THE TEACHING OF ARITHMETIC

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PREFACE

The *Teachers College Record* for March, 1903, contained an article on Mathematics in the Elementary School by the author's colleague Professor McMurry and himself. This number, however, has long since been out of print, and as a result of this fact it was thought best in the autumn of 1908 to prepare a new number of the *Record* on The Teaching of Arithmetic. This was done by the author, and the article appeared in January, 1909. Although it was thought that the unusually large edition was sufficient for all demands for some years to come, it was exhausted within a few weeks, and it became necessary to print the article in book form. In spite of the fact that the work was originally written in a popular style, to the end that it might be read by those who have no more interest in mathematics than in the various other subjects of the curriculum, it has been thought best to make but a few changes in arranging for its publication in the present form. Intended as it is for those who are teaching or supervising the work in arithmetic in the elementary schools, it would hardly serve its purpose if it departed widely from the practical and entered the domain of pure theory. As between influencing the few or the many on a topic of such general interest it has been thought better to adopt the latter course, and to prepare a book that might have place in educational reading circles generally and serve as a basis for the work in the class-room for the training of teachers.

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CHAPTER I

THE HISTORY OF THE SUBJECT

Of all the sciences, of all the subjects generally taught in the common schools, arithmetic is by far the oldest. Long before man had found for himself an alphabet, long before he first made rude ideographs upon wood or stone, he counted, he kept his tallies upon notched sticks, and he computed in some simple way by his fingers or by pebbles on the ground. He did not always count by tens as in our decimal system; indeed this was a rather late device, and one suggested by his digits. At first he was quite content to count to two, and generations later to three, and then to four. Then he repeated his threes and had what we call a scale of three, and then, as time went on, he used a scale of four, and then a scale of five. At one time he seems to have used the scale of twelve, because he found that twelve is divisible by more factors than ten, and particularly by two and three and four; but by the time he became ready to write his numbers the convenience of finger reckoning had become so generally recognized that ten became practically the universal radix. Nevertheless there remain in our language and customs numerous relics of the duodecimal idea, such as the number of inches in a foot, of ounces in a troy pound, and of pence in a shilling, all influenced by the Roman inclination to make much use of twelve in practical computation.

The writing of numbers has undergone more change than even the number names. Not only was there a notation for each language in ancient times, as to-day in the Orient, but some languages had several sets of numerals, as is seen in the three standard systems of Egypt, the two of Greece, and the somewhat varied forms in use in Rome. The Roman supremacy gave the numerals of these people great influence in Europe, and they were practically in universal use in the West until the close of the Middle Ages. Meantime there had arisen in the East, probably in India although very likely subjected to influence from without, our present system of notation, and little by little this permeated the West. When it arose it was without a zero, and hence without such place value as we use to-day; but probably about the seventh century the zero appeared, and the completed system found its way northward into Persia and Arabia, and thence in due time it was transmitted to the West.

At first the subject was purely practical, a counting of arrows or of sheep or of men. For a long time this was all that number meant to the world, until the mystic age developed and philosophy began. Then numbers were differentiated, and odd and even were distinguished, and "There's luck in odd numbers" became a tenet of faith, and the even numbers became designated as earthly and feminine. The story is long and interesting, that of the development of this mysticism with its special prominence of three and seven, particularly so as the movement led to a study of the properties of numbers, to roots, and to series. It is connected with number games, and these in turn led to the abacus, and so the practical and the mysterious are more or less blended even at times when they are generally regarded as widely separated.

The growth of topics of arithmetic is also an interesting subject for investigation. We say that there are four fundamental operations, although once there was only one, and at another time the world recognized as many as nine. We operate chiefly with decimal fractions, as in working with dollars and cents, although these fractions are scarcely three hundred years old. We are impatient that a child stumbles over common fractions, and yet, so difficult did the world find the subject that for thousands of years only the unit fraction was used. We wonder how the long division form of greatest common divisor ever had place in arithmetic, and yet it was a practical necessity in business until about 1600 A. D. We feel that "partnership involving time" could never have been practical, and yet until a couple of centuries ago it was intensely so. And thus it is with many topics of arithmetic,—they have changed from century to century, and even in our own time from year to year. It is well for a teacher to know a little of this history of the subject taught, although space does not allow for any serious consideration of the topic in this work. In the bibliography some reference will be found to sources easily available, and the teacher who wishes to see arithmetic in progress, as opposed to arithmetic stagnant and filled with the obsolete, should become acquainted with one or more of these works upon the subject. The history of arithmetic is the best single stimulus to good method in teaching the subject.

BIBLIOGRAPHY: Smith, *The Teaching of Elementary Mathematics,* New York, 1900; Smith, *Rara Arithmetica,* Boston, 1909; Smith and Karpinski, *Hindu-Arabic Numerals,* Boston, 1911; Ball, *A Primer of the History of Mathematics,* London, 1895, and *A Short Account of the History of Mathematics,* London, 4th edition, 1908; Fink, *History of Mathematics,* translated by Beman and Smith, Chicago, 1900; Cajori, *History of Elementary Mathematics,* New York, 1896, and *History of Mathematics,* New York, 1893; Jackson, *The Educational Significance of Sixteenth Century Arithmetic,* New York, 1906; Gow, *A Short History of Greek Mathematics,* Cambridge, 1884; Conant, *The Number Concept,* New York, 1896; Brooks, *Philosophy of Arithmetic,* revised edition, Philadelphia, 1902. There are numerous works in German on the history of mathematics and of mathematical teaching, and a considerable number in French and Italian.

CHAPTER II

THE REASONS FOR TEACHING ARITHMETIC

The ancients had less difficulty than we have in assigning a reason for teaching arithmetic, because they generally differentiated clearly between two phases of the subject. The Greeks, for example, called numerical calculation by the name *logistic,* and this subject was taught solely for practical purposes to those who were going into trade. A man might have been a very good philosopher or statesman or warrior without ever having learned to divide one long number by another. Such a piece of knowledge would probably have been looked upon as a bit of technical training, like our use of the slide rule or the arithmometer. On the other hand the Greeks called their science of numbers *arithmetic,* a subject that had nothing whatever to do with addition, subtraction, multiplication, or division, and that excluded all applications to trade and industry. This subject was taught to the philosopher, and to the man of "liberal education" as we still call him. It considered questions like the factorability of numbers, powers and roots, and series,—topics having little if any practical application in the common walks of life. Therefore when a Greek was asked why he taught logistic, his answer was definite: It is to make a business man able to compute sufficiently for his trade. If he was asked why he taught arithmetic, as the term was then used, his answer was still fairly a unit: I teach it because it makes a man's mind more philosophic.

In the present day we have a somewhat more difficult task when we attempt to answer this question. Arithmetic with us includes the ancient logistic, and we teach the subject to all classes of people:—to one who will become a day laborer, belonging to a class that never in the history of the world studied such a subject until very recently; to the tradesman, who never uses or cares to use the chapter on prime numbers; to the statesman, who will probably have little opportunity to employ logarithms in any work that may come to him; to the clergyman, to whom the metric system will soon be merely a name; to the housewife, the farmer, and to all those who travel the multifarious walks of our complex human life. For us to tell why we teach the American arithmetic to all these people is by no means so easy as it was for the Greek to answer his simple question.

In general, however, we may say that as we have combined the ancient logistic (calculation) and arithmetic (theory) in one subject, so we have combined the Greek purposes, and that we teach this branch because it is useful in a business way to every one, and also because it gives a kind of training that other subjects do not give.

As to the first reason there can be no question. When the great mass of men were slaves the business phase was not so important; but now that every man is to a great extent his own master, receiving money and spending it, some knowledge of calculation is necessary for every American citizen. To elaborate upon this point, is superfluous. There is, however, one principle that should guide us in the consideration of this phase of the question: *Whatever pretends to be practical in arithmetic should really be so.* We have no right to inject a mass of problems on antiquated investments, on obsolete forms of partnership, on forgotten methods of mercantile business, or on measures that are no longer common, and make the claim that these problems are practical. If we wish them for some other purpose, well and good; but as practical problems they have no right to appear. To set up a false custom of the business world is as bad as to teach any other untruth; it places arithmetic in particular, and education in general, in a false light before pupils and parents, and is unjustified by any reason that we can adduce. An obsolete business problem has just one reason for being, and this reason is that it has historical interest. We can secure the mental discipline as well by other means, and we have no right to handicap a child's mind with things that he will be forced to forget the minute he enters practical life.

There remains the side of mental discipline, which I have elsewhere called, for want of another term and following various other writers, the culture side. What mental training does a child get from arithmetic that he does not get from biology, or

Latin, or music? This is a question so difficult to answer that no one has yet satisfied the world in his reply, and no one is liable to do so. There have been elaborate articles written to show that the proper study of arithmetic has an ethical value, though exactly what there is in the subject to make us treat our neighbor better it is a little difficult to say. Others have said that arithmetic, through its very rhythm, has an aesthetic value, as is doubtless true; but that this is generally realized, or that it serves to make us more appreciative of the beautiful, is hardly to be argued with any seriousness. Still others have felt that by coming in contact with exact and provable truth an individual sets for himself a higher standard in all other lines of work, and this again is probably the case, although the measure of its influence has never been satisfactorily accomplished. And to these reasons may be added many more, such as the training of a deductive science, although elementary arithmetic is to a large extent inductive; the training in concentration, although the untangling of a Latin construction requires quite as close attention; the exaltation of mind that comes from the study of numbers that may increase or decrease indefinitely,—and others of like nature. And out of it all, what shall we say? That arithmetic has no mental discipline that other subjects do not give? No one really feels this, in spite of the fact that the exact nature of this discipline is hard to formulate. Every one is conscious that he got something out of the study, aside from calculation and business applications, that has made him stronger, and the few really scientific investigations that have been made, as to the effect of mathematical study, bear out this intuitive feeling. And this being so, we might be justified even if we did not attempt to define just what this is, any more than we should attempt too seriously to define time, or love, or God, or eternity.

Not to dismiss the mental discipline side too summarily, however, and at the same time seeking to avoid the endless verbiage that usually characterizes the discussion, it is well to set forth more clearly some of the objects to be sought on the culture side of arithmetic. In the first place, we seek an absolute accuracy of operation that differs from the kind of accuracy we seek in science or linguistics or music. The fact that we have, in thousands of problems, sought a result so exact as to stand every test, leads us to set a higher standard of accuracy in all lines than we could have set without it. This justifies the introduction of any part of theoretical arithmetic for which the pupil is mentally ready. It is one reason why cube root was formerly there, when pupils were more mature than now, and in the same way it has justified progressions and a more elaborate treatment of primes than any business need would warrant. Here then, is a reason for teaching arithmetic that is above and beyond the merely practical of the present moment.

A similar and related reason appears in the fact that mathematics in general, and arithmetic in particular, requires a helpful form of analysis that does not stand out so clearly in other studies. "I can prove this if I can prove that; I can prove that if I can prove a third thing; but I can prove that third thing; hence I see my way to proving the first." This is the analytic form that has come down to us from Plato. It more evidently appears in geometry, but is essentially the reasoning of arithmetic as well. "I can find the cost of 2½ yd. if I can find the cost of 1 yd.; but I know the cost of 6¼ yd., so I can find the cost of 1 yd.; hence I can solve my problem," is the unworded line of the child's analysis. Such a training, unconsciously received and often unconsciously given, is valuable in every problem we meet, leading us to exclude the non-essential and hold with tenacity to a definite line of argument.

These two phases of the culture side of arithmetic, the side of mental discipline, will then suffice for our present purpose, which is to show that such a side exists: (1) The contact with absolute truth; (2) The acquisition of helpful forms of analytic reasoning.

Bibliography: Smith, *Teaching of Elementary Mathematics,* pp. 1-70; Smith, *Teaching of Geometry,* Boston, 1911; Suzzallo, *Teaching of Primary Arithmetic,* Boston, 1911; Young, *The Teaching of Mathematics,* New York, 1907, pp. 9, 41-52, 202-256; Branford, *A Study of Mathematical Education,* Oxford, 1908; Stone, *Arithmetical Abilities,* Teachers College Series, 1908; Rietz and Shade, *Correlation of Efficiency in Mathematics,* etc., University of Illinois *Bulletin,* vol. VI, no. 10; Hill, *Educational Value of Mathematics, Educational Review,* vol. IX, p. 349; Schubert, *Mathematical Essays and Recreations,* Chicago, 1899, p. 27.

CHAPTER III

WHAT ARITHMETIC SHOULD INCLUDE

If we taught arithmetic only for its utilitarian value, to fit a person for the computations that the average man needs to perform or know about in daily life, the range of subject matter would not be great. Addition, particularly of money, but not involving very large or numerous amounts, is probably the most important topic. Perhaps, for it is difficult to say with certainty, the simple fractions ½ and ¼ are next in line of relative importance, including ½ of a sum of money, ¼ of a length or weight, and so on. Very likely the making of change, one of the forms of subtraction, is next in frequency of use. Then may come easy multiplications, to find the cost of 5 lb. of sugar, or of 16 yd. of cloth, given the price per pound or yard. A few of the most commonly used measures and their relations must then be known, as that 12 inches equal 1 foot and 16 ounces equal 1 pound. Given this equipment, the average run of humanity would be able to get along fairly well. But beyond this there lies a second field of work that every one may need, that a large minority will need, and that we must all at least know something about. This field includes all four fundamental operations with integers, with simple common fractions (say with denominators of one or two figures), with decimal fractions at least to hundredths, and with compound numbers of at least two denominations; the common business cases of percentage, and their applications; the common problems of business, all of which are applications of the operations above mentioned, and a little knowledge of ratio and proportion, chiefly for understanding the meaning of these terms. From the standpoint of business needs this equipment would answer the purposes of nearly every one. Whatever of applied arithmetic lies beyond this is a part of the technical training of a very small minority. Apothecary's measures form part of the technical training of the drug clerk and the physician; the average citizen has long since forgotten them, and happily so. Compound proportion is never used practically, and any mathematician if called upon to solve its problems would employ another and a better method. Duodecimals, while interesting historically and philosophically, from the practical standpoint are used by so few as to place them also in the technical training of the very small minority. Subjects like discount and interest are, of course, included under the common applications of percentage. Similarly with stocks and bonds, for although such securities are purchased by relatively few people, their nature and uses should be understood by all, particularly as we seem to have entered upon an era of extensive coöperation upon a stock basis. The general nature of applications, however, will be discussed later.

If we taught arithmetic only from the standpoint of mental discipline we might use all the material here mentioned, and any other topics that allowed for securing accurate results by clear reasoning processes. Obsolete measures, obsolete methods, progressions, cube root and even higher roots, compound proportion, —all such topics might have place if we were seeking only the discipline of arithmetic. When, however, we consider that we are seeking to unite these two considerations, and are attempting to make the subject both practical and disciplinary, then we are met by the necessity for mutual concessions. The practical side must concede to the disciplinary by having its processes clearly understood, and by developing the reason at every step; the disciplinary side must concede to the practical by selecting its topics in such way as to give no false notions of business, and as to encourage the pupils to an interest in the quantitative side of the world about them. On the one side we must not teach business arithmetic by mere arbitrary rules that are not understood, since this would be to eliminate the disciplinary nature; on the other side we must not introduce a style of time draft that is now obsolete in America, or artificial examples in compound proportion, because these inculcate wrong ideas of the business world about us, nor extensive work in equation of payments because this is part of the technical training of such a very small minority that we can use our time to better advantage by dwelling upon other topics.

A word should also be said as to the tendency of some teachers to feel that worthy results are attained when children have been drilled to unnecessary facility in one line or another. It is not difficult to train children to add two columns of figures at a time, or to multiply by cross multiplication, or to detect six or eight cubes at a glance or to display various other forms of arithmetical ability analogous to the acrobatical feats sometimes allowed in a gymnasium. Such activities have some value as games, and they attract attention on the part of visitors, but it is a question whether the pupil derives any real benefit from most of this kind of work. It is because of this feeling that such features are not found in our textbooks, the space being devoted to those things that the business man finds useful in everyday life.

Thus it happens that the modern American arithmetic is a fair compromise between the practical without theory and the theoretical without practice, the two distinct phases of the old Greek number work. To keep this balance true is one of the missions of teachers to-day. The tendency is to obtain the mental discipline of arithmetic from problems that are practical, and that this tendency is a healthy one there seems to be no room for reasonable doubt.

BIBLIOGRAPHY: Smith, *Teaching of Elementary Mathematics*, p. 19; Young, pp. 23-242; Spencer, *The Teaching of Elementary Mathematics in the English Public Elementary Schools*, London, 1911.

CHAPTER IV

THE NATURE OF THE PROBLEMS

In no way has arithmetic changed as much of late years as in the nature of the problems and the arrangement of the material. The former has come about from two causes, (1) the needs of society, and (2) the study of child psychology. The latter, the arrangement of the material, has been determined almost entirely by psychological considerations. In this chapter it is proposed to speak of the former, the nature of the problems.

It should definitely be stated, however, that this emphasis laid upon applied problems should not be construed to mean that the abstract number work is not quite so important as the concrete. To be sure we have some advocates of only the concrete, what the Germans call the "clothed problem," to the entire exclusion of the abstract drill work of the Pestalozzi school. Such extremists are, however, not numerous, and they have but a small following. Every one who looks into the subject is aware that on the score of interest a child prefers to work with abstract numbers, while as to the final results upon his education we seem to neglect altogether too much the ability to get exact results quickly and with a certainty as to their exactness. This phase of the work will be mentioned in a later chapter, and for the time being we may well consider the applied problem.

Within the last few years the question of the practical uses of arithmetic has been a vital one in educational circles, especially in Germany and America, resulting in considerable literature upon the subject. These needs, while generally similar in various countries, differ more or less in details. Thus a country whose business was chiefly farming would need to emphasize agricultural problems; one that derived its wealth from its metals or its coal would emphasize mining; a manufacturing nation would find certain lines of problems of the factory peculiarly suited to its needs, while one that derived its wealth chiefly from shipping would require those relating to foreign commerce. The mathematical foundation would be the same in all cases, but the material content of the problem would vary. Now in America we are unusually cosmopolitan in our needs in this respect, ranking high in all these particulars save only (at present) in ocean traffic. We are therefore very fortunate in having at our disposal unlimited problem material that relates to our wide range of national resources and industries. The advantage of using this material instead of the obsolete inherited problems that came down to us from Italy, through England, ought to be so evident as to require no argument. There will always be some who cry out against what they call encyclopedic information in an arithmetic, but surely if a problem is to contain any facts at all it is better that these facts be American and of the twentieth century than Italian and of the fifteenth. Can there be any doubt that an American boy or girl will get more breadth of view, more interest, and possibly more directly useful information from a problem about the mixing of plant foods for a southern farm, than one about the mixing of teas that are never mixed in the way the text-book says? If a pupil is to study about goods being transported, is it not better for him to take a practical case relating to our railroads than the old-time one of pedlars carrying their packs? It is well, however, to avoid the unfortunate tendency manifested by some recent writers to introduce problem material that no child is ready to understand and with which teachers themselves should not be expected to be familiar. Technical information of trades, scientific nomenclature that belongs to the college or to the later years in the high school, problems of the civil engineer or the food chemist,—these have no more place in the arithmetic class than has the apothecary's table or the subject of equation of payments. Common information, the subjects of interest to the general public, and those matters that are topics of conversation in the usual walks of life are the bases upon which we may reasonably build our problems. Undertaken in this spirit we need not fear if critics accuse us of making our schools encyclopedic. Every usable school arithmetic has always been an encyclopedia; what we have to determine is whether it shall now be an encyclopedia of vital, modern facts, or one of obsolete, dull, useless information. The needs of society demand the former; *vis inertiæ* holds to the latter. The earnest teacher, awake to the needs of the business community in which a school is located, can hardly fail to introduce genuine problems with local color to enliven the work in arithmetic. No text-book can fit the needs of every locality, and original problems are easily found by the children themselves if the opportunity is given. The awakening of interest in such work, however, will not come from the style of text-book of a generation ago; it will only come in connection with the study of a book that is itself filled with this spirit. Fortunately most of our modern writers are working earnestly to meet the needs of to-day, and our American arithmetics may well lay claim to being among the most progressive that are appearing in this generation.

But what as to the effect of the study of child psychology? Here too there has been made very great progress in recent years. Although a problem may represent all that business needs suggest, it still may not be suited to a particular school year. In other words, we have to consider from grade to grade the interests and powers of the child. We would not think of giving to a child in the first grade the problem, If A has 2 shares of railroad stock and B has 3 shares, how many have they together? For while the child can add 2 and 3, he has no knowledge of stocks, nor any interest in them. Change the subject to marbles or apples or tops, and it is suited to his mind, but not otherwise. Thus it has come about that teachers are trying to decide what are the larger interests, actual or potential, of the children in the various school years, to the end that the problems of arithmetic may be the better apperceived. A beginning has been made, but the future will see the work extended. We know that pride in our national resources renders interesting a style of problem in the fifth school year that would be of no value in the second, even with smaller numbers. On the other hand we are equally aware that certain problems involving children's games that are part of the genuine applied mathematics of the third year would have no interest whatever in the eighth. And so in general, teachers are seriously attempting at the present time to coördinate the interests of children, the needs of society, and the mathematical powers in each of the grades from the first year through the elementary school.

Bibliography: Smith, *Teaching of Elementary Mathematics,* p. 21; Smith, *Teaching of Geometry,* Boston, 1911; Young, pp. 97, 210. Consult also the author's text-books on arithmetic and works like Saxelby, *Practical Mathematics,* London, 1905; Consterdine and Andrew, *Practical Arithmetic,* London, 1907; Consterdine and Barnes, *Practical Mathematics,* London, 1908; Castle, *Workshop Mathematics,* London, 1907; Castle, *Manual of Practical Mathematics,* London, 1906; Cracknell, *Practical Mathematics,* 4th edition, London, 1906. For statistics for problems *The World Almanac,* New York, is an inexpensive and valuable source.

CHAPTER V

THE ARRANGEMENT OF MATERIAL

As already stated, the two most noteworthy changes in arithmetic in recent years have related to the nature of the problems and the arrangement of material. The latter has been the result of a more or less serious study of child psychology, namely of the powers of the individual in the various school years.

