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PHYSICAL

AND

CELESTIAL MECHANICS.

BY

BENJAMIN PEIRCE,

PERKINS PROFESSOR OF ASTRONOMY AND MATHEMATICS IN HARVARD UNIVERSITY,
AND CONSULTING ASTRONOMER OF THE AMERICAN EPHEMERIS
AND NAUTICAL ALMANAC.

DEVELOPED IN FOUR SYSTEMS OF

ANALYTIC MECHANICS, CELESTIAL MECHANICS, POTENTIAL PHYSICS, AND ANALYTIC MORPHOLOGY.

BOSTON:
LITTLE, BROWN AND COMPANY.
1855.

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A SYSTEM

OF

ANALYTIC MECHANICS.

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CAMBRIDGE:
ALLEN AND FARNHAM, PRINTERS.

TO

THE CHERISHED AND REVERED MEMORY OF

MY MASTER IN SCIENCE,

NATHANIEL BOWDITCH,

THE FATHER OF AMERICAN GEOMETRY,

THIS VOLUME

IS INSCRIBED.

ADVERTISEMENT.

The substance of the present volume was originally prepared as part of a course of lectures for the students of mathematics in Harvard College. But at the request of some of my pupils, and especially of my friend Mr. J. D. Runkle, I have been induced to undertake its publication. The liberality of my publishers, the well-known firm of Little, Brown & Co., who generously gave directions to the printers, that no expense should be spared in its typographical execution, seemed to impose upon me an increased obligation to perform my portion of the task to the best of my ability. I have consequently reëxamined the memoirs of the great geometers, and have striven to consolidate their latest researches and their most exalted forms of thought into a consistent and uniform treatise. If I have, hereby, succeeded in opening to the students of my country a readier access to these choice jewels of intellect, if their brilliancy is not impaired in this attempt to reset them, if in their new constellation they illustrate each other and concentrate a stronger light upon the names of their discoverers, and still more, if any gem which I may have presumed to add, is not wholly lustreless in the collection, I shall feel that my work has not been in vain. The treatise is not, however, designed to be a mere compilation. The attempt has been made to carry back the fundamental principles of the science to a more profound and central origin; and thence to shorten the path to the most fruitful forms of research. It has, moreover, been my chief object to develop the special forms of analysis, which are usually neglected, because they are only applicable to particular problems, and to restore them to their true place in the front ranks of scientific progress. The methods which, on account of their apparent generality, have usually attracted the almost exclusive attention of the student, are, on the contrary, reëstablished in their true position as higher forms of speciality.

BENJAMIN PEIRCE.

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ANALYTIC MECHANICS.

CHAPTER I.

MOTION, FORCE, AND MATTER.

§ 1. Motion is an essential element of all physical phenomena; and its introduction into the universe of matter was necessarily the preliminary act of creation. The earth must have remained forever "without form, and void;" and eternal darkness must have been upon the face of the deep, if the Spirit of God had not first "*moved* upon the face of the waters."

2. Motion appears to be the simplest manifestation of power, and the idea of force seems to be primitively derived from the conscious effort which is required to produce motion. Force may, then, be regarded as having a spiritual origin, and when it is imparted to the physical world, motion is its usual form of mechanical exhibition.

3. Matter is purely inert. It is susceptible of receiving and containing any amount of mechanical force which may be communicated to it, but cannot originate new force or, in any way, transform the force which it has received.

CHAPTER II.

MEASURE OF MOTION AND FORCE.

I.

MEASURE OF MOTION.

§ 4. *Uniform Motion* is that of a body which describes equal spaces in equal times.

5. *Velocity* is the measure of motion. In the case of uniform motion it is the distance passed over in a given time, which is assumed as the *unit* of time, and, in any case, it is at each instant the space which the body would pass over, if it preserved the same motion during a unit of time.

6. If the space described by a body in the time t is denoted by s, the expression for the velocity v is, in the case of uniform motion,

$$v = \frac{s}{t}.$$

If the *differential* is denoted by d and the *derivative* by D, the expression for the velocity is, in any case,

$$v = \frac{ds}{dt} = D_t s.$$

II.

MEASURE OF FORCE.

7. Experiments have shown that the exertion which is required to move any body, is proportional to the product of the

intensity of the effort into the space through which it is exerted. This product is, then, the proper measure of the whole amount of force which is necessary to the production of the motion; long established custom has, however, limited the use of the word *force* to designate the *intensity* of the effort, and the *whole amount of exertion* may be denoted by the term *power*. Hence, if the power P is produced by the exertion of a constant force F, acting through the space s, the expression of the force is

$$F = \frac{P}{s}.$$

But if the force is variable in its action, the expression of its intensity at any point is

$$F = \frac{dP}{ds} = D_s P.$$

8. It is found by observation that the force of a moving body is proportional to its velocity. Thus, if m is the force of a body when it moves with the unit of velocity, its force, when it has the velocity v, is mv.

9. Different bodies have different intensities of force when they move with the same velocity. The *mass* of a body is its force, when it moves with the unit of velocity; thus, m in the preceding article, denotes the mass of the body.

10. The force communicated to a freely moving body, by a force which acts in the direction of the motion, is found to be the product of the intensity of the acting force, multiplied by the time of its action. Thus, if the mass m, acted upon by the constant force F, for the time t, in the direction of its motion, has its velocity increased by v, the addition to the force of the moving body is

$$mv = Ft.$$

In case the acting force is not constant, the rate at which the force of the body increases is

$$mD_t v = F.$$

III.

FORCE OF MOVING BODIES.

11. *The power with which a body moves is equal to the produet of one half of its mass multiplied by the square of its velocity.*

For if the body, of which the mass is m, is acted upon by the force F, until from the state of rest it reaches the velocity v, the power P, which has been communicated to it, and which it consequently retains, must, by (3_{14}) * and (4_3), give the equation

$$D_s P = mD_t v.$$

The derivative of P relatively to t, is by (2_{24})

$$D_t P = D_s P.\ D_t s = vD_s P = mvD_t v.$$

The integral of this equation is

$$P = \tfrac{1}{2}mv^2,$$

to which no constant is to be added, because the power vanishes with the velocity. (*Note* A.)

12. Hence the power of a moving body is equal to one half of the product of its force multiplied by its velocity.

---

* The form of referenee here given is by means of numbers, of whieh the leading number refers to the page, and the secondary number, whieh is printed in smaller type, refers to the place upon the page, estimated from the top of the page, in lines of equal typographie interval. Printed marks, eorresponding to these intervals, aeeompany each copy of the work. Thus, (3_{14}) denotes the equation which is at the 14th typographic interval from the top of the third page.

13. It is convenient to refer the measure of force to the unit of mass as a standard. Thus, if F is the force exerted upon each unit of mass, the force exerted upon the body of which the mass is m, is mF. With the F, used in this sense, (4_3) becomes

$$D_t v = F.$$

CHAPTER III.

FUNDAMENTAL PRINCIPLES OF REST AND MOTION.

I.

TENDENCY TO MOTION.

§ 14. *A system of moving bodies may be regarded mechanically as a system of forces or powers, which must be the exact equivalent of all the forces or powers which, by simultaneous or successive communication to the bodies, are united in its formation.*

This results from the inertness of matter, and its incapacity to increase, diminish, or vary in any way, the power which it contains.

15. It also follows from its inertness, that matter yields instantaneously to every force, and cannot resist any tendency to the communication or abstraction of power. With a system which is at rest, there can consequently be no tendency to the communication of power.

16. The tendency of any body or system of bodies to move in any given way is easily ascertained. It is only necessary to suppose the system moved with the proposed motion to an infinitesimal

distance. The product of the corresponding distance, by which each body of the system advances in the direction in which each force acts, multiplied by the intensity of the force is, by § 7, the corresponding power which the force communicates directly to the body, and through it to the system.

The whole amount of power which is thus communicated by all the forces to the system, or rather its ratio to the infinitesimal element of the proposed motion is evidently the measure of the tendency of the system to this proposed motion.

It must be observed that, when a body moves in a direction opposite to that of the action of the force, the corresponding product is negative, and must be used with the negative sign in forming the algebraical sum, which represents the whole amount of power communicated to the system.

17. By a skilful use of the principles of the preceding section, all the elementary tendencies to motion in a system may be determined, and, therefore, all the elements of change of motion in the system which is actually moving, or all the conditions of equilibrium in the system which is at rest. Thus, let

m_1, m_2, m_3, &c., denote the masses of a system of bodies;
F_1, F_1', F_1'', &c, the forces which act upon each unit of m_1;
F_2, F_2', F_2'', &c., the forces which act upon each unit of m_2;
&c. &c.;
$\delta f_1, \delta f_1', \delta f_1''$, &c., the distances by which m_1 advances in the direction of the forces F_1, F_1', F_1'', &c., in consequence of any proposed motion;
$\delta f_2, \delta f_2', \delta f_2''$, &c.; δf_3, &c., the corresponding distances for the other bodies and forces of the system;
Σ', the sum of all quantities of the same kind, obtained by changing the accents;

Σ_1, the sum of all quantities of the same kind, obtained by changing the underwritten numbers;

Σ'_1, the sum of all quantities of the same kind, obtained by all admissible combinations of both changes.

The power communicated to the system by the proposed motion through m_1, m_2, &c., is

$$\Sigma' m_1 F_1 \delta f_1 = m_1 (F_1 \delta f_1 + F'_1 \delta f'_1 + \&c.)$$
$$\Sigma' m_2 F_2 \delta f_2 = m_2 (F_2 \delta f_2 + F'_2 \delta f'_2 + \&c.)$$
$$\&c.\ \&c.;$$

and the whole power communicated is

$$\Sigma'_1 m_1 F_1 \delta f_1 = \Sigma_1 \Sigma' m_1 F_1 \delta f_1$$
$$= \Sigma' m_1 F_1 \delta f_1 + \Sigma' m_2 F_2 \delta f_2 + \&c.$$

This is, therefore, the complete measure of the tendency in the system to the proposed motion, or of the change of motion which the moving system would experience in the direction of the proposed motion. But by a simple change in the values of δf_1, $\delta f'_1$, δf_2, $\delta f'_2$, &c., the tendency to any other proposed motion may be measured; and, in the same way, all the elements of the change of motion may be definitely ascertained.

II.

EQUATIONS OF MOTION AND REST.

18. If, instead of the given forces, each body were acted upon by a force in the direction of its motion, and of such an intensity as to produce the exact change of velocity which it undergoes, this new system of forces would precisely correspond to that actually imparted to the moving bodies, and would be the exact equivalent of the given system of forces. Let

v_1, v_2, v_3, &c. denote the velocities of the bodies;

δs_1, δs_2, δs_3, &c., the distances by which, in consequence of the proposed arbitrary motion of the preceding section, the bodies advance in the actual direction of this motion;

and then from (4_3)

$D_t v_1$, $D_t v_2$, $D_t v_3$, &c., are the intensities of the new forces relatively to the unit of mass.

The whole power communicated by the new system of forces with the proposed motion becomes, then,

$$\Sigma_1 m_1 D_t v_1 \delta s_1 = m_1 D_t v_1 \delta s_1 + m_2 D_t v_2 \delta s_2 + \text{\&c.},$$

and it must, therefore, be equal to the expression (7_{13}) of the power communicated by the given forces. Hence,

$$\Sigma'_1 m_1 F_1 \delta f_1 = \Sigma_1 m_1 D_t v_1 \delta s_1,$$

or by transposition

$$\Sigma_1 m_1 (D_t v_1 \delta s_1 - \Sigma' F_1 \delta f_1) = 0.$$

When the system is at rest, this equation becomes

$$\Sigma'_1 m_1 F_1 \delta f_1 = 0.$$

19. The equation (8_{18}) in the case of motion, or the equation (8_{20}) in the case of rest, although it appears to be a single equation, involves in fact as many equations as there are distinct elements of motion or rest in the system of bodies. For every such element gives a different set of values of δf_1, $\delta f'_1$, δf_2, &c., δs_1, δs_2, &c., which, substituted in (8_{18}) or (8_{20}), produce a corresponding equation. These equations, therefore, involve all the necessary conditions of motion or rest in every mechanical problem. All that remains, then, is to determine, by geometrical analysis, the various elements of motion or rest, and to integrate and interpret the algebraical

equations, into which (8_{18}) and (8_{20}) are finally decomposed. The *Mécanique Analytique* of the ever-living Lagrange contains the general forms of investigation with unequalled elegance and perspicuity. But the special modes of analysis, which are peculiarly adapted to the illustration and development of particular problems, have been too much neglected, and the attention of youthful explorers is earnestly invited to this unbounded field of research.

CHAPTER IV.

ELEMENTS OF MOTION.

I.

MOTION OF TRANSLATION.

§ 20. A single material point may be moved to an infinitesimal distance in any direction, which may be defined by either of the methods known to geometers, by the reference, for instance, to the directions of three mutually perpendicular axes. By the known theory of projections, (*Note* B,) the distance by which the point advances in the direction of its actual motion, or in any other direction, may be fully determined from the distances which it advances in these three directions. The three distances, moved in the directions of the axes, which are simply the projections of the proposed motion upon the three axes, are the *three independent elements of motion which completely define the elementary motion of the single point.*

Thus if

δp denotes the proposed elementary motion, if

$\genfrac{}{}{0pt}{}{p}{x}, \genfrac{}{}{0pt}{}{p}{y}, \genfrac{}{}{0pt}{}{p}{z},$ denote the angles which this motion makes with the three mutually perpendicular axes, called the axes of x, y, and z, and

δx, δy, δz, the projections of δp upon the axes,

the expressions for these projections are,

$$\delta x = \cos \genfrac{}{}{0pt}{}{p}{x} . \delta p,$$

$$\delta y = \cos \genfrac{}{}{0pt}{}{p}{y} . \delta p,$$

$$\delta z = \cos \genfrac{}{}{0pt}{}{p}{z} . \delta p.$$

If, in general,

$\genfrac{}{}{0pt}{}{p}{q}$ denotes the angle which the directions of p and q make with each other, the distance by which the point advances, in consequence of the proposed motion, in the direction of f is, by the theory of projections,

$$\begin{aligned} \delta f &= \cos \genfrac{}{}{0pt}{}{p}{f} . \delta p \\ &= \cos \genfrac{}{}{0pt}{}{f}{x} . \delta x + \cos \genfrac{}{}{0pt}{}{f}{y} . \delta y + \cos \genfrac{}{}{0pt}{}{f}{z} . \delta z \\ &= \Sigma_x \cos \genfrac{}{}{0pt}{}{f}{x} . \delta x\,; \end{aligned}$$

in which

Σ_x denotes the sum of all the similar terms obtained by proceeding from one axis to each of the others.

21. The most important of all the elementary motions of a system of bodies are those which, being independent of the peculiar constitution of the system, may be common to all systems. Such

motions must be possible, even if the bodies which compose the system, do not change their mutual positions, but are so rigidly fixed that the whole may be regarded as one solid body. It will be shown that there are but two distinct classes of such motions, namely, those of *translation* and those of *rotation.*

22. The motion of *translation* is that by which all the points of a body, or system of bodies, are transported through the same distance in the same direction. The projections of an elementary translation upon three rectangular axes are given by equations $(10_{10\text{-}12})$, while (10_{21}), is the expression of the distance by which the system, or any one of its bodies, advances in any direction, such as that of f, by reason of the proposed translation.

23. Any number of different elementary translations may be supposed to be given at the same time to a system, and the resulting motion will be such an elementary translation, that its projection, estimated in any direction, will be the sum of the projections of the elementary translations estimated in the same direction.

Two coëxistent elementary translations may be combined geometrically by setting off from any point two lines of the same length with the elementary motions, and in the same direction with them; and if a parallelogram is described upon these two lines as sides, the diagonal, which is drawn from the given point, will represent in distance and direction the resulting elementary translation.

In the same way the geometrical resultant of the combination of three elementary translations may be represented by the diagonal of a parallelopiped described upon the lines which represent the component translations. But this parallelopiped vanishes when the three lines are in the same plane.

II.

MOTION OF ROTATION.

§ 24. The motion of *rotation* is that by which all the points of a body or system of bodies turn about a fixed line in the body, which line is called the *axis of rotation.* If one stands with his feet against the axes of rotation, and his body perpendicular to it, and faces in the direction of the rotation, the *positive* direction of the axis of rotation is, in this treatise, regarded as lying upon his right hand, and its *negative* direction upon his left hand. It will be found convenient to represent a rotation geometrically by a distance proportional to the elementary angle of rotation, set off upon the positive direction of the axis of rotation from any point taken at pleasure in the axis. If

$\delta\theta$ denotes the elementary angle of rotation, and r the distance of a point of the body from the axis of rotation;

$r\delta\theta$ is the elementary distance through which the point moves in consequence of the rotation.

The form in which the subject of rotation will be here presented, is not greatly modified from that which it has finally assumed in Poinsot's admirable exposition of the "*Theory of the Rotation of Bodies,*" as it is printed in the additions to the *Connaissance des Temps* for 1854.

25. *When a body rotates about an axis, it is, in consequence of this rotation, simultaneously rotating about any other axis which passes through the same point, with an angle of rotation which is represented by the projection upon this new axis of the line which represents the original angle of rotation.*

For by the angle of rotation θ about the axis OA (fig. 1), the point P of the axis OB, which is at the distance

$$r = PM$$

from the axis OA, is moved through the distance $r\theta$. Although every point of the axis OA is actually at rest, it has with respect to P, a relative motion, which is the negative of that of P. A rotation θ' about the axis OB gives the point N of the axis OA, which is in the plane drawn through P perpendicular to OB, and at the distance

$$r' = PN$$

from the axis of OB, a motion through the distance $r'\theta'$ taken negatively. This rotation is, then, the same with that which the actual rotation produces about the axis OB, if

$$r'\theta' = r\theta,$$

or

$$\frac{\theta'}{\theta} = \frac{r}{r'} = \cos MPN$$
$$= \cos AOB;$$

that is, if θ' is equal to the projection of θ upon OB.

26. *Three simultaneous elementary rotations about three axes, which pass through the same point, and are not in the same plane, are equivalent to a single rotation about the diagonal of a parallelopiped, of which the three lines representing the rotations are the sides, and the length of the diagonal represents the angle of elementary rotation.*

For the algebraic sum of the projections of the sides of the parallelopiped upon any line perpendicular to its diagonal is zero, and, therefore, there is no rotation about any such line. Hence the diagonal is stationary, that is, it is the axis of rotation. The whole amount of rotation, being the sum of the partial rotations about the diagonal which arise from the several rotations about the sides, is represented by the sum of the projections of the sides upon the diagonal, which is, by the theory of projections, equal to the diagonal itself.

27. In the same way, two simultaneous rotations about the sides of a parallelogram may be combined into a single rotation about the diagonal. In short, *simultaneous elementary rotations about axes which cut each other may be combined in the same way as elementary translations.*

28. To investigate the distance by which a given rotation causes any point of a body or system to advance in a given direction, as that of f; let

$\delta\theta$ be the elementary angle of rotation about the axis of p and
r' the perpendicular let fall from the point upon the axis of rotation.

Let a line be drawn through the given point, parallel to the projection of f upon a plane, which is perpendicular to the axis of rotation, and let

ϱ be the perpendicular let fall upon this line from the point in which r' meets the axis of rotation; and
$\frac{\theta}{f}$ the angle which f makes with the direction in which the point is moved by the elementary rotation.

The distance by which the point advances in the direction of f is

$$\delta f = r' \cos\tfrac{\theta}{f} . \delta\theta = r' \cos\tfrac{\varrho}{r'} \sin\tfrac{p}{f} . \delta\theta$$
$$= \varrho \sin\tfrac{p}{f} . \delta\theta,$$

in which ϱ should be taken positively when the point is moved towards the positive direction of f.

29. If three rectangular axes are drawn through any point of the axis of rotation, and if

$\delta\vartheta_x$, $\delta\vartheta_y$, $\delta\vartheta_z$ are the projections of $\delta\vartheta$ upon these axes, the distance by which the point (x, y, z) is moved in the direction of the axis of x, is

$$\begin{aligned}\delta x &= y\delta\vartheta_z - z\delta\vartheta_y \\ &= (y\cos{}^p_z - z\cos{}^p_y)\,\delta\vartheta \\ &= (\cos{}^r_y \cos{}^p_z - \cos{}^r_z \cos{}^p_y)\, r\delta\vartheta \\ &= (\cos{}^r_y \cos{}^p_z - \cos{}^r_z \cos{}^p_y)\,\operatorname{cosec}{}^r_p . r'\delta\vartheta \\ &= (\cos{}^{r'}_y \cos{}^p_z - \cos{}^{r'}_z \cos{}^p_y)\, r'\delta\vartheta. \\ &= \cos{}^\theta_x . r'\delta\vartheta\end{aligned}$$

There are similar expressions for the distances by which the point advances in the directions of the axes of y and z, which may be found by advancing each of the letters x, y, z, and x to the following letter of the series.

30. The two last members of equation (15_5) divided by $r'\delta\vartheta$ give the following theorem;

$$\cos{}^\theta_x = \cos{}^{r'}_y \cos{}^p_z - \cos{}^{r'}_z \cos{}^p_y,$$

in which the direction of ϑ is that of the perpendicular to the common plane of r' and p, and it is taken upon that side of the plane for which, a positive rotation about it, would correspond to a motion through the acute angle from r' to p.

31. If there were another system of rectangular axes, x', y', and z', equation (15_{20}) applied to them would give

$$\cos{}^{x'}_x = \cos{}^{y'}_y \cos{}^{z'}_z - \cos{}^{y'}_z \cos{}^{z'}_y.$$

In this equation each of the letters x, y, z, and x might be advanced to the subsequent letter of the series, as well as each letter of the series x', y', z', and x'. In this way eight other equations might be found similar to equation (15_{28}).

III.

COMBINED MOTIONS OF ROTATION AND TRANSLATION.

32. *An elementary rotation, combined with an elementary translation in any direction, which is perpendicular to the axis of rotation, is equivalent to an equal elementary rotation about an axis which is parallel to the original axis of rotation. The position of the new axis is determined by the condition that each of its points is carried by the original elementary rotation as far as by the elementary translation, but in an opposite direction.*

For the given motions cancel each other's action upon each point of the new axis, and leave it stationary; while the original axis advances with the elementary translation by the exact distance which corresponds to the elementary rotation about the new axis. The common plane of the two axes is perpendicular to the direction of the translation.

33. *Any simultaneous elementary rotations about axes parallel to each other are equivalent to a single rotation, equal to their sum, and about an axis parallel to the given axes, combined with an elementary translation equal to the motion which any point of the new axis receives from their simultaneous action.*

This is a simple deduction from the preceding proposition.

34. Let there be three rectangular axes, such that the new axis of rotation may be that of z; let

x_1, y_1, x_2, y_2, &c., be the points in which the original axes cut the plane of xy; and let

$\delta\theta_1, \delta\theta_2$, &c., be the elementary angles of rotation about these axes.

The elementary rotation about the axis of z is

$$\delta\vartheta = \Sigma_1 \delta\vartheta_1.$$

The elementary translations in the directions of the axes of x and y are by (12_{19})

$$\delta x_0 = \Sigma_1 y_1 \delta\vartheta_1,$$
$$\delta y_0 = -\Sigma_1 x_1 \delta\vartheta_1.$$

The distances through which any point (x, y, z) is carried forward in the directions of the axes, are

$$\delta x = \delta x_0 - y\delta\vartheta = \Sigma_1 y_1 \delta\vartheta_1 - y\,\Sigma_1 \delta\vartheta_1,$$
$$\delta y = \delta y_0 + x\delta\vartheta = -\Sigma_1 x_1 \delta\vartheta_1 + x\,\Sigma_1 \delta\vartheta_1.$$

The points are, therefore, at rest for which

$$0 = \delta x_0 - y\delta\vartheta = \Sigma_1 y_1 \delta\vartheta_1 - y\,\Sigma_1 \delta\vartheta_1,$$
$$0 = \delta y_0 + x\delta\vartheta = -\Sigma_1 x_1 \delta\vartheta_1 + x\,\Sigma_1 \delta\vartheta_1.$$

These are, therefore, the equations of the axis of rotation, an elementary rotation about which, equal to the sum of all the elementary rotations, is equivalent to the combination of all the elementary rotations.

35. If the original elementary rotations are all equal, and if there are n axes of rotation, the equations (17_2) and (17_{11}) become

$$\delta\vartheta = n\delta\vartheta_1,$$
$$\delta x = (\Sigma_1 y_1 - ny)\,\delta\vartheta_1,$$
$$\delta y = (-\Sigma_1 x_1 + nx)\,\delta\vartheta_1.$$

The equations (17_{16}) give for the single axis of rotation

$$y = \frac{\Sigma_1 y_1}{n},$$
$$x = \frac{\Sigma_1 x_1}{n}.$$

36. If any of these rotations are about an axis lying in the opposite to the assumed direction, they may be regarded as nega-

tive rotations about axes having the same direction as the assumed one, and may be combined algebraically in the preceding sums.

37. When the second member of equation (17_2) vanishes, the resulting rotation disappears, and the given elementary rotations are equivalent to the elementary translation defined by equations (17_6).

38. Two equal rotations about axes, which are parallel, but have opposite directions, constitute a combination which Poinsot has called a *couple of rotations.*

A couple of elementary rotations is, therefore, equal to an elementary translation in a direction perpendicular to the common plane of the axes, and equal to the product of the distance between the axes multiplied by the elementary angle of rotation.

39. *Any simultaneous elementary motions of rotation and translation are equivalent to a single elementary rotation about an axis, combined with an elementary translation in the direction of the axis of rotation.*

For each rotation may be resolved into a translation and a rotation about an axis passing through any assumed point. But all the elementary rotations about axes passing through the same point are equivalent to a single rotation about an axis passing through the point, and all the translations are equivalent to a single translation. The single translation may be resolved into two translations, of which one is parallel, and the other perpendicular to the single axis of rotation. The translation, which is perpendicular to the axis of rotation, combined with the rotation, is equivalent to a single rotation about an axis, parallel to the single axis, and, therefore, having the same direction with the remaining translation.

40. *Every possible motion of a rigid system or body is equivalent to a combination of the motions of translation and rotation.*

This is evident, if it can be shown that, by such a combination of motions, any three points, A, B, and C, of the system, can be carried to any positions, A', B', and C', in which it is possible for them to be placed. For three points of a rigid system not in the same straight line completely determine, by their position, that of the whole system. Now, by a translation of the system, equal to that by which A might be directly moved from A to A', the point A is actually brought to the position A'. By a subsequent motion of rotation about an axis, which is perpendicular to each of the lines AB and $A'B'$, the point B may be moved to B'; and then by a rotation about $A'B'$ the point C may be carried to C'. Hence the whole motion is accomplished by one translation and two rotations.

Every elementary motion of a rigid system must then be equivalent to a single rotation about an axis and a translation in the direction of the axis of rotation. This motion is perfectly represented by that of the *screw*, whose helix causes it to advance in the direction of the axis about which it is turning.

41. During each instant of its motion, a rigid system rotates about an axis, which is called the *instantaneous axis of rotation*. This axis is generally varying its position in the system and in space from one instant to another, which renders it difficult to form a distinct conception of the nature of the corresponding motion of the system.

42. In attempting to conceive of the motion of a rigid system, it is expedient, at first, to neglect the translation in the direction of the axis of rotation, and to assume that the motion is solely that of rotation. The successive positions of the axis of rotation in the system form by their union a surface which turns with the system; and its successive positions in space form another fixed surface. In the motion now considered, the moving surface rolls on the fixed surface without sliding, and carries the system with it.

43. If the axis of rotation does not move perpendicularly to itself each of these surfaces is evidently a developable surface, and

in the act of rolling the line of retrogression of the one falls upon that of the other; so that these two lines are of the same length. Upon the surfaces, developed into a plane, the two lines of retrogression will be precisely alike.

In combining with this rotation the translation in the direction of the axis of rotation, the surface, generated by the instantaneous axis in the moving system, remains unchanged. But the fixed surface, generated by the instantaneous axis, is changed; it is still a developable surface obtained from that in which the translation is neglected, by adding to each element of the arc of the curve of retrogression, the elementary translation in the direction of the axis of rotation. In the actual motion, the moving surface rolls upon the fixed surface, and glides simultaneously in the direction of the line of contact, so as to keep the curves of retrogression constantly in contact.

In this general case, the whole length of the arc of the fixed curve of retrogression is equal to that of the moving curve augmented by the whole amount of translation in the direction of the axis of rotation.

When the elementary translation is equal to the elementary arc of the moving curve of retrogression, but lies in the opposite direction, there is a corresponding *cusp* in the fixed curve of retrogression.

A point of inflection in the curves of retrogression generally corresponds to a change in the direction of the rotation. A similar combination of the translation with the rotation can be introduced into the general case of motion.

44. When either of the surfaces of the instantaneous axis is a *cone*, the curve of retrogression is reduced to a point which is the *vertex* of the cone. When both of the surfaces are cones, there is no translation in the direction of the axis.

When either of the surfaces is a cylinder, both surfaces must be cylinders; and the lines of retrogression, removing to an infinite distance, cannot be used for guiding the motion of translation. But in this case, a section may be made of one of the cylinders perpendicular to its axis, and in the actual motion the moving cylinder will move so as to keep the point, in which the perimeter of this section touches the other cylinder, upon a curve properly drawn upon that cylinder.

45. The general motion of a rigid system may be conceived as a translation, equal to that of any one of its points assumed at will, combined with a rotation about an instantaneous axis of rotation passing through the point. If the translation is neglected, the rotation is effected as in § 42 by rolling a cone, of which the assumed point is the vertex, and which carries the system with it, in its motion, about a fixed cone, of which the same point is the vertex. The translation may be simultaneously effected by moving the two cones in space, with a translation equal to that which belongs to their vertex in the actual motion of the system.

46. For all the points of the instantaneous axis in each of its positions, the corresponding centres of greatest curvature of either of the conical surfaces which it describes, are all upon the same straight line passing through the vertex.

In the case of the right cone, or of the right cylinder, the axis of revolution is the line of the centres of greatest curvature. In all these investigations the plane may be regarded either as a cylinder of infinite radius, or as a cone, of which the angle at the vertex is equal to two right angles.

47. The elementary rotation of the system may be conceived as decomposed into two elementary rotations about the lines of the centres of greatest curvature as axes of rotation. By the rotation about the line, which unites the centres of the fixed surface, the

instantaneous axis receives its elementary motion in space, and is carried to its proper position upon the fixed surface. By the rotation about the line which unites the centres of the moving surface, the system receives that additional rotation which is required to turn the moving surface into that position in which it may have the proper line of contact with the fixed surface. Each of these rotations produces a sliding of the moving upon the fixed surface; but as the sliding produced by the one is just equal and opposite to that produced by the other rotation, the two rotations cancel each other's action in this respect, and there is no sliding in the combined motion, but a simple rolling of one surface upon the other.

48. Let

α_f be the acute angle which the instantaneous axis of rotation makes with the line of the centres of curvature of the fixed surface;

α_m that which it makes with the line of the cèntres of curvature of the moving surface, this angle being *positive* when the two lines of the centres are on opposite sides of the instantaneous axis, and *negative*, when they are upon the same side;

$\delta\,\omega$ the elementary angle by which the instantaneous axis changes its direction;

$\delta\,\theta_f$ the elementary angle of rotation about the line of centres of the fixed surface; and

$\delta\,\theta_m$ the elementary angle of rotation about the line of centres of the moving surface.

Since the instantaneous axis must be carried forward by the rotation about the fixed axis, and backward by the rotation about the moving axis just as far as its actual change of position, its elementary angle of change of direction is

$$\delta\,\omega = \delta\,\theta_f . \sin\alpha_f = \delta\,\theta_m . \sin\alpha_m .$$

But the combination of the two rotations about these axes gives the actual rotation about the instantaneous axis, and therefore,

$$\begin{aligned}\delta\,\theta &= \delta\,\theta_f . \cos\alpha_f + \delta\,\theta_m . \cos\alpha_m \\ &= (\cot\alpha_f + \cot\alpha_m)\,\delta\,\omega \\ &= \frac{\sin(\alpha_f + \alpha_m)}{\sin\alpha_f \sin\alpha_m}\,\delta\,\omega.\end{aligned}$$

49. When the surfaces described by the instantaneous axis are cylinders, let

ϱ_f and ϱ_m be the respective radii of greatest curvature of the fixed and moving surfaces at any point of their mutual contact; and

$\delta\,p$ the elementary distance which the instantaneous axis moves in a direction perpendicular to itself.

The conditions of the motion of the instantaneous axis give the equations

$$\delta\,p = \varrho_f\,\delta\,\theta_f = \pm\,\varrho_m\,\delta\,\theta_m ;$$

in which the upper sign corresponds to the case where the lines of the centres of curvature are upon opposite sides of the instantaneous axis, and the lower sign to that in which they are upon the same side. The rotation about the instantaneous axis is

$$\begin{aligned}\delta\,\theta &= \delta\,\theta_f + \delta\,\theta_m \\ &= \left(\frac{1}{\varrho_f} \pm \frac{1}{\varrho_m}\right)\delta\,p .\end{aligned}$$

IV.

SPECIAL ELEMENTS OF MOTION AND EQUATIONS OF CONDITION.

50. The variation of each independent element of position of a system gives an independent element of motion. But the elements of position are various, and must be selected in each case with special reference to the problem under discussion. It often occurs that parts of the system are rigidly connected; such parts are themselves rigid systems, and subject only to motions of translation and rotation, and, therefore, none but such elements are required for the investigation of their motions.

Points of the system are sometimes restrained to move upon given surfaces, and, in this case, it may be expedient to introduce elements of position dependent upon the principal lines of curvature of these surfaces, or elements, in reference to which the surfaces are peculiarly simple or symmetrical. Points of the system may be compelled to preserve simple geometrical relations to each other, which may suggest appropriate elements of position to the skilful analyst; or he may find indications to direct his choice in the very nature of the motion itself.

51. It is often desirable to adopt a combination of elements of position which are not wholly independent of each other, but are subject to certain mutual restrictions. These restrictions, when they are expressed algebraically, are called *equations of condition.* They may assume the differential form of equations between the elementary motions; or they may be finite equations between the elements of position, in which case they may be reduced by differentiation to equations between the elementary motions.

By means of the equations of condition, as many of the elements of motion may be determined in terms of the rest as there are equations of condition; and the remaining elementary motions may be regarded as independent of each other.

52. Instead of introducing into the equations (8_{18}) and (8_{20}) of motion and rest the special values of δs_1, δs_2, &c., δf_1, δf_2, &c., for each particular element of motion, their general values may be found in terms of all these elements. When the elementary motions are wholly independent, their coefficients in these equations give, when they are equalled to zero, the same equations which would have been obtained by the special investigations. But when the elements are not independent, all, except the independent elements can be eliminated by means of the values given by the equations of condition.

