line ϖ^2p drawn from the point (x, y, z) on the line whose direction-cosines are proportional to $\varpi_x, \varpi_y, \varpi_z$, p being the length of that perpendicular. Hence it is manifest that the analytical formulæ in question merely express the proposition enunciated at the commencement of this section.

It may, finally, be observed that the above results might also have been easily deduced from the formula in section 34 for the acceleration along the radius vector in exactly the same manner as the corresponding analytical formulæ for the relative motion of a particle in one plane were deduced in section 22 from the formula for the acceleration of the particle along the radius vector.

42. If the origin of coordinates also moves, it is evident that the particle's actual acceleration is the resultant of the acceleration of the origin and the acceleration relatively to the origin. Hence substituting for the latter acceleration the expression already found for it, it is easy to see that the particle's actual acceleration is, as before, the resultant of the relative acceleration, an acceleration represented by 2 det (Ω, V_1), and the particle's system-acceleration, but that the system-acceleration is now the resultant of the acceleration of the origin, and of the system-acceleration relatively to the origin, for which latter system-acceleration we have already obtained the expression. Now in whatever way a system moves, the motion may be decomposed into a motion of translation and a motion of rotation. Hence we see that a particle's absolute accelera-

tion is in all cases the resultant of the relative acceleration, the system-acceleration, and an acceleration equal to $2\det(\Omega, V)$, where Ω is the axis about which the system is turning at the time, and V is the relative velocity of the particle.

This is the most general form of CORIOLI's theorem.

43. One of the most important illustrations of the theory of relative motion is the motion of a heavy particle relatively to a system which revolves uniformly about a fixed axis; for this includes the case of a falling body and the pendulum, where the earth's motion is taken into account.

Suppose then a particle of mass m to have for its actual weight W', and for its apparent weight W, so that a force $-W$ would keep the particle in relative equilibrium or apparently at rest. Then evidently $\left(\frac{W'}{m}\right)-\left(\frac{W}{m}\right)$ is equivalent to the particle's system-acceleration.

Let then the particle be acted on by a force P over and above the weight W', and let the particle's actual acceleration be F, its relative acceleration F_1, its system-acceleration F_2. Then clearly

$$F=\left(\frac{P}{m}\right)+\left(\frac{W'}{m}\right).$$

But by CORIOLI's theorem

$$F=(F_1)+(F_2)+2\det(\Omega, V);$$

and we have just seen that the system-acceleration $F_2=\left(\frac{W'}{m}\right)-\left(\frac{W}{m}\right)$. Hence it follows that

$$(F_1)+2\det(\Omega, V)=\left(\frac{P}{m}\right)+\left(\frac{W}{m}\right): \quad \text{(I.)}$$

$\frac{W}{m}$ is the apparent acceleration of gravity, and is generally denoted by g.

44. The above formula is quite general; but in most cases g may be considered as constant both in magnitude and direction, its direction being the vertical direction at the point of reference or origin.

We have then the formula

$$F_1=\left(\frac{P}{m}\right)+(g)-2\det(\Omega, V_1). \quad \text{(II.)}$$

This simple formula enables us to solve easily all problems concerning the motion of a heavy particle relatively to a spectator on the earth. The formula shows that the relative acceleration is found, just as if the earth did not move, by substituting the apparent for the actual force of gravity, and by adding on a force $-2m\det(\Omega, V_1)$, where V_1 is the particle's apparent velocity.

45. Let us take the vertical downwards as axis of z; let the axis of x be the horizontal line drawn from north to south, and let the axis of y be the horizontal line drawn from west to east. Then the equation to the earth's axis is evidently $\frac{x}{\cos\lambda}=\frac{z}{\sin\lambda}$, if λ denote the latitude of the spectator's position.

Therefore $\varpi_x=\varpi\cos\lambda, \quad \varpi_y=0, \quad \varpi_z=\varpi\sin\lambda.$

Moreover we know that the components of det (Ω, V_1) are

$$\frac{dz}{dt}\varpi_y-\frac{dy}{dt}\varpi_z=-\varpi\sin\lambda\frac{dy}{dt},$$

$$\frac{dx}{dt}\varpi_z-\frac{dz}{dt}\varpi_x=\varpi\left(\frac{dx}{dt}\sin\lambda-\frac{dz}{dt}\cos\lambda\right),$$

$$\frac{dy}{dt}\varpi_x-\frac{dx}{dt}\varpi_y=\varpi\cos\lambda\frac{dy}{dt}.$$

Let then the force P, which, besides gravity, acts on the particle, have for its components X, Y, Z, then, as F_1, the relative acceleration, has for its components $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$, it evidently follows from equation (II.) of the preceding section, by revolving along the axes, that

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=\frac{X}{m}+2\varpi\sin\lambda\frac{dy}{dt},\\ \frac{d^2y}{dt^2}&=\frac{Y}{m}+2\varpi\left(\frac{dz}{dt}\cos\lambda-\frac{dx}{dt}\sin\lambda\right),\\ \frac{d^2z}{dt^2}&=\frac{Z}{m}+g-2\varpi\cos\lambda\frac{dy}{dt}.\end{aligned}\right\} \quad \ldots\ldots\ldots \quad \text{(III.)}$$

On multiplying these equations by $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ respectively, and adding, we find

$$\frac{dx}{dt}\frac{d^2x}{dt^2}+\frac{dy}{dt}\frac{d^2y}{dt^2}+\frac{dz}{dt}\frac{d^2z}{dt^2}=\frac{X}{m}\frac{dx}{dt}+\frac{Y}{m}\frac{dy}{dt}+\frac{Z}{m}\frac{dz}{dt}+g\frac{dz}{dt}.$$

This equation may also at once be deduced from the formula (II.) if we resolve along the direction of the particle's relative motion, and observe that det (Ω, V_1) is perpendicular on that direction which coincides with the direction of V_1.

By integrating the last equation, we see that the equation of *vis viva* applies to the particle's relative motion just as if the particle's relative motion were its actual motion, with this difference only, that for the actual force of gravity the apparent force of gravity must be substituted.

If the particle be a free particle acted on by no forces but gravity, then $X=0$, $Y=0$, $Z=0$, and the equations (III.) are linear, and are therefore easily integrated.

Moreover if v_1 be the relative velocity, the equation of *vis viva* gives

$v_1^2=2g(z-h)$, h being a constant.

46. If the particle be suspended by a string from a point fixed to the earth, then if that point be taken as the origin, and X, Y, Z be the components of the string's tension, we evidently have $Zy-Yz=0$, $Xz-Zx=0$, $Yx-Xy=0$; and substituting those values in the preceding equations (III.), we shall obtain two independent equations, which, together with the equation $x^2+y^2+z^2=$a constant, will determine the particle's relative motion.

But those resulting equations can be found far more simply and directly in the following manner.

For this purpose let us revert to the fundamental formula

$$F_1=\left(\frac{P}{m}\right)+(g)-2\det(\Omega, V). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

Now section 30 of Chapter II. shows us how to find the moment-axis with respect to the origin of $\det(\Omega, V)$. If, namely, ϖ and v be the magnitudes of Ω and V, and ϕ be the angle between Ω and the radius vector, then the moment-axis of $\det(\Omega, V)$ is $-\varpi v r \cos\phi$. Therefore the moment about the axis of x of $-2\det(\Omega, V)$ is $2\varpi r\cos\phi\frac{dx}{dt}$; and similarly, its moments about the axis of y and z are respectively $2\varpi r\cos\phi\frac{dy}{dt}$ and $2\varpi r\cos\phi\frac{dz}{dt}$. Moreover, since ϕ is the angle between the radius vector and Ω, it is evident that

$$\varpi r\cos\phi=\varpi_x x+\varpi_y y+\varpi_z z=\varpi(x\cos\lambda+z\sin\lambda),$$

λ being the latitude of the origin.

Let us now take the moments of F_1 about the axes of coordinates, and equate them to the moments of those components of F_1 which are given in the formula (I.).

The moments of $\frac{P}{m}$ about the axes are zero in this case of the pendulum. The moments of g about the axes of x, y, and z are respectively gy, $-gx$, and 0; and the moments of $-2\det(\Omega, V)$ we have just found. Hence we obtain at once the following three equations:—

$$\left.\begin{aligned} y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}&=\quad gy+2\varpi\frac{dx}{dt}(x\cos\lambda+z\sin\lambda),\\ z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}&=-gx+2\varpi\frac{dy}{dt}(x\cos\lambda+z\sin\lambda),\\ x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}&=\qquad 2\varpi\frac{dz}{dt}(x\cos\lambda+z\sin\lambda).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (II.)$$

These are the three equations given in HANSEN's elaborate 'Theorie der Pendel-Bewegung,' and which are generally obtained by means of very complicated analysis.

One simple equation can be deduced by means of the proposition contained in section 45; for the principle of *vis viva* gives

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=2g(z)+\text{a constant}.$$

Moreover $x^2+y^2+z^2$ is a constant, and these two equations, combined with any one of the equations (II.), determine the particle's motion.

Chapter IV.

47. As soon as we pass from the statics or dynamics of a particle to the statics or dynamics of a system of particles or of a rigid body, we find that two forces which are equal and parallel to one another are not equivalent to one another, and that we have to take into account the *position* as well as the magnitude and direction of a force. Notwithstanding this, we are enabled by means of an elementary principle of statics to confine our operation and our notation to lines passing through one and the same point. For suppose a force P to act at a point m of a rigid body, and apply at the origin O two equal and opposite forces P and $-$P, then P at m is equivalent to P at O and the couple whose forces are P at m and $-$P at O. Let the *axis* of this couple be denoted by G; the couple, being completely determined by G, may be called *the couple* G. It is extremely convenient to have a name for the line G, indicating briefly its connexion with the force P at m, and I shall adopt that given to it by French writers*, and shall call G *the moment-axis about* O *of the force* P *at m.*

It has been proved in section 25 of Chapter II., that the line G, being the axis of the couple whose forces are $-$P at O and P at m, is equal to det($-$P, R), where R is the radius vector of the particle. Hence we have

$$G=\det(-P, R)=\det(R, P).$$

We thus see that the force P at m is completely represented and determined by the two lines P and G drawn from the origin, G being the moment-axis with respect to the origin of P at m, and being equal to det (R, P).

48. Suppose now that we have a system of forces P_1, P_2, &c. acting respectively at points m_1, m_2, &c. of a rigid body. Then it is clear from statics that the given system of forces is equivalent to a force P at the origin O and a couple whose axis is G, where P is the *complete sum* of the forces P_1, P_2, &c. supposed to be collected at the origin, and G is the complete sum of the *moment-axes* (G_1, G_2, &c.) (about the origin) of the forces P_1 at m_1, P_2 at m_2, &c.

49. We will now apply the above considerations to dynamics. Since the acceleration is the complete differential coefficient of the velocity, it is evident that the line which represents the *moving force* of the particle is the complete differential coefficient of the line which represents the particle's momentum; or, more briefly, *the moving force is the complete differential coefficient of the momentum.*

Let now P represent the moving force, and U the momentum of a particle m, P and U denoting straight lines; then, if we treat the moving force and momentum as if they were statical forces, it is clear that P at m is equivalent to P at the origin O and a couple G, where G is the moment-axis about O of P at m; and similarly, the momentum U at m is equivalent to U at O and a couple of momenta whose axis is H, where H is the moment-axis about O of U at m. We have just seen that P is the complete differential coefficient of U, and we will now prove that in like manner G is the complete

* See Delaunay's 'Mechanics,' page 254.

differential coefficient of H. If, namely, R denotes the radius vector of the particle, G=det (R, P), and similarly H=det (R, U). Now, if we differentiate the last equation, H=det (R, U), we obtain, according to section 28,

$$D_t(H)=\det(R, D_t(U))+\det(D_t(R), U). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

But $D_t(R)$ is identical with the particle's velocity, and is therefore in the direction of the momentum U. Whence it follows, from the very definition of a determinant, that

$$\det(D_t(R), U)=0.$$

Therefore the above equation (I.) becomes, since $D_t(U)=P$,

$$D_t(H)=\det(R, D_t(U))=\det(R, P)=G.$$

This is an important result. It shows that the moment-axis about any point of the moving force of a particle is the complete differential coefficient of the momentum, and that therefore the *moment* of the moving force *about any line* is the differential coefficient of the moment of the momentum.

The above result may also be easily deduced from the identical equation

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=\frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right).$$

50. The proposition which we have just proved may be easily extended to a system of moving forces and momenta of the particles of a rigid body. For, according to section 43, the system of moving forces is reducible to a moving force at the origin O, and a couple G. And the system of momenta may be similarly reduced to a momentum U, and a momentum-couple whose axis is H. Now we have seen that P is the complete sum of the moving forces, and that each moving force is the complete differential coefficient of the corresponding momentum. It therefore evidently follows that P is the complete differential coefficient of the complete sum of the momenta, or of U. Hence $P=D_t(U)$. Moreover we have seen that G is the complete sum of the moment-axes about O of the moving forces, and that each of these moment-axes is the complete differential coefficient of the moment-axis of the corresponding momentum. Hence it follows that G is the complete differential coefficient of the complete sum of the moment-axes of the momenta. Hence $G=D_t(H)$.

This result may be also easily proved by means of the identical equations

$$\Sigma\left(m\frac{d^2x}{dt^2}\right)=\frac{d}{dt}\Sigma\left(m\frac{dx}{dt}\right),$$

$$\Sigma m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right)=\frac{d}{dt}\Sigma m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right).$$

51. The science of the dynamics of a rigid body is founded upon D'ALEMBERT'S principle, which asserts that the moving forces of a body's particles are together statically equivalent to the impressed forces acting on the body. If therefore these external forces be reduced to a force P at the origin O and a couple G, then P and G are equal

respectively to what was denoted in the preceding section by P and G; and we have therefore

$$P=D_t(U), \qquad G=D_t(H).$$

In other words, if we treat the momenta as statical forces, and reduce the system of momenta of a body's particles to a momentum U at O and a momentum-couple whose axis is G, then the external forces acting on the body are equivalent to the force $D_t(U)$ at O and the couple of forces whose axis is $D_t(H)$.

Since $G=D_t(H)$, the resolved part of G along any fixed line will be the differential coefficient of the resolved part of H along that line; or, in other words, the sum of the moments of the external forces about any line equals the differential coefficient of the sum of the moments of the momenta of the body's particles.

The above results may be easily deduced from the ordinary equations

$$\Sigma(X)=\Sigma\left(m\frac{d^2x}{dt^2}\right)=\frac{d}{dt}\Sigma\left(m\frac{dx}{dt}\right), \text{ \&c.},$$

$$\Sigma(Zy-Yz)=\Sigma m\left(\frac{d^2z}{dt^2}y-\frac{d^2y}{dt^2}z\right)=\frac{d}{dt}\Sigma m\left(\frac{dz}{dt}y-\frac{dy}{dt}z\right), \text{ \&c.}$$

But it will be generally found far better not to use those six equations at all, and simply to bear in mind the fact which they alone express, namely, that $P=D_t(U)$, $G=D_t(H)$.

52. It will be convenient to recapitulate once for all the notation and phraseology I shall constantly use in the sequel. The system of momenta of a body's particles, or what may be called *the body's momenta-system*, is reducible, if we treat the momenta as forces, to a momentum at a point O, and a couple of momenta. The former I call *the body's momentum*, and denote it by U; the latter I call *the body's momentum-couple* about O, and denote its axis by H. U and H may both be represented by straight lines through the origin O. It is to be observed that U remains the same wherever O be taken, but that H changes with the position of O.

The components of U and H along the axes of coordinates will be denoted by U_x, U_y, U_z, H_x, H_y, H_z respectively, the *magnitudes* of all these quantities being represented by the corresponding small letters. Thus h_x will be equal to the resolved part of H along the axis Ox, and will therefore equal the sum of the moments of the momenta about Ox.

The system of moving forces is reducible to a moving force $D_t(U)$ at O, and a couple whose axis is $D_t(H)$; and it follows from D'ALEMBERT'S principle that, if the external forces be reduced to a force P at O and a couple G, then

$$P=D_t(U), \qquad G=D_t(H).$$

53. The next step will be to investigate the expressions for U and H.

In the first place, U can be easily found. For, let R denote the radius vector of a particle of mass m; then, if Σ denote the operation of taking the *complete* sum of lines, we have

$$\begin{aligned} U&=\Sigma m D_t(R)\\ &=D_t\Sigma(mR). \end{aligned}$$

Now, by a well-known proposition, it is evident that $\Sigma(m\mathrm{R})=\mathrm{M}\overline{\mathrm{R}}$, where M is the mass of the body and $\overline{\mathrm{R}}$ is the radius of the body's centre of gravity. Hence

$$\mathrm{U}=\mathrm{MD}_t(\overline{\mathrm{R}});$$

and therefore, if $\overline{\mathrm{V}}$ denote the velocity of the centre of gravity in magnitude and direction, $\mathrm{U}=\mathrm{M}\overline{\mathrm{V}}$, or, in other words, U, *the body's momentum, is the momentum of the body's mass collected at the body's centre of gravity.*

54. In the next place we have to find H. The investigation will be very much facilitated by the following consideration. A body's motion is said to be compounded of motions α, β, γ, if the velocity of each of the body's particles may be considered as the resultant of the respective velocities due to the motions of α, β, γ *separately.* In such case the momentum of a particle will evidently be the resultant of the momenta due to each of the motions α, β, γ separately; and since the resultant of the momenta of all the particles will be the same in whatever way we group them together, it is evident that we have the following proposition:—

"The resultant of the momenta of a body's particles, or the body's momenta-system, is the resultant of the momenta-systems due to each of the motions α, β, γ."

Thus a body's motion may be decomposed into a motion of rotation and translation. Hence the body's momenta-system may be found by compounding the momenta-system due to the motion of rotation with that due to the motion of translation.

Again, a motion of rotation may be decomposed into rotations about three axes. Hence the momenta-system of a body which rotates about a fixed point is the resultant of the momenta-systems respectively due to the separate motions of rotation about the three axes.

55. Let us then first investigate H for a body having simply a motion of translation.

Let v be the velocity of translation in the direction of a line AB at time t, then the momentum of a particle of mass m is mv in the direction of AB. Hence the momenta-system consists of a number of momenta parallel to one another, and proportional to the masses of the respective particles. Their resultant is therefore Mv at the centre of gravity, M being the body's mass. Hence the momenta-system is reducible to Mv at the centre of gravity. Therefore H, the axis of the body's momentum-couple about O, is the moment-axis about O of Mv at the centre of gravity, and is zero, if the point O coincides with the centre of gravity. In the latter case, since $\mathrm{H}=0$, therefore $\mathrm{D}_t(\mathrm{H})=0$, therefore $\mathrm{G}=0$, or the moment of the external forces about any line through the centre of gravity is zero for a body which has simply a motion of translation.

56. The next simplest case is that of a body rotating about a line. Take that line as axis of z, and suppose the body of mass M to be revolving at time t about that line with angular velocity ϖ. It may be easily shown in the ordinary way, that the sum of the moments of the momenta about the axes of x, y, z are respectively

$$-\varpi\Sigma(mxz),\quad -\varpi\Sigma(myz),\quad \varpi\Sigma(mr^2).$$

So that, using the notation of section 52, we have

$$h_x = -\varpi\Sigma(mxz), \quad h_y = -\varpi\Sigma(myz), \quad h_z = \varpi\Sigma(mr^2).$$

If the axis of z be a principal axis, we have

$$\Sigma(mxz)=0, \quad \Sigma(myz)=0,$$

therefore

$$h_x=0, \quad h_y=0, \quad h_z=\varpi\Sigma(mr^2);$$

and consequently H is a line in the direction of Oz, and equal to the product of ϖ and the moment of inertia about Oz.

57. If the axis about which the body rotates is neither a fixed line nor a principal axis, it is more convenient to express H, the body's momentum-couple, in the following manner.

The body's rotation about the point O may be considered as compounded of motions of rotation about the principal axes at O. Let ϖ_1, ϖ_2, ϖ_3 be the angular velocities of those component rotations, and let A, B, C be the respective moments of inertia about the principal axes. We have shown in the preceding section that the body's momentum-couples due to the three separate rotations about the principal axes would have for their respective axes lines along the principal axes and equal to $A\varpi_1$, $B\varpi_2$, $C\varpi_3$. It follows, therefore, from section 54, that H, the body's momentum-couple, is the resultant of the three couples, whose axes are repectively $A\varpi_1$, $B\varpi_2$, $C\varpi_3$. In other words, H, the axis of the body's momentum-couple, has $A\varpi_1$, $B\varpi_2$, $C\varpi_3$ for its components along the principal axes.

58. The results of the last two sections may be also proved in the following more direct manner.

Take any rectangular axes as axes of coordinates. Let ϖ_x, ϖ_y, ϖ_z be the components, along those axes, of the body's angular velocity of rotation. Let v_x, v_y, v_z be the components of the velocity of a particle of mass m, whose coordinates are x, y, z.

If then h_x, h_y, h_z be the components of H, we have

$$h_x=\Sigma m(yv_z-zv_y).$$

But
$$v_z=y\varpi_x-x\varpi_y, \quad v_y=x\varpi_z-z\varpi_x.$$

Therefore
$$h_x=\varpi_x\Sigma m(y^2+z^2)-\varpi_y\Sigma(myx)-\varpi_z\Sigma(mxz).$$

If, then, the moments of inertia about the axes of x, y, z be denoted by A, B, C respectively, and if we denote $\Sigma(myz)$ by A′, $\Sigma(mxz)$ by B′, $\Sigma(myx)$ by C′, the last equation becomes

$$h_x=A\varpi_x-C'\varpi_y-B'\varpi_z;$$

and similarly,

$$h_y=B\varpi_y-A'\varpi_z-C'\varpi_x,$$
$$h_z=C\varpi_z-B'\varpi_x-A'\varpi_y.$$

These results are true, whatever rectangular axes of coordinates be taken; but if they be principal axes, then A′=0, B′=0, C′=0, and therefore we have, as before,

$$h_x=A\varpi_x, \quad h_y=B\varpi_y, \quad h_z=C\varpi_z.$$

59. We are now in a condition to solve easily the problem of the motion of a body rotating about a fixed line, or about a fixed point under the action of any forces.

First, let us take the case of a body of mass M revolving about a fixed line. Take that line as axis of z, and choose for the axes of x and y any lines *fixed in the body* which are perpendicular to one another and to Oz.

It was proved in section 52 that H, the axis of the body's momentum-couple, has for its components

$$h_x=-\varpi\Sigma(mxz), \quad h_y=-\varpi\Sigma(myz), \quad h_z=\mathrm{M}k^2\varpi, \quad \ldots\ldots \quad (\mathrm{I.})$$

Mk^2 standing for the moment of inertia about Oz.

Moreover we know from section 48 that U, the body's momentum, is M$\overline{\mathrm{V}}$, where $\overline{\mathrm{V}}$ is the velocity of the centre of gravity. Now if $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coordinates of the centre of gravity, $\overline{\mathrm{V}}$ has evidently for its projections on Ox, Oy, Oz, $-\varpi\bar{y}$, $\varpi\bar{x}$, 0 respectively. Hence the components of U are equal to

$$u_x=-\mathrm{M}\varpi\bar{y}, \quad u_y=\mathrm{M}\varpi\bar{x}, \quad u_z=0\,; \quad \ldots\ldots\ldots \quad (\mathrm{II.})$$

knowing, then, the components of U and H, we can easily find the components of $\mathrm{D}_t(\mathrm{U})$ and $\mathrm{D}_t(\mathrm{H})$. Using the notation of section 52, $\mathrm{P}=\mathrm{D}_t(\mathrm{U})$ and $\mathrm{G}=\mathrm{D}_t(\mathrm{H})$, and the components of P and G may be denoted by P_x, P_y, P_z, and G_x, G_y, G_z respectively.

In the problem now before us, the axis of z does not move; hence evidently

$$\mathrm{P}_z=\frac{d}{dt}(u_z) \text{ and } \mathrm{G}_z=\frac{d}{dt}(h_z). \quad \ldots\ldots\ldots\ldots \quad (\mathrm{III.})$$

But as to the axes of x and y, they revolve about the axis of z with an angular velocity ϖ at time t. Hence by the elementary formulæ of section 18 in Chapter I., we have

$$\left.\begin{array}{l}\mathrm{P}_x=\frac{d}{dt}(u_x)-u_y\varpi, \quad \mathrm{P}_y=\frac{d}{dt}(u_y)+u_x\varpi. \\ \text{Similarly,} \\ \mathrm{G}_x=\frac{d}{dt}(h_x)-h_y\varpi, \quad \mathrm{G}_y=\frac{d}{dt}(h_y)+h_x\varpi.\end{array}\right\} \quad \ldots\ldots\ldots \quad (\mathrm{IV.})$$

Let, then, external forces acting on the body be reduced to a force at O whose components are X, Y, Z, and to a couple the components of whose axis are L, M, N. Let the reactions of the fixed axis be similarly reduced to a force whose components are X′, Y′, Z′, and to a couple the components of whose axis are L′, M′. Then, by D'Alembert's principle,

$$\mathrm{X}+\mathrm{X}'=\mathrm{P}_x, \text{ \&c.,}$$
$$\mathrm{L}+\mathrm{L}'=\mathrm{G}_x, \text{ \&c.}$$

Therefore, substituting the values (I.) and (II.) in equations (III.) and (IV.), we obtain the following six equations:—

$$\mathrm{X}+\mathrm{X}'=-\mathrm{M}\bar{y}\frac{d\varpi}{dt}-\mathrm{M}\varpi^2\bar{x},$$

$$\mathrm{Y}+\mathrm{Y}'=\ \mathrm{M}\bar{x}\frac{d\varpi}{dt}-\mathrm{M}\varpi^2\bar{y},$$

$$\mathrm{Z}+\mathrm{Z}'=\ 0,$$

$$L+L'=-\Sigma(mxz)\frac{d\varpi}{dt}+\varpi^2\Sigma(myz),$$

$$M+M'=-\Sigma(myz)\frac{d\varpi}{dt}-\varpi^2\Sigma(mxz),$$

$$N=\quad Mk^2\frac{d\varpi}{dt}.$$

These six equations are those ordinarily given in text-books, and their full import and meaning is now apparent. The first three have for their right-hand members the components of $D_t(U)$, where U has for its components $-M\varpi\bar{y}$, $M\varpi\bar{x}$, 0.

The last three have for their right-hand members the components of $D_t(H)$, where H has for its components $-\varpi\Sigma(mxz)$, $-\varpi\Sigma(myz)$, $Mk^2\varpi$. And the six equations are at once obtained by applying the elementary formulæ of section 18.

60. It is, however, in the solution of problems, far better to avoid using those six equations, and simply to remember that the body's momentum U is $M\varpi\bar{r}$ in the direction of the velocity of the centre of gravity ($\bar{r}$ being its distance from the axis), and that the body's momentum-couple H has for its components $-\varpi\Sigma(mxz)$, $-\varpi\Sigma(myz)$, and $Mk^2\varpi$. Then the complete differential coefficients of U and H can be found at once according to the ordinary rules; and those complete differential coefficients are by D'ALEMBERT'S principle respectively identical with the force and the axis of the couple to which the forces acting on the body may be reduced.

Take for example the following well-known problem:—

"Under what circumstances will there be no pressure on the fixed axis, supposing no external forces to act on the body?"

Since there are no external forces nor pressures which act on the body, it follows that $D_t(U)$ and $D_t(H)$ must each equal zero. Therefore U and H are lines of constant magnitude and direction. Now the direction of U is that of the velocity of the centre of gravity, and would therefore vary, unless the centre of gravity were at rest. Hence the first condition is that the fixed axis passes through the centre of gravity.

Again, since H is a line of constant length and direction, its components along and perpendicular to the fixed axis Oz must be lines of constant length and direction. Hence $h_z=Mk^2\varpi$ must be constant. Therefore ϖ is constant, or the body revolves with uniform angular velocity.

Moreover the components of H perpendicular to Oz are $h_x=-\varpi\Sigma(mxz)$, $h_y=-\varpi\Sigma(myz)$; and we have just seen that the resultant of these two components must be a line of constant length and direction. But as ϖ is constant, it is clear that that resultant has always the same components along the variable axes of x and y, and would therefore move with the latter, unless those components were always zero. Hence the second condition is that $\Sigma(mxz)=0$, $\Sigma(myz)=0$; in other words, the fixed axis must be a principal axis. It is evident also that the two conditions are sufficient, for they make H and U constant lines, and therefore they make $D_t(H)$ and $D_t(U)$ vanish, and consequently, by D'ALEMBERT'S principle, there are no forces acting on the body.

61. We now come to the case of a body moving about a fixed point O. Let ϖ be, at time t, the angular velocity about the instantaneous axis, let A, B, C be the moments of inertia about the principal axes at O, and let ϖ_x, ϖ_y, ϖ_z be the components of ϖ along these axes.

H, the axis of the body's momentum-couple about O, has, we have already seen, for its components

$$h_x = \mathrm{A}\varpi_x, \quad h_y = \mathrm{B}\varpi_y, \quad h_z = \mathrm{C}\varpi_z.$$

If, then, we denote by Ω the instantaneous axis, we know, from the fundamental propositions in sections 35 and 36, that $\mathrm{D}_t(\mathrm{H})$ is equivalent to $\frac{dh_x}{dt}$ $\parallel$ to Ox, $\frac{dh_y}{dt}$ $\parallel$ to Oy, $\frac{dh_z}{dt}$ $\parallel$ to Oz, together with the determinant of Ω to H, Ω denoting the instantaneous axis; and moreover, that this determinant has for its components

$$h_z\varpi_y - h_y\varpi_z, \text{ or } (\mathrm{C}-\mathrm{B})\varpi_y\varpi_z \text{ parallel to O}x;$$
$$h_x\varpi_z - h_z\varpi_x, \text{ or } (\mathrm{A}-\mathrm{C})\varpi_x\varpi_z \text{ parallel to O}y;$$
$$h_y\varpi_x - h_x\varpi_y, \text{ or } (\mathrm{B}-\mathrm{A})\varpi_y\varpi_x \text{ parallel to O}z.$$

Therefore the components of $\mathrm{D}_t(\mathrm{H})$ are

$$\mathrm{A}\frac{d\varpi_x}{dt} + (\mathrm{C}-\mathrm{B})\varpi_y\varpi_z,$$

$$\mathrm{B}\frac{d\varpi_y}{dt} + (\mathrm{A}-\mathrm{C})\varpi_z\varpi_x,$$

$$\mathrm{C}\frac{d\varpi_z}{dt} + (\mathrm{B}-\mathrm{A})\varpi_x\varpi_y.$$

But by D'ALEMBERT's principle $\mathrm{D}_t(\mathrm{H})$ is the same as G, the axis of the couple resulting from the external forces. If, therefore, L, M, N be the components of G, L, M, N must be respectively equal to the components of $\mathrm{D}_t(\mathrm{H})$. Hence we have

$$\mathrm{L} = \mathrm{A}\frac{d\varpi_x}{dt} + (\mathrm{C}-\mathrm{B})\varpi_y\varpi_z,$$

$$\mathrm{M} = \mathrm{B}\frac{d\varpi_y}{dt} + (\mathrm{A}-\mathrm{C})\varpi_x\varpi_z,$$

$$\mathrm{N} = \mathrm{C}\frac{d\varpi_z}{dt} + (\mathrm{B}-\mathrm{A})\varpi_y\varpi_x.$$

We thus see that these well-known equations of EULER are found at once by resolving $\mathrm{D}_t(\mathrm{H})$ along the principal axes, where H is the axis of the body's momentum-couple and has $\mathrm{A}\varpi_x$, $\mathrm{B}\varpi_y$, $\mathrm{C}\varpi_z$ for its components, and that they merely express the fact that G, the resultant of L, M, N, is identical with $\mathrm{D}_t(\mathrm{H})$.

62. The theory of the motion of a body about a fixed point can be more simply investigated, and the problems connected with that theory can generally be more easily solved, by merely bearing in mind that $\mathrm{G}=\mathrm{D}_t(\mathrm{H})$ than by using EULER's equations, which merely express that fact in *one particular form*; for that form is not always the most convenient form, and is in all cases apt to conceal the fact which it embodies.

Take, for instance, the problem of a body rotating about a fixed point, no external forces acting on it. Here $D_t(H)=0$, therefore H is a line of constant length and direction. The motion of the body must therefore entirely depend upon the fact that, *whilst the body moves about the principal axes with angular velocities* ϖ_x, ϖ_y, ϖ_z, *the line* H, *whose projections on those axes are respectively* $A\varpi_x$, $B\varpi_y$, $C\varpi_z$, *remains throughout the body's motion the same in magnitude and direction.*

The length of H is evidently $\sqrt{A^2\varpi_x^2+B^2\varpi_y^2+C^2\varpi_z^2}$. But the length of H is constant, say equal to h. Therefore

$$A^2\varpi_x^2+B^2\varpi_y^2+C^2\varpi_z^2=h^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

Moreover, from section 35, we see that $D_t(H)$ is equivalent to $A\frac{d\varpi_x}{dt}$ || to Ox, $B\frac{d\varpi_y}{dt}$ || to Oy, $C\frac{d\varpi_z}{dt}$ || to Oz, together with det (Ω, H); and since the last line det (Ω, H) is perpendicular on the instantaneous axis Ω, it follows that the resolved part of $D_t(H)$ along the instantaneous axis equals

$$\frac{\varpi_x}{\varpi}A\frac{d\varpi_x}{dt}+\frac{\varpi_y}{\varpi}B\frac{d\varpi_y}{dt}+\frac{\varpi_z}{\varpi}C\frac{d\varpi_z}{dt}.$$

But this must equal zero, since $D_t(H)$ equals zero, and since consequently its resolved part along any line is zero. Hence we have

$$A\varpi_x\frac{d\varpi_x}{dt}+B\varpi_y\frac{d\varpi_y}{dt}+C\varpi_z\frac{d\varpi_z}{dt}=0.$$

Therefore

$$A\varpi_x^2+B\varpi_y^2+C\varpi_z^2 \text{ is constant, equal, say, to } k^2. \quad . \quad . \quad . \quad . \quad . \quad (II.)$$

We have already seen that H is a line of constant direction; and since its direction-cosines are proportional to $A\varpi_x$, $B\varpi_y$, $C\varpi_z$, it follows that the plane whose moment has direction-cosines which are proportional to the last three quantities is a fixed plane. This plane is the invariable plane. From this fact and the two equations (I.) and (II.), POINSOT's celebrated illustration of the motion of a body which rotates about a fixed point may be easily deduced in the ordinary manner; but it is unnecessary to discuss the problem further, as it must be already sufficiently apparent that the body's motion entirely depends upon the fact that H is a line of fixed length and direction.

63. On looking at EULER's equations, we find that, when A=B=C, they take the simple form

$$L=A\frac{d\varpi_x}{dt},$$

$$M=B\frac{d\varpi_y}{dt},$$

$$N=C\frac{d\varpi_z}{dt}.$$

Moreover, when only A=B, then the third of EULER's equations becomes $N=C\frac{d\varpi_z}{dt}$. It may be interesting to trace the real meaning of these results.

Let, as before, G be the resultant of L, M, N, and let H be the axis of the body's momentum-couple about the fixed point, and let Ω represent the instantaneous axis. Then $G=D_t(H)$, and $D_t(H)$ is equivalent to $A\frac{d\varpi_x}{dt}$ ∥ to Ox, $B\frac{d\varpi_y}{dt}$ ∥ to Oy, $C\frac{d\varpi_z}{dt}$ ∥ to Oz, together with det (Ω, H).

Now, if $A=B=C$, then H, which has for its components $A\varpi_x$, $B\varpi_y$, $C\varpi_z$, evidently coincides in direction with Ω, which has for its components ϖ_x, ϖ_y, ϖ_z. Therefore it follows, from the very definition of a determinant, that det $(\Omega, H)=0$. It is therefore because det $(\Omega, H)=0$ when $A=B=C$, that the components of $D_t(H)$ are simply $A\frac{d\varpi_x}{dt}$, $B\frac{d\varpi_y}{dt}$, $C\frac{d\varpi_z}{dt}$, and that EULER's equations take so simple a form.

Secondly, suppose only $A=B$. It is evident, from what has been just said, that $N=C\frac{d\varpi_z}{dt}+$the resolved part along Oz of det (Ω, H). Now the equation to the line H is

$$\frac{x}{A\varpi_x}=\frac{y}{B\varpi_y}=\frac{z}{C\varpi_z}.$$

If then $A=B$, the projection of H on the plane of xy evidently coincides with the projection of Ω on that plane. Therefore the lines H, Ω, and the axis of z lie in the same plane. Hence it follows that the line det (Ω, H), which by definition is perpendicular on Ω and on H, is also perpendicular on the axis of z, and has therefore no component along that axis. We thus see that the reason why $N=C\frac{d\varpi_z}{dt}$, when $A=B$, is that in that case det (Ω, H) is perpendicular to the axis of z.

64. In those cases in which there are more sets than one of principal axes at the fixed point, it is sometimes convenient to take moments about a set of principal axes, which *are not fixed in* the body.

There is no difficulty in applying the same method to such cases. Let ϖ_x, ϖ_y, ϖ_z be the angular velocities of the body about the principal axes Ox, Oy, Oz, and suppose those axes not to move with the body as if rigidly connected with it, but to move at time t about an instantaneous axis Ω' with an angular velocity equal to the length of Ω', and let the components of Ω' along the principal axes be ϖ'_x, ϖ'_y, ϖ'_z.

Let H, as before, represent the body's momentum-couple. We have seen that H has for its components

$$h_x=A\varpi_x, \quad h_y=B\varpi_y, \quad h_z=C\varpi_z.$$

Therefore, according to the fundamental proposition in section 35, $D_t(H)$ is equivalent to $\frac{d}{dt}(A\varpi_x)$ ∥ to Ox, $\frac{d}{dt}(B\varpi_y)$ ∥ to Oy, $\frac{d}{dt}(C\varpi_z)$ ∥ to Oz, together with det(Ω', H); and looking at the formulæ of section 36, we see that det (Ω', H) has for its components

$$C\varpi_z\varpi'_y-B\varpi_y\varpi'_z \text{ parallel to } Ox,$$

$$A\varpi_x\varpi'_z-C\varpi_z\varpi'_x \text{ parallel to } Oy,$$

$$B\varpi_y\varpi'_x-A\varpi_x\varpi'_y \text{ parallel to } Oz.$$

Therefore, if L be the moments of inertia of the forces about the principal axes, we have by D'ALEMBERT's principle,

$$L=\frac{d}{dt}(A\varpi_x)+C\varpi_z\varpi'_y-B\varpi_y\varpi'_z,$$

$$M=\frac{d}{dt}(B\varpi_y)+A\varpi_x\varpi'_z-C\varpi_z\varpi'_x,$$

$$N=\frac{d}{dt}(C\varpi_z)+B\varpi_y\varpi'_x-A\varpi_x\varpi'_y.$$

It is clear that, when there is more than one set of principal axes at the fixed point, either all three or at least two of the quantities A, B, C must be equal to one another. Suppose then A=B, then the axis of z, Oz is *fixed in* the body, and therefore ϖ'_x and ϖ'_y are clearly the same as ϖ_x and ϖ_y. And the last equations, therefore, become

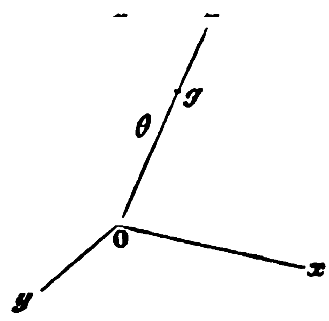
$$L=A\frac{d\varpi_x}{dt}+\varpi_y(C\varpi_z-A\varpi'_z),$$

$$M=A\frac{d\varpi_y}{dt}+\varpi_x(A\varpi'_z-C\varpi_z),$$

$$N=C\frac{d\varpi_z}{dt}.$$

If we put $\varpi'_z=\varpi_z+\frac{d\chi}{dt}$, the above equations become the same as those which are given in ROUTH's 'Dynamics,' page 134, where they are deduced from EULER's equations.

65. To the above equations, however, the same remark applies as has already been made with regard to EULER's equations. They merely express the fact that $D_t(H)$ has L, M, N for its components; and it is far better in most problems to start with that simple fact, and, without using those equations, to choose any axes which the nature of the problem may suggest.



Take, for instance, the problem of the top spinning upon a perfectly rough plane.

Let O be the fixed point, g the top's centre of gravity. Take Og as axis of z. Draw Oa vertically, and take as axis of x a line perpendicular to Oz and in the plane zOa, and take as axis of y a line perpendicular on the plane zOx, and therefore perpendicular on Og. The axes of coordinates are evidently principal axes.

The components of H, which determines the body's momentum-couple, are $A\varpi_x$, $B\varpi_y$, $C\varpi_z$, ϖ_x, ϖ_y, ϖ_z being the angular velocities about the principal axes, and A being the moment of inertia about Ox and about Oy, and C being the moment of inertia about Og. We have chosen Og so as to be perpendicular on plane aOx; consequently the resolved part of H along Oa is the sum of the resolved parts of $A\varpi_x$, $C\varpi_z$, and equals therefore, if we denote angle aOz by θ,

$$-A\varpi_x\sin\theta+C\varpi_z\cos\theta.$$

Moreover, since Oa has a fixed direction, the differential coefficient of the last expression is evidently the resolved part along Oa of $D_t(H)$. But this must equal zero by

D'ALEMBERT'S principle, since there are no forces acting on the top which have any moments about the vertical. Hence we have

$$\frac{d}{dt}(C\varpi_z \cos\theta - A\varpi_x \sin\theta) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

Therefore

$$C\varpi_z \cos\theta - A\varpi_x \sin\theta \text{ is constant.}$$

Moreover, since two of the principal moments of inertia are equal to one another, it follows from section 58 that the sum of the moments of the external forces about Oz, the axis of unequal moment of inertia, is equal to $C\frac{d\varpi_z}{dt}$. Hence, as the forces have no moment about Oz, we have $C\frac{d\varpi_z}{dt}=0$. Therefore ϖ_z is constant. This relation, together with the equation (I.) and the equation of *vis viva*, solve the problem. We, namely, obtain the three equations

$$C\varpi_x \cos\theta - A\varpi_x \sin\theta = h,$$

$$\varpi_z = \alpha,$$

$$A\varpi_x^2 + A\varpi_y^2 + C\varpi_z^2 = -2gb\cos\theta + c,$$

where $b=\mathrm{O}g$, and c is some constant.

66. In some few cases it may be convenient to take moments about lines which are fixed in the body but which are not principal axes. Let then H have for its components along the rectangular axes of coordinates h_x, h_y, h_z respectively, then we know, from section 53, that

$$h_x = A\varpi_x - B'\varpi_z - C'\varpi_y, \quad \text{where } A' = \Sigma(myz),$$

$$h_y = B\varpi_y - C'\varpi_x - A'\varpi_z, \quad B' = \Sigma(mxz),$$

$$h_z = C\varpi_z - A'\varpi_y - B'\varpi_z, \quad C' = \Sigma(myx).$$

Therefore, if L, M, N be the moments of the forces about the axes, we have, as before,

$$L = \frac{dh_x}{dt} + h_z\varpi_y - h_y\varpi_z,$$

$$M = \frac{dh_y}{dt} + h_x\varpi_z - h_z\varpi_x,$$

$$N = \frac{dh_z}{dt} + h_y\varpi_x - h_x\varpi_y.$$

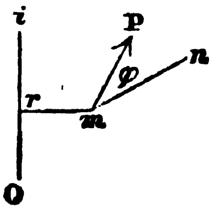
If, on the other hand, the axes of Ox, Oy, Oz are not fixed in the body, but rotate with an angular velocity whose components are ϖ'_x, ϖ'_y, ϖ'_z, then we have, in a similar manner,

$$L = \frac{dh_x}{dt} + h_z\varpi'_y - h_y\varpi'_z, \ \&c.$$

The above equations are somewhat more general than those given by LIOUVILLE in his Journal of 1858, and are, as we have just seen, at once obtained by applying the fundamental formulæ of section 36.

67. I will now show how the principle of *vis viva* may be easily proved for a body

moving about a fixed point without assuming the principle of virtual velocities, and is in fact a very simple deduction from D'ALEMBERT'S principle.



Let O i be the instantaneous axis, about which the body is rotating at time t with an angular velocity ϖ. Let m be a particle of the body, let its mass be m, its distance from O i, r, and its velocity v in the direction $m\,n$.

Suppose now a force P to act at m. Then, since $m\,n$ is perpendicular on the plane i O m, it is clear from statics that the moment of P about O i is the moment of the resolved part of P along $m\,n$. It is therefore, if ϕ be the angle which the direction of P makes with $m\,n$, equal to rP $\cos\phi$, or $\frac{v}{\varpi}$ P $\cos\phi$.

Suppose then P to be the moving force of the particle m. Then its resolved part along the velocity $m\,n$ is of course $m\frac{dv}{dt}$, so that $\text{P}\cos\phi=m\frac{dv}{dt}$, and therefore the moment of the moving force about O i equals $m\frac{v}{\varpi}\frac{dv}{dt}$. Consequently the sum of the moments of the moving forces about O i equals $\frac{1}{\varpi}\Sigma\left(mv\frac{dv}{dt}\right)$. But this sum, by D'ALEMBERT'S principle, equals the sum of the moments of the external forces about O i. Now we have already seen that the moment of any force P about O i is $\frac{v}{\varpi}$ P $\cos\phi$, so that, if P represent an external force acting on the body, the sum of the moments about O i of the forces acting on the body equals $\frac{1}{\varpi}\Sigma(\text{P}v\cos\phi)$. Hence we have

$$\frac{1}{\varpi}\Sigma(\text{P}v\cos\phi)=\frac{1}{\varpi}\Sigma\left(mv\frac{dv}{dt}\right).$$

Therefore

$$\Sigma(mv^2)\qquad=2\int dt\,\Sigma(\text{P}v\cos\phi).$$

This equation embodies the principle of *vis viva*; for it is evident that, if the components of P be X, Y, Z, then

$$\text{P}v\cos\phi=\text{X}\frac{dx}{dt}+\text{Y}\frac{dy}{dt}+\text{Z}\frac{dz}{dt}.$$

68. The same result may also be obtained by analysis; and it may be worth while to notice that each step in the analytical proof is exactly equivalent to the corresponding step in the above geometrical proof. This correspondence between the steps in analytical and geometrical demonstrations is one of the most striking features of modern analytical geometry, and would, as we have already attempted to show, present itself generally in analytical mechanics, if more attention were paid to the interpretation of the equations and formulæ which are employed.

The analytical proof is as follows:—

Let X, Y, Z be the components of any one of the forces acting on the body, and suppose that force to act at a point (x, y, z) of mass m. Let ϖ_x, ϖ_y, ϖ_z be the angular velocities of rotation about the axes of coordinates, which are here supposed to be fixed in space. Then it is clear that

$$\frac{dx}{dt}=z\varpi_y-y\varpi_z,\quad \frac{dy}{dt}=x\varpi_z-z\varpi_x,\quad \frac{dz}{dt}=y\varpi_x-x\varpi_y.$$

Therefore

$$\Sigma\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)$$

may be put into the form

$$\Sigma(Xz-Zx)\varpi_y+\Sigma(Yx-Xy)\varpi_z+\Sigma(Zy-Yz)\varpi_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad (I.)$$

But, by D'ALEMBERT'S principle,

$$\Sigma(Xz-Zx)=\Sigma m\left(\frac{d^2x}{dt^2}z-\frac{d^2z}{dt^2}x\right),$$

$$\Sigma(Yx-Xy)=\Sigma m\left(\frac{d^2y}{dt^2}x-\frac{d^2x}{dt^2}y\right),$$

$$\Sigma(Zy-Yz)=\Sigma m\left(\frac{d^2z}{dt^2}y-\frac{d^2y}{dt^2}z\right).$$

Substituting these expressions in (I.), we obtain

$$\varpi_y\,\Sigma m\left(\frac{d^2x}{dt^2}z-\frac{d^2z}{dt^2}x\right)+\varpi_z\,\Sigma m\left(\frac{d^2y}{dt^2}x-\frac{dx^2}{dt^2}y\right)+\varpi_x\,\Sigma m\left(\frac{d^2z}{dt^2}y-\frac{d^2y}{dt^2}z\right),$$

which again can be put into the form

$$\Sigma m\left\{\frac{d^2x}{dt^2}(z\varpi_y-y\varpi_z)+\frac{d^2y}{dt^2}(x\varpi_z-z\varpi_x)+\frac{d^2z}{dt^2}(y\varpi_x-x\varpi_y)\right\},$$

$$=\Sigma m\left(\frac{d^2x}{dt^2}\frac{dx}{dt}+\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt}\frac{dz}{dt}\right).$$

Hence it follows that

$$\Sigma\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)=\Sigma m\left(\frac{d^2x}{dt^2}\frac{dx}{dt}+\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt^2}\frac{dz}{dt}\right),$$

and therefore

$$\Sigma m\left(\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right)=2\int dx(Xdx+Ydy+Zdz),$$

which is the equation of *vis viva*.

It may be observed that the same proof may be quite easily extended to a body moving freely, by decomposing the original motion into a motion of translation with the velocity of the centre of gravity and a rotation about the centre of gravity.

69. The two proofs which have just been given of the principle of *vis viva* are both founded on the fact that the sum of the moments of the moving forces of the body's particles about the instantaneous axis is equal to $\frac{1}{\varpi}\Sigma\left(mv\frac{dv}{dt}\right)$, or to $\frac{1}{2\varpi}\frac{d}{dt}\Sigma(mv^2)$. This

fact is the reason why the equations of *vis viva* can be obtained by multiplying EULER'S three equations by ϖ_x, ϖ_y, ϖ_z respectively, and by adding together the products so obtained; for in performing those operations we are in reality finding the sum of the moments of the forces about the instantaneous axis whose direction-cosines are $\frac{\varpi_x}{\varpi}, \frac{\varpi_y}{\varpi}, \frac{\varpi_z}{\varpi}$.

70. If G_1 be the sum of the moments of the impressed forces about the instantaneous axis, then by D'ALEMBERT'S principle G_1 equals the sum of the moments of the moving forces about the instantaneous axis. Hence it follows from the preceding section that

$$G_1=\frac{1}{2\varpi}\frac{d}{dt}\Sigma(mv^2).$$

This equation is often useful.

For instance, since $\Sigma(mv^2)=\Sigma(m\varpi^2r^2)$, where r is the distance of a particle from the instantaneous axis, it follows that $\Sigma(mv^2)=I\varpi^2$ if I denote the body's moment of inertia about the instantaneous axis. Therefore

$$G_1=\frac{1}{2\varpi}\frac{d}{dt}(I\varpi^2)$$

$$=\frac{d\varpi}{dt}+\frac{\varpi}{2}\frac{dI}{dt}.$$

Now, if the instantaneous axis were fixed in space, we should evidently have

$$G_1=I\frac{d\varpi}{dt}.$$

Therefore the only cases in which we can take moments about the instantaneous axis as if it were fixed in space are when $\frac{\varpi}{2}\frac{dI}{dt}=0$, or when the moment of inertia about the instantaneous axis is constant. This proposition is useful in solving problems concerning rolling cones, and is usually deduced by analysis from EULER'S equations.

71. I will give two more examples of the advantage of the method I have employed in these pages.

Let ϖ_x, ϖ_y, ϖ_z be the angular velocities of rotation of a body about three rectangular axes which are fixed in and move with the body, and let a, b, c be the direction-cosines with respect to those axes of a line which is *fixed in space*. Take on the latter line a point P at a unit of distance from the origin. The velocity of the fixed point P is zero. Now, as its components along the moving axes are a, b, c respectively, it follows from one of our elementary propositions that

$$\frac{da}{dt}+c\varpi_y-b\varpi_z$$

equals the component of P's velocity along Ox, and therefore equals zero.

Hence

$$\frac{da}{dt}=b\varpi_z-c\varpi_y,$$

a formula which is generally deduced as the result of somewhat long analytical work.

72. Secondly, in order to give a striking example of the manner in which the theory of the determinants of lines explains and shortens analytical processes, I will give the following direct proof of EULER'S equations.

Take the principal axes at the fixed point as the axes of coordinates. Let v_x, v_y, v_z be the components of the velocity V, and f_x, f_y, f_z those of the acceleration F of a particle (x, y, z) of mass m.

We have seen that the fact of the acceleration being the complete differential coefficient of the velocity leads at once to the three following equations:—

$$\left.\begin{aligned} f_x&=\frac{dv_x}{dt}+v_z\varpi_y-v_y\varpi_z,\\ f_y&=\frac{dv_y}{dt}+v_x\varpi_z-v_z\varpi_x,\\ f_z&=\frac{dv_z}{dt}+v_y\varpi_x-v_x\varpi_y.\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (1.)$$

Putting then for brevity's sake f'_x for $v_z\varpi_y-v_y\varpi_z$, f'_y for $v_x\varpi_z-v_z\varpi_x$, f'_z for $v_y\varpi_x-v_x\varpi_y$, we have

$$\left.\begin{aligned} f_x&=\frac{dv_x}{dt}+f'_x,\\ f_y&=\frac{dv_y}{dt}+f'_y,\\ f_z&=\frac{dv_z}{dt}+f'_z.\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (2.)$$

Now the sum of the moments of the moving forces about Ox equals

$$\Sigma m(f_zy-f_yz)=\Sigma m\left(\frac{dv_z}{dt}\,y-\frac{dv_y}{dt}\,z\right)+\Sigma m(f'_zy-f'_yz) \quad \ldots \quad (3.)$$

Let us first investigate the expression $\Sigma m\left(\frac{dv_z}{dt}\,y-\frac{dv_y}{dt}\,z\right)$. Evidently

$$v_z=y\varpi_x-x\varpi_y, \text{ and } v_y=x\varpi_z-z\varpi_x.$$

And as the axes move with the body, x, y, z do not vary with the time, and we therefore obtain

$$\frac{dv_z}{dt}=y\,\frac{d\varpi_x}{dt}-x\,\frac{d\varpi_y}{dt},\qquad \frac{dv_y}{dt}=x\,\frac{d\varpi_z}{dt}-z\,\frac{d\varpi_x}{dt}.$$

Therefore

$$\Sigma m\left(\frac{dv_z}{dt}\,y-\frac{dv_y}{dt}\,z\right)=\frac{d\varpi_x}{dt}\,\Sigma m(y^2+z^2)-\frac{d\varpi_y}{dt}\,\Sigma(mxy)-\frac{d\varpi_z}{dt}\,\Sigma(mxz).$$

But as the axes of coordinate axes are principal axes, $\Sigma(mxy)=0$, $\Sigma(mxz)=0$. Therefore

$$\Sigma m\left(\frac{dv_z}{dt}y-\frac{dv_y}{dt}z\right)=\frac{d\varpi_x}{dt}\Sigma m(y^2+z^2)=\mathrm{A}\frac{d\varpi_x}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

A being the moment of inertia about Ox.

We have still to investigate the expression $\Sigma m(f'_z y-f'_y z)$ where $f'_z=v_y\varpi_x-v_x\varpi_y$, and $f'_y=v_x\varpi_z-v_z\varpi_x$.

We obtain by substitution

$$\Sigma m(f'_z y-f'_y z)=\varpi_x\Sigma m(v_y y+v_z z)-\varpi_y\Sigma(mv_x y)-\varpi_z\Sigma(mv_x z).$$

But since

$$v_x=z\varpi_y-y\varpi_z,$$

$$v_y=x\varpi_z-z\varpi_x,$$

$$v_z=y\varpi_x-x\varpi_x,$$

it is evident that for principal axes we have

$$\Sigma(mv_y y)=0, \qquad \Sigma(mv_z z)=0,$$

$$\Sigma(mv_x y)=-\varpi_z\Sigma(my^2), \qquad \Sigma(mv_x z)=\varpi_y\Sigma(mz^2).$$

Hence

$$\Sigma m(f'_z y-f'_y z)=\varpi_y\varpi_z(\Sigma(my^2)-\Sigma(mz^2))$$

$$=\varpi_y\varpi_z(\Sigma m(y^2+x^2)-\Sigma m(z^2+x^2))$$

$$=\varpi_y\varpi_z(\mathrm{C}-\mathrm{B}),$$

where C and B are the moments of inertia about the axes of z and y.

Substituting then this last expression and the expression in (4.) in equation (3.), we see that the sum of the moments of the moving forces about Ox equals

$$\mathrm{A}\frac{d\varpi_x}{dt}+(\mathrm{C}-\mathrm{B})\varpi_y\varpi_z.$$

Hence by D'ALEMBERT's principle, if L, M, N be the sum of the moments of the impressed forces about the axes of x, y, z, we have

$$\mathrm{L}=\mathrm{A}\frac{d\varpi_x}{dt}+(\mathrm{C}-\mathrm{B})\varpi_y\varpi_z.$$

Similarly,

$$\mathrm{M}=\mathrm{B}\frac{d\varpi_y}{dt}+(\mathrm{A}-\mathrm{C})\varpi_z\varpi_x,$$

$$\mathrm{N}=\mathrm{C}\frac{d\varpi_z}{dt}+(\mathrm{B}-\mathrm{A})\varpi_x\varpi_y.$$

I will now show the full import of each of the steps in the above analytical proof.

We have seen, in section 39 of Chapter III., that the acceleration of the particle is the result of the acceleration det $(\mathrm{D}_t(\Omega), \mathrm{R})$, and the acceleration det (Ω, V). If then the particle's mass be m, its moving force is represented by

$$m\det(\mathrm{D}_t(\Omega), \mathrm{R})+m\det(\Omega, \mathrm{V}).$$

Now the above analytical proof merely shows that the sum of the moments about the coordinate axes, of the moving forces of which m det $(D_t\,\Omega, R)$ is the type, are respectively A $\frac{d\varpi_x}{dt}$, B $\frac{d\varpi_y}{dt}$, C $\frac{d\varpi_z}{dt}$, and that the sum of the moments about the coordinate axes of the moving forces of which m det (Ω, V) is the type are respectively $(C-B)\varpi_y\varpi_z$, $(A-C)\varpi_x\varpi_z$, $(B-A)\varpi_y\varpi_x$.

73. I will next proceed to show how these results can be obtained far more briefly by applying the propositions concerning the determinants of lines. In the first place, if we put, for $D_t(\Omega)$, P, and suppose P to have for its components p_x, p_y, p_z parallel to the axes, then m det $(D_t(\Omega), R)$ has for its components parallel to z and to y,

$$m(yp_x-xp_y) \text{ and } m(xp_z-zp_x),$$

and therefore the moment of m det $(D_t(\Omega), R)$ about the axis of x equals

$$m\{(yp_x-xp_y)y-(xp_z-zp_x)z\}=m(z^2+y^2)p_x-myxp_y-mxzp_z.$$

Therefore, if we take the sum of these for all the particles and remember that the axes are principal axes, that sum will equal $p_x\Sigma m(z^2+y^2)$. Now we have already proved that p_x, the component parallel to the axis of x of $D_t(\Omega), =\frac{d\varpi_x}{dt}$. Hence we see that the sum of the moments about Ox of the moving forces of which m det $(D_t(\Omega), R)$ is the type is A $\frac{d\varpi_x}{dt}$, and that this follows from the properties of principal axes, and from the fact that the component of $D_t(\Omega)$ parallel to the axis of x is $\frac{d\varpi_x}{dt}$.

In the second place, we have already seen in section 30 of Chapter II. that the moment about the axis of x of det (Ω, V) equals $-r\varpi v_x \cos\phi$, v_x being the component of V, and ϕ the angle between the radius vector and V.

But $-r\varpi v_x\cos\phi$ equals evidently $-v_x(x\varpi_x+y\varpi_y+z\varpi_z)$, and v_x equals $z\varpi_y-y\varpi_z$. If then we observe that $\Sigma(mxv_x)=0$, $\Sigma(myv_x)=-\Sigma(my^2)\varpi_z$, $\Sigma(mzv_x)=\Sigma(mz^2)\varpi_y$, it follows easily from the above that the sum of the moments about Ox of the moving forces, of which m det (Ω, V) is the type, equals $\Sigma(mz^2)\,\varpi_y\varpi_z-\Sigma(my^2)\,\varpi_z\varpi_y=(C-B)\varpi_y\varpi_z$. This last proposition may be also proved in a different manner, which will show its connexion with the proof first given of EULER's equations.

The moment-axis, with respect to the origin, of the acceleration det (Ω, V) is det $\{R, \det(\Omega, V)\}$, which, as we see from section 30 of Chapter II., equals a line opposite to V and of length $\varpi vr\cos\phi$; but it follows from the same section that, since Ω is perpendicular on V, det $\{\Omega, \det(R, V)\}$ equals a line opposite to V and of length $\varpi vr\cos\phi$. Hence we have

$$\det\{R, \det(\Omega, V)\}=\det\{\Omega, \det(R, V)\}.$$

Let then Σ denote the operation of taking the *complete* sum of lines. Then it follows from the last equation that

$$\begin{aligned}\Sigma m \det\{R, \det(\Omega, V)\} &=\Sigma m \det\{\Omega, \det(R, V)\}\\ &=\det(\Omega, \Sigma m \det(R, V)).\end{aligned}$$

Now Σm det (R, V) equals the complete sum of the moment-axes of the momenta, or is, in other words, the axis of the body's momentum couple H whose components are $A\varpi_x$, $B\varpi_y$, $C\varpi_z$. Hence we see that

$$\Sigma m \det(\mathrm{R}, \det(\Omega, \mathrm{V})) = \det(\Omega, \mathrm{H}),$$

and therefore the sum of the moments about Ox of the moving force, of which m det (Ω, V) is the type, is the component parallel to x of det (Ω, H), and equals therefore $C\varpi_z\varpi_y - B\varpi_y\varpi_z = (C-B)\varpi_y\varpi_z$.

XXXIII. *A Supplement to Two Papers published in the Transactions of the Royal Society, "On the Science connected with Human Mortality;" the one published in* 1820, *and the other in* 1825. *By* Benjamin Gompertz, *F.R.S., F.R.A.S. &c.*

Received June 19,—Read June 20, 1861 *.

In offering to the Royal Society the ensuing Supplement to my two former papers on the Law of Mortality, with subsequent remarks on invalidism, I am anxious to acknowledge that I have derived great advantage from the encouragement and persuasion of my esteemed brother-in-law, Sir Moses Montefiore, Bart., given me to endeavour to compile and publish some of my later observations on the subject; knowing that, though I felt flattered by the attention originally shown by scientific gentlemen to these papers, they appeared to me capable of advantageous illustrations. Therefore I may venture to hope that if this Supplement merit the attention of those interested in this branch of science, I may consider that he has added a mite further to entitle him to the good wishes of those who applaud him for his constant endeavours to promote the general interest of mankind—endeavours which he has shown to extend through Europe and Asia in the cause of humanity, and to be exercised at home in various ways, among which I notice his attention to the practice of Life, Fire, and Marine Assurance; he being the President of the Alliance British and Foreign Life and Fire Assurance Company; of which I was the founding Actuary, and in which Institution, though retired from it, I feel greatly interested; it having been established about the year 1824 by the late N. M. de Rothschild, Esq., the late John Irving, Esq., the late Samuel Gurney, Esq., and Francis Baring, Esq., and himself conjointly with other gentlemen, and he being also President of the Alliance Marine Assurance Society, founded at the same time by them with him.

Art. 1. In the year 1820 the Royal Society did me the honour to publish in their Transactions a paper of mine on the Analysis and Notation applicable to the valuation of Life Contingencies, in which I introduced a new and general notation, which appeared to me far more extensively useful, and more explanatory of its object, than any other notation I had met with; and in that paper I think I introduced a new manner of dealing with the subject, by offering an analysis, with examples of the extensive use of it, applicable to some of the most intricate questions which had up to that period met with anything like a proper solution; and showed, by selections from the treatise of Life Annuities of my late learned and much-respected friend, Francis Baily, Esq., a mode of solution of all the problems in chapter 8 of that work, depending on a particular

* Subsequently revised by the author, with the insertion of some additional matter.

order of survivorship; problems previously considered many years before, and presented by my late friend WILLIAM MORGAN, Esq., of the Equitable Society, to the Royal Society, and published in their valuable Transactions; and which had been since considered, in a learned work on Life Annuities, by my late respected friend JOSHUA MILNE, Esq., with some ingenious notation with respect to those contingencies. But still, the solutions given to many of the problems, though there were but three lives concerned, were of such an intricate practical form, as to be in my opinion perfectly useless; especially on considering that it was necessary to obtain, by Tables of single and joint lives, by necessary interpolations, the required data; as the differences to be used for the interpolations, in consequence of the great irregularity of the numbers of those Tables, are so irregular as to throw great doubt on the necessary accuracy of the results. And I think the examples I gave of my method could leave no doubt as to the comparative simplicity which resulted from it, and consequently comparative utility of my analysis; an analysis which applies where there are more than three lives concerned, and, in fact, where there are any number of lives to be considered. And I may refer the reader to my solutions in that tract, to enable him to make the comparison.

There were various other subjects in that paper, and one I mention in particular, which is the problem to determine what would be the law of mortality between two lives A, B, so that, should it be known that they are both extinct, it would be an equal chance which of them had died first; because that assumption is made, in some of the solutions above alluded to, by former writers, and for a short period would, in fact, be approximatively true; and the solution of the problem showed that it could only be accurately true where there was for each life a uniform equal decrement, though not necessarily the same for both, or else a decrement for each life proceeding in geometrical proportion; the former law being in fact only an extreme case of the latter law. I may mention that there are some omissions in the printed solution of this problem, which may lead the reader, if he does not enter properly into the analysis, to think it faulty; and that the paper on the whole stands in need of some errors in the printing of it, and in one or two places of an incorrect portion of the manuscript sent to the printer, being pointed out.

Art. 2. Since that paper was written, I ventured to communicate a paper to the Royal Society, which it did me the honour to print in their Transactions of 1825, as a letter to my late friend FRANCIS BAILY, Esq., on the nature of the functions expressive of the law of human mortality expressed by the equation $L_x = d.\overline{g|}^{q^x}$; where L_x, according to my notation in my first paper, is the number of persons living, at the age x, out of the number L_0 who were born x years previously. And as I use, from great preference, the fluxional notation of our great NEWTON, instead of the furtive notation used on the Continent and now much used in England, my d does not denote the differential character. I say I use the notation in preference, because I consider the fluxional calculus far more luminous. But as strictures have been cast on me on that account by some readers of my paper, I will not apologize for a small digression from my subject, to state some among the causes of my preference. I call the differential notation furtive, on I think a moral ground, and also on the ground of its introducing an interruption and an inconvenience in practice. The moral ground is, that it appears to give LEIBNITZ a greater claim to originality, to the prejudice of NEWTON, than I think he is justly entitled to. And the other ground is, it steals from the alphabet a letter—and one which it is most convenient to retain, in order to keep up the regular order of notation—to use it for a purpose of different intent to that for which it was originally used; and may introduce confusion. And with respect to the superior advantage of the fluxional calculus over the differential calculus, I observe that if x and y be rectangular coordinates of a curve in a plane, and z the length of the curve from a given point in it to the point of which x and y are coordinates, the fluxional calculus gives $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}$, which is strictly true; and may be proved to be so without the introduction of infinitely small quantities: but the differential calculus gives $dz=\sqrt{\overline{dx}|^2+\overline{dy}|^2}$, true only on consideration of infinitely small quantities; and even with that consideration it cannot be proved luminously to be true; because $\sqrt{\overline{dx}|^2+\overline{dy}|^2}$ only expresses the length of the chord of an infinitely small arc, and not of the arc itself, as they have no part common with each other, but at the points of intersection: but in the fluxional calculus $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}$, which, I think, is a much neater and a much more commodious expression, $\dot{z}$, $\dot{x}$, $\dot{y}$ only express finite values; namely, the velocities in the several directions, at the point to which x and y are coordinates, with which the point describing the curve is moving in the relative directions parallel to the axes of x and y, and that of the tangent to the curve at the point; in the same way as if all causes which might incurvate the future path of the point were to cease; and similar observations may be made with respect to the luminous character of the fluxional analysis, when compared with the differential analysis, in the application of it to physical subjects. Thus if f be a force acting on a body, $\dot{v}$ the velocity which is generated by its momentary impulse, that is to say its single impulse, by which is meant the finite space the body would describe in consequence of it in the finite time $\dot{t}$, but not the variable portions of space it would describe if that force were considered to be an infinitely small action, as it were continually active during infinitely small portions of time; the fluxional calculus gives $f\dot{t}=\dot{v}$, and is correctly true, however large $\dot{v}$ and $\dot{t}$ are. But the differential analysis gives $f.dt=dv$, which is correct only if dt and dv are infinitely small, and is then only to be considered so in virtue of the hypothesis that infinitely small quantities of the second and higher degree may be omitted.

But whilst I am endeavouring to clear away the shadowing clouds which may obstruct the brilliant light of NEWTON'S lamp from being duly perceived by the scientific eye, I am willing to acknowledge that LEIBNITZ'S differential d has, in many instances, done great service in his own hands, in the hands of EULER, LAGRANGE, LAPLACE, and of a host of scientific men whose names cannot be pronounced without gratitude and reverence.

And I observe that other letters, such as f, ϕ, ψ, &c., when used as characteristics instead of representatives of value, have their valuable service as well as inconvenience to be attended to; though I prefer much the Continental use of a letter when used as a characteristic, to be used, as it is in many cases, as a letter underscored by some one or more letters, denoting the quantities of which that letter may be the functional characteristic; as, for instance, to write L_a to express the function of a, which may be the age of a person of whom there may be the number L_a living. And having thus intruded by a digression on the reader's attention, I will venture to hope that my still continuing the digression will be thought to have some interesting excuse for me, as a scientific amusement to the reader; as a person walking for the sake only of healthy exercise in a beautiful garden, may find a pleasure and an advantage to notice the elegant flowers, and even the noxious weeds which the ground produces. I will state that mathematicians who have enlightened the world with the most beautiful discoveries use notations which are incorrect, often ambiguous, often furtive, and often contradictory. Thus in the notation of partial fluxions or partial differentials, I consider $\frac{\dot{y}}{x}$, $\frac{\ddot{y}}{x^2}$, &c., $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ as both incorrect and furtive, and that they may mislead; instead of which I use the expressions $\underset{\dot{x}}{\overset{\dot{y}}{\frown}}$, $\underset{\dot{x}|^2}{\overset{\ddot{y}}{\frown}}$, $\underset{dx}{\overset{dy}{\frown}}$, $\underset{dx|^2}{\overset{d^2y}{\frown}}$, &c. The incorrectness of the former has been in some degree avoided by WARING, LAPLACE, and many other mathematicians of eminence, by using the expressions $\left(\frac{\dot{y}}{\dot{x}}\right)$, $\left(\frac{\ddot{y}}{\dot{x}^2}\right)$, $\left(\frac{dy}{dx}\right)$, $\left(\frac{d^2y}{(dx)^2}\right)$, &c., which removes some part of my objection, but not the whole of it; but has the disadvantage of occupying too much space, and is furtive, as it steals the excellent use of the parentheses, which have been employed for useful purposes. And now, requesting my reader to pardon me for a digression which some readers may consider uncalled for, I will proceed in the path in which I hope to lead him with some satisfaction.

Art. 3. In the equation $L_x=d.\overline{g|}^{q^x}$, where L_x signifies the number of persons living at the age x, the letters d, g, q in my paper in the 'Transactions' of 1825 express quantities apparently nearly constant during a very long period of life—say, for instance, in the Carlisle Table of Mortality between the ages 10 and 60; but if we commence our limit at a different age, and terminate it at some other far distant age, those quantities apparently constant would have different values of apparent constancy; and here the product $d.g$, which would appear in the equation by taking $x=0$, would be an arbitrary quantity depending on the arbitrary value of L_0 as a base, expressive of the arbitrary number we intend to set out with as the persons living at the age 0; and if, which is not the case accurately, the formula were universally true from birth to any other age, $d.g$ would express the number of persons born on which we intended the formula to be constructed; for we should have $L_0=d.g$; but as g is not arbitrary, d would be the arbitrary quantity, and we might take $d=1$, if we did not object to fractions in the number of the births, to set out with; and in that case the formula would stand generally $L_x=\overline{g|}^{q^x}$; if it were universally true from birth to any age; which, as I observed above, it is not; and it is never in fact true if g and q are to be considered absolutely constant; as is evident from the example of the method I used in my investigation, by what I call the vital rule of three. And as the Table from which the data are to be obtained is moreover likely to contain irregularities, and undoubtedly does contain such, which are not contained in the real law of mortality, it follows that if even the equation $L_x = d.\overline{g|}^{q^x}$, where d, g, q are supposed to be constant, did exactly express the real law of mortality, the values given to g and q by this vital rule of three would not be exactly the same for every selection, and therefore I should not have expected that any tolerable mathematician in reading my paper could suppose that I meant to state that I had given their exact values. But wherever the three lives were selected, had the Table been accurate from whence I selected the data, and were the law for constant values for d, g, q consistent throughout with the real law of mortality, their respective values would have come out the same.

Art. 4. But neither of these requisites is to be depended on, as is proved by the two formulæ given in my paper of 1825 for the Carlisle mortality, Art. 10; the one where the vital rule of three is based on the selection of the ages 20, 40, and 60, and the other where it is based on the ages 40, 60, and 100. The first of these equations is

$$\lambda L_x = 3{\cdot}88631 - \lambda^{-1}(\bar{2}{\cdot}75536 + {\cdot}0126x),$$

or its equal,

$$\lambda L_x = 3{\cdot}88631 - \lambda^{-1}({\cdot}0126x - 1{\cdot}24464), \text{ or very nearly}$$

$$3{\cdot}88631 - \lambda^{-1}{\cdot}0126(x - 100);$$

λ standing for the common logarithm of, and λ^{-1} the reverse, or the number whose common logarithm is &c., and the other, namely, from the selection of the ages 40, 60, 100, gives the equation

$$\lambda L_x = 3{\cdot}79657 - \lambda^{-1}(\bar{3}{\cdot}7467 + {\cdot}02706x)$$

$$= 3{\cdot}79657 - \lambda^{-1}({\cdot}02706x - 2{\cdot}2533).$$

And when the middle age of the three selections above is 40, we have

$$\lambda d = 3{\cdot}88631,\ \lambda q = {\cdot}0126;\ \lambda(-\lambda g) = -1{\cdot}24464,$$

sufficiently near $-1{\cdot}26$; and when the middle age is 60, of the three selections, we have $\lambda d = 3{\cdot}79657$, $\lambda q = {\cdot}02706$; sufficiently near the double of the former value; and $\lambda(-\lambda g) = -2{\cdot}2533$, also equal to nearly the double value in the former case, but not exactly in that ratio, but nearly in the ratio of 2 to $1\frac{1}{10}$; but still, considering the nature of the data which furnish the values, in order to follow the hints these numbers give, I am inclined to think but lightly of the small variation from that proportion, and to suppose that the law of mortality, instead of being $\lambda L_x = \lambda d - \lambda^{-1}(\lambda(-\lambda g) + x.\lambda q)$, when d, g, q are constant, which we have proved are only apparently constant, though for a long time exhibiting a strong appearance of constancy, should be, according to the above hints, $\lambda L_x = \lambda d_x - \lambda^{-1}(\lambda q_x.\overline{x - h})$, with the addition of some small formulæ to be sought for, where h is a constant from birth to extreme old age, and λd_x, λq_x are functions of x to be discovered to meet the cases of comparison of the formula with the Tables. Here I retain the d and q with the prefix λ for the sake of convenient reference to the old formula; and it appears that these functions λd_x and λq_x are subject to such slow variation by the variation of x, that in forty years in the above examples λd_x only varies from 3·88631 to 3·79657, and λq_x from ·0126 to about its double; the prefix λ signifying common logarithm of, and λ^{-1} the anti-common logarithm of. Now supposing t and v to be very small quantities, and that λd_x is a function of $1+vx$, and λq_x a function of $1+tx$, then, provided v and t are sufficiently small, whatever these functions may be, we consider that if in consequence of the smallness of t and v we may in the development of the functions according to the powers of x be satisfied with the first power, we may assume these functions of any form we please to suit our convenience. I will then for that end suppose $\lambda d_x = C\mathfrak{C}^x$, $\lambda q_x = \lambda q_0 . e^x$; C standing for λd_0, and $\mathfrak{C}$ and e will be quantities differing very little from unity, the one being $=1+v$, where, as above shown, v must be a very small negative fraction, and the other $=1+t$, where t is a very small affirmative fraction, and then the approximate law of mortality will be

$$\lambda L_x = C.\mathfrak{C}^x - \lambda^{-1}(e^x.\lambda q_0.(x-h)),$$

where C, $\mathfrak{C}$, e, q_0 are all four constant quantities from birth to extreme old age, quantities to be discovered from the Table of Statements of the living at different ages; C and $\mathfrak{C}$ first to be determined by two convenient statements, and e, q and h by three statements, by the method I call the vital rule of three. First, for finding $\mathfrak{C}$ and C take two values of x, one $x=m$, the other $x=n$, saying $C.\mathfrak{C}^m = \lambda d_m$, $C\mathfrak{C}^n = \lambda d_n$; and therefore by division $\mathfrak{C}^{n-m} = \frac{\lambda d_n}{\lambda d_m}$, and

$$\therefore \quad \lambda\mathfrak{C} = \frac{\lambda\lambda d_n - \lambda\lambda d_m}{n-m}; \; \lambda C = \lambda\lambda d_m - \lambda\mathfrak{C}^m = \frac{n\lambda\lambda d_m - m\lambda\lambda d_n}{n-m};$$

d_m and d_n being found from the Table of data by means of the previously stated original formula,

$$\lambda L_x = \lambda d - \lambda^{-1}(\lambda(-\lambda g) + \lambda q . x),$$

which gave when x was 20, λd_x, that is $\lambda d_{20} = 3·88631$, and when $x=60$, $\lambda d_{60} = 3·79657$;

$$\therefore \quad C\mathfrak{C}^{20} = 3·88631; \; C\mathfrak{C}^{60} = 3·79657.$$

Consequently

$$\lambda\mathfrak{C}^{40} = ·57939 - ·58954; \quad \therefore \lambda\mathfrak{C} = -\frac{·01015}{40} = -·00025375 = \bar{1}·99974625$$

and

$$\lambda C = ·589540 - 20\lambda\mathfrak{C} = ·589540 + ·005075 = ·594615.$$

Art. 5. But instead of proceeding at present with this formula, I will refer to my original formula, $L_x = d.\overline{g}|^{q^x}$, to show its value, and how g may be found by the vital rule of three by a method different from that in my paper of 1825, a mode I then pursued before I discovered by a general investigation that the above equation with d, g, q constant quantities did very approximatively express for a very long period the law of mortality.

The equation gives $\lambda L_x = \lambda d + \lambda g . q^x$, and therefore by taking x successively $=m$, $m+n$,

$m+2n$, we obtain

$$\lambda L_m=\lambda d+\lambda g.q^m,\ \lambda L_{m+n}=\lambda d+\lambda g.q^{m+n},\ \lambda L_{m+2n}=\lambda d+\lambda g.q^{m+2n},$$

whence we get

$$\lambda g.q^m\times(q^n-1)=\lambda L_{m+n}-\lambda L_m,\ \ \lambda g.q^m q^n.(q^n-1)=\lambda L_{m+2n}-\lambda L_{m+n};$$

and therefore, by division,

$$q^n=\frac{\lambda L_{m+2n}-\lambda L_{m+n}}{\lambda L_{m+n}-\lambda L_m};$$

and

$$\lambda q=\frac{\lambda(\lambda L_{m+n}-\lambda L_{m+2n})-\lambda(\lambda L_m-\lambda L_{m+n})}{n},$$

and we have

$$\lambda g=\frac{\lambda L_{m+n}-\lambda L_m}{q^m.(q^n-1)},$$

and λg in the application I am about to make will turn out to be a negative quantity, or in other words, g will be a positive fraction less than unity; and consequently to find g, as a negative number has no logarithm in the positive scale, if we are to proceed by logarithms, we must take the logarithm of the positive quantity $-\lambda g$, and say

$$\lambda(-\lambda g)=\lambda(\lambda L_m-\lambda L_{m+n})-\lambda q^m-\lambda(q^n-1);$$

and then g being found, we find d by the equation

$$\lambda d=\lambda L_m-\lambda g.q^m.$$

This is the way I have generally proceeded since I discovered the above approximative formula of mortality; but if we only consult this formula in order to find λd, which it contains, which is the only part of it necessary to find C, $\mathfrak{C}$ of the formula

$$\lambda L_x=C\mathfrak{C}^x-\lambda^{-1}(e^x.\lambda q_0.\overline{x-h}),$$

we need not take the trouble to find g or q of the preceding or old formula, and from the equations above put in the form

$$\lambda g.q^m=\lambda L_m-\lambda d,\quad \lambda g.q^{m+n}=\lambda L_{m+n}-\lambda d,\quad \lambda g.q^{m+2n}=\lambda L_{m+2n}-\lambda d,$$

multiplying the first and last together, we have

$$(\lambda g)^2q^{2m+2n}=(\lambda L_m-\lambda d)\times(\lambda L_{m+n}-\lambda d);$$

and squaring the middle, we have also

$$(\lambda g)^2q^{2m+2n}=(\lambda L_{m+n}-\lambda d)^2;$$

these put equal to each other, give

$$(\lambda L_m-\lambda d)\times(\lambda L_{m+2n}-\lambda d)=(\lambda L_{m+n}-\lambda d)^2,$$

and therefore

$$\lambda L_m.\lambda L_{m+2n}-\lambda d(\lambda L_m+\lambda L_{m+2n})=(\lambda L_{m+n})^2-2\lambda L_{m+n}\lambda d;$$

and consequently

$$\lambda d=\frac{\lambda L_m.\lambda L_{m+2n}-\overline{\lambda L_{m+n}}|^2}{\lambda L_m-2\lambda L_{m+n}+\lambda L_{m+2n}}.$$

Art. 6. The information which we have that the formula

$$\lambda L_x=\lambda d-\lambda^{-1}(\lambda(-\lambda g)+x\lambda q),$$

or its equal,

$$\lambda L_x = \lambda d + \lambda g . q^x,$$

will for a long series of years, with d, g, q constant, be a very near approximation to the values which the data afford, provided d, g, q be determined by the vital rule of three, by taking x successively equal m, $m+n$, $m+2n$, where m and $m+2n$ are somewhere near the limits of age through which the formula is meant to be applied, is a most valuable information; on account of the application of the above law to the most complicated intricacies which may be proposed, by means of what I consider rather a novel branch of mathematical investigation, which I term vital algorithm and analytical arithmetic of logarithms and anti-logarithms, in very complicated entanglements and disentanglements: for by the law of mortality we can get the value of L_x, the number living at the age x, or of L_{a+x}, or their logarithms in a series of powers of x with very converging coefficients, as will be shown further on, and thence we can obtain a series of the values of $L_{a+x} \times L_{b+x} \times L_{c+x}$, &c., however many lives there may be; or we may have the logarithms λL_{a+x}, λL_{b+x}, λL_{c+x}, &c., to which if we add the logarithm of r^x, say x into the logarithm of r, r being the present value of unity due in one year certain, at the proposed rate of interest, and deduct from this sum the sum of $\lambda L_a + \lambda L_b + \lambda L_c$, &c., we shall have the logarithm of unity to be received in x years on the condition that all these proposed lives be in existence in x years time. And then finding the analytical expression for the anti-logarithm of this, expressed by a series $A_0 + A_1 x + A_2 x^2 + A_3 x^3 +$ &c., in which A_1, A_2, A_3, &c. express a series of very converging terms, we have the present value of unity to be received in x years, and then by a Table to be presented in this tract of powers of numbers, and the sum of these numbers from $x=1$ to any proposed required limit, we can by multiplying the coefficients A_1, A_2, A_3, &c., of the series A_1, A_2, A_3, &c., which will be mostly very convergent coefficients, if not always, find the value of all the annual payments between any one time and any other; many examples of the ease of this method comparatively with the seemingly insurmountable difficulties which appear on these subjects, I hope to have time to lay before the reader, and to discuss many other branches which may be of interest. But not to keep the reader in suspense, I will continue the subject referable to the formula $L_x = d . \overline{g|}^{q^x}$, where from any proposed value of x to any far greater value, d, g, q may be considered as constants, though not actually so; and now I will show how nearly the formula

$$\lambda L_x = C \mathfrak{G}^x - \lambda^{-1}(e^x . \lambda q_0 . \overline{x-h})$$

between the ages of 10 and 80 agrees with the Tables; and will then show how to find the constants of the equation from the commencement of life to extreme old age from the more complete formula

$$\lambda L_x = C \mathfrak{G}^x + {}_{,}k_{,} \varepsilon_{,}^x + k \varepsilon^x - \lambda^{-1}(e^x . \lambda q_0 . \overline{x-h}) + \mu \nu^x,$$

where from birth to extreme old age all the quantities except x are constant, but of interesting values, so that ${}_{,}k_{,} \varepsilon_{,}^x$ commences its significance at birth, and within less than one year decreases to total insignificance, but will never absolutely vanish; the term

ke^x also commences of significance at birth, but sinks gradually, and sinks into insignificance at the age of 20, and then and after becomes of so small value that the terms sink into entire insignificance; the terms $C\mathfrak{C}^x$ and $\lambda^{-1}(\sigma^x.\lambda q_0.\overline{x-h})$ are values of significance, from birth to extreme old age; the term $\mu\nu^x$ is insignificant till x is equal about 80, "but for analytical anticipation is of significance some years before," and then it slowly increases during the remainder of life*. The value ${}_{\prime}k_{\prime}\varepsilon^x$ is particularly interesting, because in the Carlisle Table it shows a surprising agreement with the stated mortality of children from the age of birth till the age of 1 year.

Art. 7. But now continuing with respect to the original formula of near proximity to the law of mortality $L_x=d.\overline{g|}^{q^x}$, where d, g, and q may for a long period be considered as constant, their values being dependent on the three selections of ages, and putting it in the form $\lambda L_x=\lambda d+\lambda g.q^x$, and using a new and useful notation with respect to logarithms, by writing underneath a letter whose logarithm is to be expressed the prostrate small l, thus $\underset{\supset}{q}$, to denote the common logarithm of q, $\underset{\subset}{q}$ to denote the Napierian logarithm of q, and the prostrate l in the reverse position, thus $\underset{\sqsupset}{q}$, to denote the number whose common logarithm is q, and $\underset{\sqsubset}{q}$ the number whose Napierian logarithm is q, we have evidently from the equation $\lambda L_x=\lambda d+\lambda g.q^x$,

$$\underset{\supset}{L}_{a+x}=\underset{\supset}{d}+\underset{\supset}{g}.q^{a+x}=\underset{\supset}{d}+\underset{\supset}{g}.q^a\times(1+\underset{\subset}{q}.x+\tfrac{1}{2}.\overline{\underset{\subset}{q}|}^{2}.x^2+\tfrac{1}{2.3}\overline{\underset{\subset}{q}|}^{3}x^3+\tfrac{1}{2.3.4}.\overline{\underset{\subset}{q}|}^{4}.x^4, \&c.);$$

where the coefficients of x and its successive powers converge, in consequence of the smallness of the Napierian logarithm of q, which in the Carlisle mortality is about ·029, so that a very few terms of the series would be required, even if, for instance, the age a were 30, and we wished to know the value of λL_{80}. To use this theorem with advantage, we should be provided with a Table of the values of $\underset{\supset}{g}.q^a$ and of $\underset{\supset}{d}$ for every value of a, the youngest life of the three selected lives which are the foundation of the values of d, g, q, their values being different according to the ages of selection; and if λL_{a+x}, λL_{b+x}, λL_{c+x}, &c. be, for the sake of example, as it was found by the above method, represented respectively by the converging series

$$A+{}^1Ax+{}^2Ax^2+{}^3Ax^3, \&c.,\quad B+{}^1Bx+{}^2Bx^2+{}^3Bx^3, \&c.,\quad C+{}^1Cx+{}^2Cx^2+{}^3Cx^3, \&c.,$$

it is evident on taking $x=0$, that λL_a, λL_b, &c. will be represented by A, B, C, &c.; and as the logarithms of the chances of the persons of the present ages a, b, c, &c. living x years are respectively $\lambda\frac{L_{x+a}}{L_a}$, $\lambda\frac{L_{b+x}}{L_b}$, &c., the logarithm of the chance of a living x years, of b living in x years, of c, &c. will be respectively

$${}^1Ax+{}^2Ax^2+{}^3Ax^3, \&c.,\quad {}^1Bx+{}^2Bx^2+{}^3Bx^3, \&c.,\quad {}^1Cx+{}^2Cx^2+{}^3Cx^3, \&c.;$$

and if r be the value of unity discounted for one year, using $\underset{\supset}{r}$ to express its common logarithm, we shall have the logarithm of the present value of unity to be received in

* But μ, though considered now to be constant, may show after x is 100 that it is not absolutely so, though data are wanting to show the nature of its variability.

x years, provided the persons of the present ages a, b, c, &c. are then all living, represented by

$$(\underline{r}+{}^{1}A+{}^{1}B+{}^{1}C, \&c.)x+({}^{2}A+{}^{2}B+{}^{2}C, \&c.)x^{2}+({}^{3}A+{}^{3}B+{}^{3}C, \&c.)x^{3}, \&c.,$$

a very converging series; but in the use of the method it is necessary to have, instead of the common logarithm, the Napierian logarithm, and the Napierian logarithms of L_{a+x}, L_{b+x}, &c. would be expressed by

$$\underline{d}+\underline{g}\,q^{a}\times\left(1+\underline{q}.x+\underline{q}^{2}.\frac{x^{2}}{2}+\&c.\right),\ \underline{d}+\underline{g}.q^{b}\times\left(1+\underline{q}x+\underline{q}^{2}.\frac{x^{2}}{2}\ \&c.\right);$$

and if, for the sake of brevity, we represent $\underline{g}.(q^{a}+q^{b}+q^{c}, \&c.)$, multiplied respectively by $\underline{q}$, $\frac{1}{2}\overline{\underline{q}}|^{2}$, $\frac{1}{2.3}.\overline{\underline{q}}|^{3}$, &c. by A_{1}, A_{2}, A_{3}, &c., we shall have the Napierian logarithm of the present value of unity to be received in x years, if the persons of the present ages a, b, c, &c. be living, represented by $\underline{r}+A_{1}.x+A_{2}x^{2}+A_{3}x^{3}+$ &c., a very converging series. And if the anti-Napierian logarithm of this expression be represented by $1+{}^{1}Sx+{}^{2}Sx^{2}+{}^{3}Sx^{3}$, &c., in which the coefficients of the different powers of x, namely, ${}^{1}S$, ${}^{2}S$, ${}^{3}S$, form a very converging series, then on using the notation adopted in my paper of 1820, $\left.\overset{x}{\overline{1}}\right|^{w}_{z} x^{n}$ to express the sum of all the values of x^{n}, from $x=w$ to $x=z$ inclusively of both by increasing x continually by unity, the present value of an annuity of 1 on the joint lives of persons now of the age of a, b, c, &c., the first payment to be made in w years, and the last in z years, will be represented by the series

$$\left.\overset{x}{\overline{1}}\right|^{w}_{z} x^{0}+{}^{1}S\left.\overset{x}{\overline{1}}\right|^{w}_{z} x+{}^{2}S\left.\overset{x}{\overline{1}}\right|^{w}_{z} x^{2}+{}^{3}S\left.\overset{x}{\overline{1}}\right|^{w}_{z} x^{3}+\&c.,$$

which will be a converging series.

And if the annuity were intended to be entered on immediately, so that the first payment was to be in one year, and it was intended to continue as long as the joint lives continued, it would be written

$$\left.\overset{x}{\overline{1}}\right|_{1} x^{0}+{}^{1}S\left.\overset{x}{\overline{1}}\right|_{1} x+{}^{2}S\left.\overset{x}{\overline{1}}\right|_{1} x^{2}+{}^{3}S\left.\overset{x}{\overline{1}}\right|_{1} x^{3}+\&c.;$$

and if this were to be an increasing annuity, continually increasing by the additions from year to year of the payment n, the xth payment would be $1+nx$, and the present value of that xth payment would be

$$\overline{1+nx}(1+{}^{1}Sx+{}^{2}Sx^{2}+{}^{3}Sx^{3}+{}^{4}Sx, \&c.)=1+({}^{1}S+n)\times x+({}^{2}S+n.{}^{1}S)\times x^{2}+({}^{3}S+n.{}^{2}S)x^{3}+\&c.,$$

and therefore the present worth of such an increasing annuity on the joint lives would be represented by

$$\left.\overset{x}{\overline{1}}\right|_{1} x^{0}+({}^{1}S+n)\left.\overset{x}{\overline{1}}\right|_{1} x+({}^{2}S+n.{}^{1}S)\left.\overset{x}{\overline{1}}\right|_{1} x^{2}+({}^{3}S+n.{}^{2}S)\left.\overset{x}{\overline{1}}\right|_{1} x^{3}+({}^{4}S+n.{}^{3}S)\left.\overset{x}{\overline{1}}\right|_{1} x^{4}, \&c.;$$

and a similar mode may be adopted for more complicated regulations of the payments from year to year.

And to make this simple theorem easily available for the calculations, it will be convenient to have a Table of powers of numbers increasing regularly in arithmetical progression, with the common difference of 1, from 1 to 100, of the annexed form, which I call the Collecting Table, and which will be found of immense service. Such, for instance, as the following example; but a more extensive Table will be given further on.

x.	Sums of x.	x^2.	Sums of x^2.	x^3.	Sums of x^3.	x^4.	Sums of x^4.	To continue to $x=10$, or more.
1	1	1	1	1	1	1	1	&c.
2	3	4	5	8	9	16	17	&c.
3	6	9	14	27	36	81	98	&c.
4	10	16	30	64	100	256	354	&c.
5	15	25	55	125	225	625	979	&c.
6	21	36	91	216	441	1296	2273	&c.
7	28	49	140	343	784	2401	4676	&c.
8	36	64	204	512	1296	4096	8772	&c.
&c. to 100	&c.	&c.	&c.	&c.	&c.	&c.	&c.	&c.

It is evident that if this Table be continued to great values of x, say to 100, or high powers of x, which may be necessary in some cases, we shall get very high numbers for the value of $\sum_{1}^{x} x^n$; but these numbers, when multiplied by their coefficients, which will be very small, may not prevent the series from converging.

Art. 8. But the case renders a new arithmetical notation convenient, which will be explained further on, as only a few of the most significant figures will be required, and the remainder may be considered as noughts. This analysis does not only require the anti-Napierian logarithm to be taken of analytical expressions, but also the reversion of analytical expressions into Napierian logarithms to be found; for if there be a series $a+bx+cx^2+dx^3$, &c. when the coefficients a, b, c, &c. are a converging series, the Napierian logarithm of it is the Napierian logarithm of a

$$+\left(\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3, \&c.\right)-\frac{1}{2}\left(\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3\right)^2+\frac{1}{3}\left(\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3, \&c.\right)^3 \&c.,$$

which may be represented by $a+{}'bx+{}'cx^2+{}'dx^3+$ &c., and would give the values of ${}'b$, ${}'c$, ${}'d$, &c., or rather proceed as follows.

Supposing the Napierian logarithm of

$$1+B_1x+B_2x^2+B_3x^3$$

were equal to

$$A_1x+A_2x^2+A_3x^3+\&c.,$$

if these equations be put into fluxions, we shall obtain, after dividing by $\dot{x}$,

$$\frac{B_1+2B_2x+3B_3x^2+4B_4x^3 \&c.}{1+B_1x+B_2x^2+B_3x^3 \&c.}=A_1+2A_2x+3A_3x^2+4A_4x^3, \&c.,$$

and consequently

$$\left.\begin{array}{r}A_1+2A_2x\ \ +3A_3x^2\ \ +4A_4x^3\ \ +\&c.\\ +\ A_1B_1x+2A_2B_1x^2+3A_3B_1x^3\ \ \&c.\\ +\ A_1B_2x^2+2A_2B_2x^3\ \ \&c.\\ +\ A_1B_3x^3\ \ \&c.\\ \&c.\\ -B_1-2B_2x-3B_3x^2-4B_4x^3\ \ \&c.\end{array}\right\}=0.$$

Consequently

$$A_1=B_1;\ A_2=B_2-\frac{1}{2}A_1B_1;\ A_3=B_3-\frac{1}{3}(2A_2B_1+A_1B_2),$$

$$A_4=B_4-\frac{1}{4}\{3A_3B_1+2A_2B_2+A_1B_3\};\ A_5=B_5-\frac{1}{5}\{4A_4B_1+3A_3B_2+2A_2B_3+A_1B_4\},\ \&c.$$

for turning natural expressions into Napierian logarithms; and we also have for the reverse, namely, turning expressions of Napierian into anti-Napierian logarithms,

$$B_1=A_1;\ B_2=A_2+\frac{1}{2}A_1B_1;\ B_3=A_3+\frac{1}{3}(2A_2B_1+A_1B_2);\ B_4=A_4+\frac{1}{4}(3A_3B_1+2A_2B_2+A_1B_3),\ \&c.$$

Art. 9. And now supposing the Napierian logarithm of the chances of persons now of the ages α, β, γ, &c. being living in x years be turned into anti-Napierian logarithms by the theorem at the end of art. 8, that is to say, into analytical natural expressions corresponding to those Napierian logarithms, and that the value of those chances be respectively deducted from unity, they will represent the respective chances of their being extinct, and may be expressed by

$$p_\alpha x\times(1+{}^1p_\alpha x+{}^2p_\alpha x^2+{}^3p_\alpha x^3,\ \&c.);\ p_\beta x\times(1+{}^1p_\beta x+{}^2p_\beta x^2+\&c.);\ p_\gamma x(1+{}^1p_\gamma x+{}^2p_\gamma x^2+\&c.),\ \&c.,$$

and supposing the Napierian logarithms of these several multipliers

$$1+{}^1p_\alpha x+{}^2p_\alpha x^2+\&c.,\ 1+{}^1p_\beta x+{}^2p_\beta x^3,\ \&c.,$$

be taken by the same theorem of art. 8, and that there be n persons of the above said ages α, β, γ, &c., and that the sum of these Napierian logarithms thus taken be represented by ${}^1Px+{}^2Px^2+{}^3Px^3$, &c., and that the product of the n quantities p_α, p_β, p_γ, &c., be represented by P; then the present value of unity to be received in x years, provided all the persons before mentioned to be living are extinct, will be Px^n, multiplied by the anti-Napierian logarithm of $(r+{}^1P)x+{}^2Px^2+{}^3Px^3$, &c.; and if the anti-Napierian logarithm be represented by $1+R_1x+R_2x^2+R_3x^3$, &c., the said value will be

$$P.x^n+P.R_1x^{n+1}+P.R_2x^{n+2}+\&c.$$

We may execute the required calculations in this manner instead of adopting rules laid down by authors of solving such questions by aid of Tables of the values of annuities on single lives and two joint lives, and obtaining by an inaccurate interpolation the values of annuities of a larger number of joint lives, as there may be required many values of annuities on three joint lives, on four joint lives, and on more joint lives, which would not only require great labour, but leave very small confidence of an accurate result. And it is owing to this circumstance that, in addition to having a Table of the expressions of the Napierian logarithm, for every age, of the chance of persons of any age living x years, I propose also, for every age, to have the Napierian logarithm of the quotient of the expression which gives the chance of a person of any age being dead divided by its first term, namely, x multiplied by the coefficient it has in the value of the chance, as in many cases such a Table will introduce great facility.

There are much more intricate cases for calculation, which the law of mortality enables us to overcome; I allude to annuities and assurances depending on conditions of survivorships among the party who are involved in the annuity. Now I observe, from having the Napierian logarithm of the chance of each individual surviving x years, or having the Napierian logarithm of the expression after it is divided by its first term, of the chance of his being dead, and adding the coefficients of all the first terms together, all the second terms together, &c., and finding the anti-Napierian logarithm of the result, and multiplying this by all the first terms, which were directed to divide the various chances, we have the natural value of the chances compounded out of them.

Art. 10. Previously to proceeding, I venture to introduce a notation which I have found convenient with respect to vital algorithm, as the theory, and the application of it to important objects, introduces very large numbers, and also extremely small fractions, of which, in both cases, there is only a necessity to attend to a very few of the significant figures: thus, suppose we had the number 897654321, and that it answered for sufficient accuracy only to consider the number as 898000000, I would write it 898⑥; by the ⑥ I mean the six noughts which are not written down; and if we had the decimal fraction ·00000000763, in which eight noughts occur before the significant figure, I would write it ⑧763; and if these two quantities had to be multiplied together, I should write it 898⑥×⑧763; and ⑧763 consists of eleven places of decimals, and $898\times763=685174$; and this, if four significant figures were sufficient, I would write 6852②; to which add ⑥, we have 6852⑧; and adding to the left ⑪, as ⑧763 signifies ·00000000763, the product will stand ⑪6852⑧, and would signify that three figures to the right are to be cut off as a decimal, and that the product is 6·852, because eleven places of decimals, including the first significant figure being multiplied by 10^{11}, leaves $11-8$ three places of decimals; and if we had to add 68·52 to ⓪23, it would be $\left.\begin{matrix}68{\cdot}52\\+{\cdot}023\end{matrix}\right\}=68{\cdot}543$; and so of other cases. The great use of this notation will appear in the construction of the Collecting Table, and its application to the analytical anti-Napierian logarithms. The algorithm of vital statistics introducing the necessity of intricate entanglements of common and Napierian logarithms and anti-Napierian logarithms, and reverses of those operations in analytical expression, I think it expedient to enter more particularly on the nature of logarithms than has been done. In the first place, I observe that it is usual to consider the logarithm of a positive quantity and a negative quantity the same; thus, suppose g were a positive number greater than 1, its logarithm would be positive, and the logarithm of this logarithm might be written the logarithm of the logarithm of g; but if g were a positive number less than unity, its logarithm would be negative; and then $\lambda\lambda g$, if λ stood for the logarithm of, and $\lambda\lambda g$ for the logarithm of the logarithm of, would have no meaning in the positive scale of logarithms. This distinction I did not notice in the notation in my paper of 1825, though, properly, the notation ought to have been, as g was found to be a positive number less than 1, $\lambda(-\lambda g)$; but I did not neglect in my calculation to attend to the consequence; because, till the value of g is known as to its being less or greater than unity, there is a convenience, but only to be adopted with care, in writing $\lambda\lambda g$.

In this paper, as has been already explained, I use the prostrate small l, thus $\subset\!\!-$ $-\!\!\supset$, with the loop upwards, the first to denote the Napierian logarithm of, the second the common logarithm of; and with the loop downwards, thus $\subset\!\!-$ $-\!\!\supset$, to represent the anti-Napierian logarithm of, and the anti-common logarithm; and $-\!\!\supset\subset\!\!-$ would mean the common logarithm of the Napierian logarithm of, if they are placed horizontally in the line on the left of a character; thus $-\!\!\supset\subset\!\!- x$ would signify the common logarithm of the Napierian logarithm of x; but if placed below the character, which may be in some cases more convenient, thus $\underset{-\!\!\supset\subset\!\!-}{x}$, would signify the common logarithm of the Napierian logarithm of x, and $\underset{-\!\!\supset\subset\!\!-}{x}$ would signify the common logarithm of *minus* the Napierian logarithm of x.

Art. 11. It appears now time to show how to use the original formula $L_x = d.\overline{g}|^{q^x}$, in order to reduce it into a form for practice, and which may be written

$$\subset\!\!-L_x = \subset\!\!-d + \underset{\smile}{g}\left(1+\underset{\smile}{q}.x+\frac{1}{2}\overline{\underset{\smile}{q}}|^2 x^2+\frac{1}{2.3}\cdot\overline{\underset{\smile}{q}}|^3 x^3+\ \&c.\right),$$

where $\subset\!\!-$, whether put to the left of a character or below it, stands for the Napierian logarithm of, and $\underset{\smile}{q}$, $\frac{1}{2}\ \overline{\underset{\smile}{q}}|^2$ $\frac{1}{2.3}\ \overline{\underset{\smile}{q}}|^3$, &c., multiplied by $\underset{\smile}{g}$, stand for the coefficients of x, x^2, x^3, &c. in the development of $\subset\!\!-L_x$; and I add, with respect to any function $ax+bx^2+cx^3$, &c., we would represent the anti-Napierian logarithm of it by $1+b_1x+b_2x^2+b_3x^3$, &c. The value of these coefficients will be different according as the values d, g, q are different, which values, from what has been stated, will differ for every selection used in the vital rule of three of the three lives, as, though they have for a long period in every selection the appearance of constancy, they are not absolutely constant; and I observe, that I use the same notation for the development of the value of $\subset\!\!-L_x$ in the more perfect formula which I have given, namely,

$$\subset\!\!-L_x = C\mathfrak{C}^x + {}_{/}k_{/}\varepsilon^x + k\varepsilon^x - \subset\!\!-(e^x.\underset{\smile}{q}\times\overline{x-h}) + \mu\nu^x;$$

but I should state the formula to which I here refer is

$$-\!\!\supset L_x = C\mathfrak{C}^x + {}_{/}k_{/}\varepsilon^x + k\varepsilon^x - -\!\!\supset(e^x\underset{-\!\!\supset}{q}.\overline{x-h}) + \mu\nu^x.$$

But the formula I now state is more convenient for ultimate reduction; but the difference of the two being while $\lrcorner$ stands for the common logarithm of, and $\urcorner$ for the anti-common logarithm of, in the formula referred to, the values of C, $_{\prime}k$, k, μ are not the same in this as in that, but would be, if multiplied by the Napierian logarithm of 10; and I will begin by showing the great use which even the original formula $L_x = d.\overline{g|}^{q^x}$ may be, though d, g, q are, instead of being absolute constants, quantities very slowly variable from one selection of three lives to another; though it is not equally valuable in point of accuracy to the improved formula, where all the quantities except x are constant from birth to extreme old age. And now, reverting to the formula $L_x = d.\overline{g|}^{q^x}$, and observing that in the Carlisle mortality the selection of the three ages may be distant from each other by even 30 years, being 10, 40, 70, we obtain a very efficiently-useful formula, although in some cases, though rarely, L_x given by the formula, and L_x of MILNE, may even differ two years; but still, the proportion of the chances of living to those ages given by each will be but a small per cent. of each other. When the selection is 10, 40, 70, we have $\llcorner d$, the Napierian logarithm of $d = 8{\cdot}8833$; $\llcorner q$ or $\underline{q}$, the Napierian logarithm of $q = {\cdot}0377355$; $\underline{g} = -{\cdot}0786136$; $\lrcorner(-\underline{g}) = \bar{2}{\cdot}89605$; $\lrcorner \underline{q} = \bar{2}{\cdot}57675$. Here the difference of ages in the three selected lives is successively 30 years; but as that difference may not give sufficient accuracy, I do not adopt it.

If $L_x = d.\overline{g|}^{q^x}$, with constant values of d, g, q, were true throughout life, we should have the logarithm of L_x=logarithm of d+logarithm of $g.q^x$, and

$$\lambda L_{a+x} = \text{logarithm of } d + \text{logarithm of } \overline{g|}.q^a.q^x,$$

as before observed; and the logarithm of the chance of a person of the age a being living in x years = the logarithm of $\frac{L_{a+x}}{L_a}$ = logarithm of $g.q^a(q^x-1)$; and similarly the logarithm of the chances of persons of the ages b, &c. living x years will be expressed, logarithm of $g.q^b(q^x-1)$, &c.; and consequently we shall have the logarithm of the chance of persons now of the ages a, b, c, e living x years

$$= \text{logarithm of } g \times (q^x-1)(q^a+q^b+q^c+q^e),$$

and also the logarithm of the chance of a person of the age p living x years = logarithm of $g \times (q^x-1).q^p$; and therefore if p be found so that $q^p = q^a+q^b+q^c+q^e$, or, which is the same thing, if p be the value of $\left(\frac{\text{logarithm of } (q^a+q^b+q^c+q^e)}{\text{logarithm of } q}\right)$, then will the chance of a person of the age p living x years, and the chance the four persons of the ages a, b, c, e being every one surviving in x years, be exactly the same. This circumstance had induced the learned Professor AUGUSTUS DE MORGAN, when commenting on the theorem $L_x = d.\overline{g|}^{q^x}$, which was given by me in the Philosophical Transactions in the year 1825, and the ingenious Mr. SPRAGUE, and others who appreciated it, and philosophically felt pleasure in bestowing praise where they thought merit was due for the discovery of a useful theorem, with good analytical judgment to observe, that if the theorem were true through life, the value of annuities on any number of joint lives might be with ease obtained, which would be a most beautiful and useful property.

But, as I have shown in my paper of 1825, and here more fully illustrated, those values can only be considered as approximative constants for a limited period, though that period is very long, and whilst x increases from 0 to about 60, if a were $=10$. I have shown in that paper, in the Carlisle Table of Mortality of the late learned Mr. MILNE, that the theorem affords values for L_x at the different ages, say from 10 to 60, differing very triflingly from MILNE'S Tables; and having in that paper expressly stated and shown that they were not absolutely constant, but depend on the ages of the three selected lives, it need not cause surprise that the theorem does not serve for that useful purpose; because p, determined by the equation $q^a+q^b+q^c+q^e=q^p$, would come out so much larger than the limits of the applicability of these first-found constants, that, if the method availed, it would be a contradiction to the assertion that the elements were variable, as any one may convince himself; but there are cases easily pointed out where the ages may be so taken as to afford a result by the theorem approximatively true. But this observation does not deprive the theorem $L_{a+x}=d\,.\,\overline{g}|^{q^{a+x}}$ of very great and serviceable value; but to make it extensively available it will require a Table for every age a of the constants d, g, q, varying from one age to the other; though the theorem

$$\leftharpoondown L_x=C\,.\,\mathfrak{C}^x+{}_{\prime}k_{\prime}\varepsilon^x+k\varepsilon^x-\leftharpoondown(e^x\,.\,\overline{x-h\,.\,\underline{q}})+\mu\nu^x$$

appears still more valuable, which has the same constants for every age, from birth to extreme old age, and which, from the age of about 10 to 80, will take the simpler form

$$\leftharpoondown L_x=C\mathfrak{C}^x-\leftharpoondown(e^x.\overline{x-h\,.\,\underline{q}}),$$

in consequence of the portions, ${}_{\prime}k_{\prime}e^x$, ke^x and $\mu\nu^x$, between these limits being insignificantly small.

When the differences of the selected ages are only instead of 30 years assumed further on 20 years, that is, when difference $n=20$, and if m be respectively 10, 20, 30, 40, 50, 60, that is, if the youngest in the selection be respectively 10, 20, ... 60, we have, for finding d, g, q in the formula $L_x=d\,.\,\overline{g}|^{q^x}$, the following data:—

For the selection 10, 20, 30, we have

$$\rightharpoondown d=3{\cdot}88012,\quad \rightharpoondown q=\cdot0132565,\quad \rightharpoondown(-\underline{g})=\bar{2}{\cdot}71185,\quad \underline{g}=-\cdot051508.$$

$$\underline{d}=8{\cdot}9343\ ,\quad \underline{q}=\cdot030526\ ,\quad \rightharpoondown(-\underline{g})=\bar{1}{\cdot}07407,\quad \underline{g}=-\cdot11860.$$

For the selection 20, 40, 60, we have

$$\rightharpoondown d=3{\cdot}88137,\quad \rightharpoondown q=\cdot012984\ ,\quad \rightharpoondown(-\underline{g})=\bar{2}{\cdot}72607,\quad \underline{g}=-\cdot053211.$$

$$\underline{d}=8{\cdot}9374\ ,\quad \underline{q}=\cdot027897\ ,\quad \rightharpoondown(-\underline{g})=\bar{1}{\cdot}08829,\quad \underline{g}=-\cdot12254.$$

For the selection 30, 50, 70, we have

$$\rightharpoondown d=3{\cdot}8273\ ,\quad \rightharpoondown q=\cdot0192535,\quad \rightharpoondown(-\underline{g})=\bar{2}{\cdot}30241,\quad \underline{g}=-\cdot020064.$$

$$\leftharpoondown d=8{\cdot}8127\ ,\quad \leftharpoondown q=\cdot044334\ ,\quad \rightharpoondown(-\underline{g})=\bar{2}{\cdot}66463,\quad \underline{g}=-\cdot0462.$$

For the selection 40, 60, 80, we have

$\supset d=3{\cdot}75272$, $\supset q={\cdot}030345$, $\supset(-\underline{g})=\bar{3}{\cdot}46094$, $\underline{g}=-{\cdot}0028903$.

$\subset d=8{\cdot}6409$, $\subset q={\cdot}069872$, $\supset(-\underline{g})=3{\cdot}82316$, $\underline{g}=-{\cdot}0066552$.

For the selection 50, 70, 90, we have

$\supset d=3{\cdot}71469$, $\supset q={\cdot}0334825$, $\supset(-\underline{g})=\bar{3}{\cdot}18036$, $\underline{g}=-{\cdot}0015148$.

$\subset d=8{\cdot}55534$, $\subset q={\cdot}077096$, $\supset(-\underline{g})=3{\cdot}54258$, $\underline{g}=-{\cdot}0034880$.

For the selection 60, 80, 100, we have

$\supset d=3{\cdot}75272$, $\supset q={\cdot}0270605$, $\supset(-\underline{g})=\bar{2}{\cdot}10984$, $\underline{g}=-{\cdot}012878$.

$\subset d=8{\cdot}74186$, $\subset q={\cdot}062309$, $\supset(-\underline{g})=\bar{2}{\cdot}47206$. $\underline{g}=-{\cdot}029652$.

The selections give, for the analytical expression of $\subset L_{a+x}$ corresponding to the several values of a below, the expressions below:—

a	$\subset L_{a+x}$.
10	8·9343—{·160927+·004912x+④75x^2+⑥762x^3+⑧58x^4+⑩ 35x^5 &c.}.
20	8·8127—{·22279+·00662x+④97995x^2+⑥9924x^3+⑧75x^4+⑩44x^5+⑫2x^6}.
30	8·8127—{·17468+·0077443x+③171678x^2+⑤2537x^3+⑦2812x^4+ ⑨24x^5+⑪2x^6}.
40	8·6409—{·10888+·007608x+③2658x^2+⑤6191x^3+⑥108x^4+⑧12x^5+⑩17x^6}.
50	8·5534—{·16470+·012696x+③48943x^2+④1258x^3+⑥2424x^4+⑧3738x^5+⑩48x^6}.
60	8·74186—{·54134+·033930x+②1051x^2+④218826x^3+⑥3396x^4+⑤423x^5+⑩44x^6}.

Where observe were a, 10, 20, 30, 40, 50, 60, and were x taken in each case $=0$, the above expression would be respectively the values of $\subset L_{10}$, $\subset L_{20}$, $\subset L_{30}$, $\subset L_{40}$, $\subset L_{50}$, $\subset L_{60}$, which would become 8·9343—·160927=8·77337; 8·9374—·22279=8·71462; 8·8127—·17468=8·3887; 8·6409—·10888=8·5320; 8·5534—·16470=8·3887; 8·74186—·54134=8·20052: the reader is already informed that the symbol, for instance, ⑧3738, signifies, ·000000003738.

Art. 12. And now returning to the assumed formula

$$\lambda L_x=C\mathfrak{C}^x-\lambda^{-1}\{e^x.\overline{x-h}.\lambda q_0\},$$

of which, in the case of the Carlisle mortality, I have already given the value of C, e, and putting $C\mathfrak{C}^x-\lambda L_x=M_x$, we shall have the equation

$$\lambda^{-1}\{e^x.\overline{x-h}.\lambda q_0\}=M_x,$$

and consequently

$$e^x.\overline{x-h}.\lambda q_0=\lambda M_x;$$

and using the vital rule of three, that is to say, selecting three values of x to find the three unknown quantities e, h, and q_0, say $x=m$, $x=m+n$, $x=m+2n$, we have the three equations,

$$e^m.(m-h).\lambda q_0=\lambda M_m,\quad e^{m+n}(m+n-h).\lambda q_0=\lambda M_{m+n},\quad e^{m+2n}(m+2n-h).\lambda q_0=\lambda M_{m+2n}.$$

Multiply the first and third of the equations together, and square the second, and we shall have by reduction, putting

$$\frac{\lambda M_m \times \lambda M_{m+2n}}{(\lambda M_{m+n})^2}=Q_m, \quad \frac{\overline{m-h}.\overline{m+2n-h}}{(m+n-h)^2}=Q_m,$$

that is,

$$\frac{(\overline{m+n-h}-n)\times(\overline{m+n-h}+n)}{\overline{m+n-h}|^2}=Q_m;$$

that is,

$$\frac{\overline{m+n-h}|^2-n^2}{(m+n-h)^2}=Q_m;$$

consequently

$$\overline{m+n-h}|^2\times\overline{1-Q_m}=n^2,$$

and therefore

$$h=m+n+\frac{n}{\sqrt{1-Q_m}};$$

and m, n, λL_m, λL_{m+n}, λL_{m+2n} being taken from the Table for the given or assumed values of m and n, and C and Ϲ having been found, we have the value of h; and from the equation

$$e^x.\overline{x-h}.\lambda q_0=\lambda M_x$$

we have, by putting m and $m+n$ for x, the two equations

$$e^m\times\lambda q_0\times\overline{m-h}=\lambda M_m$$

and

$$e^{m+n}.\lambda q_0.\overline{m+n-h}=\lambda M_{m+n},$$

and consequently we have

$$e^n=\frac{\lambda M_{m+n}\times\overline{m-h}}{\lambda M_m\times\overline{m+n-h}},$$

and therefore

$$n\lambda e=\lambda(-\lambda M_{m+n})+\lambda(h-m)-\lambda(-\lambda M_m)-\lambda(h-m-n),$$

because λM_m, &c. is negative, and therefore we cannot take its logarithm. Now having found h and e, we find q_0 from the equation

$$e^m.\overline{m-h}.\lambda q_0=\lambda M_m,$$

which will give

$$\lambda\lambda q_0=\lambda(-\lambda M_m)-\lambda(h-m)-m\lambda e;$$

and taking for m and n 20 and 30 respectively, which will give the data for the selection (by the vital rule of three) 20, 50, 80, we are to expect, if the theorem is an approximation to the law of mortality, that it will very nearly agree with the Tables of mortality for every age, from the age of 20 to 80, which, by the example I am about to give, will be found to be the case. And to proceed, from having found

$$\lambda C=\cdot 59461, \qquad \lambda Ϲ=\bar{1}\cdot 999746,$$

we have

$\lambda CϹ^{20}=\cdot 589443$; $\lambda CϹ^{50}=\cdot 581882$; $\lambda CϹ^{80}=\cdot 57431$;

$\therefore$ $CϹ^{20}=3\cdot 88631$; $CϹ^{50}=3\cdot 81882$; $CϹ^{80}=3\cdot 75249$, and therefore

$$M_{30}=C\mathfrak{C}^{30}-\lambda L_{30}=\left\{\begin{matrix}3\cdot88631\\-3\cdot78463\end{matrix}\right\}=\cdot10168 \qquad M_{50}=C\mathfrak{C}^{50}-\lambda L_{50}=\left\{\begin{matrix}3\cdot81882\\-3\cdot64316\end{matrix}\right\}=\cdot17566$$

$$M_{80}=C\mathfrak{C}^{80}-\lambda L_{80}=\left\{\begin{matrix}3\cdot75249\\-2\cdot97909\end{matrix}\right\}=\cdot77340$$

$$\left.\begin{matrix}\lambda M_{30}=\bar{1}\cdot007326\\ =-\cdot99268\end{matrix}\right\} \quad \left.\begin{matrix}\lambda M_{50}=\bar{1}\cdot22467\\ =-\cdot77533\end{matrix}\right\} \quad \left.\begin{matrix}\lambda M_{80}=\bar{1}\cdot04747\\ -\cdot95253\end{matrix}\right\}$$

$$\lambda(-\lambda M_{30})=\bar{1}99680;\quad \lambda(-\lambda M_{50})=\bar{1}\cdot87814;\quad \lambda(-\lambda M_{80})=\bar{1}04747.$$

To find Q.

$$\begin{aligned}
\lambda(-\lambda M_{30})&=\bar{1}\cdot99680\\
\lambda(-\lambda M_{80})&=\bar{1}\cdot04747\\
&\overline{\bar{1}\cdot04427}\\
-2\lambda(-\lambda M_{50})&=\bar{1}\cdot75628\\
\lambda Q&=\overline{\bar{1}\cdot28799}\\
Q&=\cdot19408\\
1-Q&=\cdot80592\\
\lambda(1-Q)&=\bar{1}\cdot90629\\
\tfrac{1}{2}\lambda(1-Q)&=\bar{1}\cdot95336\\
\text{Its compt.}&=\cdot04686\\
\lambda(n=30)&=1\cdot47712\\
\lambda\frac{30}{\sqrt{1-Q}}&=1\cdot52398\\
\frac{30}{\sqrt{1-Q}}&=33\cdot418
\end{aligned}$$

$$\begin{aligned}
h&=m+n+\frac{n}{\sqrt{1-Q}}=83\cdot418\\
h-m&=63\cdot418\\
h-m-n&=33\cdot418\\
\lambda e&=\frac{\overline{\lambda-\lambda.M_{m+n}}+\lambda\overline{h-m}-\overline{\lambda-\lambda M_m}-\lambda\overline{h-m-n}}{30}
\end{aligned}$$

$$\begin{aligned}
\lambda(-\lambda M_{m+n})&=\bar{1}\cdot87814\\
\lambda(h-m)&\ 1\cdot80223\\
&\overline{\bar{1}\cdot68037}\\
&\ 1\cdot52081\\
30)&\ \overline{\cdot15956}=\lambda e^{30}\\
\lambda e&=\cdot00531866\\
\lambda e^{20}&=\cdot10637
\end{aligned}
\qquad
\begin{aligned}
\lambda(-\lambda M_m)&=\bar{1}\cdot99680\\
\lambda(h-m-n)&=\bar{1}\cdot52401\\
&\overline{1\cdot52081}
\end{aligned}$$

$$\begin{aligned}
\lambda\lambda q_0&=\left.\begin{matrix}\lambda(-\lambda M_{20})\\ -\lambda e^{20}\\ -\lambda\overline{h-20}\end{matrix}\right\}\\
\lambda(-\lambda M_{20})&=\bar{1}\cdot99680\\
-\lambda e^{20}&=\bar{1}\cdot89363\\
-\lambda(h-20)&=\bar{2}\cdot19777\\
\lambda\lambda q_0&=\overline{\bar{2}\cdot08820}\\
\lambda q_0&=0\cdot12252
\end{aligned}$$

Formula.

$$\lambda L_x=C\mathfrak{C}^x-\lambda^{-1}\{e^x\overline{x-h}\cdot\lambda q_0\}$$

$$\lambda C=\cdot59461 \quad h=83\cdot412$$

$$C=3\cdot93197$$

$$\lambda e=\cdot0053186$$

$$\lambda\lambda q_0=\bar{2}\cdot08822$$

Proof of work (see the Theorem).

$$\lambda(-\lambda M_{30})=\left\{\begin{matrix}\lambda\lambda q_0 & =\bar{2}\cdot0882\\ \lambda 63\cdot418 & =1\cdot80223\\ \lambda e^{30} & =\cdot10637\end{matrix}\right.\quad \overline{\bar{1}\cdot99680}$$

$$\lambda(-\lambda M_{50})=\left\{\begin{matrix}\lambda\lambda q & =\bar{2}\cdot0882\\ \lambda 33\cdot418 & =1\cdot52398\\ \lambda e^{50} & =\cdot26593\end{matrix}\right.\quad \overline{\bar{1}\cdot87811}$$

$$\lambda(-\lambda M_{80})=\left\{\begin{matrix}\lambda\lambda q & =\bar{2}\cdot0882\\ \lambda 3\cdot418 & =\cdot53377\\ \lambda e^{80} & =\cdot42548\end{matrix}\right.\quad \overline{\bar{1}\cdot04745}$$

	Carlisle.	Milne.		Carlisle.	Milne.		Carlisle.	Milne.		Carlisle.	Milne.
x.	The formula gives λL_x.	λL_x.	x.	The formula gives λL_x.	λL_x.	x.	The formula gives λL_x.	λL_x.	x.	The formula gives λL_x.	λL_x.
10	3·81323	3·8102	28	3·75759	3·75952	46	3·67106	3·66811	64	3·48635	3·49734
11	3·81113	3·80823	29	3·75397	3·75557	47	3·66569	3·66162	65	3·47298	3·47972
12	3·80809	3·80618	30	3·75619	3·75143	48	3·65771	3·65523	66	3·45128	3·46150
13	3·80564	3·80400	31	3·74623	3·74702	49	3·65051	3·64915	67	3·43176	3·44264
14	3·80237	3·80175	32	3·74222	3·74257	50	3·64309	3·64316	68	3·41090	3·42292
15	3·80089	3·79934	33	3·73870	3·73815	51	3·63513	3·63729	69	3·38863	3·40226
16	3·79694	3·79664	34	3·73398	3·73376	52	3·62707	3·63104	70	3·36540	3·38039
17	3·79354	3·79373	35	3·72957	3·72933	53	3·61853	3·62439	71	3·33865	3·35776
18	3·79070	3·79071	36	3·72548	3·72485	54	3·60917	3·61731	72	3·32065	3·33102
19	3·78792	3·78767	37	3·71984	3·72024	55	3·59321	3·60991	73	3·27909	3·30038
20	3·78454	3·78462	38	3·71501	3·71550	56	3·59022	3·60206	74	3·24834	3·26505
21	3·78145	3·78154	39	3·71112	3·71063	57	3·58953	3·59373	75	3·21177	3·22401
22	3·77822	3·77851	40	3·70380	3·70544	58	3·56774	3·58456	76	3·20301	3·18041
23	3·77501	3·77546	41	3·69939	3·69975	59	3·55672	3·57392	77	3·12399	3·13322
24	3·77167	3·77240	42	3·69491	3·69373	60	3·54427	3·56146	78	3·08460	3·08386
25	3·76828	3·76930	43	3·68929	3·68744	61	3·53269	3·54667	79	3·06272	3·03383
26	3·76483	3·76612	44	3·68313	3·68106	62	3·51806	3·53084	80	2·97895	2·97909
27	3·76125	3·76290	45	3·67643	3·67459	63	3·50241	3·51428			

Art. 13. The very near coincidence of the result of the formula

$$\lambda L_x = C\mathfrak{C}^x - \lambda^{-1}\{e^x.\overline{x-h}.\lambda q_0\}$$

with Mr. MILNE'S Table, from the age 10 years to the age 80, that is to say, for seventy years in continuance, appears strongly demonstrative of the near proximity of the above formula to the law of mortality; and from the uniformity of the progression being evident, which uniformity in MILNE'S Table does not equally appear, gives reason for a preference for the number deduced from the law, to those of MILNE'S Table, for adoption. But it will appear that notwithstanding this agreement of the result of the formula whose constants C, $\mathfrak{C}$, e, h, q_0 are obtained, the first two of them from the values of d in the two formulæ in my paper of 1825, which treats of the formula $L_x = d.\overline{g|}^{q^x}$, the one being obtained by the vital rule of three, by the selection of the three ages 20, 40, 60, the other by the selection 60, 80, 100, and the other three constants, namely e, h, q_0, by help of those constants C, $\mathfrak{C}$, by only three selected ages at thirty years' distance from each other, namely the ages 20, 50, 80; and though the vital rule of three is here constructed on a more recondite analysis than the former; still that uniformity and that interesting coincidence does not subsist with ages less than 10, nor with ages above 80, and especially for ages from birth to the age of a few months; because the more correct formula seems to require three additional terms discoverable by investigation. These I find to be of the form ${}_{\prime}k_{\prime}\epsilon^x$, $k\epsilon^x$, $\mu\nu^x$, where ${}_{\prime}k$, ${}_{\prime}\epsilon$, k, ϵ, μ, ν are all constant quantities, and of such peculiarly interesting values, that I feel it proper to draw the reader's attention to their values, their effects, and the mode of the discovery of them. The effect of the first, ${}_{\prime}k_{\prime}\epsilon^x$, commences at birth in its greatest value, but at the expiration of one month sinks to comparative insignificance, and before the end of one year leaves no appreciable signs of its existence; the second, $k\epsilon^x$, arises in its effect with birth, but continually decreases in effect till the age of about 21 in the Carlisle mortality, and then and after, through the remainder of life, becomes of total insignificance; and the third term does not come into appreciable effect for calculating the number of living till the age of about 80; though for anticipating the number of living which will result from some age to ages above 80, for the purpose of calculation, its effect cannot be overlooked when the age from which the anticipation is necessary is some years less than 80. And the methods of finding those constants which I have adopted will now be explained. Paying attention only to the additional function $k\varepsilon^x$, the formula will stand

$$\lambda L_x = C\mathfrak{C}^x + k\varepsilon^x - M_x;$$

where M stands as above for $\lambda^{-1}\{\sigma^x.\overline{x-h}.\lambda q_0\}$; and putting $\lambda L_x + M_x - C\mathfrak{C}^x = K_x$, for the sake of brevity, and taking, in order to have two equations, in order to find the two constants k, ε, two values of x, namely $x=1$, $x=2$, we shall, from MILNE's Tables having λL_1, λL_2, and from the known values of $C\mathfrak{C}=3\cdot92971$, $C\mathfrak{C}^2=3\cdot9271$, and the values of M_1 and M_2, have the value K_1 and K_2; and as we have $k\varepsilon = K_1$, $k\varepsilon^2 = K_2$, we have

$$\varepsilon = \frac{K_2}{K_1} \text{ and } k = \frac{K_1}{\varepsilon} = \frac{K_1^2}{K_2},$$

that is,

$$\lambda\varepsilon = \lambda K_2 - \lambda K_1 = \bar{1}\cdot79811, \text{ and } \lambda k = \bar{1}\cdot16855, \ \varepsilon = \cdot62822, \ k = \cdot14742.$$

These values of k and ε being now known, we introduce the term $_{\prime}k_{\prime}\varepsilon^x$, and we shall have

$$\lambda L_x = C\mathfrak{C}^x + {}_{\prime}k_{\prime}\varepsilon^x + k\varepsilon^x - M_x = K_x - M_x + {}_{\prime}k_{\prime}\varepsilon^x;$$

and consequently

$$_{\prime}k_{\prime}\varepsilon^x = \lambda L_x + M_x - K_x;$$

put this $= {}_{\prime}K_x$ for the sake of brevity, and we shall have $_{\prime}k_{\prime}\varepsilon^x = {}_{\prime}K_x$; and in order to have two equations for the purpose of finding $_{\prime}k$ and $_{\prime}\varepsilon$, take for x the two ages of $x=0$, that is, the age of birth, and $x=\frac{1}{12}$, that is, the age of one month, and we have the two equations

$$_{\prime}k = {}_{\prime}K_0 \text{ as } \varepsilon^0 = 1, \text{ and } {}_{\prime}k_{\prime}\varepsilon^{\frac{1}{12}} = {}_{\prime}K_{\frac{1}{12}},$$

and we obtain

$$k_{\prime} = \cdot02266, \quad \lambda_{\prime}k = \bar{2}\cdot35526, \quad \lambda_{\prime}\varepsilon^{\frac{1}{12}} = \bar{1}\cdot27720, \quad \lambda_{\prime}\varepsilon = \bar{9}\cdot3264^*.$$

It now remains to find μ, ν of the expression $\mu\nu^x$; which only is of appreciable value when $_{\prime}k_{\prime}\varepsilon^x$, $k\varepsilon^x$ become perfectly of inappreciably small consideration; that is, in the case

* In deducing the above values of $_{\prime}k_{\prime}\varepsilon$, there were some slips of the pen: for instance, taking $k=\cdot14042$ for $k=\cdot14742$, so that $_{\prime}k$ was taken too large, namely $\cdot02266$ by $\cdot007$, and should be taken therefore $=\cdot01566$; this change will require a change in the value of $_{\prime}\varepsilon$, which is obtained by making some slight alteration in MILNE's date for $L_{\frac{1}{12}}$, that is to say, for the age of one month, and that change I have made in my calculation to an increase of eight years, making $L_{\frac{1}{12}}=9475$ instead of 9467, which is as likely to be correct as the other. Some small alteration I thought I found necessary in order not to get into imaginary quantities, and my calculation gives $\lambda_{\prime}\varepsilon^{\frac{1}{12}}=\bar{1}\cdot43767$, $\lambda_{\prime}\varepsilon=\bar{7}\cdot25104$, and of course the formula gives $\lambda L_{\frac{1}{12}}=9475$ to be adopted, though for the first months of age the last may be retained.

where the equations may be considered

$$\lambda L_x = C\mathfrak{C}^x - \lambda^{-1}\{e^x.\overline{x-h}.\lambda q_0\} + \mu\nu^x;$$

and therefore, putting $\lambda L_x + M_x - C\mathfrak{C}^x = Q_x$, we shall have $\mu\nu^x = Q_x$.

The operations for $\lambda\varepsilon$, λk, $\lambda_{,}\varepsilon$, $\lambda k_{,}$, and $\lambda\mu$, $\lambda\nu$, the logarithms of the constants in the general formula

$$\lambda L_x = C\mathfrak{C}^x + k_{,}\varepsilon_{,}^x + k\varepsilon^x - \lambda^{-1}\{e^x.\overline{x-h}.\lambda q_0\} + \mu\nu^x,$$

are as follow:—

For $\lambda\varepsilon$, λk,

$$\lambda k\varepsilon = \lambda K_1 = \bar{2}\cdot96666,\quad \lambda k\varepsilon^2 = \lambda K_2 = \bar{2}\cdot76477$$
$$-\lambda k\varepsilon = \bar{2}\cdot96666$$
$$\lambda\varepsilon = \bar{1}\cdot79811$$
$$\lambda k\varepsilon = \bar{2}\cdot96666$$
$$\lambda k\varepsilon - \lambda\varepsilon = \lambda k = \bar{1}\cdot16855$$
$$k = \cdot14742$$

For $\lambda k_{,}$, $\lambda_{,}\varepsilon$,

$\lambda L_0 = 4\cdot00000$	$C = 3\cdot93197$	$\lambda L_{\frac{1}{10}} = 3\cdot97621$	$C\mathfrak{C}^{\frac{1}{10}} = 3\cdot93179$
$M_0 = \cdot09505$	$k = \cdot14042$	$M_{\frac{1}{10}} = \cdot09505$	$k\varepsilon^{\frac{1}{10}} = \cdot13518$
$4\cdot09505$	$4\cdot07239$	$4\cdot07126$	$4\cdot06697$
$-4\cdot07239$	$4\cdot06697$		
$k_{,} = \cdot02264$	$k_{,}\varepsilon_{,}^{\frac{1}{10}} = \cdot00429$	$\lambda k_{,}\varepsilon_{,}^{\frac{1}{10}} = \bar{3}\cdot63246$	
$\lambda k_{,} = \bar{2}\cdot35526$		$-\lambda k_{,} = \bar{2}\cdot35526$	
		$\lambda_{,}\varepsilon^{\frac{1}{10}} = \bar{1}\cdot2777$	
		$\lambda_{,}\varepsilon = \bar{9}\cdot3264$	
		$\lambda k_{,} = \bar{2}\cdot35526$	

For $\lambda\mu$, $\lambda\nu$.

$$\left.\begin{aligned}\lambda\mu + 90\lambda\nu &= \lambda Q_{90}\\ \lambda\mu + 100\lambda\nu &= \lambda Q_{100}\end{aligned}\right\}$$

$\lambda L_{90} = 2\cdot15229$	$\lambda L_{100} = \cdot95425$	$\lambda Q_{100} = \cdot33421 = \lambda\mu\nu^{100}$
$M_{90} = 1\cdot74908$	$M_{100} = 4\cdot91349$	$\lambda Q_{90} = \bar{1}\cdot23042 = \lambda\mu\nu^{90}$
$3\cdot90137$	$5\cdot86774$	$1\cdot10379 = \lambda\nu^{10}$
$C\mathfrak{C}^{90} = 3\cdot73138$	$C\mathfrak{C}^{100} = 3\cdot70336$	$\lambda\nu = \cdot110379$
$Q_{90} = \cdot16999$	$Q_{100} = 2\cdot15888$	

And we have

$C = 3\cdot931968$; $\mathfrak{C} = \cdot99942$; $\lambda C = \cdot59461$, $\lambda\mathfrak{C} = \bar{1}\cdot99974617$;

$k = \cdot4742$, $\lambda k = \bar{1}\cdot16855$; $\lambda\varepsilon = \bar{1}\cdot79811$; $k_{,} = \cdot015522$, $\lambda k_{,} = \bar{2}\cdot1824$;

$e = 1\cdot01247$, $\lambda e = \cdot005318666$; $q_0 = 1\cdot0286$, $\lambda q_0 = \cdot012252$, $\lambda\lambda q_0 = \bar{2}\cdot08822$;

$\lambda_{,}\varepsilon = 7\cdot\bar{2}5104$, $\lambda_{,}\varepsilon^{\frac{1}{10}} = \bar{1}\cdot43767$;

$h = 83\cdot418$; $\mu = (10)1978$, $\lambda\mu = \overline{11}\cdot29631$; $\nu = 1\cdot2894$, $\lambda\nu = \cdot110379$;

which will give an easy means of regularly finding the value of L_x, the number of persons living out of the number L_0, the number taken for the birth, as a base, for every year of age to 100, or beyond that if the accuracy can be depended on beyond 100; and, what will be interesting to the reader in favour of the formula, it will give the number agreeing with MILNE's Table living for the monthly portions of the deaths of children below one year of age with a very satisfactory agreement, considering that perfect accuracy in MILNE's Table cannot be expected to exist, owing to the scanty means he could have had for that purpose. In a paper I presented for consideration to the fourth section of the International Statistical Congress, already referred to, held during the week commencing on 16th July last, "On the one uniform Law of Human Mortality from the age of Birth to extreme old age, and on the Law of Sickness," which was honoured by its publication among its reports, I only offered hints, without the abstruse mathematical portions being brought forward of this paper which I am venturing to offer to this Society; which I felt was partly a duty I had to perform; namely, to add my small services to the Congress; especially in consequence of the insufficient state of my health ever since and before I attempted to search into my former published and unpublished papers on the interesting subject of Vital Statistics, and useful Application of the Results, with a view to improve, and add matter of interest to the subject; which I flatter myself I had treated on with some approbation from the scientific public; and being doubtful if I should be able to offer even the paper I had so far completed for the approbation of the Society, I therefore feel satisfied that my having given these slight hints will not be a cause to render this attempt to bring these few pages before the Society unacceptable, especially as, since the hints were written, I think I have discovered a greatly improved form of the formula of mortality, and a more satisfactory one. The formula of mortality which in these hints was given was

$$\lambda L_x = \text{constant} + ke^x - {}_ik_ie^x.x - nq^x - P_x, \quad P_x \text{ being } = \theta.(w)^{\pi^x(x-u)},$$

where all the values on the right-hand side of the equation are constant except x, from birth to extreme old age, including θ, w, and u; and θ very nearly I said unity, which value it was taken; but I stated I thought it could not be exactly unity, the reason for which doubt was that P_x having been put for the common logarithm of the number whose common logarithm is θ, raised to the power $w^{\pi^x(x-u)}$, if θ were exactly 1, the number whose common logarithm is θ would be exactly 10, and the equation, of which the aforesaid equation is the logarithm, would be

$$L_x = \text{constant} \times A^{e^x} B^{{}_ie^x.x} \times C^{q^x} \times D^{P_x};$$

A, B, C, D being constants, and $\lambda^{-1} P_x = 10^{\overline{w|}^{\pi^x(x-u)}}$, and it appeared to me that nature could not be governed by the conventional notation of the decimal arithmetic.

Art. 14. In the paper presented to the International Congress, I gave a Table, resulting from the above formula, of persons living to the extent of life, from birth to the age of 100, and for the age of 1, 2, 3, 6, 9, and 12 months, which appeared to me very satisfactory; but I do not repeat that Table here, as I have given a Table from what I consider the improved formula; but I gave in that paper the values for the four cases, namely, Carlisle, Northampton, Sweden, and De Parcieux, of the aforesaid valuable constants of the formula just stated, and there alluded to; in this I mean to subjoin the results for tables of mortality of constants for the last three places, which are satisfactory in my opinion. But I am not able to say if, before this goes to press, I shall be able to give the results which an investigation of the constants in the formula, which I consider an improvement of the other, will give for the last three mortalities.

Art. 15. The term 'expectation of life' in common use is not applied to that which I should call the orthodox expectation of life, and therefore, not venturing to discard the usual adoption of it, I will introduce the term orthodox expectation of life for that term of years to which it is an equal chance whether a person of a given age shall reach or not; and I observe, if the Napierian logarithm of the chance of a person of a given age living x years beyond that age be represented by ${}^{1}Ax+{}^{2}Ax^{2}+{}^{3}Ax^{3}+$&c., which will be a negative quantity, this must be put $=-\cdot691472$, namely, Napierian logarithm of $\frac{1}{2}$. And the value of x which will result from that equation is that which I call the orthodox expectation of life of that person; and if there be any number of joint lives, and the sum of the Napierian logarithms of the chances of each separately living x years be represented by ${}^{1}Ax+{}^{2}Ax^{2}+{}^{3}Ax^{3}+$&c., if this be put $=-\cdot691472$, namely, the Napierian logarithm of $\frac{1}{2}$, the value of x, which this equation will give, will be the period beyond the present to which the joint existence of those joint lives has an equal chance of attaining or not attaining; and as the coefficients ${}^{1}A$, ${}^{2}A$, ${}^{3}A$ are very converging, a very few terms will be sufficient, perhaps merely the first term; and if there be two separate combinations of joint lives, which I will call the A combination and the B combination, and the Napierian logarithm of the chance of the A combination lasting x years be represented by ${}^{1}Ax+{}^{2}Ax^{2}+{}^{3}Ax^{3}+$&c., and that of the B combination lasting x years be represented by ${}^{1}Bx+{}^{2}Bx^{2}+{}^{3}Bx^{3}$, &c., then if these two be equal to each other, and x comes out positive and not beyond the limit of the accuracy of the theorem, x will be the term to which it is an equal chance of one combination in particular surviving or not surviving the other; but should x come out negative, that term of years does not exist within the limits at least of accuracy of the theorem.

Art. 16. If there be two sets of lives, which I will call the A combination and the B combination, and the separate chances of their existing x years be respectively represented by

$$1+{}^{1}Ax+{}^{2}Ax^{2}+{}^{3}Ax^{3}+\text{\&c., and } 1+{}^{1}Bx+{}^{2}Bx^{2}+{}^{3}Bx^{3}+\text{\&c.}$$

respectively, then the chance that the B combination shall fail during the continuance of the A combination, and between the periods $t=$ a given quantity to $t=x$, will be the fluent of

$$(1+{}^{1}Ax+{}^{2}Ax^{2}+{}^{3}Ax^{3}\text{ \&c.})\times-\text{fluxion of }({}^{1}Bx+{}^{2}Bx^{2}+{}^{3}Bx^{3},\text{ \&c.})$$

between those limits, because the fluxion of the discontinuance is *minus* the fluxion of the continuance, that is to say, that chance is equal to

$$-({}^1B.\overline{x-t}+\tfrac{1}{2}{}^1K.\overline{x^2-t^2}+\tfrac{1}{3}{}^2K.\overline{x^3-t^3}, \&c.),$$

where $${}^1K=2\,{}^2B+{}^1A{}^1B;\quad {}^2K=3\,{}^3B+2\,{}^1A{}^2B+{}^2A\,{}^1B;$$

$${}^3K=4\,{}^4B+3\,{}^1A.{}^3B+2\,{}^2A.{}^2B+{}^3A{}^1B;\quad {}^4K=5\,{}^5B+4\,{}^1A.{}^4B+3\,{}^2A.{}^3B+2\,{}^3A.{}^2B+{}^4A.{}^1B;$$

where the law of continuation is evident. And the chance that the combination A failed previously to the failure of the combination B, is the fluent of

$$({}^1Ax+{}^2Ax^2+{}^3Ax^3, \&c.)({}^1B\dot{x}+2\,{}^2Bx\dot{x}+3\,{}^3Bx^2\dot{x}, \&c.),$$

which evidently is the excess of the chance of the combination of B failing independently of any connecting A, above the chance of the combination B failing whilst the combination A exists, and thus we may proceed to successive additional cases of conditional contingencies with respect to more combinations. Now it is observable that in consequence of the great convergency of the coefficients ${}^1A, {}^2A, {}^3A$, &c., and of ${}^1B, {}^2B, {}^3B$, &c., and of the small values of ${}^1A$ and ${}^1B$, except very shortly after birth, which result from what has been previously stated, if all of them, except ${}^1A, {}^1B$, be considered as nothing, unless x be very great, the first value will be very nearly expressed by the first term $-{}^1B.\overline{x-t}$, because ${}^1A.{}^1B$ is small of the second degree; and the second value in a similar way, for a long period in which it shall occur that both combinations fail, is very nearly $\frac{1}{2}A_1B_1.\overline{x^2-t^2}$, giving an equal chance which shall have failed first; and this is a generalization of Mr. MORGAN's hypothesis of two lives only, which enabled him to solve questions respecting the lives of three persons A, B, C, contingent on the life or death of A, provided B and C be both dead, involving contingencies of survivorship between them. But that this law cannot accurately exist for any possible continuous law of mortality, I have proved in my paper of 1825, unless of the form $L_a=e'-e''.e^a$, where e', e'', e are constant quantities at pleasure, and a the age, and which, in an extreme case of e differing infinitely little from unity, is reducible into the form

$$L_a=g'-g''a,\ \text{if}\ g'=e'-e'',\ \text{and}\ g''=e''\epsilon,$$

e' and e'' being infinitely large, differing from each other by a finite quantity, and ϵ infinitely small, and in consequence g'' a finite quantity. But with respect to the two combinations generally—(and thus is found the chance of one combination failing during the continuance of the other combination, or of its failing after the continuance of the other combination, and this is evidently considerably more general than the cases of Messrs. MORGAN, BAILY, and MILNE, in which there are only three lives concerned,)—I observe that they will give useful practical solutions however many lives are concerned, and however complicated the conditions of survivorships may be between them, whereas it may be a question if the solutions of those gentlemen, whose memory I respect, are at all practical; especially when there are only tables of single and two joint lives at hand. I therefore consider that the solutions I had given to those three-lived questions, and to questions of more lives, in my Tract of 1820, to which I refer the reader, which were to be derived from calculated Tables of annuities, are worthy of the attention of scientific inquirers. For instance, the first of them and the following, in which, instead of there being only three lives concerned, there are any number: the first question of those gentlemen is to find the value of the assurance of the life of A provided in the lifetime of B and C; and my solution, expressed in the notation of that paper, is as follows:—p representing one year, if the tables are calculated for annual payments; n the period from which the assurance is to commence; m the period to which the assurance is to continue; r the present value of unity due in one year certain at the rate of interest involved in the question; then the value calculated from the Table of annuities for every unity assured, the present ages of A, B, C being a, b, c respectively, is

$$\frac{L_{a-p,\, b-\frac{1}{2}p,\, c-\frac{1}{2}p}}{L_{a,\, b,\, c}} \times \overset{\overset{r}{\smile}}{\underset{m}{\overset{p}{\overset{n}{\big|}}}}\,\overline{a-p,\, b-\tfrac{1}{2}p,\, c-\tfrac{1}{2}p} - \frac{L_{b-\frac{1}{2}p,\, c-\frac{1}{2}p}}{L_{b,\, c}} \times \overset{\overset{r}{\smile}}{\underset{m}{\overset{p}{\overset{n}{\big|}}}}\,\overline{a,\, b-\tfrac{1}{2}p,\, c-\tfrac{1}{2}p};$$

an extremely small and insignificant quantity being neglected.

Art. 17. This, with Tables calculated for two joint lives, and the annuities for those interpolated from them, instead of being impracticable, as the other solutions are, will be found perfectly simple and easy. And in my second Tract, namely that of 1825, I gave Tables for calculating the values of annuities at any likely rate of mortality to be adopted, and at any rate of interest, and applicable when the lives are subject to different rates of mortality, and the same theorem for the value of assurance, *mutatis mutandis*, however many lives B, C, D, E, &c. are proposed to be jointly living at the death of A. And with regard to art. 7 of the paper of 1825, I will only in this place direct the reader to my solution to the question, which is one of the more intricate nature of those given, and solved by those gentlemen. The question being the value of the assurance on the first death of A and B, which shall be second or third of the deaths which shall happen of the three lives A, B, C, my solution to this question in my first paper is simply

$$\tfrac{1}{2}\cdot\overset{\overset{r}{\smile}}{\underset{m}{\overset{p}{\overset{n}{\big|}}}}\,\underline{b} + \tfrac{1}{2}\cdot\overset{\overset{r}{\smile}}{\underset{m}{\overset{p}{\overset{n}{\big|}}}}\,\underline{a} + \tfrac{1}{2}\overset{\overset{r}{\smile}}{\underset{m}{\overset{p}{\overset{n}{\big|}}}}\,\underline{b,\, c} + \tfrac{1}{2}\overset{\overset{r}{\smile}}{\underset{m}{\overset{p}{\overset{n}{\big|}}}}\,\underline{a,\, c} - \overset{\overset{r}{\smile}}{\underset{m}{\overset{p}{\overset{n}{\big|}}}}\,\underline{a,\, b,\, c};$$

that is, half the sum of the assurances of B, of A, of B, C, of A, C, less the assurance of the joint lives A, B, C, all of which are easily computed. This solution, when I obtained it, seemed to me so enormously superior to those which those learned gentlemen had given to it, that I found it necessary to show my readers that they depended on the same principles, by reducing the portions of my solutions to the portions of the earlier solutions; and the remaining questions of that section are of a similar superiority to the earlier solutions.

Art. 18. But in making this remark I wish to express myself indebted to those gentlemen for their labours, whose names remain honoured in the memory of scientific persons who have cultivated different branches of science, and to state I allude to the comparison that my readers may follow an improved path if they meet with one, and to remind those who may follow with benefit for science others who have trodden down the asperities of the path to repay with gratitude the benefit they have received, and not in self-conceit to forget that a dwarf on a giant's shoulders may see a more extended prospect of beauties than the giant could; and hoping to be excused for this digression, I will proceed by observing that the best way of solving intricate questions relative to the value of life interest, may not at all be, as is usual, to have recourse to the value of life annuities. With respect to the expression given above, namely the fluent of

$$(1+{}^1Ax+{}^2Ax^2+{}^3Ax^3, \&c.) \times -\text{ fluxion of } (1+{}^1Bx+{}^2Bx^2, \&c.),$$

and the one which follows, I observe that if in the two sets of combinations of lives all the lives of the combination A were required to be living, and a specified number of specified persons of the combination B were also required to be living, and the assurance is to be on the failure of the remaining portion of joint lives in the B combination, provided all the lives of the A combination and all the lives specified of the B combination were also living, then the solution would be the same, with the exception that all those specified lives of the B combination must first be transferred to the A combination; this must be evident; but it is mentioned that in case with respect to the B combination only a certain number specified, without specification of which lives of the B combination, are to be living, the question may be solved with this difference, that the sum of all the values must be taken which will occur by the various modes of withdrawing the specified number from the B combinations; but there are cases where a less laborious mode may apply.

Art. 19. But returning to the originally expressed combinations A and B, suppose it were required to find the value of an insurance of unity on the failure of the combination B, provided it happened during the continuance of the combination A. Let the expression $1+{}^1Ax+{}^2Ax^2+{}^3Ax^3$, &c., instead of representing, as before, the anti-Napierian logarithm of the sum of all the Napierian logarithms of the chances of existence of the different lives separately in the combination A, now represent the anti-Napierian logarithm of that sum, having the coefficient of the first power of x increased by the Napierian logarithm of the present value of unity due in one year at the rate of interest to be involved in the calculation, then the fluent of the expression

$$(1+{}^1Ax+{}^2Ax^2+{}^3Ax^3, \&c.)\times(-{}^1B\dot{x}-2\,.\,{}^2Bx\dot{x}-3\,.\,{}^3Bx^2\dot{x}, \&c.),$$

instead of being the chance of a failure of the combination B between the given period and x during the continuance of the combination A, will be the present value of unity payable on that event taking place, and so of others. For instance, if only three lives of the present ages a, b, c were concerned, then the value of the assurance on the life A, if it happened in the lifetime of B and C, which is the first of art. 6 in my paper of 1820, would be solved thus: supposing the present ages of A, B, C to be a, b, c, and the Napierian logarithms of

$$\frac{L_{a+x}}{L_a}, \frac{L_{b+x}}{L_b}, \frac{L_{c+x}}{L_c}$$

to be respectively

$$A_1x+A_2x^2+A_3x^3+\&c., \quad B_1x+B_2x^2+B_3x^3\,\&c., \quad C_1x+C_2x^2+C_3x^3+\&c.,$$

and the Napierian logarithm of the present worth of unity certain in one year to be ϱ, find the anti-Napierian logarithm of

$$\overline{\varrho+A_1}.x+A_2x^2+A_3x^3+ \&c.,$$

and the anti-Napierian logarithm of

$$\overline{B_1+C_1}.x+\overline{B_2+C_2}.x^2+\overline{B_3+C_3}.x^3+ \&c.,$$

which will be easily done by the method explained above, and will only require a few terms in consequence of the coefficients of x in the above expressions being small; and calling the first analytical anti-logarithm

$$1+{}^1Ax+{}^2Ax^2+{}^3Ax^3 \&c.,$$

and the second

$$1+{}^1Bx+{}^2Bx^2+{}^3Bx^3 \&c.,$$

and putting the fluent of *minus* the fluxion of the first

$$-({}^1A\dot{x}+2.{}^2Ax\dot{x}+3.{}^3Ax^2\dot{x}, \&c.), \times(1+{}^1Bx+{}^2Bx^2+\&c.)={}^1K\dot{x}+{}^2Kx\dot{x}+{}^3Kx^2\dot{x}, \&c.,$$

then the value of the assurance for the term commencing in t years and ending in x years will be

$${}^1K\overline{x-t}+\tfrac{1}{2}.{}^2K\overline{x^2-t^2}+\tfrac{1}{3}.{}^3K.\overline{x^3-t^3}, \&c.,$$

and will not require many terms, and the equation is thus solved without the Tables of the values of annuities; and in a similar way are other questions of this species; and the solution in the more compounded cases of art. 7 of my first paper, such as example 1, taken from Messrs. BAILY, or MILNE, or Mr. MORGAN'S original paper, which is the foundation of them, containing an intermediate conditional contingence of survivorship, and which in my paper above alluded to contains the double fluent reducible to a single fluent multiplied by a variable quantity, and a single fluent added, and the solution in much more intricate cases, and containing any number of lives, and even a long string of intermediate contingencies which would involve double, triple, quadruple, &c. fluents, &c., may be effected. As a simple example, suppose there were three separate combinations of lives, which I will call A, B, C combinations of lives, and it be required to find the value of an assurance on the failure of the C combination after the failure of the B combination, the A combination having failed previously, provided that failure should happen during certain given periods. First, find by the method above the analytical value of the contingency that the B combination shall fail after a given time, the combination A having failed previously; and suppose this to be

$$K+{}^1Kx+{}^2Kx^2+ \&c.;$$

and having found the anti-Napierian logarithm of

$$1+\frac{{}^1K}{K}x+\frac{{}^2K}{K}x^2+\frac{{}^3K}{K}x^3, \&c.,$$

and having found the Napierian logarithm of the separated chances of every life in the combination B being living, and added the sum to the Napierian logarithm of

$$1+\frac{{}^1K}{K}x+\frac{{}^2K}{K}x^2, \&c.,$$

and having found the anti-Napierian logarithm of this sum, and multiplied this by K, call this

$$\mathrm{H}+{}^1\mathrm{H}x+{}^2\mathrm{H}x^2+{}^3\mathrm{H}x^3, \&c.,$$

then on multiplying *minus* the fluxion of this by the anti-Napierian logarithm of the sum of all logarithms of the separate chances of each life in the combination C being living, increased by ϱx (previously ϱ being the Napierian logarithm of the value of unity certain due in one year at the rate of interest to be involved in the calculation), the fluent of this will give the value required, and will not require many terms of the resulting series, which I represent by

$${}^1\mathrm{I}(x-t)+{}^2\mathrm{I}(x^2-t^2)+{}^3\mathrm{I}(x^3-t^3), \&c.,$$

t being the period of commencing of the assurance.

Art. 20. And much more intricate cases may occur in valuations of property offered to the public and to reversionary companies. But instead of in this place dwelling on the intricacies which may occur, I will refer to art. 5 of my paper of 1820 with respect to calculations referring to the formulæ of the convenient notation there used,

$$\overset{r}{\smile}\;\; {}^{p}_{n}{}_{m}\Big|\,a, b, c, \&c., \qquad \overset{r}{\smile}\;\; {}^{p}_{n}{}_{m}\Big|\,a, b, c, \&c., a', b', c', \&c., \;\Big|\,\nu \quad \nu'$$

in which I have developed the various combinations and their connexions of annuities of 1, 2, 3, &c. joint lives which would come into the calculation, which I derived by the development of the expression

$$(x\mathrm{E}_{a,n}+\mathrm{D}_{a,n})\times(x\mathrm{E}_{b,n}+\mathrm{D}_{b,n})\times(x\mathrm{E}_{c,n}+\mathrm{D}_{c,n})\times \&c.=1,$$

when $x=1$, where $\mathrm{E}_{a,n}$, $\mathrm{E}_{b,n}$, &c. denote the chances of persons of the ages a, b, &c. being living in n years, and $\mathrm{D}_{a,n}$, $\mathrm{D}_{b,n}$ the chances of their being then dead; and in consequence

$$\mathrm{E}_{a,n}+\mathrm{D}_{a,n}=1,\ \ \mathrm{E}_{b,n}+\mathrm{D}_{b,n}=1, \&c.,$$

the x attached to the E being supposed $=1$, but only introduced, as x, to point out, by its exponent in the development, the various combinations of persons on which the annuities are to be calculated, if, previously to the development, $\mathrm{D}_{a,n}$, $\mathrm{D}_{b,n}$ be expunged by the equations

$$\mathrm{D}_{a,n}=1-\mathrm{E}_{a,n},\ \ \mathrm{D}_{b,n}=1-\mathrm{E}_{b,n}, \&c.$$

But had we introduced y, also a representative of 1, and written the expression

$$\overline{x\mathrm{E}_{a,n}+y\mathrm{D}_{a,n}}\times\overline{x\mathrm{E}_{b,n}+y\mathrm{D}_{b,n}}\times \&c.=1,$$

and not expunged D, the exponents of x and y in the developments would represent the combinations and connexions of the various cases which would occur of persons living and persons dead in connexion with each other in the time n; so that if we chose to have Tables constructed on a certain number of the persons being dead of the whole number, we might have a variety of modes of calculating the value of life contingencies, instead of being dependent on the value of joint lives; but I do not mention this by any means as a recommendation, but quite the contrary. But if Tables of the reversions of annuities enjoyed by any number of joint lives were calculated, they would in many cases be more convenient than the value of annuities on joint lives; as, for instance, if we had to find the reversion of an annuity on three lives now aged a, b, c, we should only have to search the Table of the reversion on three joint lives, and we should have the value at once: but if the calculation were to be made by methods above, and we represent the anti-Napierian logarithm of the present expression, as we did that of the former expression, by ${}^1\mathrm{S}x+{}^2\mathrm{S}x^2+{}^3\mathrm{S}x^3$, &c., the present value of the annuity of which the first payment is to be in the time w, and the last in the time z, will be represented by the expression

$$\mathrm{P}\,\overset{x}{\overline{1}}\Big|{}^{w}_{z}\,x^{n}+\mathrm{P}^{1}\mathrm{S}\,\overset{x}{\overline{1}}\Big|{}^{w}_{z}\,x^{n+1}+\mathrm{P}^{2}\mathrm{S}\,\overset{x}{\overline{1}}\Big|{}^{w}_{z}\,x^{n+2}+\mathrm{P}^{3}\mathrm{S}\,\overset{x}{\overline{1}}\Big|{}^{w}_{z}\,x^{n+3},\ \&c.,$$

where it may be observed that if the condition of the deaths, say of α, β, γ, did not enter the question, this formula ought to be the same as the first, which would require P to be 1, which may appear a paradox; but the paradox is solved if we consider the condition would be the same if they entered in the question or not, if they could not die; that is to say, if p_α, p_β, &c., and consequently P be $=1$. Then, by means of the Collecting Table, the value of the above series will be given.

Art. 21. The valuation by the old methods of annuities depending on such complications, by the aid of Tables of only two joint lives, as the value of the annuities on three, four, five, &c. joint lives can only be obtained from them by very troublesome interpolations of very inaccurate results, evidently calls for an improved method. For instance, if the annuity depended on the joint existence of three lives of the present ages of a, b, c after the existence of all the lives d, e, f, the labour and insufficiency in point of accuracy of that mode of valuation will immediately appear; because, the chance of each of them individually being living in x years being for the sake of ease represented by a, b, c, d, e, f respectively, by multiplying out at length the chance of the conditions required being fulfilled, that chance will be then represented by

$$abc\times\overline{1-d}\times\overline{1-e}\times\overline{1-f}=abc-abce-abcd-abcf+abced+abcfe+abcfd-abcdef.$$

It appears that by the old methods, with only Tables of two joint lives, we should have to interpolate the value of one of three annuities of three joint lives, three of four different annuities of four joint lives, three of five different annuities of five joint lives, and one annuity of six joint lives, and the results of every one of these interpolations would be inaccurate.

Art. 22. On special risk, considered with respect to single lives, and on many lives in connexion; and the valuation of certain contingencies; and of property depending on those contingencies and under influential connexion; I am not aware whether or not any one has gone before me on this important subject; but I have not, in my official practice in the science of assurance, lost sight of its existence. And to draw the reader's attention at once to the subject, I will suppose we had the two problems to consider—the first, to assure a sum on the extinction of the coexistence of two coexistent lives A, B, commonly called two joint lives, that is, to be payable at the death of the first of the two of them which may die; and the other problem, to assure the like sum on the death of A in particular, provided it happens in the lifetime of B.

Then if A and B were of equal age, and both subject to the same law of mortality, without any speciality, the value of the assurance in the first case would be just double of that in the second case, for however long or however short the assurance might be. But if A and B were both sailors, continually together in the same ship, the common risk for each by risk of shipwreck, &c. would equally attach to each; and if the assurance were only for a few hours, say, for instance, for one voyage from Dover to Calais, so that the risk for time were insignificant with respect to the common uninfluenced risk; there then would be only to be considered the risk for each not to escape the consequence of the wreck; but the chances would now come into play, of all on board being lost, of none being lost, or of a portion, and what portion of them being lost; and that the portion lost were the two A and B, or only one of them, and which one that might be, which might depend on the powers of each for swimming; but should the chance be that the wreck, if it happened, would cause the death of both A and B, then the assurance for the period of the intended voyage on the death of one of them only, of both, or of one in particular named, of this in the lifetime of the other, would be all the same. This is but one case of specialities, and specialities of influences of risks. And perhaps there are very few cases of assurances and valuations connected with conjoint lives, if any, or even of single, with reference to affections which may arise from climate, localities, epidemics, endemics, ancestorial influences, &c., and also epochs, which are not affected by speciality; but though the subject may be very interesting, I have not in this paper entered largely on it, but will only now touch on a portion of it which may be found by my readers well worthy of their attention. Suppose by the common law of mortality, uninfluenced by any speciality, out of the number of the age a, represented by L_a, there would be living in x years the number L_{a+x}; but that they each for any period become subject to an extra risk of death during an infinitely small time, $\dot{t}=\alpha\dot{x}$, where α may, as the case may be, be either a constant quantity or a function of x; then it is evident L_{a+x} will not be the number of them living in x years; and to find what that number would be, suppose it represented by M_x; then its fluxion $\dot{M}_x$, on the supposition that the lives of the then existing number were not deteriorated by the pre-existence of those specialities to that period, would evidently be

$$=M_x\frac{\dot{L}_{a+x}}{L_{a+x}}-\alpha\dot{x}M_x.$$

But it is to be observed that this supposition is not necessarily tenable, but it would be so in case the circumstance of speciality was transient after the moment of its operation; such, for instance, if it were a wreck at sea, leaving no effects; and therefore I shall at present only consider the case when the hypothesis is tenable. And resuming the equation, we have

$$\frac{\dot{M}_x}{M_x}=\frac{\dot{L}_{a+x}}{L_{a+x}}-\alpha\dot{x}\,;$$

and supposing α to be constant, that is to say, supposing the circumstance of speciality to cause a constant instantaneity of effect, we shall have

$$\subset\frac{M_x}{M_0}=\subset\frac{L_{a+x}}{L_a}-\alpha x;$$

and therefore if $\aleph$ represent the number whose Napierian logarithm is 1, that is, if $\aleph$ be put for $\subset-1$, then, because $M_0=L_a$, we have

$$M_{a+x}=L_{a+x}\times\aleph^{-\alpha x}.$$

If α were very small, which would be the case if the speciality depended on the risk of death by shipwreck, which might be but 1 or 2 per cent. per annum to a sailor either always at sea or occasionally at sea, then the above expression would give M_{a+x}, the number out of L_a living who would be living in x years

$$=L_{a+x}\times\overline{1-\alpha.x}.$$

Art. 23. No. 1. This Table of Napierian Logarithms of persons living at every Age according to MILNE'S Carlisle Mortality will be found useful; I have therefore given it.

x Age.		x Age.		x Age.	
0	9·21034	34	8·59739	68	7·88156
1	9·04322	35	8·58709	69	7·83400
2	8·95854	36	8·57678	70	7·78344
3	8·89206	37	8·56617	71	7·73061
4	8·85338	38	8·55526	72	7·66996
5	8·82434	39	8·54403	73	7·59940
6	8·80627	40	8·53208	74	7·51806
7	8·79392	41	8·51899	75	7·42357
8	8·78508	42	8·50512	76	7·32317
9	8·77848	43	8·49064	77	7·21450
10	8·77338	44	8·47595	78	7·10085
11	8·76889	45	8·46105	79	6·98564
12	8·76405	46	8·44613	80	6·85961
13	8·75904	47	8·43120	81	6·72982
14	8·75385	48	8·41649	82	6·62274
15	8·74830	49	8·40479	83	6·43455
16	8·74219	50	8·38868	84	6·27019
17	8·73536	51	8·37526	85	6·09807
18	8·72847	52	8·36077	86	5·90536
19	8·72144	53	8·34555	87	5·69036
20	8·71440	54	8·32918	88	5·44674
21	8·70738	55	8·31214	89	5·19850
22	8·70025	56	8·29405	90	4·95583
23	8·69333	57	8·27487	91	4·65396
24	8·68626	58	8·25375	92	4·31749
25	8·67914	59	8·22924	93	3·98809
26	8·67180	60	8·20056	94	3·68888
27	8·66441	61	8·16650	95	3·40129
28	8·65661	62	8·13006	96	3·13549
29	8·64788	63	8·09193	97	2·89037
30	8·63799	64	8·05293	98	2·63906
31	8·62784	65	8·01235	99	2·39790
32	8·61758	66	7·97039	100	2·19722
33	8·60749	67	7·92696		

Art. 23. No. 2. Power-collecting Table referred to at p. 540.

x and x^0.	Sx.	x^2.	Sx^2.	x^3.	Sx^3.	x^4.	Sx^4.	x^5.	Sx^5.
1	1	1	1	1	1	1	1	1	1
2	3	4	5	8	9	16	17	32	33
3	6	9	14	27	36	81	98	243	276
4	10	16	30	64	100	256	354	1024	1300
5	15	25	55	125	225	625	979	3125	4425
6	21	36	91	216	441	1296	2275	7776	12201
7	28	49	140	343	784	2401	4676	16807	29008
8	36	64	204	512	1296	4096	8772	32768	61776
9	45	81	285	729	2025	6561	15333	59049	12083 (1)
10	55	100	385	1000	3025	10000	25333	10000 (1)	22085 (1)
11	66	121	506	1331	4356	14641	39974	16105 (1)	38188 (1)
12	78	144	650	1728	6084	20736	60710	24883 (1)	63071 (1)
13	91	169	819	2197	8281	28561	89271	37129 (1)	10020 (2)
14	105	196	1015	2744	11025	38416	12769 (1)	53782 (1)	15398 (2)
15	120	225	1240	3375	14400	50625	17831 (1)	75938 (1)	22992 (2)
16	136	256	1496	4096	18496	65536	24385 (1)	10476 (2)	33468 (2)
17	153	289	1785	4913	23409	83521	32737 (1)	14199 (2)	47667 (2)
18	171	324	2109	5832	29241	10498 (1)	43235 (1)	18896 (2)	66568 (2)
19	190	361	2470	6859	36100	13032 (1)	56267 (1)	27461 (2)	91324 (2)
20	210	400	2870	8000	44100	16000 (1)	72267 (1)	32000 (2)	12333 (3)
21	231	441	3311	9261	53361	19448 (1)	91715 (1)	40841 (2)	16417 (3)
22	253	484	3795	10648	64009	23426 (1)	11514 (2)	51536 (2)	21571 (3)
23	276	529	4324	12167	76176	27984 (1)	14312 (2)	64363 (2)	28007 (3)
24	300	576	4900	13824	90000	33178 (1)	17630 (2)	79627 (2)	35969 (3)
25	325	625	5525	15625	10563 (1)	39063 (1)	21536 (2)	97658 (2)	45735 (3)
26	351	676	6201	17576	12329 (1)	45698 (1)	26106 (2)	11881 (3)	57616 (3)
27	378	729	6930	19683	14288 (1)	53144 (1)	31421 (2)	14349 (3)	71965 (3)
28	406	784	7714	21952	16484 (1)	61466 (1)	37567 (2)	17210 (3)	89175 (3)
29	435	841	8555	24389	18923 (1)	70728 (1)	44640 (2)	20511 (3)	10969 (4)
30	465	900	9455	27000	21623 (1)	81000 (1)	52740 (2)	24300 (3)	13399 (4)

TABLE (continued).

x and x^0.	Sx.	x^2.	Sx^2.	x^3.	Sx^3.	x^4.	Sx^4.	x^5.	Sx^5.
31	496	961	10416	29791	24602 (1)	92352 (1)	61975 (2)	28629 (3)	16262 (4)
32	528	1024	11440	32768	27878 (1)	10486 (2)	72461 (2)	33555 (3)	19617 (4)
33	561	1089	12529	35937	31472 (1)	11859 (2)	84320 (2)	39135 (3)	23530 (4)
34	595	1156	13685	39304	35403 (1)	13363 (2)	97684 (2)	45435 (3)	28074 (4)
35	630	1225	14910	42875	39690 (1)	15006 (2)	11269 (3)	52522 (3)	33326 (4)
36	666	1296	16206	46656	44356 (1)	16796 (2)	12949 (3)	60466 (3)	39373 (4)
37	703	1369	17575	50653	49421 (1)	18742 (2)	14823 (3)	69344 (3)	46307 (4)
38	740	1444	19109	54872	54908 (1)	20851 (2)	16909 (3)	79235 (3)	54231 (4)
39	780	1521	20540	59319	60840 (1)	23134 (2)	19223 (3)	90224 (3)	63253 (4)
40	820	1600	21140	64000	67240 (1)	25600 (2)	21782 (3)	10240 (4)	73493 (4)
41	861	1681	23821	68921	74132 (1)	28258 (2)	24607 (3)	11586 (4)	85079 (4)
42	903	1764	25585	74088	81541 (1)	31117 (2)	27719 (3)	13069 (4)	98144 (4)
43	946	1849	27434	79507	89492 (1)	34188 (2)	31138 (3)	14701 (4)	11285 (5)
44	990	1936	29370	85184	98010 (1)	37481 (2)	34886 (3)	16492 (4)	11934 (5)
45	1035	2025	31395	91125	10712 (2)	41006 (2)	38986 (3)	18452 (4)	14780 (5)
46	1081	2116	33511	97336	11686 (2)	44775 (2)	43464 (3)	20596 (4)	16840 (5)
47	1128	2209	35720	103823	12724 (2)	48797 (2)	48345 (3)	22935 (4)	19133 (5)
48	1176	2304	38024	110592	13830 (2)	53084 (2)	53652 (3)	25480 (4)	21681 (5)
49	1225	2401	40425	117649	15006 (2)	57648 (2)	59467 (3)	28248 (4)	24506 (5)
50	1275	2500	42925	125000	16526 (2)	62500 (2)	65667 (3)	31250 (4)	27631 (5)
51	1326	2601	45526	132651	17583 (2)	67652 (2)	72432 (3)	34502 (4)	31081 (5)
52	1378	2704	48230	140608	18989 (2)	73116 (2)	79743 (3)	38021 (4)	34883 (5)
53	1431	2809	51030	148877	20478 (2)	78905 (2)	87654 (3)	41820 (4)	39065 (5)
54	1485	2916	53955	157464	22052 (2)	85031 (2)	96137 (3)	45917 (4)	43656 (5)
55	1540	3025	56980	166375	23716 (2)	91506 (2)	10529 (4)	50328 (4)	48690 (5)
56	1596	3136	60116	175616	25472 (2)	98345 (2)	11512 (4)	55073 (4)	54197 (5)
57	1653	3249	63365	185193	27324 (2)	10556 (3)	12568 (4)	60179 (4)	62214 (5)
58	1711	3364	66729	195112	29275 (2)	11316 (3)	13700 (4)	65636 (4)	66778 (5)
59	1770	3481	70210	205379	31329 (2)	12117 (3)	14911 (4)	71490 (4)	73927 (5)
60	1830	3600	73810	216000	33489 (2)	12960 (3)	16207 (4)	77600 (4)	81703 (5)

Art. 24. In applying the formula $\underline{c}L_x$, the Napierian logarithm of the number of persons living at the age x,

$$=C\mathfrak{E}^x+{}_{,}k_{,}\varepsilon^x+ke^x-\underline{c}(e^x.\overline{x-h}.\underline{q_0})+\mu\nu^x,$$

to the useful purposes to which it is serviceable, it stands in need of reduction into another, for which might be represented an expression like

$$A+Bx+Cx^2+Dx^3, \&c.;$$

but such a form might not have the converging property requisite for the purpose required if x were a large number: but if a represent a certain age, and the age to which the law were meant to apply were represented by $a+x$ instead of x, then the formula would stand

$$\underline{c}L_{a+x}=C\mathfrak{E}^{a+x}+{}_{,}k_{,}\varepsilon^{a+x}+k\varepsilon^{a+x}-\underline{c}(e^{a+x}\times\overline{a+x-h}\times\underline{c}\,q_0)+\mu\nu^{a+x},$$

and would admit of the terms of each member to stand in the above form of x^0, x, x^2, with given converging coefficients, and each of the members of the last equation would be convertible into a converging series for most or all the values which x would be required to have; and therefore the right side of the last equation, which would consist of all the developments together, would form a series, as above, which would be convenient, and might be expressed by

$$A_0+A_1x+A_2x^2+A_3x^3+ \&c.;$$

and the Napierian logarithm of the number of persons living at the age a would be represented by the first term A_0 of that series; and the Napierian logarithm of $\frac{L_{a+x}}{L_a}$, that is, of the chance of a person now of the age a being living in x years, would be

$$=A_1x+A_2x^2+A_3x^3+\&c.,$$

because $\mathfrak{E}^{a+x}$ is

$$=\mathfrak{E}^a\times\mathfrak{E}^x=\mathfrak{E}^a\times(1+\underline{\mathfrak{E}}.x+\tfrac{1}{2}\overline{\underline{\mathfrak{E}}}|^2x^2+\tfrac{1}{2.3}\overline{\underline{\mathfrak{E}}}|^3x^3+\tfrac{1}{2.3.4}\overline{\underline{\mathfrak{E}}}|^4x^4+\&c.),$$

because $\underline{\mathfrak{E}}$, the Napierian $\mathfrak{E}$, is very small, and a very few terms of the above series will be sufficient. Similarly, ${}_{,}\varepsilon^x$ will be

$${}_{,}\varepsilon^a\times(1+{}_{,}\underline{\varepsilon}.x+\tfrac{1}{2}\,\overline{{}_{,}\varepsilon}|^2x^2+\tfrac{1}{2.3}\,\overline{{}_{,}\varepsilon}|^3x^3, \&c.);$$

and if x is equal to or greater than one, would be of perfectly insignificant value, and omissible.

I may observe that

$$\varepsilon^{a+x}=\varepsilon^a\times\left(1+\underline{\varepsilon}.x+\tfrac{1}{2}\overline{\underline{\varepsilon}}|^2x^2+\tfrac{1}{2.3}\overline{\underline{\varepsilon}}|^3x^3, \&c.\right)$$

is a very convergent series; and if a be equal to 20 in the Carlisle mortality, or above that value, would be of insignificant value, and the term which depends on it omissible; in like manner the term $\mu.\nu^{a+x}$, depending on ν^{a+x}, which is

$$=\nu^a\times\left(1+\underline{\nu}.x+\tfrac{1}{2}\overline{\underline{\nu}}|^2x+\tfrac{1}{2.3}\overline{\underline{\nu}}|^3x^3, \&c.\right),$$

will be converging, and will be insignificant whilst $a+x$ is less than 80, but will be significant when $a+x$ is greater than 80.

It remains now to consider the important term $\subset\!\!-\{e^{a+x}\times\overline{a+x-h\subset\!\!-q_0}\}$, which for abbreviation I call M_{a+x}; and putting $h-a=w$, we get

$$M_{a+x}=\subset\!\!-\left\{e^a.\underline{q}_0\times\overline{a+x-h}\times\left(1+\underline{e}x+\tfrac{1}{2}\underline{e}]^2x^2+\tfrac{1}{2.3}\underline{e}]^3x^3+\text{ \&c.}\right)\right\}$$

$$=\subset\!\!-\left\{e^a.\underline{q}_0\times\left(-w-w.\underline{e}x-\tfrac{1}{2}w.\underline{e}]^2.x^2-\tfrac{1}{2.3}w.\underline{e}]^3.x^3-\tfrac{1}{2.3.4}\underline{e}]^4x^4,\text{ \&c.}\right.\right.$$

$$\left.\left.+x+\underline{e}x^2+\tfrac{1}{2}\underline{e}]^2x^3+\tfrac{1}{2.3}\underline{e}]^3x^4+\text{\&c.}\right)\right\};$$

consequently, if V_a be put $=\subset\!\!--e^a\underline{q}_0w$, that is, equal to the anti-Napierian logarithm of $-e^a.\underline{q}_0w$, and we put

$$A_1=e^a\underline{q}_0\times\overline{1-w\underline{e}};\qquad A_2=e^a.\underline{q}_0\times\overline{1-\tfrac{1}{2}w\underline{e}}\times\underline{e};$$

$$A_3=\tfrac{1}{2}e^a.\underline{q}_0\times\left(1-\tfrac{1}{3}w\underline{e}\right).\underline{e}]^2;\qquad A_4=\tfrac{1}{2.3}e^a.\underline{q}_0.\overline{1-\tfrac{1}{4}w\underline{e}}.\underline{e}]^3;\text{ \&c.},$$

we shall have

$$M_{a+x}=\subset\!\!-\{-e^a.\underline{q}_0w+A_1x+A_2x^2+A_3x^3+A_4x^4+\text{\&c.}\}$$

$$=V_a\times\subset\!\!-\{A_1x+A_2x^2+A_3x^3+\text{\&c.}\};$$

or making use of the theorem for finding the anti-Napierian logarithm above given, and taking $A_0=0$, we have

$$M_{a+x}=V_a\times(1+B_1x+B_2x^2+B_3x^3+\text{ \&c.});$$

and consequently we have, for instance, in the Carlisle mortality, the Napierian logarithm of

$$L_{a+x}={}^1A_0+{}^1A_1x+{}^1A_2x^2+{}^1A_3x^3,\text{ \&c.},$$

if

$${}^1A_0=C\mathfrak{E}^a+{}_{\prime}k_{\prime}\varepsilon^a+k\varepsilon^a-V_a+\mu\nu^a;$$

$${}^1A_1=C\mathfrak{E}^a.\underline{\mathfrak{E}}+{}_{\prime}k_{\prime}\varepsilon^a.\underline{{}_{\prime}\varepsilon}+k\varepsilon^a.\underline{\varepsilon}-V_aB_1+\mu\nu^a.\underline{\nu};$$

$${}^1A_2=\tfrac{1}{2}C\mathfrak{E}^a.\underline{\mathfrak{E}}]^2+\tfrac{1}{2}{}_{\prime}k_{\prime}\varepsilon^a.\underline{{}_{\prime}\varepsilon}^2+\tfrac{1}{2}k\varepsilon^a.\underline{\varepsilon}^2-V_aB_2+\tfrac{1}{2}\mu\nu^a\underline{\nu}]^2;$$

$${}^1A_3=\tfrac{1}{2.3}C\mathfrak{E}^a.\underline{\mathfrak{E}}]^2+\tfrac{1}{2.3}{}_{\prime}k_{\prime}\varepsilon^a.\underline{{}_{\prime}\varepsilon}+\tfrac{1}{2.3}k\varepsilon^a.\underline{\varepsilon}]^2-V_a+\mu\nu^a\underline{\nu}]^2;$$

$${}^1A_4=\text{\&c.}\qquad A_5=\text{\&c.}\qquad\text{\&c.}$$

But here it is to be observed, that if a be equal to one year or more, the term affected with ${}_{\prime}\varepsilon$ is insignificant; and whilst $a+x$ is less than 80, in the Carlisle mortality, for instance, the term affected with ν is insignificant; and when a is as great as 20, the term affected with ε is insignificant; which observation shows that though the values of ${}^1A_0$, ${}^1A_1$, ${}^1A_2$ appear intricate, only some of the terms for any value of a come into play in them, and that the values have much more simplicity than is apparent, without taking that circumstance into consideration. And, for instance, when $a+x$ is between 20 and 80, or even between about 10 and 80, a being no tless than 10, in the Carlisle Table, there will only be in the values of $'A_0$, $'A_1$, $'A_2$, &c. the terms affected with $\mathfrak{C}$ and e; and when a is as great as 80, there will be only the terms affected with $\mathfrak{C}$, e, and ν, as those affected with $_{,}\epsilon$ and ϵ will be of total insignificance. And meanwhile it will be easy to have a Table calculated for every age from birth, even including the first months after birth, which will give all the values of $'A_0$, $'A_1$, $'A_2$, $'A_3$ by bare inspection. And to illustrate this observation, I will give the value of $\llcorner M_{a+x}$, according to the Carlisle mortality, for the respective values of a, 30, 40, 50, as follows:—

$$-\llcorner M_{30+x}=2{\cdot}17621-\left\{\begin{array}{l}\cdot014087x\\+\cdot0003357x^2\\+\cdot000\\\quad\cdot0033924x^3\\+\textcircled{8}\,95944x^3\\\quad\&c.\end{array}\right.\qquad -\llcorner M_{40+x}=\bar{1}{\cdot}99926-\left\{\begin{array}{l}\cdot021562x\\+\cdot00041513x^2\\+\textcircled{5}\;2839x^3\\\quad\textcircled{7}\,12245x^4\\\quad\&c.\end{array}\right.$$

$$-\llcorner M_{50+x}=1{\cdot}73924-\left\{\begin{array}{l}\cdot030745x\\+\cdot00050696x^2\\+\textcircled{5}\,26777x^3\\+\textcircled{7}\;1395x^4\\\quad\&c.\end{array}\right.$$

Art. 25. But from what has been stated, the result of the analysis of any problem may give the present value due the xth year's payment by an expression of the form of

$$A_0x^p+A_1x^{p+1}+A_2x^{p+2}+A_3x^{p+3},\ \&c.,$$

where A_0, A_1, A_2 will be determined by the methods explained by the conditions of the problem. If we wish to have the present value of the sum due to every yearly payment between the values m and n of x, we have, according to our notation, to find the sum of the series $A_0\overset{x}{\underset{n}{\overset{m}{\Big\rceil}}}x^p+A_1.\overset{x}{\underset{n}{\overset{m}{\Big\rceil}}}x^{p+1}+A_2.\overset{x}{\underset{n}{\overset{m}{\Big\rceil}}}x^{p+3}+A_3.\overset{x}{\underset{n}{\overset{m}{\Big\rceil}}}x^{p+4}$, where $\overset{x}{\underset{n}{\overset{m}{\Big\rceil}}}$ signifies, when applied to the term x for instance, the sum $m^p+\overline{m+1}|^p+\overline{m+2}|^2$ up to n^p inclusively of m and n; and I therefore give a Table (see p. 543,) which I call a Collecting Table of Powers, to facilitate the summation.

This Table for the collection of powers given does not go to the extent which may be required in complicated cases, though it may be sufficient, for instance, for finding the value of an annuity on several joint lives, or in cases not very much more intricate; but when the powers required of x are much higher than this Table comprehends, it will not be difficult, but, on the contrary, very easy to find the sum of these high powers of the series 1, 2, 3, &c. from the well-known theorem that $\overset{x}{\underset{x}{\overset{1}{\Big\rceil}}}x^p=1^p+2^p+3^p+$ &c. x^p by the well-known theorem that the sum is equal to

$$\frac{x^{p+1}}{p+1}+\frac{1}{2}x^{p}+\frac{1}{2}\cdot\frac{p}{2.3}x^{p-1}-\frac{1}{6}\cdot\frac{p.\overline{p-1}.\overline{p-2}}{2.3.4.5}\cdot x^{p-3}+\frac{1}{6}\cdot\frac{p.\overline{p-1}.\overline{p-2}.\overline{p-3}.\overline{p-4}}{2.3.4.5.6.7}x^{p-5}, \text{ \&c.},$$

as very few of the terms will be sufficient, perhaps only the first term, or the two first terms, or the three first terms, in which several cases the theorem will stand $1^{p}+2^{p}+3^{p}+$ &c.... x^{p}, respectively equal to

$$\frac{x^{p+1}}{p+1};\ x^{p}\times\left(\frac{x}{p+1}+\frac{1}{2}\right);\ x^{p-1}\times\left(\frac{x^{2}}{p+1}+\frac{x}{2}+\frac{1}{2}\cdot\frac{p}{2.3}\right).$$

And to explain why high powers may be required, I may first mention that the yearly payments which the question may require to be made may not all require to be of the same value, but of values depending on the time in which they are to be made, or in other words, some function of x: if, for example, the payments were uniformly to increase as x increased from one to two, to three, &c., and if the corresponding payments were to be one pound, two pounds, three pounds, &c., then each term of the series $A+Bx+Cx^{2}$, &c. which would express the function of the value required for every value of x from 1 to x, to be summed, would have to be multiplied by x, and so be changed by that multiplier into $Ax+Bx^{2}+Cx^{3}+$ &c.; and the conditions relative to the yearly payments may be such as to introduce multiplications of much higher powers of x. And in complicated questions, even when the complication is not great, as for instance in the case which may not unlikely occur, to find the present value of the reversion of an annuity to be granted to the joint lives A, B after the longest life of five other given lives, C, D, E, F, G; were this to be attempted by the method given by authors by Tables of joint lives, even though we might have Tables of the value of any number of joint lives for every age (which we have not, and in consequence of not having such Tables very insufficient interpolations are to be had recourse to), the labour would be very great.

We should have, simple as the question appears by its enunciation, to search for the value of thirty-two annuities, of which one would be on two joint lives, six on three joint lives, ten on four joint lives, ten on five joint lives, five on six joint lives, and one on seven joint lives; but if the question were to be solved by the method explained, we should only have to find, by inspection of the Collecting Table, if it extended to those high powers, severally the sums of the several progressions

$$1^{5}+2^{5}+3^{5} \text{ \&c.};\ 1^{6}+2^{6}+3^{6} \text{ \&c.};\ 1^{7}+2^{7}+3^{7} \text{ \&c. \&c.}$$

severally multiplied by coefficients, say A_{1}, A_{2}, A_{3}, &c., found without much trouble from the conditions of the question, such values, A_{1}, A_{2}, A_{3}, &c., being of series converging so swiftly that a few terms which the multiplication produced would be sufficient.

And in the operation we should find the notation I have proposed useful, as for instance to write (5)234, 789(6), for ·00000234, 789000000, as numbers of such description will come into operation.

As the application of the theorem requires an easy mode of finding an analytical expression for the anti-logarithm of an analytical expression, and of giving the analytical expression of a logarithm of an analytical expression, and as there are or may be in some applications of the theorems many entanglements of logarithms in analytical expressions, and anti-analytical expressions of logarithms, I will now give different methods of operation for those purposes. I will commence by stating that if the analytical logarithmic expression is that of the common logarithm, then this must be converted to that of the Napierian logarithm by the multiplication of the coefficients by the Napierian logarithm of 10, that is, by the reciprocal of ·4342944, &c.; but a very few terms of this decimal will be required. Let the Napierian logarithm be represented by

$$A_0+A_1x+A_2x^2+A_3x^3+\&c.,$$

and by the notation I have adopted because $\subset\!\!-A_0$ will express the anti-Napierian logarithm of A_0, that is to say, the natural number of which A_0 is the anti-Napierian logarithm, and the expression, namely the anti-Napierian logarithm of

$$(A_0+A_1x+A_2x^2+\&c.),$$

will stand

$$\subset\!\!-A_0\times\subset\!\!-(A_1x+A_2x^2+A_3x^3+\&c.);$$

but

$$\subset\!\!-(A_1x+A_2x^2+A_3x^3+\&c.),$$

or the anti-Napierian logarithm of

$$(A_1x+A_2x^2+\&c.),$$

is

$$=1+A_1x+A_2x^2+\&c.+\frac{1}{2}\overline{A_1x+A_2x^2+\&c.}\,\big|^2+\frac{1}{2.3}\overline{A_1x+\&c.}\,\big|^3\ \&c.$$

$$\left.\begin{array}{l}=1+A_1x+A_2x^2+A_3x^3+A_4x^4\\ \quad\frac{1}{2}A_1^2x^2+A_1A_2x^3+A_1A_3x^4\\ \quad\quad+\frac{1}{2}A_2^2x^4\\ \quad+\frac{1}{2.3}A_1^3x^3+\frac{1}{2}A_1^2A_2x^4\\ \quad\quad+\frac{1}{2.3.4}A_1^4x^4\end{array}\right\}\ \&c.;$$

and consequently if the anti-Napierian logarithm of $A_0+A_1x+A_2x^2+\&c.$ be expressed by $B_0+B_1x+B_2x^2+B_3x^3$, &c., we shall have the following equations between B_0, B_1, B_2, &c., and A_0, A_1, A_2, &c.:

$$B_0=\subset\!\!-A_0;\ B_1=B_0A_1;\ B_2=B_0(A_2+\tfrac{1}{2}A_1^2);$$

$$B_3=B_0\times\left(A_3+A_1A_2+\frac{1}{2.3}A_1^3\right);\ B_4=B_0\left(A_4+A_1A_3+\frac{1}{2}A_2^2+\frac{1}{2}A_1^2A_2+\frac{1}{2.3.4}A_1^4\right),\ \&c.;$$

so that as B_0, B_1, B_2, &c. are found by these equations from A_1, A_2, A_3, &c., and on the contrary, A_0, A_1, A_2, &c. are found from B_0, B_1, B_2, &c. from the equations

$$A_0=\subset\!\!-B_0,\ A_1=\frac{1}{B_0}\times B_1,\ A_2=\frac{1}{B_0}\times B_2-\frac{1}{2}A_1^2;$$

$$A_3=\frac{1}{B_0}\times B_3-A_1A_2-\frac{1}{2.3}A_1^3;\ A_4=\frac{1}{B_0}\times B_4-A_1A_3-\frac{1}{2}A_2^2-\frac{1}{2}A_1^2A_2-\frac{1}{2.3.4}A_1^4,\ \&c.;$$

and where, as will generally be the case, A_1, A_2, A_3, &c., or B_1, B_2, B_3, &c. are a series of terms which are very convergent, a very few terms of the said coefficients will be found necessary, which is a very advantageous circumstance for our purpose.

Another mode of finding the anti-Napierian logarithm of the expression

$$A_1x+A_2x^2+A_3x^3+\&c.$$

is to consider it

$$={\subset\!-}A_1xx{\subset\!-}A_2x^2x{\subset\!-}A_3x^3x,\ \&c.=\left(1+A_1x+\frac{1}{2}A_1^2x^2+\frac{1}{2.3}A_1^3x^3+\&c.\right)$$
$$\times\overline{1+A_2x^2+\tfrac{1}{2}A_2^2x^4+\&c.}\times\overline{1+A_3x^3+\tfrac{1}{2}A_3^2x^6+\&c.}\times\&c.;$$

this being multiplied out at length would give what might be considered an easier way than the last for effecting our purpose, if many terms were thought requisite. But another mode of finding the anti-Napierian logarithm of $A_1x+A_2x^2+$&c., which would be the anti-Napierian logarithm of $A_0+A_1x+A_2x^2+$&c. if $A_0=0$, and therefore $B_0=1$, would be by result of the equations of $A_1x+A_2x^2+$&c. $=$ the Napierian logarithm of $1+B_1x+B_2x^2+$&c., by putting the equation in fluxions, which would give

$$A_1+2A_2x+3A_3x^2+\&c.=\frac{B_1+2B_2x+3B_3x^2+\&c.}{1+B_1x+B_2x^2+\&c.};$$

this will give the following equation for finding B_1, B_2, &c.,

$$\left.\begin{array}{rrrrr} A_1+2A_2x & +3A_3x^2 & +4A_4x^3 & +\&c. \\ +A_1B_1x & +2A_2B_1x^2 & +3A_3B_1x^3 & +\&c. \\ & +A_1B_2x^2 & +2A_2B_2x^3 & +\&c. \\ & & +A_1B_3x^3 & +\&c. \\ -B_1-2B_2x & -3B_3x^2 & -4B_4x^4 & +\&c. \end{array}\right\}=0;$$

and consequently

$$B_1=A_1;\ B_2=A_2+\tfrac{1}{2}A_1B_1;\ B_3=A_3+\tfrac{2}{3}A_2B_1+\tfrac{1}{3}A_1B_2;$$

$$B_4=A_4+\tfrac{3}{4}A_3B_1+\tfrac{2}{4}A_2B_2+\tfrac{1}{4}A_1B_3;\ B_5=A_5+\tfrac{4}{5}A_4B_1+\tfrac{3}{5}A_3B_2+\tfrac{2}{5}A_2B_3+\tfrac{1}{5}A_1B_4,\ \&c.,$$

where B_1, B_2, B_3, &c. are found respectively one after the other by a very simple law; and this method may also, if many terms be thought requisite, be preferable to the first method, and the equation reversed would give

$$A_1=B_1,\ A_2=B_2-\tfrac{1}{2}A_1B_1,\ A_3=B_3-\tfrac{2}{3}A_2B_1-\tfrac{1}{3}A_1B_2,\ A_4=B_4-\tfrac{3}{4}A_3B_1-\tfrac{2}{4}A_2B_2-\tfrac{1}{4}A_1B_1,\ \&c.$$

N.B. The calculations which these sheets required were laborious, and necessitated the very frequent, in fact the continual use of a table of Logarithms, and the searching for the Anti-logarithms; and I feel pleasure in stating that I found the last edition of Anti-logarithms, published by Mr. H. E. Filipowsky, a very serviceable assistance, and I have no doubt that calculators in the same path must be glad to possess these Tables.

Art. 26. The following Table, to show the result of the formula $\lambda L_x = C\mathfrak{c}^x + {}_{,}k_{,}\iota^x + k\iota^x - M_x + \mu\nu^x$, is calculated for Carlisle Mortality.

Age x =	1	2	3	4	5	6	7	8	9	10
λL_x formula.	3·92747	3·89062	3·86653	3·85053	3·83975	3·83148	3·82612	3·82162	3·81774	3·81323
λL_x Milne ...	3·92742	3·89092	3·86177	3·84497	3·83231	3·82461	3·81915	3·81531	3·81245	3·81023
L_x formula...	8461	7773	7274	7088	6914	6784	6701	6631	6573	6504
L_x Milne ...	8461	7779	7274	6998	6797	6676	6594	6536	6493	6460
Age x =	11	12	13	14	15	16	17	18	19	20
λL_x formula.	3·81113	3·80809	3·80504	3·80237	3·80057	3·79694	3·79354	3·79070	3·78792	3·78453
λL_x Milne ...	3·80823	3·80618	3·80400	3·80175	3·79934	3·79664	3·79372	3·79071	3·78767	3·78462
L_x formula...	6473	6399	6383	6334	6299	6260	6219	6176	6136	6088
L_x Milne ...	6431	6400	6368	6335	6300	6261	6219	6176	6133	6090
Age x =	21	22	23	24	25	26	27	28	29	30
λL_x formula.	3·78145	3·77822	3·77501	3·77167	3·76828	3·76483	3·76125	3·75759	3·75397	3·75019
λL_x Milne ...	3·78154	3·77851	3·77546	3·77240	3·76930	3·76612	3·76290	3·75952	3·75557	3·75143
L_x formula...	6046	6001	5957	5911	5863	5819	5771	5722	5675	5626
L_x Milne ...	6047	6005	5963	5921	5879	5836	5793	5748	5698	5642
Age x =	31	32	33	34	35	36	37	38	39	40
λL_x formula.	3·74623	3·74222	3·73870	3·73398	3·72957	3·72548	3·71984	3·71501	3·71112	3·70380
λL_x Milne ...	3·74702	3·74257	3·73815	3·73376	3·72933	3·72485	3·72024	3·71550	3·71063	3·70544
L_x formula...	5575	5524	5442	5420	5353	5315	5246	5188	5142	5056
L_x Milne ...	5585	5528	5472	5417	5362	5307	5251	5194	5136	5075
Age x =	41	42	43	44	45	46	47	48	49	50
λL_x formula.	3·69939	3·69491	3·68929	3·68313	3·67643	3·67106	3·66569	3·65771	3·65051	3·64309
λL_x Milne ...	3·69975	3·69373	3·68744	3·68106	3·67459	3·66811	3·66162	3·65523	3·64914	3·64316
L_x formula...	5009	4954	4890	4821	4747	4689	4631	4547	4472	4396
L_x Milne ...	5009	4940	4869	4798	4727	4657	4588	4521	4458	4397
Age x =	51	52	53	54	55	56	57	58	59	60
λL_x formula.	3·63513	3·62707	3·61853	3·60917	3·59321	3·59022	3·57953	3·56774	3·55672	3·54427
λL_x Milne ...	3·63729	3·63104	3·62439	3·61731	3·60991	3·60206	3·59373	3·58454	3·57392	3·56146
L_x formula...	4316	4237	4155	4066	3919	3892	3798	3696	3603	3502
L_x Milne ...	4338	4276	4211	4143	4073	4000	3924	3842	3749	3643
Age x =	61	62	63	64	65	66	67	68	69	70
λL_x formula.	3·53269	3·51806	3·50242	3·48635	3·47298	3·45128	3·43176	3·41090	3·38863	3·36540
λL_x Milne ...	3·54667	3·53084	3·51428	3·49734	3·47972	3·46150	3·44264	3·42292	3·40226	3·38039
L_x formula...	3410	3297	3180	3045	2972	2833	2702	2576	2447	2320
L_x Milne ...	3521	3395	3268	3143	3018	2894	2771	2648	2525	2401
Age x =	71	72	73	74	75	76	77	78	79	80
λL_x formula.	3·33864	3·32065	3·27909	3·24834	3·21177	3·20301	3·12579	3·08460	3·06270	2·97895
λL_x Milne ...	3·35736	3·33102	3·30038	3·26505	3·22401	3·18041	3·13322	3·08386	3·03383	2·97909
L_x formula...	2181	2092	1901	1771	1628	1596	1336	1215	1155	
L_x Milne ...	2277	2143	1997	1841	1675	1515	1359	1213	1081	
Age x =	81	82	83	84	85	86	87	88	89	90
λL_x formula.	2·93253	2·87351	2·80613	2·73383	2·64479	2·56753	2·47561	2·37526	2·26730	2·15239
λL_x Milne ...	2·92273	2·86034	2·79449	2·72326	2·64838	2·56476	2·47129	2·36549	2·25768	2·15229
Age x =	91	92	93	94	95	96	97	98	99	100
λL_x formula.	2·03631	1·89820	1·76150	1·59630	1·47969	1·33147	1·16301	1·09266	1·00205	·95396
λL_x Milne ...	2·02119	1·87508	1·73239	1·60206	1·47712	1·36173	1·25527	1·14613	1·04139	·95424

x=	0 or Birth.	1 Month.	2 Months.	3 Months.	6 Months.	9 Months.	12 Months.	Average deaths per month.
λL_x formula.	400000	3·97622	3·96755	3·96182	3·94753	3·93494		In first month 0
λL_x Milne ...	400000	3·97622	3·97621	3·96501	3·95279	3·94027		Between 1 and 2 ......
L_x formula...	10000	9467	9279	9158	9158	8609		Between 2 and 3 ......
L_x Milne ...	10000	9467	9313	9226	9226	8715		Between 3 and 6 ......
								Between 6 and 9 ......

Art. 27. In a paper I had the honour of presenting to the International Statistical Congress, in consequence of the flattering invitation I had received to offer my assistance towards the scientific objects in view, I presented a sketch of some investigations I had been making since the publication of my paper of 1820 and that of 1825, relative to the subject of mortality, and on some investigations I had since made on the law of sickness and invalidity subjects, which I was prevented, in consequence of the absence of sufficient health, to finish for presentation to the Royal Society, as a supplement to those two papers which were honoured by a place in the Society's Transactions. But in my paper I presented to the Congress, I did not show how I obtained the formulæ of the one uniform law of mortality I presented, from birth to extreme old age, reserving that communication for the honour I intended myself to present it to this Society, should my health sufficiently recover to render it possible to me, though I presented the formula which follows, $\lambda L_x = \text{constant} + ke^x - {}_ik_ie^x - nq^x - P_x$, in which λL_x represents the common logarithm of the number of persons living at the age x, $P_x = \theta . \overline{w}|^{\pi^x(x-u)}$, k, e, ${}_ik$, ${}_ie$, n, q, θ, w, π, u constant quantities from birth to the extremity of old age, to be found from statistical tables given of the actual persons living at every age in places to which the formula is meant to apply; and these constants I found for Carlisle mortality, Northampton mortality, De Parcieux mortality, and Sweden, male and female mixed mortality, from the examination of published statistical tables; and I computed for these four stated rates of mortality from this formula by means of these constants the number of living at every age, and arranged the results opposite the statements, which show a remarkably satisfactory agreement between the formula and the statement which it is intended to represent throughout, from commencement to every year of age. And in the Carlisle Table, where there are data for comparison, for the first months after birth, where there appears great irregularity in the deaths, even the close approximate agreement seems very interesting; and the value of the constants there given are in the formula resulting when θ is taken $=1$, which, though it may not be its exact value, is very nearly so.

	u.	λk.	λe.	$\lambda_i k$.	$\lambda_i e$.	λn.	λq.	$\lambda \pi$.	λw.	Constant.
Carlisle	90·37	$\bar{1}$·2310	$\bar{1}$·76774	$\bar{1}$·59375	$\bar{4}$·34652	$\bar{2}$·75526	0126	$\bar{1}$·98952	$\bar{1}$·50837	3·8631
Northampton ...	90·131	$\bar{1}$·43172	$\bar{1}$·72758	No data.	No data.	$\bar{1}$·11526	011213	$\bar{1}$·9954	$\bar{1}$·15125	3·92650
Sweden	96·137	$\bar{1}$·23562	$\bar{1}$·7918	No data.	No data.	$\bar{2}$·87042	01296	$\bar{1}$·99608	$\bar{1}$·03727	3·87142
De Parcieux ...	86·21	$\bar{2}$·99	$\bar{1}$·8415	No data.	No data.	$\bar{1}$·3323	006005	$\bar{1}$·99293	$\bar{1}$·2250	3·19130

I gave the results for the Carlisle mortality to the Congress, which were extremely satisfactory, but I did not give the results for the Northampton mortality and for the Sweden, and De Parcieux's mortality, which (though with the exception that for the few first months of age these statistical tables give no data) I find, I think, equally satisfactory; but not having sent the results to the Congress, I presume that I am authorized, without infringing on the rule of this Society not to publish what has been already published, to present, as I have done here, the results for the Northampton mortality; and it was my wish also to insert the satisfactory results I found for the other two, but for want of time I am prevented, though I think they would be found interesting. The aforesaid formula bears a very different form from the formula of this paper, $\rightharpoonup L_x = C\epsilon^x + k_{,}\varepsilon_{,}^x + k\varepsilon^x - \rightharpoonup M_x + \mu\nu^x$, where $M_x = e^x . \overline{h-x} . q \rightharpoonup$, &c., the constants here not being represented by the same letters as in the former; these two differ interestingly in appearance, and so much so as to lead to surprise that the result of each, for the Carlisle mortality, gives such very near approximation to the Table of observation. I was therefore, as both formulæ have particular features of interest, induced to examine by analysis their analytical analogy.

Northampton Mortality, from the formula sent to the Congress.

(From the Northampton formula; the results being from the formula which was printed in the Reports of the International Statistical Congress, which results were not given.)

Age x =	0	1	2	3	4	5	6	7	8
λL_x formula .	4·06633	3·93704	3·86626	3·82676	3·80290	3·78988	3·78053	3·77361	3·76797
λL_x Morgan .	4·06633	3·93702	3·86231	3·83129	3·80929	3·79581	3·78283	3·77269	3·76545
Age x =	9	10	11	12	13	14	15	16	17
λL_x formula .	3·76295	3·75820	3·75355	3·74889	3·74418	3·73936	3·73444	3·72941	3·72425
λL_x Morgan .	3·75833	3·75397	3·74992	3·74609	3·74218	3·73822	3·73424	3·73022	3·72591
Age x =	18	19	20	21	22	23	24	25	26
λL_x formula .	3·71897	3·71353	3·70797	3·70225	3·69638	3·68737	3·68119	3·67785	3·67135
λL_x Morgan .	3·72115	3·71592	3·71029	3·70415	3·69767	3·69108	3·68440	3·67761	3·67071
Age x =	27	28	29	30	31	32	33	34	35
λL_x formula .	3·66468	3·65433	3·65080	3·64359	3·63619	3·62859	3·62073	3·61281	3·60460
λL_x Morgan .	3·66370	3·65658	3·64933	3·64197	3·63448	3·62685	3·61909	3·61119	3·60314
Age x =	36	37	38	39	40	41	42	43	44
λL_x formula .	3·59618	3·58755	3·57868	3·56958	3·56025	3·55067	3·54084	3·53075	3·52040
λL_x Morgan .	3·59494	3·58664	3·57807	3·56937	3·56500	3·55132	3·54180	3·53191	3·52192
Age x =	45	46	47	48	49	50	51	52	53
λL_x formula .	3·50977	3·49888	3·48770	3·47628	3·46446	3·45196	3·43943	3·42653	3·41327
λL_x Morgan .	3·51162	3·50106	3·49024	3·47014	3·46775	3·45591	3·44342	3·43640	3·41697
Age x =	54	55	56	57	58	59	60	61	62
λL_x formula .	3·39960	3·38543	3·37098	3·35595	3·34042	3·32433	3·30763	3·29028	3·27221
λL_x Morgan .	3·40312	3·38881	3·37401	3·35870	3·34282	3·32034	3·30920	3·29134	3·27277
Age x =	63	64	65	66	67	68	69	70	71
λL_x formula .	3·25425	3·23361	3·21286	3·19111	3·16812	3·14363	3·11771	3·09040	3·06097
λL_x Morgan .	3·25258	3·22350	3·21272	3·19089	3·16791	3·14364	3·11793	3·09061	3·06145
Age x =	72	73	74	75	76	77	78	79	80
λL_x formula .	3·02944	2·99557	2·95889	2·91834	2·87596	2·82861	2·77758	2·72252	2·63755
λL_x Morgan .	3·03019	2·99651	2·95999	2·92012	2·87622	2·82930	2·77960	2·72754	2·67117
Age x =	81	82	83	84	85	86	87	88	89
λL_x formula .	2·59479	2·51156	2·43295	2·35326	2·24725	2·15001	2·03269	1·90233	1·76507
λL_x Morgan .	2·60832	2·53908	2·46090	2·36922	2·26951	2·16137	2·04532	1·91908	1·79239
Age x =	90	91	92	93	94	95	96		
λL_x formula .	1·60203	1·44327	1·25940	1·05593	1·04019	60181	34315		
λL_x Morgan .	1·66276	1·53479	1·38021	1·20402	·95424	60206			

This Table was not given in the paper written for the (late) International Congress, but was calculated for the present, from the formula $\lambda L_x = u + ks^x - k_i s^x - nq^x - P_x$; P_x being put for $\theta(w)^{r^x \cdot \overline{x-u}}$, all the qualities except x being constant from birth.

Art. 28. It remains now slightly to touch on the law of sickness, though the subject is already mentioned in the paper I presented to the Congress, in consequence of a similarity, as far as I have had data to discover, with the law of mortality. I gave the formula, but not the investigation of that formula; but I gave Tables of comparison with the sickness stated to have occurred in different societies and in different places, which appear to be extremely satisfactory as to the approximate agreement with the stated results. The formula I gave is as follows:—If S^x be the number of weeks of sickness due to a person of x in the Society, the log of $S_x = A + BC^x$, where A, B, C are apparent constants for a long period, to be found by the vital rule of three, from the stated sickness prevailing at three selected ages; and here the apparent constants are evidently not truly constant, as their values will slightly change with the selection. The formula $S_x = A . \overline{B}|^{C^x}$ has the same form as $L_x = D . \overline{g}|^{q^x}$, the theorem of mortality, and, as in that theorem, the elements A, B, C, though apparently constant for a less time, depend on three selected ages for the determining their values, and are only apparently constant, but afford for a long time near approximations to the amount of sickness which occurs in different societies. The values of A, B, C of the formula, as determined from statements of societies, I gave in my paper presented to the International Congress, with Tables showing the near agreement with the statements of those societies. I quote from that paper as follows:—

Selected ages, 25, 45, 65, from Mr. ANSELL, all England.

Selected ages, 25, 45, 65, Town:

$\lambda A = \bar{1}\cdot72778$; $\lambda B = \bar{2}\cdot6772$; $\lambda C = \cdot020433$.

Selected ages, 25, 45, 65, Scotland:

$\lambda A = \bar{1}\cdot66237$; $\lambda B = \bar{2}\cdot41351$; $\lambda C = \cdot02433$.

Selected ages, 25, 45, 65, ANSELL, Town:

$\lambda A = \bar{1}\cdot65112$; $\lambda B = \bar{1}\cdot04297$; $\lambda C = \cdot02818$.

Selected ages, 35, 50, 65, ANSELL, Town:

$\lambda A = \bar{1}\cdot92937$; $\lambda B = \bar{1}\cdot70942$; $\lambda C = \cdot0165265$

Selected ages, 25, 45, 65, City district:

$\lambda A = \bar{1}\cdot84121$; $\lambda B = \bar{2}\cdot47741$; $\lambda C = \cdot009206$.

Selected ages, 25, 45, 65, City:

$\lambda A = \bar{1}\cdot84121$; $\lambda B = \bar{1}\cdot1553$; $\lambda C = \cdot0301685$.

Selected ages, 30, 40, 50, Rural:

$\lambda A = \bar{1}\cdot85616$; $\lambda B = \bar{2}\cdot02951$; $\lambda C = \cdot0301685$.

There seems to be a distinction between the law of mortality and that of sickness, inasmuch as g, in the law of mortality, is a positive fraction less than unity, whereas the similar term B in the law of sickness is greater than unity.

Art. 29. Here I must draw the reader's very particular attention to the formula

$$\rightharpoondown L_x = C\epsilon^x + {}_{,}k_{,}\varepsilon_{,}^x + k\varepsilon^x - M_x + \mu\nu^x, \text{ where } \rightharpoondown M_x = e^x . \overline{x - h} . q \rightharpoondown,$$

where for analytical anticipation it is put in the form

$$\subset L_{a+x} = C\epsilon^{a+x} + {}_{,}k_{,}\varepsilon_{,}^{a+x} + k\varepsilon^{a+x} - M_{a+x} + \mu\nu^{a+x};$$

and the two portions $k\varepsilon^{a+x}$ and $\mu\nu^{a+x}$ are developed into series proceeding by the powers of x, because those series for several of the first terms, when x is at all large, are so divergent that in that case they become of no practical service; and in fact, though, if a sufficient number of terms be used, they are ultimately convergent and would lead to the true value, they present so formidable an obstacle as to cause, if those series be used, a complete refusal of the aid which was expected from them to render the formula analytically anticipatory, at least at those periods when these functions have a prevailing influence over anticipation; and if this difficulty had not been overcome by the adoption of a subterfuge, which I almost despaired of finding, a great part of the highly important analytically anticipating powers of the formula would be destroyed. The series alluded to are exhibited as follows:—

$$k\varepsilon^{a+x} = k\varepsilon^a \times \left(1 + \underline{\varepsilon}x + \frac{1}{2}\overline{\underline{\varepsilon}|}^2 x^2 + \frac{1}{2.3}.\overline{\underline{\varepsilon}|}^3 x^3, \text{ \&c.}\right),$$

and

$$\mu\nu^{a+x} = \mu\nu^a \times \left(1 + \underline{\nu}x + \frac{1}{2}\overline{\underline{\nu}|}^2 x^2 + \frac{1}{2.3}.\overline{\underline{\nu}|}^3 x^3, \text{ \&c.}\right).$$

And alluding to the first of these, I observe that in the formula above, when we refer it to the Carlisle mortality, $\rightharpoondown\varepsilon = \bar{1}{\cdot}79811$; $\subset\varepsilon = -{\cdot}46427$; $\rightharpoondown k = \bar{1}{\cdot}16855$; $k = {\cdot}14742$: if we wished to anticipate only for ten years forward, $\underline{\varepsilon}x$ would be, say about $-\frac{9}{2}$, and the series $1 + \underline{\varepsilon}x + \frac{1}{2}\overline{\underline{\varepsilon}|}^2 x^2 +$ &c. would be

$$1 - \frac{9}{2} + \frac{1}{2}.\overline{\tfrac{9}{2}|}^2 - \frac{1}{2.3}.\overline{\tfrac{9}{2}|}^3 + \frac{1}{2.3.4}.\overline{\tfrac{9}{2}|}^4 - \text{\&c.} = 1 - \frac{9}{2} + \frac{9}{2}.\frac{9}{4} - \frac{9}{2}.\frac{9}{4}.\frac{9}{6} + \frac{9}{2}.\frac{9}{4}.\frac{9}{6}.\frac{9}{8} - \text{\&c.};$$

so that we should not only have to go to the fifth power of x before the series began to converge, but to go to much higher powers, whereas the real value of $k\varepsilon^{10}$ sought is but of the small value of about ·001411, of very small value, which may be considered even of insignificant value with respect to the total value of $\subset L_{a+x}$, notwithstanding its perplexing annoyance. But if we wished to anticipate for twenty years, although if a is even equal to 0, the value of the function $k\varepsilon^{a+x}$ is of such total insignificance with respect to the other portion of $\subset L_{a+x}$, being no greater than about ·000014; whilst the value of $\subset L_{a+x} = 3{\cdot}81023$, yet in the development of $k\varepsilon^{a+x}$ into the series above, the number of diverging terms alternately positive and negative of greater value before the series even began to converge would render the development of perfect impracticability, though ultimately we could arrive as near as we chose to the insignificant value ·000014; and I found that with such an annoyance it was no use to wrestle, no more than it would have been for David to wrestle with Goliath. I therefore was at the pains of finding a subterfuge by an interpolation which discards all the terms of a series alternately greatly positive and greatly negative, which so nearly destroyed each other's effect when we arrived to very nearly the true but very insignificant value ·000014. I have reason to think good mathematicians would think such a subterfuge was unattainable, but what follows will show the contrary.

Art. 30. Calculating the value $k\varepsilon^0$, $k\varepsilon^5$, $k\varepsilon^{10}$, $k\varepsilon^{20}$, we find the value to form the very swiftly converging series, ·147424, ·014424, ·001441, ·000188, ·00008, and beyond this term perfectly insignificant; so that, if we wished to express the value of ε^{5x}, where x is a whole number, by the common method of interpolation, we might easily do it thus:—

$$1+\Delta_1 x+\Delta_2 .x.\frac{x-1}{2}+\Delta_3 .x.\frac{x-1}{2}.\frac{x-2}{3}+\Delta_4 .x.\frac{x-1}{2}.\frac{x-2}{3}.\frac{x-3}{4};$$

beyond which term the differences are of perfect insignificance, Δ_1, Δ_2, Δ_3, Δ_4 being the first of the several orders of differences of the value $k\varepsilon^0$, $k\varepsilon^5$, $k\varepsilon^{10}$, &c.; and it is evident that Δ_1, Δ_2, Δ_3, Δ_4 are respectively $k(\varepsilon^5-1)$, $k(\varepsilon^5-1)^2$, $k(\varepsilon^5-1)^3$, &c., which we may adopt for the sake of proof, instead of taking the actual difference. And as the form in which we obtain the value of $k\varepsilon^x$ requires for our purpose to be expressed by a series such as $I_0+I_1x+I_2x^2+I_3x^3+I_4x^4$, if the above value of $k\varepsilon^x$ be expanded, it will give

$$I_0=k,\quad I_1=\Delta_1-\frac{1}{2}\Delta_2+\frac{1}{3}\Delta_3-\frac{1}{4}\Delta_4,\quad I_2=\Delta_2-\frac{1}{2}\Delta_3+\frac{11}{24}\Delta_4,\quad I_3=\frac{1}{6}\Delta_3-\frac{1}{4}\Delta_4,\quad I_4=\frac{1}{24}\Delta_4,$$

where, as already observed, $\Delta_1=k(\varepsilon^5-1)$, $\Delta_2=k(\varepsilon^5-1)^2$, &c.; and we find from the above values, I_0, I_1, I_2, I_3, I_4; and as $k\varepsilon^{5x}$ is therefore $=I_0+I_1x+I_2x^2+I_3x^3+I_4x^4$, if in the room of $5x$ we write x, we shall have the equation

$$k\varepsilon^x=I_0+\frac{1}{5}I_1x+\frac{1}{5^2}I_2x^2+\frac{1}{5^3}I_3x^3+\frac{1}{5^4}I_4x^4=J_0+J_1x+J_2x^2+J_3x^3+J_4x^4,$$

if $J_0=I_0$, $J_1=\frac{1}{5}I_1$, $J_2=\frac{1}{5^2}I_2$, $J_3=\frac{1}{5^3}I_3$, $J_4=\frac{1}{5^4}I_4$, and we have

$I_0=·147422$, $I_1=-·253484$, $I_2=+·158862$, $I_3=-·042453$, $I_4=+·004068$,

$J_0=I_0$, $J_1=-·050696$, $J_2=·0063545$, $J_3=-\textcircled{4}33962$, $J_4=\textcircled{5}65088$,

$\lambda-J_1=\bar{2}·70497$, $\lambda J_2=\bar{3}·80308$, $\lambda-J_3=\bar{5}·53100$, $\lambda J_4=\bar{6}·81350$.

And with respect to $\mu\nu^{a+x}$, the least value of a may be taken when $a=60$, as $\mu\nu^a$ is so extremely insignificant if a be less than 60. I will consider the value of $\mu\nu^{60+x}$ first, and consider afterwards $\mu\nu^{60+b+\nu}$, if a exceeds 60 by b; and I observe that $\mu\nu^{60}$, $\mu\nu^{60+5}$, $\mu\nu^{60+10}$, $\mu\nu^{60+15}$, $\mu\nu^{60+20}$ form a regular series; and here I stop, though this series is of a different character from the series $k\varepsilon^0$, $k\varepsilon^{5x}$, $k\varepsilon^{10x}$, &c., because the terms of the latter series dwindle to such insignificance, but those of the other, on the contrary, continually increase.

There may be cases, indeed, when we should not stop; but though we have not data for ages beyond 100, the coefficient μ in the expression $\mu\nu^{a+x}$, though taken as sufficiently expressed by the constant value till $a+x$ does exceed 100, is evidently not absolutely constant; for if so, a person would be represented to be capable to live for ever; and I therefore satisfy myself by omitting to introduce a term for which I have no data, by limiting my term of anticipation to the period when $a+x$ becomes $=100$.

I now observe that for the Carlisle mortality (where $\lambda\mu = \bar{1}\bar{1}\cdot 29927$, $\lambda\nu = \cdot 110349$), $\mu\nu^{60}$, $\mu\nu^{60+5}$, $\mu\nu^{60+10}$, $\mu\nu^{60+15}$, $\mu\nu^{60+20}$ give the following series of terms:—

$$\textcircled{4}83303, \quad \textcircled{3}10572, \quad \textcircled{2}37662, \quad 013417.$$

And adopting the same mode of interpretation as in the former case, and representing $\mu\nu^{60+x}$ by

$$'I_0 + 'I_1 x + 'I_2 x^2 + 'I_3 x^3 + 'I_4 x^4,$$

and putting

$$'I_0 = 'J_0, \quad \frac{1}{5}'I_1 = 'J_1, \quad \frac{1}{5^2}'I_2 = 'J_2, \quad \frac{1}{5^3}'I_3 = 'J_3, \quad \frac{1}{5^4}'I = {}_4'J_4,$$

we have

$$'I_0 = \cdot 000083303, \quad 'I_1 = -\cdot 00040315, \quad 'I_2 = +\cdot 00121095, \quad 'I_3 = -\cdot 00065977, \quad 'I_4 = +\cdot 0001489,$$

$$\multimap -'J_1 = \bar{5}\cdot 98816, \quad \multimap 'J_2 = \bar{5}\cdot 68517, \quad \multimap -'J_3 = \bar{6}\cdot 72248, \quad \multimap 'J_4 = \bar{6}\cdot 23825,$$

$$'J_0 = \cdot 000083303, \quad 'J_1 = -\textcircled{4}97313, \quad 'J_2 = \textcircled{4}484363, \quad 'J_3 = -\textcircled{5}52753, \quad 'J_4 = \textcircled{6}23825.$$

But here it is necessary to observe, that though these series for finding ks^{a+x}, $\mu\nu^{a+x}$ are exactly true with respect to any value of x, if it be divisible by 5 (limited with respect to $\mu\nu^{a+x}$ to the case observed above, $a+x$ not being greater than 100), still they are not identical with that expression, as they may differ both in plus and in minus with them, if x, when divided by 5, should leave a remainder, say of 1, 2, 3, 4, which are four cases of more exact identity; but as the variance from identity is quite insignificant with respect to the value of $\subset L_{a+x}$, as will be shown further on, it is perfectly allowable to use this method without paying the slightest attention to the more absolute identity of the expression; but to proceed to prove this assertion. I represent ks^x by

$$J_0 + J_1 x + J_2 x^2 + J_3 x^3 + J_4 x^4 + W_x,$$

and also represent $\mu\nu^{60+x}$ by

$$'J_0 + 'J_1 x + 'J_2 x^2 + 'J_3 x^3 + 'J_4 x^4 + 'W_x,$$

and I am to show that W_x, and $'W_x$ are either equal to nothing, or are insignificant with respect to $\subset L_{a+x}$, whether they shall turn out to be positive or negative. Now I observe in the case where x is divisible exactly by 5, the above investigation shows that W_x and $'W_x$ are both absolutely equal 0; and for the rest I will take x, by way of example, successively 11, 12, 13, 14, in which x divided by 5 leaves either 1, 2, 3, or 4, and where the anticipation is respectively for 11, 12, 13, 14 years.

By direct calculation . .	$k\varepsilon^{11}=$ ·000887,	$k\varepsilon^{12}=$ ·000537,	$k\varepsilon^{13}=$ ·000350,	$k\varepsilon^{14}=$ ·0002[illegible]
By the anticipating formula	$k\varepsilon^{11}=$ ·00106,	$k\varepsilon^{12}=$ ·00129,	$k\varepsilon^{13}=$ ·00355,	$k\varepsilon^{14}=-$·0035[illegible]
	$W_{11}=-$·00018,	$W_{12}=-$·00075,	$W_{13}=-$·00320,	$W_{14}=+$·0037[illegible]

where x is not divisible by 5; but in cases where x is divisible by 5, W_x is always equal to 0, and among the cases the three negative values are

$$\left.\begin{array}{l}-·00018\\-·00075\\-·00320\end{array}\right\}=-·00435.$$

Their sum differs very little from the one positive case ·00375, and they are all perfectly insignificant compared to the total values of $\lambda L_{11}=3·808$ &c., $\lambda L_{12}=3·806$ &c., $\lambda L_{13}=3·804$ &c., $\lambda L_{14}=3·803$ &c.; but when I say perfectly insignificant, I mean with respect to the chance of living,—though that difference would, in a small degree, cause the number answering, say to the age of 11, to make it apply to an age very triflingly differing from the exact age of 11, but this is not of the slightest importance with respect to valuing chances of anticipation. It may be observed that when $k\varepsilon^x$ is very small, as in the case above enumerated, we see that the anticipated values of $k\varepsilon^x$ may have not only a large proportion to each other, but may even be of contrary signs, as $k\varepsilon^x$ cannot be negative, though by the anticipating formula in the case of k it comes out negative; but these are cases where the value is of no importance in consequence of its smallness; and for long before the age of 20, the effect of $k\varepsilon^x$ in the anticipation can be omitted.

And now with respect to the formula which is not as imperatively required as the other,

$$\mu\nu^{a+x}=(\text{when } a=60)\ 'J_0+'J_1x+'J_2x^2+'J_3x^3+'J_4x^4+'W_x,$$

let the anticipation also be successively for 11, 12, 13, 14 years. I purposely take years not divisible by 5, because, as I have said above, when $x=5$, and $a+x$ does not exceed 100, $'W_x$ is invariably $=0$, the variability of μ not coming into play; we have

The real values of	$\mu\nu^{71}=$·0013630,	$\mu\nu^{72}=$·0017573,	$\mu\nu^{73}=$·0022656,	$\mu\nu^{74}=$ ·0029217
Anticipating formula	$\mu\nu^{71}=$·0013264,	$\mu\nu^{72}=$·0017106,	$\mu\nu^{73}=$·0022601,	$\mu\nu^{74}=$ ·0029430
	$'W_{71}=$·0000366,	$'W_{72}=$·0000467,	$W_{73}=$·0000055,	$W_{74}=-$·0000213

The mode above described of getting rid of the annoyance of the non-convergency of the developed expression $k\varepsilon^x$ is particularly worthy of attention, and is entitled to be noticed by a name which I call the Interpolation by selected terms taken *per saltum*, as it offers another and an easier mode of calculating the three analytical anticipating Tables mentioned, which are so efficient in practical valuations; and calculating those Tables independent of each other when, as in the mode pointed out first, required the calculation of the value of $\llcorner L_{a+x}$, and from that the value of L_{a+x}; and lastly, the value of $\frac{L_{a+x}}{L_a}$, say in the form $'B_1x+'B_2x+'B_3x^3$, and then the value $=(1+\frac{'B_2}{'B_1}x+\frac{'B_3}{'B_1}x^2+\&c.)$. And not only does this mode furnish the means of finding those Tables from the developed function of mortality, but, if I mistake not, it furnishes a means off-hand, from any Table of mortality formed only from statistical information, and without requiring the formulæ of mortality above pointed out, to furnish efficient analytical participating tables.

It was my wish not only to conclude this paper with the three Tables above-mentioned, which have been calculated, but which I have been prevented from examining from causes to which mortality is liable, but also to add some interesting matter. But I hope to be able to add a supplement to this paper, and to be permitted to publish it in the Royal Society's Transactions, to illustrate the practical adaptation of the analysis, with some other matters of vital statistics and invalidism.

XXIV. *On* TSCHIRNHAUSEN'S *Transformation.* *By* ARTHUR CAYLEY, *Esq.*, *F.R.S.*

Received November 7,—Read December 5, 1861.

THE memoir of M. HERMITE, "Sur quelques théorèmes d'algèbre et la résolution de l'équation du quatrième degré,"* contains a very important theorem in relation to TSCHIRNHAUSEN'S Transformation of an equation $f(x)=0$ into another of the same degree in y, by means of the substitution $y=\phi x$, where ϕx is a rational and integral function of x. In fact, considering for greater simplicity a quartic equation,

$$(a, b, c, d, e \between x, 1)^4=0,$$

M. HERMITE gives to the equation $y=\phi x$ the following form,

$$y=a\mathrm{T}+(ax+b)\mathrm{B}+(ax^2+4bx+6c)\mathrm{C}+(ax^3+4bx^2+6cx+4d)\mathrm{D}$$

(I write B, C, D in the place of his T_0, T_1, T_2), and he shows that the transformed equation in y has the following property: viz., every function of the coefficients which, expressed as a function of a, b, c, d, e, T, B, C, D, does not contain T, is an *invariant*, that is, an invariant of the two quantics

$$(a, b, c, d, e \between \mathrm{X}, \mathrm{Y})^4, \ (\mathrm{B}, \mathrm{C}, \mathrm{D} \between \mathrm{Y}, -\mathrm{X})^2.$$

This comes to saying that if T be so determined that in the equation for y the coefficient of the second term (y^3) shall vanish, the other coefficients will be invariants; or if, in the function of y which is equated to zero, we consider y as an absolute constant, the function of y will be an invariant of the two quantics. It is easy to find the value of T; this is in fact given by the equation

$$0=a\mathrm{T}+3b\mathrm{B}+3c\mathrm{C}+d\mathrm{D};$$

and we have thence for the value of y,

$$y=(ax+b)\mathrm{B}+(ax^2+4bx+3c)\mathrm{C}+(ax^3+4bx^2+6cx+3d)\mathrm{D};$$

so that for this value of y the function of y which equated to zero gives the transformed equation will be an invariant of the two quantics. It is proper to notice that in the last-mentioned expression for y, all the coefficients except those of the term in x^0, or $b\mathrm{B}+3c\mathrm{C}+3d\mathrm{D}$, are those of the binomial $(1, 1)^4$, whereas the excepted coefficients are those of the binomial $(1, 1)^3$; this suffices to show what the expression for y is in the general case.

I have in the two papers, "Note sur la Transformation de TSCHIRNHAUSEN,"† and "Deuxième Note sur la Transformation de TSCHIRNHAUSEN,"† obtained the transformed

* Comptes Rendus, t. xlvi. p. 961 (1858).

† CRELLE, t. lviii. pp. 259 and 263 (1861).

equations for the cubic and quartic equations; and by means of a grant from the Government Grant Fund, I have been enabled to procure the calculation, by Messrs. DAVIS and OTTER, under my superintendence, of the transformed equation for the quintic equation. The several results are given in the present memoir; and for greater completeness, I reproduce the demonstration which I have given in the former of the above-mentioned two Notes, of the general property, that the function of y is an invariant. At the end of the memoir I consider the problem of the reduction of the general quintic equation to Mr. JERRARD's form $x^5+ax+b=0$.

Considering for simplicity the foregoing two equations

$$(a,\ b,\ c,\ d,\ e\between x,\ 1)^4=0,$$
$$y=(ax+b)\mathrm{B}+(ax^2+4bx+3c)\mathrm{C}+(ax^3+4bx^2+6cx+3d)\mathrm{D};$$

let the second of these be represented by $y=\mathrm{V}$, the transformed equation in y is

$$(y-\mathrm{V}_1)(y-\mathrm{V}_2)(y-\mathrm{V}_3)(y-\mathrm{V}_4)=0,$$

where V_1, V_2, V_3, V_4 are what V becomes upon substituting therein for x the roots x_1, x_2, x_3, x_4 of the quartic equation respectively. Considering y as a constant, the conditions to be satisfied in order that the function in y may be an invariant are that this function shall be reduced to zero by each of the two operators

$$a\partial_b+2b\partial_c+3c\partial_d+4d\partial_e-(\mathrm{D}\partial_\mathrm{C}+2\mathrm{C}\partial_\mathrm{B}),$$
$$4b\partial_a+3c\partial_b+2d\partial_c+e\partial_d-(2\mathrm{C}\partial_\mathrm{D}+\mathrm{B}\partial_\mathrm{C}):$$

These conditions will be satisfied if each of the factors $y-\mathrm{V}_1$, &c. has the property in question; that is, if $y-\mathrm{V}$, or (what is the same thing) if V, supposing that x denotes therein a root of the quartic equation, is reduced to zero by each of the two operators. Considering the first operator, which for shortness I represent by

$$\Delta-(\mathrm{D}\partial_\mathrm{C}+2\mathrm{C}\partial_\mathrm{B}),$$

in order to obtain $\Delta\mathrm{V}$ we require the value of Δx. To find it, operating with Δ on the quartic equation, we have

$$(a,\ b,\ c,\ d\between x,\ 1)^3\Delta x+(a,\ b,\ c,\ d\between x,\ 1)^3=0,$$

or $\Delta x=-1$. In $\Delta\mathrm{V}$, the part which depends on the variation of Δx then is

$$-a\mathrm{B}+(-2ax-4b)\mathrm{C}+(-3ax^2-8bx-6c)\mathrm{D},$$

and the other part of $\Delta\mathrm{V}$ is at once found to be

$$+a\mathrm{B}+(4ax+6b)\mathrm{C}+(4ax^2+12bx+9c)\mathrm{D};$$

whence, adding,

$$\Delta\mathrm{V}=2(ax+b)\mathrm{C}+(ax^2+4bx+3c)\mathrm{D},$$

and this is precisely equal to

$$(\mathrm{D}\partial_\mathrm{C}+2\mathrm{C}\partial_\mathrm{B})\mathrm{V};$$

so that V is reduced to zero by the operator $\Delta-(\mathrm{D}\partial_\mathrm{C}+2\mathrm{C}\partial_\mathrm{B})$.

Similarly, if the second operator is represented by

$$\nabla-(2\mathrm{C}\partial_{\mathrm{D}}+\mathrm{B}\partial_{\mathrm{C}}),$$

then we have

$$(a, b, c, d \between x, 1)^3\nabla x+x(b, c, d, e \between x, 1)^3=0,$$

which by means of the equation

$$(a, b, c, d, e \between x, 1)^4=0$$

is reduced to $\nabla x=x^2$. Hence in $\nabla\mathrm{V}$ the part depending on the variation of x is

$$ax^2\mathrm{B}+(2ax^3+4bx^2)\mathrm{C}+(3ax^4+8bx^3+6cx^2)\mathrm{D},$$

and the other part of $\nabla\mathrm{V}$ is at once found to be

$$(4bx+3c)\mathrm{B}+(4bx^2+12cx+6d)\mathrm{C}+(4bx^3+12cx^2+12dx+3e)\mathrm{D};$$

and, adding, the coefficient of D vanishes on account of the quartic equation, and we have

$$\nabla\mathrm{V}=(ax^2+4bx+3c)\mathrm{B}+2(ax^3+4bx^2+6cx+3d)\mathrm{C},$$

which is precisely equal to

$$(2\mathrm{C}\partial_{\mathrm{D}}+\mathrm{B}\partial_{\mathrm{C}})\mathrm{V},$$

so that V is reduced to zero by the operator

$$\nabla-(2\mathrm{C}\partial_{\mathrm{D}}+\mathrm{B}\partial_{\mathrm{C}}),$$

which completes the demonstration; and the demonstration in the general case is precisely similar.

In the case of the cubic equation we have

$$(a, b, c, d \between x, 1)^3=0,$$
$$y=(ax+b)\mathrm{B}+(ax^2+3bx+2c)\mathrm{C};$$

and writing the second equation in the form

$$(y-b\mathrm{B}-2c\mathrm{C})+x(\ -a\mathrm{B}-3b\mathrm{C})+x^2(\qquad -a\mathrm{C})=0,$$

multiplying by x and reducing by the cubic equation, we have

$$d\mathrm{C}\qquad +x(y-b\mathrm{B}+\ c\mathrm{C})+x^2(\ -\ a\mathrm{B})=0,$$

and repeating the process,

$$d\mathrm{B}\qquad +x(\ 3c\mathrm{B}+\ d\mathrm{C})+x^2(y+2b\mathrm{B}+c\mathrm{C})=0;$$

or, what is the same thing, we have the system of equations

$$\left(\begin{array}{ccc} y-b\mathrm{B}-2c\mathrm{C}, & -\ a\mathrm{B}-3b\mathrm{C}, & -a\mathrm{C} \\ d\mathrm{C}, & y-\ b\mathrm{B}+\ c\mathrm{C}, & -a\ \mathrm{B} \\ d\mathrm{B}\quad , & 3c\mathrm{B}+\ d\mathrm{C}, & y+2b\mathrm{B}+c\mathrm{C} \end{array}\between 1, x, x^2\right)=0,$$

and the resulting equation in y is of course that formed by equating to zero the determinant formed out of the matrix in this equation. The developed expression is

$$(1, 0, \mathfrak{C}, \mathfrak{D} \between y, 1)^3=0,$$

where

$\frac{1}{3}\mathfrak{C}=$

	B^2		BC		C^2
ac	+1	ad	+1	bd	+1
b^2	−1	bc	−1	c^2	−1
	±1		±1		±1

$\mathfrak{D}=$

	B^3		B^2C		BC^2		C^3
a^2d	+1	abd	+3	acd	−3	ad^2	−1
abc	−3	ac^2	−6	b^2d	+6	bcd	+3
b^3	+2	b^2c	+3	bc^2	−3	c^3	−2
	±3		±6		±6		±3

The sum of the coefficients in each column should here and elsewhere in the present memoir be equal to zero, and I have by way of verification annexed to each column the sums (± a number) of the positive and negative coefficients. The coefficients $\mathfrak{C}$, $\mathfrak{D}$, and therefore the function in y, are invariants of the two forms,

$$(a, b, c, d \between X, Y)^3, \quad (B, C \between Y, -X);$$

or in the present case, where there are only two coefficients B, C, the coefficients $\mathfrak{C}$, $\mathfrak{D}$, and therefore also the function in y, are covariants of the single form $(a, b, c, d \between B, C)^3$, considering therein (B, C) as the facients.

It may be remarked that in the present case, assuming the invariance of the function in y, we may obtain the transformed equation in a very simple manner by writing in the first instance C=0, this gives

$$(a, b, c, d \between x, 1)^3=0,$$
$$y=(ax+b)B,$$

and thence

$$\frac{1}{a}(a, b, c, d \between y-bB, aB)^3=0;$$

or developing,

$$y^3+3y(ac-b^2)B^2+(a^2d-3abc+2b^3)B^3=0,$$

where the expressions for the coefficients are to be completed by the consideration that these coefficients are covariants of the form $(a, b, c, d \between B, C)^3$. But it is only in the case in hand of a cubic equation that the transformed equation can be obtained in this manner.

In the case of a quartic equation, we have

$$(a, b, c, d, e \between x, 1)^4=0,$$
$$y=(ax+b)B+(ax^2+4bx+3c)C+(ax^3+4bx^2+6cx+3d)D,$$

and these give the system of equations

$$\left(\begin{array}{cccc} y-b\mathrm{B}-3c\mathrm{C}-3d\mathrm{D}, & -a\mathrm{B}-4b\mathrm{C}-6c\mathrm{D}, & -a\mathrm{C}-4b\mathrm{D}, & -a\mathrm{D} \\ e\mathrm{D}, & y-b\mathrm{B}-3c\mathrm{C}+d\mathrm{D}, & -a\mathrm{B}-4b\mathrm{C}, & -a\mathrm{C} \\ e\mathrm{C}, & 4d\mathrm{C}+e\mathrm{D}, & y-b\mathrm{B}+3c\mathrm{C}+d\mathrm{D}, & -a\mathrm{B} \\ e\mathrm{B}, & 4d\mathrm{B}+e\mathrm{C}, & 6c\mathrm{B}+4d\mathrm{C}+e\mathrm{D}, & y+3b\mathrm{B}+3c\mathrm{C}+d\mathrm{D} \end{array}\right)(1, x, x^2, x^3)=0,$$

and the transformed equation is therefore found by equating to zero the determinant formed out of the matrix contained in this equation.

The developed result, which was obtained by a different process in the 'Deuxième Note' above referred to, is

$$(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}\between y, 1)^4=0,$$

where

$\frac{1}{2}\mathfrak{C}=$

	B^2		BC		BD^2	C^2		CD		D^2
ac	+3	ad	+6	ae	+2	+1	be	+6	ce	+3
b^2	−3	bc	−6	bd	−2	+8	cd	−6	d^2	−3
	.			c^2	.	−9				
	±3		±6		±2	±9		±6		±3

$\frac{1}{4}\mathfrak{D}=$

	B^3		B^2C		B^2D	BC^2		BCD	C^3		BD^2	C^2D		CD^2		D^3
							ac^2									
a^2d	+1	a^2e	+1	abe	+1	+ 4	ad^2	−6	−4	ade	−1	− 4	ae^2	−1	be^2	−1
abc	−3	abd	+2	acd	−3	−12	b^2e	+6	+4	bcd	+3	+12	bde	−2	cde	+3
b^3	+2	ac^2	−9	b^2d	+2	+ 8	bcd			bd^2	−2	− 8	c^2e	+9	d^3	−2
		b^2c	+6	bc^2			c^3			c^2d			cd^2	−6		
	±3		±9		±3	±12		±6	±4		±3	±12		±9		±3

and

$\mathfrak{E}=$

	B^4		B^3C		B^3D	B^2C^2		B^2CD	BC^3		B^2D^2	BC^2D	C^4
a^3e	+1	a^2be	+ 8	a^2ce	+12	− 6	a^2de	∞	− 4	a^2e^2	+ 2	− 4	+ 1
a^2bd	−4	a^2cd	−12	a^2d^2	−12	∞	$abce$	+60	− 12	$abde$	−16	+ 20	− 16
a^2c^2	∞	ab^2d	−20	ab^2e	− 8	+30	abd^2	−72	+ 16	ac^2e	+36	+ 36	− 18
ab^2c	+6	abc^2	+36	$abcd$	+12	−48	ac^2d	+36	+ 36	acd^2	−18	∞	+ 48
b^4	−3	b^3c	−12	ac^3	∞	+54	b^3e	−36	+ 48	b^2ce	−18	∞	+ 48
				b^3d	− 4	−48	b^2cd	+12	−192	b^2d^2	+14	−160	∞
				b^2c^2	.	+18	bc^3	.	+108	bc^2d	.	+108	−144
										c^4	.	.	+ 81
	±7		±44		±24	±102		±108	±208		±52	±164	±178

	BCD^2	C^3D		BD^3	C^2D^2		CD^3		D^4
abe^2	∞	− 4	ace^2	+12	− 6	ade^2	+ 8	ae^3	+1
$acde$	+60	− 12	ad^2e	− 8	+30	bce^2	−12	bde^2	−4
ad^3	−36	+ 48	b^2e^2	−12	∞	bd^2e	−20	c^2e^2	∞
b^2de	−72	+ 16	$bcde$	+12	−48	c^2de	+36	cd^2e	+6
bc^2e	+36	+ 36	bd^3	− 4	−48	cd^3	−12	d^4	−3
bcd^2	+12	−192	c^3e	.	+54				
c^3d	.	+108	c^2d^2	.	+18				
	±108	±208		±24	±102		±44		±7

I write

$$U'=aB^2+4bBC+c(2BD+4C^2)+4dCD+eD^2,$$

$$H'=(ac-b^2)B^2+2(ad-bc)BC+(ae-2bd+c^2)BD+4(bd-c^2)C^2$$
$$+2(be-cd)CD+(ce-d^2)D^2;$$

and I represent by Φ' the expression which has just been found for $\frac{1}{4}\mathfrak{D}$. These functions, U', H', Φ', are invariants of the two forms

$$(a, b, c, d, e \between X, Y)^4, \quad (B, C, D \between Y, -X)^2;$$

we have, moreover, the invariants

$$ae-4bd+3c^2, \; ace-ad^2-b^2e+2bcd-c^3,$$

which I represent as usual by I, J, and the invariant $BD-C^2$, which I represent by Θ'. This being so, we have

$$\mathfrak{C}=6H'-2I\Theta',$$
$$\mathfrak{D}=4\Phi',$$
$$\mathfrak{E}=IU'^2-3H'^2+I^2\Theta'^2+12J\Theta'U'+2I\Theta'H',$$

the last of which may be verified as follows:—viz. writing $a=e=1$, $b=d=0$, $c=\theta$, it becomes

$$\begin{aligned}
&(1+3\theta^2)\{B^2+\theta(2BD+4C^2)+D^2\}^2\\
&-3\{\theta B^2+(1+\theta^2)BD-4\theta^2C^2+\theta D^2\}^2\\
&+(1+3\theta^2)^2(BD-C^2)^2\\
&+12(\theta-\theta^3)(BD-C^2)\{\theta B^2+(1+\theta^2)BD-4\theta^2C^2+\theta D^2\}\\
= \;& B^4\\
&+B^3C \;\; (12\theta)\\
&+B^2C^2 \;\; (-6\theta+54\theta^3)\\
&+B^2D^2 \;\; (2+36\theta^2)\\
&+BC^2D \;\; (-4+36\theta^2)\\
&+B^4 \;\; (1-18\theta^2+81\theta^4)\\
&+C^2D^2 \;\; (-6\theta+54\theta^3)\\
&+C^3D \;\; (12\theta)\\
&+D^4,
\end{aligned}$$

which is an identical equation.

The expression for the invariant I (quadrinvariant) of the function $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \between y, 1)^4$ is $\mathfrak{E}+3(\frac{1}{6}\mathfrak{C})^2$, or $\mathfrak{E}+3(H'-\frac{1}{3}I\Theta')^2$, viz. it is

$$\begin{aligned}
&IU'^2-3H'^2+I^2\Theta'^2+12J\Theta'U'+2I\Theta'H'\\
&\qquad\;\, +3H'^2+\tfrac{1}{3}I^2\Theta'^2 \qquad\qquad\quad -2I\Theta'H',
\end{aligned}$$

or, finally, it is

$$IU'^2+\tfrac{4}{3}I^2\Theta'^2+12J\Theta'U',$$

which is equal to

$$\frac{1}{\mathrm{I}}\left[(\mathrm{IU}'+6\mathrm{J}'\Theta')^2+\tfrac{4}{3}(\mathrm{I}^3-27\mathrm{J}^2)\Theta'^2\right];$$

so that the condition in order that this invariant may be equal to zero is

$$\mathrm{IU}'+\left[6\mathrm{J}\pm2\sqrt{-\tfrac{1}{3}(\mathrm{I}^3-27\mathrm{J}^2)}\right]\Theta'=0,$$

which agrees with a result of M. HERMITE's.

There should, I think, be an identical equation of the form

$$\mathrm{JU}'^3-\mathrm{IU}'^2\mathrm{H}'+4\mathrm{H}'^3+\mathrm{M}\Theta'=-\Phi'^2,$$

which would serve to express the square of the invariant Φ' in terms of the other invariants U', H', Θ', I, J; but assuming that such an equation exists, the form of the factor M remains to be ascertained. The invariant J (cubinvariant) of the form $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \between y, 1)^4$ contains Φ'^2, and it would be necessary to make use of the identical equation just referred to in order to reduce it to its simplest form; and (this being so) I have not sought for the expression of the cubinvariant of $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \between y, 1)^4$.

For the quintic we have the equations

$$\begin{aligned}
&(a, b, c, d, e, f \between x, 1)^5=0,\\
y=\ &(ax+b)\mathrm{B}\\
&+(ax^2+5bx+4c)\mathrm{C}\\
&+(ax^3+5bx^2+10cx+6d)\mathrm{D}\\
&+(ax^4+5bx^3+10cx^2+10dx+4e)\mathrm{E},
\end{aligned}$$

and this leads to the system of equations

$$\left(\begin{array}{ccccc}
y-b\mathrm{B}-4c\mathrm{C}-6d\mathrm{D}-4e\mathrm{E}, & -a\mathrm{B}-5b\mathrm{C}-10c\mathrm{D}-10d\mathrm{E}, & -a\mathrm{C}-5b\mathrm{D}-10c\mathrm{E}, & -a\mathrm{D}-5b\mathrm{E}, & -a\mathrm{E}\\
f\mathrm{E}, & y-b\mathrm{B}-4c\mathrm{C}-6c\mathrm{D}+e\mathrm{E}, & -a\mathrm{B}-5b\mathrm{C}-10c\mathrm{D}, & -a\mathrm{C}-5b\mathrm{D}, & -a\mathrm{D}\\
f\mathrm{D}, & 5e\mathrm{D}+f\mathrm{E}, & y-b\mathrm{B}-4c\mathrm{C}+4d\mathrm{D}+e\mathrm{E}, & -a\mathrm{B}-5b\mathrm{C}, & -a\mathrm{C}\\
f\mathrm{C}, & 5e\mathrm{C}+f\mathrm{D}, & 10d\mathrm{C}+5e\mathrm{D}+f\mathrm{E}, & y-b\mathrm{B}+6c\mathrm{C}+4d\mathrm{D}+e\mathrm{E}, & -a\mathrm{B}\\
f\mathrm{B}, & 5e\mathrm{B}+f\mathrm{C}, & 10d\mathrm{B}+5e\mathrm{C}+f\mathrm{D}, & 10c\mathrm{B}+10d\mathrm{C}+5e\mathrm{D}+f\mathrm{E}, & y+4b\mathrm{B}+6c\mathrm{C}+4d\mathrm{D}+e\mathrm{E}
\end{array}\between 1, x, x^2, x^3, x^4\right)=0,$$

and the transformed equation is obtained by equating to zero the determinant formed out of the matrix contained in this equation.

The determinant in question was calculated by the formula

	Div.
$\square = -12.345$	1
$+13.245$	2
-14.235	3
$+15.234$	4
-23.145	5
$+24.135$	6
-25.134	7
-34.125	8
$+35.124$	9
-45.123	10,

where the duadic symbols refer to the first and fifth columns, viz. 12 is the determinant formed out of the lines 1 and 2 of these columns, and so for the other like symbols; and the triadic symbols refer to the second, third, and fourth columns, viz. 345 is the determinant formed out of the lines 3, 4, 5 of these columns, and so for the other like symbols.

The ten divisions were separately calculated. It is to be noticed that these divisions other than 4 and 6 correspond to each other in pairs, while each of the divisions 4 and 6 corresponds to itself, as thus:

Div. 1,	-10
2,	$-\ 9$
3,	$-\ 7$
5,	$-\ 8$
4,	$-\ 4$
6,	$-\ 6,$

viz. if in the place of

$$y;\ a, b, c, d, e, f;\ \mathrm{B, C, D, E},$$

we write

$$-y;\ f, e, d, c, b, a;\ \mathrm{E, D, C, B},$$

then division 1 becomes division 10 with its sign reversed, and so for divisions 2 and 9, 3 and 7, 5 and 8; while each of the divisions 4 and 6 is unaltered, except that the sign is reversed. But the corresponding divisions were each of them calculated, and the property in question was used as a verification. Another very convenient verification, which was employed for the several divisions, was obtained by putting

$$a=b=c=d=e=f=\mathrm{B}=\mathrm{C}=\mathrm{D}=\mathrm{E}=1,$$

upon which supposition the determinant becomes

$$\begin{vmatrix} y-15, & -26, & -16, & -6, & -1 \\ 1, & y-10, & -16, & -6, & -1 \\ 1, & 6, & y, & -6, & -1 \\ 1, & 6, & 16, & y+10, & -1 \\ 1, & 6, & 16, & 16, & y+15 \end{vmatrix}$$

and the values of the ten divisions respectively are

y^5,	y^4,	y^3,	y^2,	y,	1	
		6,	−288,	+ 4608,	−24576	1
		16,	−576,	+ 6144,	−16384	2
		26,	−544,	+ 3584,	−24576	3
1,	0,	−96,	0,	−28672,	0	4
				0,	0	5
				0,	0	6
		26,	+544,	+ 3584,	+24576	7
				0,	0	8
		16,	+576,	+ 6144,	+16384	9
		6,	+288,	+ 4608,	+24576	10
1,	0,	0,	0,	0,	0	

A verification similar to this was in fact employed at each step of the calculation of a division: viz. in forming a product such as $(\lambda X+\mu Y+\&c.)(\lambda' X+\mu' Y+\&c.)$, where λ, μ, &c., λ', μ', .. &c. are numerical coefficients, and X, Y, &c. are monomial products of a, b, c, d, e, f and B, C, D, E, the sum of the numerical coefficients of the product is $(\lambda+\mu+\&c.)(\lambda'+\mu'+\&c.)$.

It was of course necessary to employ such verifications, as a test of the correctness of the several divisions, before proceeding to collect them together, but the collection itself affords an exceedingly good ultimate verification. The following is an exemplification: the terms in y which involve the product BCDE are obtained by the collection of the corresponding terms in the ten divisions, as follows:

				1	2	3	4	5	6	7	8	9	10	
y	BCDE.	− 5	a^2f^2	+ 1	− 2	− 2	+ 1	+ 1	− 2	− 2	+ 1	− 2	+ 1	
		+ 30	$abef$	+ 49	− 16	− 20	+ 4	− 25	+ 50	− 20	− 25	− 16	+ 49	
		+ 980	$acdf$	+ 80	+ 200	+ 184	− 148	+ 100		+ 184	+ 100	+ 200	+ 80	
		− 280	ace^2	+ 80	.	+ 80	− 440							
		− 180	ad^2e	.	− 60	− 60	− 60							
		− 280	b^2df	.	.		− 440	.		+ 80	.	.	+ 80	
		− 825	b^2e^2	.	.		− 825							
		− 180	bc^2f	.	.		− 60	.		− 60	.	− 60		
		+ 740	$bcde$	.	.		+ 740							
			bd^3											
			c^3e											
			c^2d^2											
	±1750=0			+ 210	+ 122	+ 182	− 1228	+ 76	+ 48	+ 182	+ 76	+ 122	+ 210	= ±1228

where it may be remarked that the greater part, but not all, of the component coefficients are divisible by 5. I soon observed in the process of summing the ten divisions that all the resulting coefficients should be divisible by 5 (the only exception is as to the terms in y^0 which contain B^5, C^5, D^5, and E^5 respectively), and the circumstance that they are so in each particular instance is as far as it goes a verification, which, however, only applies to those of the component coefficients which are not themselves divisible by 5. But it was known *à priori* (I will presently show how this is so) that the sum of the resulting coefficients should be equal to zero, and that they are so in fact is a verification as to *all* the coefficients. The foregoing specimen term BCDE is one which remains unaltered when B, C, D, E are changed into E, D, C, B; and on making the further change a, b, c, d, e, f into f, e, d, c, b, a, the coefficient of BCDE remains, as it should do, unaltered; this is a verification of the coefficients of the terms ace^2, b^2df; ad^2e, bc^2f, which are respectively interchanged by the substitution in question, but not of the other terms a^2f^2, $abef$, $acdf$, b^2e^2, $bcdf$, which are respectively unaltered by the substitution. I did *not* employ what would have been another convenient verification of the several divisions, viz. the comparison of their values on putting therein $a=b=c=d=e=f=1$, with the corresponding values as calculated independently from the determinant

$$\begin{vmatrix} y-B-4C-6D-E, & -B-5C-10D-10E, & -C-5D-10E, & D-5E, & -E \\ E, & y-B-4C-6D+E, & -B-5C-10D, & -C-5D, & -D \\ D, & 5D+E, & y-B-4C+4D+E, & -B-5C, & -C \\ C, & 5C+D, & 10C+5D+E, & y-B+6C+4D+E, & -B \\ B, & 5B+C, & 10B+5C+D, & 10B+10C+5D+E, & y+4B+6C+4D+E \end{vmatrix}$$

The calculation of the ten divisions of this determinant would however itself have been a work of some labour.

The last-mentioned determinant is $=y^5$; in fact, equating it to zero, we have the transformed equation corresponding to the system of equations

$$(1,1,1,1,1,1\between x,1)^5=0,$$

$$y=(x+1)B+(x^2+5x+4)C+(x^3+5x^2+10x+6)D+(x^4+5x^3+10x^2+10x+4)E.$$

But the first of these equations is $(x+1)^5=0$, and the second is

$$y=(x+1)\{B+(x+4)C+(x^2+4x+6)D+(x^3+4x^2+6x+4)E\},$$

so that for each of the five equal roots $x=-1$, we have $y=0$, or the transformed equation in y is $y^5=0$.

And since upon writing $a=b=c=d=e=f=1$ the transformed equation becomes $y^5\doteq 0$, it is clear that in the coefficient of any monomial product of B, C, D, E, the sum of the numerical coefficients of the several monomial products of a, b, c, d, e, f must be $=0$, which is the property above referred to as affording a verification of the calculated expression of the transformed equation.