Formerly, say a century or more ago, it was the custom to study arithmetic from a single book, after the boy (for the girl seldom understood mathematics of any kind) could read and write. The child was mature enough to understand the subject after a fashion, and he " went through " the book. But as arithmetic began to work its way down to the earliest grades it was found impracticable to follow this plan, the subject being too difficult for young minds. The ordinary text-book was therefore preceded by a primer on arithmetic, and thus a two-book series was formed. From this beginning numerous experiments have proceeded, seeking to carry the improvement still farther. We have had three-book series, eight-book series, lesson leaflets, two-book courses arranged by grades, and so on. We have had spirals of various degrees of turning, efforts to resume the topical arithmetic, books arranged to follow certain narrow lines of manual training, and so on,—all serious efforts for betterment, but many of them too hastily considered to have any material influence.

In the midst of it all, what have the practical teachers of the country done? In every city, in several states, and by numerous associations, courses of study have been arranged setting forth the material that experience has shown can be used to advantage in the several school years, and selections have been made from the current arithmetics to supply what was needed. In other words the practical teachers of America arranged their own books to a great extent, basing their selection of material upon an empirical psychology, and giving to the child what his interests and capabilities suggested.

It is out of such a movement, spontaneous but thoroughly sound, that the later American text-books have arisen, and the care and earnestness given to their preparation by various authors and publishers should have the commendation of all.

These books are and probably will continue to be of two distinct types, each with strong merits of its own, and each capable of producing the best work. First there is the topical arithmetic, that is, a book arranged by topics, a subject like percentage being studied once for all, the pupil staying with it until it is thoroughly mastered. Such a book has two great merits; it tends to keep the child upon each subject long enough to give him a feeling of mastery that he would not have if he studied some of the scrappy books constructed on the extreme spiral system, and it allows a teacher who wishes to adopt a moderate spiral to do this in a manner that will meet the local conditions. By this latter is meant that some classes seem to need more work in a subject like simple interest than other classes do, when it is first taken up. A teacher may, therefore, select as much or as little as is necessary when a topic is taken up for the first or the second time, and this is perhaps more easily done from a topical than from another form of book.

The second general type of text-book is one that attempts to fit the course of study in a large number of schools. In general these books agree in a number of particulars: (1) Certain subjects, such as the most commonly used operations, should appear several times in the school course; (2) others, such as the business application of simple interest, may appear perhaps two or three times, with gradually increasing difficulty; (3) still others, like board measure, may be sufficiently treated once for all; (4) the closing year of arithmetic should be devoted to a study of such higher problems of business as can be understood by the children. These are some of the articles of agreement, and they go to show the common-sense principles on which our modern courses of study and text-books in arithmetic are based.

As to which of these two types is the better it is impossible to give a general decision. It depends largely upon the school and the teacher. With the one book the teacher arranges the matter to suit the pupils' needs; in the other the matter is already arranged to suit the average pupil. Neither will fit each individual case, and the great thing is not so much the arrangement of the matter as it is the modern spirit displayed in the omission of the obsolete and the introduction of new, vital, interesting, intelligible problems of to-day.

It should also be stated in this connection that a similar movement is taking place with reference to algebra and geometry, and that it is destined to reach still further up the grade and into college mathematics. Algebra is old, but algebra text-books are relatively modern. They were at first based upon the arithmetics that preceded them, and as far as possible they followed their arrangement of matter. The ordinary school algebra is, therefore, merely an old-style arithmetic with letters used instead of numerals, and with a considerable number of its problems taken from the arithmetical collections of earlier days. The arrangement of matter is confessedly not scientific, and we are even now seeing the sequence of topics challenged, and a serious effort to improve the nature of the problems. The next move of importance in the teaching of elementary mathematics will be the rearrangement of the material and the enriching of the applications of algebra and, probably to a less degree, of geometry.

Bibliography: Consult the latest text-books, comparing the two types and noting the distinctive advantages of each. Consult also those extreme forms in which the spiral arrangement is carried too far. This, and in general the other questions considered in this monograph, will be found discussed in Professor Suzzallo's noteworthy work, *Teaching of Primary Arithmetic*, Boston, 1911.

CHAPTER VI

METHOD

Of all the terms used in educational circles "Method" is perhaps the most loosely defined. Efforts have been made to limit its meaning, to divide its responsibilities with such terms as "Mode" and "Manner," but it still stands and is likely to stand as a convenient name for all sorts of ideas and theories and devices. Nevertheless it has been most often used in arithmetic to speak of the general plan of some individual for introducing the subject, as when we speak of the Pestalozzi or Tillich or Grube Methods, although it is also applied to such arrangements of material as is indicated by the expressions Topical Method and Spiral Method, and to such an emphasis of some particular feature as has given name to the Ratio Method. It is not the intention to attempt any definition of the term that shall include all of these ramifications, but to take it as it stands, to characterize briefly some of these "Methods," and then to speak briefly of the subject as a whole.

Pestalozzi's method was really a creation of his followers. What this great master did for arithmetic was to introduce it much earlier into the school course, to use objects more systematically to make the number relations clear, to abandon arbitrary rules, to drill incessantly on abstract oral work, and to emphasize the unit by considering a number like 6 as "6 times 1." For the time in which he lived (about 1800) all this was a healthy protest against the stagnant education that he found. To-day it is only an incidental lesson to the teacher, although Pestalozzi's spirit and several of his ideas may well command the admiration and respect of all who study the results of his great work.

The "Method" of Tillich, who followed Pestalozzi by a few years consisted largely of making a systematic use of sticks cut in various lengths, say from 1 inch to 10 inches. It is evident that such a collection allowed for emphasizing the notion of tens, for treating fractions as ratios, and for visualizing in a very good way the simpler number relations. On the other hand it is also evident that the use of only a single kind of material is based upon a much narrower idea than that of Pestalozzi, who purposely made use of as wide a range of material as he could.

The Grube Method, that created such a stir in America a generation ago, was not very original with Grube (1842). Essentially it was an adaptation of the concentric circle plan that had already been used, a kind of spiral arrangement of matter to meet the growing powers of the child. It contained some absurdities, such as the exhausting of the work on one number before proceeding to the next, as of studying 25 in all its relations before learning 26; but on the whole it made somewhat for progress by assisting to develop a sane form of the spiral idea.

It is not worth while to speak of other individual "Methods," since they have little of value to the practical teacher, and the student of the history of education can easily have access to them. Enough has been said to show that one of the easiest things in the teaching of arithmetic is the creation of "Method"—and one of the most useless. We may start off upon the idea that all number is measure, and hence that arithmetic must consist of measuring everything in sight,—and we have a "Measuring Method." It will be a narrow idea, we shall neglect much that is important, but if we put energy back of it we shall attract attention and will very likely turn out better computers than a poor teacher will who is wise enough to have no "Method," in this narrow sense of the term. Again, we may say that every number is a fraction, the numerator being an integral multiple of the denominator in the case of whole numbers. From this assumption we may proceed to teach arithmetic only as the science of fractions. It will be hard work, but, given enough energy and patience and skill, the children will survive it and will learn more of arithmetic than may be the case with listless teaching on a better plan. We might also start with the idea that every lesson should be a unit, and that in it should come every process of arithmetic, so far as this is possible, and we could stir up a good deal of interest in our "Unit Method." Or, again, we could begin with the idea that all action demands reaction, and that every lesson containing addition should also contain subtraction; that $6 + 4 = 10$ should be followed by $10 - 6 = 4$ and $10 - 4 = 6$; and that $2 \times 5 = 10$ should be followed by $10 \div 2 = 5$ and $10 \div 5 = 2$. By sufficient ingenuity a very taking scheme could be evolved, and the "Inverse Method" would begin to make a brief stir in the world. This in fact has been the genesis, rise, and decline of Methods; given a strong but narrow-minded personality, with some little idea such as those above mentioned; this idea is exploited as a panacea; it creates some little stir in circles more or less local; it is tried in a greater or less number of schools; the author and his pupils die, and in due time the Method is remembered, if at all, only by some inscription in those pedagogical graveyards known as histories of education.

The object in writing thus is manifest. For the teacher with but little experience there is a valuable lesson, namely, that there is no "Method" that will lead to easy victory in the teaching of arithmetic. There are a few great principles that may well be taken to heart, but any single narrow plan and any single line of material must be looked upon with suspicion. Certain of the general principles of Pestalozzi are eternal, but the Reckoning-chest of Tillich is practically forgotten.

And here it is proper to say a word as to what schools of observation and practice should stand for in these matters of method and purpose. It would be a very easy thing to concentrate on some single point, some device of teaching, some particular line of problems, and to carry the work to an extreme that would attract attention and produce results that would be remarked upon. This is the temptation of those who direct such schools. But is it a wise policy? These schools are established to train teachers to well-balanced leadership, not to be extremists when this means the neglect of essential features of education. The graduates should know the best that there is in every theory of education, but they should also avoid the worst. The prime desideratum in arithmetic is the ability to work accurately, with reasonable rapidity, and with interest, and to know how to apply number to the ordinary affairs of life. To secure accuracy alone, to secure speed alone, to have arithmetic a play spell without accuracy and speed, or to know how to apply number to life in a slovenly way,—these are extremes that should be avoided at any cost, including the tempting cost of sensationalism. It is the mission of the training school or college to make the earnest, well-balanced teacher, first of all. With this duty goes the laudable one of reasonable experiment, of trying out suggestions from whatever source; but normal schools and teachers colleges must at all hazards guard against having it appear that an experiment is an accomplished result, or of sacrificing our children in unnecessary quasi-clinical work that is doomed to failure. In this connection one of the resolutions adopted by the National Education Association in 1908 may be read with profit as voicing the sentiment of that saner element in education that is, after all, the strength of our profession:

"We recommend the subordination of highly diversified and overburdened courses of study in the grades to a thorough drill in essential subjects; and the sacrifice of quantity to an improvement in the quality of instruction. *The complaints of business men that pupils from the schools are inaccurate in results and careless of details is a criticism that should be removed.* The principles of sound and accurate training are as fixed as natural laws and should be insistently followed. *Ill-considered experiments and indiscriminate methodizing should be abandoned, and attention devoted to the persevering and continuous drill necessary for accurate and efficient training;* and we hold that no course of study in any public school should be so advanced or so rigid as to prevent instruction to any student, who may need it, in the essential and practical parts of the common English branches."

Bibliography: The author's *Teaching of Elementary Mathematics*, pp. 71-97, a rather extensive discussion; Seeley, *Grube's Method*, New York, 1888; Soldan, *Grube's Method*, Chicago, 1878; C. A. McMurry, *Special Method in Arithmetic*, New York, 1905; McLellan and Dewey, *The Psychology of Number*, New York, 1895; Young, pp. 53-150; Hornbrook, *Laboratory Method of Teaching Mathematics*, New York, 1895; Perry, *The Teaching of Mathematics*, London, 1902; Suzzallo, *loc. cit.*; Ballard, *The Teaching of Mathematics in London Public Elementary Schools*, London, 1911.

CHAPTER VII

MENTAL OR ORAL ARITHMETIC

The objection to the expression "Mental Arithmetic" is fully a generation old. It is argued that written arithmetic is quite as mental as any other kind, and that the opposite to written is oral. As to this there can be no argument, but the word "mental" has so long been used to apply to that phase of arithmetic that is not dependent upon written help that, like a person's proper name, it need not be held strictly to account for what it literally signifies. The expression "Mental Arithmetic" is therefore employed as well as "Oral Arithmetic" in this article simply because it is historical and of well-understood significance.

What, now, are the relative claims of written and mental arithmetic? Historically, the mental long preceded the written, but only in very simple problems, chiefly involving counting and easy addition. As soon as the writing of numbers was introduced, written arithmetic or else the arithmetic of some form of the abacus became practically universal. In Japan to-day a native shopkeeper will multiply 2 by 6 upon the soroban (abacus), and such mechanical aids were not only not discarded in western Europe until the sixteenth century, but they are still universal in Russia. About the beginning of the last century, however, mental arithmetic underwent a great revival, largely through the influence of Pestalozzi in Europe and Warren Colburn in this country, in each case as a protest against the intellectual sluggishness, lack of reasoning, and slowness of operation of the old written arithmetic. For a long time the mental form was emphasized, in America doubtless unduly so, and was naturally followed by such a reaction that it lost practically all of its standing. The question for teachers to-day is this, what are the fair claims of these two phases of the subject upon the time and energy of pupil and teacher?

There are two points of view in the matter, the practical and the educational or psychological, and fortunately they seem to lead to the same conclusion. Practically a person of fair intelligence should not need a pencil and paper to find the cost of 6 articles at 2 cents each, or of 5¾ yards at 16 cents a yard. The ordinary purchase of household supplies requires a practical ability in the mental arithmetic of daily life, and this ability comes to the mind only through repeated exercise. As will be seen later, it is a fair inference from statistical investigations that a person may be rapid and accurate in written work but slow and uncertain in oral solutions. Therefore, it will not do, from the practical standpoint, to drill children only in written arithmetic if we expect them to be reasonably ready in purely mental work. On psychological grounds, too, the neglect of mental arithmetic is unwise. It is a familiar law that the memory is stronger on a fact that is known in several ways (a convenient phrase, if not scientific) than on a fact that is known in only one way. A man who knows a foreign word only through the eye may forget it rather easily, but if his tongue has been taught to pronounce it, even though he be deaf, he can the more readily recall it. If in addition to this his ear has often heard it he is the more strongly fortified, and if he has also often written it, by pen or by typewriter, there is this further chain that holds it to the memory. In other words, the greater number of stimuli that we can bring to bear, the more certain the reaction. Now arithmetic furnishes merely a special case of this general law. If a child could simply see $9 \times 8 = 72$ often enough he would come to be able to write it in due time, even if he did not know the meaning. If in addition to this he knows the meaning of these symbols and recalls having taken 9 bundles of 8 sticks each and finding that he had 72 sticks, then the impression on the brain is the more lasting. If, furthermore, he has been trained to say "nine times eight are seventy-two" repeatedly, the impression is still stronger, and if he has repeatedly heard this expression (and here is one of the advantages of class recitation) he has then a still further mental grip upon the fact. In other words, mental arithmetic in the form of rapid oral work, with both individual and class recitation, is a valuable aid, psychologically, to the retention of number facts.

There is, however, a danger to be recognized. It is asserted that a child tires more quickly of abstract work than of genuine concrete problems; problems, that is, that are not manifestly "made up" but that represent some of his actual quantitative experiences. Whether he really tires more quickly of the abstract than of the concrete is by no means certain; for he seems to have more interest in the former than in the latter, probably from the added difficulty that the concrete problem presents in requiring him to know what operations he must perform. At any rate he tires of both, as he does of any other intellectual exercise. It therefore follows that if five minutes of mental work produce a certain efficiency, thirty minutes will by no means produce six times that efficiency. If, now, this mental work is valuable, how much time and energy should be allotted to it? Possibly we shall have a statistical reply to this question sometime, although it will be a sorry day for good teaching if we should ever accept such a reply as final, any more than we should accept the crude statistics of the health department as determining the prescription our physician gives us for indigestion. The statistics may help us, but they can never control us. But in absence of even their assistance, what shall we give as an empirical answer? It seems to be the experience of teachers generally that a little mental work, rapid, spirited, perhaps with some healthy, generous rivalry to add spice to the exercise, should form part of every recitation throughout the course in arithmetic. There will often be exceptions, but in general it is a pretty good rule to devote from three to five minutes daily, and sometimes much more time, to this kind of work. In this way a child never gets out of practice, save during the summer holidays, and the practical and psychological benefits can hardly be estimated.

What should be the nature of this mental work? On the applied side there is no better test for the teacher's ability to adapt herself to her environment, educationally, than this, for the answer varies with the school year, the locality, the related subjects in the course, and with many other factors. In general, however, it may be said that mental arithmetic offers the best means for correlating the subject with the pupil's other work, both within and without the school. To limit it to this field, however, would be an evident mistake, the work with abstract number demanding the major part of the time assigned to this feature. To acquire perfect mechanical reaction to a given stimulus much exercise is required, and for a child to think 72 when stimulated by the ideas 9×8 and 8×9 demands repeated practice, not merely in relatively few applications but in a multitude of questions involving abstract numbers. Nor is this practice any more irksome than is the solution of applied problems, as any teacher knows. It was almost exclusively by this abstract work that Pestalozzi developed calculators of such ability with concrete problems as astonished those who visited his school, although, if we may place confidence in the results of Dr. Stone's recent investigations, ability in either of these lines does not necessarily imply ability in the other.

In conclusion, we have two lines of work in mental arithmetic; (1) the concrete, in which the teacher has an excellent opportunity for correlation, for local color, and for stimulating the interest in the uses of arithmetic; (2) the abstract, in which the text-book may be trusted to furnish a considerable part of the material. Each must be cultivated, and ability in one does not necessarily mean a corresponding standard of ability in the other, although a failure in the abstract line must lead to a failure in the concrete. One leads to the acquisition of number-facts, the other to the ability to rationally use these facts in applied problems.

As a practical question for the teacher, how is the material for this oral work to be found? The answer is evident; it must be found exactly as we find material in geography, in history, and in written arithmetic,—from a text-book. No teacher can make up on the spur of the moment all of the oral examples necessary, and arrange them properly, and cover all of the important phases of drill work. Either, then, a book must be used that supplies both the oral and the written work, or else two books must be used, one for the oral and one for the written. In either case there should be a good supply of oral problems, to be supplemented by the teacher with such local problems and such correlation with other work as may be advisable.

Bibliography: The author's *Teaching of Elementary Mathematics,* p. 117; *Handbook to Arithmetics,* Boston, 1904, p. 6; Young, p. 230; C. W. Stone, *Arithmetical Abilities;* Wentworth-Smith, *Oral Arithmetic,* Boston, 1909.

CHAPTER VIII

WRITTEN ARITHMETIC

What has been said of mental arithmetic naturally leads to some question as to the nature of the written work. What shall this be? If the difference in longitude between two ships (since standard time by one system or another is now coming to be universal on land) is $33° 45'$, how shall a pupil find the difference in time? Here are a few possibilities:

(1)

```
        2 hr.  15 min
15)33° 45'
   30
   3° 45' = 225'
             15
             75
             75
```

(2)

```
         2 hr.  15 min.
15°)33° 45'
    30
    3° 45' = 225'
              15
              75
              75
```

(3)

```
  33° 45'
  60
  1980
    45
15)2025(135 min. = 2 hr. 15 min.
   15
   52
   45
   75
```

(4)

```
   33° 45'
15)33¾°(2¼ hr.
   30
   3¾ = 15/4
        15/4
   0
```

(5)

$33° \ 45' = 33\frac{3}{4}°$

$33\frac{3}{4} \times {}^{1}/_{15}$ hr. $= 2\frac{1}{4}$ hr.

Numerous other forms could be suggested, but these will suffice for our purposes. Which of these should be preferred? In general, should we recommend the form that gives us the result most quickly, or some other that may show clearer reasoning? In Nos. 1 and 4 the forms indicate that we divide degrees by an abstract number and get hours instead of degrees; in No. 2 we seem to divide degrees by degrees and get hours instead of an abstract number; in No. 3 we seem to multiply degrees and get an abstract number, and to divide one abstract number by another and get concrete time in the quotient; in No. 5 we omit part of work of reduction but otherwise the solution is a truthful one, with none of the errors of reasoning of the rest. This problem has been selected as the first for consideration because it opens at once such a wide range of possibilities of form, but essentially the same question repeatedly occurs from the very first grade through the pupil's school life. If 1 yard of cloth costs 15c., what will 6 yards cost?—is a simpler question to consider. Here we have these possibilities, among others:

(1)	(2)	(3)
15	15	15c.
6	6	6
—	—	—
90	90c.	90c.

(4) $6 \times 15 = 90$ (5) $6 \times 15 = 90$c.

(6) 6×15c. $= 90$c. (7) $15 \times 6 = 90$c.

Out of all these, which shall a child use in writing a solution?

In each of these problems the fundamental question is the same: Shall written work be considered from the standpoint of the answer only, as a business man would be inclined to do, or from the standpoint of the logic of the school, the often non-practical school?

The answer to such a question ought not to be dogmatic to the extent of saying that any one form is always the best, although it may say that those forms that are untrue in statement are always bad. That is to say, sometimes it is better to write

15)33 45	15
2 15)225	6
15	—
2 hr. 15 min.	90

At other times the step form, with the denomination accurately set forth, is better. We need to distinguish between two lines of work, equally important; the one relates to accuracy and speed in operation, the getting of an answer as a business man would, with no circumlocution and no superfluous symbols or operations; this is the mechanical part of the problem and there must be abundant exercise on this side. Then there is the equally important side of the reasoning, explaining why the mechanical work is performed as it is, why we multiply instead of divide, and how we know that the result is hours instead of degrees, or cents instead of yards of cloth. Here the step form of analysis may be depended upon to show the pupil's line of reasoning. These two lines of written work are, therefore, legitimate. What, then, is illegitimate in written work, and what are the dangers to be guarded against in that which we do adopt? As to the first, it may be laid down as axiomatic that a form that states or seems to state a falsehood is illegitimate. That is, $30° \div 15 = 2$ hr. is a false statement; it is not even excusable on the score of brevity, since $30 \times {}^{1}/_{15}$ hr. $= 2$ hr. is as brief, is true, and is as easily explained as any form. So $6 \times 15 = 90$c. is a false statement and should not be tolerated, although $6 \times 15 = 90$, or 6×15c. $= 90$c., is legitimate. And as to the dangers against which to guard, the following advice may be given: (1) To require that every applied problem should be solved in steps is to encourage arithmetical dawdling; the pupils should continually be exercised in rapid solution, the correct answer speedily ascertained, as a business man would get it, being the aim. A pupil who lets his mind continually dwell upon dollar signs and well written steps cannot help but drop away from strict attention to rapidity and accuracy of calculation. (2) To split hairs on questions of such forms as 9×15c. or 15c. $\times 9$ is to get away from the essential point; we must recognize the fact that there is good authority for both, although the former, writing the symbols in the order they are read, is coming into rather general use in America. The great question is to see, in these analyses, that the thought is clear, and that a pupil is not thinking in a hazy way of "15 cents times 9." (3) To require no analyses of the applied problems is an extreme that is about as bad as to require them for all, and perhaps worse. It is quite sure to result in looseness of reasoning that makes correct results mere matters of luck. (4) To require some particular form of analysis, only to meet the idiosyncrasy of the teacher, is also a danger against which we need to be on our guard. For example, always to require a solution stated in one step, if possible, is a hobby that some like to ride, because it seems to demand continued thought, although it is entirely foreign to the plan that a common-sense business man would adopt, and is not the form of reasoning that we commonly take in mathematics. So to require that a child should always take some unitary form of analysis, finding in every case what one thing costs, may be the means of checking the originality and dampening the ardor of some very promising pupil.