The equations (8_{18}) and (8_{20}) of motion and rest, on account of their differential form, are necessarily linear in reference to the elementary motions; and the differential equations of condition are likewise linear. The proposed elimination may therefore be conducted by the *method of multipliers.* By this process each differential equation, multiplied by an unknown quantity, is to be added to the given equation of motion or rest. The unknown multipliers are to be determined by the conditions that the coefficients of the elementary motions, which are to be eliminated, become equal to zero. Since the remaining elementary motions are independent of each other, their coefficients must also be equalled to zero. In the sum, therefore, obtained by the addition of the equations, each of the coefficients of the elementary motions is equal to zero. The number of unknown quantities is increased in this process by that of the unknown multipliers; but, because there are as many equations of condition as there are multipliers, the whole number of equations, including the equations of condition, in their finite form, is just sufficient to determine the values of the multipliers and of all the elements of position.

53. Let

$$L_1 = 0,$$

be one of the equations of condition in its finite form; and let its differential form be

$$\delta L_1 = 0.$$

Let also,

λ be the unknown multiplier by which it is to be multiplied.

The sum obtained by adding the similar products of all the equations of condition to equation (8_{18}) or (8_{20}) is

$$\Sigma_1 m_1 (D_t v_1 \delta s_1 - \Sigma' F_1 \delta f_1) + \lambda_1 \delta L_1 = 0,$$
$$\Sigma_1' m_1 F_1 \delta f_1 + \Sigma_1 \lambda_1 \delta L_1 = 0,$$

which is the equation of motion or rest, and in which the general values of δs_1, δf_1, &c., are to be substituted, and the coefficient of each elementary motion is to be equalled to zero.

54. Each equation of condition becomes the equation of a surface, to which any one of the points whose elements of position occur in the equation is restricted, provided that, for the moment, the variations of all the other elements are neglected. Since the point is restricted to move upon the surface, it cannot move in the direction of the normal to the surface. Let a system of three rectangular axes be adopted, and let

N be the normal to the surface.

Its variation, arising from the variation of coördinates, which may be regarded as the elements of position of the point, is

$$\delta N = \Sigma_x \delta x \cos \tfrac{x}{N}.$$

If the equation of the surface is (26_2), with the omission of the numbers written below, which may be neglected in the general discussion, its variation is

$$\delta L = \Sigma_x D_x L \delta x.$$

Let, then,

$$M^2 = \Sigma_x (D_x L)^2;$$

and the angle, made by the normal with one of the axes, is given by the equation

$$\cos \frac{x}{N} = \frac{D_x L}{M};$$

which substituted in (26_{29}) gives

$$\delta N = \frac{\Sigma_x D_x L \delta x}{M} = \frac{\delta L}{M}.$$

Hence the equation of condition with its multiplier may be written in the form

$$\lambda \delta L = \lambda M \delta N = 0;$$

and this form may be substituted in the equations (26_{12}) and (26_{13}) of motion and rest.

CHAPTER V.

FORCES OF NATURE.

I.

THE POTENTIAL AND ITS RELATIONS TO LEVEL SURFACES, THE POSITIONS OF EQUILIBRIUM, AND THE POSSIBILITY OF PERPETUAL MOTION.

§ 55. It appears, at first sight, to be inconsistent with the assumed spiritual origin of force, that the principal forces of nature reside in centres of action, which are not thinking beings, but particles of matter. The capacity of matter to receive force from mind in the form of motion, contain and exhibit it as motion, and communicate it to other matter, under fixed laws, is not, however, less difficult or more conceivable than the capacity to receive and contain it in a more refined and latent form, from which it may become manifest under equally fixed laws. It is only, indeed, when force is thus separated from mind, and placed beyond the control of will, that it can be subject to precise laws, and admit of certain and reliable computation.

56. The laws of the development of power in nature are of two classes. In the one class, the forces depend solely upon the relative positions of the bodies, and may be called *fixed*. In the other class, the forces depend, not only upon the positions of the bodies, but also upon their actual state of power, especially upon the velocities and directions of their motions; and these forces may be called *variable*.

57. The most fruitful and enlarged view of the fixed forces of

nature, and one which peculiarly corresponds to their laws of action so far as they have been observed, is to regard them as the manifestations of the *dynamic situation* of the bodies which exhibit them. The dynamic situation depends solely upon the masses and positions of the bodies; it is a condition of *form*, and its research is a problem of pure geometry. The algebraic function which embodies the idea of the dynamic state is called the *potential*. Its complete investigation and determination involves the solution of all the problems which can arise in regard to the power and the conditions of force of all systems, whether they are at rest or in motion, so far at least as the fixed forces of nature are concerned.

The amount of power of a system is not to be inferred from its situation, although there is a certain measure of power appropriate to that situation. It is this latter power which is expressed by the potential of the system, and expressed as a function of all the elements of position, by which the situation is defined.

58. *The power of a moving system increases or decreases with the power which belongs to its situation, and the increase or decrease of its power is measured by that of its potential.*

59. Hence, if a system moves from a state of rest, its power is constantly equal to the excess of its potential over the initial value of the potential; and it can never arrive at a position in which the potential would be less than its initial value. No system, indeed, can move to a situation in which the potential would be diminished more than the initial power of the system.

60. When a system is in a permanent state of rest which the actual forces do not tend to disturb, its dynamic condition is such, that the power of the system is not changed by a slight change of position. Hence,

The potential of a system which is in equilibrium, is generally a maximum or a minimum. The exceptional case of a condition of *indiffer-*

ence rarely oecurs in nature; but even this ease may be philosophically regarded as the eombination of a maximum and minimum, or as the result of several sueh eombinations.

61. When a moving system passes through a position of equilibrium, or a position whieh is one of equilibrium in reference to the element of position with whieh the system is changing its plaee, the power of the system is either a maximum or a minimum, or in a eondition of indifferenee.

62. When a system, in a state of rest, is plaeed very near the position of equilibrium, it eannot tend to move away from the position of equilibrium, if the potential of that situation is a maximum relatively to the element by which the system is removed from it; and it eannot tend to move towards the situation of equilibrium, if the potential is a minimum for the same element. On this aeeount the equilibrium is *stable*, in referenee to those elements for which the potential is a maximum, and it is *unstable* in referenee to these elements, for which the potential is a minimum.

63. As when a funetion ehanges in eonsequenee of the ehange of any one of its variables, the maxima and minima sueeeed eaeh other alternately; in the motion of a system, the positions of stable and unstable equilibrium, relatively to the element of ehange of position, suceeed each other alternately. Situations of equilibrium of indifferenee may be interposed without disturbing the order of sueeession of the situations of stable and unstable equilibrium. If the system returns to its initial position, it must have passed through an even number of sueh situations of equilibrium, relatively to the element of change of position, half of whieh must have been positions of stable, and the other half positions of unstable equilibrium. In general, these situations will not be positions of absolute equilibrium, but only sueh in referenee to the ehanging element of motion.

64. Fixed forces might easily be imagined different from those of nature, and in the action of which the power of a moving system would depend upon its previous situations as well as upon its actual position. With such forces the increase or decrease of power of a system would vary with the path which it pursued in moving from one situation to another, and would be greater by one path than by another. The change of power for each element of any given path, would still be computed by the process of § 17, and thence the whole change of power would be obtained by integration. If the motion of the system were reversed, and it were carried back through the same path to its initial position, its initial power would be restored. If, of two courses, by which a system could move from one situation to another, it were forced to go by that through which it would arrive, with the greater power at its final position, and if it were then made to return to its initial position by the other path, it would return with an increased power; if it were again to move through the same circuit, it would again return with an equal additional increase of power; and, by successive repetitions of this process, the power might be increased to any, even to an infinite amount. Such a series of motions would receive the technical name of a *perpetual motion,* by which is to be understood, that of a system which would constantly return to the same position, with an increase of power, unless a portion of the power were drawn off in some way, and appropriated, if it were desired, to some species of work. A constitution of the fixed forces, such as that here supposed, and in which a perpetual motion would be possible, may not, perhaps, be incompatible with the unbounded power of the Creator; but, if it had been introduced into nature, it would have proved destructive to human belief, in the spiritual origin of force, and the necessity of a First Cause superior to matter, and would have subjected the grand plans of Divine benevolence to the will and caprice of man.

65. A surface, for each of whose points the potential has the same value, may be called a *level surface*. A level surface may be drawn through any point in space.

Since the potential of every finite system of nature vanishes for an infinitely distant point, *all the level surfaces of nature are finite, and, returning into themselves, include a space which they wholly surround, with the exception of those level surfaces for which the potential is zero.*

66. A material point, placed upon a level surface, has no tendency to move in the direction of the surface, because there is no increase of power in such direction. *The tendency of a material point to motion is, therefore, perpendicular to the level surface upon which it is placed.*

67. If two level surfaces are drawn infinitely near to each other, *a material point, placed upon either of them, tends to move in the direction, from the surface of the less potential towards the other, with a force which is measured by the quotient of the difference of the potentials of the two surfaces, divided by their distance apart.*

Hence, if the surfaces are, throughout, at the same distance apart, the disposition to motion is everywhere the same.

If the surfaces were to intersect each other, the tendency to motion in the line of intersection would be infinite ; but, since there is no such infinite tendency to motion in nature, *each level surface of nature must be wholly included within every other level surface, within which any portion of it is included.* For the same reason, *the potential in nature is always a continuous function.*

68. Within each level surface of nature there must be a point or points of maximum or minimum potential. A continuous curved line, drawn perpendicularly to each of the level surfaces which it intersects, represents a line of action or tendency to motion, and every such trajectory must finally terminate in one of the included points of maximum or minimum potential. Each of

these points may then be regarded as a centre of action, towards, or from which, all motion tends along the various trajectories, according as the point is that of a maximum or a minimum potential.

69. *If the potential has a constant value for any portion of space, this same constant value must extend throughout all that space, including this portion, for which the potential and all its derivatives are finite and continuous functions.* For, in order that the potential may be absolutely constant for any finite extent, however small, all its derivatives must vanish. But it follows, from Taylor's Theorem, that the difference of the value of the potential for any portion of space, for which it is continuous and finite, as well as all its derivatives, is a linear function of its derivatives at any point of that space. The difference of the potential, therefore, vanishes, when all the derivatives vanish and the potential is constant.

The portion of space, for which the derivatives are originally assumed to be constant, must be a solid, having the three dimensions of extension, in order that this theorem be applicable.

70. Throughout any such portion of space, in which the potential is constant, there can be no tendency to motion in any direction. In such extent, therefore, there can be no mass of matter, for it is contrary to experience that there should be matter where there are no dynamical phenomena.

71. In all the observed laws of material action, the potential, which belongs to the action of each particle of matter, is finite and continuous, as well as all its derivatives, for the whole extent of space exterior to the particle. Hence, the potential and its derivatives, for every system of nature, are finite and continuous functions throughout any portion of space which contains no material mass.

72. Hence, it follows, that *for every finite system of nature, any portion of space, in which the potential is constant, must be finite, and bounded on all sides by material masses.* This portion of space cannot

extend to infinity, because, if it were to have such an extent, the finite mass, which would be its inner limit, would exhibit no external indication of force; whereas, it is obvious that no matter can ever have been observed, except by such a manifestation of its existence.

73. There are forces in nature which are *temporarily fixed*, and for which the potential may vanish throughout all space exterior to the limit in which the centres of action are contained.

74. The difference between the values of the potential for any two points may be computed by supposing a unit of mass to move from one point to the other upon any line taken at pleasure, and determining the change of power which it receives from this motion. The change of the potential may be computed for each force separately, and, in making the partial computations, it is sufficient to suppose the unit of mass to move from the level surface of one point to that of the other, and one of the perpendicular trajectories may be taken for the path of this motion.

75. If, in any system,

F, F', &c., are the forces;
f, f', &c., the directions in which they act; and
Ω is the value of the potential;

the general expression of the potential for any point of the system is

$$\Omega = \Sigma' \int F df,$$

in which the limits of integration extend from the values of f, f', &c., which correspond to the position of the point, to infinity. The expression for the tendency to motion in any direction, as that of p, is

$$D_p \Omega = D_p \Sigma' \int F df.$$

II.

COMPOSITION AND RESOLUTION OF FORCES.

76. No phenomenon is observed, in which a single force acts freely by itself. In all cases, various forces are combined; and it is important, therefore, to ascertain what are the dynamical results of such combinations.

77. A single force acts, at each point, perpendicularly to its level surface, with an intensity which is measured by the derivative of the potential, taken with reference to the element of direction of the force. The intensity of its action, in any other direction, is measured by the derivative, with reference to the element of that direction. If another level surface is drawn infinitely near the one which passes through the point, the action in any direction is inversely proportional to the length, intercepted by the surfaces, upon a straight line drawn in the given direction. But the surfaces may, for this purpose, be considered as reduced to their parallel tangent planes at the given point; and the length, intercepted between two parallel planes, upon a straight line, is proportional to the secant of the angle which the line makes with the perpendicular to the plane. Hence, the action of a force in the direction of any line, is proportional to the cosine of the angle which it makes with the direction of the force.

If, then, upon a straight line drawn in the direction of a force, a length is taken to represent the intensity of the force, the action in any direction is represented by the projection of this length upon that direction, or by using the word *force* for the representative of the force, the proposition becomes, that *the action of a force in any direction is the projection of the force upon that direction.*

78. When several forces act upon a point, their total action in

any direction is the algebraic sum of their projections upon that direction.

79. *When three forces, which are not in the same plane, act upon a point, their combined action is equivalent to that of a single force, which is represented in magnitude and direction by the diagonal of the parallelopiped constructed upon the three forces.*

For the algebraic sum of the projections of the forces upon any direction perpendicular to the diagonal, is zero, while that of the projections upon the diagonal is the diagonal itself.

80. *All the forces which act upon a point, are equivalent to a single force, which is called their resultant.* For a single point can only tend to move, with a certain intensity, in some one direction, however various may be the forces which act upon it; and any such tendency to motion can be produced by one force acting upon the point.

The actions of all the forces in three directions which are perpendicular to each other, can be found by § 78; and these three partial forces can then be combined by § 79 into one force which will be the resultant. But the following method of finding the resultant illustrates the use which may be made of the level surfaces.

81. In considering the action of a force upon a fixed point in space, the variable character of the force for other points of space may be neglected, and its level surfaces may be regarded as parallel planes perpendicular to the direction of the force. Thus, it may be assumed that

Ff is the potential of the force F, which acts in the direction of f; for

$D_f(Ff) = F$, is the intensity of the force; and

$Ff =$ a constant, or

$f =$ a constant,

is the equation of a plane perpendicular to f. Hence, the potential of all the forces which act upon the point, is

$$\Omega = \Sigma' Ff.$$

If then

P_q is the resulting force resolved in the direction of q; if
p is the direction of the resultant, and
P is the resultant;

the value of either of these forces is represented by the formula

$$P_q = D_q \Omega = \Sigma' F D_q f = \Sigma' F \cos\frac{f}{q}.$$

But, by putting

$$L^2 = \Sigma_x (D_x \Omega)^2 = \Sigma_x P_x^2,$$

the condition that p is perpendicular to the level surface, for which the potential is constant, gives

$$\cos\frac{p}{x} = \frac{D_x \Omega}{L} = \frac{P_x}{L}.$$

Hence the value of the resultant is

$$\begin{aligned} P &= D_p \Omega = \Sigma_x D_x \Omega \, D_p x \\ &= \Sigma_x D_x \Omega \cos\frac{p}{x} = \Sigma_x \frac{P_x^2}{L} \\ &= \frac{\Sigma_x P_x^2}{L} = \frac{L^2}{L} \\ &= L = \sqrt{(\Sigma_x P_x^2)}. \end{aligned}$$

82. By an elementary motion of translation, each point of a system is carried to the same distance in the same direction; the potential of the system is changed, therefore, precisely as if all its points were united in one, and all the forces applied at this point. *The tendency of a system to any motion of translation, is, then, the same as*

that which would arise from the action of a single force, equal to the resultant of all the forces, supposed to be applied at the same point.

83. The *moment of a force, with reference to a point,* is the product of the force multiplied by its distance from the point. The *moment of a force, with reference to a line,* is the product of the projection of the force upon a plane perpendicular to the line multiplied by the distance of the force from the line.

The moment of a force, with reference to a line, may be represented geometrically by a corresponding length taken upon the line, and the name of the moment may be given to its geometrical representative.

The moment of a force, with reference to a point, is the same with the moment, with reference to the line, which is drawn through the point perpendicular to the common plane of the point and the force.

84. *The moment of a force, with reference to a line passing through a point, is equal to the projection upon the line of the moment, with reference to the point.* For the moment, with reference to the point, is equal to double the area of the triangle, of which the base is the force, and the altitude is the distance of the force from the point; and the moment, with reference to the line, is equal to double the area of the triangle, of which the base is the projection of the force upon the plane perpendicular to the line, and the altitude is the distance of this projection from the line. But the latter of these triangles is the projection of the former upon the plane, and its area is equal to the product of the area of the former triangle, multiplied by the cosine of the angle of the planes of the two triangles. But the lines upon which the moments are represented, being respectively perpendicular to these planes, have the same mutual inclination. The moment, with reference to the line, is, therefore, equal to the product of the moment, with reference to the point, multiplied by

the cosine of the mutual angle of the moments; that is, it is equal to the projection upon the line of the moment, with reference to the point.

85. Hence it follows that the moments of forces, with reference to points, may be combined by the same processes in which the forces themselves are combined, and that *all the moments, with reference to a point, may be combined into one resultant moment.*

86. The tendency of the force F, of which the potential is Ff, to produce an elementary rotation, $\delta\vartheta$, about a line p, is

$$D_\theta(Ff) = FD_\theta f.$$

But if

ϱ is the distance of F from p,

(14_{26}) gives

$$D_\theta f = \varrho \sin{}^p_f;$$

the projection of F upon the plane perpendicular to p, being

$$F \sin{}^p_f,$$

the tendency to rotation about p becomes

$$\varrho F \sin{}^p_f = \text{the moment of } F \text{ with reference to } p;$$

that is, *the moment of a force, with reference to a line, is the measure of its tendency to produce rotation about that line.*

87. The direction of the positive moment must be assumed to be the same with that of the axis, about which the tendency to rotation of the force is positive.

88. *The resultant moment of all the forces of a system, with reference to a point, is the measure of their tendency to produce rotation about that point.* Hence, the one force, of which the moment is equal to the resultant moment, has the same tendency to produce rotation.

89. The resultant moment of all the forces which act upon a point, with reference to any line or to any other point, is the same with the moment of their resultant. For the point upon which the forces act tends to move in the direction of their resultant, with a force equal to its intensity, and its moment is, therefore, the measure of the tendency to motion.

90. The moment of a force, with reference to a line p', is equal to its moment, with reference to a parallel line p, increased by the moment of an equal and parallel force, acting at any point of the line p. For the distance of the original force from the line p', is equal to its distance from the line p, increased by the distance from p' of the parallel force passing through p.

91. *Hence the resultant moment of any forces, with reference to a line p', is equal to their resultant moment, with reference to a parallel line p, increased by the moment, with reference to p', of equal and parallel forces acting at any point of the line p.*

92. *The resultant moment of any forces, with reference to a point O', is equal to their resultant, with reference to a point O, increased by the moment, with reference to O', of equal and parallel forces acting at O.* For this proposition is true for each pair of the parallel axes of two parallel systems of three rectangular axes, of which the points O and O' are the respective origins.

93. A *couple* of forces is a system of two parallel and equal forces which act in different lines.

94. *The moment of a couple of forces has, for every point of space, the same value, which is equal to the moment of one of them for any point of the other.* For two forces, equal and parallel to them, applied at any point, destroy each other's action, and their resultant vanishes.

95. The tendency of a couple of forces to produce rotation about a point, is the same as that of any system of forces, when its moment is equal to the resultant moment of the system, with

reference to the point. But the couple has no tendency to produce a translation; whereas the resultant of a system of equal and parallel forces, acting at the point, has all the tendency of the system to produce translation, but none to produce rotation about the point. Hence, *the three forces, of which one is the resultant of the equal and parallel forces acting at a point, and the other two constitute a couple, of which the moment is the same with the resultant moment, with reference to the point, fully represent any system of forces in their tendency to produce rotation and translation.*

96. Since the position of the couple of forces is quite arbitrary, one of the pair may be taken to act at the same point with the resultant of all the forces; and, by combining it with the resultant, the system of three forces may be reduced to two.

97. A point can always be found in space, for which the moment of a given force has any assumed magnitude, and any direction which is perpendicular to the force. Because the distance of the point from the force, which is one of the factors of the moment, may vary from zero to infinity, and its direction from the force may be that of any perpendicular to the force.

Hence, if the resultant moment, with reference to a point O, of any system of forces, is decomposed into two moments, of which one has the same direction with the force, and the other is perpendicular to it, another point O' can be found, for which the moment of the resultant, acting at O, is, in amount and direction, the negative of that component of the resultant moment for O, which is perpendicular to the resultant. For the point O', therefore, the resultant moment, coincides in direction with the resultant itself; and of the three corresponding forces which represent the tendency of the system to produce rotation and translation, the plane of the couple is perpendicular to the direction of the resultant.

98. If all the forces lie in the same plane, for any point of the plane the moment of each of the forces is perpendicular to the plane, and, therefore, the resultant moment is perpendicular to the plane. But the resultant of the parallel and equal forces acting at the point must, if it does not vanish, lie in the same plane, and be perpendicular to the resultant moment. If, then, the resultant does not vanish, a point of the plane can be found for which the resultant moment vanishes.

99. If all the forces are parallel, the moment of each of them, for any point, lies in the plane which is drawn through the point perpendicular to the forces. But the resultant of the parallel and equal forces, acting at the point, has the same common direction with them, and is, therefore, perpendicular to the resultant moment. If, then, the resultant does not vanish, a point can be found for which the resultant moment vanishes.

Hence, *if all the forces of a system lie in the same plane, or if they are all parallel to each other, their tendency to produce translation or rotation is equivalent, either to that of a single force, or to that of a couple of forces.*

100. If of any system of forces, and for a point O

M is the resultant moment,
R the resultant of equal and parallel forces acting at O,
M_p and R_p the projections of M and R upon the direction of p,

and if the same letters accented denote the same quantities for the point O', and if

x, y, and z are the rectangular coördinates of O' with reference to O,

the value of the moment of the forces for either of the axes passing through O' is,

$$M'_x = M_x - zR_y + yR_z.$$

But if the direction of the axis of z is assumed to be the same with that of R, these moments become

$$M'_x = M_x + yR,$$
$$M'_y = M_y - xR,$$
$$M'_z = M_z.$$

The coördinates of the points, for which the resultant moment has the same direction with the resultant, are

$$y = -\frac{M_x}{R}, \; x = \frac{M_y}{R}.$$

101. The number of forces which is required to produce any of the special effects of a given system of forces, is usually much less than the whole number of those which actually concur in their production. The mode of analysis, by which the requisite forces may be ascertained, is, in most cases, quite as simple as that by which the effects of rotation and translation have been investigated.

III.

GRAVITATION, AND THE FORCE OF STATICAL ELECTRICITY.

102. *Gravitation* is, among all the forces of nature, conspicuous for its universality, and the grandeur of the scale upon which it is exhibited.

Each particle of matter is an elementary centre of action for the force of gravitation, and all the level surfaces for each particle are spherical surfaces, of which the particle is the centre. The value of the potential for any particle, is inversely proportional to the distance from the particle, and for different particles it is proportional to the mass of the particle.

103. Another force which seems to be equally universal with gravitation, and of which gravitation has been, perhaps justly,

regarded as a residual force, and which is subject to the same law, in respect to distance from each elementary centre of action, is that of *statical electricity*. This force, however, is endowed with *duality*, and *consists of two forces, of which one has a positive, and the other a negative potential*. Both forces are usually combined with equal intensity, in the same centre of action, so as to neutralize each other's influence, and thus lie dormant. *With each of these the potential is positive in reference to electricity of the other kind, and negative with reference to that of the same kind.* The tendency to motion, arising from one kind of electricity, is exactly equal and opposite, then, to that which arises from the action of an equal intensity of the other kind, distributed in the same way.

104. The action of electricity upon the mass of a particle is indirect; the direct action is upon the electricity associated with the mass. In most bodies the electricity yields with more or less facility to this action, leaves the particle with which it is originally combined for another particle, and finally *assumes such a form of distribution within and upon the body, that the tendency to motion shall nowhere exceed the resistance to motion.* Bodies in which there is no resistance to the motion of electricity are called *perfect conductors;* while those in which the resistance is infinite are called *perfect non-conductors.*

105. Let

dm denote the mass of a particle of matter in the case of gravitation, or the value of its potential at the unit of distance, in the case either of gravitation or electricity;

$d\sigma$, the element of volume of the mass;

k, the density of the matter, in the case of gravitation, or the intensity of the force of electricity, compared with the unit of intensity;

f, the distance from the particle ;

$d\Omega$, the value of the potential for the particle ;

the expression of the potential for the particle is

$$d\Omega = \frac{dm}{f} = \frac{k\,d\sigma}{f}.$$

The general value of the potential for the whole body is

$$\Omega = \int_m \frac{1}{f} = \int_\sigma \frac{k}{f}.$$

106. With reference to a system of three rectangular axes, let

x, y, z, be the coördinates of the point in space, for which the potential is Ω, and

ξ, η, ζ, those of the particle.

Adopt also the functional notation

$$\nabla = \Sigma_x D_x^2 = D_x^2 + D_y^2 + D_z^2.$$

The derivatives of f and f^{-1} are

$$D_x f = \cos\frac{x}{f} = \frac{x-\xi}{f}.$$

$$D_x^2 f = \frac{f^2 - (x-\xi)^2}{f^3} = \frac{\sin^2\frac{x}{f}}{f}.$$

$$D_x\frac{1}{f} = -\frac{1}{f^2}D_x f = -\frac{\cos\frac{x}{f}}{f^2}.$$

$$D_x^2\frac{1}{f} = -\frac{1}{f^2}D_x^2 f + \frac{2}{f^3}(D_x f)^2 = \frac{-\sin^2\frac{x}{f} + 2\cos^2\frac{x}{f}}{f^3}.$$

$$= \frac{-1 + 3\cos^2\frac{x}{f}}{f^3}.$$

Hence

$$\nabla f = \frac{2}{f},$$

$$\nabla\frac{1}{f} = \frac{-3 + 3\,\Sigma_x\cos^2\frac{x}{f}}{f^3} = 0\,;$$

and, therefore,

$$\nabla d\Omega = 0,$$

$$\nabla \Omega = 0.$$

This last equation, which is called LAPLACE'S *equation*, only applies to that extent of space for which the derivatives of the potential are continuous functions, that is, where there are no centres of action ; but, where there are centres of action, it requires a modification which will soon be investigated. The integration of this equation, combined with peculiar considerations in special cases, gives the value of the potential for all the problems of gravitation or statical electricity.

107. *The tendency to motion, resulting from the gravitating or electrical action of a particle of matter, being normal to the level surface, is directed in the straight line drawn to the particle.* Its intensity is the derivative of the potential, and expressed by the equation.

$$D_f d\Omega = -\frac{dm}{f^2}.$$

The force of the gravitating or electrical action of a particle of matter, is, therefore, inversely proportional to the square of the distance from the particle. It is attraction in the case of gravitation, or between electricities of opposite kinds, and repulsion between electricities of the same kind.

ATTRACTION OF AN INFINITE LAMINA.

108. The investigation of the potential of a lamina of uniform density, and included between two infinitely extended planes, is simplified by the consideration, that it must have the same value for all points of space which are at the same distance from either surface of the lamina. Because all such points are similarly situated with reference to the lamina, on account of its infinite extent. Hence, if either surface of the lamina is adopted for the plane of $y\,z$,

the derivatives of the potential, with reference either to y or z, must vanish, and LAPLACE's equation becomes

$$\nabla\Omega = D_x^2\Omega = 0.$$

The integral of this equation gives the value of the potential, for a point external to the lamina, or upon its surface,

$$\Omega = Ax + B,$$

in which A and B are arbitrary constants.

109. The level surfaces are the planes determined by the equation (47_7), when Ω is the constant value of the potential for the level surface.

110. The action of the lamina upon any external point, is in a direction perpendicular to either surface, and *its force of attraction or repulsion is constant upon all points*, for it is given by the equation

$$D_x\Omega = A.$$

111. The values of A and B in any special case must be ascertained by direct integration. The integration indicated in (45_8), gives an infinite value of the potential, whereas the integration of its derivative, with reference to x, gives A itself, in a finite form, which shows that the infinite portion of the potential belongs to B. The integration for finding the derivative of the potential is effected by putting

$$\varrho = f \sin \tfrac{x}{f},$$
$$= \text{the projection of } f \text{ upon the plane of } yz.$$
$$a = \text{the thickness of the lamina};$$

whence

$$f = (x - \xi)\sec \tfrac{x}{f},$$

$$\varrho = (x - \xi)\tan \tfrac{x}{f},$$

$$d\sigma = \varrho\, d\varrho\, d_\varrho^y\, d\xi,$$
$$= (x - \xi)^2 \sin_f^x \sec^3{}_f^x\, d_f^x\, d_\varrho^y\, d\xi.$$
$$D_x \Omega = -\int_\sigma \frac{k D_x f}{f^2} = -\int_\sigma \frac{k \cos\frac{x}{f}}{f^2}.$$
$$= -\int_0^a{}_\xi \int_0^{2\pi}{}_\rho^y \int_0^{\frac{1}{2}\pi}{}_f^x k \sin\tfrac{x}{f}.$$
$$= -2\pi a k = A.$$

This value of A corresponds to a positive value of x, but for a negative value of x its sign must be reversed.

112. For a point situated within the lamina, a plane may be drawn through it parallel to the superficial planes, and dividing the lamina into two partial laminæ, of which the thicknesses are x and $a - x$. Hence, the value of the derivative of the potential is

$$D_x \Omega = -2\pi k x + 2\pi k (a - x)$$
$$= 2\pi k (a - 2x).$$

POISSON'S MODIFICATION OF LAPLACE'S EQUATION FOR AN INTERIOR POINT.

113. The modification which is required of LAPLACE'S equation, in order that it may be applicable to any point of an acting mass, must be the same for all cases. For it would not be needed, if the point of action were contained within any extent, however small, of void space. It depends, therefore, exclusively upon the infinitesimal portion of matter at the point, and is unaffected by any variations in the form and extent of the acting body. It need be investigated, then, in only a single case. Now the derivative of (48_{16}) gives

$$D_x^2 \Omega = -4\pi k,$$

which substituted in LAPLACE'S equation gives for an internal point of the infinite lamina,

$$\nabla \Omega = -4\pi k_0,$$

which is, therefore, the required modification of this equation. This modified equation, in which k_0, denotes the value of k at the point of action, is applicable, as remarked by Sturm, even when the point is exterior to the body. This same geometer has observed that, by supposing the value of k gradually to shade off from its value within the body to zero, this graduation occurring within an infinitely small extent, so as not sensibly to interfere with the actual phenomena of nature, the potential and its differential coefficients may become continuous functions. It must be further observed, however, that this imaginary graduation must extend throughout all space, although k must have an infinitesimal value where there is no portion of active force; for if it were to vanish throughout any finite portion of space, however small, the reasoning of § 69, would prove that all the derivatives of the potential were not finite and continuous.

ATTRACTION OF AN INFINITE CYLINDER.

114. The investigation of the potential of an infinite cylinder is simplified by the consideration that its value must be the same for all points situated upon the same straight line parallel to one of the sides of the cylinder. If this direction is adopted for the axis of z, the derivative of the potential, with reference to z, must vanish, and Laplace's equation becomes

$$\nabla \Omega = (D_x^2 + D_y^2)\,\Omega = 0.$$

The integral of this equation is

$$\Omega = \mathfrak{F}(x + y\sqrt{-1}) + \mathfrak{F}_1(x - y\sqrt{-1}),$$

in which $\mathfrak{F}$ and $\mathfrak{F}_1$ are arbitrary functions, and must be determined for each case by special considerations.

115. The level surfaces are the cylindrical surfaces, of which (49_{30}) is the general equation, if Ω has the constant value belonging to that surface.

116. The attraction in the direction of the axis of x is

$$D_x\Omega = \mathfrak{F}'(x + y\sqrt{-1}) + \mathfrak{F}_1'(x - y\sqrt{-1}),$$

in which the accents denote the derivatives of the functions, with reference to their explicit variables.

The attraction in the direction of the axis of y is

$$D_y\Omega = [\mathfrak{F}'(x + y\sqrt{-1}) - \mathfrak{F}_1'(x - y\sqrt{-1})]\sqrt{-1}.$$

The whole action is, then,

$$\sqrt{[(D_x)^2 + (D_y)^2]}\,\Omega = 2\sqrt{[\mathfrak{F}'(x + y\sqrt{-1}) . \mathfrak{F}_1'(x - y\sqrt{-1})]}.$$

117. *When the point of action is so far from the cylinder that the square of the linear dimensions of the base can be neglected, in comparison with the square of the least distance of the point from the cylinder,* the problem can be greatly simplified.

Find in this case a line parallel to the axis of z, of which the coördinates a and b, with reference to the axes of x and y, are determined by the equations

$$\int_m (\xi - a) = 0 = \int_m \xi - am,$$
$$\int_m (\eta - b) = 0 = \int_m \eta - bm.$$

This line may be called the *axis of gravity* of the cylinder, and its position is wholly independent of the directions of the axes of x and y. For the conditions by which this axis is determined will give, with regard to any other axis of x, with reference to which the notation is distinguished by the subjacent numbers,

$$\int_m (\xi_1 - a_1) = \int_m (\xi - a) \cos{}^x_{x_1} + \int_m (y - b) \cos{}^y_{x_1} = 0.$$

If the axis of gravity is, then, assumed for the axis of z, the equations $(50_{25\text{-}26})$ become

$$\int_m \xi = \int_m \eta = 0,$$

or

$$\int_\xi \int_\eta \int_\zeta k\xi = \int_\xi \int_\eta \int_\zeta k\eta = 0.$$

118. Since, from the nature of the cylinder, the functions which are here to be integrated are independent of ζ, these equations give

$$\int_\xi \int_\eta k\xi = \int_\xi \int_\eta k\eta = 0.$$

119. Let the perpendicular from the point of action upon the axis of z be assumed for the axis of x, and let

f_0 be the distance of the point of action from the projection of any particle of the cylinder upon the axis of z,

ϱ the distance of the particle from the axis of z.