The final result is that the equations

$$(a, b, c, d, e, f \between x, 1)^5=0,$$

$$\begin{aligned} y= &\ (ax+\ b)\mathrm{B} \\ &+(ax^2+5bx+\ 4c)\mathrm{C} \\ &+(ax^3+5bx^2+10cx+\ 6d)\mathrm{D} \\ &+(ax^4+5bx^3+10cx^2+10dx+4e)\mathrm{E} \end{aligned}$$

give for the transformed equation in y

$$(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F} \between y, 1)^5=0,$$

where

$\frac{1}{5}\mathfrak{C}=$

	B²		BC		BD	C²		BE	CD		CE	D²		DE		E²
ac	+2	ad	+6	ae	+4	+2	af	+1	+1	bf	+4	+2	cf	+6	df	+2
b^2	−2	bc	−6	bd	−4	+10	be	−1	+15	ce	−4	+10	de	−6	e^2	−2
				c^2	.	−12	cd	.	−16	d^2	.	−12				
	±2		±6		±4	±12		±1	±16		±4	±12		±6		±2

$\frac{1}{5}\mathfrak{D}=$

	B³		B²C		B²D	BC²		B²E	BCD	C³		BCE	BD²	C²D
a^2d	+2	a^2e	+ 4	a^2f	+1	+ 1	abf	+1	+10	+ 3	acf	∽	+ 6	+ 4
abc	−6	abd	+ 2	abe	+3	+19	ace	−4	− 8	− 4	ade	−12	−22	− 46
b^3	+4	ac^2	−24	acd	∽	−52	ad^2	∽	−48	−20	b^2f	+ 8	+ 4	+ 20
		b^2c	+18	b^2d	−4	+20	b^2e	+3	+30	+25	bce	+ 4	+20	+ 70
				bc^2	.	+12	bcd	.	+16	−20	bd^2	.	− 8	− 80
							c^3	.	.	+16	c^2d			+ 32
	±6		±24		±4	±52		±4	±56	±44		±12	±30	±126

	BDE	C²E	CD²		BE²	CDE	D³		CE²	D²E		DE²		E³
adf	∽	− 6	− 4	aef	−1	−10	− 3	af^2	−1	− 1	bf^2	− 4	cf^2	−2
ae^2	− 8	− 4	− 20	bdf	+4	+ 8	+ 4	bef	−3	−19	cef	− 2	def	+6
bcf	+12	+22	+ 46	be^2	−3	−30	−25	cdf	∽	+52	d^2f	+24	e^3	−4
bde	− 4	−20	− 70	c^2f		+48	+20	ce^2	+4	−20	de^2	−18		
c^2e		+ 8	+ 80	cde		−16	+20	d^2e		−12				
cd^2			− 32	d^3			−16							
	±12	±30	±126		±4	±56	±44		±4	±52		±24		±6

$\frac{1}{5}\mathfrak{C}=$

	B^4
a^3e	+1
a^2bd	−4
a^2c^2	ꝏ
ab^2c	+6
b^4	−3
	±7

	B^3C
a^3f	+ 1
a^2be	+ 7
a^2cd	−16
ab^2d	−22
abc^2	+48
b^3c	−18
	±56

	B^3D	B^2C^2
a^2bf	+ 2	+ 9
a^2ce	+20	− 22
a^2d^2	−24	ꝏ
ab^2e	−18	+ 31
$abcd$	+32	− 56
ac^3	ꝏ	+ 96
b^3d	−12	− 70
b^2c^2		+ 12
	±54	±148

	B^3E	B^2CD	BC^3
a^2cf	+ 6	+ 10	− 2
a^2de	− 6	− 18	− 10
ab^2f	− 5	+ 17	+ 29
$abce$	+ 8	+ 96	− 92
abd^2	ꝏ	−144	+ 40
ac^2d	ꝏ	+128	+128
b^3e	− 3	−105	+ 75
b^2cd	.	+ 16	−360
bc^3	.	.	+192
	±14	±267	±464

	B^2CE	B^2D^2	BC^2D	C^4
a^2df	+12	− 6	ꝏ	− 4
a^2e^2	−12	+ 10	− 20	+ 5
$abcf$	+28	+ 38	+ 48	+ 6
$abde$	−36	− 86	− 8	− 50
ac^2e	+32	+100	+ 72	− 64
acd^2	ꝏ	− 24	+144	+160
b^3f	−28	− 14	+ 60	+ 25
b^2ce	+ 4	− 70	−240	+ 50
b^2d^2	.	+ 52	−440	ꝏ
bc^2d	.	.	+384	−320
c^4	.	.	.	+192
	±76	±200	±708	±438

	B^2DE	BC^2E	BCD^2	C^3D
a^2ef	+ 5	− 5	− 7	+ 1
$abdf$	− 8	+ 92	− 8	− 36
abe^2	−29	− 67	+ 15	− 25
ac^2f	+60	+ 12	+ 72	− 20
$acde$	−12	+ 36	+196	+ 28
ad^3	ꝏ	ꝏ	− 72	+240
b^2cf	−42	− 54	+ 78	+190
b^2de	+26	−110	−510	− 50
bc^2e	.	+ 96	− 60	−280
bcd^2	.	.	+296	−560
c^3d	.	.	.	+512
	±91	±236	±657	±971

	B^2E^2	$BCDE$	BD^3	C^3E	C^2D^2
a^2f^2	+ 1	− 1	− 1	− 1	+ 1
$abef$	− 5	+ 6	− 4	− 4	− 25
$acdf$	+20	+196	+ 28	+ 28	− 80
ace^2	−14	− 56	+100	− 28	+ 10
ad^2e	ꝏ	− 36	− 68	+ 60	+ 336
b^2df	−14	− 56	− 28	+100	+ 10
b^2e^2	+12	−165	− 75	− 75	ꝏ
bc^2f	.	− 36	+ 60	− 68	+ 336
$bcde$	.	+148	−140	−140	− 940
bd^3	.	.	+128	.	− 120
c^3e	.	.	.	+128	− 120
c^2d^2	.	.	.	.	+ 592
	±33	±350	±316	±316	±1285

	BCE^2	BD^2E	C^2DE	CD^3
abf^2	+ 5	− 5	− 7	+ 1
$acef$	− 8	+ 92	− 8	− 36
ad^2f	+60	+ 12	+ 72	− 20
ade^2	−42	− 54	+ 78	+190
b^2ef	−29	− 67	+ 15	− 25
$bcdf$	−12	+ 36	+196	+ 28
bce^2	+26	−110	−510	− 50
bd^2e	.	+ 96	− 60	−280
c^3f	.	.	− 72	+240
c^2de	.	.	+296	−560
cd^3	.	.	.	+512
	±91	±236	±657	±971

	BDE^2	C^2E^2	CD^2E	D^4
acf^2	+12	− 6	ꝏ	− 4
$adef$	+28	+ 38	+ 48	+ 6
ae^3	−28	− 14	+ 60	+ 25
b^2f^2	−12	+ 10	− 20	+ 5
$bcef$	−36	− 86	− 8	− 50
bd^2f	+32	+100	+ 72	− 64
bde^2	+ 4	− 70	−240	+ 50
c^2df	.	−. 24	+144	+160
c^2e^2	.	+ 52	−440	ꝏ
cd^2e	.	.	+384	−320
d^4	.	.	.	+192
	±76	±200	±708	±438

	BE^3	CDE^2	D^3E
adf^2	+ 6	+ 10	− 2
ae^2f	− 5	+ 17	+ 29
bcf^2	− 6	− 18	− 10
$bdef$	+ 8	+ 96	− 92
be^3	− 3	−105	+ 75
c^2ef	.	−144	+ 40
cd^2f	.	+128	+128
cde^2	.	+ 16	−360
d^3e	.	.	+192
	±14	±267	±464

	CE^3	D^2E^2
aef^2	+ 2	+ 9
bdf^2	+20	− 22
be^2f	−18	+ 31
c^2f^2	−24	ꝏ
$cdef$	+32	− 56
ce^3	−12	− 70
d^3f	.	+ 96
d^2e^2	.	+ 12
	±54	±148

	DE^3
af^3	+ 1
bef^2	+ 7
cdf^2	−16
ce^2f	−22
d^2ef	+48
de^3	−18
	±56

	E^4
bf^3	+1
cef^2	−4
d^2f^2	ꝏ
de^2f	+6
e^4	−3
	±7

$\mathfrak{F}=$

	B⁵
a^4f	+ 1
a^3be	− 5
a^3cd	∞
a^2b^2d	+10
a^2bc^2	∞
ab^3c	−10
b^5	+ 4
	±15

	B⁴C
a^3bf	+ 15
a^3ce	− 20
a^3d^2	∞
a^2b^2e	− 55
a^2bcd	+ 80
a^2c^3	∞
ab^3d	+ 70
ab^2c^2	−120
b^4c	+ 30
	±195

	B⁴D	B³C²
a^3cf	+ 30	− 10
a^3de	− 30	∞
a^2b^2f	− 15	+100
a^2bce	−100	−190
a^2bd^2	+120	∞
a^2c^2d	∞	+160
ab^3e	+ 55	−260
ab^2cd	− 80	+540
abc^3	∞	−480
b^4d	+ 20	+200
b^3c^2	.	− 60
	±225	±1000

	B⁴E	B³CD	B²C³
a^3df	+20	∞	− 10
a^3e^2	−20	∞	∞
a^2bcf	−30	+320	− 60
a^2bde	+30	−360	+ 50
a^2c^2e	∞	−400	+ 120
a^2cd^2	∞	+480	∞
ab^3f	+15	−140	+ 340
ab^2ce	−20	−440	−1020
ab^2d^2	∞	+960	− 100
abc^2d	∞	−640	+ 960
ac^4	∞	∞	− 640
b^4e	+ 5	+300	− 500
b^3cd	.	− 80	+1900
b^2c^3	.	.	−1040
	±70	±2060	±3370

	B³CE	B³D²	B²C²D	BC⁴
a^3ef	∞	+ 10	− 20	+ 5
a^2bdf	+240	− 40	+ 20	− 80
a^2be^2	−240	− 50	+ 100	− 25
a^2c^2f	−120	+300	+ 20	− 40
a^2cde	+120	−600	+ 60	+ 200
a^2d^3	∞	+360	∞	∞
ab^2cf	−140	−220	+1190	+ 20
ab^2de	+240	+540	−1980	+ 250
abc^2e	−160	−500	−3160	+ 320
$abcd^2$	∞	+120	+4080	− 800
ac^3d	∞	∞	−1280	− 320
b^4f	+ 80	+ 40	− 400	+ 500
b^3ce	− 20	+200	+ 850	−2750
b^3d^2	.	−160	+2600	∞
b^2c^2d	.	.	−2080	+5600
bc^4	.	.	.	−2880
	±680	±1570	±8920	±6895

	B³DE	B²C²E	B²CD²	BC³D	C⁵
a^3f^2	+ 5	− 5	− 5	+ 5	− 1
a^2bef	− 30	+ 30	+ 60	− 130	+ 25
a^2cdf	+220	− 40	+ 380	− 380	+ 80
a^2ce^2	−400	+ 40	− 200	+ 400	− 100
a^2d^2e	+180	∞	− 180	+ 300	∞
ab^2df	− 40	+1070	− 640	+ 380	− 250
ab^2e^2	+185	−1145	+ 25	+ 125	∞
abc^2f	−300	− 940	+1640	+ 940	− 360
$abcde$	+ 60	+1020	−4380	− 1140	+1000
abd^3	∞	∞	+3960	− 1200	∞
ac^3e	∞	− 320	−2000	− 2080	+ 960
ac^2d^2	∞	∞	+ 480	+ 960	−1600
b^3cf	+120	+ 160	− 620	+ 800	+ 750
b^3de	− 80	+ 650	+2900	− 3500	∞
b^2c^2e	.	− 520	− 100	− 2600	− 300
b^2cd^2	.	.	−1320	+14800	∞
bc^3d	.	.	.	− 7680	+4800
c^5	.	.	.	.	−2304
	±810	±2970	±9445	±18710	±7615

	B³E²	B²CDE	B²D³	BC³E	BC²D	C⁴D
a^2bf^2	+ 5	+ 15	− 10	− 25	∞	+ 5
a^2cef	− 70	− 60	+ 200	+ 80	− 210	+ 30
a^2d^2f	+100	+ 240	+ 100	− 200	− 180	+ 120
a^2de^2	− 40	− 240	− 300	+ 200	+ 600	− 150
ab^2ef	+ 30	− 30	− 90	+ 70	∞	− 125
$abcdf$	−100	+1380	− 640	+ 260	+1920	− 1180
$abce^2$	+ 70	−2820	− 500	− 360	+ 450	+ 500
abd^2e	∞	+1980	+1540	− 300	−3180	+ 1500
ac^3f	∞	−1200	+1000	− 560	+1080	− 240
ac^2de	∞	+ 240	−3000	+ 240	−8160	+ 3120
acd^3	∞	∞	+1440	∞	+5040	− 4800
b^3df	+ 40	+ 240	+ 120	+2000	−1800	+ 500
b^3e^2	− 35	+ 975	+ 500	−2125	∞	∞
b^2c^2f	.	− 60	− 400	−1060	+1620	+ 2100
b^2cde	.	− 660	+ 600	+3700	+2700	− 6500
b^2d^3	.	.	− 560	∞	+9600	∞
bc^3e	.	.	.	−1920	−5400	− 4800
bc^2d^2	.	.	.	.	−4080	+17600
c^4d	.	.	.	.	.	− 7680
	±245	±5070	±5500	±6550	±23010	±25475

	B²CE²	B²D²E	BC²DE	BCD³	C⁴E	C³D²
a^2cf^2	− 50	+ 70	+ 10	− 50	− 10	+ 20
a^2def	+ 110	+ 80	− 340	+ 160	+ 130	− 30
a^2e^3	− 80	− 200	+ 400	∞	− 100	∞
ab^2f^2	+ 80	− 70	− 40	+ 20	− 25	∞
$abcef$	− 520	− 460	+ 620	+ 380	− 20	− 500
abd^2f	+ 700	+ 460	+ 1240	− 760	−1000	+ 540
$abde^2$	− 190	+ 620	− 2140	− 200	+1000	+ 750
ac^2df	− 400	+ 200	− 1720	+ 2840	− 80	− 880
ac^2e^2	+ 280	−2000	− 2240	− 2000	+ 880	+ 1400
acd^2e	∞	+1080	+ 2520	− 6440	−1200	− 120
ad^4	∞	∞	∞	+ 4320	∞	− 3600
b^3ef	+ 160	+ 430	− 200	− 500	+ 125	∞
b^2cdf	− 20	− 440	+ 5320	− 2240	+3000	− 700
b^2ce^2	− 70	+ 650	− 2950	+ 2750	−3500	∞
b^2d^2e	.	− 420	+ 4800	+ 7400	∞	− 3000
bc^3f	.	.	− 3240	+ 1800	−1680	+ 3960
bc^2de	.	.	− 2040	−10200	+4400	− 5600
bcd^3	.	.	.	+ 2720	∞	+20400
c^4e	.	.	.	.	−1920	− 7200
c^3d^2	.	.	.	.	.	− 5440
	±1330	±3590	±14910	±22390	±9535	±27070

	B^2DE^2	BC^2E^2	BCD^2E	BD^4	C^3DE	C^2D^3
a^2df^2	+ 50	− 70	− 10	+ 10	+ 50	− 20
a^2e^2f	− 80	+ 70	+ 40	+ 25	− 20	∞
$abcf^2$	− 110	− 80	+ 340	− 130	− 160	+ 30
$abdef$	+ 520	+ 460	− 620	+ 20	− 380	+ 500
abe^3	− 160	− 430	+ 200	− 125	+ 500	∞
ac^2ef	− 700	− 460	− 1240	+ 1000	+ 760	− 540
acd^2f	+ 400	− 200	+ 1720	+ 80	− 2840	+ 880
$acde^2$	+ 20	+ 440	− 5320	− 3000	+ 2240	+ 700
ad^3e	∞	∞	+ 3240	+ 1680	− 1800	− 3960
b^3f^2	+ 80	+ 200	− 400	+ 100	∞	∞
b^2cef	+ 190	− 620	+ 2140	− 1000	+ 200	− 750
b^2d^2f	− 280	+ 2000	+ 2240	− 880	+ 2000	− 1400
b^2de^2	+ 70	− 650	+ 2950	+ 3500	− 2750	∞
bc^2df	.	− 1080	− 2520	+ 1200	+ 6440	+ 120
bc^2e^2	.	+ 420	− 4800	∞	− 7400	+ 3000
bcd^2e	.	.	+ 2040	− 4400	+ 10200	+ 5600
bd^4	.	.	.	+ 1920	∞	+ 7200
c^4f	.	.	.	.	− 4320	+ 3600
c^3de	.	.	.	.	− 2720	− 20400
c^2d^3	.	.	.	.	.	+ 5440
	± 1330	± 3590	± 14910	± 9535	± 22390	± 27070

	B^2E^3	$BCDE^2$	BD^3E	C^3E^2	C^2D^2E	CD^4
a^2ef^2	− 5	− 15	+ 25	+ 10	∞	− 5
$abdf^2$	+ 70	+ 60	− 80	− 200	+ 210	− 30
abe^2f	− 30	+ 30	− 70	+ 90	∞	+ 125
ac^2f^2	− 100	− 240	+ 200	− 100	+ 180	− 120
$acdef$	+ 100	− 1380	− 260	+ 640	− 1920	+ 1180
ace^3	− 40	− 240	− 2000	− 120	+ 1800	− 500
ad^3f	∞	+ 1200	+ 560	− 1000	− 1080	+ 240
ad^2e^2	∞	+ 60	+ 1060	+ 400	− 1620	− 2100
b^2cf^2	+ 40	+ 240	− 200	+ 300	− 600	− 150
b^2def	− 70	+ 2820	+ 360	+ 500	− 450	− 500
b^2e^3	+ 35	− 975	+ 2125	− 500	∞	∞
bc^2ef	.	− 1980	+ 300	− 1540	+ 3180	+ 1500
bcd^2f	.	− 240	− 240	+ 3000	+ 8160	− 3120
$bcde^2$	.	+ 660	− 3700	− 600	− 2700	+ 6500
bd^3e	.	.	+ 1920	∞	+ 5400	+ 4800
c^3df	.	.	.	− 1440	− 5040	+ 4800
c^3e^2	.	.	.	+ 560	− 9600	∞
c^2d^2e	.	.	.	.	+ 4080	− 17600
cd^4	.	.	.	.	.	+ 7680
	± 245	± 5070	± 6550	± 5500	± 23010	± 25475

	BCE^3	BD^2E^2	CD^2E^2	CD^3E	D^5
a^2f^3	− 5	+ 5	+ 5	− 5	+ 1
$abef^2$	+ 30	− 30	− 60	+ 130	− 25
$acdf^2$	− 220	+ 40	− 380	+ 380	− 80
ace^2f	− 40	− 1070	+ 640	− 380	+ 250
ad^2ef	+ 300	+ 940	− 1640	− 940	+ 360
ade^3	− 120	− 160	+ 620	− 800	− 750
b^2df^2	+ 400	− 40	+ 200	− 400	+ 100
b^2e^2f	− 185	+ 1145	− 25	− 125	∞
bc^2f^2	− 180	∞	+ 180	− 300	∞
$bcdef$	− 60	− 1020	+ 4380	+ 1140	− 1000
bce^3	+ 80	− 650	− 2900	+ 3500	∞
bd^3f	.	+ 320	+ 2000	+ 2080	− 960
bd^2e^2	.	+ 520	+ 100	+ 2600	+ 3000
c^3ef	.	.	− 3960	+ 1200	∞
c^2d^2f	.	.	− 480	− 960	+ 1600
c^2de^2	.	.	+ 1320	− 14800	∞
cd^3e	.	.	.	+ 7680	− 4800
d^5	.	.	.	.	+ 2304
	± 810	± 2970	± 9445	± 18710	± 7615

	BDE^3	C^2E^3	CD^2E^2	D^4E
abf^3	∞	− 10	+ 20	− 5
$acef^2$	− 240	+ 40	− 20	+ 80
ad^2f^2	+ 120	− 300	− 20	+ 40
ade^2f	+ 140	+ 220	− 1190	− 20
ae^4	− 80	− 40	+ 400	− 500
b^2ef^2	+ 240	+ 50	− 100	+ 25
$bcdf^2$	− 120	+ 600	− 60	− 200
bce^2f	− 240	− 540	+ 1980	− 250
bd^2ef	+ 160	+ 500	+ 3160	− 320
bde^3	+ 20	− 200	− 850	+ 2750
c^3f^2	.	− 360	∞	∞
c^2def	.	− 120	− 4080	+ 800
c^2e^3	.	+ 160	− 2600	∞
cd^3f	.	.	+ 1280	+ 320
cd^2e^2	.	.	+ 2080	− 5600
d^4e	.	.	.	+ 2880
	± 680	± 1570	± 8920	± 6895

	BE^4	CDE^3	D^3E^2
acf^3	− 20	∞	+ 10
$adef^2$	+ 30	− 320	+ 60
ae^3f	− 15	+ 140	− 340
b^2f^3	+ 20	∞	∞
$bcef^2$	− 30	+ 360	− 50
bd^2f^2	∞	+ 400	− 120
bde^2f	+ 20	+ 440	+ 1020
be^4	− 5	− 300	+ 500
c^2df^2	.	− 480	∞
c^2e^2f	.	− 960	+ 100
cd^2ef	.	+ 640	− 960
cde^3	.	+ 80	− 1900
d^4f	.	.	+ 640
d^3e^2	.	.	+ 1040
	± 70	± 2060	± 3370

	CE^4	D^2E^3
adf^3	− 30	+ 10
ae^2f^2	+ 15	− 100
bcf^3	+ 30	∞
$bdef^2$	+ 100	+ 190
be^3f	− 55	+ 260
c^2ef^2	− 120	∞
cd^2f^2	∞	− 160
cde^2f	+ 80	− 540
ce^4	− 20	− 200
d^3ef	.	+ 480
d^2e^3	.	+ 60
	± 225	± 1000

	DE^4
aef^3	− 15
bdf^3	+ 20
be^2f^2	+ 55
c^2f^3	∞
$cdef^2$	− 80
ce^3f	− 70
d^3f^2	∞
d^2e^2f	+ 120
de^4	− 30
	± 195

	E^5
af^4	− 1
bef^3	+ 5
cdf^3	∞
ce^2f^2	− 10
d^2ef^2	∞
de^3f	+ 10
e^5	− 4
	± 15

Upon writing (B, C, D, E)$=(x^3, xy^2, xy^2, y^3)$, the foregoing values of $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ become covariants of the quintic $(a, b, c, d, e, f \between x, y)^5$. In fact we have

$$\tfrac{1}{5}\mathfrak{C}=2(\text{Tab. No. 15}),$$

$$\tfrac{1}{5}\mathfrak{D}=2(\text{Tab. No. 18}),$$

$$\tfrac{1}{5}\mathfrak{E}=(\text{Tab. No. 13})^2(\text{Tab. No. 14})-3(\text{Tab. No. 15})^2,$$

$$\mathfrak{F}=(\text{Tab. No. 13})^2(\text{Tab. No. 17})-2(\text{Tab. No. 15})(\text{Tab. No. 18}),$$

where the Tables referred to are those of my Fifth Memoir on Quantics, Tab. No. 13 being the quintic itself. This is a further verification of the foregoing result.

I will conclude by showing how the formula may be applied to the reduction of the general quintic equation to Mr. JERRARD's form $x^5+ax+b=0$. It was long ago remarked by Professor SYLVESTER that TSCHIRNHAUSEN's Transformation, in its original form, gave the means of effecting this reduction. In fact, if the transforming equation be

$$y=\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4,$$

then the equation in y is of the form

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F} \between y, 1)^5=0,$$

where $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ are, in regard to $\alpha, \beta, \gamma, \delta, \varepsilon$, of the degrees 1, 2, 3, 4, 5 respectively. And by assuming, say α a linear function of $\beta, \gamma, \delta, \varepsilon$, we may make $\mathrm{B}=0$, and we have then $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ functions of the degrees 2, 3, 4, 5 respectively of the quantities $\beta, \gamma, \delta, \varepsilon$: and these can be determined by means of a quadric equation and a cubic equation in such manner that $\mathfrak{C}=0$, $\mathfrak{D}=0$, in which case the equation in y will be of the required form. For considering $\beta, \gamma, \delta, \varepsilon$ as the coordinates of a point in space, the equations $\mathfrak{C}=0$, $\mathfrak{D}=0$ will be the equations of a quadric surface and a cubic surface respectively, and if $\beta, \gamma, \delta, \varepsilon$ be the coordinates of a point on the curve of intersection, the required conditions will be satisfied. And by combining with the equation of this the quadric surface, the equation of any tangent plane thereto (or by the different process which is made use of in the sequel), we may, by means of a quadric equation, find a generating line of the quadric surface, and then, by means of a cubic equation, a point of intersection of this line with the cubic surface, *i. e.* a point the coordinates whereof give the required values $\beta, \gamma, \delta, \varepsilon$. And similarly for the new form of TSCHIRNHAUSEN's Transformation; the only difference being that, starting with an equation in y which contains the four arbitrary quantities B, C, D, E, we obtain in the first instance an equation $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F} \between y, 1)^5=0$ wanting the second term. And then B, C, D, E are to be so determined that $\mathfrak{C}=0$, $\mathfrak{D}=0$.

To proceed with the reduction, I write the foregoing value of $\mathfrak{C}$ in the form

$$\tfrac{2}{5}\mathfrak{C}=\left(\begin{array}{llll} 4ac-4b^2, & 6ad-6bc, & 4ae-4bd, & af-be \\ 6ad-6bc, & 4ae+20bd-24c^2, & af+15be-16cd, & 4bf-4ce \\ 4ae-4bd, & af+15be-16cd, & 4bf+20ce-24d^2, & 6cf-6de \\ af-be, & 4bf-4ce, & 6cf-6de, & 4df-4e^2 \end{array}\right)\!\!\between \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E})^2,$$

which for shortness may be represented by

$$\tfrac{2}{5}\mathfrak{C}=\left(\begin{array}{cccc} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & p \end{array}\right)(B, C, D, E)^2,$$

or say

$$\tfrac{2}{5}\mathfrak{C}=(\Omega)(B, C, D, E)^2.$$

Now, by a formula in my memoir "On the Automorphic Linear Transformation of a Bipartite Quadric Function*," if Υ denote any skew symmetric matrix of the order 4, then if

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}\Omega)(B', C', D', E'),$$

in which formula Ω^{-1}, $\Omega-\Upsilon$, $(\Omega+\Upsilon)^{-1}$, Ω are all matrices which are to be compounded together into a single matrix, we have identically

$$(\Omega)(B, C, D, E)^2=(\Omega)(B', C', D', E')^2.$$

Let Q denote the determinant $|\Omega+\Upsilon|$, then the terms of the matrix $(\Omega+\Upsilon)^{-1}$ are respectively divided by Q, and we may write

$$(\Omega+\Upsilon)^{-1}=\frac{1}{Q}.Q(\Omega+\Upsilon)^{-1},$$

where $Q(\Omega+\Upsilon)^{-1}$ is the matrix obtained from the matrix $(\Omega+\Upsilon)^{-1}$ by multiplying each term by Q, the terms of $Q(\Omega+\Upsilon)^{-1}$ being thus rational and integral functions of the terms of the matrix $(\Omega+\Upsilon)$. Hence if, instead of the before-mentioned relation between (B, C, D, E) and (B', C', D', E'), we assume

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)Q(\Omega+\Upsilon)^{-1}\Omega)(B', C', D', E'),$$

we find

$$(\Omega)(B, C, D, E)^2=Q^2(\Omega)(B', C', D', E')^2.$$

And if the matrix Υ is such that we have $Q=0$, *i. e.* Det. $(\Omega+\Upsilon)=0$ (which is a quadric relation between the terms of the skew matrix, that is, each term is contained therein in the first and second powers only), then the equation becomes

$$(\Omega)(B, C, D, E)^2=0.$$

It is clear that this can only be the case in consequence of the coefficients of transformation in the equation

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)Q(\Omega+\Upsilon)^{-1}\Omega)(B', C', D', E')$$

being such that there shall exist at least two linear relations between the quantities (B, C, D, E), and I assume (without stopping to prove it) that they are such that the number of such linear relations is in fact two. That is, the last-mentioned equation establishes between the quantities (B, C, D, E) two linear relations, in virtue whereof

* Philosophical Transactions, vol. cxlviii. (1858), see p. 44.

$\mathfrak{C}=0$. And this being so, we may, without loss of generality, write $D'=0$, $E'=0$, or put

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)Q(\Omega+\Upsilon)^{-1}\Omega)(B', C', 0, 0);$$

so that B, C, D, E are linear functions of B′, C′, such that $\mathfrak{C}=0$. And then substituting these values for (B, C, D, E), we find $\mathfrak{D}$ a cubic function of B′, C′; so that, putting $\mathfrak{D}=0$, we have a cubic equation to determine the ratio B′ : C′.

The foregoing reasoning presents no real difficulty, but it is expressed by means of a very condensed notation, and it may be proper to illustrate it by the case of the quadric function $x^2+y^2+z^2$. Considering the equations

$$\begin{aligned} x&=(1+\lambda^2-\mu^2-\nu^2)x'+2(\lambda\mu-\nu)y'+2(\lambda\nu+\mu)z', \\ y&=2(\lambda\mu+\nu)x'+(1-\lambda^2+\mu^2-\nu^2)y'+2(\mu\nu-\lambda)z', \\ z&=2(\nu\lambda-\mu)x'+2(\mu\nu+\lambda)y'+(1-\lambda^2-\mu^2+\nu^2)z', \end{aligned}$$

these equations, if the expressions for x, y, z had been divided by $1+\lambda^2+\mu^2+\nu^2$, would have given

$$x^2+y^2+z^2=x'^2+y'^2+z'^2.$$

Hence they actually do give

$$x^2+y^2+z^2=(1+\lambda^2+\mu^2+\nu^2)^2(x'^2+y'^2+z'^2);$$

or if

$$1+\lambda^2+\mu^2+\nu^2=0,$$

they give

$$x^2+y^2+z^2=0.$$

But if

$$1+\lambda^2+\mu^2+\nu^2=0,$$

then

$$\begin{aligned} &1+\lambda^2-\mu^2-\nu^2 : 2(\lambda\mu-\nu) : 2(\lambda\nu+\mu) \\ =&2(\lambda\mu+\nu) : 1-\lambda^2+\mu^2-\nu^2 : 2(\mu\nu-\lambda) \\ =&2(\nu\lambda-\mu) : 2(\mu\nu+\lambda) : 1-\lambda^2-\mu^2+\nu^2; \end{aligned}$$

so that we have

$$x : y : z=1+\lambda^2-\mu^2-\nu^2 : 2(\lambda\mu+\nu) : 2(\nu\lambda-\mu),$$

which is the same result as would have been found by writing $y'=z'=0$, and which comes to saying that x, y, z are not independent, but are connected by two linear relations.

The equation Det. $(\Omega+\Upsilon)=0$, written at length, will be

$$\begin{vmatrix} a, & h-\tau, & g+\sigma, & l+\lambda \\ h+\tau, & b, & f-\rho, & m+\mu \\ g-\sigma, & f+\rho, & c, & n+\nu \\ l-\lambda, & m-\mu, & n-\nu, & p \end{vmatrix}=0,$$

where $\lambda, \mu, \nu, \rho, \sigma, \tau$ are the arbitrary constituents of the skew matrix; or developing, this is

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & p \end{vmatrix}$$

$$+\left(\begin{array}{llllll} bc-f^2, & fg-ch, & hf-bg, & mg-nh, & fm-bn, & -fn+cm \\ fg-ch, & ca-g^2, & gh-af, & -gl+an, & nh-lf, & gn-cl \\ hf-bg, & gh-af, & ab-h^2, & hl-am, & -hm+bl, & lf-mg \\ mg-nh, & fm-bn, & -fn+cm, & ap-l^2, & ph-lm, & pg-ln \\ -gl+an, & nh-lf, & gn-cl, & ph-lm, & bp-m^2, & pf-mn \\ hl-am, & -hm+bl, & lf-mg, & pg-ln, & pf-mn, & cp-n^2 \end{array}\right|\!\!\between\!(\lambda, \mu, \nu, \varrho, \sigma, \tau)^2$$

$$+(\lambda\varrho+\mu\sigma+\nu\tau)^2=0,$$

the first term whereof, substituting for (a, b, c, f, g, h, l, m, n, p) their values, is in fact equal to the discriminant a^4f^4+ &c. of the quintic $(a, b, c, d, e, f\between X, Y)^5$. There is no loss of generality in putting all but two of the quantities $(\lambda, \mu, \nu, \varrho, \sigma, \tau)$ equal to zero; in fact this leaves in the formulæ a single arbitrary quantity, which is the right number, since the ratios B : C : D : E have to satisfy only the two conditions $\mathfrak{C}=0$, $\mathfrak{D}=0$.

Addition, Nov. 10, 1862.

I take the opportunity of remarking, with reference to my memoir "On a New Auxiliary Equation in the Theory of Equations of the Fifth Order*," that I recently discovered that the auxiliary equation there considered is in fact due to Jacobi, who, in his paper, "Observatiunculæ ad theoriam æquationum pertinentes †," under the heading "Observatio de æquatione sexti gradus ad quam æquationes quinti gradus revocari possunt," gives the theory, and observes that the equation is of the form

$$\phi^6+a_2\phi^4+a_4\phi^2+a_6=32\sqrt{\square}\,\phi,$$

and mentions that the value of a_2 is easily found to be (I adapt his notation to the denumerate form $(a, b, c, d, e, f\between v, 1)^5=0$)

$$=40ae-16bd+6c^2$$

(this ought, however, to be divided by -2), but that the values of a_4, a_6 "paullo ampliores calculos poscunt."

The value of the coefficient in question is correctly obtained (page 270 of my memoir) in the form

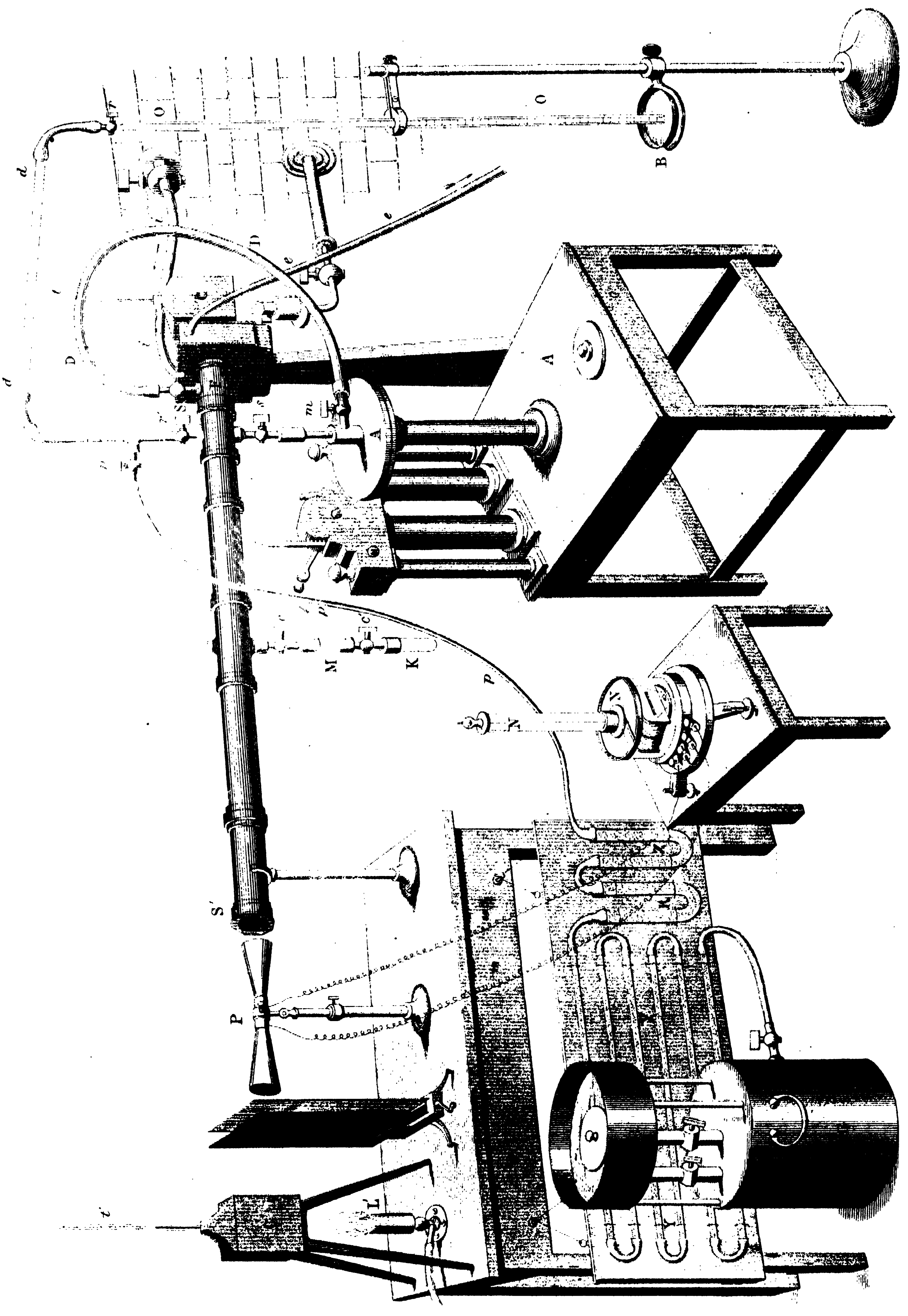
$$\begin{array}{l} -c^2 \\ +2(-16ae+4bd-c^2) \\ +12ae; \end{array}$$

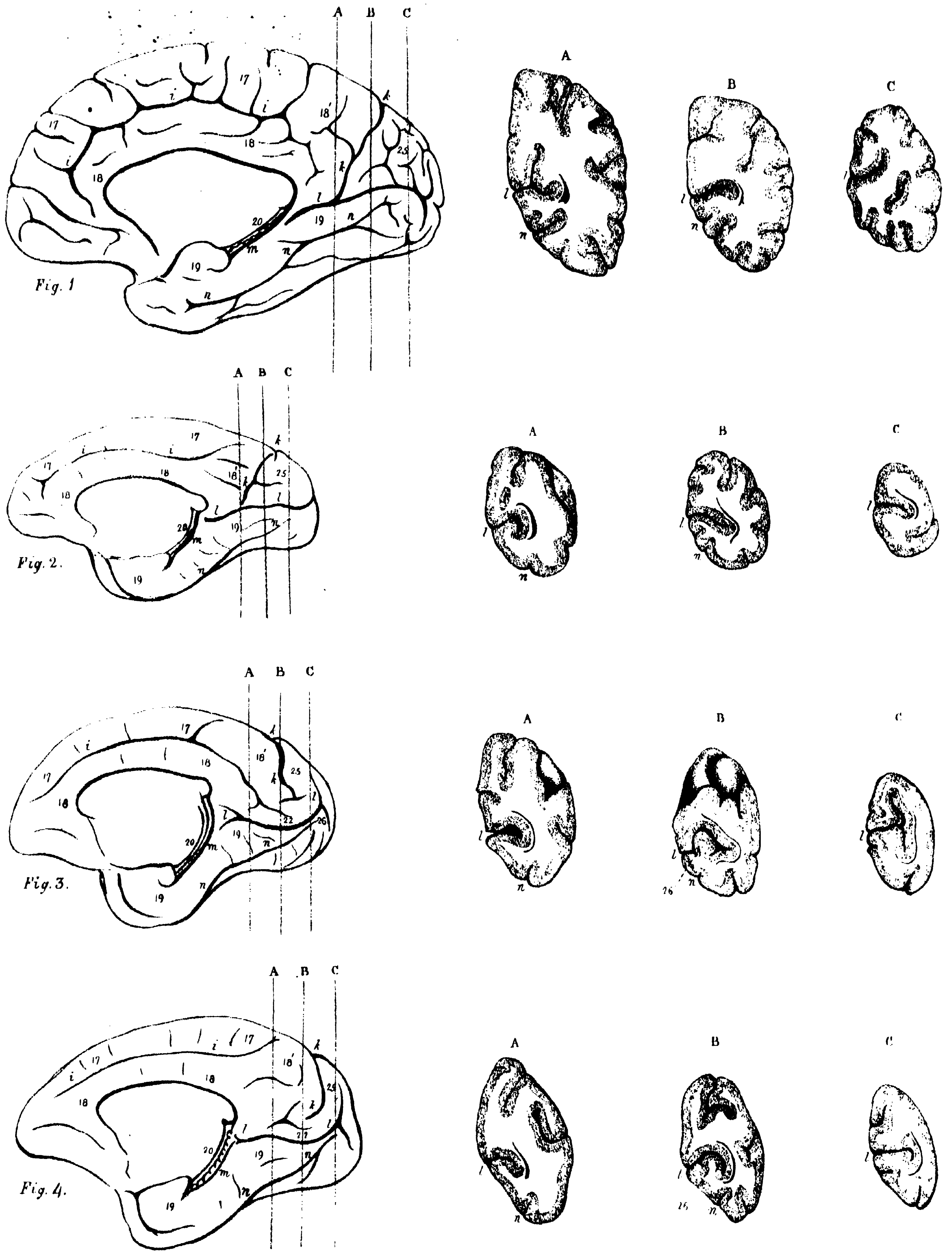
but the reduced value is given in two places (page 271) as equal to

$$\begin{array}{lcl} -32\,ae, & \text{this should be} & -20\,ae, \\ +\ 8\,bd, & \text{''} & +\ 8\,bd, \\ -\ 3c^2, & \text{''} & -\ 3c^2. \end{array}$$

The last-mentioned correct value was used in obtaining the coefficient for the standard form, which coefficient is given correctly, page 274.

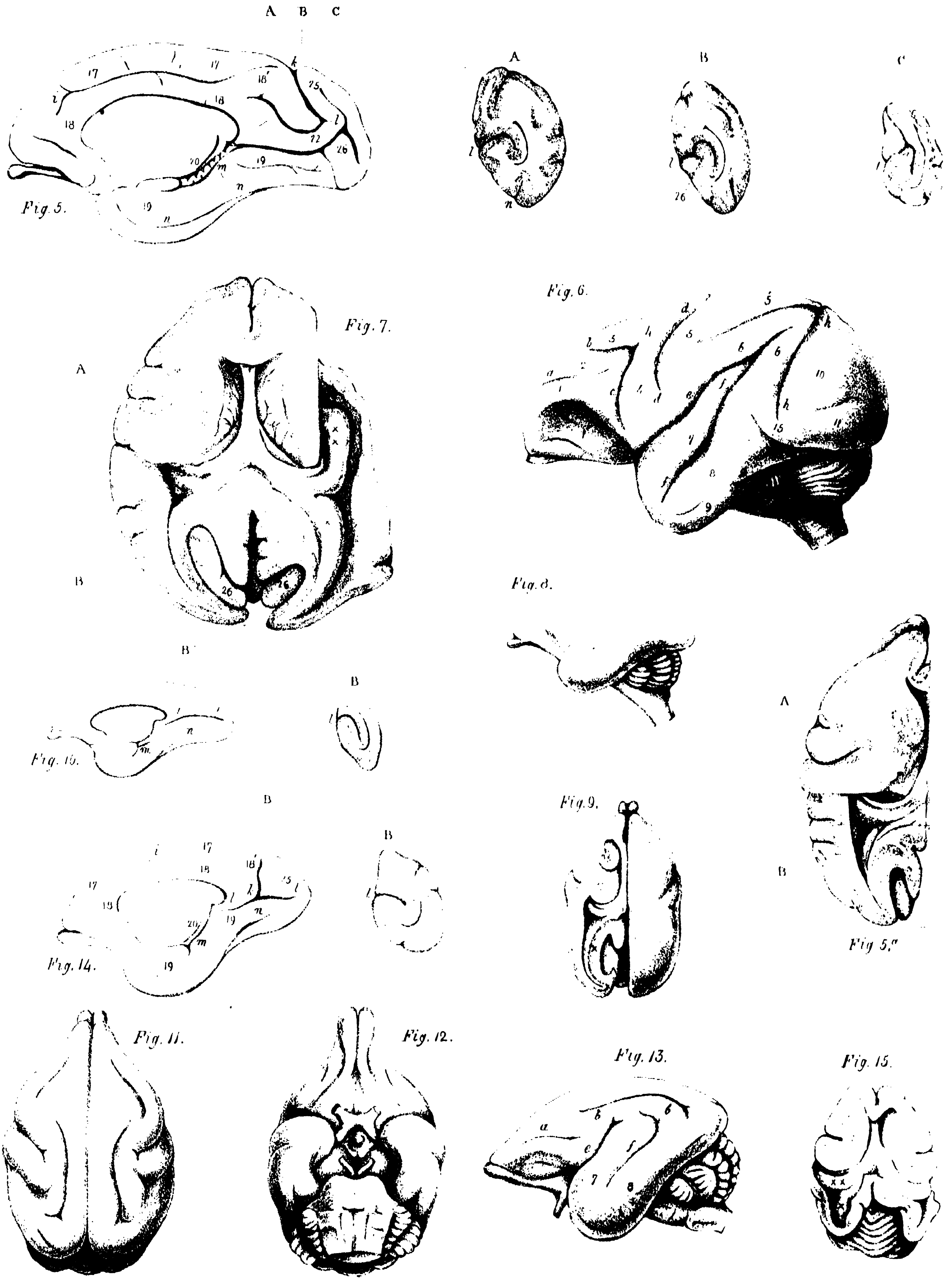
* Philosophical Transactions, vol. cli. (1861) pp. 263–276. † Crelle, t. xiii. (1835) pp. 340–352.





W.H.F. Del.

E.M. Williams Lith.



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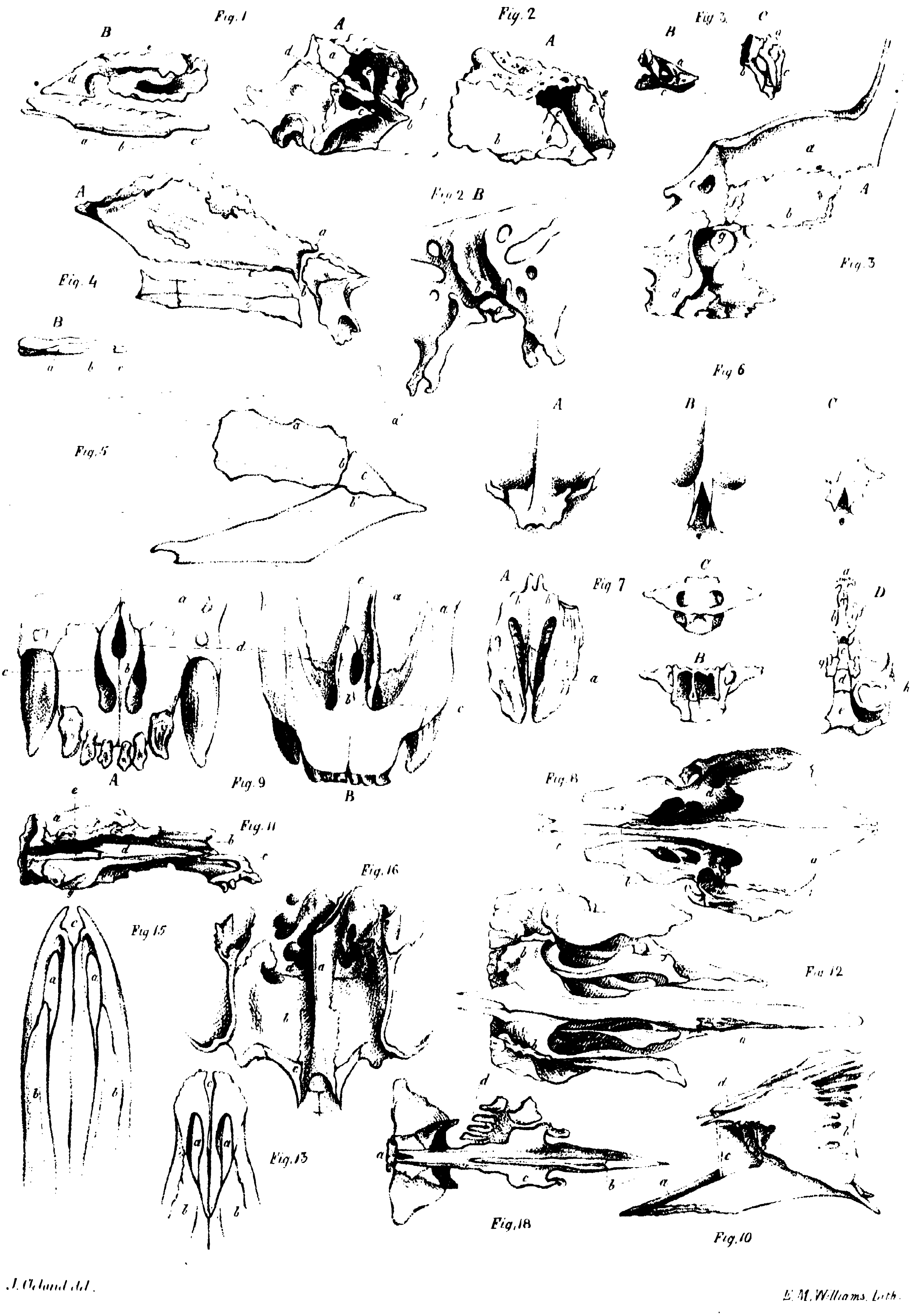
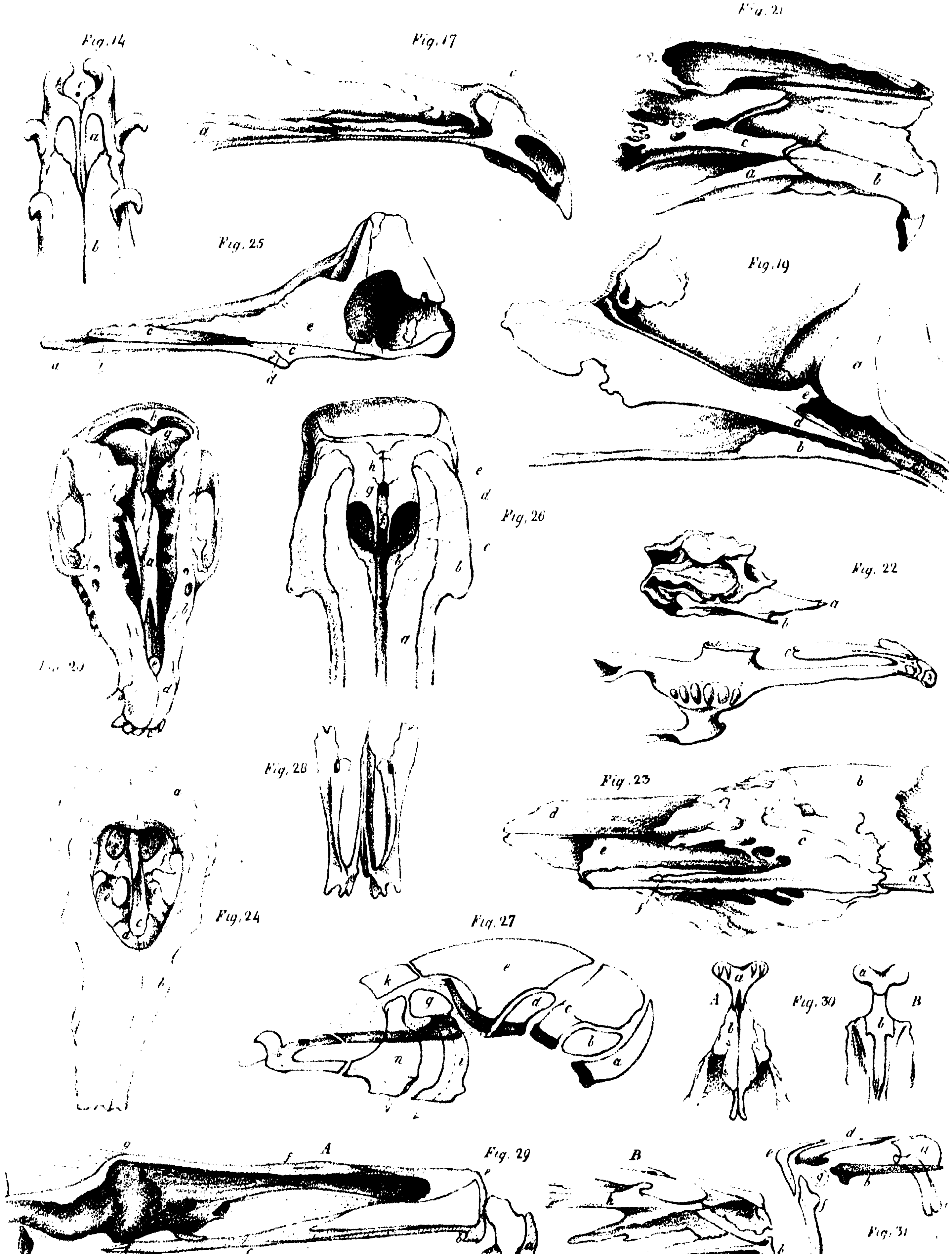


J. Cleland del.

E. M. Williams, Lith.



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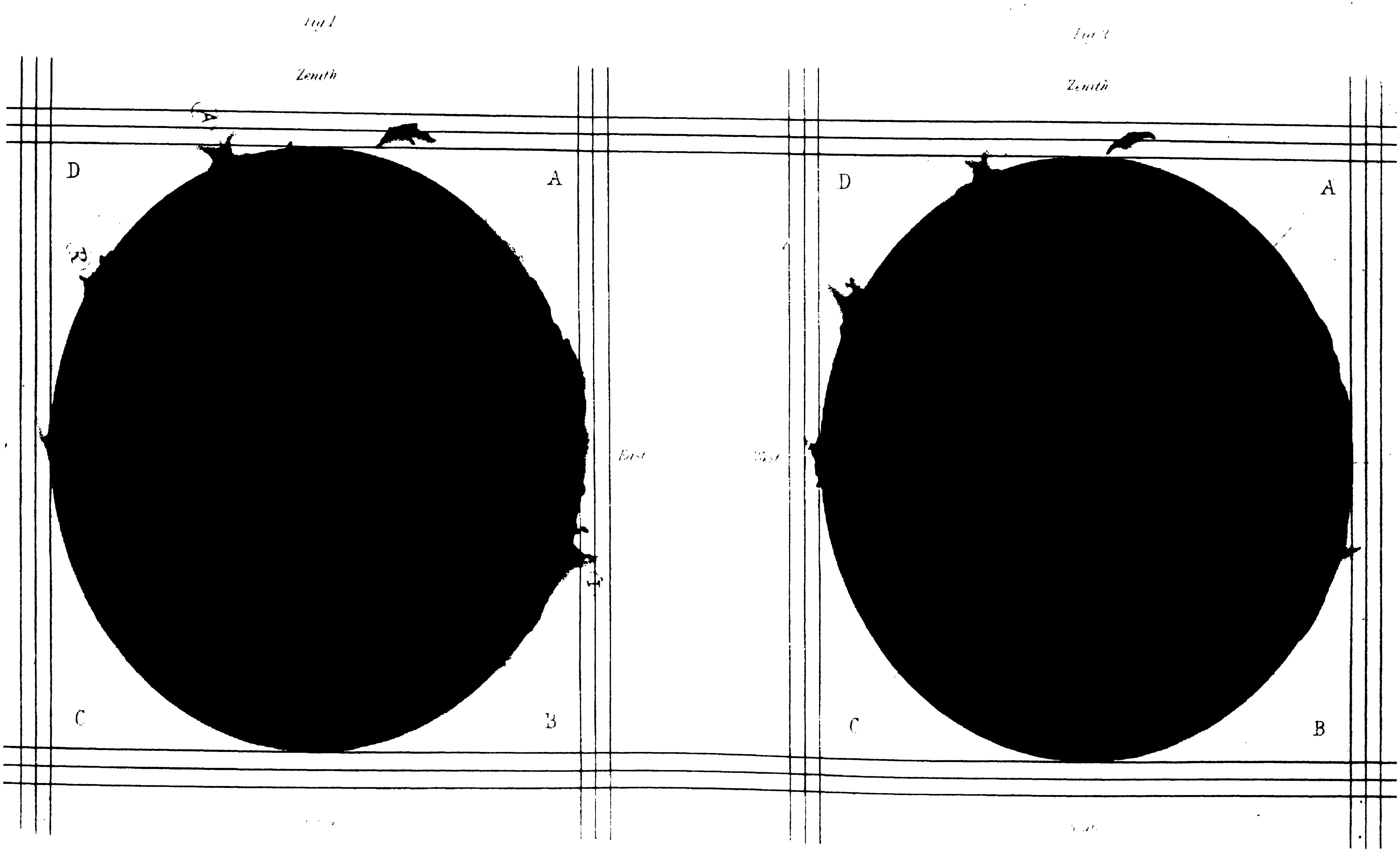
E. M. Williams Lith.



S. Russell sc.

Fac-simile of hand drawing

Vincent Brooks

APPEARANCE OF PHENOMENA

IMMEDIATELY AFTER THE BEGINNING OF TOTALITY.

Vincent Brooks.

APPEARANCE OF PHENOMENA
IMMEDIATELY PREVIOUS TO THE END OF TOTALITY.

FAC SIMILE OF Nº 25 PHOTOGRAPH - FIRST TOTALITY.

FAC SIMILE OF Nº 26 PHOTOGRAPH · SECOND TOTALITY.

J. Basire sc.

COPY OF A TOUCHED PHOTOGRAPH

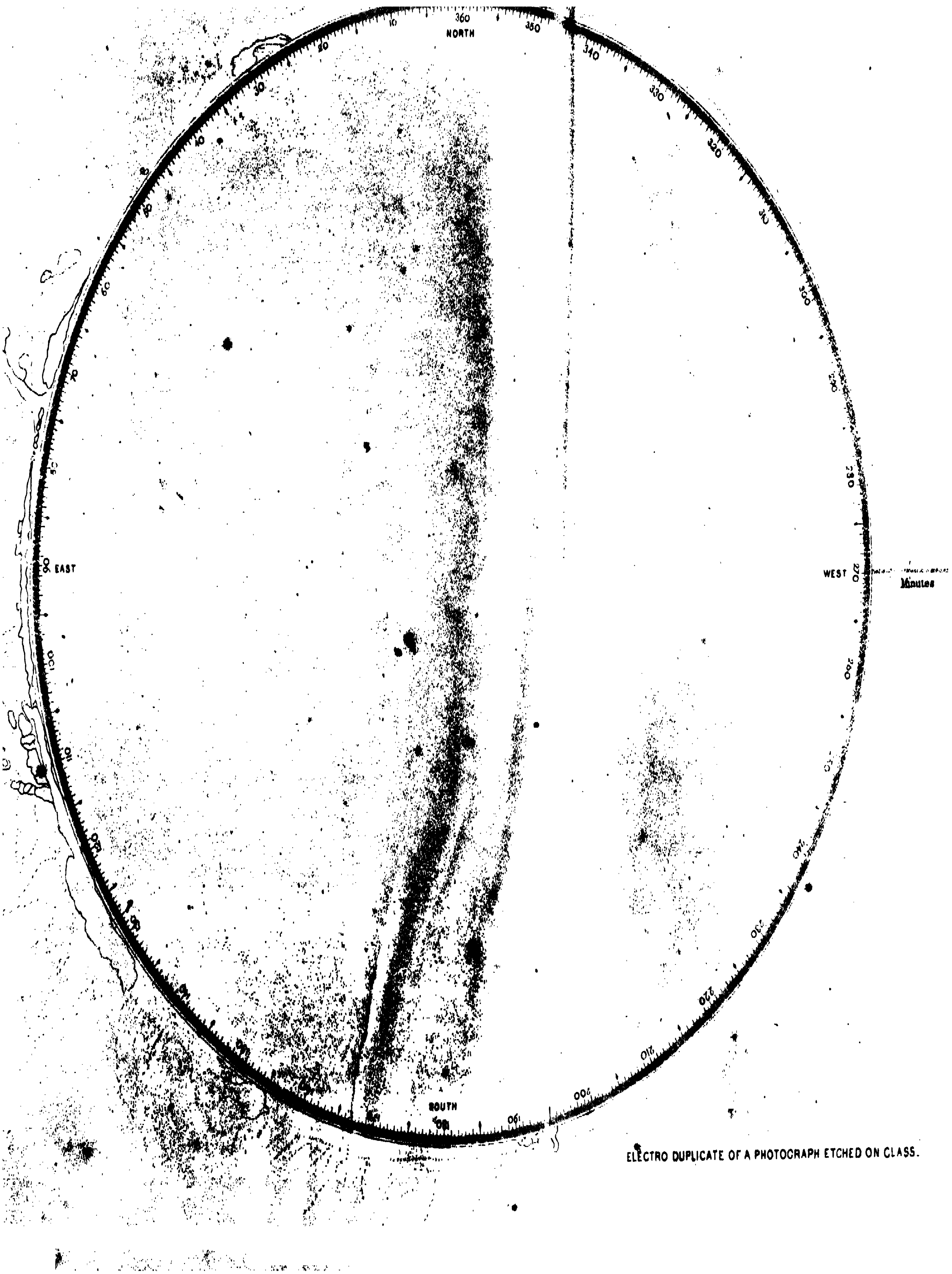
Shewing the phenomena of totality immediately after total obscuration.

COPY OF A TOUCHED PHOTOGRAPH

Shewing the phenomena of totality one minute after total obscuration.

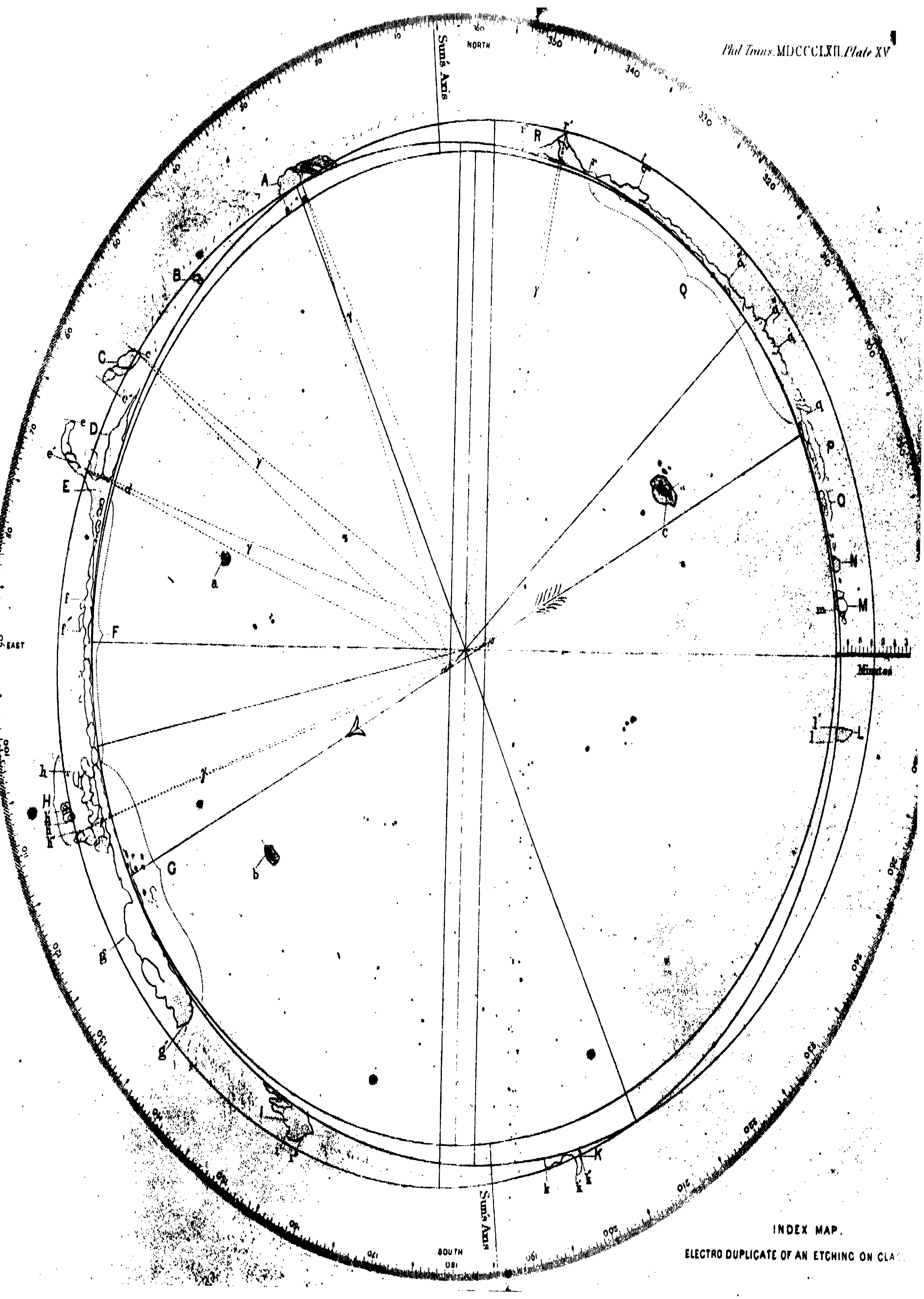
COPY OF A TOUCHED PHOTOGRAPH

Shewing the phenomena of totality immediately previous to the reappearance of the Sun.



ELECTRO DUPLICATE OF A PHOTOGRAPH ETCHED ON GLASS.

ELECTRO DUPLICATE OF A PHOTOGRAPH ETCHED ON GLASS.



INDEX MAP.

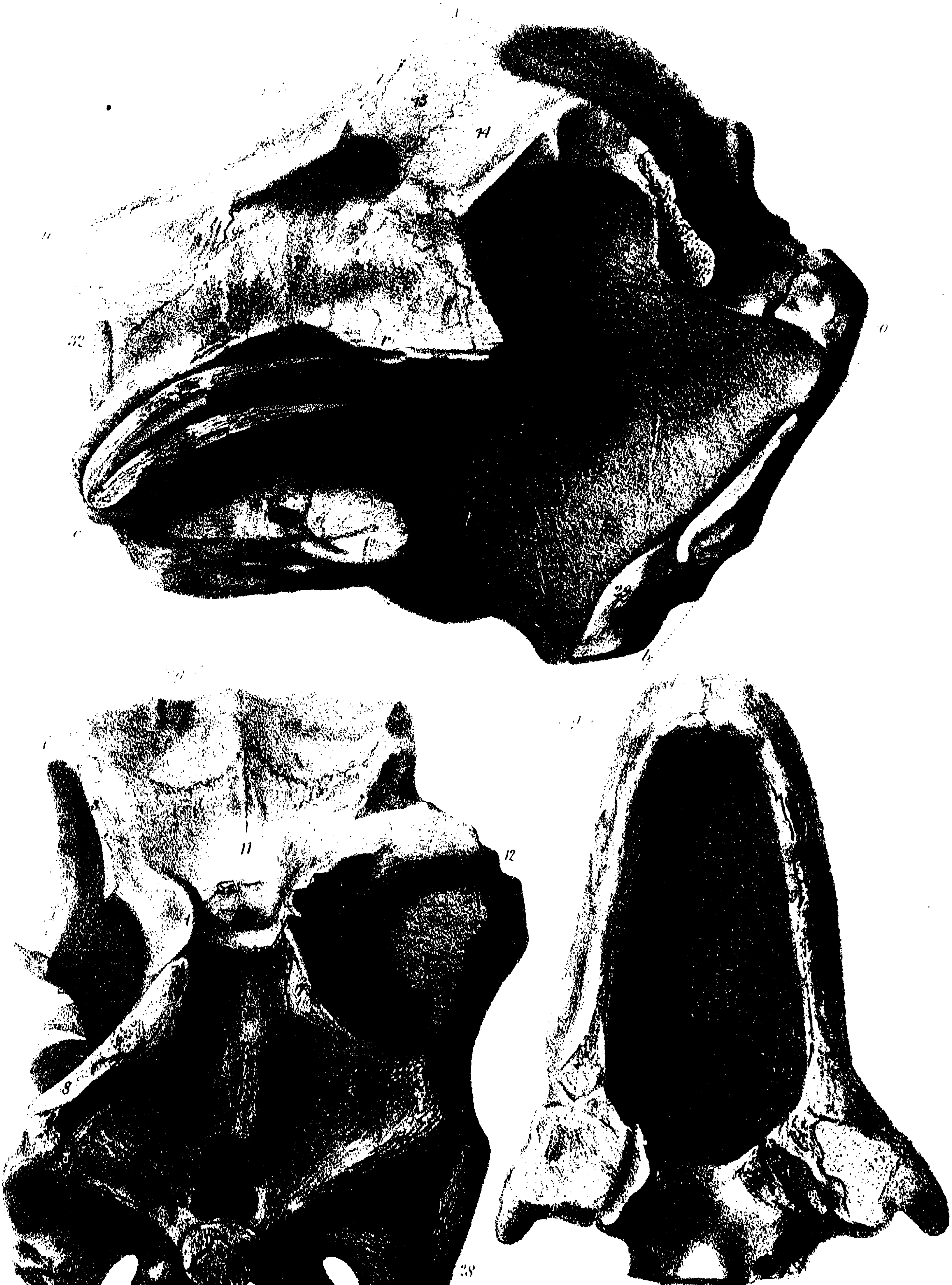
ELECTRO DUPLICATE OF AN ETCHING ON GLA[illegible]

PARTIAL PHASE
at $2^{h}\,48^{min}\,25{\cdot}6^{sec}$ *G M T*

ELECTRO DUPLICATE OF A PHOTOGRAPH ETCHED ON GLASS

Fig. 1.

Fig. 2.



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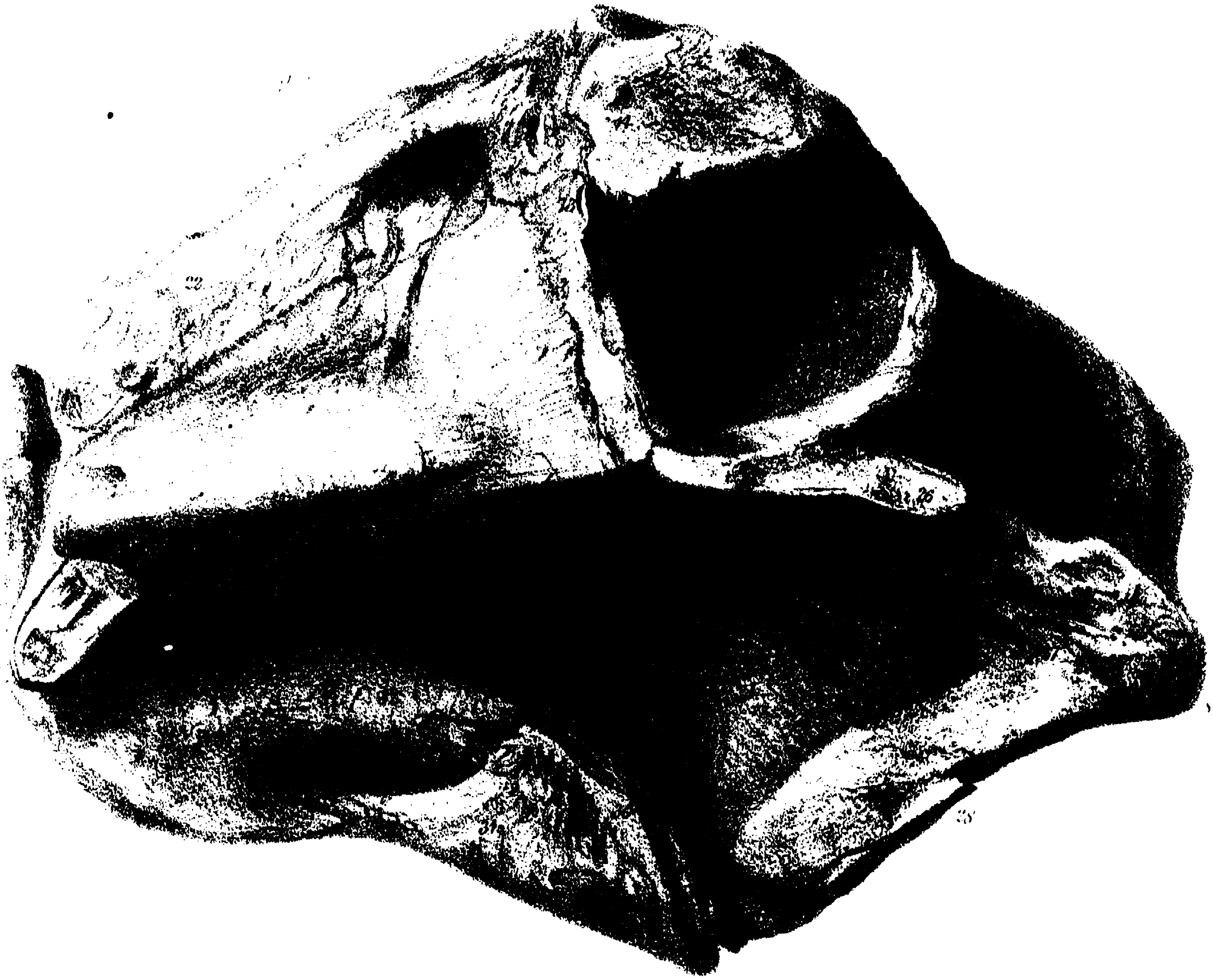

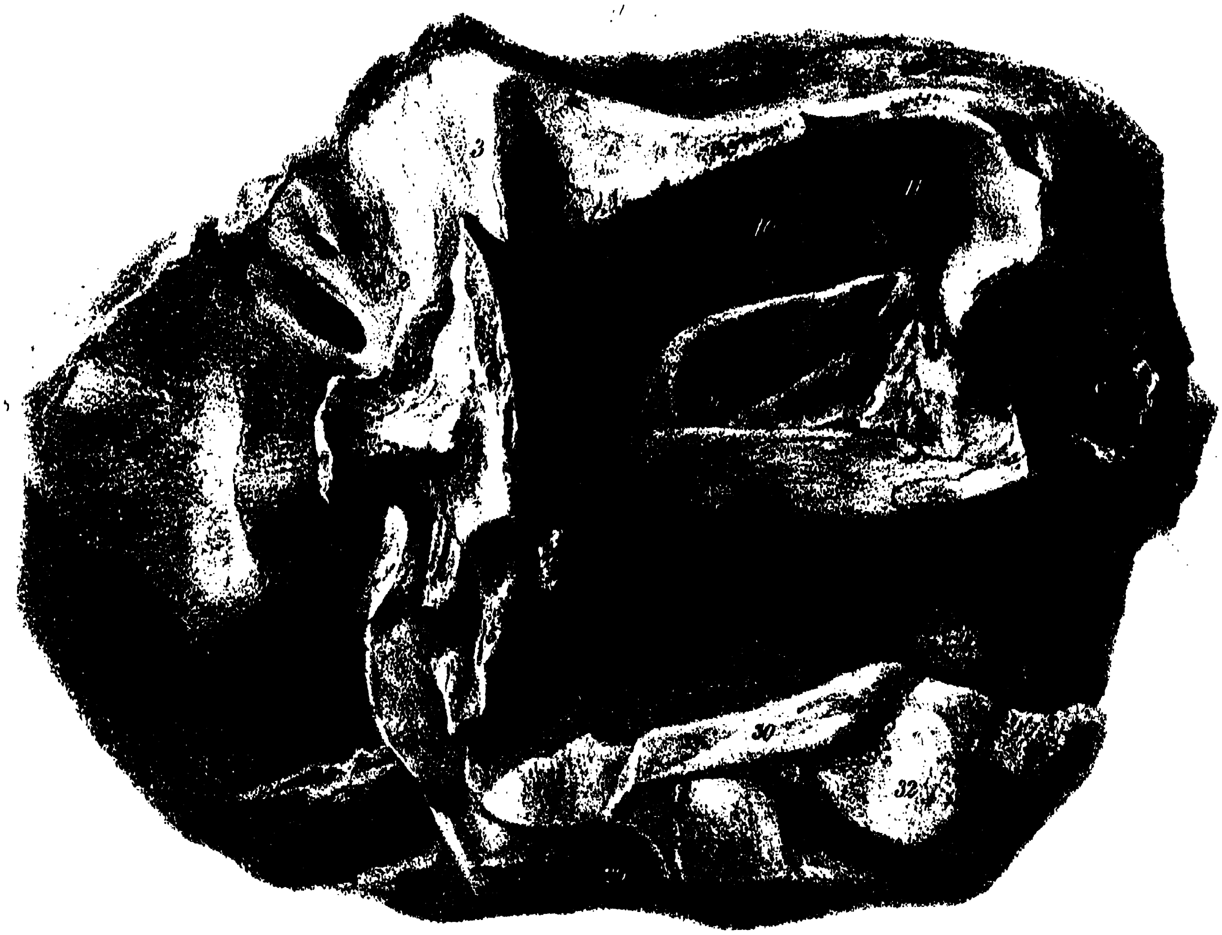
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J. Dinkel del. et lith.

J. Basire imp.

XXV. *On the Thermal Effects of Fluids in Motion.*—Part IV.

By J. P. Joule, *LL.D., F.R.S. &c., and Professor* W. Thomson, *A.M., LL.D., F.R.S. &c.*

Received June 19,—Read June 19, 1862.

In the Second Part of these researches we have given the results of our experiments on the difference between the temperatures of an elastic fluid on the high- and low-pressure sides of a porous plug through which it was transmitted. The gases employed were atmospheric air and carbonic acid. With the former, 0°·0176 of cooling effect was observed for each pound per square inch of difference of pressure, the temperature on the high-pressure side being 17°·125. With the latter gas, 0°·0833 of cooling effect was produced per lb. of difference of pressure, the temperature on the high-pressure side being 12°·844.

It was also shown that in each of the above gases the difference of the temperatures on the opposite sides of the porous plug is sensibly proportional to the difference of the pressures.

An attempt was also made to ascertain the cooling effect when elastic fluids of high temperature were employed; and it was satisfactorily shown that in this case a considerable diminution of the effect took place. Thus, in air at 91°·58, the effect was only 0°·014; and in carbonic acid at 91°·52, it was 0°·0474.

In the experiments at high temperatures there appeared to be some grounds for suspecting that the apparent cooling effect was too high; for the quantity of transmitted air was very considerable, and its temperature possibly had not arrived accurately at that of the bath by the time it reached the porous plug.

The obvious way to get rid of all uncertainty on this head was to increase the length of the coil of pipes. Hence in the following experiments the total length of 2-inch copper pipe immersed in the bath was 60 feet instead of 35, as in the former series. The volume of air transmitted in a given time was also considerably less. There could therefore be no doubt that the temperature of the air on its arrival at the plug was sensibly the same as that of the bath.

The nozle employed in the former series of experiments was of box-wood,—the space occupied by cotton-wool, or other porous material, being 2·72 inches long and an inch and a half in diameter. The box-wood was protected from the water of the bath by being enveloped by a tin can filled with cotton-wool. This was unquestionably in most respects the best arrangement for obtaining accurate results; but it was found necessary to make each experiment last one hour or more before we could confidently depend on the thermal effect. The oscillations of temperature which took place during the first part of the time were traced to various causes, one of the principal being the length of time which, on account of the large capacity for heat and the small conductivity of the box-wood nozle, elapsed before the first large thermal effects consequent on the getting up of the pressure were dissipated. No doubt the results we arrived at were very accurate with the elastic fluids employed, viz. atmospheric air and carbonic acid; but we possessed an unlimited supply of the former and a supply of the latter equal to 120 cubic feet, which was sufficient to last for more than half an hour without being exhausted. In extending the inquiry to gases not so readily procured in large quantities, it was therefore desirable to use a porous plug of smaller dimensions enclosed in a nozle of less capacity for heat, so as to arrive rapidly at the normal effect.

Various alterations of the apparatus were made in order to meet the new requirements of our experiments. A small high-pressure engine of about one horse-power was placed in gear with a double-acting compressing air-pump, which had a cylinder $4\frac{1}{2}$ inches in diameter, with a length of stroke of 9 inches. The engine was able to work the piston of the pump sixty complete strokes in the minute. The quantity of air which it ought to have discharged at low pressure was therefore upwards of 16,000 cubic inches per minute. But much loss, of course, occurred from leakage past the metallic piston, and in consequence of the necessary clearance at the top and bottom of the cylinder when the pressure increased by a few atmospheres; so that in practice we never pumped more than 8000 cubic inches per minute.

The nozle we employed will be understood by inspecting Plate XXVI. fig. 1, where *a a* is the upright end of the coil of copper pipes. On a shoulder within the pipe a perforated metallic disk (*b*) rests. Over this is a short piece of india-rubber tube (*c c*) enclosing a silk plug (*d*), which is kept in a compressed state by the upper perforated metallic plate (*e*). This upper plate is pressed down with any required force by the operation of the screw *f* on the metallic tube *g g*. A tube of cork (*h h*) is placed within the metallic tube, in order to protect the bulb of the thermometer from the effects of a too rapid conduction of heat from the bath. Cotton-wool is loosely packed round the bulb, so as to distribute the flowing air as evenly as possible. The glass tube (*i i*) is attached to the nozle by means of a piece of strong india-rubber tubing, and through it the indications of the thermometer are read. The top of the glass tube is attached to the metallic tube *l l*, for the purpose of conveying the gas to the meter.

The thermometer (*m*) for registering the temperature of the bath is placed with its bulb near the nozle. The level of the water is shown by *n n*; and *o o* represents the wooden cover of the bath.

When a high temperature was employed, it was maintained by introducing steam into the bath by means of a pipe led from the boiler. The water of the bath was in every case constantly and thoroughly stirred, especially when high temperatures were used.

The general disposition of the apparatus will be understood from fig. 2, in which A represents the boiler, B the steam-engine geared to the condensing air-pump C. From this pump the compressed air passes through a train of pipes 60 feet long and 2 inches in diameter, and then enters the coil of pipes in the bath D. Thence, after issuing from the porous plug, it passes through the gasometer E, and ultimately arrives again at the pump C. This complete circulation is of great importance, inasmuch as it permits the gas which has been collected in the meter to be used for a much longer period than would otherwise have been possible. A glass vessel full of chloride of calcium is placed in the circuit at F, and chloride of calcium is also placed in the pipe at *f*. A small tube leading from the coil is carried to the shorter leg of the glass siphon gauge G, of which the longer leg is 17 feet, and the shorter 12 feet long.

The thermometers employed were all carefully calibrated, and had about ten divisions to the degree Centigrade. We took the precaution of verifying the air- and bath-thermometers from time to time, especially when high temperatures were used, in which latter case a comparison between the thermometers at high temperature was made immediately after each experiment.

Atmospheric Air.

In the experiments described in the present paper, the air was not deprived of its carbonic acid. It was simply dried by transmitting it in the first place, before it entered the pump, through a cylinder 18 inches long and 12 inches in diameter filled with chloride of calcium, and afterwards, in its compressed state, through a pipe 12 feet long and 2 inches in diameter filled with the same substance. The experiments were principally carried on in the winter season; so that the chloride kept dry for a long time. From its condition after some weeks' use, it was evident that the water was removed, almost as much as chloride of calcium can remove it, after the air had traversed three inches of the chloride contained by the first vessel.

Table I.

No. of experiment.	Cubical inches of air transmitted per minute.	Pressure over that of the atmosphere, in inches of mercury.	Temperature of the bath.	Thermal effect.	Correction on account of conduction of heat.	Corrected thermal effect.	Thermal effect reduced to the pressure of 100 inches of mercury.	Time occupied by experiment, in minutes.	Number of observations comprised in each mean.	Extreme range of the temperature of the bath.	Extreme range of the temperature of the air.	Extreme range of the pressure.
1	3000	83·96	4°·499	−0°·711	−0°·044	−0°·755	−0°·900	14	5	0°·020	0°·015	2·25
2	3600	136·19	6·112	−1·11	−0·058	−1·168	−0·858	24	7	0·017	0·055	1·7
3	2600	156·59	6·082	−1·307	−0·094	−1·401	−0 895	15	5	0·009	0·065	8·0
4	1750	139·58	7·471	−1·137	−0·122	−1·259	−0·902	24	15	0·006	0·19	5·3
5	2250	153·9	7·640	−1·231	−0·103	−1·334	−0·867	12·5	10	0·008	0·028	3·6
6	2300	159·31	8·546	−1·252	−0·102	−1·354	−0·850	18	20	0·017	0·105	3·0
7	2060	165·73	8·2	−1·329	−0·121	−1·450	−0·875	14	15	0·034	0·128	4·8
8	1500	129·78	8·72	−1·019	−0·127	−1·146	−0·883	8	9	0·008	0·135	2·9
9	5000	128·9	24·92	−0·983	−0·037	−1·020	−0·791	12	8	0·015	0·09	7·0
10	4600	122·8	27·81	−0·874	−0·036	−0·910	−0·741	26	15	0·029	0·064	0
11	5000	123·5	42·64	−0·947	−0·036	−0·983	−0·796	8	4	0·127	0·122	7·0
12	4800	137	43·54	−0·943	−0 037	−0·980	−0·715	6	3	0·02	0·02	0
13	5000	127·5	47·92	−0·937	−0·037	−0·974	−0·764	6	4	0·058	0·09	2·0
14	3700	147	49·96	−0·969	−0·049	−1·018	−0·692	35	30	0·14	0·26	0
15	5600	146	53·375	−0·860	−0·028	−0·888	−0·608	28	30	0·05	0·08	0
16	5700	146	64·9	−0·870	−0·029	−0·899	−0·616	24	20	0·18	0·23	0
17	2700	112·43	89·901	−0·469	−0·033	−0·502	−0·446	20	10	0·112	0·23	8·6
18	1700	147	90·353	−0·821	−0·091	−0·912	−0·620	4	3	0·022	0·085	0
19	1700	153·16	92·486	−0·756	−0·083	−0·839	−0·547	19	10	0·202	0·273	7·0
20	3150	156·5	92·603	−0·674	−0·040	−0·714	−0·456	12	10	0·078	0·19	3·5
21	3800	146	93·78	−0·700	−0·036	−0·736	−0·504	24	20	0·236	0·255	0
22	4600	158·5	97·528	−0·722	−0·029	−0·751	−0·474	20	16	0·112	0·115	0
1	2	3	4	5	6	7	8	9	10	11	12	13

Oxygen Gas.

This elastic fluid was procured by cautiously heating chlorate of potash mixed with a small quantity of peroxide of manganese. In its way to the meter it passed through a tube containing caustic potash, in order to deprive it of any carbonic acid it might contain. The same drying-apparatus was employed as in the case of atmospheric air.

TABLE II.

No. of experiment.	Cubical inches of elastic fluid transmitted per minute.	Composition of the elastic fluid.	Pressure over that of the atmosphere, in inches of mercury.	Temperature of the bath.	Thermal effect.	Correction on account of conduction of heat.	Corrected thermal effect.	Thermal effect reduced to the pressure of 100 inches of mercury.	Ditto, calculated for pure oxygen.	Time occupied by experiment, in minutes.	Number of observations comprised in each mean.	Extreme range of the temperature of the bath.	Extreme range of the temperature of the elastic fluid.	Extreme range of the pressure.
1	2000	5·095 N, 94·905 O	159·28	8°·682	−1°·547	−0°·145	−1°·692	−1°·061	−1°·075	9	10	0°·007	0°·35	7·8
2	2000	54·62 N, 45·38 O	161·81	8·75	−1·373	−0·129	−1·502	−0·928	−1·074	11	10	0·017	0·046	0·9
3	1700	3·64 N, 96·36 O	151	89·466	−1·069	−0·118	−1·187	−0·786	−0·800	14	10	0·45	0·43	6·2
4	3150	22·37 N, 77·63 O	159·77	90·8	−0·840	−0·050	−0·890	−0·557	−0·580	12	10	0·326	0·336	8·0
5	3150	51·03 N, 48·97 O	154·1	92·792	−0·734	−0·043	−0·777	−0·501	−0·527	12	10	0·18	0·19	4·0
6	4500	4 N, 96 O	152	95·453	−0·795	−0·033	−0·828	−0·544	−0·570	11	8	0·135	0·158	0
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Nitrogen Gas.

In preparing this gas the meter was first filled with air, and then a long shallow tin vessel was floated under it, containing sticks of phosphorus so disposed as to burn in succession. Some hours were allowed to elapse after the combustion had terminated, in order to allow of the deposition of the phosphoric acid formed.

TABLE III.

No. of experiment.	Cubical inches of elastic fluid transmitted per minute.	Composition of the elastic fluid.	Pressure over that of the atmosphere, in inches of mercury.	Temperature of the bath.	Thermal effect.	Correction on account of conduction of heat.	Corrected thermal effect.	Thermal effect reduced to the pressure of 100 inches of mercury.	Ditto, calculated for pure nitrogen.	Time occupied by experiment, in minutes.	Number of observations comprised in each mean.	Extreme range of the temperature of the bath.	Extreme range of the temperature of the elastic fluid.	Extreme range of the pressure.
1	2050	7·9 O, 92·1 N	163·38	7°·204	−1°·448	−0°·133	−1°·581	−0°·967	−1°·034	7	8	0°·008	0°·25	6·2
2	2500	2·2 O, 97·8 N	162·65	91·415	−0·857	−0·064	−0·921	−0·567	−0·576	13	10	0·036	0·48	4·5
3	2500	12·5 O, 87·5 N	164·61	91·965	−0·869	−0·065	−0·934	−0·567	−0·691	12	9	0·337	0·378	3·0
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Carbonic Acid.

This gas was formed by adding sulphuric acid to a solution of carbonate of soda. It was dried in the same manner as all the other gases.

TABLE IV.

No. of experiment.	Cubical inches of elastic fluid transmitted per minute.	Composition of the elastic fluid.	Pressure over that of the atmosphere, in inches of mercury.	Temperature of the bath.	Thermal effect.	Correction on account of conduction of heat.	Corrected thermal effect.	Thermal effect reduced to the pressure of 100 inches of mercury.	Ditto, calculated for pure carbonic acid, calling its sp. heat for equal vol. 1·39.	Time occupied by experiment, in minutes.	Number of observations comprised in each mean.	Extreme range of the temperature of the bath.	Extreme range of the temperature of the elastic fluid.	Extreme range of the pressure.
1	2450	68·42 Air 31·58 CO_2	163·7	7°·362	−2°·699	−0°·190	−2°·889	−1°·765	−3°·166	12	10	0°	0°·16	3·2
2	2350	89·16 Air 10·84 CO_2	148·82	7·360	−1·621	−0·125	−1·746	−1·173	−2·990	14	10	0·004	0·282	9·2
3	3100	3·52 Air 96·48 CO_2	164·07	7·384	−6·719	−0·299	−7·018	−4·277	−4·367	6·5	6	0·008	0·021	1·4
4	2500	62·5 Air 37·5 CO_2	162·925	7·407	−2·839	−0·191	−3·030	−1·860	−3·052	8	8	0·007	0·11	5·8
5	2300	88·13 Air 11·87 CO_2	158·08	7·433	−1·682	−0·132	−1·814	−1·147	−2·648	10	10	0·005	0·107	5·2
6	2260	97·46 Air 2·54 CO_2	163·52	7·608	−1·407	−0·116	−1·523	−0·931	−2·753	8	8	0·007	0·064	2·0
7	3300	4·0 Air 5·286 H 90·714 CO_2	161·97	7·960	−6·131	−0·262	−6·393	−3·947	−4·215	6	8	0	0·18	4·8
8	3000	4·23 Air 46·47 H 49·3 CO_2	153·72	8·020	−2·189	−0·117	−2·306	−1·500	−2·631	5	5	0	0·19	1·6
9	1500	7·09 Air 67·05 H 25·86 CO_2	97·56	8·296	−0·543	−0·063	−0·606	−0·622	−1·940	15	15	0·012	0·146	5·4
10	2925	2·11 Air 97·89 CO_2	167·25	93·523	−3·418	−0·160	−3·578	−2·139	−2·164	10	10	0·382	0·49	4·0
11	2925	56·78 Air 43·22 CO_2	167·4	91·26	−1·746	−0·099	−1·845	−1·102	−1·674	30	20	0·292	0·49	11·0
12	2925	77·77 Air 22·23 CO_2	146·83	91·642	−1·292	−0·077	−1·369	−0·938	−2·053	9	6	0·045	0·245	3·5
13	5500	0·83 Air 99·17 CO_2	146	54·0	−4·184	−0·104	−4·288	−2·937	−2·951	24	16	0·24	0·46	0
14	5300	67·7 Air 32·3 CO_2	147	49·703	−1·832	−0·059	−1·891	−1·286	−2·225	24	16	0·025	0·17	0
15	5600	87·77 Air 12·23 CO_2	145	49·764	−1·250	−0·032	−1·282	−0·884	−2·025	20	16	0·01	0·11	0
16	5100	1·83 Air 98·17 CO_2	127·5	35·604	−4·186	−0·112	−4·298	−3·371	−3·407	18	15	0·03	0·095	0
17	5000	1·66 Air 98·34 CO_2	151	97·553	−3·11	−0·084	−3·194	−2·115	−2·135	20	16	0·292	0·272	0
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Hydrogen.

Our method in procuring this elastic fluid was to pour sulphuric acid, prepared from sulphur, into a carboy nearly filled with water and containing fragments of sheet zinc. The gas was passed through a tube filled with rags steeped in a solution of sulphate of copper, and then through a tube filled with sticks of caustic potash. The rags became speedily browned, and we therefore adopted the plan of pouring a small quantity of solution of sulphate of copper from time to time into the carboy itself. This succeeded perfectly; the rags retained their blue colour, and the gas was rendered perfectly inodorous, whilst at the same time its evolution became much more free and regular.

TABLE V.

No. of experiment.	Cubical inches of elastic fluid transmitted per minute.	Composition of the elastic fluid.	Pressure over that of the atmosphere, in inches of mercury.	Temperature of the bath.	Thermal effect.	Correction on account of conduction of heat.	Corrected thermal effect.	Thermal effect reduced to the pressure of 100 inches of mercury.	Ditto, calculated for pure hydrogen.	Time occupied by experiment, in minutes.	Number of observations comprised in each mean.	Extreme range of the temperature of the bath.	Extreme range of the temperature of the elastic fluid.	Extreme range of the pressure.
1	3000	17·635 Air 82·365 H	64·1	6°·34	−0°·144	−0°·009	−0°·153	−0°·239	−0°·104	3	3	°	°	0
2	3000	75·16 Air 24·84 H	99·86	6·355	−0·564	−0·035	−0·599	−0·600	+0·226	10	4	0	0·15	1·5
3	3900	4·866 Air 95·134 H	49·91	6·132	+0·033	+0·002	+0·035	+0·070	+0·118	12	6	0·002	0·06	1·2
4	2900	78·295 Air 21·705 H	99·657	5·808	−0·535	−0·034	−0·569	−0·571	+0·525	34	12	0·03	0·11	1·85
5	2800	9·2 Air 90·8 H	86·885	7·244	+0·041	+0·003	+0·044	+0·05	+0·143	27	10	0·034	0·033	1·75
6	3300	1·798 Air 98·202 H	79·84	7·572	+0·043	+0·003	+0·046	+0·058	+0·075	23	8	0·008	0·023	2·85
7	2950	4·795 Air 95·205 H	74·08	6·654	+0·054	+0·004	+0·058	+0·078	+0·126	17	10	0·016	0·11	6·6
8	2650	67·75 Air 32·25 H	130·97	6·717	−0·571	−0·040	−0·611	−0·466	+0·383	12	6	0·01	0·07	2·6
9	3800	4·07 Air 95·93 H	100·72	6·781	+0·039	+0·002	+0·041	+0·041	+0·08	10	10	0·012	0·078	2·6
10	2700	58·29 Air 41·71 H	144·02	6·846	−0·504	−0·035	−0·539	−0·375	+0·317	8·5	8	0·011	0·07	3·6
11	1900	91·81 Air 8·19 H	152·67	7·406	−1·002	−0·099	−1·101	−0·721	+0·904	9	8	0	0·225	9·0
12	1760	97·56 Air 2·44 H	138·55	7·474	−1·032	−0·11	−1·142	−0·825	+0·814	13	8	0·001	0·053	8·2
13	3100	4·375 Air 95·625 H	87·74	88·66	+0·178	+0·011	+0·189	+0·215	+0·248	14	6	0·08	0·17	4·6
14	3300	6·08 Air 93·92 H	91·52	92·951	+0·081	+0·005	+0·086	+0·094	+0·132	18	8	0·157	0·07	3·2
15	3000	5·043 Air 94·957 H	73·99	90·353	+0·072	+0·005	+0·077	+0·104	+0·136	20	10	0·18	0·11	1·65
16	3000	2·99 Air 97·01 H	85·15	89·242	+0·111	+0·007	+0·118	+0·139	+0·159	42	15	0·472	0·44	3·2
17	2900	4·13 Air 95·87 H	104·72	89·858	+0·073	+0·004	+0·077	+0·073	+0·098	15·5	10	0·09	0·035	6·2
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Remarks on the Tables.

The correction for conduction of heat through the plug, inserted in column 6 of Table I., and in column 7 of the rest of the Tables, was obtained from data furnished by experiments in which the difference between the temperature of the bath and the air was purposely made very great. It was considered as directly proportional to the difference of temperature, and inversely to the quantity of elastic fluid transmitted in a given time.

The 10th column of Tables II., III., IV., and V. is calculated on the hypothesis that, in mixtures with other gases, atmospheric air retains its thermal qualities without change. This hypothesis is almost certainly incorrect, since it is reasonable to expect that the effect of mixture on the physical character is experienced by each of the constituent gases. The column is given as one method of showing the effect of mixture.

Effect of Mixture on the Constituent Gases.—Although the experiments on nitrogen given in Table III. are not so numerous as might be desired, we may infer from them, and the results in Table II., that common air and all other mixtures of oxygen and nitrogen behave more like a perfect gas, *i. e.* give less cooling effect than either one or the other gas alone. We might expect the mixture to be something intermediate between the two. But this does not appear to be the case. The two are very nearly equal in their deviations from the condition of a perfect gas. Nitrogen deviates less than oxygen, but oxygen mixed with nitrogen differs less than nitrogen!

In the case of carbonic acid, which at low temperatures (7°) deviates five times as much as atmospheric air, we might expect that a mixture of CO_2 and air would deviate more than air and less than CO_2. This is the case (see Table IV.). Further, we might expect the two to contribute each its proportion of cooling effect according to its own amount, and its specific heat volume for volume. But do the mixtures exhibit such a result? No! See column 10, Table IV., in which also note, under experiments 8 and 9, the great diminution produced by the admixture of hydrogen.

If, instead of attributing to air and carbonic acid moments in proportion to their specific heats, or 1 : 1·39, as we have done in column 10, we use 1 : ·7, we obtain more consistent results.

Let δ denote the cooling effect experienced by air per 100 inches of mercury, δ' that by carbonic acid, and Δ that by a mixture of volume V of air, and V′ of carbonic acid; then we may take

$$\Delta = \frac{mV\delta + m'V'\delta'}{mV + m'V'}$$

to represent the cooling effect for the mixture, where m and m' are numbers which we may call the moments (or importances) of the two in determining the cooling effect for the mixture. The ratio of m to m' is the proper result of each experiment on a mixture, if we knew with perfect accuracy the cooling effect for each gas with none of the other mixed. Now for common air we have direct experiments (Table I.), and know the cooling effect for it better than from any inferences from mixtures. But for pure CO_2 we know the effect, for the most part, only inferentially. Hence, having tried making $m : m' :: 1 : 1{\cdot}39$ without obtaining consistent results, we tried other proportions; and, after various attempts, found that $m : m' :: 1 : {\cdot}7$, for all temperatures and pressures within the limits of our experiments, gives results as consistent with one another as the probable errors of the experiments justify us in expecting. Thus, using the formula

$$\Delta = \frac{V\delta + V'\delta' \times {\cdot}7}{V + V' \times {\cdot}7},$$

we have, for calculating the effect for CO_2 from any experiment on a mixture, the following formula,

$$\delta' = \frac{(V + V' \times {\cdot}7)\Delta - V\delta}{V' \times {\cdot}7}.$$

Hence, using the numbers in columns 3 and 9 of Table IV. which relate to mixtures of air and carbonic acid alone, we find

TABLE VI.

No. of experiment.	Proportions of mixtures.		Temperature of bath.	Thermal effect for air.	Deduced thermal effect for pure CO_2.
	Air.	CO_2.	°	°	°
1	68·42	31·58	7·36	—·88	—4·51
2	89·16	10·84	7·36	—·88	—4·61
3	3·52	96·48	7·38	—·88	—4·46
4	62·5	37·5	7·41	—·88	—4·19
5	88·13	11·87	7·43	—·88	—3·98
6	97·46	2·54	7·61	—·88	—3·89
16	1·83	98·17	35·6	—·75	—3·44
14	67·7	32·3	49·7	—· 7	—3·04
15	87·77	12·23	49·76	—· 7	—2·77
13	0·83	99·17	54	—·66	—2·96
10	2·11	97·89	93·52	—·51	—2·19
11	56·78	43·22	91·26	—·51	—2·21
12	77·77	22·23	91·64	—·51	—3·08
17	1·66	98·34	97·55	—·49	—2·16
1	2		3	4	5

The agreement for each set of results at temperatures nearly agreeing (with one exception, No. 12), shows that the assumption $m:m'::1:\cdot 7$ cannot be far wrong within our limits of temperature.

[Received subsequently to the reading of the Paper.]

Application of the preceding results to deduce approximately the Equation of Elasticity for the gases experimented on.

The "equation of elasticity" for any fluid is the most appropriate name for the equation expressing the relation between the pressure and the volume of any portion of the fluid. As this relation depends on the temperature, the equation expressing it involves essentially three variables, which, as in our previous communications on this subject, we shall denote by p, v, t. Of these, p is the pressure in units of force per unit of area, v the volume of a unit mass of the fluid, and t the temperature according to the absolute thermodynamic system of thermometry * which we have proposed. As before, we shall still adopt a degree, or thermometric unit, agreeing approximately with the degree Centigrade of the air-thermometer; according to which, as we have demonstrated by experiment †, the value of t for the freezing-point is within a few tenths of a degree of 273·7 (its value at the standard boiling-point being, by definition of the Centigrade scale, 100° more than at the freezing-point).

Instead of, as in our previous communications, taking v and t as independent variables, we shall now take p and t; and we shall accordingly consider the object of the equation of elasticity as being to express v explicitly as a function of p and t. Whatever may be the relation between these elements, the thermal effect, $d\vartheta$ (reckoned as positive when it is a rise in temperature), produced by forcing the fluid in a continuous stream through a narrow passage or porous plug by an infinitely small difference of pressures, dp, will

* Philosophical Transactions, 1854, p. 350. † Ibid. p. 352.

be given by the formula

$$\frac{d\vartheta}{dp}=-\frac{1}{\mathrm{JK}}\left(t\frac{dv}{dt}-v\right),$$

where K denotes the thermal capacity, under constant pressure, of unit of mass of the fluid. This formula may be derived from equation (15.) of our previous communication already referred to, by substituting p, v, and $-\vartheta$ for P, V, and δ in that equation, changing to p and t, instead of v and t, as independent variables, and differentiating with reference to p. It is scarcely necessary to remark that a direct demonstration of our present formula, founded on elementary thermodynamic principles, may be readily obtained.

Each experiment, of the several series recorded above, gives a value for $\frac{d\vartheta}{dp}$, which is found by multiplying the "corrected thermal effect" by $\frac{\cdot 299218}{2114}$, to reduce from the amounts per 100 inches of mercury to the amounts per pound per square foot. Now by examining carefully the series of results for different temperatures, in the cases of atmospheric air and of carbonic acid, we find that they follow very closely the law of varying inversely as the square of the absolute temperature (or temperature Centigrade with 273·7 added). Thus for air the formula

$${}^{\circ}\!\cdot 92\times\left(\frac{273\cdot 7}{t}\right)^{2}$$

and for carbonic acid

$$4^{\circ}\!\cdot 64\times\left(\frac{273\cdot 7}{t}\right)^{2}$$

express, the former almost accurately, the latter with a deviation which we shall hereafter investigate, the results through the whole range of temperature for which the investigation has been carried out.

Air.

Temperature.	Actual cooling effect.	Theoretical cooling effect.
0°	°·92	°·92
7·1	·88	·87
39·5	·75	·70
92·8	·51	·51

Carbonic acid.

Temperature.	Actual cooling effect.	Theoretical cooling effect.
0°	4°·64	4°·64
7·4	4·37	4·4
35·6	3·41	3·63
54·0	2·95	3·23
93·5	2·16	2·57
97·5	2·14	2·52

We have not experiments enough to establish the law of variation with temperature of the thermal effect for the pure gases oxygen and nitrogen, or for any stated mixture of them other than common air; but there can be no doubt, from the general character of the results, that the same law will be about as approximately followed by them as it is by air.

Hence we may presume that in all these cases the cooling effect is very well represented by the formula

$$\frac{-d\vartheta}{dp}=A\left(\frac{273\cdot 7}{t}\right)^2.$$

Comparing this with the general formula given above, we find

$$t\frac{dv}{dt}-v=AJK\left(\frac{273\cdot 7}{t}\right)^2.$$

The general integral of this differential equation, for v in terms of t, is

$$v=Pt-\tfrac{1}{3}AJK\left(\frac{273\cdot 7}{t}\right)^2,$$

P denoting an arbitrary constant with reference to t, which, so far as this integration is concerned, may be an arbitrary function of p. To determine its form, we remark in the first place, in consequence of BOYLE'S law, that it must be approximately $\frac{C}{p}$, C being independent of both pressure and temperature; and thus, if we omit the second term, we have two gaseous laws expressed by the approximate equation

$$v=\frac{Ct}{p}.$$

Now it is generally believed that at higher and higher temperatures the gases approximate more and more nearly to the rigorous fulfilment of BOYLE'S law. If this is true, the complete expression for P must be of the form $\frac{C}{p}$, since any other would simply show deviation from BOYLE'S law at very high temperatures, when the second term of our general integral disappears. Assuming then that no such deviation exists, we have, as the complete solution,

$$v=\frac{Ct}{p}-\tfrac{1}{3}AJK\left(\frac{273\cdot 7}{t}\right)^2 p.$$

This is an expression of exactly the same form as that which Professor RANKINE found applicable to carbonic acid, in the first place to express its deviations from the laws of BOYLE and GAY-LUSSAC, as shown by REGNAULT'S experiments, and which he afterwards proved to give correctly the law and the absolute amount of the cooling effect demonstrated by our first experiments on that gas*.

That more complicated formulæ were found for the law of elasticity for common air both by Mr. RANKINE and by ourselves, now seems to be owing to an irreconcileability among the data we had from observation. The whole amounts of the devia-

* Philosophical Transactions, 1854, Part II. p. 336.

tions from the gaseous laws are so small, for common air, that very small absolute errors in observations of so heterogeneous a character as those of REGNAULT on the law of compression and on the changes produced by pressure in the coefficients of expansion, and our own on the thermo-dynamic property on which we have experimented, may readily present us with results either absolutely inconsistent with one another, or only reconcileable by very strained assumptions. It is satisfactory now to find, when we have succeeded in extending our observations through a considerable range of temperature, that they lead to so simple a law; and it is probable that the formula we have been led to by these observations alone, will give the deviations from BOYLE'S law, and the changes produced by pressure in the coefficients of expansion, with more accuracy than has hitherto been attained in attempts to determine these deviations by direct observation. We must, however, reserve for a future communication the comparison between such results of our theory and experiments and REGNAULT'S direct observations. In the mean time we conclude by putting the integral equation of elasticity into a more convenient form, by taking $C=\frac{\mathfrak{h}}{t_0}$, where $\mathfrak{h}$ denotes the "height of the homogeneous atmosphere" for the gas under any excessively small pressure, at any temperature t_0, and taking t_0 to denote the absolute temperature of freezing water, in which case we shall have, as nearly as observations hitherto made allow us to determine,

$$t_0=273^\circ{\cdot}7.$$

Then, in terms of this notation, and of that above explained, in which t, p, v denote absolute temperature, pressure in pounds weight per square foot, and volume in cubic feet of one pound of air, the equation of elasticity investigated above becomes

$$v=\frac{\mathfrak{h}t}{pt_0}-\tfrac{1}{3}\mathrm{AJK}\left(\frac{t_0}{t}\right)^2,$$

where A denotes the amount of the thermal effect per pound per square foot, determined by our observations, reckoned positive when it is a depression of temperature.

XXVI. *On the Law of Expansion of Superheated Steam.*

By WILLIAM FAIRBAIRN, *Esq., LL.D., F.R.S., and* THOMAS TATE, *Esq.*

Received March 20,—Read April 3, 1862.

THE following experiments have been undertaken to verify the law of expansion for superheated steam indicated in a previous paper*.

The earliest experiments on the subject were made by Mr. FROST in America, but without sufficient accuracy to be of scientific value. Mr. SIEMENS has also experimented on the expansion of steam isolated from water; his results give a much higher rate of expansion for steam than for ordinary gases; but, owing to the obvious defects of Mr. SIEMENS's method of conducting the experiments, we consider that his results are not reliable.

For gases, the rate of expansion is expressed by the formula, for constant volume,

$$\frac{P}{P_1}=\frac{E+t}{E+t_1}, \quad \ldots \ldots \ldots \quad (1.)$$

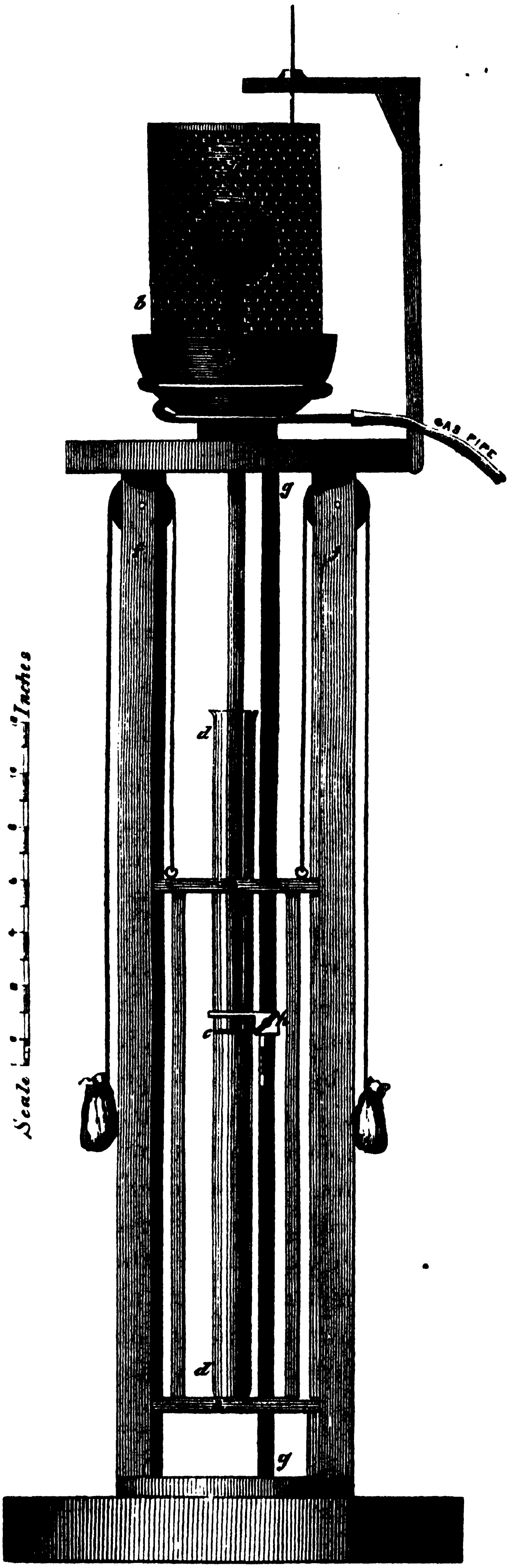
where E is a constant derived from experiment and determined by REGNAULT to be 459 in the case of air In the paper alluded to, it was shown that, with a certain proviso, the rate of expansion of superheated steam nearly coincided with that of air. Within a short distance of the maximum temperature of saturation, the rate of expansion of steam was found to be exceedingly variable; near the saturation-point it is higher than that of air, and it decreases as the temperature is increased until it becomes sensibly identical with that of air. The results on which this law was based were too limited in their range for much numerical accuracy in the constants deduced.

Hence it has been our object in the present paper to supply the deficiency in the previous one, by affording experimental data of the expansion of steam at higher temperatures and with a greater range of superheating than was possible with the apparatus employed in ascertaining the density of steam. The results obtained in these later researches, however, confirm the general law deduced from the previous ones.

The annexed diagram represents the experimental apparatus employed where the pressures did not exceed that of the atmosphere. It consists of a glass globe *a*, about three inches in diameter, and with a stem about thirty-five inches long. The capacity of this globe was known to a point *b* on the stem, where a piece of fine platinum wire was twisted round it to mark accurately the level to which the mercury column in the stem was to be brought to maintain a constant volume in the globe. The stem dipped below into the

* "Experimental Researches on the Density of Steam at different Temperatures, and to determine the Law of Expansion of Superheated Steam," Phil. Trans. 1860, p. 185.

$1\frac{1}{4}$-inch tube *d*, which was supported on a frame of wood *e e*, sliding up and down vertically on the larger frame *f f*, on which the apparatus was supported. The frame *e e*, and the tube *d*, were balanced by cords passing over pulleys in the top of the frame, and weighted by bags filled with small pieces of iron. In this way the tube *d* could be adjusted with the greatest facility, so as to maintain the upper level of the mercury column at a constant position. To read the lower and variable level of the column, a rod *g g*, graduated into tenths of an inch, was fixed vertically, carrying a finger and vernier, *h*, which could be made to coincide accurately with the top of the meniscus of mercury, *c*.



For heating the globe, a glass bath, *k*, containing oil, was provided, fixed in an outer iron mercury bath, *l*. This was heated by a coil of gas-jets, and the oil was stirred continually, excepting at the instant of adjusting the upper level of the column to the level of the platinum wire. The globe was fixed in the oil-bath by a stuffing-box.

The method of conducting the experiments was as follows. The globe *a* was filled with dry and warm mercury, the air-bubbles being extracted from time to time by an air-pump. It was then inverted to form a Torricellian vacuum. A small glass globule of water was then inserted, the platinum wire fixed in its place, and an india-rubber cap fitted over the extremity of the stem. It was then transferred to its place in the oil-bath, and fixed there. The india-rubber cap was replaced by an open glass cistern, so that the glass tube *d* could be elevated to its position. The gas-jets were lighted, and the temperature raised to 300°. From this point the levels of the column were read off at intervals of 50° until the temperature of saturation was reached. The levels were taken in a series of descending temperatures, to avoid the influence of steam boiling out of the mercury as the temperature rose, and to eliminate the effect of the cohesion of the glass on the water, as explained in our previous paper on the Density of Steam.

Capacity of the Globe.

Twelve cubic inches of mercury were measured into the globe, and a file-mark made on the stem.

Below the first file-mark, at a distance of 14·45 inches, another file-mark was made to afford a fixed point for ascertaining the correspondence of the upper file-mark with the readings on the fixed graduated rod or cathetometer.

Correction of Readings.

Let a be the reading on the fixed rod of the level of the column; b be the reading of the lower file-mark on the globe-stem. Then

$$+b-a=\text{the height of the column of mercury in the globe-stem.}$$

To correct this for temperature, $7\frac{1}{2}$ inches of mercury, included in the oil-bath and its stuffing-box, were corrected for the temperature of the oil, and the remainder for the temperature of the atmosphere at the time.

By deducting the column so corrected from the reading of the barometer at the time, the total pressure in the globe is obtained.

The readings of the thermometer are corrected for the portion out of the oil-bath.

The pressure of the vapour of mercury is calculated from data supplied with great courtesy by M. Regnault, and embodying the results of unpublished experiments. The pressure of this vapour is assumed to be the same as that in a vacuum, as the vapour in the globe remains still for a sufficient time (it is believed) for saturation to take place. In this view we have been strengthened by M. Regnault's opinion.

By deducting the pressure of mercury vapour from the total pressure in the globe, the pressure of the steam is obtained.

Experiment I.

0·285 grain of water introduced into globe.

Time.	Temperature in oil-bath, Fahr.	Cathetometer, in inches.	Column of mercury, in inches.	Temperature of air, Fahr.	Remarks.
h m					
4 46	250°	10·96	25·57	58°	Barometer read 30·33 inches, or, corrected to 32° F., 30·27 in. The lower file-mark read 22·08 inches on the cathetometer.
4 55	250	10·96	25·57		
5 8	300	11·40	25·13		
5 14	300	11·38	25·15		
5 17	300	11·40	25·13		
5 20	300	11·40	25·13		
5 37	250	10·97	25·56		
5 40	250	10·96	25·57		
6 0	200	10·63	25·90		
6 8	200	10·60	25·93		
6 13	200	10·60	25·93		
6 45	150	10·28	26·25		
6 50	150	10·25	26·28		
7 25	100	7·90	28·63		
7 30	100	7·87	28·66	57	Barometer read 30·36 inches, or, corrected to 32° F., 30·30 inches.

Summary of Results.

Temperature, Fahr.	Mean column of mercury, in inches.	Mean column corrected to 32°.	Total pressure in globe, in inches.	Volume of globe corrected for expansion of glass, in grs. of water.	Pressure of vapour of mercury, in inches.	Pressure of steam, in inches.	Value of E.
302°·88	25·13	24·89	5·38	3045·5	0·1736	5·21	474·48
251·64	25·55	25·34	4·94	3043·4	0·066	4·87	450·11
200·74	25·92	25·75	4·54	3041·2	0·024	4·52	428·88
150·18	26·26	26·13	4·17	3039·1	0·008	4·16	
100·00	28·64	28·54	1·76	3037·0	0·003	1·76	

Experiment II.

0·405 grain of water introduced into globe.

Time.	Temperature in oil-bath, Fahr.	Cathetometer, in inches.	Column of mercury, in inches.	Temperature of air, Fahr.	Remarks.
h m					
4 30	300°	13·43	23·16	59°	Barometer read 30·38 inches throughout the experiment, or, corrected to 32° F., 30·31 inches. Lower file-mark read 22·14 inches on the cathetometer.
4 40	300	13·45	23·14		
4 45	302	13·50	23·09		
4 50	300	13·49	23·10		
5 10	250	12·90	23·69		
5 16	250	12·90	23·69		
5 20	250	12·90	23·69		
5 40	200	12·43	24·16	60	
5 45	200	12·43	24·16		
5 50	200	12·43	24·16		
6 10	150	11·94	24·65		
6 15	150	11·98	24·61		
6 20	150	11·97	24·62		
6 25	150	11·97	24·62		

Summary of Results.

Temperature, Fahr., corrected.	Mean column of mercury, in inches.	Mean column corrected to 32°.	Total pressure in globe, in inches.	Volume of globe corrected for expansion of glass, in grs. of water.	Pressure of vapour of mercury, in inches.	Pressure of steam, in inches.	Value of E.
302°·88	23·11	22·87	7·44	3045·5	0·174	7·27	} 455·57 *
251·64	23·69	23·49	6·82	3043·4	0·066	6·75	
200·74	24·16	23·99	6·32	3041·2	0·024	6·30	
150·18	24·62	24·49	5·82	3039·1	0·008	5·81	443·86

In the readings at 251·64 some error seems to have crept in. In deducing E, the readings at 302·88 and 200·74 are therefore taken.

Experiment III.

0·545 grain of water introduced into globe.

Time.	Temperature of oil-bath, Fahr.	Cathetometer, in inches.	Column of mercury, in inches.	Temperature of air, Fahr.	Remarks.
h m					
3 55	299°	16·05	20·26	59°	Barometer read 30·13 inches, or, corrected to 32° F., 30·06 inches throughout the experiment. Lower file-mark read 21·86 inches on the cathetometer.
4 0	300	16·05	20·26		
4 5	300	16·05	20·26		
4 33	250	15·33	20·98		
4 40	250	15·32	20·99	60	
4 45	250	15·34	20·97		
5 3	200	14·66	21·65		
5 10	200	14·66	21·65		
5 14	200	14·65	21·66	61	
5 32	150	13·05	23·26	....	Condensation in globe.
5 37	150	13·10	23·21		
5 43	150	13·05	23·26		

Summary of Results.

Temperature, Fahr., corrected.	Mean column of mercury, in inches.	Mean column, corrected to 32°.	Total pressure in globe, in inches.	Volume of globe corrected for expansion of glass, in grs. of water.	Pressure of vapour of mercury, in inches.	Pressure of steam, in inches.	Value of E.
302°·88	20·26	20·03	10·03	3045·5	0·174	9·86	466·85
251·64	20·98	20·78	9·28	3043·4	0·066	9·21	451·94
200·74	21·65	21·49	8·57	3041·2	0·024	8·55	109·74
150·18	23·25	23·12	6·94	3039·1	0·008	6·93	

Experiment IV.

0·53 grain of water introduced into globe.

Time.	Temperature in oil-bath, Fahr.	Cathetometer, in inches.	Column of mercury, in inches.	Temperature of air, Fahr.	Remarks.
h m					
12 0	300°	16·62	20·63	65°	Barometer read 30·17 to 30·18 inches during experiment, the corrected reading being 30·09 inches. Lower file-mark read 22·80 inches on the cathetometer.
12 5	300	16·62	20·63		
12 15	300	16·61	20·64		
12 44	250	15·93	21·32	65	
12 51	250	15·92	21·33		
1 5	250	15·92	21·33		
1 38	200	15·27	21·98		
1 45	200	15·28	21·97		
1 52	200	15·28	21·97	66	
2 35	150	13·95	23·30		
2 47	150	14·00	23·25		
3 0	150	14·01	23·24		
3 5	150	14·00	23·25		
3 20	165	14·80	22·45		
3 23	165	14·79	22·46	67	

Summary of Results.

Temperature, Fahr., corrected.	Mean column of mercury, in inches.	Mean column, corrected to 32°.	Total pressure in globe, in inches.	Volume of globe corrected for expansion of glass, in grs. of water.	Pressure of vapour of mercury, in inches.	Pressure of steam, in inches.	Value of E.
302°·88	20·63	20·39	9·70	3045·5	0·174	9·53	464·83
251·64	21·33	21·12	8·97	3043·4	0·066	8·90	460·79
200·74	21·97	21·80	8·29	3041·2	0·024	8·27	378·33
165·45	22·46	22·31	7·78	3039·7	0·011	7·77	
150·18	23·25	23·11	6·98	3039·1	0·008	6·97	

Experiment V.

0·875 grain of water introduced into globe.

Time.	Temperature in oil-bath, Fahr.	Cathetometer, in inches.	Column of mercury, in inches.	Temperature of air, Fahr.	Remarks.
h m					
1 10	300°	23·25	14·03	63°·5	Barometer read 30·13 inches, or, corrected to 32° F., 30·06 inches. Lower file-mark read 22·83 inches on the cathetometer.
1 15	301	23·27	14·01		
1 20	300	23·26	14·02		
1 25	300	23·25	14·03		
1 45	250	22·14	15·14		
1 50	250	22·11	15·17		
2 0	250	22·14	15·14		
2 5	250	22·13	15·15		
2 35	200	21·00	16·28		
2 41	200	21·00	16·28		
2 45	200	21·01	16·27	63·5	
2 50	200	21·00	16·28		
3 5	180	20·50	16·78		
3 15	180	20·49	16·79		
3 20	180	20·49	16·79	64·0	
3 35	165	17·27	20·01		
3 45	167	17·85			
3 50	165	17·32	19·96		
4 0	165	17·30	19·98		

Summary of Results.

Temperature, Fahr., corrected.	Mean column of mercury, in inches.	Mean column, corrected to 32°.	Total pressure in globe, in inches.	Volume of globe corrected for expansion of glass, in grs. of water.	Pressure of vapour of mercury, in inches.	Pressure of steam, in inches.	Value of E.
165°·45	19·98	19·84	10·22	3039·7	0·011	10·21	
180·72	16·79	16·65	13·41	3040·6	0·016	13·39	
200·74	16·28	16·13	13·93	3041·2	0·024	13·91	430·90
251·64	15·15	14·97	15·09	3043·4	0·066	15·02	460·28
302·88	14·02	13·80	16·26	3045·5	0·174	16·09	

Summary of Results.

The law of expansion of gaseous bodies is expressed by the formula

$$\frac{E+t}{E+t_1}=\frac{PV}{P_1V_1},$$

$$\therefore\ E=\frac{PVt_1-P_1V_1t}{P_1V_1-PV},$$

where E is a constant. The values of E thus deduced have been placed in the last column of the foregoing Tables. They show a decreasing rate of expansion from the saturation-point upwards until (at a certain increase of temperature) the rate of expansion coincides with that of a perfect gas.

Taking from the preceding Tables the two results, which in each case represent the rate of expansion at the greatest distance from the saturation-point, we have the following values of E:—

(1)	474·48
	450·11
(2)	455·57
	443·86
(3)	466·85
	451·94
(4)	464·83
	460·79
(5)	460·28
	9)4128·71
Mean value of E=	458·74

The value of E for air, as ascertained by REGNAULT, is 459. That assumed for a perfect gas by RANKINE is 461·2.

Hence the conclusion which we suggested in our previous paper has been satisfactorily demonstrated in more carefully conducted experiments, and the rate of expansion of superheated steam is shown to be almost identical with that of air and other permanent gases, if calculated at temperatures not too close to the maximum temperature of saturation.

XXVII. *On the Long Spectrum of Electric Light.* By G. G. Stokes, *M.A.*, *D.C.L.*, *Sec. R.S.*, *Lucasian Professor of Mathematics in the University of Cambridge.*

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Introduction.

The experimental researches described in a former paper * led me indirectly to the conclusion that the electric spark, whether obtained directly from the prime conductor of an ordinary electrifying machine, or from the discharge of a Leyden jar, emits rays of very high refrangibility, surpassing in this respect any that reach us from the sun—and that these rays pass freely through quartz, while glass absorbs them, as it does also the most refrangible of the solar rays. I was induced in consequence to procure prisms and a lens of quartz, which were applied in the first instance to the examination of the solar spectrum, and which immediately revealed the existence of an invisible region extending as far beyond that previously known as the latter extends beyond the visible spectrum, and exhibiting a continuation of Fraunhofer's lines †. A map of the new lines was exhibited at an evening lecture delivered before the British Association at their Meeting in Belfast in the autumn of the same year; and I then stated that I conceived we had obtained evidence that the limit of the solar spectrum in the more refrangible direction had been reached. In fact, the very same arrangement which revealed, by means of fluorescence, the existence of what were evidently rays of higher refrangibility coming from the electric spark failed to show anything of the kind when applied to the solar spectrum. At least, the only link in the chain of evidence which remained to be supplied by direct experiment related to the reflecting power, for rays of high refrangibility, of the metallic speculum of the heliostat which was employed to reflect the sun's rays into a convenient direction; and this was shortly afterwards tested by direct experiment, on rays from an electric discharge separated by prismatic refraction.

In making preparations for a lecture on the subject delivered at the Royal Institution in February 1853, in which I had the benefit of the kind assistance of Mr. Faraday, recourse was naturally had to electric light, on account of the extraordinary richness which it had been found to possess in rays of high refrangibility. Although fully prepared to expect rays of much higher refrangibility than were found in the solar spectrum, I was perfectly astonished, on subjecting a powerful discharge from a Leyden jar to prismatic analysis with quartz apparatus, to find a spectrum extending no less than six or eight times the length of the visible spectrum, and could not help at first suspecting that it was a mistake arising from the reflexion of stray light. A similarly extensive

* "On the Change of Refrangibility of Light," Phil. Trans. for 1852, p. 463.

† Ibid. p. 559.

spectrum was obtained from the voltaic arc, and this was sufficiently bright to be exhibited to the audience, the arc passing between copper electrodes, and the pure spectrum formed by quartz apparatus being received on a piece of uranium glass cut for the purpose. The spectrum thus formed was found to consist entirely of bright lines *, whereas the spectrum of the discharge of a Leyden jar had appeared (perhaps from not having been truly in focus) to be continuous, or at least not wholly discontinuous.

The mode of absorption of light by coloured solutions, as observed by the prism, affords in many cases most valuable characters of particular substances, which, strange to say, though so easily observed, have till very lately been almost wholly neglected by chemists. Having obtained the long spectrum above mentioned, I could not fail to be interested with the manner in which substances, especially pure but otherwise imperfectly known organic substances, might behave as to their absorption of the rays of high refrangibility. But the difficulties attending the habitual use of a nitric-acid battery of 30 or 40 cells deterred me from entering on this investigation, and I determined to confine myself to the solar spectrum.

On account of some inconvenience attending the tarnishing of the speculum of my heliostat, I was induced to order a quartz plate, intended to be either silvered or coated with the usual amalgam of tin. On trying on a small scale the reflecting power of such plates with respect to the invisible rays, which may be done by means of fluorescence almost as easily as if those rays were visible †, I noticed a remarkable falling off in the reflecting power of the silvered plate for the most refrangible of the solar rays, which I readily found was due to a peculiarity of the metal silver. This metal is highly reflective for the invisible as it is for the visible rays up to about the fixed line S ‡, when its reflecting power falls off, with remarkable rapidity, and for the more refrangible rays of the solar spectrum is comparable with that of a vitreous substance rather than with that of a metal. Steel, gold, tin, &c. showed nothing of the kind, but copiously reflected the invisible rays.

A few years ago, as Dr. ROBINSON was showing me some experiments with the induction coil, it seemed worth while to try whether the spark obtained when a Leyden jar has its coatings connected with the secondary terminals might not be sufficiently strong to exhibit by projection the long spectrum shown by electric light. On projecting a spectrum formed by a prism and lens of quartz on a piece of uranium glass, the long spectrum was in fact exhibited. It was not, indeed, so bright as when formed by means of a powerful voltaic battery, but nevertheless was quite bright enough to work by. It was discontinuous, consisting of bright lines. On changing the metals between which the spark passed, we found that the lines were changed, which showed clearly that they were due to the particular metals.

* Proceedings of the Royal Institution, vol. i. p. 264.

† Philosophical Transactions for 1852, p. 537.

‡ According to the notation employed in the Map published in the Philosophical Transactions for 1859, Plate XLVII. In this Plate the group S should have been represented as three lines, of which the middle (specially named S) divides the interval between the 1st and 3rd in the proportion of 3 to 2 nearly, the spaces between the lines being a little darkened by shading.

A wide field of research was thus thrown open to any one taking the very moderate trouble attending the use of an induction coil. It remained to study the lines given by different metals and gases, and the absorbing action of various substances with respect to the invisible rays of different refrangibilities.

Various observations were made from time to time in this subject. As regards the metallic lines, it is perfectly easy to view them at pleasure; but to obtain faithful delineations of them is another matter. Even an accomplished artist would find difficulty in obtaining by mere eye-sketching a faithful representation of an object which requires to be seen in the dark. I tried different methods without being able to satisfy myself as to the accuracy of the drawings which could be thus obtained, and frequently thought of resorting to photography.

Meanwhile the mode of absorption of the rays of high refrangibility by a good number of substances was observed. Nothing is easier, to a person provided with a cell with parallel faces of quartz, than to observe by means of fluorescence the mode of absorption of these rays by a given solution; but to draw safe conclusions as to the optical character in this respect of the substance deemed to be in solution is not so easy as it might appear; for the rays of high refrangibility are liable to be absorbed by an exceedingly small amount of an impurity which may chance to be present without the observer's knowledge. Thus I found that about a quarter of a square inch of clean filtering paper sufficiently contaminated the water contained in a small cell to interfere sensibly with its transparency. Should the solution be transparent there would be no difficulty, for the effect of an impurity would not be to render transparent a solution which otherwise would be opaque. Should it, on the other hand, absorb the invisible rays, or some of them, with great energy, or in a peculiar manner, we might again conclude that we had obtained the true character of the substance deemed to be observed. The most remarkable example of this kind which I met with among inorganic colourless solutions was in the case of nitric acid and its salts, such as nitrate of potash, soda, ammonia, baryta, which absorb the rays of high refrangibility with great energy and in a peculiar manner, exhibiting a maximum of opacity followed by a maximum of transparency, beyond which the absorption becomes still more energetic than before. But if the solution should be found to absorb the rays of high refrangibility with only moderate energy, it would be left doubtful whether the observed absorption might not be due to some impurity; and I did not see how this doubt could be solved otherwise than by a laborious system of recrystallizations.

After having obtained these results, I found by conversation with my friend Dr. Miller that he also had been engaged at the same subject, working by photography, and had prepared a number of photographs of metallic spectra, and studied by the same means the absorption of the rays of high refrangibility by a great variety of substances, chiefly inorganic acids, bases, and salts, and the commoner organic bodies. Although a large part of the task which I had proposed to myself has thus been accomplished in another way, there are many results which I have met with which are not likely to have been obtained by one working by photography, and I have therefore thought it well to draw up a paper embodying these results, and thus forming, as it were, a supplement to the paper by Dr. MILLER.

Preparation of a Screen by means of a Salt of Uranium.

Few substances are more powerfully fluorescent than several of the salts of sesquioxide of uranium; and a piece of glass coloured by uranium and polished along at least two planes at right angles to each other is exceedingly convenient, from its powerful fluorescence and its permanence, for a screen on which to receive a spectrum. Nevertheless such a screen, which must be viewed in particular directions in order to get the strongest effect, is in many cases less convenient than a screen would be which was prepared by means of a highly fluorescent powder treated like a water colour, which could be viewed in all directions indifferently. This is especially the case in taking measures by a method which will be mentioned presently. Besides, I find an excellent piece of such glass defective in fluorescent power as regards the extreme lines shown by aluminium; and some specimens are defective to a much greater extent, which is doubtless due to impurities. Accordingly I have long regarded it as a desideratum to obtain by precipitation an insoluble or very sparingly soluble salt of sesquioxide of uranium which should be as fluorescent as the best salts of that base, and which might be treated like a water colour. I have now succeeded in preparing such a salt, though not by direct precipitation.

The ordinary phosphate obtained by precipitation, the composition of which, independently of water of hydration, is $PO_5(U_2O_3)_2HO$, is only slightly fluorescent. If, however, this salt, with as much water as remains when it is washed by decantation, be put into a saucer, a little free phosphoric or sulphuric acid added, and then crystals of phosphate of soda, phosphate of ammonia, microcosmic salt, or borax be added in excess, the original salt is gradually changed into one which is powerfully fluorescent. The change seems to take place most rapidly with borax; but as an excess of this salt is liable slowly to decompose the fluorescent salt first formed, it is better to employ a phosphate. The quantity of acid should be sufficient to leave a decided acid reaction when the liquid is fully saturated by the alkaline phosphate employed. The change may be watched by observing from time to time the fluorescence of the salt by daylight, with the aid of absorbing media. It is complete in a few days at furthest, when the salt is ready to be collected.

This requires precaution, as the salt is quickly decomposed by dilute acids (and accordingly by its own mother-liquor if diluted), and even, though more slowly, by pure water, with the formation apparently of the original phosphate. It is also decomposed, at least in time, by alkaline carbonates, with the formation of a beautiful yellow non-fluorescent salt resembling the precipitate given by alkaline carbonates in salts of sesquioxide of uranium. The salt may be collected by adding at once, instead of water, a saturated solution of borax, in quantity at least sufficient to destroy the acid reaction.

The salt is then poured off in suspension from any undissolved crystals of the alkaline phosphate employed, and collected on a filter. A pressed cake of this salt, or a porous tile on which the salt is spread, having been moistened with a solution of borax, forms an admirable screen, and is what I have chiefly employed of late. It shows, of course, the visible as well as the invisible rays—the former by ordinary scattering, the latter by fluorescence.

From the circumstances of its formation, the salt is probably (abstraction being made of the water of hydration) the original phosphate with the equivalent of constitutional water replaced by an equivalent of an alkali, which would make it analogous to the highly fluorescent natural yellow uranite. At any rate this hypothesis guides us to its successful preparation, the conditions of which it would not have been easy to make out by observation alone. Without the use of free acid the fluorescence is not fully developed, which is accounted for by the insolubility of the original phosphate and the fluorescent salt, which presents an obstacle to the complete conversion of the one into the other.

Metallic Lines.

These may be viewed, as already mentioned, by passing the spark of an induction coil between two electrodes formed of the metal to be examined (the secondary terminals being respectively in connexion with the coatings of a jar of suitable size), forming a pure spectrum by a prism and lens of quartz, the faces of the prism being equally inclined to the axis of the crystal, and the lens being cut perpendicular to the axis, and receiving the spectrum on a suitable screen, for which, if a fluorescent liquid be employed, it is to be placed in a quartz-faced vessel, in default of which a piece of filtering paper may be saturated with the liquid.

If the visible spectrum and the very beginning of the invisible be excepted, the lines thus seen vary from metal to metal, and therefore are to be referred to the metal and not to the air. They are further distinguished from air lines by being formed only at an almost insensible distance from the tips of the electrodes, whereas air lines would extend right across. The spectrum is far too extended to allow us to regard the whole at once as in the position of minimum deviation; and if the prism be placed at all near the electrodes, without which we should have comparatively little light to work with, the effect of the different divergency, converted by the lens into convergency, of the rays in the primary and secondary planes is very great. In order to obtain a pure spectrum, the screen must be in focus as regards the primary plane; and if a particular point P of the spectrum be at a minimum deviation, the lines immediately about P are reduced almost to points, which are the images, for light of that refrangibility, of the tips of the electrodes, or, to speak more exactly, of the part of the spark just outside the tips. But in the secondary plane the rays on one side of P have not yet reached their focus, and on the other side have passed it; so that the image of a point is a line, the primary focal line, of a length increasing on receding from P in either direction, and accordingly the spectral image of either tip, assumed to be a mere point, would be a pair of slender triangles vertically opposite, and having their common vertex at P, their lengths lying in the plane of refraction. The invisible spectrum is in fact made up of two such pairs of triangles corresponding to the two tips respectively, as may be readily seen when the electrodes are not too close. At a distance from P at which the length of the primary focal line becomes equal to that of the image of the spark, the two lines which are the images, for rays of the refrangibility answering to that distance, of the tips of the electrodes meet in the middle of the spectrum, and beyond that distance they overlap, so that a line appears to run across the spectrum, though it relates to rays which emanated only from the immediate neighbourhood of the tips of the electrodes, as may be seen by turning the prism till that part of the spectrum is at a minimum deviation, and focusing afresh.

Besides the bright lines, evidently due to metals, which have been mentioned, other weaker light is perceptible, too faint for precise observation. A portion of this is probably due to the air.

The chief part of the visible spectrum as seen by projection appears plainly to belong to the air; for the lines stretch across the interval separating the electrodes, while the lines belonging to the metals extend but a little way, even in the visible spectrum, and the former reappear when the electrodes are changed. With some metals, however, lines belonging to the metal appear in the visible spectrum which are comparable in strength with the invisible lines of high refrangibility; but in general it is rather remarkable how poor is the visible spectrum, and even the invisible region for a good distance beyond, compared with the part of the spectrum of still higher refrangibility, with respect to strong lines characteristic of the metal.

I have lately adopted a mode of laying down positions in the invisible spectrum which is extremely simple and convenient, and yields results agreeing well with one another. It might be applied to the formation of maps of the metallic lines; but this is unnecessary, as the subject has been worked out by Dr. MILLER. It is still useful, however, for laying down the positions of bands of absorption, being more convenient and exact than estimating their place with reference to the known metallic lines.

The method is as follows. The quartz prism is placed on a block, raising it to a convenient height above a long drawing-board, to which the block is screwed, and is fixed at pleasure by a screw pressing upon it from above. The lens is fixed in a blackened board screwed edgeways to the drawing-board near the prism, so as to be ready to receive the rays of all refrangibilities after refraction through the prism. The focal length of the lens actually used was about 12 inches, and its diameter $1\frac{1}{4}$ inch. A convenient distance of the spark from the prism having been selected (I chose 30 inches), the drawing-board was turned round till it attained such a position that, on placing the prism in the position of minimum deviation for the middle of the long spectrum, the rays belonging to that part fell perpendicularly, or nearly so, on the lens, which had previously been placed so that this should be a convenient position relatively to the drawing-board. The prism was then fixed by its screw, and to mark the angle of incidence a pin was placed at the edge of the shadow of one of the blocks. On account of the increasing refraction by the lens of rays of increasing refrangibility, the locus of the foci of the different rays formed an arc of a curve, or nearly a straight line, lying very obliquely to the axes of the pencils coming through the lens. The projection of this line on the board having been marked, a line was drawn bisecting this at right angles, and at a point in the latter line situated $11\frac{2}{3}$ inches from the former*, the board was pierced for the insertion of a pivot, which carried two wooden rulers, which could be clamped together at any convenient angle. The shorter of these carried a vertical needle, which as the ruler was turned moved in front of the focus of the different rays at the distance of about a quarter of an inch. The longer ruler carried a pricker, destined to mark on a sheet of paper, temporarily fastened to the drawing-board, the position of any object observed. Thus the prism, the lens, the axis of motion of the needle and pricker, and the pin for fixing the angle of incidence retained an invariable relative position when the drawing-board was moved. In observing, the electrodes were placed at the proper distance, and the board turned till the edge of the shadow fell on the pin. The rulers were then turned together till any bright line or other object was eclipsed by the needle, and its place was then pricked down. To obtain a fixed point of reference, I generally pricked down the position of the extreme red visible on a screen, such as a piece of paper; but if great accuracy were required, it might be better to employ a well-marked green air line.

The metals the spectra of which I have observed are Platinum, Palladium, Gold, Silver, Mercury, Antimony, Bismuth, Copper, Lead, Tin, Nickel, Cobalt, Iron, Cadmium, Zinc, Aluminium, Magnesium. Several of these show invisible lines of extraordinary strength, which is especially the case with zinc, cadmium, magnesium, aluminium, and lead, which last, in a spectrum not generally remarkable, contains one line surpassing perhaps all the other metals. Other metals exhibit lines which in certain parts of the spectrum are both bright and numerous; so that, in taking a rough view of the whole, certain parts of the spectrum are bright and tolerably continuous, while other parts are comparatively weak. This grouping of the lines is especially remarkable in copper, nickel, cobalt, iron, and tin. Of the metals mentioned, magnesium gives by far the shortest spectrum, ending in a very bright line, beyond which, however, excessively faint light may be perceived to a distance about as great as the extent of the longer spectra. Aluminium, on the other hand, stands at the head of the above metals for richness in rays of the very highest refrangibility; and it is to this part of the spectrum that the strong lines above mentioned belong. In calling these lines strong, it must be understood that some allowance is made for their very high refrangibility; for when observed as above described they do not appear *absolutely* quite so strong as the bold lines of zinc or cadmium. This is partly due to the defective transparency of quartz, which for this part of the spectrum shows itself by no means perfect; and indeed the highest aluminium line, which is a double line, can only be seen by rays which pass through the prism near its edge.

* A longer distance would have been better.

The following figure exhibits the principal lines of aluminium, with zinc and cadmium for comparison. In the first of the aluminium lines represented, I could not make out the division into two parts corresponding to the tips of the electrodes. R denotes the extreme red visible on a screen; the lines in the visible spectrum are omitted, as this

Fig. 1.

has been made the subject of elaborate researches by others. The horizontal distances are proportional to the distances of the several pricks from that belonging to the extreme red, and therefore vary as the chords of the arcs described by the pricker. This tends to correct to a certain extent the exaggeration of the more refrangible end of the spectrum arising from the mode adopted of laying down the positions of the lines. The lowest row of lines in the figure, which is placed here for the sake of comparison, will be referred to further on.

Besides the lens above mentioned, I sometimes employ in a different manner another of $\frac{1}{2}$ inch diameter and $2\frac{1}{2}$ inches focal length, and accordingly large for its focal length. This is used for forming an image of the spark, which is received on the substance that is to be examined, or that is used for examining the spark. The difference of focal length for the different rays is so enormous that, while one part of the spectrum is in focus, other parts are utterly out of focus, and thus we may judge in a general way of the refrangibility of the rays by which any particular effect is produced. In this way such concentration of the rays is obtained, that effects may be studied which would not bear examination by prismatic analysis. In speaking of this lens I shall call it the 2·5-inch lens, from its focal length.

Absorption of the invisible rays by Alkaloids, Glucosides, &c.

Before examining these substances it is requisite to dissolve them, and we must first inquire into the transparency of the solvent. Fortunately the most useful of all solvents, water, is transparent when pure; and as to reagents, we may employ sulphuric or hydrochloric acid for an acid, these acids being transparent, and ammonia, suppose, for an alkali. In speaking of a substance as transparent, I wish it only to be understood that it is of a transparency comparable with quartz. As to ammonia, although it absorbs the more refrangible rays when in quantity (unless the observed absorption were due to some impurity), it may be deemed transparent in the small quantity which alone it is requisite to employ. Even alcohol, which in the state in which it is to be had is defective in transparency, is sufficiently transparent to be employed as a solvent for such substances as those under consideration, provided it be used in small thickness only.

The alkaloids and glucosides which I have examined are almost without exception intensely opaque for a portion at least of the invisible rays, absorbing them with an energy comparable for the most part to that with which colouring matters (such as alizarine, &c.) absorb the visible rays. The mode of absorption also is frequently, I might almost say generally, highly characteristic; so that by this single property they might be distinguished one from another. It frequently happens too that the mode of absorption decidedly changes according as the solution is acid or alkaline, which assists still further in the discrimination.

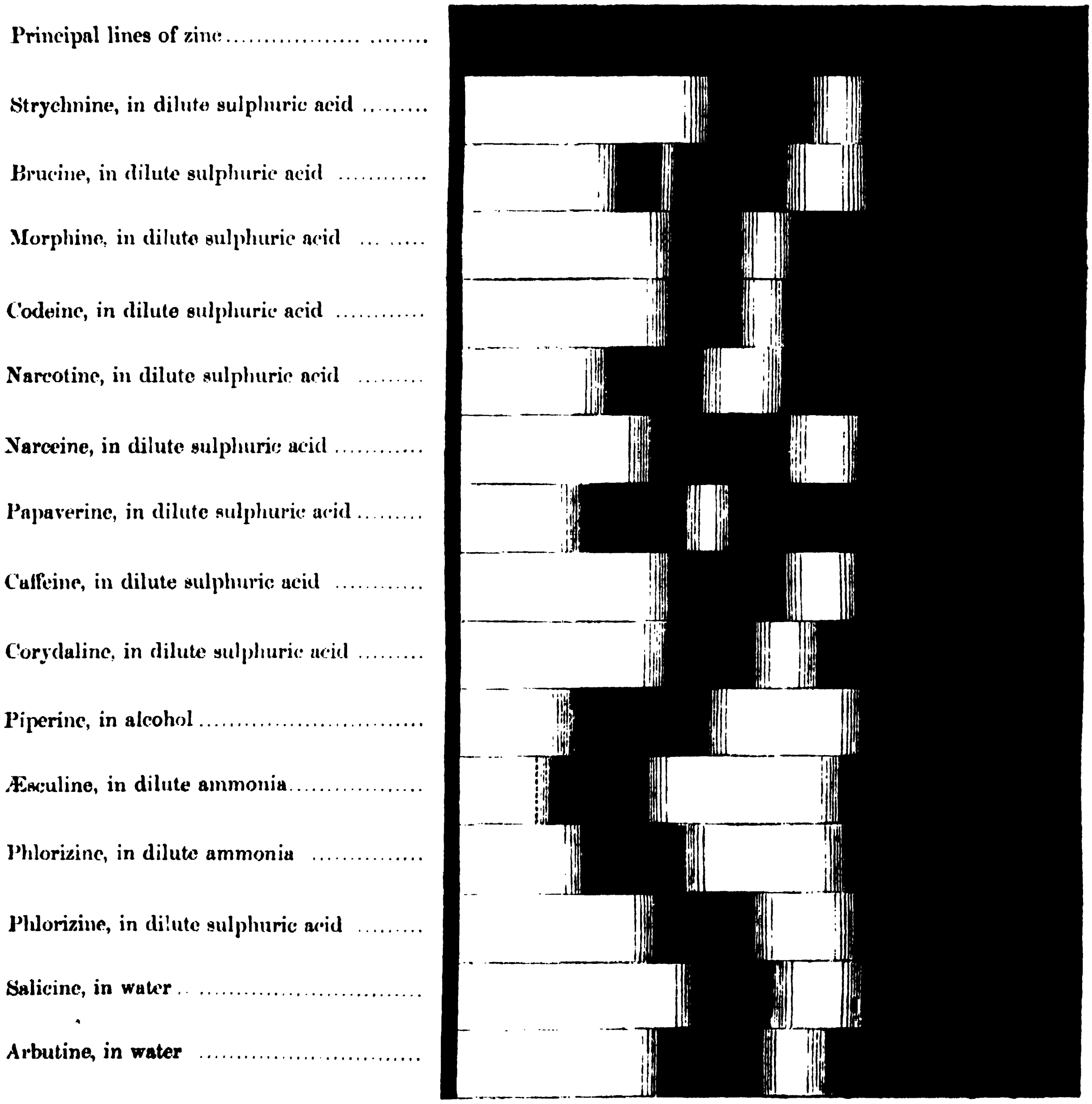
In the examination I sometimes employ a small cell with parallel faces of quartz, sometimes a wedge-shaped vessel, having its inclined faces also of quartz, but more commonly the former. The cell being filled with the solvent, a minute quantity of the substance is introduced, and the progress of the absorption is watched as the substance gradually dissolves, the fluid meantime being of course stirred up. In this way it is easy to seize the most characteristic phase of the absorption, which may be then registered by the pricking instrument. When minima of opacity occur, it is best to seize that stage of the absorption at which they are well developed. When no minima occur, a greater or less part of the more refrangible region is quickly absorbed, after which the absorption creeps on towards the less refrangible side. When once it has become tolerably stationary, the limit of the rays transmitted may be marked. It seems desirable not to go beyond this point in the absorption, lest some possible impurity in the substance examined, which if it had formed the whole of the specimen would have absorbed rays of lower refrangibility, should begin to make itself perceived, and its mode of absorption should be mistaken for that of the substance professed to be examined.

All the metallic spectra are discontinuous, which prevents the mode of absorption of even a solid or liquid from being observed quite so well as in the solar spectrum, even independently of the greater intensity of the latter, and would greatly interfere with the observation of narrow bands like those shown by the absorption of certain gases in the visible spectrum, and of which chlorous acid gas ($Cl\ O_4$) shows a splendid system in the invisible part of the solar spectrum. Should a general absorption take place in a part of the spectrum where previously a bright group of lines was seen, with weaker light for some distance on both sides, it is evident that at a certain stage of the absorption the bright group would be left isolated, and the effect might be mistaken for a maximum of transparency. In doubtful cases of this kind it is requisite to change the electrodes, so as to use the spectrum of some other metal; but practically the difficulty is not so great as might be supposed.

It is desirable to choose a metal which gives a spectrum that is bright and tolerably continuous in the region in which the distinctive features of the absorption are most likely to occur. For general use in the examination of substances such as here considered, I prefer tin—the electrodes (or one of them at least) being broad, for a reason which will be mentioned presently. Tin, indeed, is weak in the most refrangible region, though after a long interval of weakness it shows one pretty strong line between the 2nd and 3rd of the strong aluminium lines; but with these substances the distinctive features of the absorption hardly ever occur so late. For combined strength and continuity, copper answers well for the highly refrangible region in which tin is weak; while mercury, which may be employed in the form of amalgamated zinc, is the richest metal for the invisible region just beyond the visible spectrum; but I have employed tin almost exclusively.

The following figure gives the bands of absorption observed in solutions of several

Fig. 2.



alkaloids and glucosides. The bold lines of zinc are given as points of reference; but the observations were made with electrodes of tin. The border on the left is the limit of the red light visible on a screen.

Although the central part of the maxima of transparency in this figure is generally left white to save trouble, the reader must not suppose that that part of the spectrum suffers no absorption. On the contrary, it is more or less weakened when the solution has the strength to which the figure corresponds, and disappears altogether when the quantity of substance in solution is increased, while at the same time the edge of the first band of absorption creeps on *a little* towards the red, the absorption being usually pretty definite at this edge. The measurements were taken from the points where the light ceased to be sensible, which are represented in the figure by the junction of plain black and shaded white. The shading merely represents the general effect, the gradation of illumination not having been registered. It extends in the figure, as a general rule, too far to the left of the edge of the first black band, and accordingly does not represent the absorption at that limit as sufficiently definite.

A glance at the figure will show how distinctive is the mode of absorption of the rays of high refrangibility by these different substances. Indeed this one character would serve to distinguish all these substances one from another, unless it be morphine from codeine, and caffeine from salicine. The dotted line in the figure for æsculine denotes the commencement of the fluorescence, which is situated near the line G of the solar spectrum. A solution of brucine cuts off the invisible end of the solar spectrum about midway between the lines S and T, and accordingly not far from the end of the region which it requires a quartz prism and lens to see. Accordingly, when these substances are examined by solar light their distinctive characters are almost wholly unperceived, the solutions of some appearing quite transparent, and those of others merely cutting off the extreme rays to a greater or less distance. With æsculine alone the maximum of opacity lies within the solar spectrum; but even in this case we should have little idea of the great increase of transparency about to take place.

The effect of acids and alkalies on all the glucosides referred to in the figure presents one uniform feature. When a previously neutral solution is rendered alkaline, the absorption begins somewhat earlier, when rendered acid somewhat later. With salicine there is merely an indication of this change, falling within the limits of errors of observation; but in the other cases it is quite perceptible, and with phlorizine the shifting of the band of absorption produced by an acid is very large. Fraxine (or paviine) agrees remarkably with æsculine in all its optical characters; the maximum of absorption is merely situated a little nearer to the red, and the tint of the fluorescent light corresponds to a slightly lower mean refrangibility.

Quinine presents no decided maximum of transparency. With this and the other bases observed, with one exception, the absorption, if changed at all, is changed in an opposite manner to the glucosides when the base is set free by ammonia.

Bands of absorption occur also with neutral substances, for example coumarine and paranaphthaline, which last exhibits a system of such bands in the invisible part of the solar spectrum.

Aconitine, atropine, and solanine exhibit no bands of absorption, but merely a general opacity for the more refrangible rays. The last, indeed, when dissolved in dilute sulphuric acid, is, for this class of bodies, remarkably transparent; while when the base is set free the solution, contrary to what takes place with the other bases, becomes much more opaque, but the absorption is vague. I am not sure, however, how far the purity of the specimen examined may be trusted, though it was white, and regularly crystallized. It would be easy to examine more such substances; but what precedes is sufficient to show the value of the study of the absorption of the rays of high refrangibility, as affording distinctive characters of substances little known.

Minerals.

I have examined a large number of minerals by the rays from the induction spark, both as to their transparency and as to their fluorescence. The transparency of those crystals which were of such a form as to permit it, was examined by holding them in front of a pure spectrum formed on a fluorescent screen. The fluorescence was sought for by forming an image of the spark, for which aluminium electrodes were employed, by the 2·5-inch lens, holding the mineral first at the focus of the visible rays, and then moving it up towards the lens, and watching for any image which might be formed by the rays of higher refrangibility. Should such be observed, its nature was further demonstrated by interposing in the path of the rays a very thin piece of mica. This cut off the image by intercepting the invisible rays, with respect to which, except a small portion of the lowest refrangibility, mica is intensely opaque.

Carbonate of lime, the sulphates of lime, baryta, and strontia, and colourless fluor-spar, were found transparent (sulphate of strontia less so), at least in the qualified sense above mentioned, thus demonstrating the transparency of carbonic, sulphuric, and probably hydrofluoric acid, and of the bases, lime, baryta, strontia. But this subject would be better followed out by salts artificially prepared, and has been investigated by Dr. MILLER. In two cases results of considerable interest were obtained with reference to fluorescence.

At the time of writing my first paper, on the change of refrangibility of light, I had found but one mineral, yellow uranite, to the essential constituents of which the property of fluorescence plainly belongs *. In many other cases, both before and since that time, I have observed with solar light fluorescence in minerals, but always apparently having reference to unknown impurities, and therefore to my mind of much inferior interest. By means of the induction spark, employed as above described, I have found one more fluorescent mineral †.

On receiving the image on adularia, and focusing it for the rays of highest refrangi-

* Philosophical Transactions for 1852, p. 524.

† The method by which M. EDMOND BECQUEREL has examined the fluorescence of minerals (Annales de Chimie, sér. iii. tom. lvii. p. 43) does not permit of distinct vision of the specimen from the distance of a few inches, which seems to me necessary to allow the observer to judge whether the fluorescence which may be observed is due to the essential constituents of the crystal or to accidental impurities.

bility, a pair of bluish dots were seen, which were the images of the tips of the electrodes exhibited by fluorescence. As the appearance was everywhere the same, on natural faces and cleavage planes alike, and the same was observed with colourless felspars generally from different localities, it is doubtless a property of the silicate of alumina and potash constituting the crystal. Some specimens, it is true, did not show the effect so strongly as adularia or moonstone; but this is easily explained by the greater purity of the latter varieties. For the fluorescence extended to a very sensible though small depth within the crystal, and yet the rays producing it were cut off by a film of mica much thinner than paper. The intense opacity of mica is doubtless due to peroxide of iron, which nevertheless forms no more than perhaps 5 per cent. of the mineral. Hence a very small percentage of peroxide of iron, or any other impurity having a similar absorbing action, would suffice greatly to reduce the quantity of fluorescent light emitted.

In a concentrated solar beam passed through a suitable absorbing medium, adularia did not show the least sign of fluorescence, in which respect it notably differs from common glass, such as window-glass.

The other case of interest relates to a particular variety of fluor-spar found at Alston Moor in Cumberland. This variety is very pale by transmitted light, being in part of a brownish purple colour, shows a strong blue fluorescence, and is eminently phosphorescent on exposure to the electric spark. On presenting such a crystal to the spark passing between aluminium electrodes, besides the usual blue fluorescence there is seen another of a reddish colour, extending not near so far into the crystal. On receiving on the crystal the image of the spark, and moving the crystal from the focus of the invisible rays towards the lens, it was soon in best focus for the rays producing the blue fluorescence. It had to be moved much nearer to the lens before it came into focus for the rays producing the reddish fluorescence, and was then at the distance at which a well-defined image of the tips of the electrodes is formed on the uranium salt; which proves that the reddish fluorescence was produced by the rays belonging to the bright lines (considered as a whole) of aluminium of extreme refrangibility.

The crystal which showed this effect best was externally colourless for about the $\frac{1}{20}$th of an inch, which stratum showed no fluorescence when examined in this way. Then came one or two strata, parallel to the faces of the cube, showing the ruddy fluorescence, and exhausting apparently the rays capable of producing that effect. The blue fluorescence extended much deeper, and presented a stratified appearance, as Sir DAVID BREWSTER long ago observed.

On admitting a pencil concentrated by a quartz lens parallel to and almost grazing a face of the cube, so that the rays traversed the colourless stratum, the reddish fluorescence was observed in the stratum which produced it to a long distance from the face by which the rays were admitted, which demonstrates the transparency of fluoride of calcium for the rays of very high refrangibility.

The property of exhibiting such a well-marked effect under the exclusive influence of rays of extreme refrangibility, renders such a crystal a useful instrument of research.

Several other metals besides aluminium show the reddish fluorescence; but none of those examined showed it so well, partly because it is evidently produced more copiously by aluminium electrodes, and partly because it is less masked by the blue fluorescence, the spectrum of aluminium being rather wanting in brightness until the region of extreme refrangibility is reached.

If the crystal be held near the electrodes, and observed while their distance changes, it will be found that on passing from the greatest striking distance the reddish fluorescence decidedly improves. On still further diminishing the distance between the electrodes, the reddish fluorescence appears still to increase; though whether this is a real absolute increase or only an increasing preponderance over the blue, it is not easy in this way to say for certain. Hence the copiousness of rays of high refrangibility increases at first, and continues to increase *relatively* if not absolutely. It is supposed that the jar is sufficiently large to prevent the discharge from degenerating into what will be presently described as the arc discharge.

If the crystal be held close to the contact-breaker when the secondary terminals are separated, and the effect be compared with that of the secondary discharge (a jar being in connexion, as has been supposed all along), the electrodes being of platinum for fairness of comparison, it will be found that the proportion of rays of extremely high refrangibility is decidedly greater for the spark at the contact-breaker than for the secondary discharge.

On forming by the 2·5-inch lens an image of the spark from aluminium electrodes, and placing a crystal, such as that above mentioned, in the focus of the rays producing the reddish fluorescence, it is easy to determine the transparency or opacity of substances for those rays, the alteration of the focus by the introduction of a thick plate being of course borne in mind, and the crystal moved accordingly. The rays forming the image have had to pass only through air, and through a very small thickness of quartz, before reaching the crystal. In this way I have found that even quartz itself in very moderate thickness is opaque for these rays; but different specimens, or different parts of the same specimen, vary in this respect. I possess a large plate 0·42 inch thick, cut perpendicular to the axis of the crystal, which is generally transparent, but is slightly brownish on one side, to the distance of about half an inch from the face of the hexagonal prism. The *colourless* part of this plate, beyond a little distance from the brownish part, is opaque for the rays in question *, while the *brownish* part is nearly transparent. It may be inferred that the colourless part contains a minute quantity of some impurity capable of absorbing these rays, which does not exist, at least to the same extent, in the brownish part, although the latter is not perfectly pure silica, as is shown by its colour. On the

* It should be mentioned that this part contains those delicate, definitely directed, elongated laminæ or crystals, hardly visible except in a beam of sunlight, which are called by practical opticians "blue shoots." An examination of a number of cut pieces of quartz lent me by Mr. DARKER confirms me in the suspicion that such crystals are more defective in transparency than other colourless specimens for the rays of extreme refrangibility.—July 1862.

whole, I am disposed to think that quartz, if it were *rigorously* pure, would be transparent. We see at any rate how difficult it is to draw certain conclusions respecting the transparency or opacity of a substance which, in the state of purity in which it may be obtained, shows only a slight defect of transparency.

I tried reflecting the rays from the spark by a fine Munich grating, but the light was far too faint to be of any use. Possibly a large and very closely ruled plane speculum, with a concave speculum instead of a lens, might give light which it would be possible to observe. But at present I have not found any sufficiently marked effects referable to rays of still higher refrangibility to make it worth trying.

The same crystal which showed the reddish fluorescence was eminently phosphorescent, with a blue colour. The phosphorescence, like the fluorescence, was arranged in strata parallel to the faces of the cube, and, like the reddish but unlike the blue fluorescence, was not perceptible beyond a moderate distance from the surface at which the exciting rays had entered. On forming an image of the discharge by the 2·5-inch lens, focusing the crystal for the rays producing the reddish fluorescence, fixing it there, and breaking the circuit after the induction coil had worked for a little while, a dart of blue phosphorescent light was seen in the crystal at the focus of the lens. On focusing for the rays most efficient in producing the blue fluorescence, the reddish was diffused over a broad portion of the strata producing it; and on repeating the above experiment in this position of the crystal, the blue phosphorescence was seen similarly diffused. This shows that the rays of extremely high refrangibility are those most efficient in producing the blue phosphorescence.

[We may suppose that the blue fluorescence, the reddish fluorescence, and the blue phosphorescence are due to the action of the assemblage of heterogeneous exciting rays on the same substance (doubtless some impurity taken up during crystallization), or on two or three distinct substances. The blue fluorescence is produced abundantly at a depth within the crystal at which the two other effects are invisible; but this alone is no proof of a diversity in the nature of the substance acted on, because the rays producing the two latter effects would have been absorbed before arriving at such a depth. Hence it is among the early strata, in crossing which rays capable of producing each of the three effects are still vigorous, that evidence must be sought, in the coincidence or non-coincidence of the strata in which the three effects are respectively perceived, of the probable identity or certain diversity of nature of the substance acted on. At the time when this paper was read I fancied I had observed slight discrepancies as to coincidence in the strata. But a renewed examination, in which a larger number of specimens were observed, leads me to regard the fancied discrepancies as too doubtful to rely upon and to overpower the increasing weight of evidence on the other side.

The blue fluorescence may be observed in the early strata (which ordinarily, at least with electrodes of aluminium and several other metals, show a red) by absorbing the more refrangible of the exciting rays by a suitable plate of quartz, or else by substituting for aluminium some metal, such as magnesium, which is poor in rays of extreme refrangibility. On the other hand, the red fluorescence really existing in the early strata, when it is overpowered by the blue, may be seen by viewing the crystal through a solution of chromate of potash, which greatly enfeebles the blue fluorescence, while at the same time it transmits enough of the spectrum to allow the unabsorbed residue to be at once distinguishable by its colour (green) from the red fluorescence. In this way the red fluorescence may be readily perceived even with electrodes of magnesium. Again, a particular stratum which showed a blue fluorescence when acted on by rays which entered by a face of the cube, and before reaching it had to traverse some other strata showing fluorescence, exhibited a red fluorescence when acted on by rays which fell on it directly, having been admitted through an octahedral face.

It is more difficult to decide as to the identity or diversity of the strata showing respectively red fluorescence and blue phosphorescence, because the two effects are observed in a different way; but as far as I could decide, the strata appeared to correspond.

On the whole, then, I am disposed to think it probable that it is the same substance which, in consequence of the action of rays beginning with a part of the violet and extending from thence onwards, exhibits a blue fluorescence, which, in consequence of the action of rays of extreme refrangibility, exhibits a red fluorescence, and which, in consequence of the action of rays of a similar refrangibility, exhibits a powerful blue phosphorescence. At least, if the substances be different they would appear to have coexisted in solution, and so to have been taken up together in the crystallization of the mineral. I should mention, however, that it is contrary to all my experience that the fluorescence of a single substance (*i. e.* not a mixture) should thus, as it were, take a fresh start *with a totally different colour* on proceeding onwards in the spectrum; but then my experience is derived mainly from the examination of substances in the comparatively short solar spectrum.—July 1862.]

I have said that the phosphorescence was produced in certain strata within the crystal. These strata were in some places sharply terminated, so as to be foreshortened into well-defined lines. On watching the phosphorescence, there was nothing to be seen at all like conduction; the strata remained sharply defined as long as the light was strong enough to enable one to judge. This is at variance with one of the two results which, on the authority of others, I formerly mentioned as indicating a distinction between phosphorescence and fluorescence *. On trying shortly afterwards along with Mr. FARADAY, I could not obtain either of these results. One of them, that relating to apparent conduction, which was obtained by MM. BIOT and A. C. BECQUEREL, has since been explained by M. EDMOND BECQUEREL as an illusion of observation †. The other, that relating to the production of phosphorescence in CANTON's phosphorus by rays which had traversed a strong solution of bichromate of potash, I am, after a conversation with Dr. DRAPER, still unable to explain.

* Philosophical Transactions for 1852, p. 547.

† Annales de Chimie, tom. lv. (1859) p. 112.

Advantage of Broad Electrodes.

At first I employed by preference wires or sharp pieces of metal for electrodes, in consequence of the greater facility with which the discharge passed, and the larger quantity of light given out by the spark. Certain considerations, however, led me to try broad electrodes; and I accordingly procured electrodes of the common metals shaped like small watch-glasses, about an inch in diameter. These showed in some cases a most marked superiority over thin wires, exhibiting the invisible metallic lines in far greater strength, while with some metals there was not much difference. With copper, for example, the superiority was very great, with iron it was comparatively small.

Instead of electrodes of this shape, it is sufficient to take two pieces of thick foil, make them slightly cylindrical by means of a round ruler, or a pencil, and mount them with their convexities opposed and the axes of the cylinders crossed.

Besides copper, silver, tin, and aluminium show a great advantage of flat electrodes, and lead a moderate advantage, while with zinc, as with iron, sharp electrodes are nearly as good. Brass agrees in this respect with zinc, and not with copper, though it shows the copper lines very strongly.

With such electrodes, however, the spark dances about: and its unsteadiness is objectionable in some experiments. A good part of the advantage of flat electrodes is however retained if one only be flat, especially if this be negative, and the spark is now steadier. Instead of using the end of a wire to combine with a flat electrode, it seems rather better, according to a plan suggested to me by Dr. MILLER, to bend a wire to a gentle curve lying in a vertical plane passing through the prism; or the edge of a flat piece of metal may be similarly employed.

On forming an image of the spark between a sharp and a flat electrode of copper, and receiving it on a fluorescent screen, the flat electrode gave the brighter of the two images already mentioned, and that, whether the electrode were positive or negative.

On similarly forming an image of the spark between two very broad electrodes, and focusing for the rays of highest refrangibility, the image did not, as usual, consist of two separate dots; but whether it was, that, from the shortness of the spark, the two ran into one, or that the rays belonging to the metallic lines of high refrangibility were emitted throughout the whole length of the spark, I am not quite certain; but I incline to the latter opinion, as a separation of the discharge into two portions, corresponding to the immediate neighbourhood of the two electrodes respectively, could hardly have escaped detection had it existed.

Arc Discharge, and Lines of Blue Negative Light.

On diminishing the distance between the electrodes, formed suppose of copper wires, the brightness of the metallic lines at first improves, and afterwards changes but little, or, if anything, rather falls off. On still further diminishing the distance, so that the electrodes almost touch, and the discharge passes with little noise, a new set of strong lines make their appearance in the invisible region of moderate refrangibility. In this mode of discharge, in which the negative electrode, if at all thin, quickly becomes red-hot and fuses, the jar has not much influence, and the lines in question are still better seen when it is suppressed altogether. To show them to perfection, it is best to take a flat negative electrode, so as to carry off the heat, and not to hide from the prism any part of the blue negative light, and a sharp positive electrode almost touching the former. In this way the visible discharge is reduced almost wholly to an insignificant-looking star of blue light; but it is wonderful how strong an effect it is capable of producing in the invisible region. The most striking part of the invisible spectrum consists of four bright lines, numbered 1, 2, 3, 4 in fig. 1, situated not far from the visible spectrum. These are followed, after a nearly dark interval, by light arranged in masses resembling in its general aspect the groups of copper lines (from which, however, it differs), but not strong enough to be resolved or accurately measured. The figure represents also a couple of blue bands (b, b') seen by projection. These are not seen on looking at the blue light directly with a flint-glass prism of 60°, because everything is seen in too great detail. Most of the air-lines in the invisible spectrum, especially the bands beyond line 4, have an ill-defined look, and would probably be resolved did the intensity of the light permit.

The appearance just described is independent of the nature of the electrodes, and therefore is to be referred to the air, and not to the metal. On viewing in a moving mirror the star of light producing this effect, it is found to have a considerable duration.

On slightly separating the electrodes, forming an image of the discharge with the 2·5-inch lens, and receiving it on a cake of the uranium salt, a very strong fluorescence was seen over the image of the blue disk when the lens was focused for a point a little beyond the visible spectrum. On moving the lens onwards, the fluorescence produced by the rays belonging to this image spread out into a ring; and on moving still further, a tolerably well-defined image of the whole discharge was perceived. Of this the part belonging to the blue disk was the brightest, and was surrounded concentrically by the ring before mentioned, now still further widened. The image of the remainder of the discharge was brightest where it was most contracted at the positive electrode. The discharge generally was perhaps of slightly higher refrangibility than the blue disk, even excluding from the latter the rays belonging to the ring. It thus appears that the four bright lines figured were produced mainly by the blue negative light.

The mode of transition of the discharge may be studied by placing the electrodes at the greatest striking-distance and making them gradually approach. At first there passes a clean bright spark making a sharp report, and not resolved by a revolving mirror. The invisible spectrum which this shows is too faint for precise observation; the visible spectrum shows chiefly air-lines. As the electrodes approach, the spark becomes clothed by the well-known yellowish envelope capable of being blown aside, and the blue negative light begins to appear. A moving mirror, as M. LISSAJOUS has already observed*, shows an instantaneous spark at the commencement, in point of time,

* See DU MONCEL, 'Recherches sur la non-homogénéité de l'étincelle d'induction,' p. 107.

of the envelope and blue negative light, both which are drawn out, indicating a very appreciable duration. On making the electrodes approach somewhat nearer, the spark diminishes, and the envelope is formed in perfection, especially with broad electrodes. The air-lines now begin to show themselves well, but are brightest on the side of the spectrum answering to the blue negative light.

It might be supposed at first sight that the permanence of the yellowish and of the blue light only indicated a glow of appreciable duration left by a sensibly instantaneous discharge; but several circumstances indicate that the discharge itself lasts, and that it is *under* its action that the glow takes place*. The action, I am persuaded, is this: a spark first passes; and this enables a continuous discharge to pass, which is due, in part at least, to the inductive action of the still falling magnetism, just as a voltaic arc may be started in a powerful battery by passing an electric spark between the slightly separated electrodes; and the glowing of the air *under the action* of this discharge produces the yellowish envelope and blue negative light. Thus, when the electrodes are nearly at the greatest distance at which this sort of discharge takes place, the blue negative light is seen pretty sharply terminated in a moving mirror. Were it a dying glow, it ought to fade away; but if produced under a discharge, it ought to cease almost abruptly, inasmuch as at this distance of the electrodes a continuous discharge is unable to pass when the tension has sunk much below that under which it was first produced.

The same conclusion may be drawn from an effect which I once obtained, the exact conditions for the production of which it is not easy to hit off. With a jar in connexion, each discharge due to a single breach of contact appeared in a moving mirror as a bright spark joined to a spark less bright by the blue negative light, and also by the yellowish or reddish light, brightest close to the positive electrode. Were the blue light due to a glow, it ought to be reinforced instead of being put out by the second spark, whereas the explanation of the result is easy on the supposition of a continuous discharge. The first spark started a continuous discharge, which emptied the jar less fast than it was filled by the secondary coil; so that presently another discharge took place, which emptied the jar so that a continuous discharge could no longer pass.

On viewing the broad discharge formed without a jar when the electrodes are at a moderate distance, through a revolving disk of black paper with a single hole near the circumference, while the envelope was being blown aside, so as to get a succession of momentary views of the discharge, the envelope was seen *extravagantly bent*, as a flexible conductor might have been—not *torn across*, as a column might have been which was heated by a *previous* spark. The central spark, of course, was usually missing, as it is sensibly instantaneous.

I have spoken of the arc, and especially the blue negative light, as exhibiting air-lines. The arc, however, is liable to be coloured not only by casual dust (as when it passes partly through the flame of a spirit-lamp with a salted wick, when it is coloured yellow

* Although this view may be considered already established (see the work by the Vicomte Du Moncel just quoted), the observations here mentioned will not, I hope, be altogether useless.

by sodium), but also by matter torn from the positive electrode. This is well seen with electrodes of aluminium, when the arc or a portion of it is frequently coloured green. This green light has a very sensible duration, and a distinctive prismatic composition, and is brighter towards the positive than towards the negative electrode, but is not confined to the immediate neighbourhood of that electrode (extending indeed sometimes over almost the whole length of the arc), in which respect, and in its duration, it differs from the light of the spark proper*. With aluminium opposed to another metal, as copper or iron, the green light is seen only when the aluminium is positive. Even with aluminium this light may generally be got rid of by making the electrodes approach; and it is the arc in what may thus be deemed its normal state that was observed for the construction of the last line of fig. 1, though I have not at present noticed variations in the invisible corresponding with those in the visible spectrum of the arc discharge.

On the Cause of the Advantage of Broad Electrodes; and on the Heating of the Negative Electrode.

Although the spark appears instantaneous when viewed in a moving mirror, it must yet occupy a certain time; so that we have in fact a brief electric current, to which we may apply OHM'S laws. The electromotive force is here the difference of tensions of the coatings of the jar. As to the resistance, the short metallic part of the circuit may be neglected, and we need only attend to the place of the discharge. The resistance here may be divided into that due to the air and that due to the parts of the electrodes close to the points of discharge. That the latter is by no means insignificant, may be inferred from the enormous temperature to which minute portions of the electrodes are raised, as indicated by the excessively high refrangibility of the rays emitted by the metals, in the state doubtless of vapour. By the use of flat electrodes the striking-distance is materially diminished, without any change in the difference of tension of the coatings of the jar. Hence the electricity which it contains passes at a higher velocity, and therefore produces a more powerful effect on the metals.

The injurious effect of the introduction of a small resistance was very strikingly shown with broad, slightly curved copper electrodes, three inches in diameter, by leading wires from a coating of the jar into a tumbler of water, and from thence to the corresponding electrode, when the spark became quite insignificant in comparison to what it had been.

With one sharp and one flat electrode placed near together, bright sparks passed when the connexion was metallic, and the invisible spectrum then showed the copper lines, with one or two air-lines not conspicuous; but when water was interposed the spark was greatly reduced, and the invisible spectrum showed the air-lines. In both cases the spark was followed by an arc discharge, as might be seen in a moving mirror; and in the latter case the arc discharge was increased in consequence of the diminution

* The *outer* part of the jar-spark between aluminium electrodes has the same green colour and prismatic composition, though in this case the green light is sensibly instantaneous.—July 1862.

of the spark, which, though necessary to start it, was formed at its expense; and as in the arc discharge the jar was idle, the increase of resistance in a circuit already comprising the secondary coil was unimportant.

The fact that the blue negative light which appears when the arc discharge is formed shows air-lines, points to the air as the seat of the intense action which there takes place; and the very high refrangibility of some of the rays emitted, and the copiousness of those rays, indicate how intense that action is. The heating of the negative electrode seems to be a secondary effect, not due to the direct passage of the electricity through the metal (for the section through which it passes is not by any means small), but to the heat communicated from the film of air investing it. Small as is the mass of the film compared with that of the portion of the electrode adjacent to it, the rate at which heat is communicated is enormous. Thus with a positive point nearly touching underneath a negative electrode of platinum foil containing water, the foil is kept red-hot under the water, though the mere passage of electricity through the metal would be quite inadequate to produce that effect. Corresponding to the heating of the electrode by the air is the cooling of the air by the electrode; and such a powerful abstraction of heat can hardly take place without altering the state of the film of air in relation to its power of conducting electricity. This would seem to be the reason why the film of air in contact with the negative electrode behaves so differently from any arbitrary section of the column along which the discharge takes place, and from offering greater resistance becomes the seat of a more intense emission of highly refrangible rays. At the positive electrode, at which, for whatever reason, the issue of electricity is confined almost to a point, nothing of this kind takes place; but, from the contraction of the section through which the electricity has to pass in the electrode, a minute portion of the metal of which it is composed is so highly acted on that matter belonging to the electrode is liable to appear in the arc.

These views lead to curious speculations respecting the negative light in highly exhausted tubes, and respecting the remarkable reversion of heating-effect which Mr. GASSIOT has obtained according as the discharge is intermittent or continuous*, but I forbear to speculate further.

* Proceedings of the Royal Society, vol. xi. p. 329.

XXVIII. *On the Nature of the Forces concerned in producing the greater Magnetic Disturbances.* *By* BALFOUR STEWART, *M.A.*, *F.R.S.*

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1. IN a previous communication submitted to the Royal Society on June 28th, 1861, and since published in their Transactions, I ventured to make a suggestion regarding the nature of that connexion which subsists between magnetic disturbances, earth-currents, and auroras.

In this hypothesis the earth was viewed as similar to the soft iron core of a RUHMKORFF'S machine, in which a primary disturbing current was supposed to induce magnetism. Earth-currents and auroras, on the other hand, were viewed as induced or secondary currents, caused by the small but abrupt changes which are constantly taking place in the strength of the primary disturbing current, these changes being very much heightened in effect by the action of the iron core, that is to say, of the earth.

2. These small and rapid changes are very observable in the photographic traces given by the Kew magnetographs during a time of disturbance. At such a time the curves for all the three elements invariably present a serrated appearance, which, when the disturbing cause is very powerful, is magnified into a succession of sharp peaks and hollows, and these exhibit a frequency of change which makes them comparable in this respect with earth-currents and auroras.

These peaks and hollows are therefore regarded by this hypothesis as denoting the changes which take place in the action of the primary disturbing current upon the needle through the intervention of the earth's magnetism, or, since the existence of a primary current is really unnecessary as an explanation, we may dispense with it altogether *, and regard these photographic peaks and hollows as simply representing the changes which take place in the magnetism of the earth, without speculating upon the cause of such changes.

3. In the paper already alluded to, in discussing the great magnetic storm extending from August 28 to September 7, 1859, I endeavoured to show that the first effect of the superimposed disturbing force was to diminish both elements of the earth's magnetism during a period of about six hours. This grand wave, constituting the great body of force, could not, I remarked, be supposed to be due to any combination of earth-currents of which the period is only a few minutes. Under these circumstances it would seem to be the most natural supposition to regard the peaks and hollows, which occur as it were on the surface of the great disturbance wave, as representing rapid changes in the in-

* This was suggested to me by Professor TYNDALL.

tensity of the disturbing force, and therefore not as phenomena caused by earth-currents, since the great body of the disturbance of which they represent the changes cannot be due to any such cause. Auroras and earth-currents being thus removed out of the list of causes, might then be supposed to follow rather as effects of these magnetic changes, on the principle of voltaic induction already mentioned. As, however, there seems to be a tendency to regard the motions of the magnetic needle as due to the direct action of earth-currents, it may be desirable here to inquire whether this view of the origin of these small and rapid disturbances is capable of being accepted as a tenable hypothesis.

4. It is almost unnecessary to remark that the hypothesis which asserts that earth-currents are not the cause of the small and rapid magnetic disturbances, does not assert that such currents have absolutely no influence upon the needle, since we know that a current must always act upon a needle; but it maintains that these currents are induced and secondary phenomena, and therefore represent only a small fraction of the whole force in operation, and react on the magnet only to a very limited extent.

5. The interesting observations on earth-currents made under the direction of Mr. C. V. WALKER, and recorded by him in a paper recently published in the Transactions of the Royal Society, appear to furnish grounds for an hypothesis regarding these phenomena.

By reference to a map appended to his paper, it appears that a line drawn from the Ramsgate to the Margate telegraph station proceeds in a direction 12° 50′ W. of true north; or, assuming 21° 30′ W. as the approximate magnetic declination, this line proceeds in a direction 8° 40′ E. of magnetic north. In like manner, a line drawn from the Ashford to the Ramsgate telegraph station proceeds in a direction 83° 50′ E. of magnetic north; also a line drawn from the Ashford to the Margate station in a direction 77° 42′ E. of magnetic north.

By Mr. WALKER's observations from August 29 to September 2, 1859, and also from August 8 to August 10, 1860, it would appear that a current proceeding from Margate to Ramsgate was always simultaneous with one proceeding from Ramsgate to Ashford, and with one proceeding from Margate to Ashford; that is to say, simultaneous currents proceeding along these lines were either all north or all south currents.

Let us endeavour to ascertain how far this agrees with the behaviour of the magnetographs at Kew during these two disturbances on the hypothesis of direct action, assuming also with Mr. WALKER that earth-currents are derived from a stream of electricity which is drifting across the country.

Conceive now a current resolved into two components, the one proceeding in the magnetic meridian, and the other in a direction perpendicular to it; and let us agree to consider as positive those currents flowing from magnetic north to south, and from magnetic east to west. Of these two components, the first only will affect the freely suspended declination magnet, a positive current tending to increase the westerly declination; while the latter only will affect the horizontal-force magnet which has been twisted into a position at right angles to the magnetic meridian, a positive current increasing the horizontal force.

Let x be the value of the north and south, and y that of the east and west component of a current as deduced from its effects upon the magnetographs.

This will give on the Margate and Ramsgate line a current represented by

$$A\{x \cos 8^\circ 40' + y \sin 8^\circ 40'\}, \quad \ldots \quad (1.)$$

while for the Ramsgate and Ashford line we shall have

$$B\{x \cos 83^\circ 50' + y \sin 83^\circ 50'\}, \quad \ldots \quad (2.)$$

and for the Margate and Ashford line

$$C\{x \cos 77^\circ 42' + y \sin 77^\circ 42'\}, \quad \ldots \quad (3.)$$

where A, B, C are positive constants depending upon the nature of the soil and strata along the various lines.

Now in the disturbance of 8th to 10th August 1860, the declination and horizontal force were increased simultaneously, or they were diminished simultaneously; hence x and y have the same sign, and the expressions within brackets are either all positive or all negative, that is to say, the currents between these lines should have been either all north or all south simultaneously. This was observed to be the case. In this instance, therefore, there is nothing to contradict the hypothesis which makes earth-currents the cause of magnetic disturbances.

Let us now examine the disturbance of August 29 to September 2, 1859. Here the horizontal force was increased when the declination was diminished, and *vice versâ*; and if we represent by unity the current influential in disturbing the former element, that disturbing the latter is most probably represented by $-\cdot46$ (see Table V.).

Hence (1.), (2.), (3.) become

$$(1)=A\{-\cdot46 \cos 8^\circ 40' + \sin 8^\circ 40'\} = -\cdot304\, A,$$
$$(2)=B\{-\cdot46 \cos 83^\circ 50' + \sin 83^\circ 50'\} = +\cdot945\, B,$$
$$(3)=C\{-\cdot46 \cos 77^\circ 42' + \sin 77^\circ 42'\} = +\cdot879\, C.$$

We thus see that in this disturbance, for the theory to hold good, currents of one name should have traversed the line between Margate and Ramsgate, while currents of the opposite name traversed the other two lines; but this is not in accordance with the observations made, which showed that currents of the same name traversed the three lines simultaneously. In one word, the proof against the theory of the direct action of earth-currents in causing disturbances may be summed up by saying that, while the magnetic disturbances of September 1859 and August 1860 were of opposite character, the corresponding earth-currents, as determined by these three telegraphic lines, were of the same character.

6. At first view it would almost appear as if this fact, which seems conclusive against earth-currents (at least those earth-currents observed in this country) being the direct and main cause of disturbances, were equally conclusive against their being induced currents, since the effect of such upon the telegraph needle might be supposed opposite in character to that which would be produced by such currents as would directly cause magnetic disturbances, the two effects being antithetically and therefore definitely related to each other.

But this difficulty will disappear when we reflect that earth-currents may be twisted in passing over the surface of our globe. Although, therefore, we may not find that oppositeness of character which we naturally associate with induced currents, yet the study of particular earth-currents in connexion with simultaneous magnetic disturbances may perhaps serve to throw some light upon the nature of the former.

7. The following is brought forward as an instance of this. On September 1st, 11.20 A.M. G. M. T., a *strong southerly* current was noticed on the Ashford and Margate line which lasted till 11.26 A.M., while from 11.28 A.M. to 11.35 A.M. a *slight northerly* current was observed on the same line.

If we take into account a slight delay in the starting of the Kew curves (equal to about 2 minutes) due to the slackness of the toothed wheels which drive the cylinders, we find the first recorded appearance of the south current to be, in point of time, 1½ minute behind that of a very abrupt disturbance which affected all the magnetic elements at Kew, and which occurred simultaneously with the outbreak of a curious phenomenon on the sun's disk. Whatever be the cause of this apparent difference in time, I do not think that it argues any want of simultaneity between the magnetic disturbance and the earth-current, and we may safely suppose that the first outbreak of the south current was really simultaneous with the commencement of the disturbance at Kew.

The magnets at Kew were affected in the following manner. The westerly declination was at first rapidly increased; but soon the disturbing force attained its maximum, and then gradually diminished, until ultimately the needle attained nearly the same position which it had before the disturbance. The horizontal force was at first rapidly diminished, then as the disturbing force died away it also gradually came back to its previous value. A direct current which would cause a disturbance of this nature would be one at first rapidly increasing, then gradually diminishing, but always preserving the same name, whereas the induced effect of such a disturbance would be at first a strong current in one direction, and afterwards a weak current in the opposite direction. The latter character agrees very well with that of the currents between Ashford and Margate, the former character not in the least. Judging therefore from this disturbance, and, though only a single example, it appears to be an unexceptionable one, it would seem that earth-currents do not cause magnetic disturbances, but are rather the induced currents which the latter give rise to.

8. I shall now endeavour to show that we have grounds for supposing the magnetograph peaks and hollows to be due to changes in the value of the disturbing force.

The action of any such disturbing force is of a twofold nature.

1°. It raises a curve above its normal position, or depresses it below the same, visibly and for a somewhat lengthened period of time.

2°. The curve which represents this definite action of the force is at the same time studded with sharp peaks and hollows.

Now if the second of these two effects denotes the changes which occur in the force producing the first, the peaks and hollows should be similar in character to the general effects of the force. The following example will explain what is meant.

If we have a force which simultaneously and for a lengthened period of time raises up the curves of all the three elements nearer to the top of the paper, and if the peaks and hollows denote changes taking place in the intensity of this force, then all the three elements should exhibit peaks at once, or hollows at once. But if, on the other hand, the general appearance of the curves represents the declination as raised up while both the other elements are depressed, then a peak in the former should correspond to a hollow in the latter two.

9. I requested Mr. CHAMBERS, Magnetical Assistant at Kew, to note his impression of the general appearance of the disturbance curves of most importance for the years 1858, 1859, and 1860; and the following Table is the result. The sign + means that the curve is raised towards the top of the paper, the sign — means that it is lowered towards the bottom.

TABLE I.—General appearance of Disturbance Curves.

Date.	Character of disturbance.		
	Declination.	Horizontal force.	Vertical force.
1858. March 28—29	—	+	+
April 9—10	—	—	—
June 23—24	+	—	—
July 5—6	0	—	—
October 27—28	indefinite.	indefinite.	+
December 4—5	+	+	—
1859. February 9—10	+	0	—
February 9—10	+	0	+
February 26—27	—	indefinite.	—
April 21—22	—	—	—
April 21—22	+	+	+
April 29—30	—	—	—
May 19—20	—	—	—
June 8—9	—	—	—
June 8—9	indefinite.	+	+
July 11—12	0	—	—
July 18—19	—	0	—
August 28—29	—	+	+
September 1—2	—	+	+
September 2—3	—	—	—
September 3—4	—	—	—
October 12—13	—	—	indefinite.
October 17—18	+	+	+
October 17—18	—	+	+
October 18—19	—	—	—
October 18—19	—	+	+
December 13—14	—	—	—
December 13—14	indefinite.	0	—
December 14—15	—	0	+
1860. March 28—29	—	—	+
March 29—30	—	—	—
April 9—10	—	—	—
April 9—10	+	+	+
July 4—5	+	+	+
July 5—6	—	—	—
August 6—7	indefinite.	—	0
August 7—8	0	—	—
August 8—9	—	—	—
August 9—10	0	—	—
August 12—13	—	—	—
August 12—13	—	+	+
September 7—8	—	—	—

It thus appears that there are twenty-two cases in which the declination is raised or lowered along with the horizontal force, and only seven cases of an opposite description. Also there are twenty-two cases in which the declination is raised or lowered along with the vertical force, and only eleven cases of an opposite description. Finally, there are thirty-one cases in which both forces are raised or lowered together, and only two cases of an opposite description.

There is therefore a decided tendency in the curves of all the elements to be raised or lowered simultaneously; but this tendency is stronger between the horizontal and vertical-force curves than between either of these and the declination. It may at the same time be affirmed that, with the exception of the disturbance of August—September 1859, there is no very prominent case in which the three elements do not rise or fall together.

10. Having thus recorded the general appearance of the curves, I shall now give the result of the examination of the simultaneous peaks and hollows; but it will first be necessary to state the method in which this has been conducted.

Each curve has a zero-line, or line of abscissæ, along which the times are reckoned; so that if we wish to find the time corresponding to any point in the curve, we have merely to measure its abscissa, the commencement of the zero-line (denoting the moment at which the instrument was started) being the origin.

In like manner, it is equally easy to find the point of the curve corresponding to any given time.

The accuracy of this process depends, however, it will be seen, on the assumption that the time-scale is constant for the different portions of a curve; and the following is a proof that this is strictly true.

It will be noticed shortly that this system of measurement has brought out a remarkable correspondence between the peaks and hollows of the horizontal force and those of the vertical force, which may be said to have failed in no one instance. This may be received as sufficient evidence, not only of the physical fact brought to light, but also of the constancy of the time-scale, in absence of which no such phenomenon could have been observed.

11. The following Table exhibits the results derived from comparing together the peaks and hollows of the declination, with those occurring at the same instant of time in the horizontal force.

When peak and peak occur together, or hollow and hollow, this is termed a correspondence, the reverse a non-correspondence.

TABLE II.—Exhibiting the connexion between the small movements of the declination and those of the horizontal force in the various disturbances.

Date of disturbance.	Number of correspondences.	Number of non-correspondences.	Doubtful.
1858. March 12—16	18	1	5
April 9—12	24	0	5
June 22—24	13	0	1
October 27—29	15	0	0
December 4—6	12	1	1
1859. February 8—10	13	2	1
February 26—28	19	0	2
April 21—23	14	0	1
April 29—May 1	15	0	1
May 19—21	14	0	0
June 8—10	13	0	1
July 11—13	10	0	5
October 12—14	8	1	2
October 17—19	19	0	2
December 13—15	9	0	1
1860. March 26—30	29	0	2
April 9—11	22	0	5
April 13—15	15	1	1
June 29—July 6	28	0	3
August 6—13	65	12	46
September 7—8	7	0	0
Sums...	382	18	85

It will be seen from this Table that, in the great majority of cases, a peak in the declination corresponds to a peak in the horizontal force, and a hollow in the one to a hollow in the other. It is proper to mention that the peaks and hollows here observed are those which represent sudden changes of short duration; for if we take for comparison some prominent peak or hollow in the one curve having a long period, we shall be much less certain of finding a corresponding phenomenon in the other.

12. When the peaks and hollows of the horizontal force and those of the vertical force are compared together, the result is that a peak invariably corresponds to a peak, and a hollow to a hollow; and even when large prominences of long duration are taken, the correspondence between the two curves is very remarkable.

The same connexion, therefore, which subsists between the sudden movements of the declination and those of the horizontal force, holds still more strikingly between those of the two forces.

13. The disturbance of August—September 1859 has been purposely left out of Table II.; in the following Table this great disturbance has been broken up into parts, for each of which the behaviour of the peaks and hollows is compared with the general appearance of the curve.

TABLE III.—Great disturbance extending from August 28 to September 7, 1859. The behaviour of peaks and hollows compared with the general appearance of the curves.

Time.	Declination movements compared with those of the horizontal force.			General character of the disturbance.
	Correspondence.	Non-correspondence.	Doubtful.	
1859. August 28, 10 A.M. to 9 P.M.	10	0	1	The great disturbance had not yet commenced.
August 28, 10.30 P.M. to August 29, 9.52 A.M.	0	6	0	The great disturbance had now commenced depressing the declination and raising both elements of the force.
September 2, 4.50 A.M. to September 3, 11.50 A.M.	0	20	2	A second great disturbance commenced about September 2, 4.50 A.M., depressing the declination and raising both elements of the force, which changed into or was succeeded by one which seemed to impress all the elements in the same manner.
September 3, 1.40 P.M. to September 4, 10.30 A.M.	9	13	3	
September 4, 12.30 P.M. to September 6, 9.30 A.M.	16	3	7	

14. Before discussing the results in this Table, I ought to mention that in this great disturbance the rapid nature of the motion makes the comparison of simultaneous peaks and hollows a matter of some little uncertainty. On the other hand, these comparisons have been much facilitated by the exceedingly good definition which the labours of the late Mr. WELSH secured for the Kew curves, without which, indeed, an investigation of this nature would have been impossible. On the whole, I am well persuaded that the results in Table III. represent the truth.

15. To recapitulate. It appears from Table I. that, studying merely the general result of a disturbance, there is a decided tendency to raise or lower the curves of all the elements simultaneously, this being stronger between the horizontal and the vertical-force curves than between either of these and the declination.

From Table II. it appears that, if we leave out of account the great disturbance of August—September 1859, a peak of declination corresponds to a peak of horizontal force, and a hollow of the one to a hollow of the other, while the same correspondence holds still more strongly between the horizontal and vertical forces.

Again, from Table III. it appears that the great disturbance of August—September 1859 may be broken up into two. In one of these the general result was to lower the declination and raise both elements of the force, while in the second the result was to raise or lower all the three curves simultaneously. It also appears that, while the first of these disturbances prevailed, a declination hollow corresponded to a peak of either force, and that, on the other hand, while the second prevailed, a declination hollow corresponded to a hollow of either force.

This very marked correspondence between the behaviour of the peaks and hollows and that of the general disturbing force *in all cases*, leaves, I think, little doubt that the former represent sharp and sudden changes in the intensity of the latter.

16. Let us now endeavour to ascertain if any use may be made of this fact to throw light on the nature of the forces concerned in producing disturbances.

17. Two distinct suppositions may be made regarding the nature and mode of action of disturbing forces.

1°. We may suppose that forces of every imaginable variety of character are concerned together in producing disturbances.

2°. We may suppose a disturbance occasioned by one or more groups of forces the elements of which are bound together by a certain law.

With respect to the first of these hypotheses, it is refuted by the discussion of disturbing forces given by General SABINE for the different Colonial Observatories and for Kew, as well as by the results in Tables I., II., and III. The second hypothesis must, therefore, represent the mode of action of the forces concerned.

18. And, first of all, it may safely be affirmed that no disturbance of any magnitude is due to the action of a single force, merely varying in amount; for if this were the case, the distance at any moment of a point in the curve of one of the elements from its normal position should bear throughout a disturbance an invariable proportion to the distance of a corresponding point in the curve of another of the elements from its normal; but this is by no means true.

Since, therefore, a disturbance is not a phenomenon due to the action of a single force, and since at the same time it does not represent the action of a number of different forces promiscuously huddled together, it becomes a question of interest to ask ourselves how we may find the elementary forces concerned.

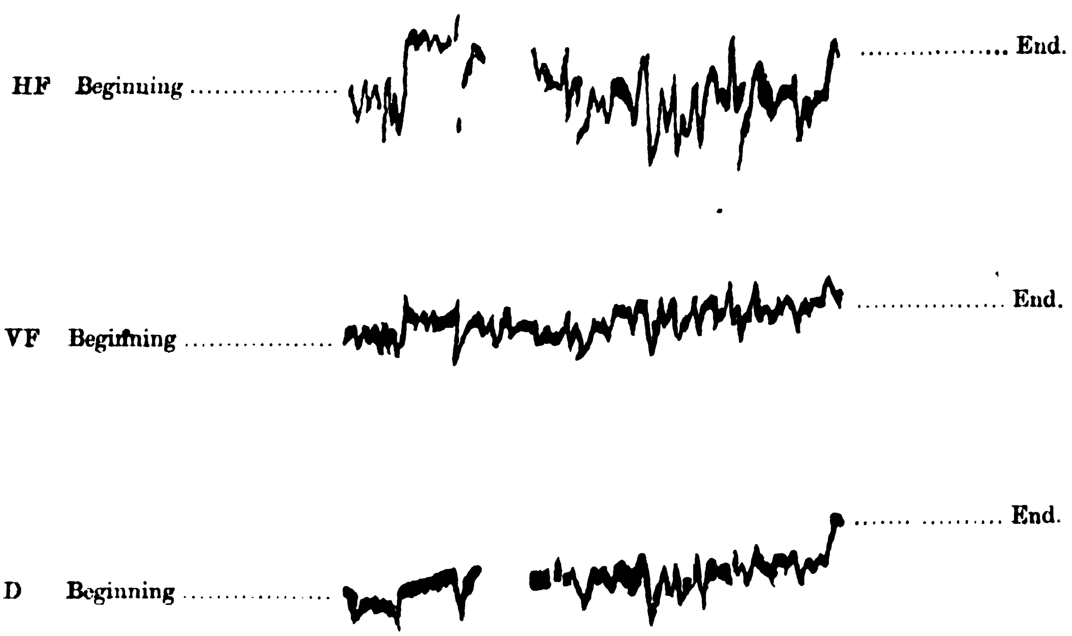
19. A little consideration will show that this is likely to be obtained by the study of small and rapid changes of force. For if several forces are at work, it is unlikely that at the same moment a sudden change should take place in all; there is thus a high probability that a sudden and rapid change is a change in one of the elementary forces concerned, and which will therefore enable us to determine the nature of that force. Even if the change is not a very abrupt one, provided that we confine ourselves to such peaks and hollows as present a similar appearance for all the curves, we may be satisfied that we are observing changes taking place in one only of the elementary disturbing forces; for it is inconceivable that two or more independent forces changing independently should produce similar appearances in all the three curves. This may be illustrated by an example. Suppose, for instance, a disturbance takes place in declination which at the end of one minute has raised the curve one-tenth of an inch, while at the end of the second minute it has fallen again to its original level. Suppose also that a change, precisely the same in nature and amount, takes place in the horizontal force, while in the vertical force the change is only one-twentieth of an inch. Here then we have in our curves three isosceles triangles which, although the last is not, strictly speaking, similar to the other two, yet all three convey the idea of only one force acting. Suppose now another force to have been superimposed which affected the declination very much more than either of the force-components, but which did not begin to work until the end of the first minute. The result would be that the declination peak would be much further

from an isosceles triangle than that of either of the forces; we might even suppose the the descending limb of the declination peak to become level, or even to ascend in consequence of the action of this second force.

The following is a more precise definition of what is requisite before we can refer the action to a pure elementary force.

Let it be supposed that we are comparing together portions of each of the three curves, commencing at the same time, then it may be asserted that these portions are due to the action of a single disturbing force when (the origin for each being the commencement of the portion of the curve under consideration) for equal abscissæ the ordinates of the one curve always bear a definite proportion to those of the other.

20. This is illustrated by the following tracings, which exhibit the movements of the three curves on August 11, 1860, from 2.34 P.M. to 5.26 P.M.



21. From all this we perceive that, in order to find what are the pure elementary forces at work, we must select suitable peaks and hollows, and measure these accurately.

The following Table exhibits the results obtained by comparing such peaks and hollows of the horizontal force with those of the vertical force, no reference being made to the declination.

TABLE IV.—Comparison in length of the Horizontal-force changes with those of the Vertical Force.

Date.	Horizontal-force change (vertical-force change = unity in each instance). Actual measurements.	Mean.
1858. March 2—3	2·4, 2·1, 2·3, 2·0	2·2
March 12—16	1·9, 1·9, 2·1, 1·8, 1·9, 2·3, 2·0, 2·1	2·0
March 27—29	2·3, 1·9, 2·0	2·1
April 9—10	1·9, 2·1, 2·0	2·0
June 23—24	2·3, 2·1, 2·0	2·1
July 5—6	2·1	2·1
1859. February 9—10	1·9, 2·2, 2·2	2·1
February 27—28	1·9, 2·1	2·0
May 19—21	2·0, 1·9, 2·2, 2·1	2·0
June 8—9	2·2, 2·0, 2·0, 2·2, 2·2, 2·0, 2·0 2·1, 1·9, 2·1, 1·9, 1·9, 2·0, 2·1	2·0
July 11—13	2·0, 2·1, 2·1, 1·9, 2·0, 2·2	2·0
August 28—September 7	1·8, 2·0, 2·0, 2·0, 1·9, 1·6, 2·0 2·1, 2·2, 2·2, 2·2, 2·2, 2·0, 2·2 2·3, 1·6, 2·0, 2·0, 2·0, 2·1, 1·8, 1·7 2·0, 1·9, 1·9, 2·1, 1·9, 2·2, 1·9, 1·7, 2·1 2·0, 2·1, 2·0, 2·1, 2·0, 1·9, 2·0, 2·2, 2·0 2·0, 2·0, 2·0	2·0
October 12—18	2·0, 1·9, 1·8, 2·2, 1·9, 2·3, 1·8, 2·0	2·0
December 13—15	2·3, 1·9, 1·8	2·0
1860. March 26—29	2·3, 1·9, 2·0, 1·9, 1·6, 1·8, 1·9, 1·6 2·2, 2·0	1·9
April 9—15	2·1, 2·3, 2·0, 2·1, 1·9, 2·0, 1·8, 2·0 1·8, 2·1	2·0
July 4—6	2·1, 1·9, 2·1, 2·0, 2·0, 2·2, 1·7, 2·3, 2·0	2·0
August 6—12	2·1, 2·0, 2·1, 1·9, 2·1, 1·9, 1·9, 2·2 1·9, 1·9, 2·0, 2·1, 2·0, 2·1, 2·0, 1·9 2·0, 2·0, 1·9, 2·2, 2·1, 1·9, 2·0, 1·9 2·0, 2·2, 1·9, 2·1, 2·0, 2·0	2·0

In this Table the actual measurements have been exhibited in order to afford an idea of the accuracy with which the proportion holds, without attempting to estimate the probable error.

22. In endeavouring to frame a similar Table between the horizontal force and declination, a curious fact presented itself.

Although it is comparatively easy to find changes in the vertical force which are similar to corresponding changes in the horizontal force, yet it is much more difficult to find similar corresponding changes in the horizontal force and declination. Indeed, for many of the cases recorded in the above Table, the declination-change would not be similar to that of either force. It thus appears that, even in cases which do not indicate the action of a pure elementary disturbance, the horizontal-force change preserves an almost invariable relation to that of the vertical force*.

23. The following would appear to be the explanation of this.

Whatever be the nature of an elementary disturbing force, its effect upon the hori-

* This proves, in addition to a physical fact, that these magnetographs are capable of recording with precision the slightest change in either element of the earth's force.

zontal component of the earth's magnetism bears always a nearly invariable proportion to its effect upon the vertical component of the same. In this case it is clear that two or more disturbing forces superimposed would, as far as regards the horizontal and vertical-force changes, behave almost in the same manner as a pure elementary disturbance. This also explains why it is almost impossible to find a peak in the one component of the force corresponding to a hollow in the other.

24. This curious fact may likewise be stated in the following language:—

Whatever be the nature of the disturbing force at work, its resolved portion, which acts in the plane of the magnetic meridian at Kew, is always in nearly the same line.

25. It is very easy to find the direction of this line.

A disturbance of the horizontal force, as it appears in the curve, is, we have seen, very nearly double that of the vertical force. Now an inch in the ordinate of the horizontal-force curve denotes a change amounting to nearly ·010 of the whole force, while for the vertical-force curve it represents a change of ·0025 of the whole; and increasing ordinates denote decreasing force for both elements. Also the absolute value of the horizontal force is 3·8, while that of the vertical force is 9·6.

Hence if we represent by 38 the value of the horizontal component of the disturbing force in the plane of the magnetic meridian, that of the vertical component will be denoted by 12, and the dip of the whole force acting in this plane will be 17°·5 nearly.

26. From what has been already mentioned (art. 22), it may be inferred that it is difficult to obtain similar corresponding changes for all the elements together. The following Table exhibits those instances in which this has been accomplished; but the results can only be regarded as very rough approximations.

In some cases, where the vertical-force disturbance was very small, it was not attempted to measure it.

TABLE V.—Similar corresponding changes for all the elements.

Date.	Greenwich Mean Time.	Character and size of change (vertical-force change = unity in each instance).		
		Declination.	Horizontal force.	Vertical force.
1858.	h m			
March 14......	2 19 P.M.	rise after =2·1	rise after =1·9	rise after =1·0
15......	9 53 A.M.	rise after =1·8	rise after =2·0	rise after =1·0
15......	9 56	fall after =1·7	fall after =1·8	fall after =1·0
15......	9 35	rise after =2·1	rise after =2·0	rise after
April 9......	11 14½	rise after =3·1	rise after =2·0	rise after
10......	4 21	rise after =3·0	rise after =2·0	rise after
10......	4 23	fall after =2·7	fall after =2·1	fall after =1·0
10......	6 32½	fall before =3·0	fall before =1·9	fall before =1·0
June 23......	3 25	fall after =1·4	fall after =2·0	fall after
23......	3 29	rise after =1·5	rise after =2·0	rise after
23......	3 33	fall after =1·6	fall after =2·0	fall after
23......	3 37	rise after =1·6	rise after =2·0	indistinct

TABLE V. (continued).

Date.	Greenwich Mean Time.	Character and size of change (vertical-force change = unity in each instance).		
		Declination.	Horizontal force.	Vertical force.
	h m			
Oct. 28......	10 59½	hollow =2·0	hollow =2·0	hollow
28......	11 27½	fall after =2·1	fall after =2·1	fall after =1·0
28......	11 42 A.M.	fall after =2·0	fall after =2·0	fall after
28......	1 8½ P.M.	rise after =2·1	rise after =2·0	rise after
Dec. 4......	11 18 A.M.	rise after =2·0	rise after =2·0	rise after
5......	8 12	fall before =2·2	fall before =2·1	fall before =1·0
5......	11 8½	fall after =2·0	fall after =2·0	fall after
5......	11 11½ A.M.	rise after =2·0	rise after =2·0	indistinct
1859.				
Feb. 26......	12 53½ P.M.	fall after =2·1	fall after =2·0	fall after
26......	1 59½ P.M.	peak =2·2	peak =2·0	peak
27......	3 12½ A.M.	hollow =2·0	hollow =2·0	hollow
27......	3 47½ A.M.	peak =2·2	peak =2·0	peak
27......	3 8½ P.M.	fall after =2·2	fall after =1·9	fall after =1·0
27......	3 12½	rise after =2·7	rise after =2·2	rise after =1·0
June 8......	11 43½	rise after =2·5	rise after =2·2	rise after =1·0
8.....	11 59½ P.M.	peak =2·3	peak =2·0	peak
9......	11 7½ A.M.	fall after =2·1	fall after =2·1	fall after =1·0
Sept. 1......	11 16 A.M.	*fall after* =1·4	rise after =1·9	rise after =1·0
4......	2 14 P.M.	rise after =1·2	rise after =2·2	rise after =1·0
Oct. 17......	11 42 P.M.	fall after =3·0	fall after =2·0	fall after =1·0
18......	12 31 A.M.	fall after =3·2	fall after =2·0	fall after =1·0
18......	12 39	fall after =3·1	fall after =2·1	fall after =1·0
18......	8 21 A.M.	fall after =2·6	fall after =2·0	fall after
Dec. 14......	12 7½ P.M.	rise after =2·4	rise after =2·0	rise after
14......	12 18 P.M.	rise after =2·5	rise after =2·0	rise after =1·0
1860.				
April 10......	5 15½ A.M.	rise after =2·7	rise after =1·9	rise after =1·0
10......	5 20½	fall after =3·2	fall after =2·0	fall after
10......	5 22	rise after =2·9	rise after =2·0	rise after =1·0
10......	6 3	fall after =3·3	fall after =2·0	fall after =1·0
10......	6 7 A.M.	rise after =3·3	rise after =2·0	rise after =1·0
13......	12 54½ P.M.	rise after =2·9	rise after =1·9	rise after =1·0
July 5......	3 20 A.M.	fall after =3·6	fall after =2·1	fall after =1·0
5......	4 34	fall after =3·5	fall after =2·1	fall after =1·0
5......	7 21½	fall after =2·8	fall after =1·7	fall after =1·0
5......	8 10 A.M.	peak =3·4	peak =2·1	peak =1·0
Aug. 10......	6 57½ P.M.	fall after =1·1	fall after =2·2	fall after =1·0
10......	7 1½	rise after =1·2	rise after =2·2	rise after =1·0
11......	4 18	fall after =1·1	fall after =2·0	fall after =1·0
11......	4 21	rise after =0·9	rise after =1·7	rise after =1·0
11......	4 25½	fall after =0·8	fall after =2·1	fall after =1·0
11......	4 29	rise after =1·1	rise after =2·0	rise after =1·0
11......	4 31	fall after =1·1	fall after =1·8	fall after =1·0
11......	4 36½	rise after =1·0	rise after =2·1	rise after =1·0
11......	4 39	fall after =1·1	fall after =1·8	fall after =1·0
11......	4 41 P.M.	rise after =1·0	rise after =1·7	rise after =1·0

27. It appears from this Table that the proportional value of the declination-element is very nearly constant during a disturbance, but that it varies much from one disturbance to another. The constancy of this element during the same disturbance will perhaps be best seen by comparing the values in the declination column with those in the column of the horizontal force, as the vertical-force changes being small, any error of measurement is very apt to alter the proportion between them and the larger changes with which they are compared.

28. It will also be noticed that the great disturbance of August—September 1859 seems to consist of two disturbances superimposed, one being of the normal type, but the other, as regards the declination-element, being decidedly abnormal. This agrees well with the results of Table III.

29. It would thus appear that in the great majority of cases only one type or group of forces operates in producing a disturbance, and that the various individual forces which compose this group have a very small range as regards their mode of action upon the three elements of the earth's magnetism at Kew. We may therefore (approximately at least) represent a disturbance by a single force; and for this purpose it is competent for us to average for each disturbance the results of Table V.

30. Now that force which affects the declination is evidently the horizontal component of the disturbing magnetic force which acts in a direction perpendicular to the magnetic meridian, and its value will be $X \tan \delta\theta$, where X denotes the absolute horizontal force, and $\delta\theta$ the angular displacement of the declination magnet. But an inch of the declination-ordinate represents 22′, hence the disturbing force which would cause a change to this amount will be $=3{\cdot}8 \tan 22'={\cdot}024$, while, as we have already seen, an inch of change in the horizontal-force ordinate denotes a disturbing force $={\cdot}038$. Again, decreasing ordinates represent increasing westerly declination; and since (except in one case) the three curves are simultaneously affected in the same direction, it follows that (with the same exception) a magnetic force acting from magnetic south to north must be compounded with one acting from magnetic east to west, and with one acting vertically downwards. But, for the anomalous disturbance of August—September 1859, a force acting from magnetic south to north must be compounded with one acting from magnetic west to east.

Applying now the ordinary rules for combining forces, we obtain the following Table, in which the astronomical azimuth and dip of the various disturbing forces are given.

31. It will, however, be first necessary to allude to a peculiarity in this method of determining the direction of the disturbing force. This is, that our results apply in strictness only to those small and rapid changes which occur as it were on the surface of the great disturbance-wave.

We might, for instance, be engaged in studying the direction of that force which would occasion a small depression which we had observed to occur simultaneously in the three curves, while at the same time these curves might be elevated above their normals, and not depressed by the main body of the disturbing force. We have, however, in art. 15 given grounds for supposing that these small and rapid peaks and hollows denote changes in the main body of the disturbing force. Our results regarding the peaks and hollows may therefore, perhaps, be viewed as applicable to the main body of the force; but we must bear in mind that while we may thus be enabled to determine the resultant line of action of this force, we do not yet learn whether it is of a positive or negative nature, whether it may be represented by a north or by a south pole. This will be rendered evident when we reflect that a small depression may be due either to the weakening of an elevating force, or to the strengthening of one of an opposite character. It thus appears that the north or south nature of the whole distubing force must be determined by reference to the general appearance of the curve. Indeed it will be afterwards shown that we have grounds for supposing two antagonistic forces to be in operation at once. To fix our thoughts, however, let us suppose for the following Table the disturbing force to be one which increases the earth's horizontal intensity at Kew, and let us consider its action on the north pole of the needle.

Table VI.—Representing the Astronomical Azimuth and Dip of the various disturbing forces.

Date of disturbance.	Direction in which the force tends to make the north pole of a needle point.	Dip which the force would give to the north pole of a needle.
1858. March 14—15	North 53° 46′·5 West	15° 30′·5
April 9—10	64 28·5	13 0·5
June 23	47 13	15 53
October 28..................	54 5·5	14 43·5
December 4—5	54 5·5	14 43·5
1859. February 26—27	56 27·5	14 23·5
June 8—9	56 10·5 West	13 53·5
September 1—4...........	3 27·5 East	16 46
September 1—4...........	40 30·5 West	15 11
October 17—18...........	64 21·5	12 52·5
December 14	59 13·5	14 1·5
1860. April 10—13	65 54	12 55
July 5........................	67 54	12 17
August 10—11	North 40 1·5 West	16 59·5

32. It has been already mentioned that we do not ascertain by our method whether the whole disturbing force is positive or negative. The following considerations, however, may serve to elucidate this subject.

We have seen in art. 18 that a disturbance cannot be represented by the action of a single force, and we have also seen (art. 29) that the various forces which compose a disturbing group have a very small range as regards their mode of action upon the three elements, this range being especially small when the two force-elements are compared together.

We might therefore hope to find, on inspecting the general appearance of the curves, that the two force-elements are simultaneously raised or depressed with reference to their normals nearly in the proportion denoted by the nature of the disturbing force whose character we have been analysing.

But this is not often the case; and although generally both elements are on the same side of their normals, yet it happens occasionally that the one element is above while the other is below; while on such occasions there is nothing in the behaviour of the peaks and hollows to indicate the action of any other than the ordinary disturbing force.

33. This may perhaps be explained by supposing two antagonistic forces to be in operation at once, the one tending to elevate, and the other to depress the curve, the absolute values of these forces bearing a somewhat large proportion to their differences, and the one force affecting the elements in a slightly different manner from the other.

Suppose, for instance, that +1 and −1 denote the position of the horizontal and the vertical-force curves at the same moment, the former being elevated above its normal, and the latter depressed below it.

Conceive this result due to the action of an elevating force represented by +40 +20 opposed by a depressing one represented by −39 −21, and we have a sufficient explanation of this anomalous circumstance, while at the same time both the antagonistic forces sufficiently present the normal type.

34. The same style of reasoning will apply in comparing the declination curve with that of either force; only here we must suppose that the one force affects the declination in a somewhat different manner from its antagonist, so that the proportion between a declination peak and one of either force is not so constant as that between peaks of the two forces. The idea of two antagonistic forces, the difference of which represents the visible action, seems also to give the simplest explanation of the fact, that sometimes a force which depresses one of the elements will change in the course of a few hours to one which elevates it—since otherwise we must suppose the disturbing force to have changed its character.

This also is the view of disturbing forces which General SABINE, who has studied the subject so long and so successfully, has lately been disposed to adopt on other grounds; and I am happy to think that the idea herein advocated is that of one whose judgment is so mature and whose information is so extensive.

35. Adopting this view of the subject, it is worthy of remark that the same element of the disturbing forces (the declination) which changes so much its comparative value from one disturbance to another, changes also the most of the three when we pass from the one to the other of the antagonistic forces concerned in the same disturbance.

In conclusion, I may be allowed to state that this paper is submitted to the Royal Society rather as indicating a method of analysis than as embodying the results of an investigation.

Note on the Electromotive Force induced in the Earth's Crust by Variations of Terrestrial Magnetism. By Prof. W. THOMSON, *A.M., L.L.D., F.R.S.*

The evidence from observation adduced in the preceding paper tending to show that some "earth-currents" which have been actually observed have been the electro-magnetically induced effects of variations of terrestrial magnetism, appears to be a very important contribution to the discovery of the complete theory of these most interesting and perplexing phenomena. It necessarily, however, suggests the question, Is the electromotive force induced by variations of terrestrial magnetism comparable in amount with that which is found in observations on earth-currents? There is scarcely occasion at present for working out a complete mathematical theory of the currents induced in the earth's crust by any fully specified magnetic variations. This can easily be done as soon as observation supplies data enough to make it useful. In the mean time a very rough theoretical estimate of the absolute amount of electromotive force induced by such magnetic variations as we know to exist, is sufficient to render it certain that this electro-magnetic induction does very sensibly influence the observed phenomena of earth-currents. For if there be, as we know there is every day, a gradual variation of terrestrial magnetism over a large portion of the earth's surface, amounting to a minute of angle in the direction of the dipping-needle, or to some thousandths, or not much less than one thousandth of force during several hours, this change of magnetism must induce in a length of a few hundred miles in the earth's crust an electromotive force, which we readily see must be comparable with that which would be induced in a linear conductor of the same length if carried across the lines of magnetic force at the rate of a minute of arc (or a nautical mile) in several hours. In two articles communicated eleven years ago to the 'Philosophical Magazine'*, I explained the principles on which such an electromotive force as this is to be compared with familiar standards, as, for instance, that of an element of DANIELL'S battery. Thus a horizontal conductor, 1,400,000 feet long (or about 270 British statute miles), carried at the rate of 600 feet (or about one-tenth of a minute of arc) per hour in a horizontal direction perpendicular to its own length across the British Isles or neighbouring Atlantic ocean (where the vertical magnetic force averages about 10 British absolute units of magnetic force), would experience an induced electromotive force amounting to $\frac{600}{3600}\times 10\times 1,400,000$, or about 2,300,000 British absolute units of electromotive force. But, as I showed in the second of those articles, the electromotive force of a single cell of DANIELL'S is about 2,500,000 British absolute units. Hence the induced electro-magnetic force in question is about equal to that of a single cell. Some such electromotive force as this, therefore, must be induced in a length of 270 miles of the earth's crust, by the ordinary diurnal variations of terrestrial magnetism; and the much more rapid variations in magnetic storms

* See 1851, Second half year. "On the Mechanical Theory of Electrolysis," and "Applications of the Principle of Mechanical Effect to the Measurement of Electromotive Forces, and of Galvanic Resistances, in Absolute Units."

must produce much greater electromotive forces, which we may conceive may not unfrequently be as much as that of ten or twenty cells, and sometimes may amount to 100 cells or more. Just such amounts of electromotive force were those which I actually observed in the Atlantic cable, as the following extract from the 'Encyclopædia Metropolitana,' article "Telegraph, electric," shows.

"In the failure of the Atlantic Cable in September 1858, the portion terminating at Valencia came to give nearly the same indications as an insulated conductor about 270 miles long, laid out westward, and connected with a copper plate sunk at a little less than that distance in the Atlantic. In these circumstances the writer found that from 1 to 9 or 10 twentieths of the electromotive force of two DANIELL's elements was generally sufficient to balance the earth-current; not unfrequently 14 or 15 were required; sometimes, although rarely, 20, or the full electromotive force of two DANIELL's elements, was insufficient; and once or twice in the course of the month of September, earth-currents were received so strong that five or six DANIELL's elements would have been required to balance them."

It seems therefore quite certain that the ordinary every-day earth-currents in that locality must be very sensibly influenced by electro-magnetic induction from the ordinary diurnal variations of terrestrial magnetism; but it is also quite certain that they are only in part due to this cause, and that some more powerful, but as yet unknown agency, is at work to produce them. For although I found that a day seldom, if ever, passes without the direction of the current changing several times, yet there was no relation between the times of such changes and the solar hours. I conclude with the following additional extract from the same article, expressing views regarding earth-currents which I think will be found to agree with the extensive and careful observations of Mr. C. V. WALKER, which have been published since it was written, although they seem quite at variance with the theory which has recently been advocated by Prof. LAMONT and Dr. LLOYD, that earth-currents, however they are themselves generated, do directly produce the magnetic variations.

"Earth-currents are certainly related to the irregular variations of terrestrial magnetism, since they are always found unusually strong during brilliant displays of aurora borealis; for it has long been known that, on these occasions, the magnetic disturbances are unusually strong. Being related to the variations of terrestrial magnetism, it is probable that the earth-currents also will be found to have daily periods; but, in the mean time, we only know that, while the diurnal variation in terrestrial magnetism is observable in general every day, and is only on rare occasions overborne by irregular disturbances, the earth-currents vary each day from hour to hour, like the wind, under some overpowering non-periodic influence, and can only show daily periodicity in residual averages derived from lengthened series of observations. It is probable that careful synchronous observations of auroras, earth-currents, and variations of terrestrial magnetism, will lead to a discovery of the primary influence, whether in the earth, or terrestrial atmosphere, or surrounding interplanetary air, which causes these phenomena."—W. T.

XXIX. *On the Analytical Theory of the Conic.* *By* Arthur Cayley, *F.R.S.*

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The decomposition into its linear factors of a decomposable quadric function cannot be effected in a symmetrical manner otherwise than by formulæ containing supernumerary arbitrary quantities; thus, for a binary quadric (which of course is always decomposable) we have

$$(a, b, c \between x, y)^2 = \frac{1}{(a, b, c \between x', y')^2} \text{ Prod. } \{(a, b, c \between x, y \between x', y') \pm \sqrt{ac-b^2}(xy'-x'y)\};$$

or the expression for a linear factor is

$$\frac{1}{\sqrt{(a, b, c \between x', y')^2}}\{(a, b, c \between x, y \between x', y') \pm \sqrt{ac-b^2}(xy'-x'y)\},$$

which involves the arbitrary quantities (x', y'). And this appears to be the reason why, in the analytical theory of the conic, the questions which involve the decomposition of a decomposable ternary quadric have been little or scarcely at all considered: thus, for instance, the expressions for the coordinates of the points of intersection of a conic by a line (or say the line-equations of the two ineunts), and the equations for the tangents (separate each from the other) drawn from a given point not on the conic, do not appear to have been obtained. These questions depend on the decomposition of a decomposable ternary quadric, which decomposition itself depends on that for the simplest case, when the quadric is a perfect square. Or we may say that in the first instance they depend on the transformation of a given quadric function $U=(* \between x, y, z)^2$ into the form W^2+V, where W is a linear function, given save as to a constant factor (that is, $W=0$ is the equation of a given line), and V is a decomposable quadric function, which is ultimately decomposed into its linear factors, $=QR$, so that we have $U=W^2+QR$. The formula for this purpose, which is exhibited in the eight different forms I, II, III, IV, I(bis), II(bis), III(bis), IV(bis), is the analytical basis of the whole theory; and the greater part of the memoir relates to the establishment of these forms.

The solution of the geometrical questions above referred to is (as shown in the memoir) involved in and given immediately by these forms. It is also shown that the formulæ are greatly simplified in the case *e. g.* of tangents drawn to a conic from a point in a conic having double contact with the first-mentioned conic, and that in this case they lead to the linear Automorphic Transformation of the ternary quadric. The memoir concludes with some formulæ relating to the case of two conics, which however is treated of in only a cursory manner.

Article Nos. 1 to 17, relating to a single conic.

1. The point-equation of the conic is

$$(a, b, c, f, g, h \between x, y, z)^2=0,$$

which expresses that the point (x, y, z) is an ineunt of the conic.

The line-equation of the same conic is

$$\begin{vmatrix} & \xi, & \eta, & \zeta \\ \xi & a, & h, & g \\ \eta & h, & b, & f \\ \zeta & g, & f, & c \end{vmatrix}=0,$$

or putting

$$(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H})=(bc-f^2, ca-g^2, ab-h^2, gh-af, hf-bg, fg-ch),$$

the line-equation is

$$(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \xi, \eta, \zeta)^2=0,$$

which expresses that the line (ξ, η, ζ) (that is, the line the point-equation whereof is $\xi x+\eta y+\zeta z=0$) is a tangent of the conic. We are thus in the analytical theory of the conic concerned with the quadrics $(a, b, c, f, g, h \between x, y, z)^2$ and $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H} \between \xi, \eta, \zeta)^2$, which are the characteristics or *nilfactums* of these equations respectively.

2. I put also

$$\mathrm{K}=\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix},$$

or, what is the same thing,

$$\mathrm{K}=abc-af^2-bg^2-ch^2+2fgh.$$

3. It may be convenient to notice that when $(a, .. \between x, y, z)^2$ breaks up into factors, the conic the equation whereof is $(a, .. \between x, y, z)^2=0$, becomes a pair of lines; and that when $(a, .. \between x, y, z)^2$ is a perfect square, the conic becomes a pair of coincident lines, or say a *twofold* line. But a pair of lines, distinct or coincident, cannot be represented by a line-equation. The analytical formulæ presently given show that in the former case $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2$ is the square of a linear function, which equated to zero gives the line-equation of the point of intersection of the two lines, or node of the conic; and the equation $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2=0$ accordingly represents such point considered as a pair of coincident points, or say a *twofold* point. But in the latter case, where the conic is a twofold line, $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2$ is identically equal to zero, and the line-equation $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2=0$ is a mere identity $0=0$, thus ceasing to have any signification at all. And the like remarks apply to the conic as represented by the line-equation $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2=0$, the conic here breaking up into a pair of distinct or coincident points, &c.

4. It is proper to remark also that

$$(a, \ldots \between x', y', z' \between x, y, z)=0$$

is the equation of the polar of the point (x', y', z') in regard to the conic, and that

$$(A, ..\between \xi', \eta', \zeta' \between \xi, \eta, \zeta)=0$$

is the line-equation of the pole of the line (ξ', η', ζ'); or, what is the same thing, the point-coordinates of the pole are

$$A\xi'+H\eta'+G\zeta' : H\xi'+B\eta'+F\zeta' : G\xi'+F\eta'+C\zeta'.$$

5. The inverse matrix is

$$\begin{pmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{pmatrix}^{-1}, = \frac{1}{K}\begin{pmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{pmatrix};$$

but it is convenient to disregard the factor $\frac{1}{K}$, and speak of (A, B, C, F, G, H) as the inverse or reciprocal coefficients. The equation just written down implies the relations $Aa+Hh+Gg=K$, $Ah+Hb+Gf=0$, &c., which may be arranged in two different ways as a system of nine equations.

6. We have also

$$(BC-F^2,\ CA-G^2,\ AB-H^2,\ GH-AF,\ HF-BG,\ FG-CH)=K(a, b, c, f, g, h),$$

and

$$ABC-AF^2-BG^2-CH^2+2FGH=K^2,$$

which are well-known theorems.

7. I notice also the theorem

$$(a, ..\between x, y, z)^2.(a, ..\between x', y', z')^2-[(a, ..\between x, y, z\between x', y', z')]^2$$
$$=(A, ..\between yz'-y'z,\ zx'-z'x,\ xy'-x'y)^2,$$

which is much used in the sequel: it may be mentioned, in passing, that this is included in the more general theorem

$$\begin{vmatrix} (a, ..\between x, y, z\between l, m, n), & (a, ..\between x', y', z'\between l, m, n) \\ (a, ..\between x, y, z\between l', m', n'), & (a, ..\between x', y', z'\between l, m, n) \end{vmatrix}$$
$$=(A, ..\between yz'-y'z,\ zx'-z'x,\ xy'-x'y\between mn'-m'n,\ nl'-n'l,\ lm'-l'm),$$

which is at once deducible from

$$\begin{vmatrix} Ll+Mm+Nn, & L'l+M'm+N'n \\ Ll'+Mm'+Nn', & L'l'+M'm'+N'n' \end{vmatrix}$$
$$=(MN'-M'N)(mn'-m'n)+(NL'-N'L)(nl'-n'l)+(LM'-L'M)(lm'-l'm),$$

by writing therein

$$(L, M, N)=(ax+hy+gz,\ hx+by+fz,\ gx+fy+cz),$$
$$(L', M', N')=(ax'+hy'+gz',\ hx'+by'+fz',\ gx'+fy'+cz').$$

8. Suppose now that

$$(a, b, c, f, g, h \between x, y, z)^2$$

breaks up into factors, or say that we have

$$(a, b, c, f, g, h \between x, y, z)^2 = 2(\alpha x+\beta y+\gamma z)(\alpha' x+\beta' y+\gamma' z),$$

the values of the coefficients $(a, ..)$ then are

$$(a, b, c, f, g, h)=(2\alpha\alpha',\ 2\beta\beta',\ 2\gamma\gamma',\ \beta\gamma'+\beta'\gamma,\ \gamma\alpha'+\gamma'\alpha,\ \alpha\beta'+\alpha'\beta),$$

and forming from these the inverse coefficients (A, ..) and the discriminant K, we find

$$(\mathrm{A, B, C, F, G, H}) = -(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta)^2.$$

$$\mathrm{K}=0.$$

9. The last-mentioned equation, $\mathrm{K}=0$, is the condition in order that $(a, .. \between x, y, z)^2$ may break up into factors; and when it does so, we have

$$(\mathrm{A}, .. \between \xi, \eta, \zeta)^2 = -[(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta \between \xi, \eta, \zeta)]^2,$$

that is, $(a, .. \between x, y, z)^2$ breaking up into factors, $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2$ is a perfect square; and equating it to zero, we have

$$[(\beta\gamma'-\beta'\gamma,\ \gamma\alpha'-\gamma'\alpha,\ \alpha\beta'-\alpha'\beta \between \xi, \eta, \zeta)]^2=0;$$

which, (ξ, η, ζ) being line-coordinates, gives (as a twofold point) the point of intersection of the lines (α, β, γ), $(\alpha', \beta', \gamma')$, that is, the lines $\alpha x+\beta y+\gamma z=0$, $\alpha' x+\beta' y+\gamma' z=0$.

10. If $(a, .. \between x, y, z)^2$ is a perfect square, then $\alpha' : \beta' : \gamma' = \alpha : \beta : \gamma$; whence not only, as before, $\mathrm{K}=0$, but the coefficients (A, B, C, F, G, H) all vanish (this implies the first-mentioned condition, $\mathrm{K}=0$); and the line-equation $(\mathrm{A}, .. \between \xi, \eta, \zeta)^2=0$ becomes the mere identity $0=0$.

11. Conversely if $\mathrm{K}=0$, then $(a, .. \between x, y, z)^2$ breaks up into factors; and if (A, B, C, F, G, H) all vanish, then $(a, .. \between x, y, z)^2$ is a perfect square. The conclusions stated *ante*, No. 3, are thus sustained.

12. I assume, first, that $(a, .. \between x, y, z)^2$ is a perfect square (No. 13); and secondly, that it breaks up into factors (No. 14); and I proceed to inquire how in the one case the root, and in the other case the factors, can be determined in a symmetrical form.

13. Considering the before-mentioned identical equation

$$(a, .. \between x, y, z)^2 . (a, .. \between x', y', z')^2 - [(a, .. \between x, y, z \between x', y', z')]^2 = (\mathrm{A}, .. \between yz'-y'z,\ zx'-z'x,\ xy'-x'y)^2,$$

if $(a, .. \between x, y, z)^2$ is a perfect square, then by what precedes, the right-hand side of the equation vanishes, and we have

$$(a, .. \between x, y, z)^2 = \frac{[(a, .. \between x, y, z \between x', y', z')]^2}{(a, .. \between x', y', z')^2};$$

and the root of $(a, .. \between x, y, z)^2$ is thus seen to be

$$= \pm \frac{(a, .. \between x, y, z \between x', y', z')}{\sqrt{(a, .. \between x', y', z')^2}},$$

an expression which involves the quantities (x', y', z'), the values whereof may be assumed at pleasure without altering the value of the expression. For instance, assuming for (x', y', z') the values (1, 0, 0), (0, 1, 0), (0, 0, 1) successively, the different values of the expression are

$$\frac{ax+hy+gz}{\sqrt{a}},\quad \frac{hx+by+fz}{\sqrt{b}},\quad \frac{gx+fy+cz}{\sqrt{c}}.$$

But if, as assumed, $(a,\ldots)(x, y, z)^2$ be a perfect square $=(\alpha x+\beta y+\gamma z)^2$, then

$$(a, b, c, f, g, h)=(\alpha^2, \beta^2, \gamma^2, \beta\gamma, \gamma\alpha, \alpha\beta),$$

and each of the foregoing values becomes equal to the root $\alpha x+\beta y+\gamma z$. It is somewhat singular that it is not possible to obtain symmetrical formulæ without employing in this manner supernumerary arbitrary quantities such as (x', y', z').

14. Next, if $(a,\ldots)(x, y, z)^2$, instead of being a perfect square, only breaks up into factors, then in the foregoing identical equation the right-hand side is a perfect square, and by the formula just obtained its value is

$$\frac{[(A,\ldots)(X, Y, Z)(yz'-y'z,\ zx'-z'x,\ xy'-x'y)]^2}{(A,\ldots)(X, Y, Z)^2},$$

where (X, Y, Z) are supernumerary arbitrary quantities. The identical equation then gives

$$(a,\ldots)(x, y, z)^2=\frac{1}{(a,\ldots)(x', y', z')^2}\left\{[(a,\ldots)(x, y, z)(x', y', z')]^2+\frac{[(A,\ldots)(X, Y, Z)(yz'-y'z,\ zx'-z'x,\ xy'-x'y)]^2}{(A,\ldots)(X, X, Z)^2}\right\},$$

and consequently

$$(a,\ldots)(x, y, z)^2=\frac{1}{(a,\ldots)(x', y', z')^2}\ \text{Product of}$$

$$\left\{(a,\ldots)(x, y, z)(x', y', z')\pm\frac{(A,\ldots)(X, Y, Z)(yz'-y'z,\ zx'-z'x,\ xy'-x'y)}{\sqrt{-(A,\ldots)(X, Y, Z)^2}}\right\},$$

a formula which exhibits the decomposition of $(a,\ldots)(x, y, z)^2$ assumed to be a function which breaks up into factors; the formula contains the two sets of supernumerary arbitrary quantities (x', y', z') and (X, Y, Z). It will be remembered that (A, . .) denotes the system of inverse or reciprocal coefficients $(bc-f^2,\ldots)$.

15. Consider the formula

$$(a, b, c, f, g, h)(\eta\zeta'-\eta'\zeta,\ \zeta\xi'-\zeta'\xi,\ \xi\eta'-\xi'\eta)^2=(\mathrm{a, b, c, f, g, h})(\xi', \eta', \zeta')^2,$$

which gives

$$\begin{aligned}
\mathrm{a}&=\ c\eta^2+b\zeta^2-2f\eta\zeta,\\
\mathrm{b}&=\ a\zeta^2+c\xi^2-2g\zeta\xi,\\
\mathrm{c}&=\ b\xi^2+a\eta^2-2h\xi\eta,\\
\mathrm{f}&=-a\eta\zeta-f\xi^2+g\xi\eta+h\xi\zeta,\\
\mathrm{g}&=-b\zeta\xi+f\xi\eta-g\eta^2+h\eta\zeta,\\
\mathrm{h}&=-c\xi\eta+f\xi\zeta+g\eta\zeta-h\zeta^2;
\end{aligned}$$

and from these we deduce

$$\left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array}\right)(\xi, \eta, \zeta) = (0, 0, 0),$$

viz. $a\xi + h\eta + g\zeta = 0$, &c.

Also

$$(bc-f^2,\ ca-g^2,\ ab-h^2,\ gh-af,\ hf-bg,\ fg-ch) = (\xi, \eta, \zeta)^2 . (A, B, C, F, G, H)(\xi, \eta, \zeta)^2,$$

that is,

$$bc - f^2 = \xi^2 (A, B, C, F, G, H)(\xi, \eta, \zeta)^2, \text{ \&c.}$$

Whence also

$$(bc-f^2, \ldots)(l, m, n)^2 \qquad = (l\xi + m\eta + n\zeta)^2 (A, \ldots)(\xi, \eta, \zeta)^2,$$

and

$$(bc-f^2, \ldots)(l, m, n)(l', m', n') = (l\xi + m\eta + n\zeta)(l'\xi + m'\eta + n'\zeta)(A, \ldots)(\xi, \eta, \zeta)^2;$$

and moreover

$$abc - af^2 - bg^2 - ch^2 + 2fgh = 0.$$

16. The last equation shows that $(a, \ldots)(\eta\zeta' - \eta'\zeta,\ \zeta\xi' - \zeta'\xi,\ \xi\eta' - \xi'\eta)^2$, considered as a function of (ξ', η', ζ'), breaks up into factors. Or since the expression is not altered by interchanging (ξ', η', ζ') and (ξ, η, ζ), the same expression, considered as a function of (ξ, η, ζ), breaks up into factors. It is in fact easy to see that any quantic whatever, $(*)(\eta\zeta' - \eta'\zeta,\ \zeta\xi' - \zeta'\xi,\ \xi\eta' - \xi'\eta)^m$, considered as a function of (ξ, η, ζ), breaks up into linear factors; for in virtue of the equation $\xi'(\eta\zeta' - \eta'\zeta) + \eta'(\zeta\xi' - \zeta'\xi) + \zeta'(\xi\eta' - \xi'\eta) = 0$, any one of the quantities $\eta\zeta' - \eta'\zeta$, $\zeta\xi' - \zeta'\xi$, $\xi\eta' - \xi'\eta$ can be expressed as a linear function of the other two; so that the quantic can be expressed as a linear function of any two of the three quantities; and *quà* homogeneous function of two quantities, it of course breaks up into factors, linear functions of these two quantities.

We may in all the formulæ interchange (x', y', z') and (x, y, z), writing (a', b', c', f', g', h') in the place of (a, b, c, f, g, h).

17. Putting, in like manner,

$$(A, B, C, F, G, H)(yz' - y'z,\ zx' - z'x,\ xy' - x'y)^2$$
$$= (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(x', y', z')^2,$$

so that

$$\begin{aligned}
\mathfrak{A} &= Cy^2 + Bz^2 - 2Fyz, \\
\mathfrak{B} &= Az^2 + Cx^2 - 2Gzx, \\
\mathfrak{C} &= Bx^2 + Ay^2 - 2Hxy, \\
\mathfrak{F} &= -Ayz - Fx^2 + Gxy + Hzx, \\
\mathfrak{G} &= -Bzx + Fxy - Gy^2 + Hyz, \\
\mathfrak{H} &= -Cxy + Fzx + Gyz - Hz^2,
\end{aligned}$$

we obtain

$$\left(\begin{array}{ccc} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C} \end{array}\right)(x, y, z) = (0, 0, 0),$$

viz. $\mathfrak{A}x+\mathfrak{H}y+\mathfrak{G}z=0$, &c.

Also

$$(\mathfrak{B}\mathfrak{C}-\mathfrak{F}^2,\ \mathfrak{C}\mathfrak{A}-\mathfrak{G}^2,\ \mathfrak{A}\mathfrak{B}-\mathfrak{H}^2,\ \mathfrak{G}\mathfrak{H}-\mathfrak{A}\mathfrak{F},\ \mathfrak{H}\mathfrak{F}-\mathfrak{B}\mathfrak{G},\ \mathfrak{F}\mathfrak{G}-\mathfrak{C}\mathfrak{H})$$
$$=(x,\ y,\ z)^2.\ \mathrm{K}(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2;$$

that is,

$$\mathfrak{B}\mathfrak{C}-\mathfrak{F}^2=x^2\mathrm{K}(a,\ b,\ c,\ f,\ g,\ h\between x,\ y,\ z)^2,\ \&c.;$$

whence also

$$(\mathfrak{B}\mathfrak{C}-\mathfrak{F}^2,\ ..\between\lambda,\ \mu,\ \nu)^2=(\lambda x+\mu y+\nu z)^2.\ \mathrm{K}(a,\ ..\between x,\ y,\ z)^2,$$
$$(\mathfrak{B}\mathfrak{C}-\mathfrak{F}^2,\ ..\between\lambda,\ \mu,\ \nu\between\lambda',\ \mu',\ \nu')=(\lambda x+\mu y+\nu z)(\lambda' x+\mu' y+\nu' z).\ \mathrm{K}(a,\ ..\between x,\ y,\ z)^2;$$

and moreover

$$\mathfrak{A}\mathfrak{B}\mathfrak{C}-\mathfrak{A}\mathfrak{F}^2-\mathfrak{B}\mathfrak{G}^2-\mathfrak{C}\mathfrak{H}^2+2\mathfrak{F}\mathfrak{G}\mathfrak{H}=0.$$

The last equation shows that $(\mathrm{A},\ ..\between yz'-y'z,\ zx'-z'x,\ xy'-x'y)^2$, considered as a function of $(x',\ y',\ z')$, breaks up into factors, or (what is the same thing) this expression, considered as a function of $(x,\ y,\ z)$, breaks up into factors; we may in all the formulæ interchange $(x,\ y,\ z)$ and $(x',\ y',\ z')$, writing $(\mathfrak{A}',\ \mathfrak{B}',\ \mathfrak{C}',\ \mathfrak{F}',\ \mathfrak{G}',\ \mathfrak{H}')$ in the place of $(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H})$.

Article Nos. 18 to 28, relating to a single conic in connexion with a point or line.

18. I apply the decomposition formula to the function $(\mathrm{A},\ ..\between yz'-y'z,\ ..)^2$, which, considered as a function of $(x,\ y,\ z)$, breaks up into factors. We have

$$(\mathrm{A},\ ..\between yz'-y'z,\ ..)^2=(\mathfrak{A}',\ ..\between x,\ y,\ z)^2$$
$$=\frac{1}{(\mathfrak{A}',\ ..\between l,\ m,\ n)^2}\ \text{Product of}$$
$$\left\{(\mathfrak{A}',\ ..\between l,\ m,\ n\between x,\ y,\ z)\pm\frac{(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between mz-ny,\ ..\between\lambda,\ \mu,\ \nu)}{\sqrt{-(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between\lambda,\ \mu,\ \nu)^2}}\right\}.$$

But we have

$$(\mathfrak{A}',\ ..\between l,\ m,\ n)^2=(\mathrm{A},\ ..\between mz'-ny',\ ..)^2,$$
$$(\mathfrak{A}',\ ..\between l,\ m,\ n\between x,\ y,\ z)=(\mathrm{A},\ ..\between mz'-ny',\ ..\between yz'-y'z,\ ..),$$
$$(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between mz-ny,\ ..\between\lambda,\ \mu,\ \nu)$$
$$=[x'(mz-ny)+y'(nx-lz)+z'(ly-mx)](\lambda x'+\mu y'+\nu z')\ \mathrm{K}(a,\ ..\between x',\ y',\ z')^2,$$
$$(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between\lambda,\ \mu,\ \nu)^2=\qquad(\lambda x'+\mu y'+\nu z')^2\mathrm{K}(a,\ ..\between x',\ y',\ z')^2,$$

and thence

$$\frac{(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between mz-ny,\ ..\between\lambda,\ \mu,\ \nu)}{\sqrt{-(\mathfrak{B}'\mathfrak{C}'-\mathfrak{F}'^2,\ ..\between\lambda,\ \mu,\ \nu)^2}}=\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix}\sqrt{-\mathrm{K}(a,\ ..\between x',\ y',\ z')^2},$$

whence we have

$$(A, \ldots \between yz'-y'z, \ldots)^2=\frac{1}{(A, \ldots \between mz'-ny')^2} \text{ Product of}$$

$$(A, \ldots \between mz'-ny', \ldots \between yz'-y'z, \ldots) \pm \left| \begin{array}{ccc} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{array} \right| \sqrt{-K(a, \ldots \between x', y', z)^2};$$

And the identical equation

$$(a, \ldots \between x, y, z)^2 . (a, \ldots \between x', y', z')^2-[(a, \ldots \between x, y, z \between x', y', z')]^2=(A, \ldots \between yz'-y'z, \ldots)^2$$

now gives.

$$\text{I.} \quad (a, \ldots \between x, y, z)^2 = \text{Quotient by } (a, \ldots \between x', y', z')^2 \text{ of } \left\{ \begin{array}{l} [(a, \ldots \between x, y, z \between x', y', z')]^2 \\ + \text{ Quotient by } (A, \ldots \between mz'-ny', \ldots)^2 \text{ of Product} \\ (A, \ldots \between mz'-ny', \ldots \between yz'-y'z, \ldots) \pm \left| \begin{array}{ccc} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{array} \right| \sqrt{-K(a, \ldots \between x', y', z')^2}, \end{array} \right.$$

where the Product part may also be written

$$\begin{array}{ll} (a, \ldots \between l, m, n \between x', y', z') . & (a, \ldots \between x, y, z \between x', y', z') \\ -(a, \ldots \between x', y', z')^2 & .(a, \ldots \between x, y, z \between l, m, n) \\ \pm\sqrt{-K(a, \ldots \between x', y', z')^2} & \left| \begin{array}{ccc} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{array} \right| . \end{array}$$

19. Writing in the formula I.

$$\left(\begin{array}{ccc} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{array} \between x', y', z')=(\xi', \eta', \zeta'),$$

we have

$$(x', y', z')=\frac{1}{K}\left(\begin{array}{ccc} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{array} \between \xi', \eta', \zeta'),$$

and thence

$$K(a, \ldots \between x', y', z') \qquad =(A, \ldots \between \xi', \eta', \zeta')^2$$

$$(a, \ldots \between x, y, z \between x', y', z')=\xi'x+\eta'y+\zeta'z.$$

Assume

$$(l, m, n)=(\nu\eta'-\mu\zeta', \ \lambda\zeta'-\nu\xi', \ \mu\xi'-\lambda\eta'),$$

then from the foregoing values of (x', y', z')

$$mz'-ny'=\frac{1}{K}\left\{(\lambda\zeta'-\nu\xi')(G\xi'+F\eta'+C\zeta')-(\mu\xi'-\lambda\eta')(H\xi'+B\eta'+F\zeta')\right\}$$

$$=\frac{1}{K}\Big\{\lambda[\xi'(A\xi'+H\eta'+G\zeta')+\eta'(H\xi'+B\eta'+F\zeta')+\zeta'(G\xi'+F\eta'+C\zeta')]$$
$$-\lambda\xi'(A\xi'+H\eta'+G\zeta')$$
$$-\mu\xi'(H\xi'+B\eta'+F\zeta')$$
$$-\nu\,\xi'(G\xi'+F\eta'+C\zeta'),$$

that is

$$mz'-ny'=\frac{1}{K}\Big\{\lambda(A,\,..\between\xi',\eta',\zeta')^2-\xi'(A,\,..\between\xi',\eta',\zeta'\between\lambda,\mu,\nu)\Big\},$$

and similarly

$$nx'-lz'=\frac{1}{K}\Big\{\mu(A,\,..\between\xi',\eta',\zeta')^2-\eta'(A,\,..\between\xi',\eta',\zeta'\between\lambda,\mu,\nu)\Big\},$$

$$ly'-mx'=\frac{1}{K}\Big\{\nu(A,\,..\between\xi',\eta',\zeta')^2-\zeta'(A,\,..\between\xi',\eta',\zeta'\between\lambda,\mu,\nu)\Big\};$$

and thence

$$(A,H,G\between mz'-ny',\,..)=$$
$$\frac{1}{K}\Big\{(A,H,G\between\lambda,\mu,\nu).(A,\,..\between\xi',\eta',\zeta')^2$$
$$-(A,H,G\between\xi',\eta',\zeta').(A,\,..\between\xi',\eta',\zeta'\between\lambda,\mu,\nu)\Big\}$$

with the like equations, writing H, B, F and G, F, C in the place of A, H, G successively: and we then have

$$(A,\,..\between mz'-ny',\,..)^2$$
$$=\frac{1}{K}\Big\{(A,\,..\between\lambda,\mu,\nu\between mz'-ny',\,..).(A,\,..\between\xi',\eta',\zeta')^2$$
$$-(A,\,..\between\xi',\eta',\zeta'\between mz'-ny',\,..).(A,\,..\between\xi',\eta',\zeta'\between\lambda,\mu,\nu)\Big\}.$$

But the foregoing values of $mz'-ny'$, $nx'-lz'$, $ly'-mx'$ give also

$$(A,\,..\between\lambda,\mu,\nu\between mz'-ny',\,..)$$
$$=\frac{1}{K}\Big\{(A,\,..\between\lambda,\mu,\nu)^2.(A,\,..\between\xi',\eta',\zeta')^2-[(A,\,..\between\lambda,\mu,\nu\between\xi',\eta',\zeta')]^2\Big\}.$$

$$(A,\,..\between\xi',\eta',\zeta'\between mz'-ny',\,..)$$
$$=\frac{1}{K}\Big\{(A,\,..\between\lambda,\mu,\nu\between\xi',\eta',\zeta').(A,\,..\between\xi',\eta',\zeta')^2-(A,\,..\between\xi',\eta',\zeta')^2.(A,\,..\between\lambda,\mu,\nu\between\xi',\eta',\zeta')\Big\}=0.$$

So that

$$(A,\,..\between mz'-ny',\,..)^2$$
$$=\frac{1}{K^2}(A,\,..\between\xi',\eta',\zeta')^2.\{(A,\,..\between\lambda,\mu,\nu)^2.(A,\,..\between\xi',\eta',\zeta')^2-[(A,\,..\between\lambda,\mu,\nu\between\xi',\eta',\zeta')]^2\}$$
$$=\frac{1}{K}(A,\,..\between\xi',\eta',\zeta')^2.(a,\,..\between\nu\eta'-\mu\zeta',\,..)^2.$$

Similarly,

$$
\begin{aligned}
&(A, \ldots \between mz'-ny', \ldots \between yz'-y'z, \ldots)\\
&\quad =\frac{1}{K}\Big\{(A, \ldots \between \lambda, \mu, \nu \between yz'-y'z, \ldots).(A, \ldots \between \xi', \eta', \zeta')^2\\
&\qquad -(A, \ldots \between \xi', \eta', \zeta' \between yz'-y'z, \ldots).(A, \ldots \between \xi', \eta', \zeta' \between \lambda, \mu, \nu)\Big\}.
\end{aligned}
$$

But

$$
\begin{aligned}
&(A, \ldots \between \lambda, \mu, \nu \between yz'-y'z, \ldots)\\
&= \quad (A\lambda+H\mu+G\nu)(yz'-y'z)\\
&\quad +(H\lambda+B\mu+F\nu)(zx'-z'x)\\
&\quad +(G\lambda+F\mu+C\nu)(xy'-x'y)\\
&= \quad x[y'(G\lambda+F\mu+C\nu)-z'(H\lambda+B\mu+F\nu)]\\
&\quad +y[z'(A\lambda+H\mu+G\nu)-x'(G\lambda+F\mu+C\nu)]\\
&\quad +z[x'(H\lambda+B\mu+F\nu)-y'(A\lambda+H\mu+G\nu)],
\end{aligned}
$$

which, substituting for x', y', z' their values

$$
(x', y', z')=\frac{1}{K}\left(\begin{array}{ccc} A, & H, & G\\ H, & B, & F\\ G, & F, & C\end{array}\right\between \xi', \eta', \zeta'),
$$

becomes

$$
\begin{aligned}
=\frac{1}{K}\Big\{ & x[(BC-F^2)(\nu\eta'-\mu\zeta')+(FG-CH)(\lambda\zeta'-\nu\xi')+(HF-BG)(\mu\xi'-\lambda\eta')]\\
&+y[(FG-CH)(\nu\eta'-\mu\zeta')+(CA-G^2)(\lambda\zeta'-\nu\xi')+(GH-AF)(\mu\xi'-\lambda\eta')]\\
&+z\,(HF-BG)(\nu\eta'-\mu\zeta')+(GH-AF)(\lambda\zeta'-\nu\xi')+(AB-H^2)(\mu\xi'-\lambda\eta')\Big\},
\end{aligned}
$$

which is

$$
=(a, \ldots \between x, y, z \between \nu\eta'-\mu\zeta', \ldots);
$$

and by merely writing (ξ', η', ζ') in the place of (λ, μ, ν), we have

$$
(A, \ldots \between \xi', \eta', \zeta' \between yz'-y'z, \ldots)=0;
$$

so that we find

$$
\begin{aligned}
&(A, \ldots \between mz'-ny', \ldots \between yz'-y'z, \ldots)\\
&\quad =\frac{1}{K}(A, \ldots \between \xi', \eta', \zeta')^2.(a, \ldots \between x, y, z \between \nu\eta'-\mu\zeta', \lambda\zeta'-\nu\xi', \mu\xi'-\lambda\eta').
\end{aligned}
$$

Now, writing the formula I. in the form

$$
(a, \ldots \between x, y, z)^2=\text{Quotient by } K(a, \ldots \between x', y', z')^2 \text{ of}
$$

$$
\left\{\begin{array}{l}
K[(a, \ldots \between x, y, z \between x', y', z')]^2\\
+\text{Quotient by } K(A, \ldots \between mz'-ny', \ldots)^2 \text{ of}\\
K^2\{[(A, \ldots \between mz'-ny', \ldots \between yz'-y'z, \ldots)]^2+\left|\begin{array}{ccc} x, & y, & z\\ x', & y', & z'\\ l, & m, & n\end{array}\right|^2 K(a, \ldots \between x, y, z)^2\},
\end{array}\right.
$$

the right-hand side is

$$=\text{Quotient by } (\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2 \text{ of } \left\{\begin{array}{l} \mathrm{K}(\xi' x+\eta' y+\zeta' z)^2 \\ +\text{Quotient by } (a, \ldots \between \nu\eta'-\mu\zeta', \ldots)^2(\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2 \text{ of} \\ \{[(a, \ldots \between \nu\eta'-\mu\zeta', \ldots \between x, y, z).(\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2]^2+\Pi^2(\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2\}, \end{array}\right.$$

where

$$\Pi=\mathrm{K}\left|\begin{array}{ccc} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{array}\right|,$$

or, what is the same thing,

$$\Pi=\left|\begin{array}{ccc} x, & y, & z \\ \mathrm{K}x', & \mathrm{K}y', & \mathrm{K}z' \\ l, & m, & n \end{array}\right|=\left|\begin{array}{ccc} x & y & z \\ \mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta', & \mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta', & \mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta' \\ \nu\eta'-\mu\zeta', & \lambda\zeta'-\nu\xi', & \mu\xi'-\lambda\eta'. \end{array}\right|;$$

More simply, the right-hand side is

$$=\text{Quotient by } (\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2 \text{ of } \left\{\begin{array}{l} \mathrm{K}(\xi' x+\eta' y+\zeta' z)^2 \\ +\text{Quotient by } (a, \ldots \between \nu\eta'-\mu\zeta', \ldots)^2 \text{ of} \\ \{[(a, \ldots \between \nu\eta'-\mu\zeta', \ldots \between x, y, z)]^2(\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2+\Pi^2\}; \end{array}\right.$$

Or restoring the left-hand side, and resolving into its linear factors the function in { }, we have

$$\text{II.}\quad (a, \ldots \between x, y, z)^2=\text{Quotient by } (\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2 \text{ of } \left\{\begin{array}{l} \mathrm{K}(\xi' x+\eta' y+\zeta' z)^2 \\ +\text{Quotient by } (a, \ldots \between \nu\eta'-\mu\zeta', \ldots)^2 \text{ of Product} \\ \Pi\pm\sqrt{-(\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2}(a, \ldots \between \nu\eta'-\mu\zeta', \ldots \between x, y, z), \end{array}\right.$$

where Π has the value given above, which may also be written

$$\begin{aligned}\Pi= & \ (\mathrm{A}, \ldots \between \xi', \eta', \zeta' \between \lambda, \mu, \nu)(\xi' x+\eta' y+\zeta' z) \\ & -(\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2(\lambda x+\mu y+\nu z).\end{aligned}$$

20. We deduce at once the inverse or reciprocal formulæ

$$\text{III.}\quad (\mathrm{A}, \ldots \between \xi, \eta, \zeta)^2=\text{Quotient by } (\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2 \text{ of } \left\{\begin{array}{l} [(\mathrm{A}, \ldots \between \xi, \eta, \zeta \between \xi', \eta', \zeta')]^2 \\ +\text{Quotient by } (a, \ldots \between \nu\eta'-\mu\zeta', \ldots)^2 \text{ of K into Product} \\ (a, \ldots \between \nu\eta'-\mu\zeta', \ldots \between \eta\zeta'-\eta'\zeta, \ldots)\pm\sqrt{-(\mathrm{A}, \ldots \between \xi', \eta', \zeta')^2}\left|\begin{array}{ccc} \xi, & \eta, & \zeta \\ \xi', & \eta', & \zeta' \\ \lambda, & \mu, & \nu \end{array}\right|, \end{array}\right.$$

where the Product part may also be written

$$\text{Product}\Big[\tfrac{1}{K}(A,..\between\xi',\eta',\zeta'\between\lambda,\mu,\nu).(A,..\between\xi',\eta',\zeta'\between\xi,\eta,\zeta)$$
$$-\tfrac{1}{K}(A,..\between\xi',\eta',\zeta')^2 \quad .(A,..\between\lambda,\mu,\nu\between\xi,\eta,\zeta)$$
$$\pm\sqrt{-(A,..\between\xi',\eta',\zeta')^2}\begin{vmatrix}\xi, & \eta, & \zeta\\ \xi', & \eta', & \zeta'\\ \lambda, & \mu, & \nu\end{vmatrix}\Big].$$

21. And also

$$\text{IV.}\quad (A,..\between\xi,\eta,\zeta)^2=\text{Quotient by }(a,..\between x',y',z')^2\text{ of}\left\{\begin{array}{l}K(\xi x'+\eta y'+\zeta z')^2\\ +\text{Quotient by }K(A,..\between ny'-mz',..)^2\text{ of Product}\\ K\Phi\pm\sqrt{-K(a,..\between x',y',z')^2}(A,..\between ny'-mz',..\between\xi,\eta,\zeta),\end{array}\right.$$

where

$$\Phi=\begin{vmatrix}\xi & \eta & \zeta\\ ax'+hy'+gz', & hx'+by'+fz', & gx'+fy'+cz'\\ ny'-mz', & lz'-nx', & mx'-ly'\end{vmatrix},$$

which may also be written

$$=\ (a,..\between x',y',z'\between l,m,n)(x'\xi+y'\eta+z'\zeta)$$
$$-(a,..\between x',y',z')^2\quad.\quad(l\xi+m\eta+n\zeta).$$

22. The geometrical signification is obvious. The formulæ I. and II. each of them show that the equation

$$(a,..\between x,y,z)^2=0$$

of the conic may be written in the form

$$W^2+\frac{1}{M}QR=0,$$

where $Q=0$, $R=0$ are any two tangents of the conic, and $W=0$ is the line joining the points of contact, or chord of contact corresponding to the two tangents; viz., in the formula I. we have

$$W=(a,..\between x',y',z'\between x,y,z),$$

$$\left.\begin{matrix}Q\\R\end{matrix}\right\}=(A,..\between mz'-ny',..\between yz'-y'z,..)\pm\sqrt{-K(a,..\between x,y,z)^2}\begin{vmatrix}x, & y, & z\\ x', & y', & z'\\ l, & m, & n\end{vmatrix}$$

(or for a different form of Q, R see the formula). The quantities (x', y', z') are the coordinates of the point of intersection of the two tangents, or pole of the chord of contact: (l, m, n) are supernumerary arbitrary quantities, the values whereof do not affect the result *. And in the formula II. we have

$$W=\xi' x+\eta' y+\zeta' z,$$
$$\left.\begin{matrix}Q\\R\end{matrix}\right\}=\Pi\pm\sqrt{-(A,..\between\xi',\eta',\zeta')^2}\,(a,..\between\nu\eta'-\mu\zeta',..\between x, y, z)$$

(for the value of Π see the formula). The quantities (ξ', η', ζ') are the line-coordinates of the chord of contact (viz. the point-equation of this line is $\xi' x+\eta' y+\zeta' z=0$); (λ, μ, ν) are supernumerary arbitrary quantities.