In general, therefore, the teacher should see to it that there is a reasonable amount of rapid, accurate solution, the "answer" being the paramount object. He should also see that there is a reasonable amount of written analysis, accurately stated, preferably in the convenient and terse form of steps, but not limited in any notional way that would destroy originality or make a solution unnecessarily long.

In the marking of papers it should be born in mind that there is only one test for a question involving a single operation. Either the answer is right or it is wrong. If the problems require some interpretation, a teacher may properly mark both for operations and for method; that is, a pupil may perform his operations correctly, but may have misinterpreted the meaning of the problem. In that case some credit may properly be given for the correct operation. In general, however, papers in arithmetic should be marked, as they are in business, largely by the accuracy of the result. In any single operation the work is *right* or it is *wrong*. A business man will not excuse a bookkeeper who writes $9250.75 instead of $90,250.75. Ony a zero is missing, but it means a difference of over $80,000. If the result is wrong, the paper is wrong. The converse of this statement is not true, for the result may be right, and yet the paper may be justly criticised for its slovenly appearance and the inaccuracy of the forms used. Where a time limit has been set, and a class has been given twenty minutes to solve as many problems as possible, teachers must use their judgment as to marking pupils who are naturally slow. If their work is accurate, and they have done a reasonable number of examples, they are entitled to credit and should receive commendation.

BIBLIOGRAPHY: The author's *Teaching of Elementary Mathematics,* pp. 121-129; *Handbook to Arithmetics,* p. 8; *Practical Arithmetic,* Boston, 1906, pp. 115, 159; Wentworth-Smith, *Complete Arithmetic,* and *Arithmetic, Book II,* Boston, 1909 and 1911, p. 191.

CHAPTER IX

CHILDREN'S ANALYSES

The questions of mental and written arithmetic lead naturally to that of the analyses to be expected on the part of children. What is their object, what should be their nature? How extensively should they be required?

As to the first, the only defensible object would seem to be that through these analyses a child makes it clear that he understands a particular problem or operation. That he acquires a habit of formal statement that is helpful in other lines of work, or that his memory is strengthened by learning set forms of analysis, has been too often disproved to require argument. To the extent that this analysis is really an explanation of his process there is an unquestionable advantage, since it enables a teacher to commend or improve the pupil's work. But how often is this the case? Indeed, how often should it be expected to be the case? Is it not the general experience that pupils too often memorize their analyses, and that teachers commend glib repetitions of their own words or those of the text-book, the matter being so imperfectly comprehended by the child that he is able to bear no questioning?

To take a concrete case, we occasionally hear some teacher say that not a child in the class can explain why, in dividing by a fraction, he inverts the divisor and multiplies. But why should he explain it? And if he does, will he do any more than repeat in a perfunctory way the analysis he learned from the book or the teacher? It took the world thousands and thousands of years to learn this process. It was a thousand years after Euclid made his great geometry before it was used, and nearly another thousand years elapsed before it appeared in a printed book. This means that maturity of mind was required to develop such a process, and still greater maturity was needed to embody it in a text-book.

But does this mean that no explanations are to be given or required? By no means. A child should know this process of dividing, and he should learn it by a teacher's questioning; he should thereby know that it is reasonable, and he should feel that for the time he understands why he proceeds in this manner. For that occasion he may be questioned as to all this, but that he should long remember the "why" of it all, or that he should be able, at any time that some teacher or supervisor thinks fit, to give a lucid explanation of such a mature process is as unnatural as it is unscientific.

So it is with the fundamental operations in general. There is no good reason why a child should remember for any considerable time an explanation for multiplying one integer by another; it is sufficient that he learned the operation as a rational one, and that he can perform it quickly and accurately as we can or as any business man does. If he does give an explanation it will usually be found to be merely a parrot-like repetition of the teacher's or the text-book's words, without any apparent mental content.

In the matter of the applied problems the case is different. So long as a pupil does not blindly recite formal analyses, there may be a good deal of value in his explanations. If allowed to state his reasons in his own language, with limitations as to tolerable English, he may acquire a habit of succinct and logical statement that will help him in many other lines of expression. This affords, moreover, a very good opportunity for the teacher's commendation and advice,—criticism in the best sense of the term, the word too often being employed to signify mere fault-finding.

My colleague, Professor Suzzallo, has properly called attention to the fact that the problem of teaching children to reason in arithmetic is twofold: (1) "It is a matter of the ability to use language; (2) It is a matter of good thinking." The former has been confused with the latter by most teachers, it being felt that if the child repeated the book language of reasoning he was satisfying the demand for honest thinking. Genuine training in reasoning is not this, however; it is a carefully thought out process, beginning with problems involving only a single step, and leading gradually to those involving two steps. This is a reasonable limit of primary work, problems involving three steps being rather matters for the intermediate grades.

In all this work it should be borne in mind that there are three things that are properly demanded at one time or another, but not necessarily for each problem that is solved. These three are: (1) to work rapidly and accurately; that is, to take the shortest road to the answer, and to be certain the answer is correct; (2) to put neatly on paper not merely the operation but a brief explanation; (3) to give a brief analysis or oral explanation.

For example, if 5 yd. of cloth cost $2.10, how much will 12 yd. cost?

(1) *The number work:*

$$\frac{12 \times \$\overset{.42}{2.10}}{5} \qquad \begin{array}{r} \$.42 \\ 12 \\ \hline 84 \\ 42 \\ \hline \$5.04 \end{array}$$

(2) *The written explanation:*

5 yd. cost $2.10. 1 yd. costs $^1/_5$ of $2.10.
12 yd. cost $12 \times {}^1/_5$ of $2.10, or $5.04.

(3) *The oral analysis:*

Since 5 yd. cost $2.10, 1 yd. will cost $^1/_5$ of $2.10, and 12 yd. will cost $12 \times {}^1/_5$ of $2.10, or $5.04.

Teachers will not need to call for all this work with every example. Sometimes it will be necessary to emphasize (2) and sometimes (3), but the important thing is that (1) should be quickly and neatly, and, above all, *accurately* done. One of the best ways to secure this accuracy and to avoid absurd answers is to *estimate the result* in advance. The pupil should write down this estimate and compare it with the answer, and if there is a great difference look over his work again.

For example, if 5 yd. cost $2.10 we know 12 yd. will cost nearly 2½ times as much, or somewhere near $5. When we solve, if we find such a result as $50.40, we see at once that there is a mistake, probably in the position of the decimal point. The correct result is $5.04.

As an example of written work, involving both the computation and the analysis, the following may be considered:

A merchant bought 800 yd. of linen lawn at 67½c. a yard, and sold 725 yd. at 80c. a yard, and the rest at a bargain sale at 65c. a yard. Find his profit.

$0.67½	$0.65	$725
800	75	.80
400	325	$580.00
536	455	48.75
$540	$48.75	$628.75
		540.
		$88.75

The written analysis is as follows:

1. 800 × $0.67½ = $540, the cost.
2. 800 yd. − 725 yd. = 75 yd., sold at bargain sale.
3. 75 × $0.65 = $48.75, received at bargain sale.
4. 725 × $0.80 = $580, received at regular sale.
5. $580 + $48.75 = $628.75, total receipts.
6. $628.75 − $540 = $88.75, profit.

Of course this solution could easily be shortened, but for beginners it is as well not to attempt too much brevity.

In conclusion it may be said that set forms of analysis seem to be rather more harmful than helpful to children, but that explanations in their own language may be the means of acquiring valuable habits and of offering to the teacher the opportunity for helpful suggestions. To acquire this power the child must be initiated gradually, first in simple examples in one-step reasoning, and second in two-step reasoning.

Bibliography: Young, p. 205; The author's *Handbook to Arithmetics*, p. 9; Suzzallo, *loc. cit.*

CHAPTER X

INTEREST AND EFFORT

There has of late years been a tendency throughout the country to make arithmetic, as other subjects, more interesting to children. What the real motive was it is hard to say, since it was probably somewhat subconscious. Such statistical information as we have shows arithmetic always to have been looked upon by children as one of the most interesting subjects of the course, so that the reason was not that it was relatively a dull study. Possibly the desire was that the work of the teacher should become easier through increased interest on the part of the pupils. But whatever the reason it cannot be questioned that, other things being always kept equal, there is a great gain in increasing the interest in any kind of work.

There is, however, a general danger accompanying this effort to increase interest. If this increase means that the subject is to become anæmic, if it is not to require the same serious effort to master it as heretofore, then it loses a considerable part of the value that has generally been assigned to it. Moreover, through this same cause it loses a considerable part of the very interest that was expected to be fostered. Boys and girls do not like to wrestle with infants or with infantile subjects, and unless a study is suitably graded as to difficulty it will appeal in vain to the interest, the vigorous attack, and the responsive mental effort of the pupils.

Our lesson, therefore, is that we should do all in our power to make arithmetic interesting and even attractive to the children, but that we must not hope to attain this result by offering a sickly substitute for the vigorous subject that has come down to us. Unfortunately we have not been free from this fault of making our arithmetic, and particularly our primary arithmetic, anæmic. Foreign critics frequently comment upon this failing, and claim with good reason that much of our work in the early grades lacks vitality. Certain it is that in spite of many points of superiority of the American school we do not at the end of eight years bring our children as far as European experience would seem to lead us to hope.

How can the interest in the applications of arithmetic be aroused and maintained? The reply has already been made. They must be real if they pretend to be so, they must relate when possible to the child's daily environment, and they must reveal the life of America to-day in such a way as to be broadly informational as well as mathematical. This can be accomplished with no less demand for mental power than was required by the obsolete problems of our old-style books. There are, however, various other channels through which we may pass to reach the required end. For example, there are the number games for children of the primary grades, games that have an interest that pleasantly conceals the mental effort required, as tennis does the muscular effort, but that accomplish the result efficiently. This subject is considered later in this work. Then there are the problems of heroic effort, then of mechanical effort, then of simple building, and later of our national resources. These resources, correlating as they do with geography, concern our supply of food, and clothing, and home comforts; they touch our mines, our farms, our transportation, and the great industries of our people.

As concrete illustrations of this type of problem the following relating to some of the metals produced in this country may be considered:

1. When 700 lb. of copper was worth $73.50, what was 250 lb. of the same quality worth?

2. A dealer sold 800 lb. of nickel at 55 ct. a pound, thus gaining 10% on the cost. How much did it cost him?

3. If a dealer buys 900 lb. of zinc at $88 a short ton, at what rate per pound must he sell it to gain 50%?

4. In one year silver averaged 54.98 ct. an ounce, which was 102.8% of the average for the preceding year. What was the average then?

5. Silver is sold by the troy ounce. This is what per cent of the avoirdupois ounce (7000 gr. = 16 av. oz., 5760 gr. = 12 troy oz.)?

6. Quicksilver (mercury) is sold by the flask of 76½ lb. If a dealer buys it at $38.25 a flask, at how much a pound must he sell it to gain 20%?

7. If a short ton of lead is worth $96, how much is 750 lb. worth? If a dealer bought it at this rate and sold it at 8 ct. a pound, what was his per cent of profit?

The interest in such topics is measurably greater than that in equation of payments, or most problems in compound proportion, or examples in the Vermont Rule of Partial Payments (a subject that, however, naturally has its place in the curriculum of that state). On the other hand the effort may be just as great as we wish to make it. It is only a matter of complicating the problem sufficiently, and using numbers and combinations of proper difficulty, to make a modern problem about the coal industry of Pennsylvania, or the silver output of Colorado, as hard as any example in the arithmetics of fifty years ago.

We have, therefore, the following points that seem fair conclusions: (1) It is possible to bring our arithmetic work to a higher plane of interest, through the game element and the applications; (2) it is possible, with this, to keep the plane of effort as high as we wish; (3) with the increased interest must necessarily come an increase of power that is vital to the improvement of our education.

Bibliography: In the matter of modern problems consult the author's arithmetics. In the matter of mathematical recreations consult Ball, *Mathematical Recreations*, London, 1896; Schubert, *Mathematical Essays and Recreations*, Chicago, 1899.

CHAPTER XI

IMPROVEMENTS IN THE TECHNIQUE OF ARITHMETIC

Nothing new goes into arithmetic without a protest, and so for what goes out. Nevertheless there has been an evolution here as everywhere else, and this evolution has made for the betterment of the subject. To take a concrete illustration, the first printed arithmetics had no symbols of operation. What we would write as "4×5" was then written "4 times 5," with the natural variation of the word "times" according to the language employed. It was half a century later, and after the symbols + and − were invented, before = was suggested, and some eighty years after that before × was used, and a long time after that before ÷ appeared for division. It was several generations after these were first used before they came into our school arithmetics for the purposes that we use them to-day, and always with strong protest on the part of those who wish to "let well enough alone." It was argued that "4×5" was more abstract than "4 times 5," that it was hard because of the symbolism, and that it took arithmetic from the written language and the customs of the common people for whom it was of greatest use. Invented for algebra, the conservatives said that all the symbols ought to remain there and not seek to enter the field of arithmetic. This struggle of symbolism seems strange to us to-day, when a child in the first grade learns at least half a dozen signs of operation and relation, and few would be found to advocate going back to the old custom. We are, however, face to face with similar questions and many of the very ones who would argue to keep ÷ (a symbol practically unknown save in England and America, strange as this may seem), are the ones who protest most vehemently against letting x stand for a phrase that is too long to write conveniently. But the question is the same. If we use ÷ as a short-hand way of writing "divided by," why should we not use x as a short-hand way of writing "what the horse cost," or "the amount due the first man," or any other phrase representing a quantity to be sought, an "unknown quantity"? Here, then, is one of the improvements suggested by algebra to assist us in reasoning out the solution of an arithmetical problem. That "$1.10x = \$3300$, therefore $x = \$3000$" is algebra instead of arithmetic, is no more true than that "4×5" is algebra while "4 times 5" is arithmetic. The symbol $\times$ was used only in algebra instead of arithmetic for a century or so, as x was for over a century longer, but the employment of each to assist in arithmetic does not make "a solution by algebra." This illustration is brought forward as one of the most prominent at the present time. It is impressed upon me by numerous letters asking for "a solution by arithmetic instead of by algebra" for some little problem that is made clearer by the use of a single symbol in place of a phrase like "the number of bushels," or "the cost of the farm." Teachers should realize that they hereby show an ignorance that is hardly pardonable at the present time, and that such improvements in symbolism are a part of the natural development of the subject.

Of course there is the danger of overdoing all this. This has often been seen, and is apparent to-day. For example, it is better to train a child's eye to see that 4 and 5 are 9, by putting the symbols in the form here given, the form in which he will usually meet them in computation,

$$\begin{array}{r} 4 \\ 5 \\ \hline 9 \end{array}$$

than to train it with the symbolism $4 + 5 = 9$, which he rarely sees in practical work. On the other hand, to neglect the latter entirely is to unfit him for reading any save the simplest mathematics, and for expressing his solutions in the condensed step-form necessary to allow the eye quickly to grasp the reasoning. So with a symbol like x; we may use it where there is not only no advantage in doing so, but a positive disadvantage. It is the part of the text-book and the teacher to suggest to the pupil the problems in which it should be employed, and to furnish a reasonable amount of exercise in the subject.

To consider a still more definite illustration, take this problem: If some goods are sold at a profit of 12½% for \$1,012.50, what did they cost? Here we have several possible lines of attack, among which are the following, (1) If there was a gain of 12½%

they must have been sold for 112½% of their cost. (But where did that 100 come from?) And since $102½ is 112½% of the cost, 1% of the cost is $\frac{1}{112\frac{1}{2}}$ (But how are we to read such a fraction?) of $1,012½, and 100% (But why do we wish 100%?) is 100 times this. Such an explanation, while it could be learned and recited, would be subject to questions like those inclosed in the parentheses, and really would have very little reasoning in it.

(2) Let 100% stand for the cost. (But why 100% instead of 200% or 50% or some other number?) Then 100% + 12½% will stand for the selling price. If 112½% = $1,012½ (But an abstract number, 1.12½, cannot equal a concrete number, $1,012½%) then 1% = $1,012½ ÷ 112½ (Why? Why not multiply? Why not divide by 112½% instead of 112½?), and 100% (why not find 500%?) = 100 times as much, or $900. This analysis is quite as superficial and unsatisfactory as the first.

(3) Let 1 = the cost. (But why not let 2 equal the cost? and why take 1 instead of 100%?) The rest of this solution might follow the lines of (1) or (2), and in that case it would be equally unsatisfactory. Like (2), it is a relic of the old and long-since forgotten method of "False Position," the discarding of which caused so many conservative teachers to feel that arithmetic was losing its mental discipline. So it would be possible to give a number of other plans of attack, some better than the above, and some much worse. Consider, however, a single other plan.

(4) Let c = the cost, a very natural thing to do since it is the initial letter. Then the selling price is $c + .12\frac{1}{2}c$.

But $c + .12\frac{1}{2}c = 1.12\frac{1}{2}c$, and therefore $1.12\frac{1}{2}c = \$1012.50$. Dividing equals by 1.12½, $c = \$900$.

Now we are not troubled here about any 100%, or 1%, or letting 1 equal something that it cannot equal. As soon as we think of 12½% as the same as 0.12½%, and know that $c + .12\frac{1}{2}c = 1.12\frac{1}{2}c$, just as $c + \frac{1}{2}c = 1\frac{1}{2}c$, the case is exceedingly simple.

Here, then, is one of the places in which the technique of arithmetic has been greatly improved by the introduction of an easy symbol from algebra, as it was years ago improved by the introduction of the symbols of operation. No teacher who has ever seriously tried to use this symbol has ever willingly abandoned it.

What has been said for the symbol x might be said for other symbols if we needed them. It is of no particular consequence that we use $4^{\frac{1}{2}}$ instead of $\sqrt{4}$ in arithmetic, because we do not make much use of square root in arithmetic, but if we did the more modern symbol would deserve a place for its influence in later work, and the same may be said of the negative number, the parenthesis, and other signs of algebra.

Outside of symbolism, however, there are certain improvements in technique that demand our consideration. One relates to subtraction, and in order to have it satisfactorily understood it becomes necessary first to speak of it historically. There have been several successful plans for teaching subtraction; each has endured for a long period, and some of them are in use to-day. Of the most prominent the following may be mentioned:

(1) The complementary method. Instead of taking 8 from 13 we may add $10-8$ to 13 and then drop 10. This depends on the relation $13-8=13+(10-8)-10$, and since $10-8$ is called the complement of 8 (to the number 10), this is known as the complementary method. It is very old, appearing in a famous Hindu work of the 12th century, in the first printed arithmetic (1478), and in numerous other text-books.

$$\begin{array}{rr} 13 & 13 \\ 8 & 2 \\ \hline 5 & 5 \end{array}$$

In a case like $452-348$ the operation would be as follows: $2+2=4$; since 10 must be dropped, we may add 1 to the 4 instead; then $5-5=0$, and $4-3=1$. That is, the complementary idea need be used only when the minuend is less than the subtrahend. This example is actually taken from the first printed arithmetic. The plan is the same as the one used in Pike's famous American arithmetic a century ago, and some teachers still employ it. It is essentially the one used when we employ co-logarithms in trigonometry.

$$\begin{array}{r} 452 \\ 348 \\ \hline 104 \end{array}$$

(2) The borrowing and repaying plan, a name that we may use for want of a better one. This may be illustrated by the annexed example, taken from the first great business arithmetic ever printed (Borghi's, of 1484). The operation is as follows: 8 from 14, 6; 8 from 15, 7; 10 from 13, 3; 3 from 6, 3. This was the plan advocated most often by

$$\begin{array}{r} 6354 \\ 2978 \\ \hline 3376 \end{array}$$

the early printed arithmetics, and the expression "to borrow" was a common one. It was known to the early Hindu arithmeticians and in Constantinople before the invention of printing.

(3) Simple borrowing,—to continue to use this old English expression. In this method the computer says, "7 from 12, 5; 2 from 3, 1." This is probably the most common plan in use to-day, and it has much to commend it. It has a long history, appearing in the oldest known manuscript on arithmetic in English, in Spain in the 13th century, in Italy in the Middle Ages, and in India still earlier. It has not, however, been nearly as popular as the second plan, although more generally used in our country to-day.

$$\begin{array}{r} 42 \\ 27 \\ \hline 15 \end{array}$$

(4) The left-to-right method, a plan that has had a long history in which many prominent advocates have appeared. It is more adapted to the needs of a professional computer, however, than to those of the average citizen, and may therefore be dismissed with this mere mention.

(5) The addition, "making change," or "Austrian" method, a vague way of naming several related plans. The efforts made to adopt it in the Austrian schools, and the consequent notice taken of it in Germany, have been the cause of its most inappropriate geographical name. As a definite method of subtraction it is not as old as the others, although it appears in the 16th century in Italy and has had occasional prominent advocates since. It consists in finding what number must be added to the subtrahend to make the minuend. Thus in thinking of $17-8$ we think: "8 and 9 are 17," writing down the 9. If the numbers are longer we may proceed in either of two ways, as in the annexed example. Here we may say: "8 and 5 are 13; 4 and 7 are 11; 1 and 2 are 3." Or we may say: "8 and 5 are 13; 5 and 7 are 12; 2 and 2 are 4." Of these two, one is as easily explained as the other. The first might naturally be approached thus:

$$\begin{array}{r} 17 \\ 8 \\ \hline 9 \end{array} \qquad \begin{array}{r} 423 \\ 148 \\ \hline 275 \end{array}$$

$$\begin{array}{l} 423 = 400 + 20 + 3 = 300 + 110 + 13 \\ \underline{148 = 100 + 40 + 8 = 100 + 40 + 8} \\ \phantom{148 = 100 + 40 + 8 = {}} 200 + 70 + 5 \end{array}$$

The arrangement for the second would be as follows:

$$\begin{array}{r} 423 = 400 + 20 + 3 \\ \underline{148 = 100 + 40 + 8} \end{array}$$

Since the difference will be the same if we add the same number to both minuend and subtrahend, we will add 10 to each, and then 100 to each, giving the following:

$$\begin{array}{r} 400 + 120 + 13 \\ \underline{200 + \ \ 50 + \ \ 8} \\ 200 + \ \ 70 + \ \ 5 \end{array}$$

Since the explanations are about equal in difficulty, we may consider the sole question of rapidity in practical use, both as to the method in general and as to these two sub-methods in particular.