The conditions of the problem under consideration give

$$f^2 = (x - \xi)^2 + \eta^2 + \zeta^2 = f_0^2 - 2x\xi + \varrho^2,$$

$$= f_0^2 \left(1 - \frac{2x\xi}{f_0^2}\right),$$

$$\frac{1}{f} = \frac{1}{f_0}\left(1 + \frac{x\xi}{f_0^2}\right) = \frac{1}{f_0} + \frac{x\xi}{f_0^3};$$

$$\Omega = \int_m \frac{1}{f} = \int_m \frac{1}{f_0} + \int_m \frac{x\xi}{f_0^3}$$

$$= \int_m \frac{1}{f_0} + \int_\zeta \frac{x}{f_0^3} \int_\xi \int_\eta k\xi = \int_m \frac{1}{f_0},$$

so that *the potential is the same as if all the particles of the cylinder were united in their projections upon the axis of gravity, when the point is at a sufficiently great distance from the cylinder.*

120. By letting

K denote the intensity of the action concentrated upon each point of the axis of gravity when the cylinder is projected upon it;

the value of the whole action of this axis is

$$D_x \Omega = -\int_{-\infty}^{\infty} {}_\zeta \frac{Kx}{f_0^3} = -Kx \int_{-\infty}^{\infty} {}_\zeta \frac{1}{(x^2+\zeta^2)^{\frac{3}{2}}}$$

$$= -\frac{K}{x} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} {}_{\frac{x}{f_0}} \cos \frac{x}{f_0} = -\frac{2K}{x},$$

or the potential is

$$\Omega = -2K \log x + B,$$

in which the arbitrary constant B is infinite.

121. *When the base of the cylinder is the space which is contained between two concentric circles,* the axis of gravity coincides with the geometrical axis, the potential is, from the symmetry of the figure, the same in all directions from the axis, and its value only depends upon the distance from the axis. Let the axes be the same as in §§ 117 and 119, except that the point of action is in the plane of $x\,y$, but not in the axis of x, and let

$r =$ the radius vector of the point of action, and
$e =$ the base of the Naperian system of logarithms.

The potential is a function of r, and does not involve the inclination of r to the axis of x. Hence

$$D_{\frac{x}{r}} \Omega = 0.$$

But by (49_{30})

$$\Omega = \mathfrak{F}\left(rc^{\frac{x}{r}\sqrt{-1}}\right) + \mathfrak{F}_1\left(rc^{-\frac{x}{r}\sqrt{-1}}\right);$$

whence

$$D_{\frac{x}{r}}\Omega = \left[\mathfrak{F}'\left(rc^{\frac{x}{r}\sqrt{-1}}\right)rc^{\frac{x}{r}\sqrt{-1}} - \mathfrak{F}_1'\left(rc^{-\frac{x}{r}\sqrt{-1}}\right)rc^{-\frac{x}{r}\sqrt{-1}}\right]\sqrt{-1} = 0;$$

and

$$\mathfrak{F}'\left(rc^{\frac{x}{r}\sqrt{-1}}\right)rc^{\frac{x}{r}\sqrt{-1}} = \mathfrak{F}_1'\left(rc^{-\frac{x}{r}\sqrt{-1}}\right)rc^{-\frac{x}{r}\sqrt{-1}}.$$

But the two members of this equation are functions of two different and independent variables, which are

$$rc^{\frac{x}{r}\sqrt{-1}} \text{ and } rc^{-\frac{x}{r}\sqrt{-1}};$$

and, therefore, neither can be contained in the value of the other, so that each of them disappears from their common value, which is, therefore, constant. With regard to any variable whatever, therefore, this equation gives

$$r\mathfrak{F}'r = r\mathfrak{F}_1'r = A,$$
$$\mathfrak{F}'r = \mathfrak{F}_1'r = \frac{A}{r};$$

and, by integration,

$$\mathfrak{F}r = A\log r + B,$$
$$\mathfrak{F}_1 r = A\log r + B_1.$$

The value of the potential is, then, if the two constants are combined in one,

$$\Omega = A\log\left(rc^{\frac{x}{r}\sqrt{-1}}\right) + A\log\left(rc^{-\frac{x}{r}\sqrt{-1}}\right) + B_2$$
$$= 2A\log r + B_2,$$

and the action upon the point is in the direction of r, and its valuc is

$$D_r \Omega = \frac{2A}{r}.$$

122. When the point of action is upon the axis, it is plain, from the symmetrical nature of the cylinder, that the action is cancelled in each direction, and in this case

$$D_r \Omega = \frac{2A}{r} = 0,$$

whence

$$A = 0.$$

For every point within the inner cylindrical boundary of this cylindrical shell, the action, therefore, vanishes, and the potential is constant.

123. When the point of action is without the cylinder, the constants are found by the condition that when the distance is very great, the value must be the same as that of (52_{13}). Hence

$$A = -K,$$

that is, *the action upon every point, without the circular cylinder, is the same as if the whole mass of the cylinder were concentrated upon its axis.*

124. No other case of the infinite cylinder is of sufficient interest to divert the current of the work from the finite masses of nature.

RELATION OF THE POTENTIAL TO ITS PARAMETER.

125. The varying value of the potential from one level surface to another, depends upon the law of the change of surface, and may be represented as a function of a variable, which may be called its *parameter*. Let

λ denote the parameter of the potential, and adopt the functional notation

$$\Box = \Sigma_x (D_x)^2 = (D_x)^2 + (D_y)^2 + (D_z)^2.$$

The derivative of the potential gives

$$D_x \Omega = D_\lambda \Omega \, D_x \lambda,$$
$$D_x^2 \Omega = D_\lambda^2 \Omega \, (D_x \lambda)^2 + D_\lambda \Omega \, D_x^2 \lambda;$$
$$\nabla \Omega = D_\lambda^2 \Omega \, \Box \lambda + D_\lambda \Omega \nabla \lambda = -4 \pi k_0,$$

which is a transformation given by LAMÉ.

126. For a point of void space, this equation gives

$$\frac{D_\lambda^2 \Omega}{D_\lambda \Omega} = D_\lambda (\log D_\lambda \Omega) = -\frac{\nabla \lambda}{\Box \lambda};$$

by which the potential may be determined for given forms of λ.

ATTRACTION OF A FINITE POINT UPON A DISTANT MASS. CENTRE OF GRAVITY.

127. In every finite mass there is a point called the *centre of gravity*, of which the coördinates are determined by equations, for each axis, which are similar to (50_{25-26}). This point is independent of the positions of the axes, for these equations give for any other axis

$$\int_m (\xi_1 - a_1) = \Sigma_x \left(\int_m (\xi - a) \cos_{x_1}^{x} \right) = 0.$$

If the centre of gravity is adopted for the origin of coördinates, these equations are reduced to (51_{8-10}).

128. When the point of action is so far from the attracting mass, that the squares of the linear dimensions of the mass may be neglected in comparison with the square of the distance of the point from the mass, the formula becomes

$$f^2 = \Sigma_x (x - \xi)^2 = \Sigma_x (x^2 - 2x\xi + \xi^2)$$
$$= r^2 + \varrho^2 - \Sigma_x (2x\xi) = r^2 - \Sigma_x (2x\xi)$$
$$\Omega = \int_m \frac{1}{f} = \int_m \left(\frac{1}{r} + \frac{\Sigma_x (x\xi)}{r^3} \right)$$

$$=\int_m \frac{1}{r} + \Sigma_x \left(\frac{x}{r^3}\int_m \xi\right)$$
$$=\frac{m}{r};$$

that is, *the potential of a finite point, for a mass which is so remote that the square of the linear dimensions of the body may be neglected, in comparison with the square of the distance of the point from the body, is the same as if the body were concentrated at its centre of gravity.*

ATTRACTION OF A SPHERICAL SHELL.

129. In the case of a shell of homogeneous matter, contained between the surfaces of two concentric spheres, the value of the potential must, from the symmetry of the figure, depend exclusively upon the distance from the centre; and for the same reason this centre is the centre of gravity. If the centre is adopted for the origin of coördinates, the parameter may be assumed to be the radius vector, or any function of it. Putting, then,

$$\lambda = r^2 = \Sigma_x x^2,$$

derivation gives

$$D_x \lambda = 2x,$$
$$D_x^2 \lambda = 2;$$
$$\Box \lambda = 4\Sigma_x x^2 = 4r^2 = 4\lambda,$$
$$\digamma \lambda = 6.$$

Hence, (55_8) becomes

$$D_\lambda (\log D_\lambda \Omega) = -\frac{3}{2\lambda}.$$

The integral of this equation is, by the introduction of the arbitrary constants A and B,

$$\Omega = B - \frac{A}{\sqrt{\lambda}} = B - \frac{A}{r}.$$

130. When the point of action is at the origin, the value of the potential is easily obtained by direct integration. Let in this case

ϱ_0 and ϱ_1 be the internal and external radii of the spherical shell,

m_0 and m_1 the masses of two homogeneous spheres of the same density with the shell, and of which the radii are respectively ϱ_0 and ϱ_1; and

$d\psi$ the elementary solid angle of which the vertex is at the point of action.

The mass of the shell is

$$m = m_1 - m_0 = 4\pi k(\varrho_1^3 - \varrho_0^3),$$

and the element of mass

$$dm = k\varrho^2 d\psi\, d\varrho.$$

The value of the potential is, therefore,

$$\begin{aligned} \Omega &= \int_m \frac{1}{\varrho} = k\int_\psi \int_\rho \varrho \\ &= \tfrac{1}{2}k\int_\psi (\varrho_1^2 - \varrho_0^2) = 2\pi k(\varrho_1^2 - \varrho_0^2) \\ &= \tfrac{1}{2}\left(\frac{m_1}{\varrho_1} - \frac{m_0}{\varrho_0}\right). \end{aligned}$$

131. *When the point of action is in the interior void space of the shell,* the constants of (56_{30}) must have the same values as at the origin, where r vanishes. Hence, for this space, the constants are

$$A = 0,$$

$$B = 2\pi k(\varrho_1^2 - \varrho_0^2) = \tfrac{1}{2}\left(\frac{m_1}{\varrho_1} - \frac{m_0}{\varrho_0}\right).$$

The value of the potential in the interior void space is, therefore, constant, and there is no tendency to motion in any direction.

132. *For an exterior point,* the potential vanishes when r is infinite, while for a point at a great distance from the origin, its value is, by § 128, the same as if the whole mass were concentrated at the origin. The value of the constants in this case are then

$$B = 0,$$
$$A = -m;$$

and the potential is

$$\Omega = \frac{m}{r}.$$

Any exterior point is, then, attracted by a homogeneous spherical shell, precisely as if the whole mass of the shell were concentrated upon its centre of gravity.

ACTION AND REACTION OF A SURFACE OR INFINITELY THIN SHELL OF FINITE EXTENT. CHÂSLESIAN SHELL.

133. An infinitely thin shell may be reduced to either of its surfaces, upon which all its acting force may be concentrated, and the intensity of the action at each point of the surface will be the product of the corresponding intensity of the force of the shell, multiplied by the thickness of the shell, and the element of the surface must be substituted for the element of volume of the shell. Let then,

$d\sigma$ be the element of the surface,
N the exterior direction of the normal to the surface,
k the concentrated intensity of action at any point of the surface,
$d\psi$ the elementary solid angle subtended by the element of the surface at the point of action;

the expression of the element of the surface is

$$d\sigma = f^2 d\psi \sec \frac{f}{N}.$$

Hence

$$-k\,d\psi = -\frac{\cos f}{f^2} k\,d\sigma.$$

The second member of this equation denotes the action exerted by each element of the surface in a direction normal to the surface, and towards the interior of the surface. If, therefore, the intensity of action is constant over the surface, the action normal to the surface is proportional for each element of the surface, to the solid angle subtended by the element, and *the total amount of the action, normal to the surface, exerted by any continuous extent of the surface, is proportional to the whole solid angle subtended by the boundary of the surface.*

134. If the surface is a plane, the direction of the normal is invariable, and the total amount of normal action exerted by any portion of the plane is the same with *the projection of the whole action of this portion of the plane upon the perpendicular to the plane, which is therefore proportional to the solid angle subtended by the portion of the plane at the point of action.*

135. *If the surface returns into itself so as to include a space, which is called a closed surface, and if the point of action is situated within the inclosed space, the whole angle subtended is the entire extent of four right angles; whereas, if the point of action is exterior to the closed surface, the whole angle vanishes; but it is two right angles when the point is upon the surface.* For, however the point of action is situated, if a line is drawn from it so as to cut the surface more than once, the successive angles which the line makes with the exterior normal, will be alternately obtuse and acute as the line cuts into the surface or out from it. The last angle, or that of which the vertex is most remote from the point of action will always be acute. The normal actions of two successive elements, therefore, upon the same line, and which subtend the same solid angle, are equal, but of

opposite signs, so that they cancel each other's effect in the total sum of the normal forces. But if the point of action is without the surface, the first angle is obtuse upon each line, and as the last angle is acute, the whole number of intersections is even, and each normal elementary action is cancelled by another, and the whole sum vanishes. If the point of action is within the surface, the first angle is acute, if there is more than one; and there are an odd number of intersections for every direction in which a line can be drawn; for each direction, therefore, one, and only one, normal action remains uncancelled, which is proportional to the elementary solid angle; and the whole sum is that of the entire extent of four right angles. But, if the point of action is upon the surface, and a tangent plane to the surface is drawn through it; every line which is drawn from the point upon the exterior side of the plane must cut the surface an even number of times, if it cuts at all, precisely as if it were drawn from an exterior point; but every line which is drawn upon the interior side of the plane cuts the surface, as if it were drawn from an interior point; the total sum, then, of the uncancelled elementary solid angles includes those for all directions which are upon the inner side of the plane, that is, it is equal to two right angles. This elegant theorem, given by Gauss, is expressed analytically in the form

$$\int_\sigma \frac{\cos \frac{f}{N}}{f^2} = \begin{cases} 4\pi \text{ for a point interior to a closed surface,} \\ 2\pi \text{ for a point upon the surface,} \\ 0 \text{ for an exterior point.} \end{cases}$$

136. The expression (59_2) represents the component in the direction of the external normal to a surface, of the action upon the element of the surface of a mass k concentrated at the point which, in that expression, was the point of action. The integral of this expression is the whole amount of such resolved action, and by

(60_{24}) its value is

$$-\int_\sigma \frac{\cos \hat{f}_N}{f^2} k = -\int_\psi k = \begin{cases} -4\pi k \text{ when the mass } k \text{ is interior to the surface,} \\ -2\pi k \text{ when the mass } k \text{ is upon the surface,} \\ \quad 0 \text{ when the mass } k \text{ is exterior to the surface.} \end{cases}$$

Neither of these values depends upon the position of the acting mass further than it is interior or exterior to the surface or upon the surface. If, then,

$M_i =$ all the mass interior to the surface,
$M_u =$ all the mass upon the surface,
$M_e =$ all the mass exterior to the surface;

the expression for the total action of the sum of all the masses upon a closed surface, resolved for each element in the direction of the external normal, is

$$-4\pi M_i - 2\pi M_u;$$

and if all the masses are exterior to the surface, this sum vanishes. If the closed surface is one of the level surfaces of the system of bodies, this sum expresses the total attraction of the masses upon the surface. This important theorem is due to Gauss, and, independently to Chasles, in almost its full extent, as well as most of the following deductions. It is applicable, even if the surface have sharp angles, because the extent of surface occupied by such angles is zero.

137. If the closed surface is one of the level surfaces of a system of bodies, but not the outer boundary of a space in which the potential is constant, the potential must at each point, by § 67, increase in passing from the interior to the exterior or the reverse, so that in this case the sum (61_{16}) does not vanish. But the term of this sum, which depends upon the mass at the surface, may be neglected at will; for the whole mass of a true geometrical surface is absolutely nothing. Hence, *every level surface must inclose masses of*

matter, unless it be the outer material boundary of a space in which the potential is constant.

138. When any masses lie upon the closed surface, the geometrical surface may, as Gauss observed, be arbitrarily assumed as being just exterior or interior to the masses, or passing through them. *If, therefore, all the masses are so distributed upon a surface that it becomes itself a level surface, the potential is constant for all the inclosed space, and there is no tendency to motion throughout this space.*

139. Around every point of maximum or minimum potential a level surface of infinitesimal dimensions may obviously be drawn; and, therefore, *every point of maximum or minimum potential must be itself a centre of action, and cannot be a void space.*

In an inclosed space, therefore, no point can be found for which the value of the potential exceeds the limits of value which are found upon the inclosing material surface; and in no point of unbounded space has the potential so great a value as its greatest value upon the exterior surface of the finite masses. This inference was drawn by Gauss.

140. *In a system of bodies, of which gravitation is the only force, there can be no point of absolute minimum potential.* For if about a point of maximum or minimum potential, as a centre, an infinitesimal sphere is described, there can be no point within the sphere, either of maximum or minimum potential, with reference to the matter external to the sphere. But, with reference to the matter of the sphere itself, the centre must be a point of maximum potential, and, therefore, cannot be a point of minimum potential, with reference to the combined action of all the masses.

This theorem is equally applicable to an aggregation of electricity, all of which is of the same kind, that is, which is *homogeneous* when the point of action is assumed to be of the opposite kind of electricity.

141. If any extent of level surface is assumed at will as a

base, and if trajectories, like those of § 68, are drawn through each point of its perimeter, their union forms a canal. The same canal cuts a base, like the assumed base, from each level surface which it intersects. *Of any canal, then, which is not extended so far as to include portions of the attracting masses, the attractions upon all the bases are equal.* For the whole amount of action, resolved in the direction of the external normal, at each point of action upon the closed surface, formed by the faces of the canal and the two terminating bases, vanishes, because there is no included mass. But there is no action perpendicular to the faces, that is, in the direction of the level surfaces; whereas the whole action upon the bases is normal to them. The actions upon one base are in the directions of its external normals, while those upon the other base are in the directions of the internal normals; but these actions balance each other in the algebraic sum, and, therefore, their absolute values must be the same. This theorem belongs to Chasles, but the brief demonstration is original.

142. In the following simple view of this whole subject, many of its propositions are condensed into a small compass. Each centre of action may be regarded as a fountain from which a stream is perpetually flowing in every direction, with an amount of discharge proportioned to the intensity of the action. The quantity which flows from each centre, for an instant, through any given elementary surface, may easily be shown to be in exact proportion to the force with which the surface is attracted by this centre perpendicularly to itself and against the current; and that which is true for each centre is also applicable to the combined action of all the centres. Upon a space, then, in which there is no spring, the amount which is flowing out must constantly be equal to that which is flowing in; while from a space which contains springs, the amount which is discharged must exceed the inward flow by all which is

supplied by the fountains. These propositions are equivalent to those of § 136, and it may be shown by an easy argument that Laplace's equation, with its modification, is merely the same proposition applied to the element of space.

By the additional hypothesis, that, to preserve the uniform flow of the stream, its density must decrease in each element of the stream with the distance from the origin, so as always to be inversely proportional to the distance from the centre, the potential represents the density of the combined streams, and the level surfaces become surfaces of equal density. The aggregate current of the combined streams is also equivalent to a single current in a direction perpendicular to the level surfaces, and having a velocity proportionate to the rate of decrease of density. But this is the well known law of the propagation of heat, when there is no radiation, and hence arise the analogies between the level and isothermal surfaces, and the identity of the mathematical investigations of the attractions of bodies and of the propagation of heat which have been developed by Chasles.

143. *If an infinitely thin homogeneous shell is formed upon each level surface of a system of bodies, having at each point a thickness proportional to the attraction at that point, the portion of either of these shells, which is included in a canal formed by trajectories, bears the same ratio to the whole shell, which the portion of another shell included in the same canal bears to that shell, provided there is no mass included between the shells.* For if the bases of the canal are infinitely small, they must be reciprocally proportional to the intensities of the actions upon them, because the whole amount of action upon the different bases is the same. But the thicknesses of the shells are proportional to the intensities of action, and, therefore, the products of the bases multiplied by the thicknesses, or the volumes of the portions of shell included in the same canal, bear a constant ratio to each other. Since the ratios

are constant the infinitesimal volumes may be added together, and their sums, which are the volumes included in a finite canal, are in the same ratio, and these sums may even be extended so as to include the whole of each shell. Hence the volume of each portion is the same fractional part of the volume of the shell to which it belongs; and, as each shell is homogeneous, the mass of each portion is the same fractional part of the mass of the whole shell. The conception of these shells, and the investigation of their acting and reacting properties was original with CHASLES, and it will be convenient, as it is appropriate, to designate them as *Chaslesian shells.*

144. The volume or mass of a Chaslesian shell has a simple ratio to the attracting mass included within it, dependent upon its own density and thickness. For each infinitesimal element of its volume or mass is proportional to the product of the element of the surface multiplied by the thickness of the shell, and the thickness at each point is proportional to the attraction at that point. The sum of all the elements, therefore, of either volume or mass, that is, the whole volume or mass, is proportional to the sum of all the attractions upon the whole surface. But, by § 136, the sum of all the attractions upon the surface is proportional to the included mass, if there is no mass at the surface. If, then,

μ is the volume of the shell,
k its density,
h the modulus of its thickness, or the thickness which corresponds to the unit of attraction;

this ratio is included in the equation

$$\frac{\mu}{hM_i} = \frac{k\mu}{khM_i} = 4\pi.$$

145. *If a Chaslesian shell which is wholly external to the acting masses of the system is assumed to be itself the attracting mass;*

1. *The potential of the shell is constant for all interior points, there is no tendency to motion within it, and its own outer surface is its level surface;*

2. *Its external level surfaces are the same as those of the original masses of the system, and the attraction of the shell upon a point external to itself has the same direction as the attraction of the original masses.*

To demonstrate these propositions, let

Ω_s be the potential of the shell for any point, and
Ω the potential of the original masses for each point of the shell;

the value of the element of the potential of the shell is

$$d\Omega_s = \frac{k\,d\mu}{f}.$$

Hence,

$$\frac{d\Omega_s}{\mu} = \frac{k\,d\mu}{\mu f}.$$

In passing along the canal of the trajectories to another shell, the ratio of $d\mu$ to μ is, by § 143, constant, whence

$$D_\lambda \frac{d\Omega_s}{\mu} = -\frac{k\,d\mu\,D_\lambda f}{\mu f^2}.$$

But

$$D_\lambda f = D_\lambda N D_N f = -D_\lambda N \cos\tfrac{N}{f},$$

$$d\mu = h\,d\sigma\,D_N\Omega;$$

and, therefore,

$$d\mu\,D_\lambda f = -h\,d\sigma\,D_N\Omega\,D_\lambda N \cos\tfrac{N}{f} = -h\,d\sigma\,D_\lambda\Omega \cos\tfrac{N}{f},$$

$$D_\lambda \frac{d\Omega_s}{\mu} = \frac{kh\,D_\lambda\Omega \cos\frac{N}{f}\,d\sigma}{\mu f^2}.$$

The integral of this equation for the whole surface of the shell is

$$D_\lambda \frac{\Omega_s}{\mu} = \frac{kh\,D_\lambda\,\Omega}{\mu} \int_\sigma \frac{\cos\frac{N}{f}}{f^2}.$$

1. For an internal point this equation becomes, by §§ 135 and 144,

$$D_\lambda \frac{\Omega_s}{\mu} = \frac{4\pi kh\,D_\lambda\,\Omega}{\mu} = \frac{k\,D_\lambda\,\Omega}{M_i},$$

the integral of which is

$$\Omega_s = \frac{k\mu\,\Omega}{M_i},$$

to which no constant need be added, because, when the dimensions of the shell are infinite, Ω and Ω_s both vanish, since all the points of action are infinitely remote from the centres of action. This equation expresses that the potential of each shell has the same value for all internal points, and, therefore, there is no tendency to motion within the shell, and the surface of the shell must be level, with reference to its own action.

2. For an external point, the equation (67_2) becomes, by § 135,

$$D_\lambda \frac{\Omega_s}{\mu} = 0.$$

Hence, by integration,

$$\frac{\Omega_s}{\mu} = \text{a constant},$$

which constant, however, depends for its value upon the position of the points of action; but since it has the same value for all the shells to which the point is external, the potential is constant for the same series of points external to one shell for which it is constant through the action of another shell; that is, all the shells have the same external level surfaces. But the external level surface, which is nearest to any shell, differs infinitely little from the level surface of the shell itself, and, therefore, the surface of each shell is a level surface for every included shell. Hence, the external level surfaces of a shell are the same with those of the original masses, and the attraction of a shell upon an external point has the same direction with the attraction of the original masses, and is normal to the level surface passing through the point. This theorem is due to Chasles.

146. *Every infinitely thin shell, of which the surface is level, from the action of the shell itself, must be a Chaslesian shell.* For, if another shell is constructed upon this level surface, which is the negative of the Chaslesian, one, namely, which is repulsive, instead of being attractive, or the reverse, and the whole mass of which is equal to that of the given shell, the two shells, having the same level surfaces, exactly cancel each other's action throughout all space. The elements of mass of the two shells must then be absolutely equal, but of opposite signs at every point. For, if they were unequal at any point, that point might be made the centre of an infinitely thin circular element of the combined shells. From the symmetry of its figure, a level surface for the action of this element alone might be made to pass through its perimeter, and which could inclose no other mass than the element itself. But such surface cannot be level for the remainder of the combined mass of the two shells, and, therefore, the value of the potential upon this surface for the combined masses of both shells, including the circular element, cannot be constant. This want of constancy in the potential is contradicted by the fact that the shells balance each other's action everywhere. There cannot, therefore, be any such want of constancy, nor any point for which the element of mass of the given shell is not absolutely equal to that of the Chaslesian shell, although it is of a contrary sign. But reversal of the sign of the action of the mass does not interfere with the Chaslesian characteristic of the shell.

147. *Two Chaslesian shells, which are constructed upon the same surface, only differ in their density and their modulus of thickness.* For the density of either of them may be increased or decreased until the value of its potential at the common surface shall be equal to that of the other shell. If, then, its action be reversed, the value of the potential for the combined shells will be zero both at the surface and at an infinite distance from the surface; and it cannot have any other value in the intermediate space, otherwise, there would be points or surfaces of maximum potential exterior to the acting masses. The combined surfaces have, therefore, neither external nor internal action, and the reasoning of the preceding article demonstrates that the component shells are identical, except in regard to their signs.

ATTRACTION OF AN ELLIPSOID.

148. *An infinitely thin homogeneous shell, of which the inner and outer surfaces are those of similar, and similarly placed, concentric ellipsoids, is a Chaslesian shell.* For, if upon the longest axes of these ellipsoids, as diameters, two concentric spheres are constructed, each sphere may be compressed into the corresponding ellipsoid, by reducing all the coördinates from the centre, as origin, parallel to either of the two shorter axes of the ellipsoid in the ratio of the longest axis to this shorter axis. But all points, which are originally in the same straight line remain upon a common straight line after this uniform compression; and all distances which are measured in the same direction are reduced in a common ratio. But the thicknesses of the spherical shell, measured upon any straight line at the two points where this line cuts the shell are equal; so that the thicknesses of the ellipsoidal shell, measured at the two points where the reduced line cuts this shell, are also equal. If, then, at a

point assumed at will, as the vertex, within the ellipsoidal shell, an infinitesimal cone is constructed and extended in each direction from the vertex, till it intersects the shell, the relative masses of the two included portions of the shell are proportional to the squares of their distances from the vertex; and, therefore, their attractions upon the vertex are equal, but in opposite directions. Hence, the action of any portion of the shell upon an internal point is balanced by the action of the opposite portion, and there is, consequently, no tendency to motion within the shell from its own action. The surface of the shell is thus proved to be a level surface, in respect to its own action, and, by § 146, it can be no other than a Chaslesian shell.

149. This proposition may be enlarged to a theorem given by Newton, for a finite shell, of which the inner and outer surfaces are those of similar and similarly placed concentric ellipsoids. Such a shell may be called a *Newtonian shell,* so that the infinitely thin Newtonian shell is a Chaslesian ellipsoidal shell. But the Newtonian shell may be subdivided by similar and similarly placed concentric ellipsoidal surfaces into an infinite number of Chaslesian ellipsoidal shells, each of which is inactive with reference to an internal point. Hence, *the whole Newtonian shell exerts no action upon an internal point.*

150. An ellipsoid may be converted into any other similar, and similarly placed, concentric ellipsoid by a process similar to that by which the sphere in § 148 was changed to an ellipsoid; that is, by increasing or decreasing the coördinates of each point, taken from the centre as origin, and parallel to either axis, in the ratio of the corresponding axes of the two ellipsoids. The points of the two ellipsoids, which correspond in this process, have been called by Ivory *corresponding points.* By this process, any Newtonian shell may be converted into another concentric and similarly placed

Newtonian shell, and at the corresponding points there will be corresponding elements of volume.

151. *The corresponding elements of volume or mass of two corresponding Newtonian shells are proportional to the volumes or masses of the shells.* For if

A_x, A_y, A_z are the semiaxes of the outer ellipsoidal surface of one shell,

B_x, B_y, B_z those of its inner ellipsoidal surface,

σ its volume,

m its mass, and

n the ratio of either axis of the inner surface, divided by the corresponding axis of the outer surface;

and if the same letters accented denote the same quantities for the corresponding shell, the construction of the shells gives for each axis

$$B_x = n A_x,$$

$$\frac{x}{A_x} = \frac{x'}{A'_x},$$

and

$$n = n';$$

and by differentiation,

$$\frac{dx}{dx'} = \frac{A_x}{A'_x}.$$

The volumes and masses are by well-known theorems of geometry

$$m = k\sigma = \tfrac{4}{3}\pi k (A_x A_y A_z - B_x B_y B_z)$$
$$= \tfrac{4}{3}\pi k (1 - n^3) A_x A_y A_z,$$
$$m' = k'\sigma' = \tfrac{4}{3}\pi k' (1 - n^3) A'_x A'_y A'_z.$$

The ratios of the elements of volume and mass are, then,

$$\frac{d\sigma}{d\sigma'} = \frac{dx\,dy\,dz}{dx'dy'dz'} = \frac{A_x A_y A_z}{A'_x A'_y A'_z} = \frac{\sigma}{\sigma'},$$

$$\frac{dm}{dm'} = \frac{k\,d\sigma}{k\,d\sigma'} = \frac{k\sigma}{k\sigma'} = \frac{m}{m'}.$$

152. *If the outer surfaces of two corresponding Newtonian shells have the same foci, their inner surfaces must also have the same foci.* For if

ε^2 is the difference of the squares of the corresponding axes of the outer surfaces,

the condition of the identity of foci gives the equations

$$\varepsilon^2 = A_x^2 - A_x'^2 = A_y^2 - A_y'^2 = A_z^2 - A_z'^2.$$

Hence, for each axis, there is the equation

$$B_x^2 - B_x'^2 = n^2 (A_x^2 - A_x'^2) = n^2 \varepsilon^2,$$

so that the foci of the inner surfaces are also identical.

153. *If the radius vector, from the centre of any point of an ellipsoid, is projected upon the radius vector of another ellipsoid which has the same foci, and if the radius vector of the corresponding point of the second ellipsoid is projected upon that radius vector of the first ellipsoid, which corresponds in direction to the projection in the second ellipsoid, the two projections are inversely proportional to the radii vectores upon which they are projected.* For if

ϱ is the radius vector of the first ellipsoid upon which the projection is made, and

ξ, η, ζ, are the coördinates of the extremity of ϱ;

the equations of the corresponding points give, for each axis,

$$\frac{\xi}{A_x} = \frac{\xi'}{A_x'}, \quad \frac{x}{A_x} = \frac{x'}{A_x'};$$

whence

$$\frac{\xi}{x} = \frac{\xi'}{x'},$$

or

$$\xi x' = \xi' x.$$

But if

p is the projection of r' upon ϱ, and
p' the projection of r upon ϱ',

these projections are

$$p = r' \cos{}^{r'}_{\varrho} = \Sigma_x \frac{x'\xi}{\varrho} = \frac{\Sigma_x (x'\xi)}{\varrho},$$
$$p' = r \cos{}^{r}_{\varrho'} = \Sigma_x \frac{x\xi'}{\varrho'} = \frac{\Sigma_x (x\xi')}{\varrho'} = \frac{\Sigma_x (x'\xi)}{\varrho'};$$

whence

$$\frac{p}{p'} = \frac{\varrho'}{\varrho}.$$

154. *The difference of the squares of the radii vectores from the centre, of two corresponding points upon the surface of two ellipsoids which have the same foci, is equal to the difference of the squares of their semiaxes.* For the equations of these surfaces are

$$\Sigma_x \frac{x^2}{A_x^2} = 1, \; \Sigma_x \frac{x'^2}{A_x'^2} = 1.$$

The difference of the squares of two corresponding radii vectores for points at the surface, is

$$r^2 - r'^2 = \Sigma_x (x^2 - x'^2) = \Sigma_x \left[x^2 \left(1 - \frac{A_x'^2}{A_x^2}\right)\right]$$
$$= \Sigma_x \frac{x^2 \varepsilon^2}{A_x^2} = \varepsilon^2 \Sigma_x \frac{x^2}{A_x^2}$$
$$= \varepsilon^2.$$

155. *The distance of any point upon the surface of an ellipsoid, from a point upon the surface of another ellipsoid which has the same foci, is equal to the distance of the two corresponding points of the ellipsoids from each other.* For if

f is the distance of the point of which
x, y, z, are the coördinates, from the point of which
ξ', η', ζ', are the coördinates, and
f' the distance of the corresponding points;

the values of these distances become, by (73_{9-10}) and (73_{24}),

$$\begin{aligned} f^2 &= r^2 + \varrho'^2 - 2\varrho' p' \\ f'^2 &= r'^2 + \varrho^2 - 2\varrho p \\ &= r^2 - \varepsilon^2 + \varrho'^2 + \varepsilon^2 - 2\varrho' p' \\ &= r^2 + \varrho'^2 - 2\varrho' p' = f^2; \end{aligned}$$

whence

$$f = f'.$$

156. *The external level surfaces of an ellipsoidal Chaslesian shell are those of ellipsoids which have the same foci with the outer surface of the Chaslesian shell.* For if

Ω_c is the potential of the given shell for any point of the external ellipsoidal surface of the same foci, and
Ω'_c the constant value of the potential of the corresponding Chaslesian shell, constructed upon the external ellipsoidal surface, for any internal point, and, therefore, for any point of the surface of the given shell;

the equations (72_1) and (74_{12}) give

$$\begin{aligned} \Omega_c &= \int_m \frac{1}{f'} = \int_m \frac{1}{f}, \\ \Omega'_c &= \int_{m'} \frac{1}{f} = \int_m \frac{m'}{mf} = \frac{m'}{m} \int_m \frac{1}{f} = \frac{m'}{m} \Omega_c. \end{aligned}$$

The value of Ω_c is, therefore, constant for all points of the surface of the external ellipsoid, so that this is one of the level surfaces of the given shell.