23. In the like manner the formulæ III. and IV. each of them show that the line-equation

$$(A,..\between\xi, \eta, \zeta)^2=0$$

of the conic may be written in the form

$$W^2+\frac{1}{M}QR=0,$$

where $Q=0$, $R=0$ are any two ineunts of the conic, and $W=0$ is the point of intersection of the corresponding tangents; viz. in the formula III. we have

$$W=(A,\ldots\between\xi', \eta', \zeta'\between\xi, \eta, \zeta),$$
$$\left.\begin{matrix}Q\\R\end{matrix}\right\}=(a,\ldots\between\nu\eta'-\mu\zeta',..)\pm\sqrt{-(A,..\between\xi',\eta',\zeta')^2}\begin{vmatrix}\xi, & \eta, & \zeta\\ \xi', & \eta', & \zeta'\\ \lambda, & \mu, & \nu\end{vmatrix}$$

(for another form of Q, R see the formula).

The quantities ξ', η', ζ' are the line-coordinates of the line through the two ineunts, or chord of contact; (λ, μ, ν) are supernumerary arbitrary quantities; and so in the formula IV. we have

$$W=x'\xi+y'\eta+z'\zeta,$$
$$\left.\begin{matrix}Q\\R\end{matrix}\right\}=K\Phi\pm\sqrt{-K(a,..\between x', y', z')^2}\,(A,..\between ny'-mz',..\between\xi, \eta, \zeta)$$

(for the value of Φ see the formula), where x', y', z' are the point-coordinates of the intersection of tangents at the two ineunts, or pole of the chord of contact; (l, m, n) are supernumerary arbitrary quantities.

24. We may, instead of the supernumerary arbitrary quantities (l, m, n) of the formula I., introduce the quantities (λ, μ, ν), where

$$(l, m, n)=\frac{1}{K}\left(\begin{matrix}A, & H, & G\\ H, & B, & F\\ G, & F, & C\end{matrix}\right\between\lambda, \mu, \nu).$$

* In a different point of view, viz. if we consider the formula I. as a transformation of the function $(a,\ldots\between x, y, z)^2$, then (x', y', z') and (l, m, n) would be each of them supernumerary arbitrary quantities: and so in the other like cases.

This gives

$$
\begin{aligned}
&(\mathrm{A, H, G}\between mz'-ny', ..)\\
&\quad =\mathrm{A}(mz'-ny')+\mathrm{H}(nx'-lz')+\mathrm{G}(ly'-mx')\\
&\quad =x'(\mathrm{H}n-\mathrm{G}m)+y'(\mathrm{G}l-\mathrm{A}n)+z'(\mathrm{A}m-\mathrm{H}l)\\
&\quad =\tfrac{1}{\mathrm{K}}\cdot \quad x'[\mathrm{H}(\mathrm{G}\lambda+\mathrm{F}\mu+\mathrm{C}\nu)-\mathrm{G}(\mathrm{H}\lambda+\mathrm{B}\mu+\mathrm{F}\nu)]\\
&\qquad\quad +y'[\mathrm{G}(\mathrm{A}\lambda+\mathrm{H}\mu+\mathrm{G}\nu)-\mathrm{A}(\mathrm{G}\lambda+\mathrm{F}\mu+\mathrm{C}\nu)]\\
&\qquad\quad +z'[\mathrm{A}(\mathrm{H}\lambda+\mathrm{B}\mu+\mathrm{F}\nu)-\mathrm{H}(\mathrm{H}\lambda+\mathrm{B}\mu+\mathrm{F}\nu)]\\
&\quad = \quad x'(g\mu-h\nu)+y'(f\mu-b\nu)+z'(c\mu-f\nu)\\
&\quad = \quad \mu(gx'+fy'+cz')-\nu(hx'+by'+fz').
\end{aligned}
$$

We have thus the system

$$
\begin{aligned}
(\mathrm{A, H, G}\between mz'-ny', ..)&=\mu(gx'+fy'+cz')-\nu(hx'+by'+fz'),\\
(\mathrm{H, B, F}\between mz'-ny', ..)&=\nu(ax'+hy'+gz')-\lambda(gx'+fy'+cz'),\\
(\mathrm{G, F, C}\between mz'-ny', ..)&=\lambda(hx'+by'+fz')-\mu(ax'+hy'+gz'),
\end{aligned}
$$

and thence

$$
\begin{aligned}
&(\mathrm{A}, ..\between mz'-ny', ..\between yz'-y'z, ..)\\
&\quad =-\begin{vmatrix} yz'-y'z, & zx'-z'x, & xy'-x'y\\ ax'+hy'+gz', & hx'+by'+fz', & gx'+fy'+cz'\\ \lambda, & \mu, & \nu \end{vmatrix};
\end{aligned}
$$

or observing that the term in λ is

$$-(zx'-z'x)(gx'+fy'+cz')+(xy'-x'y)(hx'+by'+fz'),$$

which is

$$
\begin{aligned}
&= \quad x(x'(ax'+hy'+gz')+y'(hx'+fy'+fz')+z'(gx'+fy'+cz'))\\
&\quad -x.x'(ax'+hy'+gz')\\
&\quad -y.x'(hx'+by'+fz')\\
&\quad -z.x'(gx'+fy'+cz')\\
&=-x'(a, ..\between x, y, z\between x', y', z')+x(a, ..\between x', y', z')^2,
\end{aligned}
$$

with similar expressions for the terms in μ, ν, we have

$$
\begin{aligned}
&(\mathrm{A}, ..\between mz'-ny', ..\between yz'-y'z, ..)\\
&\qquad =-(\lambda x'+\mu y'+\nu z').(a, ..\between x, y, z\between x', y', z')+(\lambda x+\mu y+\nu z).(a, ..\between x', y', z')^2;
\end{aligned}
$$

and so also

$$
\begin{aligned}
&(\mathrm{A}, ..\between mz'-ny', ..)^2\\
&\qquad =-(\lambda x'+\mu y'+\nu z').(a, ..\between l, m, n\between x', y', z')+(\lambda l+\mu m+\nu n).(a, ..\between x', y', z')^2,
\end{aligned}
$$

where

$$
\begin{aligned}
(a, ..\between l, m, n\between x', y', z')&=\lambda x'+\mu y'+\nu z',\\
\lambda l+\mu m+\nu n &=\tfrac{1}{\mathrm{K}}(\mathrm{A}, ..\between \lambda, \mu, \nu)^2,
\end{aligned}
$$

so that

$$(\mathrm{A}, ..\between mz'-ny', ..)^2=-(\lambda x'+\mu y'+\nu z')^2+\frac{1}{\mathrm{K}}(\mathrm{A}, ..\between \lambda, \mu, \nu)^2.(a, ..\between x', y', z')^2.$$

Moreover,

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ l, & m, & n \end{vmatrix} =\frac{1}{\mathrm{K}}\Big\{(\mathrm{A}\lambda+\mathrm{H}\mu+\mathrm{G}\nu)(yz'-y'z)+(\mathrm{H}\lambda+\mathrm{B}\mu+\mathrm{F}\nu)(zx'-z'x)+(\mathrm{G}\lambda+\mathrm{F}\mu+\mathrm{C}\nu)(xy'-x'y)\Big\},$$

which is

$$=\frac{1}{\mathrm{K}}(\mathrm{A}, ..\between \lambda, \mu, \nu\between yz'-y'z, ..);$$

and hence instead of the formula I. we have

$$(a, ..\between x, y, z)^2=\text{Quotient by } (a, ..\between x', y', z')^2 \text{ of}$$

$$\text{I. (bis)}\quad \left\{\begin{array}{l} [(a, ..\between x, y, z\between x', y', z')]^2 \\ +\text{Quotient by } +(\mathrm{A}, ..\between \lambda, \mu, \nu)^2(a, ..\between x', y', z')^2-\mathrm{K}(\lambda x'+\mu y'+\nu z')^2 \text{ of K Product} \\ \left\{\begin{array}{l}(\lambda x'+\mu y'+\nu z').(a, ..\between x, y, z\between x', y', z')-(\lambda x+\mu y+\nu z).(a, ..\between x', y', z')^2 \\ \pm\frac{1}{\mathrm{K}}\sqrt{-\mathrm{K}(a, ..\between x', y', z')^2}(\mathrm{A}, ..\between \lambda, \mu, \nu\between yz'-y'z, ..).\end{array}\right\}^2 \end{array}\right.$$

25. If, in like manner, in the formula II. we introduce, instead of (λ, μ, ν), the new quantities (l, m, n), where

$$(\lambda, \mu, \nu)=\begin{pmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{pmatrix}\between l, m, n),$$

or, what is the same thing,

$$(l, m, n)=\frac{1}{\mathrm{K}}\begin{pmatrix} \mathrm{A}, & \mathrm{H}, & \mathrm{G} \\ \mathrm{H}, & \mathrm{B}, & \mathrm{F} \\ \mathrm{G}, & \mathrm{F}, & \mathrm{C} \end{pmatrix}\between \lambda, \mu, \nu),$$

then we have

$$(a, h, g\between \nu\eta'-\mu\zeta', ..)=n\,(\mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta')-m(\mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta'),$$
$$(h, b, f\between \nu\eta'-\mu\zeta', ..)=l\,(\mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta')-n\,(\mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta'),$$
$$(g, f, c\between \nu\eta'-\mu\zeta', ..)=m(\mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta')-l\,(\mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta');$$

and thence

$$(a, ..\between \nu\eta'-\mu\zeta', ..\between x, y, z)=\begin{vmatrix} x & , & y & , & z \\ \mathrm{A}\xi'+\mathrm{H}\eta'+\mathrm{G}\zeta', & & \mathrm{H}\xi'+\mathrm{B}\eta'+\mathrm{F}\zeta', & & \mathrm{G}\xi'+\mathrm{F}\eta'+\mathrm{C}\zeta' \\ l & , & m & , & n \end{vmatrix}$$

$$=\ (\mathrm{A}, ..\between mz-ny, ..\between \xi', \eta', \zeta'),$$

$$(a, ..\between \nu\eta'-\mu\zeta', ..)^2=\frac{1}{\mathrm{K}}\Big\{(\mathrm{A}, ..\between \lambda, \mu, \nu)^2.(\mathrm{A}, ..\between \xi', \eta', \zeta')^2-[(\mathrm{A}, ..\between \lambda, \mu, \nu\between \xi', \eta', \zeta')]^2\Big\}$$

$$=(a, ..\between l, m, n)^2.(\mathrm{A}, ..\between \xi', \eta', \zeta')^2-\mathrm{K}(l\xi'+m\eta'+n\zeta')^2;$$

$$V=\left|\begin{array}{ccc} x\,, & y\,, & z \\ A\xi'+H\eta'+G\zeta', & H\xi'+B\eta'+F\zeta', & G\xi'+F\eta'+C\zeta' \\ \nu\eta'-\mu\zeta'\,, & \lambda\zeta'-\nu\xi'\,, & \mu\xi'-\lambda\eta' \end{array}\right.$$

$$\begin{aligned}
=&(\nu\eta'-\mu\zeta').\,y(G\xi'+F\eta'+C\zeta')-z(H\xi'+B\eta'+F\zeta')\\
&(\lambda\zeta'-\nu\xi').\,z(A\xi'+H\eta'+G\zeta')-x(G\xi'+F\eta'+C\zeta')\\
&(\mu\xi'-\lambda\eta').\,x(H\xi'+B\eta'+F\zeta')-y(A\xi'+H\eta'+G\zeta')
\end{aligned}$$

$$\begin{aligned}
=&\lambda\{(\xi'x+\eta'y+\zeta'z)(A\xi'+H\eta'+G\zeta')-x(A,\ldots)(\xi',\eta',\zeta')^2\}\\
&+\mu\{(\xi'x+\eta'y+\zeta'z)(H\xi'+B\eta'+F\zeta')-y(A,\ldots)(\xi',\eta',\zeta')^2\}\\
&+\nu\{(\xi'x+\eta'y+\zeta'z)(G\xi'+F\eta'+C\zeta')-z(A,\ldots)(\xi',\eta',\zeta')^2\}
\end{aligned}$$

$$=(\xi'x+\eta'y+\zeta'z).(A,\ldots)(\lambda,\mu,\nu)(\xi',\eta',\zeta')-(\lambda x+\mu y+\nu z).(A,\ldots)(\xi',\eta',\zeta')^2$$

$$=K(l\xi'+m\eta'+n\zeta').(\xi'x+\eta'y+\zeta'z)-(a,\ldots)(l,m,n)(x,y,z).(A,\ldots)(\xi',\eta',\zeta')^2;$$

and the formula II. thus becomes

$$\text{II. (bis)}\quad (a,\ldots)(x,y,z)^2=\text{Quotient by }(A,\ldots)(\xi',\eta',\zeta')^2\text{ of}\left\{\begin{array}{l} K(\xi'x+\eta'y+\zeta'z)^2\\ +\text{Quotient by }(a,\ldots)(l,m,n)^2.(A,\ldots)(\xi',\eta',\zeta')^2-K(l\xi'+m\eta'+n\zeta')^2\text{ of Product}\\ \left\{\begin{array}{l}K(l\xi'+m\eta'+n\zeta').(\xi'x+\eta'y+\zeta'z)-(a,\ldots)(l,m,n)(x,y,z).(A,\ldots)(\xi',\eta',\zeta')^2\\ \pm\sqrt{-(A,\ldots)(\xi',\eta',\zeta')^2}(A,\ldots)(mz-ny,\ldots)(\xi',\eta',\zeta').\end{array}\right.\end{array}\right.$$

26. And from these we at once deduce the inverse or reciprocal formulæ

$$\text{III. (bis)}\quad (A,\ldots)(\xi,\eta,\zeta)^2=\text{Quotient by }(A,\ldots)(\xi',\eta',\zeta')^2\text{ of}\left\{\begin{array}{l} [(A,\ldots)(\xi,\eta,\zeta)(\xi',\eta',\zeta')]^2\\ +\text{Quotient by }(a,\ldots)(l,m,n)^2.(A,\ldots)(\xi',\eta',\zeta')^2-K(l\xi'+m\eta'+n\zeta')^2\text{ of K into Product}\\ \left\{\begin{array}{l}(l\xi'+m\eta'+n\zeta')(A,\ldots)(\xi,\eta,\zeta)(\xi',\eta',\zeta')-(l\xi+m\eta+n\zeta).(A,\ldots)(\xi',\eta',\zeta')^2\\ \pm\sqrt{-(A,\ldots)(\xi',\eta',\zeta')^2}(a,\ldots)(l,m,n)(\eta\zeta'-\eta'\zeta,\ldots),\end{array}\right\}\end{array}\right.$$

27. And

$$\text{IV. (bis)}\quad (A,\ldots)(\xi,\eta,\zeta)^2=\text{Quotient by }(a,\ldots)(x',y',z')^2\text{ of}\left\{\begin{array}{l} K(x'\xi+y'\eta+z'\zeta)^2\\ +\text{Quotient by }(A,\ldots)(\lambda,\mu,\nu)^2.(a,\ldots)(x',y',z')^2-K(\lambda x'+\mu y'+\nu z')^2\text{ of Product}\\ \left\{\begin{array}{l}K(\lambda x'+\mu y'+\nu z').(x'\xi+y'\eta+z'\zeta)-(A,\ldots)(\xi,\eta,\zeta)(l,m,n).(a,\ldots)(x',y',z')^2\\ \pm\sqrt{-K(a,\ldots)(x',y',z')^2}(a,\ldots)(x',y',z')(\mu\zeta-\nu\eta,\ldots),\end{array}\right\}\end{array}\right.$$

which four formulæ have the same geometrical significations with the original four formulæ to which they correspond respectively.

28. The eight formulæ become all of them the same or very similar for the quadric form $(a, .. \between x, y, z)^2 = x^2+y^2+z^2$, which of course implies $(A, .. \between \xi, \eta, \zeta)^2 = \xi^2+\eta^2+\zeta^2$. Thus selecting any one of them at pleasure, *e. g.* the formula II. (bis), this becomes

$$\begin{aligned}
&\{(x^2+y^2+z^2)(\xi'^2+\eta'^2+\zeta'^2)-(\xi'x+\eta'y+\zeta'z)^2\}\\
\times&\{(l^2+m^2+n^2)(\xi'^2+\eta'^2+\zeta'^2)-(l\xi'+m\eta'+n\zeta')^2\}\\
=&\{(l\xi'+m\eta'+n\zeta')(x\xi'+m\eta'+n\zeta')-(lx+my+nz)(\xi'^2+\eta'^2+\zeta'^2)\}^2\\
&+(\xi'^2+\eta'^2+\zeta'^2)\begin{vmatrix} \xi', & \eta', & \zeta' \\ x, & y, & z \\ l, & m, & n \end{vmatrix}^2,
\end{aligned}$$

where the terms independent of $\xi'^2+\eta'^2+\zeta'^2$ destroy each other. Omitting these terms, and dividing by $\xi'^2+\eta'^2+\zeta'^2$, the resulting equation is found to be

$$\begin{vmatrix} \xi', & \eta', & \zeta' \\ x, & y, & z \\ l, & m, & n \end{vmatrix}^2 = \begin{vmatrix} \xi'^2+\eta'^2+\zeta'^2, & \xi'x+\eta'y+\zeta'z, & \xi'l+\eta'm+\zeta'n \\ x\xi'+y\eta'+z\zeta', & x^2+y^2+z^2, & xl+ym+zn \\ l\xi'+m\eta'+n\zeta', & lx+my+nz, & l^2+m^2+n^2, \end{vmatrix}$$

which is a well-known identical equation.

Article Nos. 29 to 33, relating to a single conic in connexion with an ineunt or a tangent of a conic of double contact.

29. The formulæ assume a very simple form when the point of intersection of the two tangents, or the line of junction of the two ineunts of the conic, is an ineunt or a tangent of a conic having double contact with the first-mentioned conic. Thus, if to the conic

$$(a, .. \between x, y, z)^2=0$$

tangents are drawn from a point (x', y', z') of the conic

$$(a, .. \between x, y, z)^2+(\xi'x+\eta'y+\zeta'z)^2=0,$$

then we have

$$(a, .. \between x', y', z')^2=-(\xi'x'+\eta'y'+\zeta'z')^2;$$

and using the form I. (bis), and putting therein (ξ', η', ζ') in the place of the arbitrary quantities (λ, μ, ν), the equation of the tangent divides out by $\xi'x'+\eta'y'+\zeta'z'$, and omitting this factor it becomes

$$\begin{aligned}
&(a, .. \between x', y', z' \between x, y, z)+(\xi'x'+\eta'y'+\zeta'z')(\xi'x+\eta'y+\zeta'z)\\
&\pm\frac{1}{\sqrt{K}}(A, .. \between \xi', \eta', \zeta' \between yz'-y'z, zx'-z'x, xy'-x'y)=0,
\end{aligned}$$

which is of the form

$$\left(\begin{matrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{matrix} \between x', y', z' \between x, y, z\right)=0,$$

where the matrix

$$\left|\begin{matrix} \alpha , & \beta , & \gamma \\ \alpha' , & \beta' , & \gamma \\ \alpha'' , & \beta'' , & \gamma'' \end{matrix}\right| \text{ is } =$$

$$\begin{matrix} a+\xi'^2 & , h+\xi'\eta' +\frac{1}{\sqrt{K}}(G\xi'+F\eta' +C\zeta'), & g+\xi'\zeta'-\frac{1}{\sqrt{K}}(H\xi'+ B\eta'+ F\zeta') \\ h+\xi'\eta'-\frac{1}{\sqrt{K}}(G\xi'+F\eta'+C\zeta'), & b+\eta'^2 & , f+\eta'\zeta'+\frac{1}{\sqrt{K}}(A\xi'+H\eta'+G\zeta') \\ g+\xi'\zeta'+\frac{1}{\sqrt{K}}(H\xi'+B\eta'+F\zeta'), & f+\eta'\zeta'-\frac{1}{\sqrt{K}}(A\xi'+H\eta'+G\zeta'), & c+\zeta'^2. \end{matrix}$$

30. But instead of further developing these formulæ, I prefer to consider the formulæ which give the points of contact of the tangents in question, viz. the ineunts of the conic $(a, ..\between x, y, z)^2=0$, or the tangents through the point (x', y', z') of the conic $(a, ..\between x, y, z)^2+(\xi'x+\eta'y+\zeta'z)^2=0$.

We have as before

$$(a, ..\between x', y', z')^2=-(\xi'x'+\eta'y'+\zeta'z')^2,$$

and using the formula IV.(bis) and writing therein (ξ', η', ζ') in the place of the arbitrary quantities (λ, μ, ν), the equation contains the factor $\xi'x'+\eta'y'+\zeta'z'$, and dividing by this factor, and by K, the line-equation of the ineunt is

$$x'\xi+y'\eta+z'\zeta+\frac{1}{K}(x'\xi'+y'\eta'+z'\zeta').(A, ..\between \xi', \eta', \zeta'\between \xi, \eta, \zeta)$$

$$\pm\frac{1}{\sqrt{K}}(a, ..\between x', y', z'\between \eta\zeta'-\eta'\zeta, ..)=0.$$

Selecting the positive sign, the coordinates of the corresponding ineunt are

$$x'+\frac{1}{K}(x'\xi'+y'\eta'+z'\zeta')(A\xi'+H\eta'+G\zeta')+\frac{1}{\sqrt{K}}\Big\{\eta'(gx'+fy'+cz')-\zeta'(hx'+by'+fz')\Big\},$$

$$y'+\frac{1}{K}(x'\xi'+y'\eta'+z'\zeta')(H\xi'+ B\eta'+ F\zeta')+\frac{1}{\sqrt{K}}\Big\{\zeta'(ax'+hy'+gz')-\xi'(gx'+fy'+cz')\Big\},$$

$$z'+\frac{1}{K}(x'\xi'+y'\eta'+z'\zeta')(G\xi'+ F\eta'+ C\zeta')+\frac{1}{\sqrt{K}}\Big\{\xi'(hx'+by'+fz')-\eta'(ax'+hy'+gz')\Big\};$$

and taking (X, Y, Z) for the coordinates of the ineunt in question, and putting for shortness

$$\alpha =1-\frac{1}{\sqrt{K}}(g\eta'-h\zeta'),\quad \beta = -\frac{1}{\sqrt{K}}(f\eta'-b\zeta'),\quad \gamma = -\frac{1}{\sqrt{K}}(c\eta'-f\zeta'),$$

$$\alpha' = -\frac{1}{\sqrt{K}}(a\zeta'-g\xi'),\quad \beta' =1-\frac{1}{\sqrt{K}}(h\zeta'-f\xi'),\quad \gamma' = -\frac{1}{\sqrt{K}}(g\zeta'-c\xi'),$$

$$\alpha''= -\frac{1}{\sqrt{K}}(h\xi'-a\eta'),\quad \beta''= -\frac{1}{\sqrt{K}}(b\xi'-h\eta'),\quad \gamma''=1-\frac{1}{\sqrt{K}}(f\xi'-g\eta'),$$

we may write

$$(1+P)X=(2-\alpha)x' \quad -\beta y' \quad -\gamma z' \quad +\tfrac{1}{K}(A\xi'+H\eta'+G\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$(1+P)Y= \quad -\alpha' x' +(2-\beta')y'-\gamma' z' \quad +\tfrac{1}{K}(H\xi'+ B\eta'+ F\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$(1+P)Z= \quad \alpha'' x'' \quad -\beta'' y'+(2-\gamma'')z'+\tfrac{1}{K}(G\xi'+ F\eta'+ C\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

where P, which is arbitrary, may be put

$$=\tfrac{1}{K}(A, \ldots)\!(\xi', \eta', \zeta')^2.$$

31. These equations then give

$$(x', y', z')=\left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right)\!\!\left(X, Y, Z\right),$$

which can be verified without difficulty by reversing the process; and we have thus the coordinates (X, Y, Z) in terms of (x', y', z'), and reciprocally.

32. If $(X_{,}, Y_{,}, Z_{,})$ are the coordinates of the other ineunt, we have, it is clear,

$$(x', y', z')=\left(\begin{array}{ccc} 2-\alpha, & -\beta, & -\gamma \\ -\alpha', & 2-\beta', & -\gamma' \\ -\alpha'', & -\beta'', & 2-\gamma'' \end{array}\right)\!\!\left(X_{,}, Y_{,}, Z_{,}\right);$$

or substituting for (x', y', z') their values in terms of (X, Y, Z),

$$(2X_{,}, 2Y_{,}, 2Z_{,})=\left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right)\!\!\left(X+X_{,}, Y+Y_{,}, Z+Z_{,}\right),$$

so that $(X+X_{,}, Y+Y_{,}, Z+Z_{,})$ are the same linear functions of $2X_{,}, 2Y_{,}, 2Z_{,}$ that (X, Y, Z) are of (x', y', z'); that is, we have

$$\tfrac{1}{2}(1+P)(X+X_{,})=(2-\alpha)X_{,}- \quad \beta\, Y_{,}- \quad \gamma\, Z_{,}+\tfrac{1}{K}(A\xi'+H\eta'+G\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$\tfrac{1}{2}(1+P)(X+X_{,})= \quad -\alpha'\, X_{,}+(2-\beta')Y_{,}- \quad \gamma'\, Z_{,}+\tfrac{1}{K}(H\xi'+B\eta'+F\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$\tfrac{1}{2}(1+P)(X+X_{,})= \quad -\alpha''\, X_{,}- \quad \beta''\, Y_{,}+(2-\gamma'')Z_{,}+\tfrac{1}{K}(G\xi'+F\eta'+C\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

which equations may be written

$$(1+P)(X, Y, Z)=\left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right)\!\!\left(X_{,}, Y_{,}, Z_{,}\right),$$

where the values of the coefficients are

$$\begin{array}{lll}
1-\frac{2}{\sqrt{K}}(g\eta'-h\zeta')+\frac{1}{K}(A\xi'^2-B\eta'^2-2F\eta'\zeta'-C\zeta'^2), & -\frac{2}{\sqrt{K}}(f\eta'-b\zeta')+\frac{2}{K}\eta'(A\xi'+H\eta'+G\zeta'), & -\frac{2}{\sqrt{K}}(c\eta'-f\zeta')+\frac{2}{K}\zeta'(A\xi'+H\eta'+G\zeta'), \\
-\frac{2}{\sqrt{K}}(a\zeta'-g\xi')+\frac{2}{K}\xi'(H\xi'+B\eta'+F\zeta'), & 1-\frac{2}{\sqrt{K}}(h\zeta'-f\eta')+\frac{1}{K}(B\eta'^2-C\zeta'^2-2G\zeta'\xi'-A\xi'^2), & -\frac{2}{\sqrt{K}}(g\zeta'-c\xi')+\frac{2}{K}\zeta'(H\xi'+B\eta'+F\zeta'). \\
-\frac{2}{\sqrt{K}}(h\xi'-a\eta')+\frac{2}{K}\xi'(G\xi'+F\eta'+C\zeta'), & -\frac{2}{\sqrt{K}}(b\xi'-h\eta')+\frac{2}{K}\eta'(G\xi'+F\eta'+C\zeta'), & 1-\frac{2}{\sqrt{K}}(f\xi'-g\eta')+\frac{1}{K}(C\zeta'^2-A\xi'^2-2H\xi'\eta'-B\eta'^2),
\end{array}$$

and considering (X, Y, Z) and ($X_{\prime}$, $Y_{\prime}$, $Z_{\prime}$) as quantities connected by the foregoing linear relations, we have identically

$$(a, \ldots \between X, Y, Z)^2=(a, \ldots \between X_{\prime}, Y_{\prime}, Z_{\prime})^2.$$

So that the investigation leads to the automorphic transformation of the quadric function, a transformation first effected by M. HERMITE*.

33. It is to be remarked that the foregoing formulæ show that (x', y', z') being the coordinates of a point on the conic $(a, \ldots \between x, y, z)^2+(\xi'x+\eta'y+\zeta'z)^2=0$, from which point tangents are drawn to the conic $(a, \ldots \between x, y, z)^2=0$, then the coordinates (x', y', z') enter *linearly* into the equations of the tangents, the ineunts (or points of contact), and the polar. And it may be added that the equation of the conic enveloped by the polar (that is, the polar conic of $(a, \ldots \between x, y, z)^2+(\xi'x+\eta'y+\zeta'z)^2=0$) has for its equation

$$\{K+(A, \ldots \between \xi', \eta', \zeta')^2\}(a, \ldots \between x, y, z)^2-K(\xi'x+\eta'y+\zeta'z)^2=0.$$

and that the coordinates of the point of contact of the polar with this conic are

$$x'+\frac{1}{K}(A\xi'+H\eta'+G\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$y'+\frac{1}{K}(H\xi'+B\eta'+F\zeta')(\xi'x'+\eta'y'+\zeta'z'),$$

$$z'+\frac{1}{K}(G\xi'+F\eta'+C\zeta')(\xi'x'+\eta'y'+\zeta'z');$$

so that (x', y', z') also enter linearly into the expressions for the coordinates of the last-mentioned point.

Article Nos. 34 to 37, relating to two conics.

34. Considering now the two conics

$$U=(a, b, c, f, g, h \between x, y, z)^2=0,$$

$$U=(a', b', c', f', g', h' \between x, y, z)^2=0;$$

Suppose that the conic

$$\theta U+\theta' U'=(\theta a+\theta' a', \ldots \between x, y, z)^2=0$$

represents a pair of lines.

* See my "Memoir on the Automorphic Transformation of a Bipartite Quadric Function," Phil. Trans. vol. cxlviii. (1858) pp. 39–46.

The condition for this is

$$\text{Disct.}\ (\theta a+\theta' a', \ldots \between x, y, z)^2=0,$$

which is

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \between \theta, \theta')^3=0,$$

where

$$\begin{aligned}
\mathfrak{A} &= \mathrm{K},\\
\mathfrak{B} &= \mathrm{A}a'+\mathrm{B}b'+\mathrm{C}c'+2\mathrm{F}f'+2\mathrm{G}g'+2\mathrm{H}h',\\
\mathfrak{C} &= \mathrm{A}'a+\mathrm{B}'b+\mathrm{C}'c+2\mathrm{F}'f+2\mathrm{G}'g+2\mathrm{H}'h,\\
\mathfrak{D} &= \mathrm{K}'
\end{aligned}$$

(the significations of K′, A′, B′, C′, F′, G′, H′ being of course analogous to those of K, A, B, C, F, G, H). The three roots $\theta : \theta'$ correspond, it is clear, to the three pairs of lines which can be drawn through the intersections of the two conics.

35. The equation

$$\text{Disct.}\ (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \between \theta, \theta')^3=0,$$

which is of the fourth order in $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$, and of the sixth order as regards (a, b, c, f, g, h) and (a', b', c', f', g', h') respectively, is the condition in order that the two conics may touch each other. Assuming that it is satisfied, the cubic equation in $\theta : \theta'$ has a pair of equal roots; or say there is a twofold root and a onefold root; the twofold root gives the pair of lines drawn from the point of contact to the other two points of intersection, the onefold root gives the pair made up of the common tangent and the line joining the other two points of intersection.

36. In particular, suppose that the two conics are

$$2(\varrho x+\sigma y+\tau z)(\varrho' x+\sigma' y+\tau' z)=0,$$
$$2(\lambda x+\mu y+\nu z)(\lambda' x+\mu' y+\nu' z)=0;$$

so that

$$\begin{aligned}
(a, b, c, f, g, h) &= (2\varrho\varrho', 2\sigma\sigma', 2\tau\tau', \sigma\tau'+\sigma'\tau, \tau\varrho'+\tau'\varrho, \varrho\sigma'+\varrho'\sigma),\\
(a', b', c', f', g', h') &= (2\lambda\lambda', 2\mu\mu', 2\nu\nu', \mu\nu'+\mu'\nu, \nu\lambda'+\nu'\lambda, \lambda\mu'+\lambda'\mu),\\
(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H}) &= -(\sigma\tau'-\sigma'\tau, \tau\varrho'-\tau'\varrho, \varrho\sigma'-\varrho'\sigma)^2,\\
(\mathrm{A}', \mathrm{B}', \mathrm{C}', \mathrm{F}', \mathrm{G}', \mathrm{H}') &= -(\mu\nu'-\mu'\nu, \nu\lambda'-\nu'\lambda, \lambda\mu'-\lambda'\mu)^2;
\end{aligned}$$

and thence also

$$\mathfrak{A}=\mathrm{K} \quad = \quad 0,$$

$$\mathfrak{B}=\mathrm{A}a'+\&\text{c.}=-2\begin{vmatrix} \lambda, & \mu, & \nu \\ \varrho, & \sigma, & \tau \\ \varrho', & \sigma', & \tau' \end{vmatrix}\begin{vmatrix} \lambda', & \mu', & \nu' \\ \varrho, & \sigma, & \tau \\ \varrho', & \sigma', & \tau' \end{vmatrix},$$

$$\mathfrak{C}=\mathrm{A}'a+\&\text{c.}=-2\begin{vmatrix} \varrho, & \sigma, & \tau \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix}\begin{vmatrix} \varrho', & \sigma', & \tau' \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix},$$

$$\mathfrak{D}=\mathrm{K}' \quad = \quad 0;$$

and the equation in (θ, θ') is

$$\mathfrak{B}\theta+\mathfrak{C}\theta'=0;$$

so that, writing $\theta=\mathfrak{C}$, $\theta'=-\mathfrak{B}$, the equation of the pair of lines is

$$\begin{vmatrix} \varrho, & \sigma, & \tau \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix} \cdot \begin{vmatrix} \varrho', & \sigma', & \tau' \\ \lambda, & \mu, & \nu \\ \lambda', & \mu', & \nu' \end{vmatrix} (\varrho x+\sigma y+\tau z)(\varrho' x+\sigma' y+\tau' z) - \begin{vmatrix} \lambda, & \mu, & \nu \\ \varrho, & \sigma, & \tau \\ \varrho', & \sigma', & \tau' \end{vmatrix} \begin{vmatrix} \lambda', & \mu', & \nu' \\ \varrho, & \sigma, & \tau \\ \varrho', & \sigma', & \tau' \end{vmatrix} (\lambda x+\mu y+\nu z)(\lambda' x+\mu' y+\nu' z)=0;$$

and it is easy to see that the left-hand side does in fact break up into factors, and that the equation is

$$\begin{vmatrix} x, & y, & z \\ \mu\tau'-\nu\sigma', & \nu\varrho'-\lambda\tau', & \lambda\sigma'-\mu\varrho' \\ \sigma\nu'-\tau\mu', & \tau\lambda'-\varrho\nu', & \varrho\mu'-\sigma\lambda' \end{vmatrix} \begin{vmatrix} x, & y, & z \\ \mu\tau-\nu\varrho, & \nu\varrho-\lambda\tau, & \lambda\sigma-\mu\varrho \\ \sigma'\nu'-\tau'\mu', & \tau'\lambda'-\varrho'\nu', & \varrho'\mu'-\sigma'\lambda' \end{vmatrix}=0,$$

which of course might have been obtained at once by means of the four points which are the intersection of each component line of the first conic by each component line of the second conic.

37. Suppose that the first conic is

$$(a, b, c, f, g, h \between x, y, z)^2=0,$$

while the second conic is the pair of lines

$$2(\lambda x+\mu y+\nu z)(\lambda' x+\mu' y+\nu' z)=0;$$

then putting, as before,

$$(\theta a+\theta'.2\lambda\lambda', .. \between x, y, z)^2=0,$$

we have

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \between \theta, \theta')^3=0,$$

where

$$\begin{aligned} \mathfrak{A} &= \text{K}, \\ \mathfrak{B} &= \ \ 2(\text{A, B, C, F, G, H} \between \lambda, \mu, \nu \between \lambda', \mu', \nu'), \\ \mathfrak{C} &= -(a, b, c, f, g, h \between \mu\nu'-\mu'\nu, \nu\lambda'-\nu'\lambda, \lambda\mu'-\lambda'\mu)^2, \\ \mathfrak{D} &= \ \ 0; \end{aligned}$$

and the equation in (θ, θ') is

$$\text{K}\theta^2+2(\text{A}, .. \between \lambda, \mu, \nu \between \lambda', \mu', \nu')\theta\theta'-(a, .. \between \mu\nu'-\mu'\nu, ..)^2\theta'^2=0,$$

which may be written

$$\{\text{K}\theta+(\text{A}, .. \between \lambda, \mu, \nu \between \lambda', \mu', \nu')\theta'\}^2=\{[(\text{A}, .. \between \lambda, \mu, \nu \between \lambda', \mu', \nu')]^2+\text{K}(a, .. \between \mu\nu'-\mu'\nu, ..)^2\}\theta'^2$$

$$=(\text{A}, .. \between \lambda, \mu, \nu)^2.(\text{A}, .. \between \lambda', \mu', \nu')^2.\theta'^2,$$

that is,

$$\text{K}\theta=[\pm\sqrt{(\text{A}, .. \between \lambda, \mu, \nu)^2}\sqrt{(\text{A}, .. \between \lambda', \mu', \nu')^2}-(\text{A}, .. \between \lambda, \mu, \nu \between \lambda', \mu', \nu')]\theta';$$

or we may assume

$$\theta=\pm\sqrt{(A,..\between\lambda,\mu,\nu)^2}\sqrt{(A,..\between\lambda',\mu',\nu')^2}-(A,..\between\lambda,\mu,\nu\between\lambda',\mu',\nu'),\quad \theta'=K,$$

so that the conic

$$\{\pm\sqrt{(A,..\between\lambda,\mu,\nu)^2}\sqrt{(A,..\between\lambda',\mu',\nu')^2}-(A,..\between\lambda,\mu,\nu\between\lambda',\mu',\nu')\}(a,\ldots\between x,y,z)^2$$
$$+2K(\lambda x+\mu y+\nu z)(\lambda' x+\mu' y+\nu' z)=0$$

breaks up into a pair of lines.

Putting for shortness

$$\pm\sqrt{(A,..\between\lambda,\mu,\nu)^2}\sqrt{(A,..\between\lambda',\mu',\nu')^2}-(A,..\between\lambda,\mu,\nu\between\lambda',\mu',\nu')=\Omega,$$

the coefficients on the left-hand side of the equation are

$$(\Omega a+2K\lambda\lambda',..\ \Omega f+K(\mu\nu'+\mu'\nu),..),$$

whence, after all reductions, the inverse function is

$$\{(A,..\between\lambda,\mu,\nu\between\xi,\eta,\zeta)\sqrt{(A,..\between\lambda',\mu',\nu')^2}\mp(A,..\between\lambda',\mu',\nu'\between\xi,\eta,\zeta)\sqrt{(A,..\between\lambda,\mu,\nu)^2}\}^2,$$

and the remainder of the process of decomposition is effected without difficulty.

Addition, 18 December, 1862.

The formulæ II. and II. (bis) each of them give the tangents of the conic $(a,\ldots\between x,y,z)^2=0$ at the ineunts of intersection with the line $\xi' x+\eta' y+\zeta' z=0$. A very elegant formula for these ineunts themselves was communicated to me by Mr. Spottiswoode, and I have since found that the same or an equivalent formula is made use of by M. Aronhold in his recent valuable memoir, "Ueber eine neue algebraische Behandlungsweise der Integrale irrationaler Differentiale, &c.", Crelle, t. lxii. pp. 95–145 (1862). The formula is as follows, viz. for the conic and line,

$$(a,b,c,f,g,h\between x,y,z)^2=0$$
$$\xi' x+\eta' y+\zeta' z=0,$$

then

$$x:y:z=$$
$$(l\xi'+m\eta'+n\zeta')\frac{1}{2\sqrt{\phi}}\frac{d\phi}{d\xi'}+\eta'(gl+fm+cn)-\zeta'(hl+bm+fn)+l\sqrt{\phi},$$
$$:(l\xi'+m\eta'+n\zeta')\frac{1}{2\sqrt{\phi}}\frac{d\phi}{d\eta'}+\zeta'(al+hm+gn)-\xi'(gl+fm+cn)+m\sqrt{\phi},$$
$$:(l\xi'+m\eta'+n\zeta')\frac{1}{2\sqrt{\phi}}\frac{d\phi}{d\zeta'}+\xi'(hl+bm+fn)-\eta'(al+hm+gn)+n\sqrt{\phi};$$

where

$$\phi = \begin{vmatrix} & \xi', & \eta', & \zeta' \\ \xi', & a, & h, & g \\ \eta', & h, & b, & f \\ \zeta', & g, & f, & c \end{vmatrix} = -(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{F}, \mathrm{G}, \mathrm{H})(\xi', \eta', \zeta')^2$$

So that $\frac{1}{2}\frac{d\phi}{d\xi'}$, $\frac{1}{2}\frac{d\phi}{d\eta'}$, $\frac{1}{2}\frac{d\phi}{d\zeta'}$ are respectively

$$= -(\mathrm{A}\xi' + \mathrm{H}\eta' + \mathrm{G}\zeta'),\ -(\mathrm{H}\xi' + \mathrm{B}\eta' + \mathrm{F}\zeta') - \mathrm{G}\xi' + \mathrm{F}\eta' + \mathrm{C}\zeta',$$

and where l, m, n are supernumerary arbitrary quantities.

XXX. *Appendix to the Account of the Earthquake-Wave Experiments made at Holyhead.*

By ROBERT MALLET, *C.E.*, *F.R.S.*

Received March 27,—Read May 8, 1862.

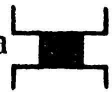
I AM now enabled to fulfill the intention expressed (vol. cli. p. 678) in concluding the above 'Account,' as read to the Royal Society, having since then completed a series of experiments upon the compression of specimens of the Holyhead Rocks, and determined their moduli of elasticity. These experiments were made upon cubes cut from solid and perfect pieces of the rocks by the lapidary's wheel, each 0·707 inch upon the edge—each side, therefore, presenting a surface of 0·5 square inch,—and the utmost care being taken to preserve perfect parallelism between the opposite boundary planes, so that, when compressed between hardened steel surfaces, fracture should not result by inequality of pressure.

The compressions were made at the Royal Arsenal, Woolwich, with the very accurate and excellent machine used for testing compression and extension of metals in the gun-factory; and I have to express my thanks to Lieut.-Col. ANDERSON, C.E., the Superintendent of that department, for the valuable assistance afforded me through his attention.

The specimens operated on consisted of two each from the following four classes, namely, the hardest and the softest slate-rock, the hardest and the softest quartz-rock, which occur within the range or neighbourhood of my experimental explosions at Holyhead; and from each of these classes or varieties of the two rocks, cubic specimens were compressed, 1st, in a direction transverse to the plane of lamination, 2nd, parallel to the same, all the cubes being so cut out of the rock that two sides were, *quam prox.*, parallel to the plane of natural lamination or jointing. The load (50 lbs.) first applied was considered zero, being only sufficient to ensure a complete bearing in all parts of the instrument. The subsequent loads advanced by 1000 lbs. per square inch of surface at a time, up to the crushing of the specimen; and at each fresh load the amount of compression was measured by beam-callipers, with instrumental arrangements that admitted of reading space to ·0005 of an inch.

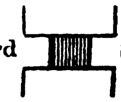
The experimental results, as obtained, are recorded in the following Tables, from No. I. to No. VIII. inclusive; and in the succeeding Tables IX. and X., the results of the former are compared, and the mean compression deduced for each 1000 lbs. of pressure applied upon a prism of each of the four classes of rock (two of slate and two of quartz), of one inch square surface, and one inch in height, and under both conditions as to the relative direction of pressure and of lamination.

Table I.—Holyhead-Rock Compression. Experiments A, on Hard Slate; pressure transverse to lamination.

Hard  Slate.

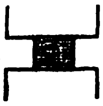
No. of experiment.	Pressure due to the unit of surface =1 square inch.	Compression readings of the column of 0·707 inch.	Compression readings due to the successive loads.	Total compressions produced by the load on column of 0·707 inch.	Total compressions reduced to a column of unit height =1 inch.
	lbs.	in.	in.	in.	in.
1.	50	·085	·000	·000	·0000
2.	1,000	·081	·004	·004	·0052
3.	2,000	·078 +	·003 −	·007	·0091
4.	3,000	·078 +	·000	·007	·0091
5.	4,000	·078	·000 +	·007	·0091
6.	5,000	·078	·000	·007	·0091
7.	6,000	·077 +	·001 −	·007	·0091
8.	7,000	·077	·000 +	·008	·0104
9.	8,000	·076 +	·001 −	·008	·0104
10.	9,000	·076 +	·000	·008	·0104
11.	10,000	·076 +	·000	·008	·0104
12.	11,000	·076	·000 +	·008	·0104
13.	12,000	·076	·000	·009	·0117
14.	13,000	·075 +	·001 −	·009	·0117
15.	14,000	·075 +	·000	·009	·0117
16.	15,000	·075 +	·000	·009	·0117
17.	16,000	·075	·000 +	·009	·0117
18.	17,000	·075	·000	·009	·0117
19.	18,000	·075	·000	·009	·0117
20.	19,000	·075	·000	·010	·0130
21.	20,000	·074 +	·001 −	·010	·0130
22.	21,000	·074 +	·000	·010	·0130
23.	22,000	·074	·000 +	·010	·0130
24.	23,000	·074	·000	·011	·0143
25.	24,000	Crushed.		·011	·0143

Table II.—Holyhead-Rock Compression. Experiments B, on Hard Slate; pressure parallel to lamination.

Hard  Slate.

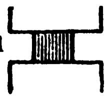
1.	50	·130	·000	·000	·0000
2.	1,000	·120	·010	·010	·0130
3.	2,000	·100	·020	·030	·0390
4.	3,000	·099 +	·001 −	·031 +	·0403 +
5.	4,000	·098	·001 +	·032	·0416
6.	5,000	·097	·001	·032	·0416
7.	6,000	·096	·001	·032	·0416
8.	7,000	·094	·002	·036	·0468
9.	8,000	·092 +	·002 −	·038 +	·0494
10.	9,000	·092 +	·000	·038 +	·0494
11.	10,000	·092 +	·000	·038 +	·0494
12.	11,000	·092	·000 +	·038 +	·0494
13.	12,000	·092	·000	·038 +	·0494
14.	13,000	·092	·000	·038 +	·0494
15.	14,000	·092	·000	·038 +	·0494
16.	15,000	·090	·002	·040	·0520
17.	16,000	·089	·001	·041	·0533
18.	17,000	·086	·003	·044	·0572
19.	18,000	·085 +	·001 −	·045 +	·0585 +
20.	19,000	·085 +	·000	·045 +	·0585 +
21.	20,000	·085 +	·000	·045 +	·0585 +
22.	21,000	·085	·000 +	·045 +	·0585 +
23.	22,000	·085	·000	·045 +	·0585 +
24.	23,000	·085	·000	·045 +	·0585 +
25.	24,000	·082	·003	·048	·0624
26.	25,000	·082	·000	·048	·0624
27.	26,000	·080	·002	·050	·0650
28.	27,000	·077	·003	·053	·0689
29.	27,000 +	Crushed.		·053	·0689

TABLE III.—Holyhead-Rock Compression. Experiments C, on Hard Quartz; pressure transverse to lamination.

Hard  Quartz.

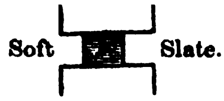
No. of experiment.	Pressure due to the unit of surface = 1 square inch.	Compression readings of the column of 0·707 inch.	Compression readings due to the successive loads.	Total compressions produced by the load on column of 0·707 inch.	Total compressions reduced to a column of unit height = 1 inch.
	lbs.	in.	in.	in.	in.
1.	50	·100	·000	·000	·0000
2.	1,000	·097	·003	·003	·0039
3.	2,000	·095+	·002−	·003	·0039
4.	3,000	·095+	·000	·003	·0039
5.	4,000	·095+	·000	·003	·0039
6.	5,000	·095+	·000	·003	·0039
7.	6,000	·095	·000+	·003	·0039
8.	7,000	·095	·000	·003	·0039
9.	8,000	·095	·000	·005	·0065
10.	9,000	·094	·001	·006	·0078
11.	10,000	·093+	·001−	·006	·0078
12.	11,000	·093+	·000	·006	·0078
13.	12,000	·093+	·000	·006	·0078
14.	13,000	·093	·000+	·006	·0078
15.	14,000	·093	·000	·006	·0078
16.	15,000	·093	·000	·006	·0078
17.	16,000	·093	·000	·007	·0091
18.	17,000	·092+	·001−	·007	·0091
19.	18,000	·092	·000+	·007	·0091
20.	19,000	·092	·000	·008	·0104
21.	20,000	·091+	·001−	·009+	·0117+
22.	21,000	·088	·003+	·012	·0156
23.	22,000	·083+	·005−	·012	·0156
24.	23,000	·083+	·000	·012	·0156
25.	24,000	·083+	·000	·012	·0156
26.	25,000	·083	·000+	·012	·0156
27.	26,000	·083	·000	·017	·0221
28.	27,000	·082+	·001−	·017	·0221
29.	28,000	·082+	·000	·017	·0221
30.	29,000	·082+	·000	·017	·0221
31.	30,000	·082	·000+	·017	·0221
32.	31,000	·082	·000	·017	·0221
33.	32,000	·082	·000	·018	·0234
34.	33,000	·081+	·001−	·018	·0234
35.	34,000	·081	·000+	·019	·0247
36.	35,000	·080+	·001−	·019	·0247
37.	36,000	·080	·000+	·020	·0260
38.	36,000+	Crushed.		·020	·0260

TABLE IV.—Holyhead-Rock Compression. Experiments D, on Hard Quartz; pressure parallel to lamination.

Hard  Quartz.

1.	50	·106	·000	·000	·0000
2.	1,000	·106	·000	·000	·0000
3.	2,000	·106	·000	·000	·0000
4.	3,000	·106	·000	·000	·0000
5.	4,000	·106	·000	·000	·0000
6.	5,000	·102	·004	·004	·0052
7.	6,000	·100+	·002−	·004	·0052
8.	7,000	·100+	·000	·004	·0052
9.	8,000	·100+	·000	·004	·0052
10.	9,000	·100	·000+	·004	·0052
11.	10,000	·100	·000	·004	·0052
12.	11,000	·100	·000	·006	·0078
13.	12,000	·098+	·002−	·006	·0078
14.	13,000	·098	·000+	·008	·0104
15.	14,000	·097	·001	·009	·0117
16.	15,000	·096	·001	·010	·0130
17.	16,000	·093	·003	·013	·0169
18.	17,000	·092	·001	·014	·0182
19.	18,000	·090+	·002−	·014	·0182
20.	19,000	·090	·000+	·016	·0208
21.	20,000	Crushed.		·016	·0208

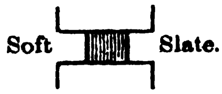
Table V.—Holyhead-Rock Compression. Experiments E, on Soft Slate; pressure transverse to lamination.



No. of experiment.	Pressure due to the unit of surface =1 square inch.	Compression readings of the column of 0·707 inch.	Compression readings due to the successive loads.	Total compressions produced by the load on column of 0·707 inch.	Total compressions reduced to a column of unit height =1 inch.
	lbs.	in.	in.	in.	in.
1.	50	·088	·000	·000	·0000
2.	1,000	·087	·001	·001	·0014
3.	2,000	·086+	·001–	·001	·0014
4.	3,000	·086	·000+	·002	·0029
5.	4,000	·085	·001	·002	·0029
6.	5,000	·085	·000	·003	·0043
7.	6,000	·079	·006	·009	·0129
8.	7,000	·077+	·002–	·009	·0129
9.	8,000	·077+	·000	·009	·0129
10.	9,000	·077	·000+	·009	·0129
11.	10,000	·077	·000	·009	·0129
12.	11,000	·077	·000	·011	·0158
13.	12,000	·075	·002	·013	·0187
14.	13,000	·060	·015	·028	·0404
15.	14,000	·050	·010	·038	·0548
16.	15,000	Crushed.		·038	·0548

Note.—The cube E was 0·693 inch on the side, and the necessary reductions have been made in column 2 and subsequent ones.

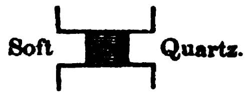
Table VI.—Holyhead-Rock Compression. Experiments F, on Soft Slate; pressure parallel to lamination.



1.	50	·107	·000	·000	·0000
2.	1000	·105	·002	·002	·0029
3.	2000	·102+	·003–	·002	·0029
4.	3000	·102	·000+	·002	·0029
5.	4000	·102	·000	·005	·0072
6.	5000	·099	·003	·008	·0115
7.	6000	·097	·002	·010	·0147
8.	7000	·089	·008	·018	·0259
9.	8000	·080	·009	·027	·0389
10.	8000+	Crushed.		·027	·0389

Note.—The cube F was 0·693 inch on the side, and the necessary reductions have been made in column 2 and subsequent ones.

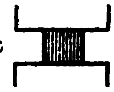
Table VII.—Holyhead-Rock Compression. Experiments G, on Soft Quartz; pressure transverse to lamination.



1.	50	·093	·000	·000	·0000
2.	1,000	·093	·000	·000	·0000
3.	2,000	·093	·000	·000	·0000
4.	3,000	·090	·003	·003	·0043
5.	4,000	·086+	·004–	·003	·0043
6.	5,000	·086+	·000	·003	·0043
7.	6,000	·086	·000+	·003	·0043
8.	7,000	·086	·000	·007	·0101
9.	8,000	·085+	·001–	·007	·0101
10.	9,000	·085+	·000	·007	·0101
11.	10,000	·085	·000+	·008	·0115
12.	11,000	·084	·001	·009	·0129
13.	12,000	·081	·003	·012	·0176
14.	13,000	·068	·013	·025	·0359
15.	14,000	·060	Crushed before being fully loaded.		

Note.—The cube G was 0·694 inch on the side, and the necessary reductions have been made in column 2 and subsequent ones.

TABLE VIII.—Holyhead-Rock Compression. Experiments H, on Soft Quartz; pressure parallel to lamination.



No. of experiment.	Pressure due to the unit of surface =1 square inch.	Compression readings of the column of 0·707 inch.	Compression readings due to the successive loads.	Total compressions produced by the load on column of 0·707 inch.	Total compressions reduced to a column of unit height =1 inch.
	lbs.	in.	in.	in.	in.
1.	50	·170	·000	·000	·0000
2.	1,000	·144	·026	·026	·0374
3.	2,000	·101 +	·043 —	·069	·0992
4.	3,000	·101	·000 +	·069	·0993
5.	4,000	·100	·001	·070	·1007
6.	5,000	·099	·001	·071	·1021
7.	6,000	·098	·001	·072	·1036
8.	7,000	·049	·049	·121	·1741
9.	7,000 +	Crushed before the increased load was applied.			

NOTE.—The cube H was 0·695 inch on the side, and the necessary reductions have been made in column 2 and subsequent ones.

TABLE IX.—Holyhead-Rock Compression. Slate Rock.—Results of compression compared. Column of unit length =1 inch.

No. of experiment.	Pressure in pounds on unit of surface =1 square inch.	A. Hard slate across lamina.	B. Hard slate with the lamina.	E. Soft slate across lamina.	F. Soft slate with the lamina.
	lbs.	in.	in.	in.	in.
1.	50	·0000	·0000	·0000	·0000
2.	1,000	·0052	·0130	·0014	·0029
3.	2,000	......	·0390		
4.	3,000	......	·0403	·0029	
5.	4,000	......	·0416	......	·0072
6.	5,000	·0091	......	·0043	·0115
7.	6,000	......	......	·0129	·0147
8.	7,000	·0104	·0468	......	·0259
9.	8,000	......	·0494	......	·0389
10.	9,000	......	......	......	Crushed.
11.	10,000				
12.	11,000	......	......	·0158	
13.	12,000	·0117	......	·0187	
14.	13,000	......	......	·0404	
15.	14,000	......	......	·0548	
16.	15,000	......	·0520	Crushed.	
17.	16,000	......	·0533		
18.	17,000	......	·0572		
19.	18,000	......	·0585		
20.	19,000	·0130			
21.	20,000				
22.	21,000				
23.	22,000				
24.	23,000	·0143			
25.	24,000	Crushed.	·0624		
26.	25,000				
27.	26,000	......	·0650		
28.	27,000	......	·0689		
29.	28,000	......	Crushed.		
30.	29,000				
Mean compression for each 1000 lbs. on unit of surface		in. ·0006217 up to 23,000 lbs.	in. ·0025000 up to 26,000 lbs.	in. ·0039144 up to 14,000 lbs.	in. ·0037000 up to 7000 lbs.

TABLE X.—Holyhead-Rock Compression. Quartz Rock.—Results of Compression compared. Column of unit length =1 inch.

No. of experiment.	Pressure in pounds on unit of surface =1 square inch.	C. Hard quartz across lamina.	D. Hard quartz with the lamina.	G. Soft quartz across lamina.	H. Soft quartz with the lamina.
	lbs.	in.	in.	in.	in.
1.	50	·0000	·0000	·0000	·0000
2.	1,000	·0039	......	......	·0374
3.	2,000	......	......	......	·0992
4.	3,000	......	......	·0043	·0993
5.	4,000	......	......	......	·1007
6.	5,000	......	·0052	......	·1021
7.	6,000	......	......	......	·1036
8.	7,000	......	......	·0101	·1741
9.	8,000	·0065	......	......	Crushed.
10.	9,000	·0078	......	......	
11.	10,000	......	......	·0115	
12.	11,000	......	·0078	·0129	
13.	12,000	......	......	·0176	
14.	13,000	......	·0104	·0359	
15.	14,000	......	·0117	Crushed.	
16.	15,000	......	·0130		
17.	16,000	·0091	·0169		
18.	17,000	......	·0182		
19.	18,000	......			
20.	19,000	·0104	·0208		
21.	20,000	·0117	Crushed.		
22.	21,000	·0156			
23.	22,000	......			
24.	23,000	......			
25.	24,000	......			
26.	25,000	......			
27.	26,000	·0221			
28.	27,000	......			
29.	28,000	......			
30.	29,000	......			
31.	30,000	......			
32.	31,000	......			
33.	32,000	·0234			
34.	33,000	......			
35.	34,000	·0247			
36.	35,000	......			
37.	36,000	·0260			
38.	37,000	Crushed.			
Mean compression for each 1000 lbs. on unit of surface.		in. ·0007085 up to 35,000 lbs.	in. ·0010947 up to 19,000 lbs.	in. ·0014666 up to 12,000 lbs.	in. ·0172666 up to 6000 lbs.

An examination of these Tables presents some remarkable and, so far as I am aware, now for the first time observed results.

As might have been expected, the quartz-rock is much less compressible generally than the slate-rock, with this exception, however, that the softest specimens of quartz-rock, and those alone, are much more compressible than the softest slate, when both are compressed in the direction of or parallel to the lamination.

In this direction of compression, the hardest slate is more than double as compressible as the hardest quartz.

When compressed transverse to the lamina, however, the hard slate and hard quartz prove to have very nearly the same coefficient of compressibility, which is very small for both; while the softest slate and the softest quartz, compressed in the same way (transverse to lamina), have also nearly the same coefficient of compressibility, but one about four times as great as for the hardest like rocks.

These facts point towards the circumstance of the original deposit and formation of these rocks as their efficient causes. Both rocks consist of particles more or less wedge-shaped and flat, angular fragments more or less crystalline, deposited together, with their larger dimensions in the planes of lamination, which lamination has been produced by enormous compression in a direction transverse to its planes. Hence the mass of these rocks has *already* been subjected to enormous compression in the *same* direction as that in which we now find their further compressibility the least. But, besides that we might from *this* cause alone anticipate a higher compressibility when the pressure is applied to them parallel to the lamination, another condition comes into play; their aggregation of flat, *wedge-shaped* particles, when thus pressed edgeways, tends powerfully to their mutual lateral expansion, and hence to their giving way in the line of pressure.

The *per-saltum* way in which all the specimens of both rocks yield, in whatever direction pressed, is another noteworthy circumstance. On examining the Tables I. to VIII. it will be seen that the compressions do not constantly advance with the pressure, but that, on the contrary, the rock occasionally suffers almost no sensible compression for several successive increments of pressure, and then gives way all at once (though without having lost cohesion, or having its elasticity permanently impaired) and compresses thence more or less for three or four or more successive increments of pressure, and then holds fast again, and so on. This phenomenon is probably due to the mass of the rock being made up of intermixed particles of several different simple minerals, having each specific differences of hardness, cohesion, and mutual adhesion, and which are, in the order of their resistances to pressure, in succession broken down, before the final disruption of the whole mass (weakened by these minute internal dislocations) takes place.

Thus it would appear that the micaceous plates and aluminous clay-particles interspersed through the mass give way first. The chlorite in the slate, and probably felspar-crystals in the quartz-rock, next, and so on in order, until finally the elastic skeleton of silex gives way, and the rock is crushed. It is observable, also, that this successive disintegration does not occur at equal pressures, in the same quality and kind of rock, when compressed transverse and parallel to the lamination. It follows from this constitution of these (and probably of all) rocks that very different powers of transmitting wave-impulses must arise when the originating forces vary considerably in amount of primary compression. It is almost superfluous also to point out the great differences in wave-transmissive power in directions transverse and parallel to lamination that these experiments disclose. They prove to us that, in an earthquake shock of given original power, the vibrations will have the largest amplitude when transmitted in the line of the lamination, but may be propagated with the greatest velocity in directions transverse to the same, *assuming in both cases the rock solid and unshattered.*

In the following Table XI. the general results are deduced, and the *mean campressions* for each of the rocks calculated, and finally the moduli of elasticity are obtained in pounds and in feet; the specific gravities adopted in calculating the latter being those given in the body of the paper, as follows:—

	sp. gr.	Weight of a prism 1 foot long and 1 inch square. lbs.
Hardest slate	2·763	1·1992
Softest slate	2·746	1·1918
Hardest quartz	2·656	1·1528
Softest quartz	2·653	1·1515
Mean for slate	2·7545	1·1955
Mean for quartz	2·6545	1·1522
General mean for both rocks	2·7045	1·1739

The load on the unit of surface (1 square inch) at which the elastic limit of the rock is passed, and that at which it is finally crushed, together with the modulus of cohesion or resistance to compression, are also given, and will be useful to the engineer and architect. In the last column the value of my own modification of PONCELET'S coefficient. T, (la force vive de rupture) is calculated in foot pounds, and represents the relative work done at fracture in each case.

TABLE XI.—Holyhead-Rock Compression.

General results reduced, Modulus of Cohesion and of Elasticity, &c.—Slate and Quartz.

No.	Class of Rock, and direction of pressure in relation to structure.	Coefficient of compression on unit surface for 1000 lbs.	Elastic limit for compression.	Crushing load on the unit of surface.	Modulus of cohesion (compression).	Modulus of elasticity.	Modulus of elasticity. L.	Coefficient. T_r.
		in.	lbs.	lbs.	ft.	lbs.	ft.	ft. lbs.
1.	Slate, *hardest*, *across* lamination	·0006217	22,000	24,000	20,014	8,042,464	6,706,524	1·2432
2.	Quartz, *hardest*, *across* lamination	·0007085	32,000	37,000	32,095	7,057,163	6,121,758	2·1830
3.	Slate, *hardest*, *parallel* to lamination	·0025000	18,000	27,000	22,515	2,000,000	1,667,778	5·6241
4.	Quartz, *hardest*, *parallel* to lamination	·0010947	17,000	20,000	17,349	4,567,461	3,962,013	1·8240
5.	Slate, *softest*, *across* lamination	·0039144	12,000	15,000	12,586	1,277,335	1,071,769	4·8930
6.	Quartz, *softest*, *across* lamination	·0014666	11,000	14,000	12,158	3,409,246	2,960,699	1·7108
7.	Slate, *softest*, *parallel* to lamination	·0037000	6,000	9,000	7,552	1,351,351	1,133,874	2·7747
8.	Quartz, *softest*, *parallel* to lamination	·0172666	7,000	8,000	6,948	289,576	251,477	11·6112
	Calculated means.							
9.	Slate, mean for hard and soft, *across* lamination	·0022680	17,000	19,500	16,311	2,204,585	1,844,069	3·6855
10.	Quartz, mean for hard and soft, *across* lamination	·0010875	16,500	25,500	22,132	4,597,701	3,990,455	2·3103
11.	Slate, mean for hard and soft, *parallel* to lamination	·0031000	12,000	18,000	15,056	1,612,903	1,349,145	4·6494
12.	Quartz, mean for hard and soft, *parallel* to lamination	·0091806	12,000	14,000	12,151	544,627	472,684	10·7100
	Calculated mean of means.							
13.	Slate, hard and soft, mean for both directions (Nos. 9 & 11)	·0026840	14,500	18,750	15,684	1,862,880	1,566,541	4·1914
14.	Quartz, hard and soft, mean for both directions (Nos. 10 & 12)	·0051340	16,750	19,750	17,141	973,899	845,252	8·4490
15.	General mean for Slate and Quartz, hard and soft, and in both directions (Nos. 13 & 14)	·0039090	15,625	19,250	16,398	1,279,099	1,089,615	6·2697

To apply the results thus obtained to those of experimental wave-transmission at Holyhead.

POISSON has shown* that the velocity of wave-transmission (sound) in longitudinal vibrations of elastic prisms is

$$V^2=\frac{glq}{p}. \quad \ldots \ldots \quad (I.)$$

When g has its usual relation to gravity, l and p are the length and weight of the prism, and $q=\frac{\Delta}{\delta}$, Δ being a weight that is capable of elongating the prism by an amount $=\delta l$, or extending it to the length

$$l(1+\delta).$$

Substituting, we have

$$V^2=\frac{gl\Delta}{p\delta};$$

but

$$\Delta : W :: \delta : 1,$$

W being the weight capable of doubling the length of the prism. Therefore

$$V^2=\frac{glW\delta}{p\delta}=\frac{glL}{l}=gL,$$

or

$$V=\sqrt{gL}. \quad \ldots \ldots \quad (II.)$$

So that L being the modulus of elasticity of the solid, expressed in feet, the velocity of wave-transmission through it, if absolutely homogeneous and unbroken, is

$$V=5{\cdot}674\sqrt{L}. \quad \ldots \ldots \quad (III.)$$

Where, owing to want of homogeneity, or to shattering, or other such condition, as found in natural rock, the experimental value of V differs from the above theoretic one, we may still express the former by the same general form of equation—

$$V'=\alpha\sqrt{L}, \quad \ldots \ldots \quad (IV.)$$

in which the coefficient α expresses the ratio to $\sqrt{g}$ that the actual or experimental bears to the theoretic (or maximum possible) velocity of wave-transmission.

In the slate- and quartz-rocks of Holyhead, I ascertained the mean *lowest* velocity of wave-transmission (for small explosions or impulses) to be 1089 feet per second (omitting decimals), the mean *highest* velocity 1352 feet per second, and the *general mean* velocity from all, 1220 feet per second.

Applying equation (IV.) to these numbers, and adopting the values of L given in Table XI. (mean of Nos. 9 and 10), we obtain

$$\alpha=\frac{V'}{\sqrt{L}};$$

* Traité de Mécanique, vol. ii. p. 319.

and for the three preceding velocities, α has the following values:—

ft. per sec.

$$1 \ldots V'=1089 \ldots\ldots \alpha=\frac{1089}{\sqrt{2917262}}=\frac{1089}{1708}=0\cdot637$$

$$2 \ldots V'=1352 \ldots\ldots \alpha=\frac{1352}{\sqrt{2917262}}=\frac{1352}{1708}=0\cdot791$$

$$3 \ldots V'=1220 \ldots\ldots \alpha=\frac{1220}{\sqrt{2917262}}=\frac{1220}{1708}=0\cdot714$$

The actual velocity of wave-transmission in the slate and quartz together, therefore, was to the theoretic velocity due to the solid material as

$$\alpha:\sqrt{g} \text{ or } 0\cdot714:5\cdot774, \text{ or } 1\cdot00:7\cdot946.$$

From which it results that *nearly seven-eighths* of the full velocity of wave-transmission due to the material is lost by reason of the heterogeneity and discontinuity or shattering of the rocky mass, as it is found piled together in nature.

This loss would be larger with still smaller originating impulses, and *vice versâ*, but in what proportion we are not at present in a position to know.

If we may for a moment allude to final causes, we cannot but be struck with this beneficent result (amongst others) arising from the shattered and broken-up condition of all the rocky masses forming the habitable surface of our globe,—that the otherwise enormous transit-velocity of the wave-form in earthquake shocks is by this simple means so reduced.

That this retardation is mainly effected by the multiplied subdivisions of the rock, and in a very minor degree by differences in the elastic moduli of rock of different species, is apparent on examining the Tables IV. and V. of the previous part of this Report referring to the experiments at Holyhead.

Although, therefore, we are now enabled, from what precedes, to calculate values for α, for the slate rocks and for the quartz of Holyhead, separately, and thus obtain separate values for V′, for *each* of those rocks; the result would probably be more or less delusive, as we have no possible means of deciding what is the relative amount of shattering and discontinuity, for equal horizontal distances, in each of these two rocks, nor what the relative retarding powers of planes of separation running in variable directions, and at all possible angles across the line of wave-transit, as compared with their retarding powers if either all transverse to, or all in the same direction as, the wave-path.

The greatest possible mean velocity of wave-propagation, *in rock as perfectly solid and unshattered* as our experimental cubes, is determinable for both slate and quartz in the two directions of transmission, viz. transverse and in the line of lamination, from equation (III.), and the mean values of L in Nos. 9 and 10, and 11 and 12, Table XI., as follows:—

ft. per sec.

Mean of slate and quartz transverse to lamination ... $V=5\cdot674\sqrt{2917262}=9691$

Mean of slate and quartz in line of lamination ...... $V=5\cdot674\sqrt{910914}=5415$

This great difference of velocity, due to the difference in the molecular properties of the material of the rocks in their opposite directions, is, as our Holyhead experiments prove, almost wholly obliterated by the vastly increased degree of discontinuity and shattering, in the directions approaching that of lamination, or transverse to the wave-path in the first case.

It is necessary to guard against any misconception as to the import of this result. The fact ascertained and just enunciated is this, that the velocity of wave-transmission is greater in the material of these rocks in a direction across their lamination than in one longitudinal to the same, provided or *assuming the material be perfectly unshattered in both*—as homogeneous, in fact, as the small specimen-cubes experimented upon. And were the whole mass of the rock, as it lies in its mountain-bed, as homogeneous as such cubes, then the velocity of wave-transmission would *actually* be greater *across* long ranges of natural lamination, than edgeways to them. The opposite, however, is often the case; the wave-transit period is slower as the range of rocky mass is more shattered, discontinuous, and dislocated.

These conditions affect rocks in nature most in or about their planes of bedding, lamination, &c., and hence most retard wave-impulses transverse to these planes; so that the *more rapid wave-transmissive power of the material* of the rock in a direction transverse to the lamination *may be more than counterbalanced by the discontinuity* of its mass transverse to the same direction.

The results of WERTHEIM, on the transmission of sound in timber, proved the velocity to be greatest in a direction longitudinal to the fibres and annual rings of wood; less in a direction perpendicular to the same, and from the centre of the tree radially towards its exterior; and least of all in a direction, *quam prox.*, parallel to the annual rings, and perpendicular to the longitudinal fibres; that is to say, that in each case the velocity of sound was rapid in proportion to the less compressibility of the wood in the same direction. His results might seem at first to conflict with those which I have announced. Any such conclusion, however, would be a mistake; on the contrary, my results perfectly analogize with those above alluded to. The difference between the cases is, that wood in mass, however large, is practically homogeneous and unshattered, and that *its direction of least compressibility is longitudinal to its laminæ* (or annual rings); whereas *the direction of least compressibility of rock is transverse to its laminæ* (which have been already powerfully compressed in this direction). In fact, as respects the point here in question, there is no true analogy *in structure* between the lamination (by annual rings) of wood, and the lamination or bedding of rock.

It follows from what precedes, that earthquakes and rocks as both actually occur in nature—the rocks being of a stratified or laminated form (generally all sedimentary rocks)—must present the following conditions as to rate of transit of shock:—

1st. If such rocks were perfectly unshattered, and the beds or laminæ in absolute contact, the shock would be transmitted more rapidly across these than in their own direction.

The difference is more in favour of the transverse line, in proportion as the rock is made up more of angular sedimentary particles of very unequal dimensions, the longest being parallel to the general lamination, and in proportion as the imbedding paste is softer in relation to such particles.

Some sedimentary rocks no doubt exist, made up of particles perfectly uniform and equal in all three dimensions, and without imbedding paste—such as the lithographic stones of Germany, the Apennine marl-beds, &c., in which (assuming the above condition as to continuity) the transit-period would probably be alike in all directions.

2nd. The actual amount of shattering and discontinuity in nature being usually greatest, upon the whole, in planes parallel to bedding or lamination, the transit-rate of shock is most generally fastest in the line of the beds or lamination, rather than across them.

Or, at least, this latter condition may interfere with the former to the extent of partial, complete, or more than complete obliteration.

I am not aware that experiments have previously been made at all upon the compressibility, &c. of the slate- and quartz-rocks of Holyhead; and as these rocks are being employed there upon a vast scale for submarine building works, it may not be out of place to draw a few conclusions of a character useful to the practical engineer from the data that have been obtained. Some conclusions may be drawn which are applicable to all classes of laminated rocks in the hands of the engineer.

It is a very prevalent belief that slate-rock (for example), in the form of the sawed roofing-slate of Anglesea or of Valentia (Ireland), will bear a much greater compressive load when the pressure is in the direction of the laminæ, than in one across them. This the preceding experiments prove to be wholly a mistake—one that has very probably arisen from some vague notion of an analogy with timber compressed the end-way of the grain.

It is now certain that Silurian slates and quartz-rock, and probably all sedimentary laminated rocks, whether with cleavage or not, are much weaker to resist a crushing force edgeways to the lamina, than across the same, and that the range of compressibility is much greater, for equal loads, in the former direction.

The fact now ascertained, as to the great relative compressibility of laminated rock in the direction of the laminæ, also points out the reason of the great bearing-power to sustain impulsive loads, which the toughest and most cohesive examples of slate-rocks, such as the slates of Caernarvonshire, present; for there can be no grounds to doubt that the high compressibility of rocks of this structure in the plane of the lamina is also accompanied with a high coefficient of extensibility, although probably confined within much narrower limits as to incipient injury to perfect continuity.

My experiments point out that the Silurian slate of Holyhead (the mean both of the hard and the soft) is crushed by a load applied across the lamina of about 1250 tons per square foot, and that its molecular arrangement is permanently injured at a little more than 1000 tons per square foot.

The quartz-rock (the mean of both hard and soft) is crushed by a load, applied in the same manner, of 1630 tons per square foot, and its molecular arrangement is permanently injured at less than 1000 tons per square foot. The quartz-rock gives the highest measure of ultimate resistance, but it is the less trustworthy material when loaded heavily.

Neither of these sorts of rock, if loaded so as to be pressed *in the direction of the lamina*, would sustain more than about 0·7 of the above loads at the crushing-point and at that of permanent injury, respectively. From the extreme inequality found within narrow limits in both rocks as quarried, neither should be trusted for safe load in practice with more than about $\frac{1}{20}$th of the mean load that impairs their molecular arrangement, as ascertained from selected specimens, or (say) not to more than 50 tons per square foot for passive or 25 tons per square foot for impulsive loads.

The high relative compressibility of laminated rocks in the direction of the lamina might probably be made advantageous use of, where they are employed as a building material, for the construction of revetment or other walls of batteries exposed to the stroke of cannon-shot, by building the work (under suitable arrangements to obviate splitting up) with the planes of the laminæ in the direction of the line of fire, *i. e.* perpendicular to the faces of the work; for on inspecting the last column in Table XI., which contains the values of T_r under the several conditions of rock and of compression, it is at once apparent how much greater is the work done in crushing the slates and the quartz in their toughest and most compressible direction, *i. e.* in the direction of the laminæ,—*twice as much work* being, upon the average, consumed in crushing the rock in this direction as suffices to destroy its coherence in the one transverse to the laminæ, and the difference in the two, in the case of the softest quartz (Nos. 6 & 8), being as much as about 5 to 1.

It would be unsuitable, however, to the present memoir here to pursue further such practical deductions suggested by the results experimentally obtained.

XXXI. *On the Theory of the Motion of Glaciers.*

By William Hopkins, *Esq., St. Peter's College, Cambridge, M.A., F.R.S. &c.*

Received April 14,—Read May 22, 1862.

Almost all the numerous discussions which have taken place during the last twenty years respecting our theories of glacial motion have had for their object the assertion of some particular view, rather than the establishment of a complete and sufficient theory founded on well-defined hypotheses and unequivocal definitions, together with a careful comparison of the results of accurate theoretical investigation with those of direct observation. Each of these views has been regarded, in my own opinion improperly, as a *Theory of Glacial Motion.* The Expansion Theory ignored the Sliding Theory, though they were capable of being combined; the latter theory was equally ignored by the Viscous Theory, in which, moreover, instead of the definitions of terms being clear and determinate, no definition of *viscosity* was ever given, though that term designated the fundamental property on which the views advocated by this theory depended. Again, the Regelation Theory is not properly a theory of the motion of glaciers, but a beautiful demonstration of a property of ice, entirely new to us, on which certain peculiarities of the motions of glaciers depend. When we shall have obtained a *Theory of the Motion of Glaciers* which shall command the general assent of philosophers, no qualifying epithet will be required for the word *theory*; it would indeed be inappropriate, as seeming to indicate the continued recognition of some rival theory. If, for instance, it should be hereafter admitted that the sliding of a glacier over its bed and the property of regelation in ice are equally necessary, and, when combined, perfectly sufficient to account for the phenomena of glacial motion, there would be a manifest impropriety, not to say injustice, in selecting either of the terms *sliding* or *regelation* by which to designate this combined theory. I make these remarks because I believe that the preservation of the partial epithets above mentioned has a tendency to prevent our regarding the whole subject in that more general and collective aspect under which it is one of the principal objects of this paper to present it.

This object must necessarily give to the present paper something of the character of a *résumé* of what has hitherto been done, whether it be our purpose to adopt or reject the conclusions of others. There are periods in the history of almost every science when its sound and healthy progress may almost as much demand the refutation of that which is erroneous as the establishment of that which is true. I shall not, however, enter into any review of the past labours of glacialists with respect to exploded theories, but shall only notice those more recent researches and speculations which appear to me either to demand refutation as erroneous, or admission into any well-founded theory as correct. This treatment of the subject must necessarily lead to a certain repetition as to results already established, and be also of too critical a character, perhaps, with reference to other results in which I may have no confidence. I can only say that, in the present state of glacial theory, such defects must be inherent in any attempt to present it under a more complete and systematic form than it has hitherto, I think, assumed. It is this circumstance, too, which I would especially offer as an apology for the repetition of certain results which I obtained many years ago. Most of them are abstract mathematical results. They are here obtained by a more general analysis of the problem than that formerly employed, and are introduced as essential steps in the development now offered of the theory of the motion of glaciers.

In the first section I shall endeavour to remove the ambiguities which have beset this subject from the want of explicit definitions of certain terms expressing properties of ice on which our theories of glacial motion must essentially depend.

In the second section I shall give a brief statement of the results of experiments which explain the sliding of a mass of ice down a plane of small inclination, with a slow and unaccelerated motion like that of an actual glacier.

In the next section certain propositions are investigated respecting the internal pressures and tensions superinduced within a solid body by external forces which slightly distort it, and produce a small relative displacement of its component particles, or what may be termed a *molecular displacement*.

In the subsequent sections these results are applied to the explanation of crevasses, and to an examination of the theories which have been proposed of the veined structure of glacial ice. Finally, I have shown the importance of the sliding motion in giving efficiency to the internal pressures and tensions to dislocate the glacier, which thus becomes relieved from its internal strain, regaining the continuity of its mass and structure by its property of *regelation* so beautifully exhibited in Dr. TYNDALL's experiment.

SECTION I.—*Definitions and Explanations of Terms.*

1. The external forms of all bodies in nature may be changed in a greater or less degree, and without producing discontinuity in their mass or destruction of their internal structure, by the action of any external forces, the original or undisturbed form from which the change of form is to be estimated being that which the body would assume if acted on by no external forces whatever. This change of form necessarily implies a change in the relative positions of the component particles of the mass, or a certain greater or smaller amount of *molecular mobility*, or power in the particles of moving *inter se*. We may speak either of the general change of the form of the whole body, or of that which takes place in each of its small elementary portions; it is, in fact, in this latter sense that we are obliged to regard it in any accurate investigations, because the change of form for different elements will usually be different. Change of form in an element may or may not be accompanied by a change of its volume. In the first case it

leads to *cubical* extension or compression; in the latter, merely to extension or compression of the surface and not the volume of the element, which may be called *superficial* extension or compression. These changes of volume and form in any element must be produced by the forces acting on it. Thus we may conceive linear extension alone produced at any interior point of the mass by two equal and opposite tensions acting on two elementary component particles there in the direction of the line joining their centres of gravity, while compression alone would result if those tensions were changed into pressures. In such cases extension or compression would be the result of forces which may be called *direct* or *normal* forces. In the case above mentioned, in which the volume and density of every element of the mass remain unaltered, there can be no such direct normal action as that just mentioned. The action must be perpendicular to the normal, and must therefore be a *transversal* or *tangential* action. There would be no tendency to make the contiguous particles approach to or recede from each other, but to cause the one to *slide* tangentially past the other.

If the body have a structure like that of any hard vitreous or crystalline mass, pressure at any point will tend to break or crush the body, and thus to destroy the continuity of its structure. This tendency will be opposed by the *resisting power* of the substance. The tendency of the direct or normal tension is to separate the contiguous particles, and thus produce a finite fissure, or a discontinuity in the mass. It is resisted by the *normal cohesive power*; and in like manner the transverse or tangential action is resisted by the *tangential cohesion*, or that which prevents the component particles from sliding past each other. Again, when the component particles at any point of a body are relatively displaced, they have always a certain tendency to regain their originally undisturbed position; and the force thus excited, considered with reference to the force of displacement at that point, affords a measure of what is called the *elasticity* of the body. Since the force of restitution may vary from zero to the corresponding force of displacement, the elasticity, when measured by their ratio, may vary from zero to unity.

2. We may now define such terms as *solid*, *plastic*, *viscous*, and the like, with all the accuracy which their definitions admit of. We may call a body emphatically a *solid* body when it possesses the following properties:—(1) small extensibility and compressibility, (2) great power of resistance and great cohesive power, both normal and tangential, and (3) great elasticity. It will thus require a comparatively great force to produce any sensible relative displacement among the constituent molecules of the body: if we conceive the force required to become infinitely great, we arrive at absolute *rigidity* as the limit of solidity. Again, we shall best, perhaps, define *plasticity* or *viscosity*, if we suppose the forces of displacement to be such as to produce only a small transverse or tangential displacement of the constituent particles, *i. e.* a superficial, not a cubical, extension or compression. Then, if the force of restitution bear only an inappreciable ratio to the corresponding force of displacement, *i. e.* if the *tangential elasticity* be not of sensible magnitude, the mass may be emphatically said to be *plastic*. This is the essential condition of what may with strict propriety be termed *plasticity*; it might also

be added that, as bodies are constituted in nature, the force required to produce the original displacement in plastic bodies will be small as compared with that required in solid bodies. *Viscosity* and *semifluidity* are terms which only express similar properties of bodies, but indicating that still smaller forces only are required to produce a given displacement in viscous or semifluid bodies than in plastic ones. The limiting case is that of *perfect fluidity*, in which both the forces of original displacement and those of restitution are indefinitely small. In these latter cases the tangential cohesion is necessarily small, and such also (as bodies are usually constituted) will be the normal cohesion. At the same time the power of resisting compression of volume may be very great, as in fact it is in nearly all masses not technically designated as *elastic masses*. In other words, the *normal* elasticity, with reference to pressure, may be of any magnitude, while the *tangential* elasticity equals zero.

It will be observed that I have here spoken of a body as held in a state of constraint by external forces, but without any kind of dislocation which should destroy its continuity or injure its structure. If, however, the external forces should be sufficiently increased, the structure of a vitreous or crystalline mass, or that of any mass possessing hardness and brittleness, will be destroyed by a pressure greater than its power of resistance can withstand; or the continuity of its mass will be destroyed by any normal tension greater than the normal cohesion, or, again, by any tangential tension greater than the tangential cohesion. The normal tension would then produce an open fissure; and the tangential tension would cause one particle of the mass to slide past another, but without producing any open discontinuity. On the contrary, in a properly plastic or viscous mass there is no definite structure for excessive pressure to destroy; there is no question as to the formation of open fissures; and the characteristic absence of *tangential* elasticity allows of any amount of change in the relative positions of the constituent particles of the mass without breach of its continuity.

It would of course be impossible to draw an exact and determinate line of demarcation between solidity and plasticity, but it is not therefore the less certain that there are bodies which do unequivocally possess the property of solidity, and others which do as unequivocally possess the property of plasticity, according to the definitions I have given of these terms. Solidity and plasticity with respect to numerous cases in nature thus become determinate properties of those aggregates of material particles which we call bodies. Ice, a vitreous or crystalline and brittle mass, which will neither bear any but the smallest extension without breaking, nor more than the smallest compression without being crushed, must be solid, and cannot be plastic, if we are to use those terms as significant of determinate properties of bodies.

3. The advocates of the Viscous Theory would not probably admit the necessity of the above rigorous definition of the term *viscous* in its application to glacial ice. But the defect of that theory has always been in the entire want of any accurate definition of that term. When such a definition was demanded, it was said that glacial ice must be viscous, because a glacier adapted itself to the inequalities of its valley as a viscous

mass would do. This was equivalent to saying that the mass was viscous because it moved in a particular manner, instead of asserting that the mass moved in that particular manner because it was viscous. Now this kind of inversion of the direct enunciation of the proposition is only admissible when there is no other physical cause, than the one assigned, to which it is conceivable that the observed phenomenon should be ascribed. Thus we may assert with perfect conviction, that gravity exists as a property of matter and acts according to a certain law, because the bodies of the solar system move as if such were the case; but the conclusiveness of this inductive proof of the proposition—that "gravity is a property of matter"—rests entirely on our conviction that matter has no other property by which we could equally account for the phenomena of the celestial motions. And so with regard to glaciers. If viscosity were the only conceivable property of ice by which we could possibly account for the observed motion of glaciers, then would the observed phenomena of that motion perfectly convince us of the existence of the property in question. But here the two cases entirely differ, inasmuch as there was no general conviction, nor even a decided probability at the time I allude to, that no physical property of ice could exist besides viscosity which might account for the observed phenomena of a glacier's motion; and at the present time it is proved that there *is* another property of ice by which those phenomena are perfectly accounted for, and the inductive proof of viscosity becomes altogether valueless. Moreover, in the case of universal gravitation, the inductive proof is the only possible one, whereas in glacial motion we are concerned with a property which, in whatever sense the definition of it may be regarded, must be as capable of being rendered patent by experiment in ice, if it exist, as in any other substance.

The answer, then, that was given to the question, what is viscosity? comprised no definition at all of that term. The viscous theory ignored the possibility of the molecular mobility of a glacial mass, united with the preservation of its continuity, being attributable to any other property than that which was designated as *viscosity*, but without giving any exact definition of the term. If it was meant to define by it the property which I have defined by the same term, the theory had a legitimate claim to be considered a *physical* theory, because it assigned a determinate physical property as the cause of certain observed phenomena. In this sense, however, I conceive that it would now be admitted to be entirely disproved by Dr. TYNDALL's experiments, in which the ice exhibits so clearly the property of solidity, and the absence of all indication of plasticity. The hypothesis of viscosity, I imagine, could only have been adopted in the first instance from the apparent absence of any other property of ice which might account equally well for the molecular mobility of the glacial mass.

4. But if the determinate property of viscosity, as I have defined it, be not recognized in ice, what, it will be asked, is really the idea which has been attached to the term plastic or viscous? The question, as I have already intimated, is difficult to answer. Perhaps the best way of doing so is to refer to the "Prefatory Note" to Principal FORBES's 'Occasional Papers' (p. xvi). He there intimates that the expressions "bruising

and re-attachment," and "incipient fissures re-united by time and cohesion," used by him in 1846, are to be regarded as having the same meaning as the expression "fracture and regelation," first introduced into the subject in 1857. Now there is no ambiguity whatever in this latter expression. "Fracture" means the breaking and splitting of the ice regarded as a brittle and crystalline *solid*, and could never be intended to have the slightest reference to viscosity. In fact the expression is altogether inapplicable to any body which can be called viscous, without what I should regard as a violation of scientific language. Still this, it may be said, may be only a want of strict accuracy of expression, rather than of accuracy of conception. But if a notion of cracking and breaking, so foreign to any idea of plasticity, should be admitted, it could not be said that a glacier moved as it is observed to move because it was plastic, but merely that it moved *as if* it were plastic. The true inference from the motion would have been that glacial ice possessed not necessarily real plasticity, a definite property of bodies, but a *quasi-plasticity*, which expresses no determinate property at all, but may consist with many different properties. It merely expresses, in fact, the power of the component elements of the mass of changing to a certain extent their relative positions. But this is not the peculiar property of ice; it is common, indeed, to all bodies exposed to disruptive forces which, as in the case of ice, the cohesive power is unable to withstand. The mass of any other substance, as well as that of a glacier, will then be broken into fragments sufficiently small to allow it to follow the impulses of the external forces acting on it. To say, therefore, that a glacier moves *as if* it were plastic is not to assign to ice any property peculiar to itself, and therefore does not properly constitute a *physical* theory of glacial motion at all.

5. But if we should pass over the difference between true plasticity and that which, as we have pointed out, is merely apparent, there would still remain the great difficulty which was only removed by the experiments of Mr. FARADAY and Dr. TYNDALL. Every one who believed ice to be a solid body, believed as a matter of necessity that a glacier must, on account of the external conditions to which it is subjected, be excessively broken and dislocated in the course of its motion. I was myself one of those who fell into the error of attributing too much influence to the larger and more visible disruptions of the mass; but the great difficulty was in the perfect subsequent reunion of portions which had thus been separated, whether by larger or smaller dislocations. And here it will necessarily be asked whether, in the expression above quoted, "*re-attachment*" and the "*re-union* by time and cohesion" of separated portions when again brought into contact, really mean the same thing as *regelation*? It can only be answered, I think, by saying that, whatever might be the intended meaning of those expressions, they failed to convey to the minds of others the most remote idea of regelation as a property of ice at a particular temperature. No better proof can be given of this than the general conviction which appeared to flash across the mind of every glacialist when he first heard of Dr. TYNDALL's experiment, that the recognition of the property of instantaneous regelation was a well-marked and important discovery, which had at once completely

removed a great stumbling-block in glacial theory. In fact, the viscous theory assigns no physical cause for the reunion in question. All we could do, before the publication of those experiments, was to infer from the observed facts that ice did possess some property which facilitated the reunion of separate pieces in contact; but this was like the attempt to define viscosity by an appeal to the phenomena which that property was intended to explain.

6. An *imperfect plasticity* in ice has sometimes been spoken of. The fact is, all solid bodies might be said to have an imperfect plasticity, if we chose to admit this vagueness in scientific language, since all are capable of greater or less extension or compression. As to the apparent plasticity inferred from the motion of glacial masses, and arising from the crevicing of the ice, I have explained that it has no relation whatever to real plasticity. Such crevices are the necessary consequences of the external forces acting on the glacier, and are as essential to the theory of regelation as they are unconnected with any property of plasticity.

I proceed, as proposed in my introductory remarks, to explain the *sliding* of glaciers.

SECTION II.—*Sliding Motion of Glaciers along the bottoms of the Valleys containing them.*

7. The sliding motion of glaciers, as first suggested by DE SAUSSURE, seemed to involve some serious difficulties. The inclination of the surfaces along which some of the Alpine glaciers descend is so small (not exceeding, perhaps, in some cases 3° or 4°) as to furnish one very obvious objection; and another was, that if glaciers did thus move at all, it must be by an *accelerated* motion, whereas their real motion was an unaccelerated one. Both these objections were founded on an entirely erroneous conception of the nature of the forces called into action in the sliding of a glacier. They were considered to be analogous to the force of friction in the ordinary case of a body sliding down an inclined plane. But in this latter case the constitution of the sliding body, and of that on the surface of which it slides, is always assumed to be such that the surfaces in contact are not affected by the sliding movement, and then we have the experimental law that the friction is independent of the velocity. Consequently the motion is an *accelerated* one. But the condition just mentioned, respecting the surfaces in contact, will not be satisfied in the case of a glacier, assuming what I shall prove shortly, that its lower surface must necessarily have a temperature not lower than that of freezing, and must consequently be always on the point of disintegration by thawing. Ice then becomes very tender, and the cohesion of its particles at its lower surface becomes insufficient to prevent the descent of a mass of such an enormous weight as that of a glacier, even along planes of the smallest inclination. This was clearly illustrated by a simple experiment which I made some years ago, which also fully explained the unaccelerated motion of a sliding glacier. The details of this experiment will be found in the Cambridge Transactions for 1844 *. It constitutes so fundamental a step in this subject, according to my own views of it, that I would beg permission to give here a brief description of it, and of the results to which it leads.

* They will also be found in the Philosophical Magazine for January 1845.

8. A mass of ice was placed on a flat rough slab of sandstone, so arranged that it could easily be placed at any proposed inclination to the horizon. When the inclination was about 20°, the ice descended with an accelerated motion as in ordinary cases, but at smaller inclinations it descended with a *slow uniform motion*, which, for inclinations not exceeding 9° or 10°, was, *cæteris paribus*, *proportional to the inclination*. The velocity was increased by an *increased weight* of the mass, the area in contact with the plane remaining the same. The motion was due to the melting of the ice in contact with the slab, for when the temperature of the air became below that of freezing, it entirely ceased.

The motion was sensible for inclinations little exceeding half a degree, and, doubtless, would also have been so for smaller inclinations. This shows how small a force was required to move the mass when its lower surface was in a state of disintegration. Let f be a retarding force applied to a mass whose weight $=$W, placed on a plane whose inclination is ι. Then will W sin $\iota - f$ be the moving force along the plane; and since the mass will require only the smallest force to move it, a force f very nearly equal to W sin ι will be necessary to hold the mass at rest. If f be removed, the ice will begin to move with a moving force nearly $=$ W sin ι, the motion being permitted by the liquefaction of successive indefinitely thin layers of ice, but then it will be retarded by the solid mass coming in contact with the surface on which it slides. A melting of another indefinitely thin layer will then take place, and the above process will be repeated, the velocity increasing till the continuous action of the plane on the mass becomes equal to the weight resolved in the direction of the plane. During this time (probably too short to be estimated) the motion will be an accelerated one, but will thenceforward become uniform, the action of the plane becoming equal to the resolved part of the weight along it. The uniform velocity is, in fact, a *terminal* velocity, similar to that of a stone descending in water, when it soon approximates to a nearly uniform motion. The action of the inclined plane on the moving mass, like that of the resistance of the water on the stone, has this property—that, while it is incapable of exerting any but the smallest force to hold the body absolutely at rest, it exerts a retarding force upon it in uniform motion, equal to that of gravity. The objections above mentioned against the sliding of glaciers have arisen from an entire misconception of this kind of mechanical action.

9. At the period when the preceding experiment was published, I was disposed to think that the greater part of the observed motion of the surface of a glacier was due to the general motion, by sliding, of the whole mass, while it was contended by other glacialists that it was principally due to an excess of the velocity of the upper strata of the mass over that of its lower strata, due to a gradual change of form of the whole mass, and that there might in fact be no sliding movement at all. In recognizing that both these causes might be *veræ causæ**, I urged the necessity of deciding on their relative claims by actual observations, which should determine the velocities of the upper and lower surfaces of a glacier at some point where the lower one was accessible. Observa-

* Philosophical Magazine and Journal for February 1845.

tions for this purpose were made by Principal FORBES in 1846, at the bottom of the Glacier des Bois at Chamouni. He found that the velocity of a point on the surface, at the height of 143 feet above the bed of the glacier, was to that of a point 8 feet above the bed, in the ratio of nearly 16 : 10 *, whence it follows that, if we divide the whole velocity of the surface into eight parts, five of them will be due to the motion of the bottom of the glacier, and three to that change of form of the mass by virtue of which its higher move faster than its lower portions. Dr. TYNDALL has also made similar observations on the flank of the Mer de Glace †, from which it appears that the motion of the upper was there rather more than twice that of the lower surface. These observations may or may not determine approximately the average ratio between the velocities of the upper and lower surfaces of a glacier; but they leave no doubt as to the fact of the sliding movement. Again, it is observed that existing glacial valleys, and those which are believed to have been such in former times, always indicate, by their smoothed and striated rocks, the sliding movements of the glaciers they formerly contained. In fact, few glacialists at present, I imagine, will doubt the existence of this sliding motion, or that it forms a considerable portion of the whole motion of a glacier; and I believe that the experiments above described afford an adequate explanation of the cause and character of that motion. I insist on this more particularly because the explanation has been singularly ignored and misunderstood. The non-applicability of the experiments has been asserted, because the sliding mass was not obstructed in its motion by lateral obstacles, like a glacier, whereas, in fact, they had no concern with lateral obstacles, being merely intended to explain the action of the bed of the valley on the superincumbent glacier. The irregularity of the sides introduces, as we shall see very shortly, no difficulty or ambiguity into my views of the subject. I may also state that, several years after my experiments and his own observation above stated were published, Principal FORBES repeats his objection of the difficulty of conceiving the possibility of the motion of sliding glaciers being unaccelerated, whereas every one now acknowledges that they do slide, and knows that their motion is unaccelerated ‡. M. AGASSIZ, on the contrary, after repeating the experiments, allows that the results remove the great difficulties of admitting the sliding motion of glaciers §. This kind of motion, as we have seen, depends very much on the temperature of the lower surface of the glacier being always equal to the freezing-temperature. That such must always be the case I proceed to show. All observations indicate that it is so; but still, since few direct observations can be made on this point, it may be well to show that it follows from the temperature of the earth, and the nature and conductivity of ice, that the temperature of the lower surface of a glacier must be that above mentioned.

* Occasional Papers, p. 175.

† Glaciers of the Alps, p. 289.

‡ "The main objection, however, is this, that a sliding motion of the kind supposed, if it commence must be accelerated by gravity, and the glacier must slide from its bed in an avalanche. The small slope of most glacier-valleys, and the extreme irregularity of their bounding walls, are also great objections to the hypothesis."—Occasional Papers; also published in 1855 in the 'Encyclopædia Britannica.'

§ Système Glaciaire, p. 568.

10. *Interior Temperature of Glaciers.*—There are two obvious causes by which the temperature in the interior of a glacier may be affected—(1) conduction of heat from the superficies of the mass according to the ordinary laws of conduction through solid bodies, and (2) infiltration of water from the upper surface. We may consider separately the operation of these causes, with the view of determining whether the lower surface of the mass is permanently at the temperature of freezing, as it must be to render the preceding experiments strictly applicable to account for the continuous and unaccelerated motion of the glacier. To investigate the effect of the first cause, let us conceive the whole surface of the earth covered with a superficial crust of ice. The temperature of this crust will be subject to periodical annual variations to a certain depth, which will depend on the annual variations of the superficial layer of the icy crust, and the conductivity of the ice. It may be considered, for our immediate purpose, sensibly the same as the temperature of the glacier itself, at all points where the glacier and our imaginary crust of ice coincide. Now the surperficial changes of temperature in the crust of ice subject to conditions similar to those of a glacier, would be much less than those which the actual rocky crust of the earth experiences; for the temperature of the ice could never rise above 32° Fahr. in the hottest summer, nor in the coldest winter could it probably fall many degrees below that temperature, on account of the covering of snow like that which on all glaciers protects their outer surface against the effect of low winter temperature. Again, the depth through which these oscillations of temperature would be perceptible, would, *cæteris paribus*, be comparatively small if the conductive power of the mass should be so. I am not aware of any experimental determination of the value of this power for ice; but that substance is known to be a very bad conductor, and probably worse than the average of the rocks which form the outer crust of the earth. For both these reasons, then, the depth of oscillatory annual temperature would be much smaller than it is found to be within the actual crust of the globe, under the same external climatal conditions. Now in the earth's crust, and in our own latitudes, this depth may be approximately estimated at 70 or 80 feet, according to the nature of the upper strata. I should therefore deem it probable that the variations of external temperature in a crust of ice like that above supposed, or therefore in an ordinary glacier, would not exceed at most perhaps some 30 feet.

Again, let us consider the probable *mean* annual superficial temperature of our hypothetical icy crust, or of a glacier of ordinary dimensions. The actual temperature could never rise, as above remarked, above 32° Fahr., and would never sink many degrees below that temperature. M. AGASSIZ has left us the only reliable observations on this subject*. He buried a self-registering thermometer in the glacier of the Aar, at the depth of 2·1 metres. It was taken up two years afterwards, and was found to have registered a minimum temperature of −2°·1 (C.) =28°·22 (Fahr.). Thus the mean temperature for the winter was probably not less than 30° (Fahr.), and that for the

* Système Glaciaire, p. 425.

whole year might perhaps not much fall short of 31°(Fahr.)*. If the conductive power of ice were equal to that of the earth's crust, the mean temperature would increase 1° (Fahr.) in descending some 60 or 70 feet; and therefore, on account of the smaller conductivity of ice, it would probably, in the case of a glacier, rise to 32° (Fahr.) at a depth of some 30 or 40 feet. This would hold, it should be observed, on the supposition that the substance below this depth should be capable, like the matter of the earth's crust, of taking any temperature higher than 32° (Fahr.). This higher temperature would be acquired, as in the actual case of the earth, by the flow of heat from the earth's interior. But in the case of a glacier this heat will be expended in melting the lower stratum of ice instead of communicating a higher temperature to the whole mass. Consequently, if the thickness of the glacier exceed some 30 or 40 feet (a depth at which, as above shown, the temperature will be invariable), the temperature of the lower surface will be constant and equal to 32° (Fahr.).

The temperature at any proposed point (P) of the interior of a glacier, at a depth greater than that estimated above at some 30 or 40 feet, will always be constant and less than 32° (Fahr.), provided the mean annual temperature of the external surface of the glacier be so. To find this constant temperature at P, take for the temperature of the upper surface its mean annual temperature. Let it $=T^\circ$ (Fahr.). Also let $a=$thickness of the glacier, $x=$distance, from the upper surface, of the proposed point, and t its required temperature. Then shall we have, according to the laws of conduction of heat,

$$\frac{t-T^\circ}{32^\circ-T^\circ}=\frac{x}{a},$$

the difference of temperatures, as is well known, being approximately proportional to the distances from the upper surface. Hence

$$t=T^\circ+\frac{x}{a}(32^\circ-T^\circ),$$

which shows that t must always be greater than T°; it must also be less than 32° (Fahr.), and must therefore lie between those quantities. Consequently, since $32^\circ-T^\circ$ is small for the Alpine glaciers, their internal temperature must be nearly uniform, but always a little below 32° (Fahr.), supposing it to depend only on the process of conduction.

But the process of infiltration will tend to raise the internal temperature more nearly to 32° Fahr.; for since the infiltrated water will have that temperature, it will constantly tend to heighten the temperature of the mass through which it passes till it rise to 32° (Fahr.), and never to lower it. This water must thus bring to the glacier (a mass of lower temperature than itself) a continual accession of heat, which it can only lose again by conduction through the upper surface during the winter; and this loss will be restricted to that small depth beyond which the annual variations of temperature cannot extend. For all points at greater depth infiltration must ultimately raise the temperature to 32° (Fahr.).

* M. Agassiz states that the temperature given by his experiment might possibly be somewhat too high. There is no probability, however, that the error would be sufficient to affect the reasoning in the text.

Hence, then, considering the combined operation of conduction and infiltration, it appears that, to the depth of perhaps 30 feet, the *interior* temperature of a glacier will be 32° (Fahr.) during the summer portion of the year, but will be rather lower than 32° (Fahr.) during the winter. For all deeper parts of the glacier it will be invariably equal to 32° (Fahr.).

These results are in exact accordance with the careful observations made by M. AGASSIZ, which have already been partially referred to. Besides the winter observations above mentioned, he also observed the temperatures in the month of July, at depths of from 3 to 5 metres, at 30, and at 60 metres. These temperatures were all exactly 32° (Fahr.) during the fortnight they were observed, with the exception of one or two very small and manifestly accidental variations in the more superficial observations.

11. It follows from the preceding articles that the temperature at the lower surface must always be the freezing-temperature, *i. e.* the ice there must be in that state in which the mutual cohesion of its constituent particles is less than in any other state. It does not follow that the glacier would not slide if the temperature of its lower surface were less than 32° (Fahr.); but that temperature is the most favourable for the motion of the glacier, because the most favourable to the disintegration of its lower surface, and the immediate conversion of the ice which forms it into water. It should be remembered, too, that it was one of the results of my experiment, that, *cæteris paribus*, the motion was increased by increasing the weight of the mass; *i. e.*, the cohesive power of the ice at the bottom of the glacier will be the more rapidly overcome by increasing the depth of the mass, the area of its base being unchanged. Consequently the tendency of a glacier to descend down its bed would be indefinitely greater, *cæteris paribus*, than that of our experimental lump of ice down its plane. It is this enormously increased tendency that enables the mass of a glacier to overcome the resistance arising from the inequalities of the sides and bottom of its valley. We shall explain in the sequel the prodigious force which may thus be exerted, and the corresponding internal tensions which would thus be produced by it. These tensions overcome the cohesion of the mass, the ice breaks, and the glacier obtains more freedom of motion than it could have in its state of greater compactness and continuity. The tendency to the sliding motion we are considering will manifestly be greater in the axial than in the marginal parts of the mass. It is there, especially, that the depth must be the greatest, and the distance from opposing lateral objects is likewise greatest; and, it may be added, the subglacial currents, by which the sliding will undoubtedly be more or less facilitated, will be generally greatest along the central parts of the valley.

Hence, then, it follows that, so far as the motion of a glacier depends on the *sliding* we have been considering, the velocity of its axial portions will generally be considerably greater than that of its marginal portions. This constitutes the most distinctive and important character of the observed motion of a glacier.

Still, though the sliding motion was perfectly consistent with this observed general character of glacial motion, it was not sufficient to account for several striking phenomena attendant upon it. The formation of crevasses was a necessary consequence of the forces acting on the glacier, and the conditions to which it was subjected; but no reason was thus assigned why such crevasses should be obliterated again, as they were frequently observed to be, and the continuity of the mass perfectly restored. Moreover, it became evident from more accurate and detailed observation, that the continuity of many parts of the general mass was preserved in a degree apparently inconsistent with the change of form to which a mass so crystalline and brittle as glacial ice did manifestly submit. It was to meet this difficulty that Principal FORBES was led to the hypothesis of the *viscosity* or *plasticity* of glacial ice. I have already explained my objection to the vagueness with which these terms appear to me to have been used, and to the total want of all experimental proof of any property in ice which could be so designated with accuracy, or with a due regard to the propriety of scientific language. Difficulties of this kind always remained on the minds of certain glacialists, till the experiments of Mr. FARADAY and Dr. TYNDALL at once explained to us that *regelation*, and not *viscosity*, was the real property of ice required for the completion of our general theory of glacial motion. This property of regelation belongs essentially to *solid* bodies, and in treating glacial masses as bodies possessing the property of regelation, we must necessarily treat them as *solid*. As such I consider them in the following investigations, the object of which is to ascertain, as far as we are able, the internal pressures and tensions to which glaciers are subjected, and the phenomena which may result from them, more especially those connected with the veined structure of glacial ice, and the formation of crevasses.

12. Before I proceed to these investigations, I would here remark that the importance of a distinct conception of the properties indicated by the terms viscosity or plasticity on the one hand, and solidity on the other, will be at once apparent if we consider the difference between the mechanical problems presented to us in the motion of glaciers, according as we conceive them to be typified by a viscous or solid mass. In the first case we have to determine the continuous motion of a mass the component particles of which move with different velocities, but without destroying its continuity. The most simple, and the limiting case, would be that in which the tangential action of contiguous particles on each other should vanish. The mass would then become a fluid mass. But even in this case we can do little by accurate mathematical investigation, and still less in the case in which the mass is viscous. Consequently, the objection against any attempt at a mathematical solution of the problem of glacial motion, founded on our ignorance of the motions of viscous masses, is perfectly valid, so long as we treat glacial ice as viscous according to our definition of that term. But this same objection has been urged against all attempts to apply accurate mathematical processes to the problem, in its complete or partial solution, under the supposition of ice having the property of solidity. The complete solution of the problem would undoubtedly be far more difficult for a solid than for a viscous mass; for it would involve conditions depending on innumerable discontinuities in the mass, resulting from its motion. All that can be attempted is a

partial solution of the problem, in which the dynamical difficulties are evaded. It has been already explained that when a solid body is acted on by external forces, it generally becomes distorted in form and changed in volume. If the forces be insufficient to overcome the cohesion of the mass and to dislocate it, they will of course continue to maintain the body in its state of constraint and distortion, and thus to produce, at different points of its interior, pressures and tensions varying both in direction and intensity. The immediate object of the first part of the investigations contained in the following pages, is the proof of certain propositions respecting these internal pressures and tensions, and the phenomena resulting from them. The problem thus considered is not a dynamical, but a statical one, in which certain results are attainable with the same accuracy as in the simplest mechanical problems; and such are the only results with which we are directly concerned in our present researches. If the distorting forces be sufficiently increased, the mass will be torn or crushed, as already stated, and will then move according to the new conditions imposed upon it in its state of dislocation. This constitutes the dynamical part of the problem, but, it must be recollected, it does not enter at all into the mathematical part of our own investigations. I have thought it necessary to point out this distinction, lest any vague objection resting on an imperfect or erroneous conception of the problem before us should exercise an undue influence on the mind of the reader.

SECTION III.—*On the Pressures and Tensions at any point of a Solid Mass held in a position of constraint by external forces.*

13. We may now proceed to the consideration of the general problem, the object of which is to investigate the nature of the internal pressures and tensions at any proposed point of a solid mass subjected to the action of impressed forces which slightly distort it from the form it would assume when acted on by no external forces at all. It will be recollected that these forces are supposed insufficient to destroy the continuity of the mass. They maintain it in its distorted form, and must therefore be in *equilibrium* with the internal forces arising from the cohesive power of the mass.

To explain the nature of the distortion produced in any small element of the mass, let us denote by s the area of an indefinitely small plane surface passing through any point (P). Generally there will be an action between the particles (M) on one side of our small plane, and those (M′) on the opposite side. Since s is indefinitely small, we may represent by f the whole action of M on M′, and suppose its direction to make an angle δ with the normal to the plane s. Then will

$$f \cos \delta \text{ and } f \sin \delta$$

be the normal and tangential actions respectively of M on M′; and

$$-f \cos \delta \text{ and } -f \sin \delta$$

will be the corresponding actions of M′ on M. If the normal force be a *pressure*, it will only tend to preserve the particles on opposite sides of s in contact, but if it be a *tension*, it will tend to separate those particles by motions parallel to the normal. Also the tangential forces $f \sin \delta$ and $-f \sin \delta$ will always tend to separate two particles on opposite sides of s and in contact, by making them move in opposite directions parallel to the plane. If we conceive s to revolve about P as a fixed point, and thus to assume all possible angular positions, the forces $f \cos \delta$ and $f \sin \delta$ will vary with the angular position of the plane, in certain positions of which they will assume their maximum and minimum values. The determination of these positions is one immediate object of the problem, with the view of determining the effect of the distorting forces on the form of each element of the mass, and thence, if the problem were completely soluble, the distortion of the whole mass. But before proceeding further, we may explain more explicitly what will be the kind of distortion to which every element of any solid body will be subjected under the action of distorting forces. Let us take a small rectangular parallelopiped as the element of the body while unconstrained by such forces. The normal forces acting on opposite sides of the element will manifestly form three pairs of equal* and opposite forces, each force of each pair acting in a direction opposite to the other force of that same pair, and thus producing compression or extension of the element according to the directions in which they act. Again, it is manifest, from what has been said respecting the small plane s, that the tangential force on any one side of the elementary parallelopiped will be equal to that on the opposite side, but will act in the opposite direction, thus tending to *twist* the element from its original rectangular form into an oblique-angled parallelopiped. Hence the primitive undistorted element will be compressed or extended according to circumstances, and will always (unless the forces acting on it be entirely normal) be twisted so as to destroy its rectangularity. In the final application of the results obtained from this our typical problem, we shall have to deduce the manner in which the continuity of the glacial mass will be destroyed when the power of resistance of the ice is no longer sufficient to equilibrate the distorting forces acting on it, and also to consider the phenomena which may result from such breach of continuity.

With respect to the solution of our abstract problem, I have little to add to that which I gave in the Transactions of the Cambridge Philosophical Society for the year 1847, and I might be content merely to refer to that solution for the results. In doing so, however, it would be necessary to give a somewhat complicated notation, and certain explanations in such detail that the space thus occupied would not differ materially from that required for the mathematical analysis of the first part of the general problem. By giving this analysis here, considerable trouble of reference will be avoided. I would request permission, therefore, to repeat a part of what appeared in the Transactions above referred to. The quotation includes the following articles, from the 14th to the 17th inclusive:—

* Omitting small quantities of a certain order, which it is not necessary in this general explanation to take into account.

14. "Taking any point (P) of the mass, let it be made the origin of coordinates xyz. Let the small plane s be conceived as above to pass through P, and let the forces upon it when in the positions specified below be denoted as follows, all being referred to a unit of surface.

"(1) When a perpendicular to the plane coincides with the axis of x, let

The *normal force* $=\mathrm{A}$; the *tangential force* $=\begin{cases}\mathrm{B}' \text{ parallel to } y,\\ \mathrm{C}' \quad\quad \text{,,} \quad\quad z.\end{cases}$

"(2) When a perpendicular to the plane coincides with the axis of y, let

The *normal force* $=\mathrm{B}$; the *tangential force* $=\begin{cases}\mathrm{C}'' \text{ parallel to } z,\\ \mathrm{A}' \quad\quad \text{,,} \quad\quad x.\end{cases}$

"(3) When a perpendicular to the plane coincides with the axis of z, let

The *normal force* $=\mathrm{C}$; the *tangential force* $=\begin{cases}\mathrm{A}'' \text{ parallel to } x,\\ \mathrm{B}'' \quad\quad \text{,,} \quad\quad y.\end{cases}$

"Between the six accented quantities there are three essential relations, which are easily found. On the three coordinate axes at P, construct an indefinitely small parallelopiped whose edges are δx, δy, δz. The six equations of equilibrium of this element will express the conditions that the sums of all the resolved parts of the forces parallel to the coordinate axes shall respectively be equal to zero; and that the moments of the forces with reference to these axes shall also severally be equal to zero. Let us take the three latter conditions, lines through the centre of gravity of the element and parallel to the coordinate axes being taken for the axes of the component couples. The tangential force parallel to the axis of x on the side $\delta x.\delta z$ being A', that on the opposite side will be $-\left(\mathrm{A}'+\frac{d\mathrm{A}'}{dy}\cdot\delta y\right)$; and the couple resulting from these forces about the axis parallel to z, will be

$$\mathrm{A}'\delta x\delta z.\frac{\delta y}{2}+\left(\mathrm{A}'+\frac{d\mathrm{A}'}{dy}\delta y\right)\delta x\delta z.\frac{\delta y}{2};$$

or, omitting small terms of the fourth order,

$$\mathrm{A}'\delta x\delta y\delta z.$$

"Similarly, the couple arising from the forces B' and $\mathrm{B}'+\frac{d\mathrm{B}'}{dx}\delta x$ about the same axis parallel to z, will be

$$-\mathrm{B}'\delta x\delta y\delta z.$$

"Also the moments of the normal forces A, B, C, with reference to the above-mentioned axes, will be zero, always omitting small quantities of the fourth order. Consequently the whole moment of the forces on the parallelopiped with reference to the axis parallel to that of z, will be

$$(\mathrm{A}'-\mathrm{B}')\delta x\delta y\delta z,$$

which must = zero by the conditions of equilibrium; and therefore

$$\mathrm{A}'=\mathrm{B}'.$$

In exactly the same way we find, by taking the moments with reference to the axes parallel respectively to those of y and x,

$$A''=C',$$
$$B''=C''.$$

By means of these three relations the six accented quantities are reduced to three independent quantities.

15. "Let us now conceive a plane to meet the three coordinate planes so as to form with them a tetrahedron, whose vertex is at the origin P. Suppose the exterior normals to the three faces formed by the coordinate planes to point respectively towards the positive directions of x, y, and z; and let α, β, γ be the angles which the normal to the base of the tetrahedron makes with the coordinate axes of x, y, z. Also, let s denote the area of the base, and s', s'', s''' the areas of the sides of the tetrahedron perpendicular respectively to the axes of x, y, z, all these quantities being indefinitely small.

"Again, let ps denote the whole resultant force on s, and let λ, μ, ν be the angles which its direction makes with lines parallel to the axes x, y, z, this direction being exterior to the tetrahedron. Then in order that the tetrahedron may be in equilibrium, we must have

$$ps.\cos\lambda=As'+A's''+A''s''',$$
$$ps.\cos\mu=Bs''+B's'+B''s''',$$
$$ps.\cos\nu=Cs'''+C's'+C''s'';$$

but

$$\frac{s'}{s}=\cos\alpha,\quad \frac{s''}{s}=\cos\beta,\quad \frac{s'''}{s}=\cos\gamma;$$

making these substitutions, and also putting

$$B''=C''=D,$$
$$A''=C'=E,$$
$$A'=B'=F,$$

we shall have

$$\left.\begin{aligned} p.\cos\lambda &= A\cos\alpha+F\cos\beta+E\cos\gamma,\\ p.\cos\mu &= B\cos\beta+F\cos\alpha+D\cos\gamma,\\ p.\cos\nu &= C\cos\gamma+E\cos\alpha+D\cos\beta,\end{aligned}\right\} \quad \ldots\ldots \quad (a.),$$

formulæ in which the notation agrees with that of M. CAUCHY*.

16. "If δ denote the angle between the direction of p and the normal to s, we shall have $p.\cos\delta$ for the whole *normal* force acting on the area s in a direction exterior to the tetrahedron, and $p.\sin\delta$ the whole *tangential* force acting on the same area. Our first object will be to determine α, β, and γ, or the position of the base s of the tetrahedron, so that the normal action upon it, $p\cos\delta$, shall be a maximum. We shall afterwards have a similar investigation with reference to the tangential force $p.\sin\delta$.

* Exercices de Mathématiques, vol. ii. p. 48.

"We have

$$\cos\delta = \cos\lambda\cos\alpha + \cos\mu\cos\beta + \cos\nu\cos\gamma,$$

whence we immediately obtain

$$p\cos\delta = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma + 2D\cos\beta\cos\gamma + 2E\cos\alpha\cos\gamma + 2F\cos\alpha\cos\beta; \quad (1.)$$

and since

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \quad \ldots \ldots \quad (2.)$$

we have (L being an arbitrary multiplier),

$$(A+L)\cos^2\alpha + (B+L)\cos^2\beta + (C+L)\cos^2\gamma + 2D\cos\beta\cos\gamma + 2E\cos\alpha\cos\gamma + 2F\cos\alpha\cos\beta = max.$$

Hence

$$\left.\begin{aligned} &\{(A+L)\cos\alpha + E\cos\gamma + F\cos\beta\}\sin\alpha = 0,\\ &\{(B+L)\cos\beta + D\cos\gamma + F\cos\alpha\}\sin\beta = 0,\\ &\{(C+L)\cos\gamma + D\cos\beta + E\cos\alpha\}\sin\gamma = 0.\end{aligned}\right\} \quad \ldots \ldots \quad (b.)$$

"To satisfy these equations together with (2.), we must equate the first brackets to zero. We thus have four equations from which L may be eliminated, and α, β, and γ determined.

"If we multiply the first factors on the left-hand sides of equations (*b.*) by cos α, cos β, cos γ respectively, and add them together, we have, by virtue of equations (*a.*),

$$L = -p\cos\delta;$$

and substituting for L in equations (*b.*), we have

$$\begin{aligned} p\cos\delta\cos\alpha &= A\cos\alpha + F\cos\beta + E\cos\gamma \\ &= p\cos\lambda \end{aligned}$$

by the first of equations (*a.*); and therefore

$$\cos\delta\cos\alpha = \cos\lambda,$$

and similarly

$$\cos\delta\cos\beta = \cos\mu,$$
$$\cos\delta\cos\gamma = \cos\nu,$$

whence

$$\cos^2\delta = 1$$
$$\delta = 0,$$

which shows that when the resultant force p is a maximum or minimum, its direction coincides with that of the normal to the plane s. Consequently, also, the tangential force, $p\sin\delta$, then becomes = zero.

"This value of δ gives $L = -p$; and substituting for L in equations (*b.*), we have

$$\left.\begin{aligned} (A-p)\cos\alpha + F\cos\beta + E\cos\gamma &= 0,\\ F\cos\alpha + (B-p)\cos\beta + D\cos\gamma &= 0,\\ E\cos\alpha + D\cos\beta + (C-p)\cos\gamma &= 0;\end{aligned}\right\} \quad \ldots \ldots \quad (c.)$$

and eliminating cos α, cos β, and cos γ by cross multiplication, we obtain

$$(A-p)(B-p)(C-p) - D^2(A-p) - E^2(B-p) - F^2(C-p) + 2DEF = 0. \quad \ldots \quad (3.)$$

"If we take the three values of p deducible from this equation, and substitute them successively in equations (*c.*), those equations combined with (2.) will give three distinct systems of values for cos α, cos β, and cos γ, belonging (as is well known) to three lines perpendicular to each other.

17. "Hence it follows that there is at every point (P) of a continuous solid mass under extension or compression, a system of three rectangular axes, such that if the small plane s at P be so placed that its normal shall coincide with one of those axes, the whole resultant action on s shall be *normal* to it, the tangential action upon it being then equal to zero. These three axes are called *the axes of principal pressure or tension* with reference to the point P.

"Of the three values of p in these directions, though they all satisfy the conditions of maximum or minimum, one is a maximum, another a minimum, and the third is neither an absolute maximum nor an absolute minimum. This is best explained, perhaps, by converting equation (1.), as CAUCHY has done, into an equation to a surface of the second order, by putting

$$p\cos\delta=\frac{1}{r^2},\quad r\cos\alpha=x,\quad r\cos\beta=y,\quad r\cos\gamma=z.$$

The inverse of the square of any radius vector will be a measure of the normal action through P perpendicular to this radius vector, the axes of this surface of the second order coinciding with the axes of principal tension or pressure. Of the three principal axes of this surface, the directions of the greatest and least will manifestly coincide with those of minimum and maximum tension; but though the tension in the direction of the mean axis of the above surface satisfies the two conditions $\frac{d.p\cos\delta}{d\alpha}=0$ and $\frac{d.p\cos\delta}{d\beta}=0$, it satisfies the one because it is maximum with respect to α, and a minimum with respect to β, or the converse, as the mean axis of an ellipsoid is a maximum in one principal section of the surface, and a minimum in the other."

18. The object of the second part of this investigation* is to determine the angular positions of the small plane (s) passing through P, so that the tangential force acting upon it shall be greatest, *i. e.* that $p\sin\delta$ may be a maximum. Our formulæ will be much simplified by taking the axes of principal tension or pressure as the coordinate axes. In this case we shall have

$$D=0,\quad E=0,\quad F=0;$$

and if A_1, B_1, C_1 now represent the principal tensions at the proposed point, and α_1, β_1, γ_1 be the values of α, β, γ referred to these new axes, equations (*a.*) will give

$$p^2=A_1^2\cos^2\alpha_1+B_1^2\cos^2\beta_1+C_1^2\cos^2\gamma_1,$$

and equation (1.) gives

$$p.\cos\delta=A_1\cos^2\alpha_1+B_1\cos^2\beta_1+C_1\cos^2\gamma_1.$$

* A solution of the problem above investigated was also given by M. CAUCHY, in his 'Exercices de Mathématiques' (vol. ii. p. 48). The solution of the problem in this second part of the investigation has only been given, I believe, by myself.

Hence we have (if $p \sin \delta = T$)

$$T^2 = p^2 \sin^2 \delta = A_1^2 \cos^2 \alpha_1 + B_1^2 \cos^2 \beta_1 + C_1^2 \cos^2 \gamma_1 - (A_1 \cos^2 \alpha_1 + B_1 \cos^2 \beta_1 + C_1 \cos^2 \gamma_1)^2,$$

which is to be a maximum subject to the condition

$$\cos^2 \alpha_1 + \cos^2 \beta_1 + \cos^2 \gamma_1 = 1.$$

The solution of this problem is longer and more complicated than that of the problem solved in the first part of the investigation. It will here be sufficient for my purpose to state the results, and, for the analytical solution, to refer the reader to the volume of the Cambridge Transactions above quoted. The results are as follows:—

α_1, β_1, and γ_1 being the angles which define the position of the small plane s (art. 13), the analytical conditions for the tangential force (T, or $p \sin \delta$) upon it being a maximum or minimum, are satisfied by the following three systems of contemporaneous values:—

$$\left.\begin{array}{lll} (1) & \alpha_1 = 90^\circ, & \beta_1 = \gamma_1 = \pm 45^\circ, \\ (2) & \beta_1 = 90, & \gamma_1 = \alpha_1 = \pm 45, \\ (3) & \gamma_1 = 90, & \alpha_1 = \beta_1 = \pm 45; \end{array}\right\} \quad \ldots \ldots \ldots \quad (d.)$$

and if T_1, T_2, and T_3 be the values of T corresponding respectively to these systems of values of α_1, β_1, and γ_1, we have

$$T_1^2 = \tfrac{1}{4}(B_1 - C_1)^2, \quad T_2^2 = \tfrac{1}{4}(A_1 - C_1)^2, \quad T_3^2 = \tfrac{1}{4}(A_1 - B_1)^2.$$

If A_1, B_1, C_1 be taken, as they always may be, in order of magnitude, T_2 will manifestly be the greatest of these values of T. It is in fact, as shown in the memoir referred to, the only value which satisfies all the conditions of a maximum. The corresponding values of α_1, β_1, and γ_1, which determine the corresponding position of the plane s, are those given by the second system of ($d.$). Now β_1 is the angle between the normal to s and the axis of y; and since it is here $=90^\circ$, the normal to s must lie in the plane of xz, and the plane s itself must pass through the axis of y. Moreover, since the corresponding values of α_1 and γ_1 are each $\pm 45^\circ$, this plane may have two positions, in both of which it bisects the angle between the coordinate planes of xy and yz, these positions being on opposite sides of the plane of yz. These considerations, however, only determine the position of the plane in which the maximum tangential force (T_2) acts; they do not determine the linear direction of the force in that plane. It is easily shown that it is perpendicular to the axis of y*. Since we have here $D=0$, $E=0$, and $F=0$ (art. 15), we have from equations ($a.$),

$$p \cos \lambda = A_1 \cos \alpha_1,$$
$$p \cos \mu = B_1 \cos \beta_1,$$
$$p \cos \nu = C_1 \cos \gamma_1,$$

p being the whole resultant force on the small plane s, referred to a unit of surface, and

* No proof of this is given in my memoir above referred to in the Cambridge Transactions.

λ, μ, ν being the angles which its direction makes with our present coordinate axes of x, y, z respectively. Hence

$$p^2 = A_1^2\cos^2\alpha_1 + B_1^2\cos^2\beta_1 + C_1\cos^2\gamma_1,$$

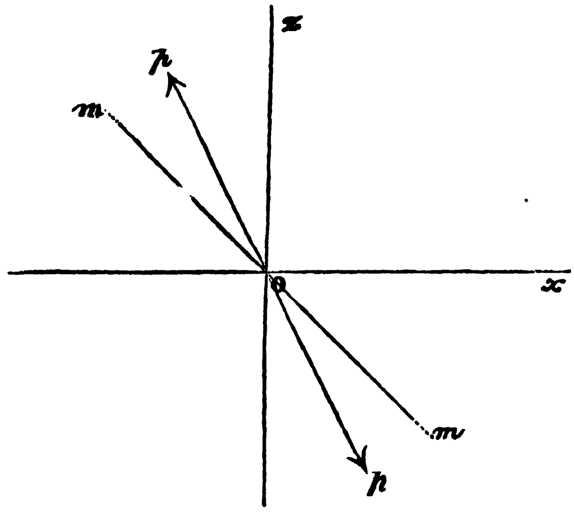
$$\cos\lambda = \frac{A_1\cos\alpha_1}{\sqrt{A_1^2\cos^2\alpha_1 + B_1^2\cos^2\beta_1 + C_1^2\cos^2\gamma_1}},$$

$$\cos\mu = \frac{B_1\cos\beta_1}{\sqrt{A_1^2\cos^2\alpha_1 + B_1^2\cos^2\beta_1 + C_1^2\cos^2\gamma_1}},$$

$$\cos\nu = \frac{C_1\cos\gamma_1}{\sqrt{A_1^2\cos^2\alpha_1 + B_1^2\cos^2\beta_1 + C_1^2\cos^2\gamma_1}}.$$

But $\beta_1 = 90°$ in the case before us, and therefore μ must $= 90°$, *i. e.* the direction of the *whole resultant force* (p) on the small plane s must be perpendicular to the axis of y, and must lie in the plane of xz. Thus, the plane of the paper representing that of xz (fig. 1), the direction Op in that plane may represent the direction of p; and if Om bisect the angle between the coordinate axes x and z, that line will be the trace of a plane on xz coinciding with the plane of s, and therefore perpendicular to that of xz. Consequently, if we resolve the whole force p normally and tangentially with reference to the plane s, the tangential part will evidently coincide with Om. But, from the particular values of α_1, β_1, and γ_1, this tangential force must necessarily be the *maximum* tangential force T_2. Consequently, if we call the axis of y the *axis of mean principal tension or pressure*, the direction of the maximum tangential force (T_2) will be perpendicular to this mean axis, and will bisect the angle between the other two axes of principal tension.

Fig. 1.



19. Hence, in our first problem, the equations (2.) and (c.) determine α, β, γ, and p. The cubic for finding p, which is deduced from them, shows that there are three values of that quantity, the three principal tensions, whose directions are defined by corresponding values of α, β, and γ, and which are at right angles to each other. These quantities being known, the value of T_2, the maximum tangential force at the proposed point, and the direction in which it acts, are immediately determinable from the results of the second part of the problem as given above. To do this we must first determine the three systems of values of α, β, and γ which have been denoted by α_1, β_1, and γ_1, and which determine the positions of the axes of principal tension; and also the values of the three principal tensions which have been above denoted by A_1, B_1, and C_1. For the greater simplicity we then take these axes of principal tension for those of x, y, and z. If A_1, B_1, and C_1 be in order of algebraical magnitude, the axis of y will be the mean axis. If A_1 be a pressure and therefore negative, the proper order will be B_1, C_1, $-A_1$,

and the axis of z will become the mean axis. Other cases must be treated in a similar manner, always preserving the order of algebraical magnitudes for determining the mean axis. The line of greatest tangential action is always perpendicular to it, and bisects the angle between the other two axes of maximum and minimum tension. In determining the magnitude T_2, we must take the same precaution, in arranging the principal pressures in their proper order of magnitude, to determine which are algebraically the greatest and least. Thus in the above case, where the order is B_1, C_1, $-A_1$, we have $T_2=\frac{1}{2}(B_1+A_1)$. We may remark that the sign of T_2 is of no importance in any application we are contemplating of these formulæ.

20. *Solution to a First Approximation.*—The complete solution of the preceding equations cannot be generally obtained. For their practical application they must be solved by approximation, when the approximate solution may be sufficient. The most important case is one in which the problem can be completely solved in consequence of its simplification arising from the particular conditions assumed. The case is that in which we suppose no forces to act at any point parallel to one of the coordinate axes, as that of z. In such case $C=0$. Also we assume the absence of any couple tending to twist a proposed element about the axis of x, or that of y, *i. e.* $D=0$, and $E=0$. Hence the equations (*c.*) (art. 16) become

$$\left.\begin{aligned}(A-p)\cos\alpha+F\cos\beta&=0,\\ F\cos\alpha+(B-p)\cos\beta&=0,\\ (\quad-p)\cos\gamma&=0;\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(c'.)$$

and the cubic for the determination of p becomes

$$+\{(A-p)(B-p)F^2\}p=0.$$

This last equation gives for the value of p,

$$p=\tfrac{1}{2}\{A+B\pm\sqrt{(A-B)^2+4F^2}\},$$
$$p=0.$$

Or putting $(A-B)^2+4F^2=M^2$,

$$\left.\begin{aligned}p_1&=\tfrac{1}{2}\{A+B+M\},\\ p_2&=\tfrac{1}{2}\{A+B-M\},\\ p_3&=0.\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(d'.)$$

For the values p_1 and p_2 of p, the third of the above equations ($c'.$) gives $\cos\gamma=0$; and the equation

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

gives

$$\cos\beta=\sin\alpha;$$

and eliminating p and β from the two first of equations ($c'.$), we obtain

$$\tan 2\alpha=\frac{2F}{A-B}.\quad\ldots\ldots\ldots\quad(e'.)$$

This corresponds to two values of α differing by 90°. These two values of α, and $\gamma=90°$ (since $\cos\gamma=0$), determine the two positions of the axes of the principal tensions p_1 and p_2. They both lie in the plane of xy.

Taking the third value of $p=p_3=0$, the equations (c'.) are also satisfied by

$$\cos\alpha=0, \quad \cos\beta=0;$$

by which the equation

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

is reduced to

$$\cos\gamma=\pm1.$$

These values of α, β, and γ correspond to the axis of z, showing that axis along which the pressure $=0$ to be the third axis of principal tension.

21. In determining the magnitude and direction of T_2, the greatest tangential action, we must bear in mind the remarks in art. 19. The quantities denoted in the preceding formulæ by p_1, p_2, p_3 are the same as those denoted in art. 18 by A_1, B_1, C_1. The former are here retained for greater distinctness. If $A+B$ be $>M$, p_2 will be positive, and the order of the principal tensions will be p_1, p_2, p_3. The axis of y will then be the mean axis, and the direction of the maximum tangential action, T_2, will bisect the angle between the other principal axes of x and z. Also we shall have

$$\left.\begin{aligned} T_2&=\tfrac{1}{2}(p_1-p_3) \\ &=\tfrac{1}{2}\{A+B+M\}.\end{aligned}\right\} \qquad (f'.)$$

If, on the contrary, $A+B$ be $<M$, p_2 will be negative, and the order of the principal tensions will be p_1, p_3, p_2. The direction of T_2 will be perpendicular to the axis of z, bisecting the angle between the principal axes of x and y. Also we shall have

$$\left.\begin{aligned} T_2&=\tfrac{1}{2}(p_1-p_2) \\ &=M.\end{aligned}\right\} \qquad (g'.)$$

These two cases hold when

$$M < \text{or} > A+B$$

respectively, or

$$\left.\begin{aligned} (A-B)^2+4F^2 &< \text{or} > (A+B)^2, \\ F^2 &< \text{or} > AB;\end{aligned}\right\} \qquad (h'.)$$

or the second case may hold when A and B are both tensions, or both pressures, provided one of them be small; and it must necessarily hold when one is a pressure and the other a tension.

22. *Solution of the General Equations to a Second Approximation.*—In proceeding to second approximation I include C and E, but regard their magnitudes as small. These magnitudes must be expressed by the ratios which they bear to some standard force. The greatest value which the tangential force F can attain in any glacier must be limited by the *tangential cohesive power* of glacial ice; for if F exceeded this latter force, dislocation, by a tangential sliding of one element past another, would instantly ensue. Let this tangential cohesion be measured by F_1; then, when it is said that C and E are small,

it is meant that the ratios $\frac{C}{F_1}$ and $\frac{E}{F_1}$ are small ratios. This is equivalent to the assuming that the force on any element parallel to the axis of z is small, and that the couple with reference to an axis parallel to that of y, is also small. The condition $D=0$ signifies, as in the first approximation, the absence of a couple on every element, with reference to an axis parallel to that of x.

Hence putting $D=0$ in our general cubic, we have

$$(A-p)(B-p)(C-p)-E^2(B-p)-F^2(C-p)=0,$$

or

$$\{(A-p)(B-p)-F^2\}(C-p)=E^2(B-p).$$

Putting $C=0$ and $E=0$, we have, as before, for the three first approximate values of p,

$$\left.\begin{aligned} p_1&=\tfrac{1}{2}\{A+B+\sqrt{(A-B)^2+4F^2}\},\\ p_2&=\tfrac{1}{2}\{A+B-\sqrt{(A-B)^2+4F^2}\},\\ p_3&=C.\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (d''.)$$

To proceed to a second approximation, put

$$p=p_1+\varpi_1$$

in the cubic, and we obtain

$$\{(\overline{A-p_1}-\varpi_1)(\overline{B-p_1}-\varpi_1)-F^2\}(\overline{C-p_1}-\varpi_1)=(B-p_1-\varpi_1)E^2,$$

or

$$\{(A-p_1)(B-p_1)-F^2-(A-p_1+B-p_1)\varpi_1+\varpi_1^2\}(C-p_1-\varpi_1)=(B-p_1-\varpi_1)E^2.$$

The first approximation gives

$$(A-p_1)(B-p_1)-F^2=0;$$

and the preceding equation shows that ϖ_1 must be of the order E^2. We may therefore neglect terms in $\varpi_1 E^2$, and ϖ_1^2, and we then obtain

$$-(A+B-2p_1)(C-p_1)\varpi_1=(B-p_1)E^2,$$

and

$$\varpi_1=-\frac{p_1-B}{A+B-2p_1}\,\frac{E^2}{p_1-C};$$

or, since C is small,

$$\varpi_1=\frac{p_1-B}{2p_1-(A+B)}\cdot\frac{E^2}{p_1^2}\left(1+\frac{C}{p_1}\right)p_1.$$

ϖ_2 and ϖ_3 may be found in the same manner.

23. Again, equations (*c.*), art. 16, become, if $D=0$,

$$\left.\begin{aligned} (A-p)\cos\alpha+F\cos\beta+E\cos\gamma&=0,\\ F\cos\alpha+(B-p)\cos\beta&=0,\\ E\cos\alpha+(C-p)\cos\gamma&=0,\end{aligned}\right\} \quad \ldots \ldots \ldots$$

The two last equations give

$$\cos\alpha=-\frac{C-p}{E}\cos\gamma,$$

$$\cos\beta=\frac{F}{B-p}\,\frac{C-p}{E}\cos\gamma.$$

If we were to substitute these expressions in the first equation, we should obtain the cubic in p used in the immediately preceding articles for the approximate determination of its value. Substituting the values of cos α and cos β in the equation

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1,$$

we obtain

$$\left\{\left(\frac{p-C}{E}\right)^2+\left(\frac{F}{p-B}\right)^2\left(\frac{p-C}{E}\right)^2+1\right\}\cos^2\gamma=1,$$

or

$$\left\{(p-C)^2\left(1+\frac{F}{(p-B)^2}\right)+E^2\right\}\cos^2\gamma=E^2.$$

The retention of E^2 in the coefficient of $\cos^2\gamma$ would only produce a term in the expression for $\cos^2\gamma$ of the order E^4, and may therefore be neglected. Also, since p only differs from its first approximate value, p_1, by a small quantity of the order E^2, we may, for the reason just assigned, write p_1 for p in the last equation. Thus we have

$$\cos\gamma=\pm\frac{E}{(p_1-C)\sqrt{1+\left(\frac{F}{p_1-B}\right)^2}},$$

and by substitution,

$$\cos\beta=\frac{F}{p_1-B}\cdot\frac{1}{\sqrt{1+\left(\frac{F}{p_1-B}\right)^2}},$$

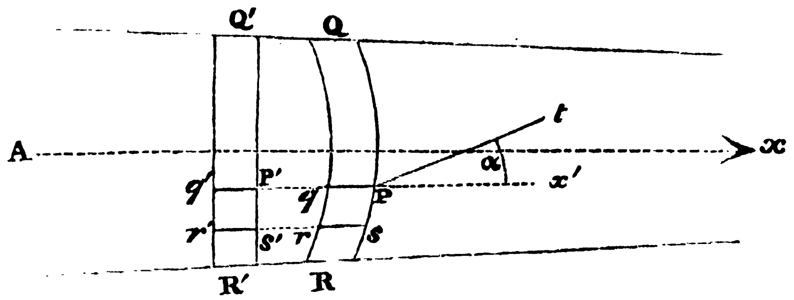
$$\cos\alpha=\frac{1}{\sqrt{1+\left(\frac{F}{p_1-B}\right)^2}}.$$

These equations show that γ is changed from a right angle to one whose difference from a right angle is of the order E, and therefore small; while cos α and cos β are changed by small quantities of the same order as the difference between p and p_1, *i. e.* of the order E^2.

24. *Nature of the Forces* A, B, C, D, E, *and* F *in the ordinary cases of Glaciers.*—In the preceding investigations we have considered the forces A, B, C, D, E and F as acting on any element of a solid body. In the case of a glacier, this body assumes a specific form, and it becomes necessary to explain what forces are represented by the above symbols in this particular and restricted case. The phenomena with which we shall be here concerned, have been observed almost entirely in those regions of glaciers in which most large ones, like those of the Alps, become much elongated in consequence of the narrowness of the valleys down which they descend. The sides of these valleys frequently approximate to parallelism with each other. The primary general characteristics of the motion of glaciers of this kind are (1) the motion is unaccelerated, (2) the axial portions move with a greater velocity than the marginal portions, and (3) the superficial portions move somewhat faster than the lower portions of the mass. These points are clearly established by observation, independently of any particular theory. In the following articles of this section their truth is assumed.

25. Let fig. 2 represent the section of a glacier by a plane parallel to its surface, and fig. 3 a vertical section through A x the axis of the glacier. Let A x be taken as the

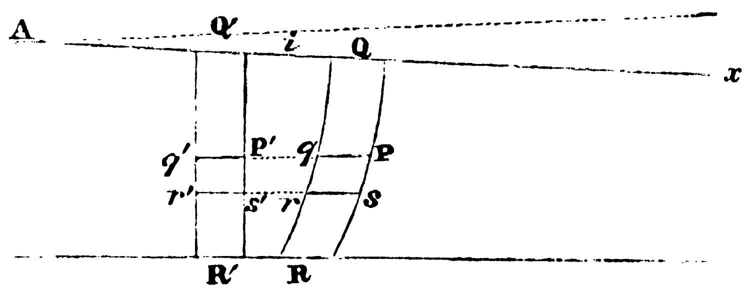
Fig. 2.



Section parallel to the surface.

axis of x, and the surface of the glacier (nearly horizontal) as the plane of xy, a transverse line perpendicular to A x being the axis of y, and a perpendicular to the surface of the glacier the axis of z. Let Q′ R′ (fig. 2) represent a section, by a plane parallel to that of xy, of an element of the mass lying between two planes perpendicular to the axis of x; Q′ R′, by the more rapid motion of the axial parts of the mass, will be brought into the position Q R. Also if Q′ R′ (fig. 3) represent a section of the same element by a vertical plane parallel to that of xz, Q′ R′ will be brought into the position Q R by the more rapid motion of the upper surface of the mass. Also the small elementary parallelopiped whose section parallel to (xy) is represented by P′ q' r' s' in fig. 2, will be brought into the position P q r s, while the section P′$q'r's'$ (fig. 3) of the same element made by a

Fig. 3.



Vertical section.

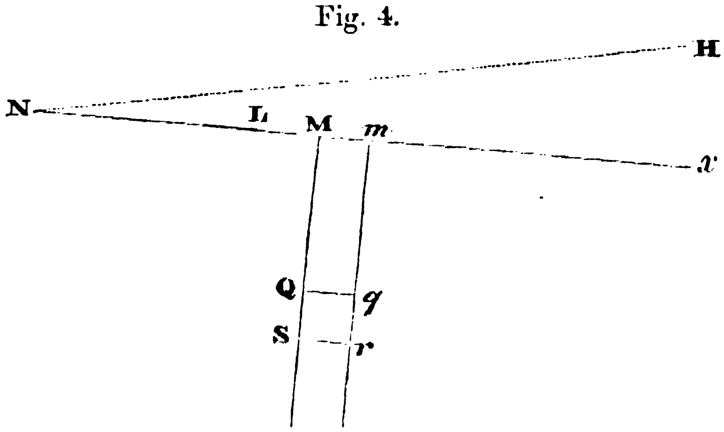
plane parallel to (xz) will be brought to the position P q r s. Thus we see that there will be an *angular distortion* of P q r s about an axis parallel to z, as represented in fig. 2, and a similar distortion about an axis parallel to y, as represented in fig. 3. These angular distortions are respectively due to the couples whose intensities are represented by F and E (art. 15). Also it is easily seen that there will be no twisting or angular distortion of P q r s about an axis parallel to that of x, and that consequently D will $=0$ in all such glaciers as we are now considering. A becomes a *longitudinal* tension, and B a *transversal* one. C will be the pressure on the element parallel to z, and due to the superincumbent weight, and the inclination of the surface of the glacier to the horizon.

26. The intensity of A and B will depend much on the form of the glacial valley. If the sides be parallel, A may be a tension or pressure according to the local variations in

the form of the bed of the valley; B will be small. If the sides be convergent in descending the valley, A will almost necessarily be a pressure, on account of the resistance which the sides will oppose, by their convergency, to the onward progress of the glacier. B will become a great pressure, greater than A, to which it will bear somewhat the same relation as the pressure on the side of a wedge bears to that applied to its back. If the valley be divergent, and if when it becomes so, its inclination is very much diminished (as at the lower end of the Rhone glacier) A may become an enormous pressure, while B may become a tension on account of the lateral expansion which will be given to the mass by the great pressure *à tergo*. C will be best considered in conjunction with E. D, as above stated, will always $=0$. F will have different values for different points in the same vertical transverse section of the glacier. It will manifestly be greatest in the marginal portions, where the angular distortion represented in fig. 2 is greatest; in the central portions, the motion of the glacier will produce very little of this angular distortion about an axis parallel to that of z, and F will be proportionally small.

27. The forces C and E are more dependent on each other than A, B, and F. I proceed to investigate expressions for them. For this purpose let fig. 4 represent a vertical section of the glacier parallel to (xz) and not too remote from its axis; then will F, as above stated, be very small, and may be neglected. Also $D=0$, and B acts perpendicular to the plane (xz), and will therefore not affect the relations between the forces acting parallel to that plane.

Fig. 4.



Let x, y, z be the coordinates of the point Q. We shall have $NM=x$, the distance of the plane NMQ from that of $xz=y$, and $MQ=z$. We might obtain the results required by considering the conditions of equilibrium of the element $\delta x \delta y \delta z$, represented by QqrS; but it will be more convenient to take an element represented by MQqm, of which the volume will be $z\delta x\delta y$. I suppose here the existence of a longitudinal pressure parallel to the axis of x, and represented in our general formulæ by A. If we draw a plane through MQ perpendicular to the axis of x, A may represent the intensity of the longitudinal pressure at any point on that plane, referred to a unit of surface. For different points in this plane x will be constant, and the pressure may vary generally with y and z; but it will answer our immediate purpose, and much simplify our problem, if we suppose A constant for every point in MQ, or independent of y and z. It will then vary only in passing from any plane MQ to a consecutive and parallel plane, *i. e.* A will be a function x alone. Hence we shall have

Pressure on one of the sides of the element Mq perpendicular to the plane of $yz = A . z\delta y$,

Pressure on the opposite side $= -\left(A+\frac{dA}{dx}\delta x\right)z\delta y$.

The algebraical sum of these pressures, estimated in the positive direction of x,

$$=-\frac{dA}{dx}z\delta x\delta y.$$

Again, if W be the weight of a unit of volume of ice,

$$\text{Weight of the element } Mq=W.z\delta x\delta y,$$

and its resolved part parallel to the axis of x

$$=Wz\delta x\delta y\sin\iota.$$

Also we shall have the tangential force on the base Qq,

$$=-E\delta x\delta y,$$

parallel to the axis of x.

Hence we must have for a condition of equilibrium of the element Mq

$$-\frac{dA}{dx}z+Wz\sin\iota-E=0,$$

and

$$E=\left(-\frac{dA}{dx}+W\sin\iota\right)z.$$

It would seem probable that A will generally vary slowly with y; it may vary more rapidly with z, especially in cases where it becomes large, as at the bottom of an ice-fall. In such case we shall have

$$A=\int_0^z\frac{dA}{d\zeta}d\zeta,$$

where ζ is taken parallel to z. In the position just mentioned (the foot of an ice-fall) the variation with regard to x may possibly be rapid, and therefore $\frac{dA}{dx}$ very considerable. Under the ordinary conditions of a glacier, away from any rapid fall, the variations of A must generally be slow, and the values of $\frac{dA}{dx}$ therefore comparatively small.

C is manifestly due to the resolved part of the pressure of Mq on the surface Qq referred to a unit of surface. The normal pressure on $Qq=W.z\delta x\delta y\cos\iota$. Whence

$$C=Wz\cos\iota.$$

This value of C shows that it must always be small when z is so. Such will also be the case with E, unless $\frac{dA}{dx}$ be very large, which is probably true only at the foot of an ice-fall. Generally, then, we see that C and E will be small for all those depths which lie within the sphere of our observation; and that for all such depths the first approximate solutions of our general equations are sufficiently accurate. The second approximate solutions give the results for greater depths, and indicate the nature of the results for those still greater depths at which C and E might be too large to render the results of the second approximation applicable with sufficient exactness.

We may now distinctly understand the interpretation of the first approximate solutions of our general equations, as applied to the case of an actual glacier. In those solutions it has been assumed that C, D, and E are so small as to be neglected. D always $=0$, and it follows from the preceding expressions for C and E that, generally, they are relatively small for all those more superficial portions of a glacier to which our observations can extend. Hence, for the same portions, our first solutions will be approximately true and practically applicable.

SECTION IV.—*On the manner in which Dislocations in the Mass of a Glacier, or in its Structure, may be produced; and on the resulting Phenomena.*

28. It will be recollected that the internal pressures and tensions which have been investigated in the preceding pages, are those which would exist in a continuous solid mass acted on by certain external forces, previous to the dislocation which must result from such forces if the intensity of the internal tensions should be sufficient to overcome the cohesion of the mass, or the pressures to overcome its resisting-power. We may, however, carry our geometrical and mechanical analysis of the problem somewhat further, and consider *how* the dislocation will take place when the forces are sufficient to produce it. It has already been remarked (art. 2) that there are three ways in which this may occur. In the first place, the cohesion may give way to the greatest normal tension, p_1; open fissures will then be formed. Again, when the maximum compression (p_2) becomes very great, it may be easily conceived how the primitive structure may break down, as it were, especially if the mass be of a crystalline structure like ice. The third kind of dislocation is that produced by the tangential action between two contiguous elements, which obviously tends to make one element slide past the other, and thus to produce what Principal FORBES has called a "differential motion." I shall consider successively these different kinds of dislocation, and the phenomena which respectively result from them, with reference, in the first place, to the more superficial portions of the glacier, in which our first approximate formulæ are applicable; and secondly, the possible formation of similar phenomena in the deeper parts of the glacial mass.

From the equations (d'.), art. 20, we have

$$p_1=\tfrac{1}{2}\{A+B+\sqrt{(A-B)^2+4F^2}\},$$

$$p_2=\tfrac{1}{2}\{A+B-\sqrt{(A-B)^2+4F^2}\},$$

$$\tan 2\alpha=\frac{2F}{A-B}.$$

The last equation gives two values (α_1 and α_2) of α, which determine the directions of p_1 and p_2 with respect to the axis of the glacier. A, B, C, and F, in the application of the formulæ to an actual canal-shaped glacier, are such as described in arts. 25 and 26. We have also the result that the lines of maximum tangential action at any point bisect the right angles between the directions of maximum tension and maximum pressure.

Of the values α_1 and α_2 of α, one will be greater and the other less than 90°; and we must determine, in each problem, which gives the maximum and which the minimum pressure p_1 and p_2. It may be convenient to consider F an absolute positive quantity; and we may suppose, for such a glacier as that represented in fig. 2, that $B=0$. In such case the preceding formula shows that one of the values (as α_1) of α must be small and positive when F is small, *i. e.* for any point P (fig. 2) near the axis A*x*. Now the maximum tension at P must be due to the tension A parallel to A*x*, and that produced by the angular distortion of the element P*qrs*. The magnitude and direction of this latter tension are obtained by putting $A=0$ and $B=0$ in the above formulæ. This gives the greatest tension $=F$, the least $=-F$, and $\tan 2\alpha=\infty$, or therefore $\alpha=45°$ or 135°. From the inspection of the element, it is manifest that the first of these values of α corresponds to the greatest tension produced by the angular distortion. It is from this tension and A that p_1 in the actual case of a glacier before us, must result. Consequently p_1 must act in some such direction as P*t* (fig. 2), where α in that figure is acute. The same result will hold if B be of finite magnitude, and algebraically less than A; and thus α_1 and α_2 are distinguished from each other in the case before us, and by similar reasoning may be distinguished in any other case.

29. *Formation of Transverse Crevasses.*—When the maximum normal tension is the force to which the cohesive power of a glacial mass first gives way, the result, as above observed, must be an open fissure, or crevasse, the direction of which must manifestly be perpendicular to that of the tension producing it. These crevasses approximate more or less to right angles with the glacial axis, and usually characterize canal-shaped valleys in which the sides are approximately parallel. In such cases B must be comparatively small; if the valley be slightly convergent, it will be a small pressure, and therefore negative. When these crevasses exist more abundantly, A will doubtless be a large tension, though not necessarily so, as we shall see, for the production of a crevasse.

(1) Taking the simplest case, let us first suppose $A=0$ and $B=0$. This may be very approximately true if the glacier descend without acceleration or retardation along a trough-like valley of uniform width and uniform inclination. We shall then have by the above equations, $p_1=F$, $p_2=-F$, and $\tan 2\alpha=\infty$. Hence the direction of maximum tension will make an angle $\alpha=45°$ with the axis of the glacier (A*x*) (fig. 2) towards which it will converge in descending. If, therefore, a fissure be formed at all, which can only be in the marginal regions where F is considerable, it must be in a direction perpendicular to P*t*, and making an angle of 45° with the axis of the glacier. We have also for the maximum tangential action (art. 21),

$$T_2=\tfrac{1}{2}(p_1-p_2)=F,$$

equal, in this case, to p_1, and making an angle of 45° with the directions of p_1 and p_2. It will therefore be parallel to the axes of x and y. Hence the maximum normal and tangential forces make equal efforts, in this case, to dislocate the mass. Let the tangential cohesive power of the mass be F_1; the greatest value which p_1 can assume will then be also F_1; and if the normal cohesive power (P_1) be less than F_1, the tension p_1 may assume a value (F) between P_1 and F_1, by which a crevasse will be formed in the position above mentioned. Transverse crevasses may therefore be formed without any of that direct longitudinal tension which might, at first sight, appear necessary to produce them.

(2) If there be a considerable longitudinal tension, but no appreciable transverse pressure, we have

$$p_1=\frac{1}{2}\{A+\sqrt{A^2+4F^2}\},$$
$$p_2=\frac{1}{2}\{A-\sqrt{A^2+4F^2}\},$$
$$\tan 2\alpha=\frac{F}{A}.$$

Hence p_1, the maximum tension, will be increased, and therefore, also, the tendency to form a crevasse. Likewise the greatest value of α will be less than 45°, and the direction of the crevasse, as we might expect, will be more nearly perpendicular to the axis of the glacier.

(3) Many glacial valleys become narrower as we descend them, and consequently the mass of the glacier may enter each part of the valley as a wedge, and may frequently become more or less compressed. In such case B will be negative, and may become very large. We shall then have

$$p_1=\frac{1}{2}\{A-B+\sqrt{(A+B)^2+4F^2}\},$$
$$p_2=\frac{1}{2}\{A-B-\sqrt{(A+B)^2+4F^2}\},$$
$$\tan 2\alpha=\frac{F}{A+B}.$$

In this instance, as well as in the preceding one, p_1 will necessarily be a tension, and greatest (A and B being constant) where F is greatest, *i. e.* in the marginal portions of the glacier. For the like reason, α will also be greatest in those portions. It will vanish at the axis, where F vanishes. Hence, if the forces be sufficient to overcome the cohesion of the mass, a curvilinear crevasse may be formed extending across the glacier and meeting its axis at a right angle. This, however, is rarely the case, the transverse crevasses being formed, in general, in the lateral portions only of canal-shaped glaciers, where they will approximate more or less to straight lines. They are most likely to be formed where A is a considerable tension, which is less likely to be the case in converging valleys.

It is important to observe that in all cases in which the expression for $\tan 2\alpha$ is positive, *i. e.* where B is algebraically less than A, α must lie between 0 and 45°, and consequently *the inclination of a curved crevasse to a transverse line perpendicular to the axis must likewise always lie between the same limits.* This rule is applicable, according to this theory, wherever transverse crevasses are likely to be formed*.

* An open curvilinear fissure in its *progressive* formation would be in some degree influenced by other causes than the maximum tension at each point through which it might pass. Moreover, the cohesive power has been above supposed to be the same in every direction from any proposed point. There *may*, on the contrary, be planes of *less cohesion*, in which case if the cohesion along any such plane bear a smaller ratio to the internal tension perpendicular to it, than the cohesion perpendicular to the maximum tension bears to p_1, the crevasse may be formed along the plane of least cohesion. I know no reason, however, to suppose that these causes are sufficient to modify in any essential degree the law enunciated in the text.

30. *Formation of Longitudinal Crevasses.*—Transverse crevasses, as above stated, are almost invariably formed in valleys the sides of which are approximately parallel; longitudinal crevasses are as generally formed in valleys in which the sides are divergent, and usually perhaps when they become rather suddenly so. They are found almost exclusively at the lower ends of glaciers. The Rhone glacier affords the best-known example of crevasses of this kind, but M. AGASSIZ refers to a number of other cases, the greater part of which, however, appertain to small glaciers*. At the foot of the great ice-fall of the Rhone glacier, the valley expands largely, as is well known, and its inclination to the horizon becomes comparatively small. Thus the ice, accumulating at the bottom of the fall, exerts an enormous pressure *à tergo* on the ice immediately before it, and this pressure is propagated onwards in directions which radiate from the bottom of the fall. It is manifest that the force along each radiating line will be a great pressure. Again, if we conceive the whole mass divided into concentric rings perpendicular at each point to the above-mentioned radiating lines, the pressure along these lines will extend the ring, and produce in it a great tension at every point, perpendicular to the direction of the radial pressure above mentioned. Hence if we take any one of these radiating lines as the axis of x, the radial pressure at any point upon it will be denoted by $-$A, and the tension upon it in the direction perpendicular to the axis of x, will be B. The former will be a *principal pressure*, and the latter a *principal tension*; and if crevasses be formed at all, they must be in directions perpendicular to that of B; *i. e.* they must be radial, as they are always observed to be. The glacier of the Rhone is only a type, as regards longitudinal crevasses, of all other glaciers in which they exist.

31. An explanation of the phenomena of transverse crevasses, essentially the same as that above given, though founded on a more restricted mechanical and mathematical analysis of the general problem, was given by me some seventeen years ago. I am not aware that any doubt was entertained as to its validity, but I am also not aware of any glacialist having recognized it previously to Dr. TYNDALL. M. AGASSIZ has given an explanation† both of the transverse and longitudinal crevasses, involving apparently the notion of the oblique tension which, I have proved, must necessarily exist in a determinate direction. The mechanical reasoning employed, however, is too vague to constitute a mechanical explanation of the phenomena. Moreover his work was published in 1847, three or four years after my memoir containing the preceding explanation was printed in the Transactions of the Cambridge Philosophical Society. Principal FORBES, also, speaks of a *drag* from the sides towards the centre of a glacier‡, but with the view apparently, not of explaining the formation of transverse crevasses, but of the veins in cases of the veined structure. His "lines of greatest strain" must therefore, I conceive, mean the same lines mechanically, not as my lines of greatest *normal* tension, to which the crevasses are unquestionably due, but my lines of greatest *tangential* action. If he supposed these two directions to be identical, it was a manifest error; for

* Système Glaciaire, p. 324.

† Ibid. p. 320 *et seq.*

‡ Occasional Papers, p. 57; also last chapter of his 'Travels.'

I have proved that they must in all cases differ by an angle of 45°. I have thought it right to say thus much to vindicate my claim to having been the first to give any explanation, founded on true mechanical principles, of the phenomena in question.

32. That the theoretical law above enunciated with respect to the directions of transverse crevasses is the actual law, is determined, I conceive, beyond all doubt. It was, in fact, so distinctly recognized by M. AGASSIZ and others*, before careful observations had been made on glacial motion, that it was regarded as a proof that the sides of a glacier moved faster than its central portion. But the best and, I believe, the only accurate recorded evidence on the subject is that afforded by the admirable map published by M. AGASSIZ, of the glacier of the Aar. There the crevasses, in a particular locality, are laid down with geometrical accuracy, and the eye recognizes at once the law in question, notwithstanding the minor deviations which must necessarily result in such a case from local and irregular causes†. Principal FORBES, however, does not seem to recognize this law; for he observes‡ that he agrees with some preceding observers in believing crevasses to be mere *accidents* of glacial motion. I have, however, demonstrated, as I had done before the publication of the paper in which this opinion is expressed, that though a glacier should be affected by no irregularities of surface along its margins or bed tending to dislocate it, the internal tensions arising from the more rapid motion of its central portions would always tend to produce oblique transverse marginal fissures. Local and irregular causes may frequently counterbalance this tendency; but the cause itself which produces these fissures is no more *accidental* than the law of the motion in which it originates. It might as well be asserted that the longitudinal or radiating crevasses of the lower part of the glacier of the Rhone were merely accidental phenomena,—an assertion which few glacialists, I imagine, would venture to maintain.

33. The positions of the crevasses as above determined, will be those in which they are originally formed. It is manifest that transverse crevasses will change their angular positions with reference to the axis of the glacier, by the more rapid motion of its axial portion. They will thus become more approximately perpendicular to the axis. Moreover, as they deviate from their original positions, less force will be exerted to keep them open, and to counteract any independent or local causes tending to close them. They do thus finally disappear, and generally at points not remote from those where they were originally formed, and where local conditions probably aided their formation. A closing up of the crevasses after a certain time, presents no difficulty; but the entire

* Système Glaciaire, p. 305.

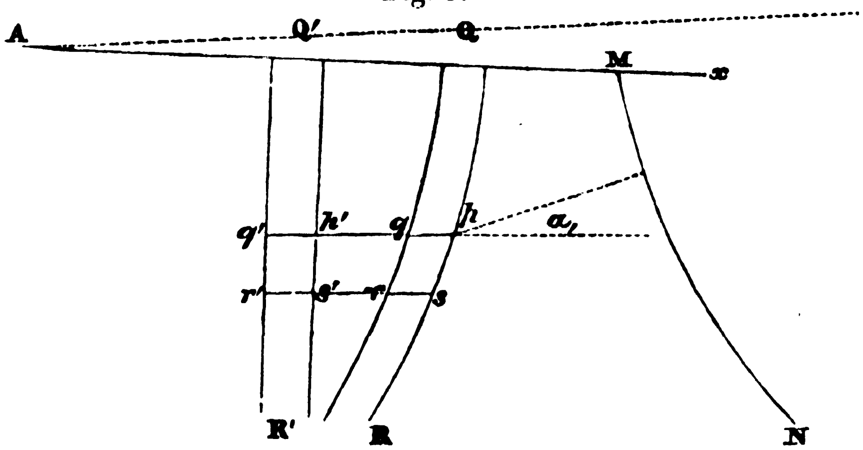
† I cannot here omit the expression of my conviction of the inadequate justice which has been rendered in this country to the 'Système Glaciaire' of M. AGASSIZ. It professes to be, as it is, principally descriptive, and contains more accurate details on many points than any other work professes to offer. The map which accompanies it, founded on the very careful trigonometrical survey of M. WILD, is full of beautifully delineated details. We possess no other glacial document of the kind at all comparable to it. I shall have other occasions to refer to it.

‡ Occasional Papers, p. 151.

disappearance of all traces of them, and the resumption of complete crystalline continuity in those parts of the mass in which these crevasses had previously existed, remained a difficulty of which no physical explanation had been given till the experiments of Dr. TYNDALL afforded so happy a solution of it by proving that the renewed continuity was attributable to regelation resulting from the renewed contact of portions of the ice which had been previously separated.

34. *Formation of Crevasses in the deeper portions of a Glacier.*—The formation of crevasses in this position must ever perhaps remain a matter of speculation; for we can have little hope of making their existence a matter of observation. The general direction of maximum tension at any point of a glacier at a great depth, cannot be exactly determined, because the general equations (*c.*), art. 16, cannot, for this case, be generally solved. It will be easy, however, to determine this direction in terms of the internal forces for any point in the vertical plane through the axis of the glacier, and, consequently, the section made by that plane with the surface to which the direction of the maximum force is a normal—the surface along which the crevasse would be formed. For this purpose let fig. 5 represent the vertical section through the axis of the glacier.

Fig. 5.



The undistorted element Q′R′ will be brought by the more rapid motion of the surface into the position QR, as described in fig. 3. The internal forces acting on the small element $pqrs$ will be A, B, C, and E, the latter producing the couple whose axis is parallel to that of y. It is manifest, however, from the conditions of symmetry, that the direction of the maximum tension must lie in the vertical plane through the axis of the glacier, and will not be affected by B, which, as in other cases of transverse fissures, must be supposed either a pressure or a comparatively small tension. Hence, putting $-$C for B and E for F in our formulæ (e'.), art. 20, we have

$$\tan 2\alpha = \frac{2E}{A+C}.$$

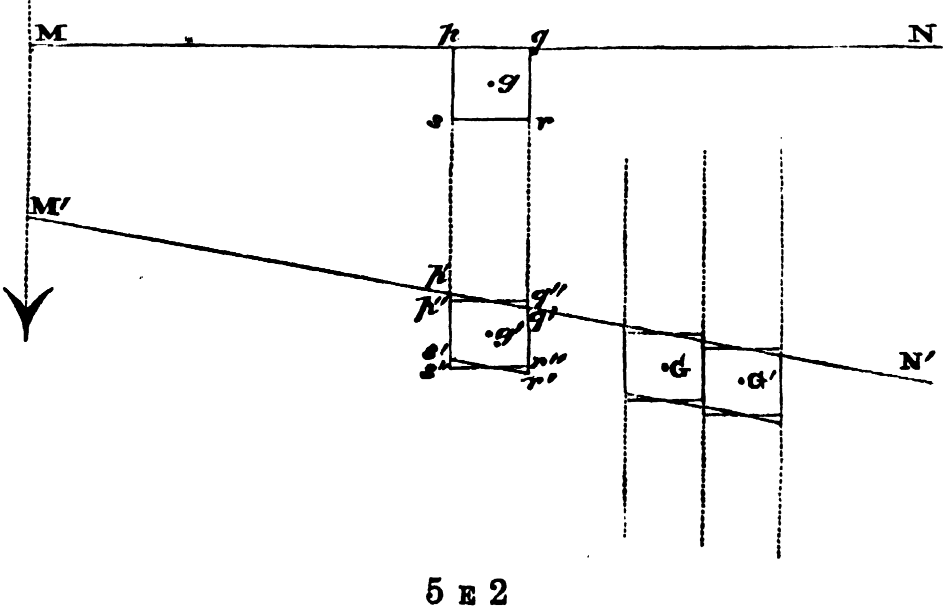
The angle α is indicated in fig. 5 (see art. 28 and fig. 2). Near the surface, E and therefore α, vanish, and the crevasse, if formed, will be vertical. At lower points of the mass α will increase with E, and the vertical section of the crevasse will resemble the curve M N in the figure.

The maximum tension in the marginal portions of the glacier will be increased by F; and it is in those portions, near the lower as well as near the upper surface of the mass, that crevasses will be most likely to be formed. The forces tending to form them may probably be much the same in both cases; but the tendency of the incipient fissures to open into wide crevasses must almost necessarily be much counteracted near the lower surface by the action of the bottom of the valley on that surface. The sides of a fissure will not there be able to separate from each other with the same facility as at the free surface of the mass. In fact any incipient fissure formed within the glacier and not extending to its external surface, will have much less of this facility of subsequently opening, than those which are formed nearer to the outer surface and directly communicate with it. I therefore conceive it to be very improbable that crevasses exist in the deeper portions of a glacier, equal in number and magnitude to those which are seen in its more superficial portions.

35. *Effects of the Crushing of the Mass.*—It may frequently happen that the maximum pressure (p_2) at any proposed point of a glacial mass shall be much greater than the maximum tension p_1, or the maximum tangential force T_2 (art. 18) at that point; and the dislocation may be produced by the crushing power of p_2. The element may thus be broken into a greater or smaller number of fragments, and the constraint of the mass removed. Its onward motion will then be continued, and its continuity restored by regelation, till a repetition of the process becomes necessary to overcome a new state of constraint. It is in this manner that the property of regelation enables us so beautifully to account for the molecular mobility of a glacial mass, and the consequent freedom of its central to move more rapidly than its marginal portions, without any reference whatever to the property of viscosity or plasticity.

36. *Dislocation produced by the Tangential Force.*—The third way in which the mass may be dislocated is, that the tangential cohesion may yield to the tangential force on any proposed element, as explained in article 28. For the greater simplicity of explanation, I shall suppose the sides of the valley parallel, and $A=0$, and $B=0$. Also we take $C=0$ and $E=0$ as in our first approximate solutions. Let $p\,q\,r\,s$ (fig. 6) be the

Fig. 6.



section of a small elementary rectangular parallelopiped (δx, δy, δz) made by the plane of the paper (that of xy); and let $p\,s$ be in the direction ($p\,p'$) of the motion of p; it will be parallel to x, and the transverse line M$p\,q$N, perpendicular to $p\,p'$, will be parallel to y. Now the physical line M N will, in a short time, come into the position M′ N′, by the assumed motion of the glacier, and the greater velocity of its axial portion. The element $p\,q\,r\,s$ will then come into the position $p'\,q'\,r'\,s'$, and will be angularly distorted. It will be acted on by the tangential force (F), forming two equal and opposite couples, with a common axis parallel to z (art. 29, (1)), and tending to destroy the continuity on every side of the element, by overcoming the tangential cohesion. The force F will be different for different angular positions of the element, and will always be greatest when the sides of the element bisect the two right angles between p_1 and p_2, the greatest tension and the greatest pressure (art. 18). In the present case the directions of p_1 and p_2 will each be inclined at an angle of 45° to the axes of x and y (art. 29, (1)); and therefore the intensity of the forces (F) of the two couples will be greatest when they act respectively parallel to x and y, i. e. when the element is in the angular position represented in the figure.

Now if tangential dislocation take place, it must necessarily do so in those directions in which the tendency of the forces F to produce it is greatest, or, in the case before us, in directions parallel and perpendicular to the axis of the glacier, the tendencies being the same in both those directions. Let us suppose, then, a complete tangential rupture to take place, and simultaneously, on each side of the element $p\,q\,r\,s$, as well as in the surrounding elements. Each element, represented by $p'\,q'\,r'\,s'$, will then regain its original rectangular form by its *elasticity*, but so moving that its centre of gravity (g') shall remain at rest, since the elastic force of restitution acts entirely within the element. Thus $p\,q\,r\,s$ will return to its original rectangular form $p''q''r''s''$; and if we take two consecutive elements whose centres of gravity are G and G′, they will be brought into the relative positions represented in the figure, in which G′ is slightly in advance of G, as much, in fact, as is necessitated by the more rapid motion of the central parts of the glacier. After the rupture, if we suppose the continuity to be instantaneously restored (as it will be by regelation according to our theory), the glacier will again be brought as a continuous mass into a position of no constraint by its general motion. By a repetition of this process, the continuity of the mass will be constantly destroyed and as constantly restored; and we thus see the *modus operandi* by which, taking any two elements situated like G and G′, the one nearest the axis of the glacier gets gradually in advance of the other, as it must do, in accordance with the general law of the glacier's motion.

37. But, it may be asked, if the dislocation takes place on the two sides of the element whose directions are transverse to the glacier, simultaneously with the rupture along the sides whose directions are longitudinal, why is it that the subsequent relative or differential motion of two contiguous elements, such as G and G′, should not be transversal as well as longitudinal? The reason is obvious. The motion which instan-

taneously takes place after the tangential dislocation is a motion of rotation *about* G, the centre of gravity of the element, by which that point is not affected; whereas the motion with which we are here concerned, as that which alone can possibly produce any finite differential motion, is the motion of the centre of gravity (G) *itself*, arising from the general onward progress of the whole mass. Any real differential motion between two particles must, in all cases (whatever, in fact, may be the directions of tangential dislocation), take place in the actual direction of the motion of the particles. Thus, if the directions of tangential dislocation be not parallel and perpendicular to the axis of the glacier, as in the above case, the true differential motion of two particles must still take place in the actual direction of their motion.

38. In the above case we have assumed the rupture to be complete and simultaneous on every side of the element, and also the absence of friction between contiguous elements. If it be otherwise, as it doubtless will be, the same reasoning will be applicable; but the relief of the constraint of the mass, or of any element, at each dislocation will be only partial, and the consequent differential motion will be somewhat less. In such case the facility with which the central portions of the mass move faster than the other parts will be diminished.

39. I am not here supposing that this tangential dislocation is the most probable mode by which the constraint of the glacial mass is commonly relieved under great pressure. The crushing of its elementary portions (art. 35) would appear, perhaps, a more likely *modus operandi*; but possibly both these processes may contribute to the actual dislocations (without finite fissures) by which the constraint of the mass is instantaneously relieved. If the dislocations were entirely tangential, it is easily seen that two contiguous elements, like G and G′ (fig. 6), would ultimately be separated, while each should preserve its physical identity. Consequently, identical particles forming at one time the continuous transverse linear element, M N, might subsequently be converted into an elongated loop which should be discontinuous in the marginal parts of the glacier, where the differential motion would be greatest, and continuous in the central portions, where that motion would be least. If, on the contrary, each element should be *crushed* in the dislocation, it would manifestly never regain its primitive form, but would become compressed or extended *as if* it were plastic or viscous (art. 4, &c.). In such case the original transverse element M N would be converted into a continuous loop. But, at all events, whether the real *modus operandi* consist of only one of the above processes, or of a combination of both, it is of the first importance for a just appreciation of the principle of regelation, that we should see distinctly how the continuity of the glacial mass is broken and again restored by it, in contradistinction to the effect of real viscosity, which would prevent that continuity from being broken at all. In the latter case also there would be no particular structure, such as the crystalline structure, to be destroyed, and no necessity for the power of regelation to restore it. It is in the difference between the *modus operandi* when the mass is viscous, and that when

the mass is crystalline and brittle, that we see best, perhaps, the distinction between the Viscous and Regelation Theories.

SECTION V.—*On the Veined Structure of Glacial Ice.*

40. Two different theories, as is well known, have been proposed to account for the curious structure frequently observed in glacial ice, and termed the laminar, veined, or ribbon structure. The preceding investigations bear intimately on these theories. One of them, that of Principal FORBES, asserts the structure to be due to the sliding of one thin lamina of ice past another, or to their differential motion—a process by which he supposes extensive portions of the mass to be divided into slices, usually nearly vertical, and not exceeding, in many cases, the fraction of an inch in thickness. His first idea was that these parallel discontinuities were, after their formation, filled with infiltrated water, which, by being frozen, formed the veins of blue ice. He afterwards appears to have abandoned this notion; but what physical process he supposed to be substituted for infiltration I have not been able distinctly to ascertain. This, however, is not material as regards the examination which I propose to give this theory in the sequel. The other theory above alluded to is that first proposed by Dr. TYNDALL, an essential point in which is that the laminæ of ice which characterize this structure must be perpendicular at each point to the direction of greatest pressure there. So far alone my researches are related to this theory, and so far only shall we be here concerned with it. I shall not discuss the manner in which this pressure is supposed to produce the lamination in question, a point on which somewhat different opinions have been propounded. It is the mechanical part only of the theory that I shall discuss. It has been called the *pressure theory* of the veined structure, while that advocated by Principal FORBES has been termed the *differential theory* of the structure. The lines in which the laminæ of blue or white ice meet the surface of the glacier will generally be curved lines, which I shall designate as *lines* or *curves of structure.*

This structure, as well as the lines depending upon it, have received different designations according to the portion of the glacier where it is found, or the directions of the lines of structure. Thus, when found in the lateral portions of the glacier, Dr. TYNDALL has termed it the *marginal structure.* It is frequently found in canal-shaped glaciers. The lines of structure in such cases are usually inclined at small angles to the sides of the glacial valley. When the lines stretch more directly and entirely across the valley, the structure is said to be *transverse,* and is more especially developed at points below an ice-fall and not remote from it. In the axial part of a glacier the lines of structure are frequently longitudinal, or parallel to the axis. The structure is then called *longitudinal.* It is generally best exhibited at the confluence of two large glaciers, as those of the Finsteraar and Lauteraar. It may be convenient to preserve these designations, though, as will be seen, they are the results, according to the pressure theory, of the same general cause, modified by the local conditions under which it acts. I take again the first approximate solutions of our equations, as applicable to all usually accessible depths.

41. *Pressure Theory of the Laminated Structure.*—We have generally,

$$p_1=\tfrac{1}{2}\{A+B+\sqrt{(A-B)^2+4F^2}\},$$

$$p_2=\tfrac{1}{2}\{A+B-\sqrt{(A-B)^2+4F^2}\},$$

$$\tan 2\alpha=\frac{2F}{A-B}.$$

In the formation of crevasses, we have been principally concerned with p_1 and its direction, we shall now be more especially concerned with p_2, when a pressure, and the corresponding value of α; but it will be convenient to bear in mind that, according to the fundamental principle of this theory, the curve of structure through any point will always coincide with the direction of p_1, the maximum tension or minimum pressure at that point, p_2 being always the maximum pressure there. Also, since the direction of p_2 is horizontal (neglecting C and E as in our first approximation), each laminar surface must be vertical, and will therefore be completely determined by the line of structure corresponding to it.

42. *Formation of the Marginal Structure.*—I have stated that this is frequently found in canal-shaped glaciers. Suppose the glacial valley to be more or less convergent, so that the transverse force B may become a pressure; and suppose its magnitude to be greater than that of A, as it probably will be generally in such a valley. We may also suppose A to be a pressure, though this is not essential for the production of the laminar structure in the case before us. The preceding formulæ now become

$$p_1=\tfrac{1}{2}\{-A-B+\sqrt{(B-A)^2+4F^2}\},$$

$$p_2=\tfrac{1}{2}\{-A-B-\sqrt{(B-A)^2+4F^2}\},$$

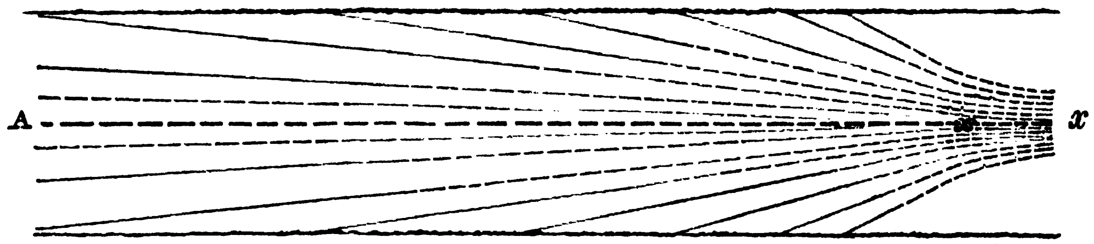
$$\tan 2\alpha=\frac{2F}{B-A}.$$

p_2 will be the maximum pressure, and the structural curve at any proposed point will be perpendicular to the direction of p_2; it will therefore coincide with that of p_1, which will be determined by the above expression for $\tan 2\alpha$. Let α_1 be the value of α which gives the direction of p_1, or that of the curve of structure at any proposed point. Then will α_1 lie between 0 and 45°. At points near the axis of the glacier, the *twisting* tendency, and therefore F also, will be very small, and may be neglected. We shall then have

$$p_2=-B.$$

Let us first suppose B too small to produce the veined structure. It will not, in such case, exist at all in the central portion of the glacier. At any point more remote from the axis, p_2 will be increased by the increase of F, and may become sufficient to produce the structure. α_1 will be greatest when F is so, *i. e.* at the sides of the glacier, supposing A and B to remain constant. Should B be much larger than F, as we should anticipate in the cases we are considering, α_1 will be much less than 45°, the greatest value to which it can attain when the transverse pressure B predominates over the longitudinal pressure A. The lines of marginal structure will be as represented in fig. 7, by the continuous portions of the oblique lines.

Fig. 7.



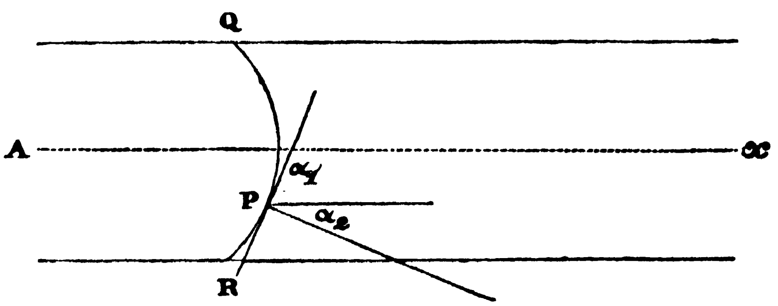
43. *Formation of the Longitudinal Structure.*—If B, the transverse pressure, be very great, p_2 may be sufficient to develope the veined structure in the central as well as the marginal parts of the glacier. The curves of structure will then be continued towards the axis of the glacier, to which they will constantly converge as an asymptote, as represented by the discontinuous lines in fig. 7. If B be very large, the lines of structure, when examined only in a limited area, will be sensibly parallel to each other and to the sides of the glacier.

44. *Formation of the Transverse Structure.*—Let us now suppose the longitudinal pressure A to be very great, as it must be, for instance, at the bottom of an ice-fall. We may also suppose it much greater than B. We shall then have

$$\tan 2\alpha = -\frac{2F}{A-B},$$

and consequently 2α will either be a small negative angle, or positive and nearly equal to 180°; and therefore α will either be small and negative, or positive and nearly equal 90°. The former must, from the nature of the case, correspond to the direction of maximum pressure, and must therefore be α_2, and the other α_1, as in fig. 8. α_2 will $=0°$ at

Fig. 8.



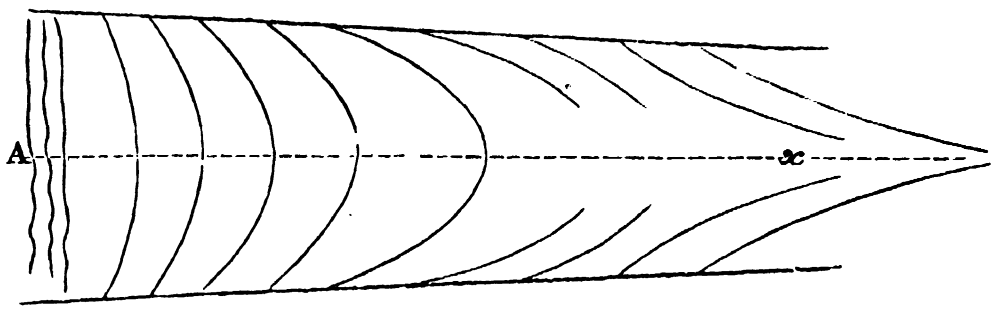
the axis, and will be greatest at Q and R. If A be sufficiently great, the curve of structure will be continued across the glacier, constituting the transverse structure.

45. There are two cases connected with the formation of the transverse structure at the bottom of an ice-fall which ought to be noticed. It may happen that the valley below the fall shall be an elongated slowly *converging* valley, or, as occurs perhaps more rarely, a rapidly diverging one like that already described at the extremity of the Rhone

glacier. In the first case, especially if the inclination of the valley should gradually increase and become considerable, the transverse pressure, B, may be much increased, and the longitudinal pressure, A, much diminished as we descend the valley, till it becomes less than B. In such case the curves of structure (so far as they depend alone on the action of the external forces) would become more and more elongated, as represented approximately in fig. 9. If B should become greater than A,

$$\tan 2\alpha=\frac{2F}{B-A},$$

Fig. 9.



and the case becomes the same as that previously considered (art. 43), in which the formation of the longitudinal structure is explained. So long, however, as A remains a pressure sufficient to produce the laminar structure at right angles to the axis, the loops will be completed; if A become too small to produce the structure, the curves will not be continued across the axis, and the structure will exist only as a marginal structure; and if B still increase (by the glacial valley becoming very narrow), so as to produce the longitudinal structure along the axis, the structure will become longitudinal, as represented by the two lower lines of structure in fig. 9.

46. In the second case above mentioned, of a rapidly divergent valley of small inclination, the mass will be urged onwards by a great radial force, as above explained (art. 30), and the curves of structure will be perpendicular to the crevasses, and consequently approximately concentric about the foot of the ice-fall, where this structure commences.

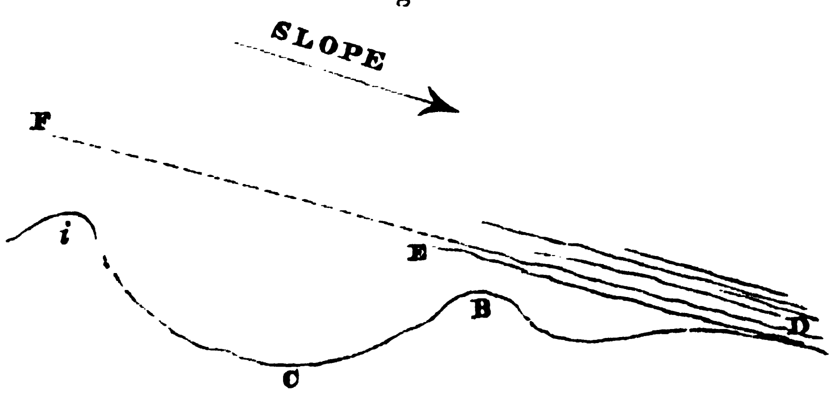
47. *Application of the preceding Theoretical Results.*—It must be recollected that the phenomena of glacial structure as they actually exist at any moment, are not merely the results of the instantaneous action of the mechanical and physical causes to which their original formation may have been due, but to those causes together with the effect produced by the motion of the glacier. In the formulæ of this section I have entirely neglected the influence of transmission by this motion, on the forms of the curves of structure, and have spoken of them as though they were dependent only on the forces originally producing them, and unaffected by the unequal motions of the central and marginal portions of the glacier. This, however, cannot be the case, unless we suppose the structure to be modified at every instant in exclusive obedience to the physical and mechanical causes in which it originates—a supposition having a great apparent improbability. The motion of a glacier being known, it is merely a geometrical problem to

point out accurately the nature of the modification which would be produced in a given curve or surface of structure by transmission alone; the amount of that modification can only be known by observation. I shall investigate this subject in a subsequent section. At present, in the application of the preceding formulæ to particular cases, my conclusions will be restricted to the hypothesis of the structure being due alone to the instantaneous action of the forces tending to produce it in each particular locality.

48. The *marginal structure* is found more or less developed in most canal-shaped glaciers. It has been best observed, perhaps, on the glacier of the Aar and its tributaries, and on the glaciers at Chamouni. It sometimes coexists with marginal crevasses. In such cases the curves of structure ought to be, according to the pressure theory, perpendicular to the crevasses. Moreover, the coexistence of the structure and crevasses implies a longitudinal tension and a transverse pressure; in which case the direction of greatest tension will make an angle of much less than 45° with the axis or sides of the glacier, and such, therefore, will also be the case with the marginal curves of structure. This seems accordant with the best observations; for though the forms and positions of these curves have not yet been observed with all the care they require, there is little doubt as to their meeting the sides of the glacier at finite angles. The law of perpendicularity between the structural curves and the crevasses was first observed, I believe, by Principal FORBES on the glacier of the Aar; and in speaking of the Talèfre glacier* he remarks, "The crevasses as they present themselves are convex towards the origin of the glacier, and here, as in other cases, perpendicular to the veined structure." The same impression appears to have been produced on other observers.

49. It is manifest that the marginal structure may be modified by any local conditions which affect the internal pressure of the glacier. Principal FORBES describes what he regards as an extraordinary case, and one to which he appeals as affording a crucial test in favour of his own views†. It is furnished by the glacier of La Brenva, and is represented by fig. 10. The line i C B D represents the side of the glacial valley,

Fig. 10.



in which projections at B and D appear to form a somewhat sudden obstacle to the onward motion of the glacier in the direction F D. Now it is certain that the maxi-

* Occasional Papers, p. 102.

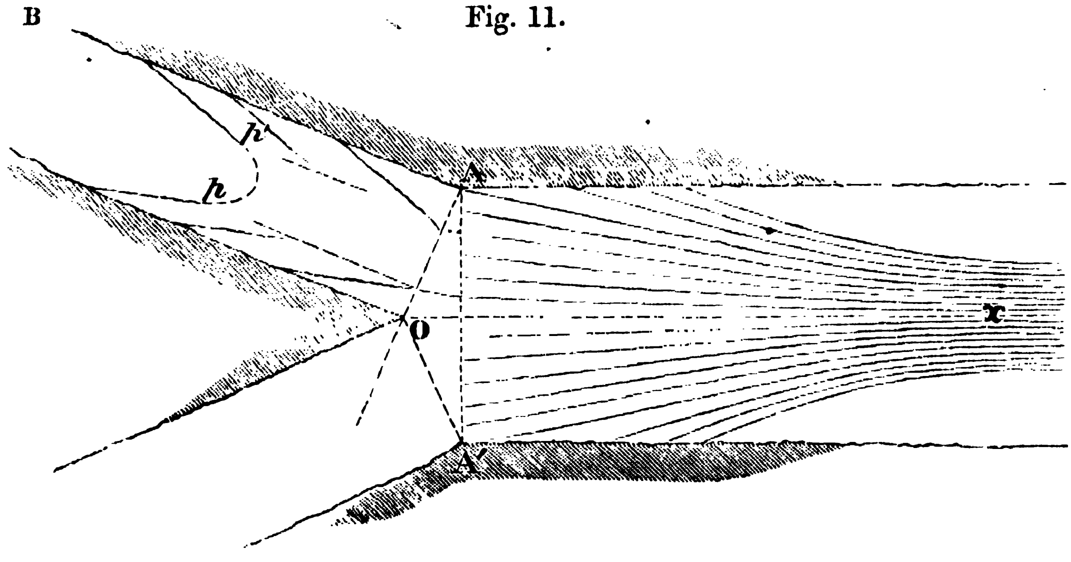
† Ibid. p. 56.

mum pressure between B and D would be very great, and that its direction would be nearly perpendicular to that of the motion of the glacier. Consequently the direction of the veins, according to the pressure theory, ought to coincide approximately with the direction of the motion, as represented by the lines between B and D. The example, as I understand it, presents no difficulty in the pressure theory*.

50. The *transverse structure* is well developed near the bottoms of the ice-falls of the upper part of the Glacier du Géant, and that by which the ice is precipitated from the glacier of Talèfre to that of Léchaud; but the most striking exhibition of it is afforded by the lower end of the Rhone glacier, which has already been mentioned (art. 30) as affording an excellent example of longitudinal, or, more properly, radiating crevasses. The curves of structure, according to the law above stated (art. 48), are perpendicular to the crevasses, and therefore coincident with the surfaces of greatest pressure, as they ought to be according to the pressure theory.

51. The *longitudinal structure* is perhaps best developed below the junction of the Finsteraar and Lauteraar glaciers, where they form the glacier of the Aar†. The marginal structure is well exhibited along the flanks of both the tributaries, and is continued along those of the combined glacier resulting from their union. It is from the point of junction O, fig. 11, and on each side of the axis Ox, that the longitudinal

Fig. 11.



structure is so finely developed. The tributary B A represents the Lauteraar glacier. The oblique lines on either flank represent the marginal structure, which is supposed to extend only to a certain distance from the sides. The whole width of the glacier of the Lower Aar is much less than the sum of the widths of its two principal tributaries, so that the transverse pressure (especially near the axis), for some distance below the

* I may remark that I see nothing distinctive in the observations made by Principal Forbes on La Brenva to determine the relative velocities of points at different distances from the side of the valley. Exactly similar results would be obtained on any glacier in which the velocity of the central should be considerably greater than that of the marginal portions.

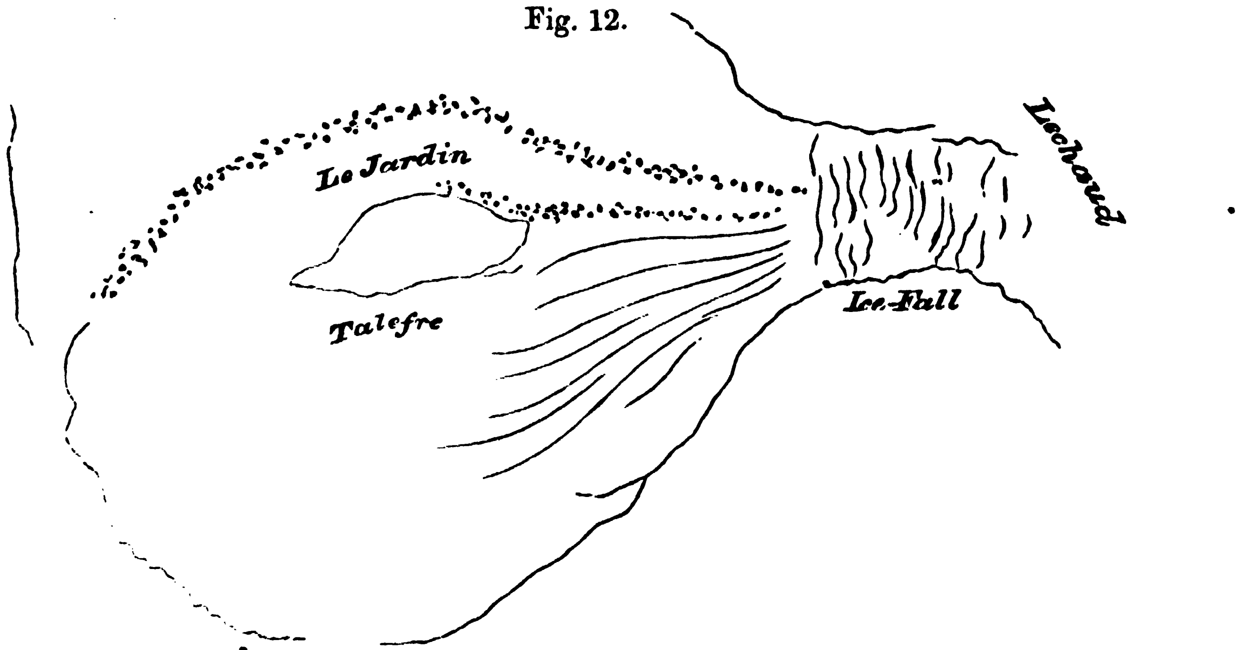
† Tyndall's 'Glaciers of the Alps,' p. 387.

junction, must manifestly be enormous. About some such section as O A this pressure will begin to be felt, and will be in full operation about A A′. Now considering the direct effects which would be produced by the forces acting on the united glacier below the section A A′ (independently of the effects of transmission), that portion will present precisely the case of longitudinal structure considered in art. 43. The transverse pressure, as above remarked, will be enormous, especially along the axis, to which it will there be perpendicular, and the general structure will be such as is represented in fig. 7, and repeated in fig. 11. The maximum pressure at any point in the axial part of the glacier will be increased by the mutual action of the two tributaries after their confluence, required to divert each from its original direction; and in the marginal parts of the great glacier the maximum pressure will be increased by the comparatively large value of F. In the region intermediate to the marginal and central portions, the pressure will probably be less than in either of those portions, but in the figure it is supposed to be sufficient to superinduce a longitudinal structure. In this case, however, the angle α at any point P would not increase regularly, and the curves in the part of the glacier now referred to might present some degree of inflexion.

According to Dr. TYNDALL, the longitudinal character of the veined structure is strongly developed, as a general rule, under all considerable central moraines; and as such moraines always originate in the confluence of two glaciers, and present conditions similar to those of the Aar, the explanation in all such cases will be precisely similar to that above given, assuming the structure to be due to the instantaneous action of the forces acting on the mass, and not to transmission.

52. In the glacier of Talèfre the ice-stream is divided by the Jardin into two separate currents, as roughly represented in fig. 12, which thus form, in fact, two separate tribu-

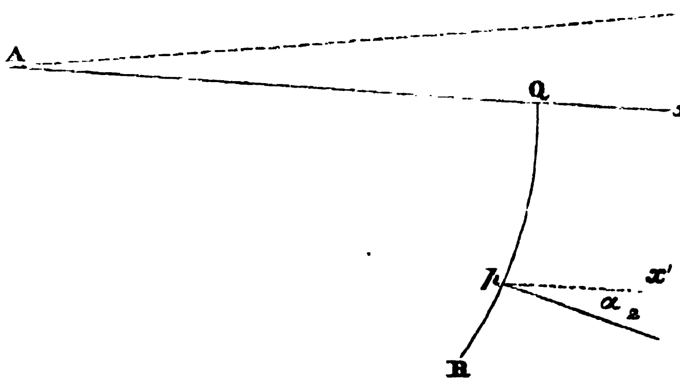
Fig. 12.



taries to the united glacier which proceeds, below the Jardin, to precipitate itself over the ice-fall of Talèfre. Principal FORBES appears to have examined the veined structure carefully, and describes it as represented in the figure by the fine lines converging to the fall. The dotted lines are moraines, and indicate the course of the ice-current. It is manifest that the mass, in approaching the fall, must suffer immense transverse compression, which accounts for the longitudinal structure in that part of the glacier.

53. *Formation of the Veined Structure in the deeper portions of a Glacier.*—We cannot determine completely the direction of the greatest pressure at any proposed point at a great depth in a glacier, for the reason above assigned (art. 34). That direction would give us the normal to the surface of greatest pressure, and therefore, also, to the surface of structure through that point, whence the differential equation to the surface would be known. In our practical inability to follow this method, we may proceed, as in art. 34, to determine the section of the required surface made by a vertical plane through the axis of the glacier. In a glacier of considerable width, the sections of this surface made by planes parallel to the one just mentioned, will be very similar to the axial section; because the velocities will be very nearly the same for all points in a transverse line on the surface of the glacier, within considerable distances of the axis*; and the same therefore must also hold for points at greater depths. The most important case is that which occurs at the base of an ice-fall in which A becomes very large and a pressure. Also B, the transverse force, may be considered much smaller than A, and a pressure. Our general formulæ will become

Fig. 13.



$$p_1=-\tfrac{1}{2}\{A+C-\sqrt{(A-C)^2+4E^2}\},$$
$$p_2=-\tfrac{1}{2}\{A+C+\sqrt{(A-C)^2+4E^2}\},$$
$$p_3=-B;$$

and we shall have, also,

$$\cos\beta=0,\ \ \tan 2\alpha=-\frac{2E}{A-C},$$

the plane of α being now that of xz. Now B being supposed small compared with A, p_2 will be the maximum pressure and p_1 the minimum pressure, or algebraically the maximum tension. The above equation gives two values of α. Now 2α must be negative and less than 90°, or positive and between 90° and 180°; and therefore α must be negative and less than 45°, or positive and between 45° and 90°. The negative value will evidently here correspond to p_2 the greatest pressure. A may possibly be much the same at different depths, in our present case, and E (art. 34) will be zero at the surface, and will increase with the depth. C also increases with the depth, as does, therefore, *cæteris paribus*, the negative angle α_2. It must always, however, be less than 45°, so long as C is less than A, whatever may be the depth; and near the surface it will vanish.

* M. Agassiz has best exemplified this in his admirable map and diagrams of the glacier of the Aar already referred to. He there delineates (Atlas, pl. 4) the forms assumed by an originally straight transverse physical line on the glacier near the junction of its two great tributaries in three successive years. The motion of different points on the central portion of the mass do not differ much throughout a breadth of nearly one-half of the glacier.

The axial section of the surface of structure may be represented by Q R in fig. 13, varying in its inclination to the nearly vertical axis of z from zero to less than 45°. A similar conclusion will hold, as above stated, for all the central portion of the glacier.

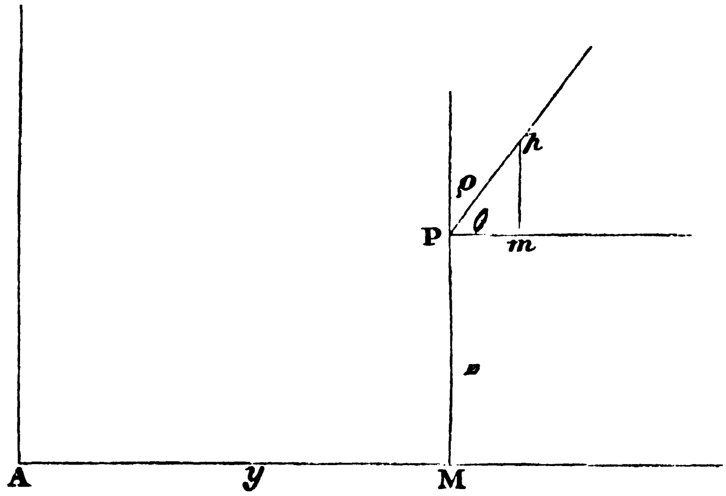
The structure here considered is manifestly a transverse structure, and is probably only formed where there is a sudden change of inclination in the bed of the glacier, like that at the foot of an ice-fall. The intensity of the longitudinal pressure to which it is there due, may possibly diminish rapidly in receding from the fall; and so far it would follow that, where the structure is continued to any considerable distance from the locality in which it was formed, it must be due in a great degree to transmission. But this is a question which I reserve for further discussion.

54. *Differential Theory of the Veined Structure.*—The idea on which this theory is founded has already been stated at the beginning of this section (art. 40); and the modes in which dislocation may take place so as to admit of the more rapid motion of the central portion of a glacier, have been explained. It has also been pointed out (art. 36) that the real differential or relative motion of two contiguous particles of the mass must necessarily be in the common direction in which the particles are constrained to move, by virtue of the external conditions to which the whole glacier is subjected. This, in fact, belongs to the definition of "differential motion" as I understand the term; nor can I conceive how any mechanical effects, such as dislocation or bruising, can be attributed to a "differential motion" in any arbitrary or conventional sense, rather than to the real relative motion which I understand to be meant by the expression. If so, it appears to me impossible to accept this theory of the veined structure without a far more explicit explanation than any which has yet been given of it. Assuming, then, what I conceive to be the only intelligible and determinate meaning of the above expression, I shall proceed to investigate certain geometrical results which flow from it. Since the investigation is entirely geometrical, we may suppose the surface of the glacier and its longitudinal axis to be horizontal. I shall also suppose the line of motion of every particle to be parallel to that axis, and the velocity to be invariable along each such line, but different for different lines, (1) because the velocity of the central is supposed to be greater than that of the marginal portions of the mass, and (2) because the velocity of its upper is greater than that of its lower surface.

55. Let P (fig. 14) be any particle of the glacier, and let the plane of the paper represent a transverse plane perpendicular to the common direction of all the lines of motion of the component particles of the mass. Also let p be any other particle likewise in the plane of the paper, its distance from P being very small and equal to ϱ; i. e. p must lie in the circumference of a circle in the plane of the paper, whose centre is P, and whose radius ϱ is extremely small. Our first object will be to determine the relative velocity, or differential motion of P and p, and the positions of p in the small circle when that relative velocity is a maximum, and when it is zero.

For this purpose let the horizontal plane of the base of the glacier be made the plane of xy, and the vertical plane parallel to the longitudinal axis of the glacier, and along its flank on the left, the plane of xz. The plane of yz will be a vertical transverse plane; let it be represented by the plane of the paper, to which, therefore, the line of motion of every particle will, by hypothesis, be perpendicular. Let P (fig. 14) represent a

Fig. 14.



material point in the plane of the paper, or that of yz. (We take it in this particular plane because we shall not be immediately concerned with the coordinate x.) Let $AM=y$, and $MP=z$. Also; take another point p supposed to be very near P, and let its coordinates be $y+\eta$ and $z+\zeta$, η and ζ being referred to P as origin. Also let V be the velocity of P perpendicular to the plane of the paper, and $V+v$ that of p; V will be some function of y and z; and v a function of η and ζ. As we have chosen the coordinate planes, V will increase with y and z, because the centre of the mass moves faster than its sides, and the upper moves faster than the lower surface; and for the like reasons, v will increase with η and ζ. Now since η and ζ are very small, we may consider any increase of v, due to an increase of η, as $=\mu\eta$, and an increase due to that of ζ, as $=\mu'\zeta$, where μ and μ' are functions of y and z, but independent of η and ζ. They express the rate at which V increases as we pass from P to contiguous points in the plane of the paper, and in directions parallel respectively to the axes of y and z. We shall then have

$$v=\mu\eta+\mu'\zeta;$$

or if

$$\eta=\varrho\cos\theta,$$
$$\zeta=\varrho\sin\theta,$$
$$v=\varrho(\mu\cos\theta+\mu'\sin\theta).$$

From what has been above stated, this quantity must be a maximum with respect to θ, considering μ, μ', and ϱ* as constants. This gives

$$-\mu\sin\theta+\mu'\cos\theta=0.$$

* The tendency of the tangential differential motion of two particles very near together, to bruise or dislocate the mass, will manifestly depend on the contortion or twisting produced by this relative motion, and therefore on the angular change of position of the line joining the two points. Thus, in fact, $\frac{v}{\varrho}$ is the quantity to be really made a maximum. This is equivalent to considering ϱ constant in differentiating as in the text.

Let θ_1 and θ_2 be the two values of θ which satisfy this equation; then

$$\tan\theta_1 = \tan\theta_2 = \frac{\mu'}{\mu};$$

and

$$\theta_2 = \tan^{-1}\frac{\mu'}{\mu} + 180°.$$

From the above expression for v, θ_1 gives v a positive value, and θ_2 the same numerical value with an opposite sign, the one being the algebraical maximum value, the other the minimum one.

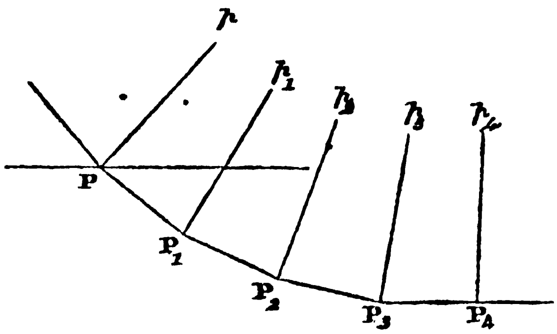
Moreover, we see from the above value of v, that $v=0$ when

$$\tan\theta = -\frac{\mu}{\mu'} = -\cot\theta_1,$$

which shows that the angular distance between the directions in which v is respectively a maximum and zero, is equal to a right angle.

56. To interpret these formulæ, draw Pp (fig. 15) making the angle $\tan^{-1}\frac{\mu'}{\mu}$ with the axis of y. The relative velocity for p and P will be greater than for any other point at the same distance from P. Draw P P_1 perpendicular to Pp, taking P P_1 very small; then since $v=0$ in the direction at right angles to Pp (by the last equation), the velocity of P_1 will equal V, that of P. At P_1 (since it is nearer the central axis of the glacier than P) the *rate* of increase of V, as depending on y, will be less than at P, and therefore μ will also be less (art. 55); and for a like reason* (since P_1 is nearer the lower surface than P) μ' will be increased. For both these reasons $\theta\left(=\tan^{-1}\frac{\mu'}{\mu}\right)$ will be greater at P_1 than at P. Draw $P_1\,p_1$ accordingly. Again, draw $P_1\,P_2$ perpendicular to $P_1 p_1$, making $P_1\,P_2$ very small. The velocity of P_2 will still $=$V, and the relative velocity of p_2 and P_2 will be a maximum in the same sense as before. We may proceed in the same manner with any number of points. Now, when we pass to the limit, by taking P P_1, $P_1\,P_2$, &c. indefinitely small, the locus of P $P_1\,P_2$, &c. will become a continuous curve, possessing the dynamical property, in the case of a glacier, that each of its particles moves with the same velocity perpendicular to the plane of the paper, and they have consequently no tendency to separate from each other.

Fig. 15.



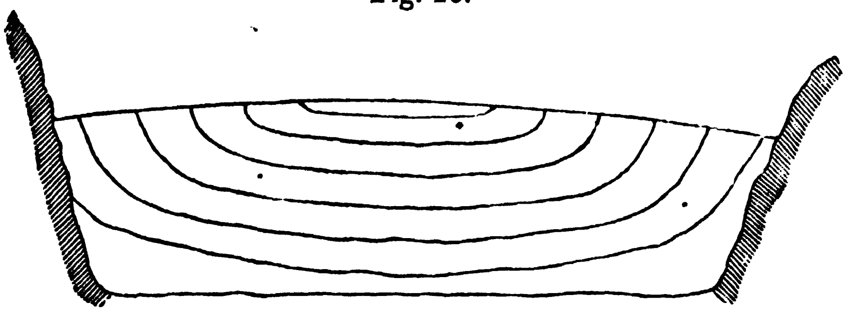
Again, taking ϱ indefinitely small, we shall have another continuous curve $p\,p_1 p_2$, &c. contiguous to the former, and so related to it that the relative velocity of two particles situated respectively on these curves, and on a common normal to them, will be a

* The increase in the *rate* at which V increases as we descend from one longitudinal line of motion to another, cannot be made a matter of observation, but the analogy with the variation of V in passing from the axis of the glacier to its sides (in which case we know that the *rate* of variation increases) would seem fully to justify the assumption. The conclusion of the text, however, is not dependent upon it.

maximum, and will consequently have a maximum tendency to separate from each other by their differential motion. I have spoken here of physical *points* as perhaps conveying more distinctly the idea of the intervening distance denoted by ρ. I might have equally spoken of indefinitely small elements of determinate forms, considering ρ as the distance between their centres of gravity. These are only different kinds of phraseology by means of which we reason on the ultimate elements of which a continuous mass must be constituted.

In the above investigation the velocity of each particle has been assumed to be the same at every point along its line of motion. Consequently whatever holds for the motions with which these particles pass through one transverse plane, will hold with respect to any other such plane. Consequently there will be two contiguous cylindrical surfaces generated by lines parallel to those of the motion of the glacier, and having for their directors the two contiguous curves above described on the plane of yz; and the motion of the particles in these cylindrical surfaces will have the same dynamical characters as those above enunciated for the particles in the two guiding curves. There will be an indefinite number of such surfaces following the same law. A section of the mass, made by a transverse vertical plane parallel to that of yz, will be represented by fig. 16. Each surface will be perpendicular to this transverse plane.

Fig. 16.



This system of surfaces would necessarily result from the differential motion in the sense in which I regard it; nor can I conceive how the veined structure can originate in that motion, in the way in which the author of the theory appears to consider it to originate, unless the veins should coincide with the surfaces here investigated. In such case every line of structure on the surface of the glacier, in the typical case above considered, would coincide with a line of motion, and would therefore be parallel to the axis and sides of the glacier. In the central parts of the glacier, where the variation of V for different points situated on a transverse horizontal line is usually very slow, μ will be very small, and therefore $\theta\left(=\tan^{-1}\frac{\mu'}{\mu}\right)$ will nearly equal 90°. Consequently the structural surfaces will be very nearly horizontal. In the marginal portions of the glacier these surfaces will meet the superficial surface of the mass at angles

$$=90^\circ-\theta=90^\circ-\tan^{-1}\frac{\mu'}{\mu},$$

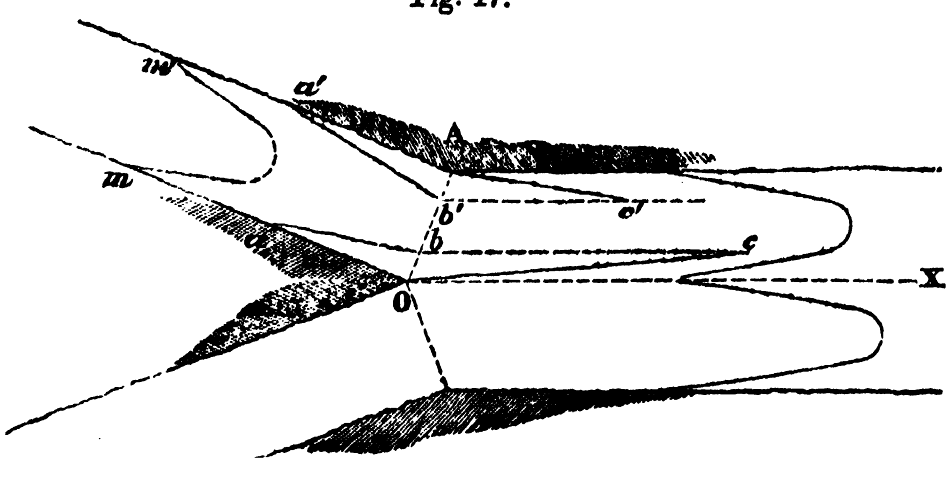
which will be nearly $=90^\circ$, since μ' will be usually small near the upper surface, and μ

will be comparatively large near the sides of the glacier. The superficial curves of structure would be all straight lines parallel to the axis of the glacier.

In actual glaciers, the forms of the curves of structure on the outer surface would be modified by the fact of the unequal thinning of the glacial mass in consequence of the more rapid thawing towards its lower extremity, so that the existing external surface is no longer parallel to the longitudinal lines of motion. Consequently the outer surface would intersect the internal structural surfaces obliquely, and the superficial curves of structure would be changed from straight parallel lines to very elongated loops, but would never approximate to straight transverse lines, as they always do at the foot of an ice-fall. We may also observe that there could be no longitudinal development of veins along the axis of the glacier, as the immediate result of differential motion. Such veins could only exist by transmission, as will be explained in the next section, where I shall revert to the forms of the surfaces of greatest differential motion which have been here investigated.

57. *Modification of the Veined Structure arising from the Motion of the Glacier.*—I now proceed, according to my intention as above expressed (art. 47), to examine the modifications produced by the general motion of a glacier, on the forms and positions of the lines and surfaces of structure as originally produced by the immediate action of the causes to which they may be due. Such modifications as can be observed on the surface of a glacier, are principally due to the more rapid motion of its central portion; the interior surfaces of structure must also be modified by the more rapid motion of the upper surface of the mass. In an ordinary canal-shaped glacier, it is manifest that the marginal curves of structure will thus be brought into more approximate parallelism with the sides, if originally inclined to them. This modification will be small. A greater one will be produced in the transverse structure, such as is usually found at the foot of an ice-fall. There the superficial curves of structure run almost directly across the glacier, by the motion of which they will be transformed into elongated curves, at distances from the fall sufficiently great, the elongation increasing with that distance. In a glacier, like the Aar, formed by the junction of two great tributaries, the modification of structure below their confluence will be more complicated. Let fig. 17 represent the glacier as

Fig. 17.



before, and let *a b*, *a′ b′* be two lines of marginal structure, representing the two systems of lines to which they respectively belong. A particle in the tributary glacier will move parallel to the side of the glacier till it arrives, for instance, at *b*, a point near the section O A, and will afterwards move parallel to the axis O X. When the particle at *a* arrives at O, suppose the particle which was simultaneously at *b* to have arrived at *c*, *b c* being parallel to O X; then will O *c* be the position in the united glacier of the physical line of structure which before occupied the position *a b*. But the velocity with which a particle will move from *b* to *c* in the central part of the united glacier, will be considerably greater than that with which a particle will move from *a* to O along the side of the tributary. Consequently the distance *b c* will be considerably greater than *a b*, and O *c* will make a proportionately smaller angle with O X than *a b* makes with the side *a* O of the tributary. Hence, if the lines of marginal structure approximate to parallelism to the sides in the tributary, the same lines, forming those of the longitudinal structure in the central portion of the united glacier, will approximate still nearer to parallelism with each other and with O X. Subsequently this approximation to parallelism will decrease, since a particle will move rather faster along O X than along *b c*; but assuming always that the divergency is small in the marginal structure of the tributary, a simple calculation shows that it will not become considerable in the united glacier except at distances from the junction equal to many multiples of the whole breadth of the glacier*. Such, at least, would be the case with the glacier of the Aar. If, on the contrary, the lines of marginal structure in the tributaries should make a considerable angle with the sides of the glacier, they would also make a considerable, though smaller angle with the axis of the united glacier, and their inclination to that axis would afterwards increase more rapidly.

Taking the opposite flank of the glacier, suppose two particles on the same line of structure to be simultaneously at *a′* and *b′*; and reasoning as before, suppose *b′* to have moved parallel to the side of the united glacier to *c′* while *a′* moves to A. Then will A *c′* be the resulting position of the *same* line of structure as previously occupied the position *a′ b′*. In this case, however, *b′ c′* will be nearly equal to *a′* A, since the velocities along the margins of the combined glacier and the tributaries will probably be much the same. Hence the directions of the marginal lines of structure with reference to the sides of the valleys, will be less changed than those of the longitudinal structure with reference to the axis O X.

I have above supposed the absence of any transverse lines of structure on the central portion of the tributary glaciers. If, however, they exist so as to complete the loops of the curves (*m m′*) (fig. 17), they will of course also exist, by the transmission we are here assuming, in the combined glacier, in the forms of similar loops, as represented in the figure.

58. If we compare the forms of the marginal structural curves in a canal-shaped glacier (represented in fig. 7, art. 42), resulting from the immediate action of the forces

* This calculation is founded on data supplied by M. Agassiz in his 'Système Glaciaire,' p. 451.

producing them, it is evident that it will generally be difficult or impossible to recognize by observation the modification which may have been produced in them by transmission. The case of structure represented in fig. 9 (art. 45), if the real structure in a glacier were sufficiently developed and carefully observed, would afford a test as to whether the forms were original, or had been modified by transmission; for, in the latter case, the loops would never change by elongation into an absolutely longitudinal structure as represented in the figure referred to. But this test is far, at present, from being a practical one. Again, in the glacier of the Aar, if the structure were perfectly developed, it would afford us the required test, as is easily seen by comparing fig. 17 and its completed loop in the main glacier, as transmitted from the tributary, with fig. 11 and the lines of longitudinal structure in the main glacier. If the glacier exhibited either of these structures complete, it would testify at once to the point in question. I am not aware, however, that the structure, to whatever cause it may be due, is sufficiently developed, or has been observed with sufficient care, to afford any positive testimony on the subject. The glacier of the Rhone, also, fails to afford us a practical test on this point; for the curves of structure may there be accounted for either by supposing them instantaneously formed or transmitted from the fall where they originate.

The supposition that the structural surfaces, at points of a glacier remote from the place at which they commence, are merely the effects of transmission, involves the conclusion that the veined structure can only be originally formed under enormous pressure (according to the pressure theory), like that at the bottom of an ice-fall, and that, when once formed, it is difficult to obliterate. In this case it would seem that the transverse curves of structure formed at the bottom of a fall ought to be distinctly preserved at a distance from it, as elongated loops extending completely across the glacier. I am not aware how far they are so, either in the case in which they proceed from the Talèfre fall, for instance, on the Mer de Glace, or from that of the Géant. The glacier of La Brenva above cited (art. 49) seems to afford an example of the instantaneous generation of the veined structure under a pressure which may not be comparable with that immediately below the junction of the Aar tributaries; and this same example at La Brenva seems also to afford an instance of a comparatively sudden obliteration of the structure where the immediate cause of it ceases to act. We may also remark that the marked and increased development of the longitudinal lamination along the central moraines proceeding from the junction of two considerable glaciers, appears to indicate the local efficiency and instantaneous effect of the causes of the structure. But particular cases of this kind require to be examined with greater accuracy of detail before we can derive from them any reliable inferences. It would seem most probable that the structure in any particular locality may frequently be due partly to the instantaneous action of physical causes, and partly to transmission. The question well deserves the consideration of those glacialists who may interest themselves about this curious structure of glacial ice, and the physical and mechanical causes to which it may be due.

I may make another remark on this doubtful question. The *dirt-bands* of the Mer de Glace, if Dr. TYNDALL's views respecting them should be proved to be correct, may afford the means, supposing their forms and positions sufficiently determinate, of deciding how far the loops of structure in the middle and lower parts of the glacier may be merely the original transverse curves at its upper end, modified by transmission, or may be attributable to the great transverse pressure to which that glacier must be subjected at certain points of its course. If the dirt-bands are entirely formed at the upper part of the glacier and are there coincident with the curves of transverse structure, and if they also remain coincident with them at distant points of the glacier, then, since the dirt-bands must necessarily be transmitted forms, the curves must almost necessarily be so likewise. Principal FORBES's theory of the dirt-bands would not lead to the same conclusion.

59. *The Pressure Theory and the Differential Motion Theory of the Veined Structure compared.*—The pressure theory of the veined structure, so far as it asserts the perpendicularity of such surfaces to the directions of maximum pressure, appears to be in perfect accordance with observed facts and mechanical deductions, whether the structure be marginal, transverse, or longitudinal. The transversal directions and approximate verticality of the structural surfaces at the bottoms of ice-falls, and the general existence of the structure wherever the glacier must, from the conditions under which it is placed, be subject to great pressure, are also perfectly consistent with this theory. We may also remark that the evidence of facts in favour of it is in a great measure independent of the degree of efficiency which may ultimately be attributed to the transmission of the structural forms. I am not, indeed, aware of any leading observed facts of the veined structure inconsistent with this theory.

With respect to the Differential Theory, the whole of the mechanical reasoning on which it is based is professedly popular, vague, and undemonstrative, and, I believe, as erroneous as such reasoning must almost necessarily be in cases as intricate as those which glaciers present to us, unless it be guided by an accurate conception and a careful analysis of problems which admit of more or less accurate solution, and are typical of those presented to us in nature. In the marginal structure, this theory could never agree with any very sensible deviation in the directions of the superficial curves of structure from parallelism with the sides of a canal-shaped glacier. The attempt to correct this defect by means of what has been called the Ripple Theory, will not now, I imagine, be maintained by any glacialist. The constant existence of the veined structure under great pressure, and its comparative absence where such pressure cannot exist, are left by this theory without satisfactory explanation. It fails altogether to assign any direct cause for the great development of the longitudinal structure along the central moraine of compound glaciers like the Aar; for along the axes of such glaciers there cannot possibly be any differential motion which could produce it. The only structure which could there exist must be a transmitted structure.

But the most conclusive objection to the differential theory is to be found, as I believe, in its total inability to account for the highly developed structural surfaces, their nearly vertical positions, and approximate perpendicularity to the axis of the glacier, at all points not far from its surface, and near to the bottom of an ice-fall (art. 56). It appears to me inconceivable that any physical or mechanical effect (such as the lamination in question) really and primarily due to the difference of motion of contiguous particles should not manifest itself, if seen at all, in those directions in which the actual difference of motion is greatest. Assuming the truth of this conclusion, the structural surfaces must necessarily be such as above represented (art. 56). The investigation of the forms and positions of these surfaces depends only on the recognized motion of the glacier, and geometrical reasoning in which there can be no ambiguity. Now superficial curves of structure at the bottom of an ice-fall, as thus theoretically determined, are extremely elongated loops (art. 56); whereas it is one of the best-established of glacial phenomena that the actual superficial lines of structure at the foot of an ice-fall are nearly straight and perpendicular to the axis of the glacier. I maintain, therefore, that it is altogether impossible that the differential theory can be true, unless the expression "differential motion" has a very different meaning from any which I have been able to attribute to it*. On the contrary, the pressure theory, so far as it asserts the perpendicularity of the structural surfaces to the lines of maximum internal pressure, appears to be in accordance with all the observations which have yet been made on the subject.

SECTION VI.—*On the Intensity of the Forces employed in dislocating and crushing the masss of a Glacier.*

60. We have seen that, according to the Pressure Theory, when a glacier is brought into a constraint which it can no longer resist, its structure, or the continuity of its mass, must be destroyed in one of the three ways already specified (art. 28). It would seem probable that this is frequently effected by the crushing of the mass, the structure being immediately restored by regelation. But here arises the question, Whence do the internal pressures and tensions derive sufficient intensity to produce these crushing effects? The enormous weight of the mass of a glacier at once presents itself as the cause of the required intensity. But suppose, after the glacier has been broken and crushed, it should be instantaneously restored to perfect continuity of crystalline structure under the pressure to which, in any part, it may at the moment be subjected; why should the pressure, apparently the same as that under which it had, the instant before, regained its crystalline structure, again destroy the continuity of that structure? The highest mountain does not *crush* the strata which form its base. Again, the superincumbent weight can only produce any great effect at depths sufficiently great, whereas the effects of the internal pressures and tensions are exhibited in the most superficial

* With respect to Principal FORBES's conclusion, that a nearly horizontal force at the bottom of an ice-fall will produce a differential motion in nearly a vertical direction, I can only say that it appears to me so obviously opposed to the simplest conception of the action of such a force, as to be entirely inadmissible. It is unquestionably opposed to all accurate mechanical investigation on the subject (art. 58).

portions of the glacier, where the superincumbent weight can have no very sensible influence. These questions may be partly answered by a careful consideration of the theoretical explanations which have been given in the preceding pages; but for their complete answer they require, I conceive, the recognition of a kind of motion which has been till recently so entirely neglected. I allude to the motion by which a glacier *slides* over the bed on which it reposes (Sect. II. art. 7). I shall shortly proceed to explain the influence of this motion on the intensities of the internal pressures and tensions.

Principal FORBES was struck with this difficulty respecting the adequacy of the internal forces to produce the crushing effects ascribed to them, and the consequent mobility of the component particles *inter se*, though, regarding ice as viscous, the difficulty may naturally be supposed to have appeared less to him than if he had regarded it as a solid substance. He puts the difficulty in the following form*:—"Were a glacier composed of a solid crystalline cake, fitted or moulded to the mountain bed which it occupies like a lake tranquilly frozen, it would seem impossible to admit such a flexibility or yielding of parts as should permit any comparison to a fluid or semifluid body transmitting pressure horizontally, and whose parts might change their mutual position, so that one part should be pushed out whilst another remained behind." The difficulty as thus stated is equivalent to that above mentioned. Principal FORBES meets it as follows:—"But we know in point of fact that a glacier is a body very differently constituted. It is clearly proved by the experiments of AGASSIZ and others, that a glacier is not a mass of ice, but of ice and water, the latter percolating freely through the crevices of the former to all depths of the glacier; and as it is a matter of ocular demonstration that these crevices, though very minute, communicate freely with one another to great distances, the water with which they are filled communicates force also to great distances, and exercises a tremendous hydrostatic pressure to move onwards, in the direction in which gravity urges it, the vast porous crackling mass of seemingly rigid ice in which it is, as it were, bound up.

"Now the water in the crevices does not constitute the glacier, but only the principal vehicle of the force which acts on it; and the slow irresistible energy with which the icy mass moves onwards from hour to hour with a continuous march, bespeaks of itself a fluid pressure. But if the ice were not in some degree ductile or plastic, this pressure could never produce any, the least, forward motion of the mass. The pressures on the capillaries of the glacier can only tend to separate one particle from another, and thus produce tensions and compressions, *within the body of the glacier itself*, which yields, owing to its slightly ductile nature, in the direction of least resistance, retaining its continuity, or recovering it by reattachment after its parts have suffered a bruise, according to the violence of the action to which it has been exposed." The author again remarks (p. 167), "If it were not for the capillarity of the ducts, it is plain that no effective hydrostatic pressure could be developed at all; the flow being equal to the supply, no part of the *vis viva* would be expended in producing internal pressures."

* Occasional Papers, p. 165. See also a memoir in the Transactions of the Royal Society for 1846, Part III.

61. The author of the preceding extracts would appear, I think, to have somewhat mistaken the nature of the problem to be solved. He asserts that if a glacier were traversed by a great number of small tubes or ducts filled with water, an *enormous hydrostatic pressure* would be the consequence. Now it is absolutely necessary for me to examine strictly the correctness of this conclusion; for, if it be true, it would be useless to seek for any cause which should give efficiency to the internal pressures, besides the hydrostatic pressure to which it is here attributed. I shall show, however, that this conclusion is not correct, and that the efficiency of the dislocating forces is derived, as above intimated, from the sliding movement of the glacier. The existence of the above assumed capillary ducts, as pervading the more compact parts of a glacial mass, appears to be rendered doubtful by the experiments of Professor HUXLEY*, though the observations of M. AGASSIZ undoubtedly prove their existence in certain superficial portions of the glacier of the Aar. But waiving any doubt of this kind, let us suppose the interior of a glacier completely pervaded by these tubes or ducts, extremely small, but sufficient to allow fluid pressure to be communicated freely through them, as Principal FORBES has assumed. When the supply at the upper surface of the mass is regular, the motion of the fluid will become *steady*; and such I shall suppose it to be in the typical case I propose to analyse. We may first confine our attention to a single tube. Suppose the area (ω) of its transverse section at any point P to be variable, but always very small, and δs to denote the length of a small element of the tube; then will $\omega\delta s$ be the volume of the element, and, if the density of the water be denoted by unity, it will also represent the mass of the water in the element δs of the tube. Let the velocity at P, any point in the tube, be v, which I shall assume to be the same for each particle in the section (ω). The motion will be retarded by the *friction* of the sides of the tube; let $f\omega\delta s$ be the retarding force on the element at P. Also let p be the fluid pressure at P; and let the coordinates x, y, z of that point be taken as heretofore; then will z be very nearly vertical, and we shall have, resolving gravity in the direction of the tube,

$$\omega\delta p = g.\omega\delta s.\frac{dz}{ds} - f.\omega\delta s - \frac{d^2s}{dt^2}.\omega\delta s,$$

or

$$\delta p = g\delta z - f\delta s - \frac{d^2s}{dt^2}\delta s;$$

or, since the motion, by hypothesis, is steady,

$$\frac{d^2s}{dt^2} = v\frac{dv}{ds},$$

and therefore we have

$$p + C = gz - \int f ds - \tfrac{1}{2}v^2.$$

Let p_1, z_1, s_1 and v_1 be the values of p, z, s, and v at the point where the tube meets the surface of the mass. Then

$$p = p_1 + g(z - z_1) - \int_{s_1}^{s} f ds - \tfrac{1}{2}(v^2 - v_1^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

where p_1 is the atmospheric pressure.

* See Dr. TYNDALL's 'Glaciers of the Alps.'

Also, since the same quantity of the fluid must pass through each transverse section of the tube in the same time, we must have

$$v\omega = v_1\omega_1,$$

and therefore

$$\tfrac{1}{2}(v^2 - v_1^2) = \frac{v^2}{2}\left(1 - \frac{\omega^2}{\omega_1^2}\right).$$

Let us interpret the above equation in different cases. We may first suppose the absence of all friction as a retarding force. We shall then have

$$p = p_1 + g(z - z_1) - \frac{v^2}{2}\left(1 - \frac{\omega^2}{\omega_1^2}\right);$$

and if the section (ω) of the tube do not contract so rapidly as to impede the motion of the water, the velocity acquired at the depth z will equal that of a body projected with velocity v_1, and falling freely by the action of gravity through the space $z - z_1$. We shall therefore have

$$\tfrac{1}{2}(v^2 - v_1^2) = g(z - z_1),$$

and therefore $p = p_1$ the atmospheric pressure, as if the tube were occupied by air. The least value of ω^2 for any value of z in this case will be given by

$$\omega^2 = \frac{v_1^2}{v^2}\omega_1^2$$

$$= \frac{v_1^2}{2g(z - z_1)} \cdot \omega_1^2.$$

If ω satisfy this condition for all values of z, the column of fluid will just fill the tube without its motion being impeded, while the pressure on the tube, as just shown, will be the atmospheric pressure p_1. But suppose ω to decrease, below a given section (Q), more rapidly as a function of z than is implied in the above equation. The motion of the fluid above the section Q would manifestly be retarded, and the retardation could only be due to the upward fluid pressure at Q being greater than the atmospheric pressure p_1 at the top of the tube. In like manner the increased downward fluid pressure at Q would accelerate the motion of the fluid immediately below that section. Any number of similar variations of ω, with consequent variations of velocity and fluid pressure, might take place, subject to the above condition that the tube shall always be full. The fluid pressure at any point cannot be less, but may be greater than p_1.

In this reasoning the tube is supposed to be independent of any other; but now suppose it to be confluent with a second and similar tube. At the point of junction we must have the condition that the pressures in the two tubes must be equal. This condition will be always satisfied, at whatever points in the surface of the mass the tubes may originate, provided each tube be such that the fluid pressure in it (as in the first case above explained) shall be equal at every point to the atmospheric pressure p_1. In such case water might pass from the upper surface through the mass, along any number of smooth tubes like those above described and communicating freely with each other.

The pressure in any tube would cause no reflux of the water, or other disturbance in a confluent one, because the pressures in them would be equal; but it must be carefully observed that there would, in this case, be no internal fluid pressure greater than that of the atmosphere, and therefore none capable of producing any effect in expanding the general mass, and urging it onwards. But if, on the contrary (as in the second case above explained), the tube be such that the pressure in each tube considered separately should be greater than p_1, the condition of an equality of pressure at the point of confluence of any two tubes would not be generally satisfied, and the whole motion would be interfered with. If the pressure p in any number of tubes became great, it would manifestly produce a reflux in any confluent tube in which the pressure was smaller, and a consequent outpouring of water from every pore and crevice on the surface of the lower regions of a glacier. Such, in fact, must necessarily be the case wherever there is an internal fluid pressure much exceeding that of the atmosphere, and where all parts of the mass are permeated by ducts of any kind through which water can flow out of the mass with little impediment.

Hence it appears that if the internal ducts were such as not to impede the currents passing through them by means of friction, or any kind of capillarity, there might indeed be large internal pressures, but they would necessarily be accompanied by overflowings from the superficial pores and crevices in many parts of a glacier. But such a phenomenon would be entirely opposed to all observation; and it is this which constitutes the proof that there can be no great fluid pressure in the interior of a glacier due to water contained within it in tubes which exert no sensible retarding effect by friction on the water running through them. This conclusion agrees with that expressed by Principal FORBES in the last of the preceding extracts; but the reason assigned is very different. The absence of great fluid pressure within the glacier in such case is not due to the absence of capillarity, but to the particular condition of its ducts and crevices being open in every part of the bounding surface of the mass. There can be no expenditure of *vis viva* in producing pressure on a *smooth* surface, as is well known, though such pressure must necessarily be produced in the case before us, as in all cases of constrained motion when the constraint is produced by the action of such surfaces. The fact of no *vis viva* being lost does not imply that no internal pressure would be produced.

62. But let us now turn to the far more important case in which the passage of the water through the small ducts of any mass, as that of a glacier, is impeded by the friction of the sides of the ducts. It follows, as shown in the preceding explanation, that, since in actual cases there is no overflowing of water from the pores and crevices, here assumed to pervade one part of the mass as well as another, the fluid pressure in all confluent ducts at the point of confluence must be the same, and, therefore, independent of the height above that point at which each duct may meet the external surface of the glacier. This pressure, when the surface is exposed to the atmosphere, can be only the atmospheric pressure. Hence it follows in this case, as in the previous one, that there can be no internal fluid pressure greater than that of the surrounding

atmosphere, and such, therefore, as can exert any influence in expanding the mass, and promoting its onward motion. It will act on the interior of the mass in directions *normal* to the tubes. But there is another force also, friction, acting on the mass of the glacier in this case, in directions *tangential* to the tubes, the magnitude of which requires to be considered. It retards the motion of the fluid, acting equally on the fluid and on the sides of the tubes containing it in opposite directions. Now in the case before us, $p=p_1$, the atmospheric pressure, and, therefore, by equation (1.), art. 61,

$$\int_{s_1}^{s} f ds = g(z-z_1) - \tfrac{1}{2}(v^2 - v_1^2);$$

and differentiating,

$$f = g\frac{dz}{ds} - \tfrac{1}{2}\cdot\frac{dv^2}{ds},$$

and

$$\int_0^{s_1} f\omega ds = \int_0^{s_1} g\omega ds \,.\, \frac{dz}{ds} - \tfrac{1}{2}\int_0^{s_1}\frac{dv^2}{ds}\,.\,\omega ds, \quad \ldots\ldots \quad (2.)$$

where $\int_0^{s_1} f\omega ds$ is the whole retarding force of friction in any one of the tubes whose length $=s_1$ (art. 61), and is therefore the amount of friction produced by the water on the sides of the whole tube.

To find the value of the last term in equation (2.), we have (the duct being always full)

$$v\omega = v_1\omega_1, \quad \ldots\ldots \quad (3.)$$

where ω_1 is the value of ω when $s=0$ and $v=v_1$. Therefore

$$v^2 = \frac{v_1^2\omega_1^2}{\omega^2},$$

$$\frac{dv^2}{ds} = -\frac{2v_1^2\omega_1^2}{\omega^3}\cdot\frac{d\omega}{ds},$$

$$\frac{dv^2}{ds}\omega\delta s = -\frac{2v_1^2\omega_1^2}{\omega^2}\cdot\delta\omega,$$

and

$$\int_0^{s_1}\frac{dv^2}{ds}\omega\delta s = 2v_1^2\omega_1^2\left(\frac{1}{\omega_2} - \frac{1}{\omega_1}\right).$$

Now we may venture to assert that when water descends through exceedingly minute ducts in the manner here supposed, the velocity with which it will permeate the lower portions of the mass will be much the same as that with which it will pass through the upper portion. Assuming this, we shall have by equation (3.) $\omega_2=\omega_1$, and the value of the above definite integral will $=0$. Hence, by equation (2.), we have

$$\int_0^{s_1} f\omega ds = \int_0^{s_1} g\,.\,\omega ds\,.\,\frac{dz}{ds}:$$

$g\omega ds$ is the weight of an element of the water in the tube, and $g\omega ds.\frac{dz}{ds}$ is a force less than that weight. Consequently the definite integral on the right-hand side of the last equation expresses a force *less* than the weight of the whole fluid contained in the tube; and if we call this last weight W_1, that equation shows that the entire effect of the friction, estimated in the tangential direction in which it acts at each point of the tube, is less than W_1.

Again, let δX_1 be the part of the friction on any element of the tube, resolved parallel to the axis of x; then

$$\delta X_1 = f\omega ds . \frac{dx}{ds},$$

$$X_1 = \int_0^{s_1} f\omega ds . \frac{dx}{ds} = \int_0^{s_1} g\omega ds . \frac{dz}{ds} . \frac{dx}{ds};$$

similarly, we have

$$Y_1 = \int_0^{s_1} f\omega ds . \frac{dy}{ds} = \int_0^{s_1} g\omega ds . \frac{dz}{ds} . \frac{dy}{ds},$$

$$Z_1 = \int_0^{s_1} f\omega ds . \frac{dz}{ds} = \int_0^{s_1} g\omega ds . \left(\frac{dz}{ds}\right)^2.$$

Since $\frac{dx}{ds}$, $\frac{dy}{ds}$, and $\frac{dz}{ds}$ are each less than unity, it follows that X_1, Y_1, and Z_1 are each less than W_1; and if, taking all the tubes,

$$X = X_1 + X_2 + \&c., \quad Y = Y_1 + Y_2 + \&c., \quad Z = Z_1 + Z_2 + \&c., \quad W = W_1 + W_2 + \&c.,$$

we shall have X, Y, and Z each less than W, the weight of water in the whole mass of the glacier.

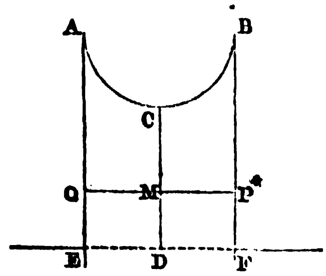
The internal pressure Z will tend to elevate the upper surface of the glacier; the pressure Y will be equilibrated by the pressures on the opposite flanking walls of the glacial valley. To interpret the meaning of X, we may suppose the vertical section of the glacier near its origin to be immoveable; then will the pressure X tend to urge onwards the whole mass in the direction of the containing valley, where there is no barrier to oppose its action.

To form a conception of the magnitudes of the pressures X, Y, Z, we may remark that the water in our supposed tubes must almost necessarily descend through the mass in approximately vertical directions. Admitting this supposition, $\frac{dz}{ds}$ will nearly equal unity, and Z_1 will nearly equal $\int_0^{s_1} g\omega ds$, or nearly the weight of the fluid contained in the single tube. Consequently the whole vertical resolved part of the friction $=Z=$ approximately the whole weight of the fluid contained in the glacier. Y will be equal only to a small portion of W, because $\frac{dy}{ds}$, in the case we have taken, will be very small; and X will be also very small, because $\frac{dx}{ds}$ will be so. And thus it follows that the whole internal pressure, arising from the friction on the sides of the small ducts through which the water descends, and tending to urge the glacier forwards, cannot exceed a small fraction of the weight of the water descending through its pores, and this latter weight again must doubtless be an exceedingly small fraction of the weight of the glacier.

63. It appears, then, from these explanations, that there will be at every point (P) of every duct, two internal forces, the fluid pressure p acting normally, and the friction acting tangentially at that point on the sides of the tube. But assuming these small ducts to pervade the superficial as well as the inner portions of the mass, the fluid pressure p can never much exceed the atmospheric pressure, which, acting externally, will very nearly counterbalance it; and this will be true whether the motion of the water in the ducts be impeded by friction or not. Also the whole resolved part of friction acting on the ducts, in the direction of the general motion of the mass, must be very small compared with the weight of the whole percolating fluid. Neither of these forces, therefore, can have any sensible influence in augmenting the onward motion of a glacier, compared with other forces which, as I shall shortly explain, tend to produce that effect.

64. In the preceding explanations I have spoken of the retardation of the motion of a fluid in small ducts, as due to friction rather than capillary action. Capillarity, when it exhibits itself in the form under which it is more usually contemplated, acts in a manner totally different to that in which friction acts. We know that a column of water, or of other fluids, of a certain length may be supported in a vertical tube of sufficiently small bore, by the attractions between the particles of the fluid, and the attraction between the fluid and the tube. These attractions produce the capillarity of the tube. If, however, the tube be perfectly full, whether the fluid be at rest or in motion, its capillary action on the fluid will counteract itself, and will produce no effect sensible to observation. It is only when the tube is partially empty that a part of the capillary action on the contained fluid is uncompensated, and exhibits itself in the column of fluid which it is capable of supporting. These partially empty tubes cannot properly belong to a system of tubes, like those of a glacier, which serve as *ducts* through which the water is constantly running, and which must therefore be constantly full; at all events, the number of tubes exhibiting the effect of capillarity as above described must probably be extremely rare in a system of ducts like those of a glacier. Supposing such tubes, however, to exist, we proceed to explain their effect in producing internal pressure.

Fig. 18.



Let A E F B be a section of the capillary tube made by a plane through its axis, the tube being considered cylindrical and vertical. Let E D F be the level of the surface of the external fluid, which we may suppose to be water, and into which the lower end of the tube is placed. Also let A C B be a section, by the above plane, of the surface of the water maintained in the tube by capillary action. Our object is to ascertain whether this water will produce any pressure on the sides of the tube tending to thrust them outwards, and thus produce an expansion of a solid mass in which any number of such capillary tubes might exist.

For this purpose we must consider how the column of water in the tube is supported.

Now, manifestly, the atmospheric pressure, acting equally in all directions, can contribute nothing to this effect; nor can there be an upward pressure on the base E F of the raised column of water; for the pressure at any point of that base can only be equal to that at any point in the surface of the external fluid, which, neglecting the pressure of the atmosphere (as we may do from what has just been stated), will be zero. Again, the resultant attraction of the tube on any single particle of the fluid must be in a direction perpendicular to the surface of the tube, and must therefore be horizontal. Consequently it can produce *directly* no force on the particle or on the whole mass of the fluid column in a vertical direction. The direct effect of gravity, on the contrary, is to drag the whole column of water downwards, and the leading point in the problem of capillary action is to explain how this tendency of gravity is counteracted.

In a question like this, which is merely subsidiary to the general problem treated on in this paper, it must suffice that I quote those results which are familiar to every one acquainted with the ordinary investigations respecting capillary action. Now it is shown by such investigations that the fluid column A E F B is supported (so far as regards any vertical action upon it) in the same manner *as if* it were suspended freely from an infinitely thin and perfectly flexible membrane accurately coinciding with the actual surface A C B, the particles of the fluid being supposed to adhere to the membrane and to each other by virtue of the cohesion or attraction between them, without which, in fact, the ordinary phenomena of capillarity could not exist.

If the column of water in the tube were entirely supported by an upward pressure on its base E F, then would gravity produce a *pressure* on any horizontal section Q M P, the whole amount of which would be the weight of the portion of the column above it, and there would be a corresponding *pressure* on the inner surface of the tube at P and Q, tending to push it *outwards*; but if the column were supported, as above supposed, at its upper surface, gravity would produce a *tension* at the horizontal plane Q M P, the whole amount of which would be equal to the weight of the fluid between Q P and the base E F. Thus the whole column would be in a state of *longitudinal tension*, and therefore also, by the fundamental property of fluids, in a state of *horizontal tension*—*i. e.* the action between the fluid and the inner surface of the solid tube, instead of being a pressure, must be a *tension*, supported by the mutual attraction between the particles of the solid and those of the fluid, supposing the contact between them not to be broken. This must be the direct effect of gravity in the actual case; and so far, therefore, its tendency is to contract and not to enlarge the diameter of the tube.

There are other causes, however, by which action may be produced between the tube and the contained fluid. First, a *pressure* will necessarily arise from the mutual attraction between the tube and the fluid; but here we have the action of the fluid on the tube, and the reaction of the tube on the fluid, the one tending to pull the surface of the tube inwards, and the other to push it outwards. The action and reaction thus counteract each other, and the pressure thus produced can manifestly have no tendency either to expand or contract the tube.

Again, in any actual case of a fluid sustained in a capillary tube, the inner surface of the tube will not generally be a surface of free equilibrium like the upper surface of the fluid itself; and therefore there must necessarily be some action between the tube and fluid, should they remain in contact, even if there were no attraction, like that just considered, between them. Now it can be shown, according to the fundamental principles on which the theory of capillarity is founded, that in all cases resembling the one we have been considering, the action in question must be equivalent to a *tension* on the inner surface of the tube. Hence this action, as well as that due to gravity, produces forces on the tube which tend to *contract* and not to *expand* it, while the mutual attractions of the tube and fluid produce directly neither the one tendency nor the other. Consequently the general tendency of capillary action in the interior of a glacier will be to contract and not to expand its dimensions. The contrary notion has probably arisen from the idea that gravity would produce a pressure on the interior surface of a capillary tube like that which it produces in a tube of larger dimensions. The fluid which may exist in these capillary tubes will, of course, add its own weight to that of the glacier, an addition probably too insignificant to produce the slightest sensible influence.

65. There is something so plausible at first sight in the idea of a great internal fluid pressure within a glacier, due to the water percolating through it, or suspended within it by capillarity, while additional weight has been given to this notion by the advocacy of the author of the Viscous Theory, that I have thought it essential to enter into more details on the subject than I should otherwise have deemed necessary, in order to prove the fallacy of the conclusions which have been deduced from erroneous views respecting it.

I now proceed to explain how the intensity of the internal forces is increased by the sliding movement of the glacier.

66. *Effect of the Sliding of a Glacier on the Internal Pressures and Tensions.*—It has frequently been objected to the sliding of a glacier, that the inequalities along the sides of the containing valley would effectually prevent such motion. This objection, however, will be entirely obviated, if we eliminate the uncertain and irregular opposing forces arising from local lateral obstacles, by substituting for the sides of the glacial valley two imaginary vertical planes near and approximately parallel to them, so that we shall thus have the tangential action of ice along these planes as the retarding force on the general mass of the glacier, instead of the action of the walls of the valley. The greatest magnitude of this retarding force must be equal to the greatest tangential cohesive power of the ice. This is, in fact, the greatest force which the walls, however irregular, could possibly exercise in opposing the sliding past them of the general mass of the glacier. To simplify our problem as much as possible, I shall also first suppose the glacier of uniform width and thickness, and the inclination of its bed to be likewise uniform. We may then consider A and B each $=0$. I shall also suppose $C=0$, and

$E=0$, as well as $D=0$. We shall thus have (art. 29 (1.)),

$$p_1=F, \quad p_2=-F,$$
$$\tan 2\alpha=\infty.$$

F will vanish along the axis of the glacier, as we have seen, and will have its greatest value along the imaginary vertical planes by which we here consider the moving mass as effectively bounded along its lateral margins. Let its value along these planes be F_1. If we suppose, as we may in this approximation, that every particle in the same vertical line moves with the same velocity, F_1 will be the same for every point of the bounding lateral planes. Moreover, let the tangential action exercised by the bed of the glacier on its lower surface be denoted by f_1. Let us now conceive the mass to be placed in a position of no constraint. It will immediately assume a position of constraint by virtue of its small extensibility, and, provided the internal forces p_1, p_2, and F are insufficient to dislocate the mass along the lateral vertical planes (where their magnitudes will be the greatest), the glacier will be held at rest in its state of constraint by the external forces acting upon it. Now F_1 and f_1 being referred to a unit of surface, if a, b, c be the length, breadth, and depth of the glacier, we shall have

$$\text{The tangential force on each flank } = F_1 ac,$$
$$\text{The tangential force on the bottom } = f_1 ab,$$

and these must be in equilibrium with the weight of the mass on the plane whose inclination is ι. Hence if ω be the weight of a unit of volume of glacial ice, we shall have

$$2F_1 ac + f_1 ab = \omega abc \sin \iota,$$

and

$$F_1 = \tfrac{1}{2}\left\{\omega b \sin \iota - f_1 \frac{b}{c}\right\}.$$

Suppose $\bar{F}$ to denote the tangential cohesive power of the mass. Then, if

$$F_1 = \bar{F},$$

the glacier will be just on the point of dislocation; and if F_1 exceed $\bar{F}$ in the smallest degree, the mass will be dislocated along the lateral planes and move onwards.

We have shown (art. 8) that the effect of f_1 *to hold the mass of the glacier at rest* is extremely small; consequently it may be neglected in the above equation, and we shall then have

$$F_1 = \tfrac{1}{2}\,\omega b \sin \iota.$$

If the lower surface of the glacier were firmly frozen to its bed, we might have $f_1=F_1$, and therefore

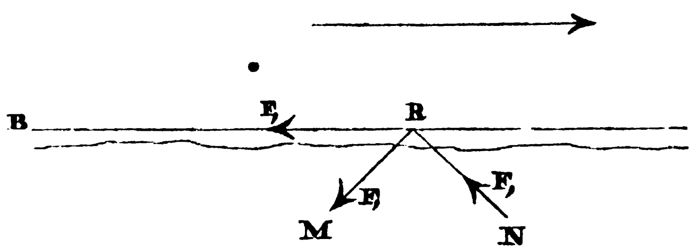
$$F_1 = \frac{\omega b \sin c}{2+\dfrac{b}{c}}:$$

c is properly the depth of the glacier along its flanks, so that $\frac{b}{c}$ may be large.

Hence we see that F_1 may become much greater when f_1 is very small, than if it were nearly $=F_1$. Consequently F_1 *will be much more likely to attain the magnitude* $\overline{F}$, *sufficient to dislocate the mass, if the glacier slide over its bed as described in art.* 8, *than if it were attached to its bed so as not to slide at all, and its whole motion should be that due to the molecular mobility of its particles alone.*

The above equation expresses the condition of the mass being on the point of dislocation in terms involving the *tangential* cohesion; it will be easy to express it in terms of the *normal* cohesion. Generally, instead of the forces A, B, F acting on any element, we may substitute the forces p_1, p_2 (art. 29), acting in their proper directions as determined by the values of α. Let fig. 19 represent one side of the glacier with the lateral plane B R; then, since at any point (R) in that plane $p_1=F_1$, $p_2=-F_1$, and $\tan 2\alpha=\infty$, we may substitute for F_1 acting tangentially along R B, two other forces, viz. a tension $=F_1$ along R M, and a pressure $=F_1$ along N R, both directions making angles of 45° with B R. Consequently an equal tension and pressure similarly applied at each point of the lateral vertical planes, would by hypothesis balance the tendency of the mass to descend in the direction B R. Now F_1 acting at an angle θ on B R will produce a force $=F_1 \cos\theta$ on a unit of surface of B R. Therefore the tension F_1 acting along R M will produce a force on the same unit, of which the resolved part along R B will

Fig. 19.



$$=F_1 \cos 45°.\cos 45°,$$

$$=\frac{F_1}{2}.$$

Similarly, the pressure F_1 along N R will produce a force the resolved part of which along R B will also $=\frac{F_1}{2}$. These together $=F_1$, and the whole supporting force on the two lateral vertical planes will $=2F_1ac$, F_1 being now the measure of a pressure or a tension acting *normally* on a unit of surface. Therefore

$$2F_1ac=\omega abc \sin\iota,$$

and

$$F_1=\omega\frac{b}{2}\sin\iota.$$

Consequently a glacier like that we have been considering, will necessarily be dislocated by its own weight resolved along the plane of its bed, if its normal cohesive power be less than the weight of a column of glacial ice whose transverse section is unity, and whose length = semiwidth of the glacier multiplied by the sine of the inclination of its bed to the horizon. If the glacier were a mile wide and its inclination about 5°, it would be necessary, in order that it should not be dislocated, that its cohesive power should be such that a vertical cylindrical column of glacial ice about 200 feet long should be capable of supporting itself when suspended by its upper extremity. This would be the measure of the greatest tension $p_1(=F_1)$ at any point along either of the lateral vertical planes, which could be produced under our assumed conditions. If the cohesion were less than this, the mass would necessarily be ruptured in some way or other in its marginal regions. The above, however, is only the lowest limit to which the force F_1 would attain. For let a represent the length of a given portion only of a glacier, such that at the lower transverse vertical section of that portion, the longitudinal *tension* shall be greater than at the higher bounding transverse section, instead of being equal to it as above supposed. It is manifest that F_1 will be increased along the lateral vertical planes of that portion of the glacier. Such will also be the case if the same portion of the glacier be acted on by a longitudinal *pressure* on its higher transverse bounding section, greater than that on its lower one. There would generally, in such cases, be a much greater effort to overcome the cohesion ($\overline{F}$) of the mass along the lateral vertical planes than in the case previously considered. If the valley of the glacier contract somewhat rapidly, the longitudinal pressure may be enormously increased on a given portion of the glacier by the action of the mass behind it, and this increased action will manifestly depend very much on the facility with which the mass behind slides over its bed.

It may be well to take another numerical example in which the conditions are similar to those of the example given above. Let the glacier be something less than half a mile broad, and its inclination about 3°. Then it appears, from the above expression for F_1, that dislocation would not take place, provided the cohesion of the mass were not less than the weight of a column of ice of about 60 feet long, instead of 200 as in the former example. This supposes the mass to slide freely over its bed, and the longitudinal pressure or tension to be the same behind as before; but if we suppose the mass to adhere to its bed, that adherence will help the tangential action along the lateral vertical planes to support the mass. Consequently the force F_1 called into play might be much less than the weight of a column 60 feet long, and might not be sufficient to overcome the cohesion $\overline{F}$, in which case the motion of the mass would be arrested. This conclusion would be strictly applicable only to our *typical* glacier, but is sufficient to explain how, in the actual case of a glacier, the facility of its sliding may increase its power to overcome the obstacles to its motion by the fracturing of its mass.

I have chosen the simplest cases for the purpose of more easily explaining the manner in which the sliding of a glacier increases the dislocating power of the forces acting upon it. In the more complicated cases, in which C and E are taken into account at considerable depths, we shall obtain greater values of the tearing and crushing forces p_1 and p_2. It is the latter which will be principally increased at greater depths. Still it must not be supposed that the mere increase of weight can explain the apparent difficulty already suggested (art. 60), viz., Why, when the mass has been crushed and then immediately regealed under the existing conditions at any proposed point, it should again be crushed at the same point. This second crushing does not, in fact, take place under the same pressure as that under which the regelation took place. This latter process occurs when the pressure depending on the angular distortion of the element, or on the force F, has been destroyed by the previous crushing; and then the angular distortion, the force F, and the increase of the maximum pressure are reproduced, and the mass is crushed again. We thus understand how the alternate processes of crushing and regelation must be repeated when the mass is in motion, though the weight alone of the mass, however great, might not, and probably would not, if unaided by the distortion arising from the motion, be sufficient to crush any even the lowest portions of the glacier, any more than the superincumbent weight of a mountain crushes the lower portion of the strata which compose it. We have seen how this power of distortion is increased by the sliding of the glacier.