Is the general plan the best one? On the side of advantages we have: (1) It is the common method of making change. If I owe $7.65 and pay $10 the merchant finds the change and I verify his work by saying: "65c. and 5c. are 70c., and 30c. more makes $1, and $2 more makes $10." That is, we find the difference by adding. This is familiar in all business and in the school, and will remain so. It is, therefore, a natural plan to use in all subtraction. (2) It avoids the necessity for learning a separate subtraction table. Everything is referred at once to the addition table, a table that unfortunately is not at present known any too well. There is, therefore, an economy of time and an increased efficiency in the very important subject of addition. (3) The facts of addition being used so much more often than those of subtraction, there is naturally an increase in speed and certainty when we employ the addition instead of the subtraction table.

On the other side, it is not desirable to change the customs of the people unless there is a decided gain by so doing. Parents who have been brought up on one plan, and who help their children more or less in their lessons, do not easily adapt themselves to a new one, and the result is a confusion in the child's mind that is most unfortunate. The fair question, therefore, is, Is it worth the while to use the better method?

If we had a standard method known by all, this argument would have much weight, but we have at least three rather common ones, with several variations, already in use in this country.

There is, therefore, bound to be more or less confusion. The only thing for the school to do, then, is to teach the method that will prove in the long run to be the most rapid and accurate, and this seems *a priori* to be the "Austrian," although a scientific investigation of the matter, on sufficient data, is desirable. And of the two or three sub-methods, the second one described on page 46 seems without doubt the better.

The question is, it should be repeated, not one of explanation, since any one of the methods is easily explained. It is purely one of practical utility, in which the teacher should divorce himself of all prejudice in favor of the method which was taught to him and with which he is most familiar,—a somewhat difficult thing to do in a discussion of this kind. It should also be remarked that it is of doubtful policy to attempt to change a method that a child already knows and handles easily, inasmuch as the difference in value is not great enough to warrant this course. For beginners, however, the plan suggested is probably the best.

These two matters of improvement in technique have been mentioned, not as so important in themselves, but as types of various changes that are worthy of sympathetic consideration. The placing of the quotient over the dividend in long division, instead of at the right as was formerly done, so as to make more clear the position of the decimal point; the multiplying of both dividend and divisor, in the division of decimals, by such a power of 10 as shall make the divisor an integer, thus avoiding the old difficulty of determining the number of integral places in the quotient; the giving of the full form of an operation before the abridged one, in explaining the process; the writing of ratios in the familiar form of fractions in the first steps in proportion; the putting of the unknown quantity first instead of last in writing a proportion and the using of x to represent this quantity,—these are some of the other improvements in technique that are getting into our books in these days and that should command our interest. To make these more clear, the space not allowing of elaborate discussion, an example of each is here given. Thus in long division it is better to use the first of these forms than the second, for the reason specified, that it makes clear the position of the decimal point,—a reason that holds equally true for the second example given on page 49:

```
     14.24
72)1025.28                72)1025.28(14.24
   72                        72
   ---                       ---
   305                       305
   288                       288
   ---                       ---
    172                       172
    144                       144
    ---                       ---
     288                       288
     288                       288
     ---                       ---
```

Similarly, the first of these forms is better than the second, the problem being to divide 102.528 by 0.72:

```
     142.4
72)10252.8                0.72)102.528(142.4
   72                            72
   ---                           ---
   305                           305
   288                           288
   ---                           ---
    172                           172
    144                           144
    ---                           ---
     288                           288
     288                           288
     ---                           ---
```

In the early stages the first of the following forms is better than the second:

```
    1424                                  1424
72)102528                             72)102528
   72000 = 1000 × 72                     72
   -----                                 ---
   30528                                 305
   28800 =  400 × 72                     288
   -----                                 ---
    1728                                  172
    1440 =   20 × 72                      144
    ----                                  ---
     288                                   288
     288 =    4 × 72                       288
     ---                                   ---
```

In introducing the idea of proportion it is better to begin with known symbols, so as not to confuse the pupil too much. Thus the first of these forms is better than the second in the early stages:

$$\frac{x}{7}=\frac{39}{91} \qquad 91 : 39 :: 7 : (?)$$

Indeed it is always better to use = than ::, and the latter is now happily passing into oblivion in this country.

There are several more important questions for the future, in relation to the technique of computation, but these can be mentioned only briefly. The first relates to limits of accuracy in results. This does not mean that the work may be inaccurate, but that if we know the circumference of a circle only to two decimal places we cannot from that find the diameter to three or more decimal places. We express this by saying that the result cannot be more accurate than the data. Suppose, for example, we have made several measurements of the circumference of a steel shaft, and their average is 7.57 in.; it is evidently useless in dividing this by 3.1416 to carry the result beyond two decimal places. Required, therefore, to divide so as to avoid unnecessary work. This is accomplished by what is known as contracted division, thus:

```
          2.41
3.1416)7.57
          6 28
          ----
          1 29
          1 25
          ----
             4
             3
          ----
```

Now how much of this kind of work are the schools called upon to teach? It is certainly not a thing that many people will need to know, and, therefore, it is properly omitted from most textbooks to-day. In some localities, however, it might very properly be taught, and when our classes in physics require the mathematics that they might properly demand, it may become necessary to teach contracted multiplication and division in the schools.

Another topic is logarithms. In all engineering computations this labor-saving device is used, and the subject is easily taught. What shall we do with it in arithmetic? So far as grade work is concerned there is nothing to be done at present, because the demand is not great. But we can hardly say what the future may bring forth, and their use may become much more common than we think if the teachers of physics and the advocates of more mathematics in manual training stir up the question of greater facility in practical calculation. In the same way we may yet see the slide rule (a simple instrument for computing by machinery) find a place in business or technical courses in the grades of the elementary school. At present our duty points to a teacher's knowledge of all these matters of technique, and a sympathetic awaiting of the time when some or all may demand more serious attention on the part of the school.

CHAPTER XII

CERTAIN GREAT PRINCIPLES OF TEACHING ARITHMETIC

Before considering the curriculum in arithmetic it is well to devote a little attention to certain great principles that teachers have as a whole agreed upon, in theory if not in practice. Some of these have already been discussed in this article; others will strike the reader as rather trite, which simply means that they are generally accepted; and others will not appeal to all. They will, however, be found to be suggestive of the thought of leaders at the present time, and they may, though stated dogmatically, form the basis for profitable discussions by teachers.

1. Arithmetic is taught both for its usefulness in daily life and for the training that it gives the mind in reasoning, in habits of application, and in exactness of statement.

2. Most of the mental discipline of arithmetic can be secured from those portions that may be called practical, and therefore the practical side of arithmetic may safely be emphasized.

3. But in emphasizing this practical side we need to offer a large amount of abstract work as well as concrete problems, skill in either not necessarily signifying skill in the other.

4. In the concrete problems, whatever pretends to be genuine, to represent practical questions of American life, should be so, all obsolete business problems being replaced by modern questions. In particular, the daily industries of our people should be drawn upon to the making of arithmetic interesting, informational, practical.

5. The topical arrangement of the curriculum has been permanently abandoned, and either a non-topical arithmetic should accordingly be used, or some good modern topical arithmetic should be adopted and selections should be carefully made so as to offer a moderate "spiral" ("concentric-circle," "recurring topic") arrangement adapted to the growing powers of the child.

6. No extreme of "method" should be adopted by any teacher or school, but the best of every "method" should be known as far as possible to all. To measure everything in sight, to base all arithmetic on sticks, to go to extremes on number charts, to put all the time on mental arithmetic, to have all written work placed in steps, to get into any narrow rut whatever,—this is to fail of the best teaching and to narrow the horizon of the children in our care. A good, usable text-book, broad in its purpose, modern in its problems, and psychologically arranged, is one of the best balance-wheels on us all, and we should depart from its sequence and methods only for reasons that have been very carefully considered, while supplementing its good features by all the problems with local color that we can find time to use.

7. Mental (oral) arithmetic should play a part in every school year, to the end that children should have not only an eye training for numerical relations, but also an ear training and a tongue training. The text-book may be expected to furnish a considerable amount of the abstract work in this line, but the concrete problems may well be correlated with local life and with the other work of the class.

8. Children's analysis instead of being memorized should be genuine statements of the reasons that prompt them to their solutions. As to problems, the analysis should proceed gradually from one-step to two-step cases. As to operations, it is not to be expected that children will long remember the reasons involved; they should understand the process when presented through a development by a series of simple questions; but they should not be expected to give a very elaborate explanation of a topic like long division after the process is once understood.

9. Written arithmetic may at one time emphasize the rapid securing of results, and at another the analysis of the problem. Both are important, but in general the accurate result, rapidly secured, is the great desideratum. To say that a child ought not to work merely for the answer is a well-sounding epigram, but if it is interpreted to mean that he may work in a slovenly way, dawdling over his problem, and getting an answer that is absurd, no amount of neatly written step-work can atone for his mental laziness.

10. Arithmetic should be as interesting as any other subject in school. To this end a teacher should know something of its interesting story, should be familiar with its best applications to local and national life, should know how to treat the mental (oral) exercises in sprightly fashion, and should have a fair stock of number recreations.

11. The improvements in the technique of arithmetic, including the use of x and the later forms of operations, should be sympathetically understood by teachers, to the end that the subject may not stagnate in our schools.

12. It is a matter of relatively little importance that we present fractions, we will say, in this way or that, by sticks or paper-folding or clay cubes or blocks; but it is a matter of great importance that we present the subject in some concrete fashion, to the end that the child does not proceed by arbitrary rules but makes up his own directions, and that he is so guided that these are the best that can be evolved at his age. What has been rather pedantically called "heuristic teaching," in its original form as old as Socrates at least, should always be in the teacher's mind—to lead the child unconsciously to feel that he is the discoverer, but to see to it that he is allowed to discover and to fix in mind only what the world has found to be the best. The carrying out of this policy, in arithmetic or any other subject, is one of the essentials of good teaching.

These are a few of the larger principles that should guide the teacher of arithmetic. The list might be extended, but these suffice to show the spirit in which the subject should be approached. Minor principles will appear as we consider the work of the various grades or school year.

Bibliography: Smith, *Teaching of Elementary Mathematics;* Young; *The Teaching of Mathematics;* De Morgan, *On the Study and Difficulties of Mathematics,* Chicago, 1898; Clifford, *Common Sense of the Exact Sciences,* 3d edition, New York, 1892.

CHAPTER XIII

GENERAL SUBJECTS FOR EXPERIMENT

It is well, before leaving this general discussion, to consider a few of the subjects for legitimate experiment in the teaching of arithmetic, that might occupy the attention of schools of observation or practice in connection with institutions for the training of teachers.

(1) It is desirable to know just how far recreations in number can be used to advantage in teaching arithmetic. Of course we have such games as bean bag, ring toss, and sometimes dominoes and number games with cards, used in the school room. This, however, is a mere beginning. There are many more games that are usable for children. For example, more people in the history of this world have learned elementary number through dice than in the public school, and this is only one of several widely used number games. It would be very easy to go to a ridiculous and even dangerous extreme in this matter, and a teacher who begins to work upon it will naturally tend to do this, and will need a balance wheel upon his endeavors. Nevertheless the work has never yet been done scientifically and it ought to be undertaken. This is, however, only a small part of the problem. There is the whole field of mathematical recreations that must sometime be examined scientifically. We have a large but undigested literature upon the subject, and no one has ever yet studied it from the standpoint of the definite needs of the grades. The result of such a study ought to add greatly to the interest in arithmetic, without going to any ridiculous and impractical extreme. Work there must always be in arithmetic, and it ought to be good hard work, but there is no reason why we should not let pupils see the amusements as well as the other interesting phases of the subject.

In this matter of the play element the following brief list of games available for primary number work may be of service.

It was prepared by Miss Julia Martin in connection with some work undertaken for the State Department of Public Instruction of Michigan:

(1) *The Game of Tag.* Every child in the room is given a number. One child is the leader. He gives orally any number below a number that has been designated by the teacher. All pupils who have numbers which are factors of the given number must change seats. They may be tagged while running or if they fail to run.

(2) *The Game of "Simon says Thumbs Up."* The teacher or one of the pupils may act as leader. The whole class take a position with thumbs up, and each pupil has a number. The leader says "Simon says 15." The thumbs of numbers 3 and 5 (the factors) must go down. "Simon says 12." The thumbs of 2, 3, 4, and 6 must go down. Occasionally the leader says "Simon says all thumbs up." This game may be used in addition and subtraction drill work.

(3) *The Guessing Game.* This game is suitable for Grade II. The materials needed are 20 counters for each pupil and the teacher, and a slip of paper and a pencil for each. The teacher picks up a handful of counters and says, "How many counters have I?" The pupil guesses, and if correct he gets the counters, and writes the number on a slip of paper. If he fails to give the correct number he must give the teacher the difference between the number and his guess. He must write this number on the paper. At the close of the game the pupils add the gains and losses, and check their work by the counters left.

Miss Martin suggests the following additional list as valuable in arithmetic: buzz; fizz; railroad station; bean bag; days of the week; hop Scotch; school ball; bird catcher; marbles. To this list may also be added certain card games of an arithmetical nature published under the editorship of the author by the Cincinnati Game Company.

(2) It is desirable definitely to map out the chief interests of children from grade to grade, with a view to ascertaining the best field for applied problems from year to year. We know these interests in a general way, and we know the child's mind well enough to judge of his arithmetical powers from year to year. But we do not yet know these interests in the exact way that we should know them. For example, when is the game element strongest? When does the interest in the heroic become most pronounced, and is the period the same for boys as for girls so that we may use this information in problem work in a mixed school? When does the interest become manifest in the food supply of our country? When in the clothing supply? When

in transportation? When in the mines? When in manufacturing? When in commercial life? We know all this in a general way, but only so,—not exactly, not as the result of any scientific investigation. And when we do know this the rational applications of arithmetic from year to year will be so much better understood that the subject will have an interest that is now but feebly developed, and a value that we at present appreciate only in part.

(3) We also need to know, statistically if possible, the result upon a group of children of emphasizing the abstract problem; upon another group of emphasizing the concrete; and upon a third of leaving the two in about the balance that experience has dictated. We have had some scientific investigation in this line, but it has been very slight and it is therefore not conclusive. To emphasize the concrete would be to diminish the number of problems very greatly, and it would seem to give less exercise in number relations, and it does seem to give less satisfactory results. On the other hand it may create interest in the work, thereby increasing the power of impression of the number relations, so that because the child multiplies only $^1/_{25}$ as many times it will not follow that he knows his subject only $^1/_{25}$ as well. At present the whole subject is in the domain of doctrinaire argument; what is needed is a scientific investigation of the problem by some school or some person who is unbiased in the case.

(4) We are at present entirely unsettled upon the question of time to be assigned to arithmetic. Two scientific investigations have been made, but each is incomplete. It would seem that excellence in arithmetic work is much less a function of the time assigned to it than has formerly been supposed. Such an investigation would probably require a number of years for its satisfactory completion, but it might be undertaken in any particular school system in a single year with some very helpful results.

(5) We are quite uncertain as to the relative amount of time to be devoted to oral and written arithmetic in our schools. The wave of oral work, beginning with Pestalozzi and culminating in this country with Warren Colburn as mentioned in Chapter VII, gradually subsided some years ago. What should

we be doing in the matter to-day? It is very easy to talk dogmatically about it, but we need, if it is possible, a scientific investigation as to the practical results of more "mental" arithmetic, and of less. Would it be well to have the work much more oral than at present? or would we gain by confining our energies more to written work? Is there any scale by which we can definitely measure this matter? and if so, what is it and what are the results of the measurement?

(6) Just how far we are justified in departing from the old plan of making the operations the basis for a course in arithmetic, and of substituting therefor the applications? Shall we ever be justified in giving up multiplication as a topic and in substituting a chapter on housebuilding that shall bring in multiplication as needed? Of course in the latter part of arithmetic we put the application to the front, as in the early part we put the operation there; but just where we should draw the line, and how far should this latter plan encroach upon the former?

To this list of general subjects for experiment my colleague Professor Henry Suzzallo contributes a number of others, and these make up the rest of this chapter. He remarks in the first place that it is not sufficient that a new way or an old way of teaching has succeeded, e. g., in the addition of fractions. The test of the worth of a given method is not alone that it gets a thing done efficiently; it must get it done as economically as possible. The method of most worth is the one that obtains the efficient result with the least possible expenditure of energy. The comparative worth of two methods must be investigated under experimental conditions, with children of about the same grade, age, and previous training, taught by teachers of fairly even strength of personality, so that approximately the only difference in the conditions of the trial is the difference in the methods involved in the test. Finally the repetition of the experiment in more than one school or system, reduces the danger always present in assuming that a special group of children and teachers is typical of school conditions in general. A sample experiment will make the method of investigation clear, as follows:

In the teaching of addition combinations it is the requirement of certain courses of study that combinations and their reverses be taught in association with each other. The teaching of "3 and 2 are 5," should be followed at once by the learning of "2 and 3 are 5." In other courses of study or texts there may be quite an interval between the learning of these two combinations. In fact the student may learn each separately without being conscious of any greater intimacy between these two combinations than between any other two, e. g., "6 and 3 are 9" and "4 and 2 are 6." Those who advocate the first method, imply, if they do not expressly state, that learning a combination and its reverse simultaneously is more efficient than learning them separately and unrelated. The problem for the investigator is to determine whether or not this is true.

For example, in any large city school there may be two, three, four, or more classes of one grade in which combinations in addition are taught for the first time. In the case where there are four classes, two could be set off against the remaining two, with such judgment as to equalize number and quality of grades, teachers' personalities, and other factors, in so far as they may be equalized within such a limited range. All four classes could then be given the same list of combinations to be learned, and a common method of general procedure could be laid down, by the experimenting principal, the only general difference in procedure being that one group of classes would always learn the reverses immediately following the original combination to which it is related, while the other group would learn them in an order that would separate the original combination and its reverse. Thus:

Group I.

2	2	3	2	4	3	2	5	3	4	3	5	4	4	5
+2	3	2	4	2	3	5	2	4	3	5	3	4	5	4
4	5	5	6	6	6	7	7	7	7	8	8	8	9	9

Group II.

2	2	2	3	2	3	3	4	5	3	4	5	4	5	5
+2	3	4	3	5	4	5	4	4	2	2	2	3	3	4
4	5	6	6	7	7	8	8	9	5	6	7	7	8	9

An equal amount of time having been spent in all the classes up to a point where they have approximately covered the complete list of combinations, a test could be given to determine how far each individual had mastered these number facts. After a lapse of a week or ten days another examination would show how stable the mastery had been in each case. Any difference between the group taught by one method and the group taught by the other method as clearly shown by statistical interpretation would then tend to indicate the relative efficiency of the two methods.

This being the method of experimentation, a few of the general questions may now be considered. First, does the effective use of objective work demand a large amount of this work with relatively young students and a smaller amount with relatively old students? Or should the distribution of objective work be determined by the fact of ignorance or immaturity in any special phase of arithmetic regardless of the age of the student? To what extent does general arithmetical maturity require less objective work in the first formal study of fractions than in the first formal study of addition?

Do all of the fundamental combinations in any given field (addition, multiplication, etc.) require objective development? How many combinations need to be developed objectively before the child will clearly know that succeeding combinations given by the teacher authoritatively stand for real relations?

Furthermore, to what extent may the constant handling of objects, pictures, and diagrams, and concrete imaging in general, interfere with rapid abstract manipulation of numbers and number combinations?

To what degree are the difficulties of children with arithmetic problems due to a failure to understand underlying concrete situations, because they do not understand the language by which it is intended to convey them? What is the relative difficulty in understanding the significance of a situation when the presentation is (1) objective, (2) oral, and (3) written? Would it be well to postpone the written presentation of problems until a specific number of school years of language training have been given?

How far is it necessary to develop a special terminology for school use in the subject of arithmetic, the terms being little used in ordinary social relations? For example, consider the case of the words multiplicand and dividend, the latter having a radically different meaning in business life. In the case of signs consider such semi-algebraic symbols as ÷, and even +. If special signs are used in examples, to stand for the process of calculation demanded by the situation (which might have been expressed in concrete problem form) how wide and varied should the language of problems describing situations be? If + may be used in an example, while the same relation is expressed in a problem by such words and phrases as "added," "and," "together," "how many," "altogether," how wide should the latter vocabulary be? To what extent is a lack of dealing with problems presented through varied language, explanatory of the failure of children in problem tests given by an outsider, another teacher, the principal or the superintendent, when the children had always "done perfectly" the problems that their own teacher gave them?

Should long forms of expressed calculations always precede short forms? Taking the specific cases of division by a one-figure divisor, or column addition involving several places, does the fact that the long form is the one really used with a two-figure divisor make the case different from addition of several two-figure addends, where the long form is really not used in business?

To what extent are so called "aids" or "crutches" real helps or final hindrances to efficient and rapid work, as in the case where numbers are crossed out and others written in their stead throughout the process of "borrowing" in column subtraction?

Which types of problems are of most concrete interest to the child, (1) those drawn from his own spontaneous play and work life, or (2) those drawn from the facts of actual social life about him? Are both essential in achieving the aims of arithmetic teaching? If so is there any special law of advantageous usage of each of the two types? Should problems from his own life be used to introduce a field showing the necessity for learning means of calculation, and those from social life be used for further, later, and final application of the formal processes of calculation that have been mastered? Does it make a problem concrete to the child merely because it is a concrete reality existing in the world? May not an imagined problem vividly within the grasp of his own imagination, be more concrete in interesting the pupil in its solution than one which actually exists in the real world?

Problems may be done by the pupil (1) silently ("mental arithmetic"), (2) orally, (3) in written form on paper or blackboard, and (4) with a mixture of any two or all three of the preceding methods. Which methods represent final forms in which efficiency is demanded in ordinary life? Which forms merely represent transitional means used by the teacher to keep track of the workings of the child's mind? What is the proper order and emphasis of these forms in the mastery of a single new line of work, say in a problem where the child needs first to divide and then to multiply?

To what extent do precise oral forms assist in the correct analyses of problems, e. g., "If two apples cost 6 cents," etc.? To what extent do precise oral forms assist in the memorization of combinations or manipulations, e. g., "3 4's are 12," "put down the three and 'carry' the 2"? On the other hand, to what extent do precise written arrangements of analyses assist the child in carrying out a strictly logical mode of thinking? Take the following case as an example: "3 pencils cost 15c.

1 pencil costs $^1/_3$ of 15c." etc.

Are certain algorisms more efficient and economical than others? In which form should a pupil learn his addition combinations, $4+6=10$ or $\begin{array}{r}6\\+4\\\hline 10\end{array}$? In the case of long division should it be 45)67836(, or $45\overline{)67836}$?