157. *The attractions of two corresponding Newtonian shells, which have*

the same foci, upon an external point, have the same direction, and are proportional to the masses of the shells. For the infinitely thin shell, this proposition is a simple corollary from (74_{26}). But the finite shells can be subdivided into corresponding infinitesimal shells, and the masses of the corresponding elementary shells will be proportional to the masses of their respective finite shells. The attractions of the corresponding elementary shells upon an external point, therefore, coincide in direction, and are proportional to the masses of the shells; and, therefore, the components of all the corresponding attractions have the same common ratio, and coincide in direction. But the components of all the elementary attractions constitute the attractions of the finite shells themselves. Several special cases of this theorem were first given by MACLAURIN, but the general form was first demonstrated by LAPLACE, and afterwards more rigorously by LEGENDRE, and *it includes the case in which the inner surfaces are reduced to the central point, and the shells become ellipsoids, having the same foci.*

158. The attraction of any Chaslesian shell upon a point at its surface is, from its construction, perpendicular to the surface, and proportional to the thickness of the shell at that point. The attraction upon the whole surface is, therefore, proportional to the mass of the surface, which corresponds to § 136. Hence, if

dN is the thickness at any point, and
p the perpendicular from the centre upon the tangent plane at that point,

the attraction of the ellipsoidal Chaslesian shell at the point is

$$4\pi k dN = 4\pi k dr \cos {}_r^N$$
$$= 4\pi k \frac{dr}{r} r \cos {}_p^r$$
$$= 4\pi k p \frac{dA_x}{A_x}.$$

The component of this action in the direction of the axis of x is

$$4\pi kp\frac{dA_x}{A_x}\cos\frac{N}{x}.$$

If, moreover, the equation of the ellipsoid is

$$L = \Sigma_x\frac{x^2}{A_x^2} - 1 = 0,$$

the general theory of contact gives

$$p = \frac{\Sigma_x(xD_xL)}{\sqrt{(\Box L)}},$$

$$\cos\frac{N}{x} = \frac{D_xL}{\sqrt{(\Box L)}} = \frac{pD_xL}{\Sigma_x(xD_xL)}.$$

Hence,

$$D_xL = \frac{2x}{A_x^2},$$

$$\tfrac{1}{4}\Box L = \Sigma_x\frac{x^2}{A_x^4},$$

$$\Sigma_x xD_xL = 2\Sigma_x\frac{x^2}{A_x^2} = 2,$$

$$\cos\frac{N}{x} = \frac{px}{A_x^2};$$

and *the attraction in the direction of the axis of x of the ellipsoidal Chaslesian shell upon a point at its surface is*

$$4\pi kp^2x\frac{dA_x}{A_x^3}.$$

159. The attraction of an ellipsoidal Chaslesian shell upon any external point is obtained by describing the corresponding Chaslesian shell, for which this point is upon the outer surface, and the attractions of the two shells for this point have the same direction, and are proportional to their masses; so that the attractions in any direction are proportional to the masses. If, then, the accented letters refer to the outer shell, the attraction of the inner

shell is

$$4\pi k \frac{\sigma}{\sigma'} p'^2 x \frac{dA'_x}{A'^3_x} = 4\pi k x \frac{\sigma}{\sigma'} \frac{p'^2}{A'^2_x} \frac{dA_x}{A_x}.$$

160. The condition that the outer surface of the exterior shell passes through the attracted point, is expressed by the equation

$$\Sigma_x \frac{x^2}{A'^2_x} = \Sigma_x \frac{x^2}{A^2_x + \varepsilon^2} = 1.$$

This is an equation of the third degree when it is reduced to its simplest form. But there are two other surfaces which can be drawn through the given point, and which depend for their definition upon the solution of the same equation. They are two hyperboloids, both of which have the same foci with the outer surface of the inner shell, one of which is a bipartite, and the other an unparted hyperboloid. For each of the hyperboloids ε^2 is negative, and its absolute value, independent of its sign, is contained, in the case of the unparted hyperboloid, between the squares of the mean and least axes of the given ellipsoid, and, in the case of the biparted hyperboloid, between the squares of the mean and greatest axes.

161. *The points in which all the ellipsoids, which have the same foci, are cut by the common intersection of the two hyperboloids which have the same foci, are corresponding points.* For if

ε'^2 is the value of $-\varepsilon^2$ for either hyperboloid,

the equation of the hyperboloid for the points of intersection with the ellipsoid is

$$\Sigma_x \frac{x^2}{A^2_x - \varepsilon'^2} = 1.$$

If the equation (77_7) of the ellipsoid is subtracted from this equation, the remainder divided by $\varepsilon^2 + \varepsilon'^2$ is

$$\frac{1}{\varepsilon^2+\varepsilon'^2}\Sigma_x\frac{(\varepsilon^2+\varepsilon'^2)x'^2}{(A_x^2+\varepsilon^2)(A_x^2-\varepsilon'^2)}=\Sigma_x\frac{x'^2}{(A_x^2+\varepsilon^2)(A_x^2-\varepsilon'^2)}=0,$$

in which x', y', and z' are accented, in order not to interfere with the notation which has been adopted for the corresponding points, and which gives for each axis

$$\frac{x}{A_x}=\frac{x'}{A'_x}=\frac{x'}{\sqrt{(A_x^2+\varepsilon^2)}}.$$

The substitution of these equations in (78_1) reduces it to

$$\Sigma_x\frac{x^2}{A_x^2(A_x^2-\varepsilon'^2)}=0;$$

the product of which by ε'^2, added to (76_6), is

$$\Sigma_x\frac{A_x^2x^2}{A_x^2(A_x^2-\varepsilon'^2)}=\Sigma_x\frac{x^2}{A_x^2-\varepsilon'^2}=1,$$

which expresses that the point (x,y,z) is upon the surface of the hyperboloid, and, therefore, all the corresponding points are upon the surfaces of both hyperboloids.

162. *The hyperboloids and ellipsoids which have the same foci, intersect each other perpendicularly.* The conditions that two surfaces of which the equations are

$$L=0,\text{ and }L'=0,$$

intersect each other perpendicularly is expressed algebraically by the equation for each point of the line of intersection,

$$\Sigma_x(D_xL\,D_xL')=0.$$

But for the hyperboloids of equation (77_{28}) and the ellipsoid of equation (76_6) this condition becomes

$$4\,\Sigma_x\frac{x^2}{A_x^2(A_x^2-\varepsilon'^2)}=0,$$

which is the same with the equation already given in (78_{10}). This same demonstration may be applied to the condition of the perpendicularity of the hyperboloids, if A_x^2 is diminished by ε'^2, and ε'^2 is changed into the difference of the squares of the semiaxes of the two hyperboloids.

163. It follows from these two theorems, which are derived from Chasles, that *each normal transversal to the ellipsoidal surfaces of level is the line of intersection of two hyperboloids which have the same foci.*

164. The lines of intersection of these three surfaces are, upon each surface, the lines of greatest and least curvature, for they are a special case of the theorem demonstrated geometrically by Dupin, that *the intersections of three surfaces which cut each other at right angles at and infinitely near their common point of intersection, are their lines of greatest and least curvature at this point.* To demonstrate this theorem, let the three normals to the three surfaces at the common point of intersection be assumed for the axes of rectangular coördinates, and let

$$L_x = 0$$

be the equation of the surface, which is perpendicular to the axis of x. This condition gives for either of the other two axes

$$D_y L_x = 0,$$

in which equation x, y, and z may be mutually interchanged, except that the same axial letter must not be repeated in the equation. Those equations satisfy of themselves the condition (78_{25}) of perpendicularity of these surfaces at the point of intersection. But the intersection of any two of these surfaces coincides with the axis which is the intersection of their tangent planes for an infinitesimal distance, and the two surfaces are perpendicular to each other for this distance. Hence, each pair of surfaces gives an equation of the form

$$D_z (D_x L_x D_x L_y + D_y L_x D_y L_y + D_z L_x D_z L_y) = 0,$$

which is reduced by (79_{20}) to

$$D_x L_x D^2_{x,z} L_y + D_y L_y D^2_{y,z} L_x = 0.$$

The other surfaces give the corresponding equations

$$D_y L_y D^2_{y,x} L_z + D_z L_z D^2_{z,x} L_y = 0,$$
$$D_z L_z D^2_{z,y} L_x + D_x L_x D^2_{x,y} L_z = 0.$$

The sum of the products obtained by multiplying the first of these equations by $D_z L_z$, the second by $-D_x L_x$, and the third by $D_y L_y$ is

$$2 D_y L_y D_z L_z D^2_{y,z} L_x = 0,$$

and the corresponding similar equations are obtained by advancing each letter to the following letter of the series, x, y, z, and x. But the factors $D_x L_x$, $D_y L_y$, and $D_z L_z$, are not zero, and, therefore, these equations may be reduced to

$$D^2_{y,z} L_x = D^2_{x,z} L_y = D^2_{x,y} L_z = 0,$$

which are the well-known conditions that the directions of the axes of x, y, and z respectively coincide with those of the lines of greatest and least curvature of the three surfaces at the origin.

165. The remarkable relations of these surfaces might be still further extended, and if it were worth while to investigate the attractions of masses of infinite extent, it might be shown that upon each series of orthogonal transversal surfaces, Chaslesian shells of infinite extent might be constructed. The level surfaces of these shells would be the orthogonal transversal surfaces of the same series, while their orthogonal transversal surfaces would be the level surfaces of the original Chaslesian shells and the other series of orthogonal transversal surfaces.

166. To investigate the attraction of an ellipsoid upon an external point, it may be supposed to be divided into an infinite

series of elementary Chaslesian shells. Let then

A_x, A_y, A_z, be the semiaxes of the ellipsoid,

a_x, a_y, a_z, those of the outer surface of either of the elementary Chaslesian shells, and let

$$n = \frac{a_x}{A_x} = \frac{a_y}{A_y} = \frac{a_z}{A_z}.$$

If, moreover, x, y, z, are the coördinates of the attracted point,

A'_x, A'_y, A'_z, are the semiaxes of the ellipsoid, which has the same foci with the given ellipsoid, and whose surface passes through the attracted point,

a'_x, a'_y, a'_z, the semiaxes of the ellipsoidal surface, corresponding to the outer surface of the Chaslesian shell, and passing through the point of action,

$$E^2 = A'^2_x - A^2_x, \text{ and}$$
$$\varepsilon^2 n^2 = a'^2_x - a^2_x = a'^2_x - A^2_x n^2;$$

the values of E and ε are the roots of the equations

$$\Sigma_x \frac{x^2}{A^2_x + E^2} = 1,$$
$$\Sigma_x \frac{x^2}{a^2_x + \varepsilon^2 n^2} = \frac{1}{n^2} \Sigma_x \frac{x^2}{A^2_x + \varepsilon^2} = 1.$$

The attraction of the Chaslesian shell upon the external point in the direction of the axis of x is by (77_2)

$$4\pi k x \frac{a_x a_y a_z p'^2}{a'^3_x a'_y a'_z} \cdot \frac{d a_x}{a_x} = 4\pi k x \frac{a_x a_y a_z p'^2}{a'^3_x a'_y a'_z} \cdot \frac{dn}{n};$$

in which the value of p' is, by equations (76_{9-18}), given in the form

$$\frac{1}{p'^2} = \frac{\Box L}{[\Sigma_x (x D_x L)]^2} = \Sigma_x \frac{x^2}{a'^4_x}$$
$$= \Sigma_x \frac{x^2}{(a^2_x + \varepsilon^2 n^2)^2} = \frac{1}{n^4} \Sigma_x \frac{x^2}{(A^2_x + \varepsilon^2)^2}.$$

The differential of (81_{21}) after it is multiplied by n^2 is

$$-\Sigma_x \frac{2x^2 \varepsilon d\varepsilon}{(A_x^2+\varepsilon^2)^2} = 2n dn;$$

whence by (81_{31})

$$dn = -\frac{n^3}{p'^2}\varepsilon d\varepsilon.$$

This value reduces the attraction of the shell in the direction of the axis of x to

$$-4\pi k x \frac{a_x a_y a_z n^2}{a_x'^3 a_y' a_z'}\varepsilon d\varepsilon = -2\pi k x \frac{A_x A_y A_z d.\varepsilon^2}{\sqrt{[(A_x^2+\varepsilon^2)^3(A_y^2+\varepsilon^2)(A_z^2+\varepsilon^2)]}}.$$

The integral of this expression is the attraction of the whole ellipsoid. The limits of integration correspond to the values of ε, for one of which the shell is evanescent, and for the other its surface coincides with the surface of the ellipsoid. But, when the shell vanishes, n is zero, and ε is infinite; and when its outer surface coincides with that of the ellipsoid, n is unity, and ε becomes E. Hence, the expression for the attraction of the ellipsoid in the direction of the axis of x is, if

M is the mass of the ellipsoid, and
K its mean density,

$$D_x\Omega = \frac{3Mx}{2K}\int_{E^2}^{\infty} {}_{\varepsilon^2}\frac{k}{(A_x^2+\varepsilon^2)\sqrt{[(A_x^2+\varepsilon^2)(A_y^2+\varepsilon^2)(A_z^2+\varepsilon^2)]}}.$$

By advancing each letter in the series x, y, z, and x to the following, the corresponding expressions are obtained for the attractions in the directions of the other two axes.

167. By the substitution of

$$b_x^2 = A_x^2 + \varepsilon^2,$$

the equation (82_{24}) becomes

$$D_x\Omega = \frac{3\,Mx}{K}\int_{A'_x}^{\infty} \frac{k}{b_x^2\sqrt{[(b_x^2+A_y^2-A_x^2)(b_y^2+A_z^2-A_x^2)]}\, b_x}.$$

168. By the substitution of

$$u_x = \frac{A_x}{b_x},\ \text{and}$$

$$U_x = \frac{A_x}{A'_x};$$

the equation (83_3) becomes

$$D_x\Omega = \frac{3\,Mx}{KA_x}\int_0^{U_x} \frac{ku_x^2}{\sqrt{[(A_x^2+u_x^2(A_y^2-A_x^2))(A_x^2+u_x^2(A_z^2-A_x^2))]}}\, u_x,$$

which formula, with transformations similar to the following, is given by LEGENDRE.

169. If A_x is assumed to be the greatest of the semiaxes of the ellipsoid, and A_z the least, let

$$\cos^2\varphi = \frac{A_z^2+\varepsilon^2}{A_x^2+\varepsilon^2},$$

$$\sin^2 i = \frac{A_x^2-A_y^2}{A_x^2-A_z^2},$$

$$\sin\vartheta = \sin i \sin\varphi,$$

$$\cos^2\Phi = \frac{A_z^2+E^2}{A_x^2+E^2},$$

$$\sin\Theta = \sin i \sin\Phi;$$

and let the first and second forms of the elliptic integrals be expressed by the notation

$$\mathfrak{F}_i\varphi = \int_0^{\phi} \sec\vartheta\, \phi,$$

$$\mathfrak{E}_i\varphi = \int_0^{\phi} \cos\vartheta\, \phi.$$

These equations give

$$\varepsilon^2 \sin^2\varphi = A_x^2 \cos^2\varphi - A_z^2,$$

$$(A_y^2 + \varepsilon^2)\sin^2\varphi = A_x^2 - A_z^2 + (A_y^2 - A_x^2)\sin^2\varphi = (A_x^2 - A_z^2)\cos^2\vartheta,$$

$$(A_x^2 + \varepsilon^2)\sin^2\varphi = A_x^2 - A_z^2,$$

$$(A_z^2 + \varepsilon^2)\sin^2\varphi = (A_x^2 - A_z^2)\cos^2\varphi;$$

$$\sin^2\varphi\, d.\varepsilon^2 = -(A_x^2 + \varepsilon^2)\, d.\sin^2\varphi = -(A_x^2 - A_z^2)\,\mathrm{cosec}^2\varphi\, d.\sin^2\varphi,$$

$$d.\varepsilon^2 = -2(A_x^2 - A_z^2)\,\mathrm{cosec}^3\varphi \cos\varphi\, d\varphi;$$

which, substituted in (82_{24}), reduce the expression for the attraction in the direction of the axis of x, when the ellipsoid is homogeneous, to the form

$$D_x\Omega = 3Mx\int_{0\,\phi}^{\Phi} \frac{\sin^2\varphi \sec\vartheta}{(A_x^2 - A_z^2)^{\frac{3}{2}}} = \frac{3Mx}{(A_x^2 - A_z^2)^{\frac{3}{2}}\sin^2 i}\int_{0\,\phi}^{\Phi}(\sec\vartheta - \cos\vartheta)$$

$$= \frac{3Mx}{(A_x^2 - A_z^2)^{\frac{3}{2}}\sin^2 i}(\mathfrak{F}_i\Phi - \mathfrak{E}_i\Phi);$$

or, if

$$P = \frac{3M}{(A_x^2 - A_z^2)^{\frac{3}{2}}} = \frac{3M}{(A_x'^2 - A_z'^2)^{\frac{3}{2}}},$$

the attraction is

$$D_x\Omega = \frac{Px}{\sin^2 i}(\mathfrak{F}_i\Phi - \mathfrak{E}_i\Phi).$$

The same substitution gives the attractions parallel to the other axes in the forms

$$D_y\Omega = Py\int_{0\,\phi}^{\Phi} \sin^2\varphi \sec^3\vartheta,$$

$$D_z\Omega = Pz\int_{0\,\phi}^{\Phi} \tan^2\varphi \sec\vartheta.$$

But the differential of the logarithm of (83_{21}) is

$$\cot\vartheta\, D_\phi\vartheta = \cot\varphi,$$

and, therefore,

$$D_\phi\vartheta = \tan\vartheta \cot\varphi,$$

$$
\begin{aligned}
D_\varphi(\tan\varphi\cos\vartheta) &= \sec^2\varphi\cos\vartheta - \sin^2\vartheta\sec\vartheta \\
&= \sec\vartheta(\sec^2\varphi\cos^2\vartheta - \sin^2\vartheta) \\
&= \sec\vartheta\sec^2\varphi - \sec\vartheta\sin^2\vartheta(\sec^2\varphi + 1) \\
&= \sec\vartheta\sec^2\varphi - \sec\vartheta\sin^2 i(\tan^2\varphi + \sin^2\varphi) \\
&= \sec\vartheta + \cos^2 i\sec\vartheta\tan^2\varphi - \sec\vartheta\sin^2\vartheta \\
&= \cos\vartheta + \cos^2 i\sec\vartheta\tan^2\varphi,
\end{aligned}
$$

$$
\begin{aligned}
D_\varphi(\sin\varphi\cos\varphi\sec\vartheta) &= \frac{D_\Phi(\tan\theta\cos\varphi)}{\sin i} = \frac{\cos\varphi\cot\varphi\sec^2\theta\tan\theta - \tan\theta\sin\varphi}{\sin i} \\
&= \cos^2\varphi\sec^3\vartheta - \sec\vartheta\sin^2\varphi \\
&= -\cos^2 i\sin^2\varphi\sec^3\vartheta + (1 - \sin^2\varphi\sin^2 i)\sec^3\vartheta \\
&\quad - \sec\vartheta\sin^2\varphi \\
&= -\cos^2 i\sin^2\varphi\sec^3\vartheta + \sec\vartheta\left(1 - \frac{\sin^2\theta}{\sin^2 i}\right) \\
&= -\cos^2 i\sin^2\varphi\sec^3\vartheta + \frac{\sec\theta}{\sin^2 i}(\cos^2\vartheta - \cos^2 i) \\
&= -\cos^2 i\sin^2\varphi\sec^3\vartheta - \cot^2 i\sec\vartheta + \operatorname{cosec}^2 i\cos\vartheta.
\end{aligned}
$$

These equations reduce the attractions to the forms

$$
\begin{aligned}
D_y\Omega &= Py\int_0^\Phi\left[\frac{\sec^2 i\cos\theta - \sec\theta}{\sin^2 i} - \sec^2 i D_\varphi(\sin\varphi\cos\varphi\sec\vartheta)\right]_\phi \\
&= Py(\tfrac{1}{4}\operatorname{cosec}^2 2i\,\mathfrak{E}_i\Phi - \operatorname{cosec}^2 i\,\mathfrak{F}_i\Phi - \sec^2 i\sin\Phi\cos\Phi\sec\Theta),
\end{aligned}
$$

$$
\begin{aligned}
D_z\Omega &= \sec^2 i Pz\int_0^\Phi\left[D_\phi(\tan\varphi\cos\vartheta) - \cos\vartheta\right]_\phi \\
&= \sec^2 i Pz(\tan\Phi\cos\Theta - \mathfrak{E}_i\Phi).
\end{aligned}
$$

170. The following values are derived from $(83_{23\text{-}24})$ and (81_{15});

$$
\begin{aligned}
\cos^2\Phi &= \frac{A_z^2 + E^2}{A_x^2 + E^2} = \frac{A_z'^2}{A_x'^2}, \\
\sin^2\Phi &= \frac{A_x^2 - A_z^2}{A_x^2 + E^2} = \frac{A_x'^2 - A_z'^2}{A_x'^2}, \\
\sin^2\Theta &= \frac{A_x^2 - A_y^2}{A_x^2 + E^2} = \frac{A_x'^2 - A_y'^2}{A_x'^2},
\end{aligned}
$$

$$\cos^2\Theta = \frac{A_y^2 + E^2}{A_x^2 + E^2} = \frac{A_y'^2}{A_x'^2},$$
$$\sin^2 i = \frac{A_x^2 - A_y^2}{A_x^2 - A_z^2} = \frac{A_x'^2 - A_y'^2}{A_x'^2 - A_z'^2},$$
$$\cos^2 i = \frac{A_y^2 - A_z^2}{A_x^2 - A_z^2} = \frac{A_y'^2 - A_z'^2}{A_x'^2 - A_z'^2}.$$

The equations of the attractions give, by means of these values, that of P and (81_{15}),

$$\Sigma_x \frac{D_x \Omega}{x} = 2\,\Sigma_x D_{x^2}\Omega = \sec^2 i\, P \sin\Phi \sec\Phi \sec\Theta (\cos^2\Theta - \cos^2\Phi)$$
$$= \frac{3M}{A_x' A_y' A_z'} = 4\pi \frac{M}{M'}.$$

This simple equation is due to Legendre, and the first of the two following equations which are obtained by the same process of reduction.

$$\Sigma_x \frac{A_x'^2 D_x \Omega}{x} = 2\,\Sigma_x (A_x'^2 D_{x^2}\Omega) = \frac{3M}{\sqrt{(A_x^2 - A_z^2)}} \mathfrak{F}_i \Phi,$$
$$\Sigma_x \frac{A_x^2 D_x \Omega}{x} = 2\,\Sigma_x (A_x^2 D_{x^2}\Omega) = \frac{3M}{\sqrt{(A_x^2 - A_z^2)}} \mathfrak{F}_i \Phi - 4\pi E^2 \frac{M}{M'}.$$

171. By putting

$$L = \tfrac{1}{4}\int_{E^2}^{\infty} \frac{1}{\varepsilon^2 \sqrt{[(A_x^2 + \varepsilon^2)(A_y^2 + \varepsilon^2)(A_z^2 + \varepsilon^2)]}} = \frac{1}{2\sqrt{(A_x^2 - A_z^2)}} \mathfrak{F}_i \Phi,$$

the attractions may assume the form

$$D_x \Omega = -3 M x D_{A_x^2} L,$$
$$D_y \Omega = -3 M y D_{A_y^2} L,$$
$$D_z \Omega = -3 M z D_{A_z^2} L.$$

in which the differentiations, relatively to A_x^2, A_y^2, and A_z^2, are performed without regard to the changes of E, dependent upon the formula (81_{15}).

172. The equation (81_{19}) may, by means of the equations (84_{2-7}) be written in the form

$$x^2 + y^2 \sec^2 \Theta + z^2 \sec^2 \Phi = (A_x^2 - A_z^2) \operatorname{cosec}^2 \Phi,$$

or by the substitution of the value of Θ from (83_{24}),

$$\frac{x^2 + z^2 \sec^2 \Phi}{A_x^2 - A_z^2} + \frac{y^2}{A_x^2 \cos^2 \Phi + A_y^2 \sin^2 \Phi - A_z^2} = \frac{1}{\sin^2 \Phi}.$$

173. *When the attracted point is upon the surface of the ellipsoid, E* vanishes, and the value of Φ becomes

$$\cos \Phi = \frac{A_z}{A_x}.$$

174. *When the attracted point is within the ellipsoid,* the Newtonian shell, of which the outer surface is that of the ellipsoid, and the inner surface passes through the point, exerts no action upon the point, and *the attraction is reduced to that of an ellipsoid similar to the given ellipsoid, and of which the surface passes through the attracted point.*

175. When the density of the ellipsoid varies in its interior, in such a way that each of its component Chaslesian shells is homogeneous, k is a function of ε, and after its substitution (82_{24}) may be integrated.

176. *When the ellipsoid is a homogeneous oblate ellipsoid of revolution* the various formulæ become

$$A_x = A_y,$$
$$i = \vartheta = 0,$$
$$x^2 + y^2 + z^2 + z^2 \tan^2 \Phi = (A_x^2 - A_z^2)(1 + \cot^2 \Phi),$$
$$z^2 \tan^4 \Phi + (r^2 - A_x^2 + A_z^2) \tan^2 \Phi = A_x^2 - A_z^2;$$

$$D_x \Omega = Px \int_0^\Phi \sin^2 \varphi \, {}_\varphi = \tfrac{1}{4} Px (2\Phi - \sin 2\Phi),$$

$$D_y \Omega = Py \int_0^\Phi \sin^2 \varphi \, {}_\varphi = \tfrac{1}{4} Py (2\Phi - \sin 2\Phi),$$

$$D_z\Omega = Pz\int_0^\Phi \tan^2\varphi = Pz(\tan\Phi - \Phi).$$

177. *When the ellipsoid is an homogeneous prolate ellipsoid of revolution,* the formulæ become

$$A_y = A_z^1.$$
$$i = \tfrac{1}{2}\pi,$$
$$\varphi = \vartheta,$$
$$x^2 + y^2 + z^2 + (y^2 + z^2)\tan^2\Phi = (A_x^2 - A_z^2)(1 + \cot^2\Phi),$$
$$(y^2 + z^2)\tan^4\Phi + (r^2 - A_x^2 - A_z^2)\tan^2\Phi = A_x^2 - A_z^2;$$
$$D_x\Omega = Px\int_0^\Phi \sin^2\varphi\sec\varphi$$
$$= Px[\log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\Phi) - \sin\Phi],$$
$$D_y\Omega = Py\int_0^\Phi \sin^2\varphi\sec^3\varphi$$
$$= \tfrac{1}{2}Py[\sin\Phi\sec^2\Phi - \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\Phi)],$$
$$D_z\Omega = Pz\int_0^\Phi \sin^2\varphi\sec^3\varphi$$
$$= \tfrac{1}{2}Pz[\sin\Phi\sec^2\Phi - \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\Phi)].$$

ATTRACTION OF A SPHEROID. LEGENDRE'S AND LAPLACE'S FUNCTIONS.

178. The investigation of the attraction of a spheroid is greatly facilitated by the introduction of certain functions which were first conceived and investigated by Legendre, but which became so fruitful in their more general form, given in the subsequent researches of Laplace, that they are usually designated by the name of the latter geometer. A method will be pursued in their development and discussion which is similar in some respects to that given by Jacobi.

179. Let

$$i = \sqrt{-1},$$
$$H = \cos\varphi + i \sin\varphi \cos\eta,$$

and if any power of H, denoted by n, is developed in a series of terms arranged according to the cosines of the multiples of η, let any one of the terms be denoted by

$$i^m \Phi_n^{[m]} \cos m\eta,$$

in which $[m]$ denotes the number of accents of Φ. The required power has then the form

$$H^n = \Sigma_{-\infty}^{\infty}{}_m \left(i^m \Phi_n^{[m]} \cos m\eta\right).$$

180. The value of H is not changed by reversing the sign of η, and, therefore, the series remains unchanged by this reversal of sign, which gives

$$i^m \Phi_n^{[m]} = i^{-m} \Phi_n^{[-m]},$$

or
$$\Phi_n^{[-m]} = \pm \Phi_n^{[m]} = (-1)^m \Phi_n^{[m]};$$

in which the upper sign corresponds to the even values of m, and the lower sign to the odd values of m. The equation (89_{11}) may also be written

$$H^n = \Phi_n + 2 \Sigma_1^{\infty}{}_m \left(i^m \Phi_n^{[m]} \cos m\eta\right).$$

181. The integral of the product of (89_{22}) by $\cos m\eta$ is, by a well known theorem

$$\int_0^{2\pi}{}_\eta \left(H^n \cos m\eta\right) = 2\pi\, i^m \Phi_n^{[m]}.$$

The derivative of this equation, relatively to φ, reduced by the condition

$$D_\phi H = -\sin\varphi + i \cos\varphi \cos\eta,$$

becomes

$$2\pi i^m D_\phi \Phi_n^{[m]} = n\int_0^{2\pi}{}_\eta [H^{n-1}\cos m\eta(-\sin\varphi + i\cos\varphi\cos\eta)]$$

$$= n\int_0^{2\pi}{}_\eta [H^{n-1}(-\sin\varphi\cos m\eta + \tfrac{1}{2}i\cos\varphi(\cos(m+1)\eta + \cos(m-1)\eta))];$$

whence, by (89_{27}),

$$D_\phi \Phi_n^{[m]} = -n\sin\varphi\,\Phi_{n-1}^{[m]} + \tfrac{1}{2}n\cos\varphi(\Phi_{n-1}^{[m-1]} - \Phi_{n-1}^{[m+1]}).$$

182. The derivative of (89_{22}), relatively to η, reduced by the condition

$$D_\eta H = -i\sin\varphi\sin\eta$$

becomes

$$inH^{n-1}\sin\varphi\sin\eta = 2\sum_1^\infty{}_m(m i^m \Phi_n^{[m]}\sin m\eta).$$

The integral of the product of this equation by $\sin m\eta$ is

$$in\sin\varphi\int_0^{2\pi}{}_\eta (H^{m-1}\sin\eta\sin m\eta) = 2\pi m i^m \Phi_n^{[m]},$$

or

$$in\sin\varphi\int_0^{2\pi}{}_\eta [H^{m-1}(\tfrac{1}{2}\cos(m-1)\eta - \tfrac{1}{2}\cos(m+1)\eta)] = 2\pi m i^m \Phi_n^{[m]};$$

which becomes by (89_{27})

$$m\Phi_n^{[m]} = \tfrac{1}{2}n\sin\varphi(\Phi_{n-1}^{[m-1]} + \Phi_{n-1}^{[m+1]}).$$

183. The equation (89_{27}) may assume the form

$$\int_0^{2\pi}{}_\eta [H^{n-1}(\cos\varphi\cos m\eta + \tfrac{1}{2}i\sin\varphi(\cos(m+1)\eta + (\cos(m-1))] = 2\pi i^m \Phi_n^{[m]},$$

which, reduced by (89_{27}), gives

$$\Phi_n^{[m]} = \cos\varphi\,\Phi_{n-1}^{[m]} + \tfrac{1}{2}\sin\varphi(\Phi_{n-1}^{[m-1]} - \Phi_{n-1}^{[m+1]}).$$

184. The remainder, if (90_8) is subtracted from the product of (90_{24}), multiplied by $\cot\varphi$, is

$$n\cos\varphi\, \Phi_{n-1}^{[m+1]} + n\sin\varphi\, \Phi_{n-1}^{[m]} = m\cot\varphi\, \Phi_n^{[m]} - D_\phi \Phi_n^{[m]} = -\sin^m\varphi\, D_\phi \frac{\Phi_n^{[m]}}{\sin^m\varphi}$$

$$= \sin^{m+1}\varphi\, D_{\cos\phi} \frac{\Phi_n^{[m]}}{\sin^m\varphi}.$$

The sum of (90_{24}) and n times (90_{31}) is

$$n\cos\varphi\, \Phi_{n-1}^{[m]} + n\sin\varphi\, \Phi_{n-1}^{[m-1]} = (n+m)\, \Phi_n^{[m]},$$

the first member of which becomes identical with that of the previous equation, when m is increased by unity. Hence,

$$\frac{\Phi_n^{[m+1]}}{\sin^m\varphi} = -\frac{1}{n+m+1} D_\phi \frac{\Phi_n^{[m]}}{\sin^m\varphi} = \frac{\sin\varphi}{n+m+1} D_{\cos\phi} \frac{\Phi_n^{[m]}}{\sin^m\varphi},$$

or, if m is diminished by unity

$$\frac{\Phi_n^{[m]}}{\sin^m\varphi} = \frac{1}{n+m} D_{\cos\phi} \frac{\Phi_n^{[m-1]}}{\sin^{m-1}\varphi}.$$

If the sign of m is reversed in this equation, it becomes by (89_{17})

$$\sin^m\varphi\, \Phi_n^{[m]} = \frac{-1}{n-m} D_{\cos\phi} (\sin^{m+1}\varphi\, \Phi_n^{[m+1]}).$$

185. It will be found convenient here and elsewhere to adopt the functional notation

$$\Gamma h = \int_x{}_0^1 (-\log.x)^h,$$

which gives, by a familiar formula, or by simple integration by parts, when h is positive, and k an integer, which is less than $h+1$,

$$\Gamma h = h(h-1)(h-2)\ldots\ldots(h-k+1) \int_x{}_0^1 (-\log x)^{h-k}$$

$$= h(h-1)(h-2)\ldots\ldots(h-k+1)\, \Gamma(h-k),$$

and

$$h(h-1)(h-2)\dots(h-k+1)=\frac{\Gamma h}{\Gamma(h-k)}.$$

When h is an integer, and k the next smaller integer, this formula becomes

$$1.2.3\dots h=\Gamma h.$$

With this notation, Taylor's theorem assumes the form

$$\mathfrak{F}(x+z)=e^{zD_x}\mathfrak{F}x=\Sigma_{m\,0}^{\,\infty}\left(\frac{D_x^m\mathfrak{F}x}{\Gamma m}z^m\right).$$

186. The equations (91_{16}) and (91_{20}) give, by successive substitutions in each other, and the use of the preceding notation,

$$\frac{\Gamma(m+n)\,\Phi_n^{[m]}}{\sin^m\varphi}=D_{\cos\phi}^{m-m'}\frac{\Gamma(m'+n)\,\Phi_n^{[m']}}{\sin^{m'}\varphi}=\Gamma(m'+n)\,D_{\cos\phi}^{m-m'}\frac{\Phi_n^{[m']}}{\sin^{m'}\varphi},$$

$$(-1)^m\Gamma(n-m)\sin^m\varphi\,\Phi_n^{[m]}=(-1)^{m'}\Gamma(n-m')\,D_{\cos\phi}^{m'-m}(\sin^{m'}\varphi\,\Phi_n^{[m']})\,;$$

in which negative differentiation must be interpreted to be integration; in the former equation, when n is negative, $m'+n+1$ and $m+n+1$ must be positive; while, in the latter equation, n, $n-m+1$, and $n-m'+1$ must all be positive. When n is positive, but $n-m+1$, and $n-m'+1$ are negative, the equation to be substituted for (92_{15}) is

$$\frac{\sin^m\varphi\,\Phi_n^{[m]}}{\Gamma(m-n-1)}=\frac{D_{\cos\phi}^{m'-m}(\sin^{m'}\varphi\,\Phi_n^{[m']})}{\Gamma(m'-n-1)},$$

which equation is also to be used when n is negative. When n and $n-m+1$ are positive, but $n-m'+1$ is negative, the combination of (92_{15}) and (92_{23}) gives, by representing by n', the greatest integer contained in $n+1$,

$$\frac{(-1)^{n'-m}\Gamma(n-m)\sin^m\varphi\,\Phi_n^{[m]}}{\Gamma(n-n')}=\frac{\Gamma(n'-n-1)\,D_{\cos\varphi}^{n'-m}(\sin^{m'}\varphi\,\Phi_n^{[m']})}{\Gamma(m'-n-1)}.$$

When n, $n+m'+1$, and $n+m+1$ are all negative, the equation to be substituted for (92_{13}) is

$$\frac{(-1)^m \varPhi_n^{[m]}}{\Gamma(-1-n-m)\sin^m\varphi} = \frac{(-1)^{m'}}{\Gamma(-1-n-m')} D_{\cos\phi}^{m-m'} \frac{\varPhi_n^{[m']}}{\sin^{m'}\varphi}.$$

When n and $n+m'+1$ are negative, but $m+n+1$ is positive, the combination of (92_{13}) and (93_3) gives, by representing by n', the greatest integer contained in $-1-n$,

$$\frac{\Gamma(m+n)\,\varPhi_n^{[m]}}{\Gamma(n+n')\sin^m\varphi} = \frac{(-1)^{n'-m'}\Gamma(-1-n-n')}{\Gamma(-1-n-m')} D_{\cos\phi}^{m-m'} \frac{\varPhi_n^{[m']}}{\sin^{m'}\varphi}.$$

There are peculiar considerations which simplify the investigations, when n is integral, whether it be positive or negative; and these are the cases to which most of the subsequent investigations are limited.