It is thus that I conceive the process of regelation in glaciers to be intimately united with their uniform sliding movement. Both these properties are proved to belong to ice at the same particular temperature (that of freezing), by clear and conclusive experiments, and form, as I have endeavoured to show, the foundation of a theory which explains all the principal observed phenomena in the motion of glaciers, without assigning to ice any property, like viscosity accurately defined, which it cannot be experimentally proved to possess. In the introduction to this paper I have shown that the property of regelation is totally distinct from that of plasticity or viscosity, if by either of those terms it is intended to designate a determinate property of bodies distinct from that of solidity. All the experimental elucidations which have been given of the Viscous Theory have been drawn from bodies, such as moist plaster of Paris, tar, treacle, unconsolidated lava, &c., all of which may with strict propriety be termed plastic or viscous; but I maintain that ice possesses no property which at all assimilates it to those substances. I have shown that the *apparent* plasticity of ice, as manifested in the compressions of glaciers under great pressure in the course of their motion, is no determinate property of ice at all, but a consequence of its fracture and regelation. Still, differing as I do from the author of the Viscous Theory, I consider the scientific world largely indebted to him for his unwearied researches respecting glacial phenomena, for the large amount of general and detailed knowledge which he has communicated to us, and the interest with which he has helped to invest the subject. Moreover, though he may not have been the first to observe that fundamental character in the motion of a glacier which consists in the more rapid motion of its axial portions, he was the first glacialist whose mind was thoroughly imbued with the necessity of recognizing the influence of molecular mobility in the mass of a glacier on its general onward motion. In like manner, though not absolutely the first to observe that interesting and characteristic structure of glacial ice, the veined structure, he was doubtless the first to recognize the existence of law in that structure, and its consequent importance as indicating the operation of causes physical or mechanical, or both, with which we were entirely unacquainted. In all this, Principal FORBES has contributed largely to the sound progress of glacial science, and his labours must always be highly appreciated. Beyond this boundary, he stepped into the region of speculative theory; and it appears to me, as may be seen from much I have said in this paper, that he did so without any distinct definition of the fundamental physical principle on which his Viscous Theory rested, and without the experimental verification which it demanded, and, moreover, without the guidance of those more refined and exact mechanical researches without which I am sure we can see our way but dimly through the more complicated of those problems which glacial theory presents to us. Hence it is that I have been led to dissent from most of his speculative views. That dissent has been unreservedly expressed in this paper, and therefore it is that, in concluding it, I would bear testimony to what I consider the great and legitimate claims which the author of the Viscous Theory (however we may differ from the theory itself) has established to be regarded as one of the leading promoters of glacial science.

After the greater part of this paper was written, a very interesting memoir by the Master of the Mint, "On Liquid Diffusion applied to Analysis," was brought under my notice. He there speaks of the "colloidal condition of matter" (of which gelatine seems to afford the type) as opposed to the "crystalloidal condition." He remarks (Philosophical Transactions, Vol. 151, Part I. 1861), "Ice itself presents colloidal properties at or near its melting-point, paradoxical though the statement may appear." Again, "Ice, although exhibiting none of the viscous softness of pitch, has the elasticity and tendency to rend seen in colloids." "It further appears to be of the class of adhesive colloids. The redintegration (regelation of FARADAY) of masses of melting ice when placed in contact, has much of a colloid character. A colloidal view of the plasticity of ice demonstrated in the glacier movements will readily develope itself."

These passages were written without any direct reference to glacial questions; but it occurred to me that a glacialist might possibly put constructions on them unfavourable to the views I have expressed respecting the *solidity* of ice, and the absence in that substance of any property which could, according to my own definition of the term, be called *plastic*. Consequently I wrote to the author requesting him to give me some further elucidation of his views on one or two points bearing on my own definitions and explanations. I proposed to him the following questions, to which he kindly gave me the subjoined clear and explicit replies.

(1) "Is the tendency to a colloidal character in ice, as opposed to a vitreous, crystalline brittle structure, sufficient to interfere materially with the *restitution* described*, by giving the ice a greater degree of plasticity?" Mr. GRAHAM's answer was, "I believe not. A colloid, on the contrary, is often as nearly perfectly elastic as possible. Take the gluey material used to form the roller by which ink is applied in book-printing, as an illustration of the elasticity of a colloid, with the entire absence (apparently) of true plasticity or viscosity."

(2) "Will the colloidal state of ice at temperature = zero (C.) materially modify the *modus operandi* (assuming the ice to be solid), rendering it approximate to the process which would take place supposing the mass to be plastic?" Mr. GRAHAM replies,

* See art. 1.

"Quite the contrary. The only influence to be looked for from the colloidal state would be greater elasticity,—that is, the matter would be likely to yield more before rending or breaking, supposing it to be colloidal, than if it were crystalline. Your mechanical conception of the *modus operandi* in the fracture of a solid (not plastic) applies equally well to a crystal or colloid. I go entirely with you up to that point."

Mr. GRAHAM further remarks, "Colloidality is invoked chiefly with the view of accounting for the ready reunion, the redintegration as I have called it, of fragments of ice brought into contact with each other. It is a very general (perhaps universal) character of colloids to adhere and reunite when two masses are pressed together. In fact all our adhesive substances, gum, glue, starch, &c., belong to the class of colloids. Even glass, which is a colloid of fusion, shows the adhesive character, two sheets of polished plate glass often adhering so thoroughly as to tear up each other's surface when forcibly separated. Another colloid adhesive like ice, is fused phosphoric acid—'glacial' phosphoric acid as it is called, in prescience, one might imagine, of this discussion! No such adhesive property is ever found in crystalline surfaces, so far as I am aware. It is *quà* colloid that ice appears to be adhesive. The discovery of FARADAY'S, of the adhesive quality of ice, is the fundamental fact of the glacier discussion. The name 'regelation' applied to it may, however, be objected to, being quite speculative, and implying, as it appears to do, that two pieces of ice come to be *cemented together* by the freezing of a film of water between them, instead of simply adhering perfectly and uniting as two pieces of plate glass might do. The great fact, however, remains, and the name is but a trivial matter."

I also thought it right to inquire of Mr. GRAHAM the exact meaning he intended to attach to the word "demonstrated" in the last sentence of the above quotation from his memoir. A glacialist might possibly, I thought, if so disposed, interpret it as meaning "proved" or "fully established." In his answer Mr. GRAHAM says, "For 'demonstrated' I should have said 'indicated' as you suggest." He also remarks, "You will, I am certain, do good service in the glacier discussion by the precision you introduce into the terminology of the subject*. I accept without reserve your definitions of plasticity, &c. In the passage you quote from my paper, I had spoken of ice as plastic in a more general sense, as plasticity is understood, I apprehend, by Professor FORBES†. A solid was looked upon as plastic of which the form can be remodelled *anyhow*—by fracture and reunion, as well as in consequence of viscosity."

These views of the Master of the Mint are so explicitly and clearly expressed that they need no comments on my part, beyond the expression of the high gratification I derive from the coincidence of the opinions stated in his letter with those which have been put forth in this memoir.

* I had communicated my definitions to Mr. GRAHAM.

† This is what I termed *quasi-plasticity*, or *apparent plasticity*.

XXXII. *On the Anatomy and Physiology of the Spongiadæ.*—Part II.

By J. S. Bowerbank, *LL.D., F.R.S., F.L.S. &c.*

Received June 17,—Read June 20, 1861 (continued from page 382, Philosophical Transactions, 1858).

Keratode

Is the substance of which the horny elastic fibres of the skeleton of the officinal sponges of commerce are composed. It has, correctly speaking, no relationship either chemically or structurally with horn, and Dr. Grant has judiciously rejected the term "horny fibre" as applied to the sponges of commerce, and has substituted that of keratose by way of distinction; and in accordance with that term I propose to designate the substance generally as keratode, whether it occurs in the elastic fibrous skeleton of true *Spongia*, which are composed almost entirely of this substance, or of those of the Halichondraceous tribe of Spongiadæ, where it is subordinate to the spicula in the construction of the skeleton, and appears more especially in the form of an elastic cementing medium. In a dried state it is often extremely rigid and incompressible, but in its natural condition it is more or less soft, and always flexible and very elastic. It varies in colour from a very light shade to an extremely deep tint of amber, and it is always more or less transparent. In its fully developed condition, in the form of fibre, it appears always to be deposited in concentric layers; but in the mode of the development of these layers there are some interesting variations from the normal course of production. As we find in *Aranea diadema*, the common Garden Spider, that the creature has the power of modifying the deposit of the substance of its web so that the radiating fibres dry rapidly while the concentric ones remain viscid for a considerable period, so we find in the production of the young fibres of the skeletons of the Spongiadæ in some species, as in those of commerce, there is no adherent power at the apex of the young fibre, excepting with parts of its own substance; while in *Dysidea*, and in some other genera, the apex of the newly-produced fibre is remarkably viscid, adhering with great tenacity to any small extraneous granules that it may happen to touch in the course of its extension; but this adhesive character appears to be confined to the earliest stages of its production only, as exhibited at the apices of the newly-produced fibres, the external surface immediately below the apex exhibiting no subsequent adhesive property.

Lehman, in his 'Physiological Chemistry,' Cavendish Society's edition, vol. i. p. 401, states that *Spongia officinalis* of commerce consists of 20 atoms of fibroin, 1 atom of iodine, and 5 atoms of phosphorus; and in treating of the physiological relations of fibroin as regards sponges, he observes, "Its chemical constitution affords one of the arguments why the *Spongia* should be classed among animals and not among plants, since in the vegetable kingdom we nowhere meet with a substance in the slightest degree resembling fibroin."

From the general physiological characters of the skeletons of the Sertularian and other Zoophytes, I had long suspected that their component parts were identical, or very nearly so, with those of the skeletons of the Spongiadæ, and I therefore applied to my friend Mr. GEORGE BOWDLER BUCKTON to assist me in determining this point, and he very kindly undertook to make comparative qualitative analyses of two species of Zoophytes, *Sertularia operculata* and *Flustra foliacea*, with the fibres of *Spongia officinalis* and of raw silk, and I cannot do better than quote entire the report of the results of his examination:—

"I have examined the Zoophytes you sent me, and have compared their deportment under chemical agency, with that shown by white silk and the fibre of ordinary sponge.

"All the specimens were treated in a similar manner, being purified from foreign matter, as far as possible, by boiling for two hours in water, and subsequently for the same period in strong acetic acid. With the exception of *Flustra*, the substances exhibited by this treatment little change in their outward appearance. Carbonate of lime enters so largely into the composition of *Flustra*, that its disintegration by acids ought to cause no surprise.

"From the results of the first seven experiments, which for convenience I have arranged in a Table (see next page), I conclude that all these bodies contain the same, or a very similar animal principle, which I suppose to be identical with MULDER'S fibroin. The varying colours of the precipitates from tannic acid and ammonia, I think are probably due to the traces of sesquioxide of iron present in the fibres, and the difference in shade is simply caused by the greater or less preponderance of that metal.

"Although I have not been able to obtain fibroin in a state of chemical purity, I would state that, to my knowledge, there is no vegetable principle which behaves itself towards reagents in a manner similar to that shown by the substance of silk, sponge, &c.

"MULDER and CROOKEWIT'S analyses show silk and sponge scarcely to differ in composition.

Fibroin from silk.		Fibroin from sponges.		
Carbon	48·5	Carbon	46·5 to	48·5
Hydrogen	6·5	Hydrogen	6·3	6·3
Nitrogen	17·3	Nitrogen	16·1	16·1
Oxygen, Sulphur, &c. &c.	27·7	Oxygen, Sulphur, Phosphorus, Iodine	31·1	29·1
	100·0		100·0	100·0

"SCHLOSSBERGER has recently expressed his doubts of the identity of composition of these bodies, from the circumstance that silk is readily soluble in strong ammonia, saturated with oxide of copper, whilst sponge is scarcely, or not at all, affected by long maceration. My own experiments prove the same fact, yet it is not impossible that the minute quantities of iodine, phosphorus, and sulphur present in sponge may modify the solubility of the fibre.

"Under the supposition that a resinous gum might act as a protection, portions of sponge were boiled in benzol, ether, and alcohol, but these solvents did not modify the characters in any noticeable degree.

	Flustra foliacea.	*Sertularia operculata.*	*Spongia officinalis.*	Silk from *Bombyx.*
Ignited.	Yields a nitrogenous odour. ——— Leaves much ash, in the form of a facsimile of the frond. Chiefly composed of carbonate of lime.	Yields a nitrogenous odour. ——— Leaves much ash, but much less than the preceding.	Yields a nitrogenous odour. ——— Leaves a white ash in some quantity.	Yields a nitrogenous odour. ——— Leaves much ash.
Boiled in water and subsequently in acetic acid.	Acid disengages much carbonic acid. ——— The zoophyte disintegrates and leaves a brown flocculent residue.	Apparently unaffected. ——— Form unchanged.	Apparently unaffected. ——— Form unchanged.	Apparently unaffected. ——— Form unchanged.
Washed and boiled in concentrated hydrochloric acid.	Greater part soluble in the acid. ——— A brown gelatinous mass remains.	Almost entirely dissolved after ten minutes' boiling.	Almost entirely dissolved. ——— The residue is gelatinous.	About $\frac{3}{4}$ soluble in the acid. ——— Gelatinous residue.
Tannic acid added to the hydrochloric acid solution.	A white precipitate, insoluble in acetic, but soluble in oxalic acid.	A white precipitate.	A white precipitate.	A rather copious white precipitate.
Potash added to the hydrochloric acid solution. ——— Ammonia gives similar reactions.	Precipitation of a few flocks.	No precipitate of any consequence.	Small quantity of a gelatinous precipitate.	A few flocks precipitated.
Tannic acid added to the above neutralized solution.	An abundant yellowish precipitate. ——— If alkali in excess, the precipitate is violet.	Abundant precipitate, ——— which turns reddish purple by excess of alkali.	Abundant precipitate. ——— Coloured by excess of alkali.	Copious precipitate, which takes a flesh tint by excess of ammonia.
Bichloride of mercury added to the hydrochloric solution.	Slight white precipitate.	Slight precipitate.	Slight precipitate.	Precipitate rather more copious than that from sponge.
Boiled in a solution of oxide of copper in ammonia.		Insoluble.	Insoluble.	Perfectly soluble; not again precipitated by acetic acid.

"I consider, however, that this difference between sponge and silk in no wise affects the question of the former substance being a product of the animal kingdom, which the other experiments, I think, satisfactorily prove."

In considering the results of these analyses with a view to proving the animal nature of the Spongiadæ, the evidence afforded by the coincidence of its structural character and its chemical constituents with those of *Sertularia operculata*, is still more conclusive than that derived from the chemical constituents of silk; and, in truth, the action of the chemical agents on the zoophyte and the sponge, as might naturally be expected, are almost in perfect accordance.

Membranous Tissues.

These structures may be divided into two classes.

1st. Simple membranous tissue.

2nd. Compound membranous tissue.

The first is a simple, apparently unorganized, thin pellucid tissue. It is evidently not composed of an extension of keratode, as it is rapidly decomposed after the death of the animal. It is found in abundance filling up the areas of the network of the skeleton in a great variety of sponges, and it appears to be capable of secreting sarcode on both its surfaces when thus situated; on the dermal membranes the sarcode is found on the internal surface only.

Compound Membranous Tissue.—These structures consist of simple membranous tissue combined more or less with primitive fibrous tissue. Their most simple forms exist in the membranes lining the interstitial cavities of the sponge, and in the dermal membranes.

It is difficult in some cases to discriminate between this class of tissues and simple membranes, unless it be by the aid of their functional characters, as the compound tissues are frequently quite as pellucid, although not so thin, as the simple ones.

In dermal membrane, and the membranous linings of the internal cavities of the sponge, they are thin and very translucent; but on careful examination with high microscopic powers and transmitted light, with the aid of polarization, we frequently detect the elastic primary fibrous tissues incorporated with the structure. In the contractile membranes forming the oscular diaphragms in *Grantia*, and in those at the base of the intermarginal cavities in *Geodia* and *Pachymatisma*, they attain a greater degree of thickness, and especially in the two latter genera of sponges. In *Alcyoncellum*, Quoy et Gaimard, the organization of their tissue is still more complex, and we there find them constructed of repeated layers of membranous structure, abounding in primitive fibrous tissue disposed in parallel lines in each layer, the fibres disposed so closely together as to completely cover the membrane beneath, and the direction of the fibres being at various angles to the axis of the great cloacal appendages of the sponge, so as most effectually to aid in the contraction or expansion of that organ. They are so closely packed together and so intermingled, that I could not ascertain their length; but from the gradual attenuation of some of their terminations, they would seem not to be continuous for any considerable distance. On some of the layers of this compound membrane the fibres were disposed in an even and continuous stratum, while on others they were gathered into broad, flat, parallel fasciculi. When the compound structure consists of several layers of fibro-membranous structure, the disposition of the fibres on the different layers is not coincident. In some cases they cross each other at right angles, while in others the angle does not exceed 45 degrees. The latter mode of arrangement appears to prevail in the membranes connecting the great longitudinal fasciculi of spicula, forming to a great extent the skeleton of the cloacal appendages of the sponge; while the arrangement at right angles appears also in the tissues immediately surrounding the great skeleton fasciculi.

This fibro-membranous tissue abounds in the dermal and interstitial structures of the sponges of commerce, but the greatest development of this structure is exhibited in the genus *Stematumenia.*

Fig. 4, Plate XXVII. represents a small portion of the lining membrane of one of the great excurrent canals of the common honeycomb sponge of commerce, in the condition in which it came from the sea. The primitive fibrous tissue is seen arranged in a single layer in parallel lines at right angles to the long axis of the canal, but partially obscured by the stratum of sarcode on the membrane.

Fig. 3, Plate XXVII. represents a small portion of the dermal membrane of a *Stematumenia*, in which the primitive fibres are seen wandering in every direction over the surface of the membrane.

Figs. 1 & 2 in the same Plate represent portions of a stouter and a more compound membranous structure, from the walls of one of the great cloacal projections from the surface of *Alcyoncellum robusta*, BOWERBANK, MS. In this case the membrane is strengthened by two or more layers of primitive fibrous structure, the parallel fibres of each crossing the others at various angles.

Fibrous Structures.

There are two well-characterized classes of fibrous structure.

1st. Primitive fibrous tissue.

2nd. The fibres of the skeleton.

1. *Primitive Fibrous Tissue.*

The first of these tissues is exceedingly minute. The fibres are cylindrical in form, and are usually of considerable length; but where they are fully developed, they occur in such numbers, and in such a matted condition, that I have been unable to separate an unbroken one from the mass. They continue through the whole of their length as nearly as possible of the same diameter, and there rarely appears to be any attenuation towards their terminations, which are usually obtuse. They are evidently very elastic and contractile. When partially separated from their attachments to the membranes, the free ends seldom remain straight, and most frequently they curl considerably in different directions. They appear to be perfectly solid; I could not by the aid of polarization discover the slightest indication of a central cavity. They vary in diameter in different species of sponge, and frequently even in the same individual. In a species of *Stematumenia* from the Mediterranean, I measured an average-sized fibre which was $\frac{1}{4166}$ inch in diameter, while a smaller one, closely adjoining, measured $\frac{1}{9875}$ inch. In this genus these fibres are more fully developed and larger in size than in any other sponges with which I am acquainted. In the sponges of commerce, in the membranes of which they are exceedingly numerous, they are much more slender. In one of the excurrent canals of the common honeycomb sponge, one of the largest measured $\frac{1}{10000}$ inch in diameter, and one of the smallest $\frac{1}{17647}$ inch. In the dermal membrane of the best Turkey sponge they were still less, not exceeding $\frac{1}{18000}$ inch.

This description of fibre is not an absolutely necessary constituent of a sponge, and in many of the Halichondraceous tribes it is exceedingly difficult to find even a single straggling fibre on the interstitial or dermal tissues, while in other genera, as in *Spongia*, *Stematumenia*, and *Alcyoncellum*, they form an important element in the structure of the compound membranous tissues, in which they are closely disposed in parallel lines, occasionally giving off branches, but never appearing to anastomose with each other like the larger fibres of the skeleton.

These fibro-membranous tissues were described by me in the 'Annals and Magazine of Natural History,' vol. xvi. p. 406, plate 14, figs. 1, 3, 4 & 5, in my description of the genus *Stematumenia*.

If a small portion of the dermal membrane of a young Stematumenia be carefully removed from the surface of the sponge, the primitive fibres will be seen projecting from the edges of the membrane in considerable numbers; and occasionally they may be seen to be furnished with a terminal bulb, the greatest diameter of which is about three times that of the fibre. The bulbs are variable in form; sometimes they are largest at the base, or pear-shaped, at other times regularly oval, or nearly globular. By far the greater number of fibres exhibit no bulbs at their terminations; those which have them are always less in diameter than the general average of the fibres. Sometimes, but not very frequently, the bulb exhibits faint traces of a nucleus. On examining the dermal membrane by transmitted light and a linear power of 666, I found numerous globular cells collected in groups on various parts of its inner surface, many of them having a well-defined central nucleus; and among these cells I found the bulbs imbedded with the fibres emanating from them, and in no respect differing in appearance from the non-fibrous cells around them (Plate XXVII. fig. 5, *a*, *a*). On carefully observing a number of these bulbous fibres that had been removed from their positions on the membrane, I found that the part of the fibre nearest to the bulb was frequently flexuous, as if in a tender and immature condition, and in these cases the marginal line of the fibre was continued without the slightest break or interruption into and around the bulb, as represented in Plate XXVII. fig. 6, *a*. At this period of the development the young fibre does not measure above half the diameter of a mature one, and there is no indication of an ultimate separation from the bulb; but when the fibre has attained nearly the full size the separation is then distinctly indicated; the basal end of the fibre immersed in the bulb becomes hemispherical, and a constriction appears at the junction of the fibre with the exhausted cell. Sometimes, when thus affording indications of their ultimate separation, the cell still retains its rotundity, but all indication of its nucleus has disappeared, and it is perfectly transparent, as represented in Plate XXVII. fig. 6, *b*; while in other cases it is visible only as a collapsed and shrivelled vesicle adherent to the hemispherical termination of the fibre, as represented in Plate XXVII. fig. 6, *c*. I could not find the slightest indication of bulbs amid the matted mass of fibres lying on the inner surface of the membrane, and it was only at the torn edges of the pieces of membrane under examination, or among the groups of cells, that the bulbs in connexion with the fibres were to be discovered.

2. *Keratose Fibrous Tissue.*

General character of the keratose fibres of the horny skeleton.—The essential character of the fibres of the horny skeleton is, that their normal form is always that of a cylinder, while the network of the skeletons of the Halichondroid sponges, which approach nearest in structure to that of spiculated keratose fibre, is always more or less irregular in shape; and in the fully developed state, generally compressed to a very considerable extent; but a careful examination of the youngest portions of the two forms of skeleton-tissue will always render the difference in the two structures apparent. In the spiculated keratose fibre the keratode is always the predominant element, and the spicula the subordinate one; while in the skeletons of the Halichondroid sponges the spicula always predominate, and the keratode is merely the secondary or surrounding medium. In the former structure, in the extension of the terminations of the skeleton, the keratode is the leading element, while in the latter the spicula take the lead.

The fibre is formed of a succession of concentric layers, its increase in diameter being apparently effected at the external surface. Its longitudinal extension appears to be caused by a progressive elongation of their terminations, and new fibres are frequently to be seen pullulating from the sides of the mature ones. In the dried state it is often extremely rigid and incompressible, but in its natural condition, notwithstanding there is frequently an internal axis of extraneous matter or of spicula, it is often remarkably soft and flexible. The spicula, although immersed in the fibre, evidently possess a considerable amount of mobility within the surrounding medium.

The colour of the fibres is always amber-yellow, varying in different species from a very light to a deep yellow brown tint, and it is always semitransparent. In the living state, when the fibres happen to touch each other, whether by their terminations or laterally, they appear at all times to unite.

The keratose skeleton-fibres vary in their organization to a very considerable extent, but the whole of them may be comprised in the following eight typical forms:—

1. Solid simple keratose fibre.
2. Spiculated keratose fibre.

3. Multi-spiculated keratose fibre.
4. Inequi-spiculated keratose fibre.
5. Simple fistulose keratose fibre.
6. Compound fistulose keratose fibre.
7. Regular arenated keratose fibre.
8. Irregular arenated keratose fibre.

1. *Solid Simple Keratose Fibre.*

The typical form of this description of fibre is that which forms the skeleton of the Turkey sponges of commerce, the structure of which I described in a paper read before the Microscopical Society of London, and published in vol. i. p. 42 of its 'Transactions.' The matûre fibre is perfectly solid, and no vestige of a central cavity can be observed in any part of it, either when viewed by transmitted light, or in transverse sections of the fibre, by the aid of a Lieberkuhn. Occasionally, but very rarely, I have seen, in young and immature fibres, faint and irregular indications of there having been a very small central cavity in perhaps the earliest period of their development, but in the mature fibre I have never been able to trace such cavities (fig. 7, Plate XXVII.).

This description of fibre is occasionally surrounded by a membranous sheath, on which is imbedded a beautiful system of hollow fibrils or vessels, which sometimes wind round the skeleton-fibre in a spiral direction, at others assume a longitudinal course, giving off short cæcoid branches, or form a complex and irregular network. In an Australian sponge in my possession, the latter mode is the only form in which they occur. In some of these minute fibrils or vessels I observed numerous minute globules, which were rendered moveable by a slight pressure on the glass under which they were exhibited. The mean diameter of these tubes or vessels was $\frac{1}{9346}$ inch. This tissue is of rare occurrence, and I have been unable to determine whether it is a specific character, or whether it is due to a peculiar condition of the sponge. Fig. 9, Plate XXVIII. represents a portion of fibre from the skeleton of one of the sponges of commerce. Fig. 10, Plate XXVIII. is from a rigid species of Australian sponge. This singular tissue is described more fully in a paper which I read before the Microscopical Society of London in 1841, and which is published in their 'Transactions,' vol. i. p. 32, plate 3.

2. *Spiculated Keratose Fibre.*

This structure is essentially a solid form of keratose fibre, no central cavity ever being visible in its axis. The normal form of the fibre is cylindrical, but it is occasionally more or less compressed, and always contains a thin central line or axis of spicula arranged in longitudinal series. The spicula are secreted within the fibre, and are nearly uniform in size, and always of the same shape in the same species of sponge. In the production of the young fibres, the projection of the new keratode and the secretion of the new spicula appear to be simultaneous. In this class of structure the keratose fibre is the predominant element, and the spicula the subordinate one, and we accordingly frequently find the fibres destitute of spicula for short distances; but these occurrences are the exceptions, and not the rule of the structure. Fig. 8, Plate XXVII. represents a portion of a longitudinal section of the skeleton of *Halichondria oculata*, JOHNSTON.

The mode of the progressive development of this form of fibre is interesting. In a young specimen of *Halichondria Montagui*, JOHNSTON, I observed that when a new fibre was projected from the skeleton it usually contained a single spiculum, thinly covered by keratode at the apex, and more thickly so towards the basal end. Another spiculum followed the first, the terminations of each overlapping the other; and at the junction of the two, the keratode was accumulated in the form of a plumber's joint, as represented in Plate XXVII. fig. 9, so as to give additional strength to the junction of the spicula, while the middle portion of the second spiculum remained very thinly covered by keratode. When the distal end of the new fibre has attained its proper length, or has become cemented to the side of another fibre, the remaining portion of the keratode is produced, and the fibre then assumes a regular cylindrical form.

3. *Multi-spiculated Keratose Fibre.*

This description of fibre is literally a cylindrical mass of spicula cemented together by keratode, and surrounded by a thin case of the same substance. The spicula are exceedingly numerous, and very closely packed in parallel lines in accordance with the axis of the fibre. They are nearly uniform in size, and always of the same shape in the same species of sponge. In this structure the spicula are the predominant element, and the keratode the subordinate one. Fig. 10, Plate XXVII. represents a fibre from the skeleton of *Halichondria ægagropila*, JOHNSTON.

4. *Inequi-spiculated Keratose Fibre.*

This form of fibre is composed of an infinite number of spicula disposed in every possible direction, cemented together by keratode, and surrounded by a sheath of the same material. The spicula agree in form in all parts of the sponge, and are nearly of the same size. In these fibres the spicula are the predominant element, the keratode the secondary one. In the only sponge in which this form of structure has yet been found, *Raphyrus Griffithsii*, BOWERBANK, MS., the fibre is very unequal in size and much varied in its form, frequently becoming very much flattened and expanded. Fig. 11, Plate A. represents a longitudinal section of a small portion of a fibre from the skeleton, showing the irregular disposition of the spicula within it.

5. *Simple Fistulose Keratose Fibre.*

This form of fibre is usually very much larger and more rigid than the solid keratose fibre. It is cylindrical, and continuously fistular. The great central cavity of the fibre usually occupies about one-third of its diameter. It is nearly uniform in its size, but occasionally it is dilated considerably for a short space, and then resumes its original diameter. In the young state the cavity is as large, or nearly so, as in the adult fibres, while the enveloping keratode assumes the form of a thin transparent amber-coloured coat, which in the mature state becomes frequently twice or three times the thickness of the diameter of the central cavity.

This great fistular space is lined with a thin pellucid membrane, which, in specimens that have been dried, appears to have been thickly covered with minute semi-opake granules. At the time of my first description of this form of fibre, published in the 'Annals and Magazine of Natural History,' vol. xvi. p. 403, I believed that in the natural condition of the fibres the central cavity was an open tube; but subsequent observations on specimens which have never been dried, have led me to the conclusion that the whole of the central space is filled with a minutely granulated substance which presents all the characteristics of sarcode.

There is no communication between the great central fistular canal and the interstitial cavities of the sponge, the projecting ends of the fibres of the skeleton being always hermetically sealed. Fig. 12, Plate XXVII. represents a fibre from the specimen of *Spongia fistularis*, LAMARCK, in the Museum at Edinburgh, given to me by Professor GRANT.

6. *Compound Fistulose Keratose Fibre.*

In its external characters this description of fibre is not, under ordinary circumstances, to be distinguished from simple fistulose fibre, and it is only when submitted to a microscopic power of about 100 linear that its peculiar character can be detected. We then find that the fibre is not only furnished with a large continuous central cavity, but that it also has numerous minute cæcoid canals radiating from the central one at irregular distances, at nearly right angles to its axis. These secondary canals are very unequal in length, and very few of them reach to near the external surface of the fibre, and none of them appear to perforate it. Their direction is usually in a straight line from the parent canal; a few assume a tortuous direction, and a still fewer number bifurcate or branch. Within the central tubes of the fibres there are frequently one or two minute simple tubular fibres; when more than one they do not unite, but they divide and traverse each a separate cavity, when they happen to reach one of the anastomosing points of the great skeleton-fibre. The structures are described more at length in the 'Annals and Magazine of Natural History,' vol. xvi. p. 405, under the head of "*Auliskia*," a new genus of sponges, founded principally on the compound fistulose structure of its skeleton-fibres. Fig. 13, Plate XXVII. represents a portion of compound fistulose keratose fibre as seen with a linear power of 100. Fig. 14, a portion of a similar fibre under a power of 300 linear.

7. *Regular Arenated Keratose Fibre.*

This description of fibre under ordinary circumstances has very much the appearance of simple fistulose fibre, but when examined by transmitted light with a linear power of about 100, we find in the centre of the fibre a series of grains of extraneous matter, occupying the place of the large continuous canals of the fistular forms of fibre. The series of extraneous matters is not always continuous, and when an interruption takes place the fibre becomes solid, or faint traces only of a central cavity remain. The mode of the inclusion appears to be due to the extreme terminations of the young fibres being viscid, and thus seizing on any extraneous particles that happen to come in contact with them. The growing keratode quickly envelopes them, and proceeding on its course of extension, seizes in like manner on other particles of sand or solid matter, and thus a continuous and regular chain of extraneous material is imbedded in the axis of the fibre, as represented by figs. 1 & 2, Plate XXVIII. This description of fibre is found in a great variety of keratose sponges, and especially among the coarse rigid skeletons of the Australian species, as represented by fig. 1, Plate XXVIII.; and among the flexible sponges, as represented by fig. 2, Plate XXVIII.

8. *Irregular Arenated Keratose Fibre.*

I have described this form of fibre in a paper descriptive of two species of *Dysidea*, read at the Microscopical Society of London, Nov. 24, 1841, and subsequently published in vol. i. p. 63 of their 'Transactions.'

The adult and fully produced fibre is frequently half a line or more in diameter. It is built up in all parts of its substance of grains of extraneous matter, each one being separately enveloped in keratode. The adhesive power in the young progressing fibre not being confined to its apex only, its sides also seize upon the surrounding grains of solid matter, and the keratode speedily passing round and enveloping them, the whole fibre becomes a solid cylinder of irregularly imbedded molecules. There is a great variety of substances imbedded in these fibres, dependent, as a matter of course, on the amount of material surrounding them at the period of their development. The skeleton of *Dysidea fragilis*, Johnston, a British species very common on the south coast of England, presents one of the best types of this form of fibre. And single grains of sand are frequently to be found among the fibres of the surface of the sponge, elevated on short pedicels of the rapidly growing young fibres, sometimes entirely, and at others only partially enveloped by the progressing keratode. Figs. 3, 4, & 5, Plate XXVIII., represent portions of fibre from the same individual.

This genus of sponges appear, to the best of my knowledge, to be the only animals that construct an internal skeleton almost entirely of extraneous materials.

Siliceous Fibre.

This structure is widely different from any of the keratose fibres which contain either secreted silex in the state of spicula, or extraneous silex in the form of sand. The whole substance of the skeleton fibre consists of solid silex, secreted and deposited in concentric layers, exactly after the manner of the secretion of pure keratode in the fibres of the sponges of commerce. When cleansed from the sarcodous matter by which they are surrounded in a living state, the fibrous skeleton bears a striking resemblance to fibres of spun glass, and is quite as pellucid and colourless as the artificial material, and the dead sponge quite as brittle. The fibrous skeleton of *Dactylocalyx pumicea*, STUTCHBURY, in its mode of arrangement strikingly resembles that of one of the sponges of commerce; it is equally complex and irregular in its structure, and the component fibres quite as much anastomosed. In that species the fibres are smooth and cylindrical, but in others they frequently abound with minute, obtuse, wart-like elevations.

There is every indication in the skeletons that the increase in diameter, and the extension in length in the fibres are effected in the same manner as in the solid keratose fibres. The free terminations of the young fibres have the same attenuated but obtuse form, and the pullulation of the young fibres from the sides of the mature ones is quite as apparent as in their keratose congeners; but, in the young state, they never appear to be viscid, as the keratose ones frequently are; and extraneous matters are never detected at their apices, or on their substance.

There are two distinct forms of this class of fibre:—

1st. Solid siliceous fibre.

2nd. Simple fistulose siliceous fibre.

The structure of solid siliceous fibre is very similar to that of solid keratose fibre. Occasionally there are indications of a former existence of a minute central canal, but in the fully developed fibre this is rarely visible. The external characters of these fibres vary in each species. In a new siliceous sponge in the British Museum, designated by Dr. GRAY *M^cAndrewsia azoïca*, the fibres are quite smooth, as represented in Plate XXVIII. fig. 6. But in the greater number of species they are more or less tuberculated, as in Plate XXVIII. fig. 7, which represents a group of fibres from the type-specimen of *Dactylocalyx pumicea*, STUTCHBURY, a portion of which is in the possession of Dr. J. E. GRAY. In other species in my possession the tuberculation is very strongly produced, as represented in a few fibres of *Dactylocalyx Prattii*, BOWERBANK, MS., Plate XXVIII. fig. 8.

Of the 2nd form, simple fistulose siliceous fibre, I know but one example, and that is the remains of the siliceo-fibrous sponge on which the beautiful specimen of *Euplectella cucumer*, OWEN, is based.

The tubulation of the skeleton-fibre is very similar to that of some varieties of simple fistulose keratose fibre, but the central cavities are not so invariably continuous as in the keratose varieties of fistulose skeleton-fibre. Fig. 11, Plate XXVIII. represents a small piece of the spinulated simple fistulose fibres of the skeleton of Dr. ARTHUR FARRE'S specimen. The spinulation of these fibres is a remarkable character. It is the only case of the production of acute spines on the skeleton-fibre of a siliceo-fibrous sponge with which I am acquainted.

Prehensile Fibre.

In the course of my examination of the fibrous skeleton-tissues, I have found but one instance in which they have developed prehensile organs to assist in the attachment of the sponge, and this is in a minute siliceo-fibrous species, parasitical on the base of a specimen of *Oculina rosea*, from the South Sea. In this sponge the basal fibres curve downward in the form of numerous small, nearly semicircular reversed arches, from the lowest portions of each of which there is a short stout portion of fibre projected; and at about the length of its own diameter downwards, a ring of stout prominent bosses, six or eight in number, is produced, very considerably increasing its diameter at that part, immediately beneath which the fibre is attenuated to a point. These singular organs are admirably calculated to penetrate the porous cavities or fleshy envelopes of the coral, and thus to securely attach the sponge to its adopted matrix (Plate XXVIII. fig. 12).

Cellular Tissue.

The cellular structures in the Spongiadæ are few and very simple in form. We find no series of conjoined cells in the body of the sponge, as in vegetable tissues. The only forms in which true cellular structures occur in the bodies of sponges, are those of detached spherical molecular cells, and of discoid or lenticular nucleated cells. Cellular structures of the first form are found in abundance on the fibres of many species of the true sponges, and are believed by Dr. JOHNSTON to be the reproductive organs of that genus. They are very minute; an average-sized one measured $\frac{1}{11000}$ inch in diameter. They are pellucid, and afford no indications of a nucleus, either single or multigranulate.

Imbedded in the sarcodous stratum on the interstitial membranes in many of the Halichondroid tribes of sponges, we frequently find numerous compressed circular cells. In the greater number of cases they are so translucent as to readily escape observation, even with a tolerably high power; but in other species, as in *Ecionemia acervus*, BOWERBANK, MS., a new genus of sponges from the South Seas, in the collection of the Royal College of Surgeons, and in *Halichondria nigricans*, BOWERBANK, MS., a British species, these tissues are developed in a more than usually distinct condition.

In the first-named sponge they are thickly dispersed on the surfaces of the interstitial membranes, but without any approach to order or arrangement. They are decidedly lenticular in form, with a well-defined transparent nucleus, which varied in size from about one-fourth to three-fourths the diameter of the cell in which it was contained. The cells varied considerably in size; the largest I could find was $\frac{1}{3500}$ inch in diameter, and one of the smallest $\frac{1}{10000}$ inch, but the greater number were about $\frac{1}{7000}$ inch in diameter (Pl. XXVIII. fig. 13). In *Halichondria nigricans* they do not appear to be quite so convex as in *Ecionemia acervus*, nor are they so numerous as in that species, but they are somewhat larger in size; one of the largest measured $\frac{1}{2800}$ inch in diameter, and a small one $\frac{1}{5800}$ inch; they are represented *in situ* in Plate XXVIII. fig. 14.

The only instance with which I am acquainted of a conjoined arrangement of such cells exists in the envelope of the ovaries * of *Spongilla Carteri*, the species described by CARTER in his "Account of the Freshwater Sponges in the Island of Bombay," which that author believed to be *Spongilla friabilis*, LAMARCK, but which proves to be a distinct species, which I have named after its discoverer, as a slight recognition of the good services he has rendered to science by his excellent and accurate observations. These cells may be detected *in situ* after the envelope of the ovary has been submitted for a very short time to the action of hot nitric acid, so as to render the coriaceous envelope semitransparent without destroying it. The structure of its walls is then seen to consist of linear series of cells, closely packed together in lines of six or eight in each, radiating from the centre of the ovary to its external surface. They do not appear to be absolutely in contact with each other, but are usually seen to be separated by a thin stratum of a transparent substance, probably an indurated membrane or sarcode. At the surface of the envelope they frequently appear to be somewhat hexagonal from mutual compression. I could not detect a nucleus in any of them (Plate XXVIII. fig. 16). CARTER and other writers on *Spongilla* have designated the granulated forms of the sarcode in those sponges, "Sponge cells," but I cannot coincide with that opinion. I have frequently tried in vain to detect a proper coat of cellular tissue on the Amœba-like granular masses into which *Spongilla fluviatilis* resolves itself at certain periods of its existence, and neither in a healthy and active condition, nor in a state of partial decomposition, have I ever been able to satisfy myself of the existence of a surrounding membrane. It appears to me that these bodies are the result of a natural resolution of the sarcode into granular masses of various sizes, each of which, on being liberated from the parent body, becomes an independent gemmule, which is capable of reproducing the species of sponge from which it emanates. And I have long suspected that the Amœbæ found in ponds and rivers, and also in sea-water, are not in reality distinct species of animals, but that they are free portions of the sarcode of various species of Spongiadæ.

Sarcode (Physical Character)

is a pellucid, semitransparent gelatinoid substance, variable in colour and insoluble in water. It dries readily, and its physical characters are restored by immersion in water with little or no apparent alteration. It is usually spread thinly and rather evenly over the internal tissues, but the surface is rarely perfectly smooth; sometimes it abounds in obtuse elevations, and occasionally separates naturally into innumerable irregularly round or oval masses which are exceedingly variable in size. When examined by transmitted light with a microscopic power of 400 or 500 linear, it is always found to contain innumerable minute molecules of apparently extraneous animal or vegetable matter, the molecules being always more or less in a shrivelled or collapsed condition, and very variable in size. Occasionally it is found abundantly furnished with lenticular nucleated cells, nearly uniform in size, and often highly coloured. Fig. 1, Plate XXIX. represents a portion of

* These bodies have hitherto been termed Gemmules. For their characters as ovaries I must beg to refer the reader to the section of this paper on Reproduction.

the interstitial membrane of the honeycomb sponge of commerce, with the sarcode in its natural condition, filled with the remains of the nutrient molecules in a collapsed state. Figs. 13 & 14, Plate XXVIII., exhibit the same tissues with the addition of nucleated cells immersed in the sarcode. In the sponges of commerce it is exceedingly largely developed, and nothing can be more different in character than their soft and flexible skeletons and the animal in its natural condition. Specimens of it in this state, which have been preserved in spirit immediately on being taken from the sea, have the whole of their interior nearly as solid and firm as a piece of animal liver, the colour being a very light grey or nearly white. While the sponge, as a whole, is sensitive and amenable to disturbing causes, the sarcode does not appear to be especially so, as I have frequently observed a minute parasitic annelid which infests the interior of *Spongilla fluviatilis*, passing rapidly over the sarcodous surfaces, and biting pieces out of its substance without apparently creating the slightest sensation to the sarcode, or at all interfering with the general action of the internal organs of the sponge; and in many cases we find foraminiferous and other minute creatures permanently located in its large cavities without appearing to cause it the slightest inconvenience.

When separated from the living sponge, it has at certain periods an inherent power of locomotion; small detached masses of it may be observed slowly but continuously changing their form, and occasionally progressing in different directions; and CARTER, in his valuable 'History of the Freshwater Sponges of Bombay,' describes such detached masses of sarcode, when progressing and encountering a fixed point, as dividing longitudinally to avoid the impediment, and again uniting when it has been passed. This gliding motion appears to be dependent on an inherent contractile power, as no cilia have been detected on the surface of such locomotive masses. DUJARDIN has recorded similar movements in portions of the sarcodous substance from specimens of his genus *Halisarca*; and similar observations have been recorded by LIEBERKUHN and other writers during their observations of the Spongiadæ. I have frequently, at different seasons of the year, taken portions of the sarcode from living and healthy specimens of *Spongilla*, in which I could not by the closest observation detect these motions, which are so readily to be seen at other periods of their existence; and even at the same period of the year the sarcode of some specimens exhibits these motions, while in others they could not be detected. I have often sought for these phenomena in portions of the sarcode of *Halichondria panicea*, *Hymeniacidon caruncula*, and other marine species, but I have never yet been fortunate enough to detect them. It is highly probable that the capability of such motions exists in the sarcode of these and other marine species for a limited period, but it does not appear that such powers of motion are a constant condition of this substance.

ORGANIZATION AND PHYSIOLOGY.

Previously to entering on the subject of the organization and physiology of the Spongiadæ in detail, it will be necessary to take a brief view of the general structure of these animals. Whatever may be their form, or however they may differ from each other in appearance, there are certain points in their organization in which they all agree. In the first place, however variable in its form and mode of structure, there is always a skeleton present, on which the rest of the organic parts are based and maintained. Amidst the skeleton, and intimately incorporated with it, are the interstitial canals, consisting usually of two series; the first appropriated to the incurrent streams of the surrounding water, and the second to the excurrent streams, which they conduct from the interior of the sponge to the oscula at its surface, through which they are discharged. In the event of the absence of the excurrent system of canals, their office is served by the great cloacal cavities that are found to exist in some forms of sponges, extending from the base to the most distant parts of the animal. Beside these large cavities, there are others of a much more limited character, the intermarginal cavities, which are situated immediately below the dermal membrane, and which receive the water inhaled by the sponge and transmit it to the mouths of the incurrent canals which have their origin in the intermarginal cavities. Enveloping the entire mass of the sponge we find the dermal membrane, in which are situated the pores, for inhalation and imbibition of nutriment, and the supply of the incurrent canals; and the oscula, through which the excrementitious matter and the exhausted streams of water are poured from the terminations of the excurrent canals. These parts are indispensably necessary, and are always present in a living sponge. The attachment of the Spongiadæ to the body to which they adhere during life, is effected by a basal membrane which presents a simple adhesive surface, following the sinuosities of the body on which it is based, entering into holes or cracks and filling them up, thus securing a firm hold of the mass on which they are fixed. When it so happens that the locality consists of sand or mud, their bases frequently assume the form of branching roots, which penetrate the mud or sand to a considerable extent; but they are never instrumental to the nutrition of the animal—they are simply the anchors by which it is fixed to its locality for life.

The Skeleton.

There are two important distinctive characters for consideration in treating of the structure of the skeleton:—1st, the material of which it is constructed; and 2nd, the mode of its arrangement.

By selecting the material substance of the skeleton as the means of dividing the Spongiadæ into Orders, we obtain three well-defined natural groups, which are again readily divisible into Families, based on the mode of the arrangement of the substance of which the skeleton is composed.

The first Order, the Keratosæ, consists of those sponges in which the primary essential material of the skeleton is keratose fibre. It may be divided into three families.

1. Those which have the skeletons constructed of keratose fibre only, as in the best cup-shaped Turkey sponges of commerce.

2. Those having skeletons of arenated keratose fibre, as in the genus *Dysidea.*

3. Those which have the skeleton formed of spiculated keratose fibre, as in *Halichondria oculata*, JOHNSTON, *Chalina*, GRANT, and in the common West Indian sponges of commerce.

In the first Order no earthy material of any kind enters into the structure of the skeleton.

The sponges of the second Order, by a natural transition, pass into the nearly allied great division of the Halichondroid skeletons; the inability of the former to secrete silex in an organized form connecting them closely with the pure keratose, while the instinctive habit of appropriating extraneous matters recognizes the necessity of other material in the skeleton beside pure keratode; and the secretion of it by its own inherent power appears to be the next natural step in the development of the animals.

In the third division, those having the skeleton formed of spiculated keratose fibre, the gradual development is also well marked, as in one group we find spicula only in the primary or radiating fibres of the skeleton, while in another group they are found in both the primary and secondary fibres, and are developed simultaneously with the keratode of the young fibres of the skeleton.

The second Order, the Siliceæ, comprises those sponges in which the primary essential material of the skeleton consists of siliceous matter; and this also may be divided into three sections or families.

1. Those sponges which have the skeleton composed of solid siliceous fibres, as in *Dactylocalyx pumicea*, STUTCHBURY.

2. Those in which the skeleton consists of spicula dispersed without order on membranous surfaces, as in *Hymeniacidon caruncula*, BOWERBANK.

3. Sponges having the skeleton consisting of spicula cemented together into a network by keratode, as in *Halichondria panicea*, JOHNSTON.

The third Order, the Calcareæ, has the primary essential material composed of calcareous matter, and this division contains but one section or family:—

Spicula dispersed without order on membranous surfaces, as in the genus *Grantia* as defined by JOHNSTON.

1. Keratosæ . . .	*a.*	Keratose fibre only.
	b.	Arenated keratose fibre.
	c.	Spiculated keratose fibre.
2. Siliceæ	*a.*	Solid siliceous fibre.
	b.	Spicula dispersed on membranes.
	c.	Spicula cemented together by keratode.
3. Calcareæ . . .	*a.*	Spicula dispersed on membranes.

Spicula of the Skeleton.

The spicula in the skeletons of the Spongiadæ appear to be the homologues of the earthy deposits in the bony structures of the more perfectly developed living forms. In the higher tribes of animals we find the disintegrated condition of the earthy deposits in the first stages of the development of the bony structures in the form of minute radiating patches, which in a more advanced stage unite and form the solid mass of bone, as in the mammalian tribes of animals, while in the cartilaginous tribe of fishes these radiating centres of bony secretion never attain a higher degree of development, but remain isolated points of bony structure during the whole of the life of the animal. And in the compound tunicated animals we find the calcareous stellate and sphero-granulate forms of spicula developed in close accordance with the similar siliceous forms in various species of sponges. Thus the stellate and cylindro-stellate spicula of the sarcode in the Spongiadæ are *apparently* the homologues of the bony centres of development in the higher animals. It is so likewise with the other forms of sponge spicula. We find isolated calcareous spicula of an irregular fusiformi-acerate shape, representing the bony skeleton of the higher animals in the outer integuments of several species of *Doris*.

Messrs. ALDER and HANCOCK, in their admirable 'History of the British Nudibranchiate Mollusca,' describe calcareous spicula occurring in *Doris aspera*, *bilamellata*, and *Triopa claviger*, which *appear* to be analogous to the rectangulated-triradiate spicula of *Grantia*; and they also state that in the first-named species crucial or dagger-shaped spicula occur in the branchiæ and margins of the cloak of the animal, and forms very similar to those occur on the interstitial membranes of *Grantia nivea*, JOHNSTON. Numerous forms of tuberculated and smooth calcareous spicula are also found in the extensive family of the Gorgoniadæ. And the siliceous simple bihamate form of retentive spiculum, so abundant on the interstitial membranes of many species of sponges, are closely represented by the calcareous bihamate spicula so numerous on the tubular suckers of *Echinus sphæra*. Thus we find in the spicula only, a series of links in the chain of animal development, intimately connecting the Spongiadæ with the higher tribes of animals.

In the solid siliceous fibres of *Dactylocalyx*, and in the tubular siliceous fibres of *Farrea*, BOWERBANK, MS., and especially in the latter, we obtain a very much closer approximation to the tubular forms of the bones of the higher classes of animals.

From our knowledge of the great scheme of the natural development of animal life, the most perfectly organized sponges appear to be those which secrete carbonate of lime as the earthy basis of their skeletons, and the least perfect those which secrete no earthy matter in the skeleton; those which secrete silex taking an intermediate position; but it must also be remembered that there is no form of spiculum found among the calcareous sponges, or in the higher tribes of animal life, that is not repeated among the siliceous forms of spicula of the Spongiadæ.

The essential Skeleton Spicula.

The general configuration of the spicula, which are essentially necessary to the structure of the reticulated skeletons, is that of simple elongate and slightly curved bodies, varying in length and stoutness in accordance with the necessities of the structure in which they form so important an element. When the areas of the reticulations are large, they are generally long and rather stout, and are usually shorter when the proportions of the network are small and close. When enclosed in keratose fibre, they are most frequently smaller and shorter in their proportions than those in the Halichondroid sponges. And in those species in which they are dispersed over the membranous tissues, as in *Hymeniacidon*, BOWERBANK, MS., they are generally long, slender, and frequently flexuous. In the sponges of this structure having siliceous spicula the triradiate form of spiculum occurs but rarely, while in the calcareous sponges, which consist of membranes and dispersed spicula, the triradiate forms of skeleton spicula are the normal ones.

When the skeleton is constructed of large fasciculi of spicula, as in *Tethea* and *Geodia*, they attain their greatest dimensions as essential spicula of the skeleton, frequently exceeding the eighth of an inch in length.

The greatest known length of spicula occurs in the prehensile ones of *Euplectella aspergillum* and *cucumer*, OWEN, where they are found to exceed three inches in length; and in *Hyalonema mirabilis*, GRAY, where, in the spiral column of the great cloacal appendage, they reach the extreme dimensions of six or seven inches in length; but in both these cases the spicula must be considered as auxiliary and not essential forms.

The larger number of forms of skeleton spicula are perfectly smooth, but in some species they are partially or entirely covered with spines.

In every case they appear in the living state to have the capability of a change of position within the fibre to a considerable extent, in accordance with the natural alterations arising from the extensions or contractions of those tissues.

The spicula are among the earliest-developed organs of the sponge. Dr. GRANT, in his valuable "Observations on the Structure and Functions of the Sponge," published in the Edinburgh New Philosophical Journal, vol. i. p. 154, states that spicula are developed in the locomotive gemmules of *Halichondria panicea* (*Hal. incrustans*, JOHNSTON) before they attach themselves for life and commence their development as fixed sponges. And in the gemmules of *Tethea cranium* they are abundantly developed even before the gemmules are detached from the parent, and some of them are of forms peculiar to the gemmule.

The growth of the spicula and their mode of extension appear to vary according to circumstances. Thus an acerate spiculum is at first short and very slender; as the development proceeds it increases in diameter, and appears to lengthen equally from the middle towards both ends; but in spinulate ones the increase in length does not appear to be effected in the same manner as in the acerate form, as we often find spinulate spicula fully developed at the base, while the shaft is exceedingly short and the apical termination hemispherical instead of acutely pointed, as in the adult state. As the shaft lengthens towards its full proportions, it attenuates; but in all the intervening stages the apical termination is usually more or less hemispherical. The progressive development from the base to the apex of the spinulate form is beautifully illustrated in the skeleton spicula of a new and very singular British sponge from Shetland, *Halicnemia patera*, BOWERBANK, MS., represented by figs. 2, 3, 4, 5, 6, & 7, Plate XXIX. The first of these (fig. 2) represents a short variety of the normal spinulate form. In the second (fig. 3) we have a bi-spinulate, and in the third (fig. 4) a tri-spinulate form. The latter two are not mere malformations, but they prevail to a great extent in the structures of the sponge, subject to variations in the distances in the development of the second and third inflations from the basal one. Figs. 5, 6, & 7 represent immature spicula in progressive stages of development, the apices having hemispherical terminations.

Auxiliary Spicula.

Beside the spicula essential to the structure of the skeleton there are several other forms of these organs, many of which, although not absolutely necessary in the structure of the skeleton, are of very frequent occurrence in subsidiary organs found in particular species and in peculiar genera. They may be conveniently classed under the following heads:—

Connecting spicula.
Prehensile spicula.
Defensive spicula.
Tension spicula.
Retentive spicula.
Spicula of the sarcode.
Spicula of the gemmules.

In the above designations of the auxiliary spicula, it must not be understood that their respective titles strictly define their offices, as it frequently occurs that under peculiar circumstances the same form of spiculum is destined to serve two, or even three distinct purposes. Thus, an external defensive spiculum will occasionally perform retentive offices for the purpose of securing prey; or internal defensive spicula will combine the offices of defensive spicula against the larger and more powerful of their enemies with that of wounding and securing their smaller ones.

The Connecting Spicula.

The normal form of the connecting spicula is that of an elongate shaft, with a ternate apical termination. But all the varieties of this form are not necessarily connecting spicula. Some of them subserve the offices of external or internal defensive organs, as I have described elsewhere. The varieties that may correctly be designated by this title are those which I have termed in the first part of this paper expando- and patento-ternate spicula, and some varieties of the recurvo-ternate form also appear to be applied to this peculiar office. Their situation in the sponge, rather than their precise form, determines their title to be thus designated. Nor is their especial purpose of connecting the dermal crust of the sponge with the great mass of the skeleton beneath, the only office they are destined to perform in the economy of the animal, as their ternate terminations are so disposed as to form a series of reticulations and areas for the support of the valvular membranes of the proximal ends of the intermarginal cavities of the sponges, in which they are best developed, as in *Geodia M^cAndrewii* and *Barretti*, *Pachymatisma Johnstonia*, and others of similar structure.

I have never seen the progressive development from a simple elongate shaft of an expando- or patento-ternate connecting spiculum, as I have those of the porrecto-ternate external defensive form, and the spinulo-recurvo-quaternate internal defensive ones, but from the great similarity that exists in their structure there can be little doubt that their mode of growth is the same; and I am very much inclined to believe that the cylindro-expando-ternate form from *Pachymatisma Johnstonia*, fig. 43 in Plate XXIII. of the Phil. Trans. for 1858, is an incompletely developed form of the mature attenuato-expando-ternate spiculum that belongs to that sponge, and which is represented by fig. 42 in the same Plate.

There is a progression of development in the ternate terminations of these spicula that is very interesting. The simplest form has three nearly straight attenuating radii. In the next stage the distal ends of the primary radii become furcated, but the secondary radii remain in the same plane as the primary ones. In the third stage of development the terminations of the secondary radii again divide into furcations, becoming dichotomo-patento-ternate (fig. 48); but in this case the radii of the extreme furcations are not all in the same plane, as appears always to be the case with those of the secondary radii, and thus we have produced an additional power for combined action. But in the whole of these varieties, in the structure of these ternate terminations, hitherto, there is no appearance, further than their general form, of their being destined to become a united structure, and in some sponges in which they do occur they rarely, or never, become thus united; but this demonstration of their destination for combined action is obtained in an irregular ternate form, as exhibited in the dermal structures of a new species of siliceo-fibrous sponge from India, *Dactylocalyx Prattii*, Bowerbank, MS., in which we have the primary radii sinuated and flattened in such a manner as to splice together and form a strong and regular reticulated structure for the support of the dermal membrane of the sponge, as in fig. 8, Plate XXIX., which represents a few of these spicula uniting to form the reticulations of the dermal tissues, while fig. 9 represents three of these spicula separated by boiling nitric acid. By this structure, as exhibited in *D. Prattii*, there is rendered apparent a more visible and common purpose in their form and mode of development, and we are gradually conducted to the still more complete and continuous form of fibro-siliceous dermal network that exists in the beautiful harrow-shaped tissue of the dermal structures of the sponge supporting the fine specimen of *Euplectella* in the possession of my friend Dr. A. Farre, and described by Prof. Owen in the 'Transactions of the Linnean Society,' vol. xxii. p. 117, plate 21, and which tissue I shall describe more fully in treating on the subject of the dermal structures of the Spongiadæ.

There are two distinct purposes in the physiological application of the ternate spicula; the simplest is that of the strengthening and connecting the dermal membrane with the mass of the animal beneath. The second and more complex one, is that of forming an internal reticulating framework for the support within its areas of the valvular tissues forming the bases of the intermarginal cavities. These offices of the ternate spicula are not demonstrated in an equal degree of perfection in all sponges in which they occur. Where the organs which they subserve are best and most abundantly developed, these forms of spicula are found in the greatest quantities, and in the most regular and perfect mode of arrangement, but where the intermarginal cavities or porous areas are in a less regularly developed state, they are deficient in a corresponding degree; thus evincing the design and purpose of their structure and presence. The most perfect and beautiful illustration of their physiological purpose, in their first mode of application, is afforded by the dermal membrane of *Dactylocalyx Prattii.* Here we find their radii overlapping each other longitudinally, and cemented together by keratode, forming a continuous and regular network, upon the upper surface of which the dermal membrane reposes, and to which it is firmly united. The mode in which the radii are united, and the material with which they are cemented together indicate a unity of firmness and elasticity in the living state that is truly admirable; and this mode of structure we perceive is especially necessary to the action of the dermal membrane, as the whole of the skeleton beneath is perfectly rigid and inelastic. Thus while their shafts are deeply plunged in, and firmly secured to, the immoveable mass beneath, their ternate apices are capable of such an amount of oscillating motion as would be required for the organic expansion and contraction of the membranous structure they support. By the action thus generated each pair of the united radii would glide in a longitudinal direction upon each other, and thus, although in each separate instance the amount of motion would appear to be exceedingly small, the aggregate of the whole would afford a very considerable range of expansion, as exhibited in fig. 8, Plate XXIX.

In their second mode of application, that is to the bases of the intermarginal cavities, it appears that as their office is different, so their form and the mode in which the radii of their apices is connected are also different. Thus at the inner surfaces of the thick dermal crust of *Geodia M^cAndrewii* and *Barretti*, we find them forming a network equally regular and continuous as that in *Dactylocalyx Prattii*, but the mode of its construction is varied. The radii do not in these cases glide upon each other longitudinally, but they cross each other at various angles; and as the whole mass of these sponges is fleshy and very elastic, so by this mode of interlacement of the radii a very considerably greater amount of expansion and contraction of the reticulated structure is provided for, while at the same time the power of maintaining the common plane of the reticulated tissue is equally as great as in the similar structure in *Dactylocalyx Prattii.* Thus far we can trace the physiological purpose of their structure; but why in one species we find their terminations simple as in *Geodia M^cAndrewii*, and furcated as in *Geodia Barretti*, or still further complicated as in the dichotomo-patento-ternate form, is a question which cannot be so readily solved without a further acquaintance with the species of *Geodia* bearing these forms in a living state.

The connecting spicula are not always an essential portion of the skeleton, and they exist only in comparatively a few genera of the Spongiadæ.

Prehensile Spicula.

I have so fully described, in the first part of this paper, the prehensile spicula found at the base of the beautiful *Euplectella*, in the possession of Dr. ARTHUR FARRE, and figured in Plate XXIII. fig. 53, Phil. Trans. for 1858, as to render it necessary to say but very little more regarding them. Fig. 44, Plate XXVI. of the same paper presents so many points of structural agreement with that from *Euplectella*, as to induce a very strong suspicion of its having had a similar office to perform in the sponge from which it was obtained; but its extremely small size is against that supposition, and in favour of its being an internal defensive one. Both sponges producing these forms were parasitical on other sponges. With respect to the larger form there is no reasonable doubt of its office in the sponge, and the smaller ones may have been basal appendages to a very small species. I have searched other species of *Euplectella* in vain for similar forms.

Defensive Spicula.

The modes of defence in the Spongiadæ by means of spicula are exceedingly various, and in many cases remarkably complex and interesting. They may be divided into two great systems,—1st, those of external defence; and 2nd, those of internal defence.

If I were to attempt to enter upon a description of every variation in the mode of the application of spicula to defensive purposes, it would extend this portion of the subject to a greater length than we can afford under the present circumstances. I shall therefore confine my observations to a description of the general principles of defence as exhibited in some of the principal genera of the Spongiadæ.

In the external defences, the mode of the application of the spicula depends in a great degree on the structure of the skeleton of the sponge. The most simple cases are those where the structure of the skeleton consists of spicula radiating from the centre or the axes of the sponge, and in these cases they usually consist of the terminations of the radial lines of the skeleton, the distal spicula of which are frequently projected for a considerable part of their length through the dermal membranes, and in many sponges the surface is thus thickly studded with them; and in species where the terminal radial lines of the skeleton contain many spicula, they are frequently found at their apices to assume a radiating direction, so as to present the greatest possible number of points to their external enemies. This mode of defence is very general in the numerous British species of the genera *Isodictya* and *Chalina*, BOWERBANK, MS. Fig. 10, Plate XXIX. represents a small portion of a section at right angles to the surface from *Halichondria seriata*, JOHNSTON, *Chalina*, BOWERBANK, MS., illustrating very distinctly this simple mode of external defence.

In the genus *Dictyocylindrus*, BOWERBANK, MS., which consists principally of slender branching sponges, many of which in their living state are exceedingly fleshy in their appearance, the skeleton is formed of a central cylinder, composed of a network of spicula, from the surface of which radiate in vast quantities long, slender and acutely pointed spicula, which in the living condition project slightly beyond the dermal membrane of the sponge, so that in the event of any small fish attempting to feed upon or suck this tempting bait, instead of a mouthful of soft and grateful gelatinous matter, he would find himself assailed in every direction with an infinite number of minute points, many of which he would carry away with him deeply imbedded in the soft lining of his mouth, as the reward of his temerity and a warning against a repetition of a like assault. Fig. 11, Plate XXIX. represents a small portion of a young branch of *Dictyocylindrus rugosus*, BOWERBANK, MS., an undescribed British species, frequently found on shells and stones dredged up at Shetland, or the Orkney Islands. In the genus *Tethea*, in which the skeleton consists of fasciculi of large, stout spicula radiating from the base or centre of the sponge, the system of defence is somewhat more complicated. It is a combination of the terminations of the skeleton fasciculi with, in some species, the addition at the surface of the sponge of porrecto-ternate and recurvo-ternate spicula; the latter two forms being probably aggressive as well as defensive, subserving the purpose of entangling prey as well as that of defence.

This mode of defence is very beautifully illustrated in *Tethea cranium*. The distal ends of the skeleton fasciculi, composed of large fusiformi-acerate spicula, are projected through the stout coriaceous surface of the sponge, and in the midst of this thick coat each of the passing fasciculi is surrounded by a cluster of stout short fusiformi-acerate spicula, their distal points closely embracing the fasciculus, while their proximal terminations are spread widely out in a circle around the lower part of the skeleton fasciculus, so as to form a strong and most efficient conical buttress to sustain it in its proper position, at the same time allowing a considerable amount of elasticity to meet pressure from without. Each skeleton fasciculus terminates with from two to eight or ten porrecto-ternate spicula, and occasionally we find one or two of the recurvo-ternate ones accompanying them; but their apices are rarely projected much beyond the dermal membrane of the sponge, while the rest of the spicula extend considerably above it (fig. 12, Plate XXIX.). The same system of defences prevails also in *Tethea simillimus*, BOWERBANK, MS., from the Antarctic regions; but in this species the recurvo-ternate spicula appear to be protruded in greater numbers and in more regular order than in our northern species, *T. cranium*.

In *Tethea muricata*, BOWERBANK, MS., the skeleton fasciculi are not protruded beyond the surface, but immediately beneath it we find the heads of numerous large furcated expando-ternate spicula, with remarkably long and acute terminal radii, while the dermal membrane is profusely furnished with attenuato-elongo-stellate spicula.

In *Tethea Norvegica* and *Ingalli*, BOWERBANK, MS., and in *T. Lyncurium*, JOHNSTON, the same protection is attained in a different manner. Instead of the spicula of the skeleton fasciculi gradually converging towards a point, they diverge considerably as they approach the surface, so as to present an infinite number of minute and nearly equidistant points, and in addition to these the dermal membrane and the coriaceous coat of the sponge is supplied with an infinite number of closely packed stellate spicula.

In some species of the genus *Geodia* the system of external defences is still more complex. Thus in *G. M^cAndrewii* and *G. Barretti* the defences are double, one system consisting of a continuation of the great radial fasciculi of the skeleton as a protection against the assaults of the larger and more powerful assailants, and then of a secondary series consisting of an infinite number of minute acerate spicula, based immediately beneath the dermal membrane and projecting to a slight extent beyond its external surface, effectually protecting it and the porous system of the sponge from the attacks of its minute and more insidious enemies.

Similar modes of external defences exist in various species of *Pachymatisma* and *Ecionemia*, but no two species appear to agree precisely in these respects.

In the genera *Microciona* and *Hymeraphia*, Bowerbank, MS., differing widely in the structure of their skeletons from any of the sponges hitherto described, and frequently not exceeding in thickness the substance of a stout sheet of paper or a thin card, the same principles of defence are carried out, although their structure is widely different from each other. In the first genus, the skeleton of which is formed of short pedestals of keratode combined with spicula, each of the pedestals, which reach nearly to the surface of the sponge, is terminated with a radiating cluster of long curved and acutely pointed spicula, the apices of which pass through the dermal membrane in every direction, and thus form a most effectual series of external defences, while their shafts beneath serve as the framework of the intermarginal cavities of the sponge (figs. 1 & 2, Plate XXX.). In *Hymeraphia*, where the sponge is less in thickness than the length of one skeleton spiculum, and where they pass from the basal membrane of the sponge through the dermal membrane, their apices acting as external defensive organs, while their shafts form the essential skeleton of the animal, there is an especial provision for their preservation from injury. Their bases are expanded in the form of large bulbs, so as not only to afford a greater surface for attachment, but to allow them at the same time to act on the principle of a ball-and-socket joint, giving them a more than usual amount of attachment, and a power of yielding in every direction to pressure on their apices from without (fig. 4, Plate XXX.). The defence of the surface of the Halichondroid sponges is less apparent, but equally efficacious; the abundantly spiculous reticulations immediately beneath and supporting the dermal membrane, would render attacks of annelids or other small predaceous creatures exceedingly unpalatable.

In the calcareous sponges the spicular defences are exceedingly interesting. In *Grantia compressa*, the distal ends of the great interstitial cells are amply protected by numerous flecto-attenuato-acuate spicula grouped around their porous terminations, with their club-shaped ends curving in every direction over them, but in no degree interfering with the freedom of their inhalant action. In *Grantia ciliata* they are grouped in circles around the distal ends of the interstitial cells, but in this species they are acutely pointed; and when the inhalant system is in a state of repose, they are concentrated at their extreme points so as to form an elongate cone, effectually enclosing and protecting the porous ends of the cells within them; but when the inhalant action is in full activity, their apices recede from each other until they assume the form of a cylinder, and then freely admit the incurrent streams of water, but effectually repel the advances of any dangerous assailant that may attempt an entrance. The distal termination of the cloaca in this species is also abundantly protected by a marginal fringe of long and very acute spicula, and is furnished with the same simple but beautiful mechanical contrivances for opening and closing in accordance with the necessities of the animal. For a more complete description of the anatomy and physiology of this highly interesting species I must refer my reader to the 'Transactions of the Microscopical Society of London,' vol. vii. p. 79, pl. 5.

In other species of *Grantia* the same principles of external defensive action exist, but the precise mode is never exactly the same in any two species.

Internal Defensive Spicula.

The internal defensive spicula of sponges are exceedingly various in their forms and modes of application to their especial purposes; and they seem naturally to resolve themselves into three distinct groups:—1st, those which are destined simply to repel; 2nd, those which wound and lacerate as well as repel; and 3rd, those which are calculated not only to destroy but also to retain intruders.

The purposes of the first class of spicula are frequently performed by the ordinary spicula of the skeleton, which are projected more or less into the cavities immediately within the oscula and other spaces requiring such protection; but when especially formed for and appropriated to defensive purposes, they are always free from spines and usually terminate acutely; and they are frequently provided with widely extended basal radii, so as to fix them rigidly and firmly in their proper positions, as exemplified in the various forms of spiculated triradiate spicula represented by figs. 14, 15, 16, & 17, Plate XXIV., Phil. Trans. for 1858.

The best illustrations of the application of the simple defensive spicula are to be found in the cloaca in several species of *Grantia*, as in *G. ciliata*, Johnston, and *G. tessellata* and *ensata*, Bowerbank, MS. In all these species this great central cavity is abundantly furnished with spiculated triradiate spicula, such as represented by figs. 15 & 16, Plate XXIV. Phil. Trans. 1858, which are so disposed that while the basal radii are firmly cemented on the surface of the cloaca, the spicular or defensive rays are projected from its surface, not at right angles to its plane, but always at such an inclination towards the mouth of the cloaca as to present a combined series of sharp points in the best possible position of defence, so that an intruding assailant could scarcely escape being seriously wounded by them, while a retiring enemy would pass with impunity over their inclined apices. In some species, as in *G. tessellata*, the defensive ray is naturally curved to the desired angle for defence (fig. 16, Plate XXIV. Phil. Trans. 1858), and it is also of such a form as to be readily released from the creature it has wounded, either by being attenuato-acuate or ensiform, as in fig. 15, Plate XXIV. Phil. Trans. 1858, from *G. ensata*, and as represented *in situ* by a small portion of a longitudinal section of the cloaca of a specimen of *Grantia tessellata* in Plate XXX. fig. 5, in which the defensive radii are all curved in the direction of the mouth of the cloaca.

In the second division the internal defensive spicula are usually short and straight, and more or less covered with strong conical acutely pointed spines, projected either at right angles to the axis of the spiculum, or recurved considerably towards its base; generally speaking the spines are dispersed on all parts of the spiculum without any approach to order, as represented in fig. 1, Plate XXIV. Phil. Trans. 1858, while in other cases, as in figs. 2 & 3 in the same Plate, they are arranged in verticillate order on all parts of the spiculum. In each of these varieties the bases of the spicula are usually profusely furnished with spines, so as to ensure a strong and somewhat rigid mode of attachment.

There is undoubtedly a special purpose in every variation of the spination of these spicula, and in their presence generally. The short strong form and acute distal termination admirably adapt them to encounter the larger description of intruding annelids, the most dangerous internal enemies of the Spongiadæ; while the spination of their shafts presents a series of minute weapons that would prove equally formidable to those intruders that were too minute to be affected by the larger weapons of defence.

The acuate entirely spined defensive spicula are of very common occurrence in sponges, and are by no means confined to particular tribes or genera. As a general rule, when the external defences are very full and sufficient, we should not expect to find the internal defences abundant, and, on the contrary, when there appears to be a paucity of external defences, the internal ones are frequently exceedingly numerous. Thus, in the genus *Dictyocylindrus*, Bowerbank, MS., where in almost every species the surface of all parts of the sponge is bristling with the acute terminations of the radiating external defensive spicula, although in most of the species we find acuate entirely spined internal defensive ones, yet in many of the species they are so rare as to be by no means readily detected.

When the skeleton is formed of keratose fibres, we find them dispersed on their surface without any approach to order, and projected at every imaginable angle. If the skeleton be formed of any of the varieties of spicular reticulations, they are based in a similar manner on the principal lines of the reticulated structure, and sometimes, but not very frequently, they occur in groups.

I will not extend this portion of my subject to an unnecessary length by describing every mode of their occurrence, but select a few of the most interesting cases as illustrations of the general principles of their application.

Fig. 6, Plate XXX. represents a small portion of the kerato-fibrous skeleton of an Australian sponge, with the attenuato-acuate entirely spined internal defensive spicula *in situ*. Fig. 7 represents a few fibres from a kerato-fibrous sponge from the West Indies, in which the verticillately spined internal defensive spicula are dispersed over the fibres; and fig. 8 represents the same description of defensive spicula from a West Indian kerato-fibrous sponge, having the defensive spicula congregated in bundles. Sometimes, but not very frequently, they are found on the interstitial or basal membranes of the sponge, and under these circumstances many of them are prostrate in place of being erect; and in one sponge, *Hymeniacidon Cliftoni*, Bowerbank, MS., a singular parasitical species from Freemantle, Australia, this prostration appears to be effected by an especial law. This singular sponge envelopes several fan-shaped portions of a *Fucus*, and systematically appropriates the minute ramifications of its stem to the purposes of an artificial skeleton; the whole sponge abounds with short stout attenuato-cylindrical entirely spined internal defensive spicula: but the remarkable circumstance attendant on their presence is, that wherever the membranes supporting them envelope and firmly embrace a portion of the vegetable stem, they assume an erect position, and exhibit all the usual characters of defensive spicula; but where the membranes merely fill up the areas of the vegetable network, they are nearly all of them perfectly prostrate, and apparently performing the office of tension, rather than of internal defensive spicula. Their form also is singular, being attenuato-cylindrical, not having the acute termination that is usual in this description of spicula.

Fig. 9, Plate XXX. represents a small portion of the fibrous stem of the *Fucus* coated by the membranes of the sponge, and covered with spicula; those immediately over the stem being erect, while those on the membrane are prostrate. (*a*) represents one of this new form of internal defensive spiculum $\times 175$ linear.

In *Hymeraphia stellifera*, Bowerbank, MS., an exceedingly thin coating British sponge, the internal defensive spicula present a singular variation from the normal form. In this case they assume the shape of an ordinary Florence oil flask, with a somewhat elongate neck, and having a beautiful star-shaped apex in place of a stopper. They occur in considerable quantities; their large bulbous bases are firmly attached to the strong basal membrane of the sponge, and they are projected thence at every possible angle upward into the interstitial spaces. Their apices are crowded with stout acutely conical spines, which radiate in all directions. Fig. 3 *a*, Plate XXX. represents a group of these spicula *in situ*, elevated by a grain of sand beneath the basal membrane; and fig. 4 *b*, Plate XXX., one of the same form of spiculum, magnified 260 linear. In this form of spiculum, as in that of *Hymeniacidon Cliftoni*, their purpose seems to be the infliction of laceration, rather than that of destruction, by deep wounds. In another species of *Hymeraphia*, *H. clavata*, these spicula have the same large bulbous bases as those of *H. stellifera*, but their apices are acute, like those of the normal forms of such spicula. In all these cases we observe in their attachments the same approximation to the structure of the ball-and-socket joints of the higher tribes of animals, rendering them capable of yielding in every possible direction to the struggles of any enemy with which they may be entangled.

In the third division of the internal defensive spicula there is an especial construction for retention as well as for destruction. Their apices are usually more or less hamate, as represented in figs. 7, 13, & 12, Plate XXIV. Phil. Trans. 1858, and their attachments to the sponge are usually such as to allow of a considerable amount of flexibility or motion.

I will not attempt to describe the whole of the numerous variations in the modes of their application to defensive purposes, but select a few of the most interesting cases as illustrations of the general principles of combined internal defence and aggression.

The spinulo-recurvo-quaternate spiculum, the growth and development of which I have described in the first part of this paper (Phil. Trans. 1858, p. 293), presents an admirable illustration of the combined defensive and aggressive character of some of these internal defensive spicula. The sponge in which they occur belongs to the Halichondroid tribe, the skeleton being composed of a network of spicula cemented together by their apices, which cross each other at the angles of the areas of the reticulations. The recurvo-quaternate spicula are not dispersed on all parts of the skeleton, but are congregated in groups, frequently consisting of as many as fifteen spicula, the whole of their bases being concentrated on one of the angles of the reticulations of the skeleton, while their shafts and apices radiate thence in every direction into the interstitial spaces of the sponge; they are thus placed on the strongest and most elastic portion of the skeleton, with their hemispherical bases firmly imbedded in the cementing keratode of the skeleton, which abounds at the angles of the network, and which by its inherent elasticity and strength renders the insertion of the base of the spiculum, in strength and extent of action, quite equivalent to the powers of the ball-and-socket joints in the higher tribes of animals. A small annelid or other minute intruder entangled amidst these numerous sharp hooks would struggle hopelessly in such a situation, as the spicula, from the nature of their attachment, would yield readily to its struggles in every possible direction, and at every new contortion arising from its efforts to escape it would inevitably receive a fresh series of punctures and lacerations.

Fig. 10, Plate XXX. represents a small portion of the skeleton of the sponge bearing the spinulo-recurvo-quaternate spicula *in situ*.

In other instances, where defence alone appears to be contemplated, we do not find these beautiful adaptations for motion in every direction prevail. The bases of the spicula in those cases are abundantly spinous, and are evidently intended to maintain a firm hold by their attachments, and are destined rather to rigidly maintain their position than to yield to any struggling body with which they may be in contact. The numerous spines with which these shafts are frequently covered are calculated to wound and lacerate, rather than to retain the enemies with which they are engaged*.

* Since I wrote the first portion of this paper, I have received from my friend, Mr. J. Yate Johnson, of Madeira, a new and very illustrative instance of the combination of defence and aggression in the structure and offices of the internal defensive spicula; and in this case it is not a new organ, but an adaptation of a well-known form to a new purpose, in the shape of a contort trenchant bihamate spiculum of unusual size and structure. In the course of my examination of the results of the deep-sea soundings in the Atlantic, I found several of these spicula, and was much interested by the singularity of their structure, which at that time I could not comprehend.

The general outline is much like that of the type-form so commonly found imbedded in the sarcode, but it is somewhat less flexuous in its curves, and the shaft and hami are very much larger and stouter than those of the spicula of the sarcode. But the most singular point in their structure is, that while the curved portion of the hami and the middle of the shaft are perfectly cylindrical, the inner portion of the hooks and those parts of the shaft immediately opposed to them present sharp trenchant edges, so that each hook assumes to some extent the form of spring hand-shears. The acute termination of the hook and the opposed trenchant edges exhibit every facility for effecting an entrance through the tough skin of the victim, while

In *Hyalonema mirabilis*, GRAY, a sponge nearly related to the genus *Alcyoncellum*, QUOY et GAIMARD, we find another extraordinary series of internal defences; one portion of the spicula appearing to be destined to wound and lacerate, rather than to retain intruding enemies, while a larger and stronger series of spicular weapons bear all the evidences of being to retain rather than to repel the assailants.

The first description of spiculum I have designated entirely spined, spiculated cruciform spicula. They consist of a short stout cruciform base with a long spicular ray, ascendingly and entirely spinous, projected at right angles from the centre of the basal radii. The spines are acutely conical, and very sharply pointed. They pass off from the spicular ray at an angle of 12 or 15 degrees in the direction of its apex. The apices of the basal radii are attenuated and slightly spined. These spicula are thickly distributed on the fasciculi of the skeleton, and frequently equally so on one side of the interstitial membranes, probably that which forms the surfaces of the interstitial spaces, and they are especially abundant near the exterior of the sponge. The four basal radii appear firmly cemented to the membrane, but not immersed in its substance, as they do not appear to leave their impressions when removed from it, nor do they bring any portion of the membrane away with them. In some parts of the tissue these spicula are very much modified in form. In the ordinary cases we find the basal radii short and stout, and not more than a fourth or a fifth of the length of the spicular ray, while in other cases the basal rays are very nearly as long as the spicular one; the only difference in their structure being that the latter is very strongly spinous, while the former have the spines comparatively very slightly produced.

The second form is a large fimbriated multihamate birotulate spiculum, which occurs dispersed amid the interstitial tissues of the large basal mass of the sponge. There are usually not more than one or two together, but occasionally they occur in groups of ten or twelve, without any approach to definite arrangement.

These spicula are comparatively large and stout. They have eight rays at each end of the shaft; the two groups of radii curving towards each other to such an extent that each

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the perfectly blunt and cylindrical state of the arch of the hook bespeaks the design of retention as well as of destruction. As soon as the hook has penetrated to the inner blunt surface of the curve, it no longer cuts, and the prey, wounded in every direction, is securely retained for the nutrition of the sponge. This result is indicated not only by the form of the spicula; their position in the structure of the sponge bespeaks their office equally unmistakeably. They are not immersed in the sarcode like their congeners in form, but are firmly cemented by one hook to the reticulating lines of the skeleton, while the other ends are projected at various angles into the interstitial cavities of the sponge in such numbers and in such a manner, that it would be next to impossible for an intruder within the sponge to escape being entangled and destroyed amongst them. Fig. 1, Plate XXXI. represents a portion of the reticulated skeleton of the sponge with the trenchant contort bihamate spicula *in situ*, magnified 50 linear; and fig. 2 one of the spicula, magnified 400 linear, to exhibit the trenchant edges and the cylindrical portions of the hami and shaft.

This sponge is allied to *Halichondria* by the structure of the skeleton, and it is described by my friend Mr. J. YATE JOHNSON as being a thin coating species, spreading over the surface of rocks and stones to the extent of two or three inches in diameter.

forms the half of a regular oval figure; the opposite apices being separated to the extent of about the length of one of the radii. Each ray is in form like a double-edged blunt-pointed knife, bent near the handle in the direction of a line at right angles to one of its flat sides*; and each ray is strengthened and connected with the shaft of the spiculum by a stout curved web of silex, which extends from a little below the inner surface of the ray to a point on the shaft about opposite to its middle. The shaft is cylindrical, and has short stout tubercles dispersed over all its parts when fully developed.

The structure of every part of this singularly beautiful spiculum is strikingly indicative of its office in the economy of the sponge; the form and mode of bending of the radii, with their thin edges at right angles to the line of force in a struggling animal, and the powerful web at the base of the ray enabling it to sustain an amount of stress that the unsupported flat ray would never otherwise be able to endure.

The spiculated cruciform spicula are exceedingly abundant in every part of the sponge, and no victim entangled and retained by the large multihamate spicula could avoid innumerable wounds while struggling to effect its escape; while the one held it secure within the sponge, the others, from the peculiarity of their form and mode of disposition of their acutely pointed spines, would readily release it after the infliction of every puncture, only that the wounds might be multiplied until the creature was pierced in every part, and bled to death for the nutrition of the sponge.

Fig. 3, Plate XXXI. represents a small portion of the skeleton of the sponge with the two forms of defensive and aggressive spicula *in situ*, magnified 50 linear. Fig. 4 represents one of the multihamate bihamate spicula with a power of 83 linear, displaying the adaptation of its structure to purposes of retention. Fig. 5 represents one of the spiculated cruciform spicula on the same scale as fig. 4, showing their relative proportions, and fig. 6 the same form of spiculum with a power of 260 linear, to exhibit the peculiarities of its spination.

It would be almost an endless task to describe every variety of these singularly beautiful contrivances for combined defence and offence in the interior of the Spongiadæ. Those which I have particularized are some of the most elaborate and beautiful that I have seen during the course of my researches. In many other cases, where all that is required is defence, the means employed are of a much more simple nature. We find in the Spongiadæ, as in other animals, that nature frequently economizes her means by the conversion of one organ to the purposes of another by slight adaptations or additions; thus in *Halichondria incrustans*, Johnston, and in other sponges, the skeleton spicula are made to perform the duties of internal defensive spicula, by being more or less furnished with spines, as represented in fig. 30, Plate XXIII. Phil. Trans. 1858, and in other cases we find them medially or apically spined, as in figs. 32, 33, & 34 of the same Plate.

In like manner we find the spicula of the sarcode, by the extreme profusion in which they occur in that substance near the surface of some sponges, are turned to good account for the general purposes of external and internal defence, as well as for their special purpose of the protection and support of the sarcode. So likewise in the tension spicula of *Spongilla lacustris* (fig. 21, Plate XXIV. Phil. Trans. 1858) they are made to serve as defensive organs as well as tension spicula; and, again, in the spicula of the gemmules of the Spongiadæ their skeleton spicula also perform the office of defensive organs as well, as represented by figs. 13 to 43, Plate XXVI. Phil. Trans. 1858.

As regards, then, their protection from their enemies, there appears to be almost a natural prohibition to the sponges becoming, to any great extent while alive, the food of other creatures. The keratode of their skeletons appears to be almost indestructible by maceration or digestion, and the abundance of the acutely pointed spicula that exist in so many of their bodies must render them anything rather than desirable or digestible food to the generality of other marine animals; and, in truth, I do not know of a single large fish, or other marine creature, that appears to prey upon them. The only animal in the stomach of which I have ever seen the spicula of any sponge was a *Doris*. But although appearing to enjoy almost an immunity from the common lot of animals, that of being eaten by others, they may yet serve, at their death by natural causes, to supply an immense quantity of animal molecules for the sustenance of the myriads of minute creatures that exist around them.

Tension Spicula.

The primary purpose of the tension spicula is that of strengthening and supporting the membranes, both external and internal. They are usually of the same form as those of the skeleton, but more slender and shorter in their proportions. On the internal membranes they are dispersed without any approach to order, and cross each other at every imaginable angle. They vary exceedingly in length and diameter, and are attached for their whole length to the tissues on which they repose. In some cases they are not readily to be distinguished from those of the skeleton, as they are frequently so nearly of the same size, and are intimately intermingled with them, as in the genus *Hymeniacidon*; but in other cases, as in some species of *Chalina* and *Isodictya*, they may always be distinguished by their position, and by the total absence of keratode around them, while those of the skeleton are always more or less coated by that substance.

In other cases they differ materially in form and proportion from those of the skeleton. Thus in *Halichondria incrustans*, while the skeleton spicula are stout, short, entirely spined and acuate, as represented by fig. 30, Plate XXIII. Phil. Trans. 1858, the tension spicula are smooth, slender mucronato-cylindrical, as represented by fig. 23, Plate XXIV. Phil. Trans. 1858. They are frequently dispersed on the dermal membranes, much in the same manner as they are on the interstitial ones, abounding most where the areas are largest, and where the areas are small they are few in number or entirely absent; but in other cases, as in the dermal membrane of *Halichondria incrustans*, they are congregated in flat broad fasciculi, which are disposed on the membrane with little or no approximation to order.

The tricurvo-acerate form in all its varieties is better calculated to effect their peculiar office in small and irregular spaces, and with greater economy in numbers, than the straight elongated forms; and they are also better adapted to membranes having unequal surfaces, such as those in *Microciona armata*, BOWERBANK, MS., where we see them following the undulations of the membranes and sustaining them in their proper positions around the columnar parts of the skeletons. The varieties of form in these spicula are well represented by figs. 26, 27, & 28, Plate XXIV. Phil. Trans. 1858. They are all out of the same sponge. In *Grantia compressa*, and other closely allied species, where the structure is systematically membranous, the skeleton spicula are triradiate, supporting the membranes in uniform planes in the most effectual manner; and they are, in fact, systematically tension spicula, as well as skeleton ones. In *Grantia nivea*, JOHNSTON, which is not symmetrical in its structure like *G. compressa* and its congeners, other forms of tension spicula are developed to suit their especial purposes, such as represented by figs. 30 & 31, Plate XXIV. Phil. Trans. 1858.

In siliceous sponges we also occasionally find triradiate spicula developed, and performing the office of tension spicula, in the midst of comparatively large membranous areas; but these forms, in every case under such circumstances in which I have seen them *in situ*, appear to belong to the exception, rather than the general rule obtaining in such sponges.

The foliato-peltate spicula, for a full account of the progressive development of which I must refer to page 298, Phil. Trans. 1858, appear to be a development of the apices of connecting spicula into dermal tension ones, bearing a strong resemblance in form and purpose to the bony scutes in the skins of some of the higher animals, while the extreme crenulation of their margins probably served the purpose of facilitating the action of the porous system.

In all the varieties in form which I have hitherto described, and with which I am acquainted, where they perform the office of tension spicula only, they are destitute of spines. In other cases the tension spicula not only fulfil their own especial office, but they subserve that of defensive spicula also. Thus in the dermal membrane of *Spongilla lacustris*, JOHNSTON, we find them dispersed rather numerously, covered with short acutely conical spines, as represented by fig. 21, Plate XXIV. Phil. Trans. 1858. In *Spongilla alba*, CARTER, we find the tension spicula as abundantly spinous as those of *S. lacustris*, but in this case the spines are truncated (fig. 22, same Plate). They have a similarly blunted, imperfectly produced character in those of *Pachymatisma Johnstonia*, as represented by fig. 24.

The production of tension spicula in the membranes of the Spongiadæ is by no means a peculiarity of that class of animals. We find them in numerous beautiful forms in the skins of the Holothuriadæ, varying in shape in the different parts of the animal to adapt themselves to the necessities of their situation; but the closest approximation, both in size and form, to those of the Spongiadæ are the bihamate ones that are found so abundantly dispersed on the membranous tubular suckers of *Echinus sphæra*; and I have also seen another variety of these spicula in the tubular tentacles of a large common species of *Actinia*, and in the latter case they were even more minute than those of the Spongiadæ.

Retentive Spicula.

In the first part of this paper (Phil. Trans. 1858, p. 300) I have described the varieties in form and modes of development of these spicula. However varied they may be in form, when they are in their normal positions their office appears to be purely retentive. They are generally produced singly, and are dispersed without any approach to regularity over all parts of the sarcodous membranes of the sponge, abounding in some situations to a very much greater extent than in others. Their positions on, and mode of attachment to, the membrane are exceedingly varied, but in almost every instance it is such as to render the spiculum obviously subservient to the retention of the sarcode on the membranes which it covers. In one instance only I have found the simple bihamate spicula congregated in loose fasciculi. In this sponge, a new and very interesting species, *Hymedesmia Zetlandica*, Bowerbank, MS., they occur in great profusion. Very few of them occur singly; nearly the whole of them are found in rather loose fasciculi, and the number is generally so great in each as to render it very difficult or impossible to count them. The mode of their disposition in the bundles is symmetrical, all the hami being in the same plane and coincident in direction, as represented in Plate XXXI. fig. 8. A few bundles of reversed bihamate spicula were observed, and these in like manner were coincident in every respect like the simple bihamate ones.

When these forms of spicula are equal in the amount of the development of their terminations, and when their hami or palms are coincident in plane and direction, their normal mode of attachment is at the middle of the bow of the shaft, and the direction of their projection is at right angles to the plane of the membrane on which they are situated, so that both terminations are rendered effective as retentive organs, as represented in fig. 8, Plate XXXI., dispersed on the membrane. But when their terminations are in different planes, or unequal in amount of development, then the normal mode of attachment to the membrane is by one end of the spiculum, while the other end is projected into the sarcode above at various angles. This mode of disposition of the inequianchorate form of spiculum is beautifully illustrated in *Halichondria lingua*, Bowerbank, MS., a new species of British sponge from the Hebrides.

In this case, as in *Hymedesmia Zetlandica*, we find these organs congregated, but in a very much more symmetrical and beautiful mode. They occur in rosette-shaped groups; the smaller palms being adherent to the membrane in a circular form, and disposed as close to each other as possible, while the larger palms radiate from the centre at angles of about 20 or 30 degrees from the plane of the membrane beneath, as represented by fig. 9, Plate XXXI.

I have selected this group for representation in consequence of its containing but a small number of spicula, and thus displaying the mode of arrangement more distinctly than a greater number would have done. In many cases these groups contain so large a number of spicula as to render any attempt to count them ineffectual, and in some instances so many are developed that the group assumes the form of a ball rather than that of a rosette. Fig. 10, Plate XXXI. represents a rosette-shaped group containing about the usual number of spicula.

Besides the rosette-shaped groups in *Halichondria lingua*, there are a considerable number of these spicula dispersed over the surfaces of the membranes; but the attachment of these spicula is more frequently at the middle of the shaft than at the smaller end of the spiculum, their normal point of attachment. In the single and separate mode of disposition they are performing the office of equi-anchorate spicula, and the mode of their attachment is varied accordingly; but under these conditions they are rarely ever so fully developed, nor do they attain the same size as those which form the radiating groups. Notwithstanding the numerous groups and dispersed spicula of the inequi-anchorate form, this sponge is also abundantly furnished with bihamate spicula of various forms, but they are never congregated like the anchorate ones.

The same radiating mode of arrangement occurs in a parasitical Australian sponge from Freemantle, but the form of the terminations of the spicula is very different from those of *Halichondria lingua*. The distal termination of each of the inequi-anchorate spicula is shortened in length, but expanded laterally to a considerable extent, and its terminal edge is furnished with three thin pointed teeth. The distal end has two small expanded and raised wings, projected in the direction of the inner curve of the spiculum, and so disposed as to cause it to resemble very closely an engineer's spanner for bringing up to their bearings projecting square-headed screws. Thus, although the forms of the termination of the two varieties of spicula vary to a considerable extent, the principles of their structure and purposes are in perfect unison. Fig. 11, Plate XXXI. represents a group of these spicula, and fig. 12, Plate XXXI. a single spiculum highly magnified to display their peculiarity of structure.

These forms of spicula appear to be peculiar to the siliceous sponges. I do not recollect having ever seen them in any species of calcareous sponge.

Spicula of the Sarcode.

The primary office of the whole tribe of multiradiate spicula is evidently that of consolidating the sarcodous substance of the sponge, nor is their presence in the exercise of this office confined to the Spongiadæ. In the soft parts of the extensive family of the Gorgoniadæ we find them in vast abundance, and in every variety of form, from an elongate tubercular spiculum to the elongo-stellate forms of the Spongiadæ, and the prevalence of the bluntly terminated radii is strongly indicative of their non-defensive character. But this latter quality does not obtain in other cases, either as regards the higher tribes of animals or the Spongiadæ. Thus we find in numerous species of compound tunicated animals their fleshy substance is crowded with sphero-granulate spicula, very closely resembling in form those of the sphero- and subsphero-stellate shapes so abundant in *Tethea Ingalli* and *T. robusta* (figs. 14 & 15, Plate XXV. Phil. Trans. 1858). In both these cases the acute termination and the peculiarities of their respective situations are indicative of their subserving the office of defensive, as well as that of consolidating spicula.

In some species of *Tethea*, where the sponge is elaborately protected by distinct systems of defensive spicules, the subsphero- or sphero-stellate forms are either entirely absent, or only represented by minute clavate or cylindro-stellate forms; but in *Tethea Ingalli* and in *Geodia carinata*, where there is an almost total absence of elongate defensive spicula at the surface of the sponge, the acutely pointed large subsphero-stellate spicula are exceedingly numerous immediately beneath the dermis, and gradually decrease in number in an inward direction until they almost cease to exist in the deeper portions of the sponge. Thus their presence in such abundance near the surface of the animal would tend materially to check the voracity of any enemy that might attempt to prey upon them. In like manner we find the smooth and abundantly porous membrane of *Tethea muricata* (figs. 14 & 15, Plate XXXI.) crowded with the elongo-attenuato-stellate form represented in Plate XXV. fig. 18, Phil. Trans. 1858; and a single glance at them, as represented *in situ*, will show how admirably they are calculated to defend the delicate tissue on which they repose from the attacks of even their most minute and insidious enemies. The mode of their disposition is also strongly indicative of their defensive functions, their long axis being, not parallel to the plane of the membrane beneath, but at right angles to it.

In *Tethea Norvegica*, BOWERBANK, MS., where the surface of the sponge is well provided with external defensive spicula, the large subsphero-stellate form is comparatively rare, but the tissues of the neighbourhood of the intermarginal spaces and canals are crowded with the minute attenuato-stellate forms, and their surfaces are bristling with the sharp points of their radii, so that no intruding annelid could either take a mouthful from their surfaces or crawl over them with impunity. Deeper in the sponge, beyond the range of penetration of such enemies, they are comparatively very few in number, and the large subsphero-stellate ones are entirely absent.

The hexradiate forms represented by figs. 24 to 36, Plate XXV. Phil. Trans. 1858, are more especially found in the siliceo-fibrous sponges. I have only seen two specimens of this class of sponges in which the sarcode was well preserved. In one of these I have observed the slender form like that of fig. 34, Plate XXV., occupying the areas of the rigid siliceous skeleton completely surrounded by sarcode, which stretched from one ray to another in thick glutinoid plates, but without touching the surrounding skeleton-fibres, excepting at one basal point connecting it with the general mass of the sarcodous tissues. From the positions and general appearances of the hexradiate spicula, it would appear that this form of spiculum has the office more especially of supporting and consolidating the sarcode, and that it is in no respect subservient to defensive purposes.

Generally speaking the slender rectangulated hexradiate spicula occur singly, but I have sometimes found them grouped together; in this case their axes were coincident, and their radii in the same plane, or very nearly so, but not always agreeing in their direction; such a framework would form a very fitting support to a large mass of sarcodous tissue partially separated from the framework of the skeleton and occupying a portion of a large interstitial space.

In the large open areas of the skeleton of *Euplectella aspergillum*, OWEN, the hexradiate forms, ranging from fig. 24 to fig. 33, Phil. Trans. 1858, are exceedingly abundant, and a considerable number of them are not developed to the extent of the full number of their radii. This may probably arise from the development of the radii being stimulated by the necessities of the mass of sarcodous tissues in which they are imbedded, and consequently where no necessity for their presence exists they would not be put forth. In the trifurcate and quadrifurcate hexradiate forms, if we may judge from the termination of their radii, they, like the simple stellate forms, are either purely consolidating, or they combine with that office that of defensive spicula also, as far as regards the sarcodous substance in which they are imbedded.

We can scarcely imagine any defensive properties in the slender and complicated but elegant forms of the floricomo-stellate spicula, and it is probable that their office is purely that of assisting in the consolidation of the sarcodous substance.

The whole of these beautiful stellate forms of spicula are siliceous, while their homologues in the Gorgoniadæ and the compound Tunicata are calcareous; and it is somewhat remarkable that hitherto none of these forms have been found in the calcareous species of sponges.

Spicula of the Ovaria and Gemmules.

We find the same laws in force regarding the spicula in the structure of the minute bodies which have been designated gemmules by previous writers on the Spongiadæ, that obtain in the sponges themselves. In some they serve the purposes of internal skeleton and defensive spicula as well. In others they combine the offices of tension and defensive organs, and frequently they are very different in form from those of the parent sponge. In the first part of this paper, in Plate XXVI. figs. 11 to 42, Phil. Trans. 1858, I have figured the varieties of form that I have hitherto found in the ovaria and gemmules, and I have shown that these bodies may be classed in three groups.

1. Those which have the spicula disposed at right angles to lines radiating from the centre of the ovarium to its surface.

2. Spicula disposed in lines radiating from the centre to the circumference of the ovarium.

3. Spicula disposed in fasciculi in the substance of the gemmule from the centre to the circumference.

In *Spongilla Carteri*, BOWERBANK, MS., and *S. fluviatilis*, JOHNSTON, our commonest British species, belonging to the first group, the external series of spicula of the ovaria are of the same form as those of the skeleton, but frequently somewhat shorter. They are disposed irregularly over the surface of the ovarium, and firmly cemented to it by the middle of the shaft, while each of their apices is projected in tangental lines. Thus their shafts perform the office of tension spicula, while their terminations become efficient weapons of defence. Fig. 11, Plate XXVI. Phil. Trans. 1858, represents the spiculum of the ovarium of *S. Carteri*.

In other cases in this group we find these spicula differing from those of the skeleton of the parent sponge; thus, the one that is represented by fig. 13, Plate XXVI. Phil. Trans. 1858, from the surface of *Spongilla lacustris*, JOHNSTON, is curved so as to accommodate it to the rotundity of the ovarium; and we do not find its apices projecting as in those of *S. fluviatilis*, but instead of the projecting apices, the whole spiculum is covered with minute spines, assimilating it in character with the general structure of those spicula which combine the office of tension and defensive spicula, but differing considerably in their proportion from the tension spicula of the same sponge, *S. lacustris*, represented by fig. 21, Plate XXIV. Phil. Trans. 1858, the one being evidently destined to sustain and protect extended membranes, while the other is especially adapted for a small curved surface by its form and small size; each of the figures being drawn with the same power, 660 linear.

On the surface of the ovarium of *Spongilla cinerea*, CARTER, we find this description of spiculum still more decidedly produced. It is of a cylindrical form and entirely spined, and has just the amount of curvature that is in unison with the curved surface on which it reposes. The spines on the middle of the shaft are cylindrical, and terminated bluntly, so as to strengthen its hold on its imbedment. Those of its apices, on the contrary, are acutely conical and recurved, and are strongly produced, so as to form very efficient weapons of defence. This spiculum is represented by fig. 17, Plate XXVI. Phil. Trans. 1858.

The birotulate and boletiform spicula of the second group appear to be more purely structural, as regards the skeleton of the ovarium. The rotulæ are very closely packed at both the external and internal surfaces of that body, and the crenulation or dentation of each rotula is as well produced on the internal as on the external ones, and it appears to be very influential in maintaining each spiculum in its proper position. In the natural condition of the ovaria these spicula are entirely imbedded in its walls, and other spicula of a truly defensive nature are superimposed for its protection. The large spines in the shafts of the birotulate spiculum from *Spongilla plumosa*, CARTER, fig. 21, Plate XXVI. Phil. Trans. 1858, are also apparently subservient to strengthening and maintaining the spiculum in its proper situation, although they are acutely terminated, as defensive spines usually are; but in the same relative position on the birotulate spicula of *Spongilla Meyeni*, CARTER, we find the spines short, stout, and cylindrical, spreading or budding at their apices, and evidently more fitted for assisting to retain the spiculum in its proper place than for defensive purposes. This spiculum is represented by fig. 29, Plate XXVI. Phil. Trans. 1858.

There is an apparent analogy between the expansions of the rotulæ and those of the folio-peltate spicula, but they do not appear, like the latter, to be derived from the ternate forms. The radiations of the canaliculi, as represented by fig. 32, Plate XXVI. Phil. Trans. 1858, are not derived from three primary rays, but each appears to emanate from a central cavity at the end of the shaft; and their number, 22, at their proximal termination is not reconcileable with any regular number of bifurcations arising from three primary rays, however short we may imagine them to be.

The progressive decline of the inner rotula in the inequi-birotulate spiculum of *Spongilla paulula* (fig. 31), and its all but total extinction in *Spongilla reticulata* and *Spongilla recurvata* (figs. 33 & 34) until the distal rotula merges in the scutulate form, with an acute external umbo in place of an internal shaft as in *Spongilla Brownii*, figs. 36 & 37, Phil. Trans. 1858, exhibit a very interesting series of gradations of development in the same description of organ.

The spicula of the third group (those having the spicula disposed in fasciculi in the substance of the gemmule) differ less in character from those of the parent sponge than either of the preceding groups. They are in reality but modifications of the external defensive spicula of the parent sponges.

The inequi-fusiformi-acerate one (fig. 39, Plate XXVI. Phil. Trans. 1858) differs from the fusiformi-acerate one of the skeleton in no other respect than in the greater proportionate attenuation towards its distal termination, which gives it a degree of flexibility that allows of its bending freely under the pressure of any comparatively large body; and I have seen them, when two gemmules have been pressed closely together, bent to the extent of semicircles without breaking. In the young gemmules these spicula are usually projected much beyond the other forms of defensive spicula that accompany them.

In like manner the small attenuato-porrecto-ternate form (fig. 43, Plate XXVI. Phil. Trans. 1858) is a modification of the similarly formed external defensive spicula of the parent sponge. In the adult gemmule the apices of these spicula rarely project beyond the dermal membrane, and it is only on pressure from without that they would be brought into effective use. The amount of the angle of their radiation at the apex of the spiculum is therefore greatly increased beyond those of the external defensive ones of like form in the parent sponge, so as to accommodate their apices to the curve of the surface of the gemmule, and to render each point equally effective; and as they are not projected beyond the dermal surface, as in the sponge, their shafts are shortened proportionally.

The unihamate, bihamate, and recurvo-ternate forms of the same gemmules (figs. 40, 41 & 42, Plate XXVI. Phil. Trans. 1858) are also modified forms of the recurvo-ternate external defensive spicula of the parent sponges, *Tethea cranium* and *simillimus*.

Of the other forms of "spicula the position of which are unknown," I can say little more than I have before stated, excepting that I have since found the subspinulato-arcuate one, represented by fig. 51, Plate XXVI. Phil. Trans. 1858, *in situ* in a new species of sponge from Freemantle, Australia, *Hymeniacidon Cliftoni*, BOWERBANK, MS., and that it is a retentive spiculum.

The Interstitial Canals and Cavities.

These organs exhibit their most complete mode of development in the genus *Spongia* and in the Halichondroid sponges, occupying nearly the whole of the masses of the animals. They consist of two distinct systems, an incurrent and an excurrent one.

The incurrent series have their origin in the intermarginal cavities immediately within the dermal membrane, and their large open mouths receive from these organs the water inhaled through the pores and convey it to the inmost depths of the sponge, ramifying continually like arteries as they proceed in their course downward until they terminate in numerous minute branches. The inhaled fluid is then taken up by the minute commencements of the excurrent series, which continually unite as they progress towards the surface of the sponge, in the manner of veins in the higher animals; until they terminate in one or more large canals which discharge their contents through the oscula of the sponge. This system is found to obtain in the whole of the genus *Spongia* and in the massive Halichondroid sponges, which have their oscula dispersed over their external surfaces. By this mode of organization the inhaled fluid, laden with nutritive particles, is poured at pleasure into the internal cavities of the sponge, flowing over extensive membranous surfaces coated with sarcode; so that the aggregated surfaces become a great system of intestinal action, fully equal in proportional extent to that of the intestines of the most elaborately organized mammal.

They do not in every genus exhibit the regularity of structure described above, and in some cases the canalicular form resolves itself into a series of irregularly formed spaces. In other cases, where a common cloaca exists, there appears to be but one system of interstitial canals, those which convey the inhaled fluid from the pores through the substance of the sponge to the parietes of the great central cloacal cavity which receives the whole of the fæcal streams, rendering the system of excurrent canals unnecessary.

In the Cyathiform sponges we find a somewhat slmilar structure. The outer portion of the cup is essentially the inhalant surface and the interior of it the exhalant one, and there accordingly we generally find a great number of small oscula dispersed on all parts of it, very often having their margins slightly elevated, that the fæcal matter that issues may be discharged free of the surrounding membrane.

The large fistular projections which form such striking and beautiful objects in the genus *Alcyoncellum* are also great cloacal organs, their dermal membranes abounding in pores, and their inner surface furnished with oscular orifices, the intervening space being occupied by the interstitial cavities, the interior forming one large cloacal cavity, which discharges its contents through a cribriform mouth at its distal end. In *Grantia* both systems, the incurrent and excurrent interstitial canals, become very nearly obsolete, the large intermarginal cavities or cells imbibing the water through their pores on the distal extremities, and becoming enlarged and elongated until they reach the parietes of the great central cloaca, into which they discharge their contents, each through a single osculum, into a short depression or cavity in the parietes of the great cloaca, and this shallow cavity represents the nearly obsolete system of excurrent canals.

The membranes lining the incurrent and excurrent canals are frequently highly organized. In the common honeycomb sponge of commerce, when in the same condition as when taken from the sea, these canals are constructed of a series of compound membranes, each consisting of simple interstitial membrane with a layer of primitive fibrous tissue beneath it; the fibrous portion consisting of a single series of fibres parallel to each other, and so closely adjoining as to touch each other through nearly their whole course (Plate XXVII. fig. 4).

When the fibres are clear of the membranous tissue they appear as simple pellucid threads, but when covered by the membrane they frequently appear as if moniliform; this character seems to be due to minute molecules arranged in linear series on the membrane immediately above them. These membranes abound in large open oval spaces, so that the tissue assumes very much the appearance of areolar tissue, as described by Professor BOWMAN in his treatise on mucous membrane in the 'Cyclopædia of Anatomy and Physiology.'

The layer of membrane forming the surface of the canal has its fibres disposed at right angles to the axis of the canal, while those of the layers beneath it assume various directions, usually in straight lines, excepting in the vicinity of the areas of communication, around which they curve to strengthen their margins.

In the canals deeply buried in the mass of the sponge, the sides frequently consist of but one layer of membrane and primitive fibrous tissue, and in this case also the fibres are always disposed at right angles to the axis of the canal, but they are neither so numerous nor so closely packed as in the sides of the great excurrent canals.

The interstitial membranes are also furnished with these fibres, sometimes in considerable quantity, but rather irregularly disposed, while in other cases a single fibre only will be observed meandering across the tissue.

The interstitial membranous tissues in a beautiful little specimen of *Alcyoncellum* from the North Sea, for which I am indebted to my friend Captain THOMAS of the Hydrographical Survey, are very similarly constituted to those of the sponges of commerce. The membranous walls of the interstitial cavities are each formed of a series of fibro-membranous layers, the fibres of each layer being disposed at angles varying from those above and below it.

Figures 1, 2, 3, & 4, Plate XXVII. represent portions of the lining membranes of the incurrent and excurrent canals, and the mode of the disposition of the primitive fibrous structure upon them.

Intermarginal Cavities.

In the Halichondroid sponges, immediately beneath the dermal membrane, there are numerous and, comparatively speaking, large irregularly formed cavities which receive the water inhaled by the pores, and convey it to the mouths of the incurrent canals, which have their origin in the deepest portions of the spaces. These organs, from their irregularity in size and form, are not always very apparent; but if a section be made at right angles to the surface in a dried specimen of *Halichondria panicea* or *Hal. simulans*, JOHNSTON, they may be readily detected and distinguished from the interstitial canals and spaces of the sponge.

Fig. 13, Plate XXXI. represents a section of *Halichondria panicea*, and fig. 1,

Plate XXXII. a similar section of a branch of *Halichondria simulans*, JOHNSTON, showing that, however varied the forms of the sponge may be, the interstitial cavities are the same in structure and position.

I have never been able, in the Halichondroid sponges, to detect valvular diaphragms separating these spaces from the interstitial canals and cavities beneath.

In the genera *Geodia* and *Pachymatisma* these organs assume a very much greater degree of regularity and a complexity in their organization that are never apparent in those of the Halichondroid sponges. In *Geodia Barretti*, BOWERBANK, MS., a highly organized species of the genus, they are found in the crustular dermis in great abundance. They are in form very like a bell, the top of which has been truncated. They are situated in the inner portion of the dermal crust; the large end of the cavity being the distal, and the smaller end the proximal one. The open mouth or distal end of the cavity is not immediately beneath the dermal membrane. There is an intervening stratum of membranes and sarcode, of about two-fifths the entire thickness of the dermal crust, which is permeated by numerous minute canals which convey the water inhaled by the pores to the expanded distal extremity of the cavity. The proximal end is closed by a stout membranous valvular diaphragm, which the animal has the power of opening and closing at its pleasure. It is usually entirely destitute of the characteristic dermal spicula that are found abundantly in the adjoining membranous tissues.

The action of the diaphragm of each cavity appears to be independent of the surrounding ones, the condition or degree of opening of no two adjacent ones being alike. In the greater number of cases they were in a closed state, and in this condition the membrane was filled with concentric circles composed of minute rugæ or thickened lines, and at the centre it was closely pressed together, completely closing the orifice. In some cases the membrane was only partially closed, and the orifice was either circular or slightly oval; in others it was nearly as large as the diameter of the basal end of the cavity. The pursing of the centre of the membrane of the diaphragm was always outward as regards the cavity, so that when viewed from within it appeared as a slightly funnel-shaped depression, the bottom of which was conical. The cavities are lined by a smooth and tolerably strong membrane, abundantly supplied with slender fibrous tissue, disposed in nearly parallel lines at right angles to the long axis of the cavity.

The adaptation of the skeleton to the support of these elaborately constructed organs is very remarkable. The sponge is furnished abundantly with large expando-ternate spicula, the radii of which are furcated at their apices. They occur in a series of bundles; the long attenuated shafts of each fasciculus approximate at their bases, and diverge thence until the ternate head of each is about equally distant from its surrounding neighbours, and the extremities of the rays touch or slightly cross each other, thus forming a beautiful and regular network, the meshes being six- or seven-sided, according to circumstances. The upper surfaces of the radii are firmly attached to or partially imbedded in the under surface of the crustular stratum, and the areas thus formed are occupied each with the proximal valvular terminations of one of the intermarginal cavities.

The progressive development of these inhalant areas, formed by combinations of the radii of the ternate forms of spicula, in different species of sponges is very interesting. In *Pachymatisma* they are so indefinite that they can scarcely be said to exist. The ternate spicula are few in number and very irregular in their mode of disposition, and a faint indication only of their future regular combination to form the dermal reticulation is apparent. In the more highly organized genus *Geodia* we find them in different species in progressive stages of combination, until, in *G. M^cAndrewii* and *Barretti*, the apices of the radii of the ternate spicula are interlaced with each other, and a continuous irregular network is formed, each area of which is filled with the proximal termination of an intermarginal cavity. In *Dactylocalyx Prattii*, BOWERBANK, MS., the structure advances another stage towards perfection.

There is the same design as that exhibited in the construction of the dermal areas in *Geodia McAndrewii* and *Barretti*, but there is a considerable difference in the application of the areas produced by the combinations of the ternate apices. In *Geodia* these areas are placed beneath the highly organized and regularly formed intermarginal cavities, and form the framework and support of their valvular proximal ends; while in *Dactylocalyx Prattii* they are situated above the distal ends of the intermarginal cavities of the sponge, which have not the regular structure and valvular appendage of those of *Geodia*, but are similar to the like organs in the Halichondroid sponges, and in this position they serve only to support and strengthen the dermal membrane, which adheres firmly to their distal surfaces. In this situation they are subject to a greater chance of pressure and disruption than the more deeply seated ones of *Geodia*, and accordingly we find extra provisions for the safety of the junctions of their radii. The shafts of these spicula are short, stout, and conical, and they penetrate but a very short distance into the substance of the sponge. They do not appear to be cemented to any part of the rigid siliceo-fibrous skeleton, but are merely plunged into a somewhat thick stratum of membranous structure reposing on the surface of the skeleton. Their radii are compressed considerably and extended laterally, so that their planes are in accordance with that of the dermal membrane, and they present a greater amount of adhesive surface than those having cylindrical radii. The ternate rays ramify irregularly. Sometimes one ramus, after slightly pullulating, remains nearly obsolete, causing the branch to assume a geniculated form like some of the ramifications of a Deer's horn, and no two appear to be exactly alike; in fact there is every appearance that each ray is influenced and modified in its development by the necessities of combination with the adjoining spicula, and their apices are directed in such a manner that they lap over each other in opposing lines, so that each two form a spliced joint, giving a much greater amount of strength than the mere crossing of the radii at various angles as in those of *Geodia*. The inhalant areas thus formed appear to differ very slightly from those of *Halichondria panicea*, in each of which several pores are opened, while those of *Dactylocalyx Prattii* seem to be devoted each to a smaller number (Plate XXIX. figs. 8 & 9).

As the ternate spicula thus united for the support of the dermal membrane would afford it little or no protection against the voracity of its smaller enemies, we find the necessary defence in innumerable short, stout, entirely spined cylindrical spicula not exceeding $\frac{1}{3000}$th of an inch in length; thus minute, there is no conceiving a predaceous creature with a mouth so small that they would not enter and become a subject of annoyance so great as to interfere seriously with its attacks on the membrane; and they are so numerous, and so closely packed together, that no portion of it equal in size to the length of a spiculum could be removed without one or two of them accompanying it.

A still further advance in this system of dermal support and defence is exhibited in the beautiful harrow tissue of Dr. A. FARRE's siliceo-fibrous sponge, to which his specimen of *Euplectella cucumer*, OWEN, is attached. In this case we have a perfect and regular quadrilateral network of smooth siliceous fibre, from the angles of which a double set of short conical spicular shafts are projected, each about $\frac{1}{120}$th of an inch in length and entirely spined. Each set are at right angles to the plane of the network, one series pointing inward, and serving the purposes of attachment to the mass of the sponge beneath, while the other set are directed outward, serving as defensive weapons; so that a small piece of this tissue beneath the microscope closely resembles an agricultural harrow, with the difference that it has two sets of teeth in opposite directions instead of one. The dermal membrane has been nearly all destroyed; but entangled with the fibres of the skeleton there are some attenuato-stellate spicula, with which it is probable the dermal membrane was amply furnished as secondary defences against its minute enemies.

I believe the surface presented to the eye in the portion represented in Plate XXXII. fig. 7 to be the external surface, as the fragments of the dermal membrane which remain all seem to cover that side of the fibres. Generally speaking there is some difficulty in detecting the double series of spicular organs at the angles of the network, but a reversal of the object beneath the microscope immediately removes all doubt on that subject.

In *Grantia compressa* and *ciliata* the intermarginal cavities appear to attain their highest degree of development, and are multiplied and expanded to such a degree as to almost supersede every other organ. The whole sponge in these species is formed of a great accumulation of elongated cells or cavities, closely adjoining each other and angular by compression. Their conical distal terminations, abounding in pores, represent the external surface of the sponge, while their valvular proximal ends form the inner surface, in conjunction with the shallow cavities, into the distal ends of which each cell discharges its contents. These shallow depressions, intervening between the intermarginal cavities and the cloaca, are all that remains to represent the incurrent portion of the interstitial systems so largely developed in the Halichondroid sponges, the great cloacal cavity entirely superseding the excurrent spaces and canals (Plate XXXIII. figs. 1 & 2).

In these species of *Grantia* there is no doubt regarding the existence of cilia, the whole of these great cavities being completely lined with them.

It is a question whether the intermarginal cavities share, in common with the interstitial canals, in the function of the assimilation of nutriment, or whether they are devoted solely to the aëration of the fluids of the animal; and this, if we consider the structure and extent of the interstitial canals in the Halichondroid sponges, is probably the case. In *Grantia* the abundant provision of cilia in those cavities at once stamps them as breathing-organs; and although cilia have never yet been satisfactorily proved to exist in the intermarginal cavities of the Halichondroid sponges, there can be no reasonable doubt of their being the homologues of the large ciliated cavities in *Grantia compressa* and other similarly constructed sponges. Now in these sponges, although the cilia may be readily seen in vivid action within the open oscula, as I have described at length in my paper "On the Ciliary Action of the Spongiadæ," published in the 'Transactions of the Microscopical Society of London,' vol. iii. p. 137, not the slightest trace of cilia exists without those organs; and this seems to indicate that the aërating functions were strictly confined in these sponges to the large intermarginal cavities.

The same mode of reasoning applies equally well to the intermarginal cavities of *Geodia* and *Pachymatisma*, to which it is probable that the cilia are in like manner confined. The great valves at the proximal ends of these cavities in this tribe of sponges appear to strongly indicate a decided separation of the functions of aëration and digestion; and if this conclusion be true in regard to the intermarginal cavities of *Geodia* and *Pachymatisma*, it will probably be so in the homologous organs in *Grantia*; and in this case we must look for the digestive surface in the shallow cavities intervening between the terminal valve of the intermarginal cavities and the parietes of the great cloaca, and of the surfaces of that organ itself. The structure and functions of the intermarginal cavities, and especially as displayed in *Geodia* and *Pachymatisma*, indicate a closer alliance with the great class Zoophyta than has hitherto been suspected to exist. In the one case we have an accumulation of individual animals conjoined in one mass; in the other a similar congregation of organs in place of individuals.

Dermal Membrane.

The dermal membrane envelopes the sponge entirely. When denuded of sarcode by partial decomposition, it has the appearance of a simple, pellucid, unorganized membrane. In the living state its inner surface is somewhat thickly coated with sarcode, and it has the appearance of, comparatively speaking, a stout, tough skin, and in many sponges it requires a considerable amount of violence to tear it. The dermal membrane of the Turkey sponge of commerce, *Spongia officinalis*, is abundantly supplied with primitive fibrous tissue. It curves round the margins of the porous areas, thickening and strengthening the whole of the dermis to a very considerable extent; but it exists to a very slight extent in the pellucid membranes of the areas in which the pores are opened. When alive, it is replete with powers of life and action of a very remarkable description.

Without the slightest appearance of nerves or muscles, it has the power of opening pores on any part of its surface and of closing them again at pleasure, without leaving a trace of their existence to indicate the spot they occupied; and there is no amount of laceration or destruction that it does not seem capable of repairing or replacing in a very short period, reproducing itself over extensively denuded surfaces in a very few hours. It also shares, in common with the interstitial membranes, the power of strongly and quickly adhering to other sponges of the same species with which it may be brought in contact, but never with those of a different species, however long the two may remain pressed against each other. In some sponges the distal extremities of the skeleton pass through and project beyond the surface of the dermal membrane, while in other cases the whole of the skeleton is confined within it.

I will not describe at length these remarkable powers of the dermal membrane, but refer the reader to a series of observations on the "Vital Powers of the Spongiadæ," published in the Reports of the British Association for 1856, p. 438, and for 1857, p. 121, in which I have described in detail a series of observations and experiments on living sponges, which demonstrate in a satisfactory manner the extent of the vital powers and capabilities of this highly sensitive membrane.

In some species of sponges the outer surface of the skeleton is especially modified to strengthen and support the dermal membrane. Thus in some of the keratose sponges of commerce, in parts of the sponge which have been in contact with other sponges, or with rocks or stones, we find a fine network of stout fibres immediately beneath the derma, as represented by fig. 9, Plate XXXII.; and *Isodictya varians*, Bowerbank, MS., is always furnished with a fine network of spicula, the reticulations consisting of a single series of spicula only, and on this framework the dermal membrane is firmly cemented. Fig. 8, Plate XXXII. represents a small portion of this dermal reticulation, magnified 108 linear.

In *Halichondria panicea* the same description of reticulation prevails, but in this sponge the fibres of the network are always composed of numerous spicula cemented together, as represented in fig. 5, Plate XXXII., illustrating the porous system of the above-named species of sponge. But this regularity of structure is not constant even in the same individual; thus in *Hal. panicea* you will often observe one portion of the dermis beautifully reticulated, while a closely adjoining spot will be supported by a series of matted spicula without any indication of areas for the pores, and these variations in structure are evidently determined by the presence or absence of those organs at particular parts of the surface. In other cases, beside a general attachment of the inner surface of the dermal membrane to the surface of the skeleton, we find it supported by numerous flat fasciculi of spicula dispersed irregularly on its inner surface, and differing materially in size and form from those of the skeleton, as in our common British species, *Halichondria incrustans*. Great variety exists in these modes of strengthening and supporting the dermal membrane; but those which I have described above will suffice to illustrate the general principles of their application. Beside the general systems of external defence, the dermal membrane is often supplied with special defences. Thus in *Tethea muricata* (figs. 14 & 15, Plate XXXI.) we find its outer surface abundantly supplied with elongo-stellate spicula, which project externally to a considerable extent; and in *Dictyocylindrus stuposus*, Bowerbank, MS., beside the numerous defensive spicula projected through the surface, we find the membrane filled with minute sphero-stellate spicula, which would effectually protect it from the assaults of any minute enemies that might attempt to prey upon it. Fig. 6, Plate XXXII. represents a small portion of the dermal membrane of this sponge. This mode of defence is very general in the genera *Geodia*, *Tethea*, and *Pachymatisma*, and it occasionally occurs in many other genera of Spongiadæ.

The Pores.

The pores in the Spongiadæ are the orifices or mouths through which the animals breathe and imbibe their nutriment. They are situated in the dermal membrane, and are exceedingly numerous when the imbibing powers are in full operation. In *Pachymatisma* and *Geodia*, and in some other highly organized genera, there is good reason to believe that they are permanent organs, opening and closing repeatedly in the same situations. But in the greater part of the Halichondroid types of sponges they are certainly not permanent orifices like the mouths of higher classes of animals, and in these sponges, when they are in a state of complete repose, there is not the slightest indication of their existence. Their usual form is circular, but they frequently assume the shape of an elongated oval, and within a limited range they vary to a considerable extent in their dimensions; on the whole they exhibit a very constant and universal type of form and size; however different may be the internal structure of the sponges, or however great may be the difference in size of the individuals, they always appear to maintain their normal characters. No definite law appears to prevail in their distribution over the surface of the sponge, and they are liable to appear to a greater or a less extent on every part of its external surface, wherever there are intermarginal cavities beneath. The situations where they may be expected to appear may in many instances be readily recognized. Thus in *Halichondria panicea*, wherever we see on the dermal membrane a well-defined reticulation of spicula with clear and distinct areas, there, when the sponge is inhaling, we may expect to find the open pores, as represented in Plate XXXII., fig. 5, while on spots perhaps immediately adjoining, where the dermal membrane is occupied by a thickly interwoven mass, a felting of spicula, the probability is that not a single pore can be detected.

In some of the West India fistulose sponges we find the large or primary area of the dermal surface composed of keratose fibre, and within these large areas the dermal membrane is strengthened and supported by a secondary reticulation of spicula, in the areas of which the pores are opened. In these secondary reticulations the spicula are abundant, while in other parts of the sponge the tension spicula are rather of rare occurrence. In *Grantia*, a sponge of a widely different construction from those of the Halichondroid type, they occupy the distal extremities of the large intermarginal cavities of the sponge, and they appear to open over the whole of those portions of the cavities not in contact with the adjoining ones.

In *Pachymatisma Johnstonia*, a British sponge closely allied to the genus *Geodia*, we find the dermal membrane perforated by innumerable pores, some as minute as $\frac{1}{1000}$th of an inch in diameter, while others attain the size of $\frac{1}{250}$th of an inch. They are nearly equidistant from each other, but without any order in their arrangement. Immediately beneath the dermal membrane there is a stratum of membranous structure and sarcode destitute of gemmules, and about equal in thickness to one-third of that of the whole of the dermal crust, the remaining two-thirds of which consist of a stratum of gemmules or ovaries closely packed together, but perforated at intervals by the intermarginal cavities. Through the upper stratum, destitute of ovaries, a small canal passes from each pore to the nearest adjacent intermarginal cavity, so that there are a series of them at various angles, all concentrating their streams of inhaled fluid at the distal end of the cavity, which is gradually expanded in diameter to receive them. In these sponges therefore each mouth appears to be furnished with a separate œsophagus, if I may be allowed the term, connecting it with a stomach-like cavity common to a group of mouths above it—a system of organization strikingly in unison with that of the higher classes of animals. In some cases, as in *Geodia M^cAndrewii* and *Barretti*, Bowerbank, MS., we find the pores systematically congregated in groups, as in Plate XXXII. fig. 4, which represents four groups from the latter species, and this congregation is accounted for by the peculiarities of the form and arrangement of the intermarginal cavities of that class of sponges.

The porous organs are still further complicated in a specimen of a branched sponge from the East Indies, presented to me by my friend Mr. S. P. Pratt. This sponge, which is a single branch about a foot in length and 9 lines greatest diameter, has nearly the whole of its surface abundantly furnished with peculiar and highly organized areas, as represented in Plate XXXV. fig. 3, each of which covers and protects a deeply depressed porous area, the depth of which in many cases rather exceeds its own diameter. The protective organ covering this depression is elaborately and beautifully constructed, very closely resembling, in many respects, the spiracula of *Dytiscus marginalis* and other similarly constructed insects. Each of the depressed areas of the sponge is furnished with ten semifollicular membranous cones, the whole of them being based on a common external marginal ring of dermal membrane, from which they are projected inward in the same plane as the dermal surface until their apices nearly meet in the centre of the inhalant area. The exterior surface of each cone is perfect and continuous from the marginal ring of membrane to its apex; but on the interior surface it is only perfect for about half its length from its apex backwards, as if half of the basal portion of a conical bag had been cut away from the remainder. Fig. 5, Plate XXXV. represents the exterior half of one of these protective organs, and fig. 6 the semifollicular structure of their conical organs. The membranes of which they are constructed are abundantly furnished with tension spicula, which are dispersed without order on every part of their surface. It is the only instance I have seen of such an elaborate mode of protection of the porous areas in the Spongiadæ.

In my "Further Report on the Vitality of the Spongiadæ," published in the Reports of the British Association for 1857, I have described at length the opening and closing of the pores in *Spongilla fluviatilis*: each operation is commenced and terminated in less than a minute; they are perfectly dependent on the will of the animal; and in neither case are they simultaneous, but follow in irregular succession, in accordance with the necessities of the animal; and when once the pores are closed, they do not appear to ever open again in precisely the same spot.

In these wonderful opening and closing operations in the dermal membrane of *Spongilla*, every movement is accomplished as systematically and accurately as if there were a perfect system of nerves and muscles present, while not a vestige of fibrous structure can be detected in the thin translucent membrane and its sarcodous lining. No cicatrix remains for an instant after closing, no indication is perceived of the spot where the opening is the next moment to be effected.

In sponges exposed to the action of the atmosphere, between high and low water marks, and in dried specimens, the pores can rarely be detected. In the first case they are carefully closed on the receding of the tide, that the water within them may be safely retained during their exposure to the atmosphere; and in the latter case the violence offered to the sponge, and the shock of its removal from its native locality, are sufficient to induce an immediate closing of those organs, as I have shown in the details of my observations on these organs in *Spongilla* in the volume of the Reports of the British Association for 1857, to which I have before alluded. But should a specimen of marine sponge, after a careful removal from its place of growth, be placed in a shallow pan of sea-water, and be allowed to die of inanition, it then frequently expires with the whole or a considerable portion of the pores open, and in that state it may be readily preserved for the cabinet.

The Oscula.

The oscula are the fæcal orifices of the sponge. They are situated at the distal terminations of the single or concentrated excurrent canals of the animal. They vary considerably in form and size; sometimes they appear as single large orifices, while at others they consist of several small orifices grouped together. When the sponges are massive and solid, they are usually to be found dispersed over the dermal surface, but occasionally they are grouped on the highest portions or on the elevated ridges of the mass. In *Geodia Barretti* they are concentrated in deep depressions or pits. In other cases they are entirely hidden from view, lining the interior of elaborately constructed cloacæ situated in the centre of the sponge, as in *Grantia compressa* and *ciliata*, *Verongia fistulosa*, and a numerous series of species of fistulose sponges from the West Indies.

They are permanent organs, and are capable of being opened or closed at the will of the animal, and are subject to a considerable amount of variation in size and form, in accordance with the variations in the actions of the sponge. Thus in littoral sponges they are frequently entirely closed, and their situation even quite indeterminable, during the period of their exposure to the air; but when immersed in water, and the sponge is in the energetic action of the imbibition of nutriment, they are expanded to their full extent; but when this action ceases and that of gentle respiration only exists, many of them close entirely, and others exhibit apertures not exceeding half their former diameter while the imbibition of nutriment was in vivid action. Their expansion or contraction is not rhythmical; each can be opened or closed at the will of the sponge without any apparent effect on the others. Nor is the habit of opening and closing the oscula the same in every species. Thus in the course of my observations on *Halichondria panicea* and *Hymeniacidon caruncula* in their natural and undisturbed localities, I have frequently observed, during their exposure to the air at low tide, that while no oscula in an open condition could be found in *Hymeniacidon caruncula*, the greater portion of those on the specimens of *Halichondria panicea* were more or less in an open state.

They appear also to be subject to a considerable amount of modification as regards situation, even in the same sponge. Thus in our common British species, *Halichondria panicea*, when of small size, they are situated on the surface of the sponge, and are scarcely, if at all, elevated above the dermal surface; while in large specimens of the same species we find them collected in the insides of large elongated tubular projections or common cloacæ, and these organs vary from a few lines only in height and diameter to tubular projections several inches in height, with an internal diameter of half or three-fourths of an inch. When they attain such dimensions their parietes are often of considerable thickness, and their external surface becomes an inhalant one, like that of the body of the sponge.

In many species the oscula are always elevated above the dermal surface, and these thin pellucid elevations are permanent, while in others, as in *Spongilla fluviatilis*, the tube exists only during the course of the energetic excurrent action; and in such cases it appears to be subject to great variation in size and form, as I have shown in the description of *Spongilla* in my "Further Report on the Vitality of the Spongiadæ," in the volume of the Reports of the British Association for 1857.

Inhalation and Exhalation.

The works of the old writers on Natural History are full of vague opinions on the nature of sponges, but none of them seem to have seriously studied their anatomy, or to have kept them alive in sea-water and examined their daily habits. They appear to have excited abundant attention in the closet, and but very little in their natural localities. The ideas of those authors are so loose and indefinite that it would really be a loss of time to seriously examine and attempt to refute them; and as Dr. JOHNSTON, in his 'History of British Sponges,' has given in his Introduction, Chapter 2, an excellent digest of the various opinions of the previous writers on the subject, I shall content myself with referring my readers to the work of that eminent author for further information on these subjects, and with briefly referring to the few actual observations that appear to have been made by naturalists.

MARSIGLI, at the beginning of the 18th century, stated that he had seen contraction and dilatation in the oscula of several sponges just removed from the sea.

After MARSIGLI, ELLIS (ELLIS and SOLANDER), Natural History of Zoophytes, pp. 184, 186, and 187 (see also Zool. Journ. pp. 375, 376), enunciated similar opinions founded on his own observations on the action of the oscula and their currents; but neither of those authors was aware of the true mode of the entrance of the water into the sponge, a much more difficult problem to solve than its exit through the oscula.

CAVOLINI in his researches, although made on sponges recently taken from the sea, failed in seeing the action of the oscula as ELLIS had done, and he accordingly disputed the truth of those opinions. At a later period, Colonel MONTAGU, although actually examining sponges in the places of their growth, arrived at similar conclusions to those of CAVOLINI, and, like that author, he believed them to be animals of a very torpid nature. MONTAGU'S reasoning to prove the animality of sponges is for the most part sound and excellent; he says, "Whether motion has ever been discovered or not in any species of sponge is not, I conceive, of so much importance as some naturalists would appear to consider. Those who are solicitous in their inquiries after the animals which they have supposed to construct the vesicular fabric of sponges, and have expressed their surprise that this in age of cultivated science no one should have discovered them, must have taken a very limited view of matter possessing vitality, and have grounded their hypothesis only upon supposed analogy." He also observes, "The true character of *Spongia* is that of a living, gelatinous flesh supported by innumerable cartilaginous or corneous fibres or spicula, most commonly ramified or reticulated, and furnished more or less with external pores or small mouths which absorb the water, and which is conveyed by an infinity of minute channels or capillary tubes through every part of the body, and is there decomposed and the oxygen absorbed as its principal nourishment, similar to the decomposition of air in the pulmonary organs of what are called perfect animals."—Wernerian Memoirs, vol. ii. pp. 74, 75.

LAMOUROUX'S conclusions regarding the nature of sponges are so thoroughly vague and supposititious as scarcely to require notice.

LAMARCK has placed the Spongiadæ in a higher position than any naturalist who had preceded him, giving them precedence of the sertularian and celliferous Corallines, and even of the Corallidæ; but I cannot concur with him to the full extent of his conclusions, which, like those of most previous writers, were derived to a much greater extent from comparative reasoning than from actual observation of the animals in a living and natural condition.

Professor SCHWEIGGER'S opinions are very much more those of a practical naturalist, and it is evident that he had closely observed them in a living condition; but he too shares the erroneous opinion of his predecessors, that the oscula were the organs of imbibition, and that no water entered through the dermal surface. Professor BELL, in the 'Zoological Journal' for June 1824, states that he saw the action of the streams from the oscula, but like previous writers concluded that they were organs of imbibition as well as excurrent organs. And it was not until the excellent and accurate "Observations and Experiments on the Structure and Functions of the Sponge" were published in the Edinburgh Philosophical Journal, vols. xiii. and xiv., by Professor GRANT, that a correct notion was entertained by naturalists of the inhalant and exhalant powers of those bodies. These details by the learned Professor are so full and complete as to leave but little room for the improvement of our knowledge of this portion of their natural history. And the facts of the imbibition of the surrounding water by the pores in the dermal membrane, its circulation through the internal cavities of the sponge, and its final ejection through the oscula, have been firmly established and acknowledged by all naturalists who have studied these animals closely in a living state. Dr. GRANT has, in truth, proved himself to have been, in regard to the aqueous circulation in the sponge, what HARVEY was to that of the blood of the higher classes of animal life, the first to discover and to publish the true mode of the circulation of the water in the animal.

This learned and accurate observer says, "I first placed a thin layer from the surface of the *S. papillaris* in a watch-glass with sea-water under the microscope, and on looking at its pores I perceived the floating particles driven with impetuosity through these openings; they floated with a gentle motion to the margin of the pores, rushed through with a greatly increased velocity, often striking on the gelatinous networks, and again relented their course when they had passed through the openings. The motions were exactly such as we should expect to be produced by cilia disposed round the inside of the pores."—Edinburgh New Philosophical Journal, vol. ii. p. 127.

The same author, in describing the excurrent action, says, "The *Spongia panicea* (*Halichondria incrustans*, JOHNSTON) presents the strongest current which I have yet seen." Two entire round portions of this sponge were placed together in a glass of sea-water with their orifices opposite to each other, at the distance of two inches; they appeared to the naked eye like two living batteries, and soon covered each other with feculent matter.

Stimulated by the recital of the observations of Dr. GRANT, I have often sought these currents flowing from the oscula, and there is no species which I have had the opportunity of examining in a fresh and vigorous condition in which I have not succeeded in seeing them. In the one observed by Dr. GRANT, *Halichondria incrustans*, JOHNSTON, the oscula being few in number and very large, the excurrent streams are more than usually powerful. In the course of my investigations "On the Vitality of the Spongiadæ," at Tenby, which are published in the Reports of the British Association for 1856, and in the "Further Report" published in the same work for 1857, I have described a long series of observations of the vital actions of the Spongiadæ as displayed in *Hymeniacidon caruncula* and *Spongilla fluviatilis*, in both of which species there was a perfect accordance in the habits and modes of exertion of these vital actions.

The power of inhalation appears to be exerted in the Spongiadæ in perfect accordance with the similar vital functions in the higher classes of animals, not involuntarily and continuously as in the vegetable creation, but at intervals, and modified in the degree of its force by the instincts and necessities of the animal. And it may be readily seen that the faculty of inhalation is exercised in two distinct modes; one exceedingly vigorous, but of comparatively short duration, the other very gentle and persistent. In the exertion of the first mode of inhalation, that is during the feeding period, a vast number of pores are opened, and if the water be charged with a small portion of finely triturated indigo or carmine, the molecules of pigment are seen at some distance from the dermal membrane, at first slowly approaching it and gradually increasing their pace, until at last they seem to rush hastily into the open pores in every direction. In the meanwhile the oscula are widely open, and pouring out with considerable force each its stream of the excurrent fluid; and if the reflexion of one of the horizontal portions of a window-frame be brought immediately over an excurrent stream, it will frequently be seen that the surface of the water is considerably elevated by its action, even although the osculum be half or three-fourths of an inch beneath its surface, and this vigorous action will sometimes be continued for several hours, and then either gently subside or abruptly terminate. Occasionally a cessation of the action may be observed in some of the oscula while in others it is proceeding in its full vigour, and sometimes it will be suddenly renewed for a brief period in those in which it had apparently ceased. These vacillations in the performance of its functions are always indicative of an approaching cessation of its vigorous action. When the vivid expulsion of the water has ceased, the aspect of the oscula undergoes a considerable change; some of the smaller ones gradually close entirely, while in the larger ones their diameter is reduced to half or one-third of what it was while in full action. Simultaneously with the decline in the force of the excurrent action the greater portion of the pores are closed, a few only, dispersed over the surface of the sponge, remaining open to enable the gentle inhalation of the fluid to be continued, which is necessary for the aëration of the breathing surfaces of the sponge. The breathing state of inhalation appears to be very persistent, and I have rarely failed in detecting it when I have let a drop of water charged with molecules of indigo quietly sink through the clear fluid immediately above an open osculum. These alternations of repose and action are not dependent on mere mechanical causes, and sponges in a state of quiescence may be readily stimulated to vigorous action by placing them in fresh cool sea-water, and especially if it be poured somewhat roughly into the pan and agitated briskly for a short period; and this will take place even in specimens that have very recently been in powerful action.

No general law seems to guide the animal in the choice of its periods of action and repose, and no two sponges appear to coincide entirely in the time or mode of their actions. In fact, each appears to follow the promptings of its own instinct in the choice of its periods of feeding and repose.

In the littoral sponges there is a third condition of the animal, and that is during its exposure to the atmosphere in the intervals between high and low water, and in some sponges the pores and oscula are both completely closed. But this condition does not obtain in all species. Thus, during the course of my investigations at Tenby, I observed that while, amidst the numerous specimens of *Hymeniacidon caruncula* and *Halichondria panicea* that covered the rocks in the neighbourhood of St. Catherine's Cave, the former rarely exhibited an open osculum in the absence of the water, those of the latter species were frequently more or less open.

The most beautiful and striking view of the differences existing between vigorous action and the comparative repose of the breathing process is exhibited in *Grantia ciliata*. In this species the pores are situated on the obtusely conical distal terminations of the intermarginal cells or cavities, each of which is furnished with a long fringe of spicula surrounding its porous end, their proximal terminations being cemented, for about a third of their length, to the slightly curved surface of the base of the cone. In the state of the comparative repose of aërating inhalation, and when the base of the conical extremity of the cavity is not distended by the incurrent action, these spicula all converge to a point at the level of their own apices, and the water thus gently inhaled passes between the shafts of the spicula, forming the protective cone to the inhalant pores, and effectually preventing any extraneous matter from approaching them. But when the vigorous feeding action commences, the distention of the base of the conical portion of the cavity brings it into lines parallel to the axis of the cell, and thus the conical fringe of spicula assumes a cylindrical form, and the molecular food of the animal is freely admitted to the pores.

A corresponding action obtains in the exhalant system of this interesting sponge. The mouth of the great central cloaca is furnished with a thick fringe of very long and slender spicula, which by the contraction of its sides near the mouth are all brought to assume a conical form like those appended to the inhalant cavities; but when the inhalant action is in vigorous operation, and the oscula are all pouring their streams into the cloaca, the force of the water thus accumulated distends the mouth of the cloaca to such an extent as to cause the fringe of long spicula to assume the form of an open cylinder, or in some cases it is expanded to such an extent as to become slightly funnel-shaped, and in this condition the fæcal stream may be seen issuing from it with considerable force. There are many other interesting points in the structure of this highly organized and interesting sponge which I will not advert to at length, but refer my reader to a fuller and more complete history of its structure published by me in the 'Transactions of the Microscopical Society of London' for 1859, vol. vii. p. 79, plate 5.

Thus we find that inhalation is the primary vital operation induced by ciliary action, and that exhalation is merely a mechanical effect arising from the primary cause. We find also that these actions are separated into two distinct modes; the one exceedingly active and vigorous, exerted only at intervals and for short periods, and the other gentle and continuous. If we combine the consideration of these peculiarities of function with those of the anatomical studies, we find that the incurrent streams are always received in intermarginal cavities, and that these organs, however modified, are always present, and in some cases can be distinctly and strikingly separated from the great mass of the interstitial canals and cavities of the sponge. If we trace the course of the inhaled fluids, we find that on their entrance through the pores they are first brought into contact with the parietes of the intermarginal cavities, and passed thence into the complicated system of digestive surfaces which line the incurrent and excurrent canals and cavities of the sponge, and that the exhausted fluids charged with fæcal matters are finally discharged without the slightest return to or intermixture with the contents of the intermarginal cavities. We may therefore, it appears to me, safely conclude that the respiratory and digestive functions are separated, and that the latter has its seat in the intermarginal cavities, and the former in the interstitial canals and cavities.

The vital energy of the Spongiadæ must be very considerable, and the quantity of oxygen consumed by their respiration great, if we may judge by the effects of their presence in the vivarium, where their introduction makes sad havoc among the other inhabitants, few being able to withstand their deleterious presence, and without a large supply of water and a frequent change of it they themselves quickly expire of exhaustion.

Nutrition.

In treating on the subjects of inhalation and exhalation, I have described the energetic period of action in the sponge during the imbibition of the surrounding fluid as equivalent to the operation of feeding in the higher classes of animals. And in my "Further Report on the Vitality of the Spongiadæ," published in the Reports of the British Association for 1857, p. 121, I have described the results of feeding a small specimen of *Spongilla fluviatilis* with finely comminuted indigo in water, and I have there stated that "many of the molecules might be readily followed, as they meandered through the interior of the sponge, and were seen flowing in every direction. During the maintenance of this action in full force, when I directed my observation to the osculum, it was pouring forth a continuous stream of water, and along with it masses of flocculent matter, and many of the larger molecules of the indigo that had entered by the pores; but it is remarkable that although the finer molecules of indigo were being imbibed by the pores in very considerable numbers, very few indeed of them were ejected from the osculum; and if the imbibition of the molecules continue for half an hour or an hour, and then cease, the sponge is seen to be very strongly tinted with the blue colour of the indigo, and it remains so for 12 or 18 hours, after which period it resumes its pellucid appearance, the whole of the imbibed molecules having undergone digestion in the sarcode lining the interior of the sponge, and the effete matter having been ejected through the osculum." If we kill the sponge immediately after being thus fed, and examine the interstitial canals and cavities, we find their sarcodous surfaces thickly dotted with molecules of indigo.

The fæcal matters discharged by the oscula exhibit all the characteristics of having undergone a complete digestion; whatever may have been the condition of molecules of organized matter when they entered the sponge, their appearance after their ejection is always that of a state of thorough exhaustion and collapse.

It is difficult to decide with any degree of certainty what is really the nature of the nutriment of the Spongiadæ, but in the greater number of species it is probably molecules of both animal and vegetable bodies, either living or derived from decomposition. This appears to be the case with the greater number of the Halichondroid sponges; but even among them, as well as other genera, there are peculiarities of structure that are strongly suggestive of carnivorous habits. Thus, in the first portion of this paper published in the 'Philosophical Transactions' for 1858, p. 293, I have described among the interior defensive spicula a remarkable form, which has been hitherto found in one sponge only, the spinulo-recurvo-quaternate spiculum, which "occurs in great profusion in the cavities of the sponge; clusters of them, consisting frequently of as many as twelve or fifteen, radiate from the angles of the reticulations of the skeleton into the interstitial cavities of the animal." I have also described, while treating on the internal defensive spicula, the recurvo-ternate forms, the heads of which are found projecting their radii, more or less, into the interstitial cavities beneath the intermarginal ones in *Geodia* and *Pachymatisma*. The spinulo-recurvo-quaternate spicula, represented *in situ* in Plate XXX. fig. 10, and the recurvo-ternate ones, figured *in situ* in Plate XXXII. fig. 2, *e*, *e*, *e*, are both admirably adapted to destroy the victims entangled among them.

I have for a long time entertained the idea that these elaborate and varied forms of defensive spicula probably subserved other purposes than that of the protection of the digestive surface against the incursions of minute annelids and other predaceous creatures. They are admirably fitted to retain and make prey of any such intruders. No small animal could become entangled in the sinuosities of the interstitial cavities of sponges thus armed without extreme injury from the numerous points of these spicula, and every contortion arising from its struggles to escape from its painful and dangerous entanglement would contribute to its destruction, and it may then, by its death and decomposition, eventually become as instrumental to the sustentation of the sponge as if actually swallowed by the animal. How far this mode of nutrimentation may obtain in the physiology of these creatures it is impossible, in the present imperfect state of our knowledge of their habits to say; but, from the complex, varied, and elaborate structure of these organs, and from their evident adaptation to retain such intruders, as well as to defend the internal surfaces from injury, it is not improbable that their office extends beyond that of the mere defensive function, and that they are, in fact, auxiliary organs for securing nutriment for the use of the sponge. If this supposition, that the elaborately formed and ingeniously disposed recurvo-quaternate spicula combine the office of securing prey with that of defending the interstitial organs of the sponge, be correct, it may afford a clue to the organic purpose of the recurvo-ternate spicula with the exceedingly long and attenuated shafts that so frequently accompany the stout patento-ternate ones in *Geodia Barretti*. The apices of these spicula (Plate XXIII. fig. 45, Phil. Trans. for 1858) rarely attain the height of the plane of the true connecting spicula, and their recurved radii are most frequently projected into the large interstitial spaces immediately beneath the plane of the proximal ends of the cells of the intermarginal cavities, and may thus form subsidiary defences to those organs. Although emanating from the fasciculi of the shafts of the true connecting spicula, their form, slender proportion and position evidently indicate a different office from the spicula with which they are associated, and no other purpose for them occurs to me so probable as the one I have suggested above. Or we may carry the supposition further, and believe them to be not only defensive but aggressive organs; also, like the recurvo-quaternate spicula, their office may be to retain soft annelids that have intruded themselves through the oscula into the digestive organs, to aid in the nutrimentation of the sponge; and this idea appears the more admissible as these spicula are never observed in the intermarginal cavities, where the decomposition of animal matters would be offensive to their especial function, but always in the spaces beneath them, which are the commencements of the digestive system.

The same course of reasoning will apply to their occurrence in such considerable quantities amidst the defensive fasciculi of spicula projected from the surface of *Tethea simillimus*, and also of *T. crania*, the latter being represented in Plate XXIX. fig. 12 *c*, in which it will be seen that the recurvo-ternate heads of the spicula are always situated beneath the level of the true defensive spicula. Thus situated they would form an admirable trap for the entanglement of soft annelids that might attempt to crawl over the surface of the sponge, and thus they would be destroyed and retained for the imbibition of their particles liberated by their gradual decomposition. If this be not their especial purpose in this situation, I must confess myself at a loss to imagine their proper function, as the surface of the sponge is effectually protected by the porrecto-ternate and large acute spicula that compose the defensive fasciculi projecting in such abundance from all parts of the sponge. If we also consider the structure and positions of the ordinary forms of internal defensive spicula, the entirely spined attenuato-acuate ones, in reference to the idea of their being offensive as well as defensive organs, we shall not fail to see that, although less striking in their forms and modes of disposition than the spicula already described, they are calculated to subserve the office of retaining prey quite as effectually as the more singular ones. The abundance in which they occur, the vast number of spines with which they are covered, the apices of which are frequently long and recurved, combined with the mode in which their bases are attached to the fibres of the skeleton, exhibiting a beautiful combination of strength and flexibility, are strongly indicative of a purpose beyond that of mere repulsion.

In the two species of sponges in which are found the acuate entirely and verticillately spined defensive spicula *in situ*, represented in Plate XXX. figs. 7 & 8, one of them has the spicula collected in groups in a manner very similar to those of the spinulo-recurvo-quaternate form, and if the latter be considered as organs for the retention of prey, the physiological purpose of the grouping together of the former can scarcely be considered in any other light.

In the isolated positions of these forms of spicula, viewed in reference to some ideas regarding their physiological purposes, there are some circumstances of a very remarkable nature. These forms of spicula occur in several distinct genera of sponges, and especially in those having a strong kerato-fibrous skeleton. Their usual locality is on the fibre of the skeleton, in which their bases are firmly imbedded, and from which they are projected at various angles into the canals and cavities of the sponge, and they are very rarely seen on the membranes. In *Hymeraphia stellifera* (Plate XXX. fig. 8, *a*) and *clavata*, BOWERBANK, MS., both exceedingly thin coating species, they occur in great quantity, but only on the basal membrane; a portion of them being erect, the remainder prostrate. But in another sponge, a remarkably curious parasitical species of *Hymeniacidon*, which, having no fibrous skeleton of its own, covers and appropriates a small fibrous *Fucus*, and converts its anastomosing vegetable stalks into an artificial skeleton, closely coating each stalk of the plant with its membranous structure, so as to cause them at first sight to be readily mistaken for keratose sponge fibre, the whole of the membranous structure abounds with attenuato-cylindrical entirely spined defensive spicula; but they are all prostrate and intermingled with the skeleton spicula of the sponge when not in contact with any part of the fibres of the vegetable, but wherever they are in contact with the plant they instinctively, as it were, assume the erect position, and the false skeleton is bristling with them to as great an extent as if it were truly a keratose fibrous structure. This feature in the habit of the sponge is very remarkable, and highly suggestive of a capability of adaptation to circumstances that we should scarcely have expected to find. By the two instinctive habits, first, that of converting the plant into an artificial skeleton, and then erecting its spinous spicula on its fibres, it at once simulates the habits of a kerato-fibrous sponge, and becomes capable of the carnivorous h bits that I have attributed to those sponges that are so strikingly adapted for preying on intruding annelids or other such small creatures. In the species above described, *Hymeniacidon Cliftoni*, BOWERBANK, MS., Plate XXX. fig. 9, the erection of the spicula on the adopted skeleton is an established habit, and it may be said to be instinctive in the species, but I have observed the same fact in sponges not habitually parasitical. I have a specimen of *Microciona carnosa*, BOWERBANK, MS., a British species, in my possession in which some small fibres of a tubular zoophyte have been accidentally included during its growth, which the sponge has coated with its own tissues, and from these adopted columns defensive spicula are projected in a similar manner to those of the columnar skeleton of the sponge. In this case we have an instinctive adaptation of an extraneous substance in a sponge in which the introduction of foreign substances is the exception, and not, as in other tribes of sponges, the rule.

In *Hyalonema mirabilis*, GRAY, a sponge nearly related to the genus *Alcyoncellum*, we find another extraordinary series of internal defensive spicula, the structure of which I have described at length under the head of 'Defensive Organs.' These elaborately and wonderfully formed weapons are evidently destined for other purposes than that of simple repulsion. The spiculated cruciform spicula, with their short stout basal radii planted firmly on the lines of the skeleton, and projecting from their centre at right angles to their own plane, the long spiculated ray furnished with numerous strong sharp recurved spines, it will be at once seen, are eminently fitted to retain annelids or other such prey, and to cause every motion of the struggling victim to contribute to its own laceration and destruction, while the structure and mode of attachment of the cruciform base is admirably calculated to resist the force and motions it has to sustain in such encounters. But these spicula, although exceedingly numerous, are not the only organs capable of retaining intruders into the body of the sponge with which it is furnished: there are, in addition, numerous large multihamate birotulate spicula dispersed in various positions on the sides of the interstitial cavities of the sponge, each of the rotulæ consisting of seven or eight stout recurved flattened radii, which if immersed in any struggling animal would be capable of sustaining a vastly greater amount of force than many of the spiculated quadriradiate ones combined could endure without injury; and that their especial office is that of auxiliary retentive organs is well demonstrated by the fact that the trenchant edges of the flattened radii are all at right angles to the line of force required to tear away their hold of any body in which they may have been inserted. Thus they appear destined by nature to secure the prey, while its own struggles among the lacerating organs contribute to its destruction (Plate XXXI. figs. 3, 4, 5, 6 & 7).

In the modification of the structure of the contort bihamate spicula, and their peculiar adaptation to the retention and destruction of intruders within the sponge, which I have described when treating on the internal defensive spicula, and which is represented in Plate XXXI. figs. 1 & 2, we have precisely the same physiological principle carried out, but by means widely different from those I have previously described.

If we consider the whole of these extraordinary organs to which I have referred in relation to each other, we cannot fail to see that, however varied their forms may be, there is every appearance of perfect harmony of design in the purposes they are destined to effect in the economy of the Spongiadæ.

The Cilia and Ciliary Action.

Our knowledge of the cilia of the Spongiadæ is, comparatively speaking, very small. Dr. Grant is, I believe, the first author who has seen and described these organs *in situ*. This learned and accurate observer, in his paper "Observations on the Structure and Functions of the Sponge," has described the origin and gradual development of the ova or gemmules of *Spongia panicea* (*Halichondria incrustans*, Johnston). After the liberation of these bodies from the sponge, he writes, "The most remarkable appearance exhibited by these ova is their continuing to swim about, by their own spontaneous motions, for two or three days after their detachment from the parent, when they are placed separately in vessels of sea-water, at perfect rest. During their progressive motions they always carry their rounded broad extremity forward, and when we examine them under a powerful microscope we perceive that these motions are produced by the rapid vibration of cilia, which completely cover over the anterior two-thirds of their surface." And he further states that they are "longest and exhibit the most distinct motions on the anterior part," and that they "are very minute transparent filaments, broadest at their base, and tapering to invisible points at their free extremities; they have no perceptible order of succession in their motions, nor are they synchronous, but strike the water by constantly and rapidly extending and inflecting themselves." The author describes the attachment and spreading out into a thin disk of the ovum or gemmule, and the cessation of action and gradual disappearance of the cilia; and he further observes, "although all visible cilia have ceased to move, we still perceive a clear space round the ovum, and a halo of accumulated sediment at a little distance from the margin." This observation is important, as tending to prove the existence of ciliary action, although the organs themselves were too minute to be detected.

DUJARDIN, in his work on the Infusoria, in plate 3, fig. 19 *b*, represents what are apparently the detached cilia and their basal cells, and which were probably from *Grantia compressa.*

If portions of a living sponge of this species be torn into small pieces, and placed in a cell in sea-water under a power of about 400 linear, groups of the detached cilia and their basal cells will be readily seen at the margins of the specimen; they are usually thus clustered together, and have a tremulous and indistinct motion. If a small specimen of the sponge be slit open and placed in a cell with fresh sea-water, with the inner surface of the sponge towards the eye so as to command a distinct view of the oscula, the cilia will be seen in the area of that organ in rapid motion, and the extraneous molecules attached to them exhibit the extent and nature of their oscillations very distinctly (Plate XXXIII. fig. 2). If the sponge be carefully torn asunder in a line at right angles to its long axis, and the torn surface be placed in a cell with a little fresh sea-water, we occasionally obtain a favourable longitudinal section of some of the large cells of the sponge, and we then see the cilia *in situ* and in motion (Plate XXXIII. fig. 1).

The whole length of the cell, from the inner edge of the diaphragm to its origin near the outer surface of the sponge, is covered with tessellated nucleated cells, which have each a long attenuated and very slender cilium at its outer end. They are oval in form, and have a distinct nucleus. When in vigorous condition their motions are rapid and cannot readily be followed; but in some in which the action was languid, the upper portion of the cilium was thrown gently backward towards the surface of the sponge, and then lashed briskly forward towards the osculum, and this action was steadily and regularly repeated. Their motions are not synchronous—each evidently acts independently of the others (Plate XXXIII. fig. 3, *a* & *b*).

The numbers, situation, and peculiarities of their actions fully account for the continuous and powerful stream that issues from the great cloacal aperture of this and other similarly constructed sponges. The natural rate of the motions of these organs must not be estimated from the sections last described, but the estimate must be made from the appearances manifested at the oscular orifices at the inner surface of the sponge. A more detailed account of these investigations is published in the Transactions of the Microscopical Society of London, vol. iii. p. 137. Figs. 1, 2, 3, & 4, plate 7, represent a longitudinal section of the intermarginal cavities of *Grantia compressa* with the cilia *in situ*. A view of the small portion of the inner surface of the sponge, exhibiting the oscular orifices and the appearance of the cilia in motion within them, and detached cilia and cells from the same sponge, are also represented by figs. 1, 2, 3 & 4, Plate XXXIII.

In the course of my endeavours to detect the cilia in Halichondroid sponges, I have frequently observed, in slices of the sponge taken from the surface, that the incurrent action has continued for a considerable period, while in sections of the same sponge taken from deep amid the tissues no such action of the currents could be detected. In sections from the surface in which the inhaling process was in vigorous condition, when the inside of the section was examined, that peculiar flickering appearance was often visible in the cavities immediately beneath the dermal membrane which is so characteristic of minute cilia in very rapid motion; and although many molecules were rushing inward with considerable velocity, others might be seen which continually waved from side to side but made no progress forward; in fact they presented precisely the appearance that I have described as taking place in the oscula of the proximal ends of the great intermarginal cells of *Grantia compressa*; and I have no doubt, in my own mind, that those of the Halichondroid sponges were also extraneous particles of matter adhering to the apices of the minute cilia, rendering their motions apparent, while the cilia themselves were perfectly invisible.

Carter, in his paper on "Zoosperms in *Spongilla*," published in the 'Annals and Mag. Nat. Hist.' vol. xiv. Second Series, p. 334, describes ciliated bodies from a *Spongilla* from the water-tanks of Bombay, somewhat similar to those of *Grantia compressa*, but the basal cell appears to be proportionally larger and the cilium shorter than in those of *G. compressa*. The author, in describing the detached cells and cilia, says, "At first the polymorphism of the cell and movements of the tail are so rapid, that, literally, neither 'head nor tail' can be made out of the little mass. Presently, however, its power of progression and motion begins to fail, and if separated from other fragments it soon becomes stationary, and after a little polymorphism assumes its natural passive form, which is that of a spherical cell. During this time the motions of the tail become more and more languid, and at length cease altogether." The author continues, "If, on the other hand, there be very large fragments in the immediate neighbourhood, or an active sponge-cell under polymorphism sweeps over the field, it may attach itself to one or the other of these, when its cell becomes undistinguishable from the common mass, and the tail floating and undulating outwards is all that remains visible." This observation is important, as it accounts in a great measure for our inability to find the cilia *in situ* in the living and active condition of the *Spongilla*; and if the structure and imbedment of the basal cell in the marine sponges be like those in that genus, the same results would probably arise in the marine species, rendering it extremely difficult, if not impossible, to detect these organs *in situ* and in action.

Lieberkuhn, in his paper in Müller's 'Archiv,' 1856, pp. 1–19, 319–414, gives an account of the cilia and their cells *in situ*. He describes them as forming a single layer of spherical cells, $\frac{1}{300}$ millim. in diameter, and which, though touching each other, are not in such contact as to lose their rounded figure. LIEBERKUHN'S description of the mode of disposition of these cells in *Spongilla* would serve equally well for those in *Grantia compressa*. Professor HUXLEY, in a paper "On the Anatomy of the Genus *Tethya*," published in the 'Annals and Mag. Nat. Hist.' vol. vii. p. 370, describes cells and cilia from an Australian sponge, which he designates spermatozoa, and which he describes as having "long, pointed, somewhat triangular heads, about $\frac{1}{300}$th of an inch in diameter, with truncated bases, from which a very long filiform tail proceeds." These bodies are figured in vol. vii. plate 14. fig. 9.

On a careful consideration of the descriptions of the ciliated cells seen by the authors I have quoted above, it strikes me forcibly that the so-called zoosperms and spermatozoa of CARTER and HUXLEY are identical in origin and purpose with the similar organs described by LIEBERKUHN and those found *in situ* and in action in *Grantia compressa*, and, in truth, that they are the homologues of the breathing and feeding organs of the zoophytes and more highly organized animals.

Reproduction.

The ovaria in sponges exhibit considerable variety in shape and structure. The most familiar form is that of *Spongilla fluviatilis*, represented in Plate XXXIII. fig. 5, in its natural condition.

These bodies have hitherto been usually designated as gemmules, but this term appears to be inappropriate. Each of them contains numerous minute vesicular, round or oval molecules, which are discharged from the foramen in succession, and each of these appears to be capable of producing a sponge. The terms ovarium and ova are therefore more in accordance with the rules of modern nomenclature, and this alteration in their designation is the more necessary, as I shall hereafter be enabled to show that, at least in *Tethea lyncurium*, propagation by true external gemmation really exists. I propose, therefore, for the future that all such large vesicular organs containing numerous molecules or ova capable of reproducing the species, and of being successively ejected from the sponge, should be designated ovaria and ova, and that the term gemmule should be restricted to the isolated bodies which pullulate from the internal or external surfaces of the parent, and by ultimate separation become each a distinct individual.

The reproductive powers of the Spongiadæ have been treated of to a considerable extent by preceding authors, and the amount of our information on this subject is, I believe, both extensive and accurate. I will not attempt a recapitulation of all that has been written on their reproduction, but content myself with a slight sketch of our knowledge of the various modes of propagation that have been well ascertained and described. From the researches of the various authors who have written on the structure and development of *Spongilla* and on the marine Spongiadæ, it appears that there are three well-established modes of propagation: 1st, by ova; 2nd, by gemmation; and 3rd, by spontaneous division of the sarcode. The terms ova and gemmule have been used so indiscriminately by authors, that it seems to me advisable to endeavour to define and limit their application in such a manner as to distinctly separate the one form of reproductive body from the other.

On a careful review of the results of the labours of previous observers and of my own researches, it appears that the following may be considered as the varieties that exist in the modes of the propagation of the Spongiadæ:—

1st. By ova without an ovarium.
2nd. By ova generated within ovaria.
3rd. By gemmules secreted within the sponge.
4th. By gemmules produced externally.
5th. By spontaneous division of the sarcode.

On the first mode of propagation, by the means of ova generated in the sponge without the presence of ovaria, very little seems to be known; and this mode appears to be confined to the true sponges, the genus *Spongia*. If we examine microscopically the fibres of the sponges of commerce in the condition in which they come into the hands of the dealers, and before they have been soaked, cleaned, and prepared for sale, we frequently find the fibres covered with innumerable minute irregularly ovoid vesicular bodies, nearly uniform in size, dispersed evenly over the surface of the fibres, and imbedded in a thin stratum of sarcode that coats the membranous sheath that surrounds them. These bodies Dr. Johnston believes to be "the matured gemmules or sporules," and I feel strongly inclined to agree with him in the conclusion that they are the reproductive bodies of that tribe of sponges, and no other reproductive bodies have, I believe, been discovered in the true sponges; but in arriving at this conclusion we must not fail to remember that our knowledge of these animals in the fleshy and solid condition in which they are when alive is so limited, and so few observations have been published regarding them in that state, that we must not attach too great a value to these conclusions.

In size and form these ovoid vesicles are very similar to the ova liberated from the well-characterized ovaria of other marine species of Spongiadæ, and, like them, they present no appearance of a nucleus. They are somewhat irregular in their form, and vary to a slight extent in size; an average-sized one measured $\frac{1}{11666}$th of an inch in diameter. Fig. 6, Plate XXXIII. represents a portion of a fibre from a Bahama sponge under a power of 400 linear, and fig. 7 a part of the same fibre ×1250 linear.

Until very recently, our knowledge of the vesicular ovaria of the Spongilladæ has been confined to two European species; but Carter, in his excellent account of the Spongillas found in the water-tanks of Bombay, has described several new and interesting varieties of these organs; and I have also become acquainted with eight new species from the River Amazon, through the kindness of Mr. Bate, and of three undescribed species from North America, through the kind and liberal assistance of Dr. Asa Gray, Professor LEIDEY, and Professor DAWSON, of M^cGill College, Montreal, Canada. The greater portion of these organs resemble each other very closely in their natural condition, presenting generally the appearance of a more or less spherical coriaceous body; but the structure of their walls, when developed by treating them carefully with hot nitric acid, is so varied and strikingly characteristic of their organic and specific differences, as to render it necessary that I should enter somewhat minutely into their history. Their structual peculiarities naturally divide them into two great groups.

1st, those in which the walls of the ovaria are strengthened and supported by birotulate or unirotulate spicula radiating in lines from the centre to the circumference of the ovarium; and 2nd, those having the walls of the ovaria supported by elongate forms of spicula, disposed on or near its surface at right angles to lines radiating from the centre to the circumference of the ovarium; and, fortunately, the types of these two forms of spicular arrangement on the cortex of the ovarium are admirably illustrated in the two European species of *Spongilla*, the first mode existing in *Spongilla fluviatilis*, and the second one in *S. lacustris*. After having described the ovaria of these two species as types of their respective groups, I shall, in my future descriptions of these organs, confine my observations rather to their anatomical structure than to their external characters, excepting when the latter are of an unusual description. These bodies occur in great profusion in the basal portions of *S. fluviatilis*; they are spherical and of an average diameter of $\frac{1}{60}$th of an inch, and they are furnished with a circular foramen at their distal extremity of about $\frac{1}{833}$rd of an inch in diameter. In their natural condition they exhibit very slight indications of the birotulate spicula imbedded in their coriaceous-looking envelope. In the dried state they become cup-shaped by the contraction of the upper half inward during the process of desiccation, and in this condition the foramen appears at the bottom of the cup. The edges of the cup being thick and round in consequence of the presence of the birotulate spicula beneath the fold of the membrane, the surface becomes pitted with numerous minute lacunæ, which are produced by the adhesion of the inner surface of the envelope to the distal extremities of the birotulate spicula. Immersion in water for an hour restores them to their spherical form, but does not obliterate the lacunæ produced by desiccation; and I have several times observed that, under these circumstances, the expansion of the ova within has forced one or more of them through the foramen.

If we take several of the ovaria, either in the living condition or in the expanded state I have described above, and place them in a test-tube with a little nitric acid, and raise the temperature of the whole until the ovaria become of a bright yellow colour and semitransparent, and then arrest the operation of the acid by immediately pouring in a quantity of cold water, we shall have preserved their form and have retained the spicula in their natural positions, and have rendered the whole so transparent, as to exhibit their form and arrangement in the walls of the ovarium, either in water or mounted in Canada balsam, in a very beautiful and satisfactory manner. They are packed very closely together, their shafts being in lines radiating from the centre of the ovarium to the circumference; their distal rotulæ supporting the outer surface of its wall, while the proximal rotulæ sustain the inner one. Fig. 8, Plate XXXIII. represents a portion of one of these prepared ovaria, and fig. 8 *a* one of the detached spicula. Two views of this form of spiculum are also represented in Plate XXVI. figs. 27 & 28, Phil. Trans. 1858, and a perfect ovarium prepared by acid by fig. 9, Plate XXXIII.

CARTER, in his paper "On the Freshwater Sponges in the Island of Bombay," in describing the birotulate spicula of the ovaria of *Spongilla Meyeni* and *plumosa*, species with ovaries of very similar structure to those of *S. fluviatilis*, states that the spaces between the rotulæ are "filled up with a white siliceous amorphous matter, which keeps them in position." I am indebted to the kindness and liberality of the author for specimens of these species, and I have frequently subjected their ovaries to the action of hot nitric acid, but I have never succeeded in finding any intervening siliceous matter, nor have I ever found any such siliceous cementing material in any other similarly constructed ovary of a *Spongilla*.

In the second group of ovaries of the Spongilladæ, represented by those of *S. lacustris*, in which the walls of the ovarium are supported by elongate forms of spicula disposed at right angles to lines radiating from its centre, the ovaria, in their natural condition, exhibit but very slight traces of the spicula imbedded in their walls. When dried, they cup inward like those of *S. lacustris*; but the margin of the cup is thin and sharp compared with that formed in a similar manner by those of *S. fluviatilis*, and they expand also in like manner when immersed in water. When treated with hot nitric acid they display an abundance of short, stout, entirely spined, subarcuate acerate spicula, one of which is represented in Plate XXVI. fig. 13, Phil. Trans. 1858. These spicula are in many instances exceedingly numerous; they are disposed without order, and overlie each other at various angles, forming, in their imbedment in the envelope, a strong and very efficient irregular network of spicula. A portion of one of these prepared ovaria is represented in Plate XXXIII. fig. 10.

In the ovaries of the different species of *Spongilla*, to be arranged hereafter in accordance with their structural peculiarities, there is a considerable amount of general resemblance, but accompanied with such permanent variations in the structure of the spicula, and in other portions of the development of these organs, as to render a somewhat detailed description of them necessary. Thus, in the development of the birotulate spicula, the ovaries of *Spongilla plumosa*, CARTER, exceed any other known species. The thick walls of these organs are filled with them in the state represented by fig. 21, Plate XXVI. Phil. Trans. 1858, and the intervals between their shafts appear to be filled with indurated sarcode or keratode. In *Spongilla Meyeni*, CARTER, the walls of the ovaria are strikingly similar in their structure to those of *S. fluviatilis*, and the form of the spicula the same, with the exception of the shafts being very much more spinous, and the size of the spiculum twice that of *S. fluviatilis*. Fig. 29, Plate XXVI. Phil. Trans. 1858, represents a spiculum from an ovary of *S. Meyeni*. The smallest and most simple development of birotulate spicula exists in *Spongilla gregaria*, BOWERBANK, MS., from the River Amazon, represented by figs. 23, 24, 25 & 26, Plate XXVI. Phil. Trans. 1858.

A gradual transition from the birotulate form to that of the unirotulate one takes place in the ovaries of *S. paulula* (fig. 31) and *S. reticulata* (fig. 33), until we obtain the perfect and beautiful unirotulate form in the ovaries of *S. recurvata*, represented by figs. 34 & 35 in the Plate quoted above. In all these species there is a general accordance in the mode of their structure.

The gradual transition from the birotulate to the unirotulate form of spiculum in the ovaries of *Spongilla reticulata* is not the only characteristic difference that exists between it and its congener. The form and structure of the ovarium also exhibit marked peculiarities of character, and it is also furnished with a beautiful reticulated spicular envelope or case. In its natural condition the ovary fills the reticulated case, and the coriaceous external surface is pressed into the areas of the network.

It is usually oviform, but it varies to some extent in its shape. When treated carefully with hot nitric acid, the outer coriaceous substance of the ovarium is dissolved, leaving the inner membrane and the boletiform spicula *in situ*; their larger terminations being applied to the distal surface of the membrane, while their smaller clavate or stellate ends are projected outward, reaching, in the natural condition, to very near the external surface of the ovarium. The foramen is situated at the small or distal end of the ovary, and differs from that of any other form of the organ with which I am acquainted, inasmuch as it exhibits a tubular elongation outward of the lining membrane equal in length to about its own diameter, causing the ovarium, when prepared with nitric acid, to appear like an oil-flask with a very short neck. Fig. 13, Plate XXXIII. represents one of the ovaria prepared with acid, and fig. 12 one of the cases in which they are contained.

In *Spongilla Brownii*, Bowerbank, MS., there is a still further deviation in the structure of the spicula of the ovary. The shaft entirely disappears, and the spiculum is reduced to the umbonato-scutulate form. They are situated on the outer surface of the inner membrane of the ovarium, with the umbones of the scutellæ outwards. This mode of disposition obviously renders them inefficient for external defence, and the ovaries have therefore been further defended by being enclosed within an elaborately constructed case of reticulated acerate spicula. The gemmule is closely embraced by this envelope, and small elongate masses of its outer surface are projected through some of its interstices, causing it to be more or less tuberculous; and, from the smallness of the interstices, the tubercles of the envelope of the ovary are much greater in length than in thickness. The spicula of the case are disposed in a close and irregular network, seldom exceeding two spicula in thickness. By a careful treatment with hot nitric acid, the thick coriaceous outer portion of the ovarium may be removed, and its thin lining membrane, with its stratum of umbonato-scutulate spicula, becomes an exceedingly beautiful object. The same mode of operation displays the structure of the reticulated case of the ovary very much more distinctly than when viewed in its natural condition. Fig. 11, Plate XXXIII. represents two of the cases after treatment with acid, one of them (*b*) having the ovary very much reduced in size by the dissolution of the thick coriaceous portion of its structure.

In the second group of the ovaries of the Spongilladæ there is also a strong general resemblance in structure to the type-form of *S. lacustris*, but each species is distinctly characterized by peculiarities of form and arrangement of the spicula.

The normal form is spherical, and the walls of the ovaries, in six out of the seven species with which I am acquainted, are comparatively thin. In the seventh species, *S. Carteri*, Bowerbank (*S. friabilis*, Carter), they are very thick and abundantly furnished with cellular structure, arranged in lines radiating from the centre to the circumference; each line consists of nine or ten cells, the length of each being about equal to the diameter. They are very closely packed together, and are irregularly angular by compression. Their combined length varies from about one-fifth to one-sixth the length of the diameter of the ovarium. This is the only species in which I have detected this description of cellular structure. Fig. 16, Plate XXVIII. represents a portion of the surface and a view of the cells *in situ.*

Although the spiculated coriaceous form of ovarium prevails so constantly among the freshwater sponges, it is one of extremely rare occurrence among the marine species; and I have met with only one instance of its occurrence, and that is in a new genus of sponges from Shetland, for which I am indebted to my indefatigable friend Mr. Barlee. The specimen incrusts a portion of the valve of a *Pecten*, covering a space about half an inch in length and the eighth of an inch in breadth, and it does not exceed half a line in thickness. The ovaries are numerous and closely packed together, and are distinctly visible to the unassisted eye, looking like very minute cocoons of some terrestrial insect. There were nearly thirty in an area equal to about a quarter of an inch. They are attached by the sides to one or more branches of the fibrous portion of the skeleton.

The wall of the ovary is very thin, and appears to consist of a single membrane profusely furnished with acerate spicula, like those of the skeleton. They cross each other in every possible direction, and occasionally appear to assume a somewhat fasciculated arrangement. The ovaries are not uniform in shape, some being regularly oval, while others are more or less ovoid. I could not detect any trace of a foramen in those I subjected to examination. I have designated this interesting species *Diplodemia vesicula* in my MS. description of it. Fig. 1, Plate XXXIV. represents two of the ovaries in their natural condition after immersion in Canada balsam, magnified 83 linear.

In the genera *Geodia* and *Pachymatisma* ovaria are produced in great abundance. They agree in form very closely with those of *Spongilla*, but their structure is widely different, and the soft animal matter that enters so largely into the structure of those of the freshwater sponges scarcely makes its appearance in the ovaries of *Geodia*, their walls being composed of closely packed spicula, firmly cemented together by silex. Their situation in the animal is also different from those of *Spongilla*, in which they are dispersed amid the interstitial tissues, but principally towards the base of the sponge, while in *Geodia* and *Pachymatisma* they are congregated in large quantities immediately beneath the dermal membrane; and when they have shed their ova they permanently retain their situation, forming a thick crustular dermis for the protection of the softer portions beneath: a few only are found dispersed in the interstitial membranes of the sponge. The progressive development of this kind of ovarium is very nearly the same in every species of *Geodia* or *Pachymatisma* in which I have had an opportunity of examining them. In an early stage they appear as a globular body of fusiformi-acerate spicula, radiating regularly from a central point in the mass. As the individual spicula increase in diameter there is a corresponding distention of the ovarium, and as the spicula do not lengthen in proportion to their increase of diameter a central cavity is produced, in which the incipient ova very shortly appear. The spicula of the wall of the ovary continue to increase considerably in diameter, but very little in length, and their distal terminations become gradually less acute as they approach the period of the full development of the ovary. When this organ has attained its greatest diameter, their distal extremities cease to lengthen, and a gradual change in the form of the spicula is effected, their apices extending in diameter and assuming a truncated form, and the whole of them becoming firmly cemented together, so as to form a common flat smooth surface to the siliceous skeleton of the ovarium, each spiculum having now changed from the acerate to the acuate form, their proximal acute terminations forming the common inner surface of the cavity of the ovarium, which is now filled with an opake mass of ova. A single conical orifice or foramen has also been produced in a portion of the wall, through which the ova are destined to be ejected. The proximal end of this foramen is very much the smaller of the two, so that, as soon as an ovum has fairly entered this conical tube, there is no longer any impediment to its ejection; and the manner in which this is effected is very interesting, and appears to be as follows. When the ova have attained maturity, the proximal terminations of the spicula which have not been cemented together like their distal ones, are progressively and simultaneously lengthened, thereby encroaching on and gradually lessening the diameter of the cavity within, so that the ova are compressed and forced through the foramen; and this process appears to be continued until the whole of them have been ejected, and the cavity becomes completely filled by the continued encroachment of the proximal ends of the spicula of the walls of the ovarium.

In fig. 6, Plate XXXIV., two ovaries from *Geodia M^cAndrewii* containing ova are represented: (*a*) contains about the greatest quantity of ova that is found within these organs. In this one the distal terminations of the spicula of the skeleton are still somewhat rounded, and slightly elevated above the common surface; while in (*b*), which has been partially exhausted of the ova, the spicula have their distal terminations flat and somewhat angular, and they are level with the general surface, thus indicating a greater age and a fuller development than obtain in the one represented by (*a*), and not a less amount of secretion of ova, as might possibly be imagined. These circumstances are strongly indicative of the fact that the ovaria, both in an active and an effete state, are permanently seated in the sponge, and that the ova only are discharged from it. So in like manner the existence of the ovarium in *Spongilla reticulata* and *Brownii*, confined within a strong spicula case firmly incorporated with the skeleton, is strong presumptive evidence of their also being permanent organs, and not of the nature of gemmules which separate from the body of the sponge when they arrive at maturity and are ejected through the great fæcal orifice.

Many other species of *Geodia* with which I am acquainted afford these ovaria in great abundance, and with some variations in size and form from those in *G. M^cAndrewii*, but in no other sponge are they so large and so completely developed.

Fig. 2, Plate XXXIV. represents an adult ovarium from *Geodia M^cAndrewii* with the conical foramen on its summit, and the distal ends of the skeleton spicula flat and angular. Fig. 3 represents a small portion of the surface of the same specimen as seen with a linear power of 308, exhibiting the flatness and angularity of their distal apices. Fig. 4 represents a portion of a young ovarium having the distal ends of the skeleton spicula disunited and acutely conical. Fig. 5 represents a portion of a section of an ovarium of *G. M^cAndrewii*, exhibiting the radial arrangement of its component spicula.

In *Pachymatisma Johnstonia*, Bowerbank, a British species common on the rocks in the neighbourhood of Torquay, and which I described in a paper read before the Microscopical Society of London in 1841, these organs assume an oval form; they are also considerably depressed. In a young specimen of this species of sponge in my possession, the progressive development of the ovaria is very strikingly illustrated. Fig. 7, Plate XXXIV. represents an adult ovarium. Fig. 8, one in a semideveloped state, and fig. 9, one of the same organs in a very early stage of development. In another species of sponge from the South Seas we find a singular variety of this class of ovarium. It is oval in form, the length being to the breadth as five to three, but it is so much depressed as to appear rather like a dermal spicula plate than an ovarium; but the radiate arrangement of its component spicula is perfectly visible with a power of 666 linear, and their distal terminations as separate and distinct as those of *Geodia* or *Pachymatisma*. The situation of the foramen is also well defined in many of them. Fig. 10, Plate XXXIV. represents a mature ovarium; fig. 11, a fragment of one to exhibit its degree of thickness; and fig. 12 represents one of the same species of ovarium in an early stage of development. I have seen four species of sponge which have this description of ovarium; in one it is very considerably longer in its proportions than that represented by fig. 10, Plate XXXIV., and in another species it is somewhat shorter.

Since the preceding portion of the account of the ovaria was written I have received a very remarkable specimen of these organs, which differs materially in its structure from any of the forms that I have previously described. The sponge consists of a small portion of basal membrane, closely resembling that of a Halichondraceous species. It was found by my friend Mr. J. Yate Johnston coating rocks and stones at Madeira.

The remains of several exhausted ovaria are dispersed over the surface of the membrane, a few only retaining their original form and proportions. They do not appear to have had a spicular skeleton, but to have consisted of a coriaceous envelope strengthened and supported by a reticulated skeleton of apparently keratose structure. They are nearly globular, and are firmly cemented to the membrane by a broad basal attachment. Although themselves apparently in an effete state, the membrane on which they are seated was in a decidedly living and active condition. It is thickly coated with sarcode, and abundantly furnished with equi-anchorate spicula. Numerous slender acuate or subspinulate spicula are also dispersed over its surface, which are occasionally fasciculated after the manner of the first indications of the formation of a Halichondraceous skeleton. But the most interesting feature of the membrane is, that at intervals over the whole of its surface, and especially at those parts most free from the dispersed spicula, there are small detached groups of spicula, each consisting of two or three irregular fasciculi crossing each other at various angles, resembling in every respect the early stages of development of the gemmules or ova so graphically described by Dr. GRANT in his account of the gemmules of the sponge he has designated *Halichondria panicea*. The presence of these early developments of the ova is precisely in accordance with the discharged and effete condition of the ovaries, and is just such an effect as might naturally be expected under such circumstances. Fig. 13, Plate XXXIV. represents one of these ovaria seen by a microscopic power of 108 linear; fig. 14, a small piece of the reticulated wall of the ovarium with a power of 308 linear; and fig. 15 represents the development of one of the ova and the surrounding equi-anchorate spicula with a power of 108 linear.

Gemmules.

If we adopt as a definition that a gemmule is a body not containing ova, but that it is a vital mass separated from the parent and capable of being ultimately developed into a single individual possessing the same specific characters and capabilities as the parent mass, we must consider the reproductive bodies so ably and minutely described by Dr. GRANT in his paper "Observations on the Structure and Functions of the Sponge*," not under the designation of ova, but rather under that of gemmules; and indeed the learned author seems to have entertained some doubt of their being correctly designated by the former term, as in speaking of them in a subsequent portion of his paper in page 14, he says, "since these germs or so-named ova are, &c.;" I have therefore been induced to arrange them under the designation of Gemmules.

Dr. GRANT describes their first appearance in the sponge in the months of October and November "as opaque yellow spots visible to the naked eye, and without any definite form, size, or distribution, excepting that they are most abundant in the deeper parts of the sponge and are seldom observable at the surface;" he also states that "they have no cell or capsule, and appear to enlarge by the mere juxtaposition of the

* Edinburgh New Philosophical Journal, vol. i. p. 16, plate 2, figs. 24–29.

monad-like bodies around them. As they enlarge in size they become oval-shaped, and at length in their mature state they acquire a regular ovate form." When they have attained a fully-developed condition, they separate from their attachment to the parent and pass out of the fæcal orifices. At this period of their existence the learned author states that they are endowed with spontaneous motion, in consequence of their larger extremity being furnished abundantly with cilia, which the author describes as "very minute transparent filaments, broadest at their base, and tapering to invisible points at their free extremities." After floating freely about for a period, they attach themselves to some fixed body, adhering firmly to it, and spreading themselves out into "a thin transparent convex circular film." The author further states that "when two ova in the course of their spreading on the surface of a watch-glass come into contact with each other, their clear homogeneous margins unite without the least interruption, they thicken, and produce spicula: in a few days we can detect no line of distinction between them, and they continue to grow as one ovum."

I have never had the good fortune to see the living gemmule with its cilia in action, as described by Dr. Grant, but I have frequently found Halichondraceous sponges with an abundance of these gemmules attached to their tissues; and I have in my possession a beautiful little specimen, dredged off Shetland, for which I am indebted to my kind friend Mr. Barlee, which is very illustrative of Dr. Grant's description of the mode of the development of the young sponge after the ovum or gemmule has attached itself. On a fragment of a bivalve shell there are more than twenty or thirty of Dr. Grant's ova or gemmules, which are all in the same early stage of development, each forming a small group of extremely slender spicula. The groups are separate from each other, but very closely adjoining. The diameter of one of the largest does not exceed $\frac{1}{300}$th of an inch, and their distance from each other is about half or once the diameter of one of them. In their present state, as represented by six of them in Plate XXXIV. fig. 16, it is evident that they are separate developments, and it is equally evident that a slightly further amount of extension would have caused them to merge in one comparatively large flat surface of sponge. We see by this instance that a sponge is not always developed from a single ovum or gemmule, but, on the contrary, that many ova or gemmules are often concerned in the production of one large individual, and this fact may probably account for the comparatively very few small sponges that are to be found; a few days probably serving by this mode of simultaneous development to form the basal membrane of the sponge of considerable magnitude, as compared with the individual ovum or gemmule, or with a sponge developed from a single ovum only. This mode of reproduction appears to have a very wide range. It is common to several distinct genera of Halichondraceous sponges; and I have observed it also in a siliceo-fibrous sponge, *Iphiteon panicea* of the Museum of the Jardin des Plantes, Paris. Fig. 17, Plate XXXIV. represents a small piece from the interior of the skeleton of *Iphiteon panicea*. Although the latter sponge is so widely different in structure from the Halichondraceous tribes of sponges, its mode of propagation by gemmation seems to be in perfect accordance with them. In *Tethea cranium* the same mode of reproduction by gemmules obtains, but the form of the organ is different, and there are other peculiarities in its growth and development that are extremely interesting.

The form of the gemmules is regularly lenticular; and there are two distinct sorts of them, which are always grouped together. The first is rather the smaller of the two, and has a nucleus of slender curved fusiformi-acerate spicula only. The bases of the spicula cross each other at the centre of the gemmule, and the apices radiate in all directions towards the external surface, but do not, in the fully developed state of the gemmule, project beyond it. The second sort of gemmule is furnished with three distinct forms of spiculum. The first are like those of the gemmule described above, slender fusiformi-acerate; the second are attenuato-porrecto-ternate, the radii being given off from the apex at about an angle of 45 degrees; and the third form is attenuato-bihamate or unihamate, and the hooked apices of this form are projected further than either of the other two forms, but do not pass beyond the inner surface of the tough dermal envelope of the gemmule when in the adult state. I have examined a great number of these gemmules, and could never find in the form first described any indication of either ternate or hamate spicula, and I am therefore satisfied that they are separate descriptions of gemmule, and that the first form is not a transition state from the young and undeveloped to the fully developed one. In like manner I have closely observed the second form, and have always found it uniform in character, and furnished with the whole three forms of spicula that characterize it. It is highly probable that this marked difference in structure is sexual, and, from the more highly developed condition of the second or large form, that it is the female or prolific gemmule; but on this point we must at present be satisfied with conjecture only, as although I have searched diligently for spermatozoa in both forms of gemmule and in the surrounding sarcode, I have not been able to detect anything resembling them. But that such bodies do occur in some species of *Tethea* appears to be the case, Professor Huxley having described and figured bodies which he believed to be spermatozoa in a paper published in the 'Annals and Mag. Nat. Hist.' Second Series, vol. vii. p. 370, plate 14, as occurring in a species of *Tethea* found in one of the small bays in Sydney Harbour, Australia. The group of gemmules represented by fig. 1, Plate XXXV., consists of (*a*) one of the larger and supposed prolific gemmules, and three (*b*, *b*, *b*) of the presumed male gemmules *in situ*, ×108. Wherever the former occurs the latter appear always to accompany them in the proportion of about two or three to one. They are not seated like the ovaria of *Geodia* at the surface of the sponge, but are always found on the interstitial membranes at a considerable depth within the sponge. The immersion of the specimen in Canada balsam has rendered the marginal lines of the gemmules undistinguishable from the surrounding sarcode, but their natural boundaries would be just beyond the extreme points of the spicula.

Fig. 2, Plate XXXV. represents one of the larger gemmules in its natural condition and separated from the sponge, by direct light and a linear power of 50. Figs. 39, 40, and 43, Plate XXVI. Phil. Trans. 1858, represent the spicula of the larger description of gemmule of *Tethea cranium*, after separation by nitric acid.

The reproductive bodies in the *Tethea* described by Professor Huxley do not resemble those in *T. cranium*; no spicula are either described or figured as existing in them, and in this respect they appear much more to resemble the reproductive organs described by Dr. Grant as existing in the Halichondraceous sponges of the Firth of Forth. But I am not surprised at this discrepancy, as in *Tethea simillima*, Bowerbank, MS., in the collection of the Royal College of Surgeons, from the Antarctic regions of the South Sea, a species very closely resembling *T. cranium*, the gemmules are so like those of the latter species as not to be readily distinguished from them in their natural condition; but when microscopically examined, not the slightest trace could be found of the smaller, and what I conceived to be the male gemmule in *T. cranium*. I have several other species of *Tethea* in my possession, but I have not yet found gemmules in the interior of any of them.

External Gemmulation.

In *Tethea Lyncurium* we have gemmules produced externally, which are perhaps much more entitled to that designation than any of the reproductive organs previously described. The fasciculi near the base of the *Tethea* are protruded considerably beyond the surface of the animal, and at the termination of each there appears a small mass of sarcode, which assumes a more or less globular form. If their bodies be immersed in Canada balsam and examined microscopically, they will be found to contain not only the spicula projected from the parent, but a second series, which have been secreted in the mass, and which have assumed the mode of disposition so characteristic of the skeleton of the parent *Tethea*. I am indebted to my friend Mr. T. H. Stewart for this interesting fact, and for the specimens illustrating it. They were found in Plymouth Sound.

Fig. 19, Plate XXXIV. represents one of these gemmules with a portion of the skeleton fasciculus on which it is produced, under a linear power of 50.

Propagation by Sarcodous Division.

The fact of the resolution of the sarcode of the interstitial tissues of *Spongilla* into small masses of unequal size and variable form has long been known to naturalists, and that when separated from the parent body each becomes capable of locomotion, and of ultimately becoming developed into a perfect sponge. Carter, in his valuable paper published in the Journal of the Bombay branch of the Royal Asiatic Society, No. 12, 1849, has given a minute account of their structure and motions when separated from the species which form the subjects of his paper, and his descriptions are in perfect accordance with the similar bodies separated from our European species *S. fluviatilis*, which I have had frequent opportunities of observing, and of confirming the history given by him of their locomotive powers and continual inherent motions. The author designates these bodies "sponge-cells," and treats of them as if they had a well-defined cell-wall, while their eccentric changes of form are perfectly inconsistent with such a structure. LIEBERKUHN, in treating of these bodies under the name of motile spores, states that he has never succeeded in discerning a "cell-membrane" around these particles, and my own observations are in perfect accordance with his experiences. The truth appears simply to be, that any minute mass of sarcode, whether separated voluntarily or involuntarily, has inherent life and locomotive power, and is capable of ultimately developing into a perfect sponge; and in the course of this process the dermal membrane is produced at a very early period, and this, surrounding an agglomeration of minute masses of sarcode, may have been mistaken by CARTER for a cell-membrane. The same author, in his observations "On the Species, Structure, and Animality of the Freshwater Sponges in the Tanks of Bombay," states, "that when the transparent spherical capsules which contain the granules within the seed-like bodies are liberated by breaking open the latter under water in a watch-glass, their first act is to burst; this takes place after the first thirty-six hours; and their granules, which will presently be seen to be the true ova of a proteaniform infusorium, varying in diameter from about the $\frac{1}{4300}$th part of an inch to a mere point, gradually and uniformly become spread over the surface of the watch-glass. On the second or third day (for this varies) each granule will be observed to be provided with an extensible pseudo-pediform base; and the day after most of the largest may be seen slowly progressing by its aid, or gliding over the surface of the watch-glass in a globular form by means of some other locomotive organs."

This description is strikingly similar to the same author's account of the masses of sarcode separated from the sarcodous lining of the interstitial canals of *Spongilla*: but it must be observed that, in the development of the egg, the first act is to liberate itself from the membranous envelope; and the contents thus hatched become moving masses of free sarcode, but without the locomotive cilia that are found on the so-called ova or gemmules of the marine sponges, so minutely and accurately described by Dr. GRANT in his papers "On the Structure and Functions of the Sponge" in the 'Edinburgh New Philosophical Journal,' vol. ii. p. 129. This author describes the ovum or gemmule of *Halichondria panicea* (*Hal. incrustans*, JOHNSTON), after having floated freely about for a period by means of the cilia around its larger extremity, as attaching itself to a fixed body by its smaller end and then gradually settling down in the form of a broad flat mass, and, after losing its cilia, being gradually developed in the form of the parent sponge. Thus every description by these close and accurate observers tends to the conclusion that the multiplication of the sponge is effected by the origination in the ovum, or by the agglomeration in the form of gemmules, of particles of sarcode. The action of the minute masses of sarcode liberated by the bursting of the envelope of the ovum, and their subsequent development, are precisely those of the so-called sponge-cell liberated from the mass of the sarcode lining the interstices of the sponge, and of the gemmules described by GRANT, when sessile: each moves independently at first; each unites with its congeners into one body; and the results, both in means and end, are precisely the same: but their origin is different. The one is a gemmation of sarcode within a proper membrane in the form of an egg, while the others are the production of a gemmule by independent growth, or by spontaneous division of the sarcodous substance of the sponge.

Both these modes of propagation occur in the same species, *Spongilla fluviatilis*, but I have never yet seen them both well developed in the same individual. Where the ovaria were abundant, the sarcode appeared even and consistent in its structure; and, on the contrary, if it exhibited manifest symptoms of granulating, very few or none of the ovaria could be detected. This double means of propagation is by no means uncommon among the Zoophytes.

I have never seen the spontaneous granulation of the sarcode in any living marine species of sponge; but as the vital powers and general physiological characters of that substance appear to be the same in all the Spongiadæ, however varied in form and structure, it is highly probable that perpetuation by spontaneous or accidental separation of minute masses of sarcode is by no means confined to *Spongilla*, and that, from the concurrent testimony of all who have investigated the subject, every molecule of sarcode, however minute, has inherent vitality, and the power of uniting with its own congeners whenever they may chance to come in contact.

Growth and Development of Sponges.

The growth of the sponge does not appear to be continuous, but periodical, as we may observe in the branching species, and especially in *H. palmata*. If the sponge be held up between the eye and a lighted candle, as many as five or six of the former pointed terminations of the sponge in succession, from near the base to the apex, may be seen; and the former lateral boundaries are also equally distinct, the oscula being most frequently, but not always, continued through the new coating of the lateral development of the spongeous structure. New branches are also frequently thrown out during the last period of development at various parts of the stem where no indication of branches existed previously. In all these newly-developed parts, it may be observed that the primary lines of the structure of the skeleton, or those radiating at nearly right angles to the axis of the sponge, are those which are first developed; and at the extreme points of the branches they are frequently seen projecting for, comparatively, a considerable distance in the form of single unsupported threads or filaments; but as we trace these lines inward, we find the secondary or connecting fibres increasing in number, and the network becoming closer and more fully developed. The same mode of development may be traced in *Halichondria oculata*, JOHNSTON, but not to such an extent as in *H. palmata*, JOHNSTON. In the sessile massive species of Halichondroid sponges the same mode of development seems to obtain, as I have frequently traced the different stages of growth in sections at right angles to the surface of the sponge.

Explanation of the Plates.

PLATE XXVII.

Fig. 1. Fibro-membranous tissue in which the layers of fibre cross each other at various acute angles, from *Polymastia robusta*, Bowerbank, MS., ×308 linear: page 751.

Fig. 2. Fibro-membranous tissue in which the layers of fibre cross each other at about right angles, from *Polymastia robusta*, Bowerbank, MS., ×666 linear: page 751.

Fig. 3. Fibro-membranous tissue from the dermal membrane of a species of *Stematumenia*. The fibres are disposed without order, ×183 linear: page 751.

Fig. 4. Fibro-membranous tissue containing a single layer of parallel fibres on a portion of the membrane from an excurrent canal of one of the common honeycomb sponges of commerce, ×666 linear: page 751.

Fig. 5. A portion of the dermal membrane of a young *Stematumenia*, with cells which produce the primitive fibres dispersed on its inner surface:—*a*, *a*, cells *in situ*, which have each produced a fibre, ×666 linear: page 752.

Fig. 6. Three fibres in progressive states of development:—*a*, exhibiting no indications of an ultimate separation from the basal cell; *b*, showing the mature termination of the fibre previous to separation; *c*, exhibiting the collapsed remains of the exhausted basal cell, ×666 linear: page 752.

Fig. 7. Solid keratose fibre from a cup-shaped specimen of the best Turkey sponge of commerce in the condition in which it came from the sea, ×175 linear: page 754.

Fig. 8. Fibres of the skeleton of *Halichondria oculata*, Johnston, illustrating spiculated keratose fibre, ×175 linear: page 755.

Fig. 9. A young fibre of *Halichondria Montagui*, Johnston: *a*, the apical spiculum, ×175: page 755.

Fig. 10. A fibre from the skeleton of *Halichondria ægagropila*, Johnston, illustrating the structure of multispiculated keratose fibre, ×108: page 755.

Fig. 11. A longitudinal section of a small fibre of the skeleton of *Raphyrus Griffithsii*, Bowerbank, MS., showing the irregular disposition of the spicula within it, ×90 linear: *a*, one of the spicula, ×175 linear: page 755.

Fig. 12. Simple keratose fistulose fibre from *Spongia fistularis*, Lamarck, ×108 linear: page 756.

Fig. 13. Compound fistulose keratose fibre from the skeleton of an *Auliskia*, ×100 linear: *a*, *a*, the minute tubular fibres traversing the central cavity of the skeleton-fibre: page 756.

Fig. 14. A portion of one of the skeleton-fibres of *Auliskia*, exhibiting the secondary canals radiating from the primary ones, ×300 linear: page 756.

PLATE XXVIII.

Fig. 1. Regular arenated keratose fibre from the skeleton of a coarse rigid Australian sponge, ×90 linear: page 757.

Fig. 2. Regular arenated keratose fibre from a flexible sponge, one of the common Bahama sponges of commerce, ×175 linear: page 757.

Figs. 3, 4, and 5. Portions of skeleton-fibre from a specimen of *Dysidea fragilis*, JOHNSTON, illustrating the varieties of form of irregular arenated keratose fibre, ×108 linear: page 757.

Fig. 5. Showing the mode by which the apex of the fibre attaches itself to a single grain of sand, ×108 linear: page 757.

Fig. 6. Smooth solid siliceous fibre, with young fibres pullulating from the adult ones at *a*. From the skeleton of *M'Andrewsia azoïca*, GRAY, ×175 linear: page 758.

Fig. 7. Tuberculated solid siliceous fibre from the skeleton of *Dactylocalyx pumicea*, STUTCHBURY, ×108 linear: page 758.

Fig. 8. Tuberculated solid siliceous fibre, very prominently tuberculated, from *Dactylocalyx Prattii*, BOWERBANK, MS., ×175 linear: page 758.

Fig. 9. Fibrillated sponge-fibre from the skeleton of one of the sponges of commerce, ×308 linear: page 754.

Fig. 10. Fibrillated sponge-fibre from the skeleton of an Australian sponge, ×175 linear: page 754.

Fig. 11. Spinulated simple fistulose siliceous fibre, from a sponge in the collection of Dr. ARTHUR FARRE, *Farrea*, BOWERBANK, MS., ×108 linear: page 758.

Fig. 12. Cidarate prehensile fibre from a parasitical siliceo-fibrous sponge from the South Sea, showing the position of the prehensile organs at the base of the sponge, ×83 linear: page 759.

Fig. 13. A group of cells on a portion of the interstitial membrane of *Ecionemia acervus*, BOWERBANK, MS., ×666 linear: page 759.

Fig. 14. Cells on a portion of the interstitial membrane of *Halichondria nigricans*, BOWERBANK, MS., ×308 linear: page 759.

Fig. 15. Detached nucleated cells from a new species of sponge from Freemantle, Western Australia, ×308 linear: page 759.

Fig. 16. A view of the upper stratum of cells of one of the ovaria of *Spongilla friabilis*, CARTER, ×308 linear: page 760.

PLATE XXIX.

Fig. 1. A piece of an interstitial membrane from the honeycomb sponge of commerce in the condition in which it came from the sea, exhibiting the sarcode on its surface, and the imbedded semi-digested minute molecules, ×666 linear: page 760.

Fig. 2. A spinulate spiculum from *Halicnemia patera*, BOWERBANK, MS., ×175 linear: page 766.

Fig. 3. A bispinulate spiculum from the same sponge, ×175 linear: page 766.

Fig. 4. A trispinulate spiculum from the same sponge, ×175 linear: page 766.

Figs. 5, 6, 7. The same forms of spicula as figures 2, 3, and 4, in progressive stages of development, the apices not having attained their acute terminations, ×175 linear: page 766.

Fig. 8. Inner surface of the dermal crust of *Dactylocalyx Prattii*, showing the manner in which the apices of the radii of the ternate spicula are spliced on each other to form the areas for the intermarginal cavities, ×108 linear: page 767.

Fig. 9. Three of the ternate spicula of *Dactylocalyx Prattii*, exhibiting the variations in form and the compressed condition of their radii, ×108 linear: page 767.

Fig. 10. A portion of a thin section at right angles to the surface of a specimen of *Halichondria seriata*, JOHNSTON, illustrating the mode of external defence by the prolongation of the radial lines of the skeleton, ×108 linear: page 769.

Fig. 11. Part of a small branch of *Dictyocylindrus rugosus*, BOWERBANK, MS., exhibiting the radiating structure of the defensive fasciculi, ×50 linear: *a*, a part of the central axis of spicula: page 770.

Fig. 12. A portion of a slice at right angles to the surface from *Tethea cranium*, showing the fasciculi of defensive spicula (*a*), and the mode in which they are supported by buttresses of spicula beneath the surface of the sponge at *b*: *c*, the recurvo-ternate spicula, ×50 linear: page 770.

PLATE XXX.

Fig. 1. A section at right angles to the surface of *Microciona atrosanguinea*, BOWERBANK, MS., showing the position of the pedestals forming the skeleton and the terminal spicula, ×108 linear: page 771.

Fig. 2. A single mature pedestal, showing its structure and the proportions and positions of the external defensive spicula, ×175 linear: page 771.

Fig. 3. A section of *Hymeraphia stellifera*, BOWERBANK, MS., showing the large bulbous skeleton-spicula *in situ*, their apices forming the external defences: *a*, the stelliferous internal defensive spicula elevated by a grain of sand beneath the basal membrane, ×108 linear: page 771.

Fig. 4. *a*, The basal portion of one of the skeleton-spicula of *Hymeraphia stellifera*, with its large bulbous base, ×260 linear: *b*, one of the stelliferous internal defensive spicula, ×260 linear: page 771.

Fig. 5. A small portion of a longitudinal section through the cloaca of a specimen of *Grantia tessellata*, BOWERBANK, MS., showing the positions of the internal defensive spicula, and their curvature towards the mouth of the cloaca, ×108 linear: page 772.

Fig. 6. A small portion of the kerato-fibrous skeleton of an Australian sponge, showing the attenuato-acuate entirely spined internal defensive spicula *in situ* dispersed on the skeleton-fibre, ×108 linear: page 773.

Fig. 7. Verticillately spined internal defensive spicula dispersed on keratose fibres of the skeleton, from a West Indian sponge, ×175 linear: page 773.

Fig. 8. Verticillately spined internal defensive spicula from a keratose sponge from the West Indies. Congregated in fasciculi, ×175 linear: page 773.

Fig. 9. A small portion of *Hymeniacidon Cliftoni*, Bowerbank, MS., exhibiting the membranous tissues of the sponge enveloping the fibres of a Fucus; the defensive spicula over the fibre being erect, while those on the adjoining membrane are recumbent, ×108 linear:—*a*, one of the attenuato-cylindrical internal defensive spicula, ×260 linear; *b*, a small portion of the surface of the Fucus showing its cellular structure, ×400 linear: page 774.

Fig. 10. A portion of the reticulated skeleton of the sponge, with the radiating fasciculi of spinulo-quaternate internal defensive spicula *in situ*, ×108 linear: page 775.

PLATE XXXI.

Fig. 1. A portion of the reticulated skeleton of a sponge from Madeira, the fibres armed with trenchant contort bihamate spicula, ×50 linear: page 776.

Fig. 2. One of the trenchant contort bihamate spicula, showing the cylindrical form at the curves of the hook and the middle of the shaft, and the trenchant edges of the rest of the inner surfaces of the spiculum, ×400 linear: page 776.

Fig. 3. A portion of the skeleton of *Hyalonema mirabilis*, Gray, showing the mode of disposition of the multihamate birotulate and spiculated cruciform spicula in the body of the sponge. In the collection at the British Museum, ×50 linear: page 777.

Fig. 4. A multihamate birotulate spiculum, magnified 175 linear, to exhibit the peculiarities of its structure: page 777.

Fig. 5. A spiculated cruciform spiculum, to show the relative proportions of the two forms of defensive spicula, ×175 linear: page 777.

Fig. 6. The same form of spiculum as fig. 5, showing the peculiarities of its spination, × 260 linear: page 777.

Fig. 7. A small portion of the skeleton of *Hyalonema mirabilis*?, Gray, from a specimen in the Bristol Museum, showing the reticulations of the skeleton to be abundantly supplied, in some parts, with a small variety of multihamate birotulate spicula: *a*, one of the large spicula, of the same form as those in fig. 5, *in situ*, × 108 linear: page 777.

Fig. 8 represents a small portion of the inner surface of the dermal membrane of *Hymedesmia Zetlandica*, Bowerbank, MS., showing the fasciculation of the simple bihamate spicula, the equi-anchorate ones dispersed singly on the membrane, and the large attenuato-acuate entirely spined defensive ones *in situ*, × 308 linear: page 780.

Fig. 9. A young circular group of inequi-anchorate spicula, situăted on one of the interstitial membranes of *Hymeniacidon lingua*, Bowerbank, MS., × 308 linear: page 780.

Fig. 10. A larger and more complete circular group of inequi-anchorate spicula, containing about the usual number of spicula, from the same sponge as the group represented by fig. 9, × 308 linear: page 781.

Fig. 11. A circular group of torqueato-tridentate inequi-anchorate spicula from the interstitial membranes of a new species of *Hymeniacidon* from Freemantle, Australia, × 308 linear: page 781.

Fig. 12. A single spiculum, from a group similar to that represented by fig. 11, exhibiting the singular structure of the base and the tridentate apex of the spiculum, × 400 linear: page 781.

Fig. 13. A section of *Halichondria panicea*, Johnston, showing the intermarginal cavities at *a*, *a*, immediately beneath the dermal surface, ×108 linear: page 787.

Fig. 14. A small portion of the dermal membrane of *Tethea muricata*, Bowerbank, MS., exhibiting the pores in an open condition, × 108 linear: page 782.

Fig. 15. The lower portion of the same piece of membrane, highly magnified, to show the positions of the elongo-stellate defensive spicula on the external surface of the dermal membrane, × 183 linear: page 782.

PLATE XXXII.

Fig. 1. A section at right angles to the surface of a branch of *Halichondria simulans*, Johnston, exhibiting the form and position of the intermarginal cavities, × 108 linear: page 787.

Fig. 2. A section, at right angles to the surface, of *Geodia Barretti*, Bowerbank, MS.:—*a*, *a*, longitudinal sections of two of the intermarginal cavities; *b*, *b*, the basal diaphragms of the intermarginal cavities; *c*, *c*, imbedded ovaria, forming the dermal crust of the sponge; *d*, *d*, the large patento-ternate spicula, the heads of which form the areas for the valvular bases of the intermarginal cavities; *e*, *e*, *e*, recurvo-ternate defensive and aggressive spicula within the summits of the great intercellular spaces of the sponge; *f*, *f*, portions of the interstitial membranes of the sponge, crowded with minute stellate spicula; *g*, *g*, portions of the secondary system of external defensive spicula, × 50 linear: page 788.

Fig. 3. View of a small portion of the inner surface of the dermal crust of *Geodia Barretti*, Bowerbank, MS., with three of the valvular membranes of the proximal ends of the intermarginal cavities:—*a*, *a*, valves closed; *b*, a valve partly open; *c*, *c*, the radii of the patento-ternate spicula, imbedded in the tissues,

and forming the areas for the support of the valvular terminations of the intermarginal cavities, × 50 linear: page 788.

Fig. 4. Four groups of inhalant pores in the dermal membrane, situated immediately above the distal ends of the intermarginal cavities of *G. Barretti*, × 83 linear: page 794.

Fig. 5. A portion of the dermal surface of *Halichondria panicea*, JOHNSTON, showing the multispicular network for the support of the dermal membrane, and the open pores in the areas, × 108 linear: page 793.

Fig. 6. A small portion of the dermal membrane from *Dictyocylindrus stuposus*, BOWERBANK, MS., exhibiting the number and position of the minute sphero-stellate defensive spicula with which it is armed, × 308 linear: page 793.

Fig. 7. A small portion of the quadrilateral siliceo-fibrous network of the dermis of the sponge upon which Dr. A. FARRE's specimen of *Euplectella cucumer*, OWEN, is based, showing the double series of entirely spined spicular organs projected from its angles, × 108 linear: page 790.

Fig. 8. A small portion of the single-seried dermal spicular network of *Isodictya varians*, BOWERBANK, MS., × 108 linear: page 792.

Fig. 9. A piece of reticulated kerato-fibrous tissue, for the support of the dermal membrane of one of the species of the common West Indian sponges of commerce, × 108 linear: page 792.

PLATE XXXIII.

Fig. 1. A longitudinal section of the intermarginal cavities of *Grantia compressa*, showing the cilia and their basal cells *in situ*, × 500 linear: page 806.

Fig. 2. A view of a small portion of the inner surface of *Grantia compressa*, exhibiting the oscula open, and the appearance presented at their orifices by the cilia within in action, × 500 linear: page 806.

Fig. 3. Two detached tessellated cells and their cilia, (*a*) in the position of inaction, (*b*) in the position of action, × 1250 linear: page 806.

Fig. 4. A group of detached tessellated cells from the interior of the intermarginal cavities of *Grantia compressa*, × 1250 linear: page 807.

Fig. 5. An ovarium of *Spongilla fluviatilis* in its natural state, exhibiting the foramen, × 83 linear: page 808.

Fig. 6. A small piece of a fibre of the skeleton of one of the common Bahama sponges of commerce, with numerous ova imbedded in its surface, × 400 linear: page 809.

Fig. 7. A small piece of the fibre represented by fig. 6, exhibiting the varieties in form and proportion of the ova, × 1250 linear: page 809.

Fig. 8. View of a section at right angles to the surface of a fragment of the skeleton of the ovarium of *Spongilla fluviatilis*, prepared with nitric acid, exhibiting the

relative positions of the spicula in the skeleton:—*a*, a spiculum from the same ovarium, detached, × 308 linear: page 811.

Fig. 9. A perfect skeleton of an ovarium of *Spongilla fluviatilis*, prepared with nitric acid, × 183 linear: page 811.

Fig. 10. A skeleton of an ovarium of *Spongilla lacustris*, prepared with nitric acid, exhibiting the spicula *in situ* and the foramen, × 183 linear: page 811.

Fig. 11. Two of the reticulated cases of the ovaria of *Spongilla Brownii*, Bowerbank, MS.:—*a*, an empty case; *b*, a case containing the skeleton of an ovarium, × 50 linear: page 813.

Fig. 12. A reticulated case of an ovarium of *Spongilla reticulata*, Bowerbank, MS., × 175 linear: page 812.

Fig. 13. Skeleton of an ovarium of *Spongilla reticulata*, Bowerbank, MS., without its case, prepared with nitric acid, × 175 linear: page 812.

PLATE XXXIV.

Fig. 1. A perfect ovarium of *Diplodemia vesiculata*, Bowerbank, MS., and a portion of a second one, showing the interior and the thickness of its walls in its natural state, × 83 linear: page 813.

Fig. 2. An ovarium of *Geodia M^cAndrewii*, Bowerbank, MS., in very nearly an adult state, showing the structure and position of the conical foramen for the discharge of the ova, natural condition, × 183 linear: page 815.

Fig. 3. A small portion of the surface of a fully-developed ovarium of *Geodia M^cAndrewii* in its natural state, showing the distal ends of the spicula flat and angular, and firmly cemented together, × 308 linear: page 815.

Fig. 4. A portion of a young ovarium of *Geodia M^cAndrewii*, with the distal ends of its spicula acutely terminated, and unconnected, × 308 linear: page 815.

Fig. 5. A portion of a section through nearly the centre of a mature ovarium of *Geodia M^cAndrewii*, showing the radiation of its spicula from near the centre to its circumference, × 308 linear: page 815.

Fig. 6. Two ovaria of *Geodia M^cAndrewii*, (*a*) containing about the maximum of ova, (*b*) after a great part of the ova have been discharged, × 108 linear: page 814.

Fig. 7. A mature ovarium of *Pachymatisma Johnstonia*, Bowerbank, exhibiting the cuneiform spicula of the foramen, × 308 linear: page 815.

Fig. 8. A young ovarium of *Pachymatisma Johnstonia* in course of development, × 308 linear: page 815.

Fig. 9. A young ovarium of *Pachymatisma Johnstonia* in a very early stage of development, × 308 linear: page 815.

Fig. 10. An ovarium from a sponge from Madeira closely allied to *Pachymatisma*, exceedingly depressed and much elongated, × 308 linear: page 815.

Fig. 11. A fragment of a similar ovarium to that represented by fig. 10, the fracture showing its extremely thin condition, × 308 linear: page 815.

Fig. 12. A young ovarium of the same species as that represented by fig. 10, in an early stage of development, × 308 linear: page 815.

Fig. 13. A reticulated ovarium *in situ*, on the fragment of a sponge from Madeira, ×108 linear: page 816.

Fig. 14. A portion of the reticulated structure from an ovarium of the same description as represented by fig. 13, × 308 linear: page 816.

Fig. 15. An ovum in course of development into a young sponge on the same membrane as that on which the ovarium represented by fig. 13 is seated, × 108 linear: page 816.

Fig. 16. A group of ova or gemmules in course of development into young sponges, found, with many others, on the inner surface of a fragment of a large *Pecten* from Shetland, × 108 linear: page 817.

Fig. 17. A small portion of the skeleton of *Iphiteon panicea* in the Museum of the Jardin des Plantes, Paris, with gemmules *in situ*, × 183 (*Dactylocalyx*, STUTCHBURY): page 817.

Fig. 18. A gemmule detached from *Iphiteon panicea*, × 666 linear: page 817.

Fig. 19. A gemmule extruded from near the base of a specimen of *Tethea Lyncurium*, on the distal extremity of one of the skeleton fasciculi, × 50 linear: page 819.

PLATE XXXV.

Fig. 1. A group of internal gemmules *in situ*, on the interstitial membranes of *Tethea cranium*:—*a*, one of the larger and most completely organized gemmules; *b*, *b*, *b*, three of the smaller and more simple gemmules which always accompany the larger ones. In Canada balsam, × 108 linear: page 818.

Fig. 2. One of the larger description of gemmules of *Tethea cranium*, in its natural state, removed from the membrane and viewed by direct light, × 25 linear: page 818.

Fig. 3 represents a portion of the sponge from the East Indies, furnished with numerous depressed porous areas with protecting organs, natural size: page 794.

Fig. 4. Two of the inhalant areas, connected by the dermal network:—*a*, the external protective organ in a perfect condition; *b*, having the external protective organ removed to exhibit the deeply depressed porous area, × 50 linear: page 794.

Fig. 5. Half of one of the external defensive organs, highly magnified, to exhibit the disposition of the spicula on the follicular radiations of the organ, × 108 linear: page 794.

Fig. 6. A view of the interior surface of half of one of the external defensive organs, exhibiting the structure of the semifollicular radiations of the organ, × 108 linear: page 794.

Supplement to Part I. "On the Anatomy and Physiology of the Spongiadæ," by J. S. BOWERBANK, *LL.D., F.R.S., F.L.S. &c.—Descriptions of New Forms of Spicula that have been discovered since the publication of the First Part (Phil. Trans.* 1858, *p.* 279).

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Spicula of the Skeleton.

FARCIMULO-CYLINDRICAL (Plate XXXVI. fig. 1).—This is the shortest and stoutest form of skeleton-spiculum I have yet seen. It forms the entire skeleton of *Spongilla coralloides*, BOWERBANK, MS. A new species from the River Amazon. In the collection of the Royal College of Surgeons of London.

INEQUI-ACERATE VERMICULOID (Plate XXXVI. fig. 2).—These spicula are found dispersed in great numbers in the basal membrane of *Hymeraphia vermiculata*, BOWERBANK, MS. A new species of British sponge from Shetland. No two of them agree in the form or amount of their contortions, but all of them are more or less inequi-acerate.

NODULATED CYLINDRICAL VERMICULOID (Plate XXXVI. fig. 3).—The sponge whence this spiculum is derived has not yet been found. It occurs along with inequi-acerate vermiculoid and other well-known forms of sponge-spicula in the soundings from the Atlantic, in 2070 fathoms, and, like the last-named form, no two of them agree in the mode or amount of their contortions.

ELONGO-EQUIANGULATED TRIRADIATE (Plate XXXVI. fig. 4).—An auxiliary skeleton-spiculum from the surface of *Grantia striatula*, BOWERBANK, MS. From Madeira. The elongate ray is always disposed in accordance with the long axis of the sponge.

EXFLECTED ELONGO-EQUIANGULATED TRIRADIATE (Plate XXXVI. fig. 5).—From the surface near the base of *Grantia striatula*, BOWERBANK, MS. From Madeira. The elongated ray in this form, as well as the one represented by fig. 4, varies considerably in its proportions.

DOLIOLATE CYLINDRICAL (Plate XXXVI. fig. 6).—From a portion of the skeleton of a sponge nearly related to *Ecionemia*, BOWERBANK, MS. Locality unknown.

BIFURCATED EXPANDO-TERNATE (Plate XXXVI. fig. 7).—From the same sponge as fig. 6; the shaft of the spiculum assisting in the formation of the skeleton, while the ternate terminations act as external defensive spicula.

PORRECTO-TERNATE (Plate XXXVI. fig. 8).—From the same sponge as fig. 6; the shaft belonging to the skeleton and the ternate apex acting as an external defensive spiculum.

EXPANDO-TERNATE (Plate XXXVI. fig. 9).—From the same sponge as fig. 6; the shaft acting as a skeleton-spiculum, while the ternate apex serves as a defensive spiculum.

GENICULATED EXPANDO-TERNATE (Plate XXXVI. fig. 10).—From *Tethea Collingsii*, BOWERBANK, MS. Sark. The shaft acts as a subsidiary skeleton-spiculum, and the ternate apex as a defensive one.

ABBREVIATO-PATENTO-TERNATE (Plate XXXVI. fig. 11).—From a sponge allied to *Pachymatisma*, in the Museum of the Royal College of Surgeons, London. External defensive. The example figured is a fully-developed spiculum.

INFLATO-FUSIFORMI-ACERATE, ASCENDINGLY HEMI-SPINOUS (Plate XXXVI. fig. 12).—From *Hyalonema mirabilis*, GRAY. British Museum. These spicula are projected in great numbers from the dermal surface of the body of the sponge, the smooth basal half being immersed in the tissues beneath the dermal membrane, and the spinous distal portion projected beyond it. The form and mode of disposition of the spines indicate its purely defensive character.

VERTICILLATELY SPINED CYLINDRICAL (Plate XXXVI. fig. 13).—This spiculum is very abundant on the dermal and interstitial membranes of an undescribed sponge from Freemantle, Western Australia. It is both externally and internally defensive.

SUB-ATTENUATO, ENTIRELY SPINED CYLINDRICAL (Plate XXXVI. fig. 14).—From *Hymeniacidon Cliftoni*, BOWERBANK, MS. Freemantle, Western Australia. Internal defensive.

SPICULATED INEQUI-ANGULATED TRIRADIATE, WITH CYLINDRICAL ENTIRELY SPINED RADII (Plate XXXVI. fig. 15).—From a fragment of a sponge presented to me by Mr. VICKERS of Dublin, who thinks it probably came from the West Indies. This spiculum is an external defensive one. The triradiate rays are imbedded immediately beneath the dermal membrane, and the spicular ray is projected through it at right angles to its plane; they are very numerous.

SPICULATED ATTENUATO-EQUIANGULAR TRIRADIATE, VERTICILLATELY SPINED (Plate XXXVI. fig. 16).—From an undescribed sponge. Freemantle, Western Australia. I have not seen the sponge whence this spiculum is derived, but, reasoning from our knowledge of the form and situation of the spiculum represented by fig. 15, there can be little doubt of its being an external defensive one.

SPICULATED CYLINDRO-EQUIANGULAR TRIRADIATE, VERTICILLATELY SPINED (Plate XXXVI. fig. 17).—From a fragment of sponge from Freemantle, Australia. This spiculum occurs in the same slide of sponge-spicula as the form represented by fig. 16. It was sent to me by my friend Mr. GEORGE CLIFTON, of Freemantle. There can be little doubt of its being an external defensive organ.

INEQUI-FURCATO-TRIRADIATE (Plate XXXVI. figs. 18 & 19).—These forms of spicula are from a new species of calcareous sponge, probably a *Grantia*. They were sent to me, mounted in Canada balsam, by my friend Mr. GEORGE CLIFTON, of Freemantle, Australia. They occur loosely fasciculated, and their mode of disposition is very similar to those represented by figs. 4 & 5, Plate XXXVI. They differ considerably from each other in length and in the width of the prongs of the fork apart, but they all have them unequal in length. It is probably an auxiliary skeleton and external defensive spiculum.

ELONGO-RECURVATE DENTATO-BIROTULATE (Plate XXXVI. fig. 20).—From *Hyalonema mirabilis*, GRAY. This spiculum is from the same sponge as that represented by fig. 4, Plate XXXI. Part II. It is an extreme variety of that form. Fig. 7, *a*, in the same plate, appears to be an intermediate variety.

ELONGO-RECURVATE DENTATO-BIROTULATE (Plate XXXVI. fig. 21).—From soundings in the Indian Ocean, 2200 fathoms. The smooth shaft and the widely-spread teeth of this spiculum render it very probable that it belongs to an unknown species of *Hyalonema.*

RECURVO-ACUTELY DENTATE BIROTULATE (Plate XXXVI. fig. 22).—From soundings in the Indian Ocean, 2200 fathoms. The thin smooth shaft and the acutely-terminated teeth of this form indicate probably a species of *Hyalonema* unknown to naturalists at present.

RECURVO-DENTATO-BIROTULATE (Plate XXXVI. fig. 23).—From soundings in the Indian Ocean, 2200 fathoms. The fragment represented is most probably from another unknown species of *Hyalonema.* It is the only specimen of this form that has, I believe, been found.

ATTENUATO-CYLINDRICAL, VERTICILLATELY SPINED (Plate XXXVI. figs. 24 & 25).—These spicula are found dispersed in abundance on the interstitial and dermal membranes of *Hymeraphia verticillata*, BOWERBANK, MS. A new British species, brought up by the sounding-line from 100 fathoms, off the Western Coast of Ireland, by the officers of H.M. ship 'Porcupine.' It is remarkable as being the only verticillately-spined spiculum that has been found in a British species of sponge, and also for exhibiting the mode of development of that class of spicula. In the earliest stage the spiculum is long, slender, and perfectly smooth; as the growth proceeds, two or three slight inflations appear near the middle of the shaft, and others are successively developed beyond them, until the spiculum assumes the moniliform appearance represented by fig. 25. As the inflations increase in number and size, a few incipient spines appear in a circumferential line at their greatest diameter; and as the growth proceeds, the spines increase in number and size, and the spaces between the inflations are filled up by the expansion of the shaft; and this mode of development is continued until the adult spiculum assumes the form represented by fig. 24. This form appears to act both as a tension and a defensive spiculum.

Spicula of the Sarcode.

TORQUEATO-BIDENTATE INEQUI-ANCHORATE (Plate XXXVI. fig. 26).—From an undescribed species of sponge. Freemantle, Western Australia. Sent to me, mounted in Canada balsam, by Mr. GEORGE CLIFTON. This is closely allied to the one represented by fig. 12, Plate XXXI. Part II.

BICALCARATE BIHAMATE (Plate XXXVI. fig. 27).—This singular and minute form of spiculum has hitherto been found only in *Isodictya Normani*, BOWERBANK, MS. A new British species.

EXPANDO-TRIDENTATE EQUI-ANCHORATE (Plate XXXVI. fig. 28).—From an undescribed sponge in the British Museum. The shaft of this minute spiculum is frequently curved to the extent of nearly a semicircle. Expando-bidentate forms are also mingled with the tridentate ones.

TRIDENTATE FIMBRIATED EQUI-ANCHORATE (Plate XXXVI. fig. 29).—From *Isodictya fimbriata*, BOWERBANK, MS. Shetland. The singular fimbriation of the shaft of the spiculum has never been observed in any other anchorate spiculum. In this sponge the spicula of this form may be traced from the earliest stage of development to the fully fimbriated form exhibited by the one represented by fig. 28. They are very abundant on the interstitial and dermal membranes, and mixed with them; there are many that are only bidentate, but are as completely fimbriated as the tridentate ones. The fimbriæ are very delicate and translucent, and require a careful management of the light to render them apparent.

QUADRIHAMATE (Plate XXXVI. fig. 30).—From *Hyalonema mirabilis*, GRAY. These very minute spicula are dispersed in considerable numbers on the interstitial membranes of the sponge.

INEQUI-TRIROTULATE (Plate XXXVI. fig. 31).—From an undescribed sponge in the cabinet of my friend Mr. GEORGE CLIFTON, of Freemantle, Western Australia. In Plate XXVI. fig. 38 I have figured a more fully developed specimen of this form, and described it in the 'Philosophical Transactions' for 1858, page 319, believing at that time that it was probably derived from the ovarium of a *Spongilla*. From the structural differences of the two specimens, it is probable that the former one is not from the same species of sponge as the latter.

ECCENTRIC TRIROTULATE (Plate XXXVI. figs. 32 & 33).—From the same sponge as fig. 31. Fig. 33 presents the fully-developed axial eccentricity, while the axis in the spiculum represented by fig 32 is both centrical and eccentrical, and these variations in the mode of the development of the rotulæ are exceedingly common.

CYLINDRO-CRUCIFORM (Plate XXXVI. figs. 34, 35, 36, 37).—From *Hyalonema mirabilis*, GRAY. British Museum. These four forms occur in considerable numbers, either imbedded in, or immediately surrounding the thick coriaceous sheath which envelopes the spiral column that is projected from the base of the sponge through its centre. All the imaginable varieties of form between figs. 34 and 37 are found mixed together; and they appear to be especially abundant around that part of the column which is imbedded in the midst of the sponge. The cylindrical form represented by fig. 34 is of rare occurrence without a slight indication near the middle of the absent third and fourth rays of the perfect cruciform spiculum.

SPICULATED CYLINDRO-CRUCIFORM (Plate XXXVI. fig. 38).—From *Hyalonema mirabilis*, GRAY. British Museum. This spiculum is from the sheath of the same sponge as the previously described forms; the ordinary cruciform spiculum being converted into an external defensive one by the projection of a spicular ray from its centre.

Dentato-cylindro-hexradiate (Plate XXXVI. fig. 39).—From a unique and very beautiful branching sponge from Nichol Bay, Australia, sent to me by Mr. George Clifton, of Freemantle. The dentation of the radii of these spicula varies considerably in form and size; the number of teeth at the apices of the rays is usually two or three, occasionally four, and very rarely five. The spicula are nearly uniform in size, and extremely abundant on all parts of the interstitial membranes.

Exter-spinulated arcuate (Plate XXXVI. fig. 40).—From a small massive sponge from the Bahamas, presented to me by my friend Mr. McAndrew. They are very abundantly dispersed over all parts of the interstitial membranes, are uniform in size, and vary to some extent in the degree of spinulation. In Part I. of this paper I described and figured a minute spiculum of the same arcuate form (Phil. Trans. 1858, p. 322, Plate XXVI. fig. 51). I was not aware at that time from what part of the sponge it had been obtained. I have since found the same form abundantly dispersed on the interstitial membranes of a new species of sponge from Freemantle, sent to me by my friend Mr. George Clifton.

Spinulo-multifurcate hexradiate stellate (Plate XXXVI. fig. 41).—This beautiful spiculum forms a connecting link between the spinulo-quadrifurcate hexradiate stellate form and the floricomo-stellate one, described and figured in the first part of this paper (Phil. Trans. 1858, p. 312, Plate XXVI. figs. 2, 3, 4). A careful examination of the specimen presents indications of there having been as many as eight secondary radii at the termination of the primary ray which exhibits the greatest number of secondary ones in the figure; and it is probable that this was the full complement of those parts.

Multiangulated cylindrical (Plate XXXVI. fig. 42).—From a sponge in the British Museum. This spiculum had been accidentally included in the sponge. It is distinctly different from one of the same form described and figured in the first part of this paper, Phil. Trans. 1858, p. 314, Plate XXVI. fig. 10. It most probably belongs to the sarcode of a *Geodia.*

Spinulo-multiangulated cylindrical (Plate XXXVI. fig. 43).—Found among the extraneous spicula of the same sponge as the spiculum represented by fig. 42. This sponge is one of the Johnstonian Collection. It is designated *Halichondria sanguinea*, and its register is 47, 9, 7, 19.

Inflato-acerate, with inscissurate terminations (Plate XXXVI. fig. 44).—From *Hymeraphia verticillata*, Bowerbank, MS. A new species of British sponge from the Western Coast of Ireland. A terminal portion only of this spiculum is represented by the figure, the inscissurate character being the only novelty in the form. The inscissuration varies in degree to a considerable extent in different spicula, in some cases being very slightly produced, in others rather beyond that represented by fig. 44. The rudiments of a third ray are sometimes apparent. This form is an auxiliary skeleton-spiculum. They are found thickly clustered around the primary spicula of the skeleton. They differ essentially from porrecto-ternate spicula in having both ends cleft or radiate, which is never the case in any of the ternate forms.

Explanation of PLATE XXXVI.

Spicula of the Skeleton.

Fig. 1. Farcimulo-cylindrical, from *Spongilla coralloides*, Bowerbank, MS., ×108 linear: page 830.

Fig. 2. Inequi-acerate vermiculoid, from *Hymeraphia vermiculata*, Bowerbank, MS., ×175 linear: page 830.

Fig. 3. Nodulated cylindrical vermiculoid, ×175 linear: page 830.

Fig. 4. Elongo-equiangulated triradiate, from *Grantia striatula*, Bowerbank, MS., ×108 linear: page 830.

Fig. 5. Exflected elongo-equiangulated triradiate, from *Grantia striatula*, Bowerbank, MS., ×108 linear: page 830.

Fig. 6. Doliolate cylindrical, ×175 linear: page 830.

Fig. 7. Bifurcated expando-ternate, ×108 linear: page 830.

Fig. 8. Porrecto-ternate, ×108 linear: page 830.

Fig. 9. Expando-ternate, ×108 linear: page 830.

Fig. 10. Geniculated expando-ternate, from *Tethea Collingsii*, Bowerbank, MS., ×108 linear: page 831.

Fig. 11. Abbreviato-patento-ternate, ×108 linear: page 831.

Defensive Spicula.

Fig. 12. Inflato-fusiformi-acerate, ascendingly hemi-spinous, ×108 linear: page 831.

Fig. 13. Verticillately spined cylindrical, ×666 linear: page 831.

Fig. 14. Sub-attenuato, entirely spined cylindrical, ×400 linear: page 831.

Fig. 15. Spiculated inequi-angulated triradiate, with cylindrical entirely spined radii, ×308 linear: page 831.

Fig. 16. Spiculated attenuato-equiangular triradiate, verticillately spined, ×666 linear: page 831.

Fig. 17. Spiculated cylindro-equiangular triradiate, verticillately spined, ×666 linear: page 831.

Figs. 18 & 19. Inequi-furcato-triradiate, ×183 linear : page 831.

Fig. 20. Elongo-recurvate dentato-birotulate, ×308 linear: page 832.

Fig. 21. Elongo-recurvate dentato-birotulate, ×308 linear: page 832.

Fig. 22. Recurvo-acutely dentate birotulate, ×308 linear: page 832.

Fig. 23. Recurvo-dentato-birotulate variety, ×308 linear: page 832

Fig. 24. ATTENUATO-CYLINDRICAL, VERTICILLATELY SPINED, ×183 linear: page 832.

Fig. 25. The same form of spiculum as that represented by fig. 24, in an early stage of development, ×183 linear: page 832.

Fig. 26. TORQUEATO-BIDENTATE INEQUI-ANCHORATE, ×308 linear: page 832.

Fig. 27. BICALCARATE BIHAMATE, ×1250 linear: page 833.

Fig. 28. EXPANDO-TRIDENTATE EQUI-ANCHORATE, ×1250 linear: page 833.

Fig. 29. TRIDENTATE FIMBRIATED EQUI-ANCHORATE, ×666 linear: page 833.

Fig. 30. QUADRIHAMATE, ×1250 linear: page 833.

Fig. 31. INEQUI-TRIROTULATE, ×666 linear: page 833.

Fig. 32. ECCENTRIC TRIROTULATE, ×666 linear: page 833.

Fig. 33. ECCENTRIC TRIROTULATE, ×666 linear: page 833.

Fig. 34. INCOMPLETELY DEVELOPED, CYLINDRO-CRUCIFORM, ENTIRELY SPINED, ×175 linear: page 833.

Fig. 35. CYLINDRO-CRUCIFORM, PARTLY DEVELOPED AND ENTIRELY SPINED, ×175 linear: page 833.

Fig. 36. CYLINDRO-CRUCIFORM, APICALLY SPINED, ×175 linear: page 833.

Fig. 37. CYLINDRO-CRUCIFORM, ENTIRELY SPINED, ×175 linear: page 833.

Fig. 38. SPICULATED CYLINDRO-CRUCIFORM, ×175 linear: page 833.

Fig. 39. DENTATO-CYLINDRO-HEXRADIATE, ×666 linear: page 834.

Fig. 40. EXTER-SPINULATED ARCUATE, ×1250 linear: page 834.

Fig. 41. SPINULO-MULTIFURCATE HEXRADIATE STELLATE, ×666 linear: page 834.

Fig. 42. MULTIANGULATED CYLINDRICAL, ×400 linear: page 834.

Fig. 43. SPINULO-MULTIANGULATED CYLINDRICAL, ×666 linear: page 834.

Fig. 44. INFLATO-ACERATE, WITH INCISSURATE TERMINATIONS, ×666 linear: page 834.

XXXIII. *On the Oxidation and Disoxidation effected by the Alkaline Peroxides.*

By B. C. BRODIE, *F.R.S.*, *Professor of Chemistry in the University of Oxford.*

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IN a former paper* I communicated to the Royal Society the results of an inquiry as to the cause of the mutual decomposition which takes place between the alkaline peroxides and the oxides of the less electro-positive metals. This decomposition, the first instance of which was discovered by THÉNARD in the case of the peroxide of hydrogen, had been regarded as of an exceptional and abnormal character, and as such had attracted the attention of chemists and been accounted for by several hypotheses.

These explanations, which attempted to show that the phenomena were caused by the repulsion of particles similarly electrified or were the consequence of the laws of mechanical vibration, assumed their abnormal character, and tended to isolate them from other chemical changes, rather than to comprehend them under the same laws.

In the paper referred to I ventured to suggest that this mode of viewing these decompositions was erroneous, that the phenomena were of a normal character, and were to be regarded as a particular case of those general laws under which all chemical changes are included. On the views there developed, every chemical change is considered as determined by the mutual attraction of particles, or groups of particles, in opposite polar conditions. No *à priori* presumption can be raised that certain particles are susceptible of this polarization, and others not; this point can be determined by experiment alone: and I brought forward several examples, the applicability of which is now very generally admitted, in illustration of the point that the elemental bodies, at the moment of chemical change, exhibit the same phenomena of polarization and are subject to the same laws of diæresis and synthesis as all other chemical substances. On these ideas, as we regard the weight of two volumes of oxygen, that is to say the weight of a molecule of oxygen, O_2, as differing from the weight of two volumes, that is, the weight of a molecule of water, H_2O, in the fact that this weight contains 16 parts of oxygen in the place of 2 of hydrogen, so do we regard the event of the synthesis or diæresis of oxygen as differing from the event of the synthesis or diæresis of water, in the fact that in the one change the two atoms of oxygen fulfil the same functions, and are respectively in the same polar conditions as the two atoms of hydrogen and the one atom of oxygen in the other.

This theory is of a purely relative character; it is connected with no special hypothesis as to the nature of oxygen or water, but it states that, if we make a certain

* See Philosophical Transactions, 1850, Part II. p. 759.

ssertion as to the molecular nature of water, we must, in consistency, make certain parallel assertions as to the molecular nature of oxygen. Our molecular hypotheses may change, but this relation will still remain.

Views as to the polarization of oxygen, and the cause of the decompositions effected by the alkaline peroxides, which to a great extent are identical with the preceding, and in which the same language and the same notation are employed, have recently been put forward, with considerable pretension, as new and originating with himself, by SCHÖNBEIN, Professor of Chemistry at Basle*. This chemist can scarcely be aware of the memoir referred to, as in his numerous publications he makes no allusion to it. A reclamation of a priority of ten years ought not to be required, but I am compelled to call the attention of chemists to these circumstances in order that I myself may not be considered to appropriate without acknowledgment the ideas and discoveries of another.

The decomposition of the alkaline peroxides by the oxide of silver and other similar bodies is complicated by the circumstance that not only is the peroxide decomposed by the metallic oxide, but the reduced silver, which is a necessary product of the action, and also probably the oxide of silver itself is capable of decomposing the peroxide by that continuous form of action which is spoken of as Catalytic. This interferes with the result; the amount of oxygen evolved depends upon the relative velocity with which these two forms of decomposition occur; and while the reduction in equal atomic proportions is never exceeded, and by certain modifications of the experiment may be very closely approximated to, it yet is never absolutely realized, for the catalytic action cannot be entirely eliminated. So that the total loss of oxygen from the oxide of silver in the experiment represents the relation which subsists between these two forms of decomposition, simultaneously occurring, and varies between the limits of the infinite, or catalytic action on the one hand, and the reduction in atomic proportions on the other.

That this is the true account of the phenomena is seen from the fact that, where these disturbing causes do not exist, the decomposition takes place in simple atomic proportions. This was shown to be the case in the decomposition of the peroxide of barium by iodine in the presence of water. The final result of this action is expressed by the equation

$$I_2 + Ba_2O_2 = 2BaI + O_2.$$

It may be considered as taking place by the decomposition and re-formation of water, according to the two equations,

$$I_2 + Ba_2O_2 + H_2O = 2HI + Ba_2O + O_2,$$
$$2HI + Ba_2O = 2BaI + H_2O.$$

* See Annalen der Chemie, vol. cviii. p. 157, SCHÖNBEIN, "Ueber die gegenseitige Katalyse einer Reihe von Oxyden, Superoxyden und Sauerstoffsäuren, &c." Also see Phil. Mag. S. 4. vol. xvi. p. 178, "Further Observations on the Allotropic Modifications of Oxygen, &c."

The parallel character of this reaction to that of the reduction of the metallic oxide is sufficiently evident. Nevertheless it was desirable absolutely to realize the normal change in the case of the oxide itself, and to discover a case in which the catalytic action should be eliminated.

In the paper referred to I gave several examples of the decomposition of the alkaline peroxides, effected by the action of oxidizing substances in aqueous solution—such, for example, as the decomposition of the peroxide of hydrogen by chlorine and by permanganic acid, and of the peroxide of barium by the alkaline hypochlorites and by a solution of ferricyanide of potassium. These reactions are free from the complicating circumstances before mentioned. The present paper contains an investigation of several of these decompositions; it will be seen that they follow the normal law of chemical action, that is, the two substances which enter into the change are decomposed in a simple atomic ratio, and that these decompositions differ from other chemical changes in no single respect, and need, to account for them, no special hypothesis.

The experiment discovered by BARRESWIL, of the reduction of the peroxide of hydrogen by chromic acid, presents points of special interest. We have in this case an action varying with the proportion present of the decomposing substances, and apparently of an abnormal character, which, however, is shown by accurate investigation to be subject to the atomic law, and to be capable of being broken up into two simple reactions.

From the effects of reduction I proceed to consider the effects of oxidation produced by the alkaline peroxides, which are of considerable theoretical importance. Certain theories have been formed as to the different nature of the oxygen in the different classes of peroxides, based on a supposed difference in the properties of this oxygen. It will be shown that the difference of properties, to account for which these hypotheses have been invented, does not exist, and that, by suitable modification of the circumstances of the experiment, the results of oxidation produced by the peroxide of manganese may equally be realized by the peroxide of barium.

Lastly, I shall give some experiments on the catalytic decomposition which this class of peroxides undergoes, instituted with the view of discovering the cause of this action, and the way in which these phenomena are connected with the ascertained properties of the peroxides. This form of decomposition I believe to be the consequence of that double function of oxidation and reduction which is peculiar to this group of substances.

In the experiments which follow, which were made by means of standard solutions, the amount of solution employed was measured by the aid of a series of carefully calibrated pipettes, which were so arranged that the capacity of each was an exact multiple of the capacity of the smallest pipette. The capacity of this pipette, which I shall designate as P, was equal to 4·55 cub. centims. The other pipettes are designated as 2 P, 3 P, ... 10 P, &c.

The burette employed for titration with permanganate was provided with a glass stopcock; it was etched and calibrated in the same manner as a tube for gas-analysis. The readings were made with a telescope.

The peroxide of barium employed was prepared by precipitation, according to a method elsewhere described. It was free from all impurities, except a trace of carbonate. When a solution of peroxide of hydrogen is spoken of, it is to be understood as the solution of this peroxide of barium in dilute hydrochloric acid.

The solutions of peroxide of sodium were prepared by digestion of the moist and freshly-precipitated hydrate of the peroxide of barium with carbonate of sodium, and filtration from the carbonate of barium formed, or, in some cases, by precipitation of the solution of peroxide of barium in hydrochloric acid by carbonate of sodium, and filtration.

In the numerous experiments and calculations which have been made in the course of the following investigation, and of which a small part only is here recorded, I have been much indebted to the skill and care of my assistant, Mr. F. SCHICKENDANZ.

1. *Decomposition of a Solution of Peroxide of Hydrogen by Permanganic Acid.*

When a solution of permanganate of potassium is mixed with an acid solution of the peroxide of hydrogen, a decomposition of both substances ensues, oxygen gas is evolved, and a colourless solution formed containing a protosalt of manganese.

The proportion in which the substances are decomposed in this reaction was determined in the following manner.

A portion of pure peroxide of barium was dissolved in very dilute hydrochloric acid; a measured quantity of this solution was decomposed by hydriodic acid, and the amount of iodine formed estimated with sulphurous acid, according to the method of BUNSEN.

A measured amount of the solution of permanganate of potassium was decomposed by hydriodic acid, and the iodine formed estimated by the same method.

A measured amount of the solution of peroxide of hydrogen was decomposed by the solution of permanganate, which was added from the burette until the solution was faintly coloured.

Now, if s be the parts of the standard iodine solution required for the decomposition of 1 cub. centim. of the solution of peroxide of hydrogen, and if s_1 be the parts of the iodine solution required for the decomposition of 1 cub. centim. of the solution of permanganate of potassium, where s and s_1 are determined according to the usual formula

$$s=\frac{nt-t_1}{p},$$

and if m be the parts of the solution of peroxide of hydrogen decomposed by the permanganic acid, and m_1 be the parts of the solution of permanganate required to effect their decomposition, and if x be the ratio of the amount of oxygen evolved from the peroxide of hydrogen to the amount of oxygen evolved from the permanganate, then

$$x=\frac{ms}{m_1 s_1}.$$

Two distinct series of experiments gave the following results:—

I. $s=14\cdot1145$, $s_1=14\cdot127$,
$m=4\cdot91$, $m_1=4\cdot9025$;

whence $x=1\cdot0007$.

II. $s=35\cdot945$, $s_1=14\cdot8$,
$m=4\cdot91$, $m_1=11\cdot919$;

whence $x=0\cdot9945$.

This experiment affords a simple and accurate method of determining by titration with a solution of permanganate the absolute amount of oxygen in a solution of the peroxide of hydrogen.

If e be the amount of oxygen contained in 1 cub. centim. of the permanganate, and y the amount of oxygen in 1 cub. centim. of the peroxide, then

$$y=\frac{m_1}{m}e.$$

In the former of the above experiments 1 cub. centim. of the iodine solution contained 0·00243 grm. of iodine, which are equivalent to 0·00017905 grm. of oxygen.

Hence

The oxygen in 1 cub. centim. of permanganate, as determined with iodine, =0·000517 grm.;

The oxygen in 1 cub. centim. of peroxide of hydrogen, as determined with iodine, =0·0005147 grm.;

and

The oxygen in 1 cub. centim. of peroxide of hydrogen, as determined with permanganate, =0·0005144 grm.

The above experiments prove that in this decomposition the decomposing substances evolve equal quantities of oxygen*; the final result of the change may be thus stated,

$$4HCl+H_2Mn_4O_8+5H_2O_2=4MnCl+8H_2O+5O_2.$$

I have varied this experiment in many ways, by adding the peroxide of hydrogen to the permanganate, and by taking the solutions excessively dilute and excessively concentrated, with the view of eliciting a variation in the reaction, but have constantly obtained one and the same result. Although the above equation expresses accurately the final result of the decomposition, we are not to believe that the five molecules of oxygen are at once eliminated, but rather that the substance passes through five successive stages of disoxidation, very rapidly succeeding one another. If to an excess of an alkaline solution of permanganate an alkaline solution of the peroxide of sodium be added, oxygen is evolved, the solution still remains clear, but becomes of the characteristic green colour of the

* This reaction has also been investigated by Aschoff with the same result.—Répertoire de Chimie pure, vol. iii. p. 296.

manganate of potassium. On the addition of a further portion of the alkaline peroxide, oxygen is again evolved, the solution becomes colourless, and hydrated peroxide of manganese is precipitated. If this precipitate be added to an acid solution of the peroxide of hydrogen, oxygen is again evolved, both the peroxides are destroyed, and a solution is formed of a protosalt of manganese, showing the successive formation and reduction of each of the oxides before the final result is attained.

2. *Decomposition of a Solution of Peroxide of Hydrogen by Ferricyanide of Potassium.*

When an acid solution of the peroxide of hydrogen is mixed with an acid solution of ferrocyanide of potassium, an oxidation takes place, and the ferrocyanide passes into ferricyanide of potassium. This action requires time, and takes place with extreme slowness in dilute solutions.

If, on the other hand, an alkaline peroxide be mixed with an alkaline or neutral solution of ferricyanide of potassium, the reverse action takes place, oxygen gas is evolved, and the ferricyanide passes into the ferrocyanide of potassium.

The proportion in which the substances are decomposed in this reaction was thus determined.

A solution was made of ferricyanide of potassium, of which the value in terms of a standard solution of permanganate of potassium was ascertained as follows. A measured amount of the solution was converted to ferrocyanide by adding to it a great excess of recently precipitated hydrated peroxide of barium, the solution was boiled until the excess of peroxide of barium was completely decomposed, and the amount of ferrocyanide formed was determined by means of the standard solution of permanganate *.

A measured amount of a solution of peroxide of hydrogen, the value of which had been estimated by the process before described in terms of the same solution of permanganate, was precipitated by an excess of baryta water. To this a measured amount of the solution of ferricyanide was gradually added by means of a pipette. After the experiment, the solution was diluted with water and acidulated. The excess of peroxide and the ferrocyanide present was determined by means of the same solution of permanganate.

It is evident that the subsequent addition of permanganate would effect two results,—the decomposition of the excess of peroxide, and the reconversion of the ferrocyanide formed to the condition of ferricyanide.

Since in an acid solution the ferrocyanide is oxidized by the peroxide of hydrogen to ferricyanide, a portion of the ferrocyanide reduced will have undergone that conversion, and an equivalent portion of the peroxide of hydrogen will have disappeared.

Now, putting s=the parts of permanganate solution equivalent to 1 part of the solution of peroxide of hydrogen employed, and s_1=the parts of permanganate solution equivalent to 1 part of the solution of ferricyanide employed,

* This affords an excellent method for the estimation of ferricyanide of potassium.

and putting m=the parts of the solution of peroxide of hydrogen, and m_1=the parts of the solution of ferricyanide respectively decomposed,

and if x=the ratio of the oxygen evolved from the peroxide of hydrogen to the oxygen evolved from the solution of ferricyanide, then

$$x=\frac{ms}{m_1s_1}.$$

The value of ms is thus given:—

Putting n=the parts of the solution of peroxide of hydrogen employed in the experiment, and p=the parts of the solution of permanganate required after the termination of the reaction,

$$ms=ns-p+m_1s_1,$$

whence

$$x=1+\frac{ns-p}{m_1s_1}.$$

In the second of the following experiments the ferricyanide was taken in excess. The experiment was conducted as before, with the exception that the solution of peroxide of hydrogen, having been rendered nearly neutral, was dropped from the pipette into the solution of ferricyanide. The whole of the peroxide is decomposed, in which case

$$n=m, \quad m_1s_1=p,$$

and

$$x=\frac{ms}{p}.$$

I. $s=1{\cdot}06, \quad s_1=0{\cdot}4,$

$n=10, \quad m_1=10, \quad p=10{\cdot}6,$

$x=1{\cdot}0000.$

II. $s=0{\cdot}76, \quad p=7{\cdot}6,$

$n=m=10,$

$x=1{\cdot}0000.$

I have varied the form of these experiments in many ways without producing any variation in the resulting value of x.

The final result of this decomposition is given in the equation

$$2(K_3\,Fe_2\,Cy_6)+Ba_2\,O_2=2(K_3\,Ba\,Fe_2\,Cy_6)+O_2.$$

3. *Decomposition of a Solution of Peroxide of Hydrogen by Hypochlorite of Barium.*

When peroxide of barium is mixed with a solution of an alkaline hypochlorite, oxygen gas is evolved, and both substances are decomposed.

The following experiments were made with the hypochlorite of barium. It was prepared by leading chlorine through a solution of hydrate of barium to complete saturation, the excess of chlorine being afterwards expelled by a current of air.

The amount of hypochlorite of barium present in the solution was estimated by decomposing the solution with hydriodic acid, according to the method of BUNSEN.

The value of the solution of peroxide of hydrogen was estimated, as before, with permanganic acid.

When the peroxide of hydrogen was taken in excess, a measured amount of the solution was precipitated by a solution of hydrate of barium. To this a measured amount of the solution of hypochlorite was added by a pipette. After the decomposition, the excess of peroxide of barium was estimated by determination with permanganic acid, the solution being first diluted and rendered acid.

When the hypochlorite was taken in excess, a measured amount of the solution was rendered strongly alkaline with baryta water, and the measured amount of the solution of peroxide of hydrogen added to this by means of a pipette. The excess of hypochlorite present after the decomposition was estimated by the iodine method.

In the former case, putting

s=the parts of permanganate required to effect the decomposition of 1 part of the peroxide of hydrogen employed,

s_1=the parts of permanganate equivalent to 1 part of the solution of hypochlorite of barium employed, as calculated from the determination with iodine,

m=the parts of the solution of peroxide decomposed,

m_1=the parts of the solution of hypochlorite decomposed,

x=the ratio of the amount of oxygen evolved from the peroxide of hydrogen to that evolved from the hypochlorite,

$$x=\frac{ms}{m_1s_1},$$

where, putting n=the parts of the solution of peroxide employed in the experiment, and p=the parts of permanganate required to decompose the excess of peroxide after the decomposition,

$$ms=ns-p,$$

and

$$x=\frac{ns-p}{m_1s_1}.$$

In the second case, where the hypochlorite was taken in excess, let

s=the parts of the standard iodine solution equivalent to 1 part of the solution of peroxide of hydrogen;

s_1=the parts of the same solution equivalent to 1 part of the solution of hypochlorite;

m=the parts of the solution of peroxide employed in the experiment;

m_1=the parts of the solution of hypochlorite decomposed;

then, as before,

$$x=\frac{ms}{m_1s_1},$$

where, putting n_1=the parts of hypochlorite employed in the experiment, and p_1=the parts of the iodine solution equivalent to the excess of hypochlorite present, as determined by experiment,

$$m_1s_1=n_1s_1-p_1,$$

and

$$x=\frac{ms}{n_1s_1-p_1}$$

I. Peroxide of barium in excess.

(1) $n=20,\quad m_1=10,$

$s=1\cdot015,\quad s_1=1\cdot113,\quad p=9\cdot2,$

$ms=ns-p=20\cdot3-9\cdot2=11\cdot1,$

$x=0\cdot997.$

(2) $n=2,\quad m_1=2,$ Two expts. Mean. $p=\left\{\begin{matrix}7\cdot923\\8\cdot005\end{matrix}\right\}7\cdot964,$

$s=4\cdot905,\quad s_1=0\cdot9314,$

$ms=ns-p=1\cdot846,$

$x=0\cdot99.$

II. Hypochlorite of barium in excess.

$m=10,\quad n_1=20,$ Mean. $p_1=\left\{\begin{matrix}32\cdot25\\32\cdot4\end{matrix}\right\}32\cdot325,$

$s=3\cdot0126,\quad s_1=3\cdot124,$

$m_1s_1=30\cdot155,$

$x=0\cdot999.$

The final result, therefore, of the decomposition is in both cases expressed by the equation

$$\text{Ba ClO}+\text{Ba}_2\text{O}_2=\text{Ba Cl}+\text{Ba}_2\text{O}+\text{O}_2.$$

I have ascertained by similar experiments that an equivalent of chlorine in aqueous solution decomposes an equivalent of the peroxide of hydrogen, according to the equation

$$\text{Cl}_2+\text{H}_2\text{O}_2=2\text{H Cl}+\text{O}_2.$$

Decomposition of Chromic Acid by Peroxide of Hydrogen.

The previous reactions are of a normal character. The peroxide of hydrogen and the permanganic acid or alkaline hypochlorite are simultaneously decomposed in simple atomic proportions, and the formation of the oxygen evolved is subject to the general law of atomic combination. The decomposition of the peroxide of hydrogen by chromic acid has a character apparently exceptional, and it is only by an attentive study of the reaction that it is seen to be of the same class as the preceding.