Is it wise to use several different algorisms for a single process, when it is possible to use but one? In other words does not a multiplication of algorisms increase the amount of memorization of forms required of the child? In division instead of having three algorisms, one for the combinations $6 \div 3 = 2$, another for short division $3\underline{)45735}$, and a third for long division $345\overline{)45735}$, would it be any better to have a similar form throughout for easy identification of the three processes as fundamentally one, thus

$$\begin{array}{r} 2 \\ 3\overline{)6} \end{array}, \qquad \begin{array}{r} 15245 \\ 3\overline{)45735} \end{array}, \qquad \begin{array}{r} 132 \\ 345\overline{)45735} \\ 345 \\ \hline 1123 \\ 1035 \\ \hline 885 \\ 690 \\ \hline 195 \end{array}$$

and in connection with all this what weight should be given to Professor Smith's objection that the universal custom of the business world does not recognize the first two forms of this latter set?

Shall the attempt to get rapidity of calculation be preceded by the attempt to get absolute accuracy, the quickening of the work being left until certainty of command over combinations is assured? Or shall rapidity in handling combinations and manipulations be a parallel activity? Take for example the attempt to have the children attack the successive combinations in column addition with a definite rhythm, the teacher pointing to each successive stage, or chorus work being utilized with the quickest students setting the pace?

Does motor activity accompanying a process of memorization of combinations require fewer repetitions than where no special provision is made for motor activity? How far does it help to repeat the numbers aloud? to write them on paper? to manipulate objects when the combination is being learned?

How far can rhythm be used in memorizing combinations or tables? Does the use of rhythm decrease the number of repetitions required for mastery? How much?

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CHAPTER XIV

DETAILS FOR EXPERIMENT

Besides these subjects that may be designated as more or less general, there are many details that demand investigation. These are partly arithmetical and partly psychological in nature, and belong quite as much in one field as another. So far as the investigation itself is concerned, the trained psychologist is the only one who could be expected to secure satisfactory results. Some of these details have already been mentioned in this article, and a dogmatic opinion has been expressed concerning them. It is proper, however, to set them forth more at length for the use of investigators.

At my request Professor Suzzallo, who has given the matter much attention, has supplemented the suggestions given in the preceding chapter by a further list of such special experiments as occur to him, and has kindly permitted me to embody them in this article. I therefore insert them without comment, it being desirable that, in this place at least, they should appear with as little expression of opinion as possible. This chapter XIV is, therefore, to be considered as Professor Suzzallo's.

The quarrels that exist as to the proper methods of teaching arithmetic are not merely theoretic. Every difference in practice in the treatment of a subject in arithmetic implies a difference of opinion, and consequently a controversy. It is in the settlement of these practical controversies that careful experimental methods can be of large service. It can scarcely be said that such an approximately scientific approach has been seriously attempted even by educational theorists, and much less by school principals and superintendents. That there are obvious reasons for this fact goes without saying. The average teacher or principal is too busy with other matters, which for the time being are exceedingly urgent. But it must be equally obvious that provision for the investigation of teaching problems must be made somewhere within the profession. Perhaps in the beginning, it must be left to those scattered workers who have at once the impulse and the opportunity to conduct investigations. In the hope of being of assistance to such as these, the following list of practical problems now existing in the teaching of primary arithmetic has been prepared. The list is confined to the handling of whole numbers in the first few grades, and is by no means complete. It pretends merely to suggest certain differences of practice, the relative value of which needs to be determined by something more than mere opinion.

(1) Which is the best way to teach young children to count serially from 1 to 100? To have them count by ones from the beginning, extending the series as fast as the child can memorize the same, without any conscious effort in the direction of showing the child that the series repeats with a certain regularity after twenty is passed? Or to have them memorize the names in their order from one to thirty (by which time the regularity is established as a basis) and then have them learn to count by tens, later using the counting by ones and the counting by tens as a double basis for learning to count serially from thirty to one hundred?

(2) Assuming that oral counting leads mainly to the association of a name (27) with a given position in a series of names (between 26 and 28), how far is it advisable for a number to be associated with a given idea of mass or grouping, as when the device of two bundles of ten sticks each and seven individual sticks is used to explain 27? Does the effort toward the association of concrete images and numbers ultimately interfere with the rapid manipulation of figures in complex calculations? How far does the material in objective work need to be varied with first-grade children (sticks, lentils, boys, etc.) so that the idea associated with a number shall be abstract rather than the image of any particular concrete thing or group of concrete things?

(3) How far is group counting (counting by 2's, 3's, etc.) really counting, that is, proceeding from one number to another by an act of absolute memory (saying 3, 6, 9, etc., exactly as one says 1, 2, 3, etc.)? How far is it really a process of consecutive adding (3 and 3 are 6, 6 and 3 are 9, etc.)? If it is a mixture of both, where does one process end and the other

begin? If group counting is really adding, should it not always be classified with the work of addition, and placed so as to assist it, rather than be operated independently as alleged counting? How far is group counting as real counting desirable? How far may it be used as another form of addition? In the latter case should it precede or follow combination work in addition (6, 9, 12, etc., precede or follow $6+3=9$, $9+3=12$, etc.)? If counting forward is an aid to addition, how far can counting backward be an aid to subtraction? How far is real counting backward (by sheer act of consecutive memory) of valid social use?

(4) How far shall the three processes of (1) oral counting, (2) reading of numbers, and (3) writing of numbers, be parallel in the first year of formal arithmetic teaching? Should counting precede reading, and reading precede writing of numbers? How far are they dependent upon each other? In relation to accomplishment in any one of these, when should the teaching of the other begin?

(5) Why do young children who know their numbers up to twenty write 16 correctly at first, and later, when they are supposed to know their numbers to 100, write 16 as 61? Would further and special drill on certain numbers of the series have prevented this error? Which numbers require this special care? Why do children sometimes say "five-teen?"

(6) In teaching children to read and write numbers, how far is it useful and how far is it confusing to have them know the place names (unit of units, tens of units, hundreds of units, etc.)? Should such a classification be given to the child finally, or not at all? Is the so-called method of "group reading" superior to the "place" method? To the method of direct memorization? In the "group" method a child reads and writes all his numbers as he would numbers of three figures or less, naming them from the commas which mark off the groups of three, as in 34,026, "34" = "thirty-four," "," = "thousand," "026" = "twenty-six." What are the special errors which are peculiar to the "place" method? What are the special errors peculiar to the "group" method?

(7) Are all numbers of from four to six places equally easy to read and write? If not, what are the types representing gradations of difficulty? Taking the following types:

4,000
80,000
13,000
257,000
900,000
120,000
304,000

In which are errors most frequent? When these same figures appear, not in "thousands place" but in "units place," would the order of difficulty be the same or different? As in

1,000
1,257
1,900
1,120
1,304
1,013
1,004
1,080?

It will be noted that there are seven types in the first list and eight in the second, due to the introduction of ,000. Note also that 4, becomes ,004 in the second list, and passes from the easiest to next to the most difficult. Are such distinctions characteristic of children's experiences with numbers?

How does some provision for equalizing drill in all types of numbers minimize the unequal distribution of errors, as opposed to the hit-and-miss methods of drilling from personal lists made up by the teacher as he needs them?

According to the types enumerated above as a result of the investigation of thousands of children's papers, would there not be 56 ($7 \times 8 = 56$) drill types for thousands, and 448 ($7 \times 8 \times 8 = 448$) for numbers in millions place?

(8) It is generally said that there are forty-five fundamental combinations which are the basis of all work in addition. What are the fundamental facts that are required as basic and which, once learned, may be applied in new forms and situations over and over again?

There are ten numbers, from 0 up to 9. Each of these may be combined with itself and the nine others, thus making 100 combinations, from $0 + 0 = 0$ up to $9 + 9 = 18$. The 19 zero combinations are left out, leaving 81 combinations. Of the 81 remaining, 36 are reverses ($2 + 7 = 9$ is a reverse of $7 + 2 = 9$). Omitting these there are 45 combinations left as fundamental. Is this procedure correct?

(9) How far does the learning of $7 + 2 = 9$ also guarantee the acquiring of its reverse, $2 + 7 = 9$? Will the second be known without further drill? With how many less repetitions

will it be learned because the other combination is mastered? Will the two combinations mentioned be learned with fewer repetitions when they are constantly learned together, as opposed to being learned as separate individual combinations the relation of which is not specially kept in mind?

(10) Is there a justification for saying that the zero combinations ($0 + 3 = 3$) may be omitted as not being basic? The contention is that they never occur in single combination. No one says, "I have nothing and three, and adding them I have three." In such a situation we merely count what we have, we do not add our count to what we do not have, for we are not conscious of the latter numerically.

But may not the zero combinations be necessary for their later application in column addition? $6 + 3 = 9$ is used as $16 + 3 = 19$ and $0 + 4 = 4$ is used as $10 + 4 = 14$. Is it true that all zeros in column addition are ignored?

In four conceivable cases,

$$0 + 4 = 4$$
$$4 + 0 = 4$$
$$10 + 4 = 14$$
$$14 + 0 = 14$$

where the zero is found in column addition, it may be said that in three the zero is treated with one attitude; it is ignored. In the case of $10 + 4 = 14$ the zero is treated as part of quantity, and must be learned. Must not the child know all the applied zero combinations from $10 + 2 = 12$ up to $10 + 9 = 19$, and must not these eight combinations be provided somewhere in the child's instruction? Which then is the most economical and efficient way of teaching the zero combinations mentioned? To teach them as $0 + 4 = 4$ and then apply as $10 + 4 = 14$, or to teach as $10 + 4 = 14$ from the very beginning? Experimentation ought to reveal the relative value of the two methods. It ought to reveal the difference between making some provision for them and making no formal provision.

(11) In the list of forty-five fundamental combinations, the zero combinations were left out (when some should probably have been left in) and the combinations with one ($6 + 1 = 7$) were left in. Should they also have been left in? As no one adds 0 to a number in a single combination in actual life, it might be asked if we ever add one? We really count one more, not add. When we have 6 and 1 more, do we not count 6, 7, nor add $6 + 1$

= 7. Counting is a more fundamental habit than adding, and it is contended that when 1 is met in any column, the mind really climbs the scale 1, it does not group it as where 3 is met. If this is so the children being able to count serially already, need not learn one as an addition. This would omit 17 combinations.

Experimentation would show how far children taught the combinations with 1 were superior in column addition where 1's occurred, to children who had not had any training in combinations with ones.

(12) In actual instruction many teachers do not drill one type of combination any more than another. The additional drill comes later when the child fails or gets confused. Additional drill is used as cure rather than as prevention of mistake. Of the four types given, which is the easiest for children? Which the hardest? If errors are more frequent in some types than in others, is this due to the innate difficulty of certain types or to the methods of teaching them? Do children add from large to small numbers (9 + 5) more readily than from small to large numbers (5 + 9 = 14)?

$$4 + 5 = 9$$
$$3 + 7 = 10$$
$$9 + 6 = 15$$
$$10 + 8 = 18$$

(13) Some courses of study require that a combination once learned (5 + 7 = 12) be applied immediately to the higher decades (15 + 7 = 22, 25 + 7 = 32, etc.). How much superior in column addition is a class thus trained to one not so trained? Is it necessary to apply all combinations learned in this way? May it not be that the general idea of application is soon acquired with the first few combinations and that special drill is not required thereafter? Are there certain combinations where special drill must be insured always (5 + 6 = eleven, 15 + 6 = twenty-one) because the sound regularity is interfered with? Or may a strictly written presentation do away with the necessity of special drill even here?

(14) Is there any increase of efficiency in drilling on combinations in columns as soon as possible? As soon as the combinations that add up 7 are learned is there a special advantage in immediately giving the child such columns in application as the following:

2	4	24
3	1	31
2	2	22
—	—	—
7	7	77

(15) In some texts and courses of study the addition combinations are presented in the order of the sizes of the sums, thus $2+2=4$, $2+3=5$, etc. In others the combinations are presented, regardless of the size of the numbers involved, so as to immediately fit into certain drill columns already prepared. Thus the annexed column would require the following combinations (beginning from the bottom), $4+6=10$, $0+3=3$, and $3+6=9$. What is the relative worth of these two methods?

6
3
6
4
—
19

(16) In column addition, where carrying is involved, some rationalize the process, and others teach it mechanically as a mere bit of habituation. In the case here given, some would add each column separately, taking a second total of the partial sums. Others would merely "put down the six and add one to the next column," writing down only the complete sum. Which is superior, in that it will result in accurate and rapid column addition in the shortest space of time?

23
47
36
—
16
9
—
106

The preceding treatment of addition will suggest similar problems as more or less recurring in subtraction, e. g., whether there is any gain in teaching $5-3=2$ immediately after learning that $5-2=3$, etc.

(17) Is there any advantage in the so-called "Austrian" method of subtraction by addition, over the method of subtracting through specially learned subtraction combinations? If so, how much considering that subtraction and addition (1) mean different concrete situations, (2) use a different written algorism, (3) use the same oral form ("6 and 4 are 10"), and (4) employ the same memorization (6 and 4 are 10)? What extent of school energy is saved, if any? Is the method of subtraction from the next digit in the

4	10
+6	− 6
—	—
10	4

top number superior or inferior to the method of adding to the next digit in the bottom number? How is it when these methods are applied to the addition-method of subtraction? to the "old" subtraction-combination method?

(18) Do children make fewer errors when they are formally taught to make a preliminary inspection of subtraction examples before proceeding to manipulate specific combinations? as when the number cannot be subtracted; as when the answer is zero. (See the two examples here given.)

389	345
−421	−345
——	——

(19) Do children make fewer errors and manifest less confusion where they are formally taught to handle the zero difficulties prior to being confronted with them in column subtraction? As in the type cases given below:

(a)	(b)	(c)	(d)
867	867	867	870
—467	—400	— 32	—650
400	467	835	220

Where borrowing from top?

(e)	(f)	(g)	(h)
128	602	612	612
— 76	—237	—318	—308
52	365	294	304

Where adding to bottom?

(i)	(j)	(k)	(l)
834	834	804	814
—406	—496	—496	—406
428	338	308	408

In subtraction, what preparation is needed in a command of zero combinations to perform the column subtraction? (Note each case given above.) Is there some general mode of handling these zeros that will not require a mastery of it in connection with each number it may be combined with? How does the above apply to the combinations with ones? Where a one is involved, is it merely counting downwards or backwards? Or is the subtraction of 1 exactly like the subtraction of 3 or 4 or any other number?

(20) What are the basic combinations required to perform any given column subtraction? In what form may they be best

mastered? Are zero subtractions ($6 - 0 = 6$) and subtractions with one ($6 - 1 = 5$) to be included or omitted? Are the reverses to be taught as basic? ($7 - 2 = 5$ and $7 - 5 = 2$.) Are subtraction combinations of varying difficulty? Which are the most difficult, as shown by children's errors?

(21) Will children have less difficulty, with fewer ensuing errors, if they approach column subtraction through a series of graded types of difficulty? Consider in the following:

(a) No borrowing.

$$\begin{array}{r} 1498 \\ -\ 964 \\ \hline 534 \end{array}$$

(b) Borrowing each time save the last.

$$\begin{array}{r} 8431 \\ -5987 \\ \hline 2444 \end{array}$$

(c) Borrowing alternately.

$$\begin{array}{r} 8431 \\ -2917 \\ \hline 5514 \end{array}$$

In what order should types (b) and (c) be given?

How rapidly may a child advance from two or three figures to seven or eight? What new difficulties present themselves in such an extension of figures?

(22) Are the first series of multiplication combinations best presented (1) by the use of objects grouped and counted, or (2) by the use of column addition? Or are these two methods best used as supplementary to each other? Are the combinations with zeros ($6 \times 0 = 0$) and the combinations with ones ($6 \times 1 = 6$) best taught in the tables, or later, in connection with their actual use in column multiplication? Is there a gain in teaching the reverses in connection with the combinations to which they are related, exactly as with the addition combinations, thus $6 \times 3 = 18$ immediately after $3 \times 6 = 18$? What gradation of steps is most economical and efficient in proceeding from combinations to their application in column multiplication? Since partial products represent but stages in calculation do they need to be understood as to their placing, or

should their placing be taught as a mechanical process through habit formation? Is it economical to allow zeros to be recorded which later will be abandoned? as in multiplying by 206? What special drill on zero difficulties (and on the manipulation of ones) is required in connection with their handling in column multiplication? How can this be best provided?

(24) Is there any need for division tables of combinations? May not the multiplication tables be used for division, precisely as the addition combinations are used for subtraction? For example, from "3 2's are 6," may we not step to the case of 2)6? "How many 2's are 6?" "3 2's are 6." Here the identification is through a common oral form of expression. Is there any need to show that a specific written form or algorism in multiplication is the equivalent of another one in division, since this is not done in subtraction by addition?

(25) In column addition, column subtraction, and column multiplication the fundamental combination is generally obvious in the process of manipulation. Is the division combination equally obvious in long and short division? Does this require special treatment?

(26) Will children do long and short divisions more efficiently if drill in division with a remainder (after $12 \div 3 = 4$, learning $13 \div 3 = 4$, with 1 remainder, and $14 \div 3 = 4$, with 2 remainder) is inserted between the learning of the combinations and their application to long and short divisions? Since the largest difficulties seem to occur in connection with long division, and since division by a one-figure divisor must precede division by a number of two figures, shall division by a one-figure number be first taught in its complete form? Shall division by a one-figure number be abridged to "short division" before or after the development of division with a divisor of two figures? Does the distinction between partition and measuring have any relation whatsoever to skill in manipulation? What is the worth of the distinction in interpreting problems and applying calculations?

(27) How many special types of zero difficulties need be anticipated and carefully drilled upon before children are allowed to attack examples where zero difficulties are likely to confront them? Is there any special order in which these zero difficulties should be attacked for purposes of economical mastery? What

special training in the handling of ones should be provided for, especially if the one combinations in the tables are omitted?

(28) The above list deals entirely with investigations in teaching which are mainly psychological. There is another large series of investigations as to the materials required in the various courses of study. These are sociological. In such investigations one would make such inquiries as the following: What is the demand for square root in ordinary business occupations? What types of fraction examples are called for frequently, and what infrequently? Which types of reasoning combinations are most used? Do we "add and multiply" within the same problem, more frequently than we "multiply and add?" Upon such social investigations should the selection and the omission, the emphasis and the subordination of specific topics be determined. Then will our courses of study represent the highest social efficiency.

BIBLIOGRAPHY: Teachers should consult, on this topic, Professor Suzzallo's work already cited.

CHAPTER XV

THE WORK OF THE FIRST SCHOOL YEAR[1]

The first question that naturally arises in connection with the arithmetic of the first grade is as to whether or not the subject has any place there at all. For several years past there has been in this country a propaganda in favor of excluding it as a topic from the first grade and even from the second. Like all such efforts, the history of which is not generally known, the very novelty of the suggestion, to many teachers, is sufficient to create a following. It is well to consider briefly the reasons for and against such a suggestion, and to attempt to weigh these reasons fairly before attempting any decision.

In favor of having no arithmetic as such in the first grade it is argued that the spirit of the kindergarten should extend farther, perhaps even through all of the primary grades; that number work should come in wherever there is need for it, all learning being made attractive and natural, and education appearing to the child as a unit instead of being made up of scattered fragments. Such a theory has much to commend it, not only in the primary school but everywhere else. Opposed to it is the rather widespread idea that most kindergarten work is superficial in aim and unfortunate in result; that children who have had this training are wanting in even the little seriousness of purpose that they should have, that they have no power of application, that they have been "coddled" mentally into a state that requires constant amusement as the condition to doing anything. The dispassionate onlooker in this old controversy probably feels that there is truth in both lines of argument, and that mutual good has been the result. Ancient education was a dreary thing, and

[1] It is impossible in the space allowed to enter very fully into details as to the work of the various grades. Teachers who desire such details may consult the author's Handbook to Arithmetics (Boston, 1905). All that can be done in this article is to give a brief survey of some of the most important topics relating to the various grades.

to the spirit of the kindergarten, although not to extreme Fröbelism, we are indebted for the brighter spirit of the modern school. On the other hand, to make children self-reliant, independent in thinking, conscious of working for a purpose, demands more thought than seems to pervade the ordinary kindergarten.

Now as to arithmetic in the first grade: Shall we leave it to the ordinary teacher to bring in incidentally such number work as he wishes, or shall we lay down a definite amount of work to be accomplished and assign a certain amount of time to it? And in answering these questions, are we bearing in mind the average primary teacher throughout the whole country? Are we also bearing in mind that arithmetic was never taught to children just entering school until about a century ago, and that it was largely due to Pestalozzi's influence that the subject was ever placed in the first grade? When, therefore, we advocate having no arithmetic in the first grade, we are going back a hundred years or so, which may be all right, but which is not a new proposition by any means.

Having thus laid a foundation for an answer to the question, it is proper to proceed dogmatically, leaving the final reply to the reader. Not to put arithmetic as a topic in the first grade is to make sure that it will not be seriously or systematically taught in nine-tenths of the schools of the country. The average teacher, not in the cities merely but throughout the country generally, will simply touch upon it in the most perfunctory way. Whatever of scientific statistics we have show that this is true, and that children so taught are not, when they enter the intermediate grades, as well prepared in arithmetic as those who have studied the subject as a topic from the first grade on.

Furthermore, while it is true that the essential part of arithmetic can be taught in about three years, it cannot, for psychological reasons, be as well retained if taught for only a short period. The individual needs prolonged experience with number facts to impress them thoroughly on the mind. We can, for example, teach the metric system in an hour to any one of fair intelligence, but for one to retain it requires long experience in its use.

But more important than all else is the consideration of the child's tastes and needs. Has he such a taste for number as shows him mentally capable of studying the subject at the age of six, and are his needs such as to make it advisable for him to do so? There can be no doubt as to the answer. He takes as much delight in counting and in other simple number work in the first grade as in anything else that the school brings to him, and he makes quite as much use of it in his games, his "playing store," his simple purchases, his reading, and his understanding of the conversation of the home and the playground, as he does of anything else he learns. If we could be certain that in the incidental teaching that is so often advocated he would have these tastes and needs fully satisfied, then arithmetic as a topic might be omitted from the first or any other grade; but since we are pretty sure that this will not be accomplished in the average school, then it is our duty to advocate a definite allotment of time and of work to the subject in every grade from the first through the eighth.

This being so, what should this allotment of work be? Of course there is no general answer for the whole country. In some schools there are many foreign born pupils who are unable to speak English when they enter and therefore the first year's work must be devoted largely to acquiring the language. In other schools the children come from homes where they have already been taught by governesses and are considerably advanced over the average. In general, however, the course here laid down may be considered a fair average for the ordinary American school.