187. By reducing m or m' to zero, the equations of the preceding section give, for positive values of n,

$$\varPhi_n^{[m]} = \frac{\Gamma n}{\Gamma(m+n)} \sin^m\varphi\, D_{\cos\phi}^m \varPhi_n = (-1)^m \frac{\Gamma n}{\Gamma(n-m)\sin^m\varphi} \int_{\cos\phi}^m \varPhi_n$$

$$= (-1)^{n'} \frac{\Gamma n\, \Gamma(m-n-1)}{\Gamma(n-n')\,\Gamma(n'-n-1)\sin^m\varphi} \int_{\cos\phi}^m \varPhi_n,$$

and for negative values of n

$$\varPhi_n^{[m]} = \frac{\Gamma(m-n-1)}{\Gamma(-1-n)\sin^m\varphi} \int_{\cos\phi}^m \varPhi_n$$

$$= \frac{(-1)^m \Gamma(-1-n-m)}{\Gamma(-1-n)} \sin^m\varphi D_{\cos\phi}^m \varPhi_n$$

$$= \frac{(-1)^{n'}\Gamma(-1-n-n')\,\Gamma(n+n')}{\Gamma(m+n)\,\Gamma(-1-n)} \sin^m\varphi D_{\cos\phi}^m \varPhi_n.$$

188. *When n is zero*, it is easily seen that

$$H^0 = 1 = \varPhi_0,$$

and that, for all other values of m

$$\varPhi_0^{[m]} = 0.$$

189. *When n is a positive integer,* and h is also a positive integer, the equation (91_8) gives

$$(2n+h)\,\Phi_n^{[n+h]} = n\cos\varphi\,\Phi_{n-1}^{[n+1]} + n\sin\varphi\,\Phi_{n-1}^{[n-1+h]}.$$

If, then, the terms of the second member vanish for any value of n, they will also vanish for the next higher value of n. But they vanish, by the preceding section, when n is zero, and, therefore, they vanish for every positive integral value of n; that is,

$$\Phi_n^{[n+h]} = 0.$$

or *the series is finite for positive integral values of n, and contains only $n+1$ terms.*

190. The substitution of the preceding equation in (91_8) gives

$$\Phi_n^{[n]} = \tfrac{1}{2}\sin\varphi\,\Phi_{n-1}^{[n-1]} = (\tfrac{1}{2}\sin\varphi)^m\,\Phi_{n-m}^{[n-m]}$$
$$= (\tfrac{1}{2}\sin\varphi)^n\,\Phi_0 = \frac{1}{2^n}\sin^n\varphi.$$

which equation, substituted in (93_{16}), gives

$$\Phi_n = \frac{(-1)^n}{\Gamma n} D^n_{\cos\phi}(\Phi_n^{[n]}\sin^n\varphi) = \frac{1}{2^n\Gamma n} D^n_{\cos\phi}(-\sin^2\varphi)^n;$$
$$\Phi_n^{[m]} = \frac{\sin^m\varphi}{2^n\Gamma(m+n)} D^{n+m}_{\cos\phi}(-\sin^2\varphi)^n$$
$$= \frac{(-1)^m}{2^n\,\Gamma(n-m)\sin^m\varphi} D^{n-m}_{\cos\phi}(-\sin^2\varphi)^n;$$

by which the coefficients of the development are obtained when n is a positive integer.

191. *When n is the negative of unity,* the equation (89_{27}) gives

$$2\pi\,\Phi_{-1} = \int_0^{2\pi}{}_\eta\,\frac{1}{H}.$$

But the value of H gives

$$\frac{1}{H} = \frac{1}{\cos\varphi + i\sin\varphi\cos\eta} = \frac{\cos\varphi - i\sin\varphi\cos\eta}{\cos^2\varphi + \sin^2\varphi\cos^2\eta}$$

$$= -\frac{i\sin\varphi\cos\eta}{1-\sin^2\varphi\sin^2\eta} + \frac{\cos\varphi}{\cos^2\eta + \cos^2\varphi\sin^2\eta}$$

$$= -\frac{\frac{1}{2}i\sin\varphi\cos\eta}{1+\sin\varphi\sin\eta} - \frac{\frac{1}{2}i\sin\varphi\cos\eta}{1-\sin\varphi\sin\eta} + \frac{\cos\varphi\, D_\eta\tan\eta}{1+\cos^2\varphi\tan^2\eta},$$

the integral of which is

$$\int_\eta \frac{1}{H} = -\tfrac{1}{2}i\log\frac{1+\sin\varphi\sin\eta}{1-\sin\varphi\sin\eta} + \tan^{[-1]}(\cos\varphi\tan\eta).$$

Hence, by passing to the limits

$$2\pi\Phi_{-1} = 2\pi,$$
$$\Phi_{-1} = 1,$$

When n and m vanish, equation (90_{31}) becomes

$$\Phi_0 = \cos\varphi\,\Phi_{-1} - \sin\varphi\,\Phi'_{-1},$$

whence

$$\Phi'_{-1} = -\frac{1-\cos\varphi}{\sin\varphi} = -\tan\tfrac{1}{2}\varphi.$$

Equation (92_{13}) gives, then,

$$\Phi_{-1}^{[m]} = -\frac{\sin^m\varphi}{2\,\Gamma(m-1)}D_{\cos\phi}^{m-1}\sec^2\tfrac{1}{2}\varphi.$$

192. *When n is any negative integer*, it is more convenient to write the formulæ with the sign of n reversed. With this change, the sum of the product of (90_{31}) multiplied by $(-n\sin\varphi)$, that of (90_8) multiplied by $\cos\varphi$, and that of (90_{24}) multiplied by $(-\operatorname{cosec}\varphi)$ becomes

$$n\Phi_{-(n+1)}^{[m+1]} = \cos\varphi\, D_\phi \Phi_{-n}^{[m]} - \left(n\sin\varphi + \frac{m}{\sin\varphi}\right)\Phi_{-n}^{[m]}$$
$$= \frac{\sin^{n-1}\varphi}{\sec\varphi}D_\phi\frac{\Phi_{-n}^{[m]}}{\sin^{n-1}\varphi} - \left((2n-1)\sin\varphi + \frac{m+1-n}{\sin\varphi}\right)\Phi_{-n}^{[m]};$$

which, when

$$m = n-1,$$

is reduced to

$$n\Phi^{[n]}_{-(n+1)} = \sin^{n-1}\varphi\cos\varphi\, D_\phi \frac{\Phi^{[n-1]}_{-n}}{\sin^{n-1}\varphi} - (2n-1)\sin\varphi\,\Phi^{[n-1]}_{-n}.$$

The successive substitution of 1, 2, 3, &c., for n, gives, by means of (91_{12})

$$D_\phi \frac{\Phi^{[n-1]}_{-n}}{\sin^{n-1}\varphi} = 0.$$

$$\Phi^{[n]}_{-(n+1)} = -\frac{2n-1}{n}\sin\varphi\,\Phi^{[n-1]}_{-n} = -\frac{2n(2n-1)}{2n^2}\sin\varphi\,\Phi^{[n-1]}_{-n}$$
$$= \frac{\Gamma(2n)}{2^n(\Gamma n)^2}(-\sin\varphi)^n.$$

The substitution of this value in (93_{21}) gives, by (94_{20}),

$$\Phi_{-(n+1)} = \frac{\Gamma n}{\Gamma(2n)} D^n_{\cos\phi}(\sin^n\varphi\,\Phi^{[n]}_{-(n+1)}) = \frac{1}{2^n\Gamma n} D^n_{\cos\phi}(-\sin^2\varphi)^n = \Phi_n;$$

and, therefore, for all values of m less than $n+1$,

$$\Phi^{[m]}_{-(n+1)} = (-1)^m \frac{\Gamma(n-m)}{2^n(\Gamma n)^2}\sin^m\varphi\, D^{n+m}_{\cos\phi}(-\sin^2\varphi)^n$$
$$= (-1)^m \frac{\Gamma(n-m)\,\Gamma(n+m)}{(\Gamma n)^2}\Phi^{[m]}_n.$$

The equation (91_8) gives, when $m-n$ vanishes, by (96_9),

$$\Phi^{[n-1]}_{-(n+1)} = -\cot\varphi\,\Phi^{[n]}_{-(n+1)} = \frac{\Gamma(2n)}{2^n(\Gamma n)^2}(-\sin\varphi)^{n-1}\cos\varphi\,;$$

whence, by (90_{24})

$$\Phi^{[n+1]}_{-(n+1)} = -2\operatorname{cosec}\varphi\,\Phi^{[n]}_{-n} - \Phi^{[n-1]}_{-(n+1)}.$$

From this equation the successive values of $\Phi^{[n]}_{-n}$ may be determined by successive substitution of 1, 2, 3, &c., for n, or they may be determined by the equation derived from (91_{20}) and (96_9),

$$\Phi^{[n+1]}_{-(n+1)} = \frac{(2n+1)\,\Gamma(2n)}{2^n(\Gamma n)^2}\,\frac{1}{\sin^{n+1}\varphi}\int_1^{\cos\phi}{}_{\cos\phi}(-\sin^2\varphi)^n.$$

The remaining coefficients, in which m is greater than n, are then to be determined by the equation derived from (92_{13});

$$\Phi^{[m]}_{-n} = \frac{\sin^m \varphi}{\Gamma(m-n)} D^{m-n}_{\cos\phi} \frac{\Phi^{[n]}_{-n}}{\sin^n \varphi}.$$

193. In order to apply the preceding investigations to the problem of attraction, it is requisite to introduce the form of polar coördinates, of which zenith distance and azimuth is the familiar instance. For this purpose let the following notation be adopted:

φ_f is the angle which a line f makes with the axis,

ϑ_f is the angle which a plane, drawn through the axis, parallel to f, makes with the primitive plane.

The distance f, between two points, of which the radii vectores are r and ϱ, is given by the equation

$$\begin{aligned} f^2 &= r^2 + \varrho^2 - 2r\varrho(\cos\varphi_r \cos\varphi_\rho + \sin\varphi_r \sin\varphi_\rho \cos(\vartheta_r - \vartheta_\rho)) \\ &= r^2 + \varrho^2 - 2\varrho r \cos^r_\varrho. \end{aligned}$$

Hence the notation

$$\begin{aligned} i &= \sqrt{-1}, \\ \Pi_f &= \cos\varphi_f + i \sin\varphi_f \cos(\eta - \vartheta_f), \end{aligned}$$

gives

$$\begin{aligned} \pm f\Pi_f &= \pm f\cos\varphi_f \pm if \sin\varphi_f \cos(\eta - \vartheta_f) \\ &= r\cos\varphi_r - \varrho\cos\varphi_\rho + i\cos\eta\,(r\sin\varphi_r\cos\vartheta_r - \varrho\sin\varphi_\rho\cos\vartheta_\rho) \\ &\qquad + i\sin\eta\,(r\sin\varphi_r\sin\vartheta_r - \varrho\sin\varphi_\rho\sin\vartheta_\rho) \\ &= r\Pi_r - \varrho\Pi_\rho; \end{aligned}$$

in which the upper sign is to be used when r is greater than ϱ, and the lower sign when r is less than ϱ.

But it follows, from § 191, that

$$\frac{1}{f} = \frac{1}{2\pi}\int_0^{2\pi}{}_\eta \frac{1}{f\Pi} = \pm \frac{1}{2\pi}\int_0^{2\pi}{}_\eta \frac{1}{r\Pi_r - \varrho\Pi_\rho}$$

$$=\frac{1}{2\pi}\int_{0}^{2\pi}{}_{\eta}\sum_{0}^{\infty}{}_{n}\frac{\varrho^{n}H_{\rho}^{n}}{r^{n+1}H_{r}^{n+1}}=\frac{1}{2\pi}\int_{0}^{2\pi}{}_{\eta}\sum_{0}^{\infty}{}_{n}\frac{r^{n}H_{r}^{n}}{\varrho^{n+1}H_{\rho}^{n+1}};$$

in which the upper sign corresponds to r, greater than ϱ, and the lower sign to r, less than ϱ, and the series represented by the fourth member corresponds to the former of these two cases, while the series represented by the last member corresponds to the latter case.

194. If, in the development of the preceding series,

Q_n is the coefficient of $\frac{\varrho^n}{r^{n+1}}$, and

Q'_n is that of $\frac{r^n}{\varrho^{n+1}}$;

the series become

$$\frac{1}{f}=\sum_{0}^{\infty}{}_{n}\left(Q_n\frac{\varrho^n}{r^{n+1}}\right)=\sum_{0}^{\infty}{}_{n}\left(Q'_n\frac{r^n}{\varrho^{n+1}}\right).$$

The values of the coefficient in these series are determined by the equations

$$Q_n=\frac{1}{2\pi}\int_{0}^{2\pi}{}_{\eta}\frac{H_{\rho}^{n}}{H_{r}^{n+1}}.$$

$$Q'_n=\frac{1}{2\pi}\int_{0}^{2\pi}{}_{\eta}\frac{H_{r}^{n}}{H_{\rho}^{n+1}}.$$

If the additional notation is adopted, corresponding to (89_7)

$$H_r^n=[r]_n+2\sum_{1}^{\infty}{}_{m}(i^m\,[r]_n^{[m]}\cos m(\eta-\theta_r)),$$

the values of the coefficients become

$$Q_n=[\varrho]_n\,[r]_{-(n+1)}+2\sum_{1}^{\infty}{}_{m}((-1)^m\,[\varrho]_n^{[m]}\,[r]_{-(n+1)}^{[m]}\cos m(\theta_\rho-\theta_r));$$

$$Q'_n=[r]_n\,[\varrho]_{-(n+1)}+2\sum_{1}^{\infty}{}_{m}((-1)^m\,[r]_n^{[m]}\,[\varrho]_{-(n+1)}^{[m]}\cos m(\theta_\rho-\theta_r)).$$

Hence, by (96_{18}) and (93_{16}),

$$Q_n = Q'_n = [r]_n [\varrho]_n + 2\sum_1^n {}_m\left(\frac{\Gamma(n-m)\,\Gamma(n+m)}{(I\,n)^2}\,[r]_n^{[m]}\,[\varrho]_n^{[m]}\cos m(\vartheta_\rho - \vartheta_r)\right)$$
$$= [r]_n [\varrho]_n + 2\sum_1^n {}_m\left(\frac{\Gamma(n-m)\sin^m\varphi_r \sin^m\varphi_\rho}{\Gamma(n+m)} D^m_{\cos\phi_r}[r]_n D^m_{\cos\phi_\rho}[\varrho]_n \cos m(\vartheta_\rho - \vartheta_r)\right).$$

195. The equation (45_8) gives for the value of the potential,

$$\Omega = \int_\sigma \frac{k}{f} = \sum_0^\infty {}_n\left(\frac{1}{r^{n+1}}\int_\sigma (k\,Q_n\,\varrho^n)\right)$$
$$= \sum_0^\infty {}_n\left(r_n \int_\sigma \frac{k\,Q'_n}{\varrho^{n+1}}\right).$$

Hence, by the notation

$$U_n = \int_\sigma (k\,Q_n \varrho^n),$$
$$U'_n = \int_\sigma \frac{k\,Q'_n}{\varrho^{n+1}},$$

the potential becomes

$$\Omega = \sum_0^\infty {}_n \frac{U_n}{r^{n+1}} = \sum_0^\infty {}_n (U'_n r^n).$$

With the notation of ($58_{23\text{-}27}$) and (97_{10}) these values become

$$d\sigma = \varrho^2 d\psi\, d\varrho = \varrho^2 \sin\varphi_\rho\, d\varphi_\rho\, d\vartheta_\rho\, d\varrho,$$
$$U_n = \int_\psi \int_\rho (k\,Q_n \varrho^{n+2}) = \int_{\varphi_\rho} \int_{\theta_\rho} \int_\rho (k\,Q_n\,\varrho^{n+2} \sin\varphi_\rho),$$
$$U'_n = \int_\psi \int_\rho \frac{k\,Q_n}{\varrho^{n-1}} = \int_{\varphi_\rho} \int_{\theta_\rho} \int_\rho \frac{k\,Q_n \sin\varphi_\rho}{\varrho^{n-1}}.$$

The first form of Ω in (99_{17}) is to be used for all values of ϱ less than r, and the second form for all values of ϱ greater than r.

If, then, k is supposed to vanish for all points of space in which there is no attracting mass, the limits of integration for the value of U_n must include all the attracting mass for which ϱ is less than r, while those for U'_n must include all the attracting mass for which ϱ is greater than r.

196. By substituting in ($99_{21\text{-}23}$) the values of Q_n given in (99_2) the resulting values of U_n and U'_n have the same form with Q_n so far as the elements of the direction of r are involved; so that the value of the term of U_n which depends upon the angle $m\theta_r$ has the form

$$[r]_n^{[m]}(A_n^{[m]}\cos m\theta_r + B_n^{[m]}\sin m\theta_r),$$

in which $A_n^{[m]}$ and $B_n^{[m]}$ are independent of the form of the body, and the number of such constants included in the most general value of U_n is $2n+1$.

197. It is expedient to introduce, at this point of the discussion, some important properties of Legendre's functions. The following theorem, given by Poisson, is of especial use in facilitating their investigation.

If N_ρ denotes any function of the elements of direction of ϱ, and if, after the performance of the integration expressed in the second member of the following equation, ϱ is made equal to r, which condition is intended to be denoted by the subsequent parenthesis, the second member will be reduced to the first member, that is,

$$N_r = \frac{1}{4\pi}\int_0^{4\pi}{}_{\psi}\frac{r(r^2-\varrho^2)N_\rho}{f^3}[\varrho = r],$$

To demonstrate this theorem, it is to be observed that all the elements of the integral vanish, except those for which

$$f = 0,$$

that is, for which

$$r = \varrho, \; \overset{r}{\underset{\varrho}{}} = 0.$$

If, then,

η denotes the angle which the plane of $r\varrho$ makes with any assumed fixed plane passing through r, the integral becomes, by (97_{16}),

$$\frac{1}{4\pi}\int_{0\,\psi}^{4\pi}\frac{r(r^2-\varrho^2)N_\rho}{f^3}=\frac{N_r}{4\pi}\int_{0\,\eta}^{2\pi}\int_{0\,\rho}^{\pi}\frac{r(r^2-\varrho^2)\sin\rho}{f^3}$$

$$=\frac{N_r}{4\pi}\int_{0\,\eta}^{2\pi}\int_{0\,\rho}^{\pi}\left(\frac{r^2-\varrho^2}{\varrho f^2}D^r_\rho f\right)=\frac{N_r}{2}\frac{r^2-\varrho^2}{\varrho f}[r=\varrho]$$

$$=\frac{N_r}{2}\frac{r^2-\varrho^2}{\varrho(r-\varrho)}[r=\varrho]=N_r.$$

198. The equation (97_{16}) gives, by means of the first form of (98_{15}),

$$r^2-\varrho^2=-f^2+rfD_rf=-f^3\left(\frac{1}{f}+rD_r\frac{1}{f}\right)$$

$$=f^3\sum_0^\infty{}_n\left((2n+1)\,Q_n\frac{\varrho^n}{r^{n+1}}\right),$$

which substituted in (100_{20}) reduces it to

$$N_r=\sum_0^\infty{}_n\left(\frac{2n+1}{4\pi}\int_{0\,\psi}^{4\pi}(Q_nN_\rho)\right)=\sum_0^\infty{}_nN_r^{[n]},$$

by adopting the notation

$$N_r^{[n]}=\frac{2n+1}{4\pi}\int_{0\,\psi}^{4\pi}(Q_nN_\rho),$$

in which it must be observed that when n is zero it must be retained in the written expression to avoid confusion. It may also be remarked that, from the comparison of the forms of (99_{12}) and (101_{21}), *the most general form of $N_r^{[n]}$ is the same with that given in* § 196 *for U_n.*

199. If the given function is such that, for every value of n' different from n

$$N_r^{[n']}=0,$$

the equations (101_{17-21}) give

$$N_r = N_r^{[n]},$$

$$N_\rho = N_\rho^{[n]},$$

$$\int_0^{4\pi}{}_\psi (Q_n N_\rho^{[n]}) = \frac{4\pi}{2n+1} N_r^{[n]},$$

$$\int_0^{4\pi}{}_\psi (Q_{n'} N_\rho^{[n]}) = 0.$$

The theorems expressed in the last two equations are of fundamental importanee, and were given by Laplace.

200. The theorem (102_7), not being limited to any special direction of r is true for all directions; and, therefore, the most general form may be substituted for $Q_{n'}$ which can be obtained by combining all its special values in any linear function. Any such general form would be the same with that of $N_\rho^{[n']}$, and if it is denoted for distinction by $M_\rho^{[n']}$, *the theorem* (102_7) *assumes the more general form given by* Laplace,

$$\int_0^{4\pi}{}_\psi (M_\rho^{[n']} N_\rho^{[n]}) = 0.$$

201. In considering *the attraction of a spheroid upon an external point, which is so remote that r is greater than any value of ϱ let*

u be the value of ϱ for the surface of the spheroid, and

$$\Sigma_0^\infty{}_m R_{\rho^n}^{[m]} = \int_0^u{}_\rho (k\varrho^{n+2}),$$

the function which is denoted by the second member of this equation being developed in the form of a series of terms of Legendre's functions, by means of (101_{21}). The equation (99_{21}) becomes, by means of (102_5),

$$U_n = \frac{4\pi}{2n+1} R^{[n]}_{r^n};$$

and the potential is

$$\Omega = \sum_0^\infty {}_n \left(\frac{4\pi}{(2n+1) r^{n+1}} R^{[n]}_{r^n} \right).$$

202. If the point is so remote that the squares of the linear dimensions of the body may be neglected in comparison with the square of the distance of the attracted point, it has been shown in § 128 that the attraction is the same as if the body were condensed upon its centre of gravity. In this case, therefore, if the origin is assumed to be the centre of gravity, the potential becomes, as in (56_2),

$$\Omega = \frac{m}{r} = \frac{1}{r}\left(U_0 + \frac{U_1}{r} \right).$$

In all cases, then, in which the origin is the centre of gravity, this equation gives for an external point which is so remote that r is greater than ϱ,

$$U_0 = m,$$
$$U_1 = 0,$$
$$\Omega = \frac{m}{r} + \sum_2^\infty {}_n \left(\frac{4\pi}{(2n+1) r^{n+1}} R^{[n]}_{r^n} \right).$$

203. *A homogeneous ellipsoid can always be found, of which the potential, for any external point, developed in the form* (103_{20}), *will be identical with this expression in its two first terms.* To demonstrate this proposition, and develop the mode of investigating the ellipsoid in a given case, it may be observed that, if the centre of the ellipsoid coincides with the centre of gravity of the given spheroid, and if the mass of the ellipsoid is the same with that of the spheroid, the potential of the ellipsoid, for an external point, has the form (103_{20}) with the same first term. The difficulty of the demonstration and investigation is thus reduced to the consideration of the second

term. The general form of this term is, by (99_4) and (100_6),

$$R_{r^2}^{[2]} = A(\cos^2\varphi_r - \tfrac{1}{3}) + \sin\varphi_r\cos\varphi_r(B\cos\theta_r + B'\sin\theta_r)$$
$$+ \sin^2\varphi_r(C\cos 2\theta_r + C'\sin 2\theta_r)$$
$$= -\tfrac{1}{3}A + A\cos^2{}^r_z + C\left(\cos^2{}^r_x - \cos^2{}^r_y\right)$$
$$+ B\cos{}^r_z\cos{}^r_x + B'\cos{}^r_z\cos{}^r_y + 2C'\cos{}^r_x\cos{}^r_y$$
$$= \Sigma_x\left(C_x(\cos^2{}^r_x - \tfrac{1}{3}) + B_x\cos{}^r_y\cos{}^r_z\right);$$

in which last form the arbitrary constants C_x, C_y, C_z, B_x, B_y, and B_z are introduced, for the sake of symmetry, and in which the six constants are only equal to five, by reason of the equation

$$\Sigma_x\cos^2{}^r_x = 1.$$

In the especial case of Q_2, these constants become, for the axis of x,

$$B'_x = 3\cos{}^\varrho_y\cos{}^\varrho_z,$$
$$C'_x = \tfrac{3}{2}\cos^2{}^\varrho_x;$$

and similarly for the axes of y and z.

The equation (76_6) of the homogeneous ellipsoid, of which the axes are the given rectangular axes, gives, for the surface of this ellipsoid,

$$\frac{1}{u^2} = \Sigma_x\frac{\cos^2{}^\rho_x}{A_x^2}$$

If, therefore, K is the density of the ellipsoid, the equation (102_{27}) becomes, in this case,

$$\sum_0^\infty{}_m R_{\rho^u}^{[m]} = \int_0^u{}_\rho K\varrho^{n+2} = \frac{1}{n+3}Ku^{n+3};$$

and hence, by (103_2),

$$U_2 = \tfrac{1}{5} K \int_{\psi 0}^{4\pi} (u^5 Q_2).$$

To obtain the value of U_2, it must be observed that, by (104_{17}),

$$\int_{\psi 0}^{4\pi} (u^5 B'_x) = 0,$$

because, from the symmetrical form of the ellipsoid, the value of u is not altered by changing either of the angles upon which it depends into its supplement, while the sign of B'_x is reversed.

The remaining terms of the integral contained in (105_2) have the form

$$\int_{\psi 0}^{4\pi} (u^5 C'_z) = \tfrac{3}{2} \int_{\psi 0}^{4\pi} \left(u^5 \cos^2 {\scriptstyle \frac{\varrho}{z}}\right) = \tfrac{3}{2} \int_{\psi 0}^{4\pi} (u^5 \cos^2 \varphi_\rho).$$

But, by the equations

$$\cos {\scriptstyle \frac{\varrho}{x}} = \sin \varphi_\rho \cos \vartheta_\rho,$$

$$\cos {\scriptstyle \frac{\varrho}{y}} = \sin \varphi_\rho \sin \vartheta_\rho,$$

(104_{25}) becomes

$$\begin{aligned} \frac{1}{u^2} &= \frac{\cos^2 \varphi_\rho}{A_z^2} + \frac{\sin^2 \varphi_\rho \cos^2 \theta_\rho}{A_x^2} + \frac{\sin^2 \varphi_\rho \sin^2 \theta_\rho}{A_y^2} \\ &= \frac{\cos^2 \theta_\rho}{A_x^2} + \frac{\sin^2 \theta_\rho}{A_y^2} + \left(\frac{1}{A_z^2} - \frac{\cos^2 \theta_\rho}{A_x^2} - \frac{\sin^2 \theta_\rho}{A_y^2}\right) \cos^2 \varphi_\rho \\ &= a + b \cos^2 \varphi_\rho, \end{aligned}$$

by putting

$$\begin{aligned} a &= \frac{\cos^2 \theta_\rho}{A_x^2} + \frac{\sin^2 \theta_\rho}{A_y^2} = \tfrac{1}{2}\left(\frac{1}{A_x^2} + \frac{1}{A_y^2}\right) + \tfrac{1}{2}\left(\frac{1}{A_x^2} - \frac{1}{A_y^2}\right) \cos 2 \vartheta_\rho \\ &= \tfrac{1}{2}(a' + b' \cos 2 \vartheta_\rho); \end{aligned}$$

$$a' = \frac{1}{A_x^2} + \frac{1}{A_y^2},$$

$$b' = \frac{1}{A_x^2} - \frac{1}{A_y^2},$$

$$b = \frac{1}{A_z^2} - a.$$

The integral (105_{15}) is, therefore,

$$\tfrac{3}{2}\int_{\psi\,0}^{4\pi} (u^5 \cos^2 \varphi_\rho) = \tfrac{3}{2}\int_{\theta_\rho\,0}^{2\pi} \int_{\varphi_\rho\,0}^{\pi} (u^5 \cos^2 \varphi_\rho \sin \varphi_\rho)$$

$$= \tfrac{3}{2}\int_{\theta_\rho\,0}^{2\pi} \int_{\varphi_\rho\,0}^{\pi} \frac{\cos^2 \varphi_\rho \sin \varphi_\rho}{(a + b\cos^2 \varphi_\rho)^{\frac{5}{2}}}.$$

But

$$\int_{\varphi_\rho} \frac{\cos^2 \varphi_\rho \sin \varphi_\rho}{(a + b\cos^2 \varphi_\rho)^{\frac{5}{2}}} = \frac{-\cos^3 \varphi_\rho}{3a(a + b\cos^2 \varphi_\rho)^{\frac{3}{2}}},$$

$$\int_{\varphi_\rho\,0}^{\pi} \frac{\cos^2 \varphi_\rho \sin \varphi_\rho}{(a + b\cos^2 \varphi_\rho)^{\frac{5}{2}}} = \frac{2}{3a(a+b)^{\frac{3}{2}}} = \frac{2A_z^3}{3a};$$

$$\int_{\theta_\rho} \frac{1}{a} = \int_{2\theta_\rho} \frac{1}{2a} = \int_{2\theta_\rho} \frac{1}{a' + b' \cos 2\theta_\rho}$$

$$= \frac{2}{\sqrt{(a'^2 - b'^2)}} \tan^{[-1} \frac{(a' - b') \tan \theta_\rho}{\sqrt{(a'^2 - b'^2)}}$$

$$= A_x A_y \tan^{[-1} \left(\frac{A_x}{A_y} \tan \theta_\rho \right),$$

$$\int_{\theta_\rho\,0}^{2\pi} \frac{1}{a} = 2\pi A_x A_y.$$

These values, with that of the mass of the ellipsoid, reduce (106_8) and (105_2) to

$$\tfrac{3}{2}\int_{\psi\,0}^{4\pi} (u^5 \cos^2 \varphi_\rho) = 2\pi A_x A_y A_z^3,$$

$$U_2 = \tfrac{2}{5}\pi K A_x A_y A_z \Sigma_x \left(A_x^2 \cos^2 {}^r_x - \tfrac{1}{3} A_x^2 \right)$$

$$= \tfrac{3}{20} m \Sigma_x \left(A_x^2 \cos^2{}^{r}_{x} - \tfrac{1}{3} A_x^2\right).$$

If the axes of the ellipsoid are not those of x, y, z, but of x', y', z', this expression, by means of the equation,

$$\cos{}^{r}_{x'} = \cos{}^{r}_{x} \cos{}^{x}_{x'} + \cos{}^{r}_{y} \cos{}^{y}_{x'} + \cos{}^{r}_{z} \cos{}^{z}_{x'},$$

becomes

$$U_2 = \tfrac{3}{20} m \Sigma_x \left[C''_x \left(\cos^2{}^{r}_{x} - \tfrac{1}{3}\right) + B''_x \cos{}^{r}_{y} \cos{}^{r}_{z} \right];$$

in which

$$C''_x = \Sigma_{x'} \left(A_x^2 \cos^2{}^{x}_{x'}\right),$$

$$B''_x = 2 \Sigma_{x'} \left(A_{x'}^2 \cos{}^{y}_{x'} \cos{}^{z}_{x'}\right).$$

This value becomes identical, therefore, with that of (104_8), if

$$C''_x = \frac{20}{3m} C_x,$$

$$B''_x = \frac{10}{3m} B_x.$$

204. If the potential and its component functions for the ellipsoid are denoted by the letter e written beneath them, the *potential of the spheroid for an external point, for which r is greater than ϱ, becomes*

$$\Omega = \underset{e}{\Omega} + \sum_3^\infty{}_n \left[\left(U_n - \underset{e}{U_n}\right) r^{-(n+1)}\right].$$

205. A transformation of coördinates, which is the reverse of that by which the equations (107_{17}) were obtained from the reference of the ellipsoid to the axes of the spheroid, would bring the equation (104_{19}) to the form (107_{16}). From the forms of the expression it is obvious that this transformation is identical with that by which the general equation of the second degree in space is referred to the

axes of the surface. Hence, if $S_{x'}$, $S_{y'}$, and $S_{z'}$, are the three roots of the equation

$$(C''_x - S)(C''_y - S)(C''_z - S) + 2B''_x B''_y B''_z - \Sigma_x [B''^2_x (C''_x - S)] = 0,$$

they are the squares of the semiaxes of the ellipsoid. But it must be observed that the mass of the ellipsoid, being the same with that of the spheroid, gives the equation

$$S_{x'} S_{y'} S_{z'} = \left(\frac{3m}{4\pi K}\right)^2.$$

The condition (104_{14}), however, shows that the values of C_x, C_y, and C_z, in (104_8), may be increased or decreased by the same quantity, without changing the value of (104_8). The values of C''_x, C''_y, and C''_z may, in like manner, be increased or decreased by the same quantity, which change will produce the opposite effect upon the roots of (108_3), until, at length, they may satisfy the equation (108_9). This common increase or decrease of all the roots of (108_3) corresponds to the performance of the same operation upon the squares of the semiaxes of the ellipsoid, that is, to a change of the ellipsoid, given by (108_3) into another ellipsoid, which has the same foci and the required mass. The change of mass is, however, more simply accomplished by an increase or decrease of the density of the ellipsoid; and, in this view of the case, it is requisite that the value of the density be determined by equation (108_9).