Barreswil made the interesting observation that chromic acid in an acid solution is oxidized by the peroxide of hydrogen, and an evanescent blue compound formed, which is rapidly decomposed with the formation of sesquioxide of chromium and the evolution of oxygen gas. The nature of this compound is unknown. Barreswil, indeed, considered that he had given reasons for believing it to be the chromic compound corresponding to permanganic acid. But he was unacquainted with the peculiar features of the reaction.

Having ascertained, by preliminary experiments which it is unnecessary to detail, that the quantity of peroxide of hydrogen decomposed by the same quantity of chromic acid was variable in amount, and depended upon the proportion in which the two substances were present, I instituted a series of experiments with the view of determining the law of this action.

The absolute amount of the solution of peroxide of hydrogen employed in each experiment was the same, namely 20 P., or very nearly 100 cub. centims. This solution was made up of three parts; of a standard solution of peroxide of hydrogen, of dilute hydrochloric acid, and of water. It is readily seen how, by means of the system of calibrated pipettes before mentioned, the bulk of this solution could be kept constant, while the amount of peroxide of hydrogen contained in it could be caused to vary. The hydrochloric acid employed was excessively dilute, and the same quantity of acid was used in each experiment. I ascertained, however, that the absolute amount of hydrochloric acid used had, within considerable limits, no appreciable influence on the reaction. Into this solution, which was contained in a small flask and kept in a state of rapid agitation during the experiment, the solution of chromic acid was allowed to run freely from the pipette in which it was measured. After the first rapid evolution of oxygen had ceased, and the blue colour had disappeared, the solution was allowed to remain for 16 or 18 hours, and the excess of peroxide of hydrogen was then determined in the manner to be described. In dilute solutions a very considerable time is required for the completion of the reaction.

When the chromic acid was in excess, the experiment was conducted in a precisely similar way, the solution of chromic acid being brought to the standard bulk of 20 P. The result was absolutely the same, whether the peroxide of hydrogen was gradually dropped into the chromic acid, or allowed to run freely from the pipette. On the effects of dilution and temperature I am not yet able to speak with precision; but the solution may at any rate be considerably diluted (for example, mixed with an equal bulk of water) with no appreciable variation in the results.

The value of the solutions of peroxide of hydrogen and of chromic acid employed, was determined by means of the same standard iodine solution and the excess present; after the decomposition, of the peroxide of hydrogen or of chromic acid*, was also estimated in the same manner.

Now, putting

s = the parts of the iodine solution equivalent to 1 part of the solution of peroxide of hydrogen employed;

s_1 = the parts of the same solution equivalent to 1 part of the solution of chromic acid employed;

* Chromic acid cannot be accurately estimated by this method if the solution be very dilute. In strong solutions the error is inappreciable, as I have ascertained by direct experiment. Similar difficulties occur if it be attempted to estimate directly a solution of peroxide of hydrogen by a standard solution of sulphurous acid. To ensure an accurate result, the peroxide of hydrogen must be first decomposed by hydriodic acid, and the iodine formed estimated by sulphurous acid.

m=the parts of the solution of peroxide of hydrogen decomposed in the experiment;

m_1=the parts of the solution of chromic acid decomposed in the same experiment;

x=the ratio of the oxygen evolved from the peroxide of hydrogen to the oxygen evolved from the chromic acid in the same experiment;

then

$$x=\frac{ms}{m_1s_1}.$$

Also, putting

n=the parts of the solution of peroxide of hydrogen employed in the experiment;

n_1=the parts of the solution of chromic acid employed in the same experiment;

r=the ratio of the oxygen contained in the peroxide of hydrogen to the oxygen contained in the chromic acid employed in the experiment;

then $$r=\frac{ns}{n_1s_1}.$$

This ratio I shall term the ratio of mass.

Further, putting

p=the parts of the iodine solution equivalent to the excess of peroxide of hydrogen after the completion of the decomposition;

p_1= the parts of the iodine solution* equivalent to the excess of chromic acid after the completion of the experiment, as determined by the formula $p=nt-t_1$;

then

$$x=\frac{ms}{m_1s_1}=\frac{ns-p}{n_1s_1-p_1},$$

where either p or $p_1=0$, according as the chromic acid or peroxide of hydrogen is in excess.

And if y be the ratio of the difference of the amount of oxygen contained in the peroxide of hydrogen and the amount of oxygen contained in the chromic acid employed, to the amount of oxygen evolved from the chromic acid decomposed, then

$$y=\frac{ns-n_1s_1}{n_1s_1-p_1}=\frac{n_1s_1}{n_1s_1-p_1}.(r-1).$$

So long as the chromic acid is not in defect, the whole of it being decomposed, $y=r-1$. If the oxygen in the peroxide of hydrogen be equal to that in the chromic acid employed, $y=0$; if this oxygen be greater in amount than that in the chromic acid, y is positive, if less, y is negative.

Also

$$\frac{x}{y}=\frac{ns-p}{ns-n_1s_1}.$$

The numerator of this fraction represents the number of atoms of oxygen evolved

* In these experiments a proportionate part of the solution was titred after the completion of the reaction, and the values of p and p_1 calculated. In strong solutions the error, from this cause, is accumulated and may become considerable. To this may be attributed the deviation from the mean in experiments 1, 3, and 4 of the following Table.

from the peroxide of hydrogen; and the denominator the difference of the number of atoms of oxygen contained in the peroxide of hydrogen, and the number of atoms of oxygen contained in the chromic acid employed in the experiment.

Also if x_1 be the ratio of the total amount of oxygen evolved from the solution, to that evolved from the chromic acid,

$$x_1=\frac{ns+n_1s_1-\overline{p+p_1}}{n_1s_1-p_1}=1+x.$$

The proportion of the oxygen evolved to the whole oxygen taken in the experiment

$$=1-\frac{p+p_1}{ns+n_1s_1}.$$

The annexed Table contains the numerical results of the experiments, and the calculated values of r, x, and y.

Table of Experiments on the Decomposition of Chromic Acid by Peroxide of Hydrogen.

	s.	s_1.	n.	n_1.	p.	p_1.		s.	s_1.	n.	n_1.	p.	p_1.
1.	17·713	51·426	2 P	10 P	0	477·31	22.	18·383	7·413	......	......	28·841	...
2.	......	......	......	......	...	478·50	23.	......	......	6 P	4 P	60·014	...
3.	16·831	17·032	2 P	18 P	...	276·05	24.	17·007	6·981	......	......	53·708	...
4.	......	......	......	......	...	276·60	25.	17·594	7·100	......	......	57·114	...
5.	......	......	2 P	16 P	...	240·23	26.	17·082	6·981	8 P	4 P	86·248	...
6.	......	......	......	......	...	239·69	27.	17·007	......	......	......	85·760	...
7.	......	......	2 P	12 P	...	171·92	28.	......	......	......	......	86·584	...
8.	......	......	......	......	...	171·92	29.	......	......	10 P	4 P	116·616	...
9.	......	......	2 P	8 P	...	103·60	30.	17·23	......	......	......	119·02	...
10.	......	......	......	......	...	103·99	31.	18·383	7·413	......	......	127·23	...
11.	......	......	2 P	6 P	...	69·62	32.	17·23	6·981	12 P	4 P	149·88	...
12.	......	......	......	......	...	69·84	33.	......	......	......	......	151·48	...
13.	17·042	6·981	2 P	12 P	...	50·26	34.	17·082	6·981	14 P	4 P	182·57	...
14.	......	......	......	......	...	50·03	35.	18·515	7·413	......	......	200·24	...
15.	......	......	2 P	6 P	...	10·72	36.	......	......	......	......	201·04	...
16.	......	......	......	......	...	11·22	37.	......	......	16 P	4 P	236·82	...
17.	16·305	7·415	2 P	4 P	2·906	0	38.	......	......	......	......	236·82	...
18.	17·183	7·023	......	......	0	...	39.	......	......	18 P	4 P	272·91	...
19.	17·082	6·981	......	......	...	...	40.	......	......	......	......	273·31	...
20.	......	......	4 P	4 P	26·213	...	41.	35·945	7·415	18 P	4 P	585·35	...
21.	17·153	7·023	......	......	27·656	...	42.	......	......	......	......	586·60	...

From these data we obtain the following values for r, y, and x:—

	r.	y.	x.	Mean value of y.	Mean value of x.
1.	0·068	−12·65	0·959	−12·65	0·975
2.	......	......	0·991		
3.	0·110	− 9·02	1·103	− 9·02	1·113
4.	......	......	1·123		
5.	0·123	− 7·34	1·042	− 7·34	1·038
6.	......	......	1·026		
7.	0·164	− 5·25	1·037	− 5·25	1·037
8.	......	......	1·037		
9.	0·247	− 3·16	1·031	− 3·16	1·037
10.	......	......	1·043		
11.	0·329	− 2·11	1·033	− 2·11	1·035
12.	......	......	1·037		
13.	0·402	− 1·61	1·017	− 1·61	1·014
14.	......	......	1·011		
15.	0·814	− 0·25	1·093	− 0·25	1·103
16.	......	......	1·111		
17.	1·100	0·10	1·001	0·10	1·001
18.	1·223	0·223	1·223	0·222	1·222
19.	1·222	0·222	1·222		
20.	2·444	1·444	1·506	1·457	1·492
21.	2·446	1·446	1·462		
22.	2·480	1·480	1·507		

	r.	y.	x.	Mean value of y.	Mean value of x.
23.	3·720	2·72	1·696	2·691	1·709
24.	3·648	2·648	1·728		
25.	3·705	2·705	1·703		
26.	4·888	3·888	1·803	3·876	1·811
27.	4·864	3·864	1·799		
28.	......	......	1·833		
29.	6·08	5·08	1·911	5·15	1·918
30.	6·16	5·16	1·905		
31.	6·20	5·20	1·909		
32.	7·39	6·39	2·034	6·39	2·000
33.	......	......	1·977		
34.	8·554	7·554	2·023	7·647	1·992
35.	8·743	7·74	1·989		
36.	......	......	1·962		
37.	9·993	8·993	2·003	8·993	2·003
38.	......	......	2·003		
39.	11·24	10·24	2·032	10·24	2·027
40.	......	......	2·022		
41.	21·816	20·816	2·079	20·816	2·061
42.	......	......	2·043		

A graphic delineation of the results is given in the curve annexed, Plate XXXVII.

It will be observed, on inspection of this line, that, so long as the chromic acid is not in defect, the two substances lose equal amounts of oxygen, according to the equation

$$2Cr_2O_3+3H_2O_2=Cr_4O_3+3H_2O+3O_2;$$

that when the amount of oxygen in the peroxide of hydrogen is greater than that contained in the chromic acid, more of the peroxide of hydrogen is decomposed, but never the whole amount taken, and that the amount decomposed increases with the proportion taken until the peroxide of hydrogen contains as much as eight and a half times the amount of oxygen contained in the chromic acid; that after this point the decomposition becomes constant, the peroxide of hydrogen losing twice the amount of oxygen lost by the chromic acid; and that no further increase in the proportion of the peroxide of hydrogen taken causes this limit to be exceeded, the final result of the change being expressed by the equation

$$2Cr_2O_3+6H_2O_2=Cr_4O_3+6H_2O+9O.$$

In what light are we to regard the decomposition between these extreme limits? Are we to consider that the two substances are capable of reacting in any proportion, and that the simple atomic decomposition is the limit of an indefinite action varying according to the mass? or are we not rather to believe that this apparently indefinite action is the sum of certain normal chemical changes which take place in simple atomic ratios, but which vary in absolute amount?

The following experiments indicate that the latter hypothesis is correct, and that the reaction between the extreme limits is not homogeneous, but consists of two chemical changes which are capable of being separated, from the circumstance that they take place with very unequal rapidity.

A definite period in the decomposition is marked by the disappearance of the blue compound. At this point the solution was titred, and the loss of oxygen estimated. The progress of the decomposition was followed by means of the same experiment until, finally, the maximum loss was attained. I give two series of experiments, in which the substances were taken in different proportions.

I.

Time of titration.	s.	s_1.	n.	n_1.	p.
1. After decomposition of blue compound	17·594	7·160	4 P	4 P	41·98 / 41·88
2. After $\frac{1}{2}$ an hour . . .	17·183	7·023	4 P	4 P	32·864
3. After 2 hours . . .	——	——	—	—	28·056
4. After 18 hours . . .	——	——	—	—	27·656

These data give the following values for r, y, and x:—

	r.	y.	x.	Mean.
1.	2·47	1·47	0·997 / 1·0033	1·001
2.	2·446	1·446	1·277	1·277
3.	——	——	1·448	1·448
4.	——	——	1·462	1·462

II.

Time of titration.	s.	s_1.	n.	n_1.	p.
1. Immediate	17·594	7·101	6 P	4 P	76·956 / 76·207
2. After 1 hour	——	——	—	—	60·12
3. After $1\frac{1}{2}$ hour . . .	——	——	—	—	——
4. After 2 days	——	——	—	—	57·114

Hence we have

	r.	y.	x.	Mean.
1.	3·705	2·705	1·007 / 1·033	1·020
2.	——	——	1·593	1·593
3.	——	——	1·593	1·593
4.	——	——	1·703	1·703

It thus appears that the decomposition takes place in two stages, the former of which is complete immediately after the destruction of the blue compound, and in which the two substances lose equal amounts of oxygen; while the latter requires several hours for its completion, the solution during this time being in a continual state of change.

It is undoubtedly difficult to speak with any great probability as to the nature of the specific changes which take place, in regard to which many hypotheses may be formed. The following view, however, is in accordance with the facts.

1. When $x=1$, we have the formation and subsequent decomposition of the substance Cr_4O_9, according to the equations

$$2Cr_2O_3+3H_2O_2=Cr_4O_9+3H_2O,$$
$$Cr_4O_9=Cr_4O_3+3O_2.$$

2. When $x=2$, after the reaction just expressed the substance Cr_4O_9 is further oxidized to the oxide Cr_4O_{12}, according to the equation

$$Cr_4O_9+3H_2O_2=Cr_4O_{12}+3H_2O.$$

The body Cr_4O_{12} further decomposes with the excess of peroxide of hydrogen present, possibly with the formation of the higher oxide of hydrogen, H_2O_3, and the former product, Cr_4O_9,

$$Cr_4O_{12}+3H_2O_2=Cr_4O_9+3H_2O_3.$$

That the product of the first action is in a continual state of formation and decomposition during the change, is probable from the greater permanence of the blue compound when an excess of peroxide of hydrogen is present: the presence of this body, which in a small excess of peroxide of hydrogen has only a momentary existence, is rendered evident by the duration of the blue colour for as long as ten minutes in the presence of an excess of the peroxide. Lastly, the substance Cr_4O_9 decomposes very much more rapidly than the substance H_2O_3, *i. e.* than the other oxidized product; so that the decomposition can be broken up into its several stages in the manner described, the chemical changes which take place being represented by the system of equations,

$$\begin{aligned}2Cr_2O_3+3H_2O_2&=Cr_4O_9+3H_2O,\\ Cr_4O_9+3H_2O_2&=Cr_4O_{12}+3H_2O,\\ Cr_4O_{12}+3H_2O_2&=Cr_4O_9+3H_2O_3,\\ Cr_4O_9&=Cr_4O_3+3O_2,\\ 3H_2O_3&=3H_2O_2+3O;\end{aligned}$$

and the result of these changes, by the equation which results from elimination between them, namely,

$$2Cr_2O_3+6H_2O_2=Cr_4O_3+6H_2O+9O.$$

Since the amount of oxygen is in all probability given off in successive stages, we cannot fix with certainty upon any one degree of oxidation as the blue compound*.

* I am unable to reconcile my own results with those of BARRESWIL, whose experiments appear to have been carefully conducted. The reaction according to this chemist is

$$2Cr_2O_3+H_2O_2=Cr_4O_7+H_2O$$
$$Cr_4O_7=Cr_4O_3+2O_2,$$

the analogue of permanganic acid being formed. I have not, however, repeated the experiment precisely in the form in which it was made by him, and it is possible that under certain circumstances the oxidation may be arrested at this point. See Annales de Chimie, vol. xx. p. 364.

It may be observed that, if the decomposition of permanganic acid were, as in this case, preceded by an oxidation, the corresponding degree of oxidation would be formed thus:

$$Mn_4 O_7 + 5 H_2 O_2 = Mn_4 O_{12} + 5 H_2 O,$$
$$Mn_4 O_{12} = 2 Mn_2 O + 5 O_2.$$

The decomposition of permanganic acid is, in a somewhat strong solution, instantaneous; but it is remarkable that, if the solution of peroxide of hydrogen be very dilute, the addition of the first drops of permanganic acid produce no apparent change. A certain time is required for the commencement of the decomposition, after which it proceeds with regularity. It may be questioned whether this decomposition also may not be effected by successive stages which escape observation, and whether the period which intervenes before the commencement of the action may not be occupied in the production of those substances by the agency of which the final result is attained, and which are successively formed and decomposed.

On the Oxidation effected by the Peroxide of Hydrogen.

We have seen that the peroxides of hydrogen, potassium, and barium possess certain chemical properties which do not belong to the analogous compounds of lead and manganese. An elaborate attempt* has been made to account for the different reactions of the two classes of peroxides, by the assumption that there are two kinds or varieties of oxygen, a positive and a negative variety; it is said that the peroxide of manganese acts as an agent of oxidation because it contains the negative variety, and that the peroxide of barium acts as a reducing agent, because it contains the positive variety of this element. This hypothesis finds its only support in an imperfect and incorrect view of the facts. In truth no such fundamental distinction exists between the properties of the different peroxides as that which it is proposed thus to characterize. The chemical properties of the alkaline peroxides, as of other chemical substances, vary with the conditions in which they are placed, and the substances with which they are associated; and it is in our power so to modify these conditions as to produce with these peroxides the very same effects of oxidation as are produced by the peroxides of the other group. This is evident from the following examples, to which it would be easy to add others:—

1. An acid solution of peroxide of hydrogen causes the conversion of a solution of ferrocyanide to ferricyanide of potassium.

2. An alkaline solution of peroxide of sodium added to a solution of a protosalt of manganese forms hydrated peroxide of manganese.

3. An alkaline solution of peroxide of sodium oxidizes an alkaline solution of sesquioxide of chromium, with the formation of chromate of potassium.

4. A strong solution of hydrochloric acid evolves chlorine with peroxide of barium †.

* SCHÖNBEIN, Annalen der Chemie, vol. cviii. p. 166.

† Professor SCHÖNBEIN lays the greatest stress on the different behaviour of the two classes of peroxides with hydrochloric acid. He says, "Die erste Gruppe ist weiter negativ dadurch charakterisirt dass kein ihr angehöriges Superoxyd mit irgend einer wasserhaltigen Säure .... Wasserstoffsuperoxyd zu erzeugen

That there are important differences in the reaction of the peroxide of barium and the peroxide of manganese is not to be denied. But these differences are perfectly conformable to analogy, and are similar in kind to those which distinguish other chemical substances. For experience teaches us that no two chemical substances, however close may be the analogies which connect them, have identical chemical properties. Hydrochloric acid has not all the properties of hydriodic acid; soda has not all the properties of potash; chlorine is not the same as iodine; and sodium is different to potassium. Is it then to be a matter of surprise to us that peroxide of manganese has not all the properties of peroxide of hydrogen? is this case to have a special explanation? and are we to refer the different properties of these substances, not to the actual and known differences in the elements of which they consist, but to an altogether hypothetical and imaginary difference in the oxygen, which is the element common to the two bodies? In each different compound, if such conventional language be admissible, oxygen has different properties; and if we are to account for this class of differences by the assumption of different varieties of oxygen, we must assume not two forms only, but an infinite number of forms of that element. Of the precise mode in which the chemical properties of the compound are connected with the chemical properties of its constituents, we are doubtless unable to give an adequate account. But the fact of this connexion is not to be doubted. We take the oxygen of the peroxide of sodium, and transfer it to the protoxide of manganese. In this new combination it will no longer produce the effects of reduction. Again, we transfer the oxygen of the peroxide of barium to anhydrous acetic acid, and we form one of the most powerful oxidizing agents with which the chemist is acquainted; while we can re-transfer this oxygen to baryta, and restore to it its original properties. It is not the oxygen which is different, but the elements differ with which the oxygen is associated.

The definite manner in which the final result is affected by modifying the conditions of the reaction, is rendered evident by the following series of experiments.

Peroxide of barium, treated with a concentrated solution of hydrochloric acid, evolves chlorine: with a dilute solution of hydrochloric acid, peroxide of hydrogen alone is formed. It was therefore probable that with hydrochloric acid of a certain degree of concentration both reactions would simultaneously occur. This is the case, and the ratio in which the two reactions occur varies with the concentration of the hydrochloric acid, according to a definite law.

A weighed quantity of peroxide of barium was placed in a small flask, to which a delivery tube could be attached by means of a caoutchouc connector. The hydrochloric acid was poured cold upon the peroxide, the delivery tube attached, and the mixture

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vermag, und die zweite Gruppe dadurch, dass keines ihrer Superoxyde unter irgend welchen Umständen aus der Salzsäure oder irgend einem salzsauren Salze Chlor zu entbinden im Stande ist."— Annalen der Chemie, vol. cviii. p. 167. This sweeping statement, in regard to which the truth could have been so readily ascertained, is quite without foundation.

rapidly boiled. The gas evolved was collected over a solution of iodide of potassium: it consisted of a mixture of chlorine and oxygen. The iodine formed was determined by a standard iodine solution in the usual manner. The hydrochloric acid was taken in very great excess; the same absolute amount was used in each experiment, but diluted with varying quantities of water. Since a solution of peroxide of hydrogen is decomposed on boiling into water and oxygen, we may regard the oxygen evolved as the measure of the peroxide of hydrogen formed.

Now, putting

α=the iodine contained in 1 cub. centim. of the standard iodine solution,

ϵ=the quantity of peroxide of barium employed in the experiment,

Z=the quantity of oxygen equivalent to the chlorine evolved,

$nt-t_1$ with its usual signification,

then

$$Z=\frac{O \times Ba_2O_2}{I \times 0\cdot2929} \times \alpha(nt-t_1).$$

In each of the following experiments 25 cub. centims. of concentrated hydrochloric acid were taken. The absolute amount of pure hydrochloric acid in this concentrated acid was experimentally determined; it amounted to 30·33 per cent. α=0·0029735.

Experiment.	Water added.	ϵ.	$nt-t_1$.	Z.	Mean value of Z.
		grm.			
I.	0	0·2929	125·5	8·015	8·015
II. (1)	5	0·3019	129·8	8·047	8·015
(2)	5	0·3008	127·5	7·983	
III.	6·25	0·3006	128·6	7·866	7·866
IV. (1)	7·5	0·2998	122·6	7·654	7·658
(2)	......	0·3080	126·1	7·662	
V. (1)	10	0·3037	105·9	6·528	6·623
(2)	......	0·3044	109·3	6·718	
VI. (1)	15	0·2964	65·7	4·188	4·092
(2)	......	0·3025	64·6	3·997	
VII. (1)	20	0·3006	31·9	1·986	1·962
(2)	......	0·2985	30·9	1·938	
VIII. (1)	25	0·2981	14·8	0·929	0·936
(2)	......	0·2990	15·1	0·943	
IX. (1)	30	0·3321	11·6	0·6538	0·677
(2)	......	0·3071	11·5	0·7005	
X. (1)	35	0·3025	7·5	0·464	0·460
(2)	......	0·2954	7·2	0·4561	
XI. (1)	40	0·3064	5·7	0·3481	0·326
(2)	......	0·3084	5·0	0·3035	
XII. (1)	50	0·3144	3·5	0·2083	0·1974
(2)	......	0·3008	3·0	0·1866	
XIII.	60	0·2973	trace.	......	......

Further. Let s= the ratio of the amount of water to the amount of pure hydrochloric acid in the solution of hydrochloric acid employed; and when water is saturated with hydrochloric acid, let s=1; s may be termed the ratio of saturation.

And let d= the amount of water added in each experiment to every 100 cub. centims. of the solution of hydrochloric acid employed.

Now it is estimated that at ordinary temperatures 100 parts of water absorb 38 parts of hydrochloric acid, while the acid employed was found to contain 30·33 per cent.

Hence s in each experiment is determined from the proportion

$$\frac{100}{38} : \frac{100+d}{30 \cdot 33} :: 1 : s,$$

whence

$$s=\frac{38(100+d)}{100\times 30 \cdot 33}=\frac{38}{30 \cdot 33}+\frac{38\times d}{3033}$$

$$=1 \cdot 25+0 \cdot 0125\times d;$$

and if y be put $=s-1$,

$$y=0 \cdot 25+0 \cdot 0125\times d;$$

and, lastly, if x be put = the oxygen equivalent to the chlorine evolved, $\frac{1}{2}$ the total oxygen in the peroxide of barium being assumed as 1, we have, assuming 16·02 as the total oxygen

$$8 \cdot 01 : 1 :: Z : x,$$

whence

$$x=\frac{Z}{8 \cdot 01}.$$

We have from the preceding experiments the following values for d, s, y, and x:—

Experiment.	d.		s.	y.	x.
I.	0	0	1·25	0·25	1·000
II.	5	20	1·5	0·50	1·000
III.	6·25	25	1·5622	0·5622	0·9814
IV.	7·5	30	1·625	0·625	0·9567
V.	10	40	1·75	0·750	0·8263
VI.	15	60	2·00	1·00	0·5104
VII.	20	80	2·25	1·25	0·2448
VIII.	25	100	2·50	1·50	0·1167
IX.	30	120	2·75	1·75	0·0844
X.	35	140	3·00	2·00	0·0574
XI.	40	160	3·258	2·25	0·0406
XII.	50	200	3·75	2·75	0·0246
XIII.	60	240	4·25	3·25	trace.

A delineation of the experiments is given in the annexed curve, Plate XXXVIII.

Catalytic Decompositions.

It has now been shown that the alkaline peroxides have a double function, and can be used as agents either of oxidation or of reduction. By certain modifications of the conditions of the experiment, we can produce separately either result. It is not unreasonable to suppose that, among the numerous and varied forms of chemical decomposition, instances would be found in which these phenomena would occur simultaneously. If this were to be the case, the result would be what is termed a contact (or catalytic) decomposition, but caused by two successive changes of a normal chemical character.

That the combination of the oxidizing with the reducing action of the peroxide of hydrogen is a cause adequate to produce the effects of catalysis, is evident from the following examples.

I. We can successively realize the chemical changes of which the result, as regards weight, is expressed by the equations

$$MnCl+KHO=KCl+MnHO,$$
$$H_2O_2+2MnHO=H_2O+Mn_2H_2O_3,$$
$$H_2O_2+Mn_2H_2O_3+2HCl=3H_2O+2MnCl+O_2.$$

The final result which ensues from the successive performance of these experiments is expressed in the following equation, derived by elimination from the preceding,

$$2KHO+2HCl+2H_2O_2=4H_2O+2KCl+O_2;$$

which equation is again equivalent to the two equations,

$$KHO+HCl=H_2O+KCl$$

and the equation expressive of the catalytic decomposition of the peroxide of hydrogen, viz.

$$2H_2O_2=2H_2O+O_2.$$

II. Again, omitting for the sake of brevity certain circumstances in the reactions, we can produce the results expressed by the equations

$$3H_2O_2+Cr_4O_3=3H_2O+2Cr_2O_3,$$
$$2Cr_2O_3+3H_2O_2=Cr_4O_3+3H_2O+3O_2,$$

whence, by elimination,

$$2H_2O_2=2H_2O+O_2.$$

III. Also

$$2Na_4Fe_2Cy_6+Na_2O_2=2Na_2O+2Na_3Fe_2Cy_6,$$
$$2Na_3Fe_2Cy_6+Na_2O_2=2Na_4Fe_2Cy_6+O_2,$$

whence

$$2Na_2O_2=2Na_2O+O_2.$$

IV. Again,

$$2HCl+H_2O_2=2H_2O+Cl_2,$$
$$Cl_2\quad+H_2O_2=2HCl+O_2,$$

whence

$$2H_2O_2=2H_2O+O_2.$$

Exp. I.—1 P. of an alkaline solution of peroxide of sodium rendered acid and titred, required 9·31 cub. centims. of permanganate solution.

A solution of protochloride of manganese was precipitated by hydrate of sodium, and 1 P. of the above solution added to it. An equivalent portion of the hydrated protoxide was converted into the hydrated peroxide of manganese.

The solution was immediately rendered acid with sulphuric acid, and 2 P. of the same solution added.

After the evolution of gas had ceased, the solution was titred. It required 9·31 cub. centims. of permanganate solution.

3 P. of peroxide had been added, and 2 P. decomposed. The manganese was in the form of protoxide, as at the commencement of the experiment.

Exp. II.—To 1 P. of a solution of ferrocyanide of potassium, which required for its titration 5·96 cub. centims. of permanganate, was added 1 P. of an acid solution of peroxide of hydrogen, which required for titration 3·681 cub. centims. of the same solution of permanganate. An equivalent portion of ferricyanide of potassium was formed.

The solution was rendered alkaline, and to it was added 1 P. of an alkaline solution of peroxide of sodium, which required for its titration 3·33 cub. centims. of the same permanganate solution.

The solution, rendered acid and titred, required 5·66 cub. centims. of permanganate.

Therefore a portion of peroxide equivalent to 7·15 of permanganate had disappeared.

The total peroxide added required 7·01 parts of permanganate.

In these experiments the final is the same as the initial condition of the chemical substances, with the single exception that the peroxide of hydrogen is decomposed. If the oxidation and reduction which in the preceding examples are separately realized had taken place simultaneously and under the same general conditions, no result would have appeared but the decomposition of the alkaline peroxide, and the action would have been termed a contact action. Now, although in the case of the catalytic decomposition of the alkaline peroxide we are undoubtedly not able to specify in each case the precise reaction by which the final result is attained, we have yet in several instances indications that the decomposition proceeds by successive stages of this kind.

When an alkaline solution of peroxide of sodium is added to a solution of protosulphate of manganese, a precipitate is formed of hydrated peroxide of manganese. If, however, a few drops of an excessively dilute solution of protosulphate of manganese be added to an excess of peroxide of sodium, there is no precipitate, but the solution remains clear, becomes brown in colour, and the peroxide undergoes the catalytic decomposition. If a great excess of a solution of the peroxide of sodium be added to a very small quantity of freshly precipitated hydrated peroxide of manganese, the peroxide of manganese dissolves, forming the same clear brown solution. It is thus seen that the peroxide of manganese can be further oxidized by the peroxide of sodium, and that the product of this oxidation decomposes in solution the peroxide of sodium.

Again, if a solution of peroxide of sodium be added to an alkaline solution of permanganate, the latter is first reduced to manganate, the solution becoming green: on further addition of the peroxide of sodium, the solution becomes of the same clear brown colour as that produced by the oxidation of the protoxide. If, however, the peroxide be sparingly added, or if permanganate be added to this solution, a precipitation takes place of peroxide of manganese. It is the compound which forms this brown solution by the agency of which the peroxide of sodium is decomposed; and it is only when the decomposition of the latter is complete, or close upon this point, that the peroxide of manganese appears. The solution then becomes turbid, and the brown flocculent peroxide is precipitated.

It appears therefore, (1) that the protoxide of manganese can be oxidized by the peroxide of sodium to an oxide, forming a clear brown solution; (2) that permanganic acid can be reduced by the peroxide of sodium to the same substance, passing through the condition of manganate; (3) that when the reduction reaches this point, in presence of an excess of peroxide the reduction is for a time arrested, and the catalytic decomposition commences: during this decomposition the brown compound is permanent; and when the peroxide of sodium is nearly decomposed, the reduction again proceeds and peroxide of manganese is formed.

These phenomena may be thus accounted for. There is a point where the oxidizing action concurs with, and as it were meets, the reducing action of the peroxide of sodium; and at this point the catalysis takes place. The peroxide of manganese is formed; but so long as a sufficient excess of the alkaline peroxide is present, it is reoxidized and destroyed as fast as it is produced. By this continuous reduction and oxidation the peroxide of sodium is gradually eliminated. Time is needed for this, as for other chemical changes; but ultimately, when but little peroxide of sodium remains, the peroxide of manganese is precipitated, being produced more rapidly than it is destroyed.

There are other examples of the same class of phenomena. When a solution of peroxide of sodium is added to a solution of a copper salt (sulphate orchloride of copper), at first there is no evolution of oxygen, but a yellowish green precipitate is formed, as was observed by THÉNARD; this precipitate may be thrown on a filter, and for a short time preserved. If a small portion of this precipitate be added to an alkaline solution of peroxide of sodium, bubbles of gas are evolved, and the peroxide of sodium rapidly decomposed. If a solution of peroxide of hydrogen be mixed with a few drops of a weak solution of chloride of copper, and the whole precipitated by baryta water, the same yellow oxide of copper is formed. The peroxide of barium is gradually decomposed; but during the decomposition this yellow oxide is permanent, and only ultimately is it decomposed into hydrated protoxide.

If a very small quantity of an ammoniacal solution of protochloride of copper be added to an alkaline solution of peroxide of sodium, the solution becomes of a yellow colour, and is gradually decomposed. During the whole time of this decomposition, which may be caused to extend over several hours, the yellow colour is permanent; but ultimately, when the whole of the peroxide is decomposed, the blue colour of the ammoniacal solution of the protoxide reappears. The yellow solution of the peroxide of copper, apart from the solution of peroxide of sodium, cannot be preserved for many minutes. The singular permanence of this compound during the decomposition is explained on the hypothesis, that it is continually reproduced as well as destroyed. In this reaction the solution of oxide of copper in potash or ammonia decomposes the alkaline peroxide into oxygen and the alkali, precisely as sulphuric acid in the process of etherification effects the decomposition of alcohol into ether and water. And we may regard the former event as determined by a series of alternate and inverse changes, analogous to those to which WILLIAMSON has shown the phenomena of etherification to be due, according to the equations

$$Cu_2O + Na_2O_2 = Cu_2O_2{}^{*} + Na_2O,$$
$$Na_2O_2 + Cu_2O_2 = Na_2O + Cu_2O + O_2.$$

It may be desirable, for the sake of clearness, to resume the points in the preceding argument. It appears, (1) that by means of the alkaline peroxides we can produce two classes of effects, oxidation and reduction, and this double function is peculiar to this group of peroxides. (2) These peroxides are decomposed by the contact of a great number of chemical substances, and this form of decomposition is also peculiar to the group. (3) The combination of an oxidizing with a reducing action is a cause adequate to produce the results of contact decomposition, and we are able, in certain cases, to imitate, as it were, the contact action by means of a successive oxidation and reduction. (4) There are instances in which we have distinct evidence that the contact decomposition is accompanied by an oxidation and subsequent reduction of the substance by which the action is determined.

In a matter so difficult to submit to the direct test of experiment, it is undoubtedly desirable to offer any view with much reservation; and it would be altogether premature to assert that this is the only form which contact decomposition can assume. Other causes may possibly lead to the same result. At the same time every new case which can be explained on these principles increases the probability of their more extended application, and raises the hope that even these obscure phenomena will ultimately be removed from the domain of conjecture and speculation, and be brought under the methods of experimental research.

* The oxide of copper here formed is probably the sesquioxide, which is also procured by the action of alkaline hypochlorites on the hydrated oxide of copper. The precise nature of this oxide is immaterial to the argument.

XXXIV. *On the Photographic Transparency of various Bodies, and on the Photographic Effects of Metallic and other Spectra obtained by means of the Electric Spark.* *By* W. A. Miller, *M.D., LL.D., Treas. & V.P.R.S., Professor of Chemistry in King's College, London.*

Received June 19,—Read June 19, 1862.

1. At the Meeting of the British Association held in Manchester last autumn, I exhibited some photographs of spectra from the electric spark obtained between wires of different metals by means of an induction-coil. Upon this occasion a hollow prism filled with bisulphide of carbon was employed, because, owing to its great dispersive power, it furnished spectra in which the lines under examination were more widely separated and exhibited with greater distinctness than by any other medium in ordinary use.

Plate XXXIX. fig. 30 exhibits a copy of the photograph of the solar spectrum obtained by means of a hollow glass prism filled with bisulphide of carbon, contrasted with the spectrum obtained through the same prism simultaneously from the spark between copper terminals of the secondary coil in the induction apparatus. In this, and in all the subsequent figures, the less refrangible end of the spectrum is upon the left-hand side of the Plate.

The great prolongation of the more refrangible portion of the spectrum beyond the part visible to the unaided eye, led me to believe that the bisulphide was a material which exerted but little absorbent action upon the chemical rays. Subsequent experiments have, however, convinced me that this opinion was erroneous, and have rendered it necessary to modify considerably the conclusions deduced from those experiments.

2. At the time that that paper was written, I believed that the photographic effects produced by the electric spectra of all the metals furnished results in a great degree similar to each other, if not actually identical. This, it will be seen from subsequent statements, is correct so far as the fact of the similarity in this portion of the spectra is concerned, but is erroneous as regards the general conclusion deduced from it. During the past winter I have renewed these experiments, substituting a quartz-train for glass and bisulphide of carbon, and have chiefly used a fine quartz prism, kindly lent to me by my friend Mr. Gassiot. The refracting-angle of this prism is about 60°; its faces are about 2 inches long and $1\frac{5}{8}$ inch broad, and are so cut as to furnish a singly refracted beam for the medium rays, by transmitting it along the axis of the crystal. It is well known, from the experiments of Prof. Stokes* and M. E. Becquerel, that quartz is remarkable for its transparency to both fluorescent and phosphorogenic rays of high refrangibility.

* Phil. Trans. 1852, p. 540.

It was soon evident that the absorbent action of the bisulphide was far greater than I had imagined, and that in reality the spectrum which it transmitted was composed of rays which did not extend beyond one-tenth or one-twelfth of the entire length of the spectrum obtained by the use of a quartz-train*.

3. The dispersive power of rock-crystal is, however, comparatively low, and the difficulty of obtaining with it a spectrum free from the effects of double refraction through its entire length is great; so that it appeared to be worth while, as a preliminary inquiry, to ascertain whether any singly refracting medium could be procured, better adapted to researches of this nature by sufficient permeability to the chemical rays, and by tolerably high dispersive power. Although no material on the whole preferable to quartz has been found, the investigation gave results of considerable interest.

4. Before proceeding to detail these results, it will, however, be convenient, as several distinct subjects will be discussed in this paper, to state the order in which I propose to arrange my remarks, and the heads to which they will be referred.

I shall commence with

(1) *The absorption of chemical rays by transmission through different media.*

a. By transmission through solids.

b. By transmission through liquids.

c. By transmission through gases and vapours.

(2) *The absorption of the chemical rays by reflexion* from polished surfaces.

(3) *The photographic effects of the electric spectra of different metals taken in air,* including

a. Pure metals.

b. Alloys.

(4) *Photographic effects of electric spectra of different metals produced by transmitting the sparks through gases other than atmospheric air.*

5. The general results of my experiments upon the absorption of the chemical rays are the following:—

(1) Colourless bodies which possess equal powers of transmitting the luminous rays vary greatly in permeability to the chemical rays.

(2) *Diactinic* solids (that is to say, solids which are permeable to the chemical rays) preserve their diactinic power both when liquefied and when converted into vapour.

(3) Colourless solids which are transparent to light, but which exert a considerable

* The absorptive power of the bisulphide for the chemical rays was, however, noticed by M. E. BECQUEREL as far back as 1843, as I find by again referring to his paper, Annales de Chimie, sér. 3. vol. ix. p. 301. In this paper M. BECQUEREL describes the absorbent action of various solids and liquids upon the chemical rays, but, from having used solar light, he failed to remark the great difference between the absorptive powers of quartz and glass. Although he used prisms of rock-salt, rock-crystal, and alum, his results do not indicate the real difference in their absorptive power; and as in all his experiments on liquids he employed a vessel with flint-glass sides to hold them, his conclusions are vitiated by the same error which affected my own earlier inquiries on the subject.

absorptive effect upon the chemical rays, preserve their absorptive power with greater or less intensity both in the liquid and the gaseous state (21).

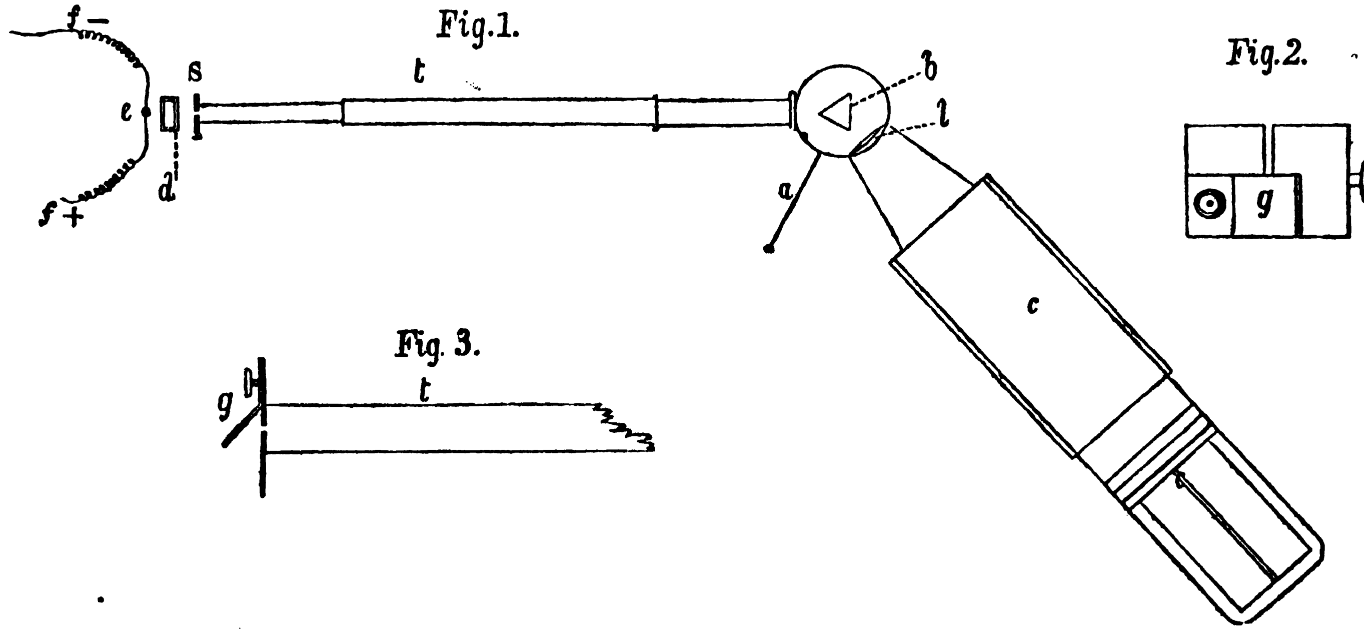
Whether the compound be dissolved in water or be liquefied by heat, these conclusions are equally true as regards liquids. Water is perfectly permeable to the chemical rays; and this circumstance, conjoined with the fact that in no instance does the process of solution seem to interfere with the special action of the substance dissolved upon the incident rays, renders it practicable to submit to trial a great number of bodies which it would otherwise be impossible to subject to experiments of this nature, owing to the extreme difficulty of obtaining them in crystals of sufficient size and limpidity.

§ 1. ABSORPTION OF THE CHEMICAL RAYS.

a. *By transmission through Solids.*

6. The general arrangement of the apparatus employed in this inquiry is represented in fig. 1, in which the observer is supposed to be looking down upon the instrument. *c*, *c* is a camera which allows of a considerable range of adjustment, and is attached to a cylindrical box, within which is a prism *b*, of rock-crystal. At *l* is a quartz lens of $1\frac{1}{4}$-inch aperture, and $17\frac{1}{2}$ inches focal length. At one end of the tube *t*, which can be

Fig. 1. Fig. 2. Fig. 3.



lengthened or shortened by a sliding joint, is a slit *s*, provided with a screw for regulating the width of the opening. This slit is arranged parallel to the axis of the prism, and in these experiments was adjusted to a distance of 37 inches from the lens *l*. The prism is placed at about its angle of minimum deviation for the mean ray, and, for facility of manipulation, can be turned round upon its own axis by means of the lever *a*. The angle formed between the camera and the tube *t* also admits of variation as circumstances may require. At *d* is placed the substance the transparency of which is to be tested; and at *e* are the metallic electrodes, which are connected with secondary wires *ff* of a 10-inch induction-coil, not shown in the figure. The wires of the coil terminate in electrodes composed of fine silver. The coil was excited by means of a battery consisting of five elements of GROVE'S construction, a condenser being included in the primary circuit, whilst a small Leyden jar, exposing about 75 square inches of metallic coating upon each of its surfaces, was introduced into the secondary circuit. In this way a torrent of sparks could be maintained between the electrodes at *e* without any sensible variation of power, for ten minutes at a time, or longer if necessary. In these experiments, an exposure of the sensitive plate for five minutes in the camera was requisite.

At a suitable distance behind the lens (about 26 inches*), a collodion plate coated with iodide, or occasionally with a mixture of iodide and bromide of silver, was supported in the camera, for the purpose of receiving the image of the spectrum †. The plate was excited by the use of a bath of nitrate of silver containing 30 grains of the nitrate to an ounce of water. The image was developed in the usual way by means of pyrogallic acid, in the proportion of one grain to the ounce of water, and fixed with cyanide of potassium.

7. The spectra of electric sparks so obtained were remarkable for their great length; indeed they extended beyond the termination of the visible rays for a space equal to five or six times the length of the luminous portion.

For the convenience of comparing the results of the various experiments together, I have adopted an arbitrary fixed scale, the fiducial point of which is the line H in the solar spectrum. Calling this 100, the more refrangible rays are numbered onwards, and the less refrangible rays backwards from it, the line B in the solar spectrum being at 84: the length of the spectrum from silver points extends from 96·5 upon this scale to 170·5. The solar spectrum for the purpose of this comparison was projected upon the collodion plate by means of a small mirror of polished steel (*g*, fig. 2) placed so as to form an angle of 45° with the surface of the plate carrying the slit, and to cover a portion of the vertical slit, as shown by an end view of the tube at *g*, fig. 3, whilst the direct image from the silver points fell simultaneously, parallel to that of the solar spectrum, upon the collodion plate in the camera.

8. The following Table contains a list of the various substances subjected to experiment. All these bodies allowed the less refrangible rays to pass, but cut off the rays of medium and extreme refrangibility wherever absorption occurred at all.

* This distance was found by experiment to give nearly a flat field, with the image of the slit formed by all the different rays in focus simultaneously. My friend and colleague, Professor J. C. MAXWELL, kindly calculated for me the relative positions of lens and prism necessary to ensure an approximatively flat field for the visible rays.

If the lens be placed between the slit and the prism, a very great difference occurs between the points of convergence of the most refrangible and the least refrangible rays, amounting with the lens and prism which I used to nearly 14 inches. When the lens is before the prism, both coincide in augmenting the convergence of the more refrangible rays; whereas when the lens is placed behind the prism, as shown in the figure, the convergence occasioned by the lens is neutralized by the prism, which now acts in the opposite direction upon the diverging rays as they fall upon it from the slit.

† My friend Mr. PIZEY, who assisted me in these experiments, prepared the collodion for me, following nearly the directions given by HARDWICH in his 'Manual of Photographic Chemistry,' 6th edition, p. 262. It was iodized with a mixture of equal parts of iodides of potassium and cadmium, and was perfectly uniform in its action, even for weeks after it had been iodized, if kept in the dark.

TABLE I.—Diactinic Power of Solids.

Name of substance.	Thickness, in inches.	Termination of spectrum.	Relative lengths of spectra.	Remarks.
Ice	about 0·5	170·5	74·0	
Diamond* (*l*)	0·032	155·5	59·0	
Diamond (*m*)	0·017	159·5	62·0	
Diamond (A)	0·182	115·5	19·0	
Sapphire (24)	0·13	116·0	19·5	
Sapphire (B)	0·093	112·0	15·5	Faint bluish tinge.
Sapphire (*n*)	0·12	111·0	14·5	
Quartz	0·16	170·5	74·0	With quartz-train.
White Topaz	0·19	162·0	65·5	Faint image of spectrum.
Mica	0·007	114·5	18·0	
Oil of Vitriol	0·75	160·5	64·0	
Sulphate of Lime (solid)	about 0·3	155·5	59·0	
Sulphate of Baryta (solid)	about 0·4	154·5	58·0	
Sulphate of Magnesia (solid)	0·34	158·0	61·5	
Sulphate of Potash	Sat. soln. 0·75 in.	159·5	63·0	
Sulphate of Soda	„	159·5	63·0	
Sulphate of Ammonia	„	145·5?	49·0 ?	
Sulphate of Zinc	„	152·5	56·0	
Alum	„	159·5	63·0	
Sulphate of Iron	„	105·0	8·5	Pale green.
Sulphate of Manganese	„	144·5	48·0	Faint pink.
Sulphate of Copper	„	112·5	16·0	Full blue.
Sulphite of Soda	„	127·5	31·0	
Hyposulphite of Soda	„	108·5	12·0	
Fluor-spar	0·17	170·5	74·0	
Fluoride of Sodium	Sat. soln. 0·75 in.	159·5	63·0	
Fluoride of Ammonium	„	166·5	70·0	
Hydrochloric Acid, sp. gr. 1·1	0·75 in.	152·5	56·0	
Rock-salt (solid)	0·75	159·5	63·0	
Chloride of Potassium	Sat. soln. 0·75 in.	159·5	63·0	
Chloride of Ammonium	„	155·0	58·5	
Chloride of Barium	„	153·0	56·5	
Chloride of Strontium	„	152·0	55·5	
Chloride of Calcium	„	147·0	50·5	
Chloride of Zinc	„	145·5	49·0	
Chloride of Manganese	„	104·5	8·0	Faint rose-colour.
Chloride of Tin (SnCl)	„	108·5	12·0	Spectrum cut off abruptly.
Chloride of Tin ($SnCl_2$)	Strong solution.	114·5	18·0	Spectrum cut off abruptly.

* I am indebted to my friend Professor W. H. MILLER, of Cambridge, for the opportunity of examining the diamonds and sapphires alluded to above. *l* was a slice of diamond bounded by cleavage-planes, from the Warburton Collection. *m* a somewhat thicker slice from the same collection. A a large octahedral diamond from the Humian Collection: all these were colourless. The sapphire 24 was a large six-sided prism from the Brooke Collection; that marked B was a smaller prism of a faint bluish tinge from Professor MILLER's own collection. *n* is a colourless crystal of sapphire from the Warburton Collection. I made an application to the Trustees of the British Museum for permission to use some of the limpid specimens in their collection, but was informed that even for such a purpose the Act of Parliament forbids them to allow any mineral to pass off their premises. Mr. Maskelyne was kind enough to lend me a fine colourless topaz from his own ollection.—[Feb. 1863.]

TABLE I. (*continued*).

Name of substance.	Thickness, in inches.	Termination of spectrum.	Relative lengths of spectra.	Remarks.
Chloride of Arsenic ($AsCl_3$)...	Liquid.	101·5	5·0	Spectrum cut off abruptly.
Corrosive Sublimate	Sat. soln. 0·75 in.	128·5	32·0	Spectrum cut off abruptly.
Bromide of Sodium	"	144·5	48·0	
Bromide of Potassium	"	144·5	48·0	
Iodide of Sodium	"	114·5	18·0	Spectra terminate abruptly.
Iodide of Potassium	"	114·5	18·0	
Cyanide of Potassium	"	105·5?	9·0?	Prepared by LIEBIG's process.
Cyanide of Mercury	"	145·5	49·0	
Sulphocyanide of Potassium...	"	112·5?	16·0?	Slightly yellowish.
Hydrate of Soda	"	131·5	35·0	From sulphate by precipitation with baryta.
Hydrate of Potash...............	"	129·5	33·0	
Hydrate of Ammonia	Sp. gr. 0·945	170·5	74·0	Rather feeble spectrum.
Hydrate of Baryta	Sat. soln. 0·75 in.	158·0	61·5	
Hydrate of Strontia	"	150·0	53·5	
Hydrate of Alumina	Strength of solution	146·0	49·5	CRUM's solution.
Hydrate of Silica	not determined.	152·0	55·5	Dialysed from hydrochloric solution.
Carbonate of Soda	Sat. soln. 0·75 in.	146·0	49·5	
Carbonate of Potash	"	146·0	49·5	From ignited bicarbonate.
Iceland Spar	0·35	160·0	63·5	
Bicarbonate of Soda	Sat. soln. 0·75 in.	145·0	48·5	
Bicarbonate of Potash	"	142·0	45·5	
Sesquicarbonate of Ammonia .	"	152·0	55·5	
Boracic Acid	"	143·0	46·5	Faint beyond 109.
Borax	"	158·5	62·0	
Phosphoric Acid	"	117·5?	21·0	
Phosphate of Soda (HO, $2NaO, PO_5$)	Solution of 60 grains of dried salt in 1 oz. of water.	156·5	60·0	Equal weights of same salt; one dried at 300° F., the other ignited.
Pyrophosphate of Soda, $2NaO, PO_5$		156·5	60·0	
Triarseniate of Soda ($3NaO, AsO_5$)	Sat. soln. 0·75 in.	127·5	31·0	
Arsenic Acid	"	119·5	23·0	
Chlorate of Potash...............	"	145·5	49·0	
Nitric Acid	Sp. gr. 1·3	106·5	10·0	Colourless.
Nitrate of Soda	Sat. soln. 0·75 in.	112·5	16·0	All the spectra of the nitrates are cut off sharply.
Nitrate of Potash	"	112·5	16·0	
Nitrate of Ammonia	"	112·5	16·0	
Nitrate of Lime..................	"	112·5	16·0	
Nitrate of Magnesia	"	112·5	16·0	
Nitrate of Baryta	"	111·5	15·0	
Nitrate of Strontia..............	"	111·5	15·0	
Nitrate of Nickel	"	absorbed	0·0	Intense green solution.
Nitrate of Lead	"	111·5	15·0	
Subnitrate of Mercury	"	111·5	15·0	
Nitrate of Silver	"	106·0	9·5	

TABLE I. (*continued*).

Name of substance.	Thickness, in inches.	Termination of spectrum.	Relative lengths of spectra.	Remarks.
Acetic Acid	Glacial, liquefied	112·5	16·0	Spectrum ends abruptly.
Acetate of Soda	Sat. soln. 0·75 in.	144·5	48·0	
Acetate of Potash	,,	113·5?	17·0?	
Acetate of Ammonia	,,	144·5	48·0	
Acetate of Baryta	,,	115·5?	19·0?	Very faint brownish tinge in liquid.
Acetate of Lime	,,	115·5?	19·0?	
Acetate of Lead	,,	130·5	34·0	
Tartaric Acid	,,	127·5	31·0	
Tartrate of Soda	,,	144·5	48·0	
Tartrate of Potash	,,	144·5	48·0	
Rochelle Salt ($NaO, KO, C_8 H_4 O_{10}$)	,,	144·5	48·0	
Tartar Emetic ($KO, SbO_3 C_8 H_4 O_{10}$)	,,	131·5	35·0	
Citric Acid	,,	133·5	37·0	
Oxalic Acid	,,	114·5	18·0	
Oxalate of Potash	,,	117·5	21·0	
Oxalate of Ammonia	,,	124·5	27·0	
Sugar-candy	60 grains in 200 grains of water.	156·5	60·0	
Milk-sugar		151·5	55·0	
Gum-arabic	Mucilage.	113·5?	16·0	Slightly opalescent.
Silicate of Soda	Sat. soln. 0·75 in.	108·5	12·0	Pale yellow.
Faraday's Optical Glass	0·54	101·5	5·0	
Flint-glass	0·68	105·5	9·0	
Window Sheet-glass	0·07	112·5	16·0	
Hard Bohemian Glass	0·18	114·5	18·0	
Plate-glass	0·22	111·5	15·0	
Crown-glass	0·74	106·5	10·0	Greenish.
Thin Glass for Microscope	0·009	116·5	20·0	

The photographic impression of each spectrum in every case quoted in this Table commences at 96·5, and the number inserted in the Table in the second column of figures indicates the point at which the most refrangible rays transmitted by the compound under examination ceased. The numbers in the third column of figures represent the length of the spectrum, the unit of the scale being one millimetre.

9. In the majority of cases of saline compounds in the foregoing Table the results given are those obtained by forming a saturated solution of the compound in distilled water, and decanting the liquid after it had become clear by standing. It is not advisable to filter in these cases, as the introduction of minute quantities of certain compounds, especially of some of organic origin, greatly impairs the transparency of the liquid to the rays which produce chemical action.

The solution, duly prepared, was then placed in a small trough made by cutting a notch in a piece of plate-glass $\frac{3}{4}$ inch in thickness, the sides of the trough being completed by thin plates of polished quartz, which were pressed by means of bands of caoutchouc against the ground surfaces of the plate-glass. No cement was employed, and the trough was taken to pieces and cleansed between each experiment—a stratum of liquid 0·75 inch thick being used in each case.

10. In the preparation of the various compounds for examination, much care was taken to employ the materials in a state of purity. In one or two instances, however, it has happened that an acid which usually forms highly diactinic salts has exhibited an anomalous and excessive absorptive power, although in combination with a base which in other instances furnishes strongly diactinic salts. Here some impurity, in quantity so small as to escape the tests in ordinary use, but sensitive to the action of light, has probably been present, and has impaired the diactinic capacity of the substance. Cases in which such impurity is suspected are indicated in the Table by the mark (?) subjoined to them. I intended to have prepared fresh portions of each of these substances with a view to their re-examination; but by the time I had arrived at this stage of my experiments I learned from my friend Professor STOKES that he had been engaged in a similar inquiry, but had been turning to account for this purpose the property of fluorescence; and as I found that my results, where both of us had employed the same substance, were in close accordance with those which he had obtained, I determined to postpone the further examination of these bodies till after the details of Professor STOKES'S experiments have appeared. I have also refrained from extending my observations to the compounds of organic chemistry, many of which I had otherwise proposed to submit to a similar investigation.

It may here be observed that the solution of a salt in water always to a certain extent impairs the diactinic quality of the liquid, however limpid the solution may be, producing an effect which may be compared to opalescence or turbidity in a liquid employed in the transmission of luminous rays.

11. I have not been able to trace any special connexion between the chemical complexity of a substance and its diactinic power. Carbon in its pure form as diamond we regard as an element. In thin slices it transmits portions of the chemical rays of nearly all degrees of refrangibility, though none of the specimens which I examined exhibited any approach to the actinic limpidity of quartz. Phosphorus, on the other hand, though transparent to light in its melted condition, and equally regarded as elementary, appears to be nearly *adiactinic*, or impermeable to the chemical rays. In many cases the peculiar diactinic or adiactinic action of an element is traceable in its simpler chemical compounds. Thus the simpler combinations of sulphur, such as sulphuretted hydrogen, sulphurous acid gas, bisulphide of carbon, and chloride of sulphur, are all powerful actinic absorbents, while in the more complicated form of sulphuric acid and the sulphates of certain bases the compounds are highly diactinic. On the other hand, the silicates are much less diactinic than silica in the form of quartz, or the bases which enter into the formation of the silicates: probably this may arise, as Professor STOKES suggests, from the difficulty of obtaining silicates, either natural or artificial, after fusion, perfectly free from iron.

12. No solid or liquid substance that I have as yet tried surpasses rock-crystal in

permeability to the rays which excite chemical action. Ice (and water), as well as white fluor-spar, rival it; and pure rock-salt approaches it very closely *. White topaz is a little inferior to the preceding bodies in diactinic capacity.

Amongst the various compounds submitted to examination, the *fluorides* rank first in diactinic power; then follow the *chlorides* of the metals of the alkaline earths. The *bromides* of the same metals appear to be less diactinic than the fluorides and chlorides, and this decline in power is still more marked in the case of the *iodides*. The short spectrum of these last-mentioned salts is interrupted by a well-marked absorption-band at a point beyond H, represented on the arbitrary scale at 103·5, beyond which the spectrum is again faintly renewed to 113·5, and then it terminates abruptly. The *cyanides* appear to be considerably diactinic; but further experiments upon these salts, as well as upon the *sulphocyanides*, are desirable. *Sulphuric*, *carbonic* and *boracic* acids furnish salts with the alkalies and alkaline earths, which are also largely diactinic: the *phosphates* seem to be less so, and the *arseniates* still less. It is remarkable that though the sulphates are so diactinic, the *sulphites* are considerably less so, and the *hyposulphites* are more opaque than the sulphites. The *hydrates* of the alkaline earths are also transparent. It is very difficult to obtain the alkaline hydrates perfectly pure; but a solution of hydrate of soda and one of potash, furnished by precipitation of their respective sulphates by means of baryta, and concentrated in a silver dish, gave a very fair result in each case.

The diactinic capacity of the *tartrates* and *citrates* is less than that of the carbonates. That of the *acetates* appears to be about the same as that of the tartrates; but the results obtained with the acetates are somewhat uncertain, as it is difficult to procure these salts absolutely free from the empyreumatic products which accompany the acid as it is usually prepared. The *oxalates* have a low diactinic power.

But the group most remarkable for its absorptive action is formed by the *nitrates*. Nitric acid, whether dissolved in water or in combination with a metallic oxide, has a specific action in arresting the chemical rays: the more refrangible portion it transmits freely, and then intercepts the spectrum abruptly at the same point whatever base be united with the acid, provided the base be capable of forming diactinic salts. The *chlorates*, on the contrary, are strongly diactinic.

13. From the observations above detailed, it appears that the following acids may be considered as possessing high diactinic capacity,—viz. the sulphuric, hydrochloric, hydrofluoric, chloric, carbonic, and boracic acids. Inferior to these are hydrobromic, phosphoric, arsenic, tartaric, citric, acetic and oxalic, hydriodic, sulphurous and hyposulphurous acids, whilst nitric acid is still less diactinic. Chromic acid arrests all the chemical rays; and the presence of a tinge of yellow or green colour in any compound is immediately apparent in a great reduction in the amount of its diactinic power.

14. Among the bases, potash, soda, ammonia, baryta, lime, strontia, magnesia, and alumina are eminently diactinic. Oxides of zinc, mercury, and lead approach them in

* A specimen of sea-water which had been standing for some months in my laboratory, furnished a resul identical with that obtained by using a strong solution of pure chloride of sodium.

power; but coloured bases, like oxide of iron, nickel, cobalt, or copper, are very inferior; and when the salts which they form are green or yellow, they are nearly opaque.

It is remarkable that, notwithstanding the high diactinic quality of silica, none of the different varieties of glass transmit rays extending beyond one-fifth or one-sixth of the range afforded by quartz. This absorptive action is produced by a lamina of glass less than the one-hundredth of an inch in thickness, which cuts off the more refrangible rays nearly as completely as a piece of glass of twenty times the thickness. All glass apparatus must therefore be abandoned in these experiments, and apparatus of quartz substituted for them.

15. I had no encouragement in my attempts to construct prisms of other materials than rock-crystal. Rock-salt offers no advantage, and it is too soft and deliquescent to yield prisms or lenses comparable with those of quartz. I did indeed make a considerable number of experiments with a hollow prism furnished with thin quartz sides, and filled with water. But the refractive power of water is less than that of rock-crystal, and its dispersive power is not higher. The addition of pure chloride of sodium till the water is saturated does not materially increase the refractive or dispersive power, whilst it appeared (in a very slight degree it is true) to diminish the amount of the more refrangible rays, so that, on the whole, I found it more convenient to work with a quartz prism, the double refraction of which in the position in which I used it, was so slight that it was not a source of any inaccuracy of importance.

b. *Absorption by transmission through Liquids.*

16. In the experiments with liquids, the same plate-glass trough with quartz sides was used as when solutions were employed, and the apparatus was arranged in exactly the same manner. Great care was taken in the purification of each specimen. The *wood-spirit* was prepared from oxalate of methyl by WÖHLER's method, and it, as well as the *alcohol* and *fousel oil* employed, was in the anhydrous state. The *glycerin* was perfectly colourless, and was prepared by Mr. GEO. WILSON by distillation with superheated steam; it retained about 4 per cent. of water. The specimen of *glycol* was the only one about which I had any doubt; it had a barely perceptible yellowish tinge, and a very slight empyreumatic odour. The *carbolic acid*, given me by Mr. CRACE-CALVERT, was a beautiful colourless specimen, which, by a slight reduction of temperature, solidified to a mass of delicate white needles. The *benzol* was a specimen prepared from benzoate of lime, purified by congelation at 32°, and carefully rectified. The *paraffin oil* was a perfectly limpid colourless specimen obtained from Rangoon petroleum; it had a specific gravity of 0·831, and boiled steadily at 360° F. The *bisulphide of carbon*, and indeed most of the other liquids, were rectified immediately before proceeding to experiment upon them.

The following Table contains a list of the liquids operated on, and the lengths of the different spectra, in terms of the scale already explained (par. 7). The compounds included in the following Table, with the exception of nitric and hydrochloric acids, are not simply solutions, but liquids to which a definite chemical formula may be assigned.

TABLE II.—Diactinic Power of Liquids.

Thickness of stratum 0·75 inch.

Name of substance.	Termination of spectrum.	Relative lengths of spectra.	Remarks.
Water	170·5	74·0	
Wood-spirit	116·5	20·0	With a faint impression of the rays about 156.
Alcohol	159·5	63·0	
Fousel Oil	116·5	20·0	
Glycol	107·5?	11·0?	Slight empyreumatic odour.
Glycerin	114·5	18·0	
Ether	112·5	16·0	
Chloroform	122·5	26·0	
Dutch Liquid	132·5	36·0	
Oxalic Ether	115·5	19·0	
Carbolic Acid	104·5	8·0	
Benzol ($C_{12}H_6$)	117·5	21·0	
Paraffin Oil, $n(C_2H_2)$	111·5	15·0	Specific gravity 0·831. Boiling-point 360°.
Oil of Turpentine	104·5	8·0	
Phosphorus (melted)	....	0	Slight yellowish tinge.
Bisulphide of Carbon	102·5	6	
Oxychloride of Phosphorus	....	0	
Terchloride of Phosphorus	....	0	Retaining a little phosphorus in solution.
Terchloride of Arsenic	101·5	5·0	
Acetic Acid	112·5.	16·0	Glacial.
Sulphuric Acid	160·5	64·0	
Nitric Acid	106·5	10·0	Specific gravity 1·3.
Hydrochloric Acid	152·5	56·0	Specific gravity 1·1.

The starting-point for each spectrum was 96·5 upon the scale already adopted.

Of all these liquids, water and alcohol are the only two, except sulphuric and hydrochloric acids, which are strongly diactinic; water is eminently so, alcohol in a much less degree. No relation in this respect is traceable between common alcohol and the other alcohols examined, viz., wood-spirit, fousel oil, glycol, glycerin, and the phenic alcohol carbolic acid. Bisulphide of carbon, the refractive medium employed in my earlier experiments, is singularly deficient in diactinic power, and is therefore eminently unfit for such researches.

c. *Absorption of Chemical Rays by transmission through Gases and Vapours.*

17. In the experiments upon the absorbent action of aëriform media, the gas or vapour under trial was introduced into a brass tube 2 feet long, blackened on the inside, and closed at the end by plates of quartz, which were fitted on so as to form air-tight joints. The tube could be attached by a stopcock to the plate of the air-pump, and after exhausting the air any gas could be easily introduced. In cases in which the gas was liable to act upon the metal, a glass tube was substituted for the metallic one, and the gas was introduced by displacement. The tube when prepared was interposed at *t*, fig. 1 (par. 6), between the slit *s* and the prism *b*, and the rays emanating from the electric spark were, after traversing the column of gas contained in the tube, received first upon the prism and lens, and then upon the excited collodion surface, in the usual manner.

When the vapour of a volatile liquid was to be examined, a few drops of it were generally allowed to fall into the tube filled with air, through which the vapour was allowed to diffuse itself at the ordinary temperature. The action of such vapours was therefore compared at a great disadvantage with that of the various gases, particularly where the volatility of the liquid was rather low. The results, however, even under these disadvantageous circumstances, were well marked, as may be seen by examining the subjoined Table of gases and vapours submitted to experiment, in which the comparative lengths of the different spectra are shown in the second column of figures.

TABLE III.—Absorbent action of Gases and Vapours on the Chemical Rays.

Length of column of gas 2 feet.

Name of gas.	Termination of spectrum.	Relative lengths of spectra.	Remarks.
Atmospheric Air	170·5	74·0	
Hydrogen	170·5	74·0	
Carbonic Acid	170·5	74·0	
Carbonic Oxide	170·5	74·0	
Olefiant Gas	162·5	66·0	
Marsh-gas	159·5	63·0	
Coal-gas	133·5	37·0	Cut off abruptly.
Protoxide of Nitrogen	159·5	63·0	
Cyanogen	159·5	63·0	
Ammonia	170·5	74·0	
Sulphurous Acid	110·5	14·0	Cut off abruptly.
Sulphuretted Hydrogen	110·5	14·0	Cut off abruptly.
Bisulphide of Carbon	101·5	6·0	A few of the strongest lines between 140 and 152 are seen.
Dichloride of Sulphur	108	10	
Benzol	131·5	35·0	Faint beyond 111·5.
Oil of Turpentine	152·0	55·5	
Chloroform	152·0	55·5	
Ether	163·5	67·0	
Terchloride of Phosphorus	131·5	35	Very feeble spectrum.
Oxychloride of Phosphorus	141·5	45	Fades out very gradually.
Hydrochloric Acid	151·5	55·0	
Hydrobromic Acid	119·5	23	Cut off abruptly.
Hydriodic Acid	111·5	15	Cut off abruptly.
Peroxide of Nitrogen	0	0	
Peroxide of Chlorine	0	0	

18. The absorbent action disclosed by the foregoing experiments on the colourless gases and vapours is very interesting, as it proves that differences exist in the diactinic power of these substances quite as marked as in the case of liquids and solids. Some of the *elementary* gases, oxygen, hydrogen, and nitrogen, appear to possess a diactinic capacity greater than any solid or liquid body. Many *compound* gases, such as ammonia, carbonic acid, and carbonic oxide, appear to rival them. Olefiant gas, cyanogen, and hydrochloric acid exhibit a decided but not great absorptive power, with which that of the vapours of ether, chloroform, and oil of turpentine at the atmospheric tension, and when diffused through air, may be compared. Doubtless if these vapours were tried at a tension of 30 inches, they would exhibit greater absorptive power. The absorptive

action of hydrobromic acid much exceeds that of the hydrochloric, and that of hydriodic acid is greater than of either.

The abrupt termination of the spectrum in coal-gas is remarkable. The absorption appears to be due not to the permanent gases, but to the vapours of benzol and other heavy hydrocarbons which it contains. The four compounds of sulphur, viz., sulphurous acid*, sulphuretted hydrogen, bisulphide of carbon, and dichloride of sulphur, are especially active in absorbing the chemical rays; and the vapours of the terchloride and oxychloride of phosphorus exhibit a similar though less intense absorptive power.

19. Coloured gases, whether elementary or compound, such as chlorine, bromine, and nitrous gas, have long been known to exert an absorptive action upon the luminous rays†; and their effect is not less marked upon the invisible prolongation of the electric spectrum.

The effects of the three halogens, chlorine, bromine, and iodine, in the form of vapour, are particularly remarkable. As a general rule, when a body exerts an absorptive influence the absorption is greatest in the most refrangible portions; but the reverse of this occurs in the case of chlorine and of bromine. A column of *chlorine* two feet in depth, cuts off the whole of the *less* refrangible portion as far as 143·5; beyond that a distinct impression is obtained as far as about 159. With *bromine* diffused in the form of diluted vapour, the impression commences at 106, and is continued distinct, though rather feeble, to the extreme end of the spectrum. The apparatus required a slight modification to adapt it for the experiment with *iodine*. I had a glass tube 6 inches long, the open ends of which were ground flat so as to admit of being closed by thin plates of quartz; this was enclosed in a brass tube; a few grains of iodine were introduced, and the quartz plates fixed by metallic caps perforated to admit the passage of the rays; this tube could then be supported as usual between the spark and the prism, and could be raised to and kept at a temperature beyond that necessary for the volatilization of the iodine. The electric light, after traversing such a column of vapour of an intensely deep violet colour, gave a strong spectrum, extending from 96·5 as far as 112, then it gradually faded till it disappeared at about 118; the impression became again rather faintly but distinctly visible at 142, and gradually disappeared at about 156. It is interesting to notice a somewhat similar interrupted absorption of the rays, though at a different part of the spectrum, in the case of the metallic iodides.

Both peroxide of nitrogen and peroxide of chlorine, in a stratum of 2 feet in depth, wholly absorb the chemical rays; but when more dilute or in shorter columns, they each give characteristic absorption-bands.

20. There appears to be little or no connexion between the absorptive power of any particular gas for the chemical rays, and its power of absorbing radiant heat as determined by the experiments of Dr. TYNDALL‡. Aqueous vapour is highly diactinic, though not diathermic; olefiant gas exhibits a similar difference; and various other instances might be pointed out.

* An aqueous solution of sulphurous acid cuts off the spectrum at the same point as the gas itself does.

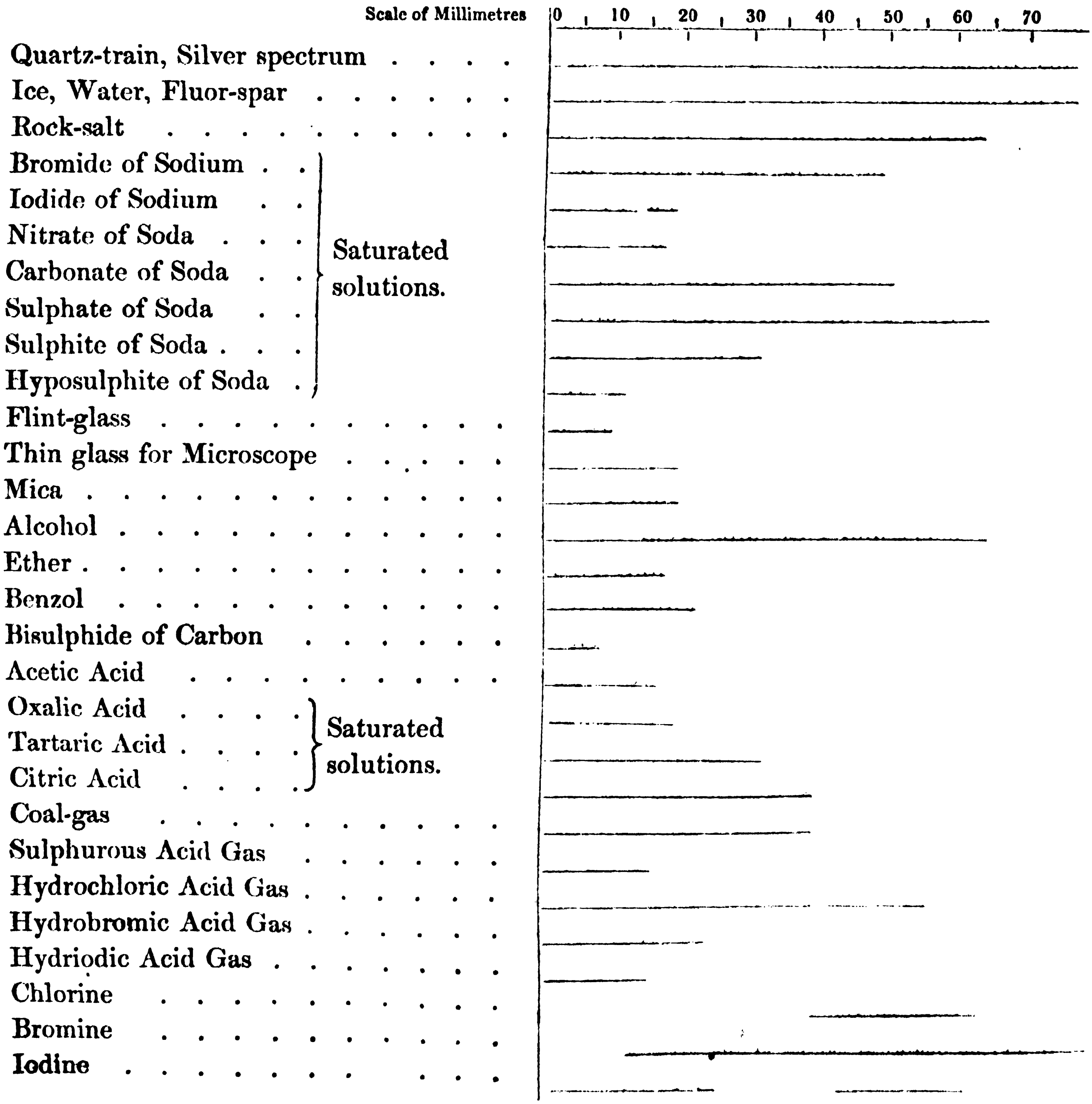
† For an historical sketch of the progress of discovery in relation to the production of bands in the spectrum, the reader is referred to a paper by the author in the 'Pharmaceutical Journal,' February 1862, p. 17 *et seq.*

‡ Phil. Trans. 1861.

21. The most interesting fact, however, disclosed by these various experiments is the persistence of either the diactinic or the absorbent property in the compound, whatever be its physical state—a circumstance which proves that the property under consideration is intimately connected with the atomic or chemical nature of the body, and not merely with its state of aggregation.

The following diagram represents approximatively the relative position of the portions of the spectrum transmitted in a few of the cases described in the foregoing section of this paper. No attempt is made to indicate partial absorption of the rays. In one or two instances, where complete absorption at a particular part of the spectrum occurs, this has been indicated by an interruption in the line.

Relative absorptive action of various Media upon the Electric Spectrum of Silver.



§ 2. THE ABSORPTION OF THE CHEMICAL RAYS BY REFLEXION FROM POLISHED SURFACES.

22. In my earlier experiments I had much difficulty in obtaining a spectrum all the parts of which were even approximatively in focus in the same plane, and, with the view of remedying this defect, I tried the effect of substituting specular reflexion for the refracting action of a lens. This led me to compare the reflecting-power of different polished surfaces for the chemical rays. With this object in view, a small polished plate of the material under experiment was supported at an angle of 45°, as shown in fig. 3 (par. 6), opposite the vertical slit of the apparatus, so that when the source of light *e*, was placed at right angles to the axis of the tube, the rays were reflected down the tube in the direction of that axis. The arrangement of the prism, lens, and camera was the same as that already described (6). As, however, much less light was reflected upon the prism from the polished surface than that which fell upon the prism when the direct rays of the spark were employed, the exposure of the sensitive plate in the camera was prolonged from 5 minutes to 10.

23. Among the metals and alloys thus submitted to trial were platinum, gold, silver, mercury, contained in a trough with quartz faces, lead, copper, tin, cadmium, zinc, aluminum, steel, brass, and speculum-metal. In addition to these, the reflecting-power of quartz, window-glass, and Iceland spar was also tried.

No judgment of the perfection of the reflecting-power could be formed from the colour of the metal. *Gold* possesses the power of reflecting all the rays, even the most refrangible, very equally, though somewhat feebly. Next to gold ranks burnished *lead*, some part of the spectrum of the electric spark reflected from lead being more intense than that from gold. The length of the spectrum obtained from the light of the electric spark between silver points, by reflexion from the surface of these two metals, extended from 96·5 to 170·5, or over the full distance of that obtained by the direct light of the spark, viz. 74 divisions of the scale which I have adopted. With all the other metals the spectrum of the same reflected spark terminated at 159·5, covering only 63 divisions of the scale.

The spectrum from a *silver* surface was remarkable. The impressed image was strong up to 112·5; then an abrupt cessation of the reflected rays occurred for a distance of 1·5 division; beyond this the reflexion gradually returned, and continued tolerably intense till it reached 159·5, covering 63 divisions of the scale.

The reflexion from *mercury* was weak in the middle, but strong towards each extremity. *Platinum*, *zinc*, and *aluminum* resembled mercury in their effects, but the spectrum was much less intense. The reflexion from *cadmium* was similar, but still weaker.

The spectrum of the rays reflected from *copper* was deficient in strength for the last half of the more refrangible portion; and that of *brass* was similar to it, but weaker. The reflexion from a surface of *steel* was more intense than that from any surface which I employed, but it ended abruptly at 159·5, or at the 63rd division of the scale. The spectrum reflected from *tin* was nearly as complete as that from steel.

A small concave mirror of *speculum-metal* gave an intense spectrum for the first half;

but the more refrangible portion was deficient in power, and no rays were reflected beyond 159·5 (63 divisions), the point at which the other metals also failed. I therefore abandoned the attempt to substitute a speculum for the lens, with which latter I succeeded subsequently in obtaining a field sufficiently flat for the purpose.

24. The reflexion from the surface of transparent objects was so scanty that, of course, no idea was entertained of using such bodies as mirrors; but it may be worthy of notice that a feeble spectrum was obtained from surfaces of *quartz*, *window-glass*, and *Iceland spar* extending to 159·5, or over a length of 63 divisions of the scale—that is to say, fully as far as the majority of the metals. The quantity of the reflected rays was small, but its quality was similar to that of the rays reflected from metallic surfaces.

§ 3. PHOTOGRAPHIC EFFECTS OF THE ELECTRIC SPECTRA OF DIFFERENT METALS TAKEN IN AIR.

a. *Pure Metals.*

25. I have spent a considerable time in endeavouring to procure exact photographs of these spectra, inasmuch as the spectrum of a metal is a constant not less important than its density or its fusing-point; and it frequently furnishes the means of identifying an element under circumstances in which no other method at present known is practicable.

KIRCHHOFF, in his elaborate and masterly researches on the constitution of the solar spectrum, has, as is well known, published in minute detail a map including the lines of a large number of the metals. He has, through a limited portion of the visible spectrum, laid down the position of the bright lines of certain metals coincident with particular dark lines of FRAUNHOFER, with a precision best appreciated by those who have followed him with most minuteness.

Much yet, however, remains to be done even for the rays which fall within the range of the visible spectrum; and for those which are beyond the limits of ordinary vision, the whole yet remains to be examined.

The lines of each spectrum are so numerous and so close together, that it would be impossible without a sacrifice of time, that would scarcely be justifiable, to obtain accurate impressions of them by eye-drawing. Indeed, except by the process of photography, these lines can only be rendered visible by the aid of a fluorescent screen, under which circumstances the minute details are almost necessarily lost even by the most careful observer.

The photographs of these spectra were obtained by an arrangement of the quartz prism and lens, identical with that already described (6), wires, plates, or irregular fragments of the metal, according to circumstances, being supported in brass forceps connected with the secondary wires of the induction-coil. The interval traversed by the spark was in each case about a quarter of an inch, and the slit was placed at a distance of half an inch from the line traversed by the spark.

The specimens of gold, silver, mercury, copper, bismuth, antimony, zinc, tellurium, thallium, and lithium employed are believed to have been pure. The tungsten, molybdenum, chromium, and manganese were reduced from pure oxides in crucibles lined with charcoal. The other metals were as they are furnished in commerce as pure.

26. Each metal gives its own distinctive spectrum; but it is remarkable that these differences are not obvious in the less refrangible end. The true metallic spectrum, when the sparks pass in air, is in fact combined with that due to atmospheric air, as has already been pointed out for the visible rays by Ångström and by Alter. The photographic lines of the air-spectrum are most marked in the less refrangible portion, whilst the characteristic lines of the metals are particularly evident in the more refrangible parts. Hence the photographs which I formerly obtained by the use of a prism of bisulphide of carbon, which transmits rays of low refrangibility only, represent, as I then correctly pointed out, lines which are chiefly atmospheric; and consequently they exhibit appearances which are almost identical whatever be the metal employed.

In describing the spectra of the different metals, I shall employ the same arbitrary scale that I have hitherto used in this paper.

27. It will be observed that generally the lines as they advance towards the less refrangible extremity become less intense in their central portion, until towards the extreme limit of the spectrum, the two marginal ends of the lines alone are visible, though these terminations are often rather intense. Indeed, throughout the whole length of the impressed photograph, the marginal extremities of the metallic lines leave a stronger image than their central portions, as though the incandescence of the volatilized portions of the electrodes, owing to their high radiating power, did not continue sufficiently intense during their transfer across the interval between the two electrodes, to enable them to produce a continuous line. Evidently the cause of this diminution of action operates more powerfully upon the more refrangible rays; and a higher temperature, as the experiments of other observers have abundantly proved, is necessary to the production of radiations of these high degrees of refrangibility.

Exceptions to this remark occur in the lines due to the atmosphere; this is well seen in the strong line at 110·5, which is in marked contrast to some of the metallic lines in its vicinity, particularly in the spectrum of silver (Plate XXXIX. fig. 9), where this nitrogen-line is included between two pairs of very intense lines due to the metal itself, and which are each interrupted in the middle.

In order to abbreviate the description of the various spectra, I shall generally speak of these interrupted lines as "*dots*"; they, indeed, constitute the characteristic features of the different metallic spectra. These dots, if the image be exactly in focus, may usually be seen to consist of groups of very short lines closely aggregated. This is well shown in some parts of Plate XL. fig. 39, which represents the spectrum of silver; and it is less distinctly shown in the spectra of palladium (fig. 38), of copper (fig. 40), of antimony (fig. 41), and of cadmium (fig. 42). These spectra were taken with the screen, lens, and prism at a distance from the slit different from those with which the other impressions were procured; some parts are consequently out of focus, but the details of other portions are shown more fully.

As might be anticipated, the spectra of the more volatile metals are the most intense—those of bismuth and antimony, of cadmium, zinc, and magnesium being especially remarkable in this respect.

A certain similarity is also observable in the spectra of allied metals, as in the case of the three metals last mentioned, also in those of iron, cobalt, and nickel, of bismuth and antimony, as well as of chromium and manganese.

It should further be observed that, in estimating the apparent length of the different spectra, considerable difficulty is frequently experienced owing to the extremely faint impressions which the most refrangible rays commonly occasion; in some experiments this portion of the spectrum with the same metal appears to be longer than in others made under apparently similar conditions.

28. *Platinum.*—The characters of the spectrum of platinum are feebly marked. The *atmospheric lines* are well developed. It is important to remark that the *continuous spectrum* which forms the background to all the metallic spectra in their less refrangible portion, appears to be due to the oxygen and nitrogen of the atmosphere. Between 96·5 and 103·6 this spectrum is intense, and is crossed by a strong pair of compound lines followed by three strong groups of lines, ending in two other groups, each successively of less intensity than the preceding ones. At 107·5 is a well-defined line, and a strong band at 110·5. Three others, nearly equidistant, follow at 114, 117·3, and 121·7: at 137 and 138·2 are two faint lines, and beyond these, at 142·5, is a very faint pair of lines. All these are atmospheric lines. The true metallic spectrum of platinum (Plate XXXIX. fig. 5) gradually fades out with a series of groups of dots, and terminates at about 162.

29. *Iridium.*—The spectrum of this metal, for a fine sample of which I am indebted to my friend Professor WHEATSTONE, scarcely differs sensibly from that of platinum. The lines apparently most prominent are atmospheric. (See Plate XL. fig. 44.)

30. *Palladium.*—This spectrum is much more uniform in intensity than that of platinum. Besides the usual atmospheric lines, groups of dots commence at 103·5, and continue at irregular intervals. Four of the strongest of these groups occur at 138·5, 143, 148, and 152·5, measuring at about the centre of each group. The spectrum terminates rather abruptly at about 162·5. (Plate XXXIX. fig. 6, & Plate XL. fig. 38.)

31. *Gold.*—The spectrum of this metal has about the same degree of intensity as that of palladium, and it is almost exactly equal to it in length. Amongst its numerous lines, the most characteristic are two pairs of dots very close together at 124, and three groups of dots, commencing at 130·0, 143·5, and 153·5 respectively; the last is faint. (Plate XXXIX. fig. 7.)

32. *Silver.*—The spectrum of this metal is very characteristic. It is intense for the first third of its length, then becomes much fainter, after which several remarkable and very strong groups of lines are observed, beyond which the impression disappears at 170·5. The most conspicuous lines are a pair of double lines interrupted in the middle groups of dots rather strong, at 126·8, 129·0, and 131·0; between 140·5 and 144·5 are three very strong groups of dots. About 148 is another broad strong group; beyond

which, and terminating at about 159·5, is another strong triple group of dots,—the spectrum terminating at 170·5, with six nearly equidistant but rather faint groups of dots *. (Plate XXXIX. fig. 9, & Plate XL. fig. 39.)

The strongly marked character of the spectrum of silver, and particularly the renewal of its intensity towards the more refrangible end, rendered it very appropriate for the purpose of testing the diactinic quality of different media; and accordingly I have used it more extensively than any other metal in the experiments already detailed upon this subject.

[33. *Thallium.*—The spectrum of thallium (for a specimen of which in a pure state I am indebted to the kindness of Mr. CROOKES, its discoverer) is particularly interesting, as in its visible portion it is remarkably simple, the single intense green line being the only one visible, even when heated in the intense flame of the oxyhydrogen jet. When, however, the sparks of the secondary coil are transmitted, not only do new lines make their appearance in the visible spectrum, but also in the extra-violet portion, and the complex impression shown at Plate XL. fig. 45 is developed. This character of its spectrum † separates thallium from the metals of the alkalies. In the less refrangible portion are two strong groups of lines at about 103 and 106; three other groups occur at 116, 121, and 126 respectively, the two first less intense, the third of about the same strength as the first pair of groups. Several feebler pairs of dots follow; and the spectrum terminates with four nearly equidistant groups, commencing respectively at 136, 141, 145, and 151: the first of these groups is very strongly marked, the others are fainter but of nearly equal intensity.—Feb. 1863.]

34. *Mercury.*—Experiments were made with this metal by soldering a platinum wire into a small glass tube which was filled with mercury and connected by means of the platinum wire with one end of the secondary wire of the coil; the other electrode consisted of a platinum wire. The spectrum obtained exhibited few lines, excepting those due to the mercurial electrode. The never-failing nitrogen line, 110·5, was evident; but there were numerous strong lines due to the mercury, the most distinct of which are those at 104, 114·5, 117·5, 119·0, 122·5, 131·0, 138, 156, 159, each of the two last forming a strong broad group of dots, the last group terminating the spectrum at 161·0. (Plate XXXIX. fig. 10.)

35. *Lead.*—This spectrum is very strong for rather more than one-third of its length; it then diminishes in strength, until the action ceases almost entirely at 146·5; at about 161 it is renewed abruptly by a strong group of dots, which terminate the spectrum at 162·5. The lines in this spectrum are numerous and complicated. It exhibits an at 109·5 and 111·5, equidistant from the nitrogen line 110·5; beyond this are three intense triple group of dots between 122 and 125, and two other strong groups at about 130 and 132. (Plate XXXIX. fig. 11.)

* In one or two instances, when using bromiodized collodion, I saw a *very* faint series of groups prolonged as far as 190·0.

† Proceedings of the Royal Society, January 1863, vol. xii. p. 407.

36. *Copper.*—The spectrum of this metal is considerably prolonged. The intensity of the light from copper points is liable to vary considerably during the course of an experiment, being apparently much influenced by the slight changes of form experienced by the electrodes in consequence of the action of the discharge upon them. In these experiments I always used thin sheets of electrotype copper. The most marked lines in the spectrum of copper are the following: viz., two close together at 111·7 and 112·3; these are followed by three strong nearly equidistant groups of dots at 128, 136·5, and 146·7; a very strong group appears at 156·7; and the spectrum ends at about 181·5, with a series of much fainter groups. (Plate XXXIX. fig. 12, & Plate XL. fig. 40.)

37. *Tin.*—The spectrum of this metal shows at 107·0 and 112·5 a pair of strong double dots, at 114 another dotted line, at 118·0 a strong double line; characteristic groups of dots commence at about 122·0 and 129·5; between 130 and 134 it exhibits three strong groups, followed by several well-marked pairs of dots to about 154, after which it terminates rather abruptly by four nearly equidistant faint groups at about 163. (Fig. 13.)

38. *Bismuth.*—This metal furnishes a strong and well-marked spectrum. Five pairs of dots show themselves between the line H and the nitrogen line 111·5; a strong pair is then seen at 115·3, followed by five strong groups, each of which is triple; the first of these is the most intense; these are followed by an intense group, which terminates at 122·0; beyond these the spectrum is continued by a numerous and intense series of groups of dots, particularly at 143 and 157, and terminates at about 158. (Fig. 14.)

39. *Antimony.*—The spectrum of antimony is also very characteristic. A triple series of groups of dotted lines commences at about 104·5; the nitrogen line at 110·5 is very intense; at 113 is a strong double line followed by seven or eight intense groups of dots which terminate at about 138·5; at 150·5 is a group of moderate intensity; and the last traces of lines disappear at about 164·0. (Figs. 15 & 41.)

40. *Arsenic.*—The characteristic dotted lines of this spectrum commence at 115·0; a series of groups occur very close together, ending at about 129·0; then follow six fainter irregular groups; and the spectrum terminates abruptly with three very strong groups—the first two of which nearly run into each other, succeeded by an interval, the last group ending at 155. (Fig. 16.)

41. *Tellurium.*—This spectrum is highly characteristic. It exhibits a close series of dots, commencing at about 106, and terminating by a very strong group at 147·5. There is then a complete blank, till at 156 another group is seen, then an interval, and the spectrum terminates abruptly with a well-defined group at 168·5. (Fig. 17.)

42. *Tungsten.*—This spectrum is not very intense. It shows a double dotted line at 114, and a stronger one at 125; it is prolonged by a distinct series of dots to about 145, and fades out at about 168. (Fig. 19.)

43. *Molybdenum.*—The spectrum of this metal resembles that of tungsten, but is more intense. It is prolonged by a very faint termination to about 157. (Fig. 20.)

44. *Chromium* shows a group of dots at 106·0, then a strong line at 114·5, then

three intense groups of dots between 122 and 131, beyond which it gradually fades till it is lost at 160·0. (Fig. 21.)

45. *Manganese.*—This spectrum is well-marked. A double group of dots appears at 107, then five strong groups of dots and lines between 119·4 and 135·5; the first is the most intense, and the last group is broad and complicated: beyond this the impression becomes gradually fainter, and disappears at about 152·0. (Fig. 22.)

46. *Iron.*—The spectrum of iron is intense, particularly in the more refrangible portion. Between 113 and 124 are three diffused groups of dots of moderate intensity; at 126 is a very intense group; then a series of numerous intense groups of dots, which terminate at 154; at 155·5 a series of fainter groups recommences, and fades away at about 162. (Fig. 23.)

47. *Cobalt.*—The spectrum of cobalt is very like that of iron in its general appearance; but it shows two or three rather strong groups of lines between 106 and 109; an increase of intensity commences at about 128, and is continued in a series of groups of very strong dots, the most intense of which are situated between 134 and 147: the spectrum terminates rather suddenly at about 163. (Fig. 24.)

48. *Nickel.*—This spectrum is not so intense as either that of iron or of cobalt, but it is like them in its intensity being greater near the more refrangible extremity (fig. 25). It is a very long one, extending to about 190, and exhibits an arrangement of groups of dots at its more refrangible end, resembling that of silver, but prolonged much further.

49. *Cadmium.*—This is a striking spectrum. The action is accumulated into intense bands with perfectly dark intervals. Two broad strong groups of lines occur at 105 and 107, then two other compound bands or groups of dots, one on each side of the nitrogen line 110·5; beyond this are two fainter lines, then at 118 and 121 two broad groups of dots, frequently prolonged into lines; at 126·3 a very strong group of dots, and another at 133·5; beyond this an almost entire cessation of action till it is resumed by three groups of dots, the first and last of which are the strongest in the entire spectrum. These groups commence at 149·5 and terminate at 155·5; beyond these are two other detached groups at 162·5 and 167, where the spectrum terminates abruptly. (Figs. 26 & 42.)

50. *Zinc.*—This spectrum is equally characteristic with that of cadmium. Between 109·5 and 112·5 is an intense double group of eight or nine distinct lines, then several well-marked lines in the interval between these and another double group, which commences at 125. An interval of diminished action follows; and then between 135 and 140 are two intense groups, the strongest in the spectrum. These are followed by a few feeble groups of dots, after which is a complete interval; and the spectrum terminates with four groups of equidistant lines, the first commencing at 167·5, the latest terminating at 183. (Fig. 27.)

51. *Magnesium.*—This is another remarkable spectrum. An intense group of lines commences at 101; at 115 a bright group of dots is seen, and between 119 and 126·5 are three remarkably intense groups of lines: the first of these comprises at least four strong

lines, the second group three, and the third consists of eight or ten separate lines. This last is the most intense group that I have met with in the course of these experiments. Beyond this the spectrum is prolonged by a faint tail, which is strongest along the edges, and nearly vanishes midway between them; this tail disappears a little beyond 150. (Fig. 28.)

52. *Aluminum.*—This spectrum exhibits a characteristic group of dots at 106, another at 115·5, followed by others at irregular intervals as far as 155, beyond which no impression is visible. I have frequently repeated this experiment, as Professor STOKES informs me he that sees in the spectrum of aluminum lines which are beyond the limit of the zinc-spectrum, when the aluminum-spectrum is received upon a screen of uranium glass, or of a particular phosphate of uranium; but either the collodion plate is insensitive to rays of this extreme refrangibility, or, as I believe, the quartz prism which I employ begins to fail in transparency at about the end of the zinc-spectrum. (Fig. 29.)

53. *Sodium.*—The lines exhibited when sodium is employed appear to be mainly atmospheric; the usual nitrogen line at 110·5 is, however, developed into a bread band or group of nine or ten distinct lines, and the other atmospheric lines are very well marked; the spectrum is prolonged into a faint nearly continuous band at about 156, when it ceases to be visible. (Plate XL. fig. 47.)

54. *Potassium.*—The spectrum produced by potassium presents the same appearances as that of sodium, both being apparently only atmospheric (fig. 46).

[55. *Lithium.*—The spectrum of this metal differs from those of the other alkali-metals in presenting a single well-marked group of dots at about 123; it is prolonged into a tail, which resembles that of magnesium, and fades out at about 150. (Fig. 48.)—Feb. 1863.]

56. It is unnecessary to give any details of experiments made with electrodes one of which consisted of one metal and the other of a different metal. Under these circumstances the lines starting from the side corresponding to each metal are identical with those furnished by the particular metal. This mode of making the experiment is therefore frequently convenient when it is desirable to compare the spectrum of any given metal with another selected for comparison. When the difference in volatility between the two is extreme, as when platinum is opposed to mercury (fig. 10), it may happen that one spectrum only is seen, the lines starting from an edge of the photographic impression, and terminating at irregular distances before they reach the opposite edge.

b. *Spectra of Alloys.*

57. The principal object of these experiments was to determine the influence which small amounts of foreign metals exercise upon the photographic image. When equal weights of the two metals are employed (tin and lead, for example, or cadmium and lead), a compound spectrum exhibiting the lines due to both metals is produced; and it is not always the more volatile metal that predominates. An alloy containing 62 parts

of copper and 38 of zinc, gave a spectrum in which the lines due to copper predominated considerably. In an alloy of about 2 parts of zinc to 1 of cadmium the zinc-spectrum was the most strongly marked.

In another experiment an alloy of 990 parts of fine gold and 10 of fine silver was prepared; on taking the spectrum obtained by an exposure of 10 minutes, a distinct but feeble impression of the more refrangible lines due to silver was procured (see fig. 8). A contamination of gold with silver to an extent not exceeding 1 per cent. could therefore be recognized by this means; but prolonged exposure was necessary in order to develope the lines due to silver. An analogous result was obtained when the spectrum of plumbago was taken. In this case, in addition to the atmospheric lines, the spectrum of iron was distinctly impressed: the total amount of metallic iron in the plumbago was 3·94 per cent. Graphite deposited in the gas-retorts, which contained 0·23 per cent. of iron, gave very feeble indications of iron. On the other hand, no indication of iron was observable in the spectrum of brass which contained 0·23 per cent. of iron, nor was lead indicated in brass which contained 0·7 per cent. of this metal.

58. All the foregoing spectra were obtained either from the metals in their uncombined form, or else from their alloys. The electric spectra of a few other metals which admit of being submitted to experiment in their isolated form, still remain to be added. A considerable proportion of the metallic elements, however, are not included in the foregoing list. These it is almost impossible to examine, except in the form of some of their saline or other compounds. As, however, this portion of the inquiry is attended with some peculiar difficulties, I shall defer what I have to add upon this subdivision of the subject to a future occasion.

§ 4. PHOTOGRAPHIC EFFECTS OF ELECTRIC SPECTRA OF DIFFERENT METALS PRODUCED BY TRANSMITTING THE SPARKS THROUGH GASES OTHER THAN ATMOSPHERIC AIR.

59. In making experiments upon the influence of various gases upon the spectra of the electric spark, the arrangement of the apparatus was modified in the following manner:—The position of the slit, prism, lens, and camera was the same as in the preceding experiments (6); but the metallic electrodes were enclosed in a stout glass tube, shown at half its real size in Plate XL. fig. 49. *a* is the tube itself, *b* a hole drilled through the side of the tube, which upon this side is ground flat in order that it may be closed air-tight by the thin plate of polished quartz *c*. This plate is kept in its place by means of an elastic band. *d*, *d* are brass forceps screwed into the brass plugs *e*, *e*, for holding the electrodes. The ends of the tube are closed by the brass plugs *e*, *e*, which are ground to fit the ends of the tube, and are pierced by small brass tubes for the conveyance of the gas. An elastic band, passing from one end of the glass tube to the other, keeps the brass plugs in their place. The tube is then connected with a gas-holder

filled with the gas under experiment (or, when practicable, the gas is disengaged during the experiment), and, after the apparatus has been connected with the induction-coil and adjusted in its proper position, a slow current of the gas at the atmospheric pressure is transmitted, the excess of gas as it passes out of the apparatus being conveyed into the chimney or out of the window by a suitable arrangement of tubes.

A simpler apparatus was admissible when the wires could, like those of platinum or of iron, be soldered into glass. Fig. 50 shows this modification. A piece of tubing *a*, about an inch and a half long and half an inch in internal diameter, is united at each extremity to a piece of quill tubing *e*, *e*; the wires *d*, *d* are then soldered through its sides. A portion of the wide tube is ground away as at *b*, leaving an opening to which the quartz plate *c* can be applied, and kept in its place by small rings of caoutchouc. The gas was transmitted through the tube as in the other form of apparatus.

60. In one or other of these modes the following gases were submitted to experiment:—hydrogen, carbonic acid, carbonic oxide, olefiant gas, marsh-gas, cyanogen, sulphurous acid, sulphuretted hydrogen, ammonia, protoxide of nitrogen, nitrogen, oxygen, chlorine, and hydrochloric acid.

The general results of these experiments on the invisible rays are in harmony with those already obtained for the visible ones by MM. Ångström *, Alter †, and Plücker ‡. The conclusions at which I have arrived may be thus summed up:—

1. Each gas tinges the spark of a characteristic colour; but no judgment can be formed from this colour of the kind of spectrum which the gas will furnish.

2. In most cases, in addition to the lines peculiar to the metal used as electrodes, new and special lines characteristic of the gas, if elementary, or of its constituents, if compound, are produced. When compound gases are employed, the special lines produced are not due to the compound as a whole, but to its constituents.

3. The lines due to the gaseous medium are continuous, not interrupted or broken into dots.

61. *Hydrogen.*—The spectrum of the spark taken in this gas is not characterized by any new lines. The most remarkable effect is the disappearance of the atmospheric lines, together with the great lowering of the photographic intensity, whether the metal employed be platinum, gold, silver, copper, iron, or zinc. It is interesting to observe that the characteristic lines of highly oxidizable metals, such as iron and zinc, are visible in hydrogen, though the impression on the plate throughout is very greatly reduced in intensity

62. *Carbonic Acid* and *Carbonic Oxide.*—The lines contained in the spectra of these two gases are identical; new lines characteristic of carbon occur in addition to the lines due to the nature of the metallic electrodes. The same lines are visible when other compounds of carbon, such as olefiant gas, marsh-gas, and cyanogen, are employed. The

* Poggendorff's Annalen, 1855, Bd. xciv. S. 141.

† Silliman's Journal, 1855, vol. xix. p. 213.

‡ Poggendorff's Annalen, 1859, Bd. cvii. S. 497.

most characteristic lines in the spectrum of carbon are the following:—At 123 a strong line, a weaker one at 127, two strong compound lines at 138 and 140, and an intense compound line at 153.

With carbonic acid the special spectrum of silver appears much intensified. Some of the lines which appeared as dots in air are continued across the spectrum in carbonic acid. In carbonic oxide the intensity of the spectrum is less than in air; and this contrast between the two gases may be observed whatever be the nature of the metallic electrodes.

Fig. 35 exhibits the spectrum obtained between platinum points in carbonic acid. Fig. 36 shows the spectrum from gold points in carbonic oxide. Unfortunately in the Plate the figure is given a little too much to the right of its true position for accurate comparison with the spectrum above it; but the principal lines will at once be recognized as coinciding, if allowance be made for this displacement.

63. *Olefiant Gas.*—Some difficulty is experienced in observing the spectrum of this gas, owing to the copious deposition of carbon which occurs immediately that a current of sparks is transmitted. The nature of the electrodes employed seems to exert considerable influence upon this decomposition. It is extremely intense when aluminum electrodes are used, but comparatively slight with gold. Observations made when gold electrodes were employed exhibited a spectrum which could not be distinguished from that of carbonic acid or of carbonic oxide.

64. *Marsh-gas.*—Sparks pass freely in this gas. The spectra obtained with gold and copper electrodes cannot be distinguished from those of the same metals in carbonic acid and carbonic oxide. A scanty separation of carbon occurs during the passage of the spark. This is particularly evident when copper electrodes are used, the bluish light of the metallic spark being frequently accompanied by reddish yellow scintillations: the deposition of finely divided carbon upon the quartz plate on the side of the gas-tube impairs the intensity of the photograph.

65. *Cyanogen.*—A difficulty was experienced in this case also in obtaining intense photographs, particularly when silver electrodes were employed; a rapid deposition of a brown matter, probably paracyanogen, took place upon the interior of the tube. When copper electrodes were used, the light of the spark was sometimes of an intense green, at others of a pale blue. The photograph showed the particular lines due to carbon as well as those of nitrogen, and the special lines due to the metallic electrode employed.

66. *Sulphurous Acid.*—This gas offers unusual resistance to the passage of the electric sparks, the electrodes requiring to be brought very close to each other before the disruptive discharge passed freely. This difference in the power of different gases to modify the striking-distance has already been examined by Dr. FARADAY*. A strong spectrum was obtained with gold wires: it terminated abruptly at 113·5, a single spot of renewed action appearing at 143·0. This result is due no doubt to the absorbent action of the gas, which has been already shown in a former section of this paper (18) to be one of the least diactinic of gaseous bodies. In this form of experiment the stratum

* Philosophical Transactions, 1838, p. 103.

of gas traversed by the rays before they entered the air amounted to about half an inch in thickness.

67. *Sulphuretted Hydrogen.*—This gas also offers considerable resistance to the passage of the electric spark. It is decomposed by the spark with deposition of sulphur. When gold electrodes are used, it furnishes lines resembling those of the same metal in air. With silver electrodes the gas was decomposed very rapidly, and no lines were produced beyond 113·5, the absorbent action of the gas being strongly manifested.

68. *Ammonia.*—Sparks pass in this gas as freely as in air; the spectrum of each metal is the same as in nitrogen; no new lines are visible in the photograph. Most of the atmospheric lines are distinct.

69. *Protoxide of Nitrogen.*—The electric sparks pass in this gas with much greater difficulty than in air. The spectrum appears to be the same as that produced in air, and no new lines are apparent in the photograph.

70. *Nitrogen.*—The spectrum of this gas, when gold or platinum electrodes are used, commences with a pale continuous spectrum, which slowly diminishes in intensity; this continuous spectrum appears to increase in intensity with the volatility of the metal, being well-marked in the case of magnesium, sodium, and potassium: at about 151 it terminates abruptly. The spectrum of nitrogen is crossed between 96·5 and 100 by two strong double lines; it shows an indistinct line at 108·5, a strong one at 110·5, three feeble lines at 113·5, 118, and 122, a faint band at 138, and another at 150. Fig. 33 shows the lines obtained from platinum points in nitrogen.

71. *Oxygen.*—This gas was obtained from black oxide of manganese heated with sulphuric acid. It gave, after purification by passing through a solution of caustic soda, lines identical with many of those obtained in atmospheric air. When the gas contained traces of carbonic acid, the lines due to the compounds of carbon were distinctly visible in the impressed spectrum. With platinum electrodes and with pure oxygen a feeble, nearly continuous spectrum extends to about 122·5; it also contains numerous lines extending as far as 142·5; beyond that, the impression is more feeble, terminating at about 156. The principal lines due to the gas are the following:—A broad line about 100, then two faint lines, beyond which, at 101·5, is a double line; a strong complex group at 103·5; a feebler one at 105·5; one rather stronger at 107·5; a double group of considerable strength at 112; another stronger at 114; between 116 and 119 is a group of six rather faint lines; after this there are no prominent lines until 138·2 and 141·5; at 153 is a strong compound line, and beyond this only the dotted lines of platinum are seen. (Fig. 32.)

72. *Chlorine* and *Hydrochloric Acid* give spectra which can scarcely be distinguished one from the other. With platinum points these spectra terminate by an abrupt band at 156·5. Their most marked features are a strong compound band at 96·5, and one still more marked at 100; then two lines, of which the first at 103 is the stronger, followed by two others, of which the second at 108·5 is the stronger: several fainter lines follow these lines. A group of six between 126 and 133, the most marked of which is a broad

band terminating at about 130. Several faint lines intervene between this and another broad band at 140, followed by several others less distinctly defined. (Fig. 34.)

I attempted to obtain the spectra of iodine and bromine by employing a current of hydriodic and hydrobromic acid; but the results were not satisfactory. It is very difficult to maintain a steady current of sparks through these gases, and not easy to keep up a continuous current of the pure and dry gases, which are immediately decomposed by the passage of the electric spark, with extrication of dense fumes of iodine or of bromine.

XXXV. *Further Observations on the Distribution of Nerves to the Elementary Fibres of Striped Muscle.*

By LIONEL S. BEALE, *M.B., F.R.S., Professor of Physiology and of General and Morbid Anatomy in King's College, London; Physician to King's College Hospital.*

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THERE are two views with regard to the peripheral distribution of nerves, which are quite incompatible with each other. According to one, it is supposed that nerves end in the tissues to which they are distributed, by free extremities; while the other doctrine is, that free ends or extremities do not exist, and it is supposed that the fibre, after perhaps a very long and circuitous course, is connected with the nervous centre from which it emanated. If this be so, every nerve-cell, central and peripheral, must have at least two fibres proceeding from it, must be bipolar or multipolar. Both these views cannot possibly be true.

That nerves terminate in ends or free extremities is the opinion of many Continental observers, and of late years this view, it may almost be said, has been generally received by anatomists.

The loops and plexuses described by the older observers have since been proved not to be *terminal.* Some of the fibres forming what appears to be a loop, have been seen to divide into finer branches, and these, after having been traced for some distance, become very thin, and are gradually lost. This change occurs abruptly in some situations, but in others a more gradual attenuation is observed.

In some tissues pale fibres have been traced a short distance beyond what appears to be the end of the dark-bordered fibres. But, it will be shown, the dark-bordered fibres differ so very much in diameter, that some of the thinnest are so very fine as scarcely to be distinguishable with very high powers, and in parts they very closely resemble some of the so-called pale fibres. In young animals the fibres corresponding to the dark-bordered fibres closely resemble the pale fibres of fully formed animals. In no tissue has it been more confidently stated that nerves terminate in free ends than in voluntary muscle; and although some differences of opinion exist as to the precise manner in which the nerves end, the existence of these ends is supported by the testimony of so many observers, that it is now almost regarded as an anatomical fact which is settled beyond question.

Yet it may be said that, as plexuses, loops, and networks * of very fine fibres in young

* In using the term network, I do not mean to imply that fine nerve-fibres unite with each other after the manner of capillaries, but merely that the *bundles* of fibres are arranged like networks. The fibres composing the bundles do not anastomose. In lace the appearance of a network of fibres is produced; but

tissues are represented by networks and plexuses of coarser fibres in adult tissues, and as plexuses of coarse and fine fibres may be seen in the adult tissues at all periods of life, it is only reasonable to assume the existence of networks composed of fibres so very fine as scarcely to be visible by the highest powers we possess, and so delicate as to be destroyed by the most careful manipulation, or to become disintegrated almost immediately after the death of the animal. All argument, however, must give way to observation; and even if the results of observation are incompatible with conclusions previously regarded as true, the new facts must be received, and our inferences must be modified accordingly. But, on the other hand, it will be admitted, in an anatomical inquiry like the present, that there may be the greatest difference of opinion as to the actual demonstration. The *ends* which one observer may consider to be natural terminations, by another may be held to be merely *apparent ends*, depending upon our being unable to see the continuation of the fibre in consequence of its extreme tenuity. The importance which all observers attach to the process of preparation they follow, is alone sufficient to show that the appearances produced are very different according to the manner in which a specimen is examined; and further, it is quite certain that many points to be readily demonstrated by one process, are quite invisible when another plan is followed. This is one explanation of the many conflicting statements as to facts of observation.

In a paper published in the Philosophical Transactions for 1860, I was led to conclude that many of the so-called connective-tissue corpuscles observed upon the muscular fibres were connected by very delicate granular fibres nearly as wide as the corpuscles, but only to be seen in specimens preserved in fluids which refracted very highly; and I also satisfied myself that these fibres were continuous with the nerve-fibres; and I stated that the nerve-fibres ramified with the capillaries external to the sarcolemma. My conclusions were terribly at variance with those of other observers; for not only was I compelled to infer (1) that the nerve-fibres formed, as it were, a network over the muscular fibres, but (2) that every muscular fibre was supplied with nerves throughout its entire length. It is usually maintained, 1st, that nerve-fibres terminate in free ends on the muscle; 2nd, that nerve-fibres only come into contact with the muscular fibre at very distant points: so that while the fibre, or the entire muscle, is freely supplied with nerves at one situation, the greater part is altogether destitute of nervous supply.

The Views of KÜHNE *and* KÖLLIKER.

Since my paper was published, an elaborate memoir on the 'Peripheral Organs at the ends of the Motor Nerves' has been written by KÜHNE; and the same inquiry formed the subject of the Croonian Lecture delivered a month since by Professor KÖLLIKER.

The conclusions of both observers are quite opposed to my own. KÜHNE and KÖLLIKER

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every apparent thread is composed of several, each of which pursues a complicated course, and forms but very small portion of the boundary of any one single space. In Plate XLI. fig. 5 a nervous network exists but each cord is compound, and composed of numerous fibres which never anastomose.

agree as to the existence of free ends. The former maintains that these are situated beneath the sarcolemma, and are connected with peculiar organs; the latter, that the free ends lie upon the surface of the sarcolemma. KÖLLIKER has failed to demonstrate the special organs described by KÜHNE. Both observers agree that the muscular fibre receives but a limited supply of nerves, and that the supply is limited to one part of the muscle.

As these observers have worked upon the muscles of the frog, I shall restrict myself entirely to the consideration of the distribution of the nerves to the muscles of this animal, and mainly to the particular muscle which these and most German observers have studied, viz. the thin pectoral muscle.

In many of KÜHNE's drawings a pale band as broad as (or broader than) the dark-bordered fibre is represented in continuity with it. The outlines of this band are irregular; and in some places it is much wider than in others. This band is five or six times as wide as many fine nerve-fibres which I have seen; and it will be shown that a fibre of the width of these pale fibres in the tissues of the frog generally, is a long distance from its finest divisions. The same observer makes the nerves of insects terminate in a collection of granular matter within the sarcolemma.

1. *The Distribution of Dark-bordered Fibres not so limited as generally supposed.*

With regard to the distribution of the dark-bordered fibres to voluntary muscle, I would remark

That, although some muscles appear to be very sparingly supplied with dark-bordered nerves, the fibres of certain muscles (for example, the inferior muscle of the eye of the frog) are crossed by bundles composed of two or more fibres at intervals of about the fiftieth of an inch *. The distribution of nerve-fibres to the pectoral muscle is by no means so limited as would appear from cursory examination. It is clear that many branches, resulting from the division and subdivision of some of the large dark-bordered fibres, supply the elementary fibres near the centre of the muscle; and to some muscular fibres four or five of the branches resulting from the subdivision appear to be distributed. These branches divide on the surface of the elementary muscular fibres, but their subdivisions gradually become too fine to be followed. Some branches divide and subdivide into very fine fibres, which are lost amongst the connective tissue upon and between the muscular fibres. But it must not be supposed that all the nerve-fibres of the bundle are arranged thus; for many pursue a much longer course. Bundles composed of two or three dark-bordered fibres leave the large trunk and pursue a long course obliquely across the surface of the muscle. The fibres of each trunk divide and subdivide; and many of the branches which result reach almost to the extremities of some of the elementary muscular fibres. Many pass between the muscular fibres, and, after pursuing a

* The muscular fibres of the diaphragm of the mouse are crossed by bundles of dark-bordered fibres at short intervals along their whole length, and nerve-fibres may be seen at the points where they are inserted into the central tendon.

very long course, gradually become finer by division, and at last terminate in a lax network of fine fibres which pass in and out between the muscular fibres; while some of the subdivisions are distributed to the elementary muscular fibres, like the set of fibres first alluded to. It appears, therefore, that in various parts of the muscle, not merely near its centre, but also on some of the fibres near their insertions, branches of dark-bordered fibres can be seen to become suddenly attenuated and give off very fine fibres which divide and subdivide on the surface of the sarcolemma, becoming gradually finer until lost, or they leave the surface of the elementary fibre and are lost in the intermuscular connective tissue—and other fibres, which gradually become attenuated and, after very numerous divisions, terminate in a network of fine fibres which may be traced for a long distance amongst the fibres of connective tissue, between or upon the elementary fibres. Many of the bundles of dark-bordered fibres above alluded to can only be detected in well-prepared specimens which have been soaked for some time in glycerine, so that the muscular fibres may be separated from each other without rupture of the nerve-fibres or capillaries.

Again, numerous branches of nerve-fibres, composed of two, three, or four dark-bordered fibres, may be seen ramifying and dividing near the tendinous extremities of the muscular fibres of the mylo-hyoid muscle of the frog. Several branches can be followed from the central tendon as far as the central part of the fibres; and often very fine fibres can be seen passing amongst the fibres for some distance and forming networks or plexuses.

But there is another fact against the view that the nerves are distributed to only a very small portion of the elementary muscular fibres of the frog. In certain parts of the pectoral muscle new muscular fibres are being developed, as KÖLLIKER has mentioned, and here in a comparatively small space may be seen elementary fibres varying very much in length and diameter. The thickest of these narrow fibres, according to my measurements, are often not more than the $\frac{1}{2000}$th of an inch in width, and the thinnest not more than the $\frac{1}{10,000}$th of an inch in diameter: some are as long as the other muscular fibres; but often single spindle-shaped fibres with two or three nuclei, or more elongated fibres the nuclei of which have divided so as to form a row of perhaps ten or more, may be found. A large nerve-fibre, generally composed of two large dark-bordered nerve-fibres in the same sheath, with finer fibres, can be followed to the oval swelling situated amongst the ordinary muscular fibres; and one or two dark-bordered fibres *leave* the swelling to be distributed on more distant muscular fibres. The dark-bordered fibres divide, and their branches pursue a very tortuous course in the oval swelling, so that one cannot be followed for any great distance; but fine fibres may be seen passing in either direction, and running parallel with the thin growing muscular fibres. Not unfrequently I have seen these nerve-fibres become so fine that they cannot be followed far; but their course can be traced by their nuclei for a considerable distance. In many instances branches of other nerve-fibres pass to these narrow muscular fibres at a point between the oval swelling and the points of insertion of the fibres. The oval swelling itself consists of very young muscular fibres in various stages of development, and nerve-fibres. My observations therefore agree in the main with those of KÖLLIKER, who has stated that at these oval swellings the development of new muscular fibres takes place in the fully formed muscle. Upon comparing the above description of these very interesting bodies ('nerve-tufts') with that of Prof. KÖLLIKER, some most important differences will be noticed. After describing the very fine transversely striated muscular fibres which exhibit the 'peculiar swellings with the coiled nerve,' he says, "in the situation of the swelling the finer component fibres of the bundle cling fast together, even after the operation of the alkali (strong solution of potash); and a certain amount of striated granular uniting matter is found between them, which may be partly the remains of fine nerve-fibres and capillaries, and some accompanying connective tissue." Prof. KÖLLIKER next states it as his opinion that the fine muscular fibres are generated by the division of thicker muscular fibres, and suggests that "the nerve-tufts arise from a simultaneous growth and division of the nerve-fibre belonging to the parent muscular fibre, in order that each of the young muscular fibres may obtain its branch of nerve." In my specimens there is no evidence of the very fine muscular fibres being formed by the splitting of a coarser fibre. On the contrary, every fine fibre seems to be developed from nuclei, and to grow in two directions from the oval swellings. With these developing muscular fibres are young nerve-fibres; and the latter seem to be carried, as it were, upon the muscular fibres as their length increases. The oval swelling exhibits a most beautiful and elaborate structure, which is not shown in Prof. KÖLLIKER'S drawings *. The minute anatomy of these 'nerve-tufts' will form the subject of further investigation. Not only do dark-bordered nerve-fibres proceed from the oval swelling and pass to their distribution to fibres at a distant part of the muscle, but fine fibres may be followed in some cases almost throughout the entire length of those very fine muscular fibres, many of which are often equal to the ordinary fibres in length. These very fine nerve-fibres divide into branches as they pass parallel to the muscular fibre; and transverse branches can be seen crossing the fibre at short intervals (Plate XLIV. fig. 30).

From these observations I think it probable that the muscular fibre is supplied with nerves in its entire length, and that the branches form a lax network of very fine fibres on the surface outside the sarcolemma. As the muscular fibre increases in size, the finest branches of the nerve are more difficult to demonstrate; and the accumulation of connective tissue external to them increases the difficulty of our seeing these very fine delicate fibres; for it is certain that many active nerve-fibres are much finer than is generally supposed, certainly less than $\frac{1}{100,000}$th of an inch in diameter.

2. *The fine Fibres extend further than* KÜHNE *and* KÖLLIKER *have traced them.*

With reference to the free terminations of KÜHNE, KÖLLIKER and others, it may be remarked,—

1. That as it is admitted that the nerve-fibre undoubtedly extends beyond the last

* Proceedings, No. 50, fig. 4, p. 78.

nucleus (Plate XLI. fig. 1, after KÜHNE, figs. 2, 3 *a*, *b*), it is possible, and indeed probable, that it is continued onwards for a greater distance than it has been followed in KÜHNE'S specimens. As a rule, it becomes too delicate and transparent to be traced further than has been represented in my drawings (figs. 14, 16, 18, 19); but I submit that the evidence adduced in favour of these ends being *actual terminations* of the nerve-fibres is very far from conclusive. Capillaries, when stretched and torn, give rise to a very similar appearance; and it is not uncommon to find an undoubted capillary vessel which gradually becomes fainter and fainter *until every appearance of it is all but lost*, and this even when the tube remains quite perfect. I have proved this in the case of capillaries, distributed to muscle, which had been injected with prussian blue injection, and so stretched that the injection was removed from the tube for a certain distance. In this space the vessel could not be recognized, and all that could be seen was a faint granular appearance. We therefore cannot conclude that nerve-fibre or capillary ceases, merely because we cannot follow it in certain specimens beyond a certain point. It may exist but be quite invisible.

2. In various tissues to which nerves are abundantly distributed, the appearance represented in fig. 3 *a* is common. This looks as if the fibre extended from the nucleus in opposite directions; while the common appearance, represented in fig. 3 *c*, in which two fine fibres are seen running together, is strongly in favour of the view that very fine nerve-fibres run for a considerable distance beyond the point at which the dark-bordered fibre can be demonstrated, without terminating in ends. It will be observed that the fine fibres represented in my figures are not one-sixth of the diameter of KÜHNE'S terminal fibres; and I have seen many fibres much finer than those represented in fig. 3.

3. The finest nerve-fibres visible by the highest powers can often be proved to divide into two fibres, by tracing them to the point where they pass into a fibre running at right angles (fig. 4 *a a*).

4. I have some preparations of the bladder of the frog, in which very distinct networks composed of very fine nerve-fibres are readily seen. One of these, magnified by a power of 1700 diameters, is shown in fig. 6. This, it might be remarked, is not completely conclusive in favour of nerves forming circuits; for it is possible that fibres may diverge from the bundle with which they run and end in free extremities or in small cells, as represented in fig. 5. Numerous observations are against this view; and I regard certain nuclei from which two or three fibres pass in different directions (figs. 2, 3, *a*, *b*) as the peripheral and active portion of the nerve-fibre in all cases. Even in the case of the finest fibres, when a nerve joins another branch running at right angles to it, fibres are given off which pursue opposite directions (apparently pursuing opposite courses), towards the centre, and towards the periphery.

5. With regard to the size of the finest nerve-fibres, I have traced fibres from true dark-bordered nerves which are certainly less than the $\frac{1}{50,000}$th of an inch in diameter, and these are probably composed of more than two fibres. Still finer fibres of this description have been traced to and from ganglion-cells. The connexion between such very fine

fibres and definite dark-bordered fibres has been demonstrated in many situations (tongue, heart, peritoneum, muscular fibre, and very clearly indeed in the bladder of the frog). I have convinced myself that in many of the ganglion-cells of the frog (abdominal cavity, ganglion on posterior roots of the nerves, heart, bladder) one fibre springs from the central part of the cell, while other fibres are in connexion with its circumference. The latter fibres very frequently are arranged spirally around the central fibre. As already stated, many of the finest fibres are less than the $\frac{1}{100,000}$th of an inch in diameter.

It will be said that these fine fibres are composed of connective tissue; but I shall show that the fibres resulting from the subdivision of those which are the direct continuations of the dark-bordered nerve-fibres exhibit the same characters, and can be distinguished from the fine fibres of yellow elastic tissue. Very many fine fibres which are undoubted nerve-fibres, and which form plexuses and networks, have altogether escaped observation, and their nuclei have hitherto been considered as belonging to the connective tissue. The fine nerve-fibres I have described cannot be seen in specimens immersed in water, or in solutions which refract like water.

The Division and Subdivision of the Dark-bordered Fibres.

The numerous divisions and subdivisions of the dark-bordered nerve-fibres have been described and figured by various observers; but it is necessary to give new drawings of some of these fibres, as their arrangement has not been accurately delineated, and the presence of *very fine nerve-fibres in the sheath* external to the white substance has not been admitted. Fig. 11, Plate XLII. shows that the dark-bordered fibres, even in the trunks, may become so thin as scarcely to be visible as distinct fibres by the $\frac{1}{12}$th of an inch object-glass. One of the fibres resulting from the division of *a* may be traced towards the right, and at *d* becomes so thin that it might be easily overlooked, and if recognized would be regarded by most authorities as a fibre of connective tissue in the sheath of the nerve.

Often a fibre resulting from the subdivision of a dark-bordered fibre is seen to pass towards another dark-bordered fibre, and run parallel with it for some distance. This arrangement, which exists in the larger fibres, and which is figured in my last paper, is also observed in the finer ramifications. The terminal dark-bordered fibre very gradually, or more or less abruptly, becomes finer, and divides into two fine fibres; but there is no *termination* or *cessation* of the white substance, as described by KÜHNE, who states that the *white substance* ceases at the sarcolemma, while the *axis cylinder* only is continued onwards as a pale fibre beneath the sarcolemma. KÖLLIKER states that, "besides the axis cylinder, they are furnished with a prolongation of the membranous sheath; indeed I have seen this so clearly in a great many favourable instances, that I can have no doubt on the point. . . . . . But whilst it is easy in most cases to perceive the membranous sheath and its enclosed matter distinct from each other at the commencement of the pale fibres, yet in their further progress these structures coalesce together, and the terminal fibres then appear as uniform pale filaments."

My specimens lead me to the conclusion that near the periphery the axis cylinder of a nerve-fibre cannot be distinguished as a separate structure from the white substance. The axis cylinder and the white substance seem to be, so to say, incorporated together to form the terminal part of the dark-bordered fibre. The dark-bordered nerve-fibre gradually becomes narrower as we approach its ultimate distribution, and it is resolved into several fine fibres; but even in threads which are so fine as only to be scarcely visible the material appears to be of the same composition as the thicker parts of the fibre, and by the slow action of acetic acid globules of fatty matter are set free (Plate XLI. figs. 7, 8, & 9). I cannot but think that the very abrupt cessation, or sudden breaking off, of the white substance indicated in some of KÜHNE's drawings is due to the pressure of the thin glass upon the surface of the specimen, which renders comparatively large dark-bordered fibres quite invisible at the point where they cross the convexity of the muscular fibre.

The white substance often diminishes in diameter rather abruptly, but it never ceases in the manner KÜHNE has represented*; and I have never seen any appearances which lead me to conclude that the axis cylinder is prolonged independently of the white substance; nor do I think that "here and there at least a thin *layer* of the white substance extends along the pale fibre," as KÖLLIKER remarks.

The ultimate Distribution of the Nerves in the Cutaneous Muscle from the Breast of the Frog.

Let me now describe more particularly the results of my own observations on the ultimate distribution of the nerves to the muscular fibres of the frog. The fibres resulting from the division of the dark-bordered fibres, which to low powers appear terminal, are often seen to pass over the surface of a muscular fibre, and then become either suddenly attenuated, or gradually thinner as they cross in succession several elementary fibres of the muscle, or pursue a longitudinal course in the intervals between two fibres. In both cases they often give off several branches. In the drawings of KÜHNE the outline of the pale fibres is irregular, and their precise connexion with the dark-bordered fibre is not clearly indicated. In Plates XLII. & XLIII. figs. 11, 12, 13, 14, 15, & 17, it is seen that the dark-bordered fibre divides into two branches, and each branch may be followed for some distance as a definite but very narrow line (less than $\frac{1}{30,000}$th of an inch in diameter). This appearance is constant, and exists in every 'terminal' fibre. Nuclei are connected with these fine 'terminal' fibres, which are but the extension of the dark-bordered fibres; and nuclei are also seen at intervals imbedded in the substance of the highly refracting matter of which the dark-bordered fibres are themselves composed. These nuclei are always to be found imbedded in the white substance near the finer divisions of the dark-bordered fibres (Plates XLI.–XLIV. figs. 8, 9, 11, 12, 14, 18, & 31). Around the fine fibres prolonged from the dark-bordered fibre, may be traced for some distance what appears to be the *outline of the sheath* (Plates XLII. & XLIII. figs. 13 & 16);

* The thickness of the dark-bordered fibre just before it divides, as for example in the case of a fibre in fig. 13, Plate XLII., is often increased by the shortening caused by the *contracted* state of the muscular fibre at the time of preparation.

but these lines may often be seen to divide and subdivide into several very fine fibres, which may run with the prolongation of the dark-bordered fibre, or at once diverge from it (Plate XLIII. figs. 17, 18, & 19; Plate XLIV. fig. 31).

Not unfrequently very fine fibres can be traced *external* to what appears to be the outline of the sheath of the nerve, and these form, with the fine fibres already described, a 'pale fibre.' The band or pale fibre divides into smaller bundles, and the fibres composing them gradually become so fine as to be invisible. Nuclei exist in connexion with the finer fibres which I have described, as well as in connexion with the prolongations of the dark-bordered fibres (Plate XLII. figs. 13, 14, & 15). Here, as elsewhere, some fibres appear to be connected with the nucleus, while others pass round it on one side (Plate XLI. figs. 2, 3 *c*). Sometimes fibres pass on both sides of the nucleus. The pale bundles of very fine fibres often give off branches, and these again divide into finer bundles, producing the appearance of a network (Plates XLII. & XLIII. figs. 13, 14, 15, 16, & 17), the meshes of which vary much in diameter. Very often several nuclei are seen at the point where the dark-bordered fibre divides. A very confused appearance often results from the number of nuclei and fibres. As already stated, in specimens examined in aqueous fluids which do not refract highly, it is not possible to trace the course of the separate bundles of fibres and their connexion with the nuclei; but in specimens mounted in glycerine, examined with high powers, the bundles of fine fibres can be separated from each other and followed (Plate XLIII. fig. 21). Some appear to come from the sheath, or are external to it; and others result from the division of the dark-bordered fibre (Plates XLII. & XLIII. figs. 15 & 17). The nuclei and fibres can be very readily demonstrated near the trunks of the dark-bordered fibres, and the network can often be seen in the same situation, but it is not easy to trace the course of the delicate fibres over the muscular fibres for any great distance. The large nerve-fibres prevent the finer ones from being ruptured; but at a distance from the coarse fibres, there being nothing to protect them, the very delicate fibres are generally destroyed by the necessary manipulation, or rendered invisible by the pressure of the thin glass. I have, however, seen such a network very distinctly on the side of some muscular fibres, upon which numerous branches from an adjacent nerve could be traced at intervals for the distance of about one-hundredth of an inch. Indications of the existence of such a network (the nuclei being perfectly distinct) are not unfrequently seen in many parts of the same elementary fibre. Nuclei and fine fibres, resembling those which have been shown to be in connexion with nerve-fibres, can often be seen just at the edge of a muscular fibre, and very frequently a true nerve can be traced, running for some distance near a capillary, and giving off branches with nuclei which are gradually lost on the surface of the muscle or amongst the connective tissue. As these branches pursue the course described, they can hardly be vascular nerves; and there are besides them other branches which lie closer to the vessel, which are probably of this nature. The distribution of very fine fibres can sometimes be traced by the course of the nuclei. In certain cases, in which no fibre connecting nuclei can be detected when the specimen is first prepared, a fibre afterwards becomes quite distinct. As I have succeeded in following very fine compound fibres for a considerable distance as they passed amongst several adjacent muscular fibres, I feel satisfied that many apparent ends result from the fibres becoming too delicate to be followed beyond this point. Very slight pressure of the thin glass renders even dark-bordered fibres quite invisible; and very moderate stretching renders undoubted nerve-fibres so very fine that they cannot be seen, or, if they remain visible, all characters which would enable us to identify them as nerve-fibres are completely lost. But even if no such appearances as those I have just referred to existed, it would be more in accordance with what we have learnt from many investigations, to conclude that the fibres really extend further than we are able to follow them, than to assume that they absolutely cease at the point at which, having very gradually become finer and finer, they are no longer visible to us. Moreover, as I have already stated, dark-bordered fibres are to be demonstrated in all parts of the muscular fibre, even upon the tendon near its connexion with the muscular fibres. It is hardly likely that this distribution of coarse fibres over the general surface of the muscles should be associated with a very partial and unequal distribution of the fine fibres which result from their division. As the broad pale fibres described by KÜHNE are really composed of several fine fibres (Plates XLII. & XLIII. figs. 14, 16, 18, & 20), which are often connected by transverse branches so as to form a network, and as the fine compound fibres gradually become finer and finer until they cease to be visible, it is still more improbable that what appeared to him and KÖLLIKER to be ends should be real natural terminations, than if the arrangement had been as KÜHNE has described it. It seems therefore to me that the only inference which can be drawn from the facts demonstrated is, that the nerves terminate in a network: but there are still two modes of arrangement possible with regard to the distribution of this network.

1. It may be confined to one part of the muscular fibre.
2. It may extend down the fibre throughout its entire length.

It is true that the network (Plate XLII. figs. 13, 14) which is so readily demonstrable near the plexus of the dark-bordered fibres very soon becomes so faint as to be lost. The meshes are smaller near the large branches of nerve-fibres than further away, and it is at this point that the rupture of the fibres takes place when the muscle is caused to contract violently soon after death. There is therefore no doubt that the fibres of the pectoral muscle receive a greater supply of nerve-fibres near their central part than nearer their extremities; but it is quite certain that the supply of nerve-fibres is not *limited* to this portion of the muscle.

Although I have never been able to demonstrate uninterruptedly the fibres of such a network over more than the $\frac{1}{100}$th of an inch of a fully-formed fibre of the breast-muscle of the frog, I have succeeded in doing so several times in the case of fibres undergoing development, and about the $\frac{1}{2000}$th of an inch in diameter (Plate XLIV. fig. 30). Upon these I have seen such a network, and I consider that the evidence in favour of the existence of a network extending from one end to the other of the muscular fibre of the voluntary muscle of the frog, though by no means equally distributed over every part of the fibre, is so strong as to justify me in concluding that such an arrangement is actually present. If the nature of the tissue under examination be considered, and the extreme delicacy of the nerve-structure which has actually been demonstrated be fairly taken into account, no surprise will be felt that the network has not been actually demonstrated throughout the entire length of the large elementary muscular fibres of the frog. The network of fine fibres I have described exactly corresponds in its arrangement with networks which are demonstrated with comparative facility in the palate, bladder (Plate XLI. fig. 6), peritoneum, cornea, skin, and other situations in the frog. In the heart and tongue, networks also exist, but the fibres are very fine, and the meshes are wider. The network, composed of very fine fibres, lies on the same plane as the capillary vessels, and just upon the surface of the sarcolemma. The fine nerve-fibres often lie very near to the capillaries.

The capillaries and the fine nerve-fibres with their nuclei are connected together by a membranous tissue so thin and delicate that its existence can only be proved after it has been coloured, or when some foreign particles have adhered to it. That it is, at least in some places, *distinct from the sarcolemma*, is easily proved in the muscles of the frog, in which animal the sarcolemma is very thick, as it can be stripped off leaving the sarcolemma. I have some specimens in which the muscular tissue has been withdrawn from the tubes of the sarcolemma, so that the nerves can be seen very distinctly. From a part of the surface the thin membrane with nerves and capillaries has been withdrawn, leaving the sarcolemma uncovered. In some specimens, on the other hand, the finest nerve-fibres and their nuclei seem in absolute contact with the sarcolemma, and appear to be adherent to or connected with it; and I incline to the opinion that some of the nuclei which appear imbedded in the sarcolemma are connected with the nerve-fibres.

The pale fibres described by Kühne and Kölliker are compound, and are composed of many very fine fibres, and they divide into bundles of fine fibres which form a network. Each 'pale fibre' consists (1) of a very fine fibre prolonged from the dark-bordered fibre, and (2) of very fine fibres continuous with those in the sheath of the nerve. The fibre which is the immediate prolongation of the dark-bordered fibre, as well as the fine fibres amongst which it runs, can be followed to nuclei, and often a lax network can be traced. The dark-bordered fibres are all continued as very fine fibres; and the subdivisions of these, a short distance from the dark-bordered fibre, are too thin to be measured, even when examined by the highest powers made. The subdivisions of all nerve-fibres pass into fibres which have hitherto been included with the fibres of connective tissue.

The same arrangement with regard to the distribution of the nerve-fibres is observed here as elsewhere. The fibres of the network are composed of numerous very fine fibres. Not one individual fibre can be followed for any distance; but certain delicate fibres can be traced from one compound fibre to another, in precisely the same manner as the coarser fibres can be followed in a plexus. In fact, from all that I have observed in examining the finest branches of the nerve-fibres in various tissues, I cannot but conclude that the general disposition of these finest fibres is the same as that of the coarser trunks and fibres. In passing from the trunks towards the ultimate distribution, it might be said that we meet with finer and still finer plexuses—the finest visible fibres being probably composed of more than a single fibre.

The chief differences observed between the arrangement of the nerves in the muscular fibres of the mouse, described by me in the Philosophical Transactions for 1860, and those of the frog, are, that the fibres are much finer in the latter than in the former, and the fibres themselves, and the nuclei in connexion with them, are much more abundant and closer together in the mouse than in the frog. The position of the fibres with regard to the sarcolemma and the muscular tissue, and their relation to the capillaries, is the same in both.

I now pass on to the consideration of the fine fibres connected with the nerves, and often seen running in the sheath with the dark-bordered fibres, which have already been alluded to in this paper. These fibres have been regarded as fibres of connective tissue, but they have not been carefully studied. The evidence in favour of these being real nerve-fibres is as strong as that upon which the nervous character of the fine fibres, which are the direct prolongations of the dark-bordered fibres, rests.

Fine Nerve-fibres imbedded in the Sheath.

The following observations are of the greatest importance with regard to the general question of the mode of distribution of nerve-fibres.

As a dark-bordered nerve crosses the muscular fibres, it often gives off several very fine branches. In Plate XLIII. fig. 19 a nerve is represented from which two branches are seen to proceed and divide upon the surface of the same muscular fibre, and other branches are given off a short distance from the point at which these branches leave the trunks. These fibres are not connected with *those portions of the dark-bordered fibres seen in the figure*, although their arrangement exactly corresponds with the fine fibres prolonged from the dark-bordered fibres. Compare the fibres marked *b*, *b* in fig. 19 with *b* in figs. 13 & 14. These fine nerve-fibres are very numerous. They often run parallel with the dark-bordered fibres for some distance, perhaps cross them several times, divide and give off fibres which pass off in different directions. The line which appears to be the outline of the tubular membrane of nerve-fibres near their periphery, is frequently found to consist of two or more very fine fibres, which are in many instances connected with nuclei, which, it is generally believed, are the nuclei of the tubular membrane. I have specimens in which five or more very fine fibres can be demonstrated between the dark-bordered fibre and the outline of the tubular membrane (Plate XLI. *et seq.*, figs. 7, 8, 9, *a*, 11, 15, 18, 21, 24, 25, 26, 27, 28, 29, & 31). These branches leave the trunk at intervals and pass to their distribution, or pursue their course with other fibres. Usually two or more pass off at the same point; and in the rare instances in which one fibre passes by itself to its distribution, one can never feel confident that it is really single. In those fortunate specimens in which it can be followed for a short distance, it can be seen to divide into two branches which pursue opposite courses; and very often at the point of division a third fibre can be seen.

The fine fibres I have described are very numerous near the point of distribution of the nerves (Plate XLIV. figs. 27 & 28), and are to be seen in relation with almost all the dark-bordered fibres, distributed not to muscle only, but to the tissues of the frog generally (palate, peritoneum, intestines, skin, heart, tongue, liver, kidney, &c.). I have also seen fine fibres corresponding to these in the mouse, and believe them to exist in vertebrate animals generally; but being so very delicate, they are more difficult to demonstrate in mammals than in the frog. These fine fibres distributed on the muscles of the frog cannot, by any peculiarity in their appearance, be distinguished from the fine fibres which are in direct continuity with the dark-bordered fibres. It has been shown that in many cases, especially in the muscles of the eye, and in those of young frogs generally, and in the palate of the frog at all ages, dark-bordered fibres can be traced, gradually becoming finer and finer until the fibres are too thin to be followed. *Some* of these fine fibres running in the sheath of the nerve, result from the division of the dark-bordered fibres (Plate XLII. fig. 11 *d*), but it is doubtful if all have this origin. At the same time there are many appearances I have seen in various peripheral nerves which are much in favour of this view. I shall be able to express myself more confidently on this question when the researches upon which I am engaged are completed. A great number of these fine fibres with nuclei, running in the same sheath with dark-bordered fibres, are represented in Plate XLIV. fig. 28, from the bladder of the frog; but they are much more numerous in this organ than in connexion with the dark-bordered fibres distributed to striped muscle. I have seen these fine fibres divide in the same direction, that is *from* centre *towards* periphery, as the coarse fibres; so that in the trunk of the nerve there are dark-bordered fibres which divide as they pass towards the periphery, and finer fibres which divide as they pass in the same direction. In the bladder of the frog there are large bundles of fine fibres passing in the same sheath with the ordinary dark-bordered fibres, as well as numerous trunks composed of fine fibres alone; but in this organ there are many ganglia connected with these fine fibres; so that it is doubtful if the fine fibres I have described in connexion with the muscular nerves are of the same character as the fine fibres in the bladder of the frog. The bundles of fine fibres ramifying in the same sheath with the dark-bordered fibres distributed to the bladder have certainly no connexion with the dark-bordered fibre; but, on the other hand, it is certain that some of the fibres we have been considering are continuous with dark-bordered fibres.

It has been assumed by many observers that a fibre which exhibits the refractive power peculiar to the white substance can alone be regarded as a true nerve; and the point beyond which the white substance cannot be traced has been looked upon as the termination of the nerve. Although pale fibres have been recognized in certain situations, and their continuity with dark-bordered fibres traced, still the general idea seems to be that all true nerves exhibit the dark contours. The pale fibres described by Kühne as lying beneath the sarcolemma, and by Kölliker upon this membrane, are regarded as the terminations of dark-bordered fibres with which they are continuous. These pale fibres, which are composed of many fine fibres and have been traced so short a distance from the dark-bordered fibres in certain instances, are, in my opinion, *only* *the commencement of that portion of the nerve-fibre which influences the muscle* and corresponds with the *fine fibres composing the network in the bladder* of the frog (Plate XLI. fig. 6). From what I have seen of the distribution of nerves to tissues generally, I feel convinced that the really important part of the peripheral nerve-fibres, by which the various tissues under the control of the nervous system are influenced, really only commences at the point where the white substance seems to cease. Beyond this point, and continuous with the dark-bordered fibres, there is a most extensive system of fine fibres with which nuclei are connected. The fibres are compound, and the arrangement of the bands is such as to produce the appearance of a network, in the meshes of which the active elements of the tissue are situated.

The relation of Nerve-fibres to the Connective Tissue of Muscle.

I have shown that the network of delicate compound nerve-fibres on the surface of the muscular fibres is continuous with certain fibres and nuclei which *seem* to belong to the connective tissue around the muscular fibres (p. 892). This connective tissue is thicker in old animals than in young ones, and may be regarded as constituted mainly of the remains of a structure which was active at an earlier period of life. These remarks do not in any way apply to the fascia of a muscle, but only to the indefinite connective tissue beneath it, and in close contact with the sarcolemma and between the muscular fibres.

Many of the fibres delineated by KÜHNE in fig. 15, plate 4 of his memoir, and termed by him fibres of connective tissue, are made exactly to resemble the pale nerve-fibres; and their nuclei are of the same size. The fibres of 'connective tissue' (?) which leave the sides of the dark-bordered fibres represented in his drawings do not accord at all with appearances I have seen, which are much more positive. KÜHNE gives a nucleus lying outside the white substance with indefinite fibrous tissue proceeding from it. In my specimens such a nucleus is seen continuous with definite fibres in the 'matrix' external to the white substance (Plates XLIII. & XLIV. figs. 17, 28, & 31). From the nucleus definite fibres can often be traced in opposite directions; and on one side of it three or four fine fibres may often be demonstrated (fig. 31). It is unnecessary here to repeat the various arguments that have been adduced in favour of the view that these fibres are veritable nerve-fibres, and not mere connective tissue. This important question is decided by some preparations I have made showing the distribution of the nerves in the bladder of the frog*; and I am sure it will be admitted, at least by those who have seen

* KÖLLIKER, speaking of the distribution of the nerves, says, "The same mode of termination as in the heart (very fine pointed extremities) prevails in the non-striated muscular tissue of the pharynx and bladder of the frog." I have fully investigated the arrangement of the nerve-fibres in the bladder of the frog, and have given drawings of the appearances I have observed in this paper, and also in No. XII. of the 'Archives of Medicine.' I can follow the finest nucleated fibres for long distances; but they always pass to other fibres, and never terminate in any ends that I can discover. In many situations there is a network with comparatively small meshes, as represented in Plate XLI. fig. 6. Amongst the delicate striped fibres of the auricle of the heart of the frog I have also seen bundles of very fine fibres forming networks with large meshes; but I have never seen ends, and feel sure that free ends do not exist in these tissues.

my specimens, that the appearances represented in the drawings accompanying this paper are incompatible with the conclusion that all the fine fibres I have described are merely fibres of connective tissue.

Since it has been shown that many of the corpuscles in the connective tissue between the muscular fibres, and in other situations which are usually considered to be connective-tissue corpuscles, are really connected with the nerve-fibres, and that the quantity of the connective tissue increases as the muscle advances in age, it remains for those who still maintain that this indefinite connective tissue is a structure developed specially for the support or nutrition of higher tissues, to explain, according to their view, its absence when the tissues are undergoing most active changes, during development, and when they are softest and therefore are in greater need of support, and how and at what period it comes into relation with the nerve-fibres,—why it is more abundant on the muscles of the tongue and diaphragm of the mouse than on the muscles of the limbs,—and many other facts which I have considered elsewhere, and which appear to me to be not only strongly opposed to this view of the origin of this form of connective tissue, but incompatible with it*.

In adult animals the connective tissue is too thick to permit our seeing the arrangement of the very much more delicate nervous structure beneath it; and the difficulty is further increased by the circumstance that the light must first traverse the muscular fibre itself. The relation of the nerves to the sarcolemma remains constant; but the connective tissue exists in greater quantity and is firmer and more fibrous in old than in young muscles, and forms a fibrous stratum external to the nerve-fibres. If we attempt to tear off this connective tissue in order to see the nerve-fibres beneath it more distinctly, we almost invariably tear away the nerves and vessels also; for these structures are undoubtedly connected with the connective tissue.

It seems to me that many of the appearances observed receive at least a partial explanation from the following considerations. The muscle grows, gradually, until it attains perhaps twenty times the size it exhibited in the young frog. That part of the nerve-fibre which in the young animal might be said to be terminal, would in the larger muscle correspond to trunks from which branches were given off in different directions; and fibres and nuclei which were in contact with the muscular fibres would be removed, as the muscle grew, further and further away from the sarcolemma, but they would still be connected with the new fibres and nuclei which are developed just external to this membrane. Thus we should have a quantity of tissue composed of modified nuclei and wasted nerve-fibres which would accumulate as the muscle advanced in growth, and through which many of the nerve-fibres would be seen to pass. Many of these nuclei and fibres having been originally continuous with the nerve-fibres would still retain connexion with them, but this connexion is not necessarily a physiological one.

Other explanations might be offered, but I desire now only to draw attention to the fact that *all* nerve-fibres at their periphery are continuous with exceedingly delicate

* "The Structure of the Simple Tissues," Lectures VI. and VII.

fibres which have hitherto been regarded as belonging to connective tissue, but which differ from very fine yellow elastic fibres in their granular appearance, in their mode of branching and the curves which they form, in refractive power, and in the alteration resulting from the prolonged action of dilute acetic acid*.

It seems quite possible that complete nervous circuits may exist, and the delicate fibres of the plexus may be connected with delicate fibres from the branches of nerve-fibres ramifying in the connective tissue. Although nuclei (connective-tissue corpuscles) appear to be connected with the nerves in great number, it by no means follows that all these are instrumental to the action of the nervous system†. The nerve-currents would of course pursue the shortest route, and as the position of the fibres must become altered in consequence of the development of new ones, some would no longer be traversed by the nerve-current. The nuclei would slowly alter but would still retain a certain anatomical connexion with the fibres, and might still absorb a certain amount of nutrient matter and slowly produce a low form of fibrous tissue.

Many fine bundles of fibres, and fibres which appear to be single, with nuclei connected with them at short intervals, may be seen ramifying in the indefinite connective tissue in all parts of the frog. The fine fibres are generally considered to be fibres of yellow elastic tissue, and the corpuscles are all included under the head of 'connective-tissue corpuscles'. These fibres may be readily followed (especially on the palate, cornea, and bladder) to undoubted nerve-fibres, and it would not be possible to distinguish one of these fibres from the fibre which is the direct continuation of the dark-bordered nerve-fibre in muscle. I am therefore forced to conclude either that nerves terminate in 'connective-tissue fibres,' or that this form of connective tissue itself results from changes occurring in nerve-fibres and the nuclei or cells which take part in their formation. As this form of connective tissue always exists where nerve-fibres are distributed—as it increases in quantity, but at the same time becomes more condensed as the tissue advances in age—as the meshes formed are smaller and less regular as we recede *from* the surface where nerves ramify—as the so-called fibres of yellow elastic tissue in connexion with the nerves become altered like the nerves themselves by the slow action of acetic acid, while other fibres more distant do not undergo this change, and as these facts have been observed, not in one organ or tissue, but in very many (palate, tongue, cornea, mesentery and other parts of the abdominal connective tissue, muscle, skin, pale muscular fibre), as well as in different animals, I venture to conclude that this so-called connective tissue lying immediately beneath sensitive surfaces, and around the elementary fibres of voluntary muscle, and in other situations where nerves are abundantly distributed, is not a tissue specially formed as a medium for con-

* See some observations on the distribution of nerves to the mucous membrane of the epiglottis of man in No. XII. of the 'Archives of Medicine.'

† KÜHNE has recently stated that the nerve-fibres are connected with the connective-tissue corpuscles of the cornea; but I have some specimens which demonstrate that this view is erroneous. The nerve-tissue is always distinct from other tissues, and possesses its own masses of germinal matter or nuclei of formation.

necting or giving support to higher tissues, but results from changes occurring in the nervous tissue itself, and that the fine fibres of elastic tissue in indefinite connective tissue were actual nerve-fibres at an earlier period of life *. Thus a low form of tissue, which results from the natural changes taking place during the life and continual growth of the highest tissue, may form a basis of support for the latter. It becomes firmer as the organ or tissue advances in growth, but it does not exist at an early period of development, nor is it a special tissue developed in a certain definite manner like tendon, true yellow elastic tissue, cartilage, muscle, &c.

Conclusions.

1. In certain muscles of the frog the distribution of dark-bordered nerve-fibres is pretty uniform in every part. Although in the case of the pectoral a greater number of nerve-fibres is distributed to the central part of the muscle, fibres may be traced from the large bundle almost to the extremities of some of the muscular fibres. Many branches which easily escape observation pass between the muscular fibres and their subdivisions and supply neighbouring fibres, or are gradually lost in the connective tissue.

2. Fine nerve-fibres are most easily demonstrated on the external surface of the sarcolemma near the nerve-trunks; but reasons have been advanced in favour of the conclusion that every elementary muscular fibre is more or less freely supplied with nerve-fibres throughout its entire length. Many of these fine nerve-fibres on the surface of the muscular fibre become gradually very faint, until from their extreme tenuity we are no longer able to follow them.

3. Fine nerve-fibres in direct continuation with the dark-bordered fibres, and less than the $\frac{1}{30,000}$th of an inch in diameter, have been seen to divide into finer branches which have nuclei in connexion with them.

4. The pale fibres delineated by KÜHNE and KÖLLIKER, and by them considered single terminal fibres, consist of—

a. Fibres about the $\frac{1}{30,000}$th of an inch in diameter, or less, resulting from the subdivision of the dark-bordered fibre.

b. Fibres resulting from the subdivision of fine nerve-fibres ramifying in the sheath of the dark-bordered fibre, or situated external to it.

5. Nuclei are found in connexion with—

a. The dark-bordered fibre itself, near its terminal ramifications.

b. The fine fibres which are the direct continuations of the dark-bordered fibres.

c. The fine fibres in the sheath, or external to it.

6. The nuclei and delicate fibres above referred to are arranged so as to form networks, the meshes of which vary much in size, situated with the capillaries on the external surface of the sarcolemna. The fibres of these networks are compound, and consist of finer fibres, which are distinct from, and do not anastomose with, each other. The

* "On the Distribution of Nerve-fibres to the Mucous Membrane of the Human Epiglottis," in my 'Archives,' No. XII. 1862.

fine fibres continued from some of the dark-bordered fibres, as well as those ramifying in the sheath of the nerves, may sometimes be followed over six or more elementary muscular fibres, and form, with other fine branches, networks, many of the meshes being as wide as a muscular fibre.

7. Fine nerve-fibres, with nuclei connected with them, exist (not unfrequently to the number of four or five) in the sheath of dark-bordered nerve-fibres near their distribution; and some are also found external to what appears to be the outline of the sheath. Some of these result from the subdivision of a dark-bordered fibre. These fine fibres and their nuclei have been included hitherto under the head of 'connective tissue.'

8. The connective tissue around the elementary muscular fibres, and in connexion with the nerve-fibres, is composed of—

a. Nuclei which might have taken part in the formation of the nerve-fibres, but which have degenerated, and a low form of fibrous tissue has alone been produced.

b. Fibres and nuclei which were once active and formed an integral part of the nervous system, but which have grown old, and have been replaced by new nuclei and fibres.

c. The remains of altered and wasted vessels and nerve-fibres distributed to them, and perhaps wasted muscular fibres themselves.

9. The nerves distributed to the voluntary muscles of the frog do not terminate in free ends, but there is reason for believing that complete nervous circuits exist.

10. In all cases the fibres resulting from the division of the ordinary nerve-fibres are so fine that many cannot be seen with a power magnifying less than a thousand diameters, and there is evidence of the existence of fibres which could only be positively demonstrated by employing a much higher magnifying power. It is probably by very fine fibres alone and their nuclei, that the tissues are influenced. The ordinary nerve-fibres are only the cords which connect this extensive peripheral system (which has been traced in different tissues far beyond the point to which the dark-bordered nerve-fibres can be followed) with the central organs of the nervous system.

11. The facts and conclusions above stated, with reference to the distribution of nerve-fibres to the voluntary muscles of the frog, are in accordance with the arrangement of the finest nerve-fibres demonstrated in many other tissues of the same animal, and agree with many appearances observed by the author in connexion with the peripheral distribution of the nerves, not only in certain tissues of man and the higher animals, but also in invertebrate animals.

12. The distribution of the finest branches of the nerve-fibres can only be demonstrated in tissues which have been immersed in fluids which refract highly, as syrup or glycerine.

EXPLANATION OF THE PLATES.

The figures represented have been copied from specimens magnified with a twelfth (700 diameters linear) and a twenty-sixth (1700 diameters linear), made by Messrs. POWELL and LEALAND. All the figures, with the exception of figs. 1, 2, 3, 4, & 5, have been carefully copied from nature. The relative sizes of the objects represented have been accurately retained. The dimensions of every object can be ascertained by measuring it upon one or other of the following scales; so that in the text I have considered it superfluous to insert the measurements.

Many of the specimens will retain their characters for some time (probably two or three years at least), and can be examined by any one desirous of seeing them. Some of the drawings were traced from my drawings, on the wood blocks, under my immediate direction, others by myself. They were engraved by Miss POWELL.

1000th ⌞______⌟ ×700
1000th ⌞____________________⌟ ×1700

PLATE XLI.

Fig. 1. Terminal fibres beneath the sarcolemma, copied from KÜHNE's paper. The fibre is as broad as the nucleus, appears smooth, and terminates in a pointed extremity.

Fig. 2. Fine fibres with nuclei, as seen in many tissues of the frog, according to the author's observations. The fine fibres delineated are less than the $\frac{1}{50,000}$th of an inch in diameter, and often become so fine that they cannot be followed; but the apparent terminations are not true ends.

Fig. 3 illustrates the manner in which fine nerve-fibres with nuclei ramify at their distribution. *a.* A nucleus with a fibre passing from either extremity; *b*, fine nerve-fibre branching; *c*, two fibres running parallel to each other.

Fig. 4. Fine nerve-fibres dividing at the points *a a*, where they meet another fibre, into two branches, which pursue opposite directions.

Fig. 5. Network of nerve-fibres to show how the finest fibres might possibly terminate in free extremities or in nuclei, consistently with the existence of networks demonstrated by the author.

Fig. 6. Network of nerve-fibres near the inner surface of the bladder of the frog; each fibre is composed of a number of very minute fibres, which do not anastomose with each other. A dark-bordered fibre was continuous with the fibres of which this network is composed, at a point a little below the lowest fibre represented in the drawing. ×1700.

Fig. 7. 'Termination' of a dark-bordered fibre, showing its continuity with fine fibres arranged to form networks as in fig. 6. A fine fibre (*a*) is seen to run parallel with the terminal part of the dark-bordered fibre (*b*). This is also lost among other fibres of the network. The granular appearance of the fibres in this

specimen results from the slow action of acetic acid. This figure, and figures 6, 8, & 9, show the relation of the fine fibres to the nuclei. ×1700.

Fig. 8. Another terminal 'extremity' of a dark-bordered fibre. A similar fibre (*a*) is seen external to the dark-bordered fibre, as in the last and also in the next specimen. Four nuclei are seen, which exhibit their relation to the bundles of fine fibres.

Fig. 9. Another terminal portion of a dark-bordered fibre. In figs. 7, 8, & 9 a nucleus is seen in the dark-bordered fibre as well as in connexion with the pale fibres. Figures 6, 7, 8, & 9 are from the bladder of the frog.

Fig. 10. Portion of muscular fibre from the leg. Several dark-bordered fibres are seen enclosed in the same sheath; and many of these are exceedingly thin, but evidently possess the same structure as the ordinary dark-bordered fibres. ×700.

PLATE XLII.

Fig. 11. Dark-bordered fibres. The nucleated dark-bordered fibre *a* is seen to divide into two branches. The one passing in the division which diverges to the right, rapidly becomes so very fine that it can hardly be followed as it runs parallel with another dark-bordered fibre and in the same sheath with it (under *d*). This very fine fibre, in direct continuity with the dark-bordered fibre, cannot be distinguished from the fine fibres seen in the sheath of the nerve on the left side of the specimen (at *b*), which leave the trunk and assist in forming a separate branch at *c*. From the pectoral. ×700.

Fig. 12. Division of dark-bordered fibre in the central part of the pectoral muscle of the frog. The fine fibres resulting from its subdivision may be followed for a considerable distance. Nuclei are connected with them at intervals, and the fibres are arranged to form a network. A more perfect example of this arrangement is represented in figure 43 (not published). This figure shows the course of the very fine fibre which is directly continuous with the dark-bordered fibre in certain cases. ×700.

Fig. 13. Division of dark-bordered fibres on the surface of a muscular fibre, from the central part of the pectoral muscle. The muscular tissue having been removed from the tube of the sarcolemma, the delicate nerve-fibres can be distinctly seen on the *external surface* of this transparent membrane. The pale granular fibres, which are here and there connected by branches so as to form networks, are composed of the delicate fibres which are the immediate continuations of dark-bordered fibres (*a* and *b*), and of finer fibres which are derived from those ramifying in the sheath or external to it. Nuclei are connected with these fibres. ×700.

Fig. 14. Division and subdivision of fine dark-bordered fibres on the surface of muscular fibre, showing numerous fine fibres arranged in the form of a network, and nuclei. Two or three of the delicate compound nerve-fibres may be traced quite to the edge of the muscle. Pectoral muscle. ×700.

Fig. 15. Division of dark-bordered fibre with fine fibres ramifying in the sheath of the nerve. From the pectoral muscle. ×700.

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Fig. 16. Distribution of finest dark-bordered fibres on the surface of a muscular fibre from the pectoral muscle. At the edge of the specimen on the left, the delicate nerve-fibres are seen lying, not only upon the external surface of the sarcolemma, but at an appreciable distance from the outer surface. ×700.

Fig. 17. Division of dark-bordered fibre and formation of fine networks. Pectoral muscle of the frog. ×700.

Fig. 18. Very fine nerve-fibres on the surface of the muscular fibre. From the pectoral of a young frog. Several of these fibres can be followed to the edge of the muscular fibre, where they are lost. ×1700.

Fig. 19. Fine fibres from the sheath of the nerve, dividing and subdividing on the surface of two elementary muscular fibres. From muscle of the neck of the frog. ×700 diameters. Compare the appearance of the fibres represented at *b*, *b* in the drawing, with fibres *b* in figs. 16, 15, 14, & 20, which undoubtedly come from true dark-bordered nerve-fibres.

Fig. 20. Division of dark-bordered fibre on surface of muscular fibre.

Fig. 21. Fine dark-bordered fibre, and fibres in sheath of the nerve, with nuclei. From muscle of the neck of the frog. ×700.

Fig. 22. Fine fibres resulting from the subdivision of a dark-bordered fibre ramifying on a muscular fibre. Pectoral muscle. ×700.

Fig. 23. Division of dark-bordered fibres crossing a muscular fibre.

PLATE XLIV.

Fig. 24. Dark-bordered fibre and fine fibre in the sheath ruptured. From muscle of under jaw of frog. ×700.

Fig. 25. Dark-bordered fibre and fine fibres, showing fibres crossing each other and forming very fine branches composed of several finer fibres. ×700.

Fig. 26. Dark-bordered fibres and fine fibres in sheath between muscular fibres of the pectoral muscle of the frog. ×700.

Fig. 27. Dark-bordered fibre, with fine fibres external to it, ramifying between the elementary muscular fibres of the pectoral muscle of the frog. ×700.

Fig. 28. Branching and division of dark-bordered nerve-fibre near terminal branches, and bundles of fine fibres. Several nuclei are seen in connexion with both sets of fibres. From the bladder of the frog. At *a* the dark-bordered fibre becomes resolved into several fine fibres. ×700.

Fig. 29. Fine fibres continued from a dark-bordered fibre, and the fibres in its sheath crossing a young muscular fibre. ×700.

Fig. 30. Young muscular fibre from the pectoral muscle of the frog, near one of the

oval swellings. At *a* and *b* are seen the fine fibres which are continuous with the dark-bordered fibres. These are gradually lost amongst very fine fibres. The compound branches divide and subdivide on the surface of the muscular fibre. × 700.

c. Another fibre at a distance from the oval swelling, showing nuclei with very fine fibres ramifying upon the muscular fibres. × 700.

Fig. 31. Dark-bordered nerve-fibre with fine fibres at the side of it. These fine fibres appear in one part like the outline of the tubular membrane, but at a short distance they are seen to give off branches which pass with other fine fibres. It will also be observed that there is no corresponding line on the other side of the dark-bordered nerve-fibre in this case. From bladder of frog. × 700.

It will be doubted by some if the fine fibres I have figured are true nerve-fibres; but it must be borne in mind, (1) that dark-bordered nerve-fibres divide into fibres as fine as these in the trunks of the nerves near their distribution; (2) that fibres as fine as these form the direct continuation of dark-bordered nerve-fibres themselves; (3) that all dark-bordered nerve-fibres terminate in fine fibres which divide and subdivide into finer branches which run in company with fine fibres apparently derived from the sheath of the nerve or from fibres external to the sheath, but which are probably true nerve-fibres and, at least in some cases, are themselves but the continuation of dark-bordered fibres which run for some distance in the sheath and then diverge.

XXXVI. *Researches on the Development of the Spinal Cord in Man, Mammalia, and Birds.* By J. LOCKHART CLARKE, *F.R.S.*

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IN the Philosophical Transactions for 1859, I showed that, in the adult spinal cord of Man and all vertebrate animals, the white columns as well as the grey substance are everywhere interspersed with granular nuclei, of which some are attached to the sheaths of the primitive nerve-fibres, while others are imbedded in the intervening connective tissue. In the grey substance these nuclei are more abundant than in the white, and have much resemblance to many of the free nuclei or cells, which are certainly in connexion with nerve-fibres, and with which they are freely intermixed. With the hope of throwing some light on the histological relation between these and the other elementary tissues of the cord, the following inquiries into their development were undertaken.

The histology of the development of the spinal cord in Birds and the higher animals had already been begun by BIDDER and KUPFFER*, and pursued a little further by KÖLLIKER. The results of their investigations are comprised in the following statements:

1. After the closing of the laminæ dorsales, the cord at first consists of a canal, the walls of which are composed of cells of one uniform kind, and disposed in a radiating form.

2. In the next place, this wall of cells divides into two layers, of which the outer forms the grey substance, while the inner one appears as the lining of the central canal.

3. The white substance makes its appearance later than the grey, by the cells of which it is without doubt furnished as an outer layer or covering. The white columns are four in number, two on each side; and to these a white commissure is added. There are no lateral columns; those which are so called are subsequently formed as an extension of the anterior columns†.

The investigations of which I now communicate the results were made on embryos of the domestic fowl, of the Sheep, Pig, Ox, and Man. The fluid employed in the process of hardening was at first a weak solution of bichromate of potash, and then a similar solution of chromic acid. When sufficiently young, the embryos were immersed in the solution without mutilation; but in the more advanced states of development the spinal cord was previously uncovered.

* Untersuchungen über die Textur des Rückenmarks, und die Entwickelung seiner Formelemente. Leipzig, 1857.

† KÖLLIKER, Entwickelungsgeschichte des Menschen und der höheren Thiere. Leipzig, 1861, pp. 259, 260.

In a fœtal sheep nine lines in length, a transverse section was made through the spinal column and cord, at the upper part of the lumbar enlargement. The section of the cord was of an ovoid form, with its longer axis before-backward, but was somewhat broader at its anterior than its posterior half (see Plate XLV. fig. 1). The grey substance occupied nearly the whole of its area, but formed an irregular outline which was not concentric with the surface of the section. The central canal was a mere slit or fissure extending backward and forward to near the surface of the grey substance, particularly at its posterior end. Around this fissure was a somewhat dark layer, which was broad along the sides, as well as in front, where it formed a nearly semicircular projection beyond the rest of the grey substance, and was covered externally by the rudiment of the anterior commissure (fig. 1, *a*). Posteriorly it reached quite to the surface of the cord, being hitherto uncovered by the white columns, and was joined to its fellow of the opposite side by a shallow bridge of the same kind of substance. When examined by means of high magnifying-powers, this layer was seen to consist of closely aggregated nuclei connected together by a continuous network of short fibres. The nuclei in size were a little unequal, but their average diameter was about equal to that of the blood-globules. In shape, also, they were somewhat diversified from the round to the oval, the pyriform or the variously angular; but were all intermingled without regularity or order. From the inner border of the layer, that is, from the verge of the canal, the fibres, although they formed an irregular network with the densely aggregated nuclei, had nevertheless a tendency to radiate outward to the rest of the grey substance. From the outer side of the layer, at its posterior extremity, a somewhat dark and curved process (*b*), forming the lateral boundary of the posterior grey substance, extended outward and terminated in a roundish but imperfectly defined mass (*b'*), midway between the anterior and posterior extremity of the section. This mass, however, was uninterruptedly continuous with the outer border of the layer surrounding the canal, as well as with the anterior grey substance, by means of a paler portion composed of similar material. The surface of the mass and concave surface of the process which joined it to the posterior extremity of the section were overlaid with the rudiments of the posterior white column (*c*, fig. 1). In section this column had an oval figure, and covered only about the two anterior thirds of the posterior grey substance, leaving the other convex third entirely uncovered, except by the rudimentary integuments. By means of a sufficient magnifying-power, it was clearly seen that at the surface of the grey substance the fine network of fibres between the closely aggregated nuclei was directly continuous with the fibres of the white column with which it was overlaid. Fig. 2 represents a portion of a transverse section of the posterior grey and white substances at their points of junction, magnified 420 diameters: *a* is an inner part of the white column, and *b* is an outer part of the grey substance, consisting of nuclei and a close network of fibres continuous with *a*. Through the same network nuclei were connected with some of the fibres of the posterior nerve-roots (*d*), which ran obliquely outward and forward, and made their exit at the anterior border of the posterior column, to enter the intervertebral ganglion (*e*).

The anterior grey substance (*f*, fig. 1) was of a somewhat triangular form, with one of its angles behind and the other two in front. The posterior angle, which projected a little into the lateral part of the antero-lateral column, consisted of a very distinct and dark group of nuclei, and was separated from the posterior grey substance (*b'*) by a much paler structure, composed chiefly of longitudinal and radiating fibres, which formed the rudiments of the lateral column. The anterior and outer angle was more obtuse (like a right angle rounded off), and consisted also of a distinct and dark, but larger group of nuclei, from which the majority of the anterior nerve-roots (*g*) originated, and then proceeded outward and forward, to join the posterior roots which escaped in the same direction from the intervertebral ganglion. At its inner side, the triangular mass, like the posterior grey substance, was continuous with the layer surrounding the central canal by a nucleated network of the same kind.

Throughout the whole of the *posterior* grey substance there was no diversity in the appearance either of the nuclei or of the close network of fibres by which they were connected. In the *anterior* grey substance, however, there was a slight diversity in the appearance of both. The nuclei contained in the separate groups already described were not *larger*, but they were less round or oval, or more irregular in shape, than those of any other part of the section, either anterior or posterior. They had also some tendency to aggregate into small, irregular and imperfectly isolated clusters, which were interspersed with granules and united by a looser and also irregular network, so that the entire structure had more or less the appearance of a sponge-like arrangement. In the central layer surrounding the canal, and of which the inner portion constitutes the epithelium, the nucleated network had a radiating tendency, which decreased on extending outward. In the parts between the separate groups or masses of grey substance, the nuclei were densely but uniformly aggregated, and the network of fibres was less distinct.

The antero-lateral white columns (*h*) were very small in proportion to the quantity of grey substance. Behind, where they joined the posterior columns, they were much shallower, and reduced, indeed, to a mere fringe on each side, sunk in the depression between the anterior and posterior grey substance, from which they were developed as the rudiments of the so-called lateral column. Hitherto the anterior median fissure had no existence, but fibres proceeding from the anterior nerve-roots on each side could be seen to cross each other transversely in front of the epithelial layer which surrounded the canal. These transverse fibres were the rudiments of the anterior commissure.

Surrounding the white columns and enclosing the intervertebral ganglia, there was a quantity of loose tissue, which in front formed a deep layer (*i*), connecting the cord with the body of the vertebra (*jj*), and consisting of a nucleated network, the meshes of which were transversely extended. The nuclei of this network, although much less numerous than those of the grey substance of the cord, differed from them but little in general appearance. Their connecting fibres, however, were thicker, coarser, less granular, and directly continuous, at the surface of the anterior commissure, with similar fibres which radiated forward from the epithelium (*a*) surrounding the front of the canal. At its opposite or anterior border—or rather anterior part, for it had no distinct border—this nucleated tissue became directly continuous with that which formed the body of the vertebra (*j j*). This body consisted of a dense mass of closely aggregated nuclei, similar at its circumference to those of the connective tissue, but somewhat different in shape toward the centre, where they were round, oval, elongated, angular, and often curiously crescentic. They were all connected by fibres which, nearer the circumference, where the structure was looser, could be very easily distinguished; but in the centre they were less distinctly seen between the densely-crowded nuclei. Near this centre was a circular spot (*k*), the section of a longitudinal cylinder, surrounded by a thick wall resembling the wall of a cartilage-cell and enclosing several closely aggregated nuclei, connected by short fibres. These nuclei were round, oval, pyriform, or otherwise irregular in shape, and rather larger and more granular than those by which they were surrounded.

The spinal cord and vertebral column of the Sheep at this period of development have a considerable resemblance to those of the chick on the fifth day of incubation. In the latter, however, the different parts of the grey substance are more distinctly marked. Fig. 19, Plate XLVII. represents a transverse section of the vertebral column and spinal cord, in the sacral enlargement of the chick, on the fifth day, and magnified 60 diameters. The section of the *cord* was nearly a perfect oval. There was no trace of the rhomboidal sinus or ventricle which in the adult bird separates its posterior lateral halves*. The epithelial layer (*a*) immediately surrounding the canal was clearly defined and easily distinguishable from the rest of the grey substance. It was somewhat broader behind, where it reached the surface in the form of an arch. On each side of it the posterior grey substance (*b*) consisted of a pyriform mass, nearly covered by the rudiments of the posterior white column (*c*), and separated by a triangular and much lighter space from the anterior grey substance (*f*), which had a nearly quadrangular form, but was separable into two different masses, of which the outer was greyer and of an oval shape. From both of these the anterior nerve-roots (*g*) originated and escaped from the anterior and outer angle of the grey substance. Along the posterior half of the epithelial layer (*a*), its nucleated network was arranged in a connected series of arched radiations, which extended into the pyriform mass of grey substance on the same side (*b*). From this substance another very evident series of separate fibres ran directly forward to that part of the *anterior* grey substance (*f*) from which its nerve-roots chiefly arose. Neither these nor the arched fibres just mentioned as radiating from the epithelium could be seen so distinctly in the Sheep. Although the grey substance consisted of the several separate masses above described, yet there was but little diversity in the appearance of its structural elements. Its nuclei, generally, were rather smaller than in the section of the fœtal sheep represented in fig. 1.

The cylindrical column (*k*) in the centre of the body of the vertebra (*jj*), and corre-

* See Philosophical Transactions, 1859, Plate XXIII. figs. 34, 35, and 36.

sponding to the *chorda dorsalis* of fishes and reptiles, was at least five times as large in diameter as that of the Sheep, and somewhat different in the arrangement of its elementary parts. It consisted of a loose honeycomb structure with large vacant spaces, formed by a coarse network of fibres, with a nucleus at each point of their junction (see fig. 45, *a*, Plate XLVIII.). Around the cylinder, elongated nuclei giving off fibres in a circular direction were arranged in concentric layers.

In a fœtal sheep a little larger than that first examined, and measuring exactly 1 inch in length, a section of the spinal cord from the same region as before (the upper part of the lumbar enlargement) presented the appearances seen in fig. 3, Plate XLV. The canal, as a sword-shaped slit, extended behind-forward through nearly the whole of the grey substance. Immediately surrounding it was a bulbous or club-shaped layer of nuclei, with its bulbous end (*l*) posterior and reaching the surface in the middle line, but partially covered on each side by a portion of the posterior column (*c*). At this latter part it was less clearly defined, and gave off a process of grey substance, which extended outward and forward beneath the rest of the white column (*c*), and terminated in a rounded but imperfectly circumscribed mass (*b*). From about its middle to its anterior extremity the narrower portion of the club-shaped layer consisted of a nucleated meshwork of epithelium with an outwardly-radiating tendency; but on passing backward to the bulbous end, this appearance was confined to the wall of the canal, and moreover became less distinct, while at the same time it was gradually lost amongst the densely aggregated nuclei in the lateral part of the bulb.

The anterior grey substance presented nearly the same aspect as in the section represented in fig. 1. There was a very evident decussation, or crossing of fibres from the opposite roots of the nerves, in front of the canal; but, as in the last section, there was no anterior median fissure.

In the elements of the grey substance there was scarcely any perceptible advance in development. Some of the nuclei in the anterior masses seemed in a trifling degree to have increased in size, become more irregular in shape, and to be connected by a network which had rather more of the sponge-like arrangement.

The antero-lateral columns had somewhat increased in depth. Behind, where they joined the posterior columns, they were much shallower than elsewhere, and formed, on each side, the so-called lateral column (*h'*), which was sunk in a depression between the anterior and posterior grey substance.

The posterior white columns (*c*, *c*) were also deeper than in the previous sections. Behind, they were bevelled on to the exposed surface of the central club-shaped layer; and in front, where they joined and overlapped the lateral columns, they terminated in prominent but rounded edges, to which the posterior roots of the nerves (*d*) were attached.

There is every reason to believe that the fibres of the white columns are developed from the grey substance as prolongations of the network by which its nuclei are connected.

In fœtal sheep of exactly 2 inches in length, it was found that the different parts of the spinal cord already described had become more or less modified in form, size, and relative arrangement, that new parts had been superadded, and that the structural elements of the grey substance had undergone considerable alterations. Fig. 4, Plate XLV. represents a transverse section of the cord from the same region as before, viz. the upper part of the lumbar enlargement. Here we at once perceive that each lateral half of the grey substance is fashioned into very distinct anterior and posterior cornua (*f*, & *l b*), which are partially separated from their fellows of the opposite side by an anterior and a posterior median fissure (*m*, *n*), both of which were absent in the former sections, and now come into existence, as we shall presently see, as a *consequence* of the development of the cornua. We also observe that the anterior half of the central slit represented in the preceding figures has dilated into an oval canal (*o*), which is near the centre of the section. Moreover it will be seen that two new portions of the posterior white columns (*p*, *p*) have made their appearance on each side of the posterior median fissure. Let us now consider the course of the developmental growth by which these changes have been gradually effected. First, then, it may be remarked that in fig. 4 the anterior cornua are formed by a growth of the grey substance forward and inward, in front of the canal, from the imaginary line (*f*), which corresponds to the line of limit of the anterior grey substance (*f*) in fig. 3. The anterior white columns (*h*, *h*) extend in the same direction around the growing ends of the cornua, until only a narrow space or fissure is left between their inner borders in the mesial line. In this way is formed the anterior median fissure (*m*) in front of the commissure, which projects into it as a conical process. During these changes in the anterior part of the cord, the posterior grey substance (*l b*, fig. 3) grows obliquely outward and backward (*l b*, fig. 4); while its anterior angle (at *b*) becomes further removed from the posterior angle of the anterior cornu (f'), and is separated from it by a broader and deeper indentation, which is the rudiment of the neck or cervix cornu posterioris. This space or indentation is filled up and overlaid with the lateral column (h'), which has reached the level of the posterior column (*c*) and assumed a convex surface. This so-called lateral column differs from the rest of the white substance in being subdivided into a much greater number of small but separate fasciculi of various shapes, by means of a remarkable system of radiating fibres which proceed both from the grey substance between the cornua and the epithelium surrounding the canal. A radiation of fibres, however, but of much less extent, takes place from the whole circumference of the grey substance.

At the commencement of these changes, the central fissure or canal represented in fig. 3 reaches the surface of the posterior grey substance. The growth of this substance is then continued not only backward, but *outward*, or divergent from the mesial line, while in the intervening angular and gradually increasing space between it and this line (fig. 4, *n*) there are developed on each side two new pyramidal columns of longitudinal fibres (*p*, p'), which increase in depth in a corresponding proportion. Of these, the outer one (*p*), which is much the larger, rests on the back of the cornu, over which it ultimately blends with the *outer* portion (*c*) of the posterior column previously developed. The inner and smaller column (*p'*) is in general more conspicuous and distinct in the dorsal and cervical than in the lumbar region, as shown in figs. 5, 6, & 7. The opening of the original canal or slit between these additional columns constitutes the posterior median fissure (*n*), which is now occupied by blood-vessels and pia mater in connexion with radiating fibres from the central epithelial layer.

We see, then, that the direction taken by the developmental growth of the posterior grey substance is just the reverse of that which is followed by the anterior. The changes which occur in the former, and the consequent production of new columns, have much resemblance to some of those that take place in the upper part of the cord during its transition and development into the medulla oblongata. In the course of those changes the central canal retreats further and further to the posterior part of the medulla, until at length it opens on the surface in the form of the fourth ventricle. In a corresponding proportion the posterior cornua diverge, while from their roots on each side of the mesial line the posterior pyramid is developed *. The cases, however, are not exactly parallel.

Returning to the interior of the grey substance, we find that a variety of changes, more or less important, have occurred,—first, in the disposition and destination of some of its parts; secondly, in the structure and arrangement of its constituent elements. First, then, we may observe that the club-shaped layer surrounding the central fissure or canal, as represented at *l*, fig. 3, has lost its defined outline and become very much diffused. Behind the oval canal, on each side of the mesial line (at *q q*, fig. 4), it forms the inner half of the cervix cornu posterioris, which subsequently contains the posterior vesicular column, or nucleus of the cervix. Hitherto, however, it consists only of a closely aggregated mass of nuclei, which become gradually more diffused on its outer side, where it joins the rest of the grey substance. Passing backward and outward and then forward in the direction *l b*, as an arched layer near the posterior border of the cornu, it terminates in the dark mass (*b*) within its outer angle or point. In the space between this arched layer and the posterior border of the cornu, the nuclei are less closely aggregated, and thus constitute, with their intervening network, a paler lamina (*r r*), which is the rudiment of the *gelatinous* substance. In the adult cord, as I showed on former occasions, the space occupied by the above-described arched layer of nuclei contains numerous bundles of nerve-fibres continuous with the posterior roots. On its inner side, near the centre of the cornu, there is a somewhat paler space, which is continued forward and outward, in a curve, to the border of the grey substance at the bottom of the lateral column. This is more conspicuous in the dorsal region, as shown in figs. 5 & 6.

On tracing the club-shaped layer forward along the side of the median line, we observe that its outer part has lost the defined outline which it possessed in section 3,

* See my memoir on the "Medulla Oblongata," Philosophical Transactions, 1858, Plate XII. figs. 13 & 14, *b* Plate XIII. fig. 16; Plate XIV.; and Plate XV. fig. 19.

and has gradually blended with the anterior grey substance; while its inner border forms the epithelial layer (*s*) surrounding the central canal. Along the *sides* of the canal, the epithelium, as regards its constituent elements, differs but little in appearance from the rest of the grey substance. The only apparent differences are, that its nuclei are less closely aggregated; that the fibres radiating through their interspaces from the verge of the canal, and which are thicker, are consequently *seen* to be more continuous and branched; and that a few of its nuclei are larger, while others are more elongated or fusiform at right angles to the axis of the canal. Nor is it bounded externally by any definite line, but is continuous, as a nucleated network, with that of the anterior cornu. From the posterior margin of the elliptical canal its radiating fibres extend outward and backward into the posterior cornua, and directly backward between them into the posterior median fissure. In *front* of the canal, however, the epithelium has a somewhat different arrangement. At this part it forms a deeper and more distinct layer, containing more of the fusiform nuclei, which are elongated in a direction forward, and terminate at each extremity in a fibre. At the sides of this layer the nuclei with their fibres are curved inward, but become progressively straighter as they approach the middle line (see figs. 4 & 8). Posteriorly their processes or fibres are thicker, and attached by their extremities to the margin of the canal; while anteriorly they converge across the anterior commissure, as a conical network, into the anterior median fissure, where they become directly continuous with the fibres of the pia mater and connective tissue enveloping numerous blood-vessels and derived from the circumference of the cord (*m*, figs. 4 & 8). In many cases the epithelium in *front* of the canal differs but little in appearance from the rest of the layer, except that the principal fibres of the network converge forward to join the process of pia mater in the anterior median fissure. In such cases it has a general resemblance to that seen in fig. 9, Plate XLV., which exactly represents a portion of the canal and surrounding grey substance of a human fœtus of nine weeks, magnified 420 diameters. It was taken from the same fœtus as the section represented in fig. 8, Plate XLVI. *o* is the anterior part of the canal; *s*, *s*, between the canal and the outer line, is the layer of epithelium, which on each side, however, is seen to be gradually continuous with the anterior grey substance, *f*; while in front it terminates in a conical network of connective tissue interspersed with nuclei, and continuous (at *m*) with the pia mater of the surface.

I have next to show that the elements of the grey substance, which in fig. 3 were nearly uniformly alike, have in fig. 4 become more or less modified both in structure and arrangement. In fig. 3 every nucleus had a plain or smooth appearance, without any trace of granular contents or of a distinct enveloping membrane. In fig. 4, however, both these modifications of structure had taken place; and although the observation of this fine distinction might seem to be trivial and unimportant, it will nevertheless be seen, as we proceed, to be well worthy of attention. It was also observed that in section 3 the nuclei were very nearly of the same size in all parts of the grey substance. In section 4, however, those of the anterior grey substance had everywhere increased in diameter; while those of the posterior grey substance, as far forward as the posterior level of the canal, had scarcely, if at all, advanced in size. In all the darker parts of the latter they were still very closely aggregated, but less so than in fig. 3; so that the network of fibres by which they were connected was conspicuous in a corresponding proportion. Fig. 10, *b*, Plate XLV., represents some of the nuclei and their connecting fibres, from the arched layer (*l b*) of the posterior cornu in fig. 4, magnified 670 diameters*. In the section, however, although it was exceedingly thin, the nuclei were rather more crowded in that part; but exactly the same appearance is presented where a very thin portion has been shaved to an edge.

Throughout the whole of the *anterior* grey substance the average size of the nuclei is about twice as great as in the *posterior* grey substance, and therefore about twice as great as in every part of section 3. Their increase in diameter begins rather abruptly near the posterior level of the oval canal. They have also become more granular, more distinctly circumscribed by well-defined walls, and lie at greater distances from each other. The tissue which supports them has still a reticular structure, but is looser, more open, and in the central portion of the cornu consists of a sponge-like network of coarser fibres, which are more or less granular and connected with the nuclei by irregular aggregations of granules. Fig. 12, at *v*, Plate XLVII., is a faithful representation of this peculiar arrangement, which can be more intelligibly represented than described. In the middle of the cornu, where it is traversed by the central fibres of the anterior nerve-roots, as they run directly backward, the structure has very much the appearance delineated in fig. 11, Plate XLVII. But in the antero-lateral parts of the cornu (*w*, *w'*, figs. 4 & 7, Plate XLV.), where the groups of large nerve-cells are developed, it has a kind of honeycomb arrangement in the form of circular or somewhat irregular cavities or cells, which are large, but variable in diameter, and frequently in close apposition, but often separated by angular interspaces containing each a nucleus encrusted with granules. Fig. 12, *x x*, Plate XLVII. is a very exact representation of a portion of the outer or lateral group (*w'*, fig. 7, Plate XLV.). The walls of the cavities and the granular network are directly connected with the tissue of the antero-lateral white columns at *y*, *y'*. Within each cavity is a nucleus, which is sometimes in or near the centre, sometimes more excentric, and sometimes close to the wall. The nucleus is imbedded in granules, which are generally seen to connect it to the wall, and which in some instances occupy the whole, in others only part of the cell; in the latter cases they aggregate in a variety of forms†.

Within the posterior angle or shoulder of the anterior cornu is a remarkable dark and nearly triangular group of nuclei (fig. 4, *f'*, Plate XLV.), which differ from the rest only in being rather larger, and more closely aggregated.

The roots of the nerves may be very distinctly traced into the grey substance (see

* By some mistake this figure is marked in the Plate as ×420 diameters.

† The separation of the granular masses from their cell-walls, and the stellate form which they sometimes assume, seem due to the action of the chromic acid.

fig. 8, Plate XLVI.). The anterior roots (g) are attached to the anterior column, which they traverse transversely inward to reach the anterior cornu (f), within which their fibres diverge and cross each other in different directions. A large number proceed backward along the lateral part of the cornu, some of them running in succession outward to the lateral column (h'), while the rest reach the triangular group of nuclei just pointed out, within the posterior angle of the cornu. Another set of fibres run more directly backward, through the central part of the grey substance (f), where they join in the general network, and appear as represented in fig. 11, Plate XLVII.; while some curve inward to decussate, in front of the canal, with their fellows of the opposite side, in company with others proceeding forward and inward around the canal from the *posterior* grey substance. (See also fig. 3, Plate XLV.)

The posterior roots (fig. 8, *d*, Plate XLVI.) have no immediate connexion with the lateral columns, and are attached solely to the posterior, through which they diverge in a direction backward and inward to reach the grey substance (*l b*). On entering this substance many of them evidently become finer by subdivision, and contribute to form the network by which the nuclei are connected. Fig. 10, Plate XLV. is a faithful representation of a very thin transverse section near the extremity of the posterior cornu, through which the roots (*d*) are entering; at *b* they are seen to become continuous with the general network. In the same way many of the fibres of the posterior columns are connected with nuclei of the grey substance. (See fig. 15, I, Plate XLVII.)

In fœtal sheep of 3 and 4 inches in length, there was no remarkable alteration in the shape and disposition of the grey and white substances in the region corresponding to fig. 4; nor was there any great difference in the appearance or arrangement of the constituent elements. Fig. 13, Plate XLVI., represents a transverse section of one, and part of the other, lateral half of the spinal cord, near the middle of the lumbar enlargement, of a fœtal sheep, 4 inches long; and fig. 14 shows a similar section of the middle of the lumbar enlargement of a fœtal ox, 5 inches long. Here we see that nearly one-half of the posterior grey substance still consists of a dark layer of closely aggregated nuclei, which differ but little in size and general appearance from those of section 4, Plate XLV. This dark layer constitutes the *caput* cornu posterioris. Fig. 15, I, Plate XLVII. is a longitudinal section of it near its outer border, or the posterior extremity of the cornu, where it is overlaid and crossed in different directions by the decussating fibres of the white column. As the posterior (*l b*, fig. 14) merges into the anterior grey substance (f), the nuclei become larger, while the network which supports them becomes coarser and looser (II, fig. 15, Plate XLVII.). In the posterior part of the section, the very irregular and granular meshwork of fibres appears to extend nearly equally in all directions; but towards the anterior part (opposite II) the meshes have a tendency to elongate in a direction forward; while many of the nuclei are elongated in the same direction, and have tapering granular masses extending from their ends. These latter appearances, however, are more conspicuous on the inner side of the cornu, nearer the median line. On proceeding forward, the network assumes the peculiar honeycomb structure already observed in section 4, and represented at x, fig. 12, Plate XLVII. Here, however, the layer is deeper; but variable in depth. Fig. 15, III, Plate XLVII. is a portion of a longitudinal section, behind-forward, through this layer of large cells of the anterior cornu (*w w*, fig. 14, Plate XLVI.). It will be remarked that these more or less globular cells are closely grouped, and frequently in actual contact; that in many instances they are seen to be filled with the granular material surrounding their nuclei; and that the nuclei themselves are sometimes larger than those in other parts of the anterior grey substance, and contain each either one large globular nucleolus or two. With the general network of this layer the anterior nerve-roots may be seen to be continuous.

In fœtal sheep of 6, 7, and 8 inches in length, further changes were observed to take place in the grey substance, but they were limited chiefly to its posterior and middle portions. In the *caput* cornu posterioris, the nuclei, although still aggregated in vast numbers, were comparatively less numerous and more widely separated from each other than in fig. 15, and somewhat larger; the network of fibres by which they were connected was also coarser and looser; while a complete system of transverse, longitudinal, and oblique nerve-fibres were very readily distinguishable.

In the middle portion of the grey substance, that is, between the caput cornu (*l b*, fig. 13, Plate XLVI.) and the groups of large nerve-cells (*w*, *w'*), the coarse network already described and represented in fig. 15, II, Plate XLVII. was now interspersed with a great number of cells, which differed from each other both in size and shape (fig. 16, Plate XLVII.). The majority were fusiform in a direction before-backward, and either more or less wavy and sigmoid, or perfectly straight, with processes which extended occasionally to an amazing distance. In some cases, as at *a*, fig. 16, a fusiform cell was bent into the shape of a crescent, and formed part of the circumference of an oval or circular space. In other cases it was converted into a triangle by giving off two processes from one end, which became broader, and embraced, as before, part of the circumference of an open space (*b*, fig. 16). In many of these cells, especially of the smaller kind, the nuclei were only faintly, or not at all, observable.

The groups of large nerve-cells (*w*, *w'*) had undergone scarcely any appreciable change, and presented nearly the appearances represented at III, fig. 15.

With regard to the changes which ensue in the structure of the cord on approaching the period of birth, I shall confine my remarks chiefly to those particular points which relate to the development of its constituent elements. During these successive changes, the *gelatinous substance* becomes more and more conspicuous as a distinct lamina around the end of the caput cornu. Along its margin the peculiar nerve-cells observable in the adult are gradually developed, after the manner of those already described in the middle of the grey substance (fig. 16, Plate XLVII.); while its arciform, transverse, oblique and longitudinal fibres increase in number in a corresponding proportion. Coincident with this development of the gelatinous substance, the nuclei in the dark and inner part of the *caput* cornu become less numerous, while its transversely-radiating, longitudinal, and variously-oblique fibres, continuous with roots of nerves, increase in a corresponding proportion and pursue a more definite course. But even at an advanced period of utero-gestation (at the fifth month, for instance, in the human fœtus) the *caput* cornu is still very thickly crowded with nuclei, and presents an opaque and uniformly dark mass, like that of the fœtal ox of 5 inches long, represented in fig. 14, Plate XLVI. By the end of the *sixth* month, however, it is much less opaque, and contains a much smaller proportion of nuclei, but a correspondingly large proportion of fibres running in the directions already indicated. Fig. 40, Plate XLVIII. represents, under a low power, a transverse section of one lateral half of the grey and white substances of the cord from the upper part of the lumbar enlargement of a human fœtus of six months*. Here the *caput* cornu posterioris (*l b*) is at once distinguishable from the *cervix*, not only by its bulbous expansion, but also by its much darker colour. The gelatinous substance, however, and arciform fibres are not very strongly marked, but were readily detected under high magnifying-powers. In the larger and darker portion of the caput, the nuclei were imbedded in a fine granular network, and were most numerous at its sides, particularly its outer side. Amongst this network were the longitudinal and oblique fibres, to which in the adult cord the opacity of this part is chiefly due. The majority of the nuclei were round, but some were oval, and a few had an angular form. They were finely granular, were enclosed in distinct envelopes, and differed from each other in size. Their average diameter was rather below that of the blood-globules of the same fœtus. Scattered irregularly amongst these were a few others, which, in addition to the fine granules, contained each a larger, central and globular nucleolus. These latter nuclei were identical in appearance with the nuclei of the larger nerve-cells.

By referring to the same figure (40), it will be seen that each inner half of the cervix cornu (*q*) is occupied and rendered convex by a remarkable group of nucleated cells, which I formerly described very fully under the name of the posterior vesicular column†. In this region of the cord (the upper part of the lumbar enlargement), not only are these columns larger than in any other region, but the cells which they contain are for the most part of the largest description, and similar in appearance to those of the *anterior* cornu: but their development is somewhat later. In the human fœtus, at the sixth month, however, they are perfectly developed, and assume a variety of stellate forms, with processes that extend in all directions. They are closely invested by a thin

* Through the kindness of Mr. HATHERLY, of Belgravia South, I was fortunate enough to obtain this fine male fœtus within about an hour after death. The brain and spinal cord were immediately removed and hardened in the most gradual and careful manner by immersion, first, in an exceedingly weak solution of bichromate of potash, then in stronger solutions of the same salt, and finally in a solution of chromic acid. The preparations, of which a series of representations are given in Plate XLVIII., are the most beautiful that I have ever succeeded in obtaining from the *human fœtus*. I must also acknowledge my obligations to Mr. R. DUNN, of Norfolk Street, Mr. WHITNEY, of Westminster, Mr. PAINTER, of Westminster, and Mr. HUNT, of South Belgravia, for their kindness in providing me with fœtuses.

† Probably a better term for this column would be the nucleus of the cervix cornu—*nucleus cervicis cornu*.

sheath or envelope, which is prolonged on to the processes and connected with the intervening reticular tissue. It is beyond all doubt (for it was distinctly seen in preparations of this fœtus) that many, at least, of the processes, by repeated subdivision, become continuous with the fine network of this intervening tissue. See fig. 17, Plate XLVII.*

In the same fœtus the *tractus intermedio-lateralis*, or tract of grey substance (*t*, figs. 41, 42 & 43, Plate XLVIII.) which projects into the lateral column between the anterior and posterior cornua, was in a nearly complete state of formation.

The epithelium, also, surrounding the canal was completely developed, and on no occasion in the *human* cord have I seen it so perfect and beautiful. In the adult human cord it is very difficult to obtain a good view of the exact form and arrangement of its constituent elements, which are often, especially in the cervical region, confusedly heaped together in a mass that entirely fills up the canal†. In the *fœtal* cord, however, the canal is larger, the epithelial layer is deeper, and its elements in general are more uniformly arranged. In the case now under consideration these elements were more than usually distinct. In some regions of the cord they were exactly alike throughout the whole circumference of the layer. In such regions they consisted partly of oval, and partly of still more elongated nuclei or cells, which gave off a process from each extremity, and were arranged with their longer axes at right angles to the axis of the canal. The *nuclei*, however, were placed at *different distances from the canal*, and were so disposed as to lie in nearly close apposition, and form a compact stratum. Their *central* processes, which reached the *inner* margin of the layer, were consequently of *different lengths*; and the length of these processes, in general, was inversely proportional to their thickness. Here and there between the rest of the nuclei, narrow interspaces were occupied by remarkably slender and fusiform bodies, of which the tapering ends reached the verge of the canal, without the intervention, apparently, of any distinct processes. At their *peripheral* or outer ends all these nuclei tapered into fine fibres which crossed each other in every direction, and frequently divided into smaller branches, to be continuous with the network surrounding the epithelial layer. In the lumbar region of the cord the nuclei were *not* exactly alike around the whole circumference of the canal. In the anterior third of the layer they were in every respect similar to those which I have just described; and at the front of the canal, the fibres proceeding from their peripheral ends were seen to cross the commissure, and become directly continuous with the process of pia mater within the anterior fissure. Frequently this process consisted almost entirely of blood-vessels containing numerous well-preserved globules, and at the bottom of the

* This statement is confirmatory of the descriptions which I first gave of the ramifications of the processes of the nerve-cells of the spinal cord.—Philosophical Transactions, 1851, p. 614.

† On a former occasion (Phil. Trans. 1859, Part I. p. 455, and Plate XXII. fig. 55) I showed that the canal in the human cord is sometimes double, or rather that two secondary canals, as it were, are hollowed through the mass of epithelium just mentioned. The same fact has since been made the subject of a paper by Dr. Joh. Wagner, in Reichert-Dubois's Archiv für Anat. &c. 1861, p. 735. Even in the fourth ventricle, at the calamus scriptorius, in Man I have frequently found on the surface a kind of short supplementary canal formed by a double layer of epithelium enclosing a narrow space.

fissure it gave off a brush-like radiation of fibres that were continuous with those of the epithelium. In other sections, blood-vessels from the same process extended right and left, as well as backward and around the canal. Some of the fibres from the epithelial cells on each side of the front layer, after crossing the anterior commissure, penetrated the anterior white columns, at the sides of the fissure, and were lost in the tissue between its longitudinal fibres. Behind, a narrower portion of the epithelial layer was composed of the same kind of elements as those which were found in front; and in a similar way fibres proceeding from their peripheral ends converged *backward*, to be continuous with blood-vessels and pia mater in the *posterior* median fissure. Other fibres, also, from the same source could be traced into the posterior white columns. Around the remaining portion of the canal the epithelial layer was narrower and somewhat different in structure. It consisted, for the most part, of round and of rather oval nuclei, irregularly disposed, and in connexion with the fibres on the outside of the layer. These nuclei, like the others just described, were finely granular, and in every way similar to a multitude of those which are scattered through the grey substance, and, as I shall presently show, through the tissue between the fibres of the white substance*.

The large nerve-cells of the anterior cornu had assumed the shape and general appearance which they present in the adult cord. Like those of the cervix cornu posterioris,

* This description of the epithelium in the human fœtus has a general resemblance to that which I gave of the same structure in the full-grown ox (Phil. Trans. 1859, p. 455, Plate XXII. fig. 53); but I have entered more particularly into details in the present case, on account of the close resemblance of the *oval* cells to the "olfactory cells" of the olfactory mucous membrane as first described by SCHULTZE in the Frog and Pike. It is believed by SCHULTZE and others, that the processes of these "olfactory-cells" are directly continuous with fine fibres of the olfactory nerves. Such a connexion, however, has never, so far as I am aware, been actually *seen*. I have myself traced these nerve-fibres quite into the epithelial layer, but have not hitherto quite satisfied myself of their actual termination. Six years ago I showed that beneath the epithelium of the pharyngeal sac of the common earth-worm, a ganglionic plexus of nerves terminates in a network of single nucleated fibres, resembling in form a capillary network (Proc. Royal Soc. Jan. 1857, No. 24. vol. viii.). It is true that AXEL KEY (Archiv für Anat. &c. 1861, p. 329) has described and figured in the tongue of the Frog, a remarkably conspicuous communication between nerve-fibres and cells which correspond to those of SCHULTZE. But no such communication was observed either by BILROTH or HOYER (Archiv für Anat. &c. 1858 & 1859) in the same organ. On the fibres proceeding from the "olfactory cells" of SCHULTZE, there are slight granular dilatations, which I have found most remarkable in the Pike. On the fine fibres which surround the canal in the human fœtus, as above described, and with which the processes of the oval epithelial cells are connected, I have also observed exceedingly minute dilatations. These fibres, as already stated, are evidently continuous, and identical in appearance with the fine fibres of the pia mater on the outside of the cord. Without, therefore, denying the *possible* continuity of the "olfactory cells" with true nerve-fibres, we must be so much the more cautious in admitting their *actual* continuity as an anatomical fact, until confirmed by actual *observation*. That the fibrous structure immediately surrounding the spinal canal is of the nature of connective tissue, was first maintained by myself (Phil. Trans. 1851) in opposition to STILLING, who described it (together with the epithelium, which he had not detected) as a "circular commissure" composed of *grey* nerve-fibres. In the human brain, also, the processes of the epithelium which extends from the aqueduct of Sylvius along the under surface of the posterior commissure, have clearly been seen by myself to pass through fissures in that commissure, to the pia mater on its opposite surface (Proceedings of Royal Society for June 20, 1861).

they were enveloped in delicate sheaths, which, however, were quite distinct from, although in connexion with, the surrounding reticular tissue (see figs. 12 & 15, Plate XLVII.). Many of their processes also were very clearly seen to subdivide, or to break up suddenly, into a multitude of fine branches to form part of the intervening network†.

I shall conclude my remarks on the development of the cord in the human and mammalian fœtus, by a few observations on the development of its nerve-fibres. In a very young fœtus it is difficult to obtain a satisfactory view of isolated nerve-fibres, and to detect the way in which their formation commences. According to my own opportunities of observation, they are not developed from nucleated cells, but rather by the extension of finely granular substance from round and oval nuclei. On this point, however, I cannot at present speak with confidence, and therefore leave it open for further inquiry. In the early stages—for instance, in a fœtal sheep, or human fœtus from 1 to 2 inches long—the fibres, in a *fresh state*, consisted of most delicately granular and nucleated bands, without any sharpness of outline or appearance of separate border. But the nuclei were far from being numerous, either in the nerves or in the white columns of the cord. In fig. 8, Plate XLVI. their average number is shown in the white columns of the left side, in a human fœtus of about nine weeks. As development, however, advances, their number increases considerably, while the fibres to which they belong acquire a more sharply defined outline or border, which in some parts of its course appears darker and thicker than in others. In fig. 9+, Plate XLV., on the left, is an exact representation of a separate fibre in a fresh and unprepared state, from the sciatic nerve of a human fœtus of four months, magnified 670 diameters; and in the same figure, on the right, is represented the appearance of several such fibres as they lie side by side in a bundle. Fig. 18, Plate XLVII. shows a small portion of a transverse section of the posterior white columns of a human fœtus of five months, magnified 670 diameters. When compared with the same parts in fig. 8, Plate XLVI., which is magnified only 50 diameters, it shows how much the nuclei have increased in number. Some of these nuclei belong to the sheaths of the nerve-fibres, others to the tissue by which their sheaths are connected. As the period of birth approaches, they are again reduced in number; but even in the adult cord, as I showed on a former occasion, they are scattered at intervals between the fibres of all the white columns. In structure they are altogether similar to the nuclei of the fœtal epithelium above described, and differ from them only in having a somewhat less average diameter.

With regard to the development of the spinal cord in Birds, I need dwell only on those particulars in which it differs from that in Man and Mammals. Fig. 19, Plate XLVII. shows the appearance of the grey and white substance in a transverse

† I have described and figured a similar ramification of the processes of the cells around the base of the peduncle of the olfactory bulb, and in the anterior perforated space, and have frequently observed the same appearance in the convolutions of the cerebral hemispheres. (See Zeitschrift für Wissensch. Zoologie, Bd. xi. Hft. 1. Taf. v. fig. 6.)

section of the upper part of the sacral enlargement of the chick, on the fifth day of incubation. The central dark layer (*a*) surrounding the canal is still uncovered by the posterior columns (*c*, *c*), and forms the posterior surface. The grey substance hitherto is not divided into distinct cornua; there is no trace of a posterior median fissure, and only the first rudiments of an anterior (*m*). In the course of four days we find that rapid and remarkable progress has been made in developmental growth. Fig. 20, Plate XVII. represents the appearances in a transverse section of the same part, at the end of the ninth day of incubation. By a succession of changes similar to those that have been described in the fœtal sheep, both the anterior and posterior cornua have become fully developed, with corresponding median fissures. In each anterior cornu (*f*) is a thick cylindrical column of large nerve-cells. These cells, however, for the most part, differ both in shape and mode of formation, from those that are found in the corresponding part of the human and mammalian fœtus. The majority are fusiform from before backward, and continuous with the antero-posterior fibres which reach the posterior cornu. They are not formed within large round and oval spaces with thick walls, like those already described in the mammal, but grow side by side in close apposition by the extension of substance from the ends of their nuclei, apparently after the manner of those in the central part of the grey substance of the Sheep and Ox, represented in fig. 16, Plate XLVII. Besides their antero-posterior processes, they send off others both outward and inward. The middle portion of the grey substance between this vesicular group and the posterior cornu contains a few isolated cells of the same kind.

The antero-lateral white columns (*h*, *h'*) have increased considerably in area, and the anterior median fissure between them is much deeper, while the canal (*o*) is reduced in size and limited to the centre of the section. The caput cornu posterioris (*l b*), on each side, consists of a dark mass of closely aggregated nuclei, which are smaller than those in any other part of the grey substance. It is entirely covered by the posterior white column (*c p*), which, however, does not extend along the inner side of the cervix (*q*). The space between the cervix cornu (*q*) of one side and that of the other, and which, in the corresponding part of the adult cord of both the bird and mammal, is occupied by the inner and deep portions of the posterior columns, is now filled up by a bell-shaped mass of connective tissue (*n'*), which consists of a loose network of fibres connected at intervals with nuclei. The fibres of this tissue are also directly continuous with the connective tissue of the inner part (*p*) of the posterior white column, and with the network in the cervix cornu (*q*), along the inner border; but its fibres are coarser, and its nuclei are larger. Its deep portion, which is divided by a central raphè, forms the posterior wall of the canal (*o*), and *constitutes its epithelium* on that side; while its superficial portion extends on each side over the posterior white column (*p*), and is connected at its convex surface with the *pia mater* which surrounds the cord. In the lower part of the sacral enlargement (fig. 21, Plate XLVII.), the cervix cornu is united along the middle line with its fellow of the opposite side, and forms with it a single mass; so that only the extremities of the cornua surrounded by the posterior columns remain apart from each other. Between these the space *n*, which here represents the posterior median fissure, is much shallower and narrower than in fig. 20, and widely separated from the central canal by the coalescence of the cervix (*q*) of each side. The loose and nucleated tissue (*n'*), however, with which it is filled is still connected, across this coalesced grey substance, with a number of fibres radiating from the epithelium around the posterior wall of the canal (*o*). On ascending the sacral enlargement, the posterior cornua become more and more deeply separated, until the division reaches the canal. Fig. 22 represents a transverse section about midway between sections 20 and 21. Here the space *n* between the cornua has, in section, the form of a deep cylinder filled with the loose nucleated tissue, of which the deepest portion itself constitutes the epithelium around the posterior wall of the canal, instead of being only connected with it, as shown in fig. 21, by a number of radiating fibres. As development advances, the inner part (*p*, fig. 20) of the posterior column gradually extends over the side of the cervix cornu (*q*), and replaces a corresponding proportion of the nucleated tissue (*n'*), until it reaches the median raphè at the back of the canal. On approaching the *middle* of the sacral enlargement, the entire lateral halves of the grey substance are more widely removed from each other, and joined only by the *anterior* commissure; for the *posterior* commissure has no existence. At the same time the canal is lengthened in a lateral direction, and opens behind, through the raphè or median fissure, to form the rhomboidal sinus, which is filled up with the remaining nucleated tissue continuous with the pia mater of the surface*.

I have now to describe the development of the intervertebral ganglia.

In the young fœtus, the first thing that strikes the observer on looking at these ganglia, is their enormous size in comparison with the diameter of the spinal cord. Fig. 8, *e*, *e*, Plate XLVI. represents them as seen in a transverse section of the vertebral column of a human fœtus of 9 weeks. In this section each ganglion appears nearly as large as the entire lateral half of the grey substance of the cord itself. It is closely invested by a fibrous sheath, which is prolonged on to the nerve-roots, and is continuous with the surrounding connective tissue, as well as with the denser nucleated tissue that constitutes the lamina (*z*) of the vertebra. In a fœtal sheep of 1 inch in length, the ganglion consisted of a mass of closely aggregated nuclei or cells, connected together by a network of fibres. Fig. 23, *a*, Plate XLVII. represents a small portion magnified 420 diameters. The fibres connecting the nuclei or cells were continuous on the one hand with the nerve-roots which entered the ganglion, and on the other hand with the surrounding connective tissue. Their course was most conspicuous in an antero-posterior direction—that is, in the direction in which the nerve-roots spread through the ganglion. The nuclei or cells were about the same size as those of the grey substance of the cord to which they belonged, but were rather more varied in shape. Like them also, at this period of development, they had a plain and smooth appearance, without any traces of granular contents and of distinct enveloping membranes.

* Compare my figs. 34, 35, & 36, from the adult bird, Philosophical Transactions, 1859, Plate XXIII.

In a fœtal sheep of about 1¾ inch long, it was found that a considerable proportion of the nuclei or cells had increased in size, but in a variable degree. Many of them were twice as large as those in the fœtus of 1 inch, while others were still about the same size. Fig. 24, Plate XLVII. represents a small portion of the ganglion as seen in a transverse section of the spinal column and cord—that is, in the direction in which the nerve-roots enter and leave it. A great number of the nuclei or cells were also more diversified in shape; they were round, oval, triangular, irregular in outline, and variously stellate. By processes of different lengths and degrees of fineness all of them were connected with each other, as well as with fibres of the nerve-roots; and at the circumference of the ganglion this common network was continuous with the surrounding nucleated connective tissue, as at *b*, fig. 23, which represents a portion of the tissue connecting the outer part of the ganglion (*a*) with the lamina of the vertebra. Many of the round and oval nuclei evidently belonged to the connective tissue, and some few to the nerve-fibres which spread through the structure. The nerve-cells, although enlarged, had a smooth homogeneous aspect, and presented scarcely any traces of internal nuclei. Here and there, however, a faintly granular appearance was visible through their surface; and in some cases, as at *a*, fig. 24, a rounded but imperfectly-defined body, resembling an indistinct nucleus, might also be observed.

As development advanced, considerable changes were observed to take place in the appearance of the ganglion. At first, the principal changes consisted in an increase in the size and granular structure of a large number of the cells, while their nuclei remained still indistinct (see fig. 24⁺). Soon after, however, a striking alteration ensued both in the structure and form of the cells: fig. 25 represents a portion of the ganglion of a fœtal sheep, about 3 inches long, transformed from those just seen in fig. 24⁺. Here the nuclei in the cells are large, well-defined, round or oval, and contain one, two, and in some instances three globular nucleoli, surrounded by numerous granules. The cells to which they belong vary considerably both in size and shape. The majority, however, are perfectly pyriform or cup-shaped, their tapering ends pointing in different directions, while the broader end of each is occupied, or as it were closed, by its nucleus. According to the position in which they lie, and their different degrees of foreshortening, in reference to the observer, they appear either round, oval or pyriform. Sometimes several of them lie side by side in a similar position, and are continuous by their tapering ends with a corresponding series of branches from one nerve-fibre. Lodged in the spaces between them are a number of small nuclei, of an angular, oval, or elongated form, and resembling those which belong to the sheath of the ganglion (see figs. 25 & 28). When exceedingly thin sections were very carefully examined under a sufficient magnifying-power, it might sometimes be observed that the nerve-fibres, in their course through the ganglion, divided into branches, which became continuous with the processes of the cells; and when the section was carefully broken up into minute fragments by means of fine needles, these appearances were found to be universal. The nerve-fibres were then frequently seen to consist of bundles of delicate fibrillæ, each of which separated in succession from the rest to be connected with one of the nerve-cells. Fig. 26, Plate XLVII. is an exact representation of a thick fibre giving off a short branch, which immediately subdivided into two and perhaps other fine filaments, to join a corresponding number of cells. The lower and broken end of the fibre seemed to be split up into its constituent parts, and resembled the hairs of a brush. Very frequently a series of pyriform cells, in close apposition, but one in advance of the other, were arranged in a compact group around a common fibre, with a fibril of which each was connected by its tapering end. Fig. 27 represents part of such a group, magnified 420 diameters. The connecting fibrils were of different lengths, and sometimes so short that the small end of the cell seemed to arise from the fibre itself. Where the relative position of the cells forming a compact group is less regular, we have the appearance represented in fig. 28. Through such a group the nerve-fibres pursue a more or less tortuous course, giving off branches or their component fibrils in succession, and in all directions, to the cells between which they pass. In fig. 29, for example, a fibre, after giving off fibrils to be connected with the smaller ends of the two uppermost cells, continued its course between them to be connected by a fibril with the next cell, along the sides of which two other branches ran in a similar way until they reached the points of other cells; and so on in different directions and planes. Between the upper and lower ends of the figure the cells were broken off, and exposed the fibre and its short branches.

When separated from the group, a large number of the cells seemed to have scarcely any investment that might be called a distinct cell-wall; but still they were frequently more or less covered by a shaggy layer of delicate fibres (*a*, *a'*, fig. 30, Plate XLVIII.), by which in their natural position they appeared to be connected. This investment seemed to be an extension from the surface of the fibres or processes with which the cells were continuous, and occasionally entangled a small nucleus (*a'*, fig. 30). In some instances, as at *b*, it assumed the appearance of a thin, loose, and delicate sheath; while in others, as at *c*, it formed a more compact investment, to which the nuclei were more closely adherent.

As development progressed the cells somewhat enlarged, while their walls increased in thickness, and, like the nerve-fibres, were studded with an increasing number of small nuclei. Fig. 31, Plate XLVIII. shows a portion of one of the intervertebral ganglia of a fœtal sheep, 8 inches long, magnified 420 diameters. Many of the cells, which had a globular appearance in their natural positions, were found to be pyriform when separated by dissection. Their walls were evidently prolongations from the surfaces of the nerve-fibres; the nuclei on the former were in every respect similar to those on the latter; and in both cases their number increased in the same proportion. Sometimes a nucleus was found at the point where a fibre was continuous with a cell; so that it was impossible to say whether it belonged to the one or the other. Fig. 32 represents a group of cells from the anterior part of an intervertebral ganglion of the chick, on the ninth day of incubation, at the point where the nerve-fibres are escaping to join the anterior roots. The free surfaces of the walls have a shaggy structure consisting of fine fibres, by which they are connected with the walls of their neighbours, and apparently with the nucleated investment of the nerve-fibres which run between them *. Sometimes, in consequence probably of the action of the chromic acid, the cell-wall is removed to a little distance from the surface of the contained cell, but is still connected with it by fine fibres or processes. Now in the case of the large nerve-cells of the spinal cord, like those represented in fig. 12, *x x*, the thick cell-walls, as development advances, form part of the surrounding reticular tissue, with which the surface of the contained cells, still retaining a thin investment, is in a similar way connected by processes. Sometimes there remains around the stellate cell a more or less circular space enclosed by what seems to be the inner surface of the original cell-wall, of which the outer portion has blended with the surrounding tissue.

In the account thus given of the development of the spinal cord, I have carefully refrained from indulging in any theoretical views, and have confined myself solely to a faithful description of what was actually seen. A few remarks, however, are required in further explanation of some of the observed facts, and in reference to the conclusions that may be drawn from them.

We have seen that in its earliest stage of development, the spinal cord consists of a canal surrounded only by one uniform or homogeneous layer of small cells or nuclei, which are not distinguishable from each other in appearance, and are so closely aggregated as to seem in actual contact. To call this single layer the epithelium of the cord, appears to me about as incorrect as it would be to call the germinal membrane of the ovum the mucous or internal of the two layers into which it immediately separates; for in the second stage of its development, we find that this single and homogeneous layer constituting the entire substance of the cord, while it continues to increase in depth,

* The connexion of nerve-cells with each other by means of prolongations of their sheaths is well seen in many of the Invertebrata. Fig. 33 shows such a connexion between five cells, from one of the groups of ganglia composing the subœsophageal mass of the common Slug. The central space between these cells is occupied by portions of nerve-fibres running in different directions, as well as by connective tissue continuous with prolongations of the cell-sheaths. It is not, however, denied that some of the processes contain prolongations from the *interior* of the cells. These cells differ considerably both in size and shape. Many of them are enormous. In fig. 33, Plate XLVIII. they are magnified only 420 diameters, and in some parts of the ganglion they reach the prodigious size represented in fig. 34 under the same magnifying-power. The nuclei within the cells are also of extraordinary dimensions, and filled with exceedingly coarse granules, amongst which are a variable number of round nucleoli enclosing some finer granules. Between the larger cells are others of a much smaller but variable size, many of them not exceeding one-fourth the diameter of the *nuclei* of those represented in fig. 33. A great number of the cells are pyriform, and lie side by side in regular rows. The cephalic or supraœsophageal ganglion has an entirely different structure. It consists of two lobes, united above the œsophagus by a thick transverse band or commissure. The portion of each lobe on the side of the median line is a globular and finely granular mass, traversed by a multitude of exceedingly fine fibres proceeding from a somewhat hemispherical layer of small and closely aggregated cells, which partially encloses the inner mass, and forms the lateral crust of the ganglion.

undergoes a differentiation into two distinct layers—an inner, constituting the true epithelium, and an outer, constituting the grey substance; and although the former does not undergo the histological and morphological changes which subsequently take place in the mucous layer of the germinal membrane, yet it is probable that *after* the production of the outer or grey substance, it differs in its histological character from the originally homogeneous layer. This differentiation of structure proceeds very gradually, and is not at first marked by any decided line of separation, or by any difference in the appearance of the structural elements. At the same time there is gradually formed around, and apparently secreted from, the small cells or nuclei, a granular substance that forms into processes or fibres, and constitutes a continuous network by which all the nuclei or cells of both layers are uninterruptedly connected. In the grey substance itself there is at first little or no apparent difference in structure between its anterior and posterior portions, although in each portion darker and more-closely aggregated groups of nuclei may be observed in connexion with the roots of the nerves. But as development progresses, a diversity of structure ensues; for while the nuclei of the *posterior* grey substance, although rather more granular than at first, have scarcely advanced in size, those of the *anterior* substance have increased to double their original diameter, and are connected by thicker fibres, which form a coarser and more granular network. At the same time, around the separate groups of the latter substance, the granular network (seen in fig. 35, Plate XLVIII.) between the nuclei assumes a more sponge-like structure, as represented at *v*, fig. 12. Meanwhile, within the group, there are formed from the nuclei a number of large, roundish or irregular but adjacent cells, with thick nucleated walls (*x x*, fig. 12). It is quite evident that the nucleated tissue constituting the walls of these cavities and the network around them is in every respect similar in appearance to that which is very commonly assumed by the connective tissue of parts external to the cord, as may be seen, for instance, at *b*, fig. 23, Plate XLVII., which represents a portion of the connective tissue on the outer surface of one of the intervertebral ganglia, with the substance of which, however, it is directly continuous.

It appears, then, that in these early stages of development there are at least two kinds of free nuclei in the grey substance of the cord. The one kind appear to develope the general network of tissue which pervades the entire structure, but proceed no further; whereas each of the other kind, while connected with this network as well as with nerve-fibres, developes a nucleated cell with a nucleated wall which is still connected, and ultimately blended, with the surrounding reticular structure. In the cells of the intervertebral ganglia, although the process appears to be *essentially* the same, there is some difference in the appearance they present in the earlier stages of development. Fig. 23, *a*, Plate XLVII. is a small portion of one of the intervertebral ganglia of a fœtal sheep, 1 inch long. Fig. 35, Plate XLVIII. is a portion of the dark group of nuclei in the anterior grey substance of the cord, destined to be developed into large nerve-cells, from the same fœtus. In the former the nuclei are joined by more sharply defined fibres, and there is an absence of the delicate granular network surrounding the nuclei in the latter. As development advances, these nuclei or small cells of the intervertebral ganglia simply enlarge, at first, but at the same time are connected with each other and with intervening granular nuclei by fine fibres, as shown in fig. 24, Plate XLVII. At a later period a very distinct and well-defined nucleus, surrounded by a variously shaped granular mass, makes its appearance within each cell; while its surface becomes the cell-wall, which at first is thin, but gradually increases in thickness. Now the surfaces of these cells are in connexion not only with the intervening nucleated fibres (as shown in fig. 24) which are continuous with the connective tissue forming the sheath of the ganglion, but also with the walls of the adjacent blood-vessels, as seen at *c*, *c* in the same figure; and it is probably through the medium of this nucleated tissue that the developing cells are supplied with nutritive fluid. Indeed, if we except the muscular fibre-cells, with which some of them are provided, the walls of the blood-vessels are only a part of the pia mater and connective tissue between them. On examining the layer of pia mater which immediately surrounds the cord, it may be seen to connect the walls of the blood-vessels which it contains with the sheaths of the intervertebral ganglia, and, through this, with the sheaths of their nerve-cells, on one side, and on the other side with the connective tissue or pia mater within the cord itself. In fig. 36, Plate XLVIII., *a a'* represent the outer surface of the cord; *b b* the outer part of one of the intervertebral ganglia; and *c c'* the intervening layer of pia mater, containing blood-vessels, *d*, *e*. On its left side, the walls of the vessels (*e*, *d*), containing some globules, are seen to be connected by a continuous and nucleated network with the sheath of the ganglion, *b b*; and from the right side of the walls a series of processes or fibres enter the cord (particularly at *a*, *a'*), in which they are continuous with the reticular tissue of both its white and grey substance. At *c'* there intervenes, between the transversely-cut vessel (*d*) and the cord, a layer of pia mater, from which similar processes are derived. These processes are not branches of the blood-vessels; and they enter the cord around the whole of its circumference. Now, in the anterior median fissure, it may be seen beyond all question, as already stated, that this layer of pia mater, with the walls of the blood-vessels which it contains, is directly continuous with the processes of the epithelium surrounding the front of the canal (see figs. 4, 8, 9, & 14); and therefore we are warranted in concluding that a similar continuity exists within and around the remaining parts of the cord. But this is only an *à priori* confirmation of what I long ago actually observed and described in the adult cord*. And since the sheaths of the nerve-cells have been shown to be continuous with, and indeed to form a part of, the reticular connective tissue of the cord, which is itself continuous with the pia mater of the surface, it is evident that the processes of the epithelium, the pia mater and connective tissue within the cord, the walls of the blood-vessels, and the sheaths of the nerve-cells must be all uninterruptedly continuous with each other. But the sheaths of the nerve-cells are certainly connected with the surfaces of their granular contents; and in the fully-developed cord, where the cell-

* Philosophical Transactions, 1851 and 1859.

sheaths are much finer and thinner, processes from the cells are continuous by fine subdivisions with the surrounding reticular structure, as shown in fig. 17, Plate XLVII. These observations, then, appear to throw important light on the question which I formerly proposed, as to whether there is any actual and essential difference between the connective and the true nerve-tissue, or "whether the connective tissue of the cord be intermediate in its nature, passing on the one hand into *nerve*-tissue, and on the other into *pia mater*"*. We have seen that the cell-sheath or wall is the product of, and indeed is constituted by, the very surface of the primitive nucleus or cell, and that, while it ever after remains in connexion with its contents, it forms a part of the surrounding connective tissue, which is itself a prolongation not only of the pia mater of the surface, as well as the walls of the blood-vessels, but also of the processes of the epithelium. But although there is this uninterrupted continuity between all the constituent elements of the cord—although, perhaps, the nerve-tissue actually changes by insensible degrees into the tissues with which it is continuous—and although the cell-wall, which forms part of the surrounding reticular structure, is a product of the primitive nucleus, there is yet no ground for believing that the connective tissue, as such, can ever develope itself into nerve-tissue, any more than that any one of the differentiated parts of a fully-developed organ can reproduce the entire structure; for the nerve-cell, although it developes from itself its own sheath, which forms part of the nucleated connective tissue, produces something more than this tissue, viz. the granular contents of the cell †.

We know that processes of the nerve-cells constitute the axis-cylinders of the vaso-motor nerves distributed to parts external to the cord; and therefore it seems probable that the finer processes which are lost by subdivision in the pia mater or connective tissue within the cord are the means of transmitting nerve-power to that tissue and to the coats of its blood-vessels, from which, by their uninterrupted connexion with them, as already shown, the nerve-cells in return receive their supply of nutriment.

* Philosophical Transactions, 1859, Part I. p. 442.

† There can be no doubt that a considerable proportion of the gelatinous substance and other parts of the posterior cornu are of the nature of pia mater; but amongst this there are numerous small nerve-cells. As I have dwelt, however, on this point in another place (Phil. Trans. 1859), I need not repeat my remarks.

EXPLANATION OF THE PLATES.

PLATE XLV.

Fig. 1. A transverse section of the spinal column and cord in the upper part of the lumbar enlargement of a fœtal sheep $\frac{3}{4}$ inch in length, magnified 60 diameters:—*h*, anterior white column; *f*, anterior grey substance; *g*, anterior nerve-roots; *c*, posterior white column; *b b'*, posterior grey substance; *d*, posterior nerve-roots; *e*, intervertebral ganglion; *a*, epithelial layer surrounding the front of the canal; *j j*, body of vertebra; *k*, chorda dorsalis; *i*, connective tissue uniting the body of the vertebra and intervertebral ganglia to the circumference of the cord.

Fig. 2. A portion of the outer surface of the posterior grey substance (*b*) in connexion with the inner surface of the posterior white column (*a*); magnified 420 diameters; from the same fœtus.

Fig. 3. Transverse section of spinal cord in the upper part of the lumbar enlargement, magnified 60 diameters, from a fœtal sheep 1 inch in length:— *h'*, lateral column.

Fig. 4. Transverse section of spinal cord in the upper part of the lumbar enlargement, magnified 60 diameters, from a fœtal sheep 2 inches long:—*h'*, lateral white column; *r*, gelatinous substance; *p*, *p'*, inner portions of posterior white column; *l b*, caput cornu; *n*, posterior median fissure; *q q*, cervix cornu; *w*, *w*, groups of large nerve-cells of anterior cornu; *o*, central canal; *m*, anterior median fissure.

Fig. 5. Transverse section of the spinal cord of the same fœtus, from the middle of the dorsal region.

Fig. 6. A similar section from the upper part of the dorsal region.

Fig. 7. Another, from the middle of the cervical enlargement.

Fig. 9. A portion of the section represented in fig. 8, magnified 420 diameters:—*o*, anterior portion of the canal; *s s*, epithelium surrounding it; *f*, nuclei forming the anterior grey substance; *g*, inner bundle of anterior roots entering the grey substance; *h*, *h*, inner parts of the anterior white columns; *m*, anterior median fissure between them. The epithelium is seen to be continuous on the one hand with the network of fibres between the nuclei of the anterior grey substance (*f*), and on the other hand with the network of pia mater (*m*) prolonged from the circumference of the cord into the anterior median fissure.

Fig. 9+. Fresh nerve-fibres from the sciatic nerve of a human fœtus of four months: on the left is a single fibre; on the right several are seen as they lie together constituting a nerve.

Fig. 10. A portion of the grey substance near the edge of the posterior cornu traversed by fibres of the posterior roots (*d*). From a fœtal sheep $2\frac{1}{2}$ inches long; magnified 670 diameters.

PLATE XLVI.

Fig. 8. A transverse section through the spinal column and cord of a human fœtus of nine weeks, from the cervical enlargement; magnified 50 diameters;—*z*, lamina of vertebra; *z'*, muscular fibres.

Fig. 13. One, and part of the other, lateral half of a transverse section through the lumbar enlargement of the spinal cord of a fœtal sheep 4 inches long:—*l b*, caput cornu posterioris.

Fig. 14. A similar portion of a transverse section through the middle of the lumbar enlargement of a fœtal ox 5 inches long:—*l b*, caput cornu posterioris, consisting of a dark mass of closely aggregated nuclei. From the outer edge of this mass, as well as from that of the rest of the grey substance, nuclei are scattered in smaller numbers through the white columns.

PLATE XLVII.

Fig. 11. Grey substance from the middle of the anterior cornu, traversed by anterior nerve-roots; from a fœtal sheep $2\frac{1}{2}$ inches long; magnified 670 diameters.

Fig. 12. Portion of the anterior grey and white substances of a fœtal sheep $2\frac{1}{2}$ inches long; magnified 420 diameters:—*y y'*, internal part of the anterior white column, bordering the grey substance; *x x*, group of large nerve-cells in process of development; *v*, granular and nucleated network nearer the middle of the anterior cornu.

Fig. 15. Portions of a longitudinal section of the grey substance of the same fœtus, in a direction before-backward; magnified 420 diameters. I. Part of the caput cornu posterioris, intersected by the deep decussating fibres of the posterior roots, and posterior white column. II. Middle portion of the grey substance (between *b* and *f*, fig. 14), in which the nuclei are much larger, and the network between them becomes gradually coarser and looser as it proceeds forward. III. A group of the large nerve-cells of the anterior cornu (*w w*, fig. 14), surrounded by thick walls, with intervening nuclei.

Fig. 16. Portion of a longitudinal section of the middle of the grey substance (corresponding to II, fig. 15), from a fœtal sheep 8 inches long; magnified 420 diameters. Numerous fusiform, triangular, and crescentic cells have become developed in it.

Fig. 17. Stellate nerve-cell from the nucleus cervicis cornu (posterior vesicular column) of a human fœtus of six months; magnified 420 diameters. Some of the processes are seen ramifying and becoming continuous with the surrounding network. It is not often seen so distinctly as in this case.

Fig. 17+. Cells from the posterior grey substance of a human fœtus of four months; ×670.

Fig. 18. A portion of one of the posterior white columns of the spinal cord of a human fœtus of five months, showing the numerous nuclei with which it is interspersed; magnified 670 diameters.

Fig. 19. A transverse section of the spinal column and cord of the chick at the end of the fifth day of incubation; magnified 60 diameters.

Fig. 20. A similar section of the cord at the end of the ninth day of incubation:—*n'*, large mass of nucleated connective tissue replacing the inner portion of the posterior columns.

Fig. 21. A transverse section of the cord of the same, through the lower part of the sacral enlargement.

Fig. 22. A similar section a little higher up, between figs. 20 & 21.

Fig. 23. Part of an intervertebral ganglion and surrounding connective tissue of a fœtal sheep 1 inch long:—*a*, nuclei or small cells of the ganglion; *b*, connective tissue on its outer surface; magnified 420 diameters.

Fig. 24. The same from a fœtal sheep $1\frac{3}{4}$ inch in length. The nerve-cells have enlarged and are connected by a continuous network with each other, with the nerve-fibres, with intervening granular nuclei, as at *b'*, and with nucleated fibres connecting the ganglion with the lamina of the vertebra, at *b*. At *a* indistinct nuclei are seen in the interior of the cells; magnified 420 diameters.

Fig. 24*. Cells from the same at a little later period of fœtal life; $\times$ 420.

Fig. 25. Portion of a transverse section of the intervertebral ganglion of a fœtal sheep about 3 inches long. The cells have increased in size, are pyriform, a well-defined nucleus has made its appearance in each, and between the cells the interspaces are occupied by small angular or elongated nuclei.

Fig. 25*. Cells from the intervertebral ganglion of a human fœtus of nine weeks; from one of the ganglia represented at *e*, *e*, fig. 8, Plate XLVI.

Fig. 26. A nerve-fibre dividing into branches to be connected with cells. From the intervertebral ganglion of a sheep 3 inches long; magnified 670 diameters.

Fig. 27. Another fibre connected with cells by division.

Fig. 28. A group of cells in their natural position connected with ramifying nerve-fibres; small nuclei occupy the spaces between them; magnified 420 diameters.

Fig. 29. Portion of such a group, showing the manner in which the nerve-fibres are connected with it.

PLATE XLVIII.

Fig. 30. Isolated nerve-cells from the same.

Fig. 31. Group of cells from the intervertebral ganglion of a fœtal sheep 8 inches long; magnified 420 diameters: each cell is enveloped in a thick nucleated sheath.

Fig. 32. Group of cells, with nerve-fibres, from the anterior part of one of the intervertebral ganglia of the chick at the end of the ninth day of incubation; magni-

fied 420 diameters. At *a* the cells are in an earlier state of development, neither their walls nor their nuclei being yet very distinct.

Fig. 33. Five large and two small cells from one of the group of subœsophageal ganglia of the common slug; magnified 420 diameters.

Fig. 34. An enormous cell from the same; magnified 420 diameters.

Fig. 35. Group of small cells or apparent nuclei, destined to be developed into the large cells of the anterior cornu, from a fœtal sheep 1 inch in length. The cells are surrounded and connected by a delicate granular network.

Fig. 36. Part of the spinal cord, intervertebral ganglion, and intervening pia mater from a fœtal sheep 3 inches long:—*a a'*, outer surface of the cord; *b b*, sheath and outer portion of intervertebral ganglion, with some of the marginal cells; *c c'*, intervening pia mater; *d*, a blood-vessel cut transversely and full of blood-globules; *e*, another blood-vessel, running round the cord: at *a* transverse processes are seen proceeding from the nucleated wall of the vessel to the surface of the cord; at *a'* similar processes are given off from the pia mater, which is merely a continuation of the nucleated walls of the vessels. On the other side, the pia mater and walls of the blood-vessels are seen to be continuous with the nucleated investment of the intervertebral ganglion, *b b*.

Figs. 37 to 44 are exact representations of transverse sections of the spinal cord of a human fœtus, all magnified about 34 diameters. The actual and relative quantities of the grey and white substance are well seen in each.

Fig. 37. A transverse section of the conus medullaris:—*l b*, caput cornu posterioris; *f*, anterior cornu; *o*, canal; *c*, posterior white column; *h'*, lateral white column; *h*, anterior white column; *m*, anterior median fissure; *n*, posterior median fissure.

Fig. 38. A transverse section through the lower third of the lumbar enlargement:—*d*, posterior nerve-roots; *q*, posterior vesicular column; *w*, increasing groups of large nerve-cells in the anterior cornu.

Fig. 39. A similar section through the middle of lumbar enlargement. The posterior vesicular column (*q*) forming the inner half of the cervix cornu has enlarged, but hitherto consists chiefly of a multitude of *small* cells. Through it and on its outer side several curved bundles of the posterior roots sweep forward and inward, and separate it from the outer half of the cervix, at the border of which are several dark spots, representing the cut ends of longitudinal bundles. In front of the canal (*o*) are the decussating fibres of the anterior commissure, and behind it is the posterior commissure. The group of nerve-cells in the anterior cornu has much enlarged.

Fig. 40. Similar section through the upper third of the lumbar enlargement. Here the *cells* of the posterior vesicular column (*q*) or nucleus of the cervix cornu have increased considerably in size: they are nearly all equal to those of the anterior cornu. The groups in the anterior cornu are much diminished.

Fig. 41. Section through the lowest part of the dorsal region, or upper end of the lumbar enlargement. Here the posterior vesicular column (q) is larger than in any other part of the cord, and consists chiefly of large, oval and stellate cells. Here also we first distinctly see the *tractus intermedio-lateralis* (t), a tract of smaller cells between the anterior and posterior cornua, and projecting in a conical form from the border of the grey substance into the lateral white column.

Fig. 42. Another section, through the *middle* of the dorsal region. The posterior vesicular column diminishes in size.

Fig. 43. A section through the *upper* part of the dorsal region; the tractus intermedio-lateralis (t) is very prominent.

Fig. 44. Another, through the middle of the cervical enlargement:—e', the posterior lateral fissure, through which the outer fibres of the posterior roots (d) are seen to reach the dark spots or longitudinal bundles on the outer side of the cervix cornu, with which they become continuous. These longitudinal bundles are more numerous here than in any other region of the cord. On the inner side of the cornu, other fibres of the roots are seen sweeping both around and within the posterior vesicular column (q). This column is here again large, but consists chiefly of a multitude of *small* nerve-cells. The groups of nerve-cells in the anterior cornu have again increased in size. These groups, and indeed the whole of the anterior cornu, as well as the outer part of the cervix of the posterior cornu, are supplied, as represented, on the left side, by a beautiful branch of a large blood-vessel (v), which enters through the anterior median fissure (m), and bifurcates right and left, at its bottom, through the anterior commissure. The parts behind the canal are supplied by other vessels running transversely and derived in part from a larger longitudinal vessel on each side of the canal, and of which the cut ends are seen in the figure. In the lateral white column is a somewhat oval space (g') occupying nearly the whole of its area, and of a lighter and greyer colour than the rest. This is apparently due to a greater abundance of blood-vessels and pia mater. It is limited chiefly to the cervical and dorsal region. The wedge-shaped column (p') on each side of the posterior median fissure, and forming part of the posterior white column, is very strongly marked in this figure. Its tapering end is gradually lost in the deep part of the column on its outer side. At this period they are limited almost entirely to the cervical enlargement; at an earlier period they may be traced lower down (see figs. 4, 5, 6, & 7).

Fig. 45. Part of the chorda dorsalis of the chick at the end of the ninth day of incubation, with some of the surrounding cartilage-cells:—*a*, chorda dorsalis, consisting of a nucleated network of fibres, having precisely the appearance of connective tissue; *b*, cartilage-cells around its circumference, in different stages of development: in one the nucleus is seen undergoing division.

XXXVII. *On Spectra of Electric Light, as modified by the Nature of the Electrodes and the Media of Discharge.* *By the Rev.* T. R. Robinson, *D.D., F.R.S., &c.*

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The important discovery of Wheatstone, that the spectra of electric sparks contain brilliant lines whose character depends on the nature of the electrodes, after being almost neglected for several years, has lately become an object of great interest; and much additional light has been thrown on it by several physicists, of whom Masson, Ångström, Plücker, and Kirchhoff are the most conspicuous. Ångström has announced as a general law that the lines in question are produced by the electric current igniting the medium in which the discharge takes place, or molecules of the electrodes which are torn off by its passage, that each of these actions produces its own spectrum, and that those spectra are simply superposed without any modification. The gases with which he worked were at the ordinary pressure. Plücker, on the other hand, used the well-known Geissler tubes, which contain minute proportions of highly rarefied gas or vapour. He attaches very little importance to the lines due to the electrodes (*metallic* lines), which he thinks are confined to those portions of the spark near the electrodes; and he maintains that in the centre of an exhausted tube of some length only gaseous lines are seen. He was embarrassed in several instances by the decomposition or absorption of the gaseous media; and there must always be some doubt as to the precise nature of these media, as the tubes are hermetically sealed. On the other hand, Kirchhoff seems to attach most importance to the metallic lines, whose influence he has exhibited to a wonderful extent by a spectrum-apparatus probably unrivalled. All hold the doctrine of an essential connexion between the character of the spectral lines and the chemical nature of the substances which are present in the track of the discharge; and the last, in conjunction with Bunsen, has based on this principle the new system of spectral analysis, which is rapidly becoming popular, and has applied it to explain the dark lines of the solar spectrum in a way which, if not absolutely certain, is singularly elegant. Yet it is impossible to overlook the fact that, in all this, much is assumed, not proved. Has it been established that these lines depend so absolutely on chemical character that none of them can be common to two or more different bodies? Has it been ascertained that, while *the chemical nature* of the bodies present remains unchanged, the lines never vary if the circumstances of mass, density, &c. are changed? And what evidence have we that spectra are superposed, so that we observe the full sum of the spectra which the

* The continuation of the paper, from p. 974 to the end, was not received complete till September 1; but the conclusions therein contained are embodied in the abstract presented to the Society on June 19, 1862.

electrodes and medium would produce separately? Lastly, is it certain that electricity produces light merely by its heating power? and may not the same action which produces thermic vibrations, also (and independently) produce luminous?

Some of these questions I have endeavoured; to investigate and the attempt can scarcely fail to be of use, were it only to direct attention to the subject.

My attention was originally turned to it by observing, some years ago, that the discharge in carbonic oxide, which is white at common pressures, becomes bright green when the gas is highly rarefied. The spectra in these cases differed so much that I determined to examine them in various gases and metals; and I procured the apparatus which seemed necessary. That of STEINHEIL had not been contrived then; and my want of experience in such researches caused at first some mistakes, but by degrees I corrected them; and though my instrument is of comparatively limited power, the results are, I hope, not without importance; and I have given details sufficient to estimate the errors which may affect them.

The source of electricity in my experiments was an induction-machine. Till July 1860 I used one by HEARDER, having three miles of secondary wire; after that, one which I have described*, with the substitution of copper wire for iron. It has 9·5 miles, of which three are not lapped, merely varnished; and when excited by three Grove's, each of which has 49 square inches of platinum acting, it gives abundantly sparks 6·8 inches long. By a simple device its two coils can be in an instant made to act collaterally instead of consecutively, thus reducing the intensity but doubling the quantity. Then the sparks are only 2·5 inches, but very dense and luminous, and possessing a far higher power of ignition. A Leyden jar, each coating 1·25 foot, was normally connected with the terminals, as the spectrum of the simple spark is far more faint and unsteady: I have given one as a specimen in Table III. With this jar there is a continuous stream of discharge at 0·6 inch.

The discharges were made in the open air, or in tubes about 0·2 inch diameter and 6 inches long. With rarefied gases they would be more luminous if the tubes were capillary; but in that case it would be very difficult to clean them from the coatings of metal or oxide which are deposited, most densely near the negative electrode, but often on the whole interval. In some metals, of which the most notable are lead, cadmium, bismuth, antimony, arsenic, and above all tellurium, this deposit was so thick that at the close of an experiment it was difficult to see the fainter lines, and I found it necessary to employ for common density a tube 1 inch diameter. Even this was coated, but not so thickly as to give much trouble. The upper electrode (which was, in all cases but one, positive) was attached to platinum wire fused in the tube. It was sometimes soldered to it, at other times twisted with it; and when I could not procure the metal as wire or foil, a globule of it was fused by a gas-blowpipe, a platinum wire inserted in this, and the heat withdrawn. For nickel and cobalt, the blowpipe was oxyhydrogen. As the electric light does not spread over the positive electrode, it was unnecessary to insulate the platinum wire; but with the negative it is otherwise; and when I had not enough of

* Philosophical Magazine, April 1859.

the body to give a length of some inches, it was inserted in a piece of quill tube, whose lower extremity was fused on platinum wire reaching up to it; and this was inserted in the discharge-tube. In tubes the distance of the electrodes was about 0·75, in open air 0·10. The discharge-tube was cemented in a cap screwed to the air-pump, or rather to a transfer piece. This is a cylinder of brass with a concave screw above, a stopcock below screwing to the air-pump, and a lateral one connected with a desiccator. The desiccator is a bottle 8 inches deep and 2 inches diameter, fitted with a ground stopper, in which are two apertures. In one of these is cemented a tube which is connected with the transfer; in the other, one descending to the bottom of the bottle, where it is drawn into a capillary point, its upper end being connected with either another desiccator or a gasometer. The desiccator is filled nearly with sulphuric acid. Supposing the transfer and the tube exhausted, shut off the air-pump cock and cautiously open that which connects the desiccator; gas bubbles up slowly through the acid and fills the transfer and tube. Shut off the desiccator, connect the pump, and exhaust. By repeating this process *ad libitum* all traces of air or any gas previously used are removed, and nothing is present but the subject of experiment. It is almost needless to say that every part of this apparatus, including the air-pump, must be *absolutely** air-tight. To ascertain whether this means of desiccation is sufficient, I used Kater's oat-beard hygrometer. In the air of the room it read at 60° 3·358 R. It was then placed under a receiver containing a capsule exposing 20 square inches of sulphuric acid for five hours at a pressure of 0·3 inch, when it read 0·455 R. The receiver was then filled with air drawn through the desiccator as rapidly as could be done without drawing acid into the pump. In 10^m it read 0·285 R. Admitting air and introducing a slip of bibulous paper moistened with water, in 40^m it was 8·480 R; and when left in the air of the room some hours it was again 3·258. It is therefore evident that in this respect nothing more can be desired. Sulphuric acid also absorbs sulphurous acid and nitric oxide; but when other acids might be present, a second desiccator was used filled with concentrated solution of potassa.

The gasometer was made of a Woulfe's bottle holding 75 cubic inches. In its central neck is cemented a siphon, one branch of which reaches to its bottom, the other to the bottom of a bottle of rather larger capacity; in the other two necks, stopcocks are cemented, one of which (A) is connected by elastic tube with the desiccator, the other (B) with any apparatus for generating gas. Suppose the Woulfe filled with water (or on a smaller scale with mercury). Let gas be supplied to B, the water will be displaced by it through the siphon into the bottle; then closing B and opening A, the transfer and its tube are filled. If the water has been boiled for a few hours and cooled without exposure to the air, I find that gas continues in this gasometer sensibly pure for a much longer period than the duration of a day's observations. For operating on vapours free from any mixture of permanent gas, I use a mercurial apparatus, consisting of a strong Woulfe's bottle in whose three necks are ground (1) a glass stopcock, (2) a tube in which

* That which I use has kept for a month a vacuum of 0·1 inch without variation of 0·01.

a platinum wire is sealed long enough to dip in the mercury and serve as the negative electrode, (3) a tube also dipping in the mercury, and with a platinum electrode at top. If the bottle be filled one-third with pure mercury, and the stopcock be connected by a caoutchouc tube* with the transfer and the apparatus exhausted, the tube can be filled with mercury, which on erecting it falls, leaving a vacuum, through which discharges can be passed by the electrodes. If the tube has been *perfectly* cleaned by filling it with nitric acid and washing it with distilled water, dried by sulphuric acid in a vacuum, and finally wiped with a morsel of linen which has been boiled in distilled water, after being fastened to a flexible wire, on inclining the tube the mercury will fill it without leaving the least speck of air, and will often adhere with considerable force. It, however, always falls at the first discharge. In this case the space is filled with mercurial vapour alone. If a few drops of any volatile fluid be introduced into the tube, by filling it with mercury the excess is expelled, and the vacuum contains only its vapour highly rarefied. With phosphorus, the tube filled with water was warmed till it fused and adhered to the upper part; the water was removed, and the tube put in its place. The apparatus was then repeatedly filled with dry nitrogen till all traces of moisture disappeared. If a platinum wire of sufficient length be introduced below, both electrodes are platinum; and if the upper part be bent so that the descending branch remains full of mercury, both are of that metal.

Prisms.—For three-fourths of the observations I used a prism by MERZ, with an angle 45° 35′·4. By sets of from eight to twelve it gives, for the deviations of FRAUNHOFER'S lines,

	° ′	μ
A	32 20·00	1·6230
B	32 32·04	1·6259
C	32 38·88	1·6285
D	32 56·50	1·6336
E	33 20·26	1·6405
F	33 41·67	1·6467
G	34 23·06	1·6587
H	35 0·06	1·6692

It is therefore nearly identical with FRAUNHOFER'S flint No. 2, but in dispersive power it is far inferior to those used by MASSON and PLÜCKER. After working with it a long time, I found that several bright but cloudy bands which it showed were resolved into two, three, or more by a prism of bisulphuret of carbon having an angle of 60°. The indices of this fluid change so much with temperature, that I did not venture to employ it; and I obtained from Mr. DUBOSCQ a prism of 60° 3′·54 angle, which, though nearly of the same density as the Merz, is more effective, in the proportion of 3 to 2. Determining with it FRAUNHOFER'S lines, and comparing with their deviations in Merz, I

* This, with which I was supplied by Messrs. SILVER, is far superior to vulcanized tube, which always leaks. The glass apparatus was made with great precision and intelligence by Mr. CASELLA.

formed by interpolation a Table of reduction to the latter, by means of which all are given on the same scale. The two glasses are so nearly similar, that it was not necessary to go beyond second differences. Even this prism leaves several bands unresolved, though often giving a suspicion of their compound character, which in some recent instances I have verified by combining *two* 60°-prisms of the bisulphuret.

It is an object of great interest to ascertain whether there be any special relations between the wave-lengths of these luminous bands. In aid of this I subjoin a Table giving the value of λ for every five minutes of MERZ's deviations within the range of my observations. It has been computed by a very simple form of interpolation given by Professor STOKES*. Assuming $\mu = A + \frac{B}{\lambda^2}$, and taking μ and $\frac{1}{\lambda^2}$ for any two of FRAUNHOFER's lines, we get, for any intermediate μ, the $\frac{1}{\lambda^2}$ *simply by proportional parts.* This is so accurate, that it gives correctly one of the intermediate lines by taking the double interval, as D from C and E; even H from F and G.

φ.	λ.	Δλ.	Δ²λ.	φ.	λ.	Δλ.	Δ²λ.	φ.	λ.	Δλ.	Δ²λ.
32° 35′	2493	−88		33° 25′	1909	−37	+2	34° 15′	1620	−21	+1
40	2405	−79	+9	30	1872	−35	+2	20	1599	−19	+2
45	2326	−69	+10	35	1837	−32	+3	25	1580	−18	+1
50	2257	−63	+6	40	1805	−31	+1	30	1562	−18	+0
55	2194	−58	+5	45	1774	−29	+2	35	1544	−18	+0
33 0	2136	−53	+5	50	1745	−27	+2	40	1526	−17	+1
5	2083	−49	+4	55	1718	−26	+1	45	1509	−16	+1
10	2034	−45	+4	34 0	1692	−25	+1	50	1493	−15	+1
15	1989	−41	+4	5	1667	−24	+1	55	1478	−14	+1
20	1948	−39	+2	10	1643	−23	+1	35 0	1464		
25	1909		+2	15	1620		+1				

Theodolite.—For the use of this instrument I am indebted to Mr. GRUBB, who made it many years since to determine the μs of the glasses for his object-glasses. It is of simple and very firm construction. A strong brass disk, supported by three screws, has on its upper surface a circle 9·5 inches diameter graduated to half degrees, and carries laterally the supports of a collimating telescope 9·5 inches focus and 1 inch aperture, which is provided with an adjustable slit. Above the disk turns a brass plate bearing two verniers in the plane of the divisions (which read to minutes), and supporting the telescope, with a triple object-glass 7 inches focus and 0·9 inch aperture. Below the disk turns another circle, similarly divided on its cylindric surface; its axis rises through that of the upper plate, and carries a table 3 inches diameter, which bears the prism. The axis of the prism is adjusted by observing the images of the slit reflected from its surfaces, which also give its angle by means of the lower circle. Some of these matters require a few remarks.

1. The telescope (and the collimator also), though sensibly achromatic on a day-object

* Report of the British Association, 1849, Trans. of Sections, p. 11.

or the moon (and very sharp), is over-corrected for G and the rays beyond it: as no provision was made for changing the distance of its object-glass from the system of wires, I at first had some difficulty from parallax at that end of the spectrum, till such an adjustment was applied. The eyepiece (positive, magnifying nine times) was also not achromatic, and had to be constantly focused. It was replaced by a microscope's objective about 0·4 inch focus, magnifying 14 with the prism. A higher power, and even a larger object-glass, avails little in comparison of an increase of the prism's dispersion. Thirdly, I found the cross of spider's lines useless, except for the brightest lines. In the spectra of rarefied gases, which are very faint, it is difficult to see them, and I substituted the point of a fine needle, carefully ground to be a sharp wedge. One can estimate very nicely the equality of the tongues of light on each side. These changes were not made till about fifty spectra had been measured, which, therefore, are not as well determined as the rest.

2. The collimator certainly possesses advantages over the simple slit which FRAUNHOFER and his predecessors used in studying the spectrum. It requires no correction for parallax, secures from any accidental shift of the theodolite, and brings the observer close to his work; but it has the great defect of diminishing the light. Unless the slit be very narrow, it is impossible to distinguish close and fine lines. I find that with the instrument which I am describing, and the Merz prism, I cannot see D double if the slit subtend more than 72″, which corresponds to a width of $\frac{1}{259}$ inch. If it be 3′ (the opening used by PLÜCKER) I cannot see any of FRAUNHOFER'S lines, and the finer parts of electric spectra are lost. Obviously the quantity of light must diminish with the slit, and the evil is made greater by the necessity of keeping the latter at some distance from the discharge. If it be nearer than 1·5 inch, or at most 1 inch, the inductive action of the spark *charges* the theodolite, and the observer, on applying his eye to the telescope, gets a stream of pungent sparks anything but pleasant. But in order that the whole object-glass may be illuminated, we must have this distance less than $\frac{f^2}{\text{A cosecant } 72''-f}$, A and f being the aperture and focus. This limit in the present instance $=0{\cdot}03$ inch, from which it is obvious that a comparatively small portion of the light can reach the object-glass. I was led to this discussion by a fact which at first startled me a little. When obtaining the deviations of FRAUNHOFER'S lines, I was surprised to find that H was not visible to me, nor any line beyond h; though in 1838, with the same prism, but with a slit in the shutter 15 feet distant, I saw several beyond K. I concluded that my eyes had become insensible to rays of short wave-length (in analogy to WOLLASTON'S inaudible sounds), or that their humours had undergone some change by which they absorbed that part of the spectrum; but never suspected the collimator. However, last year, while examining a very fine Munich grating belonging to Mr. STOKES, I was surprised to find that it showed me the missing lines perfectly: here the aperture was a slit $\frac{1}{80}$ at 18 feet. He was so kind as to entrust me with the grating; and on my return home I found that with the collimator it behaved no better than the prisms had done, but that with

the plain slit it recovered its power: the same was the case with my prisms. That the sole cause of this was the narrowness of the slit, and not any peculiar action of the collimator, I verified by attaching to the stand of the theodolite a good object-glass of 8 feet focus, and placing it at that distance from the slit ($\frac{1}{30}$). This arrangement gave vision that was first-rate, and showed lines in an electric spectrum 14′ beyond the last of those that were visible in the usual mode of observing. It follows from these facts, that the collimator's focus should be as long as possible consistent with convenience; and I will suggest that it should be a cassegrain instead of an achromatic. The *equivalent* focus of this is from six to seven times that of the large mirror, and I find by a rough trial that the image is very sharp. If the mirrors of A were glass silvered on FOUCAULT'S plan, it would have as much light as the achromatic, and be free from all chromatic error. I have recently found that the brightness of faint lines is much improved by using a cylindric lens before the slit. The slit was generally one minute wide.

3. The precision of the angles measured depends, on the determination of the index-correction, on the precision of the bisection, and on the reading of the circle. The zero was obtained at the beginning, and also often at the end of each set, by bisecting the slit when illuminated by nearly homogeneous light. At first this was done by interposing a slip of red glass; latterly, in preference, by using the flame of a Bunsen burner in which chloride of sodium was present. The bisection is sometimes doubtful from the flicker of the discharges, more frequently from faintness of the lines, whether intrinsic or relative to the ground on which they are seen, and occasionally from want of sufficient light to see the point. The verniers read only to minutes; but the half minute is easily estimated, and so 0′·25 may be considered the uncertainty of reading. It is desirable to form some estimate of the probable amount of error due to the combined action of the three. This is done by observing twice over the same spectra, and comparing the differences of the observed lines; from which may be obtained the probable error, and the probabilities of given observed differences being errors of observation, or evidences of the lines not being identical. Unless the graduation were much finer than it is, this process would not be of any real value; and a much simpler one may serve. In this work 325 such differences were observed, of which number there were,

				Comp.
Equal to 0′	86		86	239
From 0′ to ±0′·5	131	Not exceeding 0′·5	217	108
From ±0′·5 to ±1′·0	89	Not exceeding 1′·0	306	19
From ±1′·0 to ±1′·5	15	Not above 1′·5	321	4
Above 1′·5	4			

We may reason thus: if the difference of the places of a certain line in two spectra exceed 0′·5, either the lines are identical and the difference is error, or they are distinct; the probability of the first happening $=\frac{108}{325}$, and therefore that of the other $=\frac{217}{325}$, or it is 2 to 1 they are not the same. If the difference exceed 1′·0, it is 16 to 1, if 1′·5, 80 to 1. Two things, however, must be remembered: these errors may be $\pm$; and therefore, in comparing a series, the limit is nearly doubled; and further, these probabilities may be much modified by other circumstances. For instance, a line or band may be identified by some peculiarity, even if the difference be greater than the probable limit. In general I would fix the limit to $\pm 1'{\cdot}0$ from the mean.