The Leading Mathematical Feature. The introduction to the addition table, this being at the same time the simplest and the most important operation in arithmetic. It is not advisable to use a text-book in this year, on account of the children's inability to read.

Number Space. It has been found best both from the standpoint of mental ability and of needs of the children to set a different limit to the numbers used in counting and in the operations. Children like to and need to count numbers that are larger than those used in operations. For reading and writing numbers, therefore, they may profitably go as far as 100, meeting these numbers in the paging of books, the numbering of houses, the playing of games, and the counting of various objects. For the operations, however, it is sufficient if they go as far as 12. Indeed, 10 would make a good limit were it not for the fact that in measuring they so often use 12 inches.

Addition. The addition tables should be learned at least as far as sums of 10 or 12. Some prefer to go as far as $9+4=13$, but it is immaterial so long as the children know the table through 9's before the text-book is used,—ordinarily the middle or the end of Grade II. Appropriate combinations for the first year may, therefore, be taken as follows:

1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	10

1	2	3	4	5	6	7	8
2	2	2	2	2	2	2	2
3	4	5	6	7	8	9	10

1	2	3	4	5	6	7
3	3	3	3	3	3	3
4	5	6	7	8	9	10

1	2	3	4	5	6
4	4	4	4	4	4
5	6	7	8	9	10

1	2	3	4	5
5	5	5	5	5
6	7	8	9	10

1	2	3	4
6	6	6	6
7	8	9	10

1	2	3
7	7	7
8	9	10

1	2
8	8
9	10

1
9
10

This arrangement makes the sum the basis for selection. Many prefer, however, to proceed to master the table of 1's, 2's, 3's, and 4's, as mentioned above, thus giving the following combinations:

1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1
—	—	—	—	—	—	—	—	—	—
2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10
2	2	2	2	2	2	2	2	2	2
—	—	—	—	—	—	—	—	—	—
3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10
3	3	3	3	3	3	3	3	3	3
—	—	—	—	—	—	—	—	—	—
4	5	6	7	8	9	10	11	12	13
1	2	3	4	5	6	7	8	9	10
4	4	4	4	4	4	4	4	4	4
—	—	—	—	—	—	—	—	—	—
5	6	7	8	9	10	11	12	13	14

It is not a matter of great importance which of these two arrangements is adopted in any given school system, at least so far as we are able to judge from any scientific investigations thus far made. The great thing is that the complete table shall be known to 10 + 10 by the end of the second year.

Subtraction. Every fact learned in addition should, judging from general experience, carry with it the inverse subtraction case. That is, the question "3 + 2 equals what number?" should carry with it the questions "3 + what number equals 5?" and "2 + what number equals 5?" or, if preferred, "5 — 3 equals what number?" and "5 — 2 equals what number?"

Multiplication. Little attention should be given to this subject in the first grade. The idea that 2 + 2 + 2 may be spoken of as 3 times 2, and the incidental use of the word "times" in other simple number relations is desirable.

Division. Since multiplication is not taken as a topic, its inverse (division) has no place, save as it appears in the fractions mentioned below.

Fractions. Children so often hear about the fractions ½, ¼, and ⅓, that these ideas and forms may profitably be introduced at this time, although ⅓ may be postponed to the next grade. The statement that half the class may go to the blackboard, the idea of ¼ of a dollar, and that of ⅓ of a yard, are all common in the first year. In the introduction of these ideas and symbols it is well to avoid extremes that will militate against the child's future progress, such as the extreme of the ratio method, for example. We should remember that a fraction, say ½, is commonly used in three distinct ways, and, that it is our duty to see that, little by little, all these become familiar to the child. These ways are as follows: (1) ½ of a single object, the most natural idea of all, the breaking of an object into 2 equal parts; (2) ½ as large, as where a 6-inch stick is ½ as long as a foot rule,— not half of it, but half as long as it is; this is essentially the ratio notion, and it is necessary to the child's stock of knowledge, but it is not necessary to make it hard by talking about ratios at this time; (3) ½ of a group of objects, as in the case of ½ of ten children.

Denominate Numbers. Children in this grade should learn the use of actual measures. They should know that 12 in. = 1 ft., 3 ft. = 1 yd., and should employ this knowledge in making measurements. They should know the cent, 5-cent piece, dime, and the dollar as 10 times (or even 100 cents), and should use toy money in playing store. They should know the pint and quart, and use these in measuring water or other convenient substance. Other terms such as pound, week, minute, mile, and gallon may be used incidentally, but they should not be learned in tables, at present.

Objects. It is important to use objects freely wherever they assist in understanding number relations, but it is equally important to abandon them as soon as they have served their purpose. The continued use of any particular set of objects (blocks, disks, measures, picture cards, etc.) is tiresome and narrowing. Pestalozzi was wiser than many of his successors when he used anything that came to hand to illustrate most of his number work. To continue to use objects after they have ceased to be necessary is like always encouraging a child to ride in a baby carriage.

Symbols. It cannot be too strongly impressed upon teachers that the symbols that children should visualize are those that they will need in practical calculation. Thus it is much better to drill upon the annexed forms than upon $6+3=9$, $9-6=3$, $9-3=6$, since the latter are never used in calculation. For ease in printing and writing, symbols like $6 + 3 = 9$ have their important place, but the eye should become accustomed to the perpendicular arrangement so as to catch number combinations as it must do when we come to actual addition.

$$\begin{array}{r} 6 \\ +3 \\ \hline 9 \end{array} \qquad \begin{array}{r} 9 \\ -6 \\ \hline 3 \end{array} \qquad \begin{array}{r} 9 \\ -3 \\ \hline 6 \end{array}$$

Technical Expressions. While it is proper to begin by reading $6+2$ "six *and* two" and $8-6$ "eight *less* six," the words "plus" and "minus" should soon enter into the vocabulary of the child as part of the technical language of the subject. It is proper to call a cat a "pussy" for a while, and a horse a "pony," but the time soon comes for "cat" and "horse,"—and so for the technical expressions in arithmetic.

Nature of the Problems. In this grade problems of play, of the simplest home purchases, and of interesting measures should dominate. In general, for all grades, the oral problems should have a local color, relating to real things that the children know about. The building of a house near the school, the repairing of a street, the cost of school supplies—these and hundreds of similar ideas may properly suggest problems adaptable to every school year. It is the business of the text-book in the grades where it is used to furnish a large amount of suggestive written work, but it can never furnish all the oral work needed nor can it meet all local conditions.

As a specimen of the early work in this grade the following oral exercise is submitted:[1]

1. How many inches wide is the window pane?
2. How many feet long is your desk, and how many inches over?
3. How many feet and inches from the floor to the bottom of the blackboard?

[1] These and other similar sets of problems used in this article are taken from other works of the author.

4. Stepping as you usually do in walking, find how many paces in the length of the room.
5. How many paces wide do you think the room is? Pace the width and see if you are right.
6. How tall do you think you are? Measure. How many feet, and how many inches over?
7. How many inches from the lower left-hand corner of this page to the upper right-hand corner?
8. How wide do you think the door is? Measure. How many feet, and how many inches over?

Such problems suggest measurements of genuine interest to the pupil, relating as they do to his immediate surroundings. They allow for the actual handling of the measures and the forming of reasonably accurate judgments concerning distances.

Abstract Computation. It is a serious error to neglect abstract drill work in arithmetic. So far as scientific investigations have shown, pupils who have been trained chiefly in concrete problems to the exclusion of the abstract are not so well prepared as those in whose training these two phases of arithmetic are fairly balanced. Abstract work is quite as interesting as concrete; it is a game, and all the joy of the game element in education may be made to surround it. At the same time it is the most practical part of arithmetic, since most of the numerical problems we meet in life are simplicity itself so far as the reasoning goes; they offer difficulties only in the mechanical calculations involved, and constantly suggest to us our slowness and inaccuracy in the abstract work of adding, multiplying, and the like. In the first grade this work is largely but not wholly oral.

Forms. It is expected that children in this grade will become familiar with the names of the common solids and polygons needed in their work. For example, square, rectangle, triangle, oblong, cube, sphere, cylinder, pyramid, prism, and similar forms should be handled and their names should be known. Paper cutting and folding is very helpful in the study of plane figures and in the work with fractions, although like any other device, it may be used to an extreme that is to be avoided.

The Time Limit. Even in the first grade, and still more in the succeeding years, a time limit should be set on all number work. The children should see how many questions they can individually, or as a class, or as half of the class, answer in a minute, or in some other period of time. Unless this is done, or some similar plan is adopted, the tendency to dawdle over the work will begin to crystallize into a habit, and computation will take much more time than necessary. It is also to be observed that, always within reasonable limits, rapid calculation contains less errors than very slow work. The reason is apparent; we concentrate our attention more completely, and other thoughts do not take our minds from the numerical work.

BIBLIOGRAPHY: The author's *Handbook to Arithmetics*, p. 19; C. A. McMurry, *Special Method in Arithmetic*. On paper folding consult Sundara Row's work, *Geometric Paper Folding* (Open Court Publishing Co.), illustrated by photographs taken by the author of the present work a few years ago. This work is suggestive, although not adapted to grade work. Consult also Wentworth-Smith, *Stepping-Stones in Number*, Boston, 1911.

CHAPTER XVI

THE WORK OF THE SECOND SCHOOL YEAR

Whether or not arithmetic has a definite time allotment in the first grade, it usually has one in the second, although some teachers oppose it even there. The argument already advanced holds the more strongly here, especially as, in many schools, the child is quite prepared to use a text-book by the middle of this year.

The Leading Mathematical Features. In schools of average advancement, where the question of language is not as serious as in some cities in the East, children in this grade may be expected to complete the addition tables and to learn the multiplication tables to 10×5.

Number Space. Children will now take an interest in counting to 1000, first by units to 10, then by 10's to 100, then completely to 100, then by 100's to 1000, and finally completely to 1000. Their operations may also be anywhere within this space, although, of course, most of their results will involve only small numbers. In the Roman notation the limit may be set at XII, this sufficing for the reading of time and for the chapter numbers of their books.

Counting. Without going to an extreme in counting by various numbers where no definite purpose is served, there is a field in which counting is very advantageous. To count by 2's from 2 to 10 and from 1 to 11 has the pleasure of any rhythmic sequence and at the same time gives the addition table of 2's, and the counting by 2's from 2 to 20 gives the corresponding multiplication table. Similarly, counting by 3's from 3 to 30 gives the multiplication table of 3's, while the further counting from 1 and 2 to 13 and 14 gives the different addition combinations. The exercise is interesting to children, and the knowledge secured in this way is more than one would at first think.

Addition. The tables should be completed during this year, including the sums of any two one-figure numbers. There are only 45 possible combinations of numbers below 10, viz.: $1+1$, $1+2$, and so on to $1+9$; $2+2$, $2+3$, and so on to $2+9$; $3+3$, $3+4$, and so on to $3+9$; and similarly for the others to $9+9$, besides the zero combinations referred to earlier in this paper. It is better, however, to continue the sums to include 10,—a simple matter but one that is often helpful. The addition of numbers of two and even of three figures each may be taken during this year, but not more than five or six in a column should be used.

Subtraction. Subtraction may be carried far enough to include numbers of three figures each. The method to be employed has already been discussed in Chapter XI. In both addition and subtraction there should be an effort to cultivate the habit of rapidity, although never to the exclusion of accuracy. The time limit on work, mentioned on page 83, should be employed in all written work. In general in both addition and subtraction the full form should be employed until it is thoroughly understood. For example, in adding 247, 376, and 85, a problem that must have been preceded by many simpler ones, it is well to use the first of the following forms until the reasons are understood, and then to adopt the second:

$$\begin{array}{r} 247 \\ 376 \\ 85 \\ \hline 18 \\ 190 \\ 500 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 247 \\ 376 \\ 85 \\ \hline 708 \end{array}$$

Likewise, if the addition or "Austrian" method is taken for subtraction, it is better to begin a problem like $852-476$ in the full form, as follows:

$$852 = 800 + 50 + 2$$
$$476 = 400 + 70 + 6$$

The difference between these is the same if we add 10 to each, and also 100 to each, and we add them as follows, so that we can easily subtract in each order:

$$\begin{array}{l} 800 + 150 + 12 \\ \underline{500 + \ \ 80 + \ \ 6} \\ 300 + \ \ 70 + \ \ 6 = 376 \end{array}$$

After this is understood we may proceed to the ordinary arrangement.

Multiplication. The multiplication tables may be learned this year as far as 10×5. Some schools go even as far as 10×10, and others find it better to postpone all of this work until the third grade. Products should be learned both ways, i. e., 5×6 and 6×5. There is a great advantage in reciting all tables aloud, and even in chorus, since this leads to a tongue and ear memory that powerfully aids the eye memory when the pupil needs to recall a number fact. Counting enables the tables to be developed in a rhythmic fashion that is pleasing to the ear, and shows multiplication by integers to be merely an abridged addition, that is, that $3 + 3 + 3 + 3$ is more briefly stated as 4×3.

Division. The multiplication table should carry with it the division table. This need not be developed as a separate feature but may be treated as the inverse of the multiplication table exactly as subtraction is the inverse of addition. The fact that $4 \times 6 = 24$ should bring out the second direct fact that $6 \times 4 = 24$, and the two inverses, $24 \div 6 = 4$, and $24 \div 4 = 6$. These inverses may be introduced in a way that is analogous to that followed in subtraction. That is to say, after learning that $4 + 5 = 9$ we ask, "What number added to 5 equals 9?" "What number added to 4 makes 9?" Similarly, after $4 \times 5 = 20$ we ask, "What number multiplied by 4 equals 20?" "What number multiplied by 5 equals 20?" These may then be expressed as $20 \div 4 = 5$, $20 \div 5 = 4$.

In division in this grade we also have an illustration of the fact that the full form should precede the short one. A child more easily grasps the idea of $36 \div 3$ if he sees the first of these forms before he comes to use the second:

$$3\overline{)30 + 6} \qquad\qquad 3\overline{)36}$$
$$\quad 10 + 2 \qquad\qquad\quad 12$$

In the same way, when he comes to divide 36 by 2, it is better to begin with the first of the following forms:

$$2\overline{)20 + 16} \qquad\qquad 2\overline{)36}$$
$$\quad 10 + \ 8 \qquad\qquad\quad 18$$

Teachers will find it better to write the quotient below the dividend in short division, even though it is preferably written above in the long process. There is no advantage in trying to change the habit of the world on such a small matter.[1]

Fractions. Children know the meaning of ½, ¼, and often of ⅓, on entering this grade. If ⅓ is not known it should be introduced and ⅛, $^1/_6$, $^1/_5$ may also be added to the list at this time, although many successful teachers prefer to postpone them until Grade III. The use of objective work is imperative, and it is better to take various simple materials than to confine one's self to elaborate fraction disks or other similar devices. Every school has cubes to work with, and the use of cubes, paper folding, paper cutting, and the common measures is recommended as quite sufficient.

Denominate Numbers. The denominations already learned in Grade I should be frequently used, and to them should be added the relation between the ounce and pound; the pint, quart, and gallon; the quart, peck, and bushel; the reading of time by the clock, and the current dates. The idea of square measure (in square inches) is introduced. All of this work should be done with the measures actually in hand so far as this is possible. A table of denominate numbers means very little unless accompanied by the real measures. This will be felt by any American grade teacher who teaches the metric system without the measures, and who tries to think of his weight in kilos, his height in centimeters, and the distance to his home in kilometers.

Symbols. It has already been said that symbols like +, —, ×, and ÷ were invented for algebra and have only recently found place as symbols of operation in arithmetic.[2] The desire to employ them has led many teachers to use long chains of operations that are never seen in practical life and which, while serving some purpose in oral work, are vicious as written exercises. For example, $2 + 4 \div 2 + 5 \times 6 \div 3 + 3$ is a kind of work that should never appear in the grades. Arithmetically it is easy enough, and the answer is 17, but there is no use in puzzling a child to remember which signs have the preference in such a chain. This is a small technicality of algebra, of which the importance is much overrated even there, and it has no place in

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[1] See the author's Handbook, p. 27.

[2] It is true that + and — were first used in Widman's arithmetic of 1489, but not as symbols of operation. See my *Rara Arithmetica.*

the elementary school. With respect to the symbols $2 \times \$3$ and $\$3 \times 2$ there is, however, a reasonable question, since there is good authority for each. Modern usage favors the former because we more naturally say "2 times 3 dollars" than "3 dollars multiplied by 2," and it is better to read from left to right as in an ordinary sentence. It should be repeated, however, that the forms which the child needs to visualize are not these but the one he will meet in actual computation, as here shown.

$$\begin{array}{r} \$3 \\ 2 \\ \hline \$6 \end{array}$$

Objects. It is here repeated, as essential to a discussion of the work of the second grade, that objects are necessary in developing certain number relations, but that they should be discarded as soon as the result is attained. Number facts must be memorized by every one, and objects may become harmful if used too often.

Nature of the Problems. This matter begins to assume considerable importance in this grade, and it has been already discussed in Chapter IV. It may be said in general, however, that several of our recent American arithmetics are making a serious effort to improve the applications of the subject, adapting them to the mental powers and to the environment of the pupils instead of offering obsolete material of no practical value and of little interest.

Necessity for Systematic Reviews. It is proper at this time to call the attention of teachers to the matter of reviews,—not those that naturally occur from time to time during the year, but those that should deeply concern every school at the close of one year and at the opening of the next one. Any one who has ever had much to do with the supervision of the grade work in arithmetic is struck by the general complaint that children are never prepared to enter any particular grade. Every teacher seems to feel that the preceding teacher has imposed a poorly equipped lot of children upon her own grade and that her problem is therefore hopeless. Now if this were only an occasional complaint the supervisor might well be worried, but he soon recognizes it as part of the tradition of the school, and pays little attention to it accordingly. What does it mean, however, and how should we remedy the evil if evil there be?

If any teacher will himself learn, let us say, the logarithms of the first fifty integers, between September and February, how many will he know in June? And if he knows them all in June how many will he remember at the end of the summer vacation? And how will he feel, say about September 15, if some one suddenly asks him to give the logarithm of 37 to six decimal places, telling him, if he fails, that he must have been pretty poorly taught the year before? Now this is a fair illustration of the mental position of a child with respect to the multiplication table when he enters Grade IV. Psychologically it would be strange if he could rapidly and accurately give every product demanded; his brain cells have clogged up or got disarranged or gone through some similar transformation during his nine or ten weeks of careless play. What, therefore, is the teacher's duty? There are two things to do. First, at the close of each school year, in June, there should be a thorough and systematic review of those number facts and operations that are the fundamental features of the year's work. The teacher ought to be satisfied that each child leaves the grade with such a mental equipment as shall leave no chance of fair criticism. His responsibility then ceases. Second, and even more important, at the opening of each school year, in September, there should again be a thorough and systematic review by the teacher in the next grade, of these same features. But this review should be conducted in the most sympathetic spirit. The teacher should be surprised if the children have not forgotten much rather than if they have failed to remember the facts perfectly. He should think of his own fifty logarithms, for example, and the review should be patiently and helpfully extended until the children's arithmetical brain-cells resume their former state. After this has been done in the spirit mentioned, and after the teacher has gone into the next higher grade for a day to see how his own pupils of the preceding year are standing the test, then he may be justified in complaining, but not before.

It need hardly be mentioned that there are few more severe tests of the ingenuity and patience of a teacher than are found in these reviews. The " edge of interest " is already worn off in any review, and it requires all the tact a teacher possesses to maintain the enthusiasm of the pupils in such exercises. The result, however, is well worth the effort, and the school system that carries out the plan will have less of complaint and more or sympathetic coöperation than would at first be thought possible.

CHAPTER XVII

THE WORK OF THE THIRD SCHOOL YEAR

The Preparation. Since the text-book is placed in the hands of children during the latter part of the second school year or at the opening of the third, it becomes particularly important to have a systematic review of the work of Grades I and II at the beginning of this year. The text-books usually provide for this, and by their help these important things are accomplished: (1) The children's memories are refreshed as to the essential features of the preceding year's work, viz., the addition table, and the multiplication table as far as the course of study may require. (2) Children are "rounded up," brought to a certain somewhat uniform standard, so that all can begin the serious use of the text-book with approximately the same equipment. (3) The superior capacity or the defect of the individual has an opportunity to show itself early, allowing for such advancement or special attention as the case demands. In other words the "lock-step" can be broken without the usual delay. As to further argument for this autumnal review the reader may refer back to Chapter XVI.

The Leading Mathematical Features. In this year rapid written work is an important feature. The oral has predominated until now, but in Grade III the operations involve larger numbers than before, and the child begins to acquire the habit of writing his computations. Multiplication extends to two-figure multipliers and long division is begun. The most useful tables of denominate numbers are completed.

Number Space. It is usually considered sufficient if the child understands numbers to 10,000 in this grade, although he may be allowed to count by 10,000's to 100,000 or even farther. Indeed, as soon as he understands numbers to 1,000 he rather enjoys showing his prowess by counting by 1,000's and by writing large numbers. Counting always extends far beyond the needs of computation—a law that is true to-day and has been true in the historical development of all peoples. In the writing of Roman numerals there is no particular object in going beyond C in the first half-year, and M in the second half. It must be borne in mind that we use the Roman forms chiefly in chapter or section numbers, and less often in reading dates, so that all writing of very large numbers by this system in an obsolete practice and a waste of time. Indeed it is not strictly a Roman system any more, so much have we changed the numerals from their early forms.

Counting. In this grade the counting of Grade II should be continued, including the 6's, 7's, 8's, 9's, and 10's, as a basis for the multiplication tables and as a review of the addition combinations. There is no need of counting beyond certain definite limits, however. Thus in counting by 2's beginning with 0, we have 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20. This suffices for the multiplication table of 2's and even the last half of this is merely a repetition of the first half with 10 added.

The Decimal Point. It becomes necessary in this grade to write dollars and cents, and hence forms like $10.75, $25.10, and $32.02 are given. It is not necessary nor even desirable that the children should know any of the theory of decimal fractions at this time. The decimal point should be looked upon by them simply as separating dollars and dimes, and it will give no trouble unless the teacher confuses the class by the ever-present danger of over-explaining.

Forms. It is usual in Grade III to review the simple geometric forms already learned, such as the triangle, rectangle, cylinder, and sphere. Formal definitions are, however, undesirable. The chief thing is that the child should use the names correctly. Some little paper-folding may well be introduced as a basis for simple square and cubic measure.