206. If the point is without the spheroid, but near its surface, it is generally necessary to combine the forms of the potential given in (99_{16}). Thus, with the notation

$\int'_\psi =$ the integral for all directions of u greater than r,

$\int''_\psi =$ the integral for all directions of u less than r,

whence

$$\int_0^{4\pi}{}_\psi = \int'_\psi + \int''_\psi,$$

the value of the potential may be expressed in the form

$$\Omega = \sum_0^\infty {}_n \left(\frac{U_n}{r^{n+1}} + U'_n r^n \right),$$

in which

$$U_n = \left(\int'_\psi \int_0^r {}_\rho + \int''_\psi \int_0^u {}_\rho \right) (k\, Q_n \varrho^{n+2}),$$

$$U'_n = \int'_\psi \int_r^u {}_\rho \frac{k\, Q_n}{\varrho^{n-1}}.$$

But it may be observed that, by (109_2),

$$U_n = \left(\int_0^{4\pi}{}_\psi \int_0^u {}_\rho - \int'_\psi \int_0^u {}_\rho + \int'_\psi \int_0^r {}_\rho \right) (k\, Q_n \varrho^{n+2})$$

$$= \left(\int_0^{4\pi}{}_\psi \int_0^u {}_\rho - \int'_\psi \int_r^u {}_\rho \right) (k\, Q_n \varrho^{n+2}).$$

whence, by putting

$$V_{\rho^n} = \frac{1}{r^{n+1}} \int_r^u {}_\rho\, k \varrho^{n+2} - r^n \int_r^u {}_\rho \frac{k}{\varrho^{n-1}},$$

and using U_n in the signification of (99_{12}), the potential assumes the form

$$\Omega = \sum_0^\infty {}_n \left(\frac{U_n}{r^{n+1}} - \int'_\psi (V_{\rho^n} Q_n) \right).$$

207. A similar investigation may be extended to the ellipsoid of § 204, and if

Ω' is the value of Ω of (107_{24})

the value of the potential for a point which is near the surface of the spheroid may assume the form

$$\Omega = \Omega' - \sum_0^\infty{}_n \int_\psi' ((V_{\rho^n} - {}_e V_{\rho^n}) Q_n) .$$

208. *If the form of the spheroid differs but little from an ellipsoid which has the same foci with the preceding ellipsoid, and if it has a constant density for all that portion for which* ϱ *is greater than* r, *a combination of two homogeneous ellipsoids may be substituted for the single ellipsoid, both of which have the same foci, while one coincides very nearly with the spheroid in form and density throughout the portion exterior to* r *; and the other, being much smaller, has the requisite positive or negative density to give the algebraic sum of the masses of the two ellipsoids equal to that of the spheroid.* The combination of the two ellipsoids upon any external point is the same with that of the single ellipsoid, and the larger of the two may be substituted for it in the values of V in (110_3).

If, in determining the values of V for the spheroid or the ellipsoid from (109_{22}), u is supposed, for every direction in which the solid is contained within the sphere, of which radius is r, not to refer to the surface of the solid, but to coincide with r, the value of V vanishes for any such direction, and it becomes a continuous function, of which the derivatives are discontinuous. The equation (101_{22}) is applicable to such a function, for the argument by which it was established was independent of this condition. With this modification, therefore, the accent may be omitted in the integral sign of (109_{27}) or (110_3), and the limits of integration extended to every possible direction, and the result may be simplified by means of (101_{22}).

In the present case, in which k is constant, equation (109_{22}) becomes

$$V_{\rho^n} = \frac{k}{n+3}\left(\frac{u^{n+3} - r^{n+3}}{r^{n+1}}\right) + \frac{k}{n-2}\left[\frac{r^n}{u^{n-2}} - r^2\right],$$

whence

$$V_{\rho^n} - \underset{e}{V}_{\rho^n} = \frac{k}{(n+3)\,r^{n+1}}(u^{n+3} - \underset{e}{u}^{n+3}) + \frac{k\,r^n}{n-2}(u^{2-n} - \underset{e}{u}^{2-n}).$$

If, then, it is assumed that

$$z_\rho = u - r,$$

$$z'_\rho = \underset{e}{u} - r,$$

the binomial theorem gives

$$\frac{u^{n+3} - r^{n+3}}{(n+3)\,r^{n+1}} = \sum_1^\infty{}_n \left(\frac{\Gamma(n+2)}{\Gamma(n+3-m)\,\Gamma m} r^{-(m-2)} z_\rho^m \right),$$

and if n is changed into $-(n+1)$

$$\frac{r^n}{n-2}(u^{2-n} - r^{2-n}) = -\sum_1^\infty{}_m \left(\frac{\Gamma(n-3+m)}{\Gamma(n-2)\,\Gamma m} (-r)^{-(m-1)} z_\rho^m \right).$$

These values, substituted in (111_2), give

$$\begin{aligned} V_{\rho^n} - \underset{e}{V}_{\rho^n} &= \sum_1^\infty{}_m \left[\frac{k}{\Gamma m\, r^{m-1}} \left(\frac{\Gamma(n+2)}{\Gamma(n+3-m)} + \frac{(-1)^m \Gamma(n-3+m)}{\Gamma(n-2)} \right) (z_\rho^m - z_\rho'^m) \right] \\ &= (2n+1)\,k \left[\tfrac{1}{2}(z_\rho^2 - z_\rho'^2) + \frac{1}{3r}(z_\rho^3 - z_\rho'^3) + \frac{n(n+1)}{72\,r^2}(z_\rho^4 - z_\rho'^4) \right] \\ &\quad + \sum_6^\infty{}_m \left[\frac{k}{\Gamma m\, r^{m-1}} \left(\frac{\Gamma(n+2)}{\Gamma(n+3-m)} + \frac{(-1)^m \Gamma(n-3+m)}{\Gamma(n-2)} \right) (z_\rho^m - z_\rho'^m) \right]. \end{aligned}$$

This value may be substituted in (110_3), and the result reduced by means of (101_{22}).

209. *If the spheroid is not very different from a sphere, and if the difference in form between it and the larger of the two combined ellipsoids is so small that, in consideration of the large divisors, the terms of* (111_{20}) *may be neglected, in which* m *is greater than* 3, (111_{20}) *is reduced to*

$$V_{\rho^n} - \underset{e}{V}_{\rho^n} = (2n+1)\,W_\rho,$$

if

$$W_\rho = \tfrac{1}{2}\,k(z_\rho^2 - z_\rho'^2) + \frac{k}{3r}(z_\rho^3 - z_\rho'^3);$$

and, by (110_{20}),

$$W_r = 0.$$

But the value of the potential, derived from (110_3), becomes in this case by (111_{28}) and (101_{22}),

$$\begin{aligned} \Omega &= \Omega' - \Sigma_0^{\infty}{}_n \left((2n+1) \int_{\psi 0}^{4\pi} (W_\rho\, Q_n)\right) \\ &= \Omega' - 4\pi \Sigma_0^{\infty}{}_n W_r^{[n]} = \Omega' - W_r \\ &= \Omega'; \end{aligned}$$

so that, *in this case, the form* (107_{24}) *is applicable to every external point.* This conclusion, and the mode of investigation, includes Poisson's analysis of the spheroid, which differs but little from a sphere by which it was suggested.

210. The formula (107_{24}) gives, for the attraction in the direction of the radius vector, the expression

$$D_r \Omega = D_r \underset{e}{\Omega} + \Sigma_3^{\infty}{}_n \left[(n+1)(\underset{e}{U_n} - U_n)\, r^{-(n+2)}\right].$$

Hence, the equation is obtained

$$D_r \Omega + \frac{1}{2r}\Omega = D_r \underset{e}{\Omega} + \frac{1}{2r}\underset{e}{\Omega} + \tfrac{1}{2}\Sigma_3^{\infty}{}_n \left[(2n+1)(\underset{e}{U_n} - U_n)\, r^{-(n+2)}\right],$$

which, by (103_2), is reduced to

$$D_r \Omega + \frac{1}{2r}\Omega = D_r \underset{e}{\Omega} + \frac{1}{2r}\underset{e}{\Omega} + 2\pi\, \Sigma_3^{\infty}{}_n \left[(\underset{e}{R}{}_{r^n}^{[n]} - R_{r^n}^{[n]})\, r^{-(n+2)}\right];$$

or

$$D_r \left[\sqrt{r}(\Omega - \underset{e}{\Omega})\right] = 2\pi \sqrt{r}\, \Sigma_3^{\infty}{}_n \left[(\underset{e}{R}{}_{r^n}^{[n]} - R_{r^n}^{[n]})\, r^{-(n+2)}\right].$$

211. *If the spheroid is homogeneous,* having the same density with the ellipsoid, the equation (104_{31}) gives

$$R_{\rho^n} = \frac{K}{n+3} u_\rho^{n+3},$$

$$\underset{e}{R}_{\rho^n} = \frac{K}{n+3} \underset{e}{u}_\rho^{n+3}.$$

By assuming, then, the values

$$u_r z_\rho = u_\rho, \qquad u_r \underset{e}{z}_\rho = \underset{e}{u}_\rho,$$

$$V_{\rho^n} = \frac{1}{n+3}\left(z_\rho^{n+3} - \underset{e}{z}_\rho^{n+3}\right), \qquad V_{r^n} = \frac{1}{n+3}\left(1 - \underset{e}{z}_r^{n+3}\right),$$

$$\Omega' = \Omega - \underset{e}{\Omega};$$

the equation (112_{28}) becomes

$$D_r \Omega' + \frac{1}{2r}\Omega' = -2\pi K u_r \sum_3^\infty {}_n \left[V_{r^n}^{[n]} \left(\frac{u_r}{r}\right)^{n+2} \right].$$

212. *If the attracted point is upon the surface of the spheroid,* the preceding equation becomes, if Ω_0 is the potential at the surface of the spheroid,

$$D_r \Omega_0' + \frac{1}{2r}\Omega_0' = -\, 2\pi K r \sum_3^\infty {}_n V_{r^n}^{[n]}.$$

213. *If the spheroid differs so little from the ellipsoid that the square of the distance between the surfaces of these two solids may be neglected,* the notation

$$y = z - \underset{e}{z},$$

gives

$$V_{\rho^n} = z_\rho^{n+2} y_\rho = \underset{e}{z}_\rho^{n+2} y_\rho.$$

214. If, moreover, the ellipsoid differs so little from a concentric sphere, that the product of the difference between the radius vector of the ellipsoid and the radius of sphere, multiplied by the distance between the surfaces of the ellipsoid and the spheroid, may be neglected, the preceding equation is reduced to

$$V_{\rho^n} = y_\rho;$$

and (113_{17}) becomes

$$D_r \Omega'_0 + \frac{1}{2r} \Omega'_0 = -2\pi K r \sum_0^\infty{}_n y_r^{[n]}.$$

In this last form, the sum of the terms in the second member is extended to include the whole series, because the first terms which vanish in the exact formula, may become sensible in the approximate form. But,

$$y_r = \sum_0^\infty{}_n y_r^{[n]};$$

and, therefore, if R is the radius of the sphere,

$$D_r \Omega'_0 + \frac{1}{2R} \Omega'_0 = -2\pi K R y_r.$$

215. If, again,

$\underset{e}{\Omega_0}$ = the potential of the ellipsoid at its surface,

$\underset{e}{\Omega'_0}$ = the potential of the ellipsoid at the surface of the spheroid,

$$\underset{e}{\Omega''_0} = \underset{e}{\Omega'_0} - \underset{e}{\Omega_0};$$

the general equations

$$\underset{e}{\Omega} = 4\pi \sum_0^\infty{}_n ((2n+1)^{-1} \underset{e}{R}{}_{rn}^{[n]} r^{-(n+1)}),$$

$$D_r \underset{e}{\Omega} = -4\pi \sum_0^\infty{}_n \left(\frac{n+1}{2n+1} \underset{e}{R}{}_{rn}^{[n]} r^{-(n+2)}\right),$$

give

$$\underset{e}{\Omega''_0} = -4\pi \sum_0^\infty{}_n \left(\frac{n+1}{2n+1} \underset{e}{R}{}_{rn}^{[n]} r^{-(n+1)} y_r\right),$$

$$D_r \underset{e}{\Omega''_0} = 4\pi \sum_0^\infty{}_n \left(\frac{(n+1)(n+2)}{2n+1} \underset{e}{R}{}_{rn}^{[n]} r^{-(n+2)} y_r\right).$$

Since the second members of these equations are multiplied by

y_r, the values of the other factors may be reduced to those which belong to the sphere. Hence, R_{ρ^n} becomes a constant quantity, and, therefore,

$$R^{[n]}_{r^n} = 0,$$

for all values of n except zero, in which case,

$$\underset{e}{R}_{\rho^0} = \frac{K}{3} R^3 = \underset{e}{R}{}^{[0]}_{r^0};$$

and the above values become

$$\underset{e}{\Omega}{}''_0 = -\tfrac{4}{3}\pi K R^2 y_r,$$

$$D_r \underset{e}{\Omega}{}''_0 = \tfrac{8}{3}\pi K R y_r$$

which give

$$D_r \underset{e}{\Omega}{}''_0 + \frac{1}{2R} \underset{e}{\Omega}{}''_0 = 2\pi K R y_r.$$

The sum of this equation and (114_{12}) is, by (113_9) and (114_{20}),

$$D_r \Omega_0 + \frac{1}{2R}\Omega_0 = D_r \underset{e}{\Omega}_0 + \frac{1}{2R}\underset{e}{\Omega}_0.$$

216. If *the ellipsoid is itself the sphere*, the equation (58_8) gives

$$\underset{e}{\Omega}_0 = \tfrac{4}{3}\pi K R^2$$

$$D_r \underset{e}{\Omega}_0 = -\tfrac{4}{3}\pi K R,$$

$$D_r \underset{e}{\Omega}_0 + \frac{1}{2R}\underset{e}{\Omega}_0 = -\tfrac{2}{3}\pi K R;$$

which, substituted in (115_{17}), gives

$$D_r \Omega_0 + \frac{1}{2R}\Omega_0 = -\tfrac{2}{3}\pi K R.$$

This is the equation given by Laplace for a spheroid which differs but little from a sphere, and is the fundamental theorem of his investigations upon this subject.

217. *If the attracted point is within the spheroid, and at such a distance from the surface that* r *is less than the value of* u, the formula for the potential is, by § 195.

$$\Omega = \Sigma_0^\infty {}_n \left(\frac{U_n}{r^{n+1}} + U'_n r^n \right),$$

in which

$$U_n = \int_0^{4\pi}{}_\psi \int_0^r {}_\rho \, (k\, Q_n \varrho^{n+2}),$$

$$U'_n = \int_0^{4\pi}{}_\psi \int_r^u {}_\rho \frac{k\, Q_n}{\varrho^{n-1}}.$$

It may also be shown by the method of §§ 208–209, that *this same formula is applicable, if the point is quite near the surface, and if the spheroid differs so little from a sphere that the square of the difference may be neglected.*

218. The important discussions in regard to the convergency of the series, derived from Legendre's functions, are deferred, on account of their great length, to the volumes which will be devoted to the application of the Analytic Mechanics.

IV.

ELASTICITY.

219. The laws by which the elementary forces of *cohesion* and *affinity* vary with the mutual distance and direction of the particles and atoms are undetermined; and, therefore, the delicate inquiries involved in the constitution and crystallization of bodies are not yet subject to the control of geometry. But it is sufficiently apparent that these forces are *insensible at sensible distances*, and that there are peculiar laws of mechanical action corresponding to the three states

of *gasses, liquids,* and *solids.* The peculiarity of these states consists, principally, in the facility with which the particles can be moved relatively to each other, and in the phenomena which arise from such motion, but especially in those of the disruption of solid bodies. As long, however, as the relative positions of the particles are so little disturbed that they return to their initial state when the disturbing cause is removed, the precise law of molecular action is not required for the investigation of the small changes which the constitution of the body undergoes, and which are treated as phenomena of *elasticity.*

220. To analyze the changes of form of a system of material points which constitute a body, let

u be the distance by which a point, of which the coördinates are x, y, and z, is moved from its initial position,

Δ the increment of a function for another point of the body which is near the former point,

p the distance of the second point from the former point;

the notation of (42_{12}) gives

$$p_x = \Delta x = p \cos_x^p,$$

$$\Delta p_x = p_x D_x u_x + p_y D_y u_x + p_z D_z u_x.$$

Hence, if

$$p' = p + \Delta p,$$

$$\varepsilon = \frac{\Delta p}{p},$$

ε is the *linear expansion* of the body in the direction of p; and its value is given by the equation

$$(1+\varepsilon)^2 = \left(\frac{p'}{p}\right)^2 = \Sigma_x \left(\frac{p'_x}{p}\right)^2$$

$$= \Sigma_x \left(\frac{p_x + \Delta p_x}{p}\right)^2$$

$$= \Sigma_x \left[(1 + D_x u_x) \cos\frac{p}{x} + D_y u_x \cos\frac{p}{y} + D_z u_x \cos\frac{p}{z}\right]^2.$$

If, then, the reciprocal of $1 + \varepsilon$ *is laid off from the origin upon a line drawn parallel to p, its extremity will be upon the ellipsoid, of which the equation is*

$$1 = \Sigma_x [(1 + D_x u_x) \mathrm{x} + D_y u_x \mathrm{y} + D_z u_x \mathrm{z}]^2.$$

221. The expansions or contractions which correspond to the axes of this ellipsoid may be called *the principal expansions and contractions,* and one of these is *a maximum,* another is *a minimum,* and the third is *a maximum in some directions and a minimum in others.*

If the ellipsoid is referred to its axes, the expression for the expansion is, if ε_x, ε_y, and ε_z are the values of ε for the axes,

$$(1 + \varepsilon)^2 = \Sigma_x \left[(1 + \varepsilon_x) \cos\frac{p}{x}\right]^2.$$

so that for these directions the values of u_x, u_y, u_z, are such that

$$(1 + D_x u_x) D_y u_x + D_x u_y (1 + D_y u_y) + D_x u_z D_y u_z = 0.$$
$$(1 + \varepsilon_x)^2 = (1 + D_x u_x)^2 + (D_x u_y)^2 + (D_x u_z)^2.$$

222. The notation

$$\varphi = \frac{p'}{p},$$

gives

$$\cos\varphi = \Sigma_x \left(\cos\frac{p}{x} \cos\frac{p'}{x}\right);$$

$$\sin^2\varphi = 1 - \cos^2\varphi$$

$$= \Sigma_x \cos^2\frac{p}{x} . \Sigma_x \cos^2\frac{p'}{x} - \left[\Sigma_x \left(\cos\frac{p}{x} \cos\frac{p'}{x}\right)\right]^2$$

$$= \Sigma_x \left(\cos\frac{p}{y} \cos\frac{p'}{z} - \cos\frac{p}{z} \cos\frac{p'}{y}\right)^2;$$

$$(1+\varepsilon)^2 \sin^2\varphi = \frac{1}{p^2}\Sigma_x\left(p' \cos\tfrac{p}{y} \cos\tfrac{p'}{z} - p' \cos\tfrac{p}{z} \cos\tfrac{p'}{y}\right)^2$$

$$= \frac{1}{p^2}\Sigma_x\left(p'_z \cos\tfrac{p}{y} - p'_y \cos\tfrac{p}{z}\right)^2$$

$$= \Sigma_x\left[\left(\cos\tfrac{p}{x} D_x + \cos\tfrac{p}{y} D_y + \cos\tfrac{p}{z} D_z\right)\left(u_z \cos\tfrac{p}{y} - u_y \cos\tfrac{p}{z}\right)\right]^2$$

$$= \Sigma_x\left[\Sigma_x\left(\cos\tfrac{p}{x} D_x\right)\left(u_z \cos\tfrac{p}{y} - u_y \cos\tfrac{p}{z}\right)\right]^2,$$

in which the derivatives are only applicable to u_x, u_y, and u_z. Hence, *if the reciprocal of the square root of* $(1+\varepsilon)\sin\varphi$ *is laid off from the origin, upon a line drawn parallel to* p, *its extremity is upon the surface of the fourth degree, of which the equation is*

$$1 = \Sigma_x[\Sigma_x(\mathrm{x}D_x)(\mathrm{y}u_z - \mathrm{z}u_y)]^2.$$

223. When the axes are those of the ellipsoid of § 221, and the disturbance is such that for each axis the equations (118_{19}) and (118_{20}) become

$$D_y u_x = 0,$$

$$\varepsilon_x = D_x u_x,$$

$$(1+\varepsilon)^2 \sin^2\varphi = \Sigma_x\left[\cos\tfrac{p}{y} \cos\tfrac{p}{z} (D_z u_z - D_y u_y)\right]^2$$

$$= \Sigma_x\left[\cos\tfrac{p}{y} \cos\tfrac{p}{z} (\varepsilon_z - \varepsilon_y)\right]^2.$$

224. To determine the rotative effects of the disturbance about the axes, let

$$\psi = \tfrac{p}{y},$$

and

$\varphi_q =$ the projection of the angle φ upon a plane perpendicular to the direction of q.

Hence

$$\varphi_x = \psi'_x - \psi_x,$$

$$\tan\psi_x = \frac{p_z}{p_y} = \frac{\cos{}_z^p}{\cos{}_y^p},$$

$$\tan(\varphi_x + \psi_x) = \frac{p_z'}{p_y'} = \frac{D_x u_z \cos{}_x^p + D_y u_z \cos{}_y^p + (1 + D_z u_z)\cos{}_z^p}{D_x u_y \cos{}_x^p + (1 + D_y u_y)\cos{}_y^p + D_z u_y \cos{}_z^p}.$$

225. If the axis of x is perpendicular to p, the equations are

$${}_y^p = \tfrac{1}{2} p - {}_z^p$$

$$\psi_x = \psi,$$

$$\tan(\varphi_x + \psi) = \frac{D_y u_z + (1 + D_z u_z)\tan\psi}{1 + D_y u_y + D_z u_y \tan\psi}.$$

226. If the axes and conditions are those of § 223, the equation (120_3) becomes

$$\tan(\varphi_x + \psi_x) = \frac{1 + D_z u_z}{1 + D_y u_y}\tan\psi_x.$$

227. The whole expansion or contraction of the body at any time, is derived from the consideration that, by the definition of ε in § 220, any very minute portion of the body which is originally a sphere, becomes, in the disturbed state, an ellipsoid similar to that of § 221. If, then,

$$\theta = \text{the expansion of the body};$$

the sphere of which the radius is i, becomes an ellipsoid, of which the axes are $i(1 + \varepsilon_x)$, $i(1 + \varepsilon_y)$, $i(1 + \varepsilon_z)$, and, therefore, its volume becomes

$$\tfrac{4}{3}\pi i^3(1 + \theta) = \tfrac{4}{3}\pi i^3(1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z),$$

and

$$1 + \theta = (1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z).$$

228. When the disturbance is so small that the squares of the expansions may be neglected, which is the ordinary case of elas-

ticity, the equation (119_6) becomes

$$\varepsilon = \Sigma_x \left[\cos^2 {}^p_x D_x u_x + \cos^p_y \cos^p_z (D_y u_z + D_z u_y)\right]$$
$$= \Sigma_x \left(\cos^p_x D_x\right) \Sigma_x \left(\cos^p_x u_x\right).$$

229. *If, then, the reciprocal of the square root of the linear expansion in any direction is laid off from the origin upon that direction, as the radius vector of a surface, the resulting surface is a surface of the second degree, of which the equation is*

$$1 = \Sigma_x [\mathrm{x}^2 D_x u_x + \mathrm{y}\mathrm{z}(D_y u_z + D_z u_y)],$$

or

$$1 = \Sigma_x (\mathrm{x} D_x) \Sigma_x (\mathrm{x} u_x).$$

230. If the axes are those of the principal expansions and contractions, the formula for expansion becomes

$$\varepsilon = \Sigma_x \left(\cos^2 {}^p_x \varepsilon_x\right);$$

and the equations of § 221 become

$$D_x u_y + D_y u_x = 0,$$
$$\varepsilon_x = D_x u_x.$$

231. If the principal expansions and contractions are all of the same name, that is, if all are expansions, or if all are contractions, the surface of § 229 is an ellipsoid. But, in other cases, in which neither of the principal expansions is zero, the surface is the combination of two hyperboloids, of which one is one-parted, and the other is bi-parted. Both these hyperboloids have the same axes and the same asymptotic conical surface; and the asymptotic conical surface, corresponding to the directions, in which there is neither expansion nor contraction, divides the directions in which the solid is expanded from those in which it is contracted.

If one of the principal expansions is zero, the surface is reduced to a cylinder; and if two of the principal expansions are zero, the surface is reduced to two parallel planes.

232. In the present case, the formula of § 227, for the expansion of the solid, is reduced to

$$\theta = \varepsilon_x + \varepsilon_y + \varepsilon_z = \Sigma_x \varepsilon_x.$$

233. The formula (120_9) for the rotation about the axis of x becomes,

$$\tan(\varphi_x + \psi) = \tan\psi + \frac{\varphi_x}{\cos^2\psi}$$
$$= (1 + D_z u_z - D_y u_y)\tan\psi - D_z u_y \tan^2\psi + D_y u_z,$$
$$\varphi_x = \tfrac{1}{2}(D_y u_z - D_z u_y) + \tfrac{1}{2}(D_y u_z + D_z u_y)\cos 2\psi + \tfrac{1}{2}(D_z u_z - D_y u_y)\sin 2\psi$$
$$= \Pi_x + \tau_x \cos 2(\psi - \eta_x);$$

in which

$$\Pi_x = \tfrac{1}{2}(D_y u_z - D_z u_y),$$
$$\tau_x \cos 2\eta_x = \tfrac{1}{2}(D_y u_z + D_z u_y),$$
$$\tau_x \sin 2\eta_x = \tfrac{1}{2}(D_z u_z - D_y u_y).$$

234. The maximum rotation about x corresponds, then, to

$$\psi = \eta_x$$

and is

$$\varphi_x = \Pi_x + \tau_x;$$

and the minimum rotation corresponds to

$$\psi = \eta_x + \pi,$$

is

$$\varphi_x = \Pi_x - \tau_x;$$

and Π_x is the mean rotation. When the maximum and minimum rotations have opposite signs, there are two intermediate rotations which vanish, corresponding to

$$\cos 2(\psi - \eta_x) = - \frac{\Pi_x}{\tau_x}.$$

235. There are similar formulæ for rotations above the axes of y and z, and the combinations of the mean rotations give a greatest mean rotation, represented by

$$\Pi^2 = \Sigma_x \Pi_x^2;$$

the direction of which is determined by the equations represented by

$$\cos \frac{\Pi}{x} = \frac{\Pi_x}{\Pi}.$$

236. If the axes are those of § 230, the equations of § 233 become

$$\Pi_x = D_y u_z = - D_z u_y,$$
$$\varphi_x = \Pi_x + \tfrac{1}{2}(D_z u_z - D_y u_y) \sin 2\psi.$$

237. When the disturbance is such that, for each of the principal axes, there is the equation

$$D_y u_z = 0,$$

the equations of the preceding section become

$$\Pi_x = \Pi = 0,$$
$$\varphi_x = \tfrac{1}{2}(D_z u_z - D_y u_y) \sin 2\psi;$$

so that, in this case, *there is compression without any mean rotation.*

238. When the disturbance is such that for each of the principal axes

$$D_x u_x = 0;$$

the equations for compression and rotation becomes

$$\varepsilon = \theta = 0,$$
$$\varphi_x = \Pi_x = D_y u_z,$$
$$\Pi^2 = \Sigma_x (D_y u_z)^2;$$

so that, in this case, *there is rotation without compression.*

All the preceding investigations upon the internal changes produced by the disturbance of the form of a body are taken from Cauchy.

239. The elastic force which is developed by any small disturbance of the internal condition of a body is proportional to the amount of disturbance, and has, therefore, the same general form with that of the disturbance itself. But the special discussion of the relative values of the coefficients involves the consideration of the laws of equilibrium, and must be reserved to a subsequent chapter.

V.

MODIFYING FORCES.

240. Among the forces of nature, those which produce the equations of condition deserve peculiar consideration. Being merely conditional, they do not augment or decrease the power of a system, but merely modify its direction and distribution. They may, therefore, be called *modifying forces ;* and may be divided into two classes of *stationary* and *moving.*

241. *Stationary modifying forces* are perpendicular to fixed surfaces or lines, and constitute the action by which certain material points of a system are restrained to move upon those surfaces or lines. A force of this nature, being perpendicular to the motion of

its point of application, does not increase or diminish the total power of the system, but modifies its elements of direction.

Thus the equation of condition,

$$L = 0,$$

between the coördinates of a point, involves the idea of a force, acting in the direction N of a normal to the surface represented by this equation. When it is combined with its multiplier, it is equivalent, by (27_{16}), (27_5), and (54_{31}), to a modifying force, of which the magnitude is

$$\lambda \sqrt{\Box L}.$$

242. This force may be decomposed into three forces, which are parallel to three rectangular axes, either of which is represented by

$$\lambda \sqrt{(\Box L)} \cos_x^N,$$

while the point of application moves through the elementary arc ds, its advance in the direction of the axis of x is

$$ds \cos_x^s.$$

The amount of power added to the system, by the component force in the direction of the axis of x, is

$$\lambda ds \sqrt{(\Box L)} \cos_x^N \cos_x^s,$$

and there is a consequent increase or diminution of force in this direction. But the mutual perpendicularity of N and s is expressed by the equation

$$\Sigma_x (\cos_x^N \cos_x^s) = 0.$$

The whole augmentation of power arising from the three components is, therefore,

$$\lambda ds \sqrt{(\Box L)} \Sigma_x (\cos_x^N \cos_x^s) = 0,$$

which agrees with the fundamental conception of a stationary modifying force, and illustrates its mode of action.

243. *Moving modifying forces* are perpendicular to moving surfaces, which surfaces are themselves portions of the moving system, and the points of application are restrained to move upon these surfaces. In this case, the motion of each point of application may be decomposed into two parts, of which one part is perpendicular, and the other is parallel to the moving surfaces. The modifying force has the same relation to the motion which is perpendicular to it, which has been already discussed in reference to the stationary surface; put by its relation to the other component of the motion, it communicates power to the point of application, or the reverse. But the power which is thus communicated to the point is abstracted from the surface, and through it from the other portions of the system; and, therefore, the whole amount of power of the system is neither increased or decreased. Although for the purposes of theoretical speculation, it is convenient to regard the surface and the point of application as parts of one system, it is often the case in the useful arts that this transfer of power is of the highest practical importance, and is the basis of the theory of the turbine wheel.

In a rigid system of bodies, these forces constitute *the bonds of union.*

CHAPTER VI.

EQUILIBRIUM OF TRANSLATION.

244. *The conditions to which any combination of forces must be subject, in order they may not tend to produce translation in the system of material points to which they are applied,* are readily investigated. It follows immediately from §§ 18 and 20, and with the notation of those sections, that the algebraic condition that the system has no tendency to move in the direction of p is

$$\Sigma_1' m_1 F_1 \cos_f^p = 0.$$

But each term

$$m_1 F_1 \cos_{f_1}^p,$$

is the projection of the force $m_1 F_1$ upon the direction of p, and, therefore, *if the algebraic sum of the projections of all the forces upon any direction vanishes, there is no tendency to translation in that direction.*

245. It also follows from the combination of translations, given in § 23, that *if there is no tendency to translation in two different directions, which are not parallel, there is no tendency to translation in the plane of these two directions; and if there is no tendency to translation in three directions, which are not in the same place, there is no tendency to translation in any direction.*

By means of rectangular axes the algebraic conditions, which are necessary and sufficient to produce equilibrium in respect to translation, are combined in the formula

$$\Sigma_x [\Sigma_1' (m_1 F_1 \cos_f^x)]^2 = 0.$$

This formula is independent of the situation of the points of the system, except so far as the elements of position are implicitly contained in the expressions of the forces and their directions; it would remain unchanged, therefore, if all the points were condensed into one, without any variation of the magnitude and direction of the forces. The conditions of equilibrium are, then, the same as if all the forces were applied at a single point.

246. If one of the points of the system were subject to the condition of being confined to a fixed surface or line, the conditions of equilibrium of translation would simply be reduced to the condition that *the resultant of all the other forces would be perpendicular to this surface or line, and the modifying force by which the point was restrained would be equal and opposite to this resultant.*

If a point of the system was absolutely fixed, or if three different points were restrained to move upon three fixed surfaces, *there would, in general, be no possibility of translation, but the resultant of all the forces applied to the system would be equal and opposite to that of the modifying forces by which the points were confined.*

247. The theory of the equilibrium of a point is wholly included in that of its translation. But since every system is a mere combination of points, the complete theory of equilibrium can easily be evolved from that of translation. This mode, however, of arriving at the conditions of equilibrium is neither luminous nor instructive.

248. The conditions of the equilibrium of translation of a system, which is free from the action of all stationary modifying forces, may assume the form, that *each force is equal and opposite to the resultant of all the other forces.*

If, then, there are only two forces, they must be equal and opposite; and if there are three forces, they must all lie in the same plane, and be represented by the sides of a triangle formed by three lines which have the same directions with the forces; so that *each* *force must be proportional to the sine of the angle included between the other two forces.* Whatever are the forces, if we were to start from a point, and proceed in the direction of either of the forces, through a distance proportional to the intensity of that force, and proceed again, in the same way, from the point at which we arrived in the direction of another force; and so on, proceeding successively from each new station in the direction of the next force, through a distance proportional to that force, the course would finally terminate at the original point of its commencement.

CHAPTER VII.

EQUILIBRIUM OF ROTATION.

249. *The conditions to which a system of forces must be subject, in order that it may not tend to produce rotation about a point or an axis,* are directly deduced from §§ 84 and 88, and are simply, *that the resultant moment of all the forces, with reference to the point or the projection of this resultant moment upon the axis, must vanish.*

250. When there is an equilibrium of rotation about a point, the resultant of the forces may not vanish, in which case there is not an equilibrium of translation. About any other point, therefore, which is not situated in the line drawn parallel to the resultant through this point, there is not, by § 100, an equilibrium of rotation; although *there is an equilibrium of rotation about every point of that line. In order, then, that there may be an equilibrium of rotation about all*

points of space, or even about three points not in the same straight line, there must be an equilibrium of translation as well as of rotation.