I at first read both verniers; but found this consumed so much time, that I determined the excentricity of the upper circle and allowed for it. This correction for the vernier A $=0'{\cdot}75 \times \sin(160°+\phi)$.

The gases which I selected for experiment are—1, air; 2, nitrogen obtained by heating nitrite of potassa with saturated solution of sal-ammoniac; 3, oxygen; 4, hydrogen: these last two were obtained by electrolysis of *pure* oil of vitriol diluted with eight volumes of distilled water, in a voltameter of peculiar construction. A porous cell has a cover cemented on it with three tubulures; one for admitting the dilute acid; one for a strong platinum wire to which is soldered with gold a platinum sheet, exposing 19 square inches; the third carries away the evolved gas. Round this cell a larger platinum is rolled, and it is immersed in a jar filled with the same dilute acid. When connected with the three Grove's already mentioned, it gives 8 inches of hydrogen per minute. Both the gases I believe to be quite pure, except as to ozone, when thus obtained. 5. The carbonic oxide was got by heating sulphuric acid with ferrocyanide of potassium.

As to the metals, the platinum and silver were obtained from Messrs. JOHNSON and MATTHEY, the aluminium from Paris. I am indebted to the kindness of Dr. MATTHIESSEN for calcium, tellurium, and gold. The palladium was given to me by Dr. WOLLASTON; tin, lead, and bismuth reduced from oxides carefully prepared; zinc, iron, and antimony, deposited by electrolysis on platinum wires. Cadmium, nickel, cobalt, magnesium, sodium, and potassium were got from Messrs. JACKSON and TOWNSON.

The spectra at common pressure (C.P.) are in general magnificent objects. Their ground seems to be a continuous spectrum, of which, however, the brightness varies very much with different substances and at different parts. Sometimes, especially at the violet end, this ground is so faint that its presence might be questioned; but I believe it exists even there. On this arc, as it were, superposed luminous lines of every degree of brightness, from a splendour almost insupportable, to a faintness such that (at least with my optical means) the least glimmer of diffused light in the telescope totally effaces them. Most of these lines are (as might be expected) as broad as the image of the slit: a few are much broader, even to six or seven times. Such are, I think, always cloudy and ill defined, giving the impression that they are groups of finer lines, which the optical power is insufficient to separate. In several instances this is shown to be the fact by the combination of two fluid prisms of 60° (2BS.C), of which the two most remarkable are those which I call ζ, in the green, and $\varkappa''$, at the beginning of the violet. The first was seen in the Merz prism as a broad bright band, but it is a crowd of very fine lines, of which the central one is much the brightest; and some of the others are developed in different spectra: the second (not resolved in Duboscq) consists of six bright and sharp lines. Even this power fails to decompose the remarkable blue-green and violet bands which characterize the hydrogen spectra C.P. Perhaps a third prism might, but there is no room for it on the theodolite.

A third class of lines is narrower than the slit, down to the finest hairbreadth; they are mostly sharp and well defined, sometimes very bright. Their occurrence is not easily explained; for in the ordinary conditions of these observations each ray illuminates the whole slit, and the image of the line due to it ought to be as broad as that slit. The only explanation which has yet been proposed (by PLÜCKER) is that they are caused by the overlapping of two images the distance of whose centres is less than the slit. It is, however, liable to two objections—that the brightness of such an overlap cannot exceed twice that of the ground on which it is seen, and that it would be often resolved by a prism of higher dispersion: the combination 2BS.C disperses seven times as much as the Merz, and ought surely to break up some of them. This is not, as far as I have examined, the case; and some of these lines are as intense as any in the spectrum. These narrow lines are sometimes very thickly crowded, as in the green and blue of iron spectra, and in those of carbonic oxide; and possibly they may compose the bright ground when so close as to be unresolvable. There is a seeming tendency in them to be grouped in two, three, or higher numbers, which, however, might disappear with more powerful prisms; but with those I use one can scarcely avoid thinking that there is some special connexion between the components of such groups; such are enclosed in brackets.

Some of the most brilliant are double, as the orange one γ, supposed to correspond with D (though its components seem to me more separated than those of that line), the splendid yellow δ, and the greens η and θ.

At that boundary of the spectrum which corresponds to the negative electrode (and in a much less degree at the positive) extremely intense lines are seen, especially in the green, which however are short: bismuth, zinc, lead, and arsenic are the best examples of this. These are generally supposed to be metallic lines, and to proceed from the intense dispersion of metal near the electrodes, especially the negative. It has even been proposed to consider this not crossing the entire breadth of the spectrum as a test of metallic origin, and to regard the others as gaseous. I, however, find that these very lines can be traced entirely across though sometimes very faint, except when the entire part is covered with a sort of bright haze, through or on which nothing can be seen which is not very bright. It seems to me that the existence of a line and its brightness depend on different causes; and I shall give instances when the *same* line assumes very different aspects as the media of discharge are changed. I may add that the sort of haze just mentioned does not occur at the negative electrode.

The spark discharge without a jar is so much fainter that none but the brightest lines can be seen. If the surface of the jar be increased (for the converse reason) more lines are visible, but I think no new ones are produced. One or two examples will be given, If the secondary coils be arranged for quantity, the red end is not changed, the violet is much brighter.

On rarefying the gas in which the discharges are made (R) there is at first no change, except perhaps a little diminution of brightness, till at a certain pressure (varying with the media and diameter of the tube) the spectrum fades away. Sometimes, as with CuO (*i. e.* copper electrodes in oxygen), *all vanishes* except a trace of the lines at 0·75 inch; in SbN all but a trace of δ; in NiN, at 2·8 inches, all but a suspicion of violet light, and sometimes perfectly dark bands take the place of bright lines. In others the change is not so striking. In Al, Air, at 1·2 inch, the chief violet lines remain tolerably conspicuous; δ can be seen, but η and $\varkappa$ vanish: in Pt, Air, tube 1 inch diameter, at 6 inches the first half fades to a neutral-tinted haze with faint alternations of brightness, and the rest has eleven definite but faint bands. In general, however, it may be considered the rule that from 3 inches to 1 inch these spectra almost vanish. I will only mention another, PbH. In hydrogen spectra, at C.P. the most distinctive characters are a *very* bright red line (found in all spectra, but not so bright); a very bright and broad one at the confines of green and blue, and a similar one in the violet. Of these the last diappears at 5·3 inches; the red at 3·1 inches; the blue not totally till 0·15 inch; but this is doubtful, as it may have been confounded when waning with another which was near its place. It deserves notice that this disappearance of the peculiar hydrogen bands is also produced when the hydrogen is diluted with air. The electrodes were cobalt; 20 cubic inches of hydrogen were mixed with 10 of air, and the spectrum of the mixture examined; its volume was reduced to 20, 10 more of air added, and so on; the degree of dilution was easily found from the number of additions; and it was found that the violet band vanished when H is 0·2 of the mixture, the blue when 0·09. These correspond to hydrogen under the pressures 6 inches and 2·7 inches: the red could not be determined, as air has the same band.

If in any case we rarefy beyond the limits of these *transition* spectra, bright lines reappear, but not all in the same places or with the same characters. Ordinarily the brilliant lines of the C.P. spectra are wanting, though sometimes lines which are faint in them assume this type in the others. The red, yellow, and green seem diffused in cloudy light; and the violet system is replaced by a set of broad cloudy bands nearly equidistant, and more conspicuous than the less refrangible ones. The red band, which almost always begins the C.P. spectra, is often wanting, and when it does occur is insulated in darkness, but at the other end both spectra are nearly of the same length. These R spectra are in general much less luminous, and show little distinction of colours unless the metal of the electrodes be easily vaporized. That of tellurium is very bright, containing thirteen brilliant lines; and under *every* circumstance of pressure or discharge, those of potassium and sodium show the dazzling orange bands.

As might be expected from their greater faintness, they contain fewer lines, or at least fewer are visible; but it is remarkable that of this number a considerable proportion *is not found* in the C.P. spectra. The percentage deduced from fifteen metals is

Air	0·39
Nitrogen	0·32
Oxygen	0·39
Hydrogen	0·66
Carbonic oxide	0·37

Of this, two explanations may be given. It may be said that these lines are not seen at C.P. because they are overpowered by the brightness of the ground on which they are seen. In some cases (of which examples are given hereafter), especially in hydrogen, this cause does act in some degree. In that gas the centre of the C.P. spectrum was at first observed, where, as I have stated, the finer lines are not easily seen with some metals; and then the percentage was very high; but at the negative boundary this bright haze does not interfere, and when observed *there*, the proportion is almost exactly that of the other gases. Besides, it occurs frequently that in the C.P. spectrum, on each side of the place where the missing line should be, faint and narrow lines are seen with perfect distinctness. The other solution is this, that these lines do not depend entirely on the chemical character of the media and electrodes, but also on their molecular condition. On any other supposition it seems hard to conceive the passage from the C.P. to the transition spectrum, and the increase of brightness from that to the R one, the same chemical elements being present in the three.

In presenting the measures of these spectra, there is a difficulty arising from the impossibility of giving, within any practicable limits, the distinctive characters of the phenomena, though they are of considerable importance. For instance, the orange band γ, which in the sodium spectrum is "intensely bright," is in Pt, CO "faint, scarcely visible." To tabulate them I must restrict myself to a few distinctive symbols. First, the lines which are far transcendent in brilliancy, and are not less broad than the image of the slit, I denote by a *. This implies merely that they surpass the others greatly; one in red or violet may fully deserve this symbol, though it would seem dull beside δ or θ. Not very frequently the same line has this mark in C.P. and R, but, except with sodium and potassium, less bright; and sometimes one which is faint in C.P. is a * in R. Then follow

Very bright, bright	vb, b
Conspicuous (from surrounding faintness rather than intense brightness)	c
Faint, very faint	f, vf
Narrower than slit	n
Very narrow, like a hair	vn
Wider than slit (the exponent expressing how many times as wide)	w^n
When they are on an obscure ground	o

The symbol R implies that the pressure $=0·2$ inch unless a different one is stated.

I shall commence with aluminium, which I selected as the type because, from its small dispersion at the negative electrode, it may be assumed that its influence on the spectrum bears a small proportion to that of the gas.

Table I.—Aluminium.

			Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
			C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32	α	39 *	......	39 *	39·5 vf.	......	......	38·7 * o.	......	38·8 f.	......
2.		α'	40·7 c.	......	40·5 nc.	......	40·9 *	......	......	......	......	......
3.			43·6 f.	......	......	......	43·4 c.	......	43·6 f.	42·8 * o.	......	......
4.		β	46·8 f.	......	46 nf.	......	47·4 f.	......	45·5 f.	......	48 f.	......
5.		β'	48·8 *	......	48·5 *	......	49·4 *	......	......	......	49·8 f.	......
6.			......	51·5 f.	......	52 f.	......	......	......	51·8 f.	......	52 f.
7.		γ	55 *	56·5 vf.	55 *	......	56·4 b.	54 f.	......	......	55·5 b.	......
8.	32		57·7 vf.	......	......	......	......	......	57·7 f.	......	......	......
9.	33		0·3 vf.	1·3 vf.	......	59·5 nf.	......	1·5 f.	59·5 f.	59·3	1·7	......
10.			2·6 c.	......	2·5 nb.	......	......	......	3 f.	......	......	2·5 f.
11.		δ	4 *	......	3 *	......	3·4 vn.	......	3·3 b.	......	3·2 *	......
12.			......	......	......	......	5·4 bn.	......	6 c.	6·8 f.	5·5 f.	6
13.			7·3 f.	7·5 wb.	......	7 wf.	7·9 vn.	7·5 b.	8 c.	......	then 4 vn.	......
14.		ε	9·4 vb.	......	8·5 b.	......	......	......	......	......	......	......
15.		ε'	11 } b.	......	10 b.	......	10·4 f.	11 f.	10·4	......	......	......
			11·4 } nf.									
16.		ε''	12 } b	12·3 bn.	12 c.	13 nb.	13·4 b.	......	12·7 f.	......	12 f.	13·5 f.
17.			14 f.	......	......	......	......	15	......	......	then 3 vn.	......
18.		ζ {	16 } nb. 18 } nb.	} 17 vf.	17·5 w^2.	16·5 wf.	17·4 b.	18 b.	18 f.	......	18 f. two.	......
19.			20 vf.	......	20 f.	20 f.	......	......	19·7 f.	......	21 f.	......
20.			22·7 vf.	......	......	......	23·4 }	......	22·1 f.	......	......	......
21.		η	24·4 *	24·2 *w. 1 f. here.	24·5 *	23·5 *	24·4 }	24 *	24·1 c.	23·3 b.	23 b.	23·5 b.
22.			27·4 vf.	......	......	......	26·4 }	......	26·4 f.	......	25·2 b.	28
23.		{	31·8 n.vb. 32·8 vn.	} 31 vf.	30·5 vnb.	29 w^2f.	32·4 c.	30 b.	30·8 c.	......	30·2 b.	31 f.
24.		θ	34·1 *	......	33 *	35 f.	34·4 *	......	33·4 *	......	33 *	34 f.
25.			36·8 } b.	......	37 b.	37·5 c.	37·4 c.	......	35·8 b.	36·8 b.vn.	35·7 c.	......
26.			39·4 } b.	39·5 vf.	38·5 b.	......	......	39 c.	......	......	......	......
27.			41·4 } b.	......	41·5 vb.	41 nc.	41·4 b.	42 c.	40·6 *	40·8 * n.	41·5	43
28.			45·4 } bn.	44·2 wb.	45 } *	44·5 $bw^{1\cdot5}$	45·4 f.	45 b.	......	45·8 f.	45·8 f.	......
29.			47·4 } bn.	......	46 } *	......	......	......	......	......	......	......
30.			49·3 f.	......	49 } f.	......	48·9 f.	......	......	......	......	49
31.			......	51·5 f.	50·5 } f.	50·5 cn.	......	51 b.	......	49·8 f.	......	......
32.		ι	52·6 *	53·5 vf.	53·5 } f.	......	52·4 *	......	52 c.	......	52 c.	53 b.
33.			54 vnc.	......	......	......	54·4 c.	......	54 c.	......	54·8 *	......
34.		κ	56·6 *	55·5 f.	55·5 *	55 b.	56·4 *	55 b.	56 double c.	55·3 f.	56·8 f.	......
35.	33	κ'	57·6 *	......	58 *	......	58·1 f.	......	58·5 b.	......	58·2 *	59 b.
36.	34	κ''	0 *	0·7 f.	......	......	59·4 *	1 b.	0·4 f.	59·8 f.	......	......
37.			2·6 } nc.	......	1·5 } f.	1·5 b.	......	......	1·7 f.	......	......	......
38.		λ	4·6 } nb.	......	3·5 } c.	......	......	......	3·6 } c.	......	3 f.	......
40.			6·2 } nc.	......	5·5 } f.	......	......	......	4·9 } c.	......	many here n.	6 b.
41.			......	8·5 wb.	......	7·5	8·9	9 b.	7·8 } c.	7·3	......	......
42.		μ	10·5 } *	......	10·5 } b.	......	......	......	10·1 b.	......	10 f.	......
43.		μ'	12·4 } b.	......	11·5 } b.	......	11·4	......	10·8 c.	12·8 f.	......	......
44.		μ''	14 } b.	13·5 f.	14 } f.	14 b.	13·9 b.	......	13·4 c.	......	......	13 b.
45.		{	15·4 f. 18 f.	}	17 n.	......	16·9 c.	16 b.	16·7 f.	......	16·5 } b.	......
46.		ν	20 vb.	19·7 wc.	20 c.	20 b.	19·4 *	......	19·3 *	19·8 c.	18·8 } b.	18 b.
47.			21·9 f.	......	......	......	21·4 f.	21 b.	......	......	21 } c.	......
48.			26·2 f.	27·9 f.	26·5	......	25·4 b.	......	25·8	26·3 f.	25·3 b.	25 c.
49.		ξ	30·1 *	......	30 *	28 b.	29·4 f.	28 b.	29·5	......	28·2 f.	......
50.			33 f.	......	34 f.	34 nc.	34·4 *	34 b.	32·4 f.	34·3 f.	34 f.	32 f.
51.		ο	36·3 wc.	36·6 vf.	36·5 c.	......	......	......			......	......
52.			38·2 vf.		40 f.	41 f.	......	40·5 f.	......	......	38 f.	38 f.
53.			41·6 vf.	......	43 f.		42·4 f.		......	......	43 f.	......
54.			44·9 f.	......	......	......	......	......	......	......	45 f.	......
55.			48·2 f.	......	48 f.	......	48·4 f.	......	......	......	47 c.	48 f.
56.			52·1 c.	......		......		......	......	......	51 f.	55 f.
57.			57·4 vf.	......	......	......	......	......	......	......	57 f.	69 f.

The first of these, air, C.P., was taken with Duboscq, and after the telescope had been improved; when first taken it showed only twenty-five lines, of which the groups ζ and λ were seen as broad bright bands. Air R. was taken at the same time, the others, except H, C.P. (which was taken with Duboscq, and at the negative boundary) were with Merz, but with the improved telescope. As first observed, and at the centre, this last showed only eight lines.

It will be remarked that all the lines in the C.P. of nitrogen, and all but one in that of oxygen, are found in air, as might be expected; often a line is common to the three, and then that in air is of intermediate character. But the same line is also often found in the other two gases. It is generally supposed that this indicates the metallic origin of that line; but it will be found that many occur, not only in all these gases, but with all or nearly all the electrodes which I have tried. I shall return to this at another time, now making a few remarks on those before us.

The pair α, α' are of almost universal occurrence at the origin of the spectrum; α has close before it a narrow but bright companion, whose place has sometimes been taken; but outside of that there are merely shadowy traces scarcely ever bright enough to be bisected. In a very few cases, however, one of them becomes predominant, *e.g.* silver and iron. In H (hydrogen) spectra, one of this pair acquires an intense brilliancy, which is peculiar to this gas, while the other fades away. It is to be noted that α is exactly in the place of the solar line C. These and β, β' are in the red. No. 7, γ is an orange band, nearly but, I think, not exactly in the place of D. When viewed with the combination 2BS.C, it is double in all cases which I have examined; but the distance of the centres of its components is fully twice that of the components of D. No. 11, δ is yellow, generally extremely brilliant, except in one; it is seen in 2BS.C double, the second one being about half the breadth of the preceding. Before it is a narrow orange line, and before that No. 10, which *seems to attend it constantly.* Nos. 14 ϵ, the origin of green, 15 ϵ' and 16 ϵ'' are of constant recurrence. With the higher prism-power the first and third are each double, and some between ϵ' and ϵ'', of which one (as here in air, C.P.) has been occasionally observed. No. 18, ζ was at first observed as a bright cloudy band; 2BS.C shows that the whole of that region is covered with close lines, of which one is more conspicuous than the rest. In CO many more are conspicuous. No. 21, η is very brilliant (except in oxygen and CO, when its following companion is the brightest). With 2BS.C it is close double of two equal*. This is also the case with No. 24, θ, at the boundary between green and blue, which is, as a rule, the most intensely bright in all the C.P. spectra, in many of which its light is almost blinding. In R it is always faint though present. It is preceded by two narrow ones of unequal brightness, which seem to form with it a system. Another system seems to be found from 32 to κ'' 36, remarkable for its beauty and peculiar character; κ' is vivid blue, close double of equals, κ'', broad and cloudy, begins the violet. With 2BS.C κ is composed of two further apart, and κ'' of six. It is less developed in nitrogen and H; λ 37·40 was at first observed as a band; in CO the whole of its vicinity is covered with narrow violet lines. The pair 28, 29, and the triplet μ, μ', μ'', are also of very constant occurrence.

* It has the deviation of b, and is double like it. No. 27 has that of F.

TABLE II.—Platinum.

		Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.		Mercury vapour.	Phosphorus.	Bisulph. carbon.
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.			Pressure. 0^{in}·45
1.	32	38·2 onc.	......	......	......	36·8 f.	......	37·5 * o.	......	37 fo.	......	......	......	
2.	α α'	39·7 vb. 41·4 c.	....	39·5 * 41 vn.	}	40·8 f.	......	......	......	......	......	......	......	......
3.		42·4 f.	......	......	......	44·3 f.	......	......	......	......	......	......	......	......
4.	β	47·1 f.	......	48 f.	......	47·8 *	......	......	......	46 n.	......	......	......	......
5.	β'	48·9 vb.	......	......	......	48·8 nb.	......	......	......	......	......	49·4 f.	......	......
6.		50·2 f.	51·4 vf.	......	......	......	......	......	......	......	50·5 vf.	51·4 b.	50·1 f.	......
7.		52·8 f.	......	......	52 f.	......	52·8 b.	......	......	......	......	......	......	51·7 f.
8.	γ	55·2 *	......	55 *	55·5 f.	......	55·8 nf.	......	54 vf.	......	55·5 nf.	55·5 f.	......	......
9.		56·4 nf.	56·7 vf.	......	......	......	......	......	......	56 vn.	......	......	......	56·7 f.
10.	32	57·7 nf. 59·4 nf.	......	......	58 f.	......	58·8 nf.	......	......	57 f.	59·5 vf.	......	......	......
11.	33	0·7 n. 1·4 n.	......	......	......	......	......	......	......	......	......	0 *	0 f.	59·6 *
12.		2·7 nb.	2·3 nf.	2·2 n.	2·5 f.	......	......	......	......	3 b.	......	......	......	3·3
13.	δ	3·9 *	......	3 *	......	......	3·8 f.	3 f.	......	4 b.	3·5 vf,	3·4 c.	......	4 f.
14.		6·7 vf.	......	......	6 c.	......	6·8 b.	......	......	5·5 vn.	7 f.	6·7 *	......	6·7 *
15.	ε	9 vb.	......	9·6 vb.	......	......	......	......	......	many	......	......	......	8 f.
16.	ε'	10·2 } b.	10 f.	10 } b.	......	......	......	......	......	close and	......	......	......	8·7 f.
17.		11·1 bn. 11·8 vn.	very faint group	... ...	......	......	......	......	......	narrow here till	......	......	......	....
18.	ε''	12·4 } b.	here.	12 } b. two f. here	12 vf.	12·3 wb.	......	......	......	......	......	11·7 *	11·4 *	12 *
19.		14·4 vf.	15·4 f.	......	15 vf.	14·8 cn.	15·3 w^2f.	......	......	......	15 w^4f.	14·7 f.	......	......
20.	ζ	16·4 vn. 17·4 vn. 18·8 b.	17·7 vf.	17 b.	......	16·8 b.	......	......	......	......	......	16·8 f.	19·7 c.	16 7 f.
21.		......	21·7 vf.	20·5 f.	20 f.	......	19·8 f.	......	......	21 vn.	19 vf.	20·7	20·1 f.	20·7 b.
22.		24·4 *	......	24 *	24 *	23·8 fn.	24·3 b.	23 vf.	22 vf.	23 vnc.	23 b.	24·1 } *	24·1 f.	24·1 b.
23.		27·8 n.	26·1 vf.	......	......	......	......	......	......	11 or 12 here.	25 vf.	25·7 } f.	......	......
24.		31·6 nb.	30·1 } f.	30·3 nb.	29 f.	......	29·3 f.	......	......	30 vnc.	28 f.	30·1 f.	......	28·7 } f.
25.		32·8 vn.	31·4 } f.	32·3 *	33 vf.	32·3 f.	......	32·5 c.	......	32·5 *	......	......	......	31·8 } f.
26.	θ	34 *	......	......	......	......	......	......	......	35 f.	35 vf.	34 f.	......	34·1 f.
27.		37·2 c.	35·8 nf.	37 wb.	......	36·8 } bn.	......	......	36·5	......	......	36·1 f.	......	......
28.		39·5 c.	38·1 c.	......	38·5 f.	39·3 } bn.	......	39·5 * w.	39·5 *?	......	......	37·5	38·1 c.	38·1 c.
29.		41·5 cw.	41 vf. rn.	40·7 *w^3.	......	41·8 } b	40·8 nb.	......	......	40 w^2b.	38·5 f.	40·1 f.	42·1 } f.	41·4 f.
30.		45·1 } b.	44·7 b.	45 } b.	45 *	......	43·8 b.	......	......	44 vn.	43 c.	44 *	44·1 } f.	44·1 *
31.		46·6 } b.	......	46 } b.	......	......	......	......	......	45 f.	......	......	46·4 } f.	......
32.		49·4 f.	48 vn. f.	50 } c.	50 f.	......	......	......	...	48 vnf. 50 *	}	......	......	49
33.		51·9 vb. double.	51 f.	51 } c.	......	51·8 *	52·3 vf.	......	......	51 f.	51·5 vf.	......	......	three f. here.
34.		53·8 nc.	......	53 } c.	......	53·8 nc.	......	......	......	53 *	......	......	......	......
35.		56·1 *	55·3 c. double.	55 } *	55 f.	55·8 *	......	......	......	56 f.	......	......	55 f,	55 b.
36.		57·1 *	......	56 } *	......	......	......	56·5 nb.	......	57 *	......	......	......	......
37.	33	59·1 *	......	58 } *	......	58·8 *	......	......	......	......	......	58·8 f.	57·6 c.	......
38.	33	......	1·1 cw.	......	0 f.	......	0·8 f.	......	......	0 } f.	1 vf.	1 f.	......	......
39.		2·3 } b.	......	2 } b.?	......	......	......	......	......	2 } f.	......	......	......	......
40.	λ	4·1 } b.	......	3 } b.?	......	......	......	......	......	4·5 } f.	......	......	......	......
41.		5·6 } b.	......	5 } f.	5·5 } f.	......	5·8 b.	......	......	5 } f.	6 b.	......	......	......
42.		......	7·8 cw.	......	8 } f.	8·8 b.	......	......	......	7 } f.	......	6·5 *	6·2 f.	6·2 c.
42.	μ	10·8 } *	......	11 } b.	......	......	......	......	......	9 } f.	......	9·1 vf.	......	......
43.	μ'	12·3 } b.	......	12 } b.	12 } f.	12·8 b.	......	......	......	12·5 b.	......	12·4 f.	......	......
44.	μ''	13·6 } b.	13·4 cw.	13·2 } b.	14 } f.	......	......	......	......	......	......	......	......	......
45.		15·4 f.	......	......	... } ...	16·3 } f.	14·8 b.	......	......	......	......	......	......	......
46.		17·7 c.	......	16·8 b. double.	16·5 } f.	... } ...	......	......	......	18 c.	17·7 w^5. seems triple.	16 vnc.	......	......
47.		19·3 b.	19·9 cw.	20 c.	20 b.	18·8 } f.	18·8 f.	20 * w.	19·5 f.	19·5 f.	......	18·6 *	18·6 *	18·3 *
48.		22 n.	......	22·2 f.	......	21·8 } f.	22·8 f.			......	......	......		23·2 c.
49.		25·9 n.	27·2 cw.	26 c.	27·5 c.	25·8	some f. seen here.	......	......	25 *	25 f.	......	......	24·5 f.
50.		30 vb.	30·1 vnf.	30 *	......	......	......	......	......	28 f.	......	30·1 f.	......	......
51.		33·3 } n.	......	33 f.	34 f.	......	......	......	......	33 f.	33 f.	......	......	......
52.		35·1 } vf.	35 nvf.	......	......	34·8 c.	......	......	......	34 f.	......	......	......	35 f.
53.		36·2 } c.	......	36 f.	......	......	......	......	......	36 f.	......	......	......	......
54.		39·1 nc.	......	39 f.	......	......	......	......	......	38 f.	......	......	......	......
55.		41·7 nc.	......	......	42 vf.	......	41·8 f.	......	......	41 f.	41·5 f.	42·8 vf.	......	......
56.		45 nc.	......	45 f.		......		......	......	44 f.	......	......	......	......
57.		48·3 f.	49·5 vf.	48 f.	......	......	......	......	......	47 f.	48·5	46·9 fv.	......	46·7 f.
58.		51·5 f.		52·2 c.	......	......	......	......	......	50·5 f.	49·2 f.	50·9 fv.	......	50·8 f.
59.		58·2 vf.	......	56 f.	......	58·8 f.	......	......	......	56 f.			......	

In the air C.P. and R. were observed with Duboscq and are therefore comparable with the aluminium air spectra. So also were those of Hg, P, and S^2C. In the mercurial vapour the discharge was bright, greenish white, with very large cloudy strata; the spark discharge without the jar gave exactly the same spectrum. It has more *s than oxygen or carb. oxide; but the rest of it is very dark. In the phosphorus and bisulphuret the tubes were so darkened by deposit as to make observation difficult; the first was red, I suppose allotropic P, for nothing would remove it but strong nitric acid; the other is partly black (I suppose sulphuret of Hg), partly grey arranged in striæ (perhaps carbon). The P spectrum is so like that of Hg vapour, that I think it exerted very little influence, and in the other the metallic vapour seems also to predominate. I also tried olive oil in the same apparatus to ascertain whether its vapour (which may be present in air-pump experiments) could produce any effect. The flash of the discharge was bright green, as in the CO and S^2C tubes; but this probably came from the decomposition of a film of oil on the electrodes: the spectrum was identical with Hg vapour, except that it was very faint.

TABLE III.—Silver.

		Air.				Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		Spark C.P.	J. C.P.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32	37·4 vf.	36·5 * double.	......	......	......	36·5 o. double.	......	......	37 *	......	......	......
2.	α	......	......	38·8 * double.	......	......	......	38·5 *	......	......	......	......	......
3.	α′	......	40 b.	40·8 *	41 o.	40 *	......	41 f.	39·5 o.	......	......	40·5 *	......
4.		......	42·5 f.	......	......	41·5 f.	......	......	......	......	......	......	......
5.	β	......	46 f.	......	......	......	......	45·5 f.	......	......	......	......	......
6.	β′	......	49 b.	47·8 b.	......	49 f.	......	48 *	......	......	......	49·5 b.	......
7.		51·4 b.	......	......	51·5 f.	......	51·5	......	51 f.	......	50·5 f.	51·5 c.	52 *$w^{1\cdot5}$
8.	γ	......	56 *	55·8 *	......	55 *	56 f.	55 c.	......	......	......	56·5 f.	56 f.
9.	32	......	58 *	......	59 f.	......	59 f.	......	......	......	58·5 f.	57·5 *	......
10	33	......	......	......	......	......	......	......	......	......	......	0·5 f.	0 f.
11.		4·3 *	4 c.	3·8 c.	......	3 c.	3·5 f.	3 c.	3 f.	3 f.	......	......	......
12.	δ	......	5 *	4·8 *	......	4 *	......	......	......	......	......	5·5 b.	4 f.
13.		6·7 vf.	6 f.	......	6·5 *	......	......	......	6·5 *	......	7	7 } f.	7 *
14.	ε	8·7 b.	9 b.	8·8 b.	......	......	8	8 f.	......	......	......	three	......
15.	ε′	......	11 c.	10·8 } b.	......	10 b.	......	......	......	......	11 b.	9·5 f.	......
16.	ε″	not taken.	12 *	11·3 } b.	......	12 } b.	......	12·5 c. close pair	......	......	......	three	......
17.		......	14 b.	13·8 c.	14 b.	13 } b.	13 c.	......	13 f.	......	......	14·5 f.	......
18.	ζ	......	18 f.	17·8 b.	......	{ 15·5 } c. 17·5 } c.	17 b.	17 *	18·5 f.	......	......	three	16 f.
19.		20·7 b.	......	......	......	19 } c.	......	......	......	......	......	19·5 } f.	......
20.		......	22 b.	22·8 c.	......	23 vnc.	21 f.	22 }	......	......	22·5	21·5 } c.	21·5 f.
21.	η	......	24	24·8 *	23·5 *	24 *	......	24·5 }	24 *	23 f.	23·5	24·5 } c.	......
22.		......	......	......	......	......	25 *	26 }	......	......	......	26 } *	25 *
23.		30·8 nb.	29	30·8 c.	29·5 f.	32 nc.	30	30 f.	29 b.	31·5 f.	......	30 }	30 f.
24.	θ	33·5 *	33·5 *	33·8 *	......	34·5 *	35 f.	33·5 b. {	32 } b. 34 } b. {	......	......	32·4 } b. 34·5 } b.	35 f.
25.		37 f. {	35 c. 37 c. }	36·8 f.	......	38 b.	39 b.	37 b.	38 } b.	......	......	37·5 f.	36 f.
26.		41·4	41	40·8 f.	41 c.	40·5 f.	41·5 f.	41 *	40 b.	39 *	40 *	......	40 f.
27.		......	......	44·8 } b.	42·5 *	43 b.	44·5 f.	......	44 b.	......	42 f.	42·5 b.	44 f.
28.		46 b.	47 c.	46·8 } b.	......	46 } b.	......	46 f.	......	......	......	......	......
29.		......	......	......	......	47·5 } b.	......	......	......	......	......	48 c.	......
30.		50·7 b.	49 * double.	50·8 c.	51	50 f.	51	50 *	50 b.	......	......	50·5	......
31.	ι	......	......	......	......	52 } b.	......	52·5 f.	......	52·5 c.	52 f.	52·5 *	53 f.
32.		......	54 b.	......	......	54·5 } b.	55	55 *	53·5 b.	......	......	54·5 c.	......

TABLE III. (continued).

		Air.				Nitrogen,		Oxygen.		Hydrogen.		Carb. oxide.	
		Spark C.P.	J. C.P.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
33.	°κ	55·3 b. double.	......	55·8 * double.	......	56 *	......	narrow one here.	......	......	......	55 5 * double.	......
34.	κ'	......	58 *	......	......	57 *	......	......	......	......	......	57 vn.	......
35.	33κ''	58·5 b.	......	59 *	......	59 *	......	58 *	0 b.	......	......	0 *	59
36.	34	1·1 vf.	......	......	......	......	1·5	2 b.	......	......	......	......	......
37.	λ	4·2 vf.	......	3·3 b.	......	3 b.	......	......	......	......	......	......	3 f.
38.		......	......	......	6	5 *	7·5	......	6	......	5 f.	......	......
39.		9·8 b.	8 b.	......	......	......	......	......	......	......		8 f. others	7 *
40.	μ	10·8 f.	9 b.	10·8 *	......	10·5 b.	......	9·5 f.	......	......	......	10·5 f.	......
41.	μ'	12·1 f.	11 b.	11·8 b.	......	12 b.	......	11·5 f.	......	......	......	......	......
42.	μ''	14·4 f.	14 f.	14 b.	14·5 f.	......	14·5	13·5 f.	13 b.	......	......	13·5 b.	......
43.		......	16 c.	17·8 b.	......	17 f.	......	16 c.	......	15·5 *	......	18·5 c.	17 f.
44.	ν	20 f.	19 f.	19·8 f.	......	19·5 c.	20·5	19 b.	19 b.		......	20·5 c.	
45.		some seen.	22 f.	......	22 f.	......	......	22 f.	......	......	......	23 c.	......
46.		......	25 b.	24·8 f.	......	26 f.	......	25 f.	25·5 b.	......	......	......	......
47.		29·1 f.	29 f.	27·8 b.	......	......	28 f.	......	......	......	......	27·5 *	......
48.	ξ	......	31 f.	......	......	30 b.	......	29·5 *	......	......	......	......	......
49.		......	......	33·3 f.	33·5 f.	33 f.	......	......	......	......	......	......	......
50.	34	36 vf.	......			37 c.	35 vf.	35 f.	......	......	......	36 c.	......
51.			41 f.	......	......		41 vf.		39·5	......	......	39·5 vf.	......
52.		......	......	......	......	......		......	......	......	......	43·5 vf.	......
53.		......	46·5 f.	......	......	......	......	......	......	......	......	48·5 vf.	......

These C.P. spectra are of very great splendour; the ground, especially the green part, is far brighter than the preceding, and the lines are dazzling. The R ones are on the whole faint. The first is a simple spark (without jar): it differs little except in brightness. The second, headed J, was obtained with two large jars having 8·5 feet of external coating; it was far more luminous, and showed eight more lines than the normal one, besides four at the violet end, which were measured, but not tabulated, as they were not seen in the other spectra. I was struck with the outline of this spectrum, which gives an idea that the red and violet rays are less abundant near the electrodes. [See p. 973.]

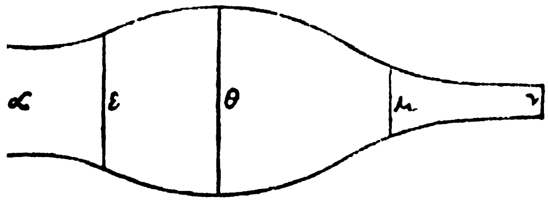

TABLE IV.—Copper.

		Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32°	37·1 nc.	......	36·5 *	......	......	......	......	......	......	......
2.	α	38·2 *	......	38·5 b.	......	38·3 b.	......	......	......	38·3 *	......
3.	α'	39·6 vb.	......	40·5 f.	......	41·3 f.	......	40 * o.	39·8 o.b.	......	......
4.		......	......	43·5 f.	......	......	......	......	......	......	......
5.	β	47·8 *	......	46 b.	......	48·3	......	......	......	47·8 f.	......
6.	β'	......	50 f.	......	49·5 f.	50·3 f.	50·3 f.	......	50·4 f.	49·5 *	51·5 c.
7.	γ	54·2 *	......	53 *	53·5 f.	54·8 vf.	......	......	......	55·5 c.	54·5 f.
8.		......	......	......	57·5 c.	......	......	......	......	56 *	......
9.	32	59 f. many here.	58·5 f.	59·5 c.	60·5 f.	......	......	......	59 c.	58·5 f.	58·5 f.
10.	33	1·5 f.	2 f.	1 *	......	......	1·8 f.	......	......	0·8 f. 2·6 f.	2·5 f.
11.		4·1 *	......	......	......	3·3 c.	......	......	3·5 c.	3·9 f.	......
12.		......	6 b.	6·5 b.	5 b.	......	6·8	......	7 b.	5·8	5·5 *
13.	ε	8·4 *	......	8·5 c.	......	8·3 f.	......	......	......	about 30 n here.	7·5 *

TABLE IV. (continued).

		Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
14.	° ι′	9·8 vb.	......	two.	......	......	......	......	......	......	......
15.	ι″	11 vb. many.	11 c.	10·5 c.	10·5 c.	12·3 b.	......	......	12 f.	......	11 c.
16.		......	......	14·2 f.	14·5 f.	14·3 f.	14·8 f.	......	14·5 bw.	......	......
17.		......	......	......	......	16·3	......	......	......	......	15·5
18.	ζ	18·5 b.	......	......	19 f.	20·8	19·8 f.	......	20 c.	19·8 b.	19·5 f.
19.		21·7 vb.	23	22 *	......	......	22·8 b.	......	......	22·3 b.	23·5 f.
20.	η	24·3 *	23·5	......	24·5 *	23·3	......	......	24·6 b.	......	24·5 f.
21.		25 c.	......	25·5 f.	25·5 f.	25·3	......	......	......	26 *	......
22.		27 c.	......	28·5 c.	27 f.	......	28·8 f.	......	28·5 vnf.	26·5 c.	28·5 f.
23.		30·5 c.	......	30·5 *	31·5 f.	......	......	......	......	30·1 c.	......
24.	θ	32·9 *	......	......	......	32·8 b.	......	33·5 f.	32·8 f.	33·3 c.	33·5 f.
25.		35·5 b.	......	35·5 f.	......	... ...	......	......	......	35·8 b.	......
26.		36·8 b.	......	......	36 f.	36·3 b.	......	......	37·4 vf.	......	......
27.		38·1 b.	.....	38·5 b.	......	... ...	......	......	......	......	38·5 f.
28.		40	......	42·5 b.	41·5	41·3 b.	39·8 c.	41·5 *	......	41 b.	......
29.		44 b.	43·5 b.	43·5 b.	......	......	43·8 c.	......	43·6 b.	......	43 *
											44 f.
30.		46 b.	......	48·5 f.	47·5 f.	......	......	......	......	46·3 f.	......
										48·9 f.	......
31.		51 b.	......	50·5 f.	51·5 f.	50·3 *	......	......	51·9 f.	50·3 *	......
32.	ι	......	......	52·5 c.	......	51·3 c.	......	......	......	52·3 c.	52 f.
33.		......	53 f.	53·5 *	......	53·8 *	......	......	......	... ...	......
34.	κ	55·4 *	......	55·5 *	......	......	......	55·5 vf.	......	54·5 *	......
35.	κ′	56·6 *	......	......	......	......	......	......	......	56·8 f.	......
36.	33κ″	58·5 w*	......	59·5 c.	58 f.	57·3 *	......	......	59·8 f.	58·9 *	59·5 f.
37.	34	......	......	1·5 c.	......	......	......	......	......	0·7 f.	......
				2·5 f.						about 20	
38.	λ	3·5 b.	......	......	......	3·3 f.	4·8 f.	......	......	here.	5·5 b.
39.		......	6·5 b.	8·5 *	6·5	8·3 f.	......	......	6·9 b.	8·5 f.	......
40.	μ	10·1 *	......	10·5 b.	10·5 f.	......	......	......	......	10·9	......
41.	μ′	11 b.	......	12·5 f.	12·5 f.	11·8 *	......	......	......	.. ...	......
42.	μ″	13·4 vb.	......	14·5 f.	15·5 f.	15·3 f.	......	......	......	13·8 f.	14·5 f.
											about20
43.		16·7 c.	17	16·5 f.	17·5 f.	17·3 f.	17·8	......	16·8 f.	16·3 f.	...... f.
44.		18·7 vb. two on each side.	some beyond.	......	......	19·3 f.		18·8 *w.	......	18·5 f.	18 f.
45.	ν	......	......	20 f.	......	......	......	......	......	20·5 f.	......
46.		22·6 w*	......	24·5 f.	24·5 b.	24·3 f.	......	......	22·3 b.	......	......
47.		25·8	......	......	25·5 b.	double.	......	......	......	25·5 *	......
48.	ξ	29 b.	......	27·5 *	......	......	......	......	......	......	......
49.		many here.	......	31·5 c.	31 f.	32·3 c.	......	......	......	32·5 c.	......
50.	ο	36 f.	......	34·5 c.	......	35·3 f.	......	......	......	37·5 f.	......
51.		......	......	......	40 f.	38·8 f.	......	......	38·8 vf.	37·5	......
52.		......	......	43 c.	......	41·3 f.	......	......		41·5 f.	42·5
53.		47·5 vf.	......	50·3 c.	46 vf.	45·8	......	45·3 vf.	......	45 f.	
										47·5 c.	......

In these the most remarkable thing is the comparative dullness of the red and yellow. In O, C.P., there seems no yellow, but the green ground is so intense that the lines there are scarcely visible. In O, R., the flash is green; in CO, C.P., is the same dullness of the red, but the faint ultra violet are visible in great numbers. In this also there are countless fine lines in the blue, and still more in the green.

TABLE V.—Nickel.

		Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32	......	......	......	33·9 vf.	......	......	......	......	......	......
2.	α	39 *	......	38·4 *	......	38·5 *	38 vnfo.	39·7 *	......	38·5	......
3.	α′	40·6 nb.	......	40	......	......	......	......	......	......	......
4.		43 c.	......	43·6 f.	......	42 vf.	42 fo.	......	......	41·5 vf.	......
5.	β	46 f.	......	45·5 f.	......	45·5 vf.	46·5 fo.	......	......	46 vf.	......
6.	β′	49 *	......	48·1 *	......	48 *	......	......	......	48·5 *	48 ovf.
7.		......	......	......	50·4 vf. 51·4 vf.	49·5 nf.	51 f.	......	50·7 f.	50·5 vb.	......
8.		......	52·2 f.	52·7 vf.	52·7 vf.	......	......	......	......	......	52
9.	γ	55 *	......	55 *	54·3 vf.	55·5 f.	54 vf.	......	......	......	......
10.		56·5 nb.	......	......	56 vf.	......	......	......	......	56·5 *	57 f.
11.	32	......	60·2 vf.	......	58·3 vf. 59·3 vf.	59 vf.	59 f.	......	59·7 vf.	58· 8f.	60 f.
12.	33	2 nb.	......	2 nb.	1·3 vf.	3 bw³.	2·5 f.	......	......	3 b.	......
13.	δ	4 *	......	4 *	......	......	5·5 *	3·7 n.	......	4·5 b.	5 f.
14.		6 f.	......	......	6 f.	6 cw.	6·5 n.	......	......	6·5 b.	......
15.	ε	8 b.	7·2 c.	8·7 vb.	7·3 f.	......	......	......	7·7 f.	8·5 b.	8 *
16.		9 n. 9·5 n.	......	......	9·7 f.	......	......	......	......	...... b.	9 n.
17.	ε′	10·5 nb.	......	10 b.	11 c.	10·5 c.	10 vf.	......	......	10·5 b.	......
18.	ε″	......	......	12 b.	12·7 f.	......	......	......	......	12·5 b.	12 vf.
19.		......	14·2 triple.	......	14·7 f. 15·1 f.	15·5 c.	14 wf.	......	14·2 wf.	14·5 b.	......
20.	ζ	16·2 bw³.	......	17·2 bw⁴.	17·4 f.	......	......	......	......	17·5 b.	17 w⁴.
21.		19 bw.	......	......	18·7 nc.	19 c.	19 wf.	......	......	20·5 b. both double.	21 f.
22.		23 *	......	......	......	21·5 *	23 *	......	......	22·5 b.	......
23.	η	25 vn.b.	24·7 c.	24·7 *	24·7 b.	25 *	......	......	......	......	......
24.		26 vn.b.	......	......	......	26 n.	......	......	......	......	......
25.		28 vn.b.	......	27·7 vf.	......	29 f.	28 vf.	......	29·2 n.	29·5 f.	......
26.		30 vn.b.	30·2 nb.	31·4 nb.	30·4 b.	......	31·5 vf.	......	......	31·5 vf.	31 w²f.
27.	θ	32·5 *	......	32·1 nf.	32·8 b.	32 f.	......	32·2 n.	......	32 f.	......
28.		35 nb.	34·7 n.	33·4 * 36·4 f.	36·1 b.	36 nb. 37 vn.	34 nf.	......	......	36·5	35 vf.
29.		38 f.	39·2 n.	39·4 f.	39·4 b.	38·5	......	......	39·2 nb.	......	39 vf.
30.		40 w²	......	......	......	40 b.	40 nb.	40·7 *w⁴.	......	......	41 vf.
31.		......	......	41·7 b.	42·1 nf.	......	42 *	......	42·2	41·5 w².	42·5 *
32.		44 b.	......	45·4 b.	......	......	......	......	......	......	44·5 b.
33.		45 b.	45·2 b.	46·7 b.	45·4 b.	46 vf.	46 f.	......	......	46 n.	......
34.		48 f.	......	48·7 f.	......	48 n.	......	......	48·2 wc.	......	......
35.		50 b.	50·7 w.	50·7 vf.	50·7 b.	50 *	50 b.	......	......	49·5 nc.	49 vf.
36.	ι	......	......	52 vf.	......	53 nc.	......	......	52·2 wc.	51·5 *	......
37.	κ	54·5 *	......	54 nf.	54·7 b.	54·5 *	54 wb.	......	......	......	53·5 f.
38.	κ′	55·5 *	55·2 w.	56 *	55·9 b.	56·5 n.	......	55·2 nb.	......	56 * double.	......
39.	κ″	57 *w.	......	57 *	......	58 *	......	......	......	58 n.	......
40.	33	......	......	59 *	......	......	......	......	59·2 wc.	59·5 *	......
41.	34	1 b.	......	2·3 b.	1·7 b.	......	0·5 b.	......	......	......	......
42.	λ	3 c.	2·2	4·2 b.	......	......	5 b.	......	......	......	......
43.		4 c.	......	5·9 b.	......	......	6 b.	......	5·7 wc.	......	6·5 *
44.		8 f.	7·7 w.	8·2 f.	8·2 b.	7·5 nb.	......	......	......	......	......
45.	μ	10·5 b.	......	......	......	9·5 nb.	......	......	......	9 b.	......
46.	μ′	12 b.	......	11·1 *	......	12 *	......	......	12·7 wc.	11 b.	......
47.	μ″	13 b.	14·7 n.	14·7 f.	14·1 b.	16 c.	15 b.	......	......	13 *	......
48.		17 f.	......	17 f.	18·3 nb.	17·5 *	18 vb.	17·4 w⁹f.	......	17 b.	17 b.
49.		19 nb.	......	19·3 f.	20 nb.	20 c.	21·5 nc.		18·7 wc.	19 vb.	......
50.	ν	22·7 f.	21·7 w.	22·6 vf.	......	23 vf.	......	......	......	22 b.	22·5 vf.
51.		25 f.	......	25·5 vf.	......	25·5 *	25 b.	......	24·2 wc.	......	
52.		28·5 b.w³.	......	27·5 vf.	27·2 b.	......	......	......	......	28 *	......
53.	ξ	......	29·7 w.	29·8 nb.	......	......	29 f.	......	......	......	......
54.		32·5 f.w³.	......	33·7 b.	34·1 b.	33 b.	33 nf.	......	31·2 vf.	31·5 vf.	......
55.	ο	......	36·2 n.	36·3 b.	......	38 f.	......	......		36 b.	......
56.		42·5 f.		40·3 vf.	40·3 vf.	41 n.	40 nf.	......	......	42·5 c.	......
57.		......	......	44·9 c.	44·9 vf.	43·5 c.	......	......	......	45 c.	......
58.		47·5 f.	......	47·9	49·5 f.	47 b.	47 f.	......	......	49 vf.	......
59.		51·5 vf.	......	52·1 c.				......	......		......

Of these the spectra N were taken with Duboscq.

The group 23–26, in air, C.P., is notable. In N, R., the remarkable set 7–12 were first recognized; they are very faint, as broad as the slit, and spread over the orange and yellow. The groups 14–16 and 18–20 are peculiar. In O, C.P., there is scarcely any yellow, and the lines from seven to twenty-eight are very indistinct, from the brightness of the ground.

TABLE VI.—Palladium.

		Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32 α	39*	......	38·4 *	......	......	......	......	38·8 of.	......	......
		double									
2.	α'	......	......	39·7 b.	......	40·3 of.	......	39·5 o*	......	39·7 *	39·5 ovf.
										very intense	
3.		41·8 vn.	......	41·6 f.	......	......	......	......	......	......	......
4.	β	......	......	45·2 f.	......	......	......	......	......	......	......
5.	β'	48·5 b.	......	47·5 *	......	48·3 f.	......	......	......	49·5 b.	48·5 ovf.
6.		......	50·3 vf.	......	51·4 f.	......	51·9 vf.	......	51·8 f.	51·5 b.	......
7.		......	......	54 *	53·7 f.	......	......	......	......	......	......
8.	32 γ	56·3 *	......	57·3 f.	56 f.	55·8 nb.	......	......	......	57 *	56 f.
9.	33	......	0·3 f.	59·6 f.	58 f.	......	0·4 vf.	......	58·8 f.	0 n.	1 f.
10.		3·3 nc.	......	1·6 b.	0 f.	......	......	3 nf. double	2·8 vf.	......	......
11.	δ	4·3 *	......	3 *	......	4·3 nb.	......	......	......	4 b.	......
12.		......	......	5·3 f.	......	......	......	6·5 nf.	......	6 b.	5 f.
13.		......	......	8·3 b.	7·3 b.	......	7·9 f.	......	7 c.	7·5 f.	8 b.
14.		9·3 vb.	......	10 b.	......	9·8 c.	......	9·5 nc.	......	10 f.	9
15.	ε'	10·8 b.	......	10·7 f.	......	...	......	......	......	not distinct f.	......
16.	ε''	12·3 b.	......	11·4 b.	11·7 f.	11·3 c.	......	......	......	...... f.	12 vf.
17.		......	13·3 c.	14 f.	......	13·8 c.	15·1 w⁴.	15·6 vn.	13·8 f.	13 f.	......
18.	ζ	17·3 wb.	17 w⁶.	16·7 b.	17 f.	17·3	......	18·5 nb.	17·8 vf.	16 f.	16 wf.
										18 f.	
19.		......	......	21·4 f.	......	......	......	21·5 fw.	......	19 f.	21 n.
										21 f.	
20.	η	24·8 *	......	24·1 *	24·7 b.	24·8 c.	25·4 b.	25·5 nc.	23 b.	24 c.	25 *
21.		......	27·3 wc.	27·4	......	......	......	28 , nc.	28 nc.	26 * double	27 f.
22.		......	......	30·8 b.	30·4 b.	......	30·4 nc.	......	......	31 f.	......
23.		31·7 nvb.	......	31·4 b.	... b.	......	......	......	......	......	......
24.		......	......	32·8 *	32·8 b.	32·3 vnb.	......	......	......	......	......
25.	θ	33·8 *	33·3 f.	......	......	34·3 *	......	33·5 nb.	33 vf.	34 vf.	34·5 vf.
26.		36·8 f.	......	36·4	36·1	37·3 nvb.	38·4 nf.	......	37·5 nc.	37 c. many	......
27.		......	......	38·8	39·4 b.	......	......	......	......	39·5 nf.	40 vf.
28.		41·3 b.	41·3 f.	40·7	41·4 f.	41·8 w³.	41·9 f.	41·2 *w².	41 b.	42 *	......
									43 b.		
29.		45·3 b.	......	44·7 b.	43·4 b.	......	......	......	45 b.	......	44·5 *
30.		46·3 b.	......	46 b.	45·4 b.	46·3 f. centre of 3	......	......	......	......	...
31.		......	47·3 wc.	48·7 f.	......	......	......	......	......	47·5	47 *
32.		......	......	50 f.	......	......	50·4c.	......	50·2 nc.	50 nc.	50·5 vf.
33.	ι	51·3 b.	......	52 b.	51·3	51·8 nc.	......	......	......	52 *	......
34.		......	54·3	54 b.	......	......	54·4 c.	......	......	54 nb.	54 vf.
35.	κ	56·3 *	......	55·3 *	55·3	55·8 *	......	......	......	55·5 *	......
36.	κ'	double	......	56·3 *	56·3	56·3 *	......	......	56 w.	... ...	......
37.		......	......	57·2 f.	......	......	......	57·5 nf.	......	58 n.	......
38.	33κ''	58·3 *	58·3	58·5 *	......	59·3 *	......	......	......	59·5 *	......
39.	34	2·3 n.	......	1·7 b.	2·3 b.	2·8 vnc.	1·4 c.	......	......	many here very sharp	......
40.	λ	3·3 n.	......	3·6 b.	......	......	......	......	......	......	......
41.		4·3 n.	5·3	4·9 b.	......	4·3 nb.	......	4·5 nf.	......	......	4 f.
42.		......	......	7·2	8·8 b.	......	7·4 c.	......	8 w.	......	7 *
43.	μ	9·8 b.	......	10·1 *	......	10·3 c.	......	......	......	10 b.	......
44.	μ'	11·8 b.	10·8	11·4 *	......	11·8 c.	......	12·5	......	11·5 b.	......
45.	μ''	13·3 b.	......	13·4 f.	14·4 b.	14·3 c.	14·4 c.	......	14 w.	14 *	13 vf.
46.		16·3 c.	......	16 f.	......	......	......	......	......	15·5 vf.	......
47.		17·3 c.	17·3	......	......	17·3 nf.	......	......	......	......	......
48.		18·3 c.	......	18·6 f.	......	... ...	......	......	......	18 b.	18 f.
49.	ν	......	......	20·9 f.	20 b.	19·3 c.	19·4	20·5 *w⁴.	20 w.	19 b.	19 f.
50.		......	......	23·2 f.	......	22·3 vf.	......		......	22 b.	......
51.		25·3 nf.	......	25·8 f.	......	25·3 f.	......	......	......	26·5 vnf.	......
52.		29·3 *w.	......	29·1 *	28·1	......	27·4 c.	......	28	27·5 *	......
										28·5 c.	
53.	ξ	......	31·3	32·4 f.	......	30·3 b.	......	......	......	31 vf.	......
54.		......	......	35·7 f.	34·4 b.	......	33·4 f.	......	......	35 nc.	......
55.	ο	37·3 vf.	38·8 vf.	......	......	37·3 vf.		......	36·5 f.	38·5 c.	......
56.				39·3 f.	40·3 f.		......	......	......	......	......
57.		......	......	43·6 c.	......	......	......	......	......	42 f.	......
58.		......	......	46·2 f.	45 b.	......	......	......	43 f.	45 f.	46 vf.
59.		......	......	50·5 c.	50·2 f.	......	......	......		48 f.	
60.	34	......	......	56·8 f.	58·1 f.	......	......	......	......		......
61.		......	......		3·4 f.	......	......	......	......	......	......
62.		......	......	......	6·5	......	......	......	......	......	......

The deposit in the tube very dense. These spectra are notable for the indistinctness of the green lines, as compared with the remarkable sharpness of those in blue and violet. The number visible in the H pair also deserves notice. This metal seems to have great illuminating power.

TABLE VII.—Iron.

		Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32	37·1 nf.	......	......	......	36·8 *	......	......	......	......	......
2.	α	38·4 *	......	38·4 *	......	......	......	......	......	38·4 vb.	38·7 vfo.
3.	α'	......	......	40 c.	40·3 of.	39·8 n.	......	39·7 *o.	......	39·7 vf.	......
4.		41·5 b.	......	42·3 c.	42·9 of.	......	......	......	......	42·7 vf.	43·3 vfo.
5.	β	......	......	45·5 c.	......	44·8 f.	......	......	......	44·9 vf.	......
6.	β'	48 *	......	47·5 *	47·5 of.	47·8 *	......	......	......	47·9 vb.	47·8 vfo.
7.		......	50·2 b.	50·1 f.	......	49 8 c.	49·8 f.	......	......	49·4 b.	50·7 *
8.		......	......	52·4 f.	51·1 f.	......	......	......	51·7 f.	51·4 f.	51·7 n.
9.		......	......	54·4 *	......	......	......	......	......	53·5 f.	......
10.	γ	55 *	......	57·3 vf.	55·3 c.	......	......	......	......	55·7 *	55·3 wb.
11.	32	......	59·7 f.	60·3 vf.	59·3 c.	0·8 c. many seen.	......	......	0·7 vn.	58 f.	58·6 b.
12.	33 δ	4·2 * double.	......	2 nb.	......	......	......	4·7	......	2·6 b.	......
				3·3 *						5 b.	
13.		......	......	......	6 *	5·8 nc.	......	......	6·2 bw.	6 b.	6 b.
14.		......	7·7	8·7 vb.	7 n	......	......	......	......	8 b.	......
15.		9·5 very many seen here.	......	10·4 b.	......	......	......	......	......	10 b.	......
16.		......	11·7	10·7 n.	11·4 f.	11·8	......	......	11·7 nc.	12 b.	10·7 nb.
				11·4 b.							
17.		14 vb.	......	14 f.	13·4 n.	......	......	......	......	... b.	13·7 nb.
18.		15 vb.	......	15·4 f.	15·4 n.	15·8 c.	14·8 w²f.	......	......	14·7 b.	15·4 nb,
19.	ζ	17 vb.	16·7	18 f.	17·7 n.	......	......	......	......	16·7 b.	17·4 nb.
20.		20 vb.	19·2 f.	20 nf.	20 f.	21·8	......	......	......	20	......
21.		22·4 vb.	......	......	......	22·8 n.	22·8 c.	......	......	21·1	23·4
22.	η	24·8 *w².	24·7 *	24·1 *	23·7 *	25·3 n.	......	......	24·2 b.	25·4 *	24·7 *
23.		......	......	26·7 vf.	......	......	......	......	......	26·1 nb.	26·1 vf.
24.		......	......	28·7 vf.	29·4 f.	......	......	......	29·7 vf.	30·1 vb.	29·4 wb.
25.		31·5 f.	30·7 f.	30·8 b.	32·8 vf.	32·3 vn.	......	32·7 c.	......	32·8 vb.	32·8 vf.
				31·8 nvf.							
26.	θ	34 * double.	......	33·4 * double.	......	......	......	......	......	......	......
27.		......	......	35·5 f.	......	35·8 vb.	......	......	......	34·8 vn.	35·1 vf.
28.		36·5 c. many.	......	36·8 f.	37·5 vf. others seen.	......	......	......	36·2 vfn.	36·1 vb.	......
29.		39 c.	39·7 f.	39·4 f.	......	38·3 nf.	......	......	......	38·8 vn.	37·5 nf.
30.		40·5 c.	......	41·1 b.	40·8 nf.	40·8 b.	39·8 nc.	40·7 *w.	......	41·2 *w⁵.	40·1 vf.
31.		......	44·7 b.	......	44·1 *	......	43·8 b.	......	43·7 nc.	44·1 f.	43·4 * 44·7 n.
32.		45·5 c.	......	45 b.	......	......	......	......	......	......	......
		46·5 c.	......	46·7 b.	......	......	......	......	......	46·7	......
33.		......	......	48·5 f.	48 vf.	......	......	......	......	......	......
34.		......	......	49·3 f.	......	......	......	......	......	49·1 nb.	49·3 nf.
35.		......	......	51·3 f.	......	50·3 *	......	......	50·7 vfw.	51·3 *	50·3 nf.
36.	ι	52·4 *	53·7 b.	53·6 f.	53·3 wb.	52·8 nc.	......	......	......	53·3 nb.	52·3 nf.
37.		55 b.	......	55·6 *	......	54·3 *	......	55·2 nc.	......	55·3 *	......
38.	κ	56·6 *	......	56·6 *	......	seen.	......	......	......	... ...	......
39.	κ'	57·3 *	......	57·2 vn.	......	57·8 *	......	......	......	57 nc.	......
40.	33κ''	59·3 *	......	58·5 *	......	......	......	......	......	58·8 *	......
41.	34	......	1·2 c.	1·7 b.	1·1 vf.	......	......	......	......	......	many.
42.	λ	4·5 c.	......	3·6 b.	......	3·8 n.	......	......	......	......	......
43.		......	......	5·2 b.	5·9 vb.	... ...	5·8 c.	......	5·7 c.	......	6·2 *
44.		......	8·7 b.	7·5 f.	......	7·3 n.	......	......	......	8·8 nb.	8·2 vf.
45.		......	......	10·1 *	......	9·8 n.	......	......	......	10·1 nb.	10·1 vf.
46.		12 * many.	......	11·4 *	......	12·8 *	......	......	......	12·7 *	12·4 vf.
47.		15·5 b.	.. ...	14 f.	......	......	......	......	......	......	15·4 nc.
48.		16·8 *	......	16·3 f.	15·4 n.	......	16·8 wf.	......	......	16·7 b.	16·7 nc.
		......	18·7	18·6 f.	18 n.	17·3 n.		17·9 *w⁷.	17·7 w.	18·6 *	18·6 nc.
	,	20 vb.	......	20·9 f.	19·3 n.	18·8 c.	......				......
		21·9 vb.	......	... ...	......	21·8 nc.	......	......	......	21·3 b.	22·6 nc.
		23·7 vb.	......	23·9 f.	......	24·8 nb.	......	......	......	......	......
		26·5 vb.	28·2 f.	26·2 f.	26·5	27·3 nb.	......	......	......	25·8 *	......
	ξ	30·8 *w³.	......	29·8 *	31·4 f.	......	......	......	......	......	30·4 f.
		32·5 b.	......	33·1 nc.	......	......	......	......	......	......	......
		......	34·7 f.	35·7 b.	34·4 f.	33·8 b.	......	......	......	34·4 c.	33·7 c.
	•	37 f.		39 f.	39 f.	37·8 n.	......	......	......	37·7 c.	36 f.
		......	......	42·3 f.		40·8 f.	......	......	......	41·6 c.	42·3
		......	......	44·2 nc.	......	... ...	......	......	......	44·2 f.	......
		......	......	47·5	......	46·8 f.	......	......	......	47·3	......
		51·5	......	52·1	......		......	......	......	......	50·2
			......		......	......	......	......	......	......	

These are beautiful spectra, the red brilliant, the violet part specially so, but the yellow often feeble. In air and CO, C.P., the green has a very great number of close narrow lines. In CO, R., the bright lines are very brilliant, the others unusually faint. N and CO were taken with Duboscq.

Table VIII.—Cobalt.

		Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.											
2.	32 α	39 *	38·4 of.	39·1 *	34·5 ? vf.	39 *	......	38·4 *o.	38·4 ovf.	38 *	......
3.	α'	41 vn.	......	41·1 b.	......	41 nb.	......	......	......	41 vf.	......
4.		......	......	43·3 c.	......	43 c.	......	......	......	......	......
5.		45 f.	......	......	......	45 c.	......	......	......	45 vf.	......
6.	β	......	......	46·5 f.	......	......	......	......	47·5 of.	47 *	......
7.	β'	49 *	......	48·8 *	......	48·5 *	......	......	......	49 nb.	......
8.		......	......	50·7 vf.	50·7 vf.	......	......	......	......	......	......
9.		......	51·4 b.	......	... vf.	......	......	......	51·1 c.	......	51 c.
10.		......	52·4 b.	52·7 f.	52·7 vf.	......	52 vf.	......	......	......	......
11.	γ	54 *	55·7 nc.	55·3 *	55·3 vf.	......	......	......	53·7 c.	......	......
12.		......	......	......	56·7 vf.	56 f.	56 vf.	......	56·3 c.	56 *	56 f.
13.		......	......	58 vf.	58 vf.	... f.	......	58 f.	58 b.	58 nf.	... ...
14.	32	......	59·7 b.	60·6 vf.	60 vf.	59 f.	59·5 vf.	......	59·3 b.	......	59·5 f.
15.	33	3 vnb.	3 nc.	2·6 nb.	2 vf.	2 f.	......	......	2·6 fn.	2 n.	... ...
16.	δ	4·5 *	......	4 *	......	... f.	4 vf.	4 vf.	......	4 b. double.	3·5 f.
17.		......	6·7 vb.	......	6 f.	6·5 f.	......	......	6·7 b.	6 f.	6 *
18.		......	......	......	7·3 vf.	......	8 c.	......	......	8 f.	7·5 n.
19.	ι	9 b.	......	9·4 vb.	9·4 vf.	9 vf.	......	......	......	9·5 f.	......
20.		......	......	10·3 b.	......	10 *	......	......	......	... f.	......
21.	ι'	11 b.	11·4 vb.	11·4 bvn.	......	......	......	one nf. 11·7 nf.			
22.	ι''	12 b.	......	12 b.	12 vb.	.. ...	12 nc.	one nf.	11·7 b.	11 f.	11·5 bn.
23.		......	......	14·7 vf.	......	......	......	......	14·7 vf.	14 f.	14 vf.
24.		......	15·1 vf.	......	......	15 n.	16 vf.	......	......	15 f.	......
25.	ζ	17 bw².	......	17·2 w⁴.	......	17 *	......	16·7 c.	17·4 vf.	......	......
26.		20 f.	20 f.	......	20·7 f.	20 b.	20·5 vf.	19·4 f.	20 vf.	20 b.	20·5 vf.
27.		22 vn.	......	......	......	22·5 f.	......	22·4 nc.	......	22 b.	......
28.	η	24 *	23·4 *	24·7 *	25·1 wb.	24·5 f.	23·5 *	24·7 vf.	23·7 c.	25 * double vb.	......
29.		27 f.	26·1 f.	28·1 vf.	......	......	......	... ...	25·7 c.	26 nc.	26 nc.
30.		30 vnb.	29·4 f.	......	30·1 b.	30·5 w³.	29 vf.	28·7 vf.	29·4 c.	29·5 b.	......
31.	θ	32 *	32·4 c.	32·1 nb.	32·4 b.	......	......	......	......	......	......
32.		......	34·8 c.	33·4 *	... ...	......	34 vf.	......	33·1 c.	......	......
33.		36	......	36·8 f.	35·5 c.	37 *	37 vf.	......	35·5 c.	36 bn.	......
34.		40·5 nb.	38·1 c.	39·4 f.	......	39 f.	40·5 nc.	......	......	40 cw².	......
35.		42 c.	41·4 c.	42·1 bw².	41·7 b.	41 b.	......	41·4 *	41·4 b.	......	42·5 *
36.		44 b.	43·7 vb.	......	43·4 b.	......	......	......	43·4 b.	......	
37.		45·5 b.	......	46	44·7 b.	......	44 vb.	......	45·4 f.	46 n.	44·5 n.
38.		......	......	47·4	......	......	......	47·4 vf.	......	... ...	......
39.		......	......	49·3 nf.	......	49 vn.	......	......	......	48 n.	50 c.
40.	ι	51 b.	51·3 b.	......	51·3 b.	51 *	......	......	50·3 c.	50·5 *	one here
41.		53 f.	52·6 n.	52·6 c.	......	53 nc.	......	......	53·3 c.	52 nc.	53·5 c.
42.	κ	54 *	... ...	54·7 nb.	54·7 b.	... ...	......	......	......	54·5 *	......
43.	κ'	55 *	55·1 b.	......	...	55 *	......	......	......	... ...	55 vf.
44.		... ...	56·6 b.	56·3 *	55·9 b.	56·5 n.	......	56·6 vf.	56·9 b.	56·5 n.	......
45.	κ''	57 *	......	57·2 *	......	... ...	......	......	......	... ...	......
46.		......	......	58·2 n.	......	... ...	......	......	......	58 vb.	......
47.	33	60·5 c.	......	59·1 *	......	59 *	......	......	60·7 b.	......	......
48.	34 λ	2 c.	1 vf.	2·3 bn.	1 b.	......	......	......	1·7 n.	......	......
49.		4 c.	......	4·6 bn.	......	......	......	......	3·9 n.	......	3·5 vf.
50.		......	6·5 b.	6·2 bn.	......	6 nb.	6 f.	......	6·9 bw³.	5 f.	6·5 *
51.		......	7·5 f.	8·5 f.	8·2 b.	8 nb.	......	......	......	8 f.	......
52.	μ	9 b.	......	......	......	10 nb.	......	......	9 8 f.	10·5 f.	......
53.	μ'	11 b.	......	11·1 *	......	......	......	......	......	......	......
54.	μ''	12 b.	......	12·7 *	......	12·5 *	......	......	......	12 *	......
55.		......	13·4 nf.	14·7 f.	14 b.	......	......	......	13·4 f.	......	......
56.		16 f.	16·7 c.	... f.	......	16 b.	......	......	15·7 f.	16 vb.	15·5 c.
57.		18 b.	18·3 c.	17·3 f.	18·6 nb.	18 *	17 vfw⁶.	......	18·6 b.	18 vb.	17·5 c.
58.	ν	20·5 f.	20 c.	19·9 f.	20·6 nb.	21 b.		20 *w⁴.	20·3 n.	21 b.	......
59.		24 f.	23·2 nvf.	22·6 f.	......	25 c.	......	......	......	25 *	22·5 n.c.
60.		......	26·8 vb.	26·5 f.	......	26·5 c.	......	26·8 vf.	......	......	......
61.		28 *	27·8 c.	......	27·8 b.	......	......	......	27·5 f.	......	......
62.	ξ	......	30·8 f.	30·1 *	......	......	......	......	......	......	......
63.		......	32·1 f.	......	.. ...	......	......	......	32·1 c.	......	.. ...
64.	ο	34 fw.	34·1 c.	33·7 b.	34 b.	34 c.	......	35 vf.	33·7 c.	33 vnf.	......
65.		......	36·3 vf.	......	......	......	......	......	36·6 c.	......	......
66.		......	38·3 vf.	......	......	38·5 n.	......	......	37·7 c.	37 f.	......
67.		40 nc.	39·6vf.	41 vnf.	40·3 f.	41 n.	......	......	39·3 c. 40·6 c.	41 f.	42 vf.
68.		43·5 vf.	42·9vf.	......	......	43 n.	......	43·9 vf.	42·9 vf.	43 nf.	
69.			......	44·6 nc.	44·9 f.	......	......	......	......	47 vf.	......
70.		......	......	47·5 vn.	......	46·5 c.	......	......	......		......
		......	50·2 f.	......	50·5 f.		......	51·5 vf.	50·5 vf.	......	......
		......		54·1	......	......	......		57·4 vf.	......	......
		......	......	57·1	57·4 vf.	......	......	......		......	......
		......	......			......	......	......	......	......	......

The spectrum O, R., has very little light, and much of it is a mere shadow. In H, C.P., the lines are visible only near the negative electrode, except the *. Above they are lost in a grey haze. Exhausting, the *s and haze vanish, and the other lines stand out conspicuous. In H, R., sometimes the tube was filled with conical strata and the lines were faint; at other times it was filled with a uniform flash, and they were bright. In CO, C.P., the sets 17–24 seem to replace the multitude of close narrow lines that fill the green in this gas in most of its spectra; they are as wide as the slit. The two of CO, the C.P. of air and H, and the R. of O were observed with Merz, the others with Duboscq.

TABLE IX.—Lead.

		Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32°	36·5 b.	......	......	......	......	......	......	......	......	......
2.	α	38·5 b.	37·8 of.	39 *	......	38·3 *	......	39 *o.	......	37·5 *	......
3.	β'	48 f.	49·1 vf.	48 vf.	......	48·8 *	......	......	......	47·5 b.	......
4.		......	......	......	......	......	......	......	......	49·5 b.	49 fo.
5.		......	51 b.	......	51 f.	......	50·8 vf.	51·4 f.	51 f.	51·5 f.	52 b.
6.	γ	55 *	......	55 *	......	......	......	......	......	54 } f. 54·5 } vn.	....
7.		57 c.	......	......	......	......	57·8 vf.	57·7 c.	......	56 vbn.	56 } f.
8.	32	......	58·5 fn.	......	59 f.	......	......	60 } f.	59·5 n.	59·5 vf.	0 } f.
9.	33	......	0·6 vf.	......	......	......	......	1 } f.	......	......	... } ...
10.		......	......	2 nc.	2·5 nf.	......	2·3 nf.	......	......	2·5 bw².	... } ...
11.	δ	4 *	3 fn.	3·5 *	......	......	......	3·3 f.	3·5 n.	......	3·5 } f.
12.		7 *	6·5 b.	......	6·5 nb.	6·8 nb.	6·8 b.	6·5 *	6 b.	5·5 } f.	7 *
13.	ε	8·5 b.	......	9 vb.	8·5 vf.	......	......	8·7 *	......	ten or twelve }	....
14.	ε'	......	10·5 nb.	10 } b.	......	......	......	......	......	here. }	......
15.	ε''	......	......	12 } b.	12 b.	12·8 nb.	......	11·7 f.	11 c.	... } ...	11 nb.
16.	ζ	16 *	......	17·5 bw³.	16 f.	16·8 b.	14·8 w.	16 *	......	17·5 } f.	16·5 bw³.
17.		......	20 f.	20 f.	......	......	19·8 nf.	19·7 f.	19·5 n.	20·5 } c.	21 f.
18.		......	......	......	......	22·8 nb.	......	22·7 f.	......	22·5 } c.	......
19.	η	24 *	24 *	24 * intense.	24 } *	24·8 bn.	23·8 b.	......	24 *	24·5 n*	24·5 * intense.
20.		......	......	......	25 } f.	......	......	25·4 f.	......	25·5 *	......
21.		......	......	27·5 f.	28 f.	......	......	28·1 f.	......	......	......
22.		......	......	30·5 vbn.	......	......	29·8 nf.	30·1 fn.	29·5 f.	30·5 b.	30·5 f.
23.		31 b.	......	......	......	......	......	31·8 *	32 f.	......	......
24.	θ	33 *	......	33 *	32·5 vf.	......	33 8	33·8 nc.	......	32·5 nb.	34 vf.
25.		36 f.	......	36·5 fw.	......	36·8 b.	......	36·1	35 f.	35·6 } b.	36·5 vf.
27.		......	......	......	37·5 f.	38·8 vn.	37·8 nf.	......	......	37·8 } b.	......
28.		41 f.	40 n.	40·5 bw³.	......	40·8 bw.	......	......	40 f.	40·5 bw⁴.	40 vf.
29.		......	43 b.	......	43·5 *	......	43·8 c.	42 *w⁶. intense.	42·5	......	43 *
30.		45 *	......	45 } nc.	......	......	......	......	......	......	......
31.		......	......	46 } nc.	......	......	......	46 bn.	......	46 f.	......
32.		48·5	......	......	......	......	......	48·3 bn.	......	48·5 vn.	......
33.		......	......	51 f.	50	50·8 *	......	......	51·5 f.	50·5 b.	......
34.	ι	52 b.	......	......	52	52·3 vn.	52·8 f.	......	......	52·5 nc.	52·5 fw⁵.
35.		......	......	54 f.	54	54·3 *	......	......	......	54·5 *	......
36.	κκ'	56 * double	......	55·5 * 56 *	......	......	......	......	......	56·5 f.	......
37.	33κ''	58 *	......	58 vb.	......	58·8 *	......	......	59·5 f.	58·5 *	......
38.	34	......	......	1·5 } nc.	0 b.	......	......	0·3 c.	......	......	0 fw.
39.	λ	4 w.	......	4 } nc.	......	......	......	3·6 c.	......	......	......
40.		......	5 b.	5·5 } nc.	......	4·8 fn.	......	......	5 b.	......	......
41.		......	......	......	6·5 } b.	......	6·8 c.	6·5 f.n	......	7·5 } vnb.	6·5 *
42.		......	......	......	9·5 } n.	8·8 n.	......	......	......	9·5 } vnb.	......
43.	μ	......	......	10·5 } *	......	10·8 f.	......	......	......	......	......
44.	μ'	11·5 bn.	......	12 } *	......	11·8 b.	......	......	......	12·5 *	......
45.	μ''	......	......	14 f.	14 f.	......	......	......	14·5 f.	14 nf.	14·5 } nc.
46.		17 *	17 f.	16 f.	16 n.	15·8 n.	15·8 fw.	16 *		15·5 nf.	17 } nc.
47.		......		19·5 f.	18 b.	19·3 b.		17·3 n.	......	......	19 } nc.
48.	ν	20 nb.	......	......	20·5 n.	......	......	20·9 *w⁷.	......	21·5 *	
50.		......	......	22·5 f.	22·5 f.	21·8 n.	......	......	......	......	......
51.		......	......	26 vf.	27 c.	24·8 bw².	......	26·5 f.	......	26·5 vf.	......
52.	ξ	30 bw.	......	29 *	......	......	......	29·5 *	......	......	......
53.		......	......	32 c.	......	32·8 nb.	......	31·8 nf.	......	33·5 vf.	......
54.	ο	35 fw.	......	35 b.	34 c.	36·8 nb.	......	35·7 vf.	......	36·5 vf.	......
55.		......	......	40 n.	40 vf.	39·3 vn.	......		......	42 vf.	......
56.		......	......	45 vf.		......	......	......	......	46·5 vf.	......
57.		50 vf.	......	51 f.	....v.	48·8 n.	......	......	......		......
			......		......		......	......	......	......	......

Some of these are peculiar. In air, C.P., No. 12 is the brightest of all, ζ is very brilliant, so is No. 46. The H, C.P., is also greatly developed. It was taken at the negative electrode only. Nos. 2, 29, and 48 reach quite across the spectrum; the others more or less, but on an average about one-third of the way from each electrode, being brightest at the negative. The deposit in the tubes was very considerable, but it does not seem to have cut off much of the spectra. To examine the effect of the diameter of tube, I took (with Duboscq) the air R. in a compound tube, of which the wider parts were 0·4 inch bore, and the narrower 0·04 and 1·5 long. The distance between the electrodes was 5 inches. The discharge was pretty bright in the narrow parts, and red, except on the negative wire, where it was blue. They are as follows:—

			Narrow.	Blue negative.	Red positive.
1.	32°	α	38·1 f.	......	......
2.			49·1 vf.	......	......
3.	32		58 } vf. many.	58 vf.	58 vf.
4.	33		0·6 } vf.	......	......
5.			5·3 } n.	6 f.	......
6.			7·3 } n.	......	7 vf.
7.		ε	9·4 vf.	......	......
8.			11 b. others.	10·7 c.	10·5 nb.
9.			14 } f. some.	......	......
10.		ζ	16·4 } f.	......	......
11.			20 f.	20·7 } c.	......
12.			22·4 } w.	23·4 } c.	23 f.
13.			25·7 } w.	25·7 } c.	......
14.			28·4 nf. one.	29·4 f.	......
15.			31·4 nf.	32·1 nc.	30·8 vf.
16.			34·4 c.	35·1 vf.	......
17.			37·5 c.	37·8 f.	37·8 nf.
18.			40·8 nb.	40·8 nc.	......
19.			44·1 w.	44·1 vf.	44·1 f.
20.			......	46·7 vf.	......
21.			50 b.	51·3 c.	49·3 vf.
22.	33		55·3 b.	54·7 f.	54·3 f.
23.	34		0·4 b.	0·1 f.	0·4 f.
24.			7·5 b.	6·9 f.	6·9 f.
25.			13·4 nb.	12·7 vf.	13·4 vf.
26.			18 } c.	......	......
27.			...... }	18·6 f.	18·6 w.
28.			19·3 } c.	......	......
29.			26·5 b.	25·8 c.	26·5 vf.
30.			33·1 c.		33·4 vf.
31.			39·6 c.	......	
32.			49·5 vf.	......	......
33.			57·8 vf.	......	......

The three evidently represent the same system of lines, for those missing in the second and third have been lost by their faintness. Nos. 12, 19 and 46, Table IX., are resolved by Duboscq into pairs. It also shows 17 which were not visible in the other. Nos. 5 and 11 of that again are not found in these: the first of these is almost peculiar to R. spectra; and its not being visible in the narrow tube here seems to argue that condensed discharge cannot show it.

Table X.—Zinc.

		Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32	37·3 f.	……	……	……	……	……	……	……	……	38 o.
2.	α	38·7 *	……	38·4 *	38·7 *	39·4	……	39 *	39·6 *o.	……	……
3.	α	40·5 vb.	……	39·7 b.	……	41·4 n.	41·3 fo.	……	……	40 *	……
4.		43 *	……	42·9 f.	42·8 f.	44·4 n.	……	……	……	……	42·5 vfo.
5.	β	……	……	45·9 f.	45·9 vf.	46·4 n.	……	……	……	……	……
6.	β'	48·2 bn.	……	48·1 vb.	……	……	……	……	……	48·2 } f.	……
7.		50·3 *	……	50·1 vf.	51·1 vf.	51·4 n.	……	……	……	50·5 } f.	……
8.		52·1 vb.	……	52·1 f.	……	……	52·6 vf.	……	……	52 f.	52 vfo.
9.	γ	54·7 *	54·8 f.	54·4 *	53·7 vf.	54·4 n.	……	……	……	55 vf.	55 fo.
10.		……	……	……	56·3 nc.	……	……	56 vf.	……	57 vb.	……
11.		……	……	……	58·3 vn.	57·4 vn.	……	……	……	……	……
12.	32	60 b.	……	……	59·6 vb.	……	……	……	……	……	59·5 f.
13.	33	……	……	1·6 nb.	2·3 vnc.	……	……	……	……	……	……
14.	δ	3·7 *	4·5 f.	3·3 *	……	3·4 n.	3·8 vf.	4·5 f.	4·6 b.	……	4 nf.
15.		……	……	……	6 b.	7·4	6·8 c.	……	……	7 } c.	7 c.
16.	ε	9·5 b.	……	8·3 b.	……	9·4 b.	……	9·5 vf.	……	9 } c.	……
17.	ε'	10·8 } b.	……	10·4 } b.	……	……	10·8 nc.	……	……	……	……
18.		…	……	11 } b.	11·7 *	……	……	……	……	11 } f.	11·5 c.
19.	ε''	12·7 } b.	13·5 } f.	……	……	12·4 b.	……	……	……	13 } f.	……
20.		… …	…	……	……	……	14·8 fw².	……	……	15·5 } f.	14·5 f.
21.	ζ	18 b.	16·3 } f.	17 bw⁴.	17·2 w².	……	……	……	……	17 } f.	16 vf.
22.		……	……	……	……	20·4 f.	19·8 n.	……	……	19 } f.	20·5 vf.
23.		……	……	……	23·7 } b.	23·4 } bn.	23·8 b.	……	……	21 } c.	……
							brightest.			23 } c.	……
24.	η	24·8 *	25·5 *	24·4 *	25·1 } b.	24·4 } bn.	……	25 vf.	……	25·5 *	25 *
25.		……	……	25·7 nf.	……	26·4 } bn.	……	……	……	……	……
26.		……	……	27·1 nf.	27·1	……	……	……	……	27·5 n.	……
27.		……	……	29·1 nf.	29·1	……	28·8 w.	……	……	29 f.	……
28.		……	31·1 f.	31·1 vb.	31·1	30·4 f.	……	……	……	……	30 fw.
29.		32·4 n.	……	32·1 vn.	……	……	……	……	……	……	……
30.	θ	33·7 *	34·7 f.	33·5 *	33·1	34·4 bn.	……	34 c.	……	34 *	34 nf.
31.		……	……	36·1 f.	35·5	……	35·8 w.	……	……	……	……
32.		37·8 *w².	……	……	……	37·4 } vb.	……	……	……	37 nb.	……
33.		……	……	……	……	38·4 } vb.	……	38 vf.	……	……	38 f.
34.		……	40 f.	39·1 f.	39·1 f.	39·4 } vb.	……	……	……	……	……
35.		……	42 f.	41·7 w.	41·7 *	42·4 w².	41·3 b.	42 *w⁴.	……	41·5 *w³.	42 *vb.
36.		44·7 *	……	……	43·7 } b.	……	43·8 bw.	……	……	44·5 vfn.	44 b.
37.		……	45·8 c.	45·7 } b.	45 } b.	45·9 b.	……	……	45·6 b.	……	……
38.		……	……	46·4 } b.	……	……	……	46 fn.	……	47 nc.	……
39.		49 *	……	48·7 nf.	……	……	47·8 n. a fine one	……	……	49·5 } fn.	48 fn.
40.	ι	50·3 *	50·5 f.	50·3 vf.	50·3 } c.	50·9 vn.	……	……	50·6 f.	… } …	50·5 f.
41.		……	……	52 f.	52·3 } c.	……	……	52 vf.	……	52 } *	……
42.		53 *	……	53·6 f.	… } …	52·9 *	……	……	53·6 vf.	53 } c.	……
43.	κ	55·6 *	55 f.	55·9 *	55 } c.	……	……	56 f.	……	… } …	56 f.
44.	κ'	56·8 *	……	57·2 *	56·6 } c.	57·4 *w.	56·8 n.	……	……	57 } n.	……
45.	33κ''	59 *	……	58·2	58·5 nf.	59·9 vn.	……	59 f.	……	59 } b.	……
46.	34	……	1 f.	2 } b.	1·1 } bn.	0·4 *	1·8 vfw.	……	……	1 } f.	1 f.
47.	λ	……	3·1 c.	3·9 } b.	2·6 } bn.	……	……	4 vfw.	……	3 } f.	……
48.		5·8 b.	……	5·9 } f.	5·9 nc.	……	……	……	……	… } …	……
49.		……	7 f.	……	7·2 vb.	6·4 f.	6·8 b.	……	……	6·5 } f.	7 bw. double.
50.		……	9·1 f.	8·5 f.	8·5 nf.	……	……	……	……	8·5 } f.	……
51.	μ	11·3 vb.	……	11·1 } *	……	10·4 } nb.	……	11 vf.	……	11 } f.	……
52.	μ'	……	……	12·7 } *	……	12·9 } nb.	……	……	……	11·5 vf.	……
53.		……	……	……	13·7 b.	… } …	13·8 } f.	……	……	13 *	13 n.
54.	μ''	14·2 b.	15·1 f.	14·4 } f.	……	14·9 } b.	a fine one	……	……	14 c.	14 vf.
55.		……	……	17·7 } f.	18·3 } b.	18·4 } n.	……	……	……	17 } b.	16·5 vf.
56.	ν	20 *	20 f.	20·3 } f.	20·4 } b.	20·9 } n.	22·8 } f.	21 *w³.	21 c.	19 } *	20 c.
57.		……	……	23·5 } f.	23·2 } nc.	23·9 } n.	23·8 f.	……	……	22 } b.	……
58.		……	……	… } …	24·9 } n.	……	……	……	……	24 } f.	24 c.
59.		……	……	… } …	25·5 } vb.	……	……	……	……	… } …	……
60.		……	27 f.	26·8 } …	27·9 } f.	28·2 bw⁴.	27·8 f.	27 nc.	……	27 } *	28 c.
61.	ξ	30 *	……	29·8 } vb.	29·5 nc.	……	……	……	……	29 } n	……
				30·5 } b.							……
62.		……	……	……	32·1 nc.	……	……	……	……	33 f.	……
					34·1 vb.						
63.	ο	……	35 f.	36·7 vn.	35·4 } nf.	……	35·8 f.	……	……	35·5 c.	35 f.
64.		……		……	36·7 } nf.	36·4 c.		……	……	……	
65.		39 f.	……	……	38·6 } nf.	……	……	……	……	……	……
66.			……	……	40·6 } nf.	40·9 nf.	……	……	……	42·5 c.	……
67.		……	……	45·2 f.	45·9 nc.	44·4 nf.	……	44 fw.	……	……	……
68.		……	……	……	49·8 c.	49·4 f.	……		……	48	……
				52·5	56·1 vf.		……	……	……		……
						……	……	……	……	……	……

This is one of the most splendid spectra, especially at the violet end. N, R., is remarkable for its brightness and the number of its lines. CO, R., has the discharge lilac-coloured instead of green, as is generally the case in this gas.

TABLE XI.—Cadmium.

		Air.		Nitrogen.		Oxygen.		Hydrogen.			Carb. oxide.	
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R. 2·55.	R. 0·2.	C.P.	R.
1.	32	35·5 o.	......	......	......	....	......	......	......	......	....	......
2.	α	38·5 * double.	......	39 *	38 f.	38·8 *	......	38·8 *o.	......	......	....	......
3.	α'	41 double.	......	41 n.	......	39·8 f.	......	......	......	......	40·7 *	......
4.		....	......	......	......	44·8 f.	......	......	......	......	......	......
5.	β'	48·5 *	......	48 f.	49 f.	47·8 *	......	......	......	......	48·7 *	......
6.		....	......	......	... ...	49·8 n.	50·8 b.	......	......	50·5 vf.	50·7 b.	......
7.		....	52 o.	......	... ...	....	53·8 fn.	53·3 vf.	......	......	......	52·7 bw².
8.	γ	54·5 *	......	55 *	... ...	56·3 n.	......	......	......	......	55·2 n.	55·7 vf.
9.	32ψ	59 n.	......	......	... ...	....	58·8 fn.	......	......	......	57·7 *	59·7 f.
10.	33 δ	3·5 *	4 fw.	3 vn. 4 *	3 f.	....	...... 5·8 vb.	......	......	......	0·2 fn. 3·7 *	3·7 f. ......
11.		....	7 c.	......	......	....	6·8 nb.	......	......	6 fw.	6·7 three.	7·7 *
12.	ι	10 b.	......	9 b.	......	....	......	......	......	......	9·7 four.	......
13.	ι'	11·5 b.	11 fn.	10·5 b.	......	11·3 bn.	10·8	......	......	......	......	......
14.	ι''	13 b.	13 vfn.	13 b.	12 b.	13·8 fn.	......	12·8 f.	14·5 vfw.	......	......	12·7 nb.
15.		16 *	......	15 *	......	16·3 bw.	14·8	17·3 f.	......	......	14·7 four.	16·7 w³f.
16.	ζ	17 *	17 f.	17 *	17 bw.	....	......	18·8 f.	......	......	......	......
17.		....	......	......	......	21·8 double. b.	......	......	......	......	20·7 c. 23·7 c.	21·2 f.
18.	η	25·5 *	24 b. 27 f.	24 *	24·5 bw³.	24·8 double. b.	23·8 *	......	24·5 vfn.	24 b.	25·7 *	24·7 vb.
19.		....	28·5 c.	......	29 c.	....	28·8 fw.	.....	29 vf.	28·5 bn.	28·7 n.	28·7 bn.
20.		30 vbn.	......	29·5 nb.	......	30·3 fw.	......	30·8 bn.	......	......	30·7 f.	......
21.		32 vn.	......	31 nb.	......	....	......	......	......	......	......	......
22.	θ	33·5 *	......	33 *	33 c.	....	32·8 f.	......	......	......	......	34·2 vf.
23.		....	......	......	35 c.	36·3 b.	......	......	......	......	......	......
24.		37·5 f.	......	38 f.	......	....	......	......	......	......	37·2 f.	38·7 f.
25.		....	......	......	39 n.	39·8 w.	39·8 nb.	......	40 nf.	39·5 f.	......	......
26.		41 f.	42 c.	41 bw.	41·5 bn. 43·5 n.	....	......	41·8 *w.	......	41·5 c.	......	41·7 nc.
27.		45 *	44 bw.	45·5 *	45·5 bn.	....	43·8 vb.	45·8 nbw.	45 vfn.	44 c.	43·7 fw².	44·7 b.
28.		....	......	......	......		......	......	......	......	47·7 n.	......
29.	ι	51·5 n.	......	50 f.	50·5	49·8 *	......	......	51·5 vfvn.	......	51·7 *	......
30.		53 *	53 cw.	53 c.	52	51·8 n.	51·8 vf.	......		52·5 cn.	53·2 nc.	54·2 fw.
31.	κκ'	56 * double.	......	56 *	55	53·8 *	......	53·8 n.	......	......	54·7 * 57·2 nf.	......
32.	33κ''	59·5 *	......	58 *	......	57·8 *	58·8 vf.	......	......	......	59·2 *	......
33.	34	....	2 f.	2 bn.	1 f.	....	......	......	......	......	......	1·7 f.
34.	λ	3·5 cw.	......	3·5 bn.	......	....	......	......	......	......	......	......
35.		....	......	4·5 bn.	......	4·8 bn.	......	......	......	......	......	......
36.		....	7 cw.	......	7 f.	7·8 bn.	6·3 vb.	......	......	......	......	6·2 *
37.		....	......	......	......	8·8 bn.	......	......	......	......	9·7 vf.	......
38.	μ	11 vb.	......	......	......	....	......	......	......	......	......	......
39.	μ'	12 b.	......	11·5 w.	......	11·8 bw.	......	......	......	......	......	......
40.	μ''	13·5 b.	......	......	14·5 f.	....	13·8 vf.	13·3 vn.	......	......	13·7 b.	......
41.		....	......	......	......	15·8 n.	16·3 vf.	......	......	......	......	15·7 f.
42.		....	18 f.	......	......	17·8 c.	...	......	......	......	17·7 n.	...
43.	ν	19 vbn.	......	......	19·5 f.	19·8 f.	19·3 vf.	19·8 *w.	......	......	19·7 c.	18·7 f.
44.		....	......	......	......	....	......		......	......	21·7 n.	23·2 f.
45.		....	27·5	26 vfn.	27 f.	25·8 bw².	......	......	......	......	26·7 *	26·7 f.
46.	ξ	29·5 *	......	30 *	......	....	......	......	......	......	......	......
47.		....	34 f.	33·5 vf.	34 vf.	32·8 nc.	......	......	......	34·5 vfw.	34·7 f.	33·7 f.
48.	ο	35 fw.		36 c.	......	....	......	......	......		......	......
49.		....	......		......	37·8 f.	......	......	......	......	38·7 vf.	......
50.		....	......	......	41 f.	41·8 f.	40·8 vf.	......	......	......	......	......
51.		46 f.	......	......		45·8 c.		......	......	......	44·7 vf.	43·7 vf.
52.		50·5 vf.	......	......			......	......	......	......	47·7 vf.	45·7 vf.
53.		53·5 vf.	......	......		....	......	......	......	......		

I have given here the H transition spectrum. The lines are scarcely visible; they are all found in the other two H. These spectra are exceedingly beautiful, but troublesome in tubes of such small diameter from the dense and opake deposits.

Table XII.—Bismuth.

		Air.		Nitrogen.			Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R.	C.P.	Ditto – end.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32°	32·7 fo.	......	....	....	33·5	....	......	......	......	....	......
2.	α	39·7 *	......	38·5 c.	....	....	39 *	......	......	......	39·2 *	......
3.	α′	41·7 b.	......	40·5 42·5 f.	....	....	42 f.	......	40·9 *o.	......	....	......
4.	β	......	......	46 f.	....	....	45 f.	......	......	......	....	......
5.	β′	48·7 b. double.	......	49 c.	....	....	49 *	......	......	48·9 vf.	47·5 *	......
6.		......	......	51 f.	... .	50·5 } vn.	50 f.	......	......	......	49·5 b.	......
7.		52·2 bn.	52·5 f.	53 f.	....	first of many.	....	......	......	......	......	52·5 f.
8.	γ	55·2 *	no other red.	55 *	....	......	....	......	56·4 vf.	......	55·5 } vn.@ 56·5 } *	56 f. vn.
9.	32	58·2 } c. 58·7 } n.	......	many close here.	....	60·5 vn.	....	......	......	......	59 f.	59·5 f. vn.
10.	33	2·2 vbn.	1·5 f.	2 nf.	....	....	....	......	......	......	2·5 }	......
11.	δ	3·7 *	......	4 *	....	....	....	4 fo.	......	......	3·5 }	4·5 vn.c.
12.		6·7 f.	......	7 vf.	....	5·5 b.	7 b.	8 f.	6·9 vfn.	......	5·5 } vn.	7·5 *
13.		9·7 b.	9·5 f.	8·5	....	9·5 } f.	....	......	......	8·4 bw.	four here.	9·5 nc.
14.	ε	......	......	10 } c.	....	...	10 vf.	......	......	......	... ...	......
15.	ε′	10·7 } b.	......	11 } c.	....	11·5 } vf.	....	......	......	11·4 n.	... ...	11·5 vf.
16.	ε″	13·2 } b.	......	12 } c.	....	....	13 fw.	12 vf.	......	......	12 vn.	......
17.		14·7 } f.	......	15 } nf.	....	....	....	......	......	15·4 fn.	four here.	15 f.
18.	ζ	15·7 } f.	......	17 } nf.	....	16·5 f.	17 w.	17 vf.	......	......	... ...	......
19.		19·7 vb.	18·5	18 } nf.	....	....	....	......	......	......	18·5 } vn.	......
20.		22·7 *	......	23 vb.	20 b.	....	21 f.	22 vf.	......	20·4 vf.	20·5 c.	20·5 f.
21.	η	24·2 f.	24·5 bw.	....	23 *	....	....	......	......	24·4 *	24·5 nb.	24·5 *
22.		25·7 *	......	....	25 vn. vb. 26 vb.	....	25	25 *	......	......	25·5 *	......
23.		27·2 *	......	27 vf.	27 vb.	....	....	27 n.	......	......	26·5 nb.	26 vnb.
24.		30·2 bn.	......	30 nb.	29 vb.	29·5 } vb.	....	30 vf.	29·4 vfn.	......	......	......
25.		32·7 c.	31·5 c.	....	31 vb.	30·5 } vb.	....	......	......	......	30·5 bn.	30 vf.
26.	θ	33·7 *	......	33 *	33·5 *	....	....	......	......	......	33·2 b.	......
27.		......	36·5 fn.	36·5 f.	....	....	36·5 vb.	35·5 vf.	36·4 vf.	......	35 n.	35·5 vf.
28.		37·7 f.	......	....	37·5	37·5 c.	38 n.	38 vf.	......	......	36·5 nb.	......
29.		40·7 f.	39·5 f.	39 nf.	....	39·5 vn.	39 n.	......	......	......	......	......
30.		40·7 f.	......	40	41 w.	....	41 f.	41 nc.	......	......	......	41 vf.
31.		......	44·5 w.	45 } c.	45 bw. double.	43·5 b.	....	......	42·4 *w.	42·9 bw.	42·5 bw².	44·5 } vb.
32.		45·2 cw.	......	46 } c.		....	44·5 vf.	45 vb.	......	......	46·7 vn.	45·5 } f.
33.		47·7 b.	......	48 f.	48 } nvc.	48·5 b.	48 bn.	48 vfn.	......	......	......	......
34.		49·7 *	49·5 nc.	....	50 } nvc.	....	....	......	......	......	49·2 vn.	50·5 f.
35.	ι	51·2 b.	......	51 f.	52 } nvc.	....	52 *	50·5 vnc.	......	......	51 *	50·5 } f.
36.		......	......	53·5 n.	....	53·5 b.	53 vbn.	......	......	......	53·5 vn.c.	52·5 } f.
37.	κ	......	55 w.	55 } *	....	....	55 *	......	......	......	54·5 *	54·5 } f.
38.	κκ′	58·2 * double.	......	56 } *	56 * double.	....	57 n.	......	......	......	57·6 vn.	......
39.	33κ″	58·7 c.	......	59·5 vb.	58·5 b.	59·5 b.	59 *	......	58·4 c.	......	58·5 *	......
40.	34	1·2 *	0·5 w.	....	2 b.	2 f.	....	......	......	......	......	2 vf.
41.		......	......	3 } n.	....	....	....	......	......	......	......	......
42.	λ	4	......	4 } n.	4 f.	....	....	......	......	......	......	......
43.		......	......	5·5 } n.	5·5 vf.	6·5 b.	5 } cn.	......	......	6·4 c.	......	6·5 vb.
44.		8·7 vf.	8·5 w.	....	....	....	8 } cn.	8 fw.	......	......	7·5 } nb.	......
45.		......	......	....	....	....	9·5 } cn.		......	......	9 } nb.	......
46.	μ	10·7	......	11 }	10 } f.	....	....	......	......	......	10·5 nb.	......
47.	μ′	12·2	......	12 }	11 } f.	12·5 bn.	....	......	......	......	11 nb.	......
48.	μ″	13·7	13·5 w.	14 }	seen. } vf.	....	....	......	......	......	13·5 *	......
49.		......	......	16 f.	....	....	16·5 } bn.	......	......	16·4 vf.	17·7 } f.	17 }
50.		......	18·5 w.	19 vn. c.	19 fw.	18·5 b.	18 } b.	......	......		19·2 } c.	18·5 }
51.	ν	20·2 b.	......	....	....	....	21 } f.	......	22·4 *w.	......	22 } f.	20·5 }
52.		23·7 *	......	....	23 nc.	....	....	......	......	......	......	23·5 c.
53.		......	25·5 w.	25 vf.	27 nb.	25·5 b.	26 b.	......	......	......	26·8 *	
54.		28·2 vb. 29·7 vf.	......	29·5 b.	29 nc.	....	29·1 } wb. 30 } vb.	......	29·4 c.	......	......	......
55.	ξ	......	32·5 vf.	33·5	34·5 f.	33 vf.	34·5 bn.	......	......	......	34·2 f.	......
56.	ο	37·7 vfw.		36 f.	....	....	....	......	......	......	38·2 f.	......
57.		......	......		....	....	....	......	......	......	42 f.	......
58.		......	......	....	....	....	....	......	......	......	46·5 f.	......
59.		48·2 vf.	......	....	....	....	....	......	......	......	48·1 f.	......

I have given N, C.P., at the negative boundary in part. It seemed identical with the central one till No. 20; so I did not take that part. In the green the lustre of the lines marked b. or vb. is fully that of θ, and they would be marked * if they were as wide as the slit. In the indigo and violet this spectrum has little, if any, superiority of brightness. In air, C.P., No. 20 is the brightest of the whole, and the companion of η is developed into first-rate brightness. In air, R., there is no red but the single line No. 7.

TABLE XIII.—Tin.

		Air.			Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R. 2·18.	R. 0·2.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32	......	......	......	......	34 vfo.	......	......	......	......	....	....
2.	32 α	38 * double.	......	......	38·5 * double.	39 fo.	39·5 *	......	37·5 *o.	......	39 b.	....
3.	32 α′	41·5 *	......	......	......	....	41·5 n.	......	......	......	....	....
4.	32 β	......	......	......	......	45 vf.	44 f.	......	......	......	....	....
5.	32 β′	48 b.	......	......	48 b.	47·5 f.	47·5 vf.	......	......	......	48	49 vf.
6.		......	......	51·5 c.	......	51 f.	50 * 51 bn.	51 f.	......	......	51	52 c.
7.	γ	54 *	53	......	53 *	....	54 f.	......	52·5 n.	......	....	....
8.		......	......	......	......	55 f.	......	55 f.	......	56 fw.	55·5 f.	55 f.
9.	32	58 b.	......	......	......	59 f.	......	...	......	......	57·5 c.	....
10.	33	1·5 vbn.	1	......	1 n.	....	0 f.	0 f.	1	......	....	0 f.
11.	δ	3 *	......	2·5 b.	3 *	....	4 f.	4 f.	......	......	....	....
12.		6 b.	......	......	......	5 c.	......	......	7	......	7 n.	7 *
13.		7·5 b.	......	......	8 b.	....	7·5 f.	8 *	......	......	many	8 *
14.	ε	9 b.	......	......	9·5 c.	....	......	9 n.	......	......	here.	....
15.	ε′	10 c.	......	10 c.	10	10 nb.	10 f.	......	......	10 b.	......	....
16.	ε″	11·5 c.	......	...	11	two	... ...	......	......	......	......	12 cn.
17.		......	......	14 c.	......	... ...	13 c.	13 cn.	......	......	......	....
18.	ζ	16 b.	......	16 c.	16	15 nb.	15·5 n.	......	15 fw.	......	......	16·5 triple.
19.		......	18·5	......	......	....	18 n.	17 f.	......	......	......	....
20.		21 n.	......	......	......	....	21 n.	21 f.	......	......	21·5 n.	21 nc.
21.	η	23 *	......	23 w.	23 *	24 b.	24 bn.	24·5 *	22·5 bn.	23·5 vf.	23 c.	24 *
22.		26·5 n. one here.	26·5	......	25 vf. 27 f.	24·5 n.	26 b.	26 n.	......	......	26 *	25 n.
23.		......	......	......	28 bn.	29	......	29 f.	......	......	....	....
24.		30·5 vn.	......	30·5 f.	faint.	31 nf.	31·5 f.	......	31 b.	......	31 c.	31 vf.
25.	θ	33 *	32 vfvn.	......	32 *	34 nf.	......	......	......	......	33·5 b.	....
26.		36 f.	......	......	35·5	....	......	35	......	......	....	36 nf.
27.		38 nf.	......	38 cn.	37·5 vf.	38 c.	37 bn. 38 cn.	......	...... one n.	37 vf.	37 c.	....
28.		......	40 vf.	......	39·5 f.	....	39 vf.	......	39 * one n.	......	....	....
29.		41 b.	......	......	42·5 f.	....	42	40·5 nc.	......	......	....	41 f.
30.		43·5 c.	......	44 n.	44 f.	43 b. compound.	......	44 *	......	......	44 b.	44 *
31.		45·5 c.	44·5 vf.	......	46 f.	....	......	one here.	......	......	....	46
32.		......	......	......	49	49 c.	49 f.	......	......	......	49 *	50 f.
33.	ι	50 b.	50 vf.	50 n.	......	......	52 *	52 f.	......	......	50·5 n.	52 f.
34.		53 c.	52 vf.	53 w.	......	54 c.	54 n.	......	53·5 f.	......	53·5 *	... ...
35.	κκ′	55 * double.	56 vf.	......	55 *	56 c.	56 *	......	...	......	55.5 n.	55 f.!
36.	33κ″	58 * double.	......	......	57 b. double.	......	58 n.	......	57·5 f.	......	57·5 *	....
37.	34	......	......	0·5 cn.	......	59·5 c.	0 *	1 f.	......	......	....	....
38.		......	2 vfw.	......	2	......	......	......	......	......	1 n.	....
39.	λ	4	......	......	3	......	......	......	......	......	3·5 n.	....
40.		......	......	......	5	5 c. triple.	......	......	......	......	4·5 n.	....
41.		......	......	8 c.	......	......	6 cn.	7 *	......	......	7·5 b.	6 b.
42.	μ	10 b.	......	......	9·5 b.	......	9 cn.	......	......	......	9·5 b.	....
43.	μ′	11 c.	......	......	11·5 b.	......	10·5 cn.	11	......	......	11·5 b.	....
44.	μ″	13 c.	10 f. many n.	13 f.	......	12·5 c. double.	13 b.	......	......	......	12·5 b.	....
45.		17 vfn.	......	......	15 f.	......	17 f.	15·5	......	......	16·5 c.	15·5
46.	ν	19 c.	19 f.	19 bw.	18 f.	18 c.	20 c.	20·5 f.	19·6 *w.	......	18·5 b.	18
47.		......	......	......	22 f.	......	22 f.			21 vf.	21·5 c.	22
48.		24·5 vfn.	......	......	......	......	......	......	......		24·5 f.	....
49.		......	27 *w.	27 *w.	......	......	27 *	......	......	......	26·5 *	26 f.
50.		28·5 *w.	29 f. double.	......	29 vb.	......	......	......	......	......	29·5 f.	....
51.	ξ	33·5 vf.		34 vf.	33 f.	32 c.	31·5 f.	......	......	......	33·5 f.	....
52.	ο	......	......		35·5 f.	....	35 bn.	......	......	......	35·5 f.	36 fw[10].
53.		......	......	......	......	39 f.	38·5 cn.	......	......	......	39·5 f.	
54.		......	......	......	43·8 f.		43 cn.	......	......	......	42·5 f.	....
55.		......	......	......		....	45	......	......	......	45·5 f.	....
56.		47 vf.	......	......	......	....	48 b.	......	......	......		....

I have given the transition spectrum for air. In it No. 7 is the beginning of light. No. 10 is the beginning of a band almost entirely dark, which occupies the place of δ. Another dark interval at No. 19 seems to replace the bright group ζ, and a third at No. 22 stands for η. No. 25, which, though very faint, is the brightest among them, is the residue of θ. The lines at the violet end are relatively less faded, but are all very obscure.

Table XIV.—Antimony.

		Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32° α	38 f.	......	39 *	39·3 vfo.	38·8 *o.	39·3 o.	39·4 *o.	......	38·7 *	......
2.	α′	41 c.	......	41 vnf.	......	......	......	......	41·4 c.	....	......
3.	β	45 f. double.	......	......	......	......	......	......	......	47·7 *	......
4.	β′	49 *	......	49 nf.	......	48·3 *	......	......	......	49·7 b.	......
5.		50 c.	......	......	......	50·3 n.	51·3 f.	......	......	....	......
6.		53 b.	52·2	......	52·5	......	......	......	......	....	52·2 *
7.	γ	55 * 56 *	....	55 *	55·5 f. 60 c.	54·8 bn.	....	....	....	54·7 * 55·7 *	55·7 f.
8.	32	59 fw.	59·2 n.	......	......	......	......	60·4 f.	59·4	....	58·7 c.
9.	33	....	......	3 vbn.	......	2·8 n.	1·8 vn.	......	......	2·7 vnc.	......
10.	δ	3·5 *	4·2	4 *	4 f.	......	......	3·9 n.	......	3·2 *	3·2 f.
11.		5 b. 7 bn.	7·2 b. double.	......	7·5	......	6·8 *	7·4 n.	7·4 c.	7·2 many.	7·2 *
12.	ε	8 n.	......	9 b.	......	8·8 f.	......	......	......	8·7	8·7 vf.
13.	ε′	10 nf.	......	10·5 b.	......	......	......	......	......	many.	......
14.	ε″	11 c.	12·2	12 b.	......	11·8 nb.	......	......	12·4	......	......
15.		13·5 b.	......	13 b.	13 b.	13·8 n.	......	......	......	13·7	......
16.	ζ	15·5 b.	......	......	16·5 vf.	16·8 *	15·3 w^7.	......	......	17·7	15·7 fw.
17.		20 c.	......	19 bw^4.	21 vf.	19·8 fn.	19·8 n.	20·9 vn.	......	20·2 c.	20·7 f.
18.		23 *	......	23 f.	......	21·8 n.	22·3 *	......	......	22·7 n.	22·7 *
19.	η	24·5 bn.	24·7 bw.	25 *	24 b.	24·8 *	......	24·5	......	23·7 nb.	24·2 * many b. and vn.
20.		26 n.	......	......	25·5 many n.	......	......	......	25·4	25·7 *w^2.	......
21.		30·5 w. some here.	30·2 vf.	......	30 cw^3.	29·8 w^2.	......	......	......	30·7 nb.	28·7 c.
22.	θ	32·5 *	......	32 vbn.	......	......	......	33 nb.	......	32·7 *	...
23.		36 f.	......	34·5 *	......	35·8 b.	......	......	......	35·7 nb. double.	34·2 c.
24.		39	......	37·5 f.	......	39·8 b.	38·8 vb.	......	......	......	......
25.		41 c.	41·2 bn.	41 cw^3.	42 nb.	......	41·8 b.	40·4 *w.	......	41·2 bw^2.	40·7 nb.
26.		44·5 b. one.	43·7 c. double.	45 b.	44·5 bw^2.	......	44·3 f.	......	43·4 bn.	....	43·7 *
27.		46 b.	......	46 b.	......	45·8 f.	......	......	......	45·2 bw^2. multiple.	......
28.		48·5 b. double.	......	......	......	......	48·3 c.	......	.....	49·2 vnc.	48·7 n.
29.		......	50·2 f.	51 nf.	50·5 vf.	49·8 *	50·3 n.	......	......	50·7 *	49·7 n.
30.	ι	......	...	52 nf.	52 vf.	51·8 bn.	52·3 n.	......	......	52·7 n.	51·2 n.
31.		54 * double.	...	53·5 nf.	54 vf.	53·8 *	......	53·6 vn.	53·4 fw.	54·7 *	53·2 n.
32.	κ′	....	56·2 f.	56 *	......	56·8 n.	......	56·4 vf.	......	56·2 vnf.	......
33.	33κ″	57·5 bw.	......	57 *	......	57·8 *	......	...	......	58·2 *	......
34.	34	....	1·2 c.	59 vb.	1 b.	......	......	1·4 vf.	......	0·7 n. three.	......
35.	λ	3·5 fw. others.	......	2·5 n. 3·5 n.	......	......	......	......	......	......	......
36.		....	......	5 n.	......	......	48 b.	......	......	5·7 n.	......
37.		6·5 vb.	7·2 bw.	......	7 cw.	6·8 bn.	......	......	7·4 bw.	8·7 n.	6·2 b.
38.		11 b.	......	11 *	......	8·8 n. others.	......	......	......	10·7 nc.	......
39.		13 b.	......	13·5 *	......	12·8 *	......	......	......	13·7 *	......
40.		16 b.	......	16 vn.	16 nc.	16·8 nc.	14·8 c.	......	. ...	16·2 nb.	15·2 nf.
41.		19 c.	......	18 f.	19 c.	17·8 c.	......	......	18·4 f.	19·7 nb.	17·7 nb.
42.		....	21·7 vf.	20·5 f.	......	19·8 f.	20·8 bn.	21·4 *w.	......	21·2 f.	......
43.		23 n.	......	24 vf.	23 f.	......	......		......	......	22·7 c.
44.		26·5 *	28·2 f.	27 f.	27 f.	25·8 *	......	......	25·4 f.	26·7 *	25·7 vf.
45.	ξ	32 fw.	......	30 *	......	......	......	......	......	30·7 vf.	31·7 f.
46.			......	33·5 n.	34 vf.	34·3 vn.	......	......	......	35·2 vf.	......
47.		....	36·7 cw.	36·5 c.		37·8 vf.	......	......	37·4 cw.	38·7 vf.	36·7 vf.
48.		....		42 vf.	......	41·8 vf.	40·8	......		43·7 vf.	41·7 vf.
50.		....	......	45 nc.	......	46·3 f.		......	......	47·7 vf.	
51.		....	......		......		......	......	......		......

The deposit on the tubes very troublesome: this is especially the case with metals of this group. In air, exhausting to 0·86 inch, No. 8 is replaced by a narrow *black* band, but reappears on further exhaustion.

Table XV.—Magnesium.

		Air.		Nitrogen.		Oxygen.		Hydrogen.		Carb. oxide.	
		C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.	C.P.	R.
1.	32 α	39 *	39 vfo.	38·5 * 39·5 nf.	......	38 f.	37·5 vfo.	39·5 *vb.	38·5 vfo.	38·2 *	......
2.	α'	41 n.	....	42 vf.	......	41 f.	......	......	......	....	......
3.		....	....	......	......	43 f.	......	......	......	....	......
4.	β	....	....	45 vf.	......	46 f.	45·3 vfo.	......	......	....	......
5.	β'	49 b.	....	47·5 f.	......	49 vb.	......	......	......	48·2 f.	......
6.		....	51 fo.	......	......	50·5 vf.	50 nf.	......	......	49·7 f.	51·2 nf.
7.		....	54 fo.	......	......	53·5 vf.	......	......	52 f.	53·2 vn.	52·7 vfn.
8.	32 γ	55	56 fo.	54 *	......	55·5 *	......	......	......	55·7 f.	57·2 vnf.
9.	33	....	0	1 nc.	1 f.	......	......	......	1 vf.	59·2 f.	......
10.		2 ne.	3	2·5 * double.	3 vf.	2·5 vnc.	2 f.	......	......	3·2 c.	2·7 vnf.
11.	δ	4 * one follows.	....	......	6 n.	3·5 *	......	4·5 f.	5 vf.	....	......
12.		....	7 vb. one n.	7·5 b.	7 f.	.....	6·5	......	......	6·2	6·2 c.
13.	ε	9 b.	......	9 c. one.	......	9 b.	9·5 vfn.	......	......	many here.	7·2 n.
14.	ε'	10 nb. 11 nf.	....	10·5 c.	10	10 b. one.	......	......	10 vf.	......	11·2 vn.
15.	ε''	13 nb.	12 bn.	......	12 vf.	12 b.	12 vfn.	......	13 c.	......	......
16.	ζ	17 w^2b.	15 f.	16 w^4. quadr.	16 vf.	16·7 w^4. triple.	15 vf.	......	17 vf.	18·2	15·7 f.
17.		....	20 f.	......	19 vf.	......	18·5 c.	......	20 vf.	19·2	20·2 f.
18.		....	....	......	......	23 vn.	......	......	......	22·2 f.	23·2 *
19.	η	24 *	24 * one in cont.	23	24	24·5 *	24·3 bw^3.	24 c.	25 vb.	24·2 vn. 25·2 *	23·7 f.
20.		27 vf.	....	26 f.	......	28 f.	......	27 vf.	26·5 f.	27·2	28·2 f.
21.		....	29·5 vf.	29 n. one.	30 f.	......	29·5 f.	......	......	29·2	...
22.		31 nvb.	....	31 * doublevb.	31 f.	31 nc.	31·5 f.	31 vf.	31 f.	....	...
23.	θ	33 *	....	......	34 f.	33 *	33·5 c.	34 nb.	......	32·7 b.	32·2 f.
24.		36 nb.	34·5 vf.	36·5	......	37 c.	35·5 c.	......	......	35·2 vn.	35·2 f.
25.		39 f.	38 f.	39·5 cw^5. triple.	38 b.	39 f.	39 c.	38 vf.	39 f.	36·2 bn.	......
26.		41 w.	....	......	......	41 c. double.	41 nc.	41·7 *w^3.	......	40·2 bw^4.	......
27.		44 n.	43 * one close.	43	44 b.	......	......	......	43 vn.	....	42·2 vb.
28.		45 n.	....	44·5	......	44·5 c.	45 bw. one.	......	......	....	......
29.		46 n.	....	......	46 vn.	46 c.	......	46 vf.	45·5 c.	....	......
30.		....	48·5 f.	......	......	48 nc.	47 fvn.	47·5 f.	48 f.	48·2 fvn.	......
31.	ι	51 c.	50 nb.	50 vf.	49 b.	51 *	......	......	......	50·2 vb.	50·7 vf.
32.		....	....	......	......	......	......	52 nc.	52	52·2 nc.	52·2 f.
33.		54 vnc.	....	54·5 v. double.	54 c.	53·5 nb.	54·5 cw. double.	.....	......	53·7 *	......
34.	κ	55·5 *	55 c. double.	......	......	55 *	......	......	......	... ...	......
35.	κ'	56·5 *	....	......	......	56 * 57 vn.	......	56 b. double.	56 double.	56·2 nf.	......
36.	33 κ''	59 b. compound.	....	57 bw^2. quadr.	......	58 *	......	59 f.	......	58·2 b.	59·2 f.
37.	34	1 n.	....	0 1	0·5 c.	1 f.	1 c.	......	......	many n. ones here.	......
38.	λ	3 n.	....	3	......	2·5 vn. 3·3 vn.	......	4 vf.	2 c.	....	......
39.		5·5 n.	....	6 c.	......	5 vn.	......	......	......	....	4·2
40.		8 nc.	7·5 double.	......	7 c.	......	7 c.	8 vb.	......	8·2 c. many.	......
41.		....	....	9 *	......	9 f.	......	......	9	....	......
42.	μ	10 b.	....	...	......	10·5 c.	......	......	......	....	......
43.	μ'	11·5 b.	....	11 b.	......	11·5 c.	......	......	......	....	......
44.	μ''	13 b.	14 n.	......	13 c.	13 c.	13·5 c.	14 vn.	......	13·2	13·7
45.		....	....	16 vn.	......	15·5 f.	......	......	15 nb.	15·2 vfn.	......
46.		16·5	....	17 fn.	......	17 nb.	......	......	......	16·2 nb.	17·2 f.
47.	ν	19 nb.	19·5 c. double.	......	19 c.	19 b.	20 c. mult.	20 *w^5.	......	18·7 nb.	20·2 f.
48.		22 vf.	....	22 vf.	......	22 nc.	......	......	21 double.	21·2 nb.	......
49.		26·5 c.	27 f.	25 nf.	27 c.	26 nc.	......	27 nc.	......	25·7 *	24·7 f.
50.		29·5 vb.	....	28 b.	......	29 cw^2.	27·5 c. mult.		28 b.	....	......
51.	ξ	....	....	31 f.	......	......	......	......	......	....	32·2 f.
52.		33 c.	....	34 f.	34 n.	......	34 c.	......	34·5	33·2 cvn.	
53.	ο	36 f.	35 f.	......	......	......	......	......	......	....	......
54.		....		......	39 f.	38·5 nf.	40 vf.	......	......	37·7 f.	......
55.		....	....	43 f.		42 f.	......	......	42 vf.	41·7 vf.	......
56.		44·5 f.	....	......	......	45 f.	45 vf.	......		....	......
		47 vf.	....	49·5 f.	......	47 c.	......	......	......	48·2 vf.	......

The discharges with this metal were extremely brilliant, so much so in O, that at first I thought the wires were in combustion. The light at the negative electrode was a dazzling green. The deposit was almost as small as that of aluminium, and intercepted very little light. In air, C.P., the comparative dulness of the κ group is peculiar; ν and ξ are scarcely perceptible.

TABLE XVI.—Tellurium.

			Nitrogen.		Hydrogen.
			C.P.	R.	C.P.
1.	32		37·4 f.	....	....
2.		α	....	38·1 b.	39 *
3.		α′	39·7 f.	41 f.	....
4.			42 b.	43·3 f.	....
5.			44·2 f.	45·5 f.	.. ..
6.		β	47·5 c.	....	....
7.		β′	49·4 f.	48·8 f.	....
8.			....	50·7	....
9.			52·1 f.	52·7	. ...
10.			54 *	54·4	....
11.		γ	....	56·3	....
12.			57 f.	58	....
13.	32		59·3 f.	59·6	....
14.	33		1·3 } b.	1·3 b.	.. ..
15.			2·6 } *	....	....
16.		δ	....	4 c.	3·6 f.
17.			....	6·7 b.	....
18.		ε	8 *	....	....
19.		ε′	9·7 fine set.	....	10·4 f.
20.		ε″	11 b.	11·7 b.	....
21.			....	....	....
22.			14 f.	....	....
23.		ζ	16·5 b.	17·7 b.	....
24.			19 f.	....	....
25.			21·4 f.	....	....
26.			....	....	....
27.		η	24·1 *	24·1 *	....
28.			....	25·4 b.	....
29.			27·4 f.	....	....
30.			30·8 b.	29·8 *	29·4 f.
31.			32·8 *	32·1 *	....
32.		θ	....	.. .. . .	33·4 b.
33.			....	34·8 b.	....
34.			36·8	....	....

			Nitrogen.		Hydrogen.
			C.P.	R.	C.P.
35.		°	....	38·4 *	38·1 f.
36.			41·4 *w6.	41·1 b.	41·4 *
37.			44·7 } n.	44·7 *	....
38.			46·4 } b.	....	....
39.			48·7 f.	....	....
40.			....	50·7 *	51 f.
41.			51·6 b.	....	....
42.		ι	53·3 b.	53·6 b.	....
43.			55·3 *	55·3 *	55·9 f.
44.		κ	56·6 *	....	....
45.		κ′	....	....	....
46.	33	κ″	58·5 *	....	....
47.	34		1 } c.	0·7 *	....
48.			......	....	....
49.		λ	3·6 } b.	....	....
50.			4·9 } c.	....	....
51.			7·5 f.	7·2 *	....
52.		μ	10·8 } *	....	....
53.		μ′	11·4 } b.	....	....
54.		μ″	13·4 f.	13·4 *	....
55.			16·7 c.	18 } b.	....
56.		ν	19·3 b.	19·9 } *	20 *w.
57.			21·3 f.	....	....
58.			23·9 f.	....	....
59.			26·5 f.	27·2 *	27·2 f.
60.		ξ	29·1 *	30·4 f.	....
61.			33·1 } b.	33·7 *	....
62.		ο	36·3 } b.	....	....
63.			39 f.	39·6 b.	....
64.			41·6 f.	....	....
65.			46·2 f.	44·6	46·9 f.
			....	49·2 b.	
			52·5 f.	....	....
				56·8 f.	....

Table XVII.—Arsenic.

		Nitrogen.		Hydrogen.
		C.P.	R.	C.P.
1.	32°	37·7 f.	....	....
2.	α	....	....	38·7 *
3.	α'	40·7 f.	....	....
4.		43·6 f.	42	....
5.		45·5 f.	....	....
6.	β	47·5 b.	....	46·2 f.
7.	β'	....	....	48·2 b.
8.		....	51·1 f.	50·1 } b.
9.		....	five follow.	50·8 } b.
10.		54·7 *	....	54·7 b.
11.	γ	....	....	....
12.		....	....	57·3 b.
13.	32	....	....	58 c.
14.	33	....	1·3 f.	....
15.		....	....	....
16.	δ	4 *	....	4·6 *
17.		....	....	....
18.	ε	9 b.	8	8·7 *
19.	ε'	10·7 } b.	....	10·7 *
		one.		
20.	ε''	11·7 } b.	11·4 } f.	....
21.		....	......	13·7 } f.
22.		13·7 f.	14 } f.	14·7 } f.
23.	ζ	17·4 b.	16·7 } f.	17 *
24.		....	18·7 c.	....
25.		20·4 f.	....	19·4 f.
26.		23 f.	22·7 f.	22·1 b.
27.	η	24·7 *	24·1 b.	....
28.		26·7 b.	25·7 f.	25·4 c.
29.		28·1 b.	29·4 c.	28·1 *
30.		30·1	30·8 f.	30·1 f.
31.		31·4 } b.	32·8 c.	32·1 f.
32.	θ	33·2 } *	....	....
33.		34·8 } *	....	34·8 vb.
34.		37·1 f.	....	....
35.		39·8 f.	39·1 b.	....
36.		42·7 f.	....	41·7 *w^{7}.
37.		44·7	44·7 b.	....
38.		46 b.	....	45·4 f.
39.		....	47·7	....
40.		50·3 } f.	....	50 } b.
41.		52·3 } f.	51·3 b.	52 } b.
42.	ι	54 } f.	54·7 } b.	54 c.
43.		....	55·9 } b.	....
44.	κ	56·6 *	....	56·9 } *
45.	κ'	....	....	......
46.	33 κ''	59 *	....	58·5 } *
47.	34	....	1·7 b.	....
48.		2·3 } c.	....	....
49.	λ	3·9 } c.	....	2·9 b.
				double.
50.		5·2 } c.	....	....
51.		....	8·2 b.	6·9 b.
52.	μ	....	....	9·5 b.
				double.
53.	μ'	11·1 } *	....	12·4 }
54.	μ''	13·1 } *	14 b.	14 }
55.		....	....	....
56.	ν	20·6 c.	20·3	....
57.		....	....	22·2 *
58.		....	....	....
59.		27·2 f.	27·8 b.	....
60.	ξ	29·8 *	....	29·1 } f.
				31·1 } f.
61.		33·7 f.	34·4 b.	32·7 } f.
				33·7 } f.
62.	ο	37 f.	....	35 f.
63.			....	....
64.		....	42·3 f.	....
65.		....	....	45·9 f.
		....	....	49·5 f.
		....	51·5 f.	52·1 f.
		....		55·4 f.

Table XVIII.—Potassium.

		Air.	Hydrogen.	
		C.P.	C.P.	R.
1.	32°	....	....	....
2.	α	38·4 *	38·7 *	....
3.	α'	39·7 b.	41·6 f.	....
4.		42·6	....	....
5.		45·5	....	....
6.	β	....	46·2 f.	46·2 f.
7.	β'	48·5 vb.	....	....
8.		50·1 vb.	49·8 f.	....
9.		52·7 f.	....	51·4 f.
10.		55 } *	53·4 f.	54·7
11.	γ	56·7 } *	56·3 *	56·7 *
12.		57·3 w^{2}b.	....	....
13.	32	....	....	59 f.
14.	33	....	....	....
15.		2 c.	2·6 c.	2
16.	δ	4·3 *	....	4
17.		....	....	6·7 b.
18.	ε	8·7 *	....	7·7 f.
19.	ε'	10·7 } b.	....	....
20.	ε''	11·4 } f.	....	....
21.		12 } b.	....	12·7 f.
22.		14 f.	....	....
23.	ζ	15·4 } n.	16	16 f.
24.		18 } n.	....	....
25.		19·4 f.	....	20·7 f.
26.		22·1 f.	....	....
27.	η	24·4 *	....	24·1 } b.
28.		....	26·1 c.	26·1 } b.
29.		28·1 b.	....	....
30.		31·4 } bn.	....	30·1 f.
31.		32·8 } n.	32·8 nb.	32·8 f.
32.	θ	33·4 } *	....	....
33.		....	....	34·8 f.
34.		36·4 c.	37·5 f.	37·8 f.
35.		39·1 c.	....	....
36.		41·4 w^{2}b.	42·1 *	40·8 f.
37.		44·1 c.	....	44·4 b.
38.		44·7 } b. 46·7 } b.	....	....
39.		....	....	....
40.		....	....	50·7 } n.
41.		52·3	....	five or six.
42.	ι	54 f.	....	.... }
43.		....	....	.... }
44.	κ	56·3 *	55·9 n.	.... }
45.	κ'	57·2 *	....	57·2 } n.
46.	33 κ''	58·8 *	59·1 c.	....
47.	34	1·7 } f.	....	1·7 f.
48.		......	....	....
49.	λ	4·2 } f.	....	....
50.		6·2 } f.	6·5	....
51.		9·1 f.	....	7·8 c.
52.	μ	10·8 }	....	....
53.	μ'	12·1 } n.	....	....
54.	μ''	13·5 b.	....	....
55.		16·3 f.	16·7 f.	....
56.	ν	17·3 f.	20 6 *	20·3 f.
		19·6 b.		
57.		....	....	....
58.		23·2 b.	24·2 f.	23·9 f.
59.		25·2 f.	....	....
		27·2 c.		
60.	ξ	29·8 b.	28·5 n.	28·2 f.
61.		....	31·8 c.	....
62.	ο	35 b.	36·7 n.	....
63.		40 f.	....	....
64.		....	42·3 f.	....
65.		44·9 f.	....	
		48·9 f.	....	....
		51·5 c.	....	....
		55·1 f.	54·8	....

These spectra are in some respects peculiar, those of tellurium most so. In N, C.P., No. 36 is so like the distinctive band of hydrogen that at first I suspected the presence of that gas. The only difference I can find between them is that at the negative electrode it separates into two sharp and bright lines. Still one can scarcely see it, without thinking that there is some close connexion between Te and H.

I have already noticed the extreme brilliancy of its R. spectrum. It is very difficult to work with from its rapid dissipation at the negative electrode, which is so great that a wire 0·01 in diameter disappears in a few minutes; and the deposit is of extreme opacity. In H few lines were seen; for I was obliged to take them at a distance from the electrode, as it became invisible in a few seconds. With arsenic, H, C.P., I was more successful; the centre of the spectrum was a bright cloud, in which only the three bright bands and two or three others were visible. But at the negative boundary I got the largest number of lines, and the brightest that I ever met in H. The arsenic of course soon lined the tube (0·5 inch diameter) with a dark mirror, through which the flash seemed crimson; but an ellipse opposite the plates which form the slit remained quite clear. I suppose the induction of the brass on the tube caused this.

The potassium (air) is of great beauty; the double orange * is of as great intensity as with sodium. No. 7 is an odd-looking olive band. In H, C.P., only the second orange * remains, but it is very bright; the first one is entirely wanting, but reappears in H, R. On exhausting, the second retains its brightness when the rest have vanished.

TABLE XIX.—Sodium.

		Air.		Nitrogen.	
		C.P.	R.	C.P.	R.
1.	32	......	......	36·5 vf.	......
2.		......	......	37·1 f.	......
3.		40·5 *	......	......	......
4.		41·5 n.	......	......	......
5.		49·5 *	......	48·1 c.	......
6.		......	......	......	53·4 *
7.		55·5 n.	56·3 *	55·3 vb.	......
8.	32	57·5 * vb.	......	......	......
9.	33	......	......	2·6 bn.	......
10.		3·5 n.	3 vf.	4 *	3·3 c.
11.		5·5 *	......	......	......
12.		8 n.	......	......	......
13.		10 n.	......	10·7 } c.	......
14.		11·5 n.	11·7 c.	12·4 } f.	12·4 nb.
15.		16·5 w.	......	17·7	15·7 f.
16.		......	21·7 } vf.	......	......
17.	η	24·5 *	25·4 } f.	25·4 *	26·1 c.
18.		......	......	28·1 vf.	......
19.		30·5 bn.	30·4 vf.	31·8 nb.	31·1 f.
20.	θ	33·5 *	33·4 f.	34·1 *	35·5 f.
21.		......	38·1 nf.	37·5 vf.	38·4 c.
22.		......	......	40·1 vf.	......
23.		41·5 w.	41·1 nf.	41·4 f.	41·7 c.
24.	33	45·5 } nc.	45·1	45·4 } c.	45·4 b.
25.		46·5 } nc.	......	47 } c.	......

		Air.		Nitrogen.	
		C.P.	R.	C.P.	R.
26.	33	51·5 nb.	51·3 c.	51·6 vf.	51·3 c. double.
27.		......	......	......	52 double.
28.		......	......	54 f.	......
29.		56·5 *	55·9 c.	56·6 f.	......
30.	33	57·5 *	......	57·2 b.	......
31.	34	0·5 *	1·1 f.	59·1 b.	......
32.		......	......	2·9 } nc.	......
33.		......	......	4·6 } nf.	......
34.		......	......	5·2 } nc.	......
35.		......	8·2 vf.	......	7·8 c.
36.	μ	10·5 c.	......	11·3 n. one here.	......
37.	μ′	12·5 nf.	......		
38.	μ″	13·5 c.	13·4 vf.	13·1 nb. many here.	14·4 c.
39.		17·5 f.	......	......	......
40.		19·5 c.	19·3 vf.	......	20·3 c.
41.		28·5 bw.	26·8 c.	......	27·8 c.
42.				30·1 bn.	31 nf.
43.		......	......	33·1 f.	34·7 nf.
44.		......	......	37·3 f.	
45.		......	......	39·6 nf.	......
46.		......	......	45·6 f.	......
47.		......	......	52·8	......

The flash of the discharge is white, with a dense yellow atmosphere, to which I think the sodium line No. 8 is due. The Air, R. spectrum is, with the exception of this line, very faint.

Table XX.—Graphite.

		Air.	Nitrogen.	Oxygen.	Carb. oxide.	
		C.P.	C.P.	C.P.	C.P.	R.
1.	32?	......	......	......	33·9 o.	......
2.		......	36·3 n.	......	......	35·8 fo.
3.		39 *	39·8 f.	37·7 o.	38·4 *	38·4 fo.
4.		......	41·8	......	40·3 f.	......
5.		44	......	43·3 f.	42·9 f.	44·2 fo.
6.		......	45·6 vf.	46·2 f.	45·5 f.	......
7.		......	48·1 *	48·5 *	48·1 f.	......
8.		50	......	......	50·1 c.	51·1 c.
9.		......	52·5 nf.	52·7 n.	......	......
10.		56 *	54·2 *	55·3 *	56 b.	56 c.
11.	32	......	......	......	58·3 nc.	59·3 c.
12.	33	......	1·8 vn.	2·3 bvn.	2·6 b.	2·6 c.
13.		5 *	4·9 *	4 *	3·6 b.	......
14.		......	6·1 f.	6·7 c.	6 nf.	6 n.
15.		......	......	......	......	7·3 *
16.		......	8·7 vb.	9 vb.	8·3 f.	8·7 } f.
17.		10 } b.	10·2 b.	10·4 c.	10·4 } f.	10·7 } f.
18.		11 } b.	......	......	11·4 } f.	11·7 nc.
19.		13 } b.	12 b.	12 c.	12·7 } f.	......
20.		......	14·6 vf.	15 }	15 n.	14·7 f.
21.		......	16·4 } vf.	... }	......	......
22.		18·5	18·1 } vf.	18·4 }	18 f.	......
23.		......	19·9 vf.	20·7	20 nc.	21 f.
24.		......	22·8 vf.	23·4 b.	22·7 nc.	......
25.		......	24·6 *	25·1 b.	......	24·1
26.		26 *	......	26·7 f.	26·1 } *	25·4 } *
27.		......	......	......	27·1 }	27·1 }
28.		......	30·2	29·4 vf.	30·8 c.	......
29.		31·5	......	32·1 nb.	......	......
30.		34 *	33·3 *	34·1 *	35·8 f.	......
31.		......	36·5 f.	37·5 nb.	36·3 } nb.	......
32.		......	39 vf.	38·1 f.	37·5 } n.	......
				38·8 f.		
33.	33	42	41·4 bw.	41·4 bw.	41·4 b.	......

		Air.	Nitrogen.	Oxygen.	Carb. oxide.	
		C.P.	C.P.	C.P.	C.P.	R.
34.	33?	45 *	45·2 } b.	45·4 } b.	......	......
35.		......	46·2 } b.	46·7 } b.	46·7 n.	46·7
36.		......	49·1 f.	49·3 f.	......	47·9 nc.
37.		......	51·6 } c.	52·6 n.	51·3) *	51·3 nc.
38.		53·5 f.	54·2 } c.	54·3 n.	53·6 nc.	52 b.
39.		56 *	55·9 *	56·3 * double.	55·3 *	......
40.		......	57·2 *	58·2 fn.	57 c.	57·9
41.	33	58·5 *	59·1 vb.	59·8 *	59·1) *	......
42.	34	...... }	2) nc.	......	......	......
43.		3·5 com-pound. }	3·8 nc.	......	......	......
44.		......	5·7) nc.	......	......	6·5
45.	μ	9 }	......	9·5 vn.	9·1 c.	......
46.	μ′	... } ...	10·7 nvb.	11·8 vn.	10·1 c.	......
47.	μ″	13 }	12·4 nb.	13·4 vb.	13·1 *	......
48.		......	seen.	......	14·7 f.	15 } f.
49.		......	16·7 f.	16 vf.	17 } b.	17·3 } f.
50.		......	19·6 f.	18 n.	19 } b.	18·6 } f.
51.		......	......	19·9 vb.	20·6 } n.	......
52.		......	22·3 f.	21·9 n.	21·9 } b.	23·2 f.
53.		......	24·3 f.	......	......	
54.		......	27 f.	25·8	27·2 *	......
55.		......	30·3 b.	29·8 c.	......	......
56.		......	33·5 nc.	......	35 c.	......
57.		36·5	36·7 nc.	35 b.	36·3 n.	......
58.			38·9 vf.	39 doubtful.	38·3 n.	......
59.		......	41·9 vf.	42·3 n.	42·3 nc.	......
60.		......	44·8 nc.	......	44·6 n.	......
61.		......	......	45·6 n.	45·6 fn.	......
62.		......	47·5 vf.	48·9 b.	47·5	......
63.		......	51·4 vf.			......

The electrodes were two of Mordan's "leads," and must have been nearly pure; for fragments of them burnt before a gas blowpipe without leaving any sensible residue. The discharge was white except in CO, R. Air, C.P., was taken with Merz, and at an early period.

The next spectra were taken in hopes of obtaining an easy mode of determining the lines due to gases. In the first, electrodes of mercury send the discharge through the vapour of that metal; its lines therefore belong to Hg *exclusively*. Then filling the same apparatus with any gas, the new lines which appear must belong to *it* exclusively; or if (as in the case of platinum, Table II.) the electrode be of a metal on which Hg does not act, *its* lines can be insulated; this, however, assumes that the lines of electrodes and media are independent.

TABLE XXI.—Mercury.

			Mercury vapour.	Nitrogen.		Hydrogen.	Carb. oxide.	
				C.P.	R.	C.P.	C.P.	R.
1.	32°		....	36·5 vfo.	....	....	....	....
2.		α	38·4 fo.	....	....	38·5 *o.	38·1 f.	....
3.		β′	48·1 f.	49·1 c.	....	48·8 b.	48·5 *	....
4.			50·9 f.	....	50·7 vf.	50·7 f.	49·8 cn.	....
5.			....	....	52·1 vf.	....	52·7 f.	52·7 vf.
6.			53·2 f.	....	....	....	53·7 f.	....
7.			54 f.	....	....	54 n.	....	....
8.		γ	....	55·7 } b.	55·3 vf.	....	56·3 b.	56 f.
9.			56 9 f.	57·3 } b.	....	....	....	....
10.			58·6 } *	....	58·6	58 c.	58·6 } *	58·6 f.
11.	32		59·7 } *	60 } *	....	59·6 *	59·3 } *	....
12.	33		2·6 nc.	0·6 } *	....	....	3 b.	2 c.
13.		δ	3·3 c.	....	....	3·3 *	....	....
14.			....	4·6 b.	....	....	4·6 nb.	....
15.			6 c.	....	6 vfn.	6 } n.	....	5·6
16.			....	7·3 f.	....	7·3 } n.	6·7 nb.	....
17.		ι	8·7 vn.	8·7 fw.	....	8·7 } f.	8·3 nb.	....
18.			....	....	10·7 *	....	10·7 vn.	10·7 f.
19.		ι′	11·5 *	....	....	11·4 *	11·4 *	....
20.		ι″	....	12·7 *	....	13 b.	12·7 nc.	13·4 b.
21.			14 vf.	14 nc.	14·7 vfw.	....	....	....
22.		ζ	....	17·4 nf.	....	16 f.	15·4 f.	17 n.
23.			19·2 vf.	18·7 nf.	19·4 f.	19·4 b.	18·7 f.	....
24.			....	20·7 nf.	....	20·7 f.	....	21·4 nf.
25.			23·3 vf.	23 4 f.	22·7 } f. several. }	22·4 b.	22·1 f.	....
26.		η	24·7 b.	....	...... }	24·7 f.	24·7 b.	....
27.			26·4 c.	26·1 b.	25·4 } f.	26·4 b.	26·1 b.	26·1 vb.
28.			27 9 c.	28·4 f.	28·1 } group. }	28·1 n.	....	....
29.			29·9 f.	....	.... }	29·8 } b.	30·1 b.	....
30.			31·1 f.	32·1 f.	32·1 }	30·8 } b.	....	31·4 vf.
31.		θ	33·2 f.	33·4 *	....	32·8 n.	....	....
32.			35 6 f.	....	34·8 vf.	35·5 b.	35·1 nb.	....
33.			....	36·8 vf.	....	....	37·1 nb.	....
34.			38·3 c.	....	38·4 c.	38·5 f.	37·8 nb.	....
35.			41·1 vb.	40·4 vf.	....	41·2 $*w^5$.	40·8 vf.	41·4 vf.
36.			43·5 b.	....	43·4 c.	....	43·1 f.	....
37.			45·4 fn.	45·4 c. double.	....	....	45 c.	....
38.			47 vf.	....	....	....	47·4 nc.	46·4 *
39.			49·3 fn.	....	....	....	49·3 } *	....
40.			50·3 vn.	50 vf.	50 fn.	....	50·7 } *	....
41.		ι	52 1 f.	....	....	....	52·6 } *	....
42.			53·1 f.	53 f.	....	....	...... }	....
43.		κ	55·3 vf.	....	54·7 w.	55·1 vf.	55·6 } *	54·1 f. many.
44.		κ′	56·9 f.	56·6 vb. double.	....	....	....	....
45.	33		....	58·5 b.	....	....	57·9 n.	....
46.	34	κ″	0·5 f.	1·7 } bn.	1 c.	....	59·1 c.	....
47.		λ	5 } c.	4·2 } bn.	....	....	4·2 vf.	....
48.			6·2 } c.	5·9 } bn.	6·9 c.	7·5 vf.	....	....
49.		μ	9·5 vf.	10·1 } b.	....	....	9·5 vf.	8·5 w.
50.		μ′	12·2 vf.	12·1 } b.	....	....	10·8 vf.	....
51.		μ″	....	....	13·1 vfn.	14 vfn.	13·4 nc.	13·7 vf.
52.			15·7 nf.	....	....	....	....	....
53.			18·2 *	18 *	17·7 *	18·3 *	18 *	18 nc.
54.		ν	22·9 nc.	....	20 f.	20·7 $*w^7$.	21·6 b.	21·3 f.
55.			25·9 f.	....	26·8 f.	....	25·8 b.	
56.			28·5 f.	29·1 vb.	....	....		....
57.		ξ	33·5 f.	33·1 nc.	34·1 f.	31·8 vf.	....	....
58.			....	36 c.		....	....	....
59.			42·9 vf.	....	....	43·6 vf.	....	....
60.			47·4 nf.	....	....	....	....	....
61.			50·9 c.	51·5 f.	....	51·5 vf.	....	....

I failed totally in getting the O spectra; for at the very first discharge the mercury was so rapidly oxidated that not a glimmer could pass the thick deposit on the tube. This singular energy is probably due to the presence of ozone in the electrolytic oxygen (though it was passed through a tube filled with silver-leaf). I shall repeat the experiment, passing it through a red-hot tube. This fact may explain the absorption of oxygen in Geissler tubes, which was observed by PLÜCKER; for Hg vapour must be present in them, as they are exhausted by a mercurial air pump. The deposit in CO was black below, grey above: the first was dissolved by nitric acid, the other not; it was probably some form of carbon from the decomposition of the gas; of that, however, there was a large residue present. The spectrum of Hg vapour has a few very brilliant lines, but the rest are narrow and faint on a very dark ground. I had expected by comparing it with Pt, Hg vapour, Table II., to obtain at once the lines of platinum, which I supposed would be *superadded* to those of Hg; but, to my great surprise, that spectrum has *fewer lines* (28, while the other has 48), and only three occur in it which the other wants. A similar deficiency occurs in the gas spectra of Table XXI.: N, C.P., wants 22 mercurial lines; N, R., 25; H, C.P., 20; CO, C.P., 12; and CO, R., 30. These observations, it must be remarked, are strictly comparable; they were all made with the Duboscq prism, nearly at the same time, and with the apparatus in precisely the same condition.

From this may be inferred, either that mercury gives different lines from its vapour, or that the gaseous media have lines which interfere with those of the electrodes so as to destroy each other.

I must reserve for a second communication the complete discussion of these and the other spectra, as it would, I fear, swell this one beyond all reasonable limits.

[P.S. (See page 954.) A more probable explanation of this peculiar form of the spectrum was suggested to me by Mr. STOKES, which is as follows:—

The aperture of the collimator was 1 inch, and its focal length 9·5 inches. The distance of the spark from the slit was probably less than 1·5 inch, say 1·25. Hence if we assume for simplicity that all the light emerging from the collimator entered the object-glass of the observing telescope, which was nearly true, each element of the slit would be capable of being illuminated by rays from a length $l=\frac{1\cdot25}{9\cdot5}\times1=\frac{1}{7\cdot6}$ inch of the spark, provided the spark reached so far, the centre of that line being in a prolongation of the line joining the centre of the object-glass with the element in question. If the length of the spark were very great compared with l, the central parts and those near the electrodes would illuminate widely separated elements of the slit, and the variations of brightness in a vertical direction would nearly correspond with the real variations of brightness of the different parts of the spark—though of course there would be a "ragged edge" above and below, where the spectrum was formed by partial pencils. If, on the other hand, the length of the spark were infinitely small compared with l, each illuminated element of the slit would receive rays from the whole spark, and the spectrum would be a band of uniform brightness in a vertical direction, and terminating abruptly above and below. In the actual experiment the length of the spark was $\frac{1}{18}$ inch, a quantity comparable with l, being a little less. Hence the spectrum would consist of a central stripe of uniform brightness, corresponding to the small portion of the slit illuminated by the whole of the spark, accompanied above and below by a ragged edge in which the illumination fades away, as more and more of the spark is cut off, till nothing is left but the rays coming from its very extremity. The distance from the central stripe to which the illumination would be perceived to extend would depend on the intrinsic brightness of the part of the spectrum regarded; and, bearing in mind the relative brightness of different parts as laid down by FRAUNHOFER for the solar spectrum, we should get for the visible boundary a curve similar to that represented in the figure. —March 1863.]

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On comparing these Tables, it is seen that the *places* of a very large proportion of these lines are identical, or at least differ only within the limits of observation, although they may be very unlike in brilliancy and magnitude. Such lines must be considered the same; for the essence of a ray is its wave's length, and the difference of intensity is but an accident depending on extrinsic causes. That this comparison may be more easily made, I have given in the following Table the mean places of those whose identity is probable: except in the spectra of iron, copper, and many of carbonic oxide, such are the great majority of what I have observed. For the conspicuous ones, which I have denoted by Greek letters, and which are recognized by peculiar characters, as also for some others which stand in definite relation to them, the means were taken without strict reference to probable errors; for the rest those limits were taken which I have already mentioned. It is possible that in several of the latter class one or two of the outliers may not belong to them; but the general agreement of each is so close as to preclude doubt. I include in the Table none beyond FRAUNHOFER'S H, though several were observed, and also omit the very faint red lines which were visible in some spectra outside the bright part*. It gives for each the mean place, the extreme deviations from that mean, the number of C.P. spectra in which it is found, in how many of these it is a *, and the same data for the R. spectra.

* Especially in manganese, A, C.P. This splendid spectrum, which had 74 lines, is not given, as it was not taken in R. or other gases.

Table XXII.

No.	Place.	Diff.	C.P.	*	R.	*	
	° ′						
1.	32 33·80	−1·10 +0·70	3	0	4	0	All the R.s in N.
2.	32 37·43	−0·92 +0·27	17	5	5	0	
3 α.	32 38·68	−1·18 +1·02	76	55	19	2	6 metals show it in all the gases, A has all in N and O.
4 α′.	32 40·62	−0·90 +0·98	55	10	7	0	Often seen, but overpowered by α.
5.	32 42·36	−0·86 +0·54	16	1	7	1	
6.	32 43·34	−0·34 +0·56	14	1	2	0	
7.	32 44·96	−0·96 +0·46	24	0	4	0	O predominates, 5 of N and 9 O wanting in A.
8	32 45·72	−0·72 +0·78	31	0	5	0	
9 β.	32 47·63	−0·55 +0·57	31	11	5	0	In this and the three preceding A has no R.
10 β′.	32 48·78	−0·78 +1·52	60	19	10	0	Rose-red. In C.P., A has all the metals' except β.
11.	32 50·35	−0·33 +0·57	24	2	21	1	10 of N and O wanting in A.
12.	32 51·43	−0·63 +0·57	13	2	43	1	Notable predominance of R.; has few common to C.P.
13.	32 52·56	−0·57 +0·43	22	2	20	4	
14.	32 53·48	−1·48 +0·52	6	0	5	0	Very faint; the companion of γ, always seen with S^2C prisms.
15 γ.	32 55·08	−2·08 +1·62	69	46	38	3	Orange; in 16 cases it was resolved in the glass prism.
16.	32 56·21	−0·51 +0·79	20	7	20	1	
17.	32 57·60	−0·70 +0·40	28	1	11	1	No A in N, or N in A.
18.	32 59·02	−0·72 +0·68	24	4	39	2	6 of A not in N or O; 7 of them not in it.
19.	33 0·36	−0·76 +0·64	22	2	26	2	Similar discordance.
20.	33 1·66	−0·66 +0·34	20	0	13	0	Mostly N; 8 of it wanting in A.
21.	33 2·85	−0·60 +0·20	37	5	20	0	Companion of δ, generally vn but vb; always I believe present.
22 δ.	33 3·55	−0·95 +1·45	77	44	35	0	Pure yellow, with 5 metals in all the gases; in A and N with all the metals.
23.	33 6·16	−0·66 +0·54	30	2	34	6	Most frequent in CO.
24.	33 7·09	−0·29 +0·81	22	2	41	9	Many fine lines shown in this region by the S^2C prisms.
25.	33 8·37	−0·37 +0·33	25	1	15	3	
26 ε.	33 9·21	−0·91 +0·79	51	5	16	2	Bright green, well characterized.
27 ε′.	33 10·29	−0·49 +0·41	50	2	12	1	Separated from ε by a dark interval.
28.	33 11·23	−0·53 +0·27	32	2	23	2	More frequent in A than N; often too faint to be surely bisected.
29 ε″.	33 12·12	−0·42 +0·68	40	0	30	2	In CO many very fine lines here.
30.	33 13·28	−0·28 +0·52	22	0	14	0	The green ground bright, especially in CO.
31.	33 14·55	−0·85 +0·45	30	1	31	0	
32.	33 15·61	−0·61 +1·09	33	3	29	0	These three are chief in a group which was at first observed as a bright band, and named ζ. The whole ground seems covered with lines, of which 33 is chief.
33.	33 17·17	−0·67 +0·85	52	6	25	0	
34.	33 18·65	−0·65 +0·35	21	1	11	0	
35.	33 19·81	−0·61 +0·39	25	1	23	0	
36.	33 20·93	−0·80 +0·60	31	0	23	0	Most common in CO.

TABLE XXII. (continued).

No.	Place.	Diff.	C.P.	*	R.	*	
37.	33° 22′·74	−0·74 +1·26	52	8	29	6	
38 η.	33 24·52	−1·02 +0·78	63	32	63	25	One of the brightest and most common. With five metals it occurs in every gas.
39.	33 25·82	−0·42 +0·88	45	13	30	3	
40.	33 27·84	−0·84 +0·96	30	1	24	0	
41.	33 29·71	−1·01 +0·79	31	2	43	1	
42.	33 30·93	−0·53 +1·37	65	0	31	1	vb, but n; all the metals in A, all but two in N.
43.	33 32·13	−1·03 +0·67	19	0	11	1	This and the preceding are evidently connected with θ.
44 θ.	33 33·48	−1·48 +1·52	72	50	42	1	Generally brightest of all, bluish green perfectly characterized.
45.	33 35·59	−0·79 +0·41	27	0	25	0	First of blue, overpowered by glare of θ.
46.	33 36·62	−0·52 +0·78	42	1	16	0	Sb is wanting in this and the next.
47.	33 38·00	−0·99 +1·00	37	0	29	1	
48.	33 39·49	−0·49 +1·01	37	3	33	1	
49.	33 41·25	−0·75 +0·85	69	18	48	3	C.P., Al, Pd, Fe have it in all the gases. It is the H blue band.
50.	33 42·86	−0·86 +0·64	11	3	19	10	Well marked; specially bright in R.
51 ς.	33 44·49	−0·99 +1·31	55	7	69	0	First of a well-marked pair, universal in A and N, C.P.; frequent in R. with all.
52 ς′.	33 46·30	−1·30 +2·30	63	2	20	3	The two extremes very faint; Cd absent in C.P.
53.	33 48·73	−0·73 +1·77	55	9	28	0	Seems connected with the next.
54 ι.	33 51·09	−1·09 +1·59	82	28	52	1	With S^2C prisms multiple; perhaps the components are sometimes unequally developed.
55.	33 53·38	−1·18 +1·32	66	8	42	0	Generally like a hair, vb, a faint one follows.
56 κ.	33 55·33	−1·33 +1·27	77	58	38	1	These are often very intense in C.P.; they are brilliant blue.
57 κ′.	33 56·58	−0·58 +1·12	46	42	7	0	
58.	33 57·32	−1·30 +1·48	30	1	2	0	vn; I think always present; the * is with Na.
59 κ″.	33 59·08	−2·08 +1·92	81	57	26	0	Beginning of violet.
60.	34 1·12	−1·12 +0·88	28	1	34	1	
61 λ.	34 2·72	−0·72 +0·88	38	1	8	0	These were at first seen as one band before the apparatus was complete. In the C.P. spectra of CO, the ground is covered with fine lines. (61–63)
62 λ′.	34 4·15	−0·65 +0·85	44	1	8	0	
63 λ″.	34 5·82	−0·92 +0·68	39	1	36	7	
64.	34 7·31	−0·51 +0·49	17	0	29	2	
65.	34 8·46	−0·46 +0·64	34	1	18	0	
66 μ.	34 10·31	−1·31 +0·99	67	13	8	0	A well-marked group in C.P., distinct in almost all the metals and the gases except H. Its parts are all found in R. but in different spectra. The * in μ″ R. belongs to Te. (66–68)
67 μ′.	34 11·93	−1·03 +0·97	57	14	10	0	
68 μ″.	34 13·71	−0·91 +1·19	64	10	43	1	
69.	34 15·75?	−1·75 +1·05	42	3	22	0	With C.P., Ni, Co; and R., Cu, in all the gases.
70.	34 17·40	−1·20 +0·80	58	9	33	3	
71.	34 19·03	−0·73 +0·87	50	5	35	2	Some of these, especially in H, probably belong to No. 72.
72 ν.	34 20·78	−1·28 +2·02	64	13	35	1	In H it is developed into the violet band, whose cloudiness and width explain the range of the measure.

TABLE XXII. (continued).

No.	Place.	Diff.	C.P.	*	R.	*	
73.	34 22·99	−1·19 +1·51	33	1	23	0	None in H, C.P.; four in H, R.
74.	34 25·56	−1·36 +0·94	55	9	22	0	Frequent in N and O.
75.	34 27·45	−0·95 +0·75	38	13	37	2	Only one metal wanting in N, R.
76 ζ.	34 30·09	−0·92 +2·01	63	20	18	0	N, C.P. with all.
77.	34 33·43	−1·63 +1·27	51	1	38	1	The two *s are Al, O; Te, N.
78 ε.	34 35·83	−0·83 +1·67	53	1	19	0	All with Pb, C.P.; the * is Ca, H.
79.	34 39·03	−1·53 +1·27	43	0	19	0	None in H, C.P. These far-violet are ill defined and hard to bisect.
80.	34 41·85	−1·85 +1·85	42	0	28	0	Few in H and A.
81.	34 44·80	−1·85 +1·85	51	0	12	0	With Co, C.P., in all the gases.
82.	34 47·73	−1·22 +1·77	42	0	13	0	Most frequent in CO.
83.	34 51·35	−2·15 +1·45	30	0	14	0	O has only Zn in C.P. and Mg in R.
84.	34 55·81	−2·31 +1·89	15	0	6	0	None in O, C.P.
85.	34 59·34	−2·24 +1·66	10	0	8	0	

Among the most notable of these are,—

No. 3 *α*. It is one of the three brilliant bands in H, C.P., and is more intense in that gas than in any other. Its mean from 22 of H, C.P., is 32° 38′·52, almost identical with the general mean. It is also extremely intense in other gases, *e. g.* A, with silver and iron; at other times dull or even faint. In H it is generally separated by a dark space from the rest of the spectrum, as if the other red rays were condensed into it. It seems to have a fine line preceding it, which in many cases is separated enough to be bisected. In R. it is of much less importance; its place is that of FRAUNHOFER'S C.

No. 12 is remarkable for its much greater display in R. than in C.P.; the same occurs, though in a less degree, in Nos. 18, 24, and 64.

No. 15 *γ*. This beautiful band is nearly, but not exactly, in the place of D, and like it is double, though my glass prisms often fail to divide it fairly. With the two prisms of S^2C the components are generally equal, but I think further apart than those of D with the same prisms. In K, A, they are seen separate as intense *s; in CO the second is often the brightest. In O it is often dull and sometimes wanting (and the same may be said of the orange and yellow lines generally).

No. 16 coincides with the yellow band of sodium. In the A spectrum of this metal, No. 15 *γ* is only nb; in N it is a *, but No. 16 is far more brilliant; it retains its brightness during exhaustion and with the spark discharge. It is exactly in the place of D.

Nos. 18 and 19 are intense orange *s in mercury vapour with electrodes of mercury.

No. 22 *δ* is another beautiful band of common occurrence. In C.P. five metals have it in all the gases; A and N have it with all the metals, though of very unequal brightness; O with 11, H with 12, and CO with 19. It seems to be connected with No. 21, the ordinary character of which is bn. When examined with the S^2C prisms it is triple, the first being very narrow but bright, with a slight tinge of orange, the second and third pure yellow, the second broader than the third, and both sharply defined.

Nos. 26 to 29 form a very conspicuous group. The first, ε (which begins the green), is marked by the very obscure interval which separates it from the others; it as well as ε'' are double, and the S^2C prisms show many fine lines in their neighbourhood. With iron, copper, and in many of the CO spectra, the number of these is very great; it may be that the bright green from this to No. 44 is composed of such lines too close to be resolved; but I think it is continuous.

No. 36 is nearly in the place of E.

No. 38 η is very conspicuous on this bright ground; it is often intensely brilliant, sometimes more so than θ, and it occurs more frequently. No. 37 seems to belong to it; with the S^2C prisms, and sometimes with the glass one, it is double of two equal. Its brightness is scarcely developed in O, C P., very much in N, but quite as much in O, R., as in N, R. CO, R. exceeds them both, though it has only 2 *s in C.P. With platinum, CO, C.P., it is replaced by a bright bluish-green band of singular appearance; this by careful focusing shows eleven or twelve fine lines, of which No. 37 is the first and No. 42 the last. In this spectrum the group ε, ε'' is also replaced by some twenty-five very fine lines from No. 23 to No. 36. The place of η is identical with the centre of the double line of *b*.

No. 44 θ, with its companions 42 and 43, is very common, though the latter, 43 especially, are hard to see in the blinding glare of the first, which is often most intense. They are always seen with S^2C prisms, which also show θ double (as Duboscq does occasionally). The components are generally, but not always, equal*. Though frequent in R. it is seldom bright. In C.P., N brings it out best; it is wanting in 8 of O, 8 of H, 5 of CO; and of its 50 * O has but 4, H 1, and CO 3; one belongs to mercury in mercury vapour. It ends the green.

No. 49 is remarkable, not only from its being in the place of F, but also from its being (or being in) the characteristic blue band of H, C.P. In the other gases and vapours it is scarcely ever a *, and often faint; here it is of great brightness and great breadth, often 6′ or 7′. It has a cloudy but irresolvable look, and its edges are not sharply defined. Even with 2 S^2C prisms I cannot resolve it; and it gives me the impression that it consists of light whose wave-length varies continuously. Its mean by 22 H, C.P., is 33° 41′·18, corresponding to a wave length 1798. It is singular that the N, C.P., spectrum of tellurium has a band possessing this peculiar type so decidedly that one who saw it without knowing its origin would undoubtingly assert the presence of H; and my first idea was that this gas must be a component of the metal. If, however,

* In this, as in similar instances, I suspect that an unequal development of the components may disturb the measures. I may at the same time notice another cause of disturbance, the flicker of the light. It might be supposed that the collimator must give an object absolutely fixed, but this is not the case. The discharge being narrower than the slit and at some distance from it, and the want of perfect achromatism in the object-glass, are capable of producing considerable unsteadiness.

the eye be directed to the negative boundary of the spectrum, two brilliant points are seen there not found in the ordinary H band. At the same time they may be due to the grosser metallic vapour which abounds there; for Te volatilizes under the discharge more than any other metal which I have tried. Iron in N, if the pressure be diminished a few inches, shows a similar band, though the other characters of the C.P. spectrum and discharge are unchanged. Zinc in CO, C.P., has one of the same sort 3′ broad.

Nos. 51 and 52 are a well-marked pair; but a much more remarkable group is that from No. 54 ι to No. 59 $\varkappa''$. Of these, 55 and 58 are very narrow, and, though bright, are often overpowered by the splendour of their neighbours; the first has a still fainter follower seen in the S^2C prisms, which also show each of the brilliant ones to be compound, ι and $\varkappa$ double, $\varkappa''$ (which begins the violet) to consist of six fine bright lines. In thirty-two cases $\varkappa$ and $\varkappa'$ were observed as one, or as close double. When taken separately, their distance was found 0′·96; and hence the place of $\varkappa$ should be increased, and that of $\varkappa'$ diminished by 0′·14. H is unfavourable to this group, which is most splendid in O and CO, except in Cu, CO, C.P., when it is but a shadow of itself.

The ground is covered with very fine lines from No. 61 to No. 68 in the CO, C.P., spectra of Cu, Pd, Fe, Bi, Sb, and Mg.

No. 72 ν, besides being important in the spectra of other gases, is apparently developed in H, C.P., to the broad violet band, the third of the three brilliant ones of that gas; 22 of H, C.P., give its place 34° 19′·82. It is not nearly so luminous as No. 49 is in H, but is quite as broad and even more undefined. Unlike other violet bands, this is the first to disappear on rarefying the gas. I have occasionally seen fine lines in it, which, however, do not seem to belong to it.

No. 73 is in the place of G, and No. 85 very nearly in that of H.

If the individual spectra be compared with this Table, the result will, I think, throw some light on the questions which I mentioned at the beginning of this paper.

1. As to the *essential* connexion of each line of a spectrum with the chemical nature of some one metal, or some one gaseous medium, I think these observations are against it. For instance, if any line, say No. 22 δ, be found with electrodes of aluminium in all the gases which we have examined, our first inference might be that it is an aluminium line. But when we find the same thing is true of nickel, palladium, antimony, and magnesium, and that all the other metals have it in some of their spectra, we must conclude that it does not belong *exclusively* to any metal; nor can it be considered a gas line. In N it is indeed found with all the twenty-two electrodes; but it also occurs nine times in O, fourteen in H, as well as in the vapours of mercury, phosphorus, and bisulphuret of carbon. There are many similar cases; and though in general the lines of this Table are not seen with all the metals, yet it will be found that, with respect to gases (omitting No. 1, which on account of its bright neighbours is seldom seen), out of the whole eighty-five only Nos. 5, 6, 9, 73, 79, and 85 are wanting* in H, C.P.; only No. 6 in CO, and that of these all but No. 5 are found in the R. spectra, and *none* are wanting in A, N,

* That is, are not visible with my means of observing.

and O. As to metals, the lowest number, 12, occurs in No. 85; but the average is far above this.

These facts may be explained in two ways. We may suppose that the action of the electric discharge on the molecules which transmit it has in itself intermittences which *tend* to produce maxima of light recurring at intervals, and which are effective in doing so when the forces inherent in these molecules are in accordance with them. If that accord be perfect, the development of light will be intense; but if imperfect or wanting, the corresponding light will be feeble or will vanish. Or it may be supposed that our metals and gases are compounds of unknown elements which are separated by the discharge, and exhibit their appropriate lines. This hypothesis is tempting, for if true it would at once lead to analysis of many of our supposed simple substances; and the facts which have been stated respecting the band No. 49 give some countenance to it. I however think the first view of the matter more probable, for reasons which I shall soon state. According to it, the existence of a luminous line merely indicates the presence of *matter* in the circuit; but its intensity depends on the nature of that matter, which may either make it extremely bright or obscure it, even to invisibility. It is generally supposed that the presence of a metal gives brilliant lines, and that those due to a gas or vapour are less bright; but as the two are always simultaneously present it is not easy to separate their influences. A promising plan of effecting this has been proposed by PLÜCKER. Two balls connected by a capillary tube were filled with any gas and then exhausted; the balls were coated with tinfoil, and when they were connected with the terminals of an induction machine *reciprocating* discharges took place, which, though very faint in the globes, were bright enough when condensed in the capillary part to give a spectrum. Here nothing but glass is in contact with the rarefied gas, and therefore he expected to obtain only gas lines. I repeated this experiment, adding a glass stopcock to one of the balls, that I might use the *same* glass with different gases. The spectra observed were the R. of N, O, and H; the first had 13, the second 23, and the last 13, besides several in each too faint to be taken. Of these were—

Common to three.	In N and O.	In O and H.	In O.	In N.	In H.
Nos. 18	Nos. 27	Nos. 49	Nos. 16	Nos. 14	Nos. 5
39	68	75	24	76	13
46			33	84	82
57			36		
61			41		
65			42		
78			44		
			48		
			52		
			54		
			72		

No. 5 is not in any other H spectrum; No. 82 only in Te, A and As, H; No. 16 only in PbO. None of these gas lines is peculiar to that one gas; thus No. 16 is found in N with nine metals and in H with five; this method therefore fails to give the true gas lines. Most of those that are common to two gases will be found in the spectra of sodium, potassium, calcium, and lead, all of which are ingredients of glass; it is hence evident that even reciprocating discharges disintegrate the surface of glass.

A plan of mine seemed more promising. With an apparatus which I have already described, I took the spectrum in mercury vapour with mercury electrodes; then filling the tube with a gas, I took the new spectrum. I tried N, O, H, and CO. O failed from its extraordinary action on the mercury, which in a few seconds blackened the tube, so that not a ray from the flash was visible. I expected that the gas lines would be thus easily obtained, but was disappointed.

Hg Hg	has 48 lines			
Hg, N, C.P.,	has 36 lines		N, C.P.,	has 10 not in Hg Hg
Hg, N, R.,	has 23 lines	Of these	N, R.,	has 4
Hg, H, C.P.,	has 33 lines		H, C.P.,	has 7
Hg, CO, C.P.,	has 41 lines		CO, C.P.,	has 10
Hg, CO, R.,	has 18 lines		CO, R.	has 8

Of these last Nos. 2, 46, 59, and 78 are found in N only (that is with Hg electrodes, for with other metals they occur in *all* the gases). Nos. 13 and 32 belong to CO, the rest are common to the three gases. It deserves notice that many of the lines of Hg Hg are wanting in the gases, the lowest number being 13 in CO, C.P., the highest (29) in CO, R. This may partly arise from mercury and its vapour having different sets of lines, the latter of which are displaced by those of the gas, fewer in number Were this the sole cause, the same lines would be missing in each gas spectrum; and the difference of the numbers shows that the presence of a gas actually *prevents* the development of some mercurial lines. Corresponding facts occur in other spectra, those of H especially.

The effect of a metal is most perceptible near the negative boundary of a spectrum, where some lines are brilliant for a short distance and then continue with much less brightness. The contrast is so great that, without measurement, one could scarcely believe the line to be the same. That its existence in the faint state is due to that metal which causes its partial splendour, is disproved by its occurring with other metals. In illustration I refer to the spectrum of bismuth (N) at the negative edge as compared to that of the centre (Table XII.), where it is seen that though lines (especially in the green) which are scarcely visible in the latter are intensely bright in the former, and faint single lines are seen as two, yet no decidedly new one appears. This is most remarkable with the volatile metals. Thus As, H, C.P., shows at the negative edge forty-four lines, of which ten are ∗s and eleven more are "bright." These retain their lustre for 10′ or 11′, and then (except the three hydrogen bands) can scarcely be traced across.

None of these, however, are peculiar, except that Nos. 11, 17, and 54* appear as double.

That the presence of a new body in the circuit developes oftener than originates lines, is further shown by moistening the electrodes with a solution of some salt; for instance, chloride of barium†. In trying it, platinum electrodes (carefully washed with nitric acid and distilled water) were lapped with *clean* cotton thread to retain the fluid, which, however, soon evaporates by the heat of the discharge. The change of the spectrum is almost startling from the instantaneous and intense development of many lines, and the increased lustre of some which before had been noted as ∗s. The most conspicuous are—

	°	′
α. Very bright	32	38·1
α′. ∗ very intense		40·0
β′. ∗ dazzling¹.		49·1
γ. Very bright		54·7
No. 16. Orange intense . . .		56·3
No. 17. Very bright		57·7
No. 34. ∗ very intense	33	18·4

Nothing striking till θ, then

No. 47. ∗ brighter than θ . . .		37·4
No. 48. Almost ∗		39·4
λ. ∗ of exceeding intensity . .	34	2·3
λ′. Very bright		4·6
No. 80. Extremely bright . . .		40·6

All these are found in the platinum air spectrum, though in most cases with a totally different aspect; so that the barium makes no change in the place of the lines, but only in their brightness.

2. That the peculiar character of a line is modified by other circumstances than the chemical nature of the bodies engaged in producing it, appears from the difference between the spectrum at common-pressure, the transition spectrum, and that in rarefied gases, to which I have already referred. The chemical conditions are unchanged; but at first sight nothing is more dissimilar than the three; the first with its numerous bright ∗s; the second a shadow rather than a spectrum, with a bare suspicion of a few lines; and the R. one much brighter, but with a seeming discrepancy of lines that marks it as peculiar. The R. has, on an average, 0·53 of the C.P. lines; it extends as far in the violet, but is cut off in the red part. There, especially in N, are often found a set of equal bands reaching from No. 11 to No. 19 inclusive; there is also another set, that

* This line is always double with S²C prisms.

† It is desirable to ascertain whether this process gives a spectrum always identical with that given by electrodes of the metal contained in the salt. The difference of aggregation and the presence of water may modify it considerably.

looks like a row of luminous pillars, from No. 53 to No. 77. In general the R. spectra are much fainter than the C.P.; the brilliant No. 3 α, 15 γ, 22 δ, 44 θ, and the κ group are inconspicuous, though often present. On the other hand, some are occasionally brilliant; thus 38 η occurs as * twenty-five times; 23 and 24 have more *s in R. than C.P.; and the R. spectrum, Te, N, is positively splendid with thirteen *s. But I think there is no instance of a line† occurring in any R. which is not found in some C.P., or *vice versâ*. These differences may be attributed to three causes. First, the discharge which at C.P. passes in a flash of, probably, evanescent section, must be very much brighter than when diffused in a tube of 0·2 inch diameter, which it fills completely. This would account for the lines being fainter, but not for the difference of character. In a compound tube of two pieces, 0·5 inch diameter connected by one 0·05 inch, with lead electrodes, A., R., the spectrum in the narrow part had more lines than in either of the wider, as being brighter; but it contained all theirs, and was quite distinct from the C.P. Secondly, it may be thought that the air is less heated, because of the less resistance and the greater heat-capacity of rarefied gas. Were this the principal cause, the R. ought *never* to contain more lines than the corresponding C.P., much less have any as a * which is faint in the other; the transition-spectrum, too, should be more luminous than the R. Thirdly, the mere increased distance of the gas molecules *may* modify the light-vibrations. This is mere guess; but so is much more of our speculation on this mysterious subject, and it may at least point out the road for future research. Experiments at pressures greater than the atmospheric, produced by mechanical force or heat, and with electrodes in various states of aggregation, would probably throw light on the subject. One thing must be remembered: in these R. discharges the light, when viewed in a revolving mirror, has a certain permanence; but in the C.P. it is, so to say, instantaneous.

3. It is generally believed that each of the bodies present in the track of a discharge has its own independent spectrum, coexisting with those of the others and in nowise interfering with them. This is not universally true, as will be seen by comparing the spectra of air, both C.P. and R., with those of its components N and O. Without going through the entire series, but taking the first twenty-five numbers of the Table, and the last, I find that out of

353	lines found in	A., C.P.,	there are neither in N nor O .	71
156	„	A., R.,	„ „	62
404	„	N, C.P.,	there are wanting in A . .	186
226	„	N, R.,	„ „	124
314	„	O, C.P.,	there are wanting in N and A.	133
133	„	O, R.,	„ „	52

Many of those wanting in A are found both in N and O. From these numbers it is

† Of course I mean the lines of this Table, without including in the statement those of special character and peculiar origin.

evident both that the spectrum of air is not formed by the mere superposition of those of N and O, and that certain electrodes can excite lines in a mixture of two gases which are not visible in either of those gases taken separately, or cannot excite them in the mixture, though they can in either or both of the components. This unexpected fact is illustrated by an attempt which I made to obtain the lines of platinum, by comparing its spectrum in mercury vapour with that of mercury. I expected, as I had done in the cases already mentioned of gases with Hg electrodes, that I should get the spectrum of Pt+that of Hg; but the result was otherwise. Hg Hg had forty-eight lines, Pt Hg only twenty-five, so that the presence of Pt instead of Hg as electrodes *put out* twenty-three. It however brought out four new ones, of which No. 15 γ is found with *every solid* metal, Nos. 33, 59 κ'', and 76 ξ with most of them. There is here no superposition, but, instead, either an antagonism of platinum and mercury, or the curious fact that the fluid electrodes act differently from the solid. It would be interesting to compare the spectra of tin in these two states.

The case is the same with a chemical compound as with a mixture. On the principle of superposition, the spectrum of CO should be merely the sum of those of C and O. To avoid all uncertainty about metallic lines, I used electrodes of graphite in the two gases. In each case nothing but C and O were present, and I expected to get the same spectrum in both. They were, however, very unlike. Of the lines which they had in common,

No. 3 α	is nc. in . .	O,	but is a * in	CO
No. 10 β'	is a * in .	O,	but only c. in	CO
No. 15 γ	is a * in .	O,	but only b. in	CO
No. 22 δ	is intense * in	O,	but only b. in	CO
No. 26 ε	is wvb. in .	O,	but one of many very fine in	CO
No. 39	is faint in .	O,	but a * as bright as η in .	CO

The κ group is not nearly so bright in O as in CO.

Besides these differences, O has ten wanting in CO, among which are No. 38 η, very bright, and No. 44 θ, in all its brilliancy; while CO has eight wanting in O, of which No. 75 is a *; the difference being greater than I have sometimes found with much less chemical agreement.

The only conclusion which such facts permit is, that the spectrum is a simple resultant of all the actions present, some of which may combine to produce an exalted effect, while others may be antagonistic in any degree.

4. As to the manner in which electricity produces the rays of these lines, whether by merely heating the medium, or by some luminiferous action analogous to its heating, nothing is really established. Heat is known to produce some brilliant lines when metallic vapours are introduced into flames; and it is possible that a temperature far above that of our hottest flame might bring out all those which I have enumerated, and the multitude of others which I did not attempt to measure. But is the temperature of the elec-

tric discharge so immensely superior to all others? and are there no means of estimating its real amount? Two of the facts which I have noted respecting these discharges may at least direct attention to this subject. In general the spectra of the simple spark of an induction machine are much fainter than when a jar, however small, is connected with it; and those with a small jar than those with a large. Thus, with silver electrodes in air, the spark gave twenty-three lines, of which δ and θ were "very bright;" a jar of 0·5 foot coating gave also twenty-three, but differently placed, more at the red, fewer at the violet, but seven of them *s; the normal one of 1·25 foot gave thirty-one, with nine *s; and two large of 8·5 feet gave thirty-six, with ten *s and many others "very bright." In these cases, were one to judge from first appearances, the spark heats air much more than the jar-discharge, for it has much greater power to burn anything which it encounters; but its section is larger, because much of it is *conducted* by the air which surrounds it; and besides, from the diminished resistance, the amount of heat produced *may* be less, as well as less concentrated. We cannot say it *must* be; we know too little of the nature of induction discharge to estimate the effect of changing resistance, for if it be increased it is possible that part of the electricity may be discharged through the coil itself. It must also be kept in mind, that while the jar-discharge is almost instantaneous, the other, at least in part, has a sensible duration. It could, however, be easily decided by experiment whether more heat is evolved in the C.P. or R. discharge.

Secondly. My induction machine, as I have already stated, can be used collaterally; in this case the quantity is double, and should have a fourfold heating power. In fact, its discharge (or rather the air which that discharge heats) fuses a piece of platinum wire, which the consecutive arrangement only reddens. Now, if the lines were produced by mere heat, the spectrum of the former discharge should be far the brightest: it is not so in the red and green; in the violet there is a difference, but I think an unpractised observer would scarcely notice it. I however saw the lines beyond H more easily and further. Unless the temperature is sufficiently raised by a weaker discharge than either of these to bring out all the lines (which seems inconsistent with the effects obtained by enlarging the jar), we might expect here, from the heat theory, a greater change. It is, I think, worth pushing the trial further, and I intend to repeat the experiment with an induction machine of much greater quantity, and at the same time to ascertain if intensity also have any influence.

These observations, on the whole, incline me to refer the origin of the lines to some yet undiscovered relation between matter in general and the transfer of electric action. According to the special properties of the molecules which are present, the brightness of these lines will be modified through a range from great intensity down to a faintness which may elude our most powerful means of observation. If several sorts of molecules be simultaneously present, there may be expected interferences which will produce alternations of brilliancy or obscurity to any extent; and if any of these be chemically united, analogy leads us to expect that such compound molecules will act with an influence of their own different from that of their elements. In even a simple mixture like air, I have shown that its spectrum, with a given metal, cannot certainly be deduced from those of its parts, and it is probable that this rule may be widely extended.

The bearing of this on electro-spectral analysis is obvious; for if the presence of one substance can be shown in any instance to disguise or transform the spectrum of another, or if the state of density, solution, alloyage, &c. have influence, it becomes necessary to eliminate such effects before we yield implicit confidence to this powerful guide. This implies a wide range of experiment and of cautious study; in fact, a complete system of spectral research through the whole range of our chemical elements *and their compounds*, conducted with strict inductive logic, and with the highest appliances of chemistry and optics. Whatever shall be so effected will be "an everlasting gift" to science, because, taking nothing for granted, it will be a real fact.

XXXVIII. *Experimental Researches on the Transmission of Electric Signals through Submarine Cables.*—Part I. *Laws of Transmission through various lengths of one Cable.* *By* FLEEMING JENKIN, *Esq.* *Communicated by* C. WHEATSTONE, *Esq.*, *F.R.S.*

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IN a paper by Professor W. THOMSON "On the Theory of the Electric Telegraph," communicated to the Royal Society in 1855*, the peculiar circumstances affecting submarine wires were especially analysed, and it is probable that all the laws regulating the transmission of signals could, by further development, be deduced from the mathematical theory there stated, if the necessary constants were known.

It is hoped that some account of an experimental research into the same subject may be found interesting, especially as the experiments not only confirmed the conclusions arrived at by Professor W. THOMSON, but led also to the discovery of several facts of considerable practical importance.

With the view of elucidating the present subject, many experiments have been made on the charge and discharge observable at the near end of a cable; but although this charge is intimately connected with the retardation of signals, the connexion is complicated, and many false deductions have been drawn from the facts observed.

The author preferred to make his experiments on the signal or current actually received at the far end of the wire, the object being to establish a direct relation between the causes operating at one end and the effects observed at the other. Some experiments of this kind have also been made, but the instruments used have been such as could only indicate some one point of the complete phenomenon, such as the presence or absence of a given current, and the conclusions so arrived at vary with the nature of the instrument employed.

The author used an instrument by which he was able to follow the phenomena throughout, observing the nature and magnitude of every change produced in the received current. An attempt was first made to obtain by experiment, from various lengths of cable, and with various battery power, the curve given by Professor THOMSON in the above-named paper, and called by him "the curve representing the gradual rise of the current in the remote instrument when the end operated on is kept permanently in connexion with the battery." This curve will in the present paper be called the arrival-curve.

The effect of continually repeated signals of various kinds was next studied, with various lengths of cable and various arrangements of battery, in order to determine in each case the practicable speed† of signalling.

* *Vide* Proceedings of the Royal Society, May 1855, and Philosophical Magazine, S. 4. vol. xi. p. 146.

† In this paper, where the words "speed of signalling" or "rate of transmission" are used, the author

Certain modifications of the usual signals were by these means discovered which materially increased the rate of legible transmission.

These experiments were all made on one cable, and do not therefore refer to the effects produced by a change of dimensions or materials; they will now be described in detail, and the connexion of the results with the mathematical theory will then be mentioned.

The Red Sea cable only was used in all the experiments; the external diameter of the insulated wire was 0·34 inch; the gutta percha weighed 212 lbs. per knot; the conductor was a copper strand of seven wires, and weighed 180 lbs. per knot. The extreme diameter of the strand was about 0·105 inch, and the ratio of the external to the internal diameter of the gutta percha sheath may be taken as 3·4. The resistance of the conductor per knot was 25×10^7 British absolute units.

The first experiment was made on July 26th, 1859, with 2168 knots disposed in ten coils, each about 26 feet in diameter, and held in a dry iron tank; the cable A C X, the battery B, a Morse key M, a galvanometer G, and the two earth-plates E E were connected as in fig. 1, Plate XLIX.

The effect at the end X produced by connecting the cable at A alternately with the battery and the earth-plate by means of the key was observed on the galvanometer G, being Professor THOMSON'S marine galvanometer.

The suspended magnet m of this galvanometer carried a little mirror n reflecting the image of a flame B on to a scale A about 26 inches off, as shown in fig. 2. The zero of the scale was in the middle, and each division was equal to $\frac{1}{40}$th of an inch. The coil C is shown in section, and the little lens L in its position in front of the mirror. The deviations of the spot of light from the centre measured on the scale were sensibly proportional to the strength of the current causing the deflection, when the deviations did not exceed 200 divisions.

The little magnet was powerfully directed by a fixed external steel magnet N S. The weight of the moveable magnet and mirror was about $1\frac{1}{2}$ grain; the inertia of the moving parts was consequently so small, and the directing force so great, that the variations of a rapidly changing current were accurately represented by the movements of the spot of light along the scale. When the Morse key (fig. 1) was pressed down, the spot of light remained apparently motionless for a short but sensible time, then shot along the scale, moving rapidly at first, but gradually losing speed, until at last it moved very slowly to a maximum deviation, at which it remained quite still: these movements truly showed the gradual change of the received current from nothing to a maximum, a change requiring fifty seconds for its completion. During the latter part of this time the movement of the spot was slow enough to allow the moment at which it passed any given division of the scale to be pretty accurately fixed. Two observers

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refers to the speed with which certain changes in the received current called signals can be made to follow each other, and not to the velocity of propagation of the electric current.

were wanted, one noting by the seconds hand of a watch the interval between the sound of the contact made by the Morse key, and the sound of a little blow struck by the other at the moment when the spot passed the given division.

Only one observation could in general be made each time the current entered the cable. The results of the observations are given in Table I.

The maximum current from a battery of 72 Daniell's cells caused a deviation of 130 divisions. The strength of this permanent current depends simply on the electromotive force of the battery and the resistance of the various parts of the circuit, being quite independent of the inductive phenomena.

The first column shows the number of seconds during which the current had been flowing into the cable when the spot had reached the division of the scale entered immediately beneath in the second column. The third column shows the percentage of the maximum deviation, or strength of current, to which the figures in the second column correspond. Thus, when the Morse key had been pressed down for eight seconds, the spot of light was just passing the 100th division of the scale, showing that the current had attained 77 per cent. of its maximum strength. One part of the arrival-curve might be constructed by using the entries in the first and third columns as coordinates.

The whole curve could not be obtained because the movement of the spot of light during the first four seconds was too rapid to allow of observation.

Table II. contains a similar set of observations made with 36 Daniell's cells instead of 72. By comparing the third lines of the two Tables, it is seen that the same percentage of the maximum strength is reached in the same time with both batteries, or, in other words, that the *electromotive force of the battery has no appreciable effect on the velocity with which the current is transmitted.*

All the variations of the received current are therefore in the Tables reduced to percentages, or to the variations which would have been observed if the permanent current due to the battery used had produced a maximum deviation of 100 divisions in each case. This condition could have been practically fulfilled, but was thought unnecessary, as no change of the battery would have altered the percentage of variation observed in the received current.

Table III. shows some observations for the arrival-curve with 1500 knots of cable in circuit.

Table VII. contains the result of a similar investigation for 1006 knots. The first and second columns of this Table contain similar entries to those in Table I.; the third column shows the division passed by the spot when the current was falling, after the Morse key had been released for the number of seconds entered above in the first column.

Thus, after the end A of the charged cable had been put in connexion with the earth-plate E (fig. 1) for 14½ seconds, the spot was just falling past the 15th division of the scale. As the maximum deviation had been 277 divisions, the spot had, during these

$14\frac{1}{2}$ seconds, traversed 262 divisions of the scale, or the received current had diminished 94·6 per cent. in that time. The fourth and fifth columns give these two numbers. It will be seen from these columns that, measuring from the starting-point, the spot of light traversed the same distance on the scale in the same time when the current was falling as when it was rising. From this we conclude that *the rate of decrease in the current received at* X *after the contact had been made for a given time with earth at* A, *was the same as the rate of increase observed after making contact with the battery at* A *for an equal time.*

Table XII. contains a very perfect set of similar observations with 2192 knots in circuit. The arrival-curves for 1006 knots and 2192 knots constructed from Tables VII. and XII. are shown in fig. 7, Plate L.

The same connexions (fig. 1) were used to test the effect of practical signals, the Morse key being moved up and down in time with a metronome. The signals sent were the dot and dash, or dot and line, singly and combined. These signals are the simplest of all, but the conclusions drawn from them are applicable to all other signals depending on the time during which a given strength of current flows.

The headings of the several columns explain the contents of Table IV., which shows the observations made when various signals were sent at various speeds through 1500 knots of cable. Tables VI., XIII., and XIV. contain similar observations made with other lengths of cable in circuit. It should be observed that the effects recorded are those which occur when the same cycle of operations has been constantly repeated for some time, and are quite distinct, as will presently be shown, from the effects produced when signals of the same kind are for the first time sent through the cable.

The first set of observations recorded in Table IV. may be explained as follows. The metronome was set so as to beat 130 times in the minute. The Morse key alternately connected the cable at A with a battery of 72 Daniell's cells, and with the earth-plate for 0·462 second. The signals sent are called dots, because the sixty-five short contacts with the battery, each followed by an equal contact with the earth, would, through a short cable or air line, have transmitted sixty-five distinct currents, each capable of printing on the common Morse receiving instrument sixty-five equally spaced dots. While the contacts were regularly repeated, the spot of light moved steadily back and forward from the 85th to the 80th division of the scale. By reducing these deviations to a percentage of the maximum deviation (200 divisions), we obtain the numbers 42·5 and 40. The variation of the strength of the received current was therefore $2\frac{1}{2}$ per cent. All these numbers are entered in the several columns of the Table.

The figures annexed to the Tables show by curves the changes of the received current. The abscissæ of the curves correspond to intervals of time, and are taken from the fourth column. The ordinates correspond to the strength of the received current at each moment, reduced to a percentage, and are taken from the eighth column. The ordinates, instead of being measured from a true base-line, are measured from a horizontal dotted line, of which the ordinate is 50.

The dots first entered (Table IV.) appear as an even wavy line; the lowest point of each wave is 10 parts, and the highest point 7½ parts below the dotted line, corresponding to the true ordinates from the base, viz. 40 and 42½. The top and bottom only of the waves are fixed by the observations.

The second kind of signals are called dashes, because if sent through a short cable or air line, they would print a succession of dashes or lines, separated by short spaces equal to those separating dots sent at the same speed. To do this the contact with earth is made the same as for a dot, but the contact with the battery is made twice as long. Examples of the dash will be found in Tables VI. and XIII.

The effect of combining these two primary signals was tried by sending alternate dots and dashes; the results are entered in the Tables. The effect of one complete cycle of operations is shown in the figures annexed to the Tables by that part of the curve drawn with a thicker black line. Thus in the second dot and dash curve of Table IV. the dot curve begins with the ordinate 53·2; the first battery-contact slightly increases the current, till the ordinate of the top of the dot curve becomes 55·5. The first earth-contact diminishes the current, so that the ordinate at the end of the dot or beginning of the dash curve is 48·0; the next battery-contact, being a long contact, raises the dash curve to 61·7. The second earth-contact lowers it to 53·2, when the cycle recommences.

The numbers in the two last columns of the Tables IV., VI., XIII., and XIV., with the figures annexed, are all directly comparable, for they all represent the results reduced to a percentage, and are consequently constant for each length, being independent of any change in the battery.

These observations do not give the retardations properly so called, *i. e.* they do not show the time separating the contact made at A from the effect produced at X, but they establish several conclusions of greater importance than the knowledge of this retardation.

It will be seen (Table VI.) that when 66 dots per minute were sent into 1802 knots of cable at A, a constant current was received at X, in which no oscillation could be seen corresponding to the signals sent*.

The same phenomenon was observed with 2192 knots in circuit, when more than 50 dots per minute were sent. The effect produced at X by these rapidly repeated dots was almost exactly that which would have been observed if the cable at A had been permanently connected with a battery of about 30 cells. In moving the Morse key up and down, a little time elapses between the contact with earth and that with the battery, during which the cable is really insulated at A. This lost time caused the received current to be equal to only 41·6 in one case and 42·9 in the other instead of 50 per cent. of the maximum permanent current, as would certainly have been the case if the sums of the battery and earth contacts had respectively occupied exactly the half of each minute.

* The immobility of the spot could not be accounted for by the inertia of the mirror and magnet, which, when in vibration, moved so rapidly that the oscillations could not be counted. Similar observations made with a galvanometer, the needle of which oscillated slowly, would lead to most erroneous results.

It is certain that in the two cases named the received current did not vary 1 per cent.; here therefore we find one positive *limit to the rate of transmission.* No doubt oscillations of considerably less than 1 per cent. might be observed and recorded by employing a more sensitive instrument and stronger batteries; but we clearly see that the useful effect produced rapidly decreases as the number of signals per minute increases, and therefore that, however sensitive the instrument and powerful the battery, signals sent at more than a certain rate will fail to cause any appreciable useful effect at the receiving end; and we may safely conclude *that in all submarine cables there is a limit to the number of signals which can be sent per minute, a limit which cannot be exceeded by any ingenious contrivance.*

Fifty-six dots per minute were distinctly visible when sent through 1802 knots, and 40 dots when sent through 2192 knots. The mean strength of the currents showing these dots is less than half the maximum permanent current, because of the time lost in moving the Morse key, as already explained. The curves showing these dots appear therefore below the dotted line representing the middle of the scale.

As the speed of sending decreased, the amplitude of oscillation of the spot or variation of the current increased, but the mean strength of the current remained nearly constant.

When dashes were sent (*i. e.* when the length of the battery-contact was made twice as long as the earth-contact), the mean strength of the received current was much higher, being above the middle of the scale.

The effect of dashes could thus, and thus only, be distinguished from that of dots sent at a lower speed. For instance, dashes sent through 1802 knots (Table VI.) when the metronome beat 84 caused a 6 per cent. oscillation, and dots sent when the metronome beat 60 caused very nearly the same oscillation; but the strength of the current due to the dashes varied from $53\frac{1}{2}$ to $59\frac{1}{2}$, whereas that due to the dots varied from 43 to 49. In the corresponding curves the dashes appear above the dotted line and the dots below. This effect will be easily understood when it is remembered that while dots are sent, the end A of the cable is altogether in contact with the earth for nearly half of each minute, but when dashes are sent, it is in contact with the earth for only about one-third of each minute.

When the dot and dash are sent alternately, the effects, differing considerably from those produced when each is sent continuously, can be best studied in the curves annexed to the Tables.

The top of the dot curve is higher, and the bottom does not go so low as when dots only are sent, whereas the top and bottom of the dash curve are both lower than when dashes only are sent; *but the bottom of the dash curve does not go so low as the bottom of the dot curve.* This was seen in every instance, but was most apparent at the higher speeds.

The strength of the received current due to the long battery-contact during the dash is naturally greater than that caused by the shorter contact during the dot; and as the connexion with earth after the two signals is only of equal duration, it is clear that the received current will not fall so low after the dash as after the dot, and this is precisely the effect shown in the curves. The beginning and end of the dot are at different heights, and similarly the beginning and end of the dash are at different heights, making altogether a very irregular curve, especially at the higher speeds.

This irregularity very seriously interfered with the distinctness of the received signals, although the change of current could be followed throughout by watching the spot. At a high speed the dot appeared as a mere pause followed by a fall, instead of a little rise followed by an equal fall; and if the dots and dashes had been irregularly combined, it would have been impossible to disentangle them, as received, by the eye. If a receiving instrument had been used like the common Morse receiver, simply marking the time during which the received current was above or below a given strength, the signals would have failed to give any intelligible record even at a very low speed: this is shown by the fact that it is impossible to draw a horizontal line intersecting both the curve of repeated dashes and that of repeated dots, even for the very lowest speeds recorded in each Table.

Thus, long before the limit is reached at which signals cease to produce any change at the receiving end, the interference of one signal with another causes so great a confusion in the currents received as to put a fresh limit to the practicable speed of transmission.

This confusion is still further increased by the effect of a pause in the signals between letters, words, or sentences.

During all intervals the cable is left in connexion with the earth at A; and if the pause lasts a little while, the current at X falls to nothing, or nearly nothing, when the effect of the first signals sent is to cause an unintelligible succession of sudden increments in the received current, until, after a certain number of contacts, a permanent mean strength of current is attained, at which regular signals might become intelligible. The higher the speed the greater the number of contacts required for this purpose; for instance when 1802 knots were in circuit, the following Table gives the number of dots required to bring the spot to the mean position in which it was maintained by regular signals.

Beats of metronome.	Dots required to raise current to 42 per cent. of maximum.
92	6
84	6
72	4
60	3
50	
40	2

The curve in fig. 3, Plate XLIX. roughly represents the variations of the received current when the operator begins to send the dots regularly through a cable which has been fully discharged. Similar but converse effects would be observed if the line were left long in contact with the battery before the signals began.

This irregularity has hitherto put the practical limit to the rate of transmission through long submarine cables. When this rate is exceeded, a dot sent may at one time cause a great rise in the received current, at another time it will barely arrest a fall, and thus similar signals at one end are found to produce utterly dissimilar effects at the other. No change in the battery has the smallest effect on this interference, nor can any delicacy in the receiving instrument disentangle the confusion, which, it should be observed, is totally distinct from the retardation of the signals.

It would matter little that signals should appear in America even a minute or so after they had been made in England, provided the same signal at the sending end always produced the same effect at the receiving end. The experiments show how far this is from being the case.

The usual mode of avoiding this confusion is to signal at so slow a rate that every dot causes a very large percentage of variation in the received current, while the dash causes nearly the maximum current which would be received from a permanent contact with the battery.

In air lines, or in short submarine cables, or in long submarine cables with large conducting wires thickly covered, this plan does not entail too slow a speed for practical work; but to avoid confusion in this manner when working through 2200 knots of the Red Sea cable, it would be necessary to reduce the speed to less than 10 dots per minute, whereas 40 dots or more could be received but for the interference.

Hence we conclude *that there is a wide margin between the limit set to the speed of transmission by the gradual diminution of the received signals and that set by their interference.*

If interference could be prevented by any change in the signals, it was clear that the capabilities of any given cable would be increased at least fourfold.

Reverse currents have been much advocated, on good authority, as a means of greatly increasing the rate of signalling, and their effect was therefore examined, although it was not thought probable that the interference would be diminished by their use. By "reverse currents" the use of alternate positive and negative currents is meant. The negative current substituted for the earth-contact is supposed by its advocates in some way to clear the line and prepare the way for a positive signal.

The connexions used during the experiments on "reverse currents" are shown in fig. 4, Plate XLIX. The first contact sent a positive, and the second contact a negative current through the line.

The effects of single and reverse currents were directly compared on 1165 knots of cable. Tables VIII. and IX. show observations of the arrival-curve and signals when a positive current from 72 cells and an earth-contact were used. Tables X. and XI. show a similar set of observations when a positive current from 42 cells and a negative current from 30 cells were used. By keeping the same total number of cells in each case, the difference of potentials between the two sources of electricity alternately in connexion with the line was maintained nearly equal. Tables VIII. and IX. are similar to those already described.

The observations in Table X. were made in almost the same manner as those in Tables VIII. and IX., but the following description may make the meaning of the entries more distinct.

The Morse key when untouched left the negative battery connected with the line, and the spot of light then stood at 113 divisions to the left of zero; when the key was permanently pressed down, the spot stood at 157 divisions to the right of zero. All deviations to the right are entered as positive, those to the left as negative. Four seconds after the key was pressed down, the spot passed the 100th division to the right; four seconds after the key was released, it passed to the 60th division on the left. In the first case it had traversed 213 divisions in four seconds, in the second case 217 divisions, or 78·8 and 80·3 per cent. respectively of the sum of the two deflections 157 and 113.

On examining the last columns of Table X. we find that in any given time after moving the key, the spot traversed an equal length of the scale whether the change was from positive to negative or from negative to positive, or, in other words, *the rate of decrease in the current received at* X *after contact had been made with the negative battery at* A *for a given time was the same as the rate of increase observed after making contact at* A *with the positive battery for an equal time*,—a conclusion exactly analogous to that arrived at when single currents were used.

Moreover, comparing the last columns of Tables VIII. and X., we find that the spot traversed the same distance in the same time in each case, and therefore that the rate of increase and rate of decrease, caused by reverse currents, is exactly the same as that observed when single currents are used.

The arrival-curves from the two Tables are shown in figs. 5 & 6, Plate XLIX.

The curves are identical in shape, and differ only in their position relatively to the zero-line.

Thus the ratio between the ordinates of the arrival-curve is the same whatever the two sources of electricity at A, or the potential of the earth at X may be.

Hence we conclude,

1st. That the absolute change in the strength of the received current during a given time after a change in the contact at A, is not influenced by the potential maintained at X, but is simply proportional to the difference of potentials or electromotive force between the two sources of electricity alternately connected with the cable at A.

2nd. That the rate of change is independent of the potentials both at A and X.

It was now clear that the alternation of negative and positive currents could have no effect on the rate of signalling, but to avoid all cavil a few experiments were made with the usual signals.

The results are given in Tables IX. and XI. The signals sent by the reverse currents were a little above the zero-line, those sent by the single currents were, as usual, near the middle of the range. This was the only difference observed; the amplitudes and relative position of the dot and dash were identical. The use of reverse currents, therefore, *does not alter the limit set by the gradual diminution of the received signals, nor that set by their interference.*

It is just possible that some small effect may be produced by a difference in the insulation when reversals are used, but there is no reason to suppose that this difference would be in their favour.

Abandoning reverse currents, the author was led to seek some other remedy for the confusion observed. One phase of this confusion might be described by saying that the mean strength of the current rose above 50 when dashes were sent, and fell below it when dots were sent, so that the dots and dashes appeared on different parts of the scale. The high position of the dash was due, as has been shown, to the comparatively short contact at A with earth after the long battery-contact. By making the second or earth-contact longer, the dash would be brought down in the scale. If the earth-contact were made equal to the battery-contact, the dash would simply become a dot sent at a slower rate. The mean strength of the current during the slow or long dots would be the same as that during the quick or short dot. The bottom of the long dot curve would be lower than the bottom of the short dot curve, and therefore, when the long and short dot were combined, there would still be considerable confusion. Moreover, if a Morse receiver or analogous instrument were used, the spaces separating the long dots would be twice as long as those separating the short dots, and this unequal spacing would cause fresh confusion.

These effects, consequent on making the second contacts always equal to the first, were well seen when the signals called A^s were sent *. The tendency observed in the original dot and dash was over-corrected. The dash now fell too low, and the dot not low enough, so that the confusion was nearly as bad as before; but the required correction was clearly enough pointed out. If, instead of keeping the *mean* strength of the current constant, the current at the *end* of each signal could be kept the same, the passage from one signal to another would cause no confusion, for the beginning or end of a dot would be exactly like the beginning or end of a dash. It was plain that the current at the end of a dash could be brought to any required point (and therefore to the point at which dots finished) by simply altering the proportion of the second to the first contact. Experiment had shown that the second contact was too short when made only half the length of the first, and too long when made equal to the first; no doubt some intermediate length would fulfil the required condition that the dashes should begin and end at the same division of the scale as the dots. The experiment required to prove this could clearly not be tried by aid of the metronome, and the apparatus shown in fig. 8 was therefore arranged so as to make contacts of any required proportion.

* So called because the dot and dash, followed by a short pause, represent the letter A in the Morse alphabet.

A strip of paper (figs. 8 & 9, Plate L.) was prepared with two parallel rows of alternate holes. This paper was joined so as to form an endless band, and was drawn by the roller R under two little bent wires *b* and *e*, placed abreast, so that each alternately came in contact with the metal plate L through one of the holes in the paper. The plate L was connected with the cable, the wire *b* with the battery, and the wire *e* with earth. As the paper was drawn along under the wires, the alternate battery and earth contacts sent signals through the cable, and the length of the holes in the paper determined the relative length of the contacts.

The first or battery-contact was made through the upper row of holes (fig. 9), the earth-contact through the lower row. Dots were sent by two equal holes, dashes by two longer holes, of which the upper bore to the lower the proportion of 5 to 4, corresponding to the relative length of contacts desired. The length of the two dash contacts was made equal to two pairs of dot contacts.

It was expected that if the right proportion between the first and second dash contacts had been adopted, all confusion from irregular combinations would be avoided; but during any pause, such as is practically required to separate groups of signals, the spot or current would still fall towards zero if the line were left in contact with the earth (or even if insulated at one end), so that the first signals sent after a pause would still cause mere irregular and unintelligible changes in the received current. To avoid this second source of confusion, it was necessary to maintain the received current, during any pause, at the constant final strength to which it returned at the end of each oscillation during a series of signals.

If this were done, the first signal would begin where the last left off, and each signal might be expected in all cases to produce one invariable and intelligible effect.

There were two obvious means of keeping up the current during a pause. The line might be left in contact with a third source of electricity* just sufficiently powerful to maintain the required strength of current, or a very rapid series of contacts might be made alternately with the full battery and with earth. Experiment had shown that such a series of contacts would maintain the current at the receiving end sensibly constant, and the strength of this constant current could be easily adjusted by varying the proportion between the first and second of these very short contacts, increasing the length of the first contact to raise the current, and increasing the length of the second contact to lower the current.

This second plan was easily carried out by the perforated paper. Where a pause was

* When reverse currents are used, if the line is put in contact with earth during a pause, the confusion of the first signals will be much lessened; the earth then acts as the third intermediate source of electricity alluded to in the text. This plan has to some extent been adopted in practice by Messrs. Siemens and Halske, by means of a key invented by Mr. L. Loeffler. This benefit, which may perhaps account for the fancied superiority of reverse currents, might be equally obtained with simple currents by connecting the line during each pause with half the battery. Mr. C. F. Varley informed the author that he also has used a similar key for a considerable time.

required, little holes were cut in the paper to make contacts in pairs, each pair half the length of those used for a dot. In fig. 9 these short openings are shown where the word "space" is written. This arrangement was perfectly successful with 1500 knots in circuit on the very first trial; and although other strips of paper were tried with other proportions between the contacts, none gave better results than those first adopted. When the paper was steadily drawn along, the signals appeared on the galvanometer with all the regularity that could be wished: during the pauses, the light stood trembling at one division on the scale, a dot caused a slight rise followed by an equal fall, and the dashes produced a greater and longer oscillation, at the end of which the spot always returned to the one constant final strength.

A still more decisive test was next made, by substituting a relay and Morse marker for the galvanometer; fig. 10, Plate L. is an exact copy of the signals which were then received, and shows faithfully the slight irregularities which did occur. The only serious flaw in the whole set of signals is shown at the beginning, between A and B, where some dots and a space contact made only unintelligible marks. This flaw invariably occurred at the same place; it was shown equally by the galvanometer and the relay, and for some time the cause could not be discovered. By carefully watching the paper strip, it was at last seen that when the joint of the endless band passed through the rollers R (fig. 8) a little hitch or pause occurred, and that this slight irregularity of speed caused a corresponding confusion in three signals. This accident showed the accuracy of proportion required between the various contacts.

Taking this hitch as the beginning, we next see six well-spaced dashes, three pairs of dots and dashes, one space, one dot, one space, one dash, a long pause, a dot, a long pause, a dash, a short pause, a dot, a dash, and a succession of dots.

In this series every possible difficulty which could arise from interference or confusion was encountered and successfully overcome. The rate was about forty-five dots per minute, and the oscillations for a dot must have been about 5 per cent.*

Signals sent at a much higher rate could have been distinctly received, but the drawing-rollers could only be driven at one speed.

The apparatus was next tried with 1800 knots in circuit: the signals could be read without difficulty on the galvanometer; but to increase the range and to facilitate the adjustment of the relay the battery power was increased, when not only could the regular dots and dashes be received, but even the short space contacts gave distinct legible signals; so that the three sets of signals of three different lengths appeared, without confusion, recorded by the relay. The shortest signals were received at the rate of ninety per minute; and from Table VI. it will be seen that even sixty dots per minute through this length reduced the oscillations to less than 1 per cent. of the permanent maximum strength. Hence we may conclude that, *by the means described*, *or by*

* When the relay was used the galvanometer could not be observed, for the motion of the soft iron used in the relay induced short currents, causing rapid vibrations of the spot of light.

analogous means, signals can be sent without confusion at any speed which will allow the shortest signals used to cause a sensible variation in the received current.

The cable was available for a few days only, the apparatus used was very imperfect, the paper frequently tore, the openings in it were cut by hand one by one, the speed of the drawing-rollers was far from regular, the relay was difficult to adjust, owing to residual magnetism in the electro-magnets, and the author could give but a small part of each day to the experiments; in spite of these disadvantages, the results obtained were so definite as to leave no doubt of the important conclusion stated above.

No further experiments could be made; but indeed no further improvement appeared possible, except in the mechanism for making the required contacts, and in the choice of signals which should give the greatest number of words with the smallest number of currents. Neither of these points can fitly be treated of in the present paper.

Before proceeding to compare the results obtained with the deduction from mathematical theory, or to apply the conclusions to cables in practical use, it will be well to consider how far the special disposition of the cable may have affected the value of the experiments.

The experiments were made through dry cables, lying in large close coils, whereas the cable when in use lies extended under water. It is found in practice that the insulation and charge of an iron-cased cable is little affected by submersion; but coiling is generally believed to cause an additional impediment to the transmission of signals, and it might certainly be expected that this difference of condition would cause some discrepancy between the experiments described and the results of practice: but since the mathematical theory is framed to meet the case of a submerged and extended cable only, whenever the conclusions experimentally deduced are found to be in accordance with the deductions of theory, it is clear that the experiments and the theory mutually confirm one another, and that the conclusions may be safely applied to the practical case of an extended and submerged cable; for it is impossible to suppose that the dry and coiled state of the cable, not contemplated in the theory, should nevertheless exactly compensate its errors, or that results due only to an accidental arrangement of the cable should by chance coincide with deductions from a defective hypothesis.

When, on the other hand, the results obtained differ from those given by theory, or even when the theory affords no confirmation of the experimental conclusions, we must forbear to extend these conclusions to the practical case of a straight cable.

The arrival-curve obtained by experiment is similar in general appearance to that given by Professor THOMSON in "The Theory of the Electric Telegraph*." The identity of the curve obtained from the increase with that obtained from the fall of the received current, follows from equation (3.) in the same paper. The described effects of using alternate currents follow from the principle of superposition, by which also the effects of the increased speed might be shown in diminishing the received signals.

* *Vide* Proceedings of the Royal Society, May 1855, and Philosophical Magazine, S. 4. vol. xi. p. 146.

An example of the confusion arising from interference between successive signals has been given by Professor THOMSON in his evidence before the Committee appointed by the Board of Trade to inquire into the construction of submarine telegraph cables. Professor THOMSON has also informed the author that the compensation derived from experiment, as explained above, is such as theory would demand, but that he has not yet verified the exact proportion required between the first and second contacts*.

Thus every conclusion hitherto stated is in accordance with the mathematical theory, and may therefore without hesitation be applied to submerged and extended cables.

The arrival-curves (fig. 7, Plate L.) do not, however, accurately correspond with that given by Professor THOMSON. The dotted line shows the calculated curve, drawn so as nearly to agree with the experimental curve from 2192 knots, at its origin; the difference between the two curves towards the end is very great. The experimental curve approaches its limiting height much too slowly after the first few seconds; one part of the curve from 1006 knots is shown on the same figure, and by theory the abscissæ of the two curves corresponding to equal ordinates should be directly as the squares of the lengths. At the origin of the curves this is approximately the case, making a small allowance for the constant resistance of batteries and instruments; but towards the end of the curves this is far from being the case. Some of the causes of the discrepancy may disappear in straight cables, but meanwhile the constants required in the mathematical theory cannot be with confidence derived from these curves or Tables.

It is generally believed that coiling increases the retardation, and the curves obtained might be pointed to as confirming this opinion. It is certain that a mutual electro-magnetic induction of considerable importance does occur between the different parts of the coils†, and probably some part of the discrepancy between the observed and calculated curves is due to this cause. But the varying resistance of gutta percha, unknown when the theory was framed, also accounts for a considerable difference between the results of observation and calculation. The allowance to be made for a constant uniform leakage such as would occur if the resistance of the gutta percha were uniform, is described by Professor W. THOMSON in the "Theory of the Electric Telegraph;" but the author of the present paper discovered that the resistance of gutta percha, such as was used for the Red Sea cable, increased more than 60 per cent.‡ during positive or

* Professor THOMSON has also stated that he has long been acquainted with some other modes of producing the required regularity, and one of his methods is alluded to in the evidence before the Board of Trade Committee. *Vide* also Proceedings of the Royal Society, Dec. 1856, and Philosophical Magazine, 1857, where the mathematical principle of the compensation is fully stated.

† *Vide* paper read by Professor THOMSON at the British Association, Aberdeen, 1859, and letter by Professor W. THOMSON and the author, published in the 'Philosophical Magazine,' 1861; also a letter from Mr. F. C. WEBB in 'The Engineer,' August 1859.

‡ These numbers are calculated from data in the paper by the author read before the Royal Society in 1860, and published in abstract in the 'Proceedings' of last year, and in the 'Philosophical Magazine' for 1861; also in full in the Appendix to the Report of the Committee of the Board of Trade on the Construction of Submarine Cables, 1861.

negative electrification, and that a great part of the observed change was completed before the end of the first minute after the cable was connected with the battery. This gradual improvement of insulation would gradually increase the received current long after all inductive phenomena had ceased. For instance, the resistance of the total gutta-percha sheath of 2192 knots at 60°, after negative electrification for about 15″, would be 104×10^{10} absolute British units; after one minute, this resistance would increase to 132×10^{10}. The resistance of the conductor was about 55×10^{10}.

If the insulation had remained constant at the first-named figure, the final maximum arriving current would have been only 78·4 per cent.* of the entering current; and if the insulation-resistance had remained constant at the second figure, the final arriving current would have been 82·4 per cent. of the entering current. It is not therefore surprising that the observed curve, subject to the influence of imperfect and varying insulation, should not more perfectly coincide with the theoretical curve, in which perfect and constant insulation is assumed. When allowance is made for the change in the entering current due to the change in the total resistance of the circuit caused by the change in the insulation-resistance from 104×10^{10} to 132×10^{10}, the final arriving current with the lower insulation would by calculation be about 97 per cent. of the arriving current with the higher insulation, and this proportion very exactly corresponds with the slow increase observed during the last forty seconds of the minute. This increase has therefore no connexion whatever with the retardation properly so called.

The identity of the curve of increase with the curve of decrease seems to show that the apparent increase of the resistance of the gutta percha is rather due to an absorption of electricity which is again given out, than to a real change in the conductivity of the material †.

The effect of varying insulation would be much less felt during the actual transmission of signals, for then the greater part of the cable is constantly electrified in one manner, and the resistance of the gutta percha under such circumstances remains sensibly constant; we might therefore here expect a better agreement between theory and observation.

In order to examine the results of the experiments on repeated signals, the number of dots sent per minute, and the corresponding amplitude of oscillation observed in the received current, might be used respectively as abscissæ and ordinates, to give a curve expressing the rate at which, for each length of cable, the effect of the signals diminished as their speed increased; but, by the mathematical theory, the time required for any electrical operation varies as the square of the length of the cable. The product of the square of the length into the number of dots producing a given amplitude of variation should therefore be constant for all lengths of the same cable; and by using this product as an abscissa, instead of the simple number of dots, one curve should give the

* Appendix, Section I.

† The truth of this conclusion has been established by the results of some experiments on the Malta-Alexandria cable, made by Dr. Esselbach, and received by the author since writing the above.

amplitudes for all speeds through all lengths of cable; and conversely, the observations made on various lengths should, when thus geometrically represented, agree in defining one curve. The agreement between theory and observation may be very readily tested by comparing the observations through various lengths in this manner. But Professor THOMSON has also stated *, as a development of the theory, that if the resistance of the battery and receiving instrument bear but a small proportion to the total resistance of the cable, their effect in retarding and weakening the signals will be sensibly the same as that of adding an equal length of actual cable with its usual electrostatic capacity. (Evidence before Board of Trade Committee.)

It will thus be necessary, in comparing the observations, to add in each case 160 knots, equivalent to the resistance of the battery and galvanometer, to the length of the cable. Table XV. and fig. 11, Plate LI. show the results of the comparison. The first and second columns of the Table give the number of beats and corresponding amplitudes, extracted from Tables IV., VI., and XIII. The third column contains the product of the number of beats into the square of the length of the cable. The fourth column gives the similar product when 160 knots has been added to the length of each cable.

Fig. 11 gives the geometrical representation of the observations made by using the entries in the second and fourth columns of the Table as ordinates and abscissæ respectively.

The star, cross, and circle respectively denote observations with 1500, 1802, and 2192 knots in circuit. All these marks fall sensibly on one curve, affording *a perfect experimental proof that the rate of transmission does vary inversely as the square of the length, whether by rate of transmission be meant that speed at which the repeated signals fail to produce any sensible effect, or the rate producing so great an amplitude that common hand signals can be received without confusion.*

Moreover, it will be found that the curve is more accurately defined by taking the abscissæ from the fourth than from the third column; verifying Professor THOMSON'S conclusion as to the effect of the resistance of the battery and receiving instrument. These points have been much debated, but no doubt should now be felt of the soundness of the theoretical conclusions.

The above is neither the only nor the most remarkable confirmation of the mathematical theory. Professor THOMSON has been so kind as to give the author a Table (XVI.†) of calculated ordinates for the curve in question‡. The full black line (fig. 11) was constructed from this Table, and coincides with the recorded observations in the most striking manner; no more perfect verification of complicated mathematical calculations was probably ever obtained by experiment.

The curve shows at once the relative number of signals per minute which will pro-

* *Vide* Evidence before the Committee of the Board of Trade on the Construction of Submarine Cables, A.D. 1861, p. 125.

† Appendix, Section II.

‡ Proceedings of the Royal Society, 1855 and 1856, and Philosophical Magazine, 1856 and 1857.

duce the various amplitudes in any one length of cable; thus we see that if an amplitude or variation in the received current of 1 per cent. will suffice for distinct signals, twice as many signals can be sent through any given cable as if an amplitude of 7 per cent. is required, or four times as many as if an amplitude of 25 per cent. is required. The latter amplitude is probably necessary for hand signalling, but our experiments have shown that less than 1 per cent. is sufficient when a proper compensation is made. The coincidence between theory and observation places it beyond doubt that the curve truly expresses the relation between the speeds and amplitudes for straight as well as for coiled cables; and if the amplitude at any one speed through any one straight cable were known, the amplitude at any other speed through any other cable of the same materials might be calculated from the curve with certainty; but unfortunately this fact is wanting. There is no proof that the absolute amplitude observed through the coiled cable would remain unaltered if the cable were extended; on the contrary, it is very generally believed that it is easier to signal through a straight than a coiled cable; and if this be so, the amplitude would increase as the cable was laid. Although, therefore, the constants for the mathematical theory might easily be calculated from the values of the coordinates of the curve given by the observations, these constants would probably be inapplicable to straight cables.

Assuming, however, for a moment the identity of a coiled and extended cable, it may be interesting to calculate the amplitudes which would correspond to the rates of signalling recorded for various cables.

For the Red Sea cable the amplitude is found by taking the ordinate corresponding to the abscissa given by the product of the square of the length into twice the number of dots per minute. The speed giving the same amplitude through any other cable of different dimensions, but of equal length, is obtained by a simple proportion.

The author has been informed that ten words per minute have been sent through 640 knots of the Red Sea cable, but that seven words was the more usual speed. The former would correspond to an amplitude of 20 per cent. for the dots*, the latter to about 35 per cent. 1·1 word per minute was sent through the Atlantic cable and received by a relay; this speed would correspond to an amplitude of about 8 per cent.; 2·4 words per minute (the ordinary rate of signalling from Newfoundland having been forty-one dots per minute) were received through the same cable by Professor THOMSON's galvanometer†, corresponding to an amplitude of little more than 1 per cent.

Ninety dots per minute, the speed of the message sent by the perforated paper, would,

* Seventeen dots per word.

† An instrument similar to that used in this research: the observer could therefore follow every change in the received current, and disentangle the meaning of signals which would have produced only hopeless confusion on a relay, or other instrument with a fixed zero.

for 1802 knots, give by the curve an amplitude of 0·3 per cent. only, and there is no reason to doubt this estimate.

In conclusion, the experiments, so far as they went, were successful. They have shown the relative effects of signals transmitted at various speeds through various lengths; they have shown how little the results are affected by changing the power or arrangement of the batteries; they have shown the nature of the ultimate limit set to the rate of signalling by the gradual diminution and disappearance of the signals, preceded by their mutual interference.

The coincidence between theory and observation on these points gives good proof of the soundness of the theory, and permits the extension of the conclusions to the case of a submerged and extended cable. The experiments have also given well-defined curves fully expressing the retardation experienced through the cable as it lay in coils; but owing to this arrangement, the observations cannot be said to fix either the retardation or the absolute effect of signals through a straight cable. A few observations made in the same manner on a sound cable in actual use would be sufficient for this purpose.

Finally, the research has proved that the rate of signalling through a given cable can be very materially increased by removing the confusion or interference of successive signals, and has led to the discovery of one method of effecting this object.

In the present paper, the phenomena depending on the length of the cable, together with the rate and manner of signalling, have alone been considered. The absolute measurement of the effects depending on the materials and dimensions of the insulated conductor will probably form the subject of another research, completing the practical examination of the mathematical theory.

TABLE I.—Arrival-curve for 2168 knots, in 10 coils, with 72 p.p. July 26, 1859.

Maximum deflection caused by permanent current from 72 p.p. ... 130^{d}.

Seconds after making contact with battery	4	5	5	6	6	7	8	11	15	21	51
Division of scale passed by spot as the current rises at far end of line	50	60	70	80	80	90	100	110	120	125	130
Reduced distance traversed by spot after contact has been made with battery for each number of seconds	38·5	46·2	53·9	61·6	61·6	69·3	77·0	84·7	92·4	96·2	100

TABLE II.—Arrival-curve for 2168 knots, in 10 coils, with 36 p.p. July 26, 1859.

Maximum deflection caused by permanent current from 36 p.p. ... 63½^{d}.

Seconds after making contact with battery	7¾	8	10¼	16¼	18	19
Division of scale passed by the spot as the current rises at far end of line	50	50	55	60	60	60
Reduced distance traversed by spot after contact has been made with battery for each number of seconds	78·5	78·5	86·4	94·2	94·2	94·2

TABLE III.—Arrival-curve for 1500 knots, in 8 coils. July 27, 1859.

Maximum deflection caused by permanent current from 72 p.p. ... 200^{d}.

Seconds after making contact with battery	6¼	6¾	8½	14	33
Division of scale passed by the spot as current rises at far end of line	170	170	180	190	200
Reduced distance traversed by spot after contact has been made with battery	85	85	90	95	100

TABLE IV.—Signals through 1500 knots, in 8 coils. July 27, 1859.

Maximum deflection caused by permanent current from 72 p.p. ... 200^{d}.

1. Beats of metronome per minute.	2. Signals sent.	3. Source of electricity in contact with line at near end.	4. Duration of contact at near end in seconds.	5. Limit of deflection at far end. First observation.	6. Second observation.	7. Mean limit of deflection caused by contact.	8. Reduced mean limit of deflection caused by contact.	9. Reduced mean amplitude of oscillation caused by two contacts.
130	Dots.	72 p.p. E.	0·462 0·462	85 80	— —	85 80	42·5 40	2·5
100	Dots and dashes.	72 p.p. E. 72 p.p. E.	0·6 0·6 1·2 0·6	112 100 122 110	111 95 125 110	111½ 97½ 123½ 110	55·2 48·4 61·7 55	Reduced amplitude of corresponding dots 4^{d}.
	Dots.	72 p.p. E.	0·652 0·652	85 75	— —	85 75	42·5 37·5	5·0
92	Dots and dashes.	72 p.p. E. 72 p.p. E.	0·652 0·652 1·3 0·652	110 95 122 105	112 97 125 108	111 96 123½ 106½	55·5 48 61·7 53·2	
	Dots.	72 p.p. E.	0·833 0·833	95 75	— —	95 75	47·5 37·5	10·0
72	Dots and dashes.	72 p.p. E. 72 p.p. E.	0·833 0·833 1·666 0·833	115 90 130 100	112 92 135 105	113½ 91 132½ 102½	56·7 45·5 66·2 51·2	
	Dots.	72 p.p. E.	1 1	100 75	— —	100 75	50 37·5	12·5
60	Dots and dashes.	72 p.p. E. 72 p.p. E.	1 1 2 1	116 85 140 100	121 90 140 100	118½ 87½ 140 100	59·2 43·7 70 50	

Diagrams showing the changes of current caused at the end of the line by the signals.
Vertical scale $\frac{1}{100}$th of an inch = 1 division (strength of current).
Horizontal scale $\frac{1}{10}$th of an inch = 1 second (time).

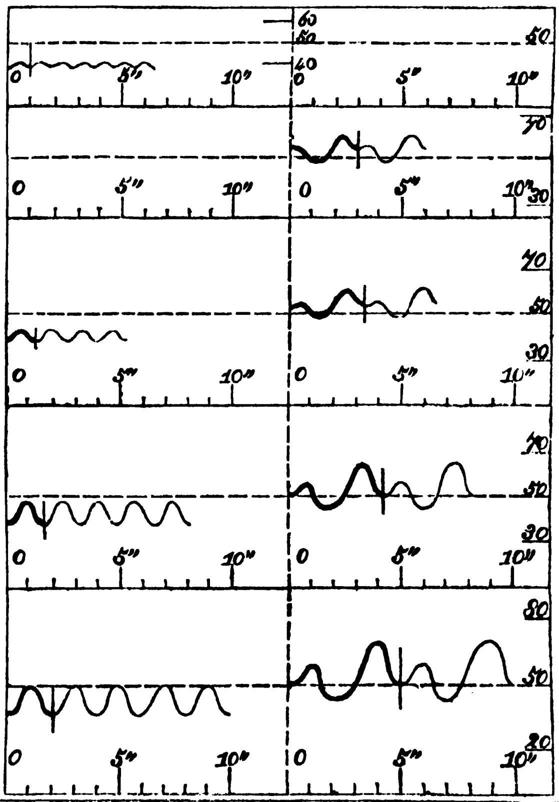


TABLE V.—Arrival-curve for 1802 knots, in 9 coils. July 28, 1859.

Maximum deflection caused by permanent current from 72 p.p. ... 168^{d}.

Seconds after making contact with battery or with earth	5¾	6¼	7	8	9½	11½	15
Division of scale passed by the spot as the current rises at far end of line	130	135	140	145	150	155	158
Reduced distance traversed by spot after contact has been made with battery	77·3	80·3	83·3	86·3	89·2	92·2	94

TABLE VI.—Signals through 1802 knots, in 9 coils. July 28, 1859.
Maximum deflection with 72 p.p. . . . 168d.

Beats of metronome per minute.	Signals sent.	Source of electricity in contact with line at near end.	Duration of contact at near end.	Limit of deflection at far end. First observation.	Second observation.	Mean limit of deflection caused by contact.	Reduced mean limit of deflection caused by contact.	Reduced mean amplitude of oscillation caused by two contacts.
132	Dots.	72 p.p. E.	0·455 0·455	70 70	... —	70 70	41·6 41·6	0 { less than 1d.
112	Dots.	72 p.p. E.	0·536 0·536	76 74	— —	76 74	45·2 44	1·2
100	Dots.	72 p.p. E.	0·6 0·6	73½ 70	— —	73½ 70	43·4 41·6	1·8
92	Dots.	72 p.p. E.	0·652 0·652	74½ 70	— —	74½ 70	44·3 41·6	2·7
	Dots and dashes.	72 p.p. E. 72 p.p. E.	0·652 0·652 1·3 0·652	90 85 85 90	85 80 90 85	87½ 82½ 92½ 87½	52 49 55 52	
84	Dots.	72 p.p. E.	0·714 0·714	75 70	80 75	77½ 72½	46·1 43·1	3·0
	Dashes.	72 p.p. E.	1·427 0·714	100 90	— —	100 90	59·5 53·5	6·0
	Dots and dashes.	72 p.p. E. 72 p.p. E.	0·714 0·714 1·427 0·714	95 85 100 93	95 86 100 94	95 85½ 100 93½	56·5 50·9 59·5 55·6	
72	Dots.	72 p.p. E.	0·833 0·833	76½ 70	81 75	79 72½	47 43·1	3·9
	Dashes.	72 p.p. E.	1·666 0·833	112 100	— —	112 100	66·6 59·5	7·1
	Dots and dashes.	72 p.p. E. 72 p.p. E.	0·833 0·833 1·666 0·833	97 85 105 95	— — — —	97 85 105 95	57·7 50·6 62·5 56·5	
60	Dots.	72 p.p. E.	1 1	80 70	85 75	82½ 72½	49 43·1	5·9
	Dashes.	72 p.p. E.	2 1	120 100	— —	120 100	71·4 59·5	11·9
	Dots and dashes.	72 p.p. E. 72 p.p. E.	1 1 2 1	99 85 110 95	95 80 112 90	97 82½ 111 92½	57·7 49 66 55	
50	Dots.	72 p.p. E.	1·2 1·2	83 67	80 65	81½ 66	48·5 39·3	9·2
	Dashes.	72 p.p. E.	2·4 1·2	120 90	125 95	122½ 92½	72·9 55	17·9
	Dots and dashes.	72 p.p. E. 72 p.p. E.	1·2 1·2 2·4 1·2	95 80 115 85	100 85 120 90	97½ 82½ 117½ 87½	58 49 69·9 52	
40	Dots.	72 p.p. E.	1·5 1·5	90 65	95 70	92½ 67½	55 40·2	14·8
	Dashes.	72 p.p. E.	3·0 1·5	125 85	130 90	127½ 87½	75·9 52	23·9
	Dots and dashes.	72 p.p. E. 72 p.p. E.	1·5 1·5 3·0 1·5	100 72 120 82	102 80 125 85	101 76 122½ 83½	60·1 45·2 72·9 49·7	

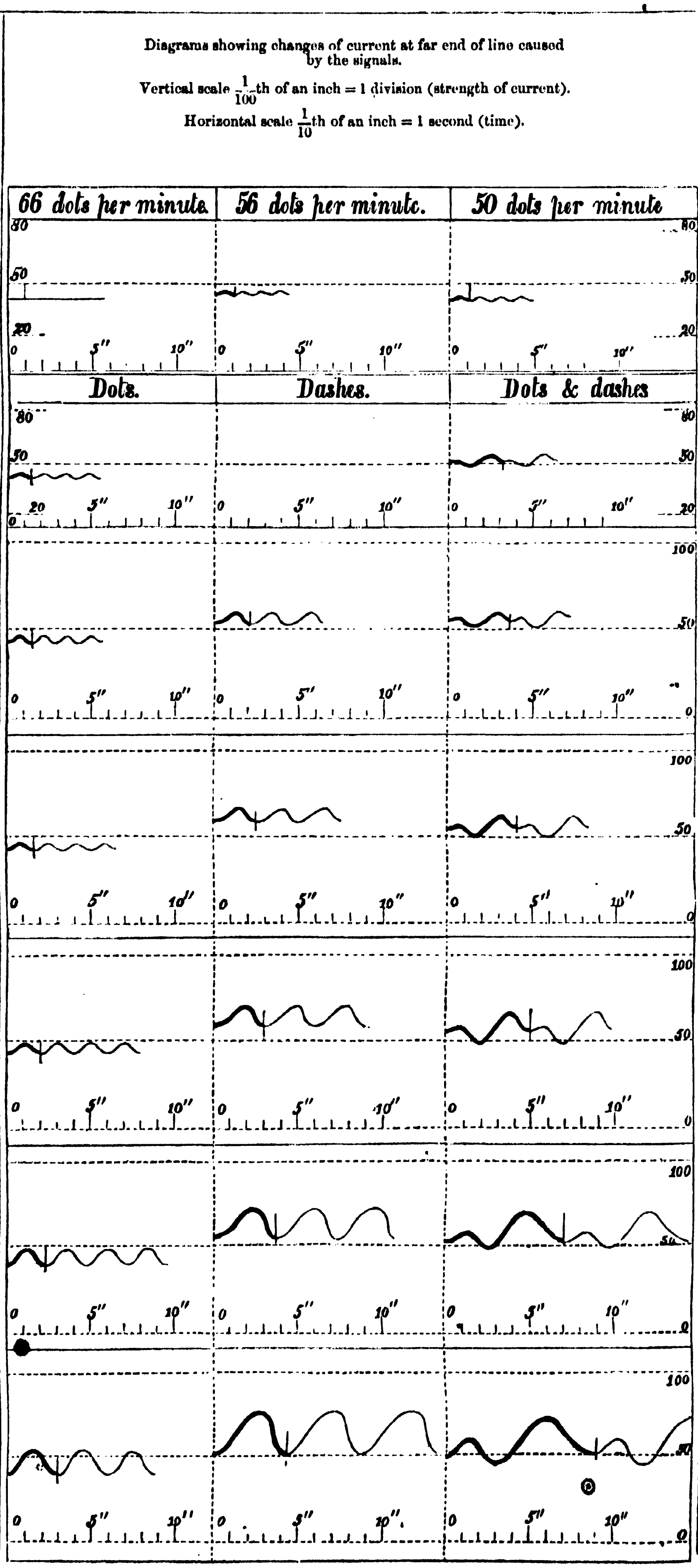


TABLE VII.—Arrival-curve for 1006 knots, in 4 coils. July 29, 1859.
Maximum deflection caused by permanent current from 72 p.p. . . . 277[d].

Seconds after making contact with battery or with earth	1¼	1½	2	2¼	3	3½	4½	4¼	4¾	6	7	9½	14½	19½
Division of scale passed by spot as current rises at far end of line	—	—	—	200	—	210	220	—	230	—	240	250	—	270
Division of scale passed by spot as current falls at far end of line	120	110	100	—	60	—	—	50	—	40	—	—	15	—
Distance on scale traversed by spot after contact has been made with battery or with earth	157	167	177	200	217	210	220	227	230	237	240	250	262	270
Reduced distance traversed by spot after contact has been made with battery or with earth	56·7	60·3	63·9	72·2	78·3	75·8	79·4	81·9	83	85·6	86·6	90·2	94·6	97·5

TABLE VIII.—Arrival-curve for 1165 knots, in 6 coils. July 29, 1859.
Maximum deflection caused by permanent current from 72 p.p. . . . 264[d].

Seconds after making contact with battery or with earth	2¾	3¾	3¾	4¼	7	9	10	17	23
Division of scale passed by spot as the current rises at far end of line	—	200	—	220	—	240	—	—	260
Division of scale passed by spot as the current falls at far end of line	70	—	60	—	30	—	20	10	—
Distance on scale traversed by spot after contact has been made with battery or with earth	194	200	204	220	234	240	244	254	260
Reduced distance traversed by spot after contact has been made with battery or with earth	73·5	75·8	77·3	83·4	88·7	91	92·5	96·3	98·5

TABLE IX.—Signals through 1165 knots, in 6 coils. July 29, 1859.
Maximum deflection caused by permanent current from 72 p.p. . . . 264[d].

Beats of metronome per minute.	Signals sent.	Source of electricity in contact with line at near end.	Duration of contact at near end.	Limit of deflection at far end. First observation.	Second observation.	Mean limit of deflection caused by contact.	Reduced mean limit of deflection caused by contact.	Reduced mean amplitude of oscillation caused by two contacts.
100	Dots.	72 p.p. E.	0″·6 0·6	130 100	125 95	127½ 97½	48·3 36·9	11·4
	Dashes.	72 p.p. E.	1·2 0·6	190 130	— —	190 130	72 49·3	22·7
	Dots and dashes.	72 p.p. E. 72 p.p. E.	0·6 0·6 1·2 0·6	150 110 180 120	145 105 180 120	147½ 107½ 180 120	55·9 40·7 68·2 45·5	
	A.	72 p.p. E. 72 p.p. E.	0·6 0·6 1·2 1·2	115 70 170 90	— — — —	115 70 170 90	43·6 26·5 64·4 34·1	

Diagrams showing changes of current caused at far end of line by the signals.
Vertical scale $\frac{1}{100}$th of an inch = 1 division (strength of current).
Horizontal scale $\frac{1}{10}$th of an inch = 1 second (time).

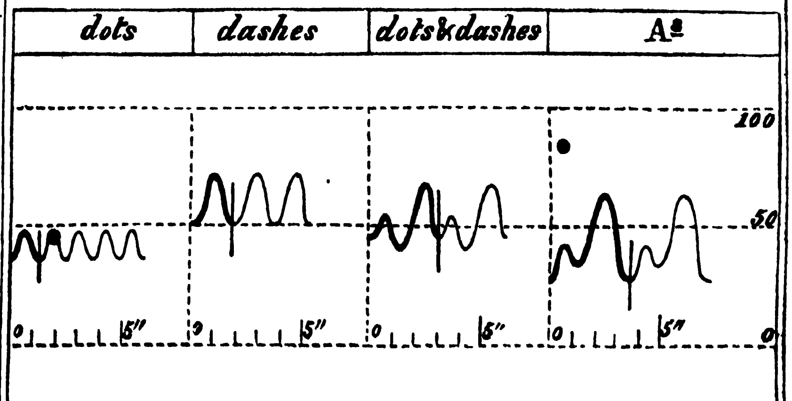


TABLE X.—Arrival-curve for 1165 knots, in 6 coils. July 29, 1859.

Maximum deflection caused by permanent current from +42 p.p.	. .	157^d
" " " " " −30 p.p.	. .	113^d
Sum of the two deflections	. . . .	270^d

Seconds after making contact with positive battery or with negative battery	3	4	4	4½	4½	7	9	9¾	13	14	17	18	25
Division of scale passed by spot as the current arises at far end of line, after contact has been made with positive battery	+90	—	+100	+110	—	—	+130	—	—	+145	—	+150	+155
Division of scale passed by spot as the current falls at far end of line, after contact has been made with negative battery	—	−60	—	—	−70	−80	—	−90	−100	—	−105	—	—
Distance on scale traversed by spot after contact with either battery	203	217	213	223	227	237	243	247	257	258	262	263	268
Reduced distance traversed by spot after contact has been made with either battery	75·1	80·3	78·8	82·5	84	87·7	89·9	91·4	95·1	95·5	96·9	97·3	99·2

TABLE XI.—Signals through 1165 knots, in 6 coils, with alternate positive and negative currents. July 29, 1859.

Maximum deflection caused by permanent current from +42 p.p.	. . .	157^d
" " " " " " " −30 p.p.	. . .	113^d
Sum of two deflections	. . . .	270^d

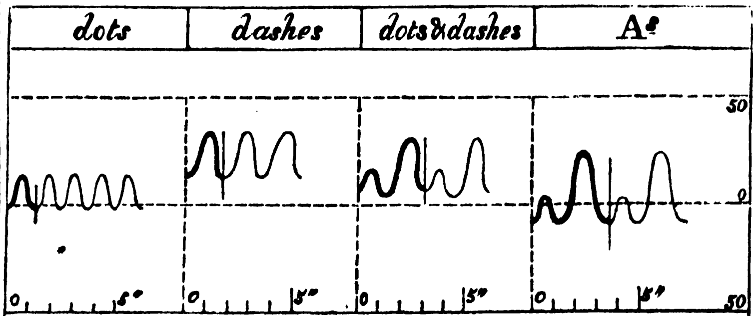
Beats of metronome per minute.	Signals sent.	Source of electricity in contact with line at near end.	Duration of contact at near end.	Limit of deflection at far end. First observation.	Second observation.	Mean limit of deflection caused by contact.	Reduced mean limit of deflection caused by contact.	Reduced mean amplitude of oscillation caused by two contacts.	Diagrams showing changes of current caused at far end of line by the signals. Vertical scale $\frac{1}{100}$th of an inch = 1 division (strength of current). Horizontal scale $\frac{1}{10}$th of an inch = 1 second (time).
100	Dots.	+42 −30	0·6 0·6	+35 0	+32 − 3	+33½ − 1½	+12·4 − 0·6	13	
	Dashes.	+42 −30	1·2 0·6	+95 +40	+95 +35	+95 +37½	+35·1 +13·9	21·2	
	Dots and dashes.	+42 −30 +42 −30	0·6 0·6 1·2 0·6	+40 +10 +80 +20	+55 +20 +85 +25	+47½ +15 +82½ +22½	+17·6 + 5·6 +30·5 + 8·3		
	A.	+42 −30 +42 −30	0·6 0·6 1·2 1·2	+10 −20 +65 −20	— — — —	+10 −20 +65 −20	+ 3·7 − 7·4 +24 − 7·4		

TABLE XII.—Arrival-curve for 2192 knots, in 10 coils. July 30, 1859.
Maximum deflection caused by permanent current from 72 p.p. ... 133[d].

Seconds after making contact with battery or with earth...	2½″	3″	3½″	4″	4½″	5″	5¾″	6″	6½″	7″	7¾″	8½″	8½″	9″	9¼″	10½″	11″	13¼″	16¾″	17″	30″	50″
Division of scale passed by spot as the current rises at far end of line	30	40	50	60	—	—	80	—	90	95	100	105	—	110	110	115	—	120	125	125	130	131
Division of scale passed by spot as the current falls at far end of line	—	—	—	—	70	60	—	50	—	—	—	—	30	—	—	—	20	—	—	—	—	—
Distance on scale traversed by spot after contact has been made with battery or earth..	30	40	50	60	63	73	80	83	90	95	100	105	103	110	110	115	113	120	125	125	130	131
Reduced distance traversed by spot after contact has been made with battery or earth..	22·6	30·1	37·6	45·1	47·4	54·9	60·2	62·4	67·7	71·4	75·2	79	77·5	82·7	82·7	86·5	85	90·2	94	94	97·8	98·5

TABLE XIII.—Signals through 2192 knots, in 10 coils. July 30, 1859.
Maximum deflection caused by permanent current from 72 p.p. ... 133[d].

Beats of metronome per minute.	Signals sent.	Source of electricity in contact with line at near end.	Duration of contact at near end.	Limit of deflection at far end. First observation.	Second observation.	Third observation.	Fourth observation.	Mean limit of deflection caused by contact.	Reduced mean limit of deflection caused by contact.	Reduced mean amplitude of oscillation caused by two contacts.
100	Dots.	72 p.p. E.	0·6″ 0·6	57 57	— —	— —	— —	57 57	42·9 42·9	0
80	Dots.	72 p.p. E.	0·75 0·75	61 59	— —	— —	— —	61 59	45·9 44·4	1·5
60	Dots.	72 p.p. E.	1 1	64 60	— —	— —	— —	64 60	48·1 45·1	3
40	Dots.	72 p.p. E.	1·5 1·5	68 58	— —	— —	— —	68 58	51·1 43·6	7·5
36	Dots.	72 p.p. E.	1·666 1·666	68 55	70 58	— —	— —	69 56½	51·9 42·5	9·4
	Dashes.	72 p.p. E.	2·333 1·666	95 74	96 75	— —	— —	95½ 74½	71·8 56·0	15·8
	Dots and dashes.	72 p.p. E. 72 p.p. E.	1·666 1·666 3·333 1·666	76 65 90 70	80 65 90 70	75 60 89 69	80 65 92 71	77 64 90 70	57·9 48·1 67·6 52·6	
	A.	72 p.p. E. 72 p.p. E.	1·666 1·666 3·333 3·333	60 50 85 45	63 53 85 47	60 51 84 45	60 51 85 46	61 51 85 46	45·9 38·4 63·9 34·6	
30	Dots.	72 p.p. E.	2 2	72 55	— —	— —	— —	72 55	54·1 41·4	12·7
	Dashes.	72 p.p. E.	4 2	100 70	— —	— —	— —	100 70	75·2 52·6	22·6
	Dots and dashes.	72 p.p. E. 72 p.p. E.	2 2 4 2	79 62 95 68	78 60 94 68	80 60 95 67	— — — —	79 61 95 68	59·2 45·9 71·4 51·1	
	A.	72 p.p. E. 72 p.p. E.	2 2 4 4	65 50 91 41	64 50 91 42	63 50 92 42	— — — —	64 50 91 42	48·1 37·6 68·4 31·6	
18	Dots.	72 p.p. E.	3·333 3·333	85 47	— —	— —	— —	85 47	63·9 35·3	28·6

Diagrams showing changes of current at far end of line caused by the signals.
Vertical scale $\frac{1}{100}$th of an inch = 1 division (strength of current).
Horizontal scale $\frac{1}{10}$th of an inch = 1 second (time).

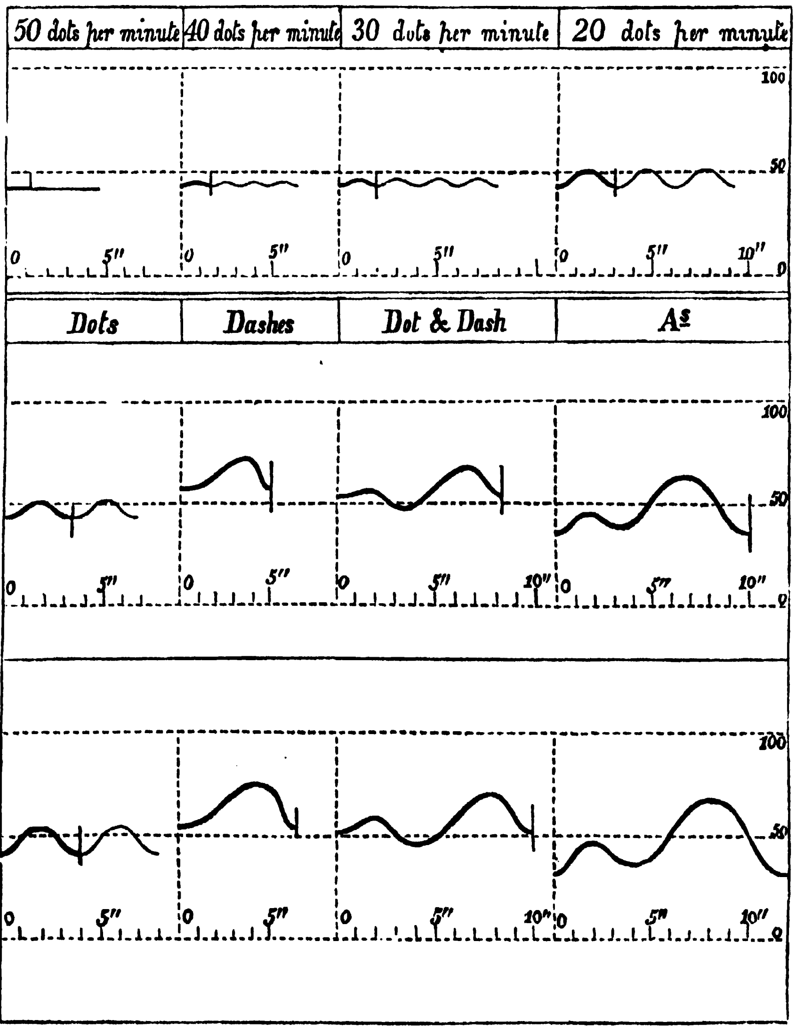


TABLE XIV.—Signals sent through 1812 knots, in 9 coils. July 30, 1859.

Maximum deflection caused by permanent current from 72 p.p. ... 166[d].

Beats of metronome per minute.	Signals sent.	Source of electricity in contact with line at near end.	Duration of contact at near end.	Limit of deflection at far end. First observation.	Second observation.	Mean limit of deflection caused by contact.	Reduced mean limit of deflection caused by contact.
92	Dots and dashes.	72 p.p.	0″·652	88	90	89	53·6
		E.	0·652	83	83	83	50
		72 p.p.	1·304	95	95	95	57·2
		E.	0·652	88	90	89	53·6
	A.	72 p.p.	0·652	66	—	66	39·7
		E.	0·652	65	—	65	39·1
		72 p.p.	1·304	85	—	85	51·2
		E.	1·304	65	—	65	39·1
84	Dots and dashes.	72 p.p.	0·714	91	—	91	54·8
		E.	0·714	85	—	85	51·2
		72 p.p.	1·429	100	—	100	60·2
		E.	0·714	90	—	90	54·2
	A.	72 p.p.	0·714	72	—	72	43·3
		E.	0·714	70	—	70	42·1
		72 p.p.	1·429	90	—	90	54·2
		E.	1·429	70	—	70	42·1
72	Dots and dashes.	72 p.p.	0·833	92	—	92	55·4
		E.	0·833	80	—	80	48·2
		72 p.p.	1·666	100	—	100	60·2
		E.	0·833	90	—	90	54·2
	A.	72 p.p.	0·833	75	75	75	45·1
		E.	0·833	70	70	70	42·1
		72 p.p.	1·666	95	100	97½	58·7
		E.	1·666	70	68	69	41·5
60	Dots and dashes.	72 p.p.	1	95	93	94	56·6
		E.	1	80	78	79	47·6
		72 p.p.	2	105	105	105	63·2
		E.	1	90	90	90	54·2
	A.	72 p.p.	1	70	—	70	42·1
		E.	1	65	—	65	39·1
		72 p.p.	2	100	—	100	60·2
		E.	2	60	—	60	36·1
50	Dots and dashes.	72 p.p.	1·2	95	—	95	57·2
		E.	1·2	75	—	75	45·1
		72 p.p.	2·4	110	—	110	66·2
		E.	1·2	85	—	85	51·2
	A.	72 p.p.	1·2	80	72	76	45·8
		E.	1·2	68	60	64	38·6
		72 p.p.	2·4	115	110	112½	67·7
		E.	2·4	60	55	57½	34·6
40	Dots and dashes.	72 p.p.	1·5	100	110	105	63·2
		E.	1·5	75	80	77½	46·7
		72 p.p.	3	120	130	125	75·3
		E.	1·5	85	90	87½	52·7
	A.	72 p.p.	1·5	80	75	77½	46·7
		E.	1·5	65	60	62½	37·6
		72 p.p.	3	115	115	115	69·2
		E.	3	50	45	47½	28·6

Diagrams showing changes of current caused at far end of line by the signals.

Vertical scale $\frac{1}{100}$th of an inch = 1 division (strength of current).

Horizontal scale $\frac{1}{10}$th of an inch = 1 second (time).

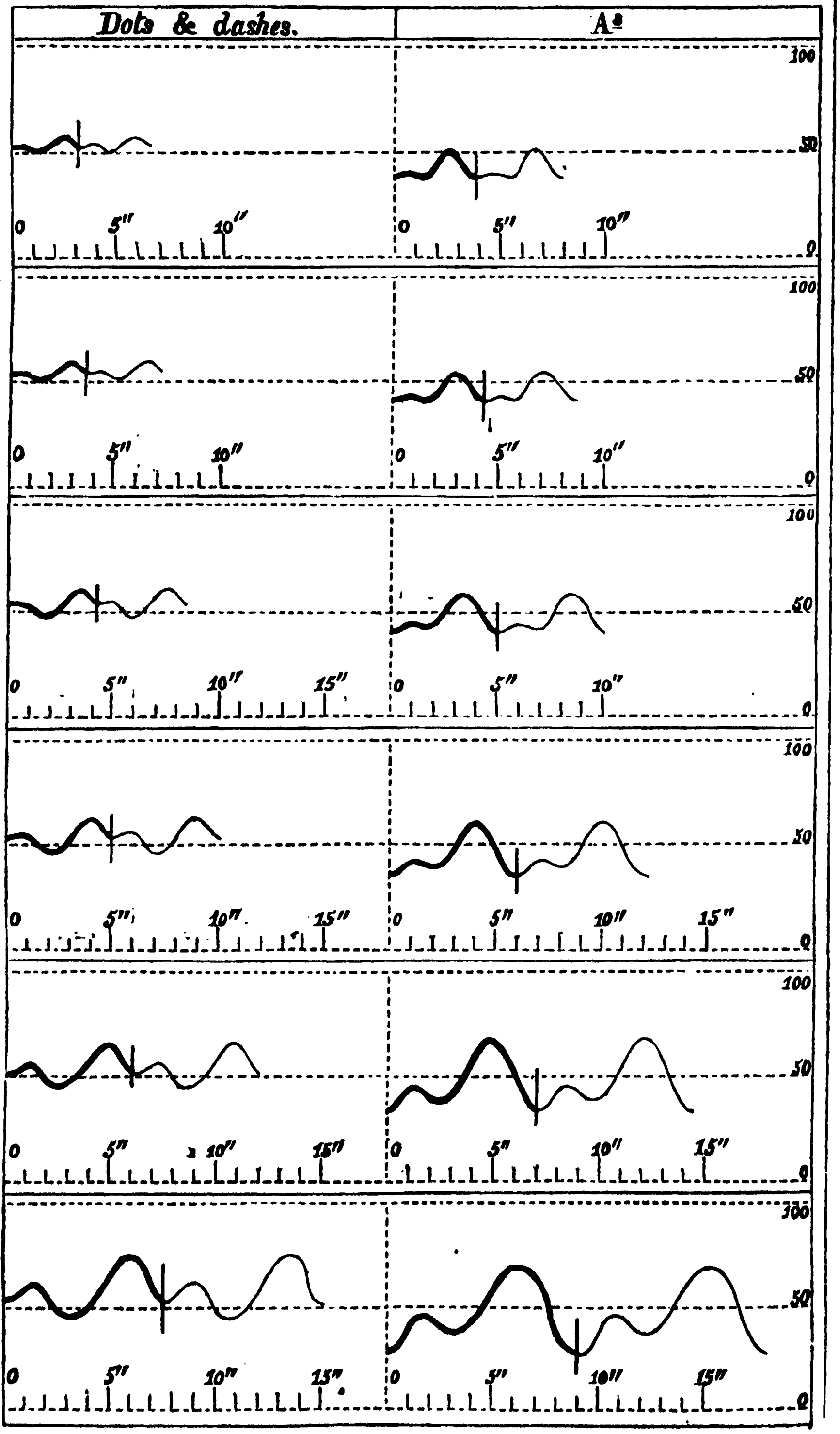


TABLE XV.—Speeds and Amplitudes for various lengths.

	1.	2.	3.	4.
	Number of beats per minute $=N$.	Observed amplitude reduced to percentage.	Product of number of beats into square of length $=N\times L^2$.	Product of number of beats into square of length corrected for battery and galvanometer $=N\times(L+160)^2$.
$L=1500$…	130	2·5	297×10^6	364×10^6
	92	5	207 "	254 "
	73	10	162 "	198 "
	60	12·5	135 "	165 "
$L=1802$…	132	0	429×10^6	508×10^6
	112	1·2	364 "	431 "
	100	1·8	325 "	385 "
	92	2·7	299 "	354 "
	84	3	273 "	323 "
	72	3·9	234 "	277 "
	60	5·9	195 "	231 "
	50	9·2	162 "	192 "
	40	14·8	130 "	154 "
$L=2192$…	100	0	480×10^6	553×10^6
	80	1·5	384 "	443 "
	60	3	288 "	332 "
	40	7·5	192 "	221 "
	36	9·4	173 "	199 "
	30	12·7	144 "	166 "
	18	28·6	86 "	100 "

APPENDIX.

Received January 14, 1863.

SECTION I.—*Effect of conduction across the sheath of an insulated wire, or of uniformly imperfect insulation on the permanent received current.*

The author is indebted to Professor W. THOMSON for the substance of the following theory.

Consider a cable extending infinitely in one direction from an origin O.

Let the distance of any point P from the origin be called x.

Let i denote the resistance of the unit length of the sheath to conduction across it, *i. e.* the measure of the insulation.

Let m denote the resistance of the unit length of the wire to conduction along it.

Let the potential at O be called V, and the potential at any point P be called V_x.

Let the strength of the current entering at O be called Q, and the strength of the current at P be called Q_x. Then it is not difficult to prove that

$$Q_x=Q\,\varepsilon^{-x\sqrt{\frac{m}{i}}}, \qquad (1.)$$

$$V_x=V\,\varepsilon^{-x\sqrt{\frac{m}{i}}}, \qquad (2.)$$

$$Q=\frac{V}{(im)^{\frac{1}{2}}}. \qquad (3.)$$

We can pass from this case to that of a finite cable of the length l with one end in connexion with the earth by the device of electrical images.

Superimpose two potential curves satisfying equation (2.) with equal but opposite potentials $+V$ and $-V$ at their origins O and O_1. Let these points be situated at a distance $2l$ apart, and let the curves extend from their origin to meet and cross one another. The resultant potential will necessarily be zero halfway between O and O_1, and the resultant curve between this point and each origin will represent the variation of potentials in a cable of the length l with one end in connexion with the earth and the origin at a certain positive or negative potential.

