Definition 1:

$$\mathbf{x + 0} \\ = \mathbf{x}$$

Definition 2:

$$\mathbf{x + Sy} \\ = \mathbf{S(x + y)}$$

Definition 3:

$$\mathbf{x \cdot 0} \\ = \mathbf{0}$$

Definition 4:

$$\mathbf{x \cdot Sy} \\ = \mathbf{x \cdot y + x}$$

Lemma 5:

$$\mathbf{0 + x} \\ = \mathbf{x}$$

Proof by induction on x:
Base case:
$0 + 0$
$= 0$ by Def. 1
Inductive case:
$0 + Sx$
$= S(0 + x)$ by Def. 2
$= Sx$ by I.H.

Lemma 6:

$$\mathbf{Sx + y} \\ = \mathbf{S(x + y)}$$

Proof by induction on y:
Base case:
$Sx + 0$
$= Sx$ by Def. 1
$= S(x + 0)$ by Def. 1
Inductive case:
$Sx + Sy$
$= S(Sx + y)$ by Def. 2
$= ss(x + y)$ by I.H.
$= S(x + Sy)$ by Def. 2

Lemma 8:

$$\mathbf{(x + y) + z} \\ = \mathbf{x + (y + z)}$$

Proof by induction on z:
Base case:
$(x + y) + 0$
$= x + y$ by Def. 1
$= x + (y + 0)$ by Def. 1
Inductive case:
$(x + y) + sz$
$= S((x + y) + z)$ by Def. 2
$= S(x + (y + z))$ by I.H.
$= x + S(y + z)$ by Def. 2
$= x + (y + sz)$ by Def. 2

Lemma 10:

$$\mathbf{0 \cdot x} \\ = \mathbf{0}$$

Proof by induction on x:
Base case:
$0 \cdot 0$
$= 0$ by Def. 3
Inductive case:
$0 \cdot Sx$
$= 0 \cdot x + 0$ by Def. 4
$= 0 + 0$ by I.H.
$= 0$ by Def. 1

Lemma 7:

$$\mathbf{x + y} \\ = \mathbf{y + x}$$

Proof by induction on y:
Base case:
$x + 0$
$= x$ by Def. 1
$= 0 + x$ by Lem. 5
Inductive case:
$x + Sy$
$= S(x + y)$ by Def. 2
$= S(y + x)$ by I.H.
$= Sy + x$ by Lem. 6

Lemma 9:

$$\mathbf{x \cdot (y + z)} \\ = \mathbf{x \cdot y + x \cdot z}$$

Proof by induction on z:
Base case:
$x \cdot (y + 0)$
$= x \cdot y$ by Def. 1
$= x \cdot y + 0$ by Def. 1
$= x \cdot y + x \cdot 0$ by Def. 3
Inductive case:
$x \cdot (y + sz)$
$= x \cdot S(y + z)$ by Def. 2
$= x \cdot (y + z) + x$ by Def. 4
$= (x \cdot y + x \cdot z) + x$ by I.H.
$= x \cdot y + (x \cdot z + x)$ by Lem. 8
$= x \cdot y + x \cdot sz$ by Def. 4

Lemma 11:

$$\mathbf{Sx \cdot y} \\ = \mathbf{x \cdot y + y}$$

Proof by induction on y:
Base case:
$Sx \cdot 0$
$= 0$ by Def. 3
$= 0 + 0$ by Def. 1
$= x \cdot 0 + 0$ by Def. 4
Inductive case:
$Sx \cdot Sy$
$= Sx \cdot y + Sx$ by Def. 4
$= (x \cdot y + y) + Sx$ by I.H.
$= S((x \cdot y + y) + x)$ by Def. 2
$= S(x \cdot y + (y + x))$ by Lem. 8
$= S(x \cdot y + (x + y))$ by Lem. 7
$= S((x \cdot y + x) + y)$ by Lem. 8
$= (x \cdot y + x) + Sy$ by Def. 2
$= x \cdot Sy + Sy$ by Def. 4

Lemma 13:

$$\mathbf{(x \cdot y) \cdot z} \\ = \mathbf{x \cdot (y \cdot z)}$$

Proof by induction on z:
Base case:
$(x \cdot y) \cdot 0$
$= 0$ by Def. 3
$= x \cdot 0$ by Def. 3
$= x \cdot (y \cdot 0)$ by Def. 3
Inductive case:
$(x \cdot y) \cdot sz$
$= (x \cdot y) \cdot z + x \cdot y$ by Def. 4
$= x \cdot (y \cdot z) + x \cdot y$ by I.H.
$= x \cdot (y \cdot z + y)$ by Lem. 9
$= x \cdot (y \cdot sz)$ by Def. 4

Lemma 12:

$$\mathbf{x \cdot y} \\ = \mathbf{y \cdot x}$$

Proof by induction on y:
Base case:
$x \cdot 0$
$= 0$ by Def. 3
$= 0 \cdot x$ by Lem. 10
Inductive case:
$x \cdot Sy$
$= x \cdot y + x$ by Def. 4
$= y \cdot x + x$ by I.H.
$= Sy \cdot x$ by Lem. 11

Legend:
$S(x)$ Successor of x
Def. Definition
Lem. Lemma
I.H. Induction Hypothesis
Binding Priorities:
$Sx \cdot y + z$ denotes $((S(x)) \cdot y) + z$
Used Induction Scheme:
If $P(0)$
and $P(x)$ always implies $P(Sx)$,
then always $P(x)$.

Red arrow: use of lemma
Definition-uses omitted