251. In the same way, it appears, *that if there is an equilibrium of rotation about parallel axes lying in the same plane, there is an equilibrium of translation in the direction perpendicular to the plane; and if there is equilibrium of rotation about parallel axes which are not in the same plane, there is an equilibrium of translation in every direction except that of the parallel axes.*

252. If there is a fixed point in a system, *it is necessary and sufficient for the equilibrium of rotation that the resultant moment for this point should be nothing;* and, in this case, *the resultant moment vanishes for every point of the straight line which is drawn through the fixed point parallel to the resultant, and also for every axis which is in the same plane with this straight line.*

253. If there are two fixed points in a system, it is *necessary and sufficient for the equilibrium of rotation that the moment of the forces should vanish for the line which joins the two points.*

254. If all the forces are parallel and equal, there is, by § 99, combined with § 250, a line parallel to the common direction of the forces for which the resultant moment vanishes. If the common direction of the forces is assumed for that of the axis of z, the moment of the force acting upon a particle dm, with reference to an axis drawn parallel to that of y at the distance a, from the plane of yz, is

$$(x - a) F dm,$$

and the whole moment of the system is

$$\int_m (x - a) F = F \int_m (x - a).$$

The condition therefore that the moment vanishes for this axis is

$$\int_m (x - a) = 0\,;$$

and the plane which is thus drawn at the distance a from the plane of yz, includes, by § 127, the centre of gravity. Hence, *the axis, for which the resultant moment of the parallel, and equal forces acting upon a system vanishes, passes through the centre of gravity ; and if the system has an equilibrium of rotation, and if there is a fixed point in it, the centre of gravity must be in the straight line which is drawn through the fixed point in the common direction of the forces ; or, if there is a fixed axis, the centre of gravity must lie in the plane which includes this axis and the direction of the forces.* It is also apparent that, *if the centre of gravity is advanced beyond the fixed point or axis in the direction of the forces, the equilibrium is stable ; but if the centre of gravity is not so far advanced as the fixed point or axis, the equilibrium is unstable.*

The ordinary case of gravitation at the surface of the earth, in which its variation in intensity and deviation from parallelism is insensible for the small system of bodies discussed in the usual investigations of mechanics, is the familiar type of this species of force.

255. In the motions of translation and rotation there is no motion of the parts of the system among themselves. There is no change, therefore, in the mutual distance of the origin and point of application of each of the forces which arise from the action of the parts of the system upon each other. The origin, regarded as a point of application of the same force, acting in the opposite direction, moves just as far in the direction of the force as the actual point of application ; so that such a force acts precisely as a moving, modifying force, and has no tendency to affect the equilibrium of translation or rotation. *All the forces, therefore, between the different parts of the system may be neglected in determining the conditions of the equilibrium of translation or rotation.*

This mutual relation of the origin and point of application of the force, by which either may be regarded, at pleasure, as being the origin or the point of application, by a simple reversal of the direction of the force without any change of its intensity, is commonly expressed by the proposition that *action and reaction are equal.*

CHAPTER VIII.

EQUILIBRIUM OF EQUAL AND PARALLEL FORCES.

I.

MAXIMA AND MINIMA OF THE POTENTIAL.

256. In order to give precision to the modes of expression, and have the benefit of well-known terms and forms of speech, the force considered in this chapter, is assumed to be the typical force of *gravitation at the surface of the earth,* acting within a space small enough to admit of the *neglect of its variation of intensity and deviation from parallelism.*

The level surfaces of this force are horizontal planes, and the potential decreases uniformly with the increase of height above the earth's surface.

257. Let the three rectangular axes be so assumed that the plane of xz is horizontal, the axis of y, the upward vertical, that of x, the northern horizontal line, and that of z, the western horizontal line. If, then,

g is the intensity of the force of gravity,

G the distance of the centre of gravity from the origin, and

Ω_0 the value which the potential would assume, if all the points were in the plane of xz;

the actual value of the potential is, by the property of the centre of gravity,

$$\Omega = \Omega_0 - \int_m y = \Omega_0 - \int_m (y - G_y + G_y)$$
$$= \Omega_0 - \int_m G_y = \Omega_0 - m\,G_y.$$

Hence *the potential is a maximum, when the height of the centre of gravity is a minimum, and such a position of the system corresponds, by* § 62, *to that of stable equilibrium; but the potential is a minimum, when the height of the centre of gravity is a maximum, and such a position corresponds to that of unstable equilibrium.*

258. Since the direction of gravity is the same for all the points of the system, *there cannot be an equilibrium of translation, unless there are stationary modifying forces, the resultant of which must be exactly equal to the whole weight of the system, and have a vertical, upward direction.*

259. The resultant moment of all the forces of gravity vanishes for the centre of gravity; *and, therefore, the resultant moment of all the stationary modifying forces must vanish for the same point.*

260. If there is but one modifying force in the system, it *must be vertically directed upwards, have an intensity equal to the whole weight of the system, and its line of action must pass through the centre of gravity.*

261. If there are but two stationary modifying forces, *they must lie in a common plane, which is vertical, and includes the centre of gravity, their resultant must have an upward direction, and be equal to the weight of the system, and they must be reciprocally proportional to the distances of their directions from the centre of gravity.* This last condition is involved in the necessity that the resultant moment must vanish for the centre of gravity.

262. If the intensity of the force of gravity were to be increased or diminished, the conditions of the position of equilibrium would not be changed, but intensity of the modifying forces would be proportionally increased or diminished. Even if the force of gravity were to be made negative, that is, if the direction of its action were to be reversed, the conditions of the position of equilibrium would still remain unchanged, provided that the modifying forces were of such a nature that the direction of their action would also be reversed; but, in this case, the position of stable equilibrium becomes that of unstable equilibrium and the opposite. This reversal of the direction of gravity is relatively accomplished by the rotation of the whole system about a horizontal axis.

II.

THE FUNICULAR AND THE CATENARY.

263. When the points of application of a system of forces are united by a single continuous chord which is destitute of mass, the polygon, which is formed in the situation of equilibrium, is called a *funicular*. The general conditions of such a system involve a mere repetition of the principles of equilibrium; and the present discussion is limited to the case, in which the points of application are masses acted upon by gravity.

264. When there is but one fixed point to the system which may, without any essential loss of generality, be assumed to be either extremity of the chord, *in every position of equilibrium the chord must be vertical.*

But if the idea of the incompressible rod is supposed to be included in that of the inextensible chord, each portion of the chord included between two successive masses may be assumed to have a vertical direction, either upwards or downwards; so that, if

n is the number of masses,

2^n is the number of positions of equilibrium,

all of these positions, except that one in which every portion of the cord is directed downwards, involves an element of instability, and must, therefore, be regarded as *absolutely unstable. The tension of each portion of the chord is, in every case, equal to that of all the weight which it has to sustain; that is, to the sum of all the subsequent masses which lie upon the portion of the chord not attached to the point of suspension.*

265. When there are two fixed points, the whole included chord must hang in the same vertical plane with these two points. The tensions of the various portions of the chord represent modifying forces; and the surfaces at which these forces act are those of spheres, all the centres of which are movable, except those of the two fixed points. In the position of equilibrium, however, all the centres become stationary, and the conditions of equilibrium of each mass or portion of the chord admit of independent discussion.

The forces which act upon each mass are gravity and the tensions of the two portions of chord upon each side. The horizontal projections of these two tensions must, therefore, be equal and opposite in order to balance each other; so that *the horizontal projection of the tension of the chord is invariable throughout its whole length, and equal to the horizontal projection of the sustaining force of each of the fixed points.*

The algebraic sum of the upward vertical projections of the tensions at the two extremities of any portion of the chord must be equal to the weight of all the intermediate masses in order to support them against the force of gravity.

266. These two conditions are necessary and sufficient to produce an equilibrium of translation in any portion of the chord, and, therefore, of the whole chord. The condition of the equilib-

rium of rotation of each portion of the chord, although included in the preceding conditions, is an interesting and useful modification of them.

With reference to the centre of gravity of the masses of each portion of the chord, the moment of the gravity of the masses is zero, and therefore the moment of the tensions applied at the extremities must also vanish. But the directions of these tensions are not parallel, and therefore their lines of tension produced must meet at a point, at which both the tensions may be regarded as applied without affecting their tendency to produce rotation. At this new point of application they may be combined into a resultant, which is vertical, because the horizontal projections of the tensions are equal and opposite. This resultant has the same tendency to produce rotation with the tensions themselves, and therefore it must pass through the point for which this tendency vanishes, that is, through the centre of gravity of the masses. *The point of meeting, therefore, of the lines of extreme tension of any portion of a chord is in the same vertical with the centre of gravity of the intermediate masses.*

267. If the two extremities of any portion of the chord are in the same horizontal line, the equal horizontal projections of the extreme tensions are exactly opposed, and therefore the moments of the vertical projections of these tensions must be equal with reference to the centre of gravity. *The vertical projections of the extreme tensions of any portion of the chord, of which the extremities are upon the same horizontal line, are, then, reciprocally proportional to their distances from the vertical drawn through the centre of gravity of the intermediate masses.*

268. Since the horizontal projection of the tension of the chord is the same throughout its whole extent, no portion of the chord can become vertical. If any portion of the chord is horizontal, the vertical projection of its tension vanishes, and, therefore,

the vertical projection of the chord at any other point is equal to the sum of the weights of all the masses intermediate between this point and the horizontal portion. If then

T is the tension of the chord at any point,

and if the axis of x is horizontal, and that of y vertical, directed upwards, so that

T_x is the horizontal projection of T, and
T_y its vertical projection; and if
s is the arc of the chord at any point, and
m the sum of all the masses included between the point and the horizontal portion of the chord;

the following equations express the preceding conditions:

$$T \cos{}^{s}_{x} = T_x,$$
$$T \cos{}^{s}_{y} = T_y = m,$$
$$\tan{}^{s}_{x} = \frac{m}{T_x}.$$

The inclination of the chord to the horizon, therefore, increases as the distance recedes from the horizontal portion.

If the chord has actually no horizontal portion, the preceding equations are still applicable by assuming for m, such a value as would be required to correspond to the vertical tension of any given portion of the chord.

269. If, in proceeding from the horizontal portion in either direction, the chord is everywhere ascending or descending, its horizontal direction must also be away from the extremity of the horizontal portion to which it is attached so as to form a portion of a convex polygon, which cannot be intersected more than once by any vertical line. Such a position of the chord corresponds to that

of the perfectly stable state, or to that of the most unstable state; and each state is always possible.

If, in proceeding from the horizontal portion, the direction of motion changes from ascent to descent, or the reverse, the horizontal direction must be reversed at the same time, and so that the subsequent portion of the chord will form an arc of a polygon which will include the preceding portion within its concavity, and the concavities of both portions will be turned the same way.

270. The difference of equation (137_{18}) applied to two different portions of the chord gives the following equation between the intermediate masses, the horizontal tension, and the directions of tension at the two points,

$$\frac{m' - m}{T_x} = \frac{\sin\left(\frac{s'}{x} - \frac{s}{x}\right)}{\cos\frac{s'}{x}\cos\frac{s}{x}}.$$

271. If the masses are infinite in number, and arranged in unbroken continuity so as to form the chord itself, the curve is called the *catenary*. In this case, if

k is the weight of an unit of length of the chord, the mass of an element is

$dm = k\,ds$; and if

$\varrho =$ the radius of curvature,

the equation (138_{13}), applied to the extremities of the element, gives, for the equation of the catenary,

$$\varrho = D_{\frac{s}{x}}\, s = \frac{T_x}{k} \sec^2 \frac{s}{x}.$$

If
$$A = \frac{T_x}{k},$$

this equation becomes

$$D_{\frac{s}{x}}\, s = \varrho = A \sec^2 \frac{s}{x}.$$

272. *If the chord is of uniform thickness and density throughout its length,* k and A are constant, and the integral of (138_{31}) is

$$s = A \tan {}^{s}_{x},$$

to which no constant is added, because the arc is supposed to be measured from the point at which it is horizontal.

273. The curve of the uniform chord is easily referred to rectangular coördinates, for the equations

$$D_{{}^{s}_{x}} y = D_{{}^{s}_{x}} s \sin {}^{s}_{x} = A \sin {}^{s}_{x} \sec^2 {}^{s}_{x},$$
$$D_{{}^{s}_{x}} x = D_{{}^{s}_{x}} s \cos {}^{s}_{x} = A \sec {}^{s}_{x};$$

give, by integration, and determining the constants, so that the origin may be at the point of horizontality,

$$y = A(\sec {}^{s}_{x} - 1),$$
$$x = A \log \tan \tfrac{1}{2} (\tfrac{1}{2}\pi - {}^{s}_{x}).$$

These equations give, by elimination and the use of the notation of potential functions,

$$\text{Sin} \frac{x}{A} = \tan {}^{s}_{x} = \frac{s}{A},$$
$$\frac{y}{A} = \text{Cos} \frac{x}{A} - 1 = \sqrt{\left(\frac{s^2}{A^2} + 1\right)} - 1,$$
$$\frac{\varrho}{A} = \text{Cos}^2 \frac{x}{A} = \left(\frac{y+A}{A}\right)^2 = 1 + \frac{s^2}{A^2}.$$

274. The vertical tension of the uniform chord is

$$T_y = sk = \frac{s}{A} T_x = T_x \tan {}^{s}_{x} = T_x \text{Sin} \frac{x}{A};$$

and the whole tension is

$$T = T_x \sec {}^{s}_{x} = T_x \text{Cos} \frac{x}{A} = T_x \left(\frac{y}{A} + 1\right) = T_x \sqrt{\frac{\varrho}{A}}.$$

275. *If the chord were required to be of such a variable thickness as to assume a given form of curve,* the law of this variable thickness is given by the equation

$$k = \frac{T_x}{\varrho \cos^2 {}^{s}_{x}}.$$

The vertical tension is

$$T_y = sk = \frac{T_x s}{\varrho \cos^2 {}^{s}_{x}},$$

and the whole tension is

$$T = T_x \sec {}^{s}_{x}.$$

276. *If the thickness of the chord were required to be proportional to its tension,* so that

$$\frac{T}{k} = B,$$

the following equations are successively obtained by easy transformations

$$D^{s}_{x} s = B \sec {}^{s}_{x},$$

$$D^{s}_{x} x = B, \quad x = B\left({}^{s}_{x}\right),$$

$$\text{Sin} \frac{s}{B} = \tan {}^{s}_{x} = \tan \frac{x}{B},$$

$$\frac{y}{B} = \log \sec \frac{x}{B} = \log \text{Cos} \frac{s}{B},$$

$$\varrho = B \sec \frac{x}{B} = B\, \text{Cos} \frac{s}{B} = c^y,$$

$$T = T_x \sec {}^{s}_{x} = T_x \sec \frac{x}{B} = T_x \text{Cos} \frac{s}{B} = T_x c^y.$$

277. *If the thickness be such as to give an uniform horizontal distribution of the weight,* that is, such a distribution that the weight of each portion of the chord is proportional to its horizontal projection, the equations are

$$C = \frac{T_x \cos {}^{s}_{x}}{k},$$

$$D_x^s s = \varrho = C \sec^3 \tfrac{s}{x},$$

$$x = C \tan \tfrac{s}{x},$$

$$y = \tfrac{1}{2} C (\sec^2 \tfrac{s}{x} - 1) = \tfrac{1}{2} C \tan^2 \tfrac{s}{x} = \frac{x^2}{2C};$$

and the curve is a parabola, of which the transverse axis is vertical.

278. If the chord were compressible and extensible, it would be compelled to assume that thickness, in which it would have the requisite tension; and the form of the curve would, with this condition, be the same as if it were incompressible and inextensible. Thus, if F denotes the function which expresses the given law of the relation of the thickness to the tension, so that

$$\frac{k}{T_x} = F \frac{T}{T_x},$$

the form of the curve is given by the equations

$$D_x^s s = \varrho = \frac{1}{\cos^2 \frac{s}{x} F(\sec^2 \frac{s}{x})},$$

$$D_x^s y = \frac{\sin \frac{s}{x}}{\cos^2 \frac{s}{x} F(\sec^2 \frac{s}{x})},$$

$$D_x^s x = \frac{1}{\cos \frac{s}{x} F(\sec^2 \frac{s}{x})}.$$

279. If the chord or any portion of it is confined to a given surface, the resultant of gravity and the tension of the chord on each point must be normal to the surface, and is balanced by the modifying force by which the point is fixed to the surface.

If, then, the tangent plane to the surface is, at each point, assumed as the plane of $x'y'$; if the axis of x' is horizontal, and that of y' directed upwards, and if

ϱ' is the radius of curvature, at this point, of the projection of the chord upon this plane;

the curve and tension may be determined by means of the equations

$$\varrho' = \frac{T}{k\sin{}^{s}_{y'}\cos{}^{y}_{y'}} = \frac{T_{x'}}{k\cos{}^{2s}_{x'}\cos{}^{y}_{y'}},$$

$$= \frac{T_{y'}}{k\sin{}^{s}_{y'}\cos{}^{s}_{y'}\cos{}^{y}_{y'}} = \varrho\sec{}^{\rho}_{\rho'},$$

$$D_s T = k\cos{}^{y}_{y'}\cos{}^{s}_{y'} = k\cos{}^{s}_{y} = kD_s y,$$

$$T = \int_y k.$$

280. The pressure upon the surface is determined by the consideration that it must exactly balance the tendency of each point of the chord to move in the direction of the normal to the surface. But the tendency of the tension to move any point of the chord in any direction, as that of p, is

$$D_s T_p = D_s(T\cos{}^{s}_{p})$$
$$= \cos{}^{s}_{p} D_s T - T\sin{}^{s}_{p} D_s {}^{s}_{p}.$$

In the case of the direction N of the normal to the surface, this expression becomes, because s is perpendicular to N,

$$D_s T_N = -\, T D_s {}^{s}_{N} = \frac{T}{\varrho''}$$
$$= \frac{T\cos{}^{\rho}_{N}}{\varrho};$$

in which

ϱ'' is the radius of curvature of the projection of the chord upon the common plane of the normal to the surface, and the tangent the chord.

Hence the pressure sustained by the surface in the direction of the normal is

$$R = \frac{T\cos{}^{\rho}_{N}}{\varrho} + k\cos{}^{y}_{N}.$$

281. *If the chord is destitute of weight upon any portion of the sur-*

face, ϱ' becomes infinite, and the curve is that of the shortest line which can be drawn upon the surface.

The tension, in this case, is constant, and the pressure upon the surface becomes

$$R = \frac{T}{\varrho}.$$

282. *In the case of a cylinder, of which the axis is vertical,* the equations become

$$\begin{aligned} {}^{y'}_{y} &= 0, \\ \varrho' &= \frac{T}{k \sin {}^{s}_{y'}}, \\ R &= \frac{T}{\varrho''}, \end{aligned}$$

so that *the curve is the same when it is developed with the cylinder into a plane, which it assumes when it hangs freely.*

283. *In the case of a surface of revolution about a vertical axis and a chord of uniform thickness,* the equations become

$$\begin{aligned} T &= k(y + y_0), \\ \varrho' &= \frac{y + y_0}{\sin {}^{s}_{y'} \cos {}^{y}_{y'}}; \end{aligned}$$

in which the angle which y makes with y' is determined by the meridian curve of the given surface, the plane of xz passes through the lowest point of the curve, and y_0 is the length of the chord which is equal in weight to the tension at the lowest point.

284. A special solution of the preceding problem is given by the equations

$$\begin{aligned} y &= 0, \quad {}^{s}_{y'} = \tfrac{1}{2}\pi, \\ \varrho' &= \frac{y_0}{\cos {}^{y}_{y'}}. \end{aligned}$$

The curve is the circumference of the circle formed by the intersection of a horizontal plane with the surface of revolution. The tension of the *chord is the weight of a length of the same chord which is equal to the distance of the plane of the curve from the vertex of the cone drawn, through the curve, tangent to the surface.*

285. If

ψ is the angle which the projection of y' upon the plane of xz makes with the axis of x, and if

$d\psi'$ is the elementary angle which two successive positions of y' make with each other,

this elementary angle and the radius of curvature are given by the equations

$$d\psi' = \sin_{y'}^{y} d\psi,$$
$$\frac{1}{\varrho'} = -D_s \, {}_{y'}^{s} + D_s \psi' = \sin_{y'}^{y} D_s \psi - D_s \, {}_{y'}^{s}$$
$$= \sin_{y'}^{y} D_s \psi - \cos_{y'}^{s} D_{y'} \, {}_{y'}^{s} = \sin_{y'}^{y} D_s \psi - D_{y'} \sin_{y'}^{s}.$$

If, moreover,

u' is the length of the tangent drawn to the meridian curve at any point of the chord, and

u the projection of u' upon the axis of y,

the following equations are obtained,

$$\sin_{y'}^{s} = u' . \sin_{y'}^{y} D_s \psi = u' D_s \psi',$$
$$\frac{1}{\varrho'} = \frac{\sin_{y'}^{s}}{u'} - D_{y'} \sin_{y'}^{s} = \sin_{y'}^{s} \cos_{y'}^{y} \left(\frac{1}{u} - D_y \log \sin_{y'}^{s}\right);$$

which substituted in (143_{19}) gives, by dividing by $\sin_{y'}^{s} \cos_{y'}^{y}$, and transposing,

$$D_y \log \sin_{y'}^{s} = \frac{1}{u} - \frac{1}{y + y_0}.$$

286. *In the case of the right cone, with the circular base,* the sum of

y' and u' is constant; if, then,

$$a' = u' + y',$$
$$a = u + y = a' \cos_{y'}^{y};$$

the curve is determined by the equation

$$D_{y'} \log \sin_{y'}^{s} = \frac{1}{u'} - \frac{1}{y' + y'_0} = \frac{1}{u'} + \frac{1}{u' - a' - y'_0}$$
$$= - D_{u'} \log . \sin . _{u'}^{s}.$$

The integral of this equation is

$$\sin_{u'}^{s} = - \frac{a' y'_0}{u'(u' - a' - y'_0)} = \frac{4 a' y'_0}{(a' + y'_0)^2 - (a' + y'_0 - 2u')^2}$$
$$= \frac{(a' + y'_0)^2 - (a' - y'_0)^2}{(a' + y'_0)^2 - (a' + y'_0 - 2u')^2},$$

in which the constant is determined, so that u' may be equal to a' when the chord is perpendicular to u'.

The chord is also perpendicular to u', when

$$u' = y'_0,$$

and also when

$$u' = \tfrac{1}{2}(a' + y'_0) \pm \tfrac{1}{2} \sqrt{[(a' + y'_0)^2 + 4 a' y'_0]}.$$

When u' is contained between a' and y'_0, the expression for the sine of the angle which the chord makes with u' is less than unity, so that the angle is real. This angle is also real when u' surpasses the greater of the roots of (145_{21}), or when it is algebraically inferior to the smaller of those roots; but the angle is not real when u' is included between these roots, but is exterior to the preceding limits u' and y'_0. *The curve of the catenary upon the vertical right cone consists, therefore, of three distinct portions, of which one is finite, and included between two intermediate points, at which the curve is perpendicular to the side* *of the cone; while the other two portions, commencing respectively at the two points, which are the highest and lowest of those at which the curve is perpendicular to the side of the cone, extend to an infinite distance. These portions have two of the sides of the cone for their asymptotes,* because the angle which s makes with u' vanishes, when u' is infinite.

287. *The finite portion of the catenary upon the vertical right cone* may be investigated by adopting the notation

$$\sin\varphi = \frac{a' + y'_0 - 2u'}{a' - y'_0},$$
$$\sin^2 i = \cos 2\beta,$$
$$n = \frac{\sin i}{\cos\beta} = \frac{a' - y'_0}{a' + y'_0},$$
$$\sin\theta = \sin i \sin\varphi;$$

and that of elliptic integrals, of which the third form may be represented by

$$\mathfrak{P}_i(n,\varphi) = \int_0^\phi \frac{\sec\theta}{1 + n\sin^2\varphi} {}_\phi.$$

These equations give

$$u' = \tfrac{1}{2}(a' + y'_0)(1 - \sin i \sec\beta \sin\varphi)$$
$$= \tfrac{1}{2}(a' + y'_0)(1 - \sec\beta\sin\theta),$$
$$\sin{}^s_{u'} = \frac{\cos^2\beta - \sin^2 i}{\cos^2\beta - \sin^2\theta} = \frac{\sin^2\beta}{\cos^2\theta - \sin^2\beta},$$
$$\cos{}^s_{u'} = \frac{\cos\theta\sqrt{(\cos^2\theta - 2\sin^2\beta)}}{\cos^2\theta - \sin^2\beta} = \frac{\sin i\cos\theta\cos\varphi}{\cos^2\theta - \sin^2\beta},$$
$$\tan{}^s_{u'} = \frac{\sin^2\beta}{\sin i\cos\theta\cos\varphi},$$
$$D_\phi u' = -\tfrac{1}{2}(a' + y'_0)\sin i\sec\beta\cos\varphi,$$
$$D_\phi s = -\frac{D_\phi u'}{\cos{}^s_{u'}} = \frac{\tfrac{1}{2}(a' + y'_0)(\cos^2\theta - \sin^2\beta)}{\cos\beta\cos\theta}$$
$$= \tfrac{1}{2}(a' + y'_0)(\sec\beta\cos\theta - \tan\beta\sin\beta\sec\theta),$$

$$s = \tfrac{1}{2}(a' + y'_0)(\sec\beta\, \mathcal{E}_i\varphi - \tan\beta \sin\beta\, \mathcal{F}_i\varphi)\,;$$

$$D_\phi \psi' = -\frac{\tan^{s}_{u'} D_\phi u'}{u'} = \frac{\tan\beta\sin\beta}{(1 - \sec\beta\sin\theta)\cos\theta} = \frac{\tan\beta\sin\beta}{(1 - n\sin\varphi)\cos\theta}$$

$$= \frac{\tan\beta\sin\beta\sec\theta}{1 - n^2\sin^2\varphi} + \frac{\sin i \tan^2\beta \sin\varphi\sec\theta}{1 - n^2\sin^2\varphi},$$

$$\psi' = \tan\beta\sin\beta\, \mathcal{P}_i(-n^2, \varphi) + \tan^{[-1]}\frac{\tan\theta}{\tan\varphi}\,;$$

for it is found, by differentiation, that

$$D_\varphi \tan^{[-1]}\frac{\tan\theta}{\tan\varphi} = D_\varphi \tan^{[-1]}\frac{\sin i\cos\varphi}{\cos\theta}$$

$$= \frac{\sin i \sin\varphi(\cos^2\theta - \sin^2 i\cos^2\varphi)}{(\cos^2\theta + \sin^2 i\cos^2\varphi)\cos\theta}$$

$$= \frac{\sin i\tan^2\beta\sin\varphi\sec\theta}{1 - n^2\sin^2\varphi}.$$

288. The preceding value of the angle ψ' admits of geometrical expression by means of the arc of the spherical ellipse in the form given by Booth.

A spherical ellipse is the intersection of a cone of the second degree with a sphere of which the centre is the vertex of the cone. Let

α and β be the two principal semiangles of the cone, of which α is the greater, and

ω the angular distance of any point of the arc of the ellipse from its centre;

and its equation is obviously

$$\cot^2\omega = \frac{1}{\tan^2\omega} = \frac{\cos^2\frac{\omega}{a}}{\tan^2\alpha} + \frac{\sin^2\frac{\omega}{a}}{\tan^2\beta}.$$

Adopt the notation

$\sigma =$ the arc of the spherical ellipse,

$i =$ the angle which the perpendicular to either of the circular sections of the cone makes with the axis, which perpendicular is called the *cyclic axis*,

$\varepsilon =$ the angle which the *focal* of the cone makes with the axis,

$\eta =$ the angle of eccentricity of the elliptic base of the cone.

If, then, through the centre O (fig. 2) of the spherical ellipse, the axes AOA' and BOB' are drawn, and B joined to the foci F and F', the sides and angle of the spherical triangle BOF, are

$$BF = \alpha, \quad BO = \beta, \quad OF = \varepsilon,$$
$$OBF = \eta, \quad BFO = \tfrac{1}{2}\pi - i,$$

which are connected by the equations

$$\begin{aligned}
\cos\alpha &= \cos\beta\cos\varepsilon = \cot\eta\tan i,\\
\sin\beta &= \sin\alpha\cos i = \cot\eta\cot\varepsilon,\\
\sin\varepsilon &= \sin\alpha\sin\eta = \tan i\tan\beta,\\
\cos\eta &= \cos i\cos\varepsilon = \cot\alpha\tan\beta,\\
\sin i &= \sin\eta\cos\beta = \cot\alpha\tan\varepsilon.
\end{aligned}$$

Let C and C' be the points at which the cyclic axes cut the surface of the sphere. Draw OP to any point of the ellipse, CE perpendicular to OP, CH perpendicular to CE, OH perpendicular to CH, $F'K$ perpendicular to OH; take $F'K$ equal to OC, and draw LM perpendicular to OA'. If, then,

$$\theta = LM, \quad \varphi = LF'M,$$
$$\lambda = HOC, \quad \lambda' = OCE,$$

the following equations are readily obtained,

$$\begin{aligned}
\cos i &= \cos OC = \cot\lambda'\tan{}^{\omega}_{a},\\
\tan{}^{\omega}_{a} &= \cos i\tan\lambda' = \cos^2 i\tan\lambda\\
&= \cos^2 i\cos\varepsilon\tan\varphi = \cos i\cos\eta\tan\varphi,
\end{aligned}$$

$$\sin\vartheta = \sin i \sin\varphi,$$

$$\sec^2\tfrac{\omega}{\alpha} = 1 + \cos^2 i\cos^2\eta\tan^2\varphi = \sec^2\varphi(\cos^2\varphi + \cos^2 i\cos^2\eta\sin^2\varphi)$$

$$= \sec^2\varphi(1 - \sin^2\eta\sin^2\varphi + \sin^2\eta\sin^2\vartheta)$$

$$= \sec^2\varphi(\cos^2\vartheta - \sin^2\eta\cos^2 i\sin^2\varphi),$$

$$\cos^2\omega = \cos^2\alpha\cos^2\tfrac{\omega}{\alpha}(1 + \cos^2 i\tan^2\varphi) = \frac{1 + \cos^2 i\tan^2\varphi}{\tan^2\alpha + \tan^2\beta\cos^2 i\tan^2\varphi},$$

$$\sec^2\omega = \frac{\cos^2\alpha(1 + \cos^2 i\tan^2\varphi)}{1 + \cos^2\eta\tan^2\varphi} = \frac{\cos^2\alpha\cos^2\theta}{1 - \sin^2\eta\sin^2\varphi},$$

$$\sin^2\omega = \frac{\sin^2\alpha\cos^2\varphi\sec^2\tfrac{\omega}{\alpha}}{1 - \sin^2\eta\sin^2\varphi},$$

$$D_\phi\tfrac{\omega}{\alpha} = \frac{\cos\eta\cos i\cos^2\tfrac{\omega}{\alpha}}{\cos^2\varphi} = \frac{\cos\eta\cos i\sin^2\alpha}{\sin^2\omega(1 - \sin^2\eta\sin^2 q)},$$

$$D_\phi\omega = -\frac{\cos^2\alpha\sin^2\eta\sin^2\beta\sin\varphi\cos\varphi}{(1 - \sin^2\eta\sin^2\varphi)^2\sin\omega\cos\omega},$$

$$D_\phi\omega^2 = \frac{\sin^2\beta\cos^2\eta\sin^2\eta\sin^2 i\sin^2\varphi\cos^2\tfrac{\omega}{\alpha}}{\cos^2\theta(1 - \sin^2\eta\sin^2\varphi)^2},$$

$$\sin^2\omega D_\phi\tfrac{\omega}{\alpha}^2 = \frac{\sin^2\beta\cos^2\eta\cos^2\tfrac{\omega}{\alpha}}{\cos^2\varphi(1 - \sin^2\eta\sin^2\varphi)},$$

$$D_\phi\sigma^2 = D_\phi\omega^2 + \sin^2\omega D_\phi\tfrac{\omega}{\alpha}^2$$

$$= \frac{\sin^2\beta\cos^2\eta}{\cos^2\theta(1 - \sin^2\eta\sin\varphi)^2}\left(\frac{\cos^2\theta\sin^2\eta\sin^2\varphi\cos^2\theta + \sin^2\eta\sin^2\theta}{\cos^2\varphi\sec^2\tfrac{\omega}{\alpha}}\right)$$

$$= \frac{\sin^2\beta\cos^2\eta}{\cos^2\theta(1 - \sin^2\eta\sin^2\varphi)^2}\left(\frac{\cos^2\theta - \sin^2\eta\cos^2 i\sin^2\varphi}{\cos^2\varphi\sec^2\tfrac{\omega}{\alpha}}\right)$$

$$= \frac{\sin^2\beta\cos^2\eta}{\cos^2\theta(1 - \sin^2\eta\sin^2\varphi)^2},$$

$$D_\phi\sigma = \frac{\sin\beta\cos\eta\sec\theta}{1 - \sin^2\eta\sin^2\varphi},$$

$$\sigma = \sin\beta\cos\eta\,\mathfrak{P}_i(-\sin^2\eta, \varphi) = \frac{\tan\beta\sin\beta}{\tan\alpha}\mathfrak{P}_i(-\sin^2\eta, \varphi).$$

289. In the particular case in which

$$\alpha = \tfrac{1}{4}\pi,$$

this equation is, by (148_{17}), reduced to

$$\sigma = \tan\beta\sin\beta\,\mathfrak{P}_i(-n^2, \varphi),$$

which substituted in (147_5) gives,

$$\psi' = \sigma + \tan^{[-1]} \frac{\tan\theta}{\tan\varphi}.$$

290. For the length of the arc of the chord which extends from its lowest to its highest point, this equation becomes

$$2\psi'_1 = 2\sigma_1;$$

and if the magnitude of this angle is commensurate with the total developed angle of the cone, *the chord returns into itself, after passing around the cone once, twice, or several times, dependent upon the magnitude of the angle of the cone.*

291. *To investigate the infinite portions of the chord,* let

l'_0 and l'_1 be the roots of the equation (145_{21}),

and the equation gives

$$l'_0 + l'_1 = a' + y'_0,$$
$$l'_0 l'_1 = -a' y'_0.$$

Adopt also the notation of § 287 and

$$\sin\varphi' = \frac{(a' - y'_0)}{\sin i\,(a' + y'_0 - 2u')} = \frac{a' + y'_0}{\cos\beta\,(a' + y'_0 - 2u')} = \frac{(l'_0 + l'_1)\sec\beta}{l'_0 + l'_1 - 2u'},$$
$$\sin\theta' = \sin i \sin\varphi',$$

and the following reductions are obtained, by the substitution of $\operatorname{cosec}\varphi'$ for $\sin\theta$,

$$u' = \tfrac{1}{2}(l'_0 + l'_1)(1 - \sec\beta\operatorname{cosec}\varphi'),$$
$$D_{\varphi'}s = \tfrac{1}{2}(a' + y'_0)\left(\frac{\tan\beta\sin\beta}{\cos\theta'} - \frac{\sec\beta\cos^2\varphi'}{\sin^2\varphi'\cos\theta'}\right)$$
$$= \tfrac{1}{2}(a' + y'_0)[\tan\beta\sin\beta\sec\theta' + \sec\beta\cos\theta' + \sec\beta D_{\varphi'}(\cos\theta'\cot\varphi')],$$
$$s = \tfrac{1}{2}(a' + y'_0)(\sec\beta\,\mathfrak{E}_i\varphi' + \tan\beta\sin\beta\,\mathfrak{F}_i\varphi' + \sec\beta\cos\theta'\cot\varphi'),$$

$$D_{\varphi'}\psi' = \frac{\sin^2\beta\cos\beta\sin^2\varphi'\sec\theta' - \sin^2\beta\sin\varphi'\sec\theta'}{1 - \cos^2\beta\sin^2\varphi'}$$

$$= \frac{\tan\beta\sin\beta\sec\theta'}{1 - \cos^2\beta\sin^2\varphi'} - \tan\beta\sin\beta\sec\theta' + D_{\varphi'}\tan^{[-1]}\frac{\cos\theta'}{\cos\varphi'},$$

$$\psi' = \tan\beta\sin\beta\,\mathfrak{F}_i(-\cos^2\beta_1\varphi') - \tan\beta\sin\beta\,\mathfrak{F}_i\varphi' + \tan^{[-1]}\frac{\cos\theta'}{\cos\varphi'}.$$

292. The term of the preceding value of ψ', which depends upon elliptic integrals of the third order, may be constructed by means of a spherical ellipse, of which the parameter is the reciprocal of that employed in the construction of the similar term of the finite portion of the chord. The parameter of the spherical ellipse of § 287 being $\sin\eta$, the reciprocal parameter is

$$\frac{\sin i}{\sin\eta} = \cos\beta,$$

and the length of the arc of the corresponding spherical ellipse for the amplitude φ' is

$$\sigma' = \sin\beta\cos\eta\,\mathfrak{F}_i(-\cos^2\beta,\ \varphi') = \frac{\tan\beta\sin\beta}{\tan\alpha}\mathfrak{F}_i(-\cos^2\beta,\ \varphi').$$

This arc is reduced, in the case of

$$\alpha = \tfrac{1}{4}\pi,$$

to

$$\sigma' = \tan\beta\sin\beta\,\mathfrak{F}_i(-\cos^2\beta,\ \varphi').$$

293. *The finite portion is exactly circular when*

$$a' = y'_0.$$

In this case

$$i = 0,\quad \beta = \tfrac{1}{4}\pi = \alpha,$$

and the equations of the infinite portion become

$$u' = a'(1 - \sqrt{2}.\operatorname{cosec}\varphi'),$$

$$s = a\sqrt{2}\cot\varphi'$$

$$\psi' = \sqrt{2}(\tfrac{1}{2}\pi - \varphi') - 2\tan^{[-1]}[(\sqrt{2} - 1)\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}\varphi')].$$

294. As y'_0 diminishes from the value a', the finite portion becomes more and more eccentric, until when

$$y'_0 = 0,$$

both the finite and the infinite portions degenerate into straight lines, which are the sides of the cone.