By putting these results into a mathematical form, we obtain from equations (1.) and (2.),

$$V_x=V\left(\varepsilon^{-x\sqrt{\frac{m}{i}}}-\varepsilon^{-(2l-x)\sqrt{\frac{m}{i}}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

$$Q_x=Q\left(\varepsilon^{-x\sqrt{\frac{m}{i}}}+\varepsilon^{-(2l-x)\sqrt{\frac{m}{i}}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

But V and Q do not represent the potential and current at the origin of the finite cable, but the potential and current at the origin of the hypothetical infinite curves superimposed. In order to obtain V_x and Q_x in function of any given potential V_0 at the origin of the finite cable, we must obtain the value of V in function of V_0 by putting $x=0$ in equation (4.), and substitute the value of V in function of V_0 in the general equations (4.) and (5.).

Then we have

$$V_0=V\left(1-\varepsilon^{2l\sqrt{\frac{m}{i}}}\right),$$

and hence

$$V_x=V_0\,\frac{\varepsilon^{-x\sqrt{\frac{m}{i}}}-\varepsilon^{-(2l-x)\sqrt{\frac{m}{i}}}}{1-\varepsilon^{-2l\sqrt{\frac{m}{i}}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

and

$$Q_x=\frac{V_0}{(im)^{\frac{1}{2}}}\,\frac{\varepsilon^{-x\sqrt{\frac{m}{i}}}+\varepsilon^{-(2l-x)\sqrt{\frac{m}{i}}}}{1-\varepsilon^{-2l\sqrt{\frac{m}{i}}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

To obtain the current flowing into the cable at its origin, make $x=0$; then

$$Q_0=\frac{V_0}{(im)^{\frac{1}{2}}}\,\frac{1+\varepsilon^{-2l\sqrt{\frac{m}{i}}}}{1-\varepsilon^{-2l\sqrt{\frac{m}{i}}}}$$

and

$$Q_x=Q_0\,\frac{\varepsilon^{-x\sqrt{\frac{m}{i}}}-\varepsilon^{-(2l-x)\sqrt{\frac{m}{i}}}}{1+\varepsilon^{-2l\sqrt{\frac{m}{i}}}};$$

or, for brevity, writing $\theta=l\sqrt{\frac{m}{i}}$ and $\theta_{\prime}=(l-x)\sqrt{\frac{m}{i}}$,

$$Q_0=\frac{V_0}{lm}\,\frac{\theta(\varepsilon^{\theta}+\varepsilon^{-\theta})}{(\varepsilon^{\theta}-\varepsilon^{-\theta})}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

and

$$Q_x=Q_0\,\frac{\varepsilon^{\theta_{\prime}}+\varepsilon^{-\theta_{\prime}}}{\varepsilon^{\theta}+\varepsilon^{-\theta}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

When $x=l$,

$$Q_l=Q_0\frac{2}{\varepsilon^{\theta}+\varepsilon^{-\theta}}. \qquad (10.)$$

The equations (8.) and (10.), applied to the case of the Red Sea cable, give the results in the text,

$$l=2192,\ \dots\ m=250\times10^{6}\ \dots\ i=228\times10^{13},$$

$$\varepsilon^{\theta}=\varepsilon^{0\cdot7258}=2\cdot0664\ \dots\ \varepsilon^{-\theta}=0\cdot48394,$$

$$Q_0=1\cdot1694\frac{V_0}{lm}=\text{the entering current},$$

$$Q_l=0\cdot784Q_0=0\cdot9168\frac{V_0}{lm}.$$

Similarly, when $i=289\times10^{13}$,

$$Q_l=0\cdot824Q_0=0\cdot938\frac{V_0}{lm}.$$

If the cable were perfectly insulated, by OHM's law,

$$Q_l=\frac{V_0}{lm}$$

SECTION II.—TABLE XVI.

1.	2.	3.	4.	5.	6.
Period of dot in seconds in function of α.	Factor used in the formula given in the third column.	Reciprocal of amplitude of variation in the received current produced by harmonic variation of potential at the operating end.	Reciprocal of amplitude of variation in the received current produced by the dot signal.	Amplitude produced by dot signals.	Number of dots per minute.
θ in function of α. $\alpha=\frac{ckl^2}{\pi^2}\times\log_\varepsilon\left(10^{\frac{1}{10}}\right)$.	$i=\sqrt{\frac{\theta}{\alpha}}\times\sqrt{\frac{\pi^2}{\left(\log_\varepsilon\frac{1}{e}\right)^2}}$ $e=10^{\frac{1}{10}}$.	$\left\{2\sqrt{2}.\,i\log_\varepsilon\left(\frac{1}{e}\right)e^{i}\right\}^{-1}$.	Third column multiplied into $\frac{\pi}{4}$.	Reciprocal of fourth column multiplied by 100.	$\frac{60}{\theta}$.
$0\cdot25\times\alpha$	100·7926	182,840,000	143,600,000	0·000,000,696	240 $\times\frac{1}{\alpha}$
0·5 „	71·2711	288,070	226,250	0·000,442	200 „
0·75 „	58·1926	17,402	13,668	0·007,315	80 „
1·0 „	50·3963	3,337·9	2,631·5	0·038,00	60 „
1·1 „	48·0510	2,040·0	1,602·2	0·062,42	54·5 „
1·2 „	46·0053	1,330·3	1,044·8	0·095,7	50·0 „
1·3 „	44·2005	913·82	717·71	0·139,3	46·2 „
1·4 „	42·5926	654·97	514·41	0·194,5	42·9 „
1·5 „	41·1496	486·24	381·89	0·261,8	40·0 „
1·6 „	39·8418	371·60	291·86	0·342,6	37·5 „
1·7 „	38·6522	291·26	228·76	0·437,1	35·3 „
1·8 „	37·5632	233·24	182·40	0·548,2	33·3 „
1·9 „	36·5617	190·28	149·44	0·669,3	31·6 „
2·0 „	35·6356	157·73	123·88	0·807,1	30·0 „
2·1 „	34·7777	132·65	104·17	0·959,7	28·6 „
2·2 „	33·9782	112·90	88·67	1·127,8	27·3 „
2·3 „	33·2299	97·21	76·35	1·309,8	26·1 „
2·4 „	32·5305	84·52	66·38	1·506,5	25·0 „
2·5 „	31·8742	74·17	58·25	1·716,7	24·0 „
2·6 „	31·2535	65·57	51·50	1·941,7	23·1 „
2·7 „	30·6696	58·41	45·88	2·179,6	22·2 „
2·8 „	30·1180	52·39	41·15	2·430,1	21·4 „

TABLE XVI. (continued).

1.	2.	3.	4.	5.	6.
Period of dot in seconds in function of a.	Factor used in the formula given in the third column.	Reciprocal of amplitude of variation in the received current produced by harmonic variation of potential at the operating end.	Reciprocal of amplitude of variation in the received current produced by the dot signal.	Amplitude produced by dot signals.	Number of dots per minute.
θ in function of a. $a=\frac{ckl^2}{\pi^2}\times\log_\epsilon\left(10^{\frac{1}{10}}\right)$.	$i=\sqrt{\frac{\theta}{a}}\times\sqrt{\frac{\pi^3}{\left(\log_\epsilon\frac{1}{\epsilon}\right)^3}}$ $\epsilon=10^{\frac{1}{10}}$.	$\left\{2\sqrt{2}\,.\,i\log_\epsilon\left(\frac{1}{\epsilon}\right)\epsilon^i\right\}^{-1}$.	Third column multiplied into $\frac{\pi}{4}$.	Reciprocal of fourth column multiplied by 100.	$\frac{60}{\theta}$.
2·9 × a	29·5944	47·26	37·12	2·694,0	20·7 × $\frac{1}{a}$
3·0 ,,	29·0964	42·86	33·66	2·970,9	20·0 ,,
3·1 ,,	28·6232	39·07	30·69	3·195,9	19·4 ,,
3·2 ,,	28·1724	35·78	28·10	3·558,7	18·7 ,,
3·3 ,,	27·7422	32·91	25·85	3·868,5	18·2 ,,
3·4 ,,	27·3312	30·39	23·87	4·189,4	17·6 ,,
3·5 ,,	26·9380	28·16	22·12	4·520,8	17·1 ,,
3·6 ,,	26·5612	26·19	20·57	4·861,5	16·7 ,,
3·7 ,,	26·1998	24·43	19·19	5·211,0	16·2 ,,
3·8 ,,	25·8528	22·86	17·53	5·704	15·8 ,,
3·9 ,,	25·5192	21·44	16·84	5·938	15·4 ,,
4·0 ,,	25·1981	20·17	15·84	6·313	15·0 ,,
4·1 ,,	24·8900	19·02	14·94	6·693	14·6 ,,
4·2 ,,	24·5909	17·97	14·11	7·087	14·3 ,,
4·3 ,,	24·3033	17·02	13·37	7·479	14·0 ,,
4·4 ,,	24·0255	16·15	12·68	7·886	13·6 ,,
4·5 ,,	23·7570	15·35	12·06	8·292	13·3 ,,
4·6 ,,	23·4974	14·62	11·48	8·711	13·0 ,,
4·7 ,,	23·2461	13·95	10·96	9·124	12·8 ,,
4·8 ,,	23·0021	13·33	10·47	9·551	12·5 ,,
4·9 ,,	22·7667	12·75	10·01	9·990	12·2 ,,
5·0 ,,	22·5379	12·22	9·60	10·417	12·0 ,,
5·1 ,,	22·3158	11·73	9·21	10·858	11·8 ,,
5·2 ,,	22·1002	11·27	8·85	11·299	11·5 ,,
5·3 ,,	21·8907	10·84	8·51	11·751	11·3 ,,
5·4 ,,	21·6871	10·44	8·20	12·195	11·1 ,,
5·5 ,,	21·4890	10·07	7·91	12·642	10·9 ,,
5·6 ,,	21·2963	9·72	7·634	13·099	10·7 ,,
5·7 ,,	21·1087	9·39	7·375	13·559	10·5 ,,
5·8 ,,	20·9259	9·08	7·131	14·023	10·4 ,,
5·9 ,,	20·7478	8·79	6·904	14·484	10·2 ,,
6·0 ,,	20·5742	8·52	6·732	14·854	10·0 ,,
6·1 ,,	20·4049	8·260	6·487	15·415	9·8 ,,
6·2 ,,	20·2396	8·017	6·296	15·883	9·66 ,,
6·3 ,,	20·0784	7·786	6·115	16·353	9·52 ,,
6·4 ,,	19·9209	7·569	5·944	16·824	9·37 ,,
6·5 ,,	19·7670	7·362	5·782	17·295	9·23 ,,
6·6 ,,	19·6167	7·166	5·628	17·590	9·09 ,,
6·7 ,,	19·4698	6·980	5·482	18·241	8·96 ,,
6·8 ,,	19·3261	6·803	5·443	18·372	8·82 ,,
6·9 ,,	19·1855	6·635	5·211	19·190	8·69 ,,
7·0 ,,	19·0480	6·474	5·085	19·666	8·57 ,,
7·1 ,,	18·9136	6·322	4·965	20·141	8·45 ,,
7·2 ,,	18·7817	6·176	4·851	20·614	8·33 ,,
7·3 ,,	18·6525	6·036	4·701	21·272	8·22 ,,
7·4 ,,	18·5260	5·903	4·636	21·575	8·11 ,,
7·5 ,,	18·4021	5·775	4·536	22·046	8·00 ,,
7·6 ,,	18·2807	5·652	4·439	22·528	7·90 ,,
7·7 ,,	18·1616	5·537	4·349	22·994	7·80 ,,
7·8 ,,	18·0448	5·425	4·260	23·474	7·69 ,,
7·9 ,,	17·9302	5·317	4·176	23·946	7·60 ,,
8·0 ,,	17·8178	5·214	4·095	24·420	7·50 ,,
9·0 ,,	16·7988	4·374	3·435	29·112	6·67 ,,
10·0 ,,	15·9367	3·780	2·969	33·681	6·00 ,,

Explanation of TABLE XVI.

Table XVI., given to the author by Professor W. THOMSON, will now be shortly explained. The author will not enter into any detailed explanation of the theory by which the results contained in the Table were obtained, as these results are the direct mathematical consequences of the equations given in the papers already alluded to, and as Professor THOMSON will probably himself publish the full mathematical development of the theory.

The first column headed Θ contains a series of "times" occupied by the full periods of electric operations, each of which, when continually repeated, produces a succession of equal and similar rises and falls in the received current, or of "dots" as they have been hitherto called in this paper. The series begins with the shorter and ends with the longer times, or, in other words, begins with the more rapid and ends with the slower speeds. The numbers in the first column are numbers measuring the times of the periods in terms of a certain quantity α taken as unity. The actual time in seconds occupied by each period or cycle of electric operations corresponding to the series in the first column is equal to the numbers entered there multiplied into α. This quantity α is equal to $\frac{ckl^2}{\pi^2}\cdot\log_\epsilon\left(10^{\frac{1}{10}}\right)$, where

c = the electrostatical capacity of the insulated wire per unit of length in absolute electrostatic measure;

k = the resistance of the conductor per unit of length in absolute electrostatic measure;

l = the length of the conductor.

α varies therefore for every cable and for every length of the same cable. The meaning of this quantity will be best explained by the following extract from a letter of Professor THOMSON'S to the author:—

"α, in definite absolute measure, means the tenth part of the time in seconds in which a simple harmonic electrification established in the wire and left to itself (two ends to earth) would subside to $\frac{1}{10}$th of its amount. Thus in the time α, an harmonic electrification subsides to $\frac{1}{10^{\frac{1}{10}}}$ of its amount. The subsidence here spoken of is the gradual loss of charge by conduction out through the ends to earth."

Now let the electrical operation producing the dot be a simple harmonic variation of the potential at one end while the other is connected to earth $\left(i.\,e.\text{ let } V=A\sin\frac{2\pi t}{\Theta}\right.$, where V= the varying potential, A= a constant, and $\frac{t}{\Theta}$ = any function of the whole period$\left.\right)$. Then, measured as a fraction of the maximum current which would be received if the maximum potential A were constantly maintained at the sending end of the cable, the difference between the maximum and minimum received current will be the reciprocal of the number entered in the third column; or translating this into the language used in the paper, the reciprocals of the numbers in the third column give the amplitude of the dot measured as a function of the maximum permanent current*. For instance, if each dot occupied a period α, the amplitude of variation in the received current would be $\frac{1}{3338}$th of the maximum permanent current, or would be 0·02956 per cent. of that current.

The fourth column is obtained by multiplying the numbers in the third column into a constant $\frac{\pi}{4}$. The numbers so obtained express the reciprocals of the amplitudes which would result if, instead of being subjected to the harmonic variation previously described, the end operated on had been maintained at the maximum potential for the first half of the periods Θ, and at the minimum during the second half. This was precisely the condition fulfilled when dots were sent in the experiments described. The potential of one pole of the battery was maintained at the sending end of the cable during one half of each dot, and the potential of the earth was maintained during the other half.

The fifth column contains the product of the reciprocals of the numbers in the fourth column multiplied into 100, and gives therefore the amplitudes produced by dots made as in the experiments occupying the various periods in the first column, and these amplitudes are moreover expressed in percentages of the maximum received current, as in the rest of the paper.

The product of the reciprocals of the times entered in the first column, multiplied into 60, will give a series of numbers corresponding to the number of dots per minute in terms of α. This series, expressing "speeds," is entered in the sixth column.

The numbers in the fifth and sixth columns used as coordinates give the curve, shown in fig. 11, Plate LI., corresponding most accurately with the observed speeds and amplitudes.

The scale of amplitudes, shown in full lines, corresponds to the numbers in the fifth column, and the scale of times, shown by dotted lines, corresponds to the values of $\frac{60}{\Theta}$ when α is taken as unity.

When in any case the amplitude corresponding to a given number of dots is known, the actual value of α can at once be determined from this curve. Thus observation (Table IV.) showed that for a length of 1500 knots of cable +160 knots of resistance, with 92 beats or 46 dots per minute, the amplitude was 5 per cent., and by the curve we find that this amplitude corresponded to a speed of $16{\cdot}4\times\frac{1}{\alpha}$; hence $46=16{\cdot}4\times\frac{1}{\alpha}$, or $\alpha=0''{\cdot}3565$†. From this value the electrostatical capacity per unit of length, and the specific inductive capacity of the dielectric could be determined. These points will, however, be more fully treated of in the second part of this paper.

* The second column only contains the value of a certain quantity i used in the formula by which the third column is calculated.

† It should be observed that inasmuch as this observation does not exactly fall on the curve, so the value of α differs a little from that which would be calculated by the curve alone, as presently to be described.

Quite similarly, if α be known, the number of beats per minute corresponding to any amplitude could be determined; but, practically, α need not enter into the calculation when treating of any given cable. The speed, amplitude, and length are here the three elements of every problem, and when two of these are known the third can be determined; but here it may be observed, that as the speed, multiplied into α, is constant for each amplitude, so will the speed, multiplied into the square of the length, be constant for each amplitude, and the scale of abscissæ may be so chosen as for any one cable to give directly this product by simple inspection.

It is this scale for the Red Sea cable which is drawn at the foot of the curve, fig. 11, and which enables the number of dots corresponding to every amplitude to be ascertained directly, and it is by this scale that the dots, crosses, or circles from Table XV. are put on the figure.

When, as in the present paper, the speed is taken as twice the number of dots, and the unit length is one knot, the ratio of the two scales must clearly be such that if

d = the number of divisions in the upper scale,

D = the corresponding number on the lower scale,

L = the length in knots,

then $$\alpha = \frac{2dL^2}{D}.$$

Thus, taking a length of 1000 knots and a speed of 100 dots per minute, $D = 2 \times 10^8$, $d = 12{\cdot}6$, and hence $\alpha = 0{\cdot}126$; and the same value would be obtained whatever number of dots had been chosen.

This may be looked on as the mean value of α determined from twenty observations, since this ratio of the scales brought all the various circles, crosses, and stars into the closest approximation with the curve. The values of α for any other length are inversely proportional to the square of the lengths.

The algebraic headings of the different columns will allow them to be still further extended by those who may require to use the Table, as they virtually contain the equation of the curve.

XXXIX. *The Lignites and Clays of Bovey Tracey, Devonshire.*

By William Pengelly, *F.G.S.* *Communicated by Sir* Charles Lyell, *F.R.S.*

Received November 16,—Read November 21, 1861.

The little town or village of Bovey Tracey, in Devonshire, nestles at the foot of Dartmoor, very near its north-eastern extremity; it is situated on the left bank of the river Bovey, about two miles and a half above the point at which it falls into the Teign, and is about eleven miles from each of the towns Exeter, Torquay, and Totnes*,—bearing south-westerly from the first, north-westerly from the second, and northerly from the last.

A considerable plain stretches away from it in a south-easterly direction, having a length of six miles from a point about a mile west of Bovey to another nearly as far east of Newton; its greatest breadth, from Chudleigh Bridge on the north-east to Blackpool on the south-west, is four miles. It forms a lake-like expansion of the valleys of the Teign and Bovey rivers, especially the latter, whose course it may be said to follow in the higher part, where it is most fully developed; whilst the Teign constitutes its axis below the junction of the two streams. Its upper, or north-western portion, immediately adjacent to the village, is known as "Bovey Heathfield," and measures about 700 acres.

On its west and north-west, rise the lofty granite hills of Dartmoor, with their border of metamorphic rocks; on the north, the trappean elevations of Hennock; on the north-east and east, the Greensands of the Haldons, and the traps and limestones of the Chudleigh and Kingsteignton districts; and on the south, the traps, Devonian limestones, and associated rocks extending from Newton towards Ashburton†.

Contrasted with this rugged and elevated country, the so-called "plain" is not without some claim to the appellation, though by no means characterized by evenness of surface.

Shafts and other excavations have shown that the deposits in this basin consist of an accumulation of coarse gravel (mixed with sand and clay), of variable thickness, unconformably covering distinct strata of lignite, clay, and sand, which are familiar to geologists as the "Bovey deposit," whilst the lignite is equally well known as "Bovey coal."

This deposit not only occupies the plain which has been described, but is continued, in a narrow southerly prolongation, from Newton to near Kingskerswill, about three and a half miles from Torbay. This entire prolongation is a divergence from the Teign. Where it crosses the estuary of that river it is about four miles from the coast.

The most important of the excavations is that known as the "Coal-pit," which is situated on the Heathfield, somewhat less than a mile south of the village, and about the same distance from the western margin of the deposit. It is open to the day, and

* The distances throughout are measured in straight lines on the Ordnance Maps.

† See Map, Plate LII.

is, in form, a rude parallelogram, having its longest side, about 960 feet, in the direction from S. 75° E. to N. 75° W., whilst the shortest measures 340 feet, and has a bearing of N. 35° E. to S. 35° W.* Its greatest depth, at the western end, is nearly 100 feet.

Subterranean excavations have been carried on very extensively, in various directions, by means of tunnels opening out of the pit at its bottom. At present the working is confined to one tunnel, extending 190 fathoms, almost in a straight line, in the direction N. 65° W. from the western end of the pit.

The lignite was formerly used in large quantities in an adjoining pottery; at present but little is employed there, and its use is almost entirely confined to the poorer cottagers of the immediate district. A very offensive sulphurous smell, which it emits during combustion, prevents its general domestic use.

The refuse matter, consisting of clay and waste or valueless lignite, is lodged on the surface around the pit. Iron-pyrites occurs in it in considerable quantities; and spontaneous combustion is common in fresh refuse, especially after much rain. The fire is not generally visible near the surface in the day-time, but its presence is indicated by smoke and the very offensive odour previously mentioned. Cracks, having their sides lined with flowers of sulphur, cross the burning mass in various directions. Occasionally, crystals of sulphate of alumina are also formed.

The attention of both the scientific and the commercial world has long been called to this deposit; several accounts of it have been laid before various learned societies, and otherwise given to the world†. Many of these, besides descriptions of the characters

* The bearings are in all cases magnetic when expressed, as above, with an appearance of numerical exactness.

† The following is a list of the principal writers on the Bovey beds:—

Rev. Dr. JEREMIAH MILLES, in the Philosophical Transactions, vol. li. Part II. p. 534, &c., in 1760.

Mr. KIRWAN, in his 'Elements of Mineralogy,' vol. ii. p. 60, &c., published in 1794.

Dr. MATON, in his 'Observations on the Western Counties of England,' made in the years 1794 and 1796, vol. i. p. 106, &c.

Mr. HATCHETT, in the Transactions of the Linnean Society, vol. iv. p. 129, &c., in 1797.

Mr. BRICE, in his 'History and Description, Ancient and Modern, of the City of Exeter,' p. 141, &c., published in 1802.

Mr. PARKINSON and Mr. SCAMMELL, in PARKINSON'S 'Organic Remains,' vol. i. p. 104, &c., in 1804.

Mr. HATCHETT, in the Philosophical Transactions, vol. for 1804, Part I. p. 396, &c.

Mr. VANCOUVER, in his 'Agriculture of the County of Devon,' p. 70, &c., published in 1808.

Dr. J. McCULLOCH, in the Transactions of the Geological Society, 1st Series, vol. ii. p. 1, &c., in 1814.

Rev. D. LYSONS, in 'Magna Britannica,' vol. vi., "Devonshire," p. ccxlix, published in 1822.

Rev. W. D. CONYBEARE and Mr. W. PHILLIPS, in the 'Outlines of the Geology of England and Wales,' pp. 328 and 364, published in 1822.

Mr. KINGSTON, in the 'Teignmouth Guide,' vol. ii., published about 1832 or 1833.

Mr. GODWIN-AUSTEN, in the Transactions of the Geological Society, 2nd Series, vol. vi. part 2, p. 439, &c., in 1834, and subsequently.

Sir H. DE LA BECHE, in his 'Report of Cornwall, Devon, &c.' pp. 248, 255, 515, &c., in 1839.

Mr. F. VAUX, in Quart. Journ. Chem. Soc., London, vol. i. p. 318, &c., in 1849.

Dr. HOOKER, in Quart. Journ. Geol. Soc. vol. xi. p. 566, &c., in 1855.

Dr. CROKER, in Quart. Journ. Geol. Soc. vol. xii. p. 354, in 1856.

and arrangement of the strata, contain speculations and discussions on various topics, especially the mineral or vegetable origin of the lignite, the mode in which the materials of the deposit were accumulated, and the place of the formation in the chronological series of the geologist. These may be said to have resulted in a settled conviction that the lignite is of vegetable origin, that the clays and sands had been furnished by the disintegration of the Dartmoor granite, and that the whole is of supracretaceous age—and a general belief that the plants had not grown on, but had been transported to, the area they now occupy. The exact chronology of the formation was by no means agreed on, further than that there seemed to be a prevalent but vague opinion that, geologically speaking, it was very modern.. The only definite expression that had been given on the question was the provisional one that the deposit belonged to the Post-pliocene epoch; this was based on a cone said to have been found in one of the uppermost beds of lignite, and which was identified by Dr. HOOKER as belonging to the Scotch fir (*Pinus sylvestris*)*. This and some small seed-vessels, described by Dr. HOOKER under the name of *Folliculites minutulus*, were the only identifiable fossils which, prior to the late explorations, had been found at Bovey. Indeed, so recently as 1839, Sir H. DE LA BECHE stated that, "excepting the lignite itself, *no* organic remains had been found in the deposit"†. Many geologists, however, were unwilling to accept this chronology as conclusive, nor were they without hopes that, on a careful and thorough examination being made, the beds might be found to contain fuller and more reliable evidence on the question.

During the spring of 1860, Dr. FALCONER made several visits to Bovey and various localities in its neighbourhood where clay-works were, or had been, carried on. The result was a strong impression that the deposit would be found to belong to the Miocene period. In one of these visits he was accompanied by the Rev. R. EVEREST, F.G.S., and in another by Sir C. LYELL, when they had an opportunity of examining the large collection of specimens of the lignite made by the late Dr. CROKER. Soon afterwards Dr. FALCONER introduced the subject to Miss BURDETT COUTTS as one which, for the credit of British geology, it was eminently desirable to have very fully investigated. After a visit to the Bovey "Coal-pit," Miss COUTTS, with characteristic liberality, furnished me with means to undertake the work. I received the most prompt and cordial co-operation from the proprietor, JOHN DIVETT, Esq., and was so fortunate as to secure the services of Mr. H. KEEPING, the well-known fossil-collector, of the Isle of Wight.

The lignite-beds, having suffered less from the weather than the interstratified clays and sand, stand out in relief, like a series of rude mouldings, on the wall of the pit, especially on its southern side; so that it is not difficult to make out, in a rough way, the succession of the beds. Nevertheless the clay and sand have been so much washed over the surface of the wall that it is impossible to do more than this; hence it was decided to make a fresh section—in fact, to cut a series of steps, on a large scale, by which to descend the face of the artificial cliff from top to bottom, and thereby accomplish the double work of collecting fossils and disclosing the geology of the deposit.

* Quart. Journ. Geol. Soc. vol. xi. p. 566, &c.

† Report of Cornwall, Devon, &c., p. 257.

As we descended, the thickness of each bed and the amount and direction of its *dip* were carefully measured; a sample, and when necessary more than one, of every bed was taken, each in a separate box, and every important fact, as to the character of the bed and the occurrence or not of fossils in it, was carefully noted. The mean of the several measurements, all very near the average, gave a *dip* of 12½° towards S. 35° W. (magn.).

The results of this systematic exploration are exhibited below.

Section 1, *of the Bovey Deposit, in the south wall of the "Coal-pit," near its western end. Dip* 12½° *towards S.* 35° *W. (magn.).*

Beds.	Thickness. ft.	Thickness. in.	Totals. ft.	Totals. in.	
1	7	6	7	6	SANDY CLAY.—Contains a large number of angular and subangular stones, without anything like regularity in their arrangement. This is locally termed "The Head." The upper two inches is a peaty soil, also containing stones.
2	2	6	10	0	CLAY.—Plastic. Contains a few fragments of lignite. The uppermost four and the lowest two inches are of a buff colour; the middle band dark, approaching to black.
3	6	3	16	3	SAND.—Quartzose, with a ferruginous clay at the base.
4	2	9	19	0	CLAY.—The uppermost ten inches more or less dark, sometimes approaching to black, in colour.
5		7	19	7	LIGNITE.—Woody and brittle.
6		11	20	6	CLAY.—Very dark. Contains much broken lignite.
7	1	3	21	9	LIGNITE.—In some places woody, in others a mass of *Sequoia Couttsiæ*, HEER, and fern débris.
8		5	22	2	CLAY.—Dark. Some broken lignite. Graduates into sand at the base.
9	2	0	24	2	SAND.
10	2	0	26	2	CLAY.—Tough. Light lead-colour. Contains lenticular patches of sand, and, at the base, much fragmentary lignite.
11		8	26	10	SAND.—Sometimes ferruginous; in some cases cemented into a coarse grit or very fine conglomerate.
12	2	6	29	4	CLAY.—Light lead-colour in the upper part, darker towards the base. Contains fragments of lignite.
13	1	0	30	4	LIGNITE.—Woody, loose, very brittle. The bed ill-defined, graduating into clay at each surface.
14	2	9	33	1	CLAY.—Sandy and brittle. Contains a few fragments of lignite near the base. The uppermost and lowest parts of the bed are dark, the middle lighter, in colour.
15		7	33	8	LIGNITE.—Woody, loose, and brittle.
16	4	0	37	8	CLAY.—Rather light in colour towards the top. Contains two bands of almost continuous broken lignite, one twelve, the other thirty inches above the base, the uppermost being the least persistent. The clay is not laminated; it breaks into irregular-shaped fragments, generally quite angular and with plane faces.
17	1	5	39	1	LIGNITE.—The lowest part of this bed abounds, in some places, with dicotyledonous leaves; where they do not occur, the lignite is very woody.
18	2	9	41	10	CLAY.—Not laminated. Light in colour. Contains fragments of lignite.

Beds.	Thickness. ft.	in.	Totals. ft.	in.	
19	2	1	43	11	LIGNITE.—Contains much clay in almost continuous bands.
20	1	2	45	1	CLAY.—Laminated. Brittle. Rich in fragments of lignite.
21	3	4	48	5	LIGNITE.—Hard. Brittle. Broken. Contains a few seeds.
22	2	0	50	5	CLAY.—Dark. Rather brittle. Coarsely laminated.
23	1	0	51	5	LIGNITE.—Contains much clay.
24	1	0	52	5	CLAY.—Dark. Brittle. Contains a few portions of lignite.
25	6	2	58	7	LIGNITE.—Contains patches of dark clay. Seeds, flat in form, occur in the uppermost portion. The base is commonly a mass of large leaf-like forms and fronds of ferns; the former are Dr. CROKER's "flabelliform leaves"*.
26	2	1	60	8	CLAY.—Light drab colour. Extremely rich in fossil stems, leaves, and fruits of *Sequoia Couttsiæ*, seeds of various kinds, and dicotyledonous leaves. An almost continuous band of broken lignite occurs at the base of the bed.
27	11	1	71	9	SAND.—Quartzose. Very coarse in the uppermost part, but becomes gradually finer towards the base. Contains somewhat large lenticular patches of clay. Ferruginous stains and bands are common.
28	5	9	77	6	CLAY.—Light colour. Near the top it is somewhat sandy.
29	3	2	80	8	CLAY.—Dark. Contains a considerable number of fragments of lignite.
30	1	0	81	8	LIGNITE.—Rather brittle. Not very woody. Contains ferns.
31		8	82	4	CLAY.—Very dark. Some broken lignite.
32		11	83	3	LIGNITE. Very woody, tough, and extremely hard.
33	2	2	85	5	CLAY.—Dark lead-colour. Contains broken lignite.
34		10	86	3	LIGNITE.—Very woody and rather tough. Contains seeds.
35	2	2	88	5	CLAY.—Dark lead-colour. Contains pieces of lignite lying at all angles to the plane of the bed.
36		1	88	6	LIGNITE.
37	1	11	90	5	CLAY.—Very dark lead-colour. Very brittle. Rich in pieces of lignite; the thickness of the bed somewhat variable.
38		4	90	9	LIGNITE.—Woody, and rather tough.
39		10	91	7	CLAY.—Readily falls into fragments, which generally have plane, but sometimes curved surfaces. Contains pieces of lignite.
40	1	0	92	7	LIGNITE.—This bed is very uniform in thickness, and well-defined at each surface. Very compact and rather tough. Contains a few seeds. One specimen of *Sequoia Couttsiæ* was found here.
41	1	6	94	1	CLAY.—Dark. So brittle as to fall in pieces at the least touch. Contains pieces of lignite.
42		9	94	10	LIGNITE.—Extremely brittle. The bed is somewhat irregular in thickness and obscure in definition.
43		9	95	7	CLAY.—Lead-colour.
44		6	96	1	LIGNITE.—Irregular in thickness. Occasionally intersected by veins of clay, commonly at right angles to the plane of the bed.
45		7	96	8	CLAY.—Dark. Fragments of lignite very abundant.
46		9	97	5	LIGNITE.—Very compact and tough. Abounds in seeds.
47	1	4	98	9	CLAY.—Dull lead-colour in the upper part, changing into a darker hue towards the base; the upper band is tougher than the lower; the latter contains many pieces of lignite.
48		7	99	4	LIGNITE.—Compact. Contains seeds, and a few young circinate fern-fronds.

* Quart. Journ. Geol. Soc. vol. xii. p. 354.

Beds.	Thickness. ft.	in.	Totals. ft.	in.	
49		10	100	2	CLAY.—Rather light lead-colour. Somewhat tough. Contains pieces of lignite smaller than usual.
50	1	0	101	2	LIGNITE.—Compact. Very tough. Yields large slabs of "Board Coal"*.
51		9	101	11	CLAY.—Very dark. Contains a few pieces of lignite.
52		3	102	2	LIGNITE.—Loose and soft.
53		7	102	9	CLAY.—Brown. Pieces of lignite abundant in the upper part, less so towards the base.
54	2	3	105	0	LIGNITE.—Compact, tough, woody. Yields large slabs of "Board Coal" having a mottled appearance. Contains a few seeds.
55		10	105	10	CLAY.—Very dark blue. Of resinous aspect. Contained a piece of lignite six feet long and two inches broad; smaller fragments rather numerous.
56	3	2	109	0	LIGNITE.—Very woody. Frequently has a "charred" appearance.
57		5	109	5	CLAY.—Blue. Brittle.
58	1	8	111	1	LIGNITE.—Woody, "charred" and mottled. Contains a few seeds; none occur where the lignite appears "charred."
59		4	111	5	CLAY.—Some parts blue, others dark drab.
60	1	8	113	1	LIGNITE.—Woody, hard, brittle. Has a fracture resembling that of ordinary coal. Contains seeds.
61		4	113	5	CLAY.—Blue and dark drab.
62	2	4	115	9	LIGNITE.—This bed consists of two bands: the upper, nine inches thick, breaks into irregularly shaped "glassy" pieces; the lower is very hard, light-brown, less heavy than the lignite usually is, brittle, woody, and has a fracture resembling that of ordinary coal. This band contains seeds, none of which appear in the upper. The bands graduate into one another through a thin layer of "charred" lignite.
63		6	116	3	CLAY.—Light lead-colour. Contains seeds, probably more than one species. One stem of *Sequoia Couttsiæ* was found here.
64	1	3	117	6	LIGNITE.—Very hard and compact; not quite so tough as some of the beds above it, but by no means brittle; possesses traces of the "charred" character.
65		3	117	9	CLAY.—Lead-colour. Resinous in aspect. Contains numerous pieces of lignite.
66	1	4	119	1	LIGNITE.—In all respects like the 64th bed.
67		2	119	3	CLAY.—Resinous appearance.
68	1	4	120	7	LIGNITE.—In all respects like the 64th and 66th beds.
69		2	120	9	CLAY.—Very brittle, laminated, and resinous in aspect.
70		3	121	0	LIGNITE.
71		1	121	1	CLAY.
72	4	0	125	1	LIGNITE.—Termed by the workmen the "Last Bed"†.

In order to ascertain whether the succession and characters of the beds are the same in other parts of the pit, two other sections were made, also in the southern wall, one about 460, and the other 680 feet eastward from the first. The results are given below.

* So named from having an appearance resembling "deal boards."

† This section is exhibited in Plate LIII.

Section 2, *of the Bovey Deposit, in the south wall of the "Coal-pit,"* 460 *feet eastward from Section* 1. *Dip* 12½° *towards S.* 35° *W.* (*magn.*).

Beds.	Thickness. ft.	in.	Totals. ft.	in.	
1	6	2	6	2	SANDY CLAY.—With angular and subangular stones; the upper six inches peaty soil, in which the stones also occur.
2	4	10	11	0	CLAY.—Contains some fragments of lignite.
3	7	10	18	10	SAND.—Quartzose and ferruginous.
4	10	5	29	3	CLAY.—In some parts sandy and of a buff colour, in others dark with vegetable débris. Much fragmentary lignite.
5		9	30	0	LIGNITE.—Woody and brittle.
6	1	7	31	7	CLAY.—Dull lead-colour.
7	1	5	33	0	LIGNITE.—A matted mass of débris of *Sequoia Couttsiæ* and ferns.
8	2	0	35	0	CLAY.—Light drab. Roughly laminated. Contains broken lignite.
9					Do not occur in this section.
10					
11					
12	1	3	36	3	CLAY.—In some parts black with vegetable matter, in others light drab; the former most prevalent.
13		4	36	7	LIGNITE.—Much broken.
14	2	1	38	8	CLAY.—Light drab. Much broken lignite near the top.
15		6	39	2	LIGNITE.—More compact than in the higher beds.
16	2	2	41	4	CLAY.—Dull lead-colour. Somewhat resinous aspect. Broken into fragments having more or less curved surfaces.
17	1	0	42	4	LIGNITE.—Contains dicotyledonous leaves.
18	1	9	44	1	CLAY.—Dark drab colour. Much broken into angular fragments. Contains pieces of lignite.
19	1	2	45	3	LIGNITE.—Contains a considerable quantity of clay, in almost continuous bands.
20	1	2	46	5	CLAY.—Rich in fragments of lignite.
21	2	5	48	10	LIGNITE.—Compact.
22	1	3	50	1	CLAY.—Dull drab colour.
23		10	50	11	LIGNITE.—Much broken; the fragments having well-defined clay-stained surfaces.
24		9	51	8	CLAY.—Light drab. Broken. Resinous in aspect.
25	5	10	57	6	LIGNITE.—Contains patches of clay. Ferns and the so-called "flabelliform leaves" occur near the base.
26	3	6	61	0	CLAY.—Light drab. Rich in stems, leaves, and fruits of *Sequoia Couttsiæ*.
27	1	7	62	7	SAND.—Quartzose and ferruginous. Contains lenticular patches of clay.
28		6	63	1	CLAY.—Light colour.
29	3	9	66	10	CLAY.—Dark colour.

The nature of the ground prevented the second and third sections being satisfactorily continued below the 29th bed. There is reason to believe, however, that the still lower beds are uniform in character and order throughout the pit.

In the third section the materials comprising the various beds will be named without remark, since those which agree numerically agree very closely geologically.

Section 3, *of the Bovey Deposit, in the south wall of the* "*Coal-pit,*" 680 *feet eastward from Section* 1, *and* 220 *feet from Section* 2. *Dip* 12½° *towards S.* 35° *W.* (*magn.*).

Beds.	Thickness. ft.	Thickness. in.	Totals. ft.	Totals. in.	
1	10	9	10	9	Sandy Clay.—With angular and subangular stones.
2	1	0	11	9	Clay.
3	12	6	24	3	Sand.
4	2	0	26	3	Clay.
5		6	26	9	Lignite.
6	1	0	27	9	Clay.
7	1	9	29	6	Lignite.
8					Do not occur in this section.
9					
10					
11					
12	1	8	31	2	Clay.
13		10	32	0	Lignite.
14	2	4	34	4	Clay.
15	1	4	35	8	Lignite.
16	3	0	38	8	Clay.
17	2	0	40	8	Lignite.
18	2	0	42	8	Clay.
19	2	0	44	8	Lignite.
20		6	45	2	Clay.
21	3	1	48	3	Lignite.
22	1	0	49	3	Clay.
23		11	50	2	Lignite.
24		6	50	8	Clay.
25	5	3	55	11	Lignite.
26	3	2	59	1	Clay.
27		10	59	11	Sand.
28		7	60	6	Clay.
29	3	2	63	8	Clay.

On comparison, it will be found that the 9th, 10th, and 11th beds of Section 1—the first and last being sand and the second clay—do not occur in either of the other two sections, and that in Section 3 another clay-bed is also missing. This last is supposed to be the 8th, but it would probably be difficult to determine between it and the 12th. It is possible, moreover, that the bed numbered 12 in the third may represent both the 8th *and* 12th of the two other sections. The same numbering has been retained in all the sections to facilitate comparison.

The total thickness of the missing beds amounts to no more than about 5 feet; the fact, however, that they are not present may be significant. It amounts to this: by removing 460 feet further into the ancient Bovey lake (for such I assume the area to have been)—460 feet further from its ancient shore, that shore being the granitic region of Dartmoor,—we leave behind two thin beds of sand, which do not reappear when we advance 220 feet further in the same direction.

Eliminating these beds, we have no sand in the series below the uppermost bed of lignite (No. 5) excepting No. 27, which occurs in all the sections, and, indeed, constitutes a marked feature in the deposit as exposed at the "Coal-pit." It is 133 inches thick in the first section, no more than 19 in the second, and dwindles to 10 inches only in the third; but no bed is more continuous or better marked: its comparatively bright colour catches the eye, and indicates its presence along the entire length of the excavation. It attains a still greater thickness in the western wall of the pit. Between the first and third sections it forms an inclined plane 680 feet long and 123 inches high; or base : height $=8160:123=$ rad : tan $52'$; so that, great as the attenuation is, it merely produces a gradient of 1 in about 66, or an inclination of less than one degree.

The sections show that a similar eastward diminution of thickness characterizes the 28th bed.

This attenuation, like the thinning out of the beds previously mentioned, (which, it may not be out of place to remark, is in the direction of the *Strike* of the deposit,) is probably an indication, were one needed, that the detrital layers were formed at the expense of the Dartmoor granite.

The sections agree in naturally dividing themselves into three parts or series, viz.—

1st. The bed No. 1, of SANDY CLAY, containing angular and subangular stones.

No stones of any kind were met with below this.

2nd. The beds from the 2nd to the 27th, both inclusive, composed of sand, clay, and lignite.

3rd. All the beds below the 27th, consisting of clay and lignite only*.

It appears that that portion of the age of the deposit, which is represented by the first (that is, lowest) forty-five beds was unmarked by the deposition of sand within that area. Forty-four beds of lignite and clay, having an aggregate thickness of upwards of 47 feet, succeed each other alternately in regular unbroken order; the next bed, however (28th in the sections), instead of being a mass of vegetable matter, as was due, is a second bed of clay, and, in the first section, of unequalled thickness; this is followed by a thick bed of sand, the first which presented itself. Clearly some change must have occurred. Had the accumulated deposit so far shallowed the waters of the ancient lake? Had it conveyed its western margin so far eastward, that sand was to be henceforward deposited in the area hitherto appropriated to clay, instead of further west as heretofore? If so, it might have been expected that the change would have been less sudden. No sand whatever had previously occurred. Moreover, on this hypothesis it is reasonable to suppose that sand would have been largely deposited in future, instead of which the old order (of clay and lignite alternately) is continued for eighteen additional beds, increasing the depth of the deposit by nearly 40 feet; indeed, omitting the two arenaceous beds (9th and 11th) which occur only in the first section, we have no more sand until the uppermost bed of lignite in the sections had been formed. In no instance does the lignite rest on or support sand, but always clay.

* See Plate LIII.

Almost all the clay-beds contain fragments of lignite, which are commonly, at least approximately, parallel to the plane of stratification. One or two exceptions to this were met with, the most marked being that of the 35th bed, where they occur at all angles to that plane.

Though when first dug the clay is not generally characterized by lamination, exposure to the atmosphere, in most cases, developes this quality.

Fossils were found in only fifteen of the beds, namely, one of clay and four of lignite in the second series, and one of the former and nine of the latter in the third or lowest. It is only necessary to particularize the 7th, 25th, 26th, and 46th beds.

The 7th is chiefly remarkable as being a mat composed of fragments of the coniferous tree *Sequoia Couttsiæ*, HEER, and of the fern *Pecopteris lignitum*, GIEB.

The 25th is that in which the so-called "flabelliform leaves" chiefly occur; a few were also met with in the 17th bed. Professor HEER has identified them as the rhizomes of ferns. Some of them were fully 5 feet in length, but too brittle to be got out entire. In most cases the large specimens have a curved outline. The lowest three inches of the bed is commonly a mat of fragmentary fronds of the ferns *Pecopteris lignitum* and *Lastrea stiriaca*, UNG.,—the first being the most prevalent. Above this lie the rhizomes, in a continuous band about 6 inches thick. Though these bands generally preserve a well-marked separation, they sometimes inosculate, but never so as to show whether the fossils were parts of the same plant. The uppermost portion of the bed consists of slabs of "board coal" of great length, and of a width indicating the existence of trees (probably *Sequoia Couttsiæ*) fully 6 feet in diameter. Bodies occur in this bed having the appearance of roots, with rootlets passing into the clay below. Mr. KEEPING reported one such "root" having a part of the stem of a tree still attached to it, the latter being almost perpendicular to the plane of stratification. The lignite in this stratum not unfrequently presents a fretted aspect, as if from some kind of corrosive action; in these cases it is crossed by cracks or fissures of variable width, having rugged walls, and filled with yellow ochre.

The 26th is the most important bed in the series, being rich in both the number and the variety of its fossils. The lowest six inches contain a large number of dicotyledonous leaves, most of them crushed and valueless; occasionally, however, nests or patches of such leaves occur in a better condition. A few twigs of *Sequoia Couttsiæ* are also found in this lowest band; whilst quite at its base are numerous branches of the same plant, measuring in some instances 3 feet in length and from a quarter of an inch to 4 inches wide. In most cases the large specimens are extremely brittle.

The next fifteen inches constitute a middle band, containing some *Sequoia*-débris and a considerable number of crushed leaves, the latter suggesting the idea that they had been deposited on a very uneven surface. A thin layer of "charred" lignite, several feet in length, was found in the middle of this band.

The remaining part of the bed (the uppermost four inches) abounds in seeds of various kinds; but it is chiefly marked by remains of *Sequoia*. It is not too much to say that every museum in the world might readily be supplied with thousands of specimens of this plant from this band. It is represented by branches, twigs covered with leaves, fruits (sometimes, but more frequently not, attached to the twigs), and seeds. A few dicotyledonous leaves occur here also.

Though most abundant in this bed, the *Sequoia* occurs also in the 7th, 40th, and 63rd—that is, the highest and lowest beds which have yielded fossils. The Bovey deposit evidently represents but one flora.

The 46th bed yielded a very large number of small seed-vessels (*Carpolithes nitens*, Heer), which, like those described by Dr. Hooker, in 1855, under the name of *Folliculites minutulus*, but which Professor Heer has identified as *Carpolithes Websteri*, Br., are "thickly strewed over the surfaces of the laminæ of lignite, and slightly imbedded in them as if the latter had been soft when the deposit was formed. They lie in all directions, but always on their flat surfaces"*. They are by no means confined to this bed, though more abundant in it than elsewhere.

Nothing resembling the cone of *Pinus sylvestris*, described by Dr. Hooker, was found during our exploration. But for its complete "carbonization and bituminization," I should believe that it belonged to a neighbouring bog, mentioned by several writers, "from which have been taken, several feet below the surface, many trees of the fir kind; several 18 inches in diameter, together with pine-nuts, but no coal"†.

In some of the lower beds, close-fitting joints not unfrequently occur in the lignite, the surfaces of which (rarely planes) have a high polish: the workmen call such pieces "glassy;" and the term aptly expresses the character. They also call them "slides," believing them to be "Slickensides." There do not appear to be any "faults" in the beds at the pit.

It has already been stated that the lignite often has a "charred" appearance; and indeed it is somewhat difficult to believe that it has not ignited spontaneously; nor are we without facts which give some support to this opinion. It is well ascertained that the combustion so prevalent in the heaps of refuse is spontaneous, and the lignite beds are sometimes found to be on fire in the tunnels or "ends." Mr. Divett, writing me on this question, says, "Some ten or twelve years since, I found a fire raging in an 'end' at the western extremity of the pit, which had been abandoned for some months. I enclosed the main western 'end' with a dam of timber and clay, in the hope of extinguishing the fire, but only succeeded in checking it. This part of the pit was buried by a run of clay from the north for many years, and was excavated again about twelve months since, when the fire was still burning. It is now again buried by 'run' sand. I have never doubted that this ignition was spontaneous." Mr. Hatchett, however, who gave much attention to the chemistry of the lignite, was of opinion that there was no evidence of true combustion‡.

* Quart. Journ. Geol. Soc. vol. xi. p. 566.

† Mr. Scammell, in Parkinson's 'Organic Remains,' vol. i. Letter 12, p. 129.

‡ Trans. Linn. Soc. vol. iv. p. 141, &c.; also Philosophical Transactions for 1804, Part I. p. 396, &c.

Newly exposed surfaces of the laminæ of the lignite are sometimes more or less covered with stellate crystals of selenite. Their beauty is very striking, and is enhanced by contrast with the dull dark surface on which they lie; unfortunately it is by no means durable; the stars first lose their brilliancy, after which many of them disappear altogether.

Fragmentary pieces of lignite occasionally occur in the "coal-beds" as well as in the clay; some of them are perfectly flat slabs, of various sizes, having sides and ends as true and angular as if they had been something more than rough-hewn in a carpenter's shop. Others have an appearance resembling stranded drift-wood; I found a well-marked piece of this character in the 72nd bed.

The flattened form which the "board coal" commonly assumes is by no means confined to the lowest beds; it is as characteristic of the 5th, or uppermost, and of that portion of it which most nearly reaches the surface, as of any bed in the pit sections. As pressure must be regarded as essential to this flatness, though probably not its sole cause*, it seems impossible to avoid the conclusion that much of the superior portion of the deposit has been removed by denudation. It must not be supposed, however, that all samples of "board coal," taken from any one bed, are equally flattened. Examples occasionally present themselves, of portions of stems and branches, in which the original curvature of outline is not entirely obliterated—the transverse section distinctly showing the rings of annual growth converted into ellipses of great excentricity. Good instances of this have been met with in the lowest beds.

The stones so abundant in the "Head," or uppermost division of the pit sections, are sufficient to show that it was formed under conditions dissimilar to those which produced the two lower series. Moreover, it lies unconformably on them. Nowhere in the excavation do the lignite and interstratified beds reach the surface; they are cut off at distances varying from 3 to 7 feet below it, as is shown in Plate LIII.

It has already been stated that the stones of the "Head" are generally angular or subangular; occasionally, however, some occur that are much rounded. They vary in size, from blocks upwards of a foot in mean diameter to pieces not larger than hazel-nuts. On Bovey Heathfield they are fragments of granite, metamorphic rock, carbonaceous grit, and trap, with a very few of flint and chert. The two last increase in number eastward—that is, with increased proximity to the Cretaceous district,—and in some localities are even more abundant than other detritus.

In no instance have I found or heard of limestone-fragments on the Bovey Heathfield. A transporting current from the north or north-north-east seems to be required to meet the facts of the case. Were it not for the samples of flint and chert, a movement from the west or north-west, or even south-west, might have supplied the materials. No agent progressing from any part of the compass between the north-east and south-west, through south, could have furnished the granite-blocks or failed to transport large quantities of limestone-débris. On the whole I incline to a transportation from a

* See Hatchett in Philosophical Transactions for 1804, Part I. p 397.

northerly direction, rather east than west of true north,—that is, a current, or other agent, moving in a line nearly at right angles to that in which the sands and clays of the true Bovey beds travelled from Dartmoor—a fact, concurring with those previously mentioned, in favour of a great chronological interval between the "Head" and the deposit it covers.

Nor are we without *organic* evidence of the lapse of time between these formations. During our exploration at Bovey, I had an opportunity of examining and measuring a section made by workmen digging clay, in the "Head," on the Heathfield, about a quarter of a mile east of the pit. The results were as below.

Section 4, of the "Head" at Bovey Heathfield.

Beds.	Thickness. ft.	in.	Totals. ft.	in.	
1*		6		6	PEAT.
2	2	6	3	0	SAND.—Fine, white, quartzose.
3	3	0	6	0	CLAY AND SAND.—In separate masses, but not distinctly stratified. The clay more abundant than the sand.
4	4	0	10	0	CLAY.—Very white.
5 unknown.					SANDY CLAY.—With angular and subangular stones.

Some time afterward we found a considerable number of dicotyledonous leaves, lying in the white clay, nine feet below the surface of the plain, and immediately below them lay some large roots. Professor HEER assigns the leaves to a period much more modern than that represented by the lignite-beds, yet to one characterized by a "colder climate than Devonshire has at the present day," thus confirming Mr. GODWIN-AUSTEN's opinion, that the "Head" belongs to the "period prior to the most recent change of climate"†. The position of the leaves is indicated in fig. 1, which is drawn on the scale of 0·2 inch to 1 foot.

Fig. 1.

* The beds in this section and in those which succeed, do not represent those which bear the same numerals in the pit sections.

† Geol. Trans. 2nd Series, vol. vi. p. 437, &c.

Though in the pit section No. 1 the 5th is the highest, and the 72nd the lowest bed of lignite, it would manifestly be unsafe to conclude that no higher or lower beds exist; and indeed Mr. Divett, in a letter on this point, says, "In a shaft sunk 135 fathoms south of the pit" (*i. e.* in the direction of the *dip* of the beds) "I had, in 99 feet sinking, some good beds of coal. When we ceased to sink we had, I believe, some six or seven fathoms between us and the top of the uppermost bed of lignite in your sections." This estimate, as to the depth at which the "5th" bed would have been cut, is fully borne out by the *dip* of the beds and the distance of the shaft, if we assume that no "fault" exists in the interval and the *dip* remains constant. On both these points we have direct confirmatory evidence for the distance of sixty-three fathoms south of the pit, as subterranean workings have been carried so far, "by driving down the dip," and show that the beds exist in unbroken continuity and uniform inclination; that is, the beds have been followed to a depth of 80 feet below the bottom of the "Coal-pit."

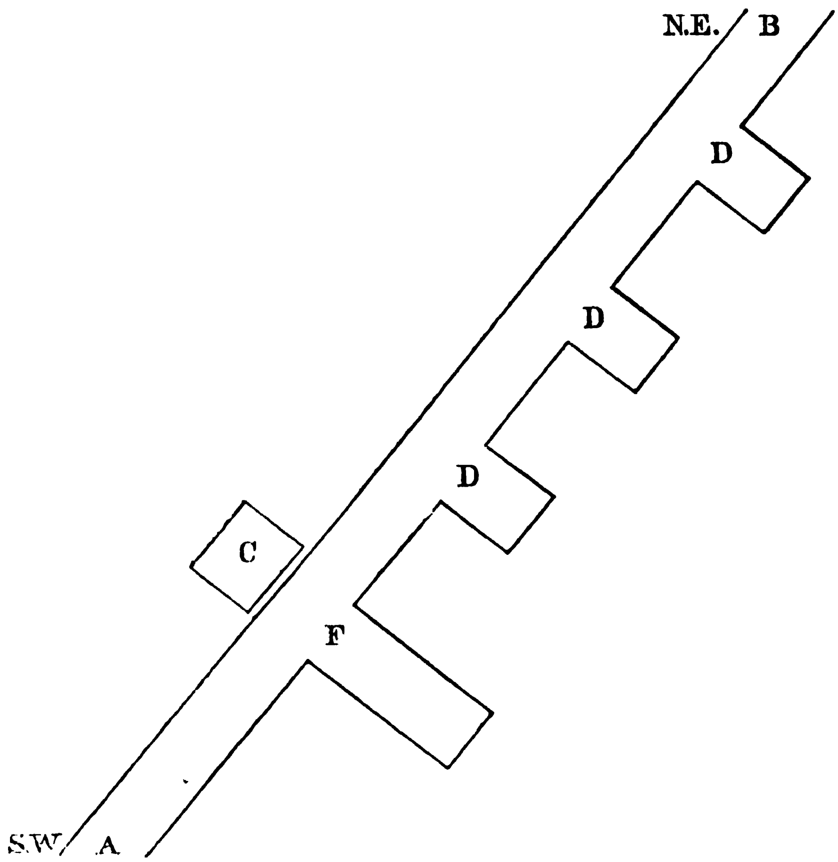
Though the workmen have named the 72nd the "last bed," it is no more than an expression of the fact that it is the last or lowest they work. That there are still lower beds is certain, since Mr. Divett says, "I sank a shaft about 13 feet" (below the bottom of the pit) "and cut two tolerable beds of coal." The workmen speak of still earlier and deeper borings, and state that thin layers, or "shells," of lignite were found separated by thick beds of "muddy clay." Omitting these traditions, however, we are now in possession of the following figures. The 72nd, or "last" bed is, at the western end of the pit, about 100 feet vertically below the surface of the plain; or, measured at right angles to the plane of stratification, we have, to the base of this bed, a thickness for the deposit of 125 feet*; the pit beds have been followed 80 feet lower, and lignite has been cut 13 feet below the so-called "last" bed; giving an aggregate of 218 feet, inclusive of the "Head," or upwards of 35 fathoms for the true Bovey deposit; exclusive of the beds mentioned by Mr. Divett as occurring in his shaft 135 fathoms south of the pit, and irrespective of the facts that the bottom has certainly not been reached, and that there are sufficient reasons for believing, as I shall now proceed to show, that denudation has swept away very much of the superior portion of the formation.

Though no trace of a "fault" exists at the pit, one has been detected a short distance east of it. "It runs," says Mr. Divett, "about N.E. and S.W., crossing close to the old engine-shaft" (56 fathoms east of the pit). "I drove towards it in many places, and always found the 'coal' fail and replaced by hard and wet 'deady' clay. At one place I drove further and cut into a bank of sand full of water, which ran into the shaft and 'starved' the pump for some time. I never got through the sand, and had great difficulty in keeping it out of the shaft. This was at about 80 feet from the surface. The section of the beds in the" (old engine-) "shaft is identical with those in the pit." The accompanying diagram (fig. 2) may serve to illustrate the foregoing facts. Let the surface of the paper represent a horizontal plane, on the level of the bottom of the coal-

* See Section 1, page 1024.

pit, at its eastern end, about 80 feet below the surface. Let A B be a portion of a tunnel, or "working," on that level, on the eastern side of the pit, running, about N.E.

Fig. 2.



and S.W., near and parallel to the plane of the "fault." C the old engine-shaft, 56 fathoms east of the pit. D and F various places at which Mr. Divett drove horizontally from the eastern side of the tunnel. In doing so he "always found the coal fail and replaced by hard and wet 'deady' clay;" at F he "drove further, and cut into a bank of sand;" in fact he seems here to have cut *through*, but elsewhere *into*, a dyke composed of heterogeneous materials; beyond which he encountered a bed of "sand full of water;" doubtless the 13th bed in the following section (No. 5), which, it will be seen, exists at the required depth ("80 feet below the surface"*).

Mr. Divett proceeds to say, "About 70 fathoms east of the shaft, I bored, in 1855, 99 feet, when I obtained the following section."

Section 5. 70 *fathoms east of the Fault on Bovey Heathfield* (furnished by J. Divett, Esq.).

Beds.	Thickness. ft.	in.	Totals. ft.	in.	
1	13	0	13	0	Gravels and Clays.—("Head.")
2	11	0	24	0	Sand.
3	4	0	28	0	Black Clay.
4	2	0	30	0	"Deady Ground."
5	6	0	36	0	Sand.
6	3	6	39	6	Clay.
7	4	6	44	0	Sand.

* See Plate LIV.

Beds.	Thickness. ft.	in.	Totals. ft.	in.	
8	8	0	52	0	CLAY.
9	13	0	65	0	SAND.
10	7	0	72	0	CLAY.
11		2	72	2	COAL.
12		10	73	0	CLAY.
13	11	0	84	0	BLUE SAND.
14	3	0	87	0	WHITE CLAY.
15		6	87	6	COALY CLAY.
16	1	6	89	0	SAND.
17	10	0	99	0	COALY CLAY*.

I learn from the workmen, that the 13th bed of sand was "full of water," like that encountered by Mr. DIVETT, and that it gave them great trouble by running into the boring whenever the instrument was withdrawn.

The foregoing section is situated, from the first three, as nearly as possible in the direction of the *Strike* of the formation. It has a depth about the same as the pit at its western end, where a vertical line cuts 27 "coal" beds having an aggregate thickness of nearly 36 feet; instead of the solitary layer of 2 inches only in the Table just given. Those beds are known to exist, in unbroken continuity, along the entire length of the western tunnel and the coal-pit, and onwards to the old engine-shaft, a distance of nearly half a mile; here they suddenly cease and their place is supplied by a series of beds having the characteristics of the uppermost portion of the second division of the pit sections. The contrast of the two will be seen in Plate LIV.

There can be no doubt that these facts are evidence of a great fault; that the beds on the east of it are an upper portion of the Bovey deposit, preserved, through the intervention of a vertical displacement of at least 100 feet, from the denuding action which swept it away on the west, after it had, by its pressure, assisted to flatten the timber in the uppermost stratum of lignite at present existing there; and that this denudation occurred before the deposition of the "Head," since this is found covering the deposit alike, without considerable variation in its thickness, on each side of the "*fault.*" It will be understood that it is by no means intended to intimate that this is the only fault in the Bovey formation; the occurrence of beds of lignite, near the surface, in various parts of the Heathfield, renders it probable that there is, at least, another. Nor is it meant to express the opinion that the "Head" itself may not have lost much by denudation; so far as they are at present understood, certain facts seem to imply that it may have suffered much in this way.

Though the neighbourhood of Bovey was necessarily regarded as the head-quarters of the formation, it was felt to be desirable that some attention should be given to certain other localities, in various parts of the basin, where clay-pits exist.

The clay-works at Aller, in the parish of Abbotskerswill, adjacent to the road from Torquay to Newton, and about two miles from the latter, have been abandoned some years. Lignite seems to have been found there in considerable quantity. Samples of

* This section is exhibited in Plate LIV. (eastern section).

it, shown me by one of the old workmen, displayed the common woody character so usual at Bovey, but no traces of leaves or other fossils. There appear to have been seven distinct beds of lignite alternating with as many of clay, the latter from 2 to 4 feet thick. Of the former, the lowest were the most compact, and were from 3 to 3½ feet in thickness, whilst the upper ones were thinner and contained a considerable admixture of clay. The whole were covered, unconformably, with gravel to the depth of 20 feet.

Clay is largely dug at the Decoy, in the parish of Woolborough, about half a mile south-west of the Newton railway station. In an artificial dyke, or water-course, two beds of lignite, separated by a layer of black clay, are well exposed. The entire cutting is about 10 feet deep, the uppermost three feet being coarse gravel surmounted by a thin layer of peat. The larger stones in the gravel are commonly flint and chert, the smaller are partly, perhaps mainly, Dartmoor débris. Beneath this are the Bovey beds dipping towards north 80° East, at an angle of 60° at the top and 50° at the bottom, the beds having somewhat curved surfaces. The western, or lowest bed of lignite is 9 feet thick, the eastern 6 feet, and the intermediate clay about 5 feet; the whole lies between clay similar to the interstratified bed. Further west, or still lower, is a valuable bed of "pipeclay," whilst on the eastern side is a good bed of the black, or "potter's clay."

Though we spent some days seeking fossils here, the only things found were two small bodies, probably seeds, and one undoubted twig, with leaves, of *Sequoia Couttsiæ*. The latter, though a very inferior specimen of this fossil, is valuable as a link of identification between the lignite of the Decoy and that of Bovey Tracey.

Considerable quantities of both white and black clay are also excavated in the parish of Kingsteignton, about two miles north of Newton, very near the eastern margin of the deposit. I measured the following section in one of Mr. WHITEWAY's "black pits."

Section 6, of the Bovey Deposit near Kingsteignton.

Beds.	Thickness. ft.	Thickness. in.	Totals. ft.	Totals. in.	
1	15	0	15	0	"HEAD," consisting of angular and subangular flint, chert, and Dartmoor débris.
2	2	8	17	8	CLAY.—Black. "Not Saving."
3	1	6	19	2	CLAY WITH LIGNITE.
4	4	0	23	2	CLAY AND SAND.—Contains root-like portions of lignite.
5	4	0	27	2	"SHOVEL" SAND.—Thickness variable.
6	8	0	35	2	CLAY.—Black. "Saving."
7	3	0	38	2	CLAY.—Black. "Short." Thickness variable.
8	12	0	50	2	"BOTTOM" CLAY.—"Saving."
9	Thickness unknown.				SAND.

The workmen denominate the clay "Saving" or "Not Saving," according as it has or has not a commercial value. Sand so loose as to be capable of removal by the use of a spade or shovel only, is termed "Shovel" Sand. Clay but slightly plastic is spoken of as "Short;" and "Bottom" Clay expresses the fact that no argillaceous deposit, having commercial value, is found below it.

None of the workmen appear to have found or heard of anything in the shape of gravel at the base of the clay series *. Sometimes, but not frequently, small, well-rounded, smooth quartz pebbles, about the size of a common pea, occur in the best clay; and mundic is said to be not uncommon.

The *dip* of the beds at this pit is about 8° towards N. 50° W. It is said to vary somewhat both in amount and direction, but is generally less northerly than the above. As at the Decoy, the white clay underlies the black.

Though lignite occurs here, it is less abundant than in either of the other areas which have been mentioned; from the report of the workmen, however, it is occasionally found, especially in the pits more removed from the margin of the deposit, in larger bodies than in the section just given. Our search for fossils was altogether without success.

At the suggestion of Dr. FALCONER, and also of Sir CHARLES LYELL, it was decided to submit the collection of fossils, which we had made at Bovey, to Professor HEER of Zürich, in the hope that he would succeed in extracting from them their chronological secret. Accordingly, the necessary arrangements having been made by Sir CHARLES LYELL, I sent him all the drawings of the fossils, prepared by Mr. FITCH of Kew, together with a large and, so far as I could judge, characteristic series of the specimens themselves, and in a short time had the gratification of learning that he had determined forty-five species of plants, of which forty-one were from the lignite series and four from the "Head," the former being decidedly of the lower miocene age, whilst the latter were much more modern.

Though, when he subsequently reached this country, Professor HEER failed to detect, in the remainder (the bulk) of the collection, any species which he had not previously seen, he was more fortunate at Bovey; where, in the few days he was able to devote to the deposit, he added nine new species of fossil plants to the list, and, by the discovery of an insect, *Buprestes Falconeri*, detected the first evidence of animal life which has been exhumed there.

From the decision just mentioned, it appears that the Bovey lignites are the contemporaries of the "Hempstead Beds" in the Isle of Wight, first discovered by the late lamented Professor EDWARD FORBES in 1852, and described by him in the following year †. Though their discoverer always regarded them as Upper Eocene, they have recently been grouped amongst the Lower Miocene ‡; this, however, is a question of classification; wherever they find a resting-place, the Bovey beds must accompany them, since they are on, or very near, the same horizon.

The ancient miocene lake of Devonshire which we have been considering, must have been of great depth; the lowest figures mentioned in an earlier page give at least 35 fathoms, whilst, if to this we add those obtained from the "fault," it amounts to fully 50 fathoms; indeed the clay-workers assert that their borings sometimes amount to quite this depth. The present surface of the plain, however, is, at the pit, no more

* See Mr. GODWIN-AUSTEN, in Trans. Geol. Soc. 2nd Series, vol. vi. part 2, p. 448; also Sir H. DE LA BECHE, in his 'Report,' p. 257.

† Quart. Journ. Geol. Soc. vol. ix. p. 259, &c.

‡ Sir C. LYELL's 'Supplement' to the fifth edition of his 'Manual,' p. 6, &c., 1857.

than 15 fathoms above the ordinary level of spring-tide high water, so that the bottom of the lake would be at least 35 fathoms below the level of the sea. Yet, says Professor Heer, and apparently on unimpeachable data, "it was a fresh-water lake." The country, then, must have stood at a much higher level than at present, or a barrier existed between the lake and the sea. Unless, however, there have been very local changes of level, the former hypothesis is disposed of by the fact that the Hempstead beds are of fluvio-marine origin, and must therefore have been formed at a level much below that which they at present occupy. A barrier, then, must have existed somewhere in the present tidal estuary of the Teign, over which the surplus waters of the lake passed to the ocean, or which, by its superior height, caused the waters to find an outlet in Torbay. Judging from the physical features of the two valleys leading from Newton to the English Channel, one by Teignmouth and the other by Torquay, the former is far more likely than the latter to have been the course followed.

The period represented by the Bovey beds must have been of considerable duration. So far as the strata themselves show, it was, in the district under consideration, one of great tranquillity. A long series of beds, alternately vegetable matter and fine clay, succeed each other in scarcely interrupted order; the three intruded arenaceous layers probably mark nothing more than a somewhat increased velocity in the current, or river, which conveyed the detritus of the granite hills of Dartmoor into the area of deposition, but which, instead of being permanent, was as short-lived as it was unusual.

The late investigations at Bovey, then, have been so far successful that they have settled the vexed question of the age of the deposits occurring there,—added forty-nine species to the fossil flora of this country, of which twenty-six are new to science,—recognized the first traces of animal life which the deposit has yielded,—detected another British fragment of the miocene page of the earth's history, which, until 1857, was supposed to be totally unrepresented in England,—taken us back to a remote period when the slopes of Devonshire were clothed with a luxuriant subtropical vegetation,—and separated, by a wide chronological hiatus, the lignite and associated beds from the gravels overlying them—a hiatus evidenced by the dissimilarity and unconformability of the two series, by a change in the direction by which detrital matter reached the Bovey area, by great vertical displacements of the lower series, followed by denudation of the consequent surface-inequalities prior to the deposition of the upper, and by the exchange of an extinct flora, requiring a high temperature, for an existing one, which is now confined to arctic and alpine regions.

Remote, however, as was the earliest of the two periods thus represented, the great leading geographical features of the district were pretty much as at present. The Teign and Bovey rivers were then in existence, but instead of the latter being tributary to the former, their mouths were three miles apart, and both fell into the same deep, sluggish, fresh-water lake; occupying the site of the present Bovey plain, and guarded by Dartmoor and the other hills which still constitute the prominent characteristics of the district.

EXPLANATION OF THE PLATES.

PLATE LII.

Is copied, with very slight alterations, from the twenty-sixth sheet of the Map published by the Geological Survey of Great Britain; and, like the original, is on the scale of 1 inch to a mile.

PLATE LIII.

Is a section of the Bovey formation, in the plane of the *Dip* of the beds. It is drawn, from the measurements obtained in the "first section," given in the text (see p. 1022, &c.), on the scale of $\frac{1}{168}$, or 1 inch to 14 feet. The *Dip* amounts to $12\frac{1}{2}°$, and is in the direction S. 35° W. magnetic. The beds in which fossils were found are those the numbers of which are placed opposite them in the margins.

PLATE LIV.

Contains two sections of the formation, in the plane of the *Strike* of the beds, and is intended to show the nature of the evidence for the existence of the "*Fault.*" The scale of thickness, in each, is $\frac{1}{120}$, or 1 inch to 10 feet; and the total depth below the surface is 99 feet.

The symbols have the same meaning in both.

a is the "old engine-shaft."

b. The eastern end of the "Coal-pit;" 56 fathoms west of *a*.

c. Mr. DIVETT'S "boring;" 70 fathoms east of *a*.

d, 80 feet below the surface, is a horizontal excavation very near the engine-shaft, and opening eastward out of a "working" which runs parallel and adjacent to the "Fault" (see fig. 2).

e f is the hypothetical *plane* of the "Fault."

With the omission of a few unimportant local differences, the western section represents the ascertained succession and thickness of the beds from *a* to nearly half a mile westward. They are probably continued much further in this direction, but are known to terminate eastward abruptly at the "Fault," *e f*, immediately east of *a*. The lowest bed shown is the 62nd in the "pit" sections.

The eastern section is drawn from the data obtained in Mr. DIVETT'S "boring" at *c*, (p. 1033,) and shows all the beds cut there. These are assumed to extend, in the same order, westward to *e f*; and though no excavations have been made at the surface in the intermediate space, the assumption is by no means gratuitous, since a bed of sand, having the same characters and at the same depth below the surface, has been met with both in the "boring" *c* and the excavation *d*.

Though the existence and situation of the "Fault" has been well ascertained, the angle which its plane (?), *e f*, makes with the horizon is not so well known. In the absence of complete evidence on this point, it has been thought best to draw it at right angles, more especially as the evidence, so far as it goes, is to that effect.

XL. *On the Fossil Flora of Bovey Tracey. By Dr.* OSWALD HEER, *Professor of Botany, and Director of the Botanical Gardens in Zürich. Communicated by Sir* C. LYELL.

Received November 16,—Read November 21, 1861.

IN the middle of the extensive plain which is bounded by the slopes of Bovey, are the potteries of Mr. DIVETT, for which fuel was formerly supplied by the lignite excavated there. In order to obtain this lignite a deep cutting has been made, and a sort of small ravine formed, on the sides of which the stratification is exposed. The surface-covering consists of a light-coloured quartzose sand, which contains here and there considerable beds of white clay. By the plants contained in it this formation is assigned to the Diluvium. Immediately under it come the beds of clay and lignite described by Mr. PENGELLY in the foregoing paper, which are all referable to *one* formation, as several kinds of plants are common to the different beds. The *Sequoia Couttsiæ* and *Pecopteris lignitum* occur in the 7th, 17th, 26th, 40th, and 63rd beds. *Carpolithes Websteri* is certainly found in the greatest abundance in the 54th bed, yet occurs also, though very rarely, in the 25th bed; *Cinnamomum Scheuchzeri* and *C. lanceolatum* in the 17th and 26th. The formation to which these strata belong is far older than that of the overlying white clay; the plants found in the former prove them to belong unquestionably to the *Miocene* period, and accordingly we must treat of them separately.

A. *The Miocene Formation of Bovey.*

Of the fifty species of plants which have hitherto been discovered in the lignite beds of Bovey, twenty-one occur also on the Continent in the Miocene formation. The lignite of Bovey Tracey is therefore undoubtedly Miocene; and it is worthy of special remark, that the species of *Cinnamomum* which are so characteristic of the Miocene and so generally distributed through it make their appearance in Bovey precisely as in the lignites and molasse of the rest of Europe. Equally characteristic are the *Lastræa Stiriaca,* the fern of most universal distribution over Miocene Europe, the ornate striated seeds of the *Gardenia Wetzleri,* and the fruits of *Carpolithes Websteri,* which are known to us from Germany, Switzerland, and Italy.

The following conspectus exhibits the proportions in which some of the species found at Bovey have been observed in other districts:—

	Tongrian.	Aquitanian.	Mayencian.	Helvetian.	Œningian.
Lastræa Stiriaca, *Ung.*, sp.	....	Monod; Hohe Rhonen; Paudèze; Wetterau; Ménat; Cadibona.	Radoboj; Eriz; Riethhäugli and Ruppen, St. Gallen.	......	Albis, very rare; Parschlug; Sarzanello.
Pecopteris lignitum, *Gieb.*	Weissenfels	Thôrens, Savoy; Manosque, Provence; Wetterau.	....	......	....
Sequoia Couttsiæ, *m.* ...	Hempstead, Isle of Wight.	Armissan, near Narbonne.	....	......	....
Palmacites Dæmonorops, *Ung.*, sp.	....	Salzhausen	....	......	....
Quercus Lyelli, *m.*	....	Altsattel in Bohemia.	....	......	....
Laurus primigenia, *Ung.*	Sotzka; Bornstedt; Weissenfels; Salzedo; Novale.	Rivaz; H. Rhonen; Spebach; Cadibona.	Eriz; St. Gallen	......	....
Cinnamomum Rossmässleri, *Hr.*	Haering; Mt. Promina; Salzedo.	Rothenthurm; Altsattel; Reut; Westerwald; Sagor.	Lausanne; Solitude near St. Gallen; Radoboj; Bilin.	Turin	Albis; Wangen; Parschlug; Senegaglia.
—— Scheuchzeri, *Hr.* ...	Sotzka	Common everywhere.	Common everywhere.	Common ...	Common everywhere.
—— lanceolatum, *Ung.*, sp.	Common	Common everywhere.	Common everywhere.	Rare	Very rare; Albis; Ischel.
Daphnogene Ungeri, *Hr.*	Sotzka	Lignite of Bonn; Westerwald; Manosque.	Develier	......	Irchel; Schrotzburg; Wangen.
Dryandroides hakeæfolia, *Ung.*	Sotzka; Mt. Promina; Haering; Salzedo.	Monod; Rivaz; Rochette; H. Rhonen; Rufi; Baltenschweil; Sagor; Peissenberg; Foveagedo.	....	......	....
—— lævigata, *Hr.*	Weissenfels	Monod; Rivaz; H. Rhonen; Peissenberg; Cadibona; Manosque.	....	......	....
Nyssa europæa, *Ung.* ...	....	Salzhausen; Nidda.	....	......	....
Vaccinium acheronticum, *Ung.*	Sotzka; Sieblos; Haering; Salzedo; Chiavon; Taurus.	Monod; H. Rhonen; Sagor; Lignite of Bonn.	Develier; Schangnau; Radoboj; Rhön.	Petit Mont, Lausanne; St. Gallen.	Œningen; Schrotzburg; Parschlug; Senegaglia.
Andromeda vacciniifolia, *Ung.*	Sotzka; Taurus ...	Monod.	Mönzlen	......	....
—— reticulata, *Ett.*	Hempstead; Haering; Sieblos.	....	....	......	....
Echitonium cuspidatum, *Hr.*	....	Manosque	....	......	Locle.
Gardenia Wetzleri, *Hr.* ..	....	Salzhausen; Samland.	Rhön; Günzburg; Ruckers.	......	....
Nymphæa Doris, *m.*	Hempstead	....	....	......	....
Eugenia Haeringiana, *Ung.*	Haering; Sieblos; Reut.	Ralligen	Lausanne; Calvaire; St. Gallen; Rhön.	Petit Mont, Lausanne; Turin.	....
Celastrus pseudo-ilex, *Ett.*	Haering	....	Rhön	......	Locle; Œningen; Hohen Krähen; Bischofsheim.
Carpolithes Websteri, *Br.*	Hempstead	Rochette; Conversion; Westerwald; Salzhausen; Laubach; Cadibona.	Rhön; Eisgraben; Zeche Einigkeit; Kaltennordheim.	......	....

A glance at this Table will at once satisfy us that the lignites of Bovey must be referred to the Lower Miocene division, and to the Aquitanian stage of it. It is true that nine of the species are found also in the Upper Miocene of other places, but these are all species which occur also in the lower stages, and which had a very extensive distribution both in time and space. Twelve species have been observed in the Mayencian and sixteen in the Tongrian stage; but nineteen have been identified in the Aquitanian stage in various localities. Certain species (*Palmacites Dæmonorops*, *Quercus Lyelli*, and *Nyssa europæa*) are not yet known in other districts as belonging to any but the Aquitanian stage; two (*Andromeda reticulata* and *Nymphæa Doris*) are known only in the Tongrian; others (*Pecopteris lignitum*, *Sequoia Couttsiæ*, *Dryandroides hakeæfolia*, and *D. lævigata*) only in the Tongrian and Aquitanian.

In this conspectus we have omitted the doubtful species (*Phragmites Œningensis*, *Dryandroides Banksiæfolia*, *Eucalyptus oceanica*, and *Pterocarya denticulata*). Should these be established by specimens in a better state of preservation, no disturbance would ensue to the above result, inasmuch as the *Phragmites* reaches back to the Lower Miocene, the *Eucalyptus* and *Dryandroides* belong to the Tongrian and the Aquitanian, the *Pterocarya* to the Aquitanian.

If we compare the Bovey flora with the several Continental floras, we shall find a great coincidence between it and that of Salzhausen in the Wetterau. A couple of species, viz. *Palmacites Dæmonorops* and *Nyssa europæa*, were previously known only from that district, while others, as *Pecopteris lignitum*, have been rarely found in other localities. With the Aquitanian stage of the Swiss Molasse (Hohe Rhonen, Ralligen, Monod and Rochette) Bovey has eleven species in common,—all species which occur in other parts, but two of them (*Dryandroides hakeæfolia* and *D. lævigata*) are especially frequent in Switzerland.

Of the French tertiary floras, it approximates most to that of Manosque in Provence. Here, as at Bovey, the most frequent fern is *Pecopteris lignitum*; and here also are found *Cinnamomum lanceolatum*, *Daphnogene Ungeri*, *Echitonium cuspidatum*, and *Dryandroides lævigata*. The water-lily of Bovey is probably one with the *Nymphæa calophylla*, Sap., of Manosque; but this cannot be determined, as we have only the leaves in the latter case, and the seeds in the former.

In the composition of the soft clay that contains the plants in the 26th bed at Bovey, and in the mode of their deposition in the same, there is great resemblance to the clays of Samland near Königsberg. A further connexion may also be traced through *Gardenia Wetzleri*.

It is remarkable that Bovey has no species in common with Iceland, although the tertiary flora of Iceland belongs to the same period, and two of its species (*Corylus MacQuarrii*, Forbes, and *Platanus aceroides*, Gp.) extend into Great Britain, having been found in Mull, in the Miocene of Ardtun Head. Even the genera are distinct, with the exception of two, *Sequoia*, and *Quercus*, which genera have each a single but distinct species in Iceland and in Bovey. The Bovey flora has a much more southern character, corresponding entirely with that of the Lower Miocene of Switzerland. Bovey had three species of cinnamon, one laurel, evergreen fig-trees, one palm, and large ferns, thus manifesting a subtropical climate.

If we compare the Bovey flora with that of the Eocene beds of the Isle of Wight, we find certainly some points of connexion, but, on the whole, an essentially different character. As connecting points, we may observe that one species (viz. *Laurus primigenia*, Ung.) is common to Alum Bay and Bovey, and, moreover, that the genera *Quercus*, *Ficus*, *Dryandroides*, *Daphnogene*, and *Sequoia* appear in both places, although differing in species. The fact of only one species being found in common at so short a distance —that in the Eocene formations of the Isle of Wight the highly characteristic Cinnamon and Lastræas are wanting—above all, the fact that Bovey has many more species in common with the more remote Miocene formations of the Continent than it has with Alum Bay and Bournemouth, satisfies us that it belongs to a different horizon *.

In this summary we have noticed only the species already known. Among the new species, however, of which I have described twenty-six, several interesting forms are found. The first place belongs to the *Sequoia Couttsiæ*, m., a Conifer, which we can illustrate by branches of every age, and by the cones and seed. It supplies a highly important link between *Sequoia Langsdorfi* and *Sequoia Sternbergi*, the widely distributed representatives of *Sequoia sempervirens*, Lamb., and *Seq. gigantea*, Lindl. (*Wellingtonia*), which latter species are at present confined to California.

Of great interest also are two species of *Vitis*, of which the grape-stones lie in the clays of Bovey. They belong to different species from the tertiary vine of the Continent (*Vitis teutonica*, A. Br.); but it is not improbable that they may be identified with that of Iceland (*Vitis islandica*, m.), though we must leave this indeterminate for the present, as we have obtained only the leaves from Iceland, and only the grape-stones from Bovey. The three remarkable species of fig, the seeds of three new species of *Nyssa* and two of *Anona*, one new water-lily (*Nymphæa*), and many highly ornate *Carpolithes*, are important additions to our knowledge of tertiary plants.

If from the relics of Bovey plants, which are still far from numerous, we attempt to represent the vegetation of Bovey as it existed in the tertiary period, we shall have to sketch it somewhat in the following manner:—The woods that covered the slopes which surrounded the beds of lignite consisted mainly of a huge coniferous tree (*Sequoia Couttsiæ*), whose figure resembled in all probability its highly admired cousin the *Sequoia* (*Wellingtonia*) *gigantea*, Lindl., of California. It had just the same graceful slender appearance in its vernal shoots, thickly studded with leaflets; and the similarity continued in the older shoots and branches, which were clothed with scales. But it presented a distinct character in its shorter leaves, which were even more closely appressed to the shoots, and in its smaller cones. The leafy trees of most frequent occurrence were

* I received lately from Mr. Pengelly a collection of plants from the Tongrian stage of Hempstead (Isle of Wight); it contains four Bovey species, viz. *Sequoia Couttsiæ*, *Andromeda reticulata*, *Nymphæa Doris*, and *Carpolithes Websteri*.

the cinnamons (*Cinnamomum lanceolatum* and *C. Scheuchzeri*) and an evergreen oak (*Quercus Lyelli*, m.) like those which now are seen in Mexico. The species of evergreen fig were rarer, as were also those of *Anona* and of *Gardenia*. The trees of the ancient forest were evidently festooned with vines, beside which the prickly Rotang-palm (*Palmacites Dæmonorops*) twined its snake-like form. In the shade of the forest throve numerous ferns, one species of which (*Pecopteris lignitum*) seems to have formed trees of imposing grandeur; besides which there were masses of underwood belonging to various species of the genus *Nyssa*, which is at present confined to North America. On the surface of the lake in which were formed the deposits of clay and sand that lie between the lignite-beds, were expanded the leaves of those water-lilies the ornate seeds of which are preserved for our examination.

If we inquire further how far the plants help us to a definite view of the course of events by which these lignite-beds were formed, our conclusions will be somewhat of the following kind:—

It is highly probable that at the period of the Lower Miocene the Bovey basin was occupied by an inland lake. The entire absence of freshwater shells, and indeed of aquatic animals generally, is certainly very extraordinary; and so is the absence of fruits of *Chara*, which abound elsewhere in Miocene freshwater deposits; the *Nymphæa* seeds, however, afford positive proof of fresh water. We must not omit to notice that the parts of the basin hitherto explored, and the only parts which are accessible to investigation, lie at a great distance from the hills. Accordingly they were far from the bank, more in the middle of the lake, and, in the case of the lower beds, at a considerable depth. This explains the absence of bog plants, so numerous in other instances, as well as the absence of mammalian relics. These would not have drifted so far out into the lake, and probably they are to be found on the edge of the lignite formation, where the vegetation also may be expected to present a somewhat different character. The lignite-beds of the under series consist almost entirely of tree-stems (probably belonging in great measure to *Sequoia Couttsiæ*); these alternate with masses of a brownish-black clay, the dusky colour of which has doubtless been produced by the decomposition of the softer portions of the plant. No leaves offer themselves for recognition, but here and there twigs and seeds of *Sequoia Couttsiæ*, and little fruits, as *Carpolithes Websteri* and *C. nitens*. The tree-stems, which are here piled one over the other in huge masses (none of them stand upright), and which every here and there stretch their branches and roots in the layer of clay which has covered them up, have apparently been floated hither, not only from the immediate circuit of hills, but doubtless also from greater distances. Such a mass of timber could hardly have been furnished by the former. Accordingly we learn from the structure of these lignite-beds that they did not originate in a tertiary peat-deposit, but from a colluvies of wood uniting in a lake; and hence they differ widely from those of Paudèze, of Hohe Rhonen, of Käpfnach, and other localities of Switzerland. At the same time the lignites of Bovey must have taken a long period in the process of formation, as the repeated alternations of clay-beds sufficiently show.

When a mass of timber and mud had been deposited in the bottom of the Bovey lake, some natural disturbance (whether owing to extensive landslips falling into the lake, or to the river undermining its banks) must have occasioned the contribution of a mass of quartzose sand, which thickly covers the under set of lignites, and which must at the time have helped largely to fill up the lake. Immediately above it lies a soft clay with numerous leaves of plants (the 26th bed), just as they were drifted together from the woods in the fall of the year. It seems, then, that this bed was formed in autumn, and that the plants it contains are due to the driftings of that season; in further confirmation of which, is the frequent recurrence of the seed and ripe cones of *Sequoia Couttsiæ*. Higher up follows the bed with the fern-rhizomes, among which occasionally can be recognized the pinnules of *Pecopteris lignitum*, which, somewhat higher up, amidst the branches of the *Sequoia*, appear in great abundance, being here and there compacted together in dense masses.

Above this bed come strata of clay and comparatively inconsiderable deposits of lignite, which last were all formed from the collection of wood and plants drifted to the spot.

As this Lower Miocene formation is immediately succeeded by quartzose sand with White Clay, we have here a great hiatus. Either the Middle and Upper Miocene, as well as the Pleiocene periods, must have passed without the formation of deposits in this place, or the latter must have been removed during the Diluvial period.

B. *The White Clay.*

While the lignites and their alternating clays at Bovey present us with a vegetation which is subtropical, the plants of the White Clay exhibit a totally different character, and must have had their origin in a period altogether distinct. The collection of Mr. PENGELLY contains four species from this formation—three of *Salix* and one of *Betula*; and, what is the most remarkable, none of these appear to me to differ from species now living. The little birch-leaves are not to be distinguished from those of *Betula nana*, Linn., nor the willow-leaves of one species from those of *Salix cinerea*, Linn.; while those of a second species come very near *Salix repens*, Linn., and also resemble strongly those of *Salix ambigua*, Ehrh. So variable is the form of these leaves, that it is hard to fix the species with positive certainty. At all events these leaves prove to us that those white clays must be much more recent than the lignite deposit; while the presence of *Betula nana*, Linn., which is in the highest degree remarkable, is conclusive for a diluvial climate, that is, a colder climate than Devonshire has at the present day; for this dwarf birch is an Arctic plant, which has no British habitat south of Scotland, and which occurs in Mid Europe only on mountains and subalpine peat-mosses. The evidence of the willow-leaves is to the same effect, indicating that at this period Bovey was a cold peat-moor. We may remark that *Salix cinerea*, Linn., is one of the most prevalent species of the diluvial travertine of Kannstatt.

I. *Descriptions of the Miocene Species of Plants.*

I. CRYPTOGAMÆ.

I. FUNGI.

1. Sclerotium, Tode.

1. Sclerotium Cinnamomi, m. (Plate LXVII. fig. 17; fig. 19, magnified; diameter magnified, 19 *b.*)

Scl. perithecio orbiculato, duro, plano, margine elevato.

On the leaf of *Cinnamomum Rossmässleri* there are several flat circular umbos. They are 1 millim. in diameter. They are quite smooth and flat in the centre, and surrounded by a very sharp edge.

They very much resemble *Sclerotium pustuliferum*, Heer, which is often found on *Quercus neriifolia* in Oeningen. Of the living species, *Scl. pustula* may be compared with it. Rossmässler has figured a closely resembling, but somewhat larger fungus, in his 'Beiträgen zur Versteinerungskunde,' taf. 8. fig. 27.

2. Sphæria, Hall.

2. Sphæria socialis, m. (Plate LXV. fig. 13, *c*; fig. 13, *c c*, magnified.)

Sph. peritheciis congregatis, minutissimis, orbiculatis, ostiolo rotundato pertusis.

In the 17th bed at Bovey.

There are many circular and convex little bodies close together on a leaf of *Dryandroides lævigata*, Heer. In the centre they are furnished with a pretty large aperture.

3. Sphæria lignitum, m. (Plate LV. fig. 1; figs. 2 & 3, magnified.)

Sph. peritheciis gregariis, liberis, conicis, nigris, apice nitidis, papillatis, ostiolo minuto, orbiculato.

I found this *Sphæria* on the bark of several branches, which perhaps belong to *Sequoia Couttsiæ*. They were lying in the 26th bed, beside some young branches of *Sequoia*. The perithecia form little black warts, which are clustered together in great numbers; the largest $\frac{1}{2}$ millim. in diameter; many of them are much smaller, and appear as black points. The largest of them are slightly conical, and furnished on the top with a very small though distinctly separated cicatricule. When this cicatricule falls away, a small aperture is left. In many of them a transverse slit has originated near the base, and the upper part of the perithecium has fallen away; thus we have a large aperture, surrounded by the pretty thick coat of the perithecium.

This belongs to the group of *Sphæriæ pertusæ*, Fries (Systema Mycolog. ii. p. 460), and has a great resemblance to *Sphæria umbrina*, which, however, is much larger and flatter.

II. FILICES.

1. LASTRÆA, Bory, Alex. Braun. (*Phegopteris*, Mettenius.)

4. LASTRÆA (GONIOPTERIS) STIRIACA. (Plate LVI. figs. 12–15.)

L. fronde pinnata, pinnis linearibus, prælongis, grosse crenatis serratisve, nervis secundariis e nervo primario angulo subacuto egredientibus, pinnatis, nervis tertiariis utrinque plerumque 6–7, curvatis, subparallelis, angulo acuto egredientibus, soriferis; soris rotundatis, biseriatis.

Heer, Flora Tertiaria Helvetiæ, i. p. 31; iii. p. 151; pl. 6, 7, 143, figs. 7 & 8.

Polypodites stiriacus, Unger, Chloris Protogæa, p. 121, pl. 36.

This is the tertiary fern which has the widest distribution. It appears rarely in the Upper Molasse, however (on the Albis, in Parschlug and Sarzanello), but very often in the Aquitanian stage of the Lower Miocene formation—thus at Monod, the Paudèze, Hohen Rhonen, &c., in France at Ménat, and in Italy at Cadibona. Several portions of leaves have been found at Bovey, the determination of which is undoubted. The leaf figured in fig. 15 represents a pinnule in almost its whole length; the other figures represent parts of leaves, the nervation of which is well preserved (Plates LVI. fig. 12; LVII. fig. 8). The specimen Plate LVI. fig. 14 is a portion of a leaf with the rachis, on the side of which the pinnules are attached. The pinnule is long and narrow, with parallel sides, deeply toothed, the teeth bent towards the apex; their long side forms an arch, the sinus is acute. The principal nerve of the pinnule is strong, the secondary nerves are fine, springing at acute angles and forming a slight arch; from each secondary nerve spring, on the inner side five or six, and on the exterior side six or seven, but seldom eight, tertiary nerves. They are strongly bent upwards, and united exactly in the same manner as in the specimens I have described in the 'Flora Tertiaria Helvetiæ' (vol. i. p. 31). At Bovey, hitherto, only sterile leaves have been found, whilst Monod has furnished pinnæ covered with sori (cf. Plate LVI. fig. 13). The sori are in the middle, or a little outside the middle of the tertiary nerves; they are round, small, and ranged in two lines, converging towards the apex. We see, from the specimens discovered at Monod, that this species was a very large one. The leaves attained probably a length of 3 feet and a diameter of at least 1 foot. The pinnules are very distant from each other at the base of the leaf, whilst above they approach and get gradually shorter. Most of the specimens are from the 17th bed of Bovey, I saw, however, one in the clay of the 26th bed.

Lastræa Stiriaca most resembles *L. prolifera*, Kaulf. (*Phegopteris prolifera*, Metten.), of tropical America, and belongs to the genus *Lastræa* (div. *Goniopteris*), as ALEX. BRAUN has established it (cf. Flora Tertiaria Helvet. i. p. 30, and iii. p. 150).

5. LASTRÆA BUNBURII, m. (Plate LXIII. fig. 1, *b*; magnified, *c*, *d*.)

L. fronde pinnata (?), pinnis linearibus, apicem versus angustatis, argute serratis, nervis secundariis flexuosis, e nervo primario angulo subacuto egredientibus, pin-

natis, nervis tertiariis utrinque 2–4, subflexuosis, curvatis, angulo acuto egredientibus.

Only a portion of a leaf, of 71 millims. in length; it occurred in the clay of the 26th bed. At the base it was probably 19 millims. in diameter; towards the apex it gradually tapers; therefore it must probably have had a long apex, which however is not preserved. It very much resembles the former species, but differs by the tapering form, the smaller and very acute teeth, and the fewer tertiary nerves, which are undulated. A comparison with fig. 4, pl. 8 of the 'Flora Tertiaria' shows that it is not the exterior part of the former species. It approaches more to *L. helvetica*, Hr. (Flora Tertiaria, i. p. 33, iii. p. 151), in the tapering of the pinnules, notwithstanding the fewer tertiary nerves and the undulated secondary nerves. It differs from *Aspidium dalmaticum*, Ett., in the smaller teeth, and fewer tertiary nerves.

The margin is provided with very small and sharp teeth, which are very much bent towards the apex. The secondary nerves are thin, strongly undulated, and have on the under side mostly four, sometimes but three, on the upper one two or three tertiary nerves, which are also undulated. The lowest one is united with the lowest of the next secondary nerves, and forms with it a triangular acute areole, out of which springs a branch that advances to the next sinus. A little higher it is joined by two united tertiary nerves, also at an acute angle, and immediately under the sinus another one. This nerve directed to the sinus is also undulated.

2. Pecopteris, Br.

6. Pecopteris (Hemitelia?) lignitum, Gieb. (Plate LV. figs. 4–6; LVI. figs. 1–11; LVII. figs. 1–7, magnified.)

P. fronde pinnata, pinnis linearibus, longis, apice valde attenuatis et acuminatis, basi plerumque breviter petiolatis, profunde inciso-serratis, nervis tertiariis furcatis, inferioribus valde curvatis, sinum attingentibus.

Pecopteris lignitum, *P. crassinervis*, *P. leucopetræ* et *P. angusta*, Giebel, "Paleontolog. Untersuchungen," Zeitschrift für die gesammten Naturwissenschaften, 1857, p. 305, pl. 2. fig. 2.

Aspidium lignitum, Heer, Beiträge zur nähern Kenntniss der sächsisch-thüringisch. Braunkohlenflora, p. 424, pl. 9. figs. 2 & 3.

Aspidium Meyeri, Ludwig (non Heer!), Palæontograph. viii. 2. p. 63, pl. 12. fig. 3.

This is the commonest fern at Bovey; it and *Sequoia Couttsiæ* are the plants most commonly met with in this locality; the petioles and pinnules are heaped up in the 17th and 26th bed. They are only separated by thin layers of clay. On dissolving this, one is able to take the leaves out of it and preserve them in fluid (spirit, glycerine, water). They decompose in the air.

This species had a large distribution in the tertiary period, but is confined to the

Tongrian and Aquitanian stages. It has been found at Weissenfels in Saxony, at Salzhausen and Münzenberg, at Thôrens in Savoy, and at Manosque in Provence.

Hitherto no perfect leaves or pinnules have been found at Bovey, but so many portions of pinnules that one can easily put together the entire pinnule, as we have tried to do in Plate LVI. fig. 8. The pinnule is in length 173 millims., its extreme breadth 21 millims.; it gradually tapers towards the apex, and forms a long sharp point (Plate LVI. figs. 2–4, and magnified fig. 1). The pinnule is also narrower at the base than in the middle; at least several pieces evidently belonging to the base are narrower than those from the middle of the pinnule (fig. 5). The pinnule is shortly petioled, and has often unequal sides at the base. There are many such pinnules around a finely striated rachis; portions of which latter common rachis are often found (cf. Plate LV. figs. 5, 4 *a*, & 8) between the pinnules, and in several cases I have seen the lateral pinnules attached.

The pinnules are of a strong, almost leathery consistence, and, examined with a lens, they appear to be finely punctate. Their margin is deeply toothed, the broad portions in the middle of the pinnule are nearly pinnatifid, the teeth on the outer margin smaller, and closer together. All these teeth are strongly bent towards the apex; the long side forms a strongly-curved arch; they are entire, and provided at the tip with a distinctly separated little tooth. The midrib is pretty strong; the secondary nerves spring at the base of the leaf in slightly acute angles, and in more acute ones in the upper narrower part of the pinnule. They are mostly more or less curved, and on each side they send forking tertiary nerves, on the upper part (nearer the tip) mostly one or two less than on the lower one. The broad pieces of pinnules have on the lower side seven or eight tertiary nerves; the number lessens nearer to the tip, as the pinnules gradually taper (we count there 6, 5, 4, 3, 2), and quite near the tip they are undivided (fig. 1). Each of the tertiary nerves soon divides into two branches, which do not again divide; only the exterior ones remain entire. Those tertiary nerves are everywhere equally strong, and sometimes the fourth or fifth advances to the margin, whilst the following one is again forking. The lowest ones are very strongly curved, and enter the sinus always between two teeth. The lowest tertiary nerves do not generally join the nerve of the neighbouring pinnules (Plate LVII. figs. 1, 4 & 5), or only in the sinus, forming a very acute-angled triangle (figs. 3 & 7). In some cases they unite like *Goniopteris*, already a little lower (Plate LVII. fig. 2), forming several very acute-angled arches. The tertiary nerves mostly spring from the secondary; but sometimes a fine forking nerve immediately springs from the principal nerve (Plate LVII. figs. 1–5), and runs to the sinus of the teeth.

A portion of a leaf very different in the nervation is figured in Plate LVII. fig. 6 (magnified). The tertiary nerves spring at very acute angles; they are very numerous, and some of them are twice forked. The teeth seem to have been narrower and longer. This fragment may belong to another species, but is too imperfect to pronounce any opinion upon. Beside the pinnules of the leaves, we often find at Bovey circinate young shoots of ferns (Plate LVI. figs. 9, 10 & 11), which probably belong to the species in question, as being the commonest one.

Professor GIEBEL, who first established this species, has given it four names: his *Pecopteris leucopetræ* represents the apex of the leaf, *P. angusta* a portion of the pinnule a little below the apex, where the tertiary nerves spring at acute angles, and *P. lignitum* and *P. crassinervis* the broader lower part of the pinnule.

LUDWIG has confounded this species with *Aspidium Meyeri*, Heer. The portion of a leaf from Münzenberg, figured by him in pl. 12. fig. 3 of the 'Palæontographica,' entirely differs from *A. Meyeri*, in the shallower incision and different nervation. It belongs to *P. lignitum*. It seems to me that the portion of a leaf figured by him in pl. 10. fig. 2, belongs to another species.

I have formerly assigned this fern to the genus *Aspidium*, because the secondary nerves are usually forked, and the tertiary nerves often jointed like *Goniopteris*; but a close examination of the numerous and well-preserved fragments from Bovey has convinced me that they do not belong to *Aspidium*. The pinnules had a hard, nearly leathery structure; and the peculiarly curved lower tertiary nerves, which run in large arches, differ from *Aspidium*, and very much remind one of *Hemitelia*. The tertiary nerves of the Hemitelias are most of them jointed by small nervules; but still there are species in which this is not the case (ex. gr. *H. integrifolia* and *H. speciosa*), and in *H. Karsteniana* (cf. Mettenius, Icones Filicum, pl. 29. fig. 2) there is a variety the nervation of which has more likeness to that of *Pecopteris lignitum* than to any other species of fern known to me; therefore the species in question probably belongs to the genus *Hemitelia*. It may, however, be better in the meanwhile to preserve the name *Pecopteris* till the fruits are found, which certainly will be soon, this species being so common at Bovey. I have, however, sought in vain for sori amongst many hundreds of pinnules. Sometimes little round spots are seen which look like sori (Plate LV. fig. 4 *d*); but a careful examination shows that they are accidental markings, some of them upon the tertiary nerves, and others beside them.

In the 25th bed at Bovey (rarely in the 17th) we often find large rhizomes quite covered with petioles, which, I suppose, for the following reasons, to belong to *Pecopteris lignitum*.

1. In several pieces, I found between the petioles the pinnules of *Pecopteris lignitum*, though not attached.

2. In the lignites of Salzhausen quite similar rhizomes are found with pinnules of *Pecopteris lignitum* (cf. Ludwig, in the 'Paleontographica,' viii. p. 64, pl. 10. fig. 3); therefore LUDWIG has compared it with this species (his *Aspidium Meyeri*).

3. The petioles are striated in the same manner as the petioles which are so often lying between the pinnules of *Pecopteris lignitum*, and which undoubtedly belong to one plant, as in some cases I saw them attached (Plate LV. fig. 4).

4. The rhizomes and the petioles are mostly curved towards one side; therefore the

rhizomes probably lay horizontally on the earth, and the leaves arched upwards, as we see in existing ferns.

The stems, and the petioles which cover them, are converted into coal; therefore a microscopic examination is not possible. The considerable thickness of the organs surrounding and covering the stems shows us that they are petioles, and not leaves. Though they are very much compressed, the thickness of the mass of coal is nearly always several millims.

The large specimens remind us directly of the pinnated leaves of palms (Plate LVIII.). On examining them minutely, we see that the supposed pinnules are not fastened in two rows on the rachis, but that they are placed around it spirally; therefore they cannot be pinnules of leaves; they are organs fastened on a stem; and as they are tapered at the base, and inserted on the stem with a tapering but not sheathing base, they cannot at all events be the leaves of a monocotyledonous plant. The leaves of ferns are inserted in that manner on the stem, and taper also sometimes towards the base. In most of the Aspidieæ, Asplenieæ, and Cyatheæ the petioles are continuous with the stem; they remain after the withering of the leaves, and form a thick and dense cover over the rhizome, as they do in the Bovey plant. It is remarkable that roots are seldom seen on these rhizomes; but in several specimens thread-like bodies can be seen between them, which probably were fibrils; and there are, further, some specimens which indicate that the numerous fine undulated striæ, which at some places are lying in heaps, are probably hairy scales which have covered the petioles.

The largest specimen is 7½ decim. long and 2 decim. broad. It is pretty strongly curved. The petioles are lying in heaps one upon another, and therefore must have covered the stem. In some specimens the stem is denuded here and there. It is quite converted into coal, and it is therefore as impossible to examine its anatomical structure as that of the petioles. These are of a considerable length, and always irregularly broken; their length therefore varies. I have collected great numbers of specimens at Bovey, and hoped to have been able to find the connexion of the petioles with the pinnules of *Pecopteris*, but was unsuccessful, though these rhizomes are in heaps in the 25th bed. The petioles gradually taper towards the base, which is rounded; they never sheath. Many specimens distinctly show that the petioles are not distichous, but are imbricated all round the stem. They are compressed, but the edges are defined.

At the same place there are portions of rhizomes provided with the cicatricules of roots (Plate LVIII. fig. 3). These are orbicular, 3 millims. in diameter, and consist of an elevated margin, surrounded by an orbicular depression, or, instead of this, they present a central wart, evidently originating from the central fibre. They are in pretty large numbers together, without being ranged in fixed order. The stems of living ferns have similar cicatricules of roots. The *Stigmaria*? of Altsattel, according to ROSSMÄSSLER*, probably represents also a portion of the stem of a fern with the cicatricules.

* Beiträge zur Versteinerungskunde, pl. 12. fig. 58.

7. PECOPTERIS HOOKERI, m. (Plate LVIII. fig. 3.)

P. pinnis elongato-lanceolatis, anguste serratis, nervis secundariis furcatis.

I have seen only a drawing (Plate LVIII. fig. 3) which Mr. FITCH has made. He assured me that the nervation and the natural size of the leaf were truly represented. The leaf seems to be lost; for it could be found neither in London nor at Mr. PENGELLY'S, at Torquay. It consists of a pinnule*, the base and point of which are wanting. It is toothed, the teeth sharp and bent towards the apex, the secondary nerves partly alternate, partly opposite, each of them divided into a simple fork. The branches of the fork run to the teeth.

II. PHANEROGAMÆ.

A. Gymnospermæ.

Order CONIFERÆ.

Fam. ABIETINEÆ, Rich.

1. SEQUOIA, Endl.

8. SEQUOIA COUTTSIÆ, m. (Plates LIX., LX., LXI.)

S. ramis alternis, rarissime verticillatis, ramulis junioribus elongatis, gracilibus; foliis squamæformibus, imbricatis, subfalcatis, medio dorso costatis, basi decurrentibus; strobilis globosis vel subglobosis; squamis peltatis, medio brevissime mucronulatis, rugosis; seminibus alatis, compressis, nucleo paulo curvato.

This and *Pecopteris lignitum* are the commonest plants of Bovey, and their stems certainly contribute the greatest amount of lignite. Larger and smaller branches of this tree occur in the 17th and 26th beds of the clay. Entire cones (as represented in Plate LIX. figs. 1, 14, 16 & 18), seeds, and scales of cones have been found in great numbers. It is certain that the cones and seeds belong to one plant; for they not only agree with those of *Sequoia*, but in several cases I have seen the seeds lying in their natural position under the cone. But it might be questioned if all those branches the principal forms of which are represented in Plates LIX. and LX. belong to this same tree, because the young twigs so closely resemble those of *Glyptostrobus europæus*. A very minute comparison, however, of many specimens has persuaded me that this is not the case, and that all the figured branches and cones belong to one plant. In comparing the leaves of the twigs which bear the cones (Plate LIX. figs. 14, 16 & 18), we see that their form agrees with the loose twigs. As the principal character of these leaves, we may observe that they are nearly always somewhat falcate (cf. Plate LX. figs. 14–20). This is not the case with *Glyptostrobus europæus* (cf. Plate LX. fig. 49, magnified fig. 49 *b*,

* This form very much reminds us of *Lastræa Bunburii*, the nervation of which is however quite different.

where a twig of *Glyptostrobus* of the Hohe Rhonen is represented for comparison). The adnate scale-like leaves are straight in *Glyptostrobus*, or only somewhat curved outwards and obtuse, as in *Glyptostrobus heterophyllus*, Br. The spreading leaves are often provided with an acute and usually straight point, and, but seldom, a little curved; when this is the case, they have at all events a great likeness to our *Sequoia*.

As we have the branches, fruits, and seeds of this tree, the determination of the genus is undoubted. It resembles in all its principal points *Sequoia*, Endl. It has decurrent leaves, globose cones with peltate scales. They are provided on the surface with wrinkles radiating from a small mucro.

Several flat-winged seeds are lying under the scales. The scales and seeds very much resemble those of *Sequoia sempervirens*, Lamb., of California (cf. this cone, Plate LX. fig. 48, and the seeds of this species, fig. 47, magnified 47 *b*); but the nucleus of the seed is somewhat curved in the fossil species. The leaves are quite different, those of the sterile branches of *Sequoia sempervirens* being distichous and long linear, almost as in *Taxus baccata*, Linn. The *Sequoia Couttsiæ* approaches *S. gigantea* (*Wellingtonia*, Lindl.) in the form and position of the leaves, but differs in the much smaller cones. The Bovey species is in some measure intermediate between the two existing species.

In comparing the species of Bovey with the tertiary species of *Sequoia*, the *S. Langsdorfi*, Br., will be first taken into consideration. The cones are very similar (cf. Flora Tertiaria Helvetiæ, pl. 21. fig. 4 *d*, pl. 146. fig. 16; and Ludwig, Palæontographica, Band viii. pl. 15. fig. 1); but this species has the leaves of *S. sempervirens*. Our species still more resembles *S. Hardtii* *, the scales of the cones being of the same length and form, but Unger and Ettingshausen describe the cones as subconical, and the seeds as provided with a mucro; further, the leaves of the fertile twigs are more acute, and those of the sterile ones are linear and spreading. The Bovey species differs from *S. Sternbergi* (*Araucarites*, Gp.) in the much more slender twigs, and the different construction of the leaves. It differs from *S. Ehrlichi*, Ung., in the shorter leaves and the globose cones.

If we compare all the Sequoiæ now known, we have to place them in the following manner:—

1. *Sequoia gigantea*, Lindl.; California.
2. *Sequoia Ehrlichi*, Ung.; tertiary formation near Spital in Austria.

* *Cupressites Hardtii*, Goeppert, Monogr. der Fossilen Coniferen, p. 184.

Cupressites taxiformis, Unger, Chloris protogæa, p. 18, pl. 8. figs. 1–8, pl. 9. figs. 1–4.

Chamæcyparites Hardtii, Endl. Synopsis Conifer. p. 277; Ettingshausen, Flora von Haering, p. 35, pl. 6. figs. 1–21.

Endlicher and Ettingshausen have wrongly referred this species to *Chamæcyparis*; the leaves of which genus are opposite, and ranged in four rows round the branch, whilst they are alternate in *S. Hardtii*, as in *Sequoia*; the cones of *Chamæcyparis* are much smaller, and the seeds have a thicker nucleus, the base and point of which are not surrounded by the thin wing. This species has no doubt been referred to *Chamæcyparis* by Unger from a comparison with *Cupressus thurifera*, Humb. et Bonpl. (*Chamæcyparis*, Endl.).

3. *Sequoia Sternbergi*, Gp.*; has a wide distribution in the Miocene formation, from Senegaglia to Iceland.

4. *Sequoia Couttsiæ*, Hr.; Bovey.

5. *Sequoia Hardtii*, Gp.; Haering and Armissan.

6. *Sequoia Langsdorfi*, Br.; spread over the whole Miocene formation, in Italy, Germany, Switzerland, France, Isle of Mull, Greenland, Bear-lake River, Vancouver Island, from the Unga at the shore of Aleski, Russia near Orenburg.

7. *Sequoia sempervirens*, Lamb.; California.

The genus *Sequoia* probably begins in the cretaceous formation; for *Geinitzia* (*Cycadopsis*, Deb.) is so nearly related to *Sequoia*, that, according to Dr. Debey, it can scarcely be separated from this genus, and may be considered as the predecessor of it. This genus most abounded in the Miocene time; it was spread over the whole Continent as far as we know. In the present creation we have but two remains of this type, both found only in California. These two living species represent the two extremes of all the known forms. Of the fossil species, *S. Langsdorfi* especially approaches to *S. sempervirens*, and *S. Ehrlichi* and *S. Sternbergi* to *S. gigantea*. *S. Couttsiæ* is the intermediate species between these two principal types.

I now proceed to describe the Bovey specimens.

The annual twigs (Plates LX. figs. 9–20; LIX. and LX. figs. 7 & 8, magnified) are very slender, often of considerable length (figs. 14–16), without producing lateral twigs. The leaves cover the twigs like scales, which are mostly very close together (Plate LX. fig. 10); on the long twigs they are more distant. At the base of the young shoots they are always closer and shorter (fig. 7), a little more outwards they are more distant. The leaves are alternate, though sometimes two are nearly opposite, but never exactly so. All the leaves are decurrent at the base. The very short scale-like leaves are somewhat falcately curved (cf. Plate LX. fig. 13, magnified), and still more so are the leaves which are more distant from one another (Plate LX. figs. 14, 15 & 17). These leaves are

* I have already tried to show, in my 'Flora Tertiaria Helvetiæ,' iii. p. 317, note, that *Araucarites Sternbergi*, Gp., belonged to *Sequoia*. Massalonga has given, in his 'Specimen Photographicum,' pl. 21, a photograph of his *Araucarites venetus* of Chiavon. He said that the leaves and twigs were not different from those of *Araucarites Sternbergi*; he took it for a different species, because he thought, with Unger and Ettingshausen, that the cone presented under the name of *Araucarites Gœpperti*, St. (Sternberg, Pflanzen der Vorwelt, pl. 89. fig. 4; Goeppert, Fossile Coniferen, pl. 44. fig. 2), was to be referred to *A. Sternbergi*. But we have shown, in the 'Flora Tertiaria,' that there are no sufficient reasons for it; and the circumstance that cones are found, in Chiavon, on a twig which is not to be distinguished from *A. Sternbergi*, and also immediately beside similar twigs in Iceland (quite different from *A. Gœpperti*), confirms this opinion. The cone of Chiavon is much compressed, and lies in a lateral position, while the cones of Iceland represent the transverse section (as do the pieces represented by Sternberg, 'Flora der Vorwelt,' ii. pl. 57. figs. 1–3, as *Steinhauera subglobosa*, Presl, which, according to my opinion, belong to this species). The scales of the cones are small and in great numbers, and very different from *Araucaria*, while the cone figured by Massalonga has a great resemblance to that of *Sequoia gigantea*. The seeds represented as *Steinhauera* quite agree with *Sequoia*.

acuminate, and the point is curved outwards (Plate LX. fig. 14 *b*, somewhat magnified). Amongst the great number of twigs I have seen, there was only one with much longer linear leaves (Plate LX. fig. 12, magnified fig. 12 *a a*) than they commonly appear in *Sequoia Hardtii*. Twigs with such long leaves therefore must be very rare. The small portion of a twig which is represented in fig. 9 (magnified) forms the intermediate form of this leaf. The leaves are always rigid, and provided at the back with an elevated edge, which runs to the apex of the leaf. The biennial twigs (Plate LX. figs. 1 & 2) are much thicker and also quite covered with leaves, which are scale-like, applied to the stem; they are broader than the leaves of the annual twigs, and closer together at the base of the twigs. They show at different places the scars on which the alternate shoots have been fastened. I only saw one branch with whorled twigs (Plate LIX. fig. 13). The branches of three years (Plate LIX. figs. 9 & 11) are 5 to 6 millims in breadth. We see on them numerous cicatrices of branches, which indicate the insertion of the twigs. The leaves are nearly of the same size as those of the biennial ones; they are, however, not so close together; they are scaly, adhering to the twigs, and the epidermis is provided with many longitudinal wrinkles. These twigs are thicker at the base (Plate LIX. fig. 11). As there are many twigs found thus thicker at the base, they appear to have separated themselves very easily from the stem at the place of insertion. The leaves disappear on still thicker and therefore older twigs, and only small scars remain on the bark. Beside these branches of different ages, there are trunks which probably belonged to this tree.

The cones are solitary or in pairs (Plate LIX. fig. 14, restored fig. 15), on rather slender twigs quite covered with scaly leaves. They are globose (Plate LIX. fig. 16, restored fig. 17; fig. 14), or shortly oval (fig. 19), from 15 to 24 millims. in length, and from 15 to 17 millims. in breadth. The scales are peltate, the footstalk is short, and seems to be central (Plate LX. figs. 29, 30 & 33); the upper side is polygonal, but this form is not constant; in the middle is a very short mucro, from which originate several wrinkles that radiate to the margin; the surface is therefore pretty roughly wrinkled. Cones in which the scales are closed are rare (Plate LIX. figs. 14, 16 & 18). They must have been enveloped by the clay when still fresh. They are often spread open and the scales separated from one another, and the spaces filled with clay (Plate LX. figs. 27 & 28); or we have but solitary scales or portions of cones (Plate LX. figs. 29–35).

There are several seeds beneath every scale (Plate LX. fig. 25); they are also very often scattered between the twigs. The seed is usually 5 millims. long and 3½ millims. broad, and flat; it is somewhat emarginate at the point of insertion, obtusely rounded and a little tapered towards the tip. The nucleus is somewhat curved and pretty flat: It is surrounded by a flat wing (Plate LX. figs. 37–41; fig. 41 *b*, magnified).

In the lignite of Bovey both small and very large pieces of resin are found, which were probably secreted by the *Sequoia Couttsiæ*.

The lignite of Bovey contains very large stems, the zones of which are mostly distinct and crowded; in several stems I could count a hundred of them. One can easily split them longitudinally; but they become quite hard and brittle in the air, and crumble if cut. They are then brown or black. The microscopical examination did not give me satisfactory results. I certainly recognized the elongated woody fibres, but in most cases I was not able to distinguish the structure of their walls, which is very obscure. However, in some cases I saw pores, which are ranged in one row (cf. Plate LXXI. figs. 8 & 9), and, further, the medullary rays, which are formed from a single row of cells (Plate LXXI. fig. 8). There is no trace of spiral or reticulated vessels. The structure of the medullary rays and of the fibres proves it to be coniferous wood. As *Sequoia Couttsiæ* is the only coniferous tree hitherto found in the lignite-beds of Bovey, and was the commonest tree of that country, it is very probable that most of the wood belonged to it.

B. Monocotyledones.

Order I. GLUMACEÆ, Bartl.

Fam. I. GRAMINEÆ, Juss.

1. PHRAGMITES, Trin.

9. PHRAGMITES ŒNINGENSIS, A. Br.? (Plates LXIV. fig. 1 *d*; LXV. fig. 13 *a*; and LXVIII. fig. 2.)

Phr. foliis latis, multinervosis, nervis interstitialibus tenuissimis.

Heer, Flora Tertiaria Helvet. i. p. 64, pl. 22. fig. 5, pl. 24.

Only some small parts of leaves; the determination is therefore uncertain. We observe very slender secondary nerves between the strong longitudinal nerves. We can count twelve of them, all equally strong, between two longitudinal nerves, while in *Phr. œningensis* the median nerve is always a little stronger. In the same layer (in the 17th bed of Bovey) there are also some indistinct remains of culms, which probably belong to these fragments of leaves; likewise the pieces figured in Plate LXVIII. fig. 2, which all represent horizontal sections of culms. Fig. 2 *a* represents the section of a a knot, where, as in *Phragmites communis*, we have a middle part, which appears as a circular umbo, and around it the wall of the culm.

2. POACITES, Br.

10. POACITES, sp. (Plate LXVIII. fig. 3.)

Bovey (Dr. Falconer).

It is a thin, finely striated grass-culm, on which we perceive a knot. It is $2\frac{3}{10}$ millims. in breadth. It is not sufficient for a more exact determination, but seems to prove that small grasses existed in that country.

Fam. II. Cyperaceæ, DC.

3. Cyperites, Hr.

11. Cyperites deperditus, m. (Plate LX. fig. 54.)

C. fructibus parvulis, ovatis, apice acuminatis, tenuissime striolatis.

In the 26th bed of Bovey; fragments only.

Under the name of *Cyperites*, I have comprised in my 'Flora Tertiaria' those remains (leaves, culms, and fruits) belonging to plants of the family of the Cyperaceæ which it is not possible now to refer to any genus. The fruits in question very much resemble those of *Carex*, and belong probably to this genus. They are like those of *Carex recognita*, Hr., from Rochette. The fruit is 4 millims. in length, and at the base $2\frac{2}{5}$ millims. in breadth; it is there obtusely rounded, but tapered and acuminate at the apex. It is provided with very fine longitudinal striæ.

Order II. PRINCIPES, Linn.

(Palmæ.)

1. Palmacites, Hr.

12. Palmacites Dæmonorops. (Plates LXII., LV. figs. 7-15.)

P. spatha coriacea, longitudinaliter tenuissime striata granulataque, aculeata, aculeis crebris seriatim in lineis oblique transversis conjunctis, compressis, subulatis, rectis, simplicibus, binis, trinis, vel ad summum senis, adpressis; caudice gracili, aculeato, fasciculis vasorum rigidis, interne planis vel sulco exaratis.

Palæospathe Dæmonorops, Unger, Sylloge Plantar. Fossil. p. 9, pl. 2. figs. 9–12.

Chamærops teutonica, Ludwig, in Meyer's Palæontogr. viii. p. 86, pl. 20. figs. 2 & 3.

Pretty common in the clay of the 26th bed of Bovey.

Chiefly the prickles are found at Bovey. They are black as coal, brilliant, very thin, and taper to a fine point. Their length varies from 4 to 50 millims. The longest are only 3 millims. in breadth at the base (cf. Plate LV. fig. 13). They are flat, and provided with a very shallow longitudinal depression (Plate LXII. fig. 9 *b*, a specimen highly magnified), which in some cases becomes almost a furrow. There are mostly three of them together (Plates LXII. fig. 1, and LV. figs. 11 & 15), and the median one is the longest (Plate LXII. figs. 2, 10 & 11, magnified). Sometimes there are but two together, or they are single; or, on the contrary, there are four, five (Plate LXII. fig. 3), or more prickles forming a group. In many pieces I am persuaded that they are not fastened on the margin of an organ, but pretty regularly distributed on a plane surface. Therefore they cannot be simple prickles of the petioles of leaves, as of *Chamærops*, for which Ludwig has wrongly taken them, as I was convinced on looking at the pieces represented in Plate LXII. fig. 1, and still more when I washed them in water. We see then that the bundles of prickles are fastened on and pressed against a very finely striated plane surface, which is provided with minute warts (Plates LXII. fig. 7, and LV. figs. 11 & 12). They are broader at the base. The form of these prickles, their position (mostly three together, and the median of them the longest), and their direction (all are directed forwards and pressed against the surface) agree so well with the organs represented by UNGER from Laubach in the Wetterau, and by LUDWIG from Salzhausen and Hessenbrucken, that we may be persuaded of their belonging to the same plant. UNGER compares these prickles with those which appear on the spatha of *Dæmonorops*. We have indeed in *D. polyacantha*, Martius (Palmæ, pl. 160), and *D. melanochætes*, Blume (Martius, pls. 117 & 125), very similar flat prickles, which are broader at the base. There are often three together, the median one of which is also the longest. As at Bovey, these bundles of prickles are often fastened on a flat, spreading, finely striated organ, which is several millims. in thickness, which may be easily taken for a spatha. But it is quite unsuitable to found the name of the genus upon this spatha, and to describe this plant as *Palæospathe*, for we find the same prickles also attached to the stems. I have found at Bovey, in the 26th bed, the stem of a palm, on which such a bundle of prickles was attached. Unhappily it was ruined on the journey. On the contrary, several of these stems, which I collected in the same layer containing the prickles, have remained entire. Some of them are represented in Plate LV. figs. 7–10. They are from 15 to 17 millims. in thickness, and consist of a bundle of fibres which are converted into coal, and appear like brilliant coal-black threads. Each fibre (or bundle of vessels) is flat on the inner side, which is turned towards the centre of the stem, or furrowed longitudinally; on the outer side it is convex; it represents, therefore, on a transverse section, more or less, a half circle or a crescent. These fibres are in some pieces close together, and in others further apart; the latter are probably from the middle, and the former from the periphery of the stem. Some of the fibres attain a thickness of 1 millim., but many are much thinner. In many pieces these bundles of vessels form a stem (cf. figs. 8 & 9). Sometimes they are free, and appear like long black fibres scattered in the clay; or beside the united ones we see numerous scattered bundles (cf. Plate LV. figs. 7 & 10). These organs very much resemble the bundles of vessels which appear in the lignite of Käpfnach (*Palmacites helveticus*, Flora Tertiaria Helvetiæ, p. 94, pl. 40. fig. 1); but such thick pieces have not hitherto been found at Bovey. The stems were thin, and probably the petioles may have been so too. The construction of the bundles of vessels shows indeed that they belonged to a monocotyledonous plant, and confirms the conclusion derived from the prickles attached to the spatha.

I found at the same place several fragments of leaves (Plate LXVIII. fig. 1) which certainly belong to a monocotyledonous plant, and perhaps, therefore, to the species in question. They are broad portions of leaves, with numerous fine longitudinal parallel striæ of equal strength; nine of them occupy 1 millim. They are therefore very close.

Mr. PENGELLY found a fruit at Bovey* which probably belongs to the plant in ques-

* I have seen only a drawing of this.

tion. It is oval (cf. Plate LX. fig. 50; magnified, figs. 51–53), 13 millims. in length and 10 millims. in breadth, and covered with scales. It resembles the fruit of the Rotang palm; and as the prickles and the construction of the stems point to these palms, we may combine these organs, and conclude that this palm grew at Bovey. In connexion with this I may mention that large pinnatifid leaves (cf. Flora Tertiaria Helvetiæ, iii. pl. 149), which belong to a Rotang palm, have been found at Oeningen. I have described them as *Calamopsis*. The palm of Bovey belongs perhaps to this genus; but I thought it better to place it in the collective genus *Palmacites*, which contains the different organs of palms which cannot yet be ranged in a fixed genus. These prickles have not been found at Oeningen up to the present time; and the petioles of *Calamopsis* are without prickles. Similar black prickles, however, appear in different genera of palms (for instance *Bactris*), but they are certainly most frequent in the group of the *Calameæ*, Kunth (*Lepidocaryinæ*, Mart.).

C. Dicotyledones.

Coh. I. *APETALÆ.*

Order I. AMENTACEÆ.

Fam. I. Cupuliferæ, Rich.

1. Quercus, Linn.

13. Quercus Lyelli, m. (Plates LXIII. figs. 2–9; LXIV. figs. 1–4; LXV. fig. 12 *b*; LXVI. figs. 1 & 2; LXVIII. figs. 4 & 5.)

Q. foliis subcoriaceis, petiolatis, lanceolatis vel oblongo-lanceolatis, basi attenuatis, margine undulatis, apice acuminatis, nervo primario valido, recto, nervis secundariis numerosis, curvatis, apice furcatis, ramulo superiore margini valde approximato.

Phyllites cuspidatus, Rossmässler, Beiträge zur Versteinerungskunde, p. 36, pl. 9. figs. 38 & 39.

It frequently appears in the 17th bed at Bovey.

Three forms are distinguishable:—1. narrow lanceolate leaves, which are strongly tapered at the base, and the margin of which is slightly undulated (Plates LXIV. fig. 1 *b*, *c*; LXIII. figs. 3 & 8); 2. narrow leaves, with almost parallel sides in the middle (Plate LXV. fig. 12 *b*); 3. broad leaves, which are distinctly undulated at the margin (Plate LXVI. figs. 1 *a* & 2; LXIII. figs. 5–7). This last form is the most frequent, and is to be considered as the typical. These leaves are of a pretty hard texture, but they seem to have been less leathery than those of *Q. furvinervis*. There are no entire leaves preserved; however, with the aid of the different pieces we can complete one. They were broadest in the middle, and gradually tapered towards the petiole and apex. In most of the fragments the petiole is not preserved. The piece represented in Plate

LXIII. fig. 9 shows us a petiole 14 millims. in length, in which it differs from *Q. furvinervis.* The pieces represented in Plate LXVIII. figs. 4 & 5 show us that the apex of the leaf was very long. The piece represented in fig. 5 had a very tapered apex. The broad leaves have a pretty strongly undulated margin (Plates LXIV. fig. 1 *a*, and LXIII. fig. 7, magnified 7 *b*) without forming teeth. The median nerve is strong and straight. Numerous secondary nerves spring from it at an angle of 50°. They are rather strongly curved, and reach nearly to the margin, where they fork, and the upper branch bends forward and, running parallel with the margin to the next following secondary nerve, joins it. The lower branch of the fork is very small and slender, or quite wanting in slightly undulated leaves, and at the places where the undulation is wanting. This course of the secondary nerves is very characteristic of our species, as of *Q. furvinervis* and *Q. undulata.* The areas are divided into secondary areas by continuous, sometimes forking nervules. A polygonal reticulation may be perceived in them, which encloses a still more delicate one (cf. Plate LXIII. fig. 7 *b*).

Rossmässler has described two forms of leaves from Altsattel as *Phyllites furvinervis* and *Ph. cuspidatus.* The latter differs from the former in the undulated margin (which is not toothed), and the long tapering point. As I also got from Weissenfels the toothed form with an elongated point, I formerly united *Ph. cuspidatus* with *Ph. furvinervis* (cf. Beiträge zur nähern Kenntniss der sächsisch-thüring. Braunkohlen-flora, von O. Heer; Abhandlungen des naturw. Vereins für die Provinz Sachsen und Thüringen, ii. p. 424), and I described the form with the narrow leaves as *Q. furvinervis cuspidata.* But as at Bovey there is found only the form with the undulated leaves, and as there is a petiole (which is not preserved with the leaves of Altsattel) which by its length gives us a new distinctive character for *Phyllites cuspidatus*, I have separated it from *Ph. furvinervis.* The name given by Rossmässler could not be retained, as there is already a *Quercus cuspidata*, Thunb. The principal distinction between *Q. Lyelli* and *Q. furvinervis* consists in the length of the petiole, and in the latter species having undulated leaves.

Further examination, and a comparison of more copious material, will show if these differences are specific or not. They belong, at all events, to the same type, which was very frequent in the Lower Miocene period, and of which similar species are still living in Mexico (for instance, *Q. xalapensis*, Thunb.). *Quercus undulata*, O. Weber* (Palæontograph. ii. p. 170, pl. 19. figs. 1, 2 *a* & *b*), is also a very similar species. In the undulated margin, and the manner in which the secondary nerves run, it is like *Q. Lyelli*; but the median nerve of this species is straight and much stronger, not undulated; the secondary nerves are more numerous, and spring at less acute angles.

I still doubt if the leaf represented in Plate LXIII. fig. 7 belongs to the species in

* The leaves (pl. 19. fig. 2 *a* & *b*) in Weber's 'Abhandlung' belong, according to my opinion, also to *Quercus undulata*, and are different from *Q. Gœpperti* (pl. 19. fig. 2 *c*), in which the leaf is rounded at the base. I may mention that the leaf which Ludwig (Palæontogr. viii. pl. 34. figs. 1–4) has represented as *Q. furvinervis*, belongs to another species.

question. The form and kind of tapering at the base agree, but it has more numerous and more crowded secondary nerves.

I found in the 26th bed at Bovey several fragments of bark which probably belonged to an oak, and therefore might belong to the species in question. The bark is very thick and corky, provided with deep longitudinal furrows, and in some places furrowed transversely (cf. Plate LXVIII. fig. 6).

Fam. II. Moreæ, Endl.

2. Ficus, Tournef.

I have already mentioned in my 'Flora Tertiaria' (ii. p. 64) that the leaves of many living species of fig-trees (I allude to the species which have leaves similarly shaped to *Ficus fulva*, Spr., *F. rubra*, Spach, *F. ferruginea*, Desf., and *F. phytolaccæfolia*, Hort. Berol.) have a granulated appearance and a rough surface due to numerous little warts which cover the epidermis, and that the tertiary flora of Switzerland also possessed two species, the leaves of which had the same rough surface. It is very curious that there are in the 26th bed at Bovey leaves which are granulated in this manner; and one of them is very near to *Ficus scabriuscula*, Hr. I therefore believe that I am not mistaken in ranging these three species under the genus *Ficus*. The following facts indicate that the granulation belongs to the leaf and not to the stone:—1. The stone certainly is finely granulated; but these granules are larger than the small points of the leaves, which can only be seen with the aid of a lens, and which are all of the same size; 2. they are regularly spread over the surface of the leaf; 3. they are on all the pieces of this species, but not on the leaves of other plants (e. g. *Cinnamomum lanceolatum*, *Daphnogene Ungeri*, and *Lastræa Bunburii*), which are lying in the same clay.

14. Ficus Falconeri, m. (Plates LXIII. fig. 1 *a*; LXIV. figs. 6 & 7; LXVI. fig. 4.)

F. foliis coriaceis, magnis, confertissime granulatis, elliptico-lanceolatis, apice longe acuminatis, nervis secundariis subtilissimis, remotis, valde curvatis.

The leaf represented in Plate LXIV. fig. 7 is the best preserved. The upper part is quite perfect, but the base is wanting. It is entire, and 12 millims. wide in the middle; it is gradually tapered towards the apex, forming a long narrow point. The secondary nerves are very delicate, and there are but few to be discovered with the lens. They are strongly curved, and form arches, which are far from the margin. The areas are divided by very delicate nervules, and filled up with a very delicate reticulation. The rigid points which cover the whole surface are very small, and only to be seen with a strong lens. Plate LXIV. fig. 6 is the point of the leaf, the secondary nerves of which are more distinct.

It is still doubtful if the leaf represented in Plate LXIII. fig. 1 *a* belongs to this species. It shows the same peculiar sculpture (cf. fig. 1 *a a*), and also very delicate secondary nerves; but they approach nearer to the margin, and are much less directed towards the apex. This leaf is very like *Ficus obtusata*, Heer*, and belongs perhaps rather to this species; but as the specimen is incomplete, it is better to unite it with *Ficus Falconeri*. It may be that the areoles of the small areas are not so long (cf. fig. 1 *a a* magnified, and 1 *a a a* still more magnified) as those of *Ficus obtusata*. They are quite covered with little warts. The secondary nerves of the leaf in question are also very delicate, and united in strongly curved arches at the end. The areas are divided by means of delicate nervules into secondary areas, in which is the fine polygonal reticulation (fig. 1 *a a*).

15. FICUS PENGELLII, m. (Plates LXV. figs. 7 & 8, and LXVI. fig. 3.)

F. foliis coriaceis, longe petiolatis, confertissime granulatis, ellipticis, basi apiceque attenuatis, nervis secundariis remotis, angulo acuto egredientibus, valde curvatis.

The leaf represented in fig. 8 shows the same peculiar sculpture as the former species (cf. fig. 8 *b*, where the portion of a leaf is magnified); it is more indistinct in the second leaf (fig. 7), which belongs to the same species. The species in question differs from the former one in the elliptical form of the leaf, which has no long point, and the acute angle of the secondary nerves. The leaf is broadest in the middle and equally tapered at both ends, so that the sides form pretty regular curved lines. The secondary nerves are delicate, springing from acute angles, strongly bent towards the apex, and thus form long arches. The principal areas are divided by means of delicate nervules into secondary areas, in which the polygonal areas are covered with very small rigid points that are only to be seen with the aid of a strong lens. The petiole is very long, and rather slender. It belongs to the same section as *Ficus phytolaccæfolia*, Hort. Berol.

16. FICUS EUCALYPTOIDES, m. (Plate LXV. figs. 3, 4 & 5.)

F. foliis coriaceis, lanceolatis, confertissime granulatis, basi apiceque attenuatis, nervis secundariis subtilissimis, valde curvatis.

This is another hard leathery shining leaf, but much smaller than that of the former species. It is broadest in the middle, and almost equally tapered towards both ends. It is doubtful if a petiole existed. Very delicate secondary nerves spring from the median nerve; they are jointed in strong arches. The surface is also covered with very delicate warts. It is like the leaf of *Eucalyptus oceanica*, Ung., in form, but differs especially in the surface, because *Eucalyptus* has quite a smooth leaf.

* Flora Tertiaria, ii. p. 65, pl. 82. figs. 5 & 6, pl. 100. fig. 14.

Order II. PROTEINÆ.

Fam. I. LAURINEÆ, Vent.

1. LAURUS, Linn.

17. LAURUS PRIMIGENIA, Ung. (Plate LXV. fig. 6.)

L. foliis subcoriaceis, late lanceolatis, acuminatis, nervo primario valido, nervis secundariis tenuibus, sparsis, sub angulo acuto egredientibus.

Unger, Fossile Flora von Sotzka, p. 38, pl. 19. figs. 1–4.

Heer, Flora Tertiaria Helvetiæ, ii. p. 77, pl. 89. fig. 15; iii. p. 184, pl. 153. fig. 3.

In the 26th bed.

The leaf represented in Plate LXV. fig. 6 agrees very well with the leaf of the Hohe Rhonen, which is represented in pl. 153 in my 'Flora.' It is dark brown, rather hard, smooth and entire, and gradually tapered towards the base. The secondary nerves are at a distance from each other, strongly curved, and form long arches near the margin. The large principal areas are divided into secondary areas by fine nervules. The apex of the leaf is not preserved.

2. CINNAMOMUM, Burm.

18. CINNAMOMUM ROSSMÄSSLERI. (Plate LXVII. figs. 17 & 18.)

C. foliis ellipticis vel oblongo-ellipticis, triplinerviis, nervis lateralibus acrodromis apicem attingentibus, nervatione in areis reticulata.

Flora Tertiaria Helvetiæ, ii. p. 84, pl. 93. figs. 15–17.

Phyllites cinnamomum, Rossmässler, Versteinerungen von Altsattel, p. 23, taf. 1. fig. 4.

Daphnogene cinnamomifolia, Unger, Genera et Spec. Plant. Foss. p. 424.

Bovey Tracey, in the 17th bed.

The two fragments figured are the only ones which were found; their apices are wanting, the determination therefore cannot be considered as quite sure. The leaf, however, agrees in its size, form, and nervation with *C. Rossmässleri*. Only a little piece is wanting in the leaf represented in fig. 18, as the raised sides show; the two lateral nerves run, however, to the end of the preserved part, and are so distinct there that undoubtedly they reached to the apex, which forms the principal character of the leaf, which is nearly related to *C. zeylanicum*, Bl., and still more to *C. eucalyptoides*, Nees (*C. nitidum*, Hook.). The leaf has a tolerably slender petiole; it is broadest in the middle, and equally tapered towards both ends. The two lateral nerves springing above the base are pretty strong, and run parallel with the margin. The very delicate nervules spring at right or at least not very acute angles.

19. Cinnamomum Scheuchzeri. (Plates LXVII. figs. 9–16; LV. fig. 4 *e*; LXVIII. fig. 12.)

C. foliis petiolatis, ellipticis, ovalibus et oblongis, triplinerviis, nervis lateralibus margine parallelis vel subparallelis, apicem non attingentibus; pedunculis articulatis, pedicellis apice incrassatis, fructibus ovatis.

Flora Tertiaria Helvetiæ, ii. p. 85, pl. 91. figs. 4–24, pl. 92, pl. 93. figs. 1–5.

Rather frequent in the 17th and 26th beds at Bovey.

These are leathery, entire, and three-nerved leaves, which are broadest in the middle, and gradually and equally tapered towards both ends; they are acuminated at the base in the same manner as at the apex, without running into the petiole; the sides form almost regularly curved lines. The two strong lateral nerves are jointed above the base of the leaf; they are mostly opposite (Plate LXVII. figs. 11, 13 & 16), the one seldom a little higher than the other (Plate LXVII. figs. 10 & 15); they run parallel with the margin. They do not reach to the apex of the leaf, but they unite above the middle of the leaf with a secondary nerve of the midrib (figs. 10 & 11). The areas are provided with very delicate nervules, which spring at almost right angles. I found the two flowers which are represented in Plate LXVIII. figs. 13 & 13 *c* (magnified 13 *b* and 13 *d*) in the 26th bed of Bovey. They very much resemble the flower from Oeningen represented in the 'Flora Tertiaria Helvetiæ,' pl. 91. fig. 23 *b*, and therefore probably belong to *Cinnamomum Scheuchzeri*. The leaflets of the perianthium of the flower represented in fig. 13 *c* are indistinct; they are much better preserved in fig. 13 (magnified 13 *b*). There are six leaflets, standing in two whorls around a circular wall. The leaflets are hard, shortly oval, obtusely rounded at the apex, and provided with longitudinal striæ. They are shorter and broader, and more obtusely rounded than in *Cinnamomum polymorphum*, A. Br. (cf. Flora Tertiaria Helvetiæ, pl. 94. figs. 1–5). We have beautiful branches from Oeningen with leaves of this species. We lately got a branch with the inflorescence, which confirms the systematic position which I assigned to the leaves. I therefore represent this branch in Plate LXVII. fig. 12. Numerous flowers, still in the state of buds, but nearly breaking off, are lying together, and sometimes one above the other, so that it is not possible to find out their insertion with certainty. One sees, however, that the peduncles spring from the axils of the leaves, and we have therefore an axillary inflorescence. The peduncles are club-shaped above, and this part forms with the bud an almost globose body, on which the leaflets of the perianthium are indicated at some places. These peduncles lengthen afterwards and become more sharply articulated, as the peduncles and fruit-stalks which I have represented in my 'Flora Tertiaria' (pl. 91. figs. 4–7) show.

It is very like *C. pedunculatum*, Thb., from Japan.

20. Cinnamomum lanceolatum. (Plates LXVII. figs. 1–8; LXVIII. figs. 14 & 15.)

C. foliis petiolatis, lanceolatis, basi apiceque acuminatis, triplinerviis, nervis lateralibus margine parallelis, approximatis, acrodromis, apicem non attingentibus.

Flora Tertiaria Helvetiæ, ii. p. 86, pl. 93. figs. 6–11.

Daphnogene lanceolata, Unger, Fossile Flora von Sotzka, p. 37.

Rather common in the 17th and 26th beds; it is also a leaf of the 54th bed.

Is like *C. Scheuchzeri*, but differs from it in the leaves being narrower, longer, elongated into a long point, and tapering towards the petiole.

The leaf is also leathery, strongly tapered towards the base, and gradually towards the apex, forming a long sharp point. The two strong lateral nerves are near the margin and run parallel with it; delicate secondary nerves proceed from the median nerve above the middle of the leaf and join them; several delicate secondary nerves follow more outwards. The areas are traversed with very fine nervules, and the secondary areas which these form are filled up with a fine polygonal reticulation. The nervules spring at almost right angles, and the upper secondary nerves at acute angles. Fig. 1 represents a leaf of the 26th bed, the others are of the 17th; figs. 5 & 6 represent young leaves.

3. Daphnogene, Ung.

21. Daphnogene Ungeri, Hr. (Plate LXV. figs. 1 & 2.)

D. foliis lanceolatis, basi subrotundatis, longe petiolatis, triplinerviis, nervis basilaribus margine subparallelis, nervulis obsoletis.

Flora Tertiaria Helvetiæ, ii. p. 92, pl. 96. figs. 9–13.

Ceanothus lanceolatus, Ung., Fossile Flora von Sotzka, p. 49.

Bovey, in the 26th bed.

This leaf differs from those of *Cinnamomum Scheuchzeri* and *C. lanceolatum* in being widest below the middle, and in the delicate nervules being absent in the large areas. In most of the leaves of this species from our molasse, and also from Manosque in Provence, the base of the leaf is rounded, which is not the case in the Bovey leaves; but since also at Sotzka (cf. Unger, plate 31. fig. 14) leaves occur of which the base is not so much rounded off, we may without hesitation unite the Bovey leaves with the above-mentioned species. The leaf (fig. 2) otherwise agrees very well with that which is represented by Unger.

The leaves are hard, coriaceous; they attain the greatest breadth below the middle, and taper gradually to the apex. The strong lateral nerves run near the margin and parallel with it; they have no visible nervules in the areas.

Fam. II. Proteaceæ, R. Br.

4. Dryandroides, Ung.

22. Dryandroides hakeæfolia, Ung. (Plate LXV. fig. 12 *a.*)

Dr. foliis coriaceis, firmis, lanceolatis lineari-lanceolatisque, in petiolum attenuatis, apice acuminatis, plerumque apice dentatis, dentibus remotis, inæqualibus, nervo medio valido, nervis secundariis subtilissimis, camptodromis, areis marginem fere attingentibus, nervatura firma, hypodroma, areolis magnis scrobiculatis.

Unger, Fossile Flora von Sotzka, p. 39, pl. 20. figs. 7–10.

Heer, Flora Tertiaria Helvetiæ, ii. p. 100, pl. 98. figs. 1–13, plate 99. figs. 4–8.

Only fragmentary leaves have been found at Bovey (in the 17th bed), but they cannot be mistaken, since they have the characteristic nervation of this species. In one piece (which in fig. 12 *a* lies upon the slab with the apex pointing downwards) part of the margin is still complete, and shows the sharply indented teeth which *Dr. hakeæfolia* commonly has. In the same leaf the nervation is pretty well preserved. The secondary nerves are very delicate, and project but very little beyond the other reticulation. To the naked eye, or when but slightly magnified, the leaf appears dotted, owing to the distinctly projecting small arches which form the reticulation.

23. Dryandroides lævigata. (Plate LXV. figs. 9, 10 & 11.)

Dr. foliis coriaceis, firmis, lanceolatis, basi in petiolum attenuatis, apice attenuatis plerumque integerrimis, nervo medio valido, nervis secundariis manifestis camptodromis.

Flora Tertiaria Helvetiæ, p. 101, pl. 99. figs. 5–8.

In the 17th bed of Bovey.

It is very like the preceding species; but the margin is generally entire, and the nervation is more delicate, being less prominent.

The leaf of Bovey which has been represented in fig. 10 is coriaceous, lanceolate, and tapering gradually towards the tip, where it ends in a point; it is entire; the median nerve is rather strong, but the secondary nerves are very delicate; they reach to near the margin, where they form arches. The more delicate reticulation, which is formed by very small and densely crowded areoles, can be made out by the aid of a lens, but it is much less distinct than in the foregoing species.

The leaf agrees well with that which has been represented in plate 99. fig. 7 of my 'Flora Tertiaria.'

24. Dryandroides Banksiæfolia. (Plate LXVIII. fig. 7.)

Dr. foliis petiolatis, firmis, linearibus vel lanceolato-linearibus, undique argute serratis, basi apiceque acuminatis, nervis secundariis approximatis, subrectis, simplicibus, parallelis.

Flora Tertiaria Helvetiæ, ii. p. 102.

Myrica Banksiæfolia, Ung., Gen. et Spec. Pl. Fossil. p. 395.

Bovey, in the 26th bed.

Only the portion of a leaf represented in Plate LXVIII. fig. 7 has been found at Bovey. It cannot be determined with certainty, as the characters which the above diagnosis (copied from my 'Flora') contains are not recognizable in them.

It was certainly a hard leathery leaf with a very delicate nervation. The secondary nerves are but slightly indicated and run towards the teeth, which are very sharply cut. The fragment probably represents part of the leaf near the base, like plate 100. fig. 10

of the 'Flora Tertiaria Helvetiæ'; and that is the reason why the teeth are there wanting. We find them, however, in most of the leaves of *D. Banksiæfolia.*

Fam. III. SANTALACEÆ, R. Br.

5. NYSSA, Linn.

25. NYSSA EUROPÆA, Ung. (Plate LXIX. figs. 11–17.)

N. putamine 4½–7 mm. longo, 3–6½ mm. lato, ovali, rarius subgloboso, basi truncato, extus striis longitudinalibus rugosis exarato.

Unger, Sylloge Plantar. Fossil. p. 16, pl. 7. figs. 25–27.

Frequent in the 26th bed of Bovey.

The fruit most probably agrees with the fragments which UNGER described as found at Nidda in the Wetterau, and which I got from Salzhausen; the latter are, however, somewhat more compressed, and consequently more flattened; hence the wrinkled longitudinal ribs do not project so much. Very similar also is a fruit represented as *Nyssa rugosa* by O. WEBER in the 'Palæontographia,' ii. plate 20. fig. 10 *c*; this fruit differs, however, in its shape and smaller dimensions, from two other fruits that were mentioned by WEBER under the same name.

We distinguish amongst the Bovey fruits three forms:—

a. Oval fruits, 4½ to 5½ millims. long and 3 to 4 millims. wide; they are truncated at the base, in the fore part obtusely rounded and marked with numerous rather deeply wrinkled longitudinal furrows (figs. 11–14, magnified 12 *b*).

b. Oval fruits, 6–7 millims. long and 4–5 millims. wide; they are also truncated at the base, but somewhat rounded, marked with numerous wrinkled longitudinal furrows. They are shaped like the stones of *Nyssa sylvatica*, but are somewhat smaller (figs. 15 & 16).

c. Nearly globose fruits, 7 millims. long and 6 to 6½ millims. wide; at the base abruptly truncate, with strongly marked irregular wrinkled furrows (fig. 17, magnified 17 *b*).

The form *a* is the most frequent; the form *c* has been met with in but a few pieces; the form *b* is not rare, and corresponds most to the fruit-stones of the Wetterau. Whether these forms belong to one or more species, I dare not decide.

They are very like the fruit-stones of the species of *Nyssa* which grow at present in the United States, especially of *Nyssa biflora* and *denticulata*, which are represented by GÄRTNER (De Fructibus, iii. p. 216).

26. NYSSA LÆVIGATA, m. (Plate LXIX. fig. 18, magnified 18 *b*).

N. putamine 5–7 mm. longo, 4–5 mm. lato, ovali, basi truncato, lævigato, extus bistriato.

In the same bed of Bovey; rare.

It has the same dimensions and shape as the foregoing species, but it is smooth, shining, and without wrinkled furrows, notwithstanding we observe on the outside two longitudinal lines, which are formed, as in the fruit-stones of *Nyssa*, by delicate striæ.

27. Nyssa microsperma, m. (Plate LXIX. fig. 24, magnified 24 *b*.)

N. putamine 4 mm. longo, globoso, extus striis longitudinalibus rugosis exarato.

Only a single specimen from the same bed.

It is upon the whole very like the small forms of *Nyssa europæa*, but it is still smaller, perfectly globose, truncated at the base and deeply furrowed.

At the base of the fruit-stone appears a small hole; the sides are marked with deep longitudinal furrows, and the intervening ribs are wrinkled.

28. *Nyssa striolata*, m. (Plate LXIX. figs. 20–23, magnified 20 *b*.)

N. putamine 10–12 mm. longo, 6–8 mm. lato, compresso, ovali, tenuiter longitudinaliter striato.

Several specimens from the 26th bed.

It is very like the *Nyssa ornithobroma*, Ung. It is of the same shape and the same dimensions as the specimen represented by Unger (fig. 16), from which we must separate it, because the striæ are more delicate and more densely arranged. The small nuts are perfectly flattened, and at the base obtusely rounded, but they terminate in the apex in a small point. Along the middle of each specimen runs a longitudinal fissure, which was probably caused by pressure. The side is furrowed by numerous very delicate longitudinal striæ, while the intervening spaces are smooth.

Coh. II. *GAMOPETALÆ.*

Order I. BICORNES, Endl.

Fam. I. Ericaceæ, Dec.

1. Andromeda, Linn.

29. Andromeda vacciniifolia, Ung. (Plate LXVIII. fig. 9.)

A. foliis coriaceis, oblongis, integerrimis, apice obtusis, basi rotundatis vel subrotundatis, petiolatis.

Unger, Fossile Flora von Sotzka, p. 43, pl. 23. figs. 10–12.

Heer, Flora Tertiaria Helvetiæ, iii. p. 7, pl. 101. fig. 25.

In the 17th and 26th beds of Bovey.

I found several pieces, but only the one represented in fig. 6 has remained entire, the others have been ruined on the journey. This leaf certainly agrees with the leaves represented in my 'Flora,' figs. 25 *b* and 25 *c*. The leaf has a petiole and a pretty strong median nerve; the lateral nerves are obsolete, and the reticulation is not preserved.

30. Andromeda reticulata, Ett. (Plate LXVIII. figs. 10 & 11.)

A. foliis coriaceis, lineari-lanceolatis, acuminatis, basi in petiolum attenuatis, integerrimis, nervatione dictyodroma.

Ettingshausen, Tertiäre Flora von Haering, p. 65, pl. 22. figs. 9 & 10.

Bovey; several pieces in the 26th bed.

The leaf is hard, leathery, and tapers towards the petiole, narrow-lanceolate, the sides almost parallel; the apex is not preserved in the leaves of Bovey. The median nerve is pretty strong; the whole leaf is covered with a fine polygonal reticulation (fig. 10 *b*, a portion of a leaf magnified), which reaches from the median nerve to the margin and has almost everywhere meshes of the same size. Only at some few places in this reticulation there appear delicate secondary nerves, which rise at a very acute angle. It very well agrees in the form and nervation with pl. 22. fig. 9 of Ettingshausen's work.

This species is nearly related to *Andromeda protogæa*, Ung., and is perhaps a variety of it; the reticulation is, however, much more delicate, and can only be seen with the lens. The leaf of *Eucalyptus oceanica*, Ung., is similarly shaped, but the nervation is different.

Fam. II. Vaccinieæ, Dec.

2. Vaccinium, Linn.

31. Vaccinium acheronticum, Ung. (Plate LXVIII. fig. 8.)

V. foliis subcoriaceis, ovalibus vel oblongis, integerrimis, petiolatis.

Unger, Flora von Sotzka, p. 43 (ex parte), pl. 24. figs. 1, 3, 4 & 6.

Heer, Flora Tertiaria Helvetiæ, iii. p. 10, pl. 101. fig. 29.

Bovey, in the 26th bed.

A small longitudinal, oval and entire leaflet with a short petiole. The median nerve is thin, and several delicate curved secondary nerves proceed from it.

This leaflet is longer and more tapered at the base than those of the 'Flora Tertiaria Helvetiæ,' but it certainly agrees with the leaf represented by Unger in the 'Flora of Sotzka,' pl. 26. fig. 6, and may therefore be referred to this species.

Order II. CONTORTÆ, Endl.

Fam. I. Apocyneæ, R. Br.

1. Echitonium, Ung.

32. Echitonium cuspidatum, Hr. (Plate LXIV. figs. 3 *b* & 5; Plate LXV. fig. 12 *c*.)

E. foliis lineari-lanceolatis, apice cuspidatis, integerrimis, nervis secundariis numerosis, camptodromis, areis margini approximatis, reticulatis.

Heer, Flora Tertiaria Helvetiæ, iii. p. 192, pl. 154, figs. 4–6.

Several pieces of leaves from the 17th bed of Bovey.

The more delicate nervation is almost obliterated in the fragments of the leaves from Bovey, but these agree well with our species as to their forms and the direction of the secondary nerves; nevertheless this determination cannot be considered to be quite certain. Very similar is a leaf from the lignite of the Rhine, which is represented by O. Weber as *Labatia salicites* (Palæontogr. iv. p. 154, pl. 28. fig. 1); but the leaves of Bovey, like those of Locle, taper into a more elongated apex, and the arches of the secondary nerves approach nearer the margin.

The leaf is very long and narrow, tapering gradually towards the base; and also elongating into a very long point. From the median nerve spring rather numerous secondary nerves, which are highly incurved, forming near the margin arches that run nearly parallel with the latter; these arches are very delicate, and can only be traced by the aid of a lens. The reticulation of the areas is only slightly indicated.

Order III. RUBIACINÆ.

Fam. I. Rubiaceæ, Juss.

1. Gardenia, Ell.

33. Gardenia Wetzleri, Hr. (Plate LXIX. figs. 1–8.)

G. fructibus lignosis, oblongo-ovalibus vel ovato-lanceolatis, subcostatis et multistriatis, polyspermis, seminibus nigro-brunneis, nitidis, striis spiralibus notatis.

Heer, Flora Tertiaria Helvetiæ, iii. p. 192, pl. 141. figs. 81–103.

Passiflora Braunii, Ludwig, Palæontograph. viii. p. 124, pl. 48. figs. 1, 4, 5 & 16.

Bovey Tracey, in the 34th bed.

At Bovey no complete fruits have been found like those which we know from near Königsberg, from the brown-coal of the Rhön and Wetterau and of Günzburg, but we have got rather numerous seeds, partly still in the position which they held within the fruits (cf. Plate LXIX. figs. 1 & 2). These seeds agree, in respect to their position, their forms, and the peculiar structure of the testa, so entirely with those of the Continent, that no doubt can arise about the question whether they belong to the same plant or not. They are arranged in series, but in such a manner that they partly overlie each other; and in consequence of pressure the shape has been here and there somewhat altered, resulting in more or less deep impressions. Figures 3–5 (magnified) show the principal forms which the seeds of Bovey display. At the apex they taper into a small point, at the base they are somewhat rounded; the colour is a shining brownish black, furrowed by very delicate striæ forming distinct spirals, the deeper striæ alternating with the more delicate (cf. especially the highly magnified piece of a seed, fig. 6).

In order to illustrate this remarkable species, which has such a very wide range, I have represented in fig. 7 a fine fruit that was sent me by Director Albrecht of Königsberg out of the clays of Samland; fig. 8 represents a similar fruit from Günzburg, remarkable for its great dimensions.

In the 'Flora' (*l. c.*) I have described this species at full length, and I have tried to show that it belongs to the genus *Gardenia*, while Ludwig refers it to *Passiflora*. It cannot belong to the latter genus, since *Passiflora* has fleshy fruits on long slender stalks, and seeds which are always distinguished by peculiar little pits ("semina impresso-scrobiculata" is mentioned by Endlicher as a characteristic feature of this family). Even the genus *Vareca*, to which Ludwig refers, has fleshy hexagonal fruits, which are only about half an inch long, and furnished at the base with a small round cup with six notches; of the seeds each rests in a separate cell: all these are peculiarities which are not met with in the fossil fruit (cf. Gärtner, De Fructib. et Seminib. i. p. 290).

In the 'Flora' I have enlarged upon the reasons why I united these fruits and seeds with *Gardenia*. The *Gardenia lutea*, Fres. and *G. Thunbergii*, L. fil., have also ligneous fruits with a pericarp furrowed by similar longitudinal fibres, and they are also supported on short and thick stalks that pass gradually into the fruit; besides we remark, exactly in the same manner as in the fossil fruit of Königsberg, represented in fig. 7 *b*, the placentæ parietales forming ridges, which project from the inner surface, and which constituted most probably soft membranaceous partitions proceeding towards the interior. The seeds of some Gardenias are arranged in series of which the number varies—in one fruit, known from the Brazils, in four, in others in more series; the number of the placentæ also varies from two to six. Thus we have in *G. Thunbergii* five, in *G. lutea* six, while the seeds lie together in great numbers and are indistinctly arranged in series; they are horizontal; in the fossil fruit they are also horizontal, but generally somewhat obliquely turned towards the bottom, with the point directed towards the base of the fruit. They are arranged in series, which are, however, but very seldom as distinctly marked as is the case in the pieces represented by LUDWIG. I thought I could distinguish five series, but LUDWIG mentioned six; and if so, we should in these fruits assume three placentæ, to each of which two series of seeds are fixed. Yet in the fruit from Königsberg there seem to exist four placentæ; two are visible in the interior of the one-half of the fruit of which the outside is represented in fig. 7 *b*, and two seem to be on the other half (7 *a*); one, which is very distinct, runs somewhat beyond the middle throughout the fruit, and another seems to have existed on the right side at the edge; this latter, however, is indistinct and uncertain. In fig. 7 *c* I have represented the transverse section; the line marks the place where the fruit was broken into halves: also from Salzhausen I got only fruits which were thus split in the middle into halves. I do not know upon what grounds M. LUDWIG supposed that the fruit was divided into three valves. It was certainly an indehiscent fruit.

Coh. III. *POLYPETALÆ.*

Order I. UMBELLIFLORÆ.

Fam. AMPELIDEÆ, Kunth.

1. VITIS, Linn.

34. VITIS HOOKERI. (Plate LXIX. figs. 27, 28 & 29.)

V. seminibus parvis, $3\frac{1}{2}$ mm. longis, 3 mm. latis, brevibus, ovato-acuminatis, læviusculis, dorso convexiusculis, tuberculo chalazino rotundato magno.

Bovey Tracey, in the clay of the 26th bed.

I have got but one single seed of this species, which, however, cannot be mistaken. It is very like those of the grape of Salzhausen, which A. BRAUN described as *Vitis* *teutonica*, and of which UNGER (Syllog. Plantarum, p. 23, pl. 9. figs. 1–8) and LUDWIG (Palæontogr. viii. p. 118, pl. 45. figs. 1–5, pl. 46. fig. 6) lately gave descriptions and drawings; it is a species which had a very wide range, and which I pointed out as occurring at Oeningen, in the brown coals of the Rhön and at Schossnitz. The seed of the grape of Bovey is, however, smoother, comparatively wider and shorter; more flattened on the dorsal part, and the little pits on the ventral surface are less deep; while also the small wart (verruca) on the back is somewhat larger, for which reasons we must separate the seed of this grape as a different species.

It is most obtusely rounded at the base, and terminates at the apex in a little short point; upon the dorsal part (fig. 29, six times magnified fig. 29 *d*) it is only slightly convex; the chalazal tubercle (tuberculum chalazinum) is large and nearly circular; upon the ventral part (fig. 29 *b*, *c*) appears a rather prominent ridge, along the middle of which runs, as in *Vitis teutonica*, a delicate longitudinal stria; on each side of the ridge we observe a small pit (scrobicula), which, however, is much less deep than in *Vitis teutonica*.

To this or the following species belong probably the grapes from the 17th bed of Bovey (figs. 27 & 28), which are very like those of *Vitis teutonica*. They are flattened, nearly circular fruits, in the interior of which no grains can be distinguished, as in those from the brown coals of Langenaubach (cf. LUDWIG, *l. c.* p. 120).

Fig. 29 represents the dorsal surface of the seed; fig. 29 *b*, the ventral surface; fig. 29 *c*, *d*, magnified; fig. 29 *c*, section; figs. 27 & 28, grape.

35. *Vitis britannica*, m. (Plate LXIX. figs. 25 & 26.)

V. seminibus parvulis, 5½ mm. longis, 3 mm. latis, ovato-ellipticis, dorso planis, tuberculo chalazino obsoleto.

From the 26th bed of Bovey.

Of this species I got two seeds, which belong probably to *Vitis* or *Cissus*; but the chalazal tubercle is not distinct, and the consequent determination cannot be considered to be quite certain. These seeds are longer and comparatively narrower than those of the foregoing species; on the ventral side we observe the prominent ridge; the pits on both sides are very flat and oblong; the dorsal part is flattened. On one seed (the other is much compressed) we observe an oval tuberculum chalazinum, which appears, however, flat, imperfectly stamped out, and therefore indistinct.

Order II. POLYCARPICÆ, Endl.

Fam. ANONACEÆ, Dun.

1. ANONA, Linn.

36. ANONA (?) DEVONICA, m. (Plate LXX. figs. 1–3.)

A. seminibus ovalibus, 19–21 mm. longis, 12–14¼ mm. latis, compressis, læviusculis.

A good many specimens from the 26th bed of Bovey.

Belongs probably to *Anona altenburgensis*, Ung. (Sylloge Plantar. p. 26, pl. 10. figs. 8–11). The representation agrees pretty well with the seeds of Bovey; but as UNGER in the diagnosis calls the seeds "ovato-oblongis," and since those of Bovey are no wider at the base than at the top, I dare not to unite them with those of Altenburg.

Most of the specimens are 20 to 21 millims. long, widest in the middle, narrowing symmetrically towards both ends, where they are most obtusely rounded. They are flat, the diameter only amounting to 2½ millims.; throughout they are formed of an homogeneous mass of coal, which has obliterated all traces of the contents of the seeds. On the outside they are smooth and rather shining when rubbed; sometimes they appear reticulated by delicate fissures, or furrowed with very fine longitudinal and transverse striæ, which are, however, much obliterated, and can only be perceived by the aid of a lens. In some pieces the middle portion is perfectly flat and slopes towards a rather sharp margin (cf. section, fig. 2 *b*), thus forming a flatly bordered seed like those of *A. paludosa*, Kit.; in others the margin is blunted. In one piece (fig. 3) the middle portion has fallen out, thus producing a broad and deep furrow; this is probably only accidental.

In fig. 1 the seed still lies within the stone, convincing us that it is not surrounded by a shell. The perfectly homogeneous substance of the coal indicates a seed and not a fruit; and since the Anonas (especially the *A. paludosa* and *Asimina triloba*) have quite similar seeds, I agree with the determination of UNGER, although it cannot be denied that it must be considered as somewhat doubtful till other organs shall be discovered.

37. ANONA CYCLOSPERMA, m. (Plate LXX. fig. 4.)

A. seminibus suborbiculatis, 14 mm. longis, 11–12 mm. latis, compressis, rugulosis.

At the same spot; several specimens.

They are very like the preceding, but much smaller, especially much shorter; and as the breadth is nearly the same, they are almost orbicular; a network is constituted by numerous irregular fissures, which cause the outside to be somewhat wrinkled (fig. 4 *b*). Since no intermediate forms occur between this and the foregoing, they seem to belong to different species.

Order III. HYDROPELTIDEÆ.

Fam. NYMPHÆACEÆ, Salisb.

1. NYMPHÆA, Linn.

38. NYMPHÆA DORIS, m. (Plate LXX. figs. 32–37.)

N. seminibus ovalibus, 2½–3½ mm. longis, subtilissime crenulato-striolatis, apice poro perforatis.

From the 54th bed.

Tolerably numerous flattened seeds lying close together: they are oval, 2½ to 3½ millims.

long and 2 to 3 millims. wide; at both ends they are obtusely rounded, and at one end are furnished with a small round hole (fig. 33, magnified fig. 34; fig. 35, magnified fig. 36), which is only visible in those specimens that are compressed from above. The sides are furrowed by very delicate and elegant longitudinal striæ, formed by the cells, which are arranged in lines. They are neatly crenated and very delicately dotted (cf. fig. 37, a highly magnified piece, of which only part of the testa has been preserved). This elegant sculpture can only be made out by the aid of a lens or microscope.

They are very much like the seeds of *Nymphæa alba*, especially the form with the larger seeds, the *Nymphæa alba melocarpa*, Casp. In the recent seeds the longitudinal striæ appear much less distinct; but in such seeds as are found in the 'Pfahlbauten' of Robenhausen, they look exactly like those of the fossil species. In these also the hole on the top of the seed is somewhat more widened, as in the fossil species. As to the shape, the latter differs in so far as the seeds are comparatively somewhat wider (those of *N. alba melocarpa* are 3½ millims. long and 2 millims. wide): this more considerable breadth may have been caused by pressure, since they are all highly compressed. To the same circumstance we may ascribe the fact that the raphe cannot be pointed out distinctly.

The seeds of Bovey are very like those of my *Nymphæa Charpentieri* (Flora Tertiaria, iii. p. 195, pl. 155. fig. 20 *b*, *c*), the latter being only somewhat narrower. It would therefore be very desirable that the leaves be sought for at Bovey, since from these alone can it be decided with certainty whether this species is really distinct from *N. Charpentieri*, to which it comes very near. The seeds of the living *Nymphææ* are variable in their dimensions.

Order IV. MYRTIFLORÆ.

Fam. Myrtaceæ, R. Br.

1. Eucalyptus, Hérit.

39. Eucalyptus oceanica, Ung.? (Plate LXIX. figs. 9 & 10.)

E. foliis coriaceis, 2–5-pollicaribus, lanceolatis vel lineari-lanceolatis, acuminatis, subfalcatis, in petiolum attenuatis, integerrimis, petiolis semipollicaribus, sæpius contortis, nervo primario distincto, secundariis subtilissimis.

Unger, Flora von Sotzka, p. 58, pl. 36. figs. 1–13.

Heer, Flora Tertiaria Helvetiæ, iii. p. 34, pl. 108. fig 21, pl. 154. figs. 14 & 15.

In the clay of the 26th bed.

As at Bovey only fragments of leaves have been found, no sure determination can be given. The leaves are smooth, shining, coriaceous, somewhat incurved, and taper towards the tip. The secondary nerves are very delicate and form arches, which run nearly parallel with the margin. Close by the leaf (represented in fig. 9) lies the top of a leaf of a quite different plant, which seems to belong to *Quercus Lyelli*.

2. EUGENIA, Mich.

40. EUGENIA HÆRINGIANA, Ung. (Plate LXVIII. figs. 16, 17 & 18.)

E. foliis coriaceis, lanceolato-linearibus, in petiolum brevem crassumque attenuatis, integerrimis, nervis secundariis distantibus, simplicissimis, camptodromis, duobus inferioribus valde elongatis.

Unger, Flora von Sotzka, p. 52, pl. 35. fig. 19.

Heer, Flora Tertiaria Helvetiæ, p. 34, pl. 2. fig. 1, pl. 108. fig. 16, pl. 154. fig. 13.

Bovey, in the 26th bed.

The leaves of Bovey are very like the leaf of Ralligen represented in my 'Flora,' pl. 108. fig. 16. They are leathery, entire, long and narrow, and tapered towards the petiole, the sides being almost parallel. The two long secondary nerves are very near the margin, and run nearly parallel with it. We cannot distinguish the finer nervation.

Resembles *Cinnamomum lanceolatum*, but the leaf is longer, relatively narrower, and has parallel sides.

Order V. FRANGULACEÆ.

Fam. CELASTRINEÆ, R. Br.

1. CELASTRUS, Linn.

41. CELASTRUS PSEUDO-ILEX, Ett. (Plate LXVIII. fig. 19.)

C. foliis coriaceis, lanceolato-linearibus, sessilibus, integerrimis, nervo medio debili, secundariis camptodromis.

Ettingshausen, Tertiäre Flora von Haering, p. 70, pl. 24. figs. 30–36.

Heer, Flora Tertiaria Helvetiæ, p. 69, pl. 121. fig. 57.

I found several leaflets in the 26th bed of Bovey; unhappily most of them have been ruined in the journey, and only the piece represented in fig. 19 *b* has remained entire. I drew the piece represented in fig. 19 (magnified 19 *b*) on the spot. These are small, stiff, entire and narrow leaflets, obtusely rounded at the apex. Short lateral nerves proceed from the delicate median nerve. It certainly agrees with the leaves of Haering, and those of Locle and Oeningen.

It may be compared to *Celastrus integrifolius*, Thunb., from the Cape.

Order VI. TEREBINTHINEÆ.

Fam. JUGLANDEÆ, DeC.

PTEROCARYA, Kunth.

42. PTEROCARYA DENTICULATA? (Plate LXX. fig. 5.)

Pt. foliis pinnatis, foliolis sessilibus, lanceolatis, acuminatis, argute et creberrime serratis, nervis secundariis numerosis.

Heer, Flora Tertiaria Helvetiæ, iii. p. 94, pl. 131. figs. 5–7.

Juglans denticulata, O. Weber, Palæont. p. 211.
Bovey, in the 26th bed.

This species remains still very doubtful, since nothing but the fragments of leaves represented was found at Bovey. The diagnosis has therefore not been made from these fragments, but from the very fine leaves that were found in our molasse; further discoveries alone can prove whether the fragments of leaves found at Bovey belong really to this species.

The leaf of Bovey is very acutely toothed, and the teeth (augmented in fig. 5 *b*) are highly inclined towards the apex. The secondary nerves are delicate, and for the most part perfectly obliterated.

Order VII. LEGUMINOSÆ, Endl.

Fam. PAPILIONACEÆ, Linn., R. Br.

LEGUMINOSITES, Bow., Heer.

43. LEGUMINOSITES AREOLATUS, m. (Plate LXVIII. fig. 20, magnified fig. 20 *b*).

L. foliolis subcoriaceis, integerrimis, ovalibus, basi valde inæquilateris, nervis secundariis subtilissimis, camptodromis, areis argute reticulatis.

Bovey, in the 26th bed.

Is very like *Leguminosites sclerophyllus* (Flora Tert. Helvet. p. 123), but much more delicate, smaller, and with a sharper reticulation. It may belong to *Copaifera*.

A small leaflet, which has unequal sides at the base. The median nerve is curved and delicate. The secondary nerves are very delicate, at a great distance from each other, and jointed in large arches. The areas are filled up with a very delicate, distinctly projecting polygonal reticulation. The areoles are of equal size.

Incertæ Sedis.

CARPOLITHES, Sternb., Heer, Flora Tertiaria Helvetiæ, iii. p. 139.

44. CARPOLITHES WEBSTERI. (Plate LXX. fig. 6, magnified 6 *b*.)

C. fructibus oblongis, subcompressis, utrinque obtusis, supra basim incurvatam paulo constrictis, longitudinaliter rugoso-striatis, rima longitudinali dehiscentibus, monospermis, membrana interna tenera, libera (testa seminis?).

Carpolithes thalictroides, var. *Websteri*, A. Brongniart, Recherches sur les Ossemens Fossiles, par Cuvier, ii. pl. 11. fig. 5; Mémoires du Muséum, p. 317, pl. 14. fig. 6.

Carpolithes Kaltennordheimensis, Flora Tertiaria Helvetiæ, iii. p. 141, pl. 21. fig. 14, pl. 141. figs. 68 & 69.

Folliculites Kaltennordheimensis, Zenker in Leonhard und Bronn's Jahrbuch, 1833, p. 177, pl. 4. figs. 1–7.

Carpolithes minutulus, Sternb. Versuch. i. 4. p. 41.

Folliculites minutulus, Bronn, Lethæa, p. 849, pl. 35. fig. 11.

J. D. Hooker, "On some small Seed-vessels from the Bovey Tracey Coal," Quart. Journ. Geol. Soc. Lond. for Nov. 1855, p. 566, pl. 17.

Unger, Sylloge Plant. Fossil. p. 17, pl. 7. figs. 10–23.

Hippophaë disperma, Ludwig in Palæontogr. viii. p. 112, pl. 43. figs. 14–18 & 20.

Not rare in the 54th bed; very rare in the rhizome-bed.

The fruits of Bovey agree so perfectly with those of Kaltennordheim, of Rochette, &c., that they must be referred to that species. Geologically they are of great importance, as they have such a wide range in the Miocene formation; it is hence the more to be regretted that their systematic position remains still very doubtful, and that it has even not yet been shown whether these small bodies, which can so easily be distinguished by their characteristic form and structure, are fruits or seeds. Most authors (BRONGNIART, ZENKER, BRONN, and HOOKER) consider them to be fruits, while UNGER (Sylloge, p. 18) declared recently that they must be seeds, referring to the seeds of coniferous trees: as such I considered them formerly, when I described them (though granting the doubts concerning their character) as *Pinus rhabdosperma*. In all those specimens which I had then seen, the peculiar constricted volva was not distinct; but it is also wanting in all the seeds of conifers which I know, as is also the prominent edge and the sculpture. More probably they are comparable with the seeds of *Samyda*, which have such a volva, and which agree besides pretty well in respect to their form; but then we must bear in mind that the organs in question are ligneous, that they are dehiscent by a longitudinal fissure, and that such facts are much against an interpretation as seeds, though in favour of the hypothesis that they might have had the organization of a fruit.

HOOKER supposed them to be cryptogamic fruits, although he was not able to point out an analogy amongst the cryptogamic plants. He maintains this view by stating that in the interior of the fruit a delicate sac is frequently found, which he considers to be a sporangium. This interpretation seems to me not quite correct; the sac seems to result from the testa of the seeds. The dotted and the spiral fibrous cells covering the interior of the cavity are much more in favour of a phanerogamic than of a cryptogamic plant; and in respect of the sac remaining in the interior of the cavity, we know exactly the same organization in cherry-stones which have lain for a long time in water or in wet places. I opened a good many cherry-stones from the 'Pfahlbauten' of Robenhausen; in all, without exception, the seed had disappeared (although the shell was left intact), and in its stead a delicate sac remained, which lay close to the interior surface of the stone, and which could easily be removed. A perfect dissolution of the kernel had taken place, and only the testa remained. The organic contents of the seed must hence have become dissolved, and escaped, in a remarkable manner, through both the testa and the putamen, that no traces of them were left in the cavity of the cherry-stone. Only the ligneous cells of the stone and the testa resisted and were preserved. In the same manner, without doubt, we must account for the existence of the sac of *Carpolithes Websteri*. Sometimes an inorganic substance has been deposited in the place formerly occupied by the seed, and the interior of the cavity of the fruit is then partly filled with it.

LUDWIG (*l. c.*) has united these fruits with *Hippophaë*, adopting, without necessity, a new specific name. These fruits do not, however, agree in the least with those of *Hippophaë*, and decidedly cannot be referred to that genus. He believes that great numbers of such fruits were arranged close together upon the boughs, which circumstance probably led him to refer them to *Hippophaë*. But it seems to me doubtful whether those little heaps of fruits which he represented were really fixed on the boughs that lie close by, or whether they lie only accidentally near the boughs. If they are fixed on the boughs, it will be proved that they are phanerogamic fruits and not seeds, while the interpretation as *Hippophaë* is not justified, considering the other contradictory marks.

The *C. Websteri* has been described so frequently and so carefully, that any more detailed description would be superfluous. I refer the reader especially to the very exact description of HOOKER (*l. c.*). Investigation with the microscope showed me that the sac in the interior of the fruit is formed by cells with very thin walls and of different lengths (cf. fig. 6 *d*, *c*), as represented by HOOKER in fig. 7, and that the pericarpium, on the contrary, is formed by ligneous cells, which partly show a very curious undulating circumscription, and are dotted (fig. 6). They are arranged in series*.

45. CARPOLITHES SCUTELLATUS, m. (Plate LXIX. fig. 30, magnified 30 *b*.)

C. fructibus complanatis, margine acutis, rotundatis, basi truncatis, dorso costulatis.

A few specimens in the 26th bed at Bovey.

It is a perfectly flat fruit (or seed?) with an acute margin (cf. the section of fig. *c*); at the base it is truncated, at the sides much rounded, obtusely rounded above, nearly in the middle somewhat emarginate. From the base originate a few delicate, partly ramified ribs, which reach somewhat above the middle.

It is rather like the fruits of *Panax*, but the middle partition is wanting.

Since I had only one well-preserved fragment at my disposal, I could not cut it open in order to see whether it had one or two furrows. Perhaps it may be a seed, and not a fruit.

46. CARPOLITHES BOVEYANUS, m. (Plate LXX. figs. 7–14.)

C. nucula 3–4½ mm. longa, ovata, apice mucronulata, dorso leviter sulcata, monosperma.

Frequent in the 26th bed at Bovey.

* The drawing M. A. BRONGNIART has given of this species is very imperfect. I did not myself recognize his drawing as belonging to this species; but Dr. FALCONER convinced me that it was so. I compared, in the British Museum, the *Carpolithes thalictroides*, var. *Websteri*, Brongn., of the Isle of Wight with the fruit from Bovey and Kaltennordheim, and am assured that they form one species. But the *Carpolithes thalictroides*, Brongn., must be separated from *Carpolithes Websteri*, as the fruit of the former species is cylindrical and acuminated, whilst the fruit of *C. Websteri* is obtusely rounded. Unhappily the place where these specimens of the Isle of Wight were found is not mentioned in the British Museum. The stone is dark, and different from the White Clay of Alum Bay, where plants are lying. Perhaps they are from the Bembridge series. Latterly I have received many seeds of this species from the Hempstead beds of the Isle of Wight.

It occurs of two different dimensions; some are 3 millims. long and 2½ millims. wide; others are 4–4½ millims. long and 3½ millims. wide. They agree, however, with the exception of the dimensions, so perfectly, that they cannot be separated.

These fruits are short, oval, rounded at the base, and furnished with a small round hole (fig. 7); the portion around the latter is somewhat wrinkled; at the apex it is furnished with a short little point. The sides are rather flat, and over each side runs a flat longitudinal furrow (fig. 8, magnified 8 *b* & 12 *b*). As the seeds lie in the soft clay, they have hardly been much compressed; the longitudinal furrows have therefore probably not been caused by the circumstance that the pericarpium was pressed in along the cavity of the fruit, but probably are characteristic of the fruit. If we open the fruit in a longitudinal section (fig. 9, and magnified 9 *b* & 10, and of the smaller fruits 13 & 14), we perceive an elliptical partition, which contained, doubtless, one seed, of which, however, nothing is preserved. The wall of the pericarpium is relatively very thick; it must therefore have been ligneous. In a few of the smaller fruits there was a longitudinal fissure.

The whole organization shows that we have here fruits, and not seeds. It is very like the fruit of *Potamogeton* (cf. *Potamogeton Eseri*, Heer, Flor. Tertiar. Helvet. i. p. 102, pl. 47. fig. 8); but the shape of the cell of the fruit (loculamentum) differs and makes it doubtful whether it belongs to this genus. In *Potamogeton* the dorsal part of the fruit is generally much more convex than the ventral part, while the apex of the fruit is somewhat incurved.

47. Carpolithes nitens, m. (Plate LXX. figs. 15–23.)

C. fructibus subglobosis, nigro-nitidis, subtilissime et obsolete striolatis, monospermis, basi truncatis, cicatrice orbiculata vel angulata ornatis.

Taxus margaritifera, Ludwig, Palæontograph. viii. p. 73, pl. 60. fig. 19?

Very frequent in the coal of the 46th bed.

Ludwig gives only a short description, which agrees, however, pretty well, but for the expression "a circular plate with a thin margin and a short stalk*," which I do not understand, and the dimensions, since his fruits are 6 millims. long, whilst those of Bovey are only 5 millims. The seeds of *Taxus* display a similar organization of the cicatrix; it is always regularly orbicular or oval; and the delicate longitudinal stripes are also met with; but the walls are much thicker, and the internal cavity is much smaller. Nevertheless I do not know any genus with seeds and fruits which are so similar to those of Bovey as the seeds of *Taxus*. If it really belong to this genus, we should have a thick ligneous testa, and but a small cavity.

These little shining black bodies have nearly the same breadth and height (of 5 millims.), and are therefore nearly orbicular, although they are frequently compressed in different directions. On the fore part of the perfectly preserved specimens we observe a little point or apex; the base is truncate, and in the middle appears a small round hole,

* Dünnrandige und kurzgestielte Kreisfläche.

surrounded by a cicatrix of a dull colour, which is frequently orbicular, but sometimes polygonal. If we cut them open we observe a small cavity, surrounded by a ring of shining black brittle coal (figs. 19 *b*, 20, 21 & 22). Fig. 22 is a magnified transverse section of the middle; fig. 20 (magnified 20 *b*) a similar section, but nearer the top of the fruit, which is not compressed; and fig. 19 (magnified 19 *b*, 21) a section of a compressed fruit. The wall of the fruit measures 1 to $1\frac{1}{2}$ millim.; the cavity has a diameter of about 2 millims. The black shining crust which covers the internal surface is very thin. It is formed by the testa, if the organization belong to a fruit; but if it be the seed of a *Taxus*, or of some tree resembling the latter, then it ought to be considered as the remains of the albumen, while the wall forms the ligneous testa. The cavity is always filled with a whitish-grey clay. As in the *Carpolithes Websteri*, the softer internal parts of the seed were also in this case dissolved, while the cavity became afterwards filled with mineral substance. The crust of coal corresponds to the cuticula of *C. Websteri*.

48. CARPOLITHES EXARATUS. (Plate LXX. fig. 27, highly magnified in figs. 24–26.)
C. putamine subgloboso, $3\frac{1}{2}$ mm. longo, nigro, nitido, sulcis rugoso-punctatis exarato.
In the 26th bed at Bovey.

A remarkable little object, which represents, without doubt, the stone of a fruit. It is $3\frac{1}{2}$ millims. long and $3\frac{3}{10}$ millims. wide, being thus nearly orbicular, although it is furnished with a rather blunt apex, which projects but very little, while the lower end is rounded very obtusely. At one place we observe a narrow fissure, which reaches from the apex to somewhat below the middle of the little stone, which is bordered by a narrow margin. It is, without doubt, the place where the raphe passed, and it may therefore be described as the umbilical fissure. The furrows and ribs upon the little stone are very elegant. On the truncated end (fig. 26, highly magnified) we observe a great many parallel furrows and ribs, which run very near one another to the lower termination of the umbilicus, and which pass also opposite the umbilicus across the back of the little stone; the sides too are marked by similar ribs and furrows; but here the first two are distant, and the furrows are therefore wider; the ribs run much more irregularly, forming ramifications, so that the furrows seem to be interrupted by numerous tubercles. The furrows and the ribs are covered besides with innumerable dots. (In fig. 24 *b*, a piece with a few ribs has been represented as seen under a microscope.)

Similar fruit-stones occur in *Celtis*, but we miss the umbilical fissure and the regularity of the furrows. As regards the character, the stone is more like those of *Prunus*, in which we observe on the suture a channel, within which the raphe runs till it passes through a hole below the apex of the stone towards the seeds. On the fossil we find no such small hole resembling a dot, but only the above-mentioned fissure; besides it differs in the very peculiar sculpture, and must therefore belong to another type of plants.

Fig. 27 represents the little stone of the natural size; fig. 24, the same magnified and viewed laterally; fig. 25, the same from the side of the umbilicus; fig. 26, from the base.

49. Carpolithes vinaceus, m. (Plate LXX. figs. 28 & 29, magnified 29, *b*).

C. semine gigantoideo, tenuissime longitudinaliter striolato, medio unisulcato.

Some pieces from the 26th bed.

This has the form and dimensions of the seeds of the grape (*Vitis vinifera*, Linn.), but it does not belong to the genus *Vitis*; the delicate longitudinal striæ, and the absence of the umbilicus within the fissure, are against such an hypothesis. It seems to me to be the seed of a monocotyledonous plant.

The seed is 6½ millims. long, and at the thicker portion 4 millims. wide; at one end it is much thickened and obtusely truncated, towards the other it is much narrowed. It has a brown, rather shining testa, marked by very delicate striæ. In some places where the testa is removed (fig. 29, magnified fig. 29 *b*), the striæ (or rather the casts of the striæ) are more distinctly visible. They form very delicate longitudinal lines, which are here and there united by transverse lines.

50. Carpolithes lividus, m. (Plate LXX. figs. 30 & 31.)

C. follicularis, membranaceus, brunneus, ovatus.

In the 26th bed; several pieces.

A very doubtful species. Small light-brown oval bodies, with a thin membrane. One perfectly preserved specimen (fig. 30) is $4\frac{8}{10}$ millims. long and 4 millims. wide; it is quite flattened; on the dorsal part it is furrowed by transverse wrinkles (fig. 30 *b*), which were, however, produced by shrinking; otherwise it is smooth, shining, and of a yellowish brown; on the other side there is a longitudinal fissure (fig. 30 *c*, magnified), through which probably the seed fell out, leaving the thin membrane of the pericarp. In such a case it would be a small folliculus, with probably but one seed. It is also possible that it may be the testa of a seed, and that the fissure marks the raphe.

II. *Descriptions of the Diluvial Species of Plants.*

1. Salix, Linn.

1. Salix cinerea, Linn. (Plate LXXI. figs. 1 *a*, *b*, 2 & 3.)

S. foliis lanceolato-obovatis, basi angustatis, apice breviter acuminatis, serratis vel subtiliter undulato-serrulatis, rugulosis, nervis secundariis valde camptodromis nervillisque validis.

Several leaves lie upon the soft white clay, which cannot be distinguished from those of *Salix cinerea*, Linn.; they are like the leaves which occur in the tuffs of Cannstadt. The leaves represented on Plate LXXI. fig. 1 *a*, *b*, are distinctly serrated; the leaf *a* with a rather sharp point; the same is seen on the leaf fig. 3, while the larger leaf, which is represented in fig. 2, has only a slightly undulated and indistinctly denticulate margin, as occurs frequently in *Salix cinerea*. The secondary nerves are perfectly like those of *S. cinerea*; they are inclined forwards, and united, forming long arches, which approach the margin. The nervules are strong, and impart to the leaf a rather wrinkled aspect.

From the midrib spring shortened secondary nerves, which are united with each lower secondary nerve, a character peculiar to the leaves of the willows.

2. SALIX, spec. (Plate LXXI. figs. 4 & 5.)

S. foliis petiolatis, oblongis, basi rotundatis, integerrimis (?), rugulosis, nervis secundariis valde camptodromis nervillisque validis.

This is doubtless the leaf of a willow, since it has the same characteristic nervation; the base of the leaf is obtusely rounded, and the margin seems to be entire. The spe cimen represented in fig. 4 has nearly parallel sides, but the base is obtusely rounded; on the margin no teeth are visible. The secondary nerves are also highly inclined towards the apex, and numerous nervules cause the wrinkled appearance of the surface. The small leaf which is represented at fig. 5 belongs undoubtedly to the same species with the foregoing, although the sides are less parallel. It is very like the leaves of *Salix amygdalina*, Linn., which vary very much in shape; but the secondary nerves and the nervules are strong, and the margin seems to be entire.

3. SALIX REPENS, Linn.? (Plate LXXI. figs. 1 *c–h*, 6 & 7 *b*.)

S. foliis breviter petiolatis, ovalibus, oblongis et oblongo-lanceolatis, integerrimis, nervis secundariis valde curvatis.

The most frequent leaf of the white clays. Numerous fragments lie confused in all directions.

Some forms are rather similar to those of *Salix cinerea*, Linn., but the nerves are not nearly so strongly developed: the surface of the leaf is much smoother, and the margin is not dentate.

The dimensions and the form of these leaves are very variable. The length varies from 6 to 45 millims.: some are short, oval, and obtusely rounded at the apex; others (and they are the most numerous) are oblong, and rather obtuse at the apex; whilst others are lanceolate, and have an acuminate apex. There are so many intermediate forms, that these cannot be separated. The midrib is rather slender; from it rise the secondary nerves at a rather acute angle; they are highly curved towards the apex, and near the margin they are united in arches. The areas are divided by very delicate nervules. At some places the shortened secondary nerves, which pass over to the lower secondary nerves, are visible, but in most of the leaves this important mark is not quite distinct.

As to the shape and nervation of the leaves, it seems to me most nearly allied to *Salix repens*, Linn.; but the incurvate apex is wanting, which, however, does not always exist in *S. repens*. *Salix ambigua*, Ehrh., and *S. ambigua*, Sendtner (*S. aurito-myrtilloides*), have very similar leaves.

2. BETULA, Linn.

4. BETULA NANA, Linn. (Plate LXXI. figs. 1 *k* & 7 *a*.)

B. foliis parvulis, orbiculatis, profunde crenatis, nervis secundariis flexuosis, craspedodromis, basalibus approximatis, areis nervulis reticulatis.

Very pretty little leaves, only 9 millims. long, and nearly orbicular. They have simple teeth, which are, however, deep and acute. From the median nerve arise on each side four secondary nerves, of which the two lowest are very much approximated. Each of them runs in a zigzag line, and terminates in a tooth; from the second spring tertiary nerves; and all are united, forming elegant polygonal reticulations, that can be traced upwards to within the teeth (cf. fig. 1 *k k*, where the leaf has been three times magnified). I saw several entire little leaves in an excellent state of preservation; there have been found besides several fragments of the leaves of this species in the white clay.

The nervation indicates the genus *Betula*. Similar forms occur in the young leaves of *Populus alba*; but in these, five primary nerves spring from the base of the leaf; while the nerves of *Betula* are pinnated, but the first ones approach the base, as seen in the fossil leaf; the reticulation also is that of *Betula*. A comparison of the different species of *Betula* leads us to *Betula nana*, Linn. The fossil leaves agree, indeed, in respect of dimensions, shape, dentation, and nervation so entirely with those of the living species, that no difference can be found; the petiole, however, is somewhat thicker than is generally the case with *Betula nana*.

Betula nana, Linn., is a boreal plant, which is at home throughout the whole arctic zone; it is found also here and there on the highland moors in Middle Europe, as, for instance, near Einsiedeln in Switzerland, and in the Jura. In the British islands it is found in Scotland only.

5. PINUS SYLVESTRIS, Linn.?

Dr. J. D. HOOKER* speaks of a pine-cone which, according to the late Dr. CROKER, was found in the upper layers of lignite. Dr. HOOKER says that it so closely resembles that of a Scotch fir (*Pinus sylvestris*, Linn.), that it might be referred to this species. I have seen it in the collection of the Geological Society in London, and another specimen in Dr. CROKER'S collection at Bovey. I am of the same opinion as Dr. HOOKER; but these cones look much more modern than the plants of the lignite beds, and they are, I believe, from the diluvial formation of Bovey.

III. *Insects from Bovey.*

During my stay at Bovey I carefully searched for the remains of insects. I found indeed some traces, but they are but fragmentary, and there is only one fragment which can be determined. It is the partly destroyed elytron of a beetle, probably a *Buprestites*, and which I shall describe as *Buprestites Falconeri* (Plate LXVIII. fig. 21, magnified 21 *b*). It was 8 millims. long and 3 millims. wide. The angle of the shoulder is somewhat rounded, and beneath it the elytron is somewhat curved inwards. It is remarkably sculptured; with the aid of a lens we can perceive numerous round points, which are ranged in rows, and so close together that the whole elytron gets quite a *sculptura alutacea*. The species of *Agrilus*, *Lampra*, and *Anthaxia* present a similar sculpturing.

* Quart. Journ. Geol. Soc., Nov. 1855, p. 566.

Explanation of the Plates.

PLATE LV.

Fig. 1. *Sphæria lignitum*, m.; fig. 2, magnified; fig. 3, more magnified. Fig. 4 *a*, *b*, *c*, *d*. *Pecopteris lignitum*, Gieb. Fig. 4 *e*. *Cinnamomum Scheuchzeri*, Hr. Figs. 5 & 6. *Pecopteris lignitum*. Figs. 7–10. *Palmacites Dæmonorops*, the fibrous bundles of the stem; figs. 11 & 12, spatha with spines; fig. 11 *b*, magnified; fig. 13, a spine; figs. 14 & 15, spines; fig. 15 *b*, magnified.

PLATE LVI.

Figs. 1–11. *Pecopteris lignitum*, Gieb.:—Fig. 1, the point of a leaflet (pinnula), magnified; figs. 2–4, parts of leaflets near the apex; fig. 5, many fragments of leaflets; fig. 5 *b*, seed, and *c*, the scales of a cone of *Sequoia Couttsiæ*; figs. 6 & 7, base of the leaflets; fig. 8, leaflet restored; figs. 9–11, young fronds. Figs. 12–15. *Lastræa Stiriaca*, Ung., sp.; fig. 12, fragment of a leaflet; fig. 13, leaflet with the sori, from Monod; fig. 14, pinnated leaf; fig. 15, leaflet.

PLATE LVII.

Figs. 1–7. *Pecopteris lignitum*, magnified. Fig. 8. *Lastræa Stiriaca*, magnified.

PLATE LVIII.

Fig. 1. Rhizome of *Pecopteris lignitum*; fig. 2, rachis, with scars of the roots. Fig. 3. *Pecopteris Hookeri*, m.

PLATE LIX.

Sequoia Couttsiæ, m.:—Fig. 1, cone and seeds; fig. 2, cone, magnified; fig. 3, scales of the cone, magnified; fig. 4, scales of the cone; fig. 5, young shoots; figs. 6–8, young shoots, magnified; fig. 9, biennial branch; figs. 10 & 11, older branches; fig. 12, young shoot with the leaves, magnified; fig. 13, branch with three shoots in a verticil; figs. 14, 16 & 18, cones; figs. 15, 17 & 19, these cones restored.

PLATE LX.

Figs. 1–46. *Sequoia Couttsiæ*, m.:—Figs. 1 & 2, biennial branches; figs. 3–6, young shoots; fig. 6 *b*, magnified; fig. 7, base of a young shoot, magnified; fig. 8, young shoot, magnified; figs. 9, 11 & 12 *a*, young shoots with spreading leaves; figs. 11 *b* & 12 *a a*, magnified; fig. 10, young shoot with adnate leaves; figs. 12 *b b* & 13, part of a branch with short adhering leaves, magnified; figs. 14–20, young shoots; fig. 14 *b*, leaf, magnified; figs. 21, 23 & 24, scales of a cone; fig. 22, base of cone; fig. 25, scale with the seeds; figs. 26–28, cones with separate scales; figs. 29–34, scales of cones; fig. 35 *a*, scales; fig. 35 *b*, seed; figs. 36–42, seeds; fig. 41 *b*, magnified. Fig. 43. *Amentum masculinum*; fig. 43 *b*, magnified; figs. 44 & 45, young shoots from Armissan; fig. 46, magnified. Figs. 47 & 48. *Sequoia sempervirens*; fig. 47, seed; fig. 47 *b*, magnified; fig. 48, cone. Fig. 49. *Glyptostrobus europæus*, Br., sp., from Hohe Rhonen; fig. 49 *b*, magnified. Fig. 50. *Palmacites Dæmonorops*, Ung., sp., fruit; figs. 51–53, magnified. Fig. 54. *Cyperites deperditus*, Hr.; fig. 54 *b*, magnified.

PLATE LXI.

Sequoia Couttsiæ, m., restored.

PLATE LXII.

Palmacites Dæmonorops, Ung., sp.:—Fig. 1, a large specimen, spatha?; fig. 2, three of the spines, magnified; fig. 3, a specimen with five spines; figs. 4–6, little spines; fig. 7, fasciculi of spines; fig. 8, spines lying in different directions; figs. 9–11, spines, magnified; fig. 9 *b*, a portion more magnified.

PLATE LXIII.

Fig. 1 *b*. *Lastræa Bunburii*, m., nat. size; *c*, *d*, magnified. Fig. 1 *a*. *Ficus Falconeri*, m.; fig. 1 *a a*, a portion of the surface of the leaf, magnified; fig. 1 *a a a*, more magnified. Figs. 2–9. *Quercus Lyelli*, m.; fig. 2, apex of the leaf; figs. 3, 8 & 9, base; figs. 4–7, middle piece; fig. 7 *b*, portion of leaf, magnified.

PLATE LXIV.

Fig. 1 *a*, *b*, *c*. *Quercus Lyelli*, m. Fig. 1 *d*. *Phragmites œningensis*, A. Br.? Figs. 2, 3 & 4. *Quercus Lyelli*. Fig. 3 *b* & 5. *Echitonium cuspidatum*, Hr. Figs. 6 & 7. *Ficus Falconeri*, m.

PLATE LXV.

Figs. 1 & 2. *Daphnogene Ungeri*, Hr. Figs. 3–5. *Ficus eucalyptoides*, m. Fig. 6. *Laurus primigenia*, Ung. Figs. 7 & 8. *Ficus Pengellii*; figs. 7 *b* & 8 *b*, a portion of leaf, magnified. Figs. 9-11. *Dryandroides lævigata*, Hr. Fig. 12 *a*. *Dryandroides hakeæfolia*, Ung. Fig. 12 *b*. *Quercus Lyelli*. Fig. 12 *c*. *Echitonium cuspidatum*, Hr. Fig. 13 *a*. *Phragmites œningensis*, A. Br.; 13 *a a*, magnified. Fig. 13 *b*. *Dryandroides lævigata*. Fig. 13 *c*. *Sphæria socialis*, m.; fig. 13 *c c*, magnified.

PLATE LXVI

Figs. 1 & 2. *Quercus Lyelli*, m., restored. Fig. 3. *Ficus Pengellii*, m., restored. Fig. 4. *Ficus Falconeri*, m., restored.

PLATE LXVII.

Figs. 1–8. *Cinnamomum lanceolatum*, Ung., sp. Figs. 9–16. *Cinnamomum Scheuchzeri*, Hr.; fig. 12, branch with the flowers, from Oeningen. Figs. 17 & 18. *Cinnamomum Rossmässleri*, Hr. Fig. 19. *Sclerotium Cinnamomi*, m., magnified; fig. 19 *b*, transverse section.

PLATE LXVIII.

Fig. 1. *Palmacites Dæmonorops*? portion of leaf; fig. 1 *b*, five times magnified. Fig. 2. *Phragmites œningensis*, A. Br.? Fig. 3. *Poacites*, spec. Figs. 4 & 5. *Quercus Lyelli*; fig. 6, bark. Fig. 7. *Dryandroides Banksiæfolia*, Ung., sp.? Fig. 8. *Vaccinium acheronticum*, Ung. Fig. 9. *Andromeda vacciniifolia*, Ung. Figs. 10 & 11. *Andromeda reticulata*, Ett.; fig. 10 *b*, magnified. Figs. 12 & 13. *Cinnamomum Scheuchzeri*; figs. 13 & 13 *c*, flowers; fig. 13 *b*, *d*, magnified. Figs. 14 & 15. *Cinnamomum lanceolatum*. Figs. 16–18. *Eugenia Hæringiana*, Ung. Fig. 19. *Celastrus pseudo-ilex*, Ett. Fig. 20. *Leguminosites areolatus*, m.; fig. 20 *b*, magnified. Fig. 21. *Buprestites Falconeri*, m.; fig. 21 *b*, magnified.

PLATE LXIX.

Figs. 1–8. *Gardenia Wetzleri*; figs. 3–6, seeds, magnified; fig. 7 *a*, *b*, seed-vessel from Samland; fig. 7 *c*, transverse section; fig. 8, fruit from Günzburg. Figs. 9 & 10. *Eucalyptus oceanica*, Ung.? Figs. 11–17. *Nyssa europæa*; figs. 12 *b* & 17 *b*, magnified. Fig. 18. *Nyssa lævigata*; fig. 18 *b*, magnified. Fig. 20–23. *Nyssa striolata*; fig. 20 *b*, magnified. Fig. 24. *Nyssa microsperma*, m.; fig. 24 *b*, magnified. Figs. 25 & 26. *Vitis britannica*, m.; fig. 26 *b*, magnified. Figs. 27 & 28. *Vitis Hookeri*, m., fruit; figs. 29 & 29 *b*, seed; fig. 29 *c*, *d*, magnified; fig. 29 *e*, transverse section. Fig. 30. *Carpolithes scutellatus*, m.; fig. 30 *b*, magnified.

PLATE LXX.

Figs. 1–3. *Anona devonica*, m.; figs. 1 *b* & 2 *b*, transverse section. Figs. 4 & 4 *b*. *Anona cyclosperma*, m. Fig. 5. *Pterocarya denticulata*, O. Web., sp.?; fig. 5 *b* & *c*, magnified. Fig. 6. *Carpolithes Websteri*, Br.; fig. 6 *b*, magnified; fig. 6 *c*, *d*, *e*, portion of the membrane, highly magnified. Figs. 7–14. *Carpolithes Boveyanus*, m.; figs. 8 *b*, 12 *b* & 13 *b*, magnified; figs. 9 *b* & 13 *b*, vertical sections. Figs. 15–23. *Carpolithes nitens*, m.; figs. 17 *b* & 15 *b*, base of the fruit, magnified; fig. 19, transverse section; fig. 19 *b*, magnified; fig. 20, transverse section near the apex; fig. 20 *b*, magnified; figs. 21 & 22, transverse section from the middle of the fruit; fig. 23, many fruits, of natural size. Figs. 24–27. *Carpolithes exaratus*, m.; figs. 24–26, magnified. Figs. 28 & 29. *Carpolithes vinaceus*, m.; fig. 29 *b*, magnified. Figs. 30 & 30 *c*. *Carpolithes lividus*, m.; fig. 30 *c*, magnified. Figs. 32–37. *Nymphæa Doris*, m.; figs. 34 & 36, magnified; fig. 37, highly magnified.

PLATE LXXI.

Fig. 1 *a*, *b*. *Salix cinerea*, Linn.; fig. 1 *c–h*. *Salix repens*, Linn.? fig. 1 *k*. *Betula nana*, Linn.; fig. 1 *k k*, magnified. Figs. 2 & 3. *Salix cinerea*, Linn. Figs. 4 & 5. *Salix*, sp. Fig. 6. *Salix repens*, Linn.? Fig. 7 *a*. *Betula nana*, Linn. Fig. 7 *b*. *Salix repens*, Linn.? Figs. 8 & 9, very highly magnified portion of wood.

XLI. *On the Anatomy and Physiology of the Spongiadæ.*—Part III. *On the Generic Characters, the Specific Characters, and on the Method of Examination.*

By J. S. Bowerbank, *LL.D.*, *F.R.S.*, *F.L.S.* *&c.*

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While the arrangement of other branches of natural history has occupied the attention of some of the most laborious and talented naturalists of every age, the Spongiadæ appear to have scarcely attracted sufficient attention to excite any writer on natural history to a serious attempt at a systematic classification. This neglect has not arisen from any incapacity for a definite arrangement on the part of the Spongiadæ, as the organic differential characters of the numerous groups into which, by careful examination, they may be readily divided are as varied and as widely removed from each other as are the strikingly distinct and well defined divisions of the Corallidæ; and the number of species I believe to be very much greater than those of the latter class. Of British species alone I am already acquainted with 150 or more; and new ones are continually being discovered by the aid of the dredge. It becomes therefore a matter of necessity that we should classify their permanent varieties of structure, and found on them a series of orders, suborders, and genera, and through these subdivisions become enabled to recognize more readily the very numerous species of these animals which abound in all parts of the world.

De Blainville proposed to include the whole of the Spongiadæ under the designation of Amorphozoa; but this term is objectionable, as all sponges cannot be considered as shapeless—on the contrary, many genera and species exhibit much constancy in their form. Neither can the term be justly applied to their internal structure, as we find in *Grantia*, *Geodia*, *Tethea*, and other genera regular and systematical structures which are very far removed from shapelessness. I have therefore thought it advisable to adopt Dr. Grant's designation of Porifera, a term which embraces the whole of the Spongiadæ, and which is truly descriptive of the most essential general action of the animal's power and mode of imbibing nutriment, which in every species with which I am acquainted is, by a series of minute pores distributed over the external membrane of the sponge.

Besides this universally existent character there are others which are strikingly characteristic of the class, although not so universally prevalent as the porous one. Thus the skeletons of the Spongiadæ are always internal, but in the material and mode of construction they vary to a very considerable extent. Sponges may therefore be defined as fixed, aquatic, polymorphous animals, inhaling and imbibing the surrounding element through numerous contractile pores situated on the external surface; conveying it through internal canals and ejecting it through appropriate orifices; having an internal flexible or inflexible skeleton composed of either carbonate of lime, silex, or keratode; with or without either of these earthy materials. Calcareous skeletons always spicular. Siliceous skeletons either spicular or composed of solid, laminated, and continuous siliceous fibre.

Propagation by ova, gemmulation, or spontaneous division of its component parts.

Dr. GRANT, in his learned and elaborate "Tabular View of the primary divisions of the Animal Kingdom," published in 1861, has divided the Porifera into three orders, based on principles which I have adopted. The first order is *Keratosa*, in which the skeletons are essentially keratose and fibrous; the second, *Leuconida*, is composed of the calcareous sponges; and the third, *Chalinida*, consisting of the siliceous sponges. I have not adopted the full and precise definition of each of these Orders as given by the learned Professor, as, if the whole of the distinctive characters in the first and third of them were insisted on in the determination of the orders to which many exotic species belong, it would lead in numerous cases to inextricable confusion. The term *Leuconida* is also objectionable, as all calcareous sponges are not white, and colour is at best but a very uncertain character even in the determination of a species; I have therefore adopted the principles of the arrangement of Professor GRANT, with the following modifications of position and descriptions of the characteristics of each order.

1. CALCAREA. Sponges the skeletons of which have as an earthy base carbonate of lime.

2. SILICEA. Sponges in which the earthy base consists of siliceous matter.

3. KERATOSA. Sponges in which the essential base of the skeleton consists of keratose fibrous matter.

While thus assuming the principles of arrangement enunciated by the learned Professor, I have been induced to vary the mode of the disposition of his Orders from the following considerations.

In the highest vertebrated animal types we invariably find the skeleton principally composed of phosphate of lime with a small portion of carbonate of lime and other substances, the whole consolidated by cartilage. As we descend the scale of the Vertebrata we find the salts of lime decrease in proportional quantity until they occur in minute detached patches only, and cartilage becomes the essential base of the skeleton.

In the great tribe of Mollusca we find carbonate of lime prevailing in their shells to the exclusion of phosphate of lime, and in the compound Tunicata we have a structure analogous to that of the cartilaginous tribe of Fishes. In the massive subcartilaginous body of this tribe there is no continuous or connected earthy deposit; this material of the skeleton exists only in the form of detached masses of radiating spicula. As we descend in the animal scale we find carbonate of lime entirely absent, and silex replacing it in the elaborate and beautifully constructed loricæ of the marine and freshwater infusoria.

If we are to reason from these gradations of structure and apply our reasoning to the

Spongiadæ, we should then give precedence to the calcareous sponges as representing in the class the highest order of secretive power; and if we add to these considerations the regularity of structure and function and the full development of ciliary action that exists in *Grantia ciliata* and *compressa* and the allied species, I think it scarcely allows of a doubt that this order should take precedence of the others in an arrangement of the Spongiadæ.

The siliceous sponges naturally follow in succession, and the Keratosæ, as indicated by their imperfect secretive powers and their low order of organization in other respects, would stand the last in the series.

On the Generic Characters of the Spongiadæ.

The foundation of the genera of the Spongiadæ has hitherto been based principally upon form and other external characters of an equally unstable description, and in many instances genera have been named without the slightest attempt to characterize them. As a generic character, form is inadmissible, inasmuch as each variety of it is found to prevail indiscriminately in genera differing structurally to the greatest possible extent.

I will not enter on the history of the genera that have been proposed by previous writers on the Spongiadæ, as the greater portion of those which have been published will hereafter be found to have been adopted, with certain revisions of their characters, in the series of genera I propose to establish, but I shall beg to refer such of my readers as may be desirous of further information on that subject to page 70 of Dr. Johnston's admirable introduction to his 'History of British Sponges and Lithophytes.'

Having thus rejected form and other external characters as the foundation of generic descriptions, we naturally resort to the anatomical peculiarities of the animal for these purposes; and here fortunately we find a variety in structure and form, and a constant adherence to their respective types that admirably adapt them to our purpose.

If any portion of the animal remains whereby we may recognize it as one of the Spongiadæ it is always the skeleton; and it is therefore advantageous to adopt this most persistent portion of the animal as the foundation of our generic descriptions. But this is not the sole reason for such a conclusion, as it is not only the most enduring portion of the animal, but it is also the most undeviatingly regular in the form and arrangement of its component structures. However great may be the variations that exist in size and form between different species of the same genus, or between individuals of the same species, the characteristic tissues of their skeletons are always found to harmonize in their structural peculiarities. It appears, therefore, advisable in these animals, as well as in the higher classes, to select the skeleton as the primary source of generic distinctions. Other portions of the permanent organs may be occasionally resorted to when necessary as auxiliary characters, such as the incurrent and excurrent canals, the intermarginal cavities, the cloaca, and the various modes of reproduction. Each of these characters is of use in generic descriptions to a certain extent; but none of them is absolutely necessary to the determination of a genus, and occasionally we find one or more of these modes of organization entirely absent; we may therefore consider them not as primary, but rather as secondary or auxiliary generic characters.

I therefore propose to consider the varieties in the construction of the skeleton as the foundation or primary source of division into genera, and to dedicate that portion of the animal especially to that purpose, the auxiliary or secondary characters being resorted to only when required to aid and assist the primary ones; and it is only to a very limited extent that they are in reality available. Thus the cloaca in the Order Calcarea becomes a very important means of generic distinction, and in some cases in the Order Keratosa it is also a prominent character, while in Silicea it is generally absent. In some species of this order, as in *Alcyoncellum*, *Polymastia*, and *Halyphysema*, it assumes a normal character, while in several species of *Halichondria*, as in *H. panicea*, it assumes very striking proportions in excessively developed specimens, whilst in others it is either an occasional, uncertain, and progressive organ, or is altogether absent.

The mode of propagation is also an uncertain character. Thus in *Tethea cranium* we find it to be by internal gemmulation, in *T. Lyncurium* by external gemmules, and in other species of the genus no gemmules of any description have hitherto been detected. In *Geodia*, *Pachymatisma*, and *Spongilla* the general structure and mode of disposition of the ovaria render them valuable auxiliary generic characters, but in other cases they are of little or no value.

The intermarginal cavities are available as generic characters in *Geodia* and the nearly allied species; and in the same sponge the relative position of the connecting spicula form good distinctive characters in the genera *Geodia*, *Ecionemia*, and also some of the siliceo-fibrous sponges. In *Alcyoncellum*, *Polymastia*, and *Geodia* the position and appendages of the oscula are also available; but generally speaking those organs are so mutable as to render them of little value as generic characters.

The following tabular view of the arrangement I propose to adopt will perhaps render the details regarding the distinctive characters and natural affinities of the genera more readily comprehensible.

Tabular View of Systematic Arrangement.

Class.	Order.	Suborder.	Genera.
PORIFERA.	I. CALCAREA.		Grantia, *Fleming.*
			Leucosolenia, *Bowerbank.*
			Leuconia, *Grant.*
			Leucogypsia, *Bowerbank.*
	II. SILICEA.	1. Spiculo-radiate skeletons	Geodia, *Lamarck.*
			Pachymatisma, *Bowerbank.*
			Ecionemia, *Bowerbank.*
			Alcyoncellum, *Quoy et Gaimard.*
			Polymastia, *Bowerbank.*
			Halyphysema, *Bowerbank.*
			Ciocalypta, *Bowerbank.*
			Tethea, *Lamarck.*
			Halicnemia, *Bowerbank.*
			Dictyocylindrus, *Bowerbank.*
			Phakellia, *Bowerbank.*
			Microciona, *Bowerbank.*
			Hymeraphia, *Bowerbank.*
			Hymedesmia, *Bowerbank.*
		2. Spiculo-membranous skeletons	Hymeniacidon, *Bowerbank.*
		3. Spiculo-reticulate skeletons	Halichondria, *Fleming.*
			Hyalonema, *Gray.*
			Isodictya, *Bowerbank.*
			Spongilla, *Linnæus.*
		4. Spiculo-fibrous skeletons	Desmacidon, *Bowerbank.*
			Raphyrus, *Bowerbank.*
		5. Compound reticulate skeletons	Diplodemia, *Bowerbank.*
		6. Solid siliceo-fibrous skeletons	Dactylocalyx, *Stutchbury.*
		7. Canaliculated siliceo-fibrous skeletons	Farrea, *Bowerbank.*
	III. KERATOSA.	1. Solid non-spiculate kerato-fibrous skeletons	Spongia, *Linnæus.*
			Spongionella, *Bowerbank.*
		2. Solid semispiculate kerato-fibrous skeletons	Halispongia, *Blainville.*
		3. Solid entirely spiculate kerato-fibrous skeletons	Chalina, *Grant.*
		4. Simple fistulo-fibrous skeletons	Verongia, *Bowerbank.*
		5. Compound fistulo-fibrous skeletons	Auliskia, *Bowerbank.*
		6. Regular semi-areno-fibrous skeletons	Stematumenia, &c., *Bowerbank.*
		7. Irregular and entirely areno-fibrous skeletons	Dysidea, *Johnston.*

Order I. CALCAREA.

The number of species of calcareous sponges that are known is comparatively so small, and the four genera into which I have divided them are naturally so well characterized, as to render the establishment of suborders unnecessary. Hereafter, when we are acquainted with a greater number of species, and other varieties of organization become known, the genera now established may become the types of suborders, for which office their distinctly different modes of construction render them eminently efficient.

Although the calcareous structure of the species of this order appears to entitle it to precedence in the arrangement of the Spongiadæ, it does not maintain in the structure of its skeleton throughout the whole of the genera the same high type of formation that is exhibited in *Grantia compressa* and the allied species, and we observe a progressive decline in regularity of structure in its genera very analogous to what we find existing among the Halichondroid tribe of sponges; but in this respect they only follow the same laws of gradual degradation that obtain in every other class of created beings; and therefore this gradual decline in regularity of structure should not militate against the claim of even the lowest in organization of the tribe from taking precedence of the siliceous sponges.

Dr. GRANT was the first naturalist who decided that the spicula of a certain group of small sponges were composed of carbonate of lime, and he separated them accordingly from those the spicula of which were siliceous, and assigned to them the generic name of *Leucalia* (Edinburgh Encyclopædia, vol. xviii. p. 844); and subsequently, in his 'Outlines of Comparative Anatomy,' he changed that name to *Leuconia*. In 1828 Dr. FLEMING gave to the group the name of *Grantia*, in compliment to the learned naturalist who had first pointed out their peculiar structure.

A careful examination of the British species of this Order will very soon satisfy a naturalist that there are at least four distinct forms in the organization of the skeleton, and that each is fully entitled to generic distinction. Thus in *Grantia ciliata* and *compressa*, Johnston, we find the sponge to be constructed of a series of cells, each having separate parietes, and extending from the dermal surface to near the inner surface of the sponge, where they discharge the fæcal streams into a common cloacal cavity. In *Grantia botryoides*, Johnston, the system of cells is entirely wanting; the sponge is composed of a single thin stratum of membranous structure and spicula, surrounding a large cylindrical cloacal cavity, from the terminations of which the fæcal streams are discharged. In *Grantia nivea*, Johnston, we find the sponge massive and irregular in form, containing numerous capacious cloacal cavities, each terminated by a single large mouth, the interstitial structures between the sides of these great cavities and the dermal surfaces of the sponge consisting of irregularly disposed membranes and spicula, permeated by contorted interstitial cavities, terminating in simple orifices or oscula in the sides of the great fæcal cavity into which they discharge their excurrent streams; and in *Leucogypsia Gossei*, Bowerbank, MS., the sponge is massive, without cloaca, formed of irregularly disposed membranous tissues and spicula, and with oscula at the external surface, thus simulating to a great extent the mode of structure of the Halichondroid tribes of sponges.

The sponges of this Order appear to possess a high degree of vital power, and I have rarely failed in finding the excurrent orifices in vigorous action in either *Grantia compressa*, *ciliata*, or *botryoides* when recently taken from the sea. In *G. compressa*, especially, I have often observed the inhalant and exhalant actions remarkably vigorous; and if a drop of water containing finely comminuted indigo be mixed with the water in which they are immersed, they will become deeply tinctured with it in a very few seconds. This vigorous action is accounted for by the highly developed ciliary system, which may be readily seen in action if the sponge be carefully split open and immersed in fresh cold sea-water, and examined with a power of about five or six hundred linear by transmitted light. The cilia will be seen in rapid action just within the oscula which terminate each of the large angular interstitial cells of the sponge. This action, and the mode of the disposition of the cilia within the cells, I have described at length in the Transactions of the Microscopical Society of London, vol. iii. p. 137, plate 19. In accordance with these variations in structure I purpose dividing the British species into four genera.

Class **PORIFERA**, Grant.

Order I. CALCAREA.

Genera. *Grantia.*
Leucosolenia.
Leuconia.
Leucogypsia.

GRANTIA, Fleming.

Sponge furnished with a central cloaca, parietes constructed of interstitial cells, more or less regular and angular in form, disposed at right angles to the external surface, and extending in length from the outer to very near the inner surface of the sponge, where each terminates in a single osculum.

Type, *Grantia compressa*, Johnston.

The cloaca varies in its form and proportion. In some species it has invariably one large terminal mouth, while in others it is furnished with several mouths, from which the excurrent fæcal streams are discharged.

The interstitial structures of the sponges of this genus assume a greater amount of regularity than is found to exist in any other genera of these animals. The whole of the parietes of the sponge are formed of somewhat angular cells, the sides of which belong to the individual cell, and are not common to the adjacent cells. The length of the cells in proportion to their diameter varies in different species, and also in the same species in proportion to the age and thickness of the parietes of the sponge. The cell-walls are formed of comparatively stout transparent membrane, strengthened and supported by numerous triradiate spicula; and the whole length of the cell, from the inner edge of the osculum to near the outer surface of the sponge, is closely studded with tessellated nucleated cells, each of which is furnished with a long attenuated cilium. Each interstitial cell terminates in a single osculum, slightly within the plane of the inner surface of the sponge. I do not remember to have ever seen these oscula entirely closed. When the inhalant action of the sponge is in vigorous operation, the excurrent streams may be seen issuing from them with considerable force, and the cilia appear in action immediately within them.

Hitherto the mouths of the great cloacal cavity of the sponges of this tribe have been described as oscula; but if we carefully examine the structure of these and similarly formed sponges, we shall find in all cases that those organs exist only on the inner surface of the great cloacal cavities.

The construction of the interstitial cells is best demonstrated in a longitudinal section of a dried specimen of *Grantia ciliata*, mounted in Canada balsam; and in a specimen so prepared, spaces are seen between the cells, which are often nearly half the size of the cells. These spaces are most probably produced by the contraction of the tissues induced by the mode of the preparation of the object, and do not exist in the living sponge; but they serve admirably to demonstrate the fact that each interstitial cell has its own special parietes, and that the divisions between the cells are not common to adjacent cells. Plate LXXII. fig. 1, and (Part II.) Plate XXXIII. figs. 1 & 2.

LEUCOSOLENIA, Bowerbank.

Grantia, Fleming and Johnston.

Sponge fistular, formed of a single layer of triradiate and other spicula surrounding a large central cloaca, which extends into all parts of the sponge.

Type, *Grantia botryoides*, Fleming.

The structure of *Grantia botryoides*, Fleming, differs essentially from that of *Grantia compressa* of that author, inasmuch as there is a total absence of the interstitial cells which are so characteristic of the latter sponge; and its structure is equally discrepant when compared with that of *Grantia nivea* of Fleming; for although it possesses cloacæ in common with that species, it has no approximation whatever to the massive Halichondroid form of the substance of that sponge. On the contrary, its parietes consist of a single thin layer of spicula and membranous tissues surrounding a large central sinuous cloaca. Plate LXXII. fig. 2.

LEUCONIA, Grant.

Grantia, Fleming and Johnston.

Sponge furnished with cloacæ, one or more. Parietes of sponge formed of a mass of irregularly disposed interstitial membranes, and triradiate and other spicula; permeated by sinuous excurrent canals, the oscula of which are irregularly disposed over the surfaces of the cloacæ.

Type, *Grantia nivea*, Fleming.

Grantia nivea of Dr. Fleming is very different in its structure from either *G. compressa* or *ciliata*, or of *G. botryoides* of that author. It has not the regular interstitial structure of either of the first two, nor the simple fistulose form of the latter one, but, with the exception of the form of the spicula, it closely simulates the structural character of the siliceous genus *Halichondria*, while it is allied with the before-named calcareous sponges by the possession of cloacæ. In consequence of these marked differences in the structure of the skeleton, I have separated it from *Grantia* as defined by Dr. FLEMING, and constituted it a genus, adopting the term *Leuconia*, which was proposed by Dr. GRANT as a general designation of the whole tribe of calcareous sponges. Plate LXXII. fig. 3.

LEUCOGYPSIA, Bowerbank.

Sponge massive, without cloacæ; formed of irregularly disposed membranous tissues and spicula. Oscula at the external surface.

The sponges of this genus are still further removed in structural character from the more highly organized genera of calcareous sponges *Grantia* and *Leucosolenia* than the genus *Leuconia* is. In the arrangement of the interstitial membranes, and the mode of dispersion on them of the skeleton-spicula, there is a manifest similitude to the structural peculiarities of the genus *Hymeniacidon* among the Silicea, and we find a corresponding simplicity in the characters of the spicula, in *Leucogypsia* the type of this genus. There are no regularly determined cloacæ projected from the surface as in *Leuconia*; and the excurrent canals of the sponge merge into each other, until they unite in one large canal immediately beneath the osculum, in the manner generally prevailing in the great mass of Halichondroid sponges. These large canals have defensive spicula similar in structure to those of the other genera of calcareous sponges. The only known British species of this genus is *L. Gossei*, Bowerbank, MS.; but I am acquainted with an exotic species, *L. algoaensis*, Bowerbank, MS., which is not uncommon on specimens of Zoophytes and Fuci from Algoa Bay and its neighbourhood. Plate LXXII. fig. 4.

Synopsis of the Suborders of the SILICEA *and* KERATOSA.

Order II. SILICEA.

Suborder I. Spiculo-radiate skeletons. Not reticulate. Composed of spicula radiating in fasciculi or separately from the base or axis of the sponge.

1. *Geodia*, Lamarck.
2. *Pachymatisma*, Bowerbank.
3. *Ecionemia*, Bowerbank.
4. *Alcyoncellum*, Quoy et Gaimard.
5. *Polymastia*, Bowerbank.
6. *Halyphysema*, Bowerbank.
7. *Ciocalypta*, Bowerbank.
8. *Tethea*, Lamarck.
9. *Halicnemia*, Bowerbank.
10. *Dictyocylindrus*, Bowerbank.
11. *Phakellia*, Bowerbank.
12. *Microciona*, Bowerbank.
13. *Hymeraphia*, Bowerbank.
14. *Hymedesmia*, Bowerbank.

Suborder II. Spiculo-membranous skeletons. Composed of membranous structure, having spicula irregularly dispersed on their surfaces.

Hymeniacidon, Bowerbank.

Suborder III. Spiculo-reticulate skeletons. Skeletons continuously reticulate in structure, but not fibrous.

1. *Halichondria*, Fleming.
2. *Hyalonema*, Gray.
3. *Isodictya*, Bowerbank.
4. *Spongilla*, Linnæus.

Suborder IV. Spiculo-fibrous skeletons. Regularly fibrous. Fibres filled with spicula.

1. *Desmacidon*, Bowerbank.
2. *Raphyrus*, Bowerbank.

Suborder V. Compound reticulate skeletons, having the primary reticulations fibro-spiculate, and the interstices filled with a secondary spiculo-reticulate skeleton.

Diplodemia, Bowerbank.

Suborder VI. Solid siliceo-fibrous skeletons, reticulate. Fibres composed of concentric layers of solid silex; without a central canal. Reticulations unsymmetrical.

Dactylocalyx, Stutchbury=*Iphiteon* of the Museum at the Jardin des Plantes, Paris.

The structure of the fibre in this suborder of siliceo-fibrous sponges is equivalent to that in the first suborder of the third order, Keratosa.

Suborder VII. Canaliculated siliceo-fibrous reticulated skeletons. Fibres composed of concentric layers of solid silex, with a continuous central canal. Reticulations symmetrical.

Farrea, Bowerbank.

The construction of the fibre of the skeletons of this suborder of siliceo-fibrous sponges is the equivalent of the fibrous structure of the fourth suborder of the third order, Keratosa.

Order III. KERATOSA.

Suborder I. Solid non-spiculate kerato-fibrous skeletons.

1. *Spongia*, Linnæus.
2. *Spongionella*, Bowerbank.

No spicula are secreted in any of the parts of the sponges of this suborder.

Suborder II. Solid semispiculate kerato-fibrous skeletons. Skeleton partially symmetrical; primary lines of fibre radiating from the proximal to the distal parts of the sponge; fibres containing spicula. Secondary lines of fibres unsymmetrical, destitute of spicula.

The Bahama sponges of commerce are most of them members of this suborder.

Halispongia, Blainville.

Suborder III. Skeletons kerato-fibrous; fibres solid, entirely interspiculous. Skeleton symmetrical.

Chalina, Grant.

In this suborder the keratode is the primary material in the structure of the fibre, and the spicula the secondary or auxiliary agent. The reverse is the case in the spiculo-fibrous tissues of the fourth suborder of Order II. Silicea.

Suborder IV. Simple fistulo-fibrous skeletons. Cavity of the fibre simple, central, and continuous.

Verongia, Bowerbank.
Spongia fistulosa, Lamarck.

The genus *Verongia* was described by me in the Annals and Magazine of Natural History for May and December 1845.

The same relative differences exist between the fibrous structures of the suborders six and seven of the second order, Silicea, that we observe between those of the first and fourth suborders of the third order, Keratosa.

Suborder V. Compound fistulo-fibrous skeletons. Central cavity of the fibre single and continuous, having secondary cæcoid branches radiating from it at nearly right angles.

Auliskia, Bowerbank.

This genus was described by me in the Annals and Magazine of Natural History for May and December 1845.

Suborder VI. Regular semi-areno-fibrous skeletons. Skeleton regularly areno-fibrous, having a well-defined central line of grains of extraneous matter within the fibres.

Stematumenia, Bowerbank.

In the sponges of this suborder the extraneous material is subordinate to the keratose fibre, in which it exists in the form of a central line of sand or other extraneous matters, constituting an axial line in the fibre surrounded by a thick coat of pure keratode. The axial line of sand is generally confined to the primary fibres of the skeleton, the secondary ones being usually without it. A portion of the commonest Bahama sponges of commerce belong to this order.

Suborder VII. Irregular and entirely areno-fibrous skeletons. Skeleton irregularly areno-fibrous, having the skeleton-fibre filled from the centre to the surface with grains of extraneous matter.

Dysidea, Johnston.

In the skeletons of the sponges of this suborder the keratode appears subordinate to the extraneous matter; the fibres frequently appearing to consist almost entirely of sand.

On the Arrangement of the Genera.

The genus *Halichondria*, as established by FLEMING and adopted by Dr. JOHNSTON, when applied to the arrangement of exotic as well as British species, embraces so wide a range as to afford little or no assistance in the determination of species. Under this designation every known sponge would be arranged having silex as the earthy basis of its skeleton, however varied their anatomical structure might be, excepting the few species contained in the genera *Geodia*, *Tethea*, and *Spongilla*.

Dr. JOHNSTON, in his 'History of British Sponges,' has divided the British species into three sections, dependent on their form, a character so mutable among the Spongiadæ, as to render it of little value, under any circumstances, when unaccompanied by structural peculiarities. I have therefore thought it advisable to distribute the genera included in the order Silicea among seven suborders, founded on the most striking peculiarities of the structure of the skeleton.

The first of these will consist of sponges having spiculo-radiate skeletons. Skeletons not reticulated, but composed of spicula radiating in fasciculi or separately from the base or axis of the sponge. This order will contain as many as fourteen distinct genera, the whole of which have skeletons the spicula of which are arranged in radial order. The mode of the radiation in these fourteen genera is not precisely the same, but they form three closely according groups, of which the leading genus of each of the first two may be considered as the type.

1. *Geodia*, Lamarck.
2. *Pachymatisma*, Bowerbank.
3. *Ecionemia*, Bowerbank.
4. *Alcyoncellum*, Quoy et Gaimard.
5. *Polymastia*, Bowerbank.
6. *Halyphysema*, Bowerbank.
7. *Ciocalypta*, Bowerbank.

The second group contains:

1. *Tethea*, Lamarck.
2. *Halicnemia*, Bowerbank.
3. *Dictyocylindrus*, Bowerbank.
4. *Phakellia*, Bowerbank.

In the whole of the first two groups, excepting *Halyphysema*, the skeleton-radiations are fasciculated to a greater or a less amount in the different genera.

The third group will comprise:

1. *Microciona*, Bowerbank.
2. *Hymeraphia*, Bowerbank.
3. *Hymedesmia*, Bowerbank.

The most striking general character in these three genera is the extremely thin coating-form of the sponge, and the radiation of the skeleton-spicula, either singly or in an irregularly fasciculated form, from a common basal membrane, the thickness of the sponge in some of the species being less than the length of one of the radiating skeleton-spicula.

Order II. SILICEA.

Suborder I. Spiculo-radiate skeletons. Not reticulate. Composed of spicula radiating in fasciculi or separately from the base or axis of the sponge.

Geodia, Lamarck.

Skeleton: spicula fasciculated, radiating from the base or central axis of the sponge to the surface. Dermis crustular, furnished abundantly with closely packed ovaria. Ovaria siliceous, composed of cuneiform spicula, firmly cemented together by silex, in lines radiating from the centre of the ovary. Pores furnished with œsophageal tubes *

* *i. e.* tubes resembling in their office the œsophageal tubes of the higher animals. The expression "pyloric valve" used further on is to be understood in a similar sense.

terminating in the distal extremity of the intermarginal cavities. Intermarginal cavities separate, symmetrical, subcylindrical; each furnished with a membranous valve at its proximal extremity.

The genus, as described by LAMARCK*, is so loosely characterized that I have thought it better to reconstruct it entirely than to endeavour to amend it. I have therefore given a new series of characters, founded solely on its structural and organic peculiarities. I am acquainted with seven species, all of which perfectly agree in the essential generic characters as thus constructed.

The type specimen of LAMARCK'S *Geodia gibberosa* in the Museum of the Jardin des Plantes of Paris, the organization of which, through the kindness of Professors MILNE-EDWARDS and VALENCIENNES, I have had an opportunity of thoroughly examining, is unfortunately in so deteriorated a condition in many respects, and especially in regard to the dermal membrane and pores, that I have been induced to select *G. Barretti* from which, to a great extent, to describe the interesting and highly organized structures of this genus; and I have the advantage also in this species of having a portion of a specimen which has never been deteriorated by drying, having been pickled in strong salt and water immediately on being taken from the sea, by my friend Mr. M^cANDREW, and in this state it closely resembles a mass of somewhat indurated animal liver.

The skeleton is composed of continuous fasciculi of stout long spicula, which in massive specimens radiate from the base to the outer surface of the sponge, or, if the species be of an elongated form, from the central axis to the circumference, where in either case they terminate at the inner surface of the crustular dermis, intermixing with, and being firmly cemented to, the shafts of the expando-ternate connecting spicula, which are attached to and firmly support the inner surface of the crustular dermis.

The organization of this external crust is exceedingly interesting. The outer surface is composed of a uniform thin pellucid dermal membrane, perforated with innumerable minute pores, variable in their diameter, and apparently possessing the power of opening or closing at the will of the animal. Immediately beneath the dermal membrane there is a stratum of sarcode of variable thickness in different species; and this stratum is permeated by numerous short canals, connecting the external pores with the intermarginal cavities which occupy, at nearly equidistant points, the thick stratum of ovaria forming the inner layer of the crustular dermis. In dried specimens, the positions of the intermarginal cavities are usually indicated on the surface of the sponge by a series of dimples or pits, frequently assuming, by the contraction of the dermal membrane, more or less of a stellated appearance. The proximal extremities of these organs is at the inner surface of the stratum of ovaria, and the distal extremities at the outer surface of the same stratum; and this termination has usually a greater diameter than the proximal end, which is furnished with a stout contractile diaphragm or pyloric valve.

* Polyparium liberum, carnosum, tuberiforme, intùs cavum et vacuum, in sicco durum; externâ superficie undique porosâ. Foramina poris majora in areâ unicâ orbiculari et laterali observata. (LAMARCK, Anim. s. Vert. 2de édit. ii. p. 593.)

The expando-ternate spicula, which are situated at the distal extremities of the radial fasciculi of the skeleton, diverge slightly from each other from their basal extremities, so that their triradiate heads, when firmly cemented to the inner surface of the ovarian stratum, form a strong and regular siliceous network, the points of the radii of each being cemented by keratode to those of its next neighbour; and within the area of each of these meshes of the network there is the proximal end of an intermarginal cavity, the diaphragm of which frequently occupies the greater portion of the area, having a much greater diameter than that of the proximal orifice of the cavity, so that when fully opened its orifice is quite equal to that of the intermarginal cavity. The ovaries vary considerably in size in different species. In the adult and prolific condition they have the form of a strong, thick-shelled, more or less globose ovarium, having a funnel-shaped orifice at the apex, which communicates with the central cavity, which, in the prolific state, is filled with closely-packed minute vesicular bodies, very similar in appearance to those contained in the ovaria of the Spongilladæ, but apparently more minute. In this condition of the ovary its parietes are formed of acutely cuneiform spicula, firmly cemented together by siliceous matter, the united apices forming the inner surface of the ovarium, while the united truncate bases form the external surface. In the early and immature state of the ovaria these truncated bases are not produced, and the young ovary has its outer surface bristling with pointed spicula, which are most acute in the youngest specimens, and become gradually more obtuse as they approach maturity. After the prolific contents of the adult ovary have been liberated, the internal cavity is gradually filled up by the extension inwards of the apices of the cuneiform spicula, until it becomes eventually a solid body; and a similar secretion of siliceous matter is also frequently continued at the outer surface until it often assumes an irregular tuberous and quite abnormal appearance.

The ovarian stratum of the crustular dermis is principally composed of exhausted solid ovaria; but occasionally near the outer surface of the stratum a few prolific ones may be observed, but the greater number of these bodies and of those in an early stage of development are situated amid the deeply-seated portions of the sponge, scattered irregularly over the sarcodous membranes and deeply immersed in the sarcode. In the young state they each appear to be surrounded by a firm stratum of sarcode, which, from its perfectly smooth and circular form, is apparently contained within a proper membrane, but in the fully developed and in the exhausted ovaria this sarcodous envelope is not observable. This description of the organization of the genus will apply equally well to any one of the seven species with which I am acquainted, and also to the nearly allied genus *Pachymatisma*, excepting the mode of the arrangement of the skeleton in the latter.

Both the type specimens of *Geodia* in the Museum at the Jardin des Plantes appear to have had large central cavities; but I have not found similar excavations in other species of the genus, excepting in one instance, a *Geodia* from Port Elliot, Australia; the internal surface in each of the three cases presents precisely the same appearance—a simple irregularly matted surface of spicula and membranes without any thickening of the tissues, and differing in no respect from the surfaces of any of the smaller internal cavities of the sponge. I am therefore inclined to consider such excavations as abnormal occurrences, which are not entitled to be considered as of either generic or specific value. Part II. Plate XXXII. figs. 2, 3, & 4; and Plate LXXII. fig. 5.

PACHYMATISMA, Bowerbank.

Skeleton composed near the external surface occasionally of short fasciculi of siliceous spicula, disposed in lines at about right angles to the surface of the sponge. Central portion of the sponge unsymmetrical. Dermis crustular, furnished abundantly with closely packed ovaria. Ovaria siliceous, formed of cuneiform spicula, firmly cemented together in lines radiating from the centre of the ovary. Pores furnished with œsophageal tubes, terminating in the distal extremity of each intermarginal cavity. Intermarginal cavities symmetrical, subcylindrical, with a pyloric valve at the proximal end of each.

Since the first publication of my description of the sponge on which this genus is founded in the Synopsis Spongiarum of Dr. JOHNSTON'S 'History of British Sponges,' p. 243, I have found it necessary to base the generic characters of the Spongiadæ on the structural peculiarities of the skeleton and reproductive organs. I have therefore reconstructed the character of the genus in accordance with this rule.

This genus is closely allied to *Geodia* in its organic structure, but the difference in the arrangement of the skeleton readily distinguishes them. The general aspect of the species of each genus is also strikingly distinct. I am acquainted with six species of *Geodia* and three of *Pachymatisma*; and in every case the species may be readily referred to its proper genus even by its general aspect. All the species of either genus have a crustular dermis, and the structures of the ovaria are also alike in each. I have described the anatomical peculiarities of the latter organs so fully in the description of the generic characters of *Geodia* as to render it unnecessary to treat of them here. Plate LXXII. fig. 6.

ECIONEMIA, Bowerbank.

Sponge having a strong axial column or centre of closely packed siliceous spicula disposed in lines parallel to the long axis of the sponge, from which axial column or centre a peripheral system of spicula radiates at about right angles. Distal ends of the radii furnished more or less with ternate connecting spicula, the radii of which are disposed immediately beneath the dermal membrane.

This genus differs from *Dictyocylindrus* in having the axial column composed of a dense mass of parallel spicula instead of a column formed of an open network of spicula; and the peripheral system is also different, inasmuch as it is essentially a portion of the interstitial system of the sponge, and not more especially a defensive system as it appears in *Dictyocylindrus*, in no species of which genus have there ever yet been found ternate spicula at the surface, while in *Ecionemia acervus*, the type species of the genus, they are abundant.

The structure of the peripheral system exhibits a close alliance with the genera *Pachymatisma* and *Tethea*. *Ecionemia* differs from *Geodia* and *Pachymatisma* in the total absence of the siliceous ovaries, and of the crustular dermal coat formed principally of those bodies in the last-named genera. There are also no cylindrical valvular intermarginal cavities, and the ternate apices of the connecting spicula appear always to be applied to the inner surface of the dermal membrane. This arrangement of the tissues therefore forms a natural transition from *Pachymatisma* to *Tethea*, in some species of which genus the ternate spicula are found without the dermal membrane in the porrecto-ternate form, and are adapted to defensive purposes, while in others they occur immediately beneath it as patento-ternate connecting spicula. I have therefore assigned this genus a position between *Pachymatisma* and *Dictyocylindrus*. Plate LXXIII. fig. 1.

We have no British species of this genus; the type species, *Ecionemia acervus*, Bowerbank, MS., is in the Museum of the Royal College of Surgeons of London.

On the Genus ALCYONCELLUM, Quoy *et* Gaimard (*Euplectella*, Owen).

Professor OWEN, in his paper on *Euplectella aspergillum*, Owen, communicated to the Zoological Society January 26, 1841, and published in the Transactions of the Zoological Society of London, vol. iii. part 2. p. 203, pl. 13, appears to have fallen into a singular number of errors in the course of his description of this beautiful sponge. He has, in the first place, designated it as belonging to the Alcyonoid family, apparently only because it is cylindrical in form and reticulate in structure, but without the slightest reference to the polyps that must necessarily characterize an Alcyonium; and he proceeds in his description to describe the base of the sponge as its apex and the apex as its base. The author then notices the first specimen of this genus that was made known to us by MM. QUOY and GAIMARD, in the 'Zoologie de l'Astrolabe,' 8vo, 1833, p. 302, planches fol. Zoophytes, pl. 26. fig. 3, but unfortunately mistakes the generic name *Alcyoncellum*, applied to the sponge by the French authors, for *Alcyonellum*; and having mistaken its name, its base, and its apex, he proceeds to reason on its generic characters thus:—"If the basal aperture of the cone were open, the resemblance to some of the known reticulate Alcyonoid sponges would be very close, especially to that called *Alcyonellum gelatinosum* by M. DE BLAINVILLE, 'Manuel d'Actinologie,' 8vo, 1834, p. 529 (*Alcyonellum speciosum*, Quoy et Gaimard): its closure by the reticulate convex frilled cap, in the present instance, establishes the generic distinction; and in the exquisite beauty and regularity of the texture of the walls of the cone, the species surpasses any of the allied productions that I have yet seen or found described. I propose, therefore, to name it *Euplectella aspergillum*." In note 5 appended to this paper, Professor OWEN also says, "If the recognition of the generic or specific identity of the specimen here figured be impracticable by reason of its mutilated condition, the generic name applied to it cannot be adopted while the Lamarckian genus of freshwater polyps, *Alcyonella*, is retained in Zoology." Now as it is manifest that the reasoning of

Professor Owen in favour of his proposed genus *Euplectella* is based, not upon one only, but upon a series of errors; and as he has not attempted to characterize his own genus, while that of *Alcyoncellum*, Quoy et Gaimard, is regularly described in the 'Histoire Naturelle des Animaux sans Vertèbres' by Lamarck, 2nd edit. vol. ii. p. 589, printed in 1836, it is evident that the generic name of the French authors must take precedence of that proposed by Professor Owen.

The following is the generic description of MM. Quoy et Gaimard:—

"Genre Alcyoncelle (*Alcyoncellum*).

Spongiaire lamelleux, dont la charpente est formée de filets très déliés, accolés les uns aux autres et entre-croisés de manière à former des mailles nombreuses, arrondies, assez régulières, et semblables à celles d'une dentelle."

In this generic description the material of which the sponge is formed is not in the slightest degree indicated, and the description of its structural peculiarities is so general that it will apply equally well to almost every known fistulose sponge. I have therefore thought it necessary to arrange the sponges of this genus with their congeners in material and mode of construction, and to reconstruct the generic characters so as to endeavour to limit the genus within definite bounds. I propose therefore to substitute the following characters for those of the French authors.

Alcyoncellum, Quoy et Gaimard.

Euplectella, Owen.

Sponge fistulate; fistula single, elongate, without a massive base. Skeleton: primary fasciculi radiating from the base in parallel straight or slightly spiral lines; secondary fasciculi at right angles to the primary ones. Oscula congregated, with or without a marginal boundary to their area.

The congregation of the oscula in *Alcyoncellum corbicula* and *A. aspergillum* is not a character peculiar to those sponges. A similar mode of arrangement exists in several species of *Geodia*. In *G. gibberosa*, in the Museum of the Jardin des Plantes at Paris, they are congregated in an area with a well-defined boundary, and in specimens of *G. Barretti* in my possession they are situated in deep depressions or cavities on the surface of the sponges; and these cavities or areas are not uniform in either shape or size; so we may infer that the presence in some species of *Alcyoncellum* of a well-defined marginal boundary to the oscular area, and its absence in other species, amounts to a specific difference rather than to a generic distinction; but in either case the oscula are congregated at the distal extremity of the sponge, and the areas of its parietes are the inhalant portions of the animal. The inhalation and exhalation of water is precisely on the same principle as that which obtains in *Grantia ciliata*; the whole of the parietes are appropriated to inhalation, the incurrent streams are passed through the interstitial cavities and discharged into a common cloaca, and the effete stream ejected at the distal extremity of the sponge,—the essential difference being that in *Grantia* the distal end of the cloaca is open, and in *Alcyoncellum* it is partially closed by a cribriform veil, the orifices of which appear to be the true oscula of the sponge. And this opinion is justified by the structure of the numerous cloacæ in the closely-allied genus *Polymastia*, where we find the orifices through which the incurrent streams are poured into the cloaca permanently open.

All the known species of this genus appear to consist of a single fistulose body, and some of them are apparently of a parasitical habit. *Alcyoncellum aspergillum* especially is furnished with numerous recurvo-quaternate spicula at its base, by which it attaches itself to sponges or other bodies. These prehensile organs do not appear in all the species of the genus; and in one perfect and beautiful specimen in the Museum of the Jardin des Plantes at Paris the base is closed, and is entirely destitute of prehensile spicula. The attachment of the sponge is partly, on one side, in the form of a thick incrustation, and partly, close to the base, by a similar patch of thickened tissue. There is also another striking difference in its structure; and that is, the absence of the raised margin to the oscular area at the apex of the sponge. In other structural characters it agrees exceedingly closely with *A. aspergillum.*

Polymastia, Bowerbank.

Skeleton a basal mass; central portion consisting of a plexus of contorted anastomosing fasciculi, resolving themselves near the surface into short straight bundles disposed at nearly right angles to the surface. Oscula congregated, elevated on numerous long fistulæ. Fistulæ composed of numerous parallel fasciculi, radiating from the base to the apex of each in straight or slightly spiral lines. Plate LXXIII. fig. 2.

This genus is closely allied to *Alcyoncellum*, Quoy et Gaimard, the principal difference being that in the latter the sponge always consists of a single fistula, while in the former it is constructed of a basal mass from which numerous fistulæ emanate. The fistular organs in each genus very closely resemble each other in their form and structure. Besides these structural differences, there are others, of a less striking description, that strongly indicate the necessity for generic separation. Thus in *Alcyoncellum corbicula*, in the Museum at Paris, and *Euplectella aspergillum*, Owen, there are an abundance of interstitial spicula of rectangulated sexradiate forms, which are very characteristic of those species, while the British species of *Polymastia* with which we are acquainted appear to be totally destitute of these complicated and beautiful forms of spicula. I have therefore thought it desirable, notwithstanding the close agreement that exists in the structure of their fistulæ, that a generic distinction should be established between them.

Halichondria mammillaris, Johnston, is the best type of the genus *Polymastia.* The whole of the parietes of these elongated fistulæ are inhalant. In some specimens of *P. mammillaris* dredged in Vigo Bay by my friend Mr. McAndrew, the open pores are exceedingly numerous, and the exhalant organs are as distinctly shown to be confined to the distal extremities of the fistulæ.

HALYPHYSEMA, Bowerbank.

Sponge consisting of a hollow basal mass from which emanates a single cloacal fistula. Skeleton: spicula of the base disposed irregularly; spicula of the fistula disposed principally in lines parallel to the long axis of the sponge, without fasciculation.

In its form and habit the type of this genus closely resembles *Polymastia brevis*; but the total absence of fasciculi in its construction at once marks it as a distinct genus, although a closely allied one. The type species, *H. Tumanowiczii*, is remarkable as being the smallest known British sponge; it rarely exceeds a line in height. The base of the sponge resembles in form the half of an orange cut at right angles to its axis; and the fistular cloaca is usually dilated at its distal extremity. I have been unable to detect either oscula or pores in any of the numerous specimens I have examined; but, from the general accordance in structure with the genera *Alcyoncellum* and *Polymastia*, there is a strong presumption that the oscula will prove to be congregated at the distal extremity of the cloacal fistula, as in those genera. Plate LXXIII. fig. 3.

CIOCALYPTA, Bowerbank.

Skeleton composed of numerous closed columns, each consisting of a central axis of compact, irregularly clongated reticulated structure, from the surface of which radiate, at about right angles, numerous short simple cylindrical pedicels, or stout fasciculi of closely packed spicula; the distal ends of each pedicel separating and radiating in numerous curved lines, which spread over the inner surface of the dermal membrane, separating and sustaining it at all parts, at a considerable distance from the central axis of the skeleton.

This genus is allied by its structural peculiarities, to a certain extent, to *Dictyocylindrus*, Bowerbank, *Hyalonema*, Gray, and *Alcyoncellum*, Quoy et Gaimard. The central axial column of the skeleton is composed of clongated stout reticulations of siliceous spicula, closely resembling the corresponding tissues of the axial column of a *Dictyocylindrus*; but the space between the surface of the column and the inner surface of the dermis is not filled, as in that genus, by the usual interstitial structures of the sponge, it is completely and widely separated from the dermis in a manner very similar to that of the structure of the greatly elongated cloacal appendage of *Hyalonema mirabilis* as it appears in its present condition in the most perfect specimens in the British Museum and in the collection of Dr. GRAY. There is this difference between the structures of the two genera. The coriaceous dermis surrounding the beautiful spiral axial column of *Hyalonema* is very thick, and is abundantly furnished with projecting oscula; and it does not present any indications of lateral pedicels, either on its inner surface or on the surface of the axial column, while these organs are abundant in *C. penicillus*, and its dermis also is comparatively thin and delicately reticulated.

The dermal portion of the sponge in *C. penicillus*, and the reticulated tissues on its inner surface, closely resemble the corresponding tissues in *Alcyoncellum* in their structure. The pores, in number, size, and mode of distribution, are very similar to those of *Polymastia robusta*, Bowerbank; but the stratum of these reticulated skeleton-structures is not so thick in proportion, and in *Alcyoncellum* and *Polymastia* there is no central axial column. I could not detect interstitial membranes in any part of the space intervening between the axial column and the dermis in *C. penicillus*; but the skeleton-column is permeated by numerous interstitial canals.

The structure of the short pedicels passing from the axial column to the inner surface of the dermis is different from that of the axis; the spicula composing them are parallel to each other, and they are firmly packed together. The bases of the pedicels arise from the surface and from within the substance of the central column, with which they appear to have no further connexion than that which is necessary to secure them firmly in their respective positions. Their apices present a very beautiful appearance, spreading out towards the inner surface of the dermis in curves diverging at angles of about 45 degrees in every direction over it,—which, when viewed with a microscopic power of about 100 linear, resembles an elaborate and beautifully groined roof of a Gothic crypt. Plate LXXIII. figs. 4 & 5.

TETHEA, Lamarck.

The following are the generic characters given by LAMARCK, in his 'Anim. sans Vert.' 2nd edit. ii. 384:—

"TÉTHIE (*Tethea*).

"Polypier tubéreux, subglobuleux, très fibreux intérieurement; à fibres subfasciculées, divergentes ou rayonnantes de l'intérieur à la circonférence et agglutinées entre elles par un peu de pulpe; à cellules dans un encroûtement cortical quelquefois caduc. Les oscules rarement perceptibles."

Dr. JOHNSTON'S version of the generic characters differs slightly from LAMARCK'S. It is as follows:—

"Sponge tuberous, suborbicular, solid and compact, invested with a distinct rind or skin, the interior sarcoid loaded with crystalline spicula collected into bundles and radiating from a more compact nucleus to the circumference. Marine."

It is much easier to find faults in the generic characters of both the authors quoted, than it is to improve them. The extreme simplicity of the structural characters of *Tethea* is a strong temptation to endeavour to multiply them; but in doing so, Dr. JOHNSTON has introduced two—the structure of the dermal portion of the sponges, and the tuberous nature of its surface—which are not common to all the known species. If we consider the word "tuberous" in the usual English acceptation of the word, as a body "full of knobs or swellings," then very few or perhaps none of the species of *Tethea* would, in their natural condition, exhibit this character; but all of them would be in a greater or less degree subglobular. Dr. JOHNSTON'S description of *Tethea* was founded on the structure of *T. Lyncurium* only; and in this species the "thick rind" is very distinctly to be seen, but in other species this structure is totally wanting. It therefore ceases to be of value as a generic character, and becomes a specific one only. Under these circumstances I propose the following modification of the previously published generic characters:—

Sponge massive, suborbicular. Skeleton consisting of fasciculi of spicula. Fasciculi radiating from a basal or excentrical point to the surface. Intermarginal cavities unsymmetrical, confluent. Propagation by internal or external gemmulation.

This genus affords us one of the few instances in which we may avail ourselves of external form as a generic character; but even in *Tethea* we approach exceptions to the rule in the depressed form of *T. Collingsii*, Bowerbank, MS., as exhibited in the only perfect specimen of that species which I have seen, and in the still more depressed form of *T. spinularia*, Bowerbank, MS.

Although the skeleton-structures in the species of this genus differ to an exceedingly slight extent, the subsidiary spicula vary exceedingly in the different species. In some, ternate spicula are numerous, and in others they are entirely absent; and stellate forms of spicula occur in many varieties of form.

The sponges of this genus appear to be highly organized. AUDOUIN and MILNE-EDWARDS saw the oscula open and the excurrent streams in action, and I have seen the same myself in a specimen of *T. Lyncurium*. My friend Mr. GEORGE CLIFTON, of Freemantle, Western Australia, in a letter dated 25th January, 1861, writes, "I have sent you several fine specimens of *Tethea*. When these animals are first taken out of the water they are of a brilliant orange-colour, and commence squirting water from the oscula situated on the centre of the upper surface; they also contract considerably, but on being replaced in their native element they regain their natural size and reabsorb water."

The mode of propagation varies in different species. In *T. cranium* and *T. simillimus*, Bowerbank, MS., it is by internal gemmulation, in *T. Lyncurium* by external gemmulation; and in some other species the mode is not apparent. Plate LXXIII. fig. 6, and Part II. Plate XXIX. fig. 12.

HALICNEMIA, Bowerbank.

Skeleton formed of a single superior stratum of spicula radiating from the centre to the circumference of the sponge at about its middle, and of an inferior stratum of spicula distributed without order.

The nearest alliance to this genus appears to be *Tethea*, in which the skeleton is formed of numerous fasciculi of spicula radiating from the centre to all parts of a spherical or elliptical mass; while in *Halicnemia* the radiating fasciculi are confined to a common plane, beneath which there is a second stratum of spicula, which fills the space beneath the radial stratum and the lower surface of the sponge, but without being disposed in order, and the spicula of the inferior stratum differ materially in form and proportions from those of the superior one.

In the only two specimens of this genus that I have seen, there is a small pebble imbedded in the centre of each sponge, from the surface of which the basal fasciculi of the radial series emanate; but although this appears to be the established habit of this species, it is advisable not to consider it as a generic character, although it may eventually prove to be that the pebble is as much a portion of the skeleton of the animal as the grains of extraneous matter which are taken up by and become imbedded in the keratose fibres of the genus *Dysidea*. Plate LXXIV. figs. 4 & 5.

DICTYOCYLINDRUS, Bowerbank.

Skeleton without fibre, composed of a loosely compacted columnar axis of spicula, disposed principally in the direction of the line of the axial column, from which a peripheral system of long single or fasciculated defensive spicula radiate at right angles to the axial column.

Halichondria hispida, Johnston, and *Spongia stuposa*, var. *damicornis*, Montagu, are excellent types of the peculiar mode of arrangement of the spicula which characterizes this genus. The skeleton consists of a central column of large elongate spicula, disposed principally in the line of the axis of the sponge and at a slight angle to it, approaching in form an irregular cylinder of network of elongated meshes, rarely exhibiting an appearance of horny fibre, but formed for the most part of spicula cemented together near their terminations. Towards the base of the sponge the horny substance surrounding the spicula is sometimes so thick as to simulate a proper horny fibre; but if it be carefully traced, it will always be found to be dependent on the spicula: where their course is abruptly terminated the horny structure also terminates, whereas in true horny fibrous structures which contain spicula the course of the fibre is continuous and uniform whether the spicula be present or deficient, and in the newly produced fibre the latter is generally the case.

The structure of the skeleton in this genus differs from that of *Halichondria oculata*, Johnston, or *Chalina oculata*, Bowerbank, in the regularly elongate disposition of the spicula of the skeleton; and the spicula are necessarily very much larger and longer than those included in the close fibrous network of *H. oculata*; and it is still further removed from the horny fibrous structure of *Halichondria cervicornis*, Johnston, Hist. Brit. Sponges, pl. 4. The axial column of this genus differs strikingly from that of the strong closely packed axis of *Ecionemia*; and the peripheral system of spicula are never furnished with ternate connecting spicula. All the species of this genus I have hitherto seen are more or less ramous in form. Part II. Plate XXIX. fig. 11; and Plate LXXIII fig. 7.

PHAKELLIA, Bowerbank.

Skeleton composed of a multitude of primary cylindrical axes, radiating from a common base and ramifying continuously, from which emanate at about right angles to the axes a secondary series of ramuli, which ramify continuously as they progress towards the surface, but never appear to anastomose.