Square and Cubic Measure. The ideas of area (square inches or square feet) and volume (cubic inches or cubic feet) may enter into the work of this grade, although some successful teachers prefer to introduce them in Grade IV, finishing this work in Grade V. If introduced here, they are of course treated objectively, usually with paper-folding, drawing, or inch cubes of wood. There is hardly any trouble with this work unless the teacher enlarges upon its difficulties. If there is accuracy of language, spoken and written, from the beginning, this will continue; but if the teacher allows expressions like "3 inches times 3 inches equals 9 square inches," instead of "3 times 3 square inches equals 9 square inches," there will be produced loose habits of thought and expression that will lead to great trouble.

Devices for Fractions. It is still necessary in this grade to make a good deal of use of objective work in treating fractions, and to make the work largely oral during the first half year. There is also an advantage in using columns of figures like those here shown. Here it is very easy to see that $\frac{1}{2}$ of 8 is two 2's, or 4; that $\frac{1}{4}$ of 12 is 3; that $\frac{3}{4}$ of 16 is three 4's or 12, and that $\frac{1}{2}$ of 20 is the same as $^2/_4$ of 20, or two 5's, or 10. From the second arrangement it is easy to see that 2 is $\frac{1}{2}$ of 4, $\frac{1}{3}$ of 6, $\frac{1}{4}$ of 8, and $^1/_5$ of 10; that 4 is $\frac{2}{3}$ of 6, $\frac{1}{2}$ of 8; that 6 is $\frac{3}{4}$ of 8 and $^3/_5$ of 10, and so on. Devices of this kind add both to the interest in and clear comprehension of the subject, and when not carried to an extreme are valuable.

2	3	4	5
2	3	4	5
2	3	4	5
2	3	4	5
—	—	—	—
8	12	16	20

			2
		2	2
	2	2	2
2	2	2	2
2	2	2	2
—	—	—	—
4	6	8	10

Addition. The 45 combinations of one-figure numbers should be reviewed, and in the first half year oral work of the types of $20 + 30$, $25 + 30$ should be taken, to be followed in the second half year by cases like $25 + 32$ and $225 + 32$, where no "carrying" is involved. Written work with four-figure numbers including dollars and cents, should be given, but long columns of figures should be avoided at present.

As already stated there is an advantage in introducing any difficulty in operation by using the complete form. While, for example, the annexed problem in addition is not designed as an introduction to the addition of three-figure numbers, it illustrates what is meant by the complete form. The teacher need have no fear that children cannot easily be brought to use the abridged form; "the line of least resistance" will bring that about, while on the

427
326
452
49
——
24
130
1100
——
1254

score of a clear understanding of the operation this complete form is far superior to the other. It should also be mentioned that the pupil should at this early stage be taught to recognize his own liability to error and to do what every computer has to do, add each column twice, in opposite directions, to be sure of his result,—to " check " it, as we say.

Subtraction. This subject has been sufficiently treated under Chapter XI. The extent of the work is suggested by the work in addition, and of the various methods the addition or "Austrian" seems at present to be the best.

Multiplication. This, with division, constitutes the special work of the year, addition and subtraction offering no essentially new difficulties. In the first half year it is customary to complete the tables through 10 × 10, and the products must be thoroughly memorized not merely in tabular form but when called for in any order. The plan of carrying the tables to 12 × 12, while necessary in England on account of the monetary system used there, has generally been discarded in America, it being felt that the time required for this extra work could be better employed. In the first half year multiplication may be carried so far as to include three-figure multiplicands and one-figure multipliers, and the work may at first be arranged in the complete, and later in the common abridged form as here shown. Since all such work is done in the classroom where the teacher can supervise it, there should be a time limit placed upon it, to the end that habits of rapidity as well as of accuracy should be acquired. In the second half year the work may usually be extended to two-figure multipliers, in which the complete form should again precede the common abridgment, as here shown. There is also introduced in this year such multiplications as that of $2.75 by 7, thus preparing the way for decimal fractions. The lat-

```
298
  3
---
 24 = 3 ×   8
270 = 3 ×  90
600 = 3 × 200
---       ---
894 = 3 × 298
          298
            3
          ---
          894
```

```
  298
   43
   --
  894 =  3 × 298
11920 = 40 × 298
-----   --
12814 = 43 × 298
```

ter are not, however, treated in this grade, and the work should not be made difficult by any unnecessary theorizing upon this subject.

Division. In this year oral division by one-figure divisors is introduced for such simple cases as $484 \div 2$, $484 \div 4$, $481 \div 2$, etc. Short division of numbers like $522 \div 6$, should be introduced by some such form as the annexed. Such separations of the dividend are made for the purpose of having the process seen in its simplest form, and teachers should write problems of this kind on the board often enough to make sure that the process is understood. The children should not be required to use this form, but should get to the practical work of division as soon as possible.

```
6)522
6)480 + 42
   80 +  7
       = 87
```

In the second half year the two-figure divisor may be introduced, but since the greatest difficulty in division consists in the estimating of the successive quotient figures, it is well to confine the divisors, for this year, to those whose unit places are at first 0, then 1, and finally 2. As an early form for long division, the annexed algorism is suggested. The use of United States money again brings in the decimal point so naturally that the difficulty of decimal fractions is much diminished when that topic is reached. The full form that may properly precede the common abridgment is here set forth.

```
      75
21)1575
   1470 = 70 × 21
   ----
    105
    105 =  5 × 21
   ----

   $0.37
21)$7.77
   6.30 = 21 × $0.30
   ----
   1.47
   1.47 = 21 × $0.07
```

Scope of Work with Fractions. The pupil is now able to use halves, thirds, fourths, fifths, sixths, and eighths, or, if not, this work should be introduced at this time. Oral addition and subtraction of fractions with a common denominator. The reduction of halves to fourths, sixths, and eighths, and of thirds to sixths, is introduced by means of objects, the objects being discarded as soon as they have served their purpose. Fractional parts of numbers of three figures or less, these being selected so as to be multiples of the denominator.

Denominate Numbers. Here as in other grades it is necessary to review and frequently use the tables already learned. The table of square units is often introduced and extended to the square yard, although it may be postponed a year. The gill is added to the table of liquid measure, and the table of time is completed. Modern teaching finds it advisable to introduce the units of measure only as rapidly as the child develops the need for them and can therefore understand them. In all cases it is desirable to have the measures where they can be seen or in some other way appreciated. For example, when the acre is introduced, somewhat later, a piece of land near the school, approximately an acre in size, should be shown to the class. In the same spirit they should see a ton of hay or a ton of coal, a cord of wood where this is possible, a rod, a gill measure, and so on. It is very important that the great basal units used by our people should be visualized by the children, so that bushel, mile, ton, etc., shall not be mere words.

Typical Problems. The following are suggested as two practical sets of problems, adapted to this grade, each telling a story that may suggest other topics for original work by the class.

Oral Exercise

Some Home Meals

1. The coffee for our breakfast cost 6c., the potatoes 4c., the meat 32c., and the bread 4c. How much did the bread and meat cost? How much did all the food cost?

2. The oatmeal for a breakfast cost 8c., the milk 4c., the fruit 10c., the rolls and butter 5c., and the eggs 8c. How much did this food cost?

3. For a dinner the meat cost 30c., the vegetables 20c., the dessert 20c., the coffee 15c., and the other food 15c. Find the total cost.

4. The meals for a small family cost $1.70 on one day and $2.20 on another day. How much did they cost for these two days?

Written Exercise

The toad is one of man's best friends. One toad will keep a garden of 800 sq. ft. free from harmful insects.

1. At this rate, how many toads would protect from insects a garden 80 ft. wide and 100 feet long?

2. The eggs of 4 toads were counted and found to be 7547, 11,540, 7927, and 9536. How many were there in all?

3. If one out of 50 hatched, how many hatched? (Divide all by 50.) If 715 of these were destroyed by other animals, how many survived?

4. If each of these survivors destroys insects that would cause $10 worth of damage, how much are they all worth to a village?

BIBLIOGRAPHY: The author's text-books on arithmetic, and the Wentworth-Smith series, set forth his views in detail.

CHAPTER XVIII

THE WORK OF THE FOURTH SCHOOL YEAR

The Leading Mathematical Features. In this the last year of the primary grades it is well to feel that the essentials of arithmetic have all been touched upon. It is, therefore, desirable to review the four fundamental operations, extending the multiplication and division work to include three-figure multipliers and divisors. The common business fractions should also be included, with simple operations as far as multiplication.

Number Space. In the first half year the numbers may extend to 100,000, and in the second half year number names may be given as high as a billion. The operations, however, should be confined to the smaller numbers of such business as can be appreciated by children of this age.

Counting. The prime object of the counting exercises, the developing of the tables of addition and multiplication, has now been accomplished, except when it is desired to carry the multiplication table to 12×12. In that case the counting may now be continued by 11's to 132 and by 12's to 144. Otherwise the only use of counting in this grade is for the purpose of review.

Addition and Subtraction. There should be much rapid oral work with numbers like the following:

$$\begin{array}{r} 7 \\ +\ 4 \\ \hline \end{array} \quad \begin{array}{r} 17 \\ +\ 4 \\ \hline \end{array} \quad \begin{array}{r} 37 \\ +\ 4 \\ \hline \end{array} \quad \begin{array}{r} 37 \\ +\ 14 \\ \hline \end{array} \quad \begin{array}{r} 47 \\ +\ 24 \\ \hline \end{array}$$

$$\begin{array}{r} 11 \\ -\ 7 \\ \hline \end{array} \quad \begin{array}{r} 21 \\ -\ 7 \\ \hline \end{array} \quad \begin{array}{r} 41 \\ -\ 17 \\ \hline \end{array} \quad \begin{array}{r} 51 \\ -\ 27 \\ \hline \end{array} \quad \begin{array}{r} 71 \\ -\ 47 \\ \hline \end{array}$$

The written work should be undertaken with the aim of (1) accuracy, secured by always checking the result; (2) rapidity, secured by setting a time limit upon all work. Children should by no means neglect this matter of checks, since it is used in all the business world. Much of the complaint of business men, that boys from the schools are always inaccurate in arithmetic, would be obviated if pupils were always required to check their additions by adding in the opposite directions, and their other results in some appropriate manner. In subtraction, for example, if the result is obtained by the "Austrian" method it should be checked by adding it to the subtrahend in the opposite direction.

Multiplication and Division. No new principles are involved here, and the work of the preceding year is simply extended to include larger numbers. In some schools the multiplication table is extended to 12×12, although this is not important enough for most people to make it worth the while. It is a good plan, however, to learn all products less than 50, as 2×13, 3×15, 4×12, and so on, since these are so often used in the purchases of the household. Even a child ought to know the cost of 2 lb. of meat at 18 cents a pound, without using pencil and paper. The practical checks on multiplication and division are not advantageously discussed as early as this.

Fractions. Here as later the work in common fractions should be confined to those needed in ordinary business, and at present to those from $\frac{1}{2}$ to $\frac{7}{8}$. Of course there is no objection to an occasional example with denominators of two or three figures, but the day of fractions like $\frac{243}{1284}$ is past, decimal fractions having taken the place of all such forms. Children in this grade should also know that $\$\frac{1}{2} = 50$ cents, and $\$\frac{1}{4} = 25$ cents. The operations may extend as far as easy multiplications of an integer and a fraction, two fractions, or an integer and a mixed number. Unusual forms of operation, not practical in business, should not be given, and the teacher should resist all temptation to depart from this principle on any supposed ground of mental discipline.

Decimal Fractions. A brief introduction to this subject, based on the work already given in United States money, may be allowed in this grade, although the serious treatment of decimals belongs later in the course.

Denominate Numbers. The tables needed in business life are completed in this grade by adding that of land measure, and completing long and cubic measure. In the work of adding and subtracting compound numbers children should feel that there is no principle involved that is not found in integers. For example, consider these two cases:

37	3 ft. 7 in.	3 lb. 7 oz.
25	2 ft. 5 in.	2 lb. 5 oz.
62	6 ft.	5 lb. 12 oz.

In the first, because $7 + 5 = 12$, which is 1 ten and 2 units, the 1 ten is added to the 10's. In the second, because 7 in. + 5 in. = 12 in. or 1 ft., the 1 ft. is added to the feet. In the third, because 7 oz. + 5 oz. = 12 oz., which does not equal a pound, it is written under ounces. In every case the principle is the same, to add to the next order any units of that order that are found. In general we use compound numbers of only two denominations, and it is on such numbers that we should lay the emphasis. The use of numbers of four or five denominations is now obsolete, and there is not enough disciplinary value in the subject to warrant using them instead of the numbers of actual business.

As heretofore mentioned, there should be an effort to have children visualize the standard measures of our country, such as the acre, mile, ton, and bushel.

Teachers should be careful at this time that slovenly methods of statement do not become habits. Such forms as the following, for example, are inexcusable:

60 in. ÷ 12 = 5 ft.
60 ÷ 12 = 5 ft.
60 in. ÷ 12 in. = 5 ft.

If we wish to reduce 60 in. to feet we have three correct forms, any one of which is easily explained:

60 × $^1/_{12}$ ft. = 5 ft.
60 in. ÷ 12 in. = 5, the *number* of feet,
60 ÷ 12 = 5, the *number* of feet.

If slovenly forms are allowed here they must be expected in all subsequent grades, and they must be expected to lead to slovenly thought in the treatment of all kinds of problems.

Review. At the close of the year there should be a review of all the essential features of the work in the primary grades. This requires skill on the part of the teacher lest it become stupid and so wearisome as to lose its chief value. Original local problems to test the children in the four fundamental operations with integers and (as far as they have gone) with fractions, will usually render the work interesting and will hold the attention.

Nature of the Problems. Here as elsewhere the problems should touch the children's interests and be adapted to their mental abilities. The following may be taken as types:

Oral Exercise

1. Tell the cost of some kind of cloth. How much will $10\frac{1}{2}$ yd. cost?

2. Tell the cost of a pair of shoes. How much will 2 pairs cost?

3. If a man earns $3 for 10 hours' work, how many hours must he work to earn enough to buy his daughter a pair of shoes at $1.50?

4. How many hours must he work to earn enough to buy a $6 suit of clothes for his son?

Written Exercise

1. Sarah's mother bought $4^1/_5$ yd. of cloth for a cloak, at $1.25 a yard. What did she pay for it?

2. She also bought $3\frac{1}{2}$ yd. of lining at 50c. a yard, and $4\frac{1}{4}$ yd. of braid at 20c. a yard. How much did these cost?

3. She also bought 6 pearl buttons at $1.50 a dozen, and 2 spools of silk at 8c. a spool. How much did these cost?

4. The dressmaker charged $5 for making the cloak. What did materials and making cost?

5. John's mother bought $2\frac{1}{2}$ yd. of goods for a coat, at $1.20 a yard, and $2\frac{1}{4}$ yd. of lining at 48c. a yard. How much did these cost?

6. She also bought a dozen buttons at 25c. a dozen, and 2 spools of silk at 8c. a spool, and paid $3 for making. How much did the coat cost?

CHAPTER XIX

THE WORK OF THE FIFTH SCHOOL YEAR

The Leading Mathematical Features. There should in this year be a thorough review of the fundamental operations with integers. This should be followed by the same operations with the common fractions and denominate numbers of business. Percentage may be begun, although in some places it is better to postpone this until the following year.

Review. There is usually a new text-book begun in this grade, and this, if properly arranged, offers plenty of material for the review above mentioned, with numbers that are appropriately larger. Teachers should undertake this review in the spirit and for the reason suggested in Chapter XVI.

Text-book. The new text-book begun in this grade will naturally be topical in its arrangement, that is, each general topic like percentage being treated once for all; or it will be on the plan of recurring topics, a subject like percentage being met two or three times. As has already been said, each of these types has its advantages. If the school chooses one with recurring topics that can probably be followed rather closely. If on the other hand it adopts one arranged by topics there are two courses open: (1) the teacher may select from the various chapters such material as fits the course of study in use in the particular locality, a task of no great difficulty; (2) the book may be followed closely, the pupils' work becoming purely topical. We are apt to condemn the latter plan because it is old, but perhaps on that very account it should be commended. The world has used it, and used it successfully, and it has the merit that it brings a feeling of mastery, a sense of thoroughness, and a development of habit that is sometimes lacking with more modern text-books. In general it may be said to depend upon the school as to which type of book is the better, and as to which plan of using the topical book is to be preferred. In a school system with a reasonably permanent staff of teachers, with adequate supervision, and with teachers' meetings that allow classes to keep in touch with one another, the book with recurring topics, or at least the course arranged on this plan, is undoubtedly the better. It is more psychological and it allows for a better grading of material. On the other hand where teachers change frequently, as in rural schools, it is safer to use the topical book and to follow it rather closely. In this and the following chapters the arrangement by recurring topics is followed, and any topical text-book can easily be adapted to the sequence suggested.

Number Space. This number space is now unlimited, but names beyond billion are of no particular importance. Large numbers should always represent genuine American conditions. It is better to perform several operations on the ordinary numbers of daily life than to perform one on an absurdly long number; but on the other hand, a reasonable number of operations on large numbers that represent real business cases are to be commended.

Addition. Larger numbers and longer columns may now be used, but there is a limit to this matter. In general, the numbers used by the average citizen are the ones to drill children upon. Children should be encouraged to read columns as nearly as possible as they read a word. When we seek the word "book" we do not think "b," "o," "o," "k,"—we think "book" without any spelling; so when we see the annexed column we should not think, "6 and 3 are 9, 9 and 3 are 12, 12 and 5 are 17," nor even "6, 9, 12, 17," if we can do better than this. Probably we cannot train our eyes to see 17 at a glance, as we seek "book," but it is well to encourage children to look at this as $9 + 8$, thinking of the 6 and 3 as 9, and the 3 and 5 as 8. But, however we think of such a column, we should always check our result by adding in the reverse order. If teachers do not think this necessary, let them add twenty sets of say ten five-figure numbers each, working rapidly, and see how many mistakes they themselves will make.

5
3
3
6
—

Subtraction. This subject has been sufficiently discussed on page 45. The important matter is not now the explanation, for the technique has already been learned; the operation, accurately and rapidly performed, is the desideratum, the check being of great importance in securing the essential accuracy.

Multiplication. It is now advisable to let the children know some good, practical check on their work in multiplication, such as computers actually use. Of the checks, the simplest is that of "casting out 9's."[1]

Division. The children are now old enough to understand the two forms of division illustrated by the following:

$$\$125 \div \$5 = 25$$
$$\$125 \div 25 = \$5$$

There are no generally accepted names to distinguish these, "measuring" and "partition" not meaning much to children. It suffices that it is clear that there are these two forms, and to see that we avoid such inaccuracies as $\$125 \div \$25 = 5$ cows.

Factors and Multiples. This subject formerly played a very important part in arithmetic, when large fractions had to be reduced to lower terms. With the introduction of the decimal fraction about 1600, however, it lost much of its former importance and need play but a small part in the arithmetic of to-day.[2]

Common Fractions. Some objective work will still be necessary in treating common fractions but it should be dispensed with as soon as possible and the material should not be of one kind alone. In the operations children should not be required to give very elaborate explanations, although they should see clearly the reasons at the time they learn the processes. This has been discussed already in Chapter IX and may, therefore, be dismissed at this time.

Denominate Numbers. The operations with these numbers should be a part of the work of the year, but only practical cases should be taken. To divide a compound number of four

[1] The explanation of this process is too long to be given here. The reader may consult the author's *Handbook*, p. 57, Beman and Smith's *Higher Arithmetic*, or the appendix to the Wentworth-Smith *Complete Arithmetic* and the *Arithmetic, Book III.*

[2] For a theoretical treatment of the subject from the advanced standpoint, consult Beman and Smith's *Higher Arithmetic.*

denominations by another one of three, for example, consumes time and patience to no worthy purpose.

How to Solve Problems. Inasmuch as the children now begin to consider problems of more than two steps, it becomes necessary to devote more attention to the methods of solving examples. The step form of analysis, therefore, has a legitimate place in this year's work. If teachers hope for exactness of thought they must insist upon accuracy of statement in these written exercises.

Percentage and Decimals. The study of decimal fractions may safely be undertaken in this grade, and this may be followed, if desired, by an elementary treatment of percentage. If at the outset children understand that 6% is only another way of writing $\frac{6}{100}$ and 0.06, there will be but little difficulty in introducing percentage. One important feature is the interchange of the per cent forms, decimal fractions, and common fractions, as for example, in $\frac{1}{4} = \frac{25}{100} = 0.25 = 25\%$. It is better not to introduce any formulas or rules in such work in percentage as may be taken at this time, but to analyze each problem as it arises. In the next school year it is allowable to reverse this policy.

Discount. Of all the applications of percentage the most common is discount, and it is at the same time the simplest. This topic may, therefore, be introduced in this grade, the other applications being reserved for the sixth year.

Nature of the Problems. The great industries of the country may be taken up at this time as a profitable field for the applications of arithmetic. Children now begin to know enough geography to permit of this wider view, and problems that relate to their own country have an interest that the traditional ones about the man who "owned a field of corn" lacked. Such problems are not statistical to the extent that their data are to be memorized, but they state real conditions instead of false ones. Of problems suited to this grade the following are types relating to the production of corn, one of the great food products of the country. The greatest corn-producing states are Iowa, Illinois, Nebraska, Missouri, Kansas, and Indiana, and such work may be taken in connection with the study of the geography of these sections whenever this can be brought about without too great change in the curriculum.

1. When this country produced 2,105,102,400 bu. of corn a year, averaging 25 bu. to the acre, how many acres had we in corn?

2. If 3 bu. of corn could then be bought for $1, what was the total value of this yield of 2,105,102,400 bu?

3. When Iowa's annual product amounted to 305,800,000 bu., this was how many times the 440,000 bu. produced by Maine?

4. To transport 1000 lb. of corn from St. Louis to New Orleans by river costs $1. How much will it cost to transport 1750 tons?

5. If the average value of corn for each of the 46,610 acres given to it in Connecticut in a certain year was $21, and for each of the 4,031,600 acres in Indiana $13, what was the entire value of the corn crop of each state?

6. If the average annual corn crop per acre is 40 bu. in Wisconsin, 36 bu. in Maine, 37 bu. in New Hampshire, 38 bu. in Massachusetts, 38 bu. in Indiana, and 38 bu. in Iowa, find the average by adding and dividing by 6.

CHAPTER XX

THE WORK OF THE SIXTH SCHOOL YEAR

The Leading Mathematical Features. The leading features of this year should be percentage and its applications, particularly to discount, profit and loss, commission, and interest. Ratio and simple proportion may also be included.

The General Solution of Problems. Since the work in percentage introduces the pupil to the problems of business, some of which become rather intricate in the later school years, it is well at this time to take up rather systematically questions of the solution of problems in arithmetic. To this end there should be considered exercise in analysis in general and in unitary analysis in particular, and the equation may well begin to find place in the mental equipment of the child. As to the matter of analysis no question will be raised, but as to introducing the letter x some teachers are in doubt. When, however, we come to consider that it merely replaces an awkward symbol that has long been used, and makes the work much clearer, the objection cannot be maintained. For example, $2 + (?) = 7$ is sometimes used as early as the first school year; this, however, is only a complicated way of writing $2 + x = 7$, the two meaning exactly the same thing. Similarly, $4 : 7 = 12 : (?)$ is only an awkward way of writing what is equivalent to

$$\frac{x}{12} = \frac{7}{4}$$

the latter being in every way simpler of understanding and easier of solution. There are several classes of problem in percentage that are made clearer by the use of this convenient x, and its use is quite as arithmetical as algebraic.

Percentage. In this work special attention should be given to the common per cents and fractions of business, such as $\frac{1}{2} = 50\%$, $\frac{1}{4} = 25\%$, $\frac{1}{8} = 12\frac{1}{2}\%$, $\frac{1}{3} = 33\frac{1}{3}\%$, and so on.

In the matter of solution, the x should be used in those inverse cases where it makes the problem clearer. Such is the case of finding the cost of goods that sell at 10% above cost, and sell for \$126.50. Here we have, if x represents the cost,

$$x + .10x = \$126.50$$
$$1.10x = \$126.50$$
$$x = \$126.50 \div 1.10$$
$$x = \$115$$

Other forms of solution might be used, but this is the most satisfactory.

Discount. This being the first and most important of the applications of percentage, considerable attention should be devoted to it. The case of several discounts may, however, be postponed until the following year.

Profit and Loss on Purchases. This topic, so closely connected with the business world with which the child is now coming into closer contact, may claim to rank second in importance among the applications of percentage. The principles involved are very simple, particularly if one allows the letter x to throw light upon all inverse problems. The examples should follow as closely as possible the common business customs of the mercantile world.

Commission. This topic ranks possibly third in importance among the applications of percentage. A considerable field of applications exists, particularly in relation to the sending of farm produce to the cities. The problems can, therefore, be made to seem real to the children, whether they live in the country or see farm products for sale in the city.

Interest. This subject may already have been met by the children. It is now taken up and extended to more difficult questions. Only real cases should, however, be considered. For example, in this school year, at least, there is little advantage in trying to find the capital, given the rate, time, and interest. It is better to spend time in writing promissory notes and in computing the interest, than to put it on questions that seldom arise in business life. If we wish more complicated problems they are easily secured from genuine mercantile sources.

Ratio. This may be introduced this year or reserved for the seventh grade. It was formerly introduced merely as an introduction to proportion. It is easy, however, to see that it may be of some use by itself, and teachers are advised to consider this phase of the subject. We mix fertilizers on the farm in a given ratio, we find ratios of attendance to absence in the school, and the term is used in the same way in business life.

Proportion. This subject may also be delayed another year. It has lost a good deal of its importance of late. A proportion is, as shown on page 107, merely one method of writing a simple equation, and with the letter x allowed in school, the equation form is likely to replace that of proportion. When this is not the case, ordinary analysis is likely to be substituted for proportion. For example, consider this problem: If a shrub 4 ft. high casts a shadow 6 ft. long at a time that a tree casts one 54 ft. long, how high is the tree?

Here we may write a proportion in the form, 6 ft. : 4 ft. = 54 ft.: (?), not attempting to explain it, but applying only an arbitrary rule. This is the old plan. Or we may put the work into equation form.

$$\frac{x}{54} = \frac{4}{6}$$

and deduce the rule for dividing the product of the means by the given exteme. Or we may take the same equation and get our result easily by multiplying these equals by 54, giving

$$x = 36$$

Or we may say: If a 6 ft. shadow is cast by a 4 ft. object, a 1 ft. shadow would be cast by a $^4/_6$ ft. object, and a 54 ft. shadow would be cast by a $54 \times {}^4/_6$ ft. object, or a 36 ft. object.

Of these plans the first is the most difficult to explain; the rest are equally easy, and the third is the shortest.

Measures. The work in measures this year may be confined to simple surfaces and solids, and may properly include practical cases of house building, plastering, carpentering, and the like. Here is a real field, interesting and profitable. Proportion leads to exercises in similar figures, and this has some excellent applications in lumbering and in carpenter's work.

Nature of the Problems. In each succeeding year the problems now come to relate more and more to the industries of the people, and the range of applications becomes very great. The farm child learns not only of his own surroundings but of the great industries of the city, while to the city child the great story of the soil and its products opens up a new world. The following farm problems may be taken as types of the problems suited to this grade:

1. A farmer puts 5 acres into celery, setting out 20,000 plants to the acre. The yield being 1,500 doz. heads to the acre, what is the ratio of the plants matured to the others?

2. He pays $95 an acre for seeds, fertilizers, labor, and other expenses, and sells the crop at 15c. a dozen heads. What is his profit on the 5 acres?

3. Another farmer tries setting out 30,000 plants to the acre, but only 80% mature, and these are so small that he has to put 16 in a bunch to sell for a dozen, and then gets only 14c. a bunch. His expenses are $100 an acre. At this rate what is his profit on 5 acres?

4. A farmer has a 30-acre meadow yielding 1½ tons of hay to the acre. If by spending $300 a year for fertilizers, he can bring the yield to 4 tons to the acre, how much more will he make a year, hay being worth $8 a ton?

5. A farmer reads that a good mixture of seed for his meadow is, by weight, as follows: timothy 40%, redtop 40%, red clover making up the rest. At 40 lb. of seed to the acre, how many pounds of each should he sow?

6. The following is, by weight, a good mixture of seed for a pasture: Kentucky blue grass 25%, white clover 12½%, perennial rye 28⅛%, red fescue ⅜%, redtop 25%. At 32 lb. to the acre, how many pounds of each are used?

7. A cow weighing 1000 lb. consumes the equivalent of 3¼ tons (2000 lb. to the ton) of dry fodder a year; a 100-lb. sheep, 770 lb.; every ton of live pork, 12 tons; and every ton of live horseflesh, 8.4 tons. Each class of animals consumes what per cent of its own weight of dry fodder a year?

CHAPTER XXI

THE WORK OF THE SEVENTH SCHOOL YEAR

The Leading Mathematical Features. As in the preceding grade, it is well to begin by a general review of the fundamental processes from a higher standpoint than before. Ratio and proportion are usually completed in this year, whether introduced here for the first time or not, and the applications naturally cover a broader field. Percentage is the leading topic of the year.

Our Numbers. The children are now ready to consider the writing of numbers from a higher standpoint, to know something of the interesting history of the numerals they use and of the science of arithmetic that they are studying. The story of the Roman numerals,[1] and that of the Arabic numerals make these subjects seem more real at this time. The difference between a uniform scale, as seen in our system of money, and a varying one, as seen in the English system, should also be explained, and the advantage of the former understood. The relation between integers, the various kinds of fractions, and compound numbers, may now understandingly be taken up.

The Fundamental Operations. These may now be reviewed in such way, that is by the introduction of such new material, as to maintain the interest even in an old subject. This is particularly true if the teacher will now and then suggest such short methods as may be found in most of the advanced arithmetics. The check of casting out nines should now be used for all products and quotients. It is simple, it takes but a moment, and it checks most of the errors that are liable to arise. It cannot be too much impressed upon teachers and pupils that both are very liable to errors in all kinds of calculation, and that they, like business computers, should always apply some kind of check to every result obtained. Some teachers feel that

[1] This is told in condensed form in the author's *Handbook to Arithmetic.* See also the Bibliography at the close of Chapter I.

the work should be so accurately done that checks should be unnecessary. This is a good theory, but practically it will not work even with the ones who advocate it. No good professional computer would think of leaving his results without checking them, and if a professional will not do this, why should we expect a child to be so infallible as to do it?

Measures. All tables of measure in common use should be reviewed in this year. If, in this review, some historical notes are given on the origin of such measures as the yard, inch, foot, mile, quart, gallon, and acre, the pupils will find the work taking on a new interest. Teachers are advised that it is of little value to memorize facts that will not be used in practical life. If we wish to know the number of cubic inches in a bushel we may go to an encyclopedia or a dictionary; it is surely inadvisable to burden our minds with such details.

Longitude and Time. This subject has greatly changed within a few years. To-day most of the civilized world uses some form of standard time. Therefore, our attention may properly be confined to the geographical principle involved, to the problem of standard time, and to the question of longitude at sea. Teachers are urged not to allow slovenly work in this subject under the plea that bad forms bring true results in a shorter time than good forms. This matter has been sufficiently discussed on page 30, and the chapter on longitude and time seems to be one of the worst offenders in all arithmetic. A form like $45 \div 15 = 3$ hrs. is false and serves to undo all of the good to be derived from the topic.

Percentage. This topic, so vital in business life to-day, should be touched upon several times in the elementary school. If the work is sufficiently progressive the pupils will not find that "the edge of interest" is worn off. In this year there should be a good deal of oral work in the common per cents of business, pupils coming to feel that pencil and paper are unnecessary in finding 12½%, 25%, 33⅓%, 50%, 66⅔%, and 75% of ordinary numbers. As to the use of terms like "base," "rate," "percentage," "amount," and "difference," there is little that can be said in their favor. They were invented in the rule stage of arithmetic, and have served their purpose. Of course, we need "rate," it being a stock term of the business world. "Percentage" is, however, rather confusing than otherwise, (1) because it is understood by the pupils as the name of the subject as a whole, and (2) because the business world does not use it quite as the school does. "Base" means so many things in mathematics that its use is equally confusing, while of "amount" and "difference" this is still more noticeably the case. On the whole, therefore, it is as well not to use these terms, although they are found in most of our leading books to-day because of the demands of teachers.

It should also be remarked that, if the use of x is allowed, there is no excuse for the old formulas of percentage. They are nothing but condensed rules; if they are not explained they defeat part of the purpose of studying arithmetic; if they are explained they are much harder than the equation form with the single letter x.

It is well to bear constantly in mind, in the midst of the large number of possible cases of percentage, that the two important things in the subject are these: (1) to find some per cent of a given number, and (2) to find what per cent one number is of another. All the rest is relatively unimportant, and on these two the emphasis should accordingly be laid.

Simple Interest. This is the leading application of percentage in this year, and the attention of pupils should be concentrated on the single problem of finding interest in practical cases. To find the time, given the principal, rate, and interest, is of very slight importance, and so for other similar cases; but to find the interest, that is the great point.

Ratio and Proportion. This work should, as stated in the preceding grade, be confined largely to the treatment of practical questions, and there are only a few where this subject can be used to real advantage. These are chiefly related to similar figures, although some other questions, like those of simple physics, enter. Compound proportion has little reason to claim a place in our schools to-day. If explained, the process is a very hard one; if not, it is a useless one, since we now have better methods of solving problems.

Nature of the Problems. With each succeeding school year the children develop new interests and come nearer to the great world that they are soon to enter. The range of topics is now practically unlimited, and the opportunities for offering series of related problems are excellent. As a type of such problems the following may be given, appealing this time to the girls, who are usually rather neglected in the matter of applied arithmetic:

Dressmaking Problems

Written Exercise

1. A dressmaker bought 16 yd. of velvet at $3 a yard, selling 9 yd. at a profit of 16⅔% and the rest at a rate of profit half as great. What was the rate of gain on the whole?

2. She bought a 25-yd. box of chiffon velvet at $4 a yard, with 10% off for cash, selling it at $4.35 a yard. What was her gain per cent?

3. She bought a 75-yd. piece of silk skirt lining at 65c. a yard. She sold 28 yd. at 90c., 15 yd. at 95c., and the remainder, at the close of the season, at 70c. What was her per cent of gain?

4. She bought a 50-yd. piece of silk waist lining at 75c. a yard. She sold 12 yd. at $1 and 10 yd. at 95c., but the remainder, being kept in stock over the season, had to be sold at 65c. What was her per cent of gain or loss?

5. She bought a 20-yd. silk dress pattern at $2.10 a yard, being allowed, as a dressmaker, a discount of 5%, and 6% off for cash. She charged her customer the marked price, $2.10. What was her per cent of profit?

6. She charged her customer $25.50 for 3 yd. of Honiton lace, which had cost her $7 a yard. What was her per cent of profit?

7. She charged her customer $2 for findings for the dress. These consisted of 4 spools of silk at 10c. each, 1 spool of thread at 5c., 3 yd. of featherbone at 10c., a card of hooks and eyes at 8c., skirt braid 16c., plaiting 30c., waist binding 30c., and collar 10c. What was her gain per cent on the findings?

CHAPTER XXII

THE WORK OF THE EIGHTH SCHOOL YEAR

The Leading Mathematical Features. The work this year is in the line of business applications, including advanced mensuration.

Business Applications. The boy and girl should now begin to feel that the world of business and of life is opening before them. It should therefore be the duty of the school, even more than in the preceding grades, to apply arithmetic to the genuine problems of life, particularly with reference to the common occupations of the people.

Banking. In banking, for example, we should not seek to train accountants or bookkeepers or cashiers, but we should seek to give a fair idea of the duties of these men in the ordinary savings bank and bank of deposit. A girl, for example, needs to know how to deposit money in a bank and how to draw checks as well as a boy, and such operations should become as real as the school can make them. School banks, with deposit slips, checks, bank book, cashier, paying teller, and receiving teller, should assist in this work.

Partial Payments. This subject has not the practical value that it had when banks were not so numerous as now, and when their machinery was not perfected. The old-style problem in partial payments should therefore give place to the more practical cases found in our best modern books.

Partnership. This is another subject that has entirely changed within a short time. The stock company (corporation) has largely supplanted it, save in its simplest form. The work of the schools should therefore be confined to this common form, the obsolete ones being supplanted by work on corporations.

Simple Accounts. It is not worth while to teach an elaborate form of bookkeeping to the average citizen. On the other hand it is necessary that every one should know how to keep simple accounts, and this work should be taken up in this year. It should relate to the income and expenditures of daily life, in the home, on the farm, or in the shop, rather than to the technical needs of the merchant, the latter being part of the special training of the individual who enters this line of trade.

Exchange. Here again there has been a great change within a few years. The form of time draft given in most of the old-style arithmetics has given place either to sight drafts or to another kind of time draft. Teachers should therefore be particular to use only those types that the ordinary citizen meets to-day, about which girls and boys alike should be informed. In connection with this work a short talk upon the clearing house, upon which any bank will gladly inform the teacher, will add new interest.

The Metric System. This system might be taught much earlier than the eighth school year, and there would be some advantage in so doing. But when we consider that it is not yet used practically by many Americans, it seems as well to postpone it until this time. There are three chief reasons for teaching it now: (1) General information requires us to know a system that is used by a large part of the civilized world, excluding the English-speaking portion; (2) it is used in all scientific laboratories in America; (3) our people should be sympathetic with a system that is liable to replace our own before long in all matters relating to our growing foreign trade; if we sell machines abroad, the measurements must be metric in most cases, and to foster this trade many of our skilled workmen will eventually need to use these instead of the awkward ones with which we are familiar.

At the same time we must not go to an absurd extreme, but must remember that our common system is the one that the people use and that the children must know before all others. In teaching the metric system the results will be poor unless the children use the actual measures and come to visualize the basal units as they should in their own system.

Taxes. This topic, like others of practical life, should be treated from the standpoint of local conditions as far as possible. It should include the question of tariff, and a few brief talks on civics should make the whole question a real one for the pupils.

Insurance. This subject has become so technical that all that the schools can hope to do is to give a general conception of the work of the various kinds of companies, and to confine the problems to the simplest practical cases that the people need to know about. We should not attempt to enter upon the technicalities of agent work, nor to do more than explain briefly some of the common types of policy.

Corporations. As remarked under Partnership, the corporation has, for good or evil, replaced the individual in large business ventures. Our schools must, therefore, adjust their work to this change. Pupils should know what a corporation is, its chief officials, how it is legally organized, what stocks and bonds are, how dividends are declared and paid, and the legitimate work of stock exchanges. On the other hand the schools cannot be expected to teach the technicalities of the stock broker's office, nor to supply information beyond that needed by the general citizen. The newspaper stock reports furnish an excellent basis for the practical problems that the case demands.

Powers and Roots. For purposes of mensuration square root is necessary. Cube root may well be delayed until the pupil studies algebra, because it has so few practical applications. Even square root is more valuable as a bit of logic than as a practical subject, since those who use it most employ tables. The explanation, therefore, is even more important than the technique of the work, and children of this age can easily comprehend it, either by the use of the diagram or by the formula, the latter being quite easily understood by this time.

Mensuration. This work is now completed so far as the needs of the average person are concerned. The teacher should use simple models that can be made in the school room, as suggested in the best arithmetics. It is not expected that strict geometric demonstrations can be given, but it is entirely possible to avoid arbitrary rules by giving enough objective work to make the matter clear. It is not advisable to introduce work that is not used in ordinary life, such as finding the volume of a frustum of a cone, there being a sufficient amount of more important work to occupy the time and attention of pupils.

Nature of the Problems. The problems should appeal to the business needs that are soon to come to the children, and the following are suggested as types:

1. A boy who has been working this year at $25 a month is offered either an increase of 20% for next year or a salary of $7

a week. Which will bring the more income, and how much more per year? (Use 52 wk.)

2. A girl who has been working in a factory at $21.67 a month is offered an increase of 10% where she is or a salary of $5.60 per week elsewhere. Which will bring the more income, and how much more per year? (Use 52 wk.)

3. A boy went to work at 90c. a day. The second year his wages were increased 20%, the third year they were 42c. a day more than the second, and the fourth they were increased 33⅓%. At 300 working days to the year, what was his total income for each year?

4. A girl entering a trade school finds that graduates from the dressmaking department receive on an average $4.60 a week the first year; those from the millinery department, 5% less; those from the embroidery department, 5% more than the dressmakers; and those from the operating department 66⅔% as much as the last two together. Find the average wages of each, and tell which department the girl probably entered. (Use 52 wk.)

5. A girl leaving the public school finds she can enter a city shop at a salary of $3 a week the first year, with 16⅔% more the second year, and a $14^2/_7$% increase the third year. Instead of this she enters a trade school for a year, tuition free. She then receives a salary of $5 a week the first year and 20% more the second year. Counting 50 working weeks a year, how much more does she receive in three years by the plan she follows after leaving the public school than she would have received without the trade-school training?

A Comparison of Eighth Grade Work. It is well known that in Europe the specialization of schools is carried much farther than has even been thought of here, or than seems possible or desirable in the future. To speak of the arithmetic of these various forms of schools—for foresters, builders, watchmakers, barbers, and so on—would therefore be unprofitable in an article like the present. It will not, however, be out of place to give an outline of the work in arithmetic in the eighth school year in a girls' school in Munich, because this shows the tendency at present, in one of the most progressive cities of Europe, to have arithmetic touch the interests and needs of the people. The work is as follows:

1. Simple domestic bookkeeping.

2. Calculation of the prices of foods, bought in large or in small quantities, together with the question of discounts.

3. Cost of meals for the home.

4. Daily, monthly, and yearly supplies for the kitchen, together with the keeping of kitchen accounts.

5. Simple measurements as needed in the household.

6. Food values of different food stuffs as necessary for a complete meal, with doubtless the application of ratio and percentage.

7. Cost of furnishing a kitchen.

8. Measurements of material, and the cost of buying, renovating, and washing clothing made of various goods, as of woolen or linen.

9. Relative cost of different systems of heating.

10. Relative cost of different systems of lighting.

11. Maintenance of the house, including questions of rent, water, taxes, insurance, and interest on a mortgage.

12. Elementary commercial arithmetic, including such general topics as percentage and its application in discount.

Such a course is highly to be commended. It meets the needs of girls as we are not meeting them in our American schools. Indeed it becomes a serious question if, in a subject like mathematics, we are not bound to have separate classes for girls and boys after the seventh grade and through our high school.

It may be well, also, to consider the work done in "Obertertia" in a Prussian Gymnasium, corresponding in years to our eighth grade. This correspondence is not exact in some respects, because the Prussian school year is somewhat longer than ours, but allowing three years before entering the lowest class (Sexta) this becomes the eighth school year. There three hours a week are allowed to mathematics, the arrangement allowing two to geometry and one to algebra in one week, and two to algebra and one to geometry the next week, and so on. The algebra includes equations of the first degree with two unknown quantities, and the geometry finishes the treatment of the circle, finding the value of pi. The arithmetic work, as we call it, is practically completed at the end of the sixth school year (in Quarta).

This gives an idea of what could be done in America if we should care to set about it. As it is, the question is a fair one if we would not be justified in materially reducing the arithmetic of Grade VII so as to include a considerable part of the work of Grade VIII, thus allowing an elementary course in algebra in that year.

Algebra. If algebra be introduced in Grade VIII, what is the purpose and what should be its nature? Aside from the general information thus given, and from the discipline that comes from this or any other subject, there is a need that a few years ago was hardly felt in this country. Boys are apt to leave school after the work of Grade VIII is finished. They go into the shops, into trade, into various occupations. Algebra was a few years ago of no practical value to them, but to-day the formula and the graph of a function are common features in our trade journals. Here then is a suggestion to our schools. Why should not elementary algebra be introduced by a study of formulas, so that the simple algebraic expressions of our trade journals or our artisans' manuals can be read easily? Why should we not introduce graphs of functions very early, not in complicated forms but as used in the journals, the manuals, and the workshop to-day?

The author's views as to details have been set forth in the Wentworth-Smith *Vocational Algebra,* Boston, 1911, and in his *Algebra for Beginners,* Boston, 1906. The whole question has recently been discussed in a masterly little monograph written by Mr. C. Godfrey, one of the present leaders in English education,—*The Algebra Syllabus in the Secondary School,* London, 1911.

As to the geometry, the work in mensuration in arithmetic probably suffices for the present, although it is possible that we may come to adopt the German plan of introducing the scientific treatment of the subject into the elementary grades in the future.

Such are some of the problems in the teaching of arithmetic to-day. Many are solved and many still await solution and are occupying the attention of a large number of teachers. It is with the hope of suggesting some of the larger problems that this book is written, rather than with any desire to treat the minor details that are sufficiently discussed in any good text-book. If the work shall lead to sane experiment, to a conservative view of the reforms to be accomplished, and to a sympathy with the effort to improve the problems of arithmetic, it will have served its purpose.