295. *When y'_0 is negative, a' and y'_0 cease to be the limits of the finite portion, and become the limits of the infinite portion, while l'_0 and l'_1 become the limits of the finite portion. But l'_0 and l'_1 are imaginary, if y'_0 is included between the values*

$$y'_0 = (-3 \pm 2\sqrt{2})\, a',$$

so that between these limits the finite portion disappears, and the chord consists only of the two infinite portions; and at the limits the finite portion is circular.

To investigate the infinite portions between the limits, in which the finite portion disappears, let

$$\tan i' = \sin i \sqrt{-1},$$

$$\varphi'' = \tfrac{1}{2}\pi - \varphi',$$

$$\sin \theta'' = \sin i' \sin \varphi'';$$

and the following equations are obtained by simple transformations,

$$\sin\beta \cos i' = \sqrt{\tfrac{1}{2}},$$

$$u' = \tfrac{1}{2}(a' + y'_0)(1 - \sec\beta \sec\varphi'') = \tfrac{1}{2}(a' - y'_0)(\cos\beta - \sec\varphi''),$$

$$\begin{aligned} \cos^2\theta' &= 1 + \tan^2 i' \cos^2\varphi'' \\ &= \sec^2 i' (1 - \sin^2 i' \sin^2\varphi'') \\ &= \sec^2 i' \cos^2\theta''; \end{aligned}$$

$$D_{\varphi''} s = \tfrac{1}{2}(a' + y'_0)\left(\frac{\cos i' \sec\beta \sin^2\varphi''}{\cos^2\varphi'' \cos\theta''} - \frac{\tan\beta}{\sqrt{2}\,.\cos\theta''}\right),$$

$$s = \tfrac{1}{2}(a' + y'_0)\left(\frac{\sec\beta}{\cos i'} \cos\theta'' \tan\varphi'' - \frac{\sec\beta}{\cos i'} \mathcal{E}_{i'}\varphi'' - \frac{\tan\beta}{\sqrt{2}} \mathcal{F}_{i'}\varphi''\right);$$

$$D_{\varphi''}\psi' = -\frac{\sqrt{\frac{1}{2}}\tan\beta\sec\theta''}{1-\cos^2\beta\cos^2\varphi''} + \sqrt{\tfrac{1}{2}}\tan\beta\sec\theta'' - D_{\varphi''}\tan^{[-1]}\frac{\tan i'}{\tan\theta''}$$

$$= -\frac{\cos i'\cos\beta\sec\theta''}{1-\frac{1}{2}\sin^2\varphi''} + \cos i'\cos\beta\sec\theta''$$

$$- D_{\varphi''}\tan^{[-1]}\left(\cos\beta\cos\varphi''\frac{\tan\theta''}{\tan i'}\right) - D_{\varphi''}\tan^{[-1]}\frac{\tan i'}{\tan\theta''},$$

$$\psi' = -\cos i'\cos\beta\,\mathfrak{P}_{i'}(-\tfrac{1}{2},\varphi'') + \cos i'\cos\beta\,\mathfrak{F}_{i'}\varphi''$$

$$-\tan^{[-1]}\left(\cos\beta\cos\varphi''\frac{\tan\theta''}{\tan i'}\right) - \tan^{[-1]}\frac{\tan i'}{\tan\theta''}$$

$$= -\cos i'\cos\beta\,\mathfrak{P}_{i'}(-\tfrac{1}{2},\varphi'') + \cos i'\cos\beta\,\mathfrak{F}_{i'}\varphi''$$

$$-\tan^{[-1]}\frac{\tan^2 i' + \cos\beta\cos\varphi''\tan^2\theta''}{\tan i'\tan\theta''(1-\cos\beta\cos\varphi'')};$$

in which the elliptic integral of the third form admits of interpretation by means of the arc of the spherical ellipse.

296. When the negative of y'_0 is equal to a' the equations may be greatly simplified and reduced to the following forms,

$$\beta = \tfrac{1}{2}\pi, \quad i' = \tfrac{1}{4}\pi,$$

$$\cos\beta = 0, \quad \cos i' = \sqrt{\tfrac{1}{2}},$$

$$D_{\varphi''}\psi' = \frac{\cos\varphi''}{\sqrt{(2-\sin^2\varphi'')}},$$

$$\sin\psi' = \frac{\sin\varphi''}{\sqrt{2}},$$

$$\cos^2\varphi'' = 1 - 2\sin^2\psi' = \cos 2\psi',$$

$$u'^2 = \frac{a'^2}{\cos^2\varphi''} = \frac{a'^2}{\cos 2\psi'};$$

which is the polar equation of the equilateral hyperbola. In this case, therefore, the curve of the chord upon the developed cone is an equilateral hyperbola; this case was recognized by BOBILLIER in an imperfect investigation of the catenary upon the surface of the vertical cone of revolution.

297. *When the surface of revolution is an ellipsoid,* of which the equation of the vertical section made by the plane yx is,

$$\frac{y^2}{A^2} + \frac{x^2}{B^2} = 1;$$

let a sphere be constructed upon the axis of revolution as a diameter, and let

φ be the angle from the vertical point of the sphere to a point of which y is the ordinate, so that

$$y = A\cos\varphi, \quad x = B\sin\varphi,$$
$$u = A(\sec\varphi - \cos\varphi) = A\sin\varphi\tan\varphi,$$
$$D_\phi y = -A\sin\varphi.$$

These equations, substituted in (144_{29}), with proper regard to the different position of the origin of coördinates, give

$$D_\phi \log \sin{}^s_{y'} = -A\sin\varphi D_y \log\sin{}^s_{y'} = -\cot\varphi + \frac{\sin\varphi}{\cos\varphi + M},$$
$$\sin{}^s_{y'} = \frac{N}{\sin\varphi(\cos\varphi + M)} = \frac{N}{\frac{1}{2}\sin 2\varphi + M\sin\varphi},$$

in which N and M are arbitrary constants.

298. The maximum and minimum of $\sin{}^s_{y'}$ are determined by the roots of the equation

$$\cos 2\varphi + M\cos\varphi = 0.$$

If these roots are φ' and φ'', the equation gives

$$\cos\varphi'\cos\varphi'' + \tfrac{1}{2} = 0,$$
$$M = -2(\cos\varphi' + \cos\varphi'') = -4\cos\tfrac{1}{2}(\varphi' + \varphi'')\cos\tfrac{1}{2}(\varphi' - \varphi'')$$
$$= \sec\varphi' + \sec\varphi''.$$

Of the two roots, therefore, one is obtuse, while the other is acute; if one is contained between $\frac{1}{3}\pi$ and $\frac{2}{3}\pi$, the other is impossible; and when both are real, one is confined between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, while the other is without these limits. The corresponding minimum and maximum values of $\sin^{s}_{y'}$ are

$$\frac{N}{\tan\varphi'\sin^2\varphi'} \text{ and } \frac{N}{\tan\varphi''\sin^2\varphi''}.$$

Both these, independently of their signs, are minimum values, and when they are both absolutely greater than unity there is no catenary; but if either is less than unity, there is a corresponding portion of the catenary. When both values are less than unity, the catenary consists of two separate portions, because there is between φ' and φ'' a value φ''' of φ which satisfies the equation

$$\cos\varphi''' = -M,$$

and the values of

$$\cos\varphi' - \cos\varphi''' = \frac{\cos^2\varphi' - \cos 2\varphi'}{\cos\varphi'} = \frac{\sin^2\varphi'}{\cos\varphi'},$$

$$\cos\varphi''' - \cos\varphi'' = -\frac{\sin^2\varphi''}{\cos\varphi''} = 2\sin^2\varphi''\cos\varphi',$$

are positive.

299. *The especial case* of

$$\varphi' = \tfrac{1}{4}\pi,$$

gives

$$\varphi'' = \tfrac{3}{4}\pi,$$

$$M = 0;$$

$$\sin^{s}_{y'} = \frac{2N}{\sin 2\varphi};$$

and each of the minimum values of $\sin^{s}_{y'}$ is

$$2N,$$

which, being less than unity, may be expressed by

$$2N = \sin 2\alpha.$$

This equation gives

$$\sin{}^{s}_{y'} = \frac{\sin 2\alpha}{\sin 2\varphi}.$$

If, then, λ is determined by the condition

$$\cos 2\lambda = \frac{\cos 2\varphi}{\cos 2\alpha},$$

simple reductions give

$$\cos{}^{s}_{y'} = \frac{\sqrt{(\cos^2 2\alpha - \cos^2 2\varphi)}}{\sin 2\varphi} = \frac{\cos 2\alpha \sin 2\lambda}{\sin 2\varphi},$$

$$\tan{}^{s}_{y'} = \frac{\tan 2\alpha}{\sin 2\lambda};$$

$$D_\lambda \varphi = \frac{\cos 2\alpha \sin 2\lambda}{\sin 2\varphi} = \cos{}^{s}_{y'},$$

$$\frac{D_\phi s}{A} = \sqrt{\left(\sin^2\varphi + \frac{B^2}{A^2}\cos^2\varphi\right)} \sec{}^{s}_{y'},$$

$$\frac{D_\lambda s}{A} = \sqrt{\left(\sin^2\varphi + \frac{B^2}{A^2}\cos^2\varphi\right)}$$

$$= \sqrt{\left[\tfrac{1}{2}\left(1 + \frac{B^2}{A^2}\right) + \tfrac{1}{2}\left(\frac{B^2}{A^2} - 1\right)\cos 2\alpha \cos 2\lambda\right]};$$

$$D_\phi \psi = \frac{D_\phi s \sin{}^{s}_{y'}}{B \sin\varphi} = \frac{\sin 2\alpha\, D_\phi s}{B \sin\varphi \sin 2\varphi},$$

$$D_\lambda \psi = \frac{\sin 2\alpha\, D_\lambda s}{B \sin\varphi \sin 2\varphi}.$$

In the case of the prolate ellipsoid, the notation

$$\sin^2 i = \frac{2(B^2 - A^2)\cos 2\alpha}{B^2 + A^2 + (B^2 - A^2)\cos 2\alpha},$$

$$\sin\vartheta = \sin i \sin\lambda,$$

gives the equation

$$s = \sqrt{(B^2\cos^2\alpha + A^2\sin^2\alpha)}\, \mathcal{E}_i \lambda.$$

In the case of the oblate ellipsoid, the notation

$$\lambda' = \tfrac{1}{2}\pi - \lambda,$$
$$\sin^2 i' = \frac{2(A^2 - B^2)\cos 2\alpha}{B^2 + A^2 + (A^2 - B^2)\cos 2\alpha},$$
$$\sin\theta' = \sin i' \sin\lambda',$$

gives the equation

$$s = \sqrt{(A^2\cos^2\alpha + B^2\sin^2\alpha)}\, \mathfrak{E}_{i'}\lambda'.$$

In the case of the sphere the equations become

$$B = A, \quad i = i' = 0,$$
$$s = A\lambda;$$

and this result of this case is obtained by Bobillier. This case also gives the equation

$$D_\lambda \psi = \frac{\sin 2\alpha}{\sin\varphi \sin 2\varphi}$$
$$= \frac{\sin 2\alpha}{(\sin^2\alpha + \cos 2\alpha \sin^2\lambda)\sqrt{(\cos^2\alpha - \cos 2\alpha \sin^2\lambda)}},$$

which by the notation

$$\cos i'' = \tan\alpha,$$
$$\sin\theta'' = \sin i'' \sin\lambda,$$

becomes

$$D_\lambda \psi = \frac{2\sec\theta''}{\sin\alpha(1 + \tan^2 i'' \sin^2\lambda)},$$
$$\psi = \frac{2}{\sin\alpha}\mathfrak{P}_{i''}(\tan^2 i'', \lambda)$$
$$= 2\sin\alpha\tan^2\alpha\,\mathfrak{P}_{i''}(-\sec^2\alpha\sin^2 i'', \lambda) + 2\sin\alpha\,\mathfrak{F}_{i''}\lambda$$
$$+ 2\tan^{[-1]}\frac{\sin i'' \tan\theta'' \cos\lambda}{\sin\alpha}.$$

300. Returning to the general case of the ellipsoid, let

α and β be the limiting values of φ for the upper portion of the curve, and

α' and β' the limiting values for the lower portion; and let

$$\eta = \tfrac{1}{2}(\alpha + \beta), \quad \varepsilon = \tfrac{1}{2}(\beta - \alpha),$$
$$\eta' = \tfrac{1}{2}(\alpha' + \beta'), \quad \varepsilon' = \tfrac{1}{2}(\beta' - \alpha').$$

Hence the following values of M and N are obtained

$$N = \tfrac{1}{2}\sin 2\alpha + M\sin\alpha = \tfrac{1}{2}\sin 2\beta + M\sin\beta,$$
$$-N = \tfrac{1}{2}\sin 2\alpha' + M\sin\alpha' = \tfrac{1}{2}\sin 2\beta' + M\sin\beta',$$
$$-M = \frac{\cos\varepsilon\cos 2\eta}{\cos\eta} = \frac{\cos\varepsilon'\cos 2\eta'}{\cos\eta'};$$
$$N = \tan\eta\,(\cos^2\eta\cos 2\varepsilon - \cos^2\varepsilon\cos 2\eta)$$
$$= \tfrac{1}{2}\tan\eta\,(\cos 2\varepsilon - \cos 2\eta) = \tan\eta\,(\cos^2\varepsilon - \cos^2\eta)$$
$$= \tfrac{1}{2}\tan\eta'\,(\cos 2\eta' - \cos 2\varepsilon') = \tan\eta'\,(\cos^2\eta' - \cos^2\varepsilon')$$
$$= \tan\eta\sin\alpha\sin\beta = -\tan\eta'\sin\alpha'\sin\beta'$$
$$\sin_{y'}^{s} = \frac{\sin\eta\sin\alpha\sin\beta}{\sin\varphi\,(\cos\eta\cos\varphi - \cos\varepsilon\cos 2\eta)} = \frac{-\sin\eta'\sin\alpha'\sin\beta'}{\sin\varphi\,(\cos\eta'\cos\varphi - \cos\varepsilon'\cos 2\eta')}$$
$$\cos_{y'}^{s} = \frac{\sqrt{[-(\cos\varphi - \cos\alpha)(\cos\varphi - \cos\beta)(\cos\varphi - \cos\alpha')(\cos\varphi - \cos\beta')]}}{\sin\varphi(\cos\varphi + M)}$$
$$= \frac{\sqrt{[-(\cos^2\varphi - 2\cos\eta\cos\varepsilon\cos\varphi + \cos\alpha\cos\beta)(\cos^2\varphi - 2\cos\eta'\cos\varepsilon'\cos\varphi + \cos\alpha'\cos\beta')]}}{\sin\varphi\,(\cos\varphi + M)}$$
$$= \frac{\sqrt{[\sin^2\varphi\,(\cos\varphi + M)^2 - N^2]}}{\sin\varphi\,(\cos\varphi + M)}.$$

The numerators of the first and last values of $\cos_{y'}^{s}$ give, by direct comparison,

$$-2M = \cos\alpha + \cos\beta + \cos\alpha' + \cos\beta' = 2\cos\eta\cos\varepsilon + 2\cos\eta'\cos\varepsilon',$$

whence

$$\cos\eta\cos\eta'\cos\varepsilon' = (\cos 2\eta - \cos^2\eta)\cos\varepsilon = -\sin^2\eta\cos\varepsilon,$$
$$\cos\eta\cos\eta'\cos\varepsilon = -\sin^2\eta'\cos\varepsilon',$$

$$\cos^2\eta\cos^2\eta' - \sin^2\eta\sin^2\eta' = \cos(\eta+\eta')\cos(\eta'-\eta) = 0,$$

The comparison of the values of N, shows that the value of η' must be obtuse, whence

$$\eta' = \eta + \tfrac{1}{2}\pi,$$
$$\cos\varepsilon' = \tan\eta\cos\varepsilon,$$
$$\cos\varepsilon = -\tan\eta'\cos\varepsilon'.$$

301. *The general case of the surface of revolution* admits of one integration, by denoting by v the ordinate of the meridian curve of revolution, which gives

$$\frac{1}{u} = -\frac{D_y v}{v} = -D_y \log v,$$

this equation, substituted in (144_{29}), gives, by integration,

$$\sin{}^s_{y'} = \frac{v_0 y_0}{v(y+y_0)},$$

in which

v_0 is the ordinate of the meridian curve at the origin.

This form of the equation is, however, limited to the case in which the curve has a point, in which its direction is horizontal. But every case is included in the form

$$\sin{}^s_{y'} = \frac{M}{v(y+y_0)},$$

in which M is an arbitrary constant.

302. *In the case of the surface formed by the revolution of the equilateral hyperbola about its asymptote,* which may be called *the equilateral asymptotic hyperboloid,* if the equation of the revolving hyperbola is

$$v(y+y_0) = b^2;$$

the equation (159_{24}) becomes

$$\sin{}^{s}_{y'} = \frac{M}{b^2},$$

and, therefore, *the inclination of the curve of this catenary to the arc of the meridian is constant.*

When M is greater than b^2, the curve is impossible, but when

$$M = \pm b^2,$$

the catenary becomes a horizontal circle, and

$$^{s}_{y'} = \pm \tfrac{1}{2}\pi.$$

303. It may be inferred from the comparison of the two preceding sections, that, *upon the circle of intersection of any surface of revolution with the equilateral asymptotic hyperboloid of equation* (159_{31}), *the arc of the catenary of either surface makes the same angle with the meridian curve of that surface.* Hence, *the limiting horizontal planes of the catenary of equation* (159_{16}) *are the intersections of the surface of revolution upon which it lies with the equilateral asymptotic hyperboloid, of which the equation is*

$$v(y + y_0) = v_0 y_0.$$

The catenary extends over that portion of surface which lies exterior to the asymptotic hyperboloid, and does not extend over that portion of surface which is included within the hyperboloid.

304. *To complete the solution of the catenary upon the equilateral asymptotic hyperboloid,* the equation (159_{31}) gives

$$\tan{}^{y}_{y'} = -D_y v = \frac{b^2}{(y + y_0)^2},$$

whence the following equations are obtained;

$$(y + y_0)^2 = b^2 \cot{}^{y}_{y'},$$

$$D_{\frac{y}{y'}}y = -\frac{b^2}{2(y+y_0)\sin^2\frac{y}{y'}},$$

$$D_{\frac{y}{y'}}\psi = -\frac{b^2 D_y\psi}{2(y+y_0)\sin^2\frac{y}{y'}}.$$

But it is found by § 285 that

$$D_y\psi = \frac{\sec\frac{y}{y'}\tan\frac{s}{y'}}{v} = \frac{(y+y_0)\tan\frac{s}{y'}}{b^2\cos\frac{y}{y'}},$$

whence

$$D_{\frac{y}{y'}}\psi = -\frac{\tan\frac{s}{y'}}{2\sin^2\frac{y}{y'}\cos\frac{y}{y'}};$$

of which the integral is

$$\psi = \frac{\tan\frac{s}{y'}}{2\sin\frac{y}{y'}} - \tfrac{1}{2}\tan\tfrac{s}{y'}\log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\tfrac{y}{y'}\right).$$

305. If the chord is not strictly confined to the surface so as to be incapable of removal from it, but if it simply lies upon the surface, without the power of penetrating it, it must leave the surface whenever the pressure becomes negative, that is, when the sign of R, computed by (142_{29}), is reversed. The points at which the chord leaves the surface are, therefore, determined by the equation

$$R = 0.$$

CHAPTER IX.

ACTION OF MOVING BODIES.

CHARACTERISTIC FUNCTION.

306. Related to the idea of the potential, and, in some respects including it, is that of the action of a system as proposed by MAUPERTUIS. Every moving body may be regarded as constantly expending an amount of action, equivalent to the power which its motion represents, that is, to the product of the force of the moving body multiplied by the space through which the body moves. Hence, with the notation of Chapters II. and III., if V designates the whole action expended by the system, the action expended at each instant is

$$dV = \Sigma_1 (m_1 v_1 ds_1),$$

and the total expenditure of action is

$$V = \Sigma_1 \int_{s_1} m_1 v_1.$$

The function V is called by HAMILTON the *characteristic function* of the moving system, and he has resolved the problem of dynamics into the investigation of its form and properties.

307. If the power, with which a system is moving at any instant, is denoted by T, its expression becomes, by (4_{20}),

$$T = \tfrac{1}{2} \Sigma_1 (m_1 v_1^2).$$

The preceding expressions for the expended action give, therefore,

$$D_t V = \Sigma_1 (m_1 v_1 D_t s_1) = \Sigma_1 (m_1 v_1^2) = 2T,$$
$$V = \int_t 2T.$$

PRINCIPLE OF LIVING FORCES, OR LAW OF POWER.

308. If Ω denotes that function which, in the case of the fixed forces of nature, is the potential of the moving system, its change for any instant is, by (34_{24}) and § 58,

$$d\Omega = dT = \Sigma'_1 (m_1 F_1 df_1).$$

Hence, in the case of the fixed forces of nature, if H is an arbitrary constant,

$$T = \Omega + H,$$

which is only the analytical form of the proposition of § 58, and is called *the principle of living forces.* The term *living force* denotes the power of a system, so that this principle may, with equal propriety, be called *the law of power.*

CANONICAL FORMS OF THE DIFFERENTIAL EQUATIONS OF MOTION.

309. The equation (8_{15}) may be written in the form

$$\begin{aligned}\delta\Omega &= \Sigma_1 (m_1 D_t v_1 \delta s_1) \\ &= D_t \Sigma_1 (m_1 v_1 \delta s_1) - \Sigma_1 (m_1 v_1 \delta D_t s_1) \\ &= D_t \Sigma_1 (m_1 v_1 \delta s_1) - \Sigma_1 (m_1 v_1 \delta v_1).\end{aligned}$$

If, then, $\eta_1, \eta_2, \eta_3, \ldots\ldots$ etc., are assumed to be the independent elements of position of the n bodies of the moving system, s_1, s_2, etc., may be regarded as expressed in terms of these elements, so that

$$v = D_t s = \Sigma_\eta (D_\eta s D_t \eta).$$

With the notation

$$D_t\eta = \eta',$$

this equation is resolved into the equations represented by

$$D_{\eta'}v = D_\eta s.$$

The substitution of these values give, if $T_{\eta,\eta'}$ denotes T expressed by means of $\eta_1, \eta_2, \eta'_1, \eta'_2, \ldots\ldots$ etc.,

$$\Sigma_1(m_1 v_1 D_\eta s_1) = \Sigma_1(m_1 v_1 D_{\eta'} v_1) = D_{\eta'} T_{\eta,\eta'},$$
$$\Sigma_1(m_1 v_1 D_\eta v_1) = D_\eta T_{\eta,\eta'};$$

whence

$$D_\eta \Omega = (D_t D_{\eta'} - D_\eta) T_{\eta,\eta'}.$$

This expression represents the elegant forms of the differential equations of motion given by LAGRANGE; but the mode of investigation is adopted from HAMILTON.

310. In the special case, in which the independent elements of position are the rectangular coördinates, x, y, z, of the different points of the system, these equations become

$$v^2 = x'^2 + y'^2 + z'^2,$$
$$T_{x,x'} = \tfrac{1}{2}\Sigma_1 m_1(x_1'^2 + y_1'^2 + z_1'^2),$$
$$D_{x'} T_{x,x'} = mx' = mD_t x,$$
$$D_x T_{x,x'} = 0,$$
$$D_x \Omega = mD_t x' = mD_t^2 x.$$

When the coördinates of the system are subject to conditions, these equations are still applicable, provided that the forces, by which the conditions are maintained, are included in the forces of Ω, or more properly of $\delta\Omega$. The values of $D_x\Omega$ and $D_\eta\Omega$ can be obtained from the given differential expression of Ω, even when such expression is incapable of integration ; for this form gives

$$D_\eta \Omega = \Sigma'_1 (m_1 F_1 D_\eta f_1).$$

311. By means of the notation

$$\omega = D_{\eta'} T_{\eta, \eta'},$$

η'_1, η'_2, ... etc., may be eliminated from the value of T, and $T_{\eta, \omega}$ may denote the resulting value, expressed by means of η_1, η_2, ω_1, ω_2, ... etc.

Since T is a homogeneous function of two dimensions in respect to η'_1, η'_2, etc., it satisfies the equation

$$2\,T = \Sigma_\eta (\eta' D_{\eta'} T_{\eta, \eta'}) = \Sigma_\eta (\eta' \omega)\,;$$

whence

$$2\,\delta\,T = \Sigma_\eta (\omega\,\delta\,\eta' + \eta'\,\delta\,\omega).$$

But the variation of T, derived by the usual method, is

$$\delta\,T = \Sigma_\eta (D_\eta T_{\eta, \eta'} \delta\,\eta + \omega\,\delta\,\eta')\,;$$

which, subtracted from the previous value of $2\,\delta\,T$, leaves

$$\delta\,T = \Sigma_\eta (\eta'\,\delta\,\omega - D_\eta T_{\eta, \eta'} \delta\,\eta).$$

This equation is equivalent to the two equations

$$D_\omega T_{\eta, \omega} = \eta',$$
$$D_\eta T_{\eta, \omega} = - D_\eta T_{\eta, \eta'},$$

and Lagrange's *canonical form assumes the following expression given by* Hamilton,

$$D_t \omega = D_\eta (\Omega - T_{\eta, \omega}).$$

312. But Ω is, in the case of the fixed forces of nature, a function of η_1, η_2, etc., without other variables. If, then, in this case,

$$H_{\eta, \omega} = T_{\eta, \omega} - \Omega\,;$$

the preceding equations assume the simple form

$$D_t \omega = - D_\eta \Pi_{\eta,\,\omega},$$
$$D_t \eta = D_\omega \Pi_{\eta,\,\omega};$$

which are given by Hamilton, in which Ω may involve the time.

VARIATIONS OF THE CHARACTERISTIC FUNCTION.

313. The variation of the characteristic function, taken upon the hypothesis that the time does not vary, is

$$\delta V = \int_t 2 \delta T.$$

But, from the preceding equations,

$$\begin{aligned} \delta T &= \Sigma_\eta (\omega \delta \eta' + D_\eta T_{\eta,\,\eta'} \delta \eta) \\ &= \Sigma_\eta (\omega \delta \eta' - D_\eta T_{\eta,\,\omega} \delta \eta) \\ &= \Sigma_\eta (\omega \delta \eta' + D_t \omega \delta \eta - D_\eta \Omega \delta \eta) \\ &= \Sigma_\eta (\omega \delta \eta' + D_t \omega \delta \eta) - \delta \Omega, \end{aligned}$$

the sum of which and of the equation

$$\delta T = \delta \Omega + \delta H,$$

is

$$\begin{aligned} 2 \delta T &= \Sigma_\eta (\omega \delta \eta' + D_t \omega \delta \eta) + \delta H \\ &= D_t \Sigma_\eta (\omega \delta \eta) + \delta H. \end{aligned}$$

The variation of the characteristic function is, therefore,

$$\delta V = \Sigma_\eta (\omega \delta \eta - \omega_0 \delta \eta_0) + t \delta H,$$

in which ω_0 and η_0 are the initial values of ω and η. If, then, V is expressed as a function of the initial and final coördinates, η, ω, η_0, and ω_0, and of the constant H, its derivatives are

$$D_\eta V = \omega, \quad D_{\eta_0} V = -\omega_0,$$
$$D_H V = t.$$

By means of these equations, the problem is resolved by Hamilton into the determination of the single function V.

314. In the case in which the independent elements of position are the rectangular coördinates, these equations become

$$\omega = m x' = m D_t x = D_x V,$$
$$\omega_0 = m x'_0 = m D_t x_0 = - D_{x_0} V.$$

315. If the expression of the forces involves the velocities the final expression of δT in § 313 is incomplete, and the present mode of investigation is not easily and simply applicable to such cases, which is of less importance, because these cases are not, in the most comprehensive view of the subject, the cases of nature.

PRINCIPLE OF LEAST ACTION.

316. When, in the case of the fixed forces of nature, the initial and final positions of the system are given as well as the initial power with which the system is moving, the variation of the characteristic function vanishes, and, therefore, the function is generally a maximum or a minimum. The action expended by the system, which is measured by this function, is also a maximum or a minimum; or, in other words, the course by which the system is compelled to move from its initial to its final position through the action of the dynamic laws, is that in which the total expenditure of action is a maximum or a minimum. But it is obvious that, in most cases, and always, when the paths in which the various bodies

move are quite short, the described course cannot correspond to the maximum of expended action; and, therefore, in most cases *the system moves from its given initial to its given final position with the least possible expenditure of action.*

Many examples can, however, be given, in which the expended action is, in some of its elements, a maximum; although, even in these cases, the expenditure is a minimum at each instant, or for any sufficiently short portions of the paths of the bodies.

317. This *principle of least action* was first deduced by MAUPERTUIS, through an *à priori* argument from the general attributes of Deity, which he thought to demand the utmost economy in the use of the powers of nature, and to permit no needless expenditure or any waste of action. This grand proposition, which was announced by its illustrious author, with the seriousness and reverence of a true philosopher, is the more remarkable that, derived from purely metaphysical doctrines, and taken in combination with the law of power which likewise reposes directly upon a metaphysical basis, it leads, at once, to the usual form of the dynamical equations.

318. To deduce the dynamical equations from the combination of the principles of least action and living forces, add together the two variations of T,

$$\delta T = \delta \Omega,$$

$$\delta T = \Sigma_\eta (\omega \delta \eta' + D_\eta T_{\eta,\eta'} \delta \eta)$$

$$= \Sigma_\eta (D_\eta T_{\eta,\eta'} - D_t \omega) \delta \eta + D_t \Sigma_\eta (\omega \delta \eta).$$

If the sum is introduced into the variation of V, the result, reduced by the condition that at the limits of integration,

$$\delta \eta = 0,$$

becomes

$$\delta V = \Sigma_\eta \int_t (D_\eta T_{\eta,\eta'} - D_t \omega + D_\eta \Omega) \delta \eta = 0.$$

The factor of $\delta\eta$, in this expression, must vanish by the principles of the method of variations, which gives immediately the general expression of LAGRANGE's canonical forms.

PRINCIPAL FUNCTION AND OTHER SIMILAR FUNCTIONS.

319. The function S determined by the equation

$$S = V - Ht = \int_t (T + \Omega),$$

is called by HAMILTON *the principal function,* and its variation deduced from that of V is, obviously,

$$\begin{aligned}\delta S &= \delta V - t\delta H - H\delta t\\ &= \Sigma_\eta(\omega\delta\eta - \omega_0\delta\eta_0) - H\delta t.\end{aligned}$$

Hence, if S is regarded as a function of η, η_0, ω, ω_0, etc., ... with the time t, its derivatives are

$$\begin{aligned}&D_\eta S = \omega, \quad D_{\eta_0} S = -\omega_0,\\ &D_t S = -H.\end{aligned}$$

The principal function may, therefore, be used in the same way with the characteristic function in the determination of the motion of the system.

320. Many other functions, as suggested by HAMILTON, can be substituted for the principal and characteristic functions. Thus the function

$$W = \int_t \Sigma_\eta (\eta\,\omega'),$$

gives

$$\begin{aligned}\delta W &= \int_t \Sigma_\eta (\omega'\delta\eta + \eta\,\delta\omega')\\ &= \int_t \Sigma_\eta (\eta\,\delta\omega' + D_\eta\Omega\,\delta\eta - D_\eta T_{\eta,\,\omega}\delta\eta)\end{aligned}$$

$$=\int_t \Sigma_\eta(\eta\delta\omega' + D_\eta T_{\eta,\eta'}\delta\eta) + \int_t \delta\Omega$$
$$=\int_t \Sigma_\eta(\eta\delta\omega' + \eta'\delta\omega) + \int_t(\delta\Omega - \delta T)$$
$$=\Sigma_\eta\int_t D_t(\eta\delta\omega) - \int_t \delta H$$
$$=\Sigma_\eta(\eta\delta\omega - \eta_0\delta\omega_0) - t\delta H.$$

Hence,

$$D_\omega W = \eta, \quad D_{\omega_0} W = -\eta_0,$$
$$D_H W = -t.$$

321. The introduction of

$$Q = W + tH = \int_t(\Sigma_\eta(\eta\omega') + H),$$

gives, in like manner,

$$D_\omega Q = \eta, \quad D_{\omega_0} Q = -\eta_0,$$
$$D_t Q = H.$$

322. Other functions can be formed by the combination of V and W, or S and Q. The combination may be such that for some of the coördinates, the function shall have the same form as V or S, while for the remaining coördinates it shall have the form of W or Q, and the function

$$U = V' - W'',$$

or

$$P = S' - Q'',$$

can be substituted for V or S.

PARTIAL DIFFERENTIAL EQUATIONS FOR THE DETERMINATION OF THE CHARACTERISTIC, PRINCIPAL, AND OTHER