I know of no other species, either British or foreign, that possesses the peculiar conformation that distinguishes the sponge that is the type of this genus. The primary cylindrical axes very closely resemble those of *Dictyocylindrus*; but in that genus the spicula radiating from the axes are separate and distinct, each having its proximal end based on the primary cylinders of the skeleton, and its distal one reaching nearly to, or passing through, the dermal membrane of the sponge; or if they be fasciculated, the fasciculi are simply plumose, and in no case with which I am acquainted at all ramulose. In *Phakellia* the secondary skeleton is formed of distinct slender branches, each composed of numerous spicula ramifying continuously, and each ramulus increases in size and the number of its spicula as it approaches the surface of the sponge. Single spicula are frequently projected from the ramuli in an ascending direction at an angle of a few degrees, and at their distal terminations at the surface of the sponge; the whole of the terminal spicula radiate more or less at angles from their axial line, and, passing through the dermal membrane, form the external defences of the sponge. Although constantly ramifying and freely intermingling, I have never detected them anastomosing. The term *Phakellia* is applicable to both the primary and secondary ramifications of the skeleton. The type of this genus is *Halichondria ventilabrum*, Johnston. I have not yet met with an exotic species of the genus. Plate LXXIV. fig. 1.

The genera *Microciona*, *Hymeraphia*, and *Hymedesmia* form a group essentially different in structural character from the other genera of the Spongiadæ; but they are closely allied to each other by the peculiar characters of their basal membranes, in conjunction with the other parts of the skeleton. From the nature of their structures, the species generally assume a thin coating-form and are often very minute.

In most of the genera of Spongiadæ the basal membrane of the sponge ceases to be of marked importance after the earliest stages of its development, but in these genera it continues throughout the whole existence of the sponge to form an important part of its skeleton-structure. It is a common base whence spring the whole of the other component parts of the skeleton; and its importance is further indicated by its also being in some species the common base of the internal as well as the external defensive spicula of the sponges in which those organs occur.

MICROCIONA, Bowerbank.

Skeleton a common basal membrane, whence spring at or about right angles to its plane numerous separate columns of spicula intermixed with keratode, furnished externally with spicula which radiate from the columns at various angles towards the dermal surface of the sponge.

The skeleton of the type of this genus, *M. atrosanguinea*, is different from that of any other genus of sponges that I have hitherto seen. It consists of numerous nearly equidistant, short, straight separate columns of spicula and keratode, from all parts of the sides of which spring stout, long, curved, fusiformi-attenuato-subspinulate spicula, the convex side of each spiculum being outward; and each column terminates with five or six of these spicula disposed in the same manner and at the same angle to the axial line of the column, that is, from about twenty to forty-five degrees. The proportions of the skeleton-columns vary in different species. In *M. atrosanguinea* they are short, stout, and exceedingly well defined. In *M. ambigua* they are short and indistinctly produced, and in *M. carnosa* they are long, slender, flexuous, and frequently branched; but however they may vary in their proportions in different species, their normal character, both as regards structure and position in the sponge, is always preserved. Plate LXXIV. fig. 2, and Part II. Plate XXX. figs. 1 & 2.

Genus Hymeraphia, Bowerbank.

Skeleton a single basal membrane, whence spring numerous large separate spicula, which pass through the entire thickness of the sarcodous stratum, to or beyond the dermal surface of the sponge.

This genus is nearly allied to *Microciona*, but is more simple in its structure, as, in place of the columns of the skeleton compounded of keratode and spicula cemented together and emanating from a common basal membrane as in the latter genus, we find single spicula only, devoid of keratode and based on a common membrane, whence they pass through the entire substance of the sponge; and in all the species at present known they penetrate the dermal membrane and project beyond its surface to a considerable extent, thus combining the two offices of skeleton and external defensive spicula. These organs are therefore, as compared with the skeleton-spicula of other members of the Spongiadæ, and to the entire mass of the sponges to which they belong, of exceedingly robust proportions, their length being frequently twice that of the entire thickness of the sponge.

These peculiarities of structure indicate a common habit of extreme thinness in the species; and such is in reality the condition of those with which we are acquainted. Part II. Plate XXX. fig. 3.

Hymedesmia, Bowerbank.

Skeleton a common basal membrane sustaining a thin stratum of disjoined fasciculi of spicula.

The species on which this genus is founded very closely resembles in habit and general appearance those of the genera *Microciona* and *Hymeraphia*, and in regard to the special offices of the basal membrane it assimilates with them completely. But it differs from them inasmuch as the spicular portions of the skeleton do not emanate immediately from the basal membrane, but are recumbent on it in the form of disjoined fasciculi of spicula. But although different from them in this important respect, the close alliance with them is indicated by the common habit of the possession by the basal membrane of the whole, or nearly so, of the defensive spicula of the sponge, indicating the common property of extreme thinness of structure which exists in these genera.

The free condition of the fasciculi of the skeleton connects this genus in some degree with the Halichondroid genera of sponges, but there are none of the species of those genera in which the fasciculi of the skeleton are separate from each other. The nearest allied genus in that direction appears to be *Hymeniacidon*. Plate LXXIV. fig. 3, and Part II. Plate XXXI. fig. 8.

Suborder II. Spiculo-membranous skeletons. Composed of interstitial membranes having the skeleton-spicula irregularly dispersed on their surfaces.

The prominent character of this Order is, that the spicula of the sponges composing it do not assume either the radiate, fasciculate, or reticulate structural arrangement, the distribution of the spicula on the interstitial membranes being without any approximation to order.

HYMENIACIDON, Bowerbank.

Skeleton without fibre; spicula without order, imbedded in irregularly disposed membranous structure.

In *Hymeniacidon* the spicula are subordinate to the membranous structure, they follow its course and are imbedded without order on its surface. The contrary is the case in *Halichondria*. The network of spicula in that genus, although irregular, is decidedly the predominant structure, and the membranous tissues are secondary to it and exist only as interstitial organs. The larger and stouter of the spicula in *Hymeniacidon*, although dispersed amid the slender ones, may be considered as the representatives of the skeleton-spicula, while the slender ones are truly those of the membranes.

In some species the interstitial tissues are constructed diffusely, as in *H. caruncula*, while in other species, as in *H. subereum* (*Halichondria suberea*, Johnston) and a few other closely allied species, they are more than usually compact, so that in the dried state the texture of these sponges is very like that of fine hard cork. From this peculiarity of their appearance in the dried condition, and the exceeding compactness of their structure, I was formerly inclined to believe them to be generically different from the great mass of the species of *Hymeniacidon*, and I accordingly inserted them in the list of British sponges, published in the Report of the Dredging Committee in the Reports of the British Association for 1860, under the titles of *Halina suberea*, *H. ficus*, &c.; but a closer examination of their internal structure has convinced me that their only real difference from the other species of *Hymeniacidon* is in their greater compactness of skeleton-structure, and I have accordingly removed those species to the genus *Hymeniacidon*.

In the greater number of the species of this genus the tension spicula are of the same form as those of the skeleton, and are only to be distinguished from them by their greater degree of tenuity; but in a few of the known species they are different both in size and form.

The mode of propagation in all the species in which I have found the reproductive organs appears to be by internal gemmulation. In *H. carnosum* and several other species of the genus the gemmules are simple, spherical, aspiculous membranous vesicles, filled with round or oval vesicular molecules. The genus *Halisarca*, Dujardin, was supposed by both that author and Dr. JOHNSTON to be entirely destitute of spicula; but I have, since the publication of the 'History of the British Sponges,' found them in *H. Dujardinii* in abundance. They are so minute and so completely obscured by the surrounding sarcode, that they can rarely be detected in either the living or the dead specimens when examined in water; but if a portion of the sponge be dried on a slip of glass and covered with Canada balsam, they may be detected by transmitted light and a power of 400 linear in considerable numbers, dispersed on the interstitial membranes of the sponge. This genus will therefore merge in that of *Hymeniacidon*, with which it agrees in every structural peculiarity. Plate LXXIV. fig. 6.

Suborder III. Spiculo-reticulate skeletons. Skeletons continuously reticulate in structure, but not fibrous.

Halichondria.
Hyalonema.
Isodictya.
Spongilla.

The sponges of this suborder vary in the different genera to a great extent in the mode of the construction of the skeleton, but in all cases the spicula are the dominant material; their terminations overlap each other, and they are cemented together by keratode. The reticulations thus formed sometimes consist of a single series of spicula, at other times they are very numerous and are crowded together in the manner of elongated fasciculi.

The genera *Halichondria* and *Isodictya* are exceedingly rich in species; but the inconvenience attending their discrimination, arising from their number, may be remedied to a great extent hereafter by subdivisions of each genus, based on the characteristic forms of the spicula of their respective skeletons. The structural distinction between *Halichondria* and *Isodictya* is so well marked as to render the recognition of each comparatively certain and easy. The skeletons of the species of the latter genus, generally speaking, are very much more slight and fragile than those of the former one; and the same rule obtains to a great extent as regards the comparative size of their spicula, and in many species of *Isodictya* they are very minute. *Hyalonema* and *Spongilla* are readily to be distinguished by the peculiarities of their structure and localities.

The genus *Halichondria* as constituted by Dr. FLEMING in his 'History of British Animals,' and adopted by Dr. JOHNSTON in his 'History of British Sponges,' contains species which differ exceedingly in their mode of organization. Thus, if we take *H. panicea* of JOHNSTON, which is undoubtedly the "sponge-like crumb of bread" of ELLIS and the older authors, and therefore the proper type of the genus, we find the skeleton destitute of fibre, but composed of an irregular network of spicula cemented together at their apices by keratode. If we examine the well-known branching sponge so common on all our coasts, *Halichondria oculata* of the same author, we find an abundance of keratose fibre containing spicula deeply imbedded in its substance, but not necessarily uniting at their apices; and the network of the skeleton is not irregular as in the first instance, but on the contrary is more or less symmetrically disposed in all parts of the sponge. If we take *Halichondria suberea* of the same authors, we find neither network of spicula nor a keratose fibrous structure, but apparently an amorphous sarcoid mass containing spicula and membranes, on which the former are dispersed without any order or connexion. As we extend our researches among the other British species of FLEMING's genus *Halichondria*, other striking and permanent variations in the arrangement of their skeleton-tissues present themselves. Their great differences in structure therefore afford ample grounds for the division of the species comprehended under *Halichondria* as constituted by FLEMING into a series of genera having each for its base a separate type of organization; and as the variations in structural character, some of which are mentioned above, are both numerous and strikingly characteristic, I propose to limit the genus *Halichondria* to those species only which agree in their organization with *H. panicea* of JOHNSTON, and to distribute the remaining species in other genera, the distinctive characters being in all cases based primarily on the different modes of the organization of the skeleton of the animal, and when necessary taking in aid such other organic characters as may be found available for the purpose of accurate discrimination. I therefore propose to limit the genus *Halichondria* to those sponges only that exhibit the following characters.

HALICHONDRIA, Fleming.

Skeleton without fibre, composed of an irregular polyserial network of spicula cemented together by keratode.

Type, *Halichondria panicea*, Johnston.

The anatomical structure of the group included under this genus is distinct and unmistakeable. There is no fibre whatever, the skeleton being formed of spicula collected into bundles of a greater or less number, cemented together by keratode, which substance, however, does not extend beyond the space occupied by the respective bundles; and when parts of the reticulated skeleton are formed of single series of spicula only, they are simply cemented together at their points, and the reticulated skeleton thus formed has no definite arrangement. Plate LXXIV. fig. 7, and Part II. Plate XXXII. figs. 1 & 5.

HYALONEMA, *Gray*.

Dr. GRAY has characterized this genus in his descriptions of genera of Axiform Zoophytes, or Barked Corals, as "coral subcylindrical, rather attenuated, and immersed in a fixed sponge. Axis in the form of numerous elongated, slender, filiform, siliceous fibres, extending from end to end of the coral, and slightly twisted together like a rope. Bark fleshy, granular, strengthened with short cylindrical spicula. Polypiferous cells scattered, rather produced, wart-like, with a flat radiated tip." (Proceedings of the

Zoological Society of London for 1857, page 279.) This description applies only to the singular cloacal appendages to the sponge from amidst which it springs, the structure of the body of the animal being evidently considered by the author as an extraneous mass. The basal sponge is undoubtedly a portion of the animal to which the part described by Dr. GRAY belongs, the spicula of the elongated cloacal portion being also abundant in the basal mass of sponge; and the basal mass of the specimen described by Dr. GRAY is identical in its structural character with that of the specimen of *Hyalonema mirabilis* in the Bristol Museum. It becomes necessary therefore to remodel the generic characters so as to embrace the leading distinctive structures of the skeleton of the animal; and I propose the following form of description:—

Skeleton an indefinite network of siliceous spicula, composed of separated elongated fasciculi reposing on continuous membranes, having the middle of the sponge perforated vertically by an extended spiral fasciculus of single, elongated and very large spicula, forming the axial skeleton of a columnar cloacal system.

The construction of the skeleton of the mass of the sponge is intermediate between that of *Halichondria panicea* and *Hymeniacidon caruncula*, the respective types of those genera. The network of fasciculated spicula appears never to be definite and continuous as in the former, nor are the skeleton-spicula in a dispersed condition on the continuous membranes as in the latter, but are gathered into elongated fasciculi which cross each other in the same plane in every imaginable direction, but without ever appearing to anastomose. The fasciculi vary exceedingly in the number of spicula of which their diameter is formed, sometimes consisting of two or three spicula only, and at other times of more than it is possible to count. They often divide, the branches passing in different directions, but they never reunite or anastomose with other fasciculi. A portion of this network of spicula is represented by figure 3, Plate XXXI. Part II. The columnar axis of the cloacal system consists of one large spiral fasciculus of spicula, each of which extends from the base or very near that part of the sponge, to near or quite to the apex of the column, the direction of the spiral being from right to left.

There is a close approximate alliance to this form of the cloacal appendage of *Hyalonema* in the corresponding organs of the British genus *Ciocalypta*, Bowerbank, MS.

ISODICTYA, Bowerbank.

Spongia, Montagu.
Halichondria, Fleming.
Halichondria, Johnston.

Skeleton without fibre; composed of a symmetrical network of spicula; the primary lines of the skeleton passing from the base or centre to the surface, and the secondary lines disposed at about right angles to the primary ones. Propagation by internal, membranaceous, aspiculous gemmules.

This genus, in the structure and arrangement of its skeleton, is intermediate between

Halichondria and *Chalina*, as defined in the present work. Like the former, the spicula of the network composing the skeleton are merely cemented together, not enclosed within a regular horny fibre; but the disposition of the network is not entirely irregular, but like that of the latter genus, more or less composed of a primary series of lines radiating from the axis or base of the sponge, and of secondary series connecting the primary ones at about right angles to them—in fact simulating very closely the arrangement of the skeleton of *Chalina oculata*, but without the keratose fibre surrounding the spicula of the skeleton in that sponge.

In some of the species of this genus the symmetrical arrangement of the lines of the skeleton is distinct only near the surface of the sponge, while in the more deeply seated parts the irregular character of a *Halichondria* is simulated. In determining the species of this genus, the sponge requires to be carefully examined by sections at right angles to the surface, where the distinctive character rarely fails to be readily detected. On the contrary, in *Halichondria panicea*, the type of that genus, I have never succeeded in finding such a linear arrangement of the skeleton as marks that of *Isodictya*. In a hasty examination a single linear series of spicula will therefore often prove an excellent guide to the discrimination of this genus.

In most of the species with which I am acquainted there is a generally prevailing character of fragility—the primary lines being composed of very few spicula, while the secondary ones are most frequently unispicular. Most of the species are thin-coating or encrusting sponges, and rarely appear to rise in tuberous masses, as the numerous species of *Halichondria* are in the habit of doing.

Isodictya infundibuliformis is perhaps the most perfect type of the genus, as in it we have the primary and secondary lines of the skeleton distinctly separated by the difference in the form of their spicula. In some species of the genus, as in *I. simulo*, the cementing keratode of the skeleton is so abundant in some parts as to cause it to simulate very closely the structure of a *Chalina*; but the irregularity and compressed form of this pseudo-fibre is readily to be distinguished from true keratose fibre by a careful observer. In other species, as in *I. mammeata*, the sarcode surrounding the skeleton is so abundant as to cause it to simulate a delicate form of *Chalina*; but on immersion in Canada balsam the fibre-like form disappears, the sarcode contracting into a mere granulated coating, and the skeleton assumes the normal appearance of *Isodictya*. Plate LXXIV. fig. 8.

SPONGILLA, Linnæus, Lamarck, and Johnston.

Halichondria, Fleming.

The structural peculiarities of the skeleton of *Spongilla* are the same as those of *Isodictya*; and if there had not existed a striking distinctive difference in their reproductive organs, the two genera must have been united. Under these circumstances I propose the following as the characters of the genus *Spongilla*.

Skeleton without fibre, composed of a symmetrical network of spicula; the primary lines of the skeleton passing from the base or centre to the surface, and the secondary lines disposed at about right angles to the primary ones. Reproductive organs ovaries, coriaceous and abundantly spiculous.

All the species are inhabitants of fresh water. The best type of the genus is *Spongilla fluviatilis*, Johnston. As an illustration of the form of the skeleton in this genus, see the figure of that of *Isodictya Normani*, Plate LXXIV. fig. 8.

Suborder IV. Spiculo-fibrous skeletons. Regularly fibrous. Fibres filled with spicula.

Desmacidon.

Raphyrus.

The spiculo-fibrous skeletons differ from the fibro-spicular ones in this respect. In the first the form and proportions of the fibre are dependent on the greater or the less development of spicula, and the keratode serves only as a cementing and coating material. In the latter the keratode is the primary agent in the formation of the fibre, and the spicula the secondary or auxiliary agent only.

DESMACIDON, Bowerbank.

Halichondria, Johnston.

Skeleton fibrous, irregularly reticulated. Fibres composed entirely of spicula arranged in accordance with the axis of the fibre, cemented together and thinly coated with keratode.

The structure of the skeleton-fibre in this genus readily distinguishes it from all others. The form and size of the tissue is entirely dependent on the greater or less quantity of spicula present, the keratode serving only as a cementing and coating material. *Halichondria ægagropila* and *H. fruticosa*, Johnston, are the only two British species of the genus known. Part II. Plate XXVII. fig. 10.

RAPHYRUS, Bowerbank.

Skeleton fibrous, but not horny. Fibre composed of a dense mass of siliceous spicula mixed together without order.

The structure of this genus is singular. The fibre in the only species with which I am acquainted, *Raphyrus Griffithsii*, is comparatively very coarse, frequently attaining the size of a line in diameter near the anastomosing parts, or expanding into a broad plate-like form. The spicula composing it are closely thrown together without any approach to the longitudinal disposition which prevails in the skeleton of *Desmacidon*. The same absence of definite arrangement obtains in the interstitial membranes, which have precisely the mode of structure which characterizes the genus *Hymeniacidon*, which has "spicula without order, imbedded in irregularly disposed membranous structure." Part II. Plate XXVII. fig. 11.

Suborder V. Compound reticulate skeletons, having the primary reticulations fibro-spiculate, and the interstices filled with a secondary spiculo-reticulate skeleton.

Diplodemia.

This genus forms a connecting structural link between the orders Silicea and Keratosa. The structure of the keratose fibre would indicate its place to be in the third suborder of the latter; but the presence of the Halichondroid secondary skeleton in such force, in conjunction with the irregular spiculated structure of the keratose fibrous primary skeleton, has induced me to place it among the Silicea. For more minute information regarding its structural peculiarities, I must refer my readers to the following description of the generic characters of *Diplodemia.*

DIPLODEMIA, Bowerbank.

Skeleton fibrous; fibres keratose, hetero-spiculous; combined with a secondary skeleton of irregular network of spicula; rete unispiculate, rarely bispiculate. Ovaries membranous and spiculous.

The fibres in the skeleton of the only known species in this genus are very remarkable. They are smooth and cylindrical, having an axial line of, generally speaking, single spicula united at their points, running throughout the whole length of the fibre. But when it is of more than ordinary diameter, there are frequently other spicula at intervals imbedded in the fibre, parallel to the axial series. Throughout the whole length of the fibres, at short intervals, there are similar spicula to the axial ones, imbedded at right angles to the axis of the fibre, frequently projecting from the surface for half, or more than half, their length. Some of these projecting spicula originate small lateral branches of the keratose skeleton; but by far the greater portion of them are the connecting points of the keratose fibres and the reticulo-spiculate secondary skeleton, the former being thus completely imbedded amidst the latter.

The structure of the ovaria in this genus is also peculiar to it. The wall is very thin, and appears to consist of a single membrane profusely furnished with spicula which cross each other in every direction, and occasionally appear to assume a somewhat fasciculated arrangement. They are not uniform in shape, some being regularly oval, while others are more or less ovoid.

But one species of this singular genus is known, *D. vesicula*, Bowerbank, MS., from deep water at Shetland. Plate LXXIII. fig. 8, and Part II. Plate XXXIV. fig. 1.

Suborder VI. Solid siliceo-fibrous skeletons. Skeletons reticulate. Fibres composed of concentric layers of solid silex, without a central canal. Reticulations unsymmetrical.

Dactylocalyx, Stutchbury (*Iphiteon*, French Museum).

The structure and mode of growth in this suborder of siliceo-fibrous sponges appear to be precisely the same as that of the kerato-fibrous sponges of the first suborder of Order III. Keratosa.

Dactylocalyx pumicea, Stutchbury, was described in the 'Proceedings of the Zoological Society, part ix. 1841, p. 86, October 26, 1841. The author describes it thus: "Sponge fixed, siliceous; incurrent canals uniform in size; excurrent canals large, forming deep sinuosities on the outer surface, radiating from the root to the outer circumference."

The sponge was received by the Bristol Museum from Dr. CUTTING of Barbadoes.

The genus *Dactylocalyx* was established by Mr. STUTCHBURY to designate this fine siliceo-fibrous sponge. Half of the type specimen is in the Museum at Bristol, and the remaining portion in the possession of Dr. J. E. GRAY of the British Museum. Although the sponge was designated *Dactylocalyx pumicea*, no generic characters were given. I propose therefore to characterize it as follows:—

DACTYLOCALYX.

Skeleton siliceo-fibrous. Fibres solid, cylindrical. Reticulations unsymmetrical. Part II. Plate XXXIV. fig. 17.

Suborder VII. Canaliculated siliceo-fibrous skeletons. Skeletons reticulate, symmetrical. Fibres composed of concentric layers of solid silex, with a continuous central canal.

Type, *Farrea occa*, Bowerbank, MS.

I have seen in the organic remains from deep-sea soundings several varieties of fragments of siliceous fibres with simple central canals, having every appearance of being from unknown species of siliceo-fibrous sponges; but the only satisfactory specimen of this genus of sponges is the one at the base of Dr. ARTHUR FARRE's specimen of *Euplectella cucumer*, Owen, described in the Transactions of the Linnean Society of London, vol. xxii. p. 117, plate 21.

The fibres in *Farrea occa* are rather coarse, abundantly tuberculated, and the mode of reticulation is rectangular. Their construction is exactly like those of *Verongia*, the type of the fourth suborder of the third order, Keratosa. Part II. Plate XXVII. fig. 11.

Order III. KERATOSA.

Suborder I. Solid non-spiculate kerato-fibrous skeletons.

The greater number of the sponges of commerce belong to this suborder. How many species are comprised under the designation of "the sponges of commerce" it is very difficult to decide, as we rarely obtain them in their natural condition; but it is certain, from their well-washed skeletons, that their number is considerable, and that at least two distinct genera occur among them. If we assume that the well-known cup-shaped sponge, usually sold as the best Turkey sponge, is the one entitled to the designation of

Spongia officinalis, we shall then have the type of the first suborder of the order Keratosa, distinguished by the above characters. There are two genera belonging to this suborder; the first of these is *Spongia*, Linnæus. Its character is as follows:—

SPONGIA, Linneus.

Skeleton kerato-fibrous. Fibres solid, cylindrical, aspiculous. Rete unsymmetrical.

Type, *Spongia officinalis*, Linnæus.

The number of species of *Spongia* appear to be very considerable; and in all of them the irregular meandering character of the skeleton-fibre readily serves to distinguish them. Plate LXXIV. fig. 9, and Part II. Plate XXVII. fig. 7.

The second genus is founded on the specimen described by SOWERBY in the 'British Miscellany,' p. 87, plate 48, and named by him *Spongia pulchella*. I fortunately have this specimen; and on carefully examining it I find it to possess all the characters of the genus *Spongia*, excepting that the reticulations of the skeleton are very symmetrical; and this is so important a structural difference that I have thought it advisable to constitute it the type of a new genus, the characters of which are as follows:—

SPONGIONELLA, Bowerbank.

Spongia, Sowerby and Johnston.

Skeleton kerato-fibrous. Fibres solid, cylindrical, aspiculous. Rete symmetrical; primary fibres radiating from the base to the apex. Secondary fibres disposed at nearly right angles to the primary ones. Plate LXXIV. fig. 10.

Type, *Spongia pulchella*, Sowerby.

Suborder II. Solid, semispiculate, kerato-fibrous skeletons.

The sponges of this suborder closely resemble in general appearance those of the genus *Spongia*, but they differ very considerably in the structural characters of their skeletons, which consist of a somewhat irregular radiation of primary fibres from the base towards the apex of the sponge, with an unsymmetrical series of secondary fibres emanating from and connecting together the series of primary ones.

The primary fibres are compressed and broad in their form, frequently three or four times the width of the diameter of the surrounding cylindrical secondary ones. But their most striking character is their possessing a considerable number of siliceous spicula, which are irregularly imbedded in their centres; sometimes the series of spicula within the fibre consists of but one or two beside each other, and at other times they are numerous and very irregularly disposed. This central series of spicula appears to exist only in the primary fibres; and I have never been able to detect the slightest indication of their presence in any of the secondary series. I first described these structural peculiarities in a paper read before the Microscopical Society of London, January 27, 1841; and it is published in vol. i. p. 32, plate 3, of their 'Transactions.'

I have met with numerous instances of the occurrence of this structural arrangement of the skeleton in sponges from Australia and the Mediterranean; but their well-washed condition has left them with but very few capabilities for specific distinction.

I propose to adopt DE BLAINVILLE'S name *Halispongia* to designate this genus, the characters of which are as follows:—

HALISPONGIA, De Blainville.

Skeleton kerato-fibrous. Fibres solid; primary fibres compressed, containing an irregularly disposed series of spicula. Secondary series of fibres unsymmetrical, cylindrical, without spicula. Plate LXXIV. fig. 11.

Suborder III. Solid, entirely spiculate, kerato-fibrous skeletons.

CHALINA, Grant.

Skeleton fibrous. Fibres keratose, solid, cylindrical, and interspiculate. Rete symmetrical; primary lines radiating from the basal or axial parts of the sponge to the distal portions. Secondary lines of fibre at about right angles to the primary ones.

The type of this genus, *Halichondria oculata*, Johnston, differs so materially in the structure of its skeleton from that of the type of *Halichondria*, *H. panicea*, Johnston, that it becomes necessary that a distinct genus should be established to receive it and other closely allied British species. The skeleton consists of a solid cylindrical keratose fibre, enclosing a single or compound series of spicula, imbedded at or near its centre, and disposed in lines parallel to its axis, thus forming a structural group intermediate between that of *Halichondria panicea* and *Spongia officinalis*.

In the sponges of this genus the spicula are decidedly subservient to the fibre, which is always cylindrical, and generally very uniform in its diameter throughout the whole of a section made at right angles to its surface; while in the nearly allied genus, *Isodictya*, the reverse is the case, the spicula being the essential basis of the skeleton, while the surrounding keratode, although often abundant, is still only the subservient cementing medium of the skeleton, and never assumes the decidedly cylindrical form of that of the fibre of *Chalina*.

In the Edinburgh Encyclopædia, vol. xviii. p. 844, Dr. GRANT proposed the name *Halina* to represent those species which were designated *Halichondria* by Dr. FLEMING, and subsequently by Dr. JOHNSTON in his 'History of British Sponges;' but as I have already proposed to restrict the term *Halichondria* to those species which agree in structure with the original type of that genus (*H. panicea*, Johnston), it becomes necessary to select other names to represent the sponges which differ essentially in their structure from that type, and I therefore propose to adopt Dr. GRANT'S genus *Chalina*, designated in his 'Tabular View of the Animal Kingdom,' published in 1861, to represent that portion of them which agree in structure with the well-known species described in the 'History of the British Sponges' as *Halichondria oculata*. Part II. Plate XXVII. fig. 8.

Suborder IV. Simple fistulo-kerato-fibrous skeletons.

The type of this suborder is LAMARCK'S *Spongia fistulosa*. The anatomical structure and the general habits of the sponges of this description are so widely different from the true Spongias, that I was induced to establish them as a separate genus, and I accordingly designated and described them as such in the Annals and Magazine of Natural History for December 1845, vol. xvi. p. 400, plate 13. fig. 7. It is unnecessary to enter here into a detailed account of these tissues, as I have described the peculiarities of the structure of the simple fistulo-keratose fibrous skeletons at length in the second part of this paper at p. 755, and figured the tissue in Plate XXVII. fig. 12.

The genus may be characterized as follows:—

VERONGIA, Bowerbank.

Spongia, Lamarck.

Skeleton kerato-fibrous. Fibres cylindrical, continuously fistulose, aspiculous. Rete unsymmetrical.

Suborder V. Compound fistulo-fibrous skeletons.

This suborder is founded on the peculiarities in the structure of the skeleton-fibre of a sponge described by me in the Annals and Magazine of Natural History for December 1845, vol. xvi. p. 405, plate 13. figs. 1 & 2, and also in the second part of this paper, p. 756, and figured in Plate XXVII. figs. 13 & 14.

The genus *Auliskia* is the only one in which compound fistulo-keratose fibres have been found, and it may be thus characterized:—

AULISKIA, Bowerbank.

Skeleton kerato-fibrous. Fibres aspiculous, cylindrical, continuously fistulose; primary fistulæ having minute cæcoid canals radiating from them in every direction. Rete unsymmetrical.

Suborder VI. Regular semi-areno-fibrous skeletons.

The sponges of this suborder have the faculty of appropriating extraneous matter, such as grains of sand, or the spicula of other sponges, which become imbedded in the centre of the cylindrical fibres of their skeletons. The fibres in these cases are regular and cylindrical, and the space between their surfaces and the central line of extraneous matter is frequently one-fourth or one-third of their own diameter. The central axis of extraneous matter usually consists of a series of single grains, but occasionally we find two or three compressed together. In some genera belonging to this suborder the arenation of the fibres is confined to the primary or radial ones, and the fibres of the secondary system are destitute of extraneous matters. In other genera they occur occasionally in the secondary system as well as in the primary one. In *Stematumenia* the primary fibres are frequently somewhat compressed, and are abundantly arenated. The smaller or secondary series of fibres are usually cylindrical, and most frequently without either grains of sand or spicula. Several of the common Bahama sponges of commerce belong to this suborder; but the best type is the genus *Stematumenia*, described by me in the Annals and Magazine of Natural History for December 1845, vol. xvi. p. 406, plate 14. figs. 1 & 2. The genus may be characterized as follows:—

STEMATUMENIA, Bowerbank.

Skeleton. Primary fibres solid, more or less compressed, containing a central axial line of spicula and grains of extraneous matters. Interstitial structures abundantly fibro-membranous. Part II. Plate XXVII. figs. 3 & 5, and Plate XXVIII. figs. 1 & 2.

Suborder VII. Irregular and entirely areno-fibrous skeletons.

Types, *Dysidea fragilis*, Johnston.
Dysidea Kirkii, Bowerbank.

The peculiarity of this suborder is, that the fibre of the skeleton is a full and complete but elongate aggregation of particles of sand, each separately coated by keratode, forming a series of stout anastomosing fibres, consisting of innumerable extraneous molecules encased by a thin coat of keratode.

In *Dysidea Kirkii*, an Australian species, both the primary and secondary fibres of the skeleton are comparatively large, frequently exceeding half a line in diameter. In our British species, *Dysidea fragilis*, Johnston, the primary fibres are often as abundantly arenated as those of the Australian species, while the secondary ones are only partially filled with extraneous matter, and in this condition they are frequently more or less tubular. Part II. Plate XXVIII. figs. 3, 4 & 5.

The structure and peculiarities of the above-named two species are described in detail in vol. i. p. 63, plate 6 of the Transactions of the Microscopical Society of London.

On the Discrimination of the Species of the Spongiadæ.

One of the reasons why so little progress has been made in our knowledge of the Spongiadæ is, that the generic and specific characters that are visible to the unassisted eye, such as form and colour, are in this class of animals remarkably uncertain and delusive, while all those that are definite and constant require not only a high degree of microscopical power to make them visible, but frequently also a peculiar mode of treatment to render them apparent even beneath the microscope. Thus it is with many of the finer forms of stellate spicula, which are very characteristic in *Tethea*, *Geodia*, *Spongilla*, and other genera. When we search for them by the dissolution of the tissues in nitric acid, they are so minute that by far the greater part of them, even with the most careful treatment, are washed away; and when the tissues in which they are imbedded are examined in water, they are totally invisible in the sarcode in which they are immersed; and it is only when small portions of such tissues are mounted in Canada balsam that they become distinctly visible *in situ*. The correct classification therefore, as well as the anatomy and physiology, is really a microscopical science; and it is only since we have possessed instruments of high defining and penetrating powers, that we have been properly prepared for the investigation of the structures and the correct determination of the generic and specific characters of these interesting and curiously constructed animals. A careful and patient examination of their component parts is therefore absolutely necessary for the determination of species; and the whole of the structures present should be noted and their peculiarities accurately described.

In the first place we will consider what are the parts of the organization of the Spongiadæ that may be used for the purposes of specific distinction; and secondly, endeavour to form an estimate of their relative values.

The parts of the sponge to be thus employed are as follows:—1. The Spicula. 2. The Oscula. 3. The Pores. 4. The Dermal Membrane. 5. The Skeleton. 6. The Interstitial Membranes. 7. The Intermarginal Cavities. 8. The Interstitial Canals and Cavities. 9. The Cloacal Cavities. 10. The Sarcode. 11. The Ovaria and Gemmules.

1. *The Spicula.*

The spicula in the descriptions of the Spongiadæ are of about the same relative value that the leaves of plants are in botanical descriptions. I have shown in the first part of this paper, published in the Philosophical Transactions for 1858, that they are exceedingly various in form in the different species; and even when of the same shape in two different sponges, as represented in fig. 9 *a* & *b*, Plate XXIII. Phil. Trans. 1858, their relative proportions are frequently so distinctly different as to render them almost as valuable as if they varied from each other in form. Wherever therefore spicula form a component part of the skeleton, they become a leading character in the discrimination of species. But it is not only those of the skeleton that are thus available, as in different sponges they vary in shape and size in each separate organ belonging to the animal; and in some cases we find as many as five or six distinct descriptions of spicula, each of which affords an invariable and excellent character. Thus, in the descriptions of sponges, it is not only the forms and relative proportions of the skeleton-spicula which have to be taken into consideration, but those also of the dermal and interstitial membranes (the external and internal defensive ones), those of the sarcode, and of the ovaries and gemmules. Those of the latter three organs named frequently afford the most determinative characters. Thus in the genus *Spongilla* but one form of spiculum, the acerate, prevails in the skeletons of all the known species; but the minute and beautiful spicula of the ovaria vary in form and size, in passing from one species to another, in a perfectly unmistakeable manner, so that, if the organs of reproduction be present, which is most frequently the case, the species may be readily recognized from their spicula only. But in other cases, and even in the same genus in the absence of the ovaria, the differences between two nearly allied species are equally well determined by the spicula of the dermal and interstitial membranes. Thus, in our two species of British *Spongilla*, *S. fluviatilis* has no tension spicula different from those of the skeleton, while in *S. lacustris* we find the fusiformi-acerate entirely-spined spiculum, represented in fig. 21, Plate XXIV. Phil. Trans. 1858, in abundance. So likewise in two species of *Tethea*, *T. cranium* from Shetland, and *T. simillima*, Bowerbank, MS., from the Antarctic regions, the only well-determined difference that exists is, that the sarcode of the former is profusely furnished with exceedingly minute sigmoid spicula, while that of the latter is entirely destitute of them. It will therefore be seen that these exceedingly minute organs frequently afford the most valuable and certain means of discriminating species. But although so minute, we must not imagine that it is very difficult to obtain these characteristic evidences; for, as I shall show more at length hereafter, it requires but the dissolution of a small piece of the sponge in hot nitric acid to at once furnish us with a general view of the whole of the spicular contents of the sponge under examination; so that, to one who has become familiarized with the general characteristics of the forms and sizes of the different classes of spicula peculiar to each organ of the sponge, such a preliminary observation at once indicates the nature and especial seat of the principal specific characters of the subject under examination.

In some sponges the relative variation in size of the adult skeleton-spicula is greater than in others; but this variation, although sometimes a substantial character, must not be always assumed to be correct, as in young sponges with simple forms of skeleton it is very difficult to discriminate between the young and only partially developed spicula and the adult ones. Thus in a young specimen of *Spongilla fluviatilis*, I found in the same field of view one spiculum perfectly well proportioned which measured $\frac{1}{324}$th of an inch in length and $\frac{1}{10000}$th of an inch in diameter, another $\frac{1}{241}$st of an inch in length and $\frac{1}{7500}$th of an inch in diameter,—the length and diameter of an average-sized spiculum of the species in a fully developed condition being, length $\frac{1}{75}$th of an inch, and diameter $\frac{1}{2000}$th of an inch.

Abnormal or immature forms must not be mistaken for fully developed and normal ones, as we find in some of the more complicated forms of spicula that the development of form is quite as progressive as that of size, as instanced in figs. 4, 5, 6 & 7, Plate XXIV. Phil. Trans. 1858, which represent the progressive stages of development of the spinulo-recurvo-quaternate form of spiculum, and also in figs. 4, 5, 6, 7 & 8, Plate XXV. Phil. Trans. 1858—where the first four figures represent the progressive development of the dentato-palmate inequianchorate spiculum, and the last an abnormal form, probably arising from arrested development.

2. *The Oscula.*

The oscula frequently afford good specific characters. Their peculiarities are, first, those of position; and secondly, those of form. Thus it should always be noted whether they are dispersed or congregated, whether disposed on the exterior surface or on the parietes of internal cloacæ. In form they are either simple orifices, or they assume a tubular shape to a greater or a less degree, and sometimes they are bounded by a slightly elevated marginal ring. All these characters are subject to a considerable amount of variation, which is sometimes dependent on peculiarities of locality, and at others on age or the amount of their development; but a comparison of several specimens of the same species will generally lead the observer to a correct conclusion regarding their normal characters.

In some species these organs are always more or less open; in others, especially littoral ones, they are entirely closed during exposure to the atmosphere, or while in a state of repose, during which condition they are frequently completely inconspicuous.

3. *The Pores.*

The pores afford but very few available characters. They are either dispersed or congregated, very rarely in the latter state. They are also either conspicuous or inconspicuous: that is, in the former condition their presence, and the areas within which the groups of them are situated, may be readily detected by the aid of a hand-lens; in the latter case they are perfectly undistinguishable without high microscopic power.

4. *The Dermal Membrane.*

The dermal membrane affords many important specific characters. In the greater number of the Spongiadæ it is a simple pellucid membrane, which invests the whole of the mass of the sponge; but in other cases it is of much more complex structure, sometimes furnished abundantly with primitive fibrous tissue, or a network of spicula or kerato-fibrous tissue for its especial support; and in the areas of such network there are frequently tension spicula differing in construction from those of the skeleton, and its interior surface is often supplied with anchorate retentive spicula of various forms. In its sarcodous lining there are occasionally an infinite number of stellate or sphero-stellate spicula to protect it from the ravages of minute enemies, and its surface is also often penetrated by large or small defensive spicula. Occasionally its external surface is profusely supplied with elongo-stellate defensive spicula. It has also frequently a thick stratum of cellular structure of various colours.

These peculiarities of structure have no generic value. They are essentially specific differences; and it is rarely the case that any two species, even in an extensive genus, are found to agree in the possession of the number, form, or mode of disposition of these peculiarities of the dermal tissues. They form therefore a constant and highly valuable series of characters, and claim the especial attention of the student in either the recognition or description of an unknown species.

5. *The Skeleton.*

Although the material, mode of structure, and arrangement of the skeleton is more especially devoted to the formation of the orders and suborders, it still presents us with a sufficient number of minor peculiarities to render it a source of valuable specific characters. Thus, as I have already shown, in treating of the relative value of the spicula for the distinction of species, the difference in their size affords a good character. The closer or more diffuse mode of their arrangement modifies to a great extent the form and size of the areas in spiculo-reticulated skeletons, and their habitually greater or less number in the thread of the reticulations produces a distinctly different aspect in the skeletons of two otherwise closely allied species. The presence or absence of defensive spicula, the mode of armature, and the forms of the defensive and other auxiliary spicula also afford a very extensive and valuable series of specific characters. In the kerato- and siliceo-fibrous sponges there are peculiarities of a similar description, such as the presence of a reticulo-fibrous sheath, as represented in figs. 9 & 10, Plate XXVIII. Part II., or the possession of spines or tubercles of various forms, as represented in the same Plate, figs. 7 & 8, or of extraordinary modifications for prehension, as in the cidarate siliceo-fibrous skeleton, represented also in the same Plate, fig. 12. These and other similar structural peculiarities afford a series of characters which are usually of a permanent and very striking description.

6. *The Interstitial Membranes.*

The peculiarities of the interstitial membranes consist principally in the shape and proportions of their tension spicula, or of the forms and varieties of structure, and mode of disposition, of the retentive spicula. The latter class of organs especially present a very extensive series of striking characters that are essentially specific. In the genera *Halichondria*, *Isodictya*, *Hymeniacidon*, and others containing numerous species, often very closely resembling each other in all the principal structural characters, they frequently, from the strongly marked peculiarities in their form and proportions, present most valuable and decisive specific characters. Plate XXVII. figs. 1, 2, 3 & 5.

In *Alcyoncellum* and other genera the interstitial membranes are strengthened and supported by layers of primitive fibrous tissue, arranged in parallel lines; and in *Stematumenia* the same fibres abound, but they are not disposed in the same symmetrical manner, and in some sponges cellular structures are present in considerable quantities. These tissues are all more or less valuable as aids in specific distinction.

7. *The Intermarginal Cavities.*

The intermarginal cavities in the greater portion of the Spongiadæ are so indefinite in their form as to render but little service in the distinction of species; but in *Geodia*, *Pachymatisma*, and a few other genera their structure is very much more regular, and their form, proportions, and mode of disposition afford good characters. But although of no extensive essential value themselves, their subsidiary ternate spicula present a great number of strongly marked specific distinctions, arising not only from their varieties of form and proportion, but also from their relative positions in the dermal crusts of those genera where they most abound; and their modes of disposition and connexion with each other are also very characteristic.

8. *The Interstitial Canals and Cavities.*

These organs themselves present very few characters that are of much service in specific descriptions, but their subsidiary spicula are often very suggestive of the nature and character of the species. Of this description, are the recurvo-ternate spicula in the interstitial cavities immediately beneath the dermal crust of some species of *Geodia*, and just without the dermal membrane of *Tethea cranium*; the remarkable groups of recurvo-quaternate spicula, represented by fig. 10, Plate XXX., Part II.; the trenchant bihamate spicula of *Hymedesmia Johnsoni*, figs. 1 & 2, Plate XXXI.; and many other instances of offensive or defensive spicula, either disposed in groups or singly, in these canals or cavities.

9. *The Cloacal Cavities.*

The cloacal cavities are especially valuable and characteristic in the calcareous sponges. Their position, number, extent, and form—the number and position of their excurrent orifices—the mode in which those orifices are armed and the nature of that armature, or the entire absence of such defences—the internal defensive spicula, their varieties of form, and mode of arrangement,—all these characters are highly effective and valuable as specific descriptions. In other genera of sponges the cloacæ afford striking and very effective distinctions, especially in *Alcyoncellum*, *Polymastia*, *Halyphysema*, and *Hyalonema*. Among the Keratosa also they avail to a considerable extent; but the latter order does not afford us the same wide range of striking characters that exist so abundantly in the cloacæ of the order Calcarea.

10. *The Sarcode.*

The universal presence and similarity in structure of the sarcode of the Spongiadæ renders the range of its use as a specific character very limited; but the spicula imbedded in its substance so abundantly in many species are so various in form, and so strikingly distinct from each other, as to afford a most valuable series of discriminative characters.

The greater portion of these spicula are more or less stellate in form. They vary in shape to a considerable extent in each group, in consequence of incomplete or complete development, and the number of the radii in the stellate forms is in many cases very uncertain; but although this amount of variation exists in each of the separate forms, there is always a limit to these differences, and a normal character present which renders it by no means difficult to decide to which class they belong. Independently of the peculiar characters of their own form and modes of radiation, their radii are frequently peculiarly and abundantly spinous, and these secondary organs are equally as constant and determinative of specific character as the primary radii. The characters derived from the spines are frequently very minute, and require the application of a high microscopic power to render them available; but they are in many cases so decisively valuable, that they should never be neglected when present. In truth, the modes of spination of these and all other forms of spicula are of considerable value as specific characters, and the shape and direction of the spines are often indicative of the character and purpose of the spiculum on which they are based.

The range of the stellate spicula is very considerable. They are found abundantly and constantly in *Geodia*, *Pachymatisma*, *Tethea*, *Dactylocalyx*, and *Alcyoncellum*, and in some species of *Spongilla*, *Dictyocylindrus*, and other genera.

11. *The Ovaria and Gemmules.*

Where the ovaria exist they afford excellent descriptive characters. Their construction is the same throughout the whole of the known species of *Geodia* and *Pachymatisma*. The varieties in their form, although not always easy of description, are yet readily distinguishable by a practised eye; and the difference in the degree of stoutness of the radiating spicula of which they are constructed, and the consequent fineness or coarseness of the reticulations on their surface, very often afford good discriminative characters.

In *Spongilla*, the varieties in their shape, and the strikingly distinct forms of their component spicula, render them exceedingly efficient for specific descriptions; and without them it would in several instances, among the exotic species, be very difficult to find descriptive characters to separate one species from another.

Excepting in *Diplodemia*, where the structural peculiarities of the ovarium are widely different from the preceding instances, we know very little more of these organs; but there is good reason to believe, from certain forms of spicula detected in the deep-sea soundings, the sources of which are at present unknown, that other marine sponges possess ovaria with which we are at present unacquainted.

The gemmules afford very efficient specific characters in some species of *Tethea*; but in the greater number of Halichondroid genera, although frequently present in abundance, they agree so closely in structure with each other as to render them of very little use as specific characters.

We thus find that we possess eleven distinct varieties of organic specific characters, many of which are exceedingly prolific in materials for descriptive purposes. A long familiarity with them has assured me of their value, and of their constancy in each species. However protean the form and colour may be, the organic structures can always be recognized with certainty, provided the specimen under examination has been dried in the condition in which it has been taken from the sea. To the organic characters may be added the less definite and valuable ones of form and mode of growth, which, although less to be depended on than the organic ones, are frequently of service in conjunction with them, as leading and suggestive in the first stage of investigation.

A dependence on the specific characters to be derived from form alone inevitably leads to erroneous conclusions. Thus, from trusting too implicitly to it in the descriptions of his species, Dr. JOHNSTON, in his 'History of British Sponges,' has made two species out of one in the case of *Dysidea fragilis*, the thin-coating form of this sponge being also described as *Halichondria areolata*. *Halichondria incrustans* has also been described a second time as *H. saburrata*. An elongated form of *Halichondria ficus* has also been again described as *H. virgultosa*. The type specimen of *Halichondria sevosa*, Johnston, in the British Museum proves to be merely a thin-coating variety of *Halichondria panicea*; and the type specimen of MONTAGU'S *Spongia digitata* in the possession of Professor GRANT, *Halichondria cervicornis*, Johnston, on being microscopically examined, proved not to be a sponge, but an alga. Numerous other instances of error arising from a dependence on form alone as a specific character might be cited; but those I have given above are sufficient to prove the ineligibility of so mutable a character unaccompanied by organic structure.

Nearly the whole of this extensive series of specific characters have hitherto not been applied in the descriptions of the Spongiadæ, excepting in my own manuscripts. This omission has occurred, not from any doubt of their value, but simply because they were unknown to naturalists. It now remains to be proved how they may be rendered available in future descriptions of those animals. I cannot, perhaps, better attain this end than by detailing the order and mode of employing them in the description of species contained in my own Manuscript History of the British Sponges. The following is the order in which these characters have been taken for examination and description:—

1. Form. 2. Mode of Growth. 3. Surface. 4. Oscula. 5. Pores. 6. Dermis, and Dermal Membrane and its Spicula. 7. Skeleton and its Spicula. 8. Connecting Spicula. 9. Defensive Spicula—external, internal. 10. Spicula of the Membranes—tension spicula, retentive spicula. 11. Sarcode and its Spicula. 12. Ovaria and Gemmules, and their Spicula.

Colour.

Habitat.

Condition when examined.

This order of description, or any other that the student may prefer, should always be adhered to, and no part of the specimen under examination that is present, and which affords specific characters, should be omitted in the description; so that, when no mention is made of particular organs or classes of spicula, it may be presumed that they are not present in the sponge in course of description. A certain portion of these characters are always available. Thus the skeleton, incurrent canals or cells, the sarcodous system, the dermal and interstitial membranes, the pores, and the oscula are always present, while the excurrent canals or the cloaca are occasionally absent. The intermarginal cavities, if present, are not always distinguishable; and the external and internal defensive organs are, either one or both of them, frequently absent.

Specific characters should always be of a positive nature, such as the presence and form of particular spicula or other organs. It is a great mistake in writing specific descriptions, to make the differences between species to consist of one or two striking essential characters only. Such a practice may answer tolerably well when there are but two or three species of a genus known; but it frequently occurs when new species are found, that they also have the most striking essential characters of the previously known ones equally strongly developed. Much confusion is thus likely to occur from this paucity of description; whereas, if the whole of the essential characters of each species be carefully investigated and accurately recorded when it is first characterized, that description will most probably suffice permanently to distinguish it as a species, however numerous the subsequently discovered members of the genus may be.

Differential characters should never be intermingled with essential ones in characterizing the species. They should be reserved for the amplified history; and here they are of much value, as they lead to the relative consideration of two or more nearly allied species, and frequently assist the student in their discrimination when the essential characters are minute or somewhat obscure.

In the description of species the adjectives long, short, stout, slender, &c., must always be understood as in comparison with the congenerous organs of the species under consideration, and not as in relation to any fixed standard of size.

In the description of a new species it should always be stated whether the characters are given from a dried specimen, or whether from one fresh from the sea, as it frequently happens that many of the *natural* characters become completely obliterated and sometimes reversed by drying; thus the surface smooth in the live state become villous when dried; inconspicuous oscula become conspicuous when contracted and dry, and conspicuous oscula are often destroyed by desiccation; and so on with other characters. It is therefore absolutely necessary that the condition of the specimen should be stated along with its description.

On the Preservation and Examination of the Spongiadæ.

The greater portion of specimens in natural history may be readily examined and their species determined in the field; but this is rarely the case with the Spongiadæ. It becomes necessary therefore to preserve them in such a manner as to effectually retain their natural characters for examination at some future period. Small specimens may be preserved in spirit of wine; but this destroys their colour. If they are not likely to be permanently lodged in the cabinet immediately, it is better that they should be laid on blotting-paper, or a soft cloth, to absorb as much as possible of the water from within them, and then dry them rapidly before a fire, or in a slack oven, without any previous washing in fresh water. By this mode they retain a sufficient amount of moisture and flexibility to allow of their being handled and operated on for examination with impunity; but the amount of salt thus left within them will in time cause considerable mischief to the specimen. After such specimens have been once thoroughly dried and their examination has been completed, they may be plunged into cold water for a few minutes, and the water then ejected by a rapid centrifugal motion of the arm, and this operation repeated two or three times; the specimen should be again rapidly dried, and it will then keep well in the cabinet and preserve all its characteristic features. It is a bad habit to soak marine specimens for a considerable time in fresh water to extract the salt, as by this mode of proceeding the minute and delicate characters of the object are to a great extent destroyed.

The most advisable mode of proceeding, in the examination of an unknown species, is, first to note the general peculiarities of form and surface as presented to the unassisted eye. After the noting of the external character, the next step should be to cut a slice out of the sponge, to about half an inch or more in depth, at right angles to the surface, taking special care that a due proportion of the dermal membrane is included; this should be placed in a long narrow test-tube, in about an inch deep of nitric acid, in which it should be gently and cautiously boiled over a very small flame until the sponge is entirely dissolved, and then set by until the acid is quite cold and the spicula have subsided to the bottom of the test-tube, so that the greater portion of the acid may be decanted off and its place be supplied with distilled water; and this operation should be repeated three or four times with much care. The spicula thus prepared should be placed in a watch-glass with a little distilled water, and the whole stirred up so that an average sample can be obtained for microscopical examination. By this mode of procedure a general view of the whole of the spicula belonging to the species will be obtained, which will serve as a guide to the subsequent modes of examination.

The boiling in nitric acid should not be continued beyond the time of the piece of sponge falling completely separated to the bottom. If stopped at this period by the addition of a little distilled water, it frequently occurs that undissolved gemmules and portions of the membranes are found, that are very suggestive for the further examinations of the specimen.

The next step should be to take a thin slice from the surface of the sponge, and place it in a cell in a little distilled water, for the purpose of the examination of the structural peculiarities of the dermal membrane. Then take a thin slice from the body of the sponge at right angles to its surface, and mount it in a similar manner for the purpose of ascertaining the nature and peculiarities of its skeleton and other internal organs. These two sections should be carefully examined with the microscope; and if they be not sufficiently characteristic, fresh ones should be mounted. If the specimens thus treated be taken from sponges properly preserved, their tissues will expand and assume very much the appearance of those of the living sponge, and they will as nearly as possible exhibit the natural positions and proportions of the internal organs.

The general characters of these sections should be observed with a half-inch or two-thirds combination, and again with not less than a quarter-inch object-glass, and the characters of the various tissues in their natural condition be immediately noted. But the whole of their minute organs will not be visible by this mode of examination; and it is therefore necessary to mount the same or similar sections in Canada balsam, by which means the spicula of the sarcode and other minute organs will become completely visible *in situ*; and the specimens thus mounted will serve as permanent records for the cabinet.

The following are a few examples of the mode of specific description that I propose for adoption by naturalists who may investigate the Spongiadæ.

GRANTIA CILIATA, Fleming.

G. ciliata, Johnston.

G. pulverulenta, Johnston.

Sponge elongately oval, rarely globular, slightly pedicelled; surface papillated, hispid. Clóaca central, cylindrical, nearly as long as the sponge; armed internally with spiculated equiangular triradiate spicula; spicular ray attenuated. Mouth of the cloaca armed with a thick ciliary fringe of very long and slender acerate spicula; base of the fringe supported by large, short and stout fusiformi-acerate spicula. Oscula simple, very slightly depressed from the surface of the cloaca, as numerous as the interstitial cells. Pores inconspicuous. Interstitial cells: distal terminations more or less obtusely conical, furnished with a ciliary fringe of slender acerate spicula. Skeleton-spicula equiangular triradiate.

Colour cream-white.

Hab. Coasts of Great Britain; parasitical on fuci; littoral to 8 or 10 fathoms or more. Examined alive.

PACHYMATISMA JOHNSTONIA, Bowerbank.

Halichondria Johnstonia, Trans. Mic. Soc. London, vol. i. p. 63, pl. 6; Johnston's Hist. Brit. Sponges, p. 198.

Sponge massive, sessile; surface smooth, undulating into ridges. Oscula simple, congregated on the elevations. Pores inconspicuous. Dermis crustular, filled with ovaria. Dermal membrane pellucid, abundantly spicular; spicula fusiformi-cylindrical, tuberculated, minute. Intermarginal cavities immersed in the dermal crust, separate, symmetrical, subcylindrical, valvular at proximal end. Connecting spicula attenuato- or cylindro-ternate; radii variable in form and proportions. Skeleton-spicula cylindrical, variable in form and proportions. Spicula of sarcode attenuato-stellate; radii incipiently spinous, rarely fully spinous, or obtuse. Ovaria oval, depressed.

Colour: littoral specimens light to dark slate-grey; deep-sea specimens pink or red. (Captain F. W. L. THOMAS, R.N.)

Hab. Rocks between high and low water mark. Torquay. Gouliot Caves, Sark. Coast of Ireland (J. S. BOWERBANK). Orkney Islands, thirty-five fathoms (Captain F. W. L. THOMAS, R.N.). Wick, Scotland (C. W. PEACH).

Examined in the live state.

TETHEA CRANIUM, Lamarck.

Tethea cranium, Johnston.

Sponge ovoid or subspherical, sessile surface even, strongly hispid. Dermal coat thick, abundantly furnished with short, stout, fusiformi-acerate spicula surrounding the large defensive fasciculi at various angles to their axes; also profusely furnished with minute sigmoid bihamate spicula, dispersed irregularly. Dermal membrane thin, pellucid. Oscula and pores inconspicuous. Spicula of the skeleton fusiformi-acerate, large and long. Defensive spicula external, collected in fasciculi; fusiformi-acerate, large and long, fusiformi-porrecto-ternate, and a few fusiformi-recurvo-ternate very long and slender. Sarcode abundantly furnished with minute sigmoid bihamate spicula. Gemmules lenticular, surface smooth, very tough and strong; of two distinct sorts: the first furnished abundantly with slender fusiformi-acerate spicula radiating in fasciculi from the centre to near the surface of the gemmule; the second furnished abundantly with slender fusiformi-acerate, slender unihamate attenuated, and with short slender porrecto-ternate spicula, mixed in fasciculi which cross each other irregularly.

Colour pallid green.

Hab. Island of Fulah (JAMESON). Haaf Banks, Shetland (BARLEE and BOWERBANK).

Examined in the fresh state.

In the specimens of specific descriptions which I have given above, there are fortunately numerous characteristic points by which we may readily separate them from their congeners; but this abundance of characters does not always exist. Thus in *Halichondria caduca* we find the structures so few and simple, as to render their description exceedingly difficult and unsatisfactory.

HALICHONDRIA CADUCA, Bowerbank.

Sponge massive, sessile, surface rugged. Oscula simple, small, dispersed, numerous. Pores inconspicuous. Dermal membrane aspiculous, pellucid, thin. Spicula of skeleton acerate, rarely acute. Interstitial membranes thin, translucent; spicula acerate, slender. Sarcode abundant.

Colour light grey.

Hab. Tenby (Mrs. BRETT).

Examined in the dried state.

Fortunately these cases of extreme paucity of characters are very few in number.

Explanation of the Plates.

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CLAUSIUS (R.) Ueber die Anwendung des Satzes von der Æquivalenz der Verwandlungen auf die innere Arbeit. 8vo. *Zürich* 1862.	The Author.
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Nederlandsch Kruidkundig Archief, Deel V. Stuk 2. 8vo. *Amsterdam* 1861.	The Minister of the Interior of the Netherlands.
Norton's Literary Letter. No. 4. New Series, No. 1. (2 copies.) 4to. *New York* 1859–60.	Mr. C. B. Norton.
The American Journal of Science and Arts. 2nd Series. Vol. XXXII. Nos. 94–99. 8vo. *Newhaven* 1861–62.	The Editors.
The Athenæum. June to December 1861; January to May 1862. 4to. *London.*	The Editor.
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The Entomologist's Annual for 1862. 12mo. *London* 1862.	H. T. Stainton, Esq.
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The Geologist, edited by S. J. Mackie. Vol. IV. Nos. 43–48. 8vo. *London* 1861.	The Editor.
The Horological Journal. July 1861 to June 1862. Nos. 35–46. 8vo. *London.*	The Editor.
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The Natural History Review. Vol. VI. No. 4; Nos. 25–28. 8vo. *London* 1859–60.	Dublin Univ. Zoological and Botanical Association.
The Philosophical Magazine. June to December 1861; January to June 1862. 8vo. *London.*	W. Francis, Esq.
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LIHARZIK (F.) Das Gesetz des Wachsthumes und der Bau des Menschen, die Proportionslehre aller menschlichen Körpertheile für jedes Alter und für beide Geschlechter. folio. *Wien* 1862.	The Author.
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SEGUIN (—.). Considérations sur les Lois qui président à l'Accomplissement des Phénomènes naturels rapportés à l'Attraction Newtonienne. 8vo. *Paris* 1861.	Sir John Herschel, F.R.S.
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SHARP (G.) Prize Essay on the Application of recent Inventions collected at the Great Exhibition of 1851 to the purpose of Practical Banking. 8vo. *London* 1852.	J. W. Gilbart, F.R.S.

Presents.	Donors.
SMITH (Edward, F.R.S.) Health and Disease as influenced by the Daily, Seasonal and other Cyclical Changes in the Human System. 8vo. *London* 1861.	The Author.
——— Consumption, its early and remediable stages. 8vo. *London* 1862.	———
SMYTH (Admiral W. H., F.R.S.) An Additional Word on the pristine establishment of the Royal Society Club. 4to. 1861.	The Author.
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STAINTON (H. T.) Natural History of the Tineina. Vol. VI. 8vo. *London* 1861.	The Author.
STEPHENSON (Rev. N.) A Plea for Rural Schools. 8vo. *London* 1861.	The Author.
STEVENSON (A.) Biographical Sketch of the late Robert Stevenson. 4to. *Edinburgh* 1861.	The Messrs. Stevenson.
STRUVE (F. G. W., For. Mem. R.S.) Arc du Méridien de 25° 20′ entre le Danube et la Mer Glaciale mesuré depuis 1816 jusqu'en 1855. Two vols. and Plates. 4to. *St. Pétersbourg* 1857.	The Observatory, Pulkowa.
STRUVE (O.) Beobachtungen und Elemente des Cometen II. 1860. 8vo. 1860.	The Author.
——— Ueber einen vom General Schubert an die Akademie gerichteten Antrag, betreffend die Russisch-Scandinavische Meridian-Gradmessung. 8vo. 1861.	———
——— Tabulæ Quantitatum Besselianarum, pro annis 1840 ad 1864. 8vo. *Petropoli* 1861.	———
STUART (Sir J.) Apologie du sentiment de Mr. le Chevalier Newton sur l'ancienne Chronologie des Grecs. 4to. *Francfort* 1759.	Dr. Sharpey, Sec. R.S.
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TWINING (A. C.) and NEWTON (H. A.) The August Meteors. Observations; Elements of the Ring. (Excerpt from Amer. Journ. Sci.). 8vo. 1861.	The Authors.
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VOLPICELLI (Paolo) Del Moto rettilineo lungo un Sistema di Piani diversamente inclinati e contigui, Memoria. 4to. *Roma* 1860.	The Author.
VOLTA (A.) Relazione di un suo Viaggio letterario nella Suizzera, pubblicata in occasione delle Nozze Stabilini-Reina. 8vo. *Milano* 1827.	Unknown.
WAUGH (Sir A. Scott, F.R.S.) Instructions for Topographical Surveying. 8vo. *Roorkee* 1861.	The Author.
——— Report on the Survey of India for the three Years ending 1858–59. folio. *London* 1861.	The Secretary of State for India.
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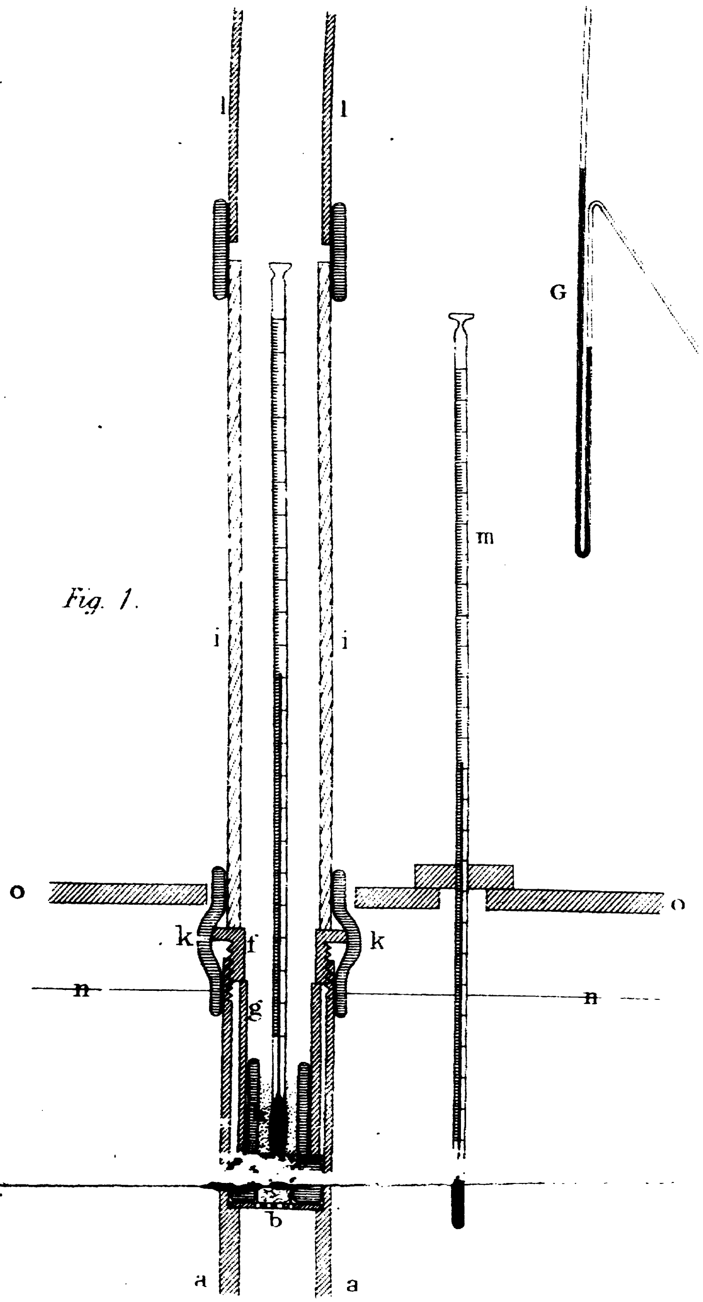


Fig. 1.

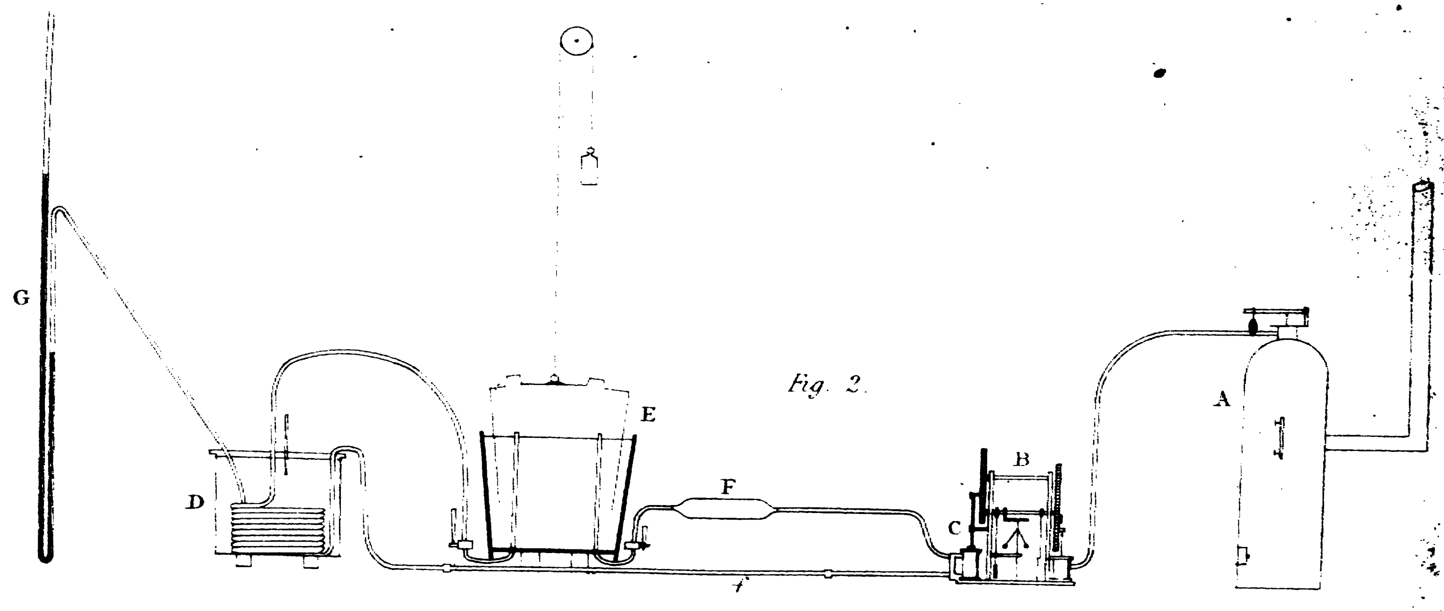


Fig. 2.

Fig. 3.

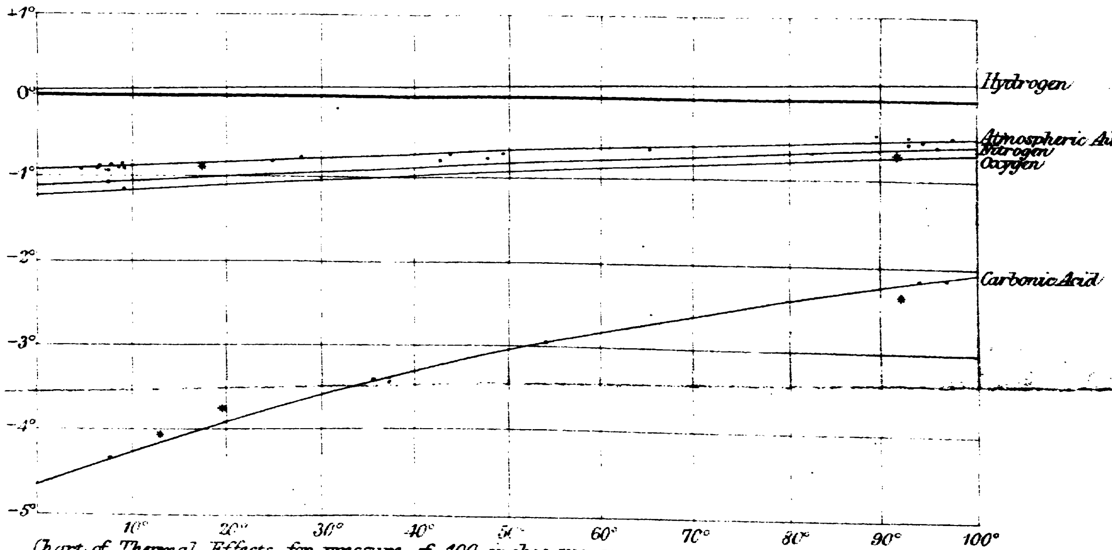
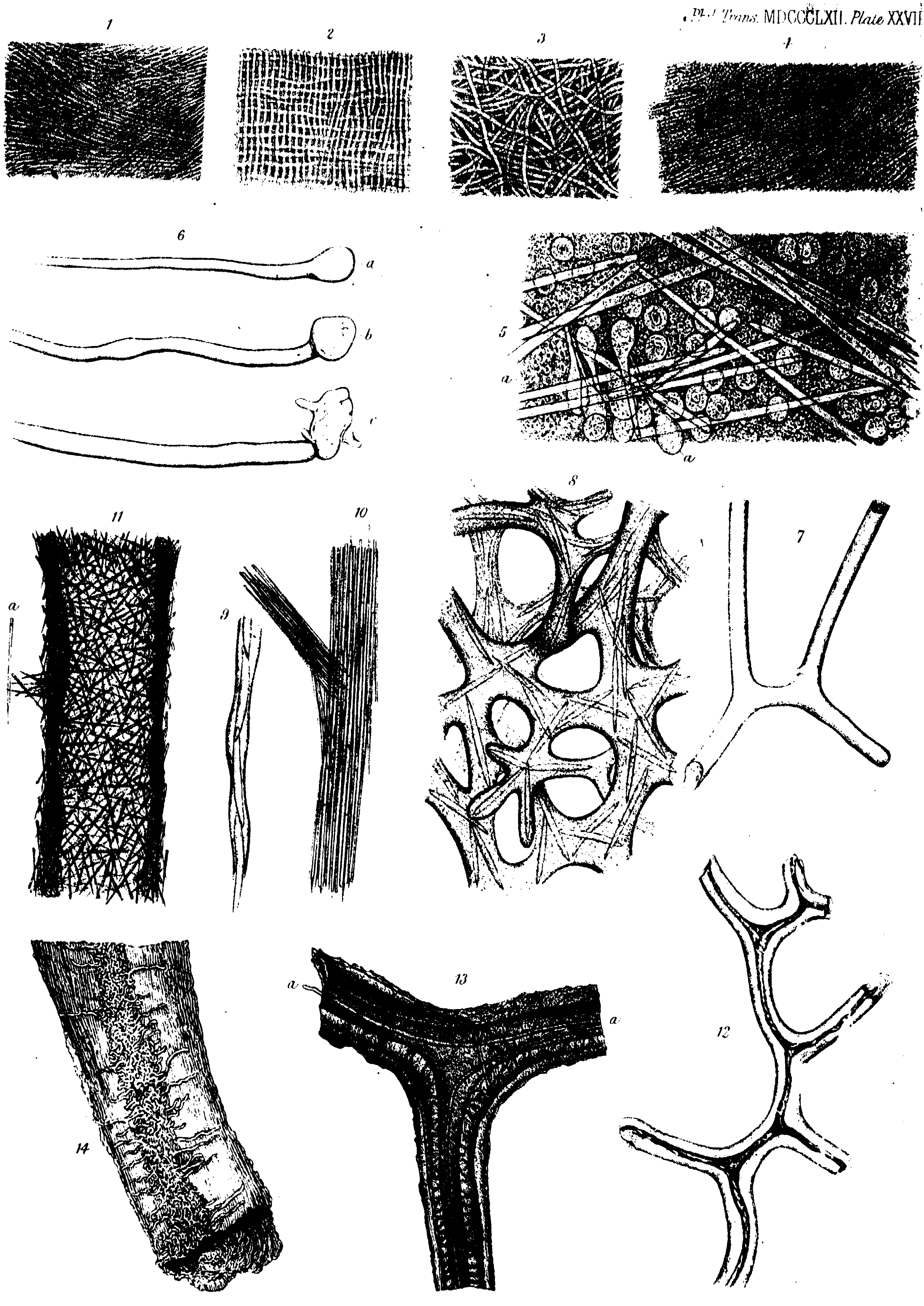


Chart of Thermal Effects for pressure of 100 inches mercury.
The asterisks show the results of the



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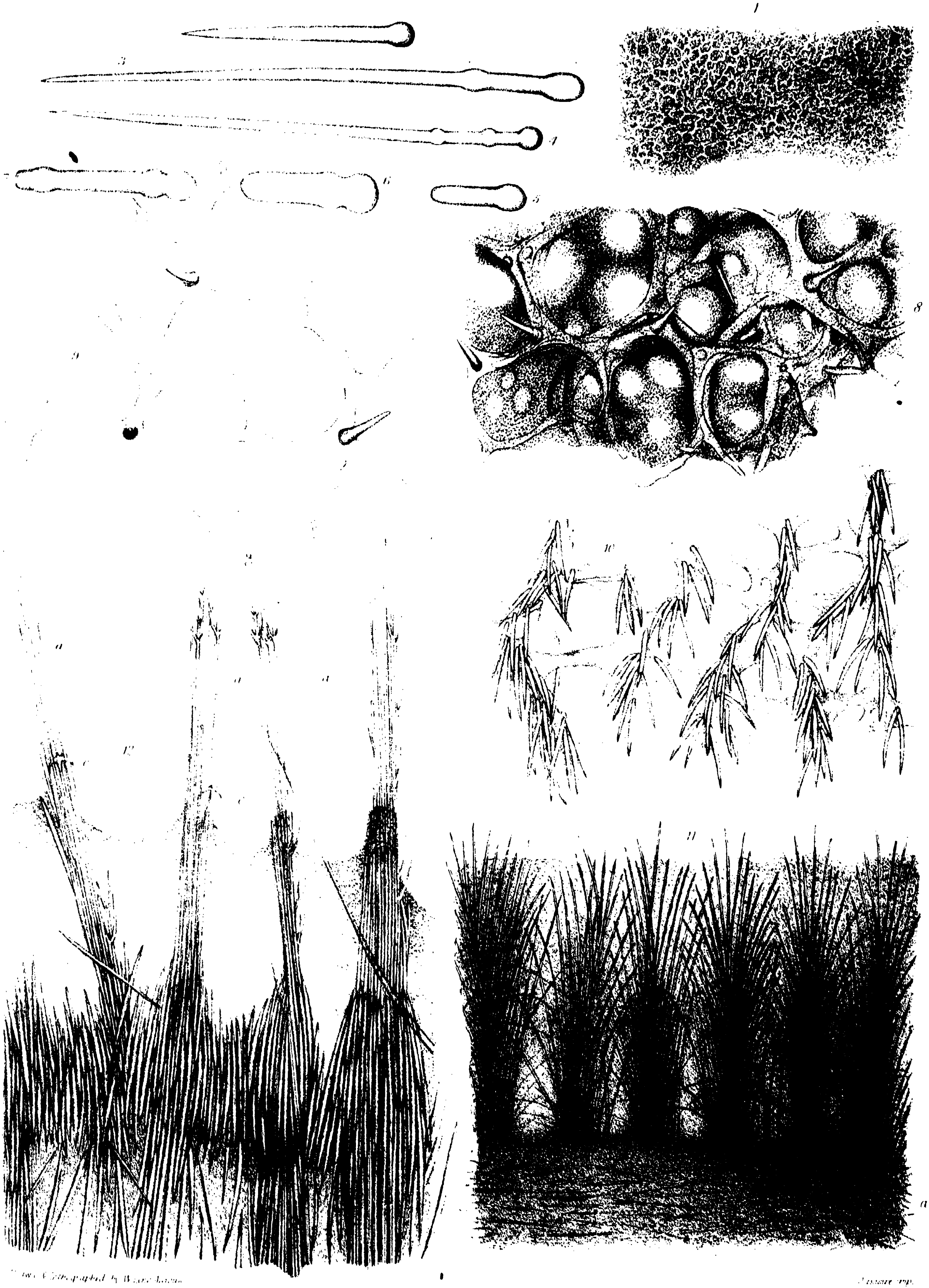

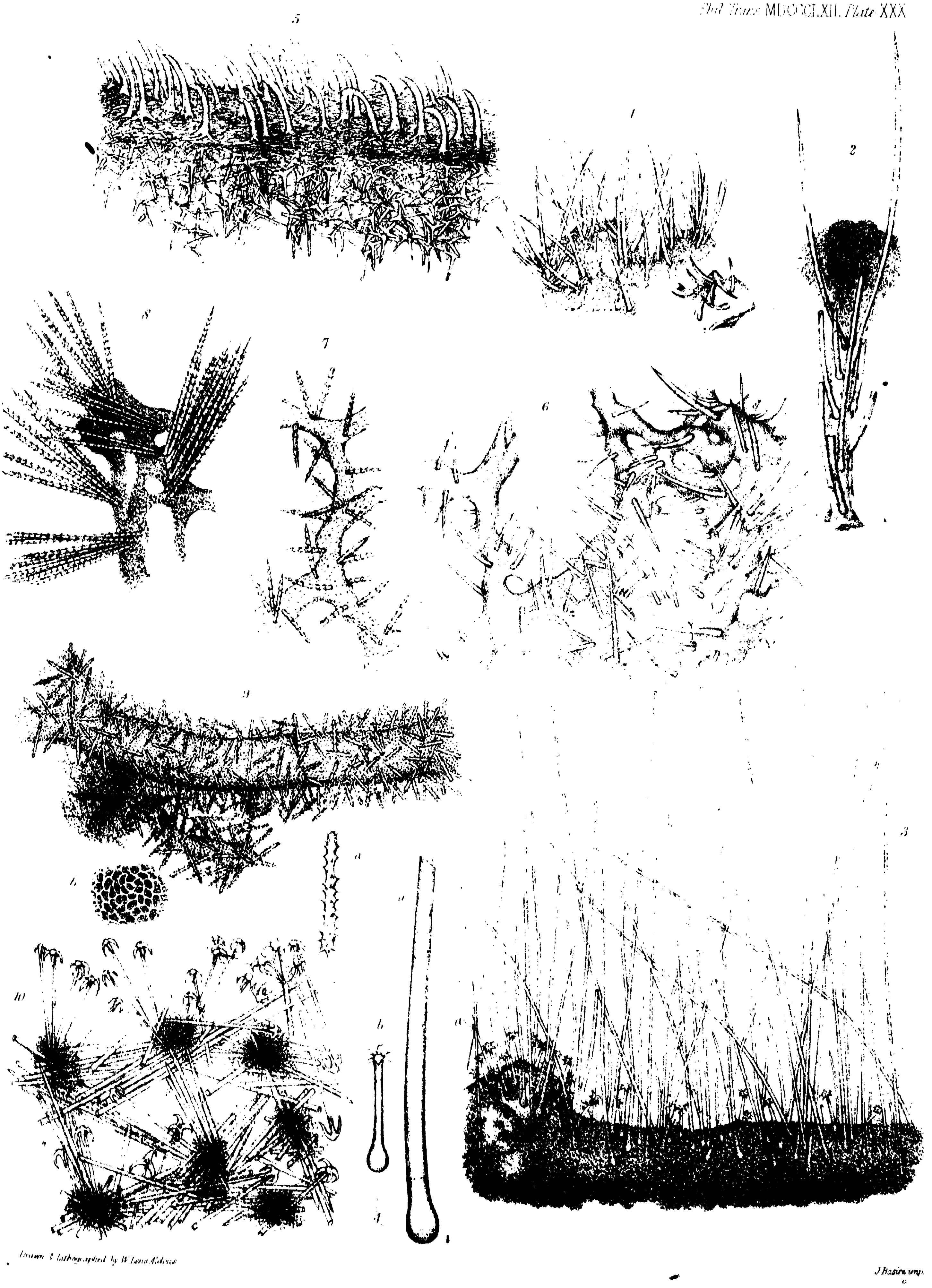

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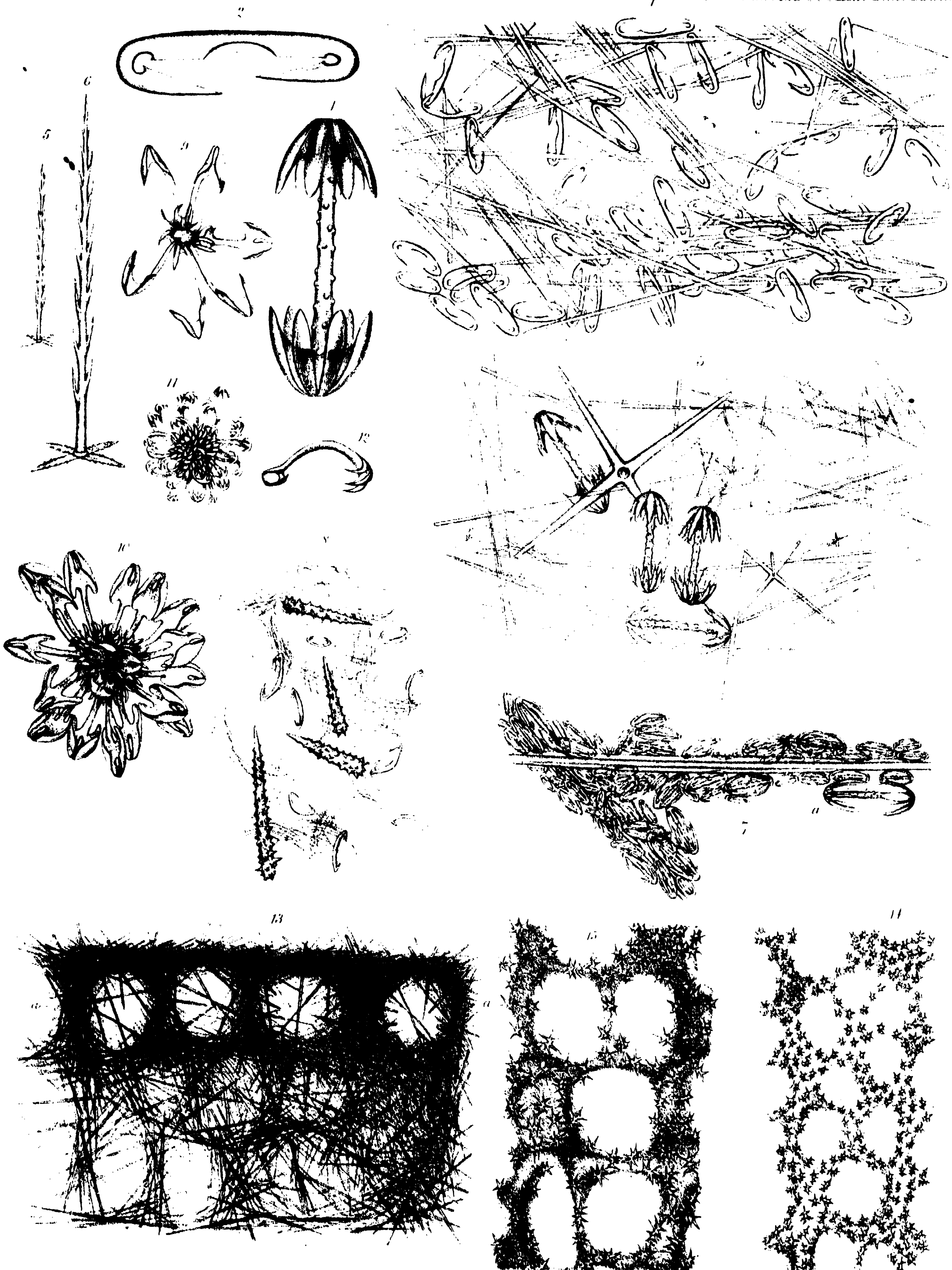

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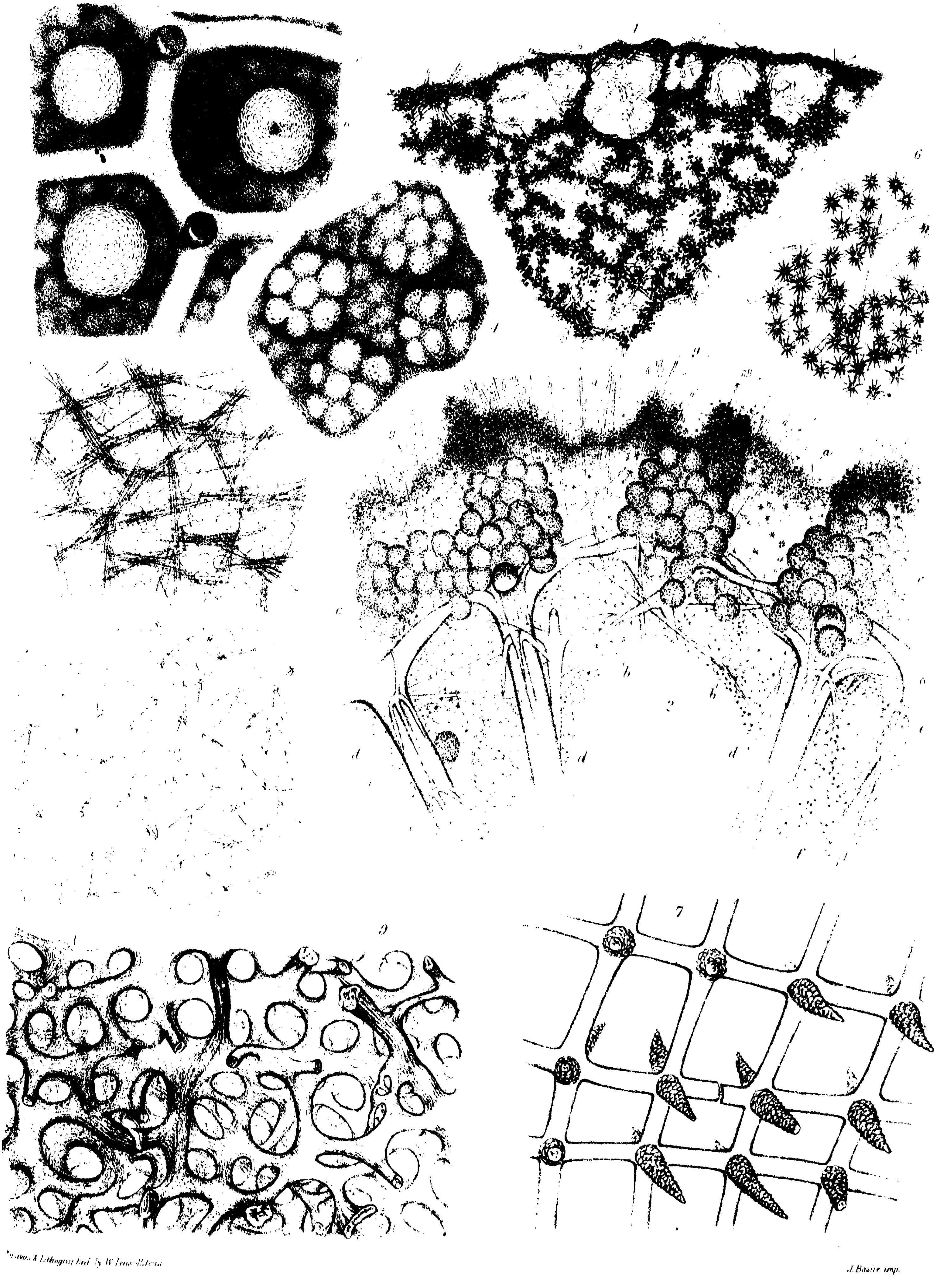

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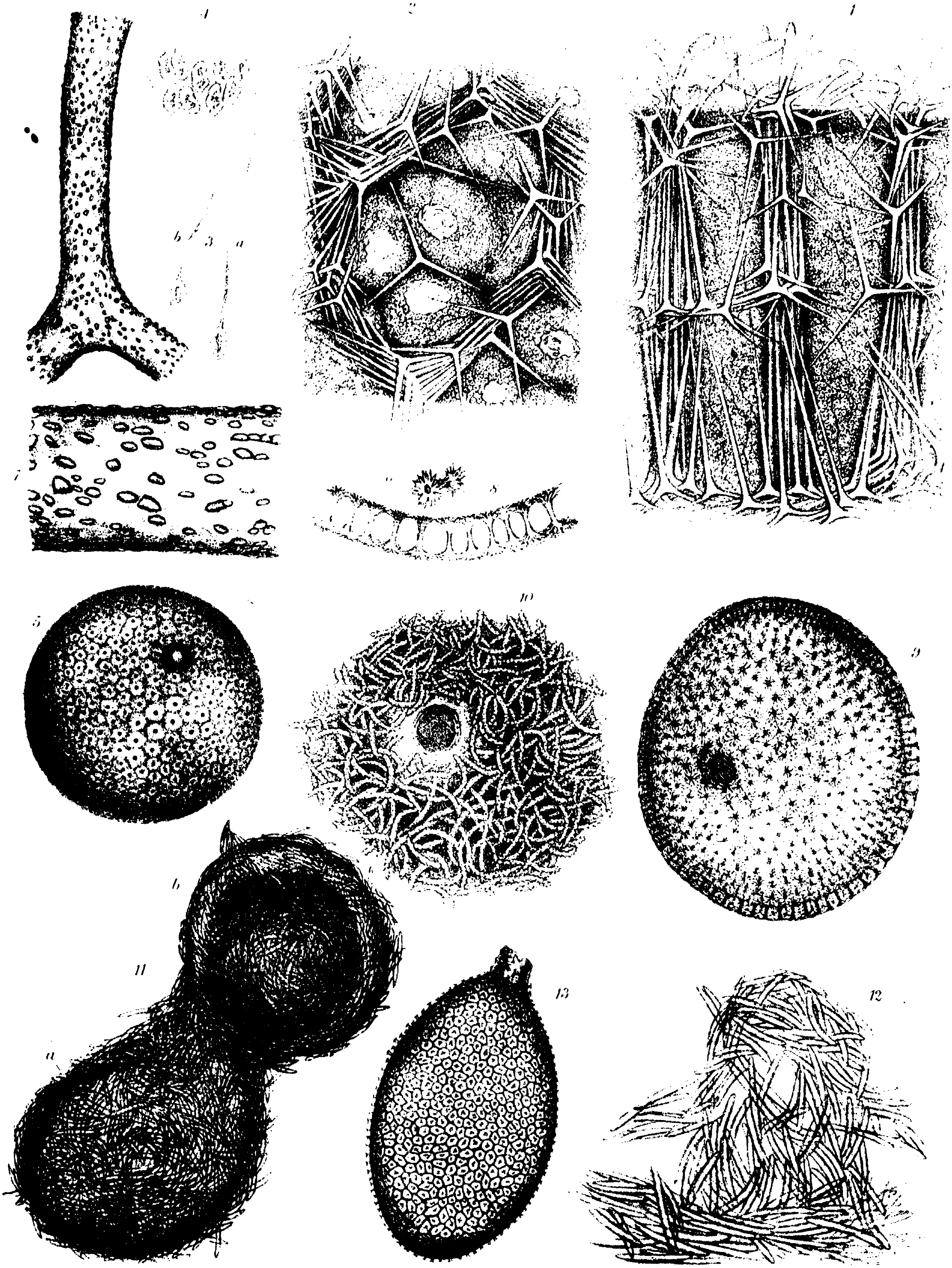

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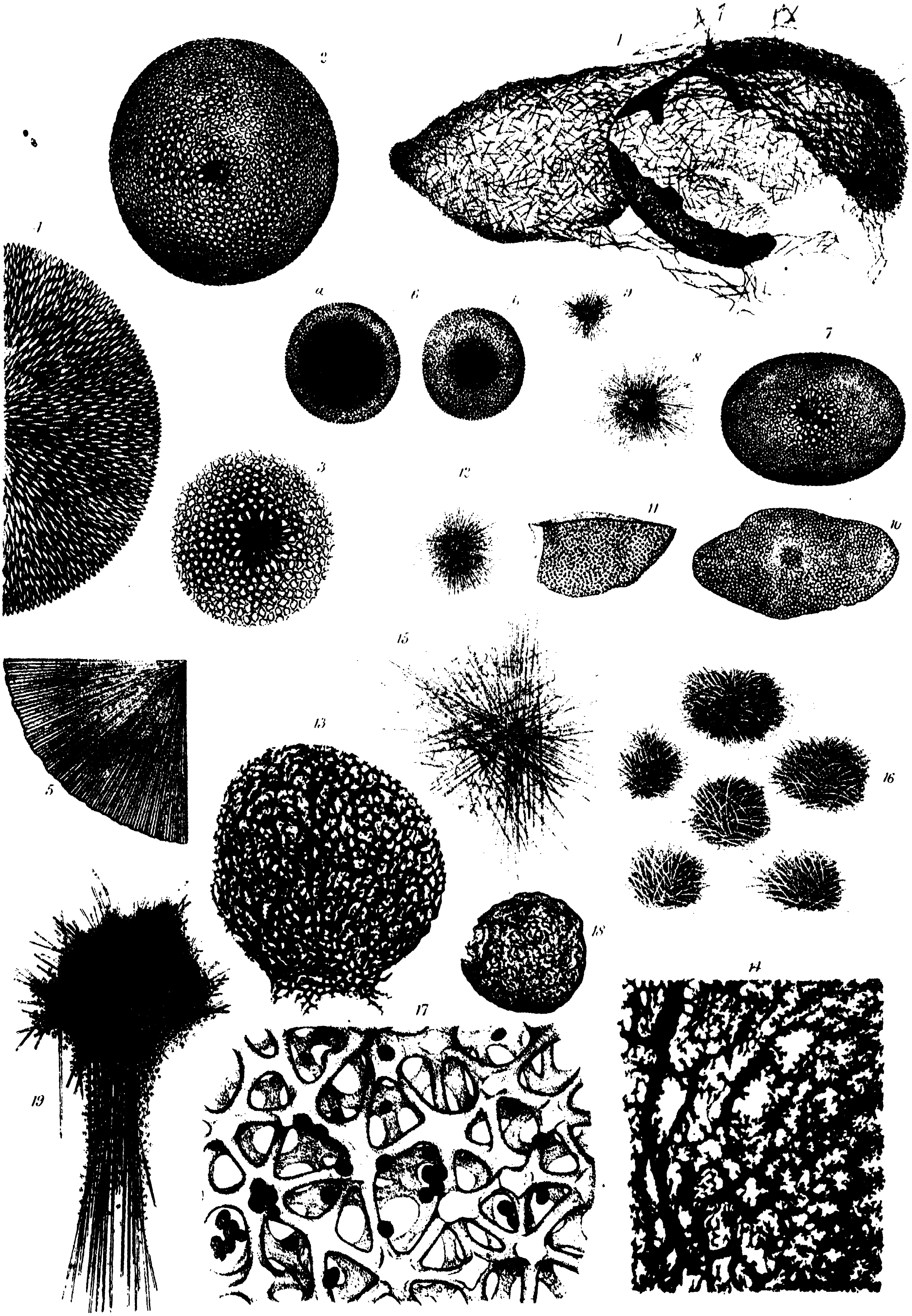

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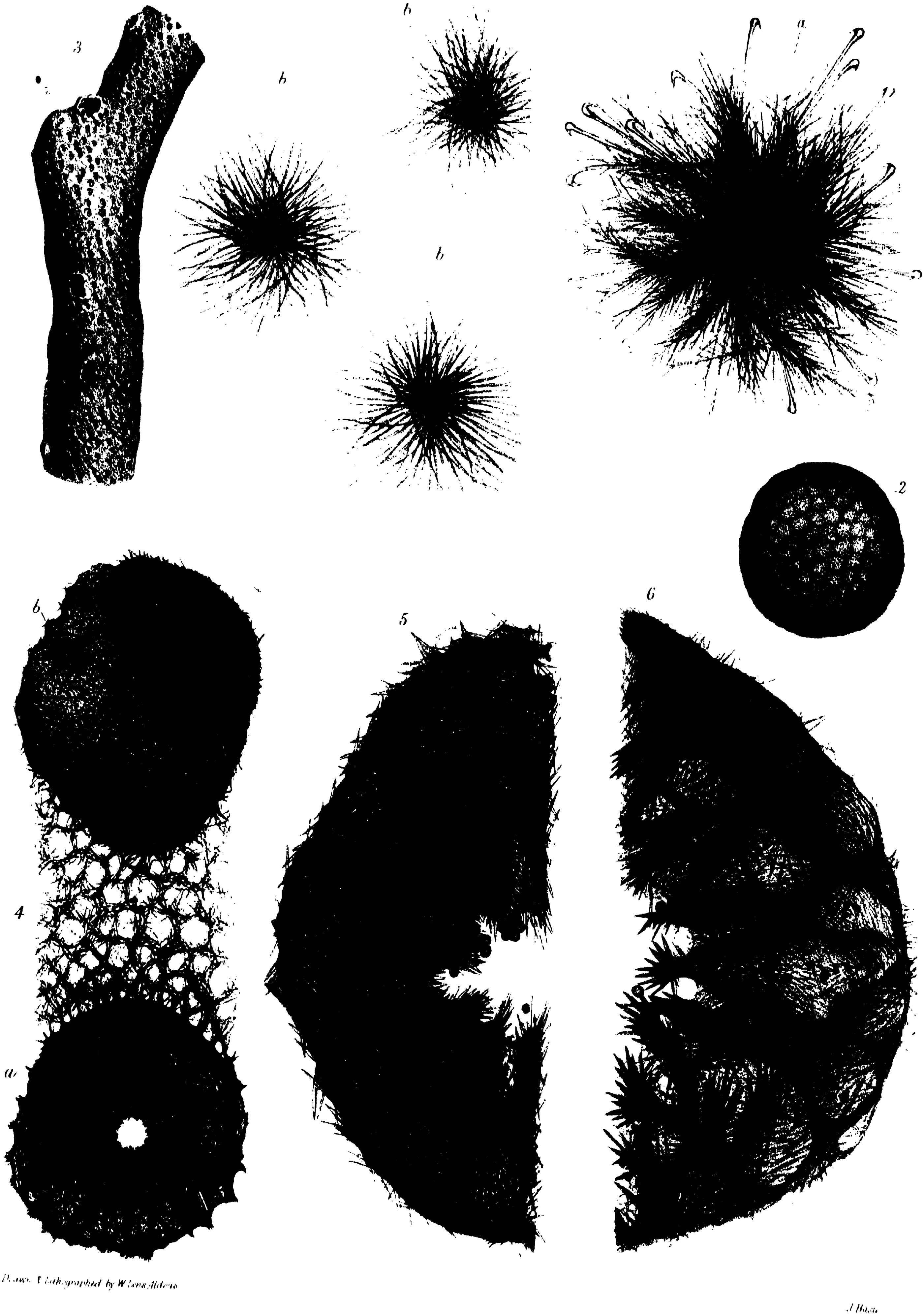

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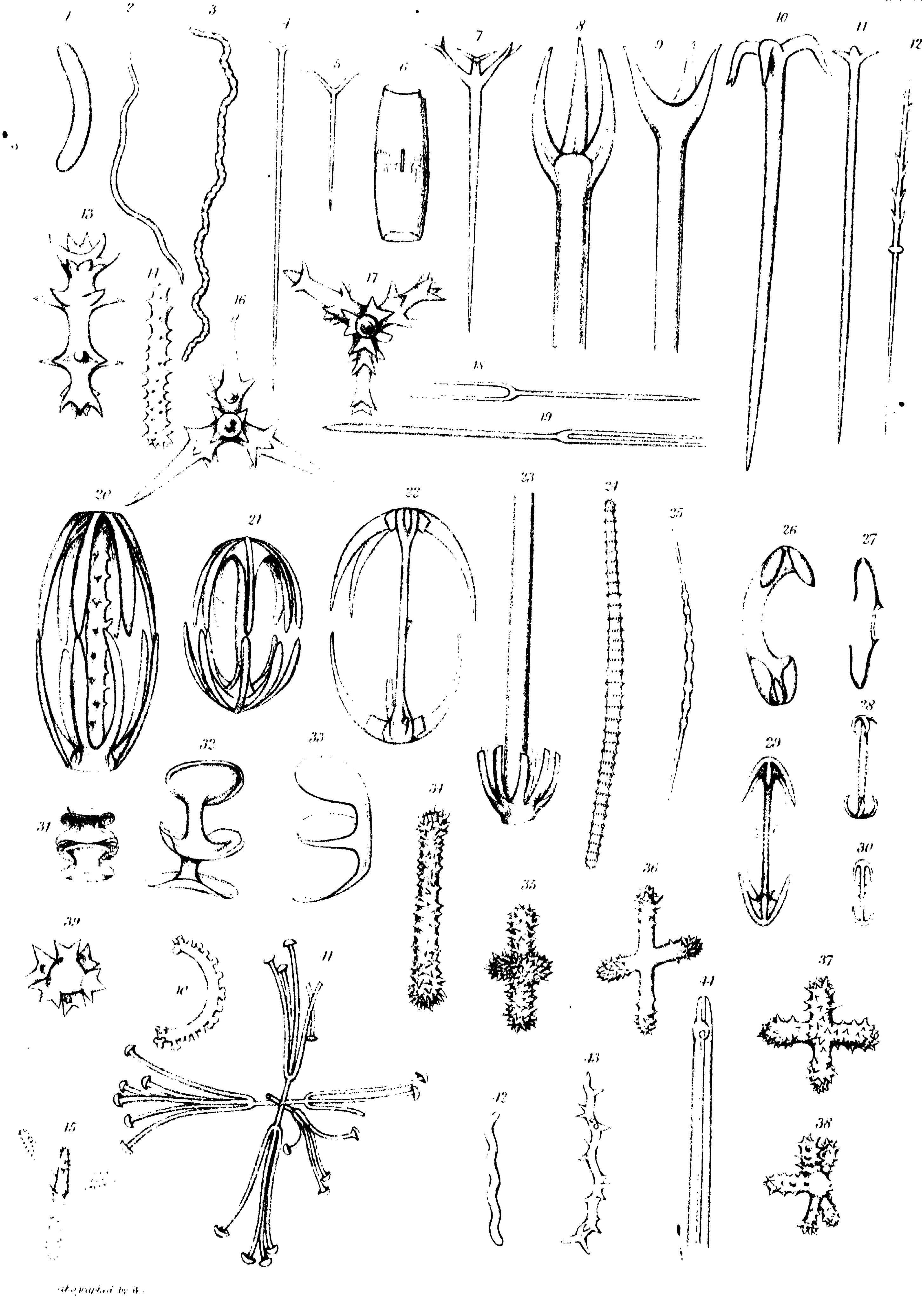
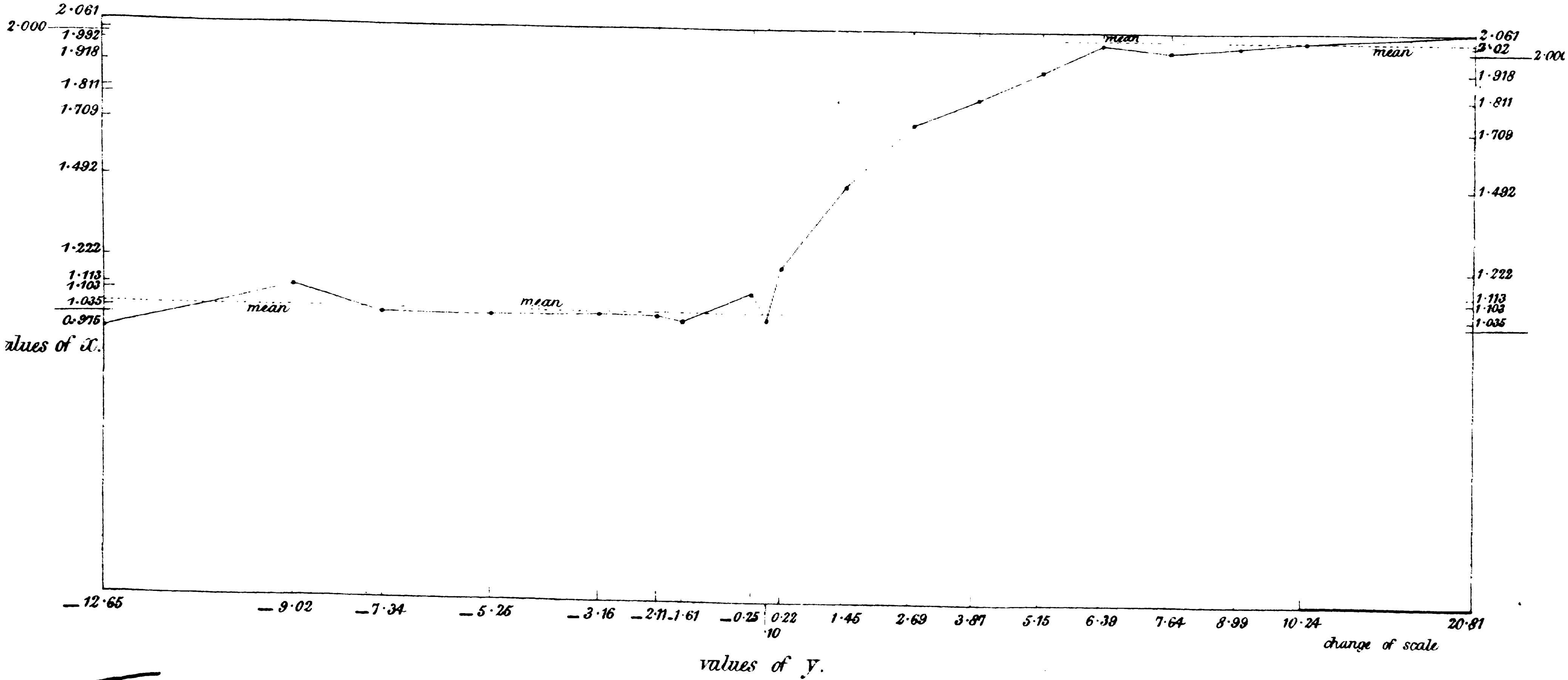


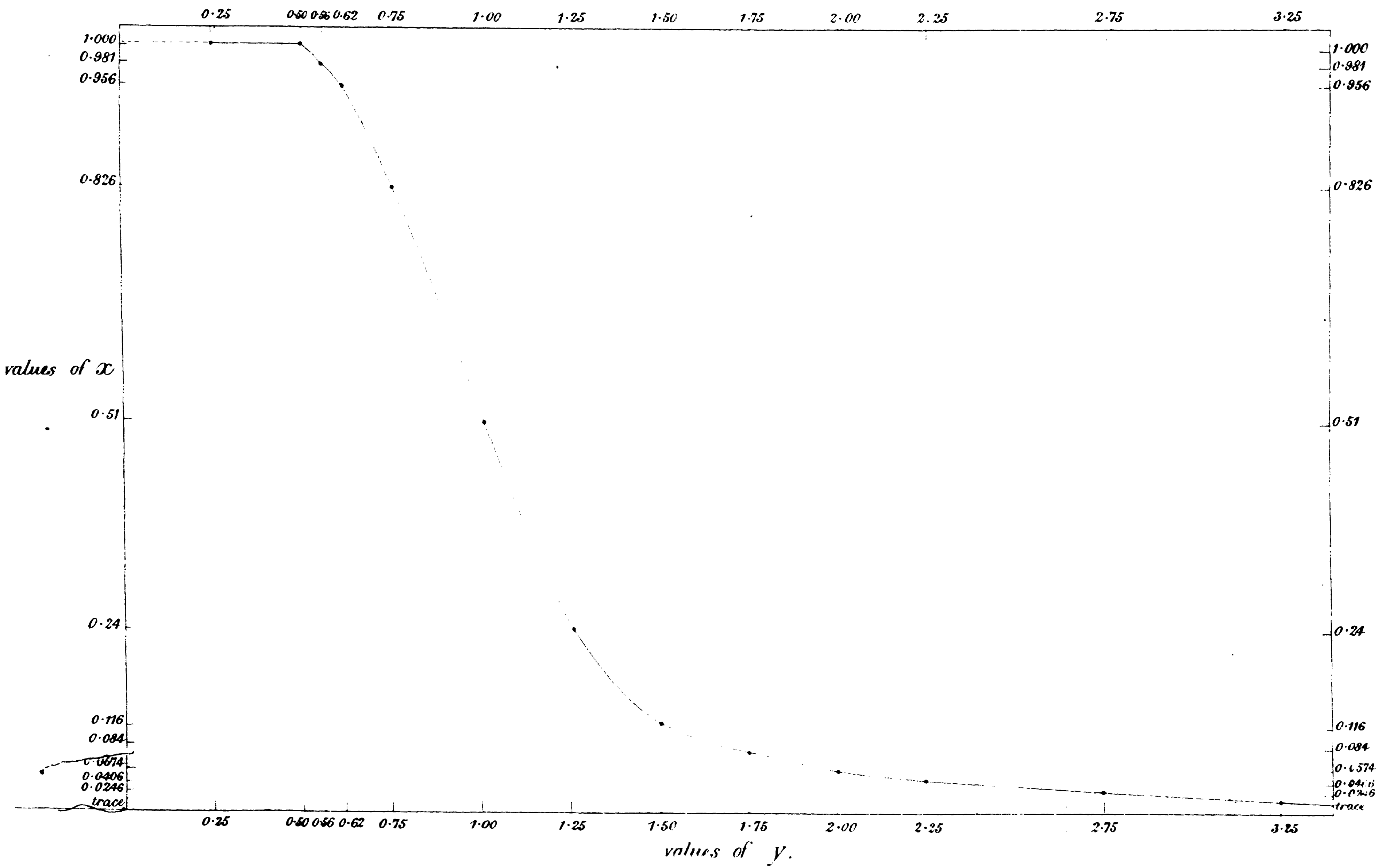
Lithographed by [illegible]

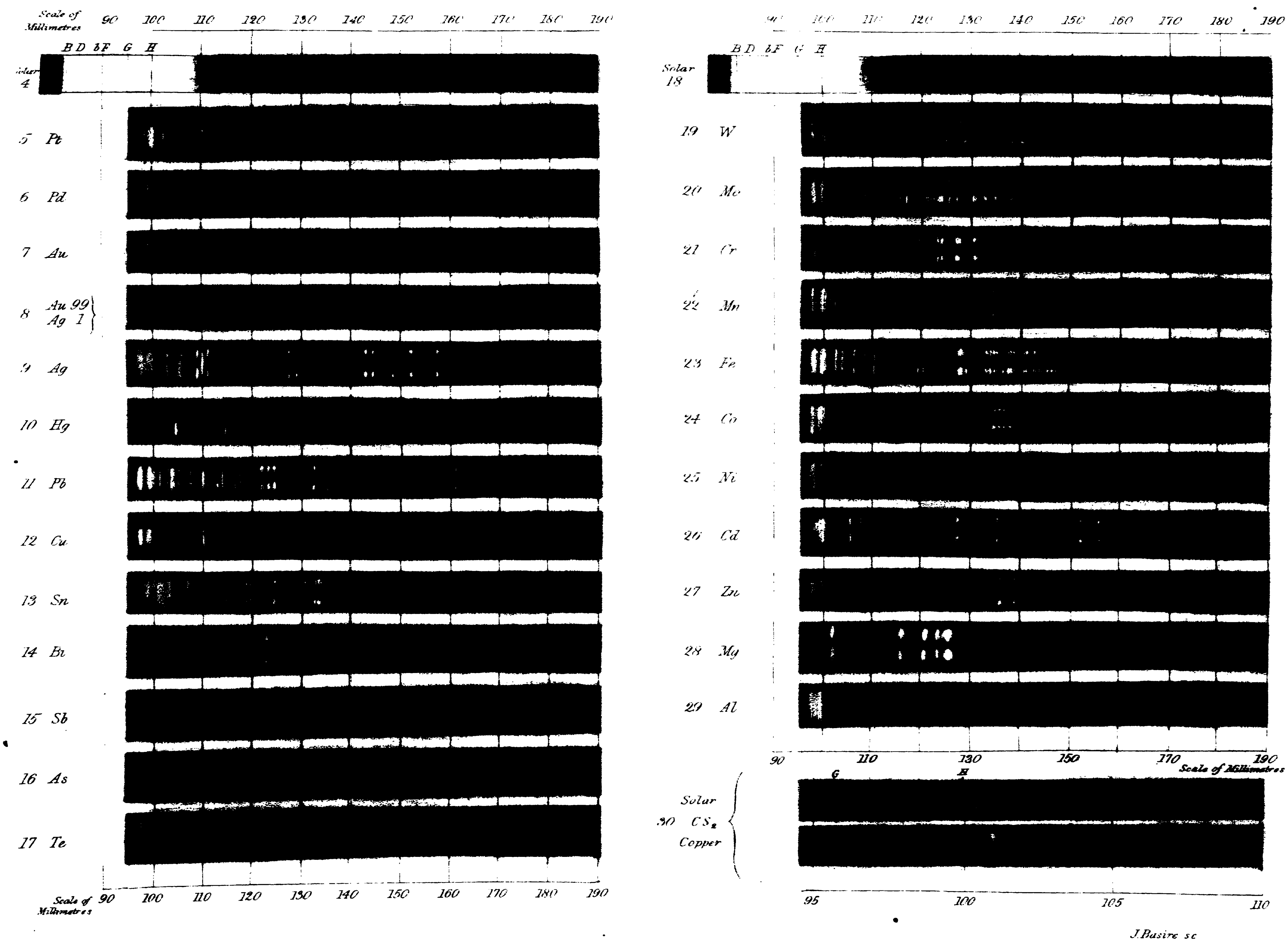
Curve of the decomposition of chromic acid by peroxide of hydrogen.

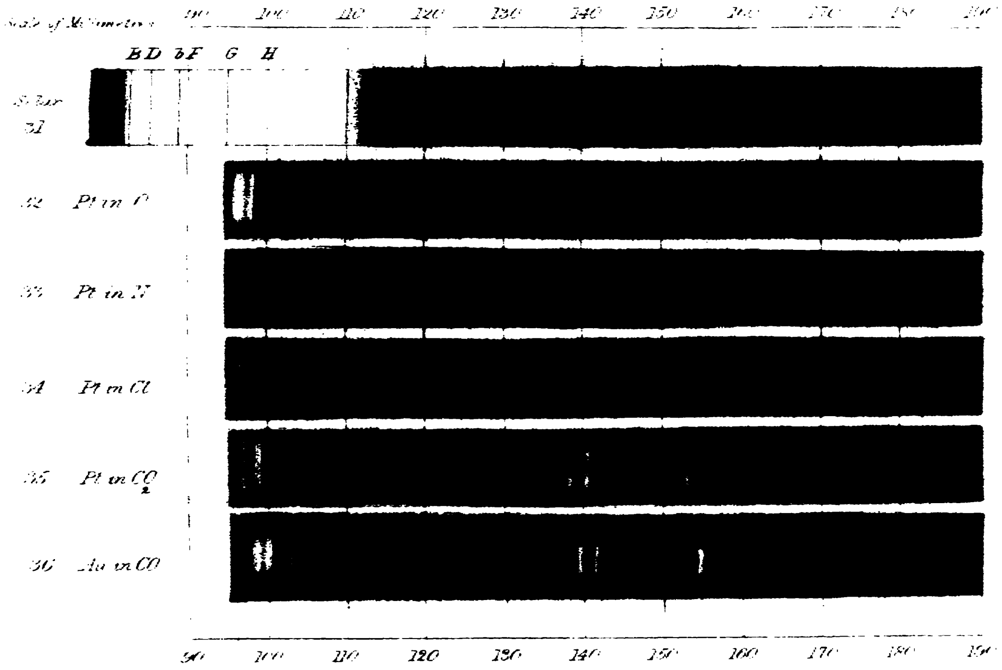


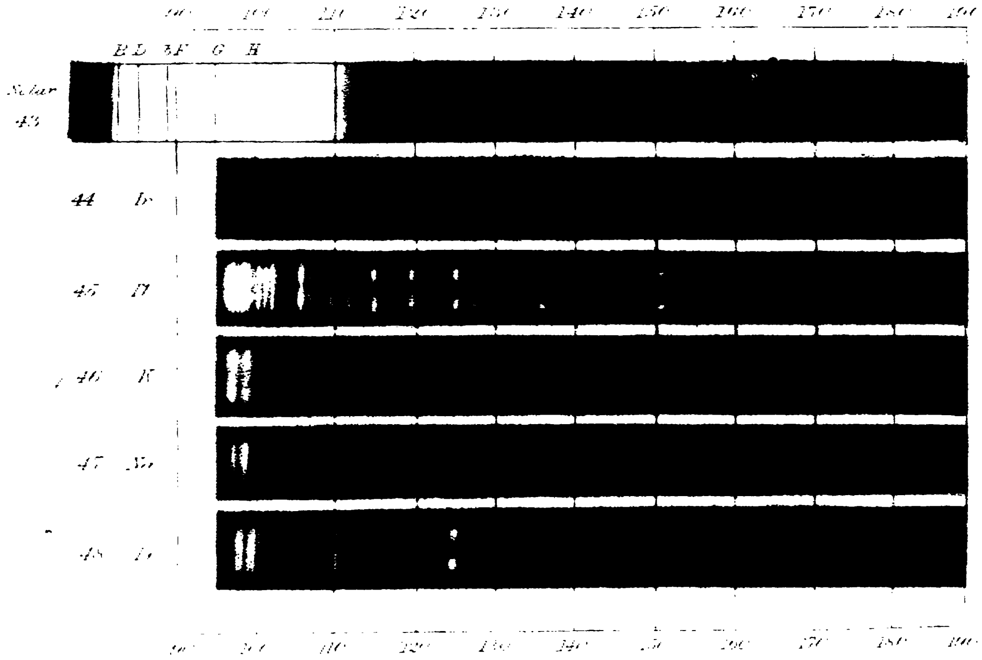
J. Basire lith.

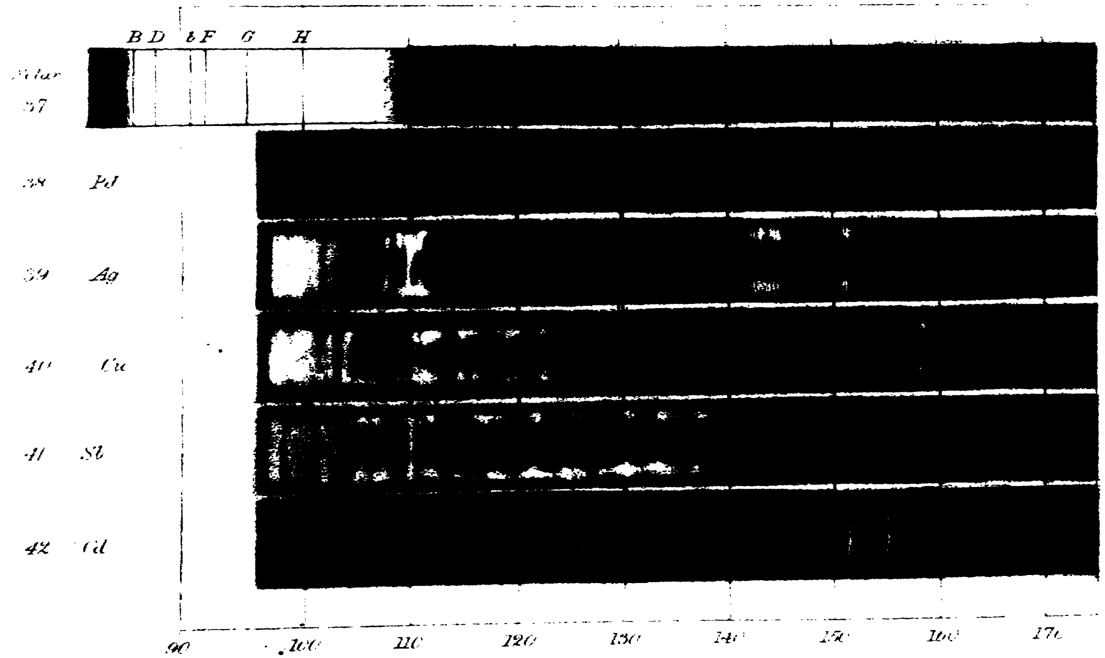
Curve of the decomposition of hydrochloric acid by peroxide of hydrogen.

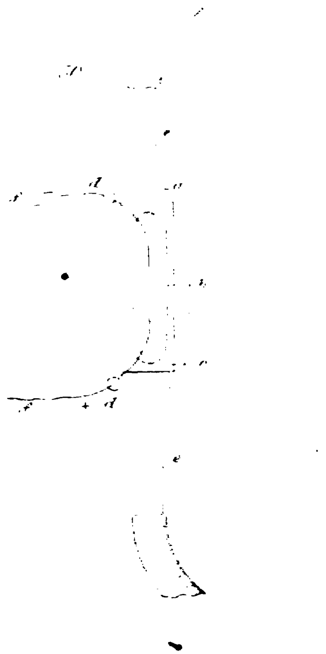



Scale of Millimetres
90 100 110 120 130 140 150 160 170 180 190
B D bF G H
Solar 4
5 Pt
6 Pd
7 Au
8 Au 99 Ag 1
9 Ag
10 Hg
11 Pb
12 Cu
13 Sn
14 Bi
15 Sb
16 As
17 Te
Solar 18
19 W
20 Mo
21 Cr
22 Mn
23 Fe
24 Co
25 Ni
26 Cd
27 Zn
28 Mg
29 Al
Scale of Millimetres
Solar
30 CS2
Copper
95 100 105 110
J.Basire sc


Scale of Millimetres
90
100
110
120
130
140
150
160
170
180
190
B D b F G H
Solar 31
32 Pt in O
33 Pt in N
34 Pt in Cl
35 Pt in CO_2
36 Au in CO


B D b F G H
Solar 43
44 Ir
45 H
46 K
47 Na
48 Li


B D b F G H
Solar 37
38 Pd
39 Ag
40 Cu
41 Sb
42 Cd
90
100
110
120
130
140
150
160
170


50
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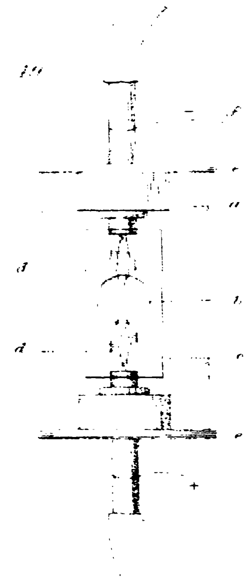

49
f
a
d
b
d
+

a
Terminal portion of dark-bordered fibre.
– Bladder of frog. × 1700

Terminal portion of dark bordered fibre.—Bladder of frog. × 1700

Dark-bordered fibres crossing a muscular fibre. – leg of frog

1000th × 700

1000th × 1700

S. B. ad nat. del

Harrison's Impt.

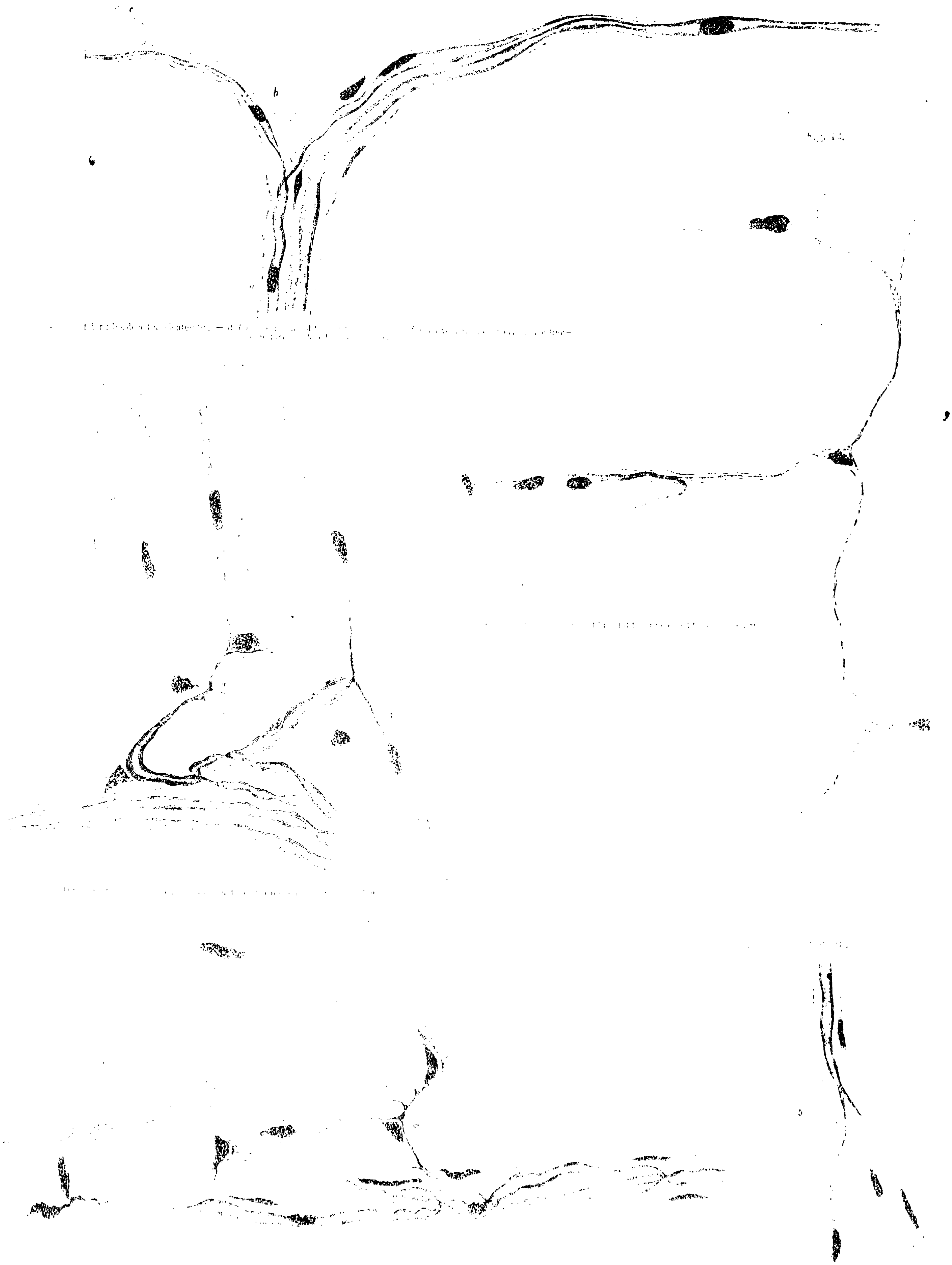
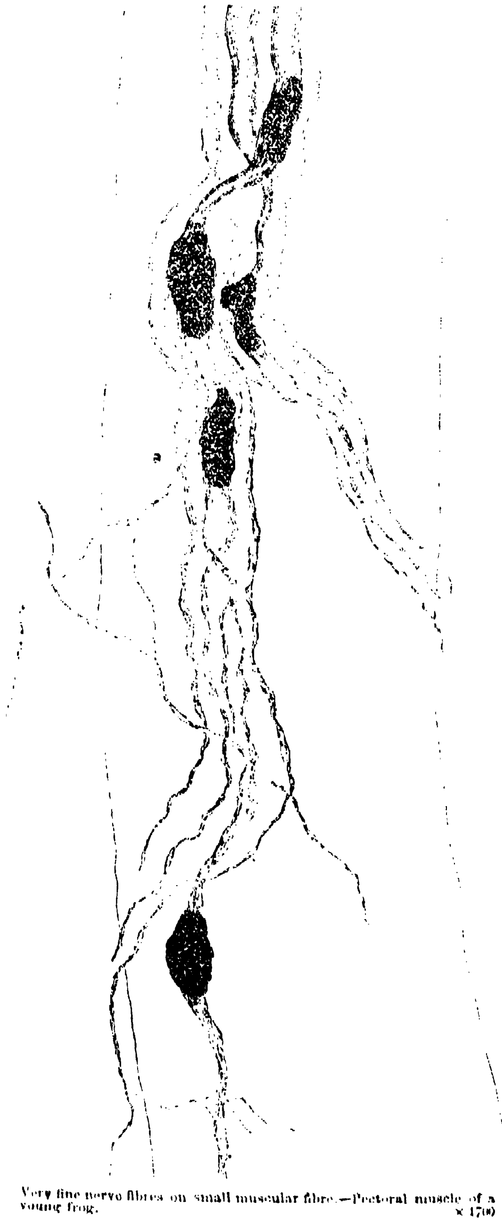


Fig. 16.

Terminal portion of finest dark bordered fibres.—Small muscular fibre.—Pectoral. × 700

Fig. 17.

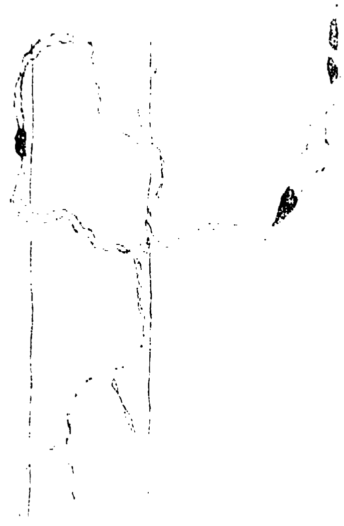
Division of dark bordered fibre, and fibres of sheath.—Formation of network.—Pectoral. × 700



Very fine nerve fibres on small muscular fibre.—Pectoral muscle of a young frog. × 1700

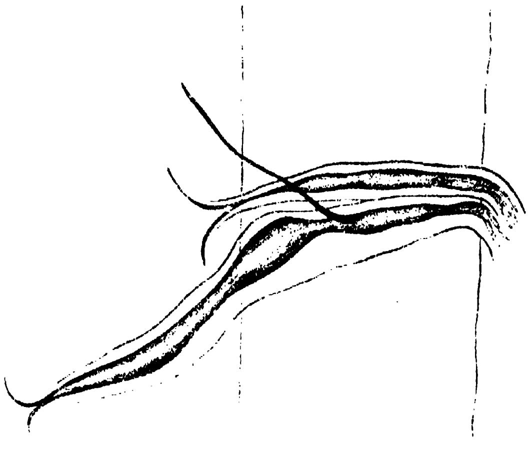
Fig. 21.

Thin dark-bordered fibre.—Fine fibres in sheath, crossing it and forming a branch which leaves the fibre.—From muscle of neck of frog. × 700



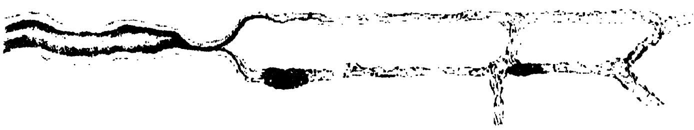
Fine fibres resulting from the subdivision of a dark-bordered fibre. × 700

Fine fibres ramifying in sheath, dividing and forming networks.—Muscle from the neck of the frog. × 700



Division of dark bordered fibres while crossing a muscular fibre. × 700

Fig. 20.



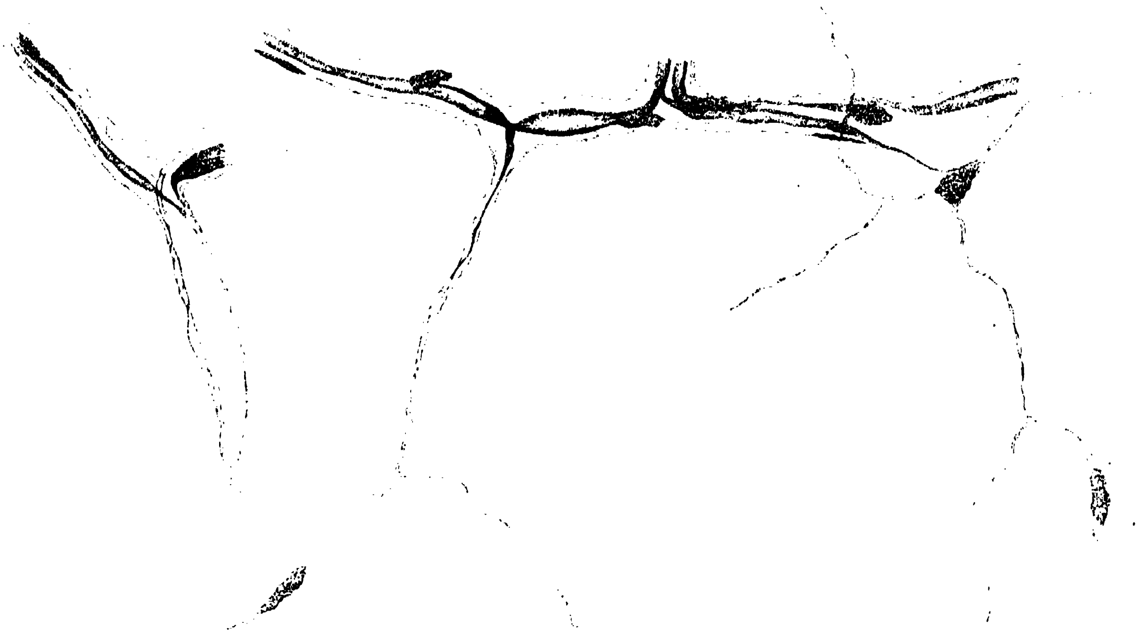
Division of dark-bordered fibre and network of fine fibres.—Pectoral muscle. × 700

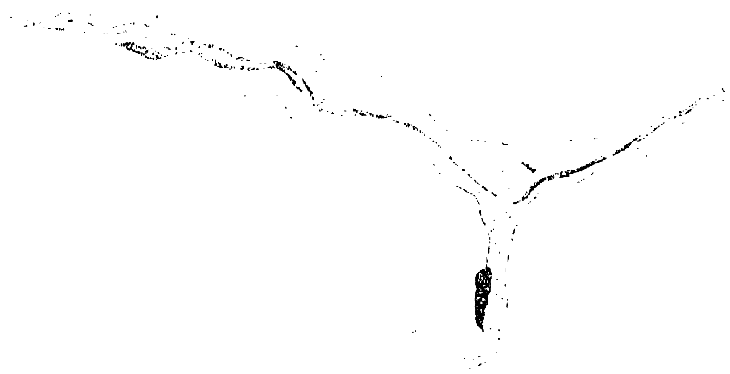
1000 th × 700

1000 th × 1700

L. S. B. ad nat. del.

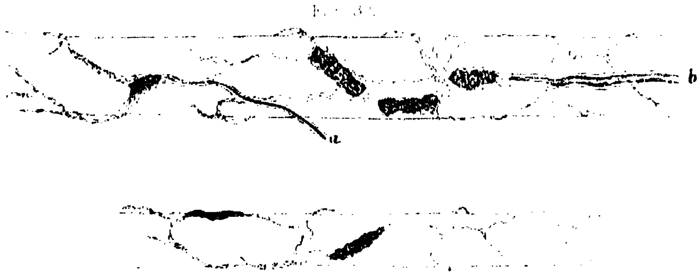
Harrison's Impt.





Branching and division of dark bordered fibres and pale fibres in same sheath.
From bladder of the frog. × 700

Terminal portion of dark-bordered fibre with fine fibres crossing a young muscular fibre. × 700

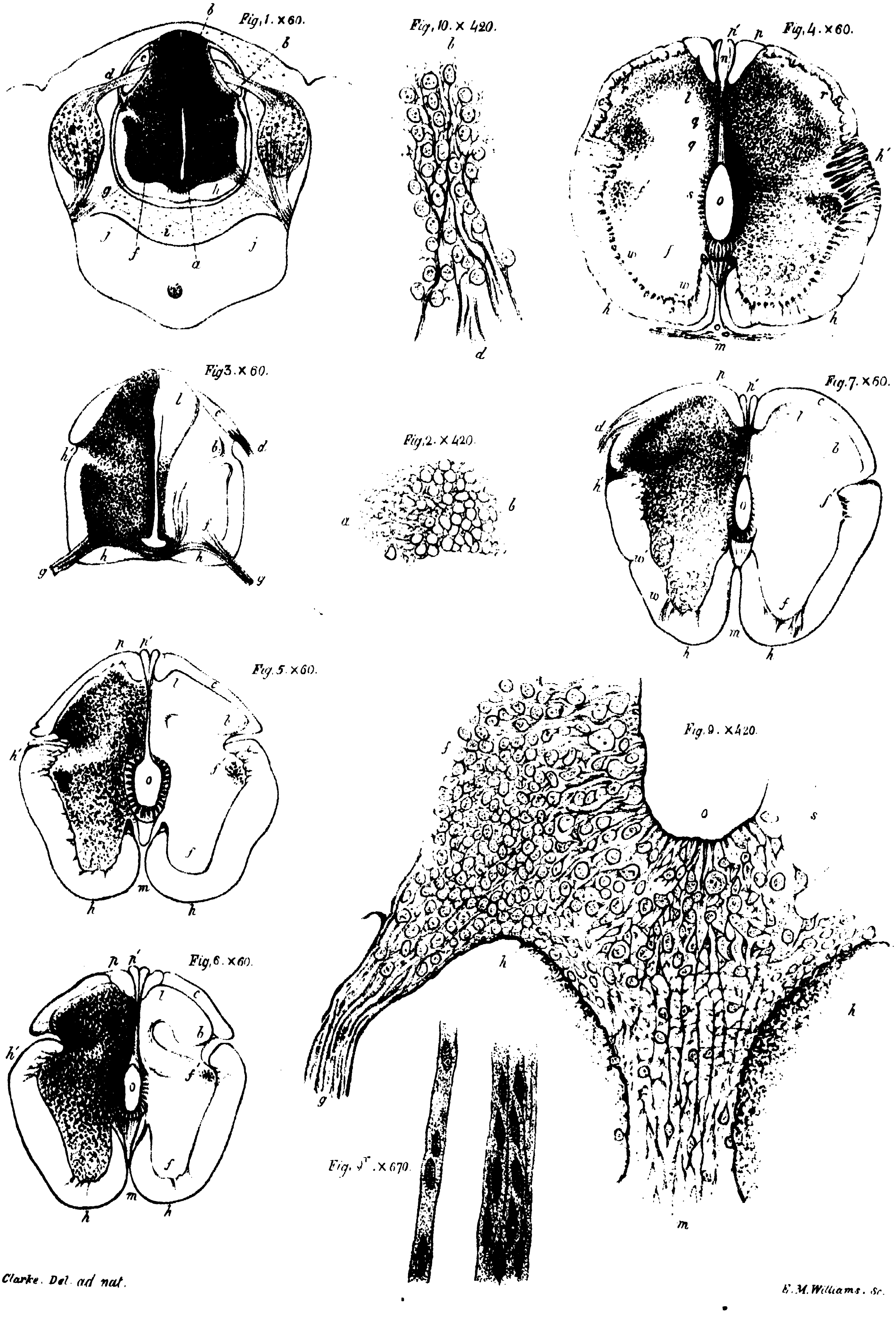


Young muscular fibres near a nerve swelling. Pectoral of frog.—Shewing division of nerve fibres. c another fibre at a distance from the nerve swelling. × 700

Dark bordered nerve fibre with a bundle of fine fibres by the side of it. Bladder of frog. × 700

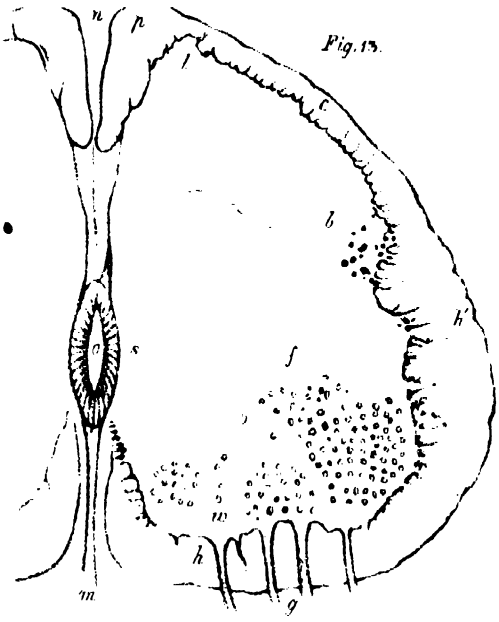
1000th × 700

1000th × 1700



J. L. Clarke. Del. ad nat.

E. M. Williams. Sc.



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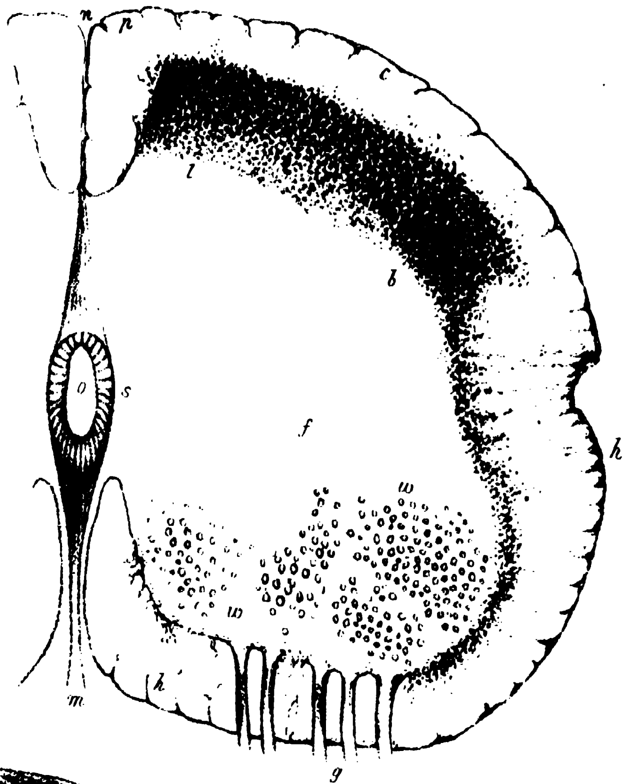

Fig. 14.

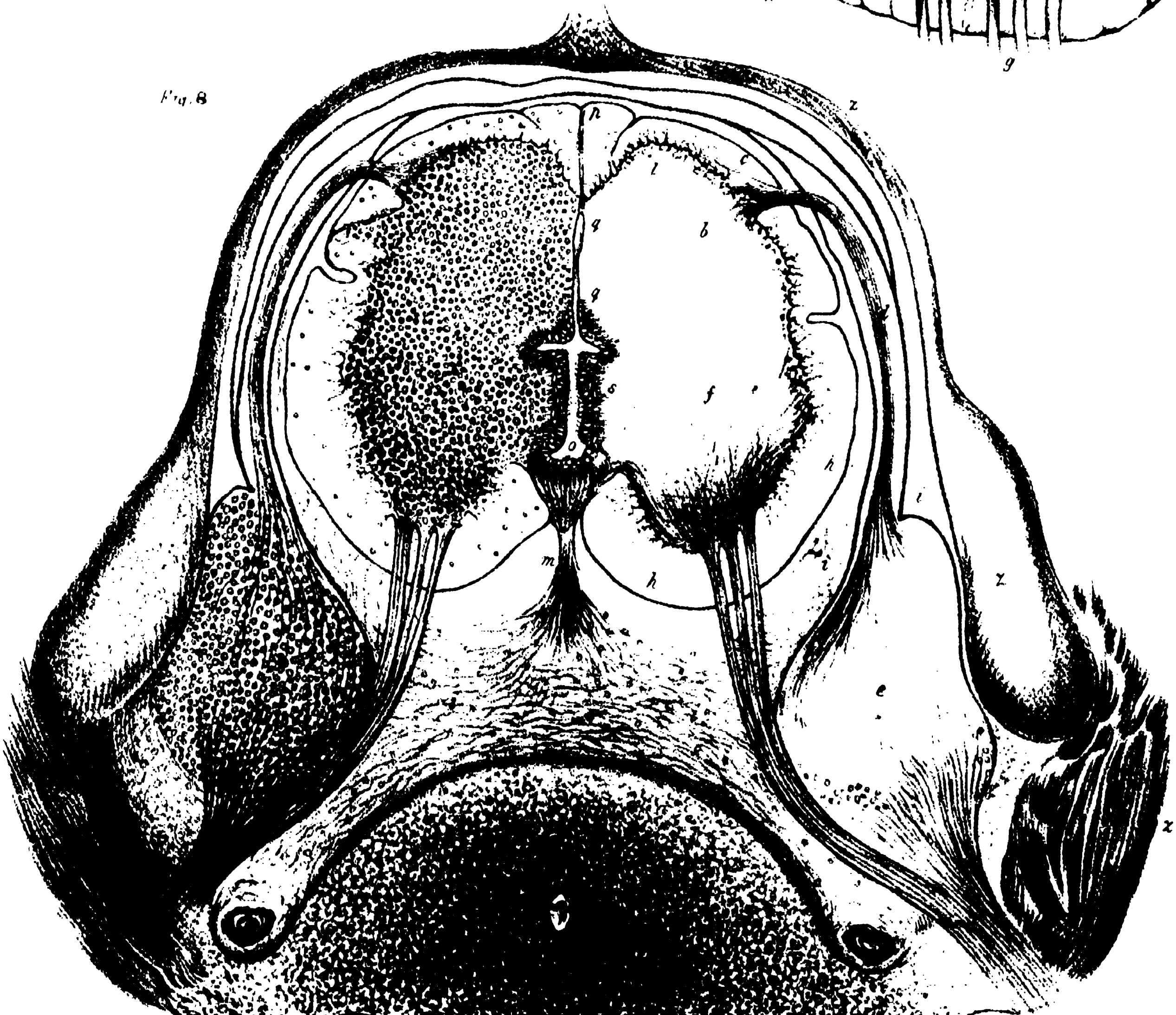

Fig. 8.

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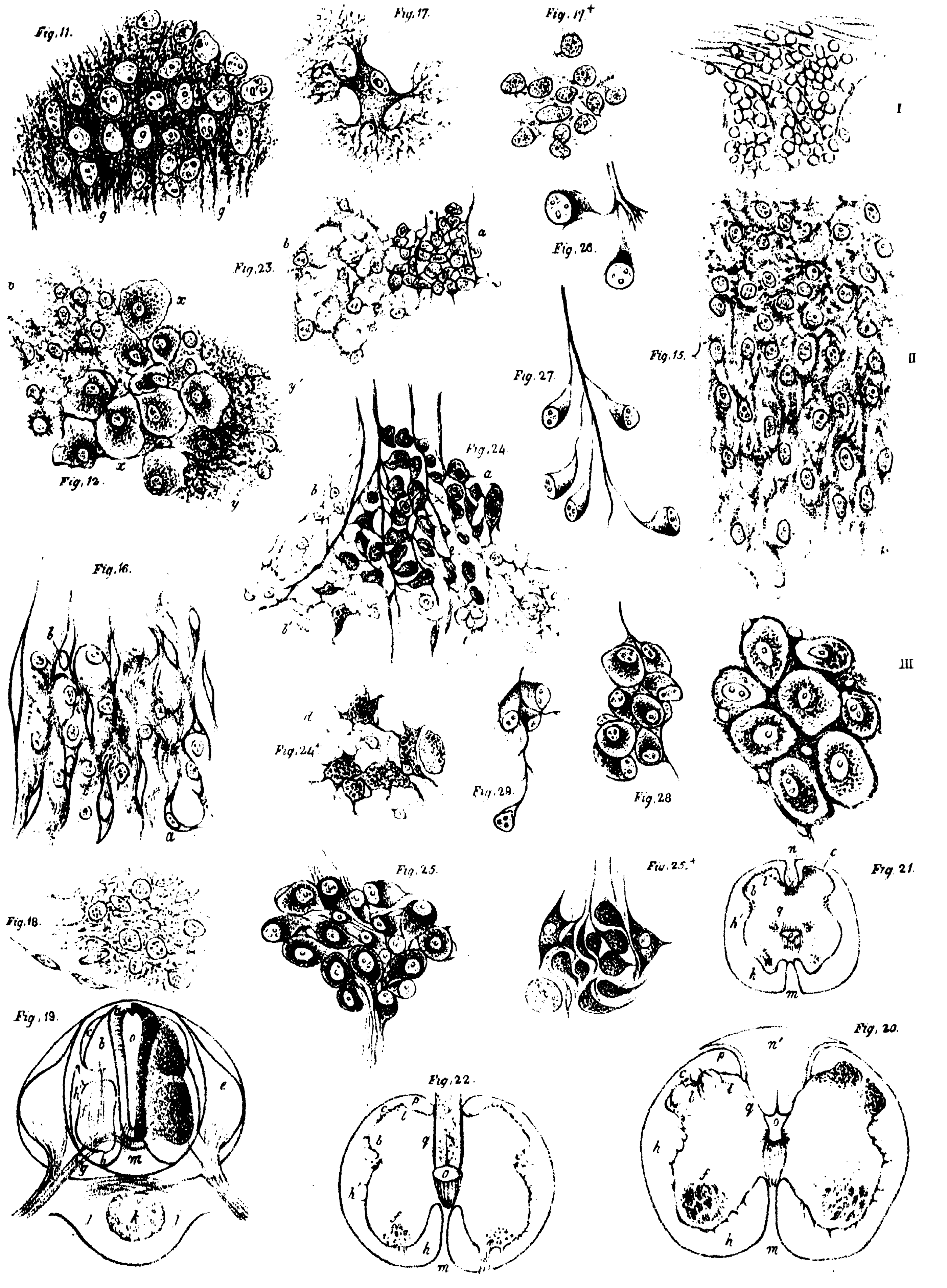
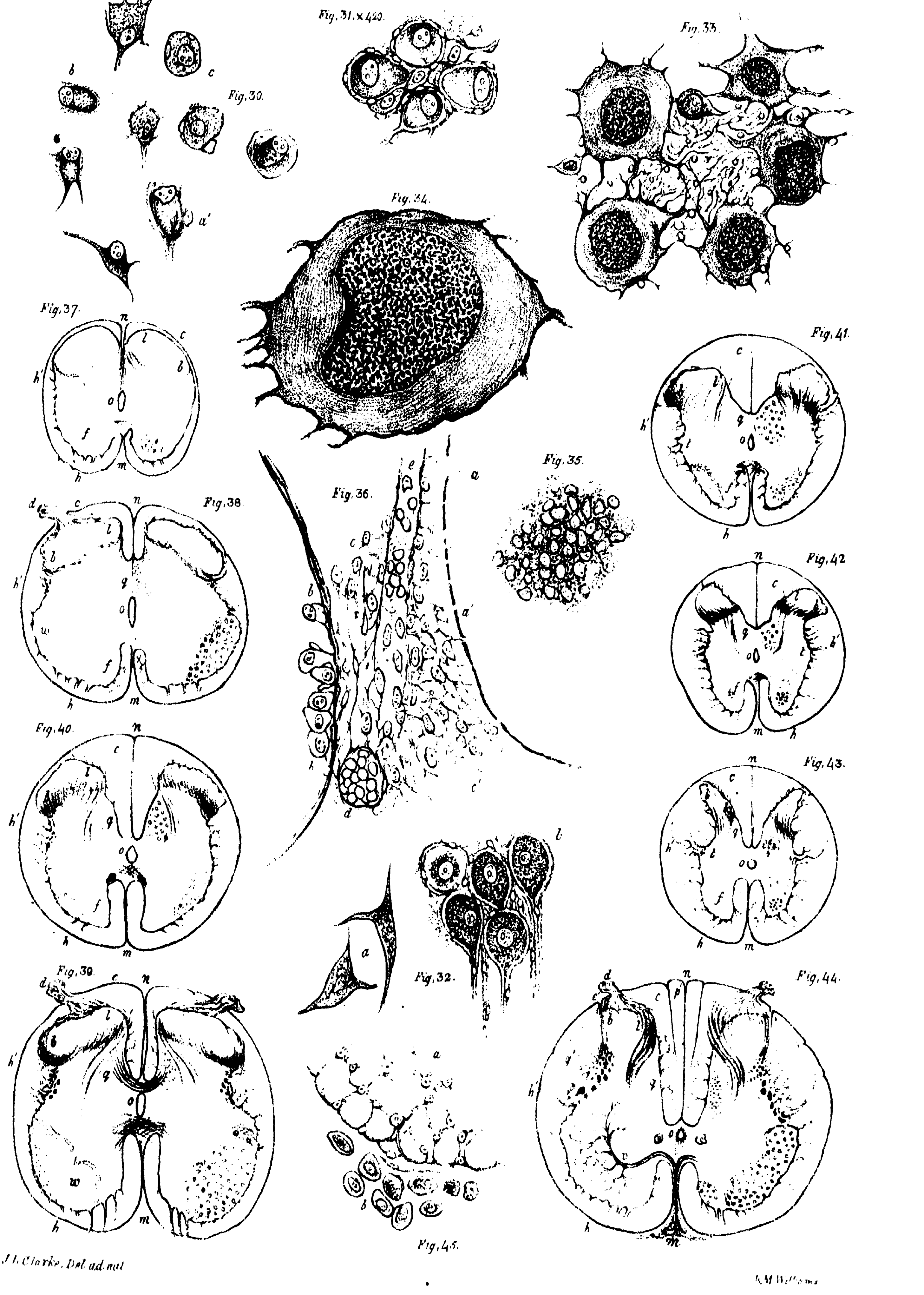


J. L. Clarke Del. ad nat.

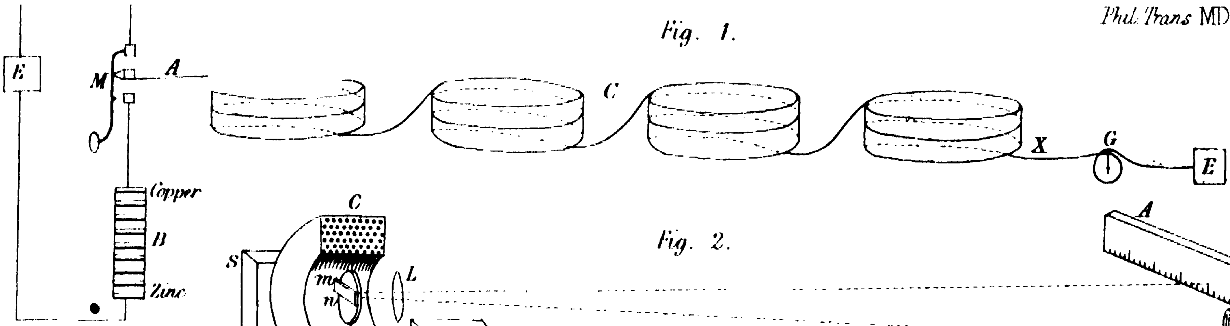
F. M. W... sc.



J. L. Clarke, Del. ad nat.

W. M. Williams

Fig. 1.



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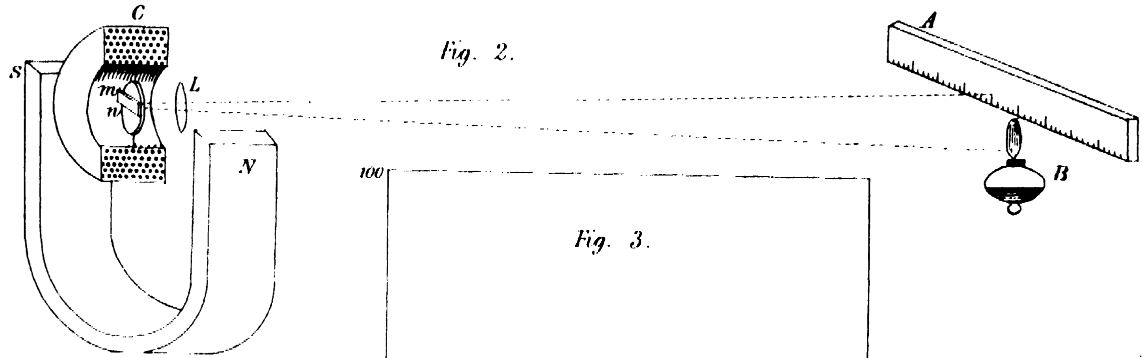


Fig. 3.

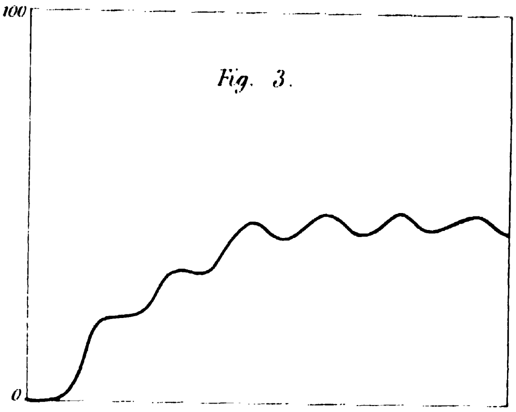


Fig. 4

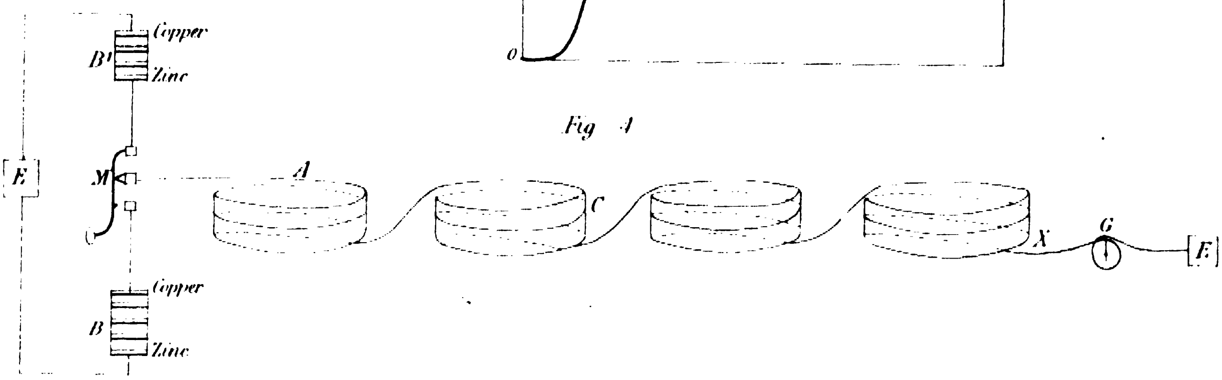


Fig. 5. Arrival Curve for 1165 knots from Positive currents & Earth contacts.

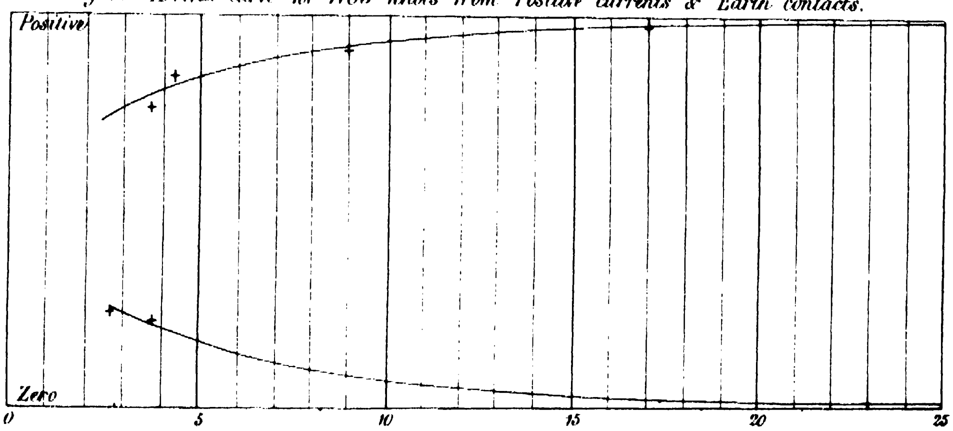


Fig. 6. Arrival Curve for 1165 knots from Positive and Negative Currents.

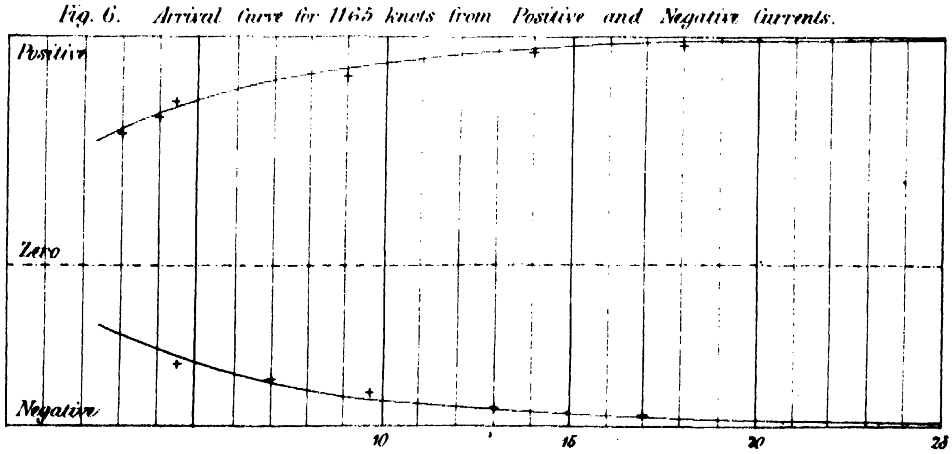


Fig. 7. Arrival-Curve for 1006 knots in 4 Coils and 2192 knots in 10 Coils.

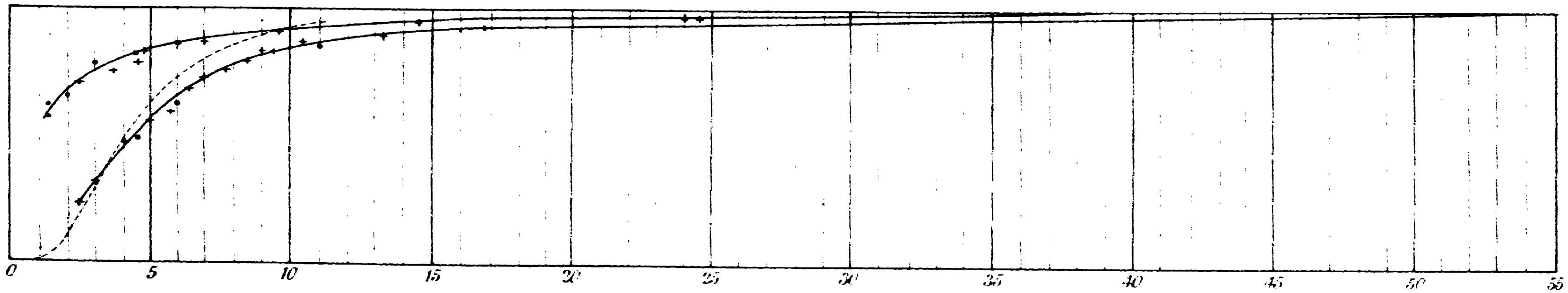


Fig. 8.

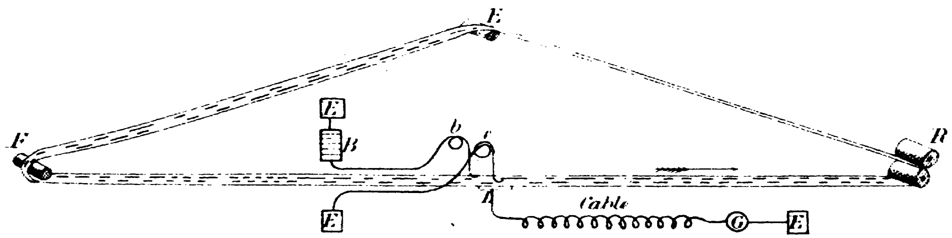


Fig. 9. Paper used to make contacts. ⅕ full size.

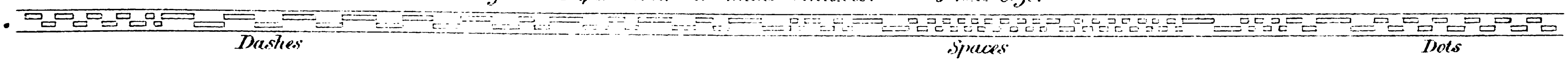
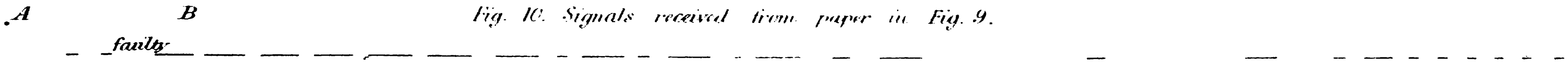
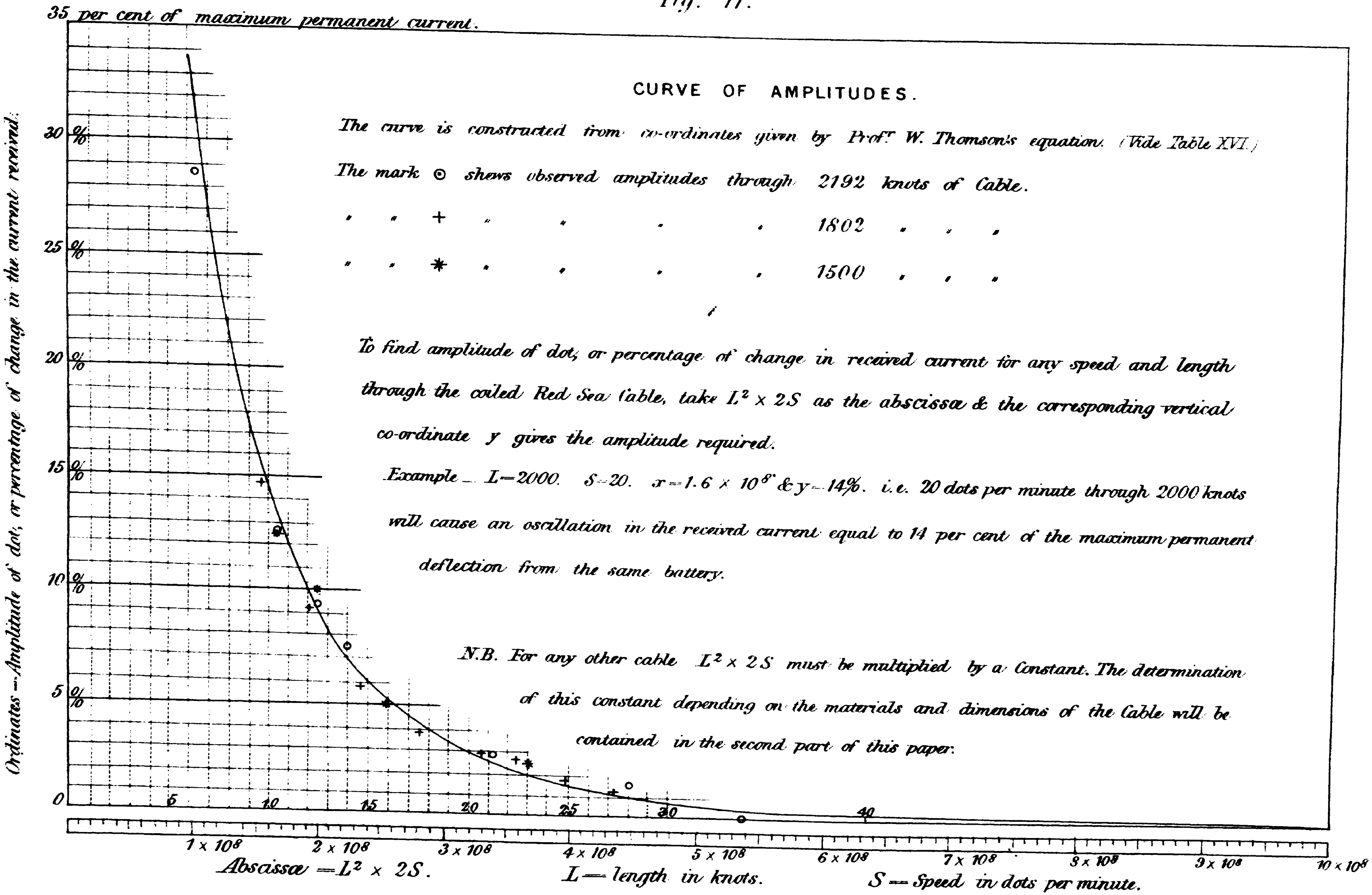


Fig. 10. Signals received from paper in Fig. 9.



J. Basire lith.

Fig. 11.

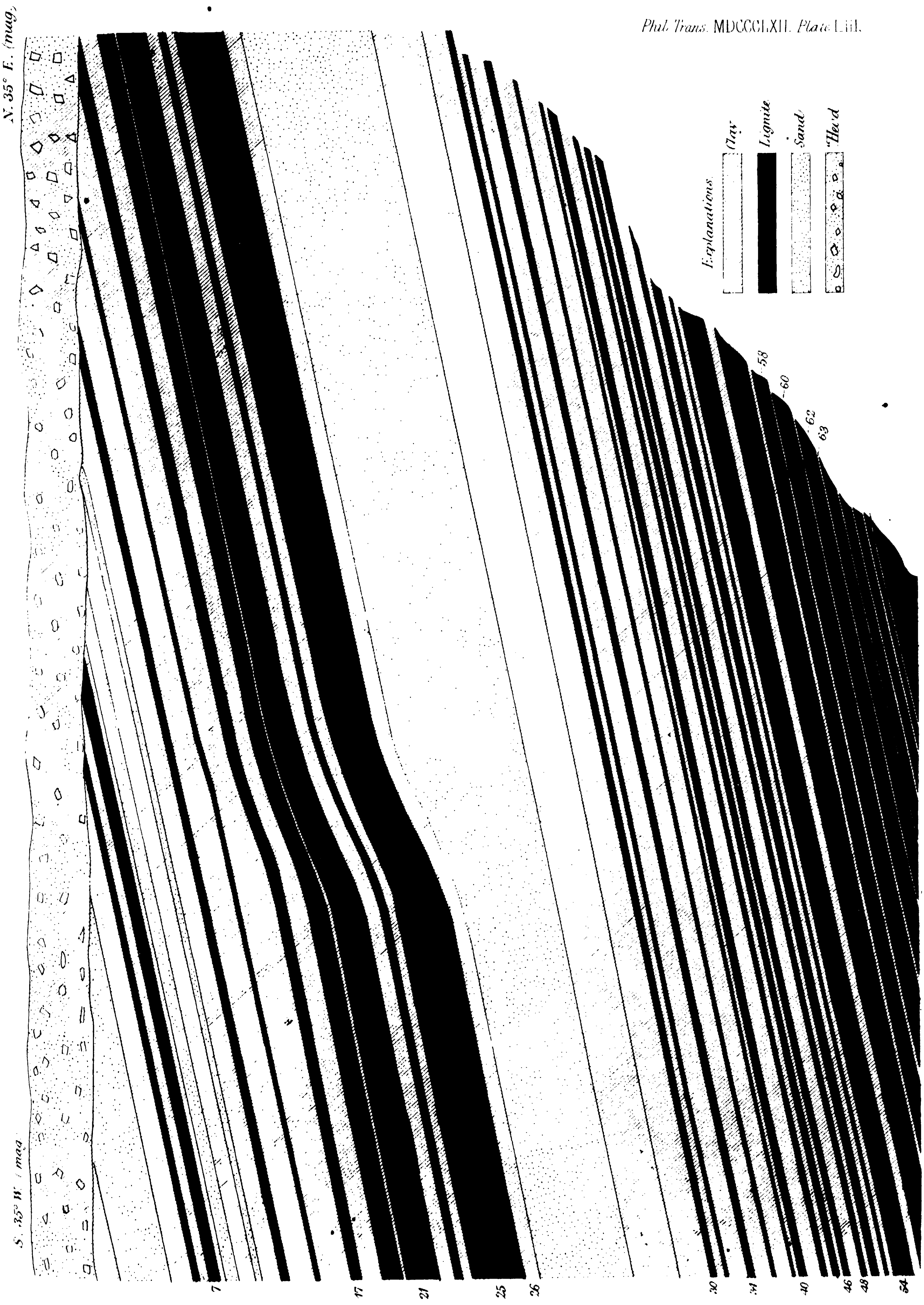


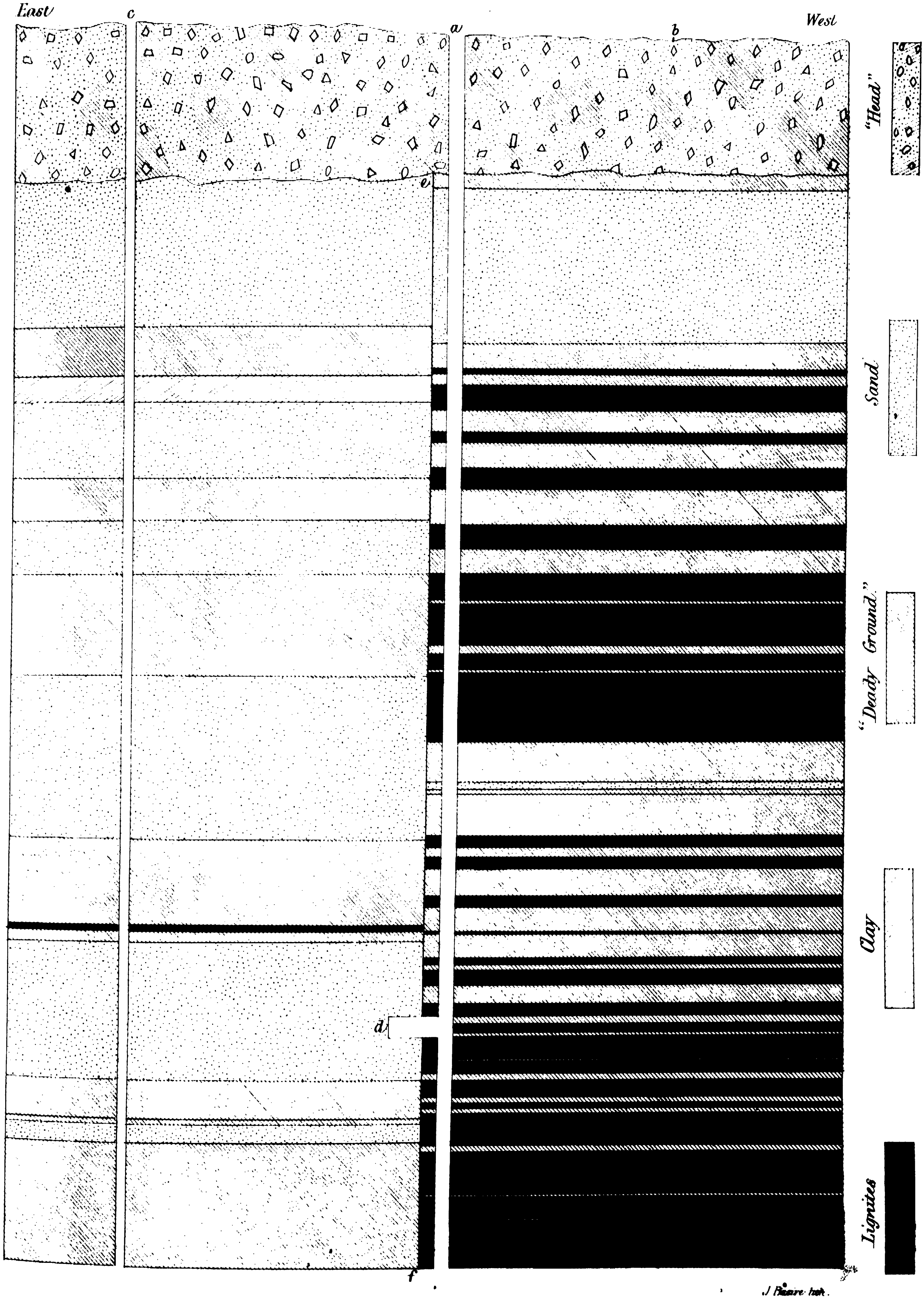
The dotted lines give the number of dots per minute in function of x (vide Appendix II.) & Table XVI.

J. Basire lith.


Chudleigh
Bovey River
Bovey Tracey
DARTMOOR
Coal Pit
Bovey Heathfield
CHUDLEIGH
Chudleigh Bridge
Teign
Black Snod
Kingsteignton
NEWTON
ASHBURTON
Kings Kerswell
Explanations.
Limestone
Devonian
Carboniferous
Granite
Greenstone
Altered Palæozoic Rocks
New Red Sandstone. White Lines...
Upper Green Sand. Faults.
Bovey deposit.

Phil. Trans. MDCCCLXII. Plate LIII.





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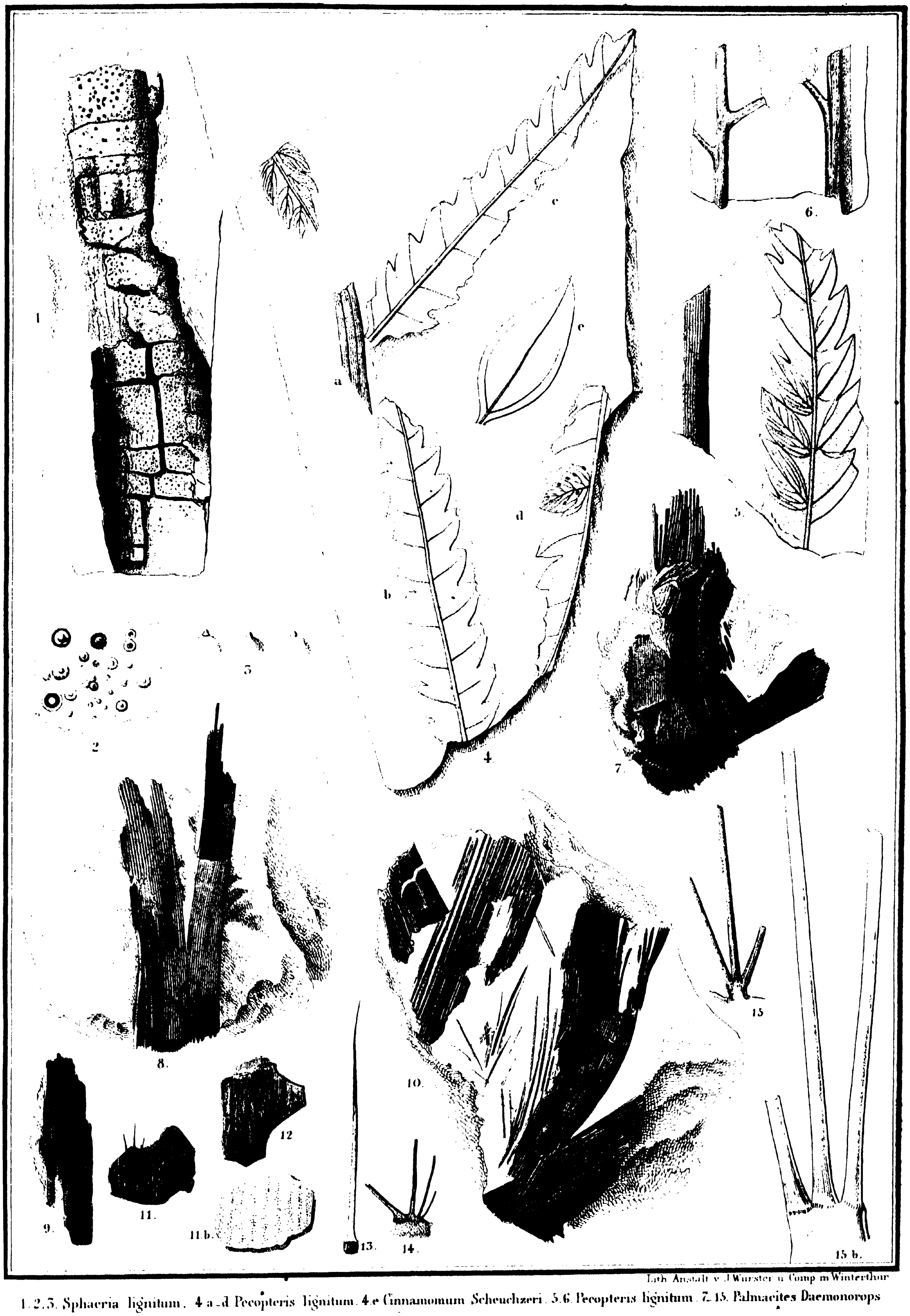
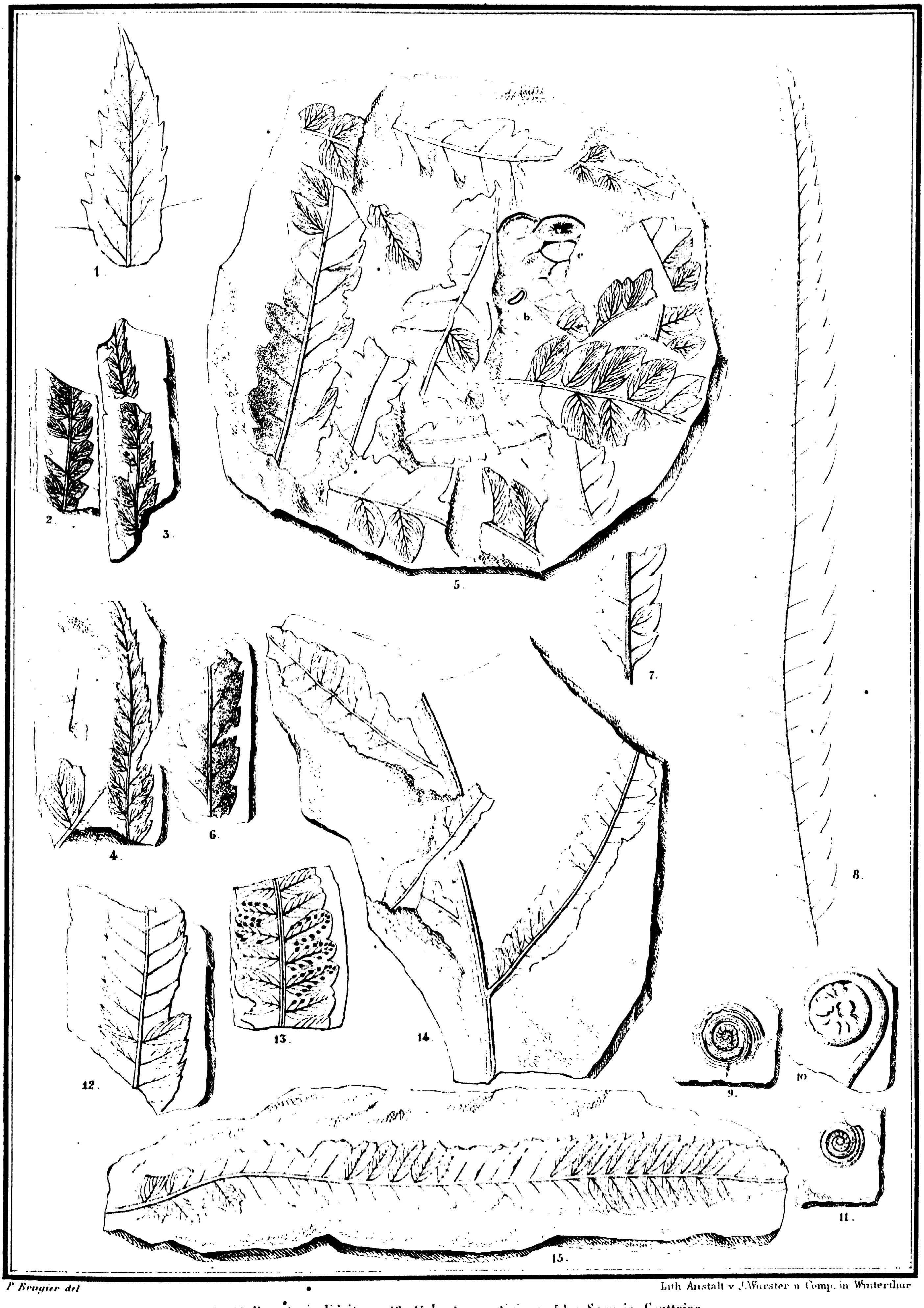


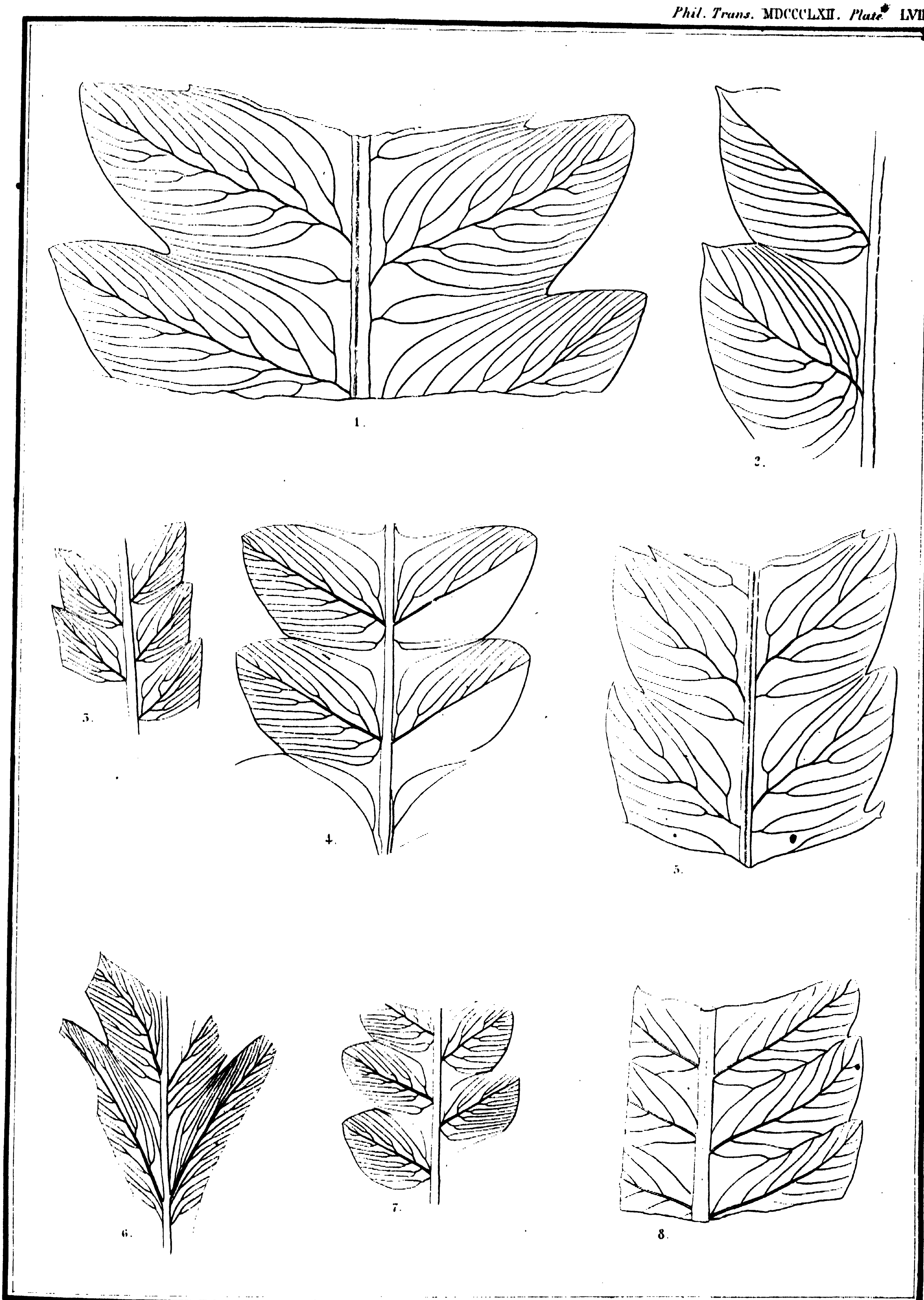
1. 2. 3. Sphaeria lignitum. 4 a–d Pecopteris lignitum. 4 e Cinnamomum Scheuchzeri. 5. 6. Pecopteris lignitum. 7–15. Palmacites Daemonorops



P. Brugier del.

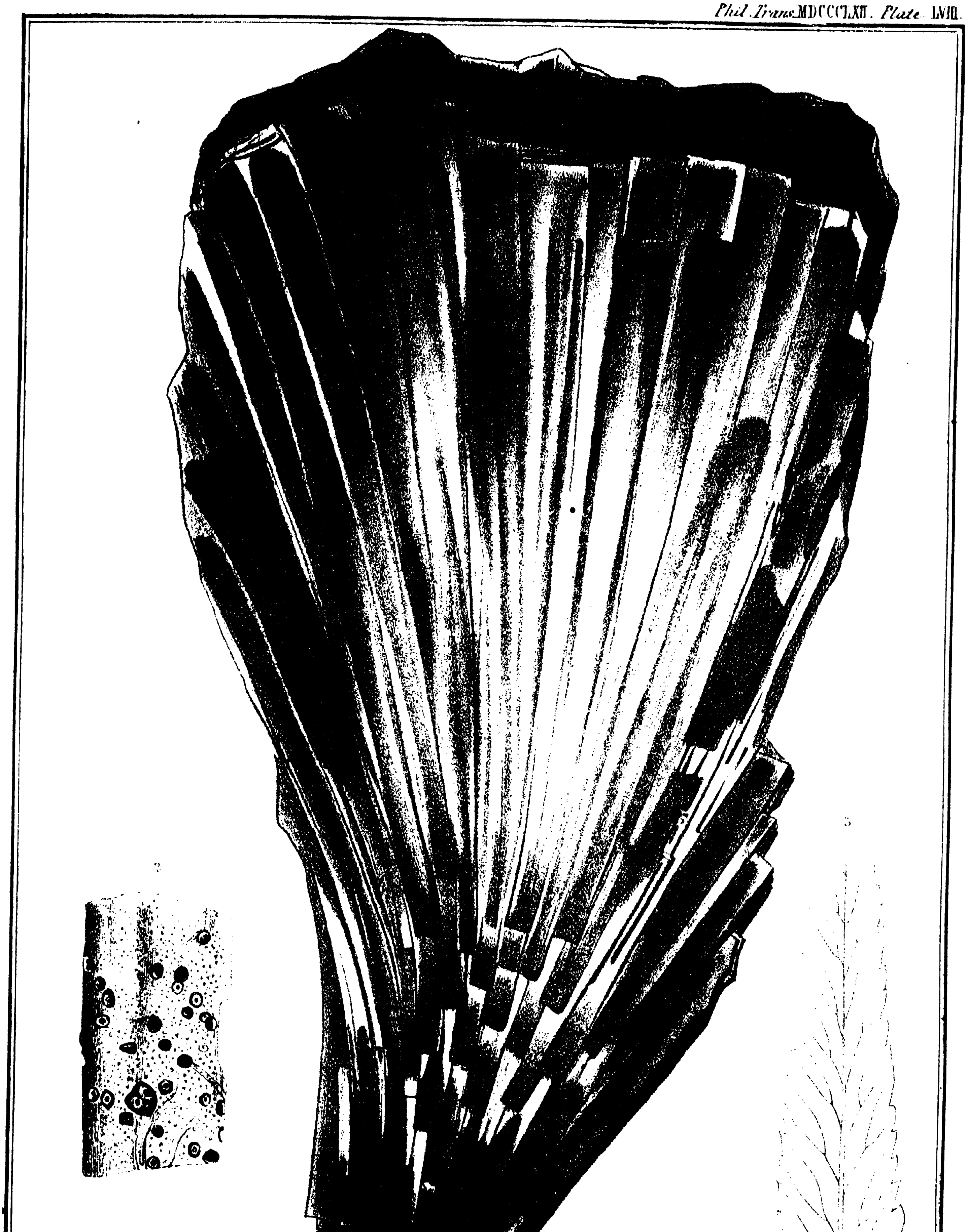
Lith Anstalt v. J. Wurster u. Comp. in Winterthur

1 – 11. Pecopteris lignitum. 12 – 15. Lastraea stiriaca. 5. b c. Sequoia Couttsiae.



Lith. Anstalt v. J. Wurster u. Comp. in Winterthur

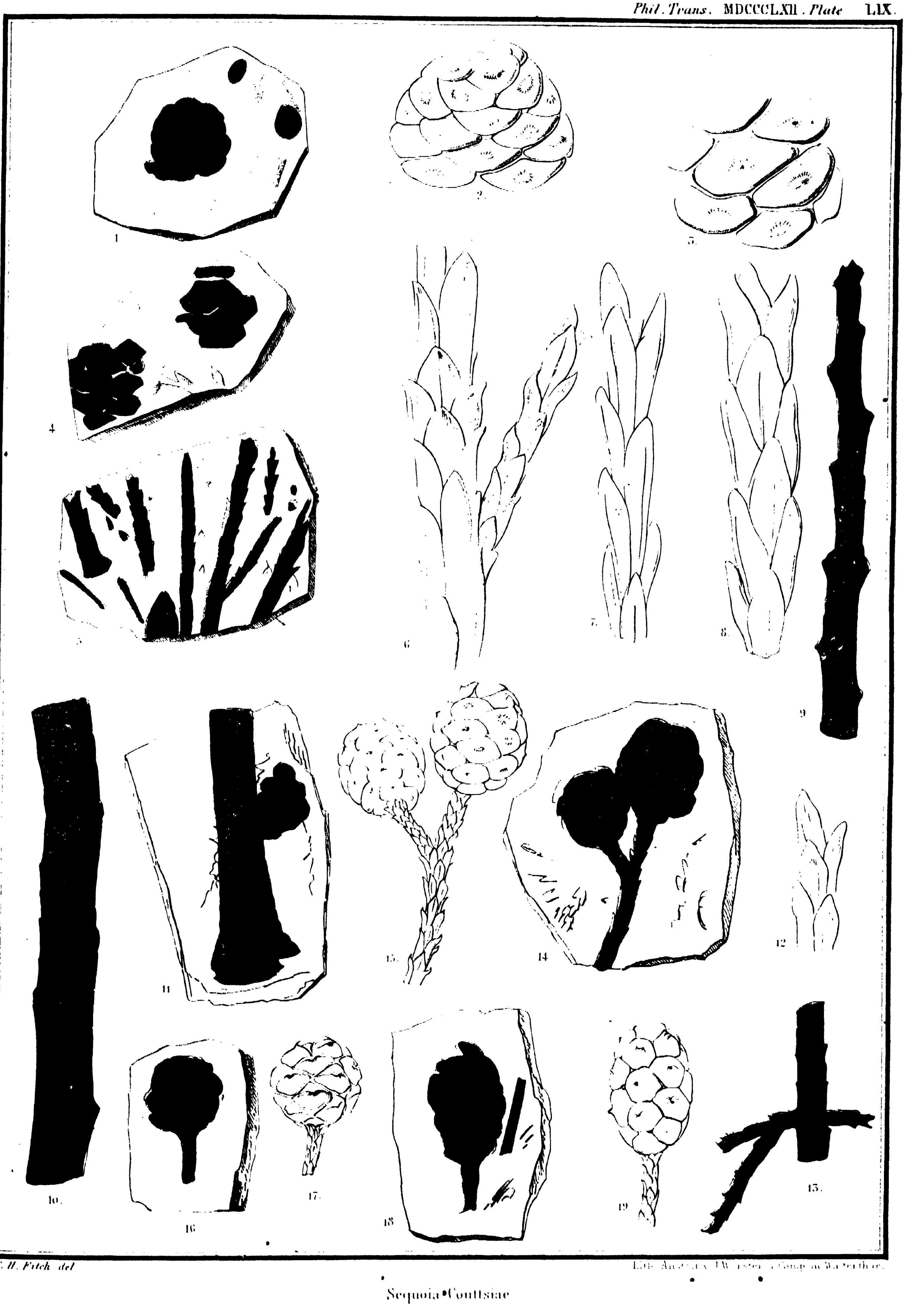
1–7. Pecopteris lignitum. 8. Lastraea stiriaca.



W. H. Fitch del.

Lith. Anstalt v. J. Wurster u Comp. in Winterthur

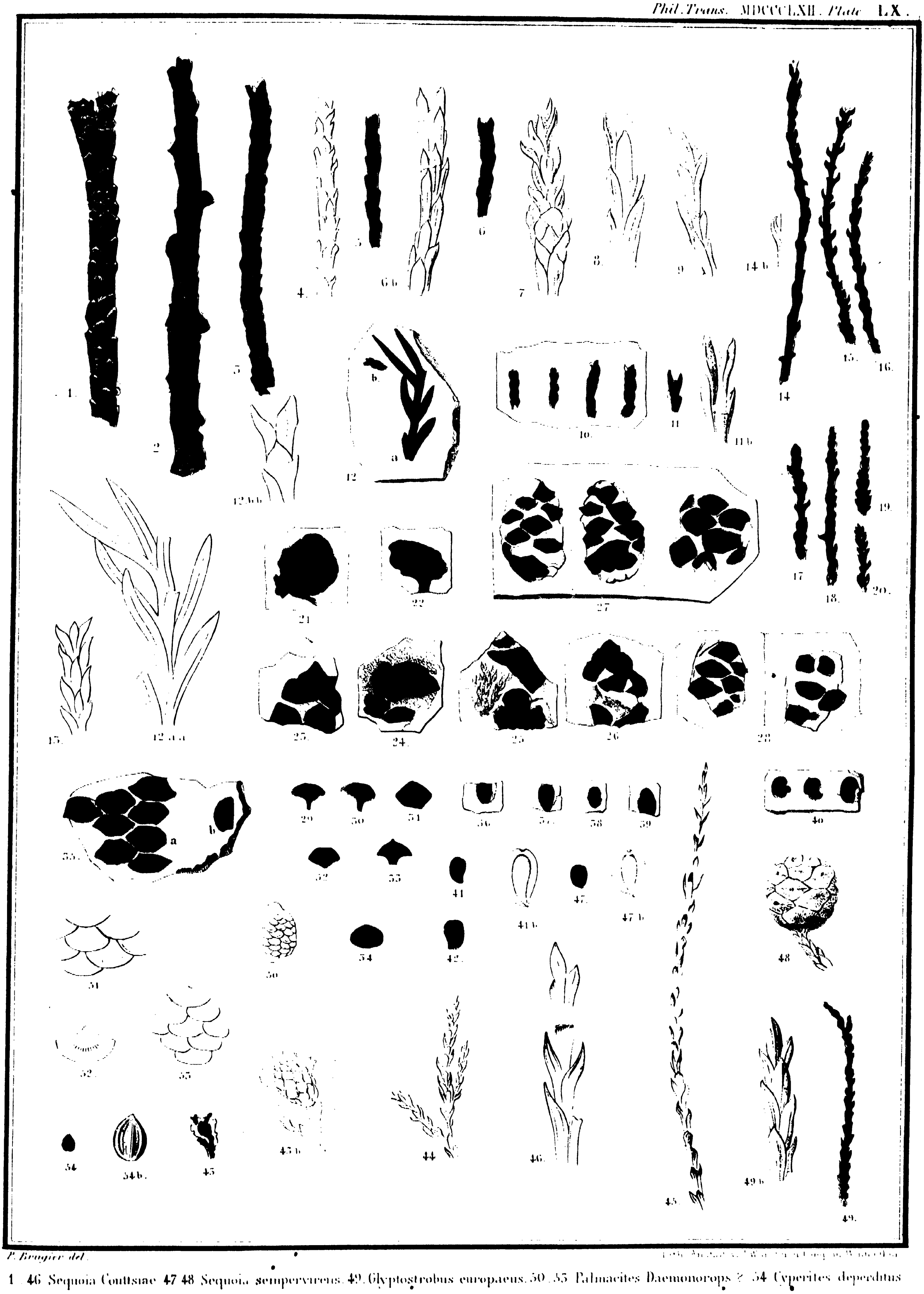
1, 2 Pecopteris lignitum. 3. Pecopteris Hookeri.



W. H. Fitch del.

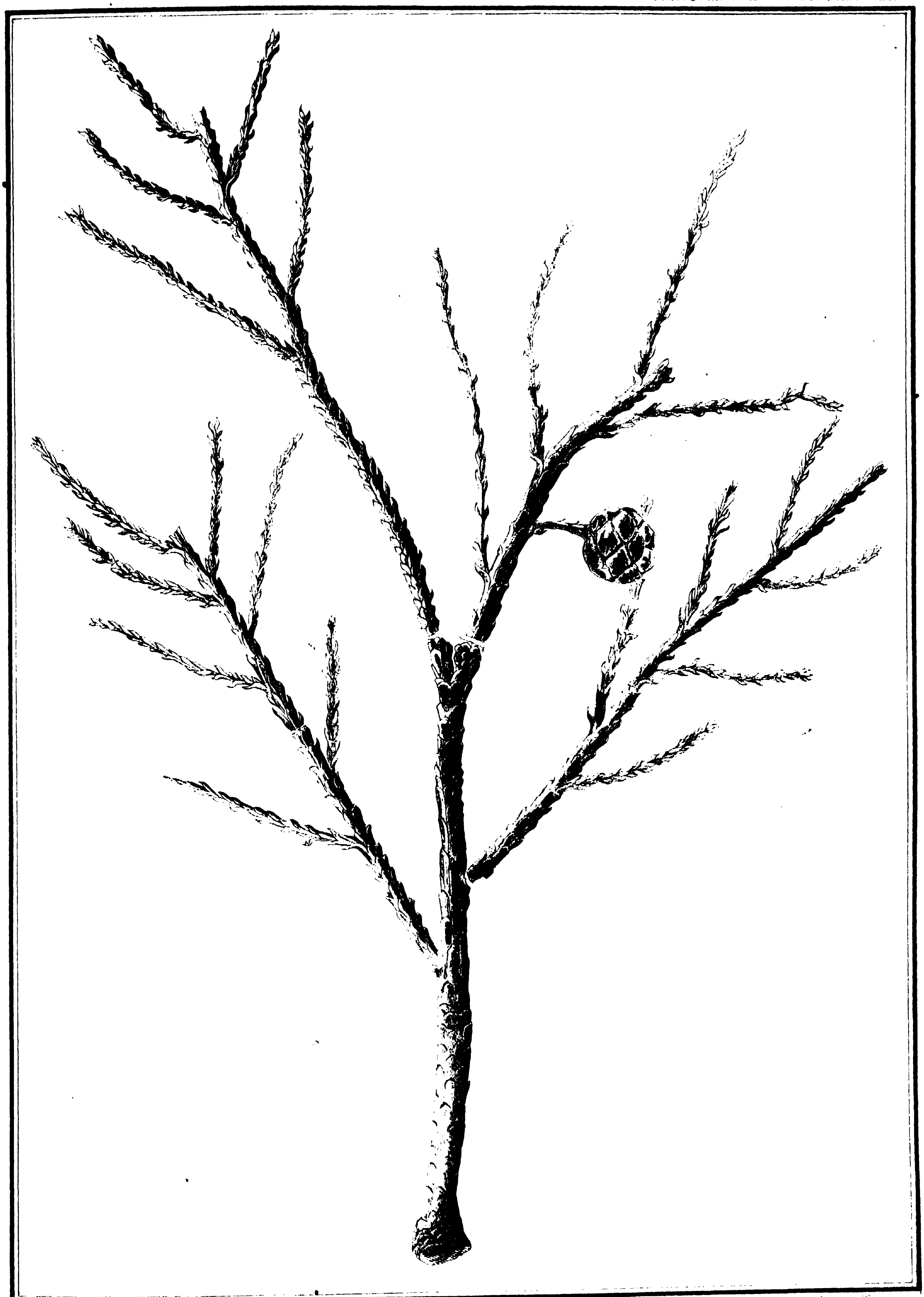
Lith. Anstalt v. J. Wurster & Comp. in Winterthur.

Sequoia Couttsiae



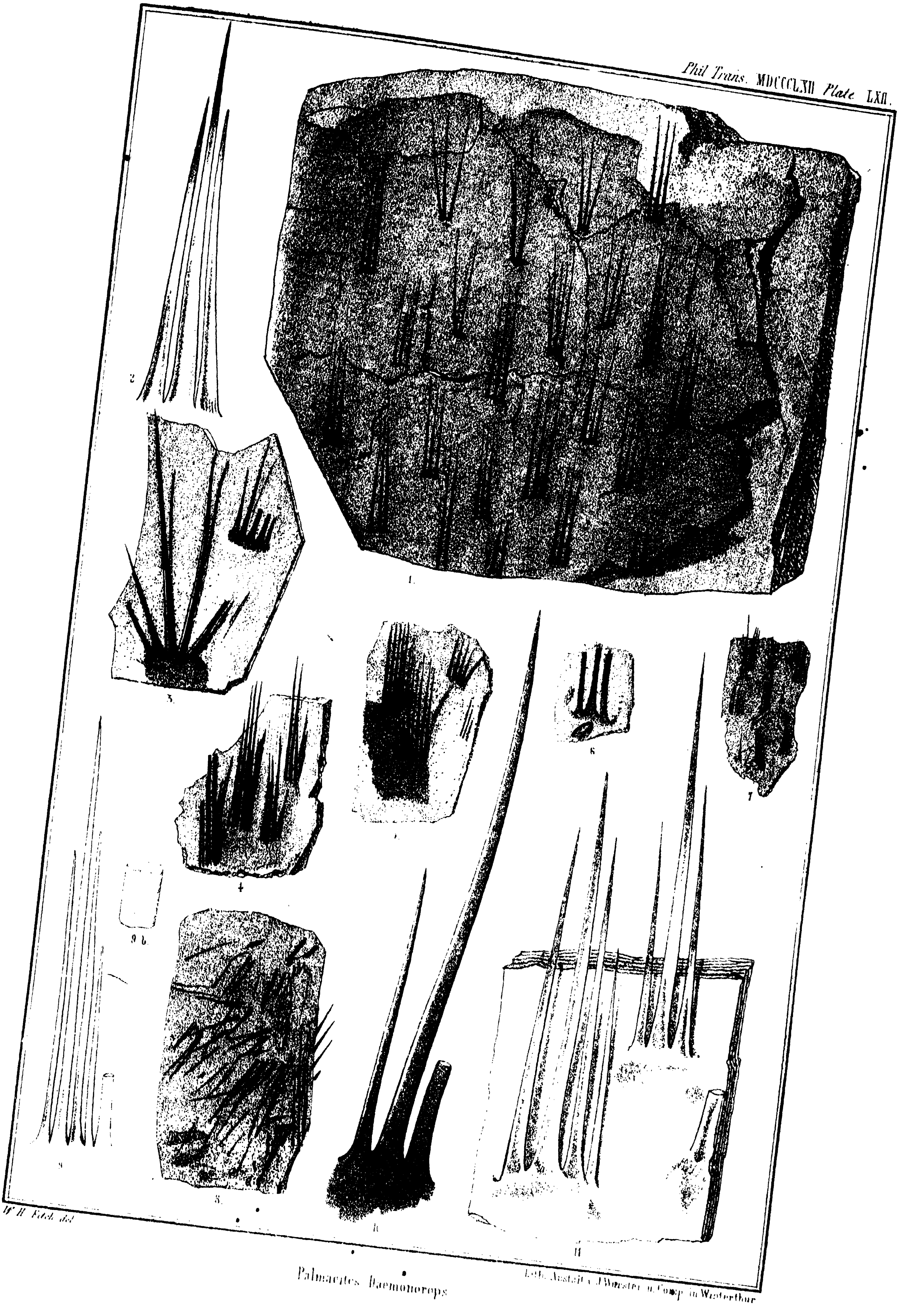
P. Bougier del.

1 . 46 Sequoia Couttsiae 47 48 Sequoia sempervirens. 49. Glyptostrobus europaeus. 50 . 53 Palmacites Daemonorops ? 54 Cyperites deperditus



P. Brugier del.

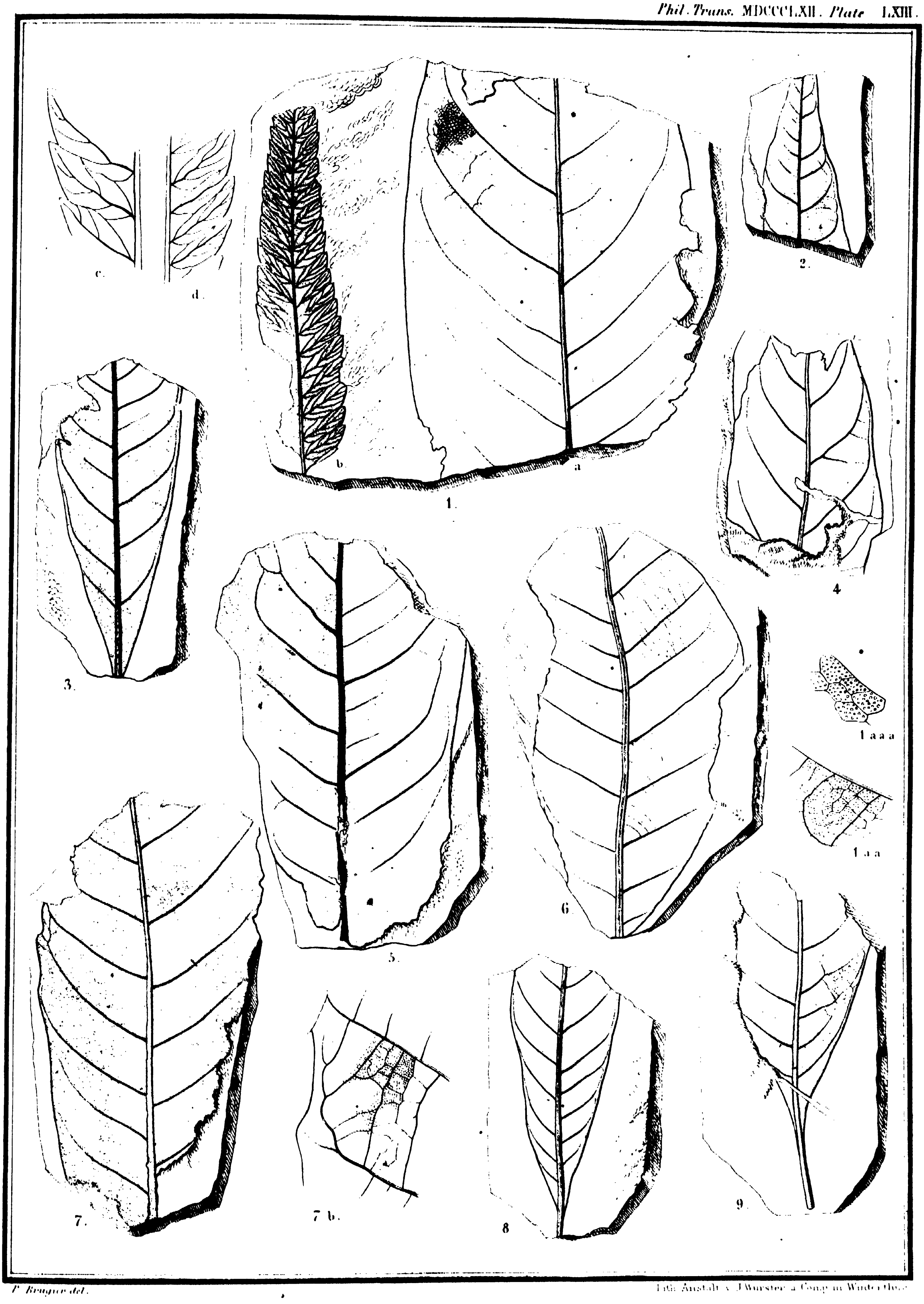
Sequoia Couttsiae



W. H. Fitch del.

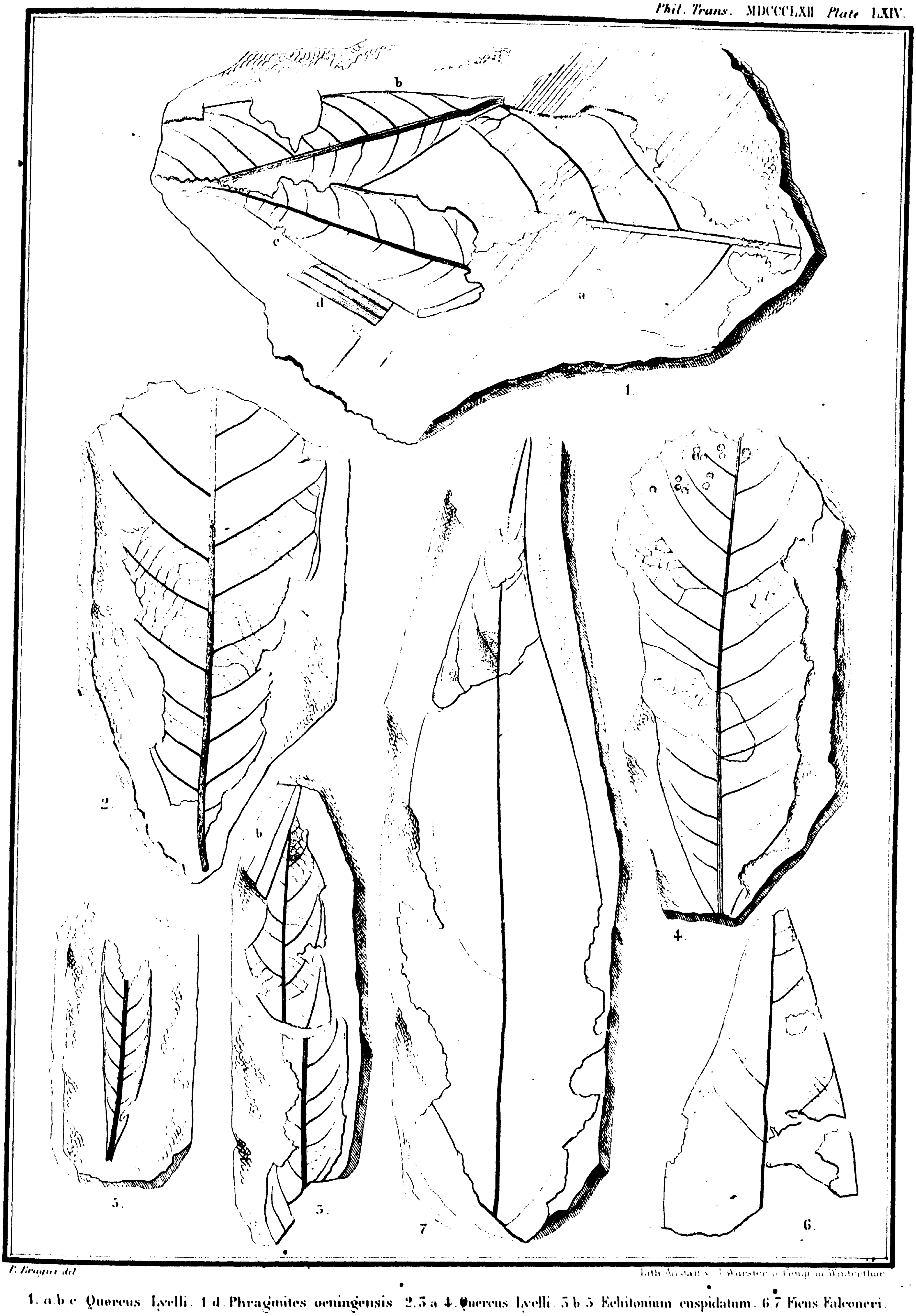
Palmacites Daemonorops

Lith. Anstalt v. J. Wurster u. Comp. in Winterthur



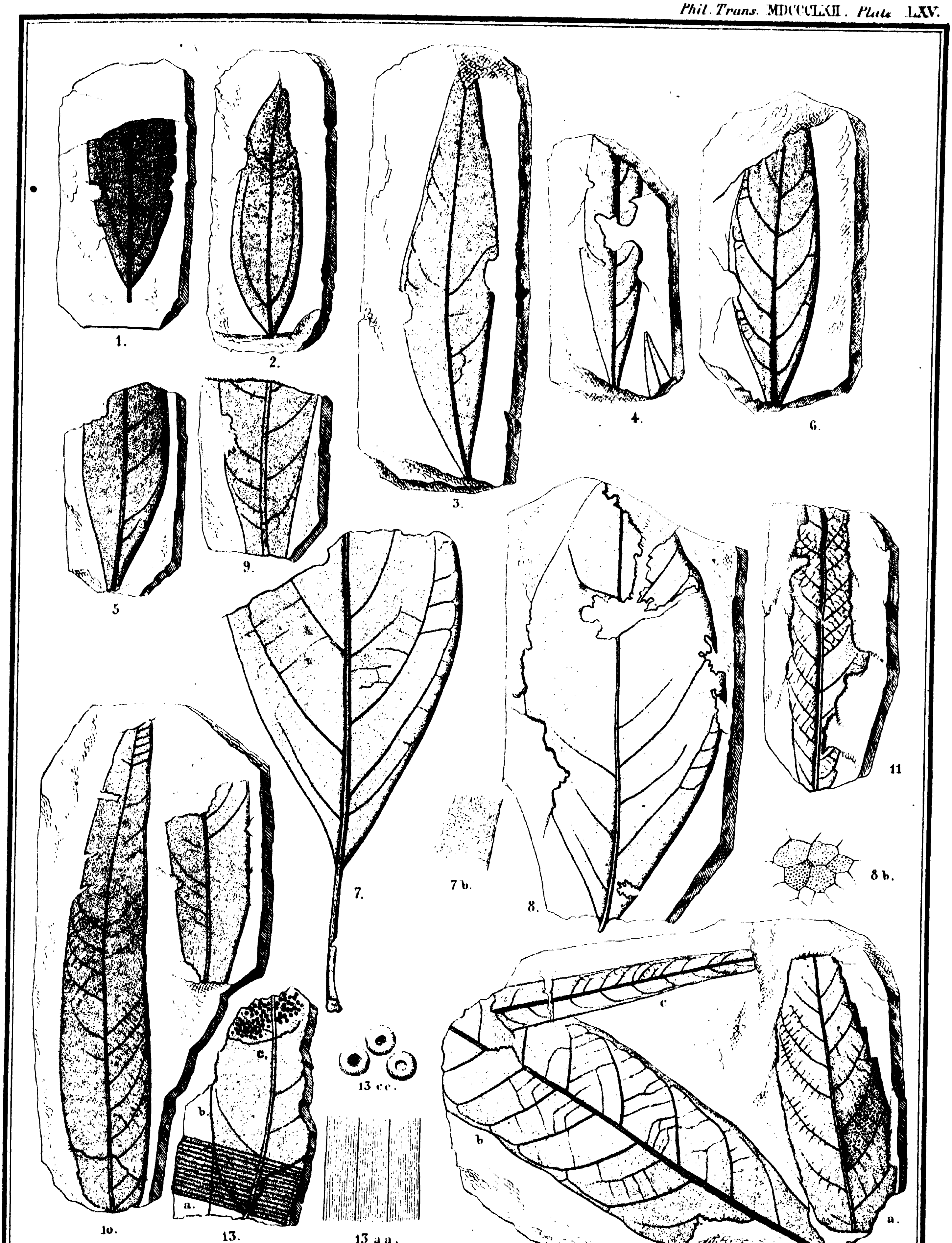
P. Bruguer del.   Lith. Anstalt v. J. Wurster & Comp. in Winterthur.

1. b. d. Lastraea Bunburii. 1 a Ficus Falconeri 2..9 Quercus Lyelli.



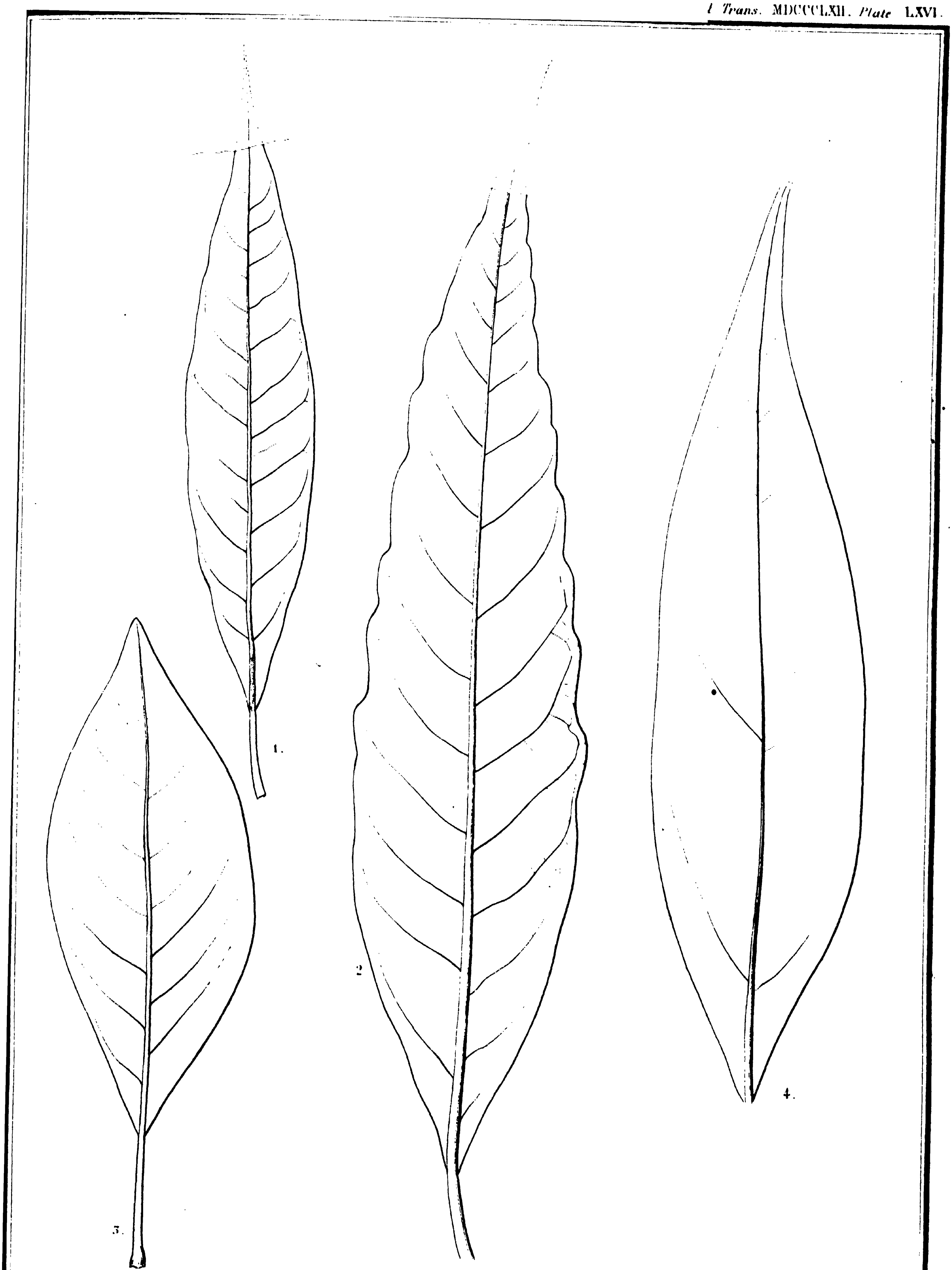
F. Bruguer del. Lith Anstalt v. J. Wurster u Comp. in Winterthur

1. a. b c Quercus Lyelli. 1 d. Phragmites oeningensis 2. 3 a 4. Quercus Lyelli. 3 b 5 Echitonium cuspidatum. 6. 7 Ficus Falconeri.



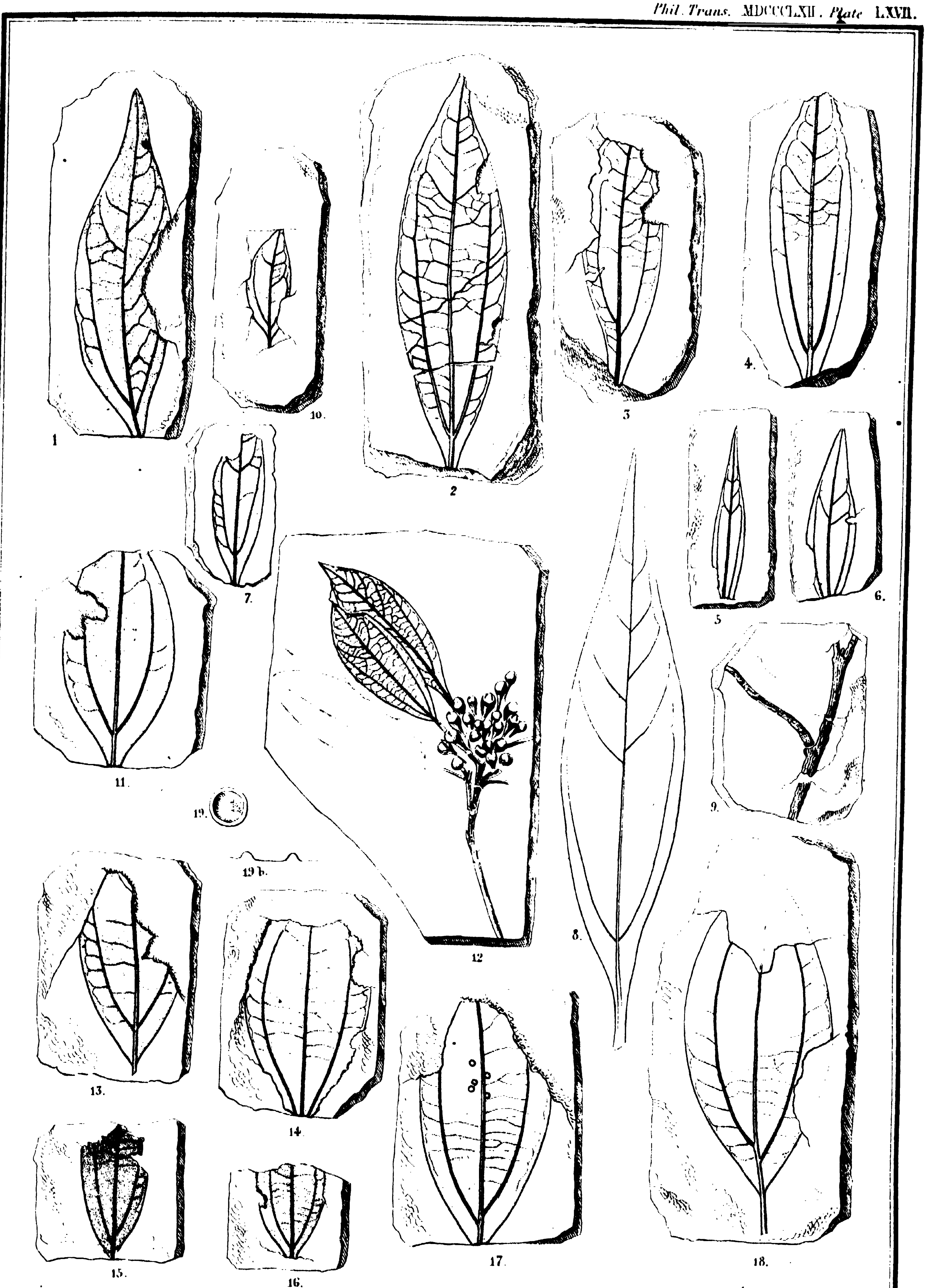
P. Bruqier del. Lith. Anstalt v. J. Wurster u. Comp. in Winterthu

1. 2. Daphnogene Ungeri 3. 4. 5. Ficus eucalyptoides. 6. Laurus primigenia 7. 8. Ficus Pengellii. 9–11. Dryandroides laevigata. 12.a Dryandroides hakeaefolia. 12.b. Quercus Lyelli 12.c. Echitonium cuspidatum. 13 a. Phragmites oeningensis ? 13. b. Dryandroides laevigata. 13.c. Sphaeria socialis.



Bromer del. [illegible]

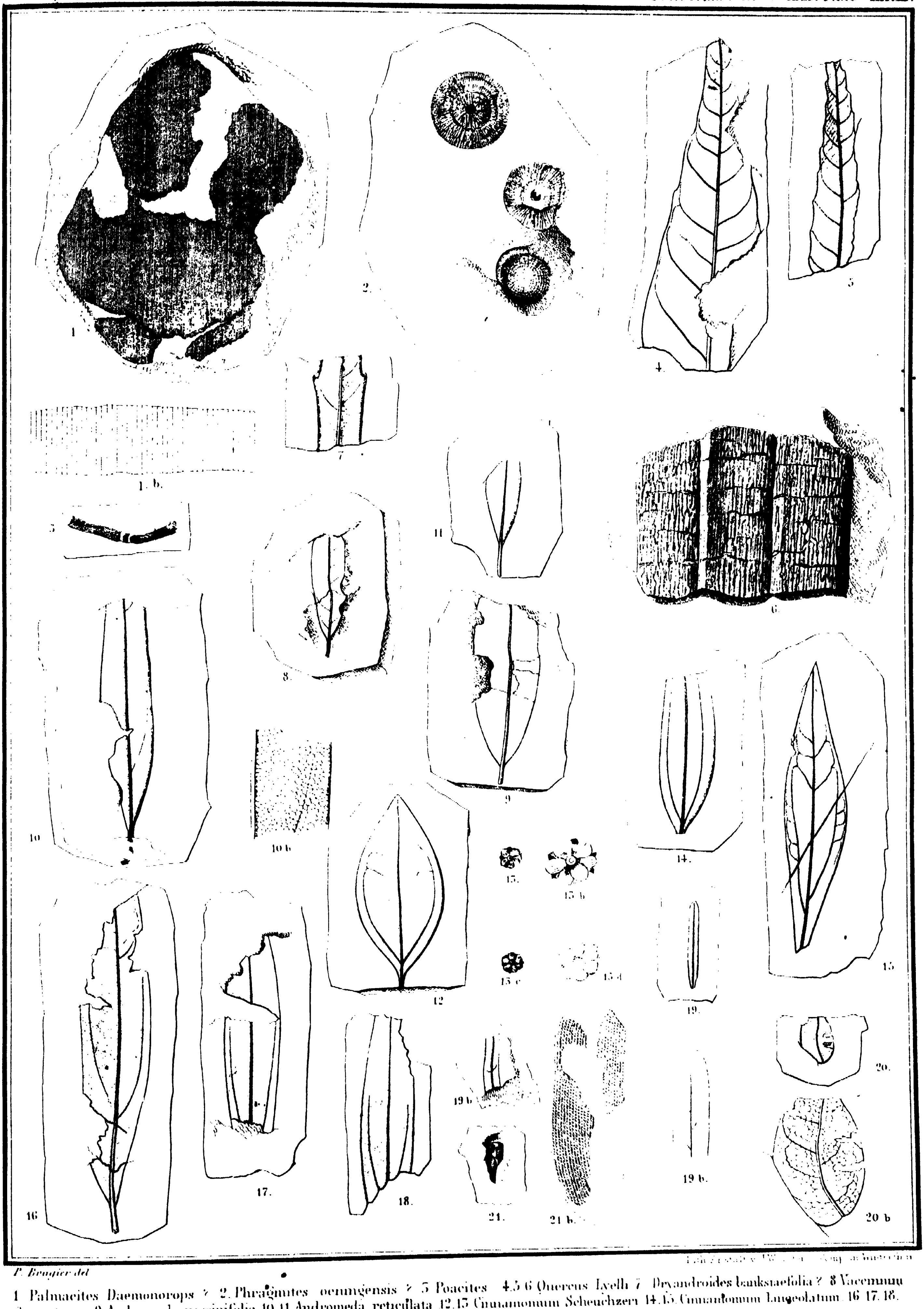
1. 2 Quercus Lyelli. 3. Ficus [illegible] 4. Ficus [illegible]



P. Bruger del.

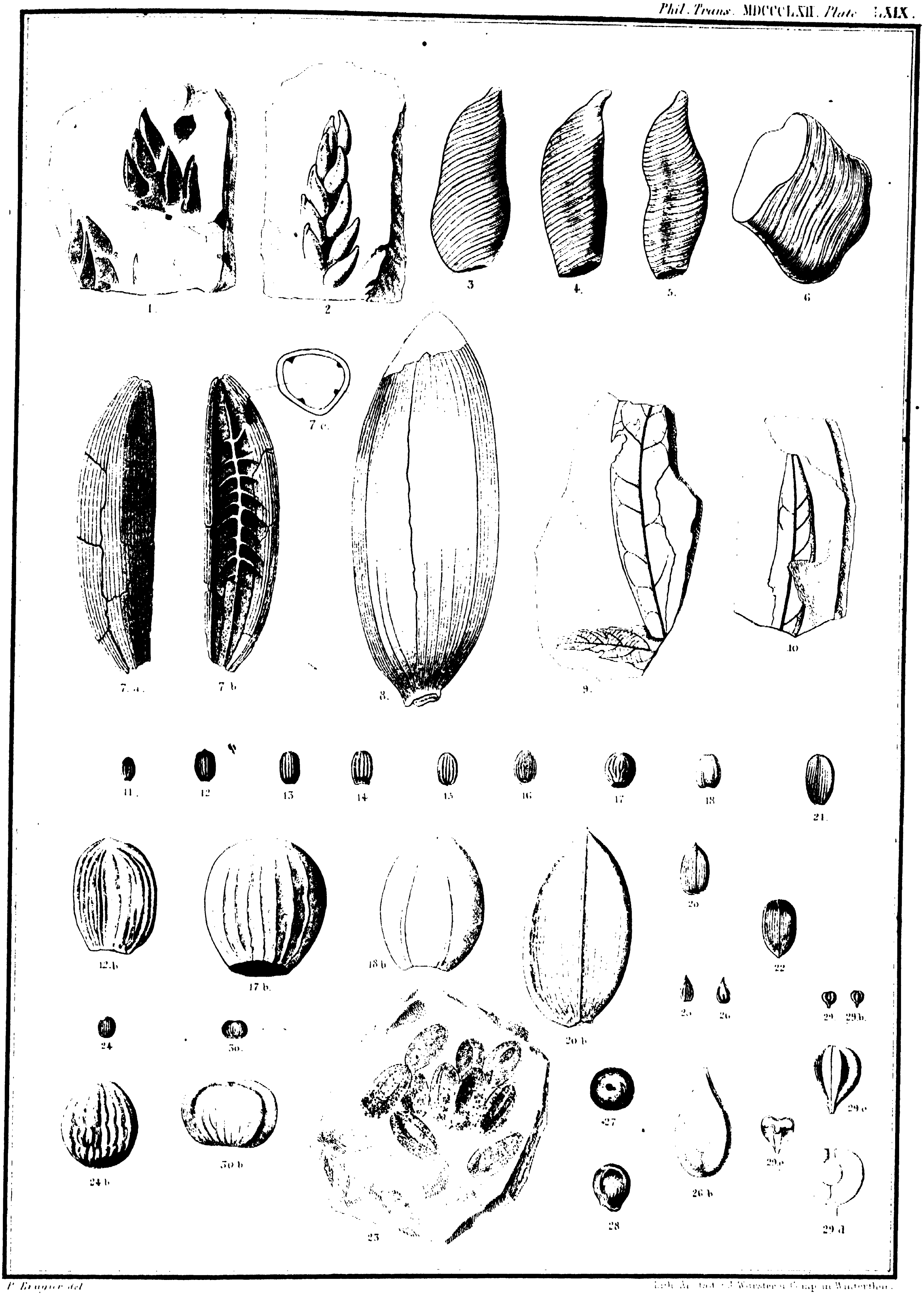
Lith. Anst. v. J. Wurster u. Comp. in Winterthur.

1 _ 8. Cinnamomum lanceolatum. 9 16. Cinnamomum Scheuchzeri. 17.18. Cinnamomum Rossmässleri. 19. Sclerotium Cinnamomi.



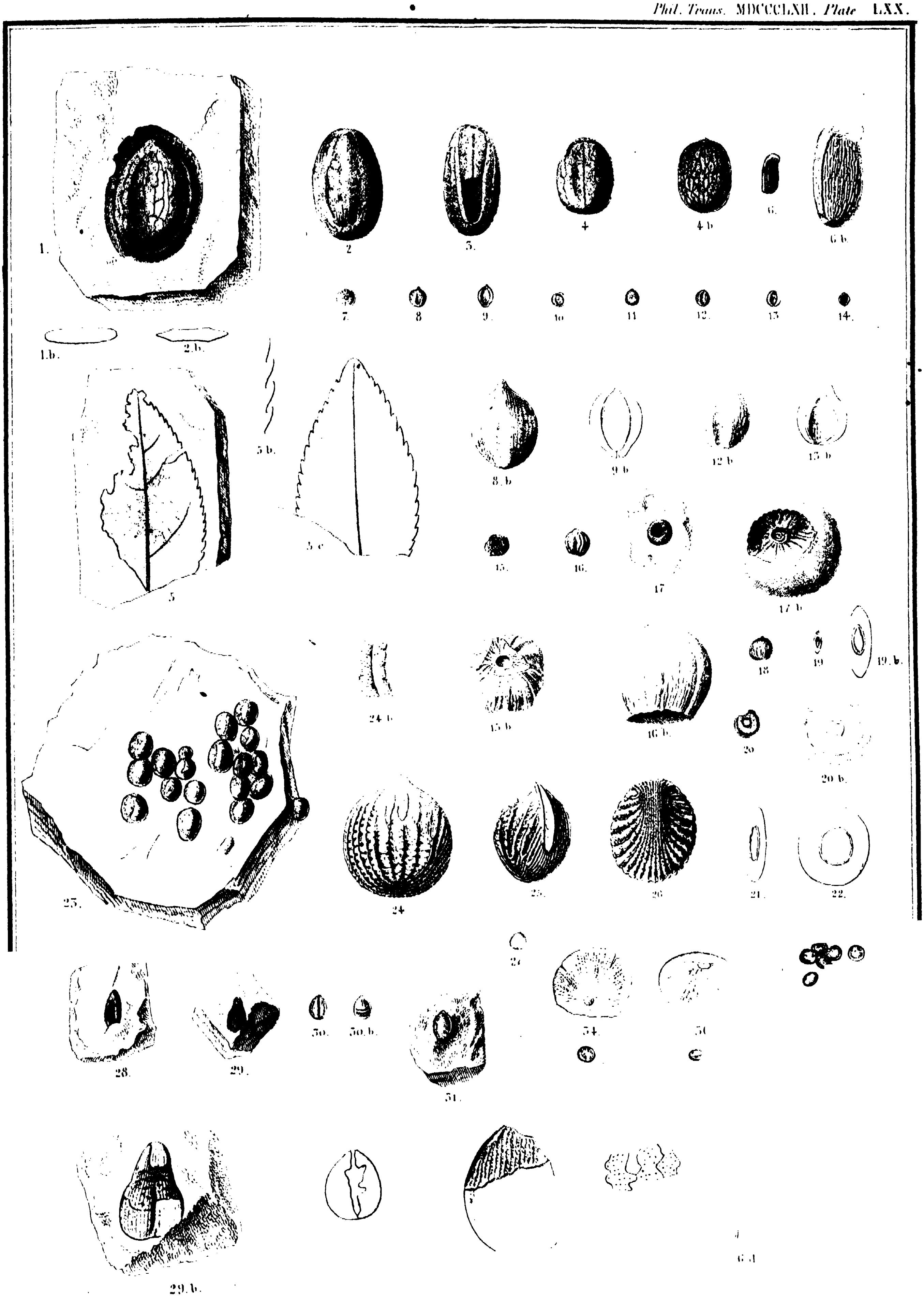
P. Brugier del

1 Palmacites Daemonorops ? 2. Phragmites oeningensis ? 3 Poacites 4.5 6 Quercus Lyelli 7 Dryandroides banksiaefolia ? 8 Vaccinium acheronticum. 9 Andromeda vaccinifolia. 10 11. Andromeda reticulata 12.13 Cinnamomum Scheuchzeri 14.15. Cinnamomum lanceolatum 16 17. 18. Eugenia haeringiana. 19. Celastrus pseudoilex. 20. Leguminosites areolatus. 21 Buprestites ? Falconeri.



P. Kruger del. Lith. Art. Inst. v. J. Wurster u. Comp. in Winterthur.

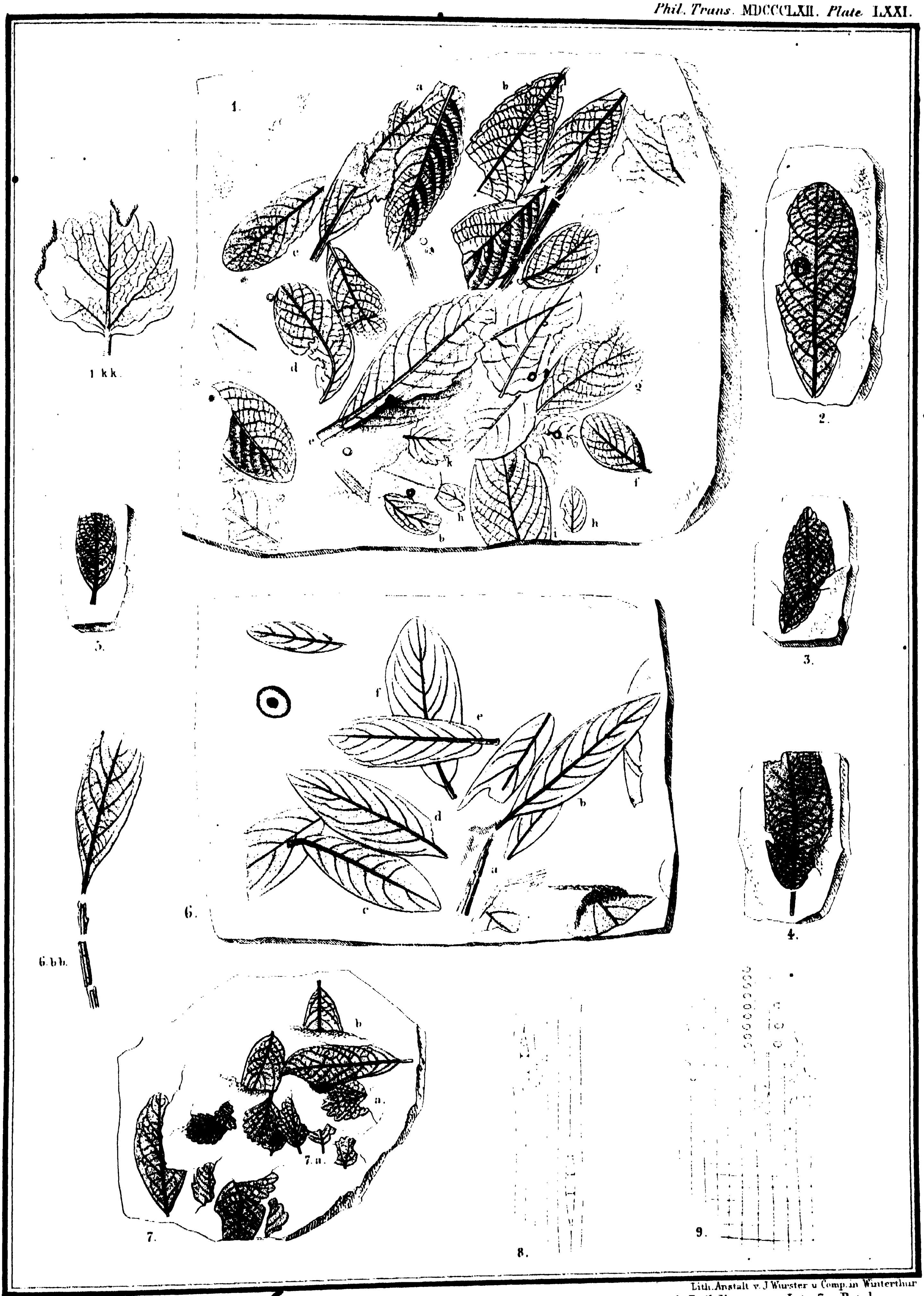
1–8. Gardenia Wetzleri. 9. 10. Eucalyptus oceanica? 11–17. Nyssa europaea. 18. Nyssa laevigata. 20–23. Nyssa striolata. 24. Nyssa microsperma. 25. 26. Vitis britannica. 27. 28. 29. Vitis Hookeri. 30. Carpolithes scutellatus.



P. Brugger del.

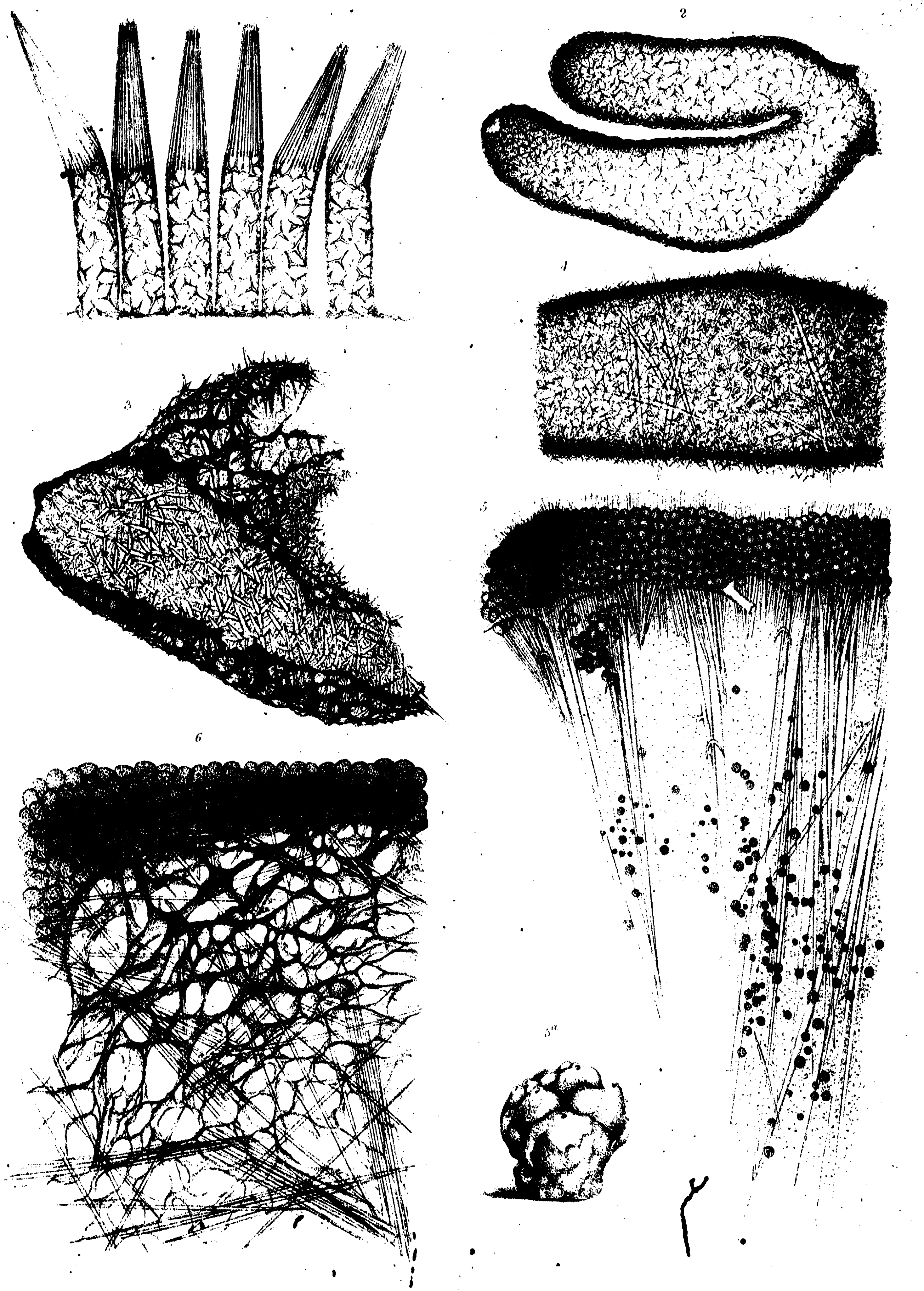
Lith. Anstalt v. J. Wurster u. Comp. in Winterthur

1–3. Anona devonica. 4. Anona cyclosperma. 5. Pterocarya denticulata ? 6. Carpolithes Websteri. 7–14. Carpolithes Boveianus. 15–23. Carpolithes nitens. 24–27. Carpolithes exaratus. 28. 29. Carpolithes vinaceus. 30. 31. Carpolithes lividus. 32–37. Nymphaea Doris.



Lith. Anstalt v. J. Wurster u Comp. in Winterthur

1. k. Betula nana. 1. a. b. Salix cinerea. 1. c–h. Salix repens L.? 2. 3. Salix cinerea 4. 5. Salix sp. 6. 7. Salix repens L.? 7. a Betula nana.


2
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6
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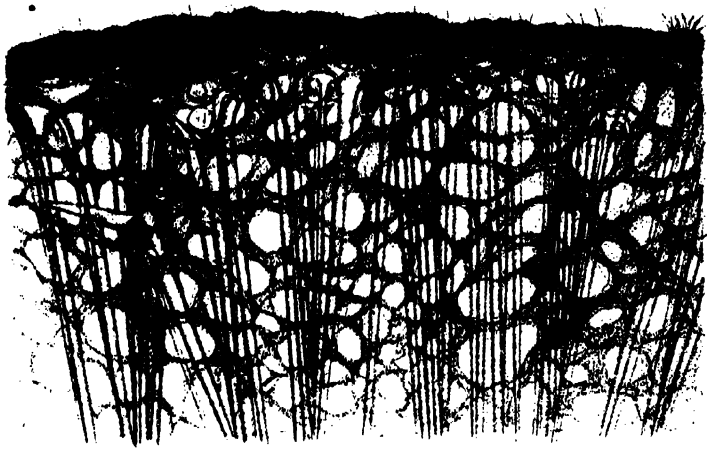

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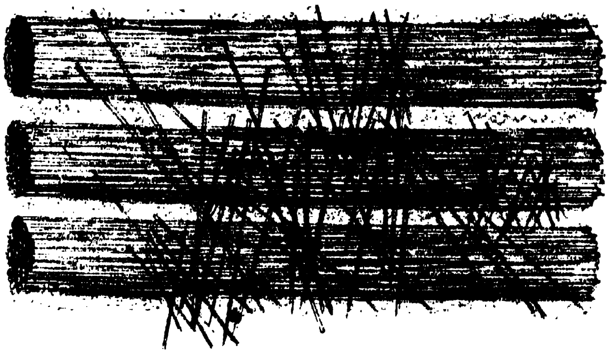

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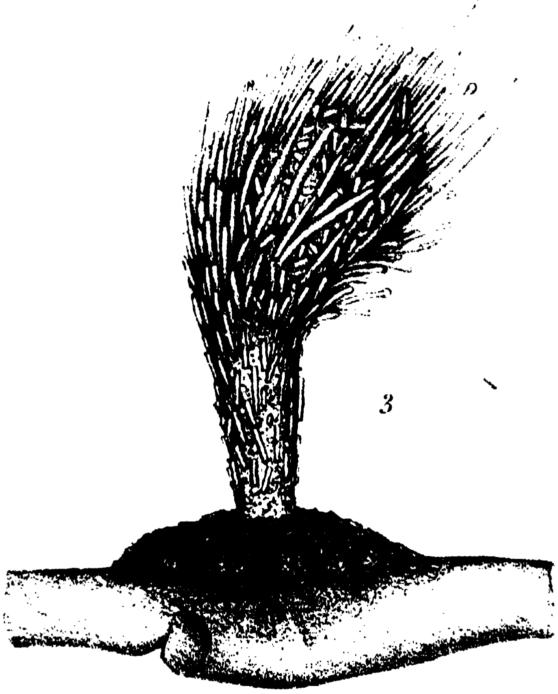

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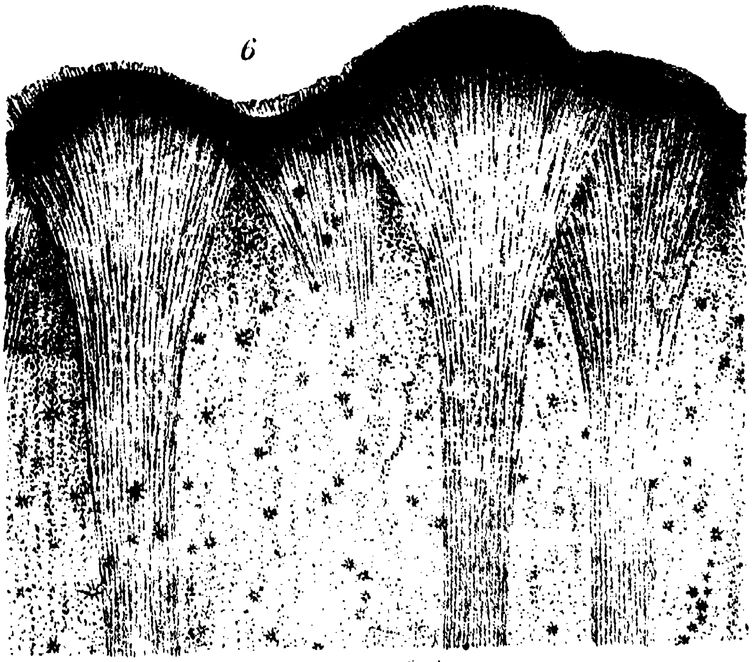
6



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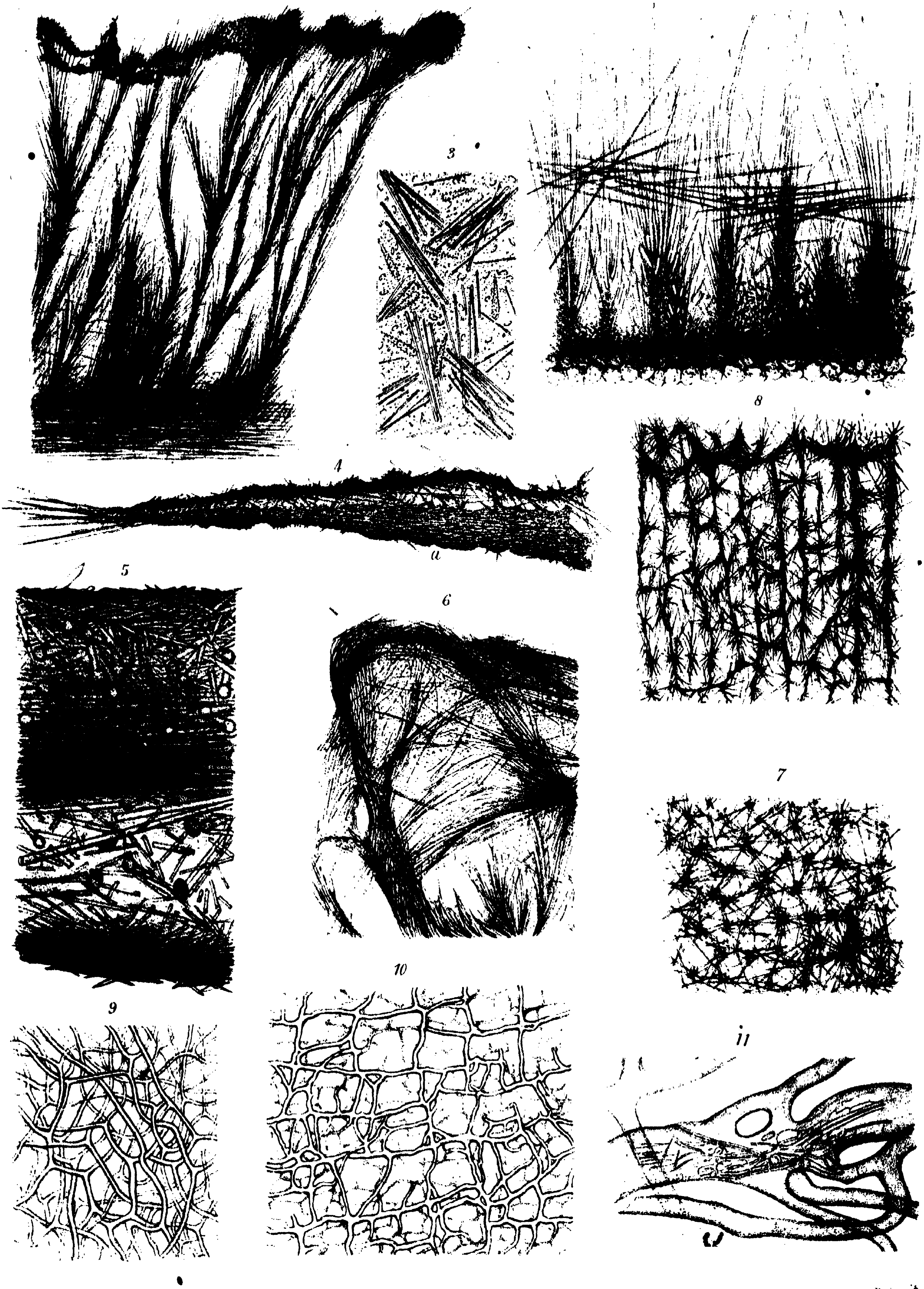
Drawn & lithographed by W. Lens Aldous.

J. Basire imp.



Drawn & Lithographed by W. Lens Aldous

...Basire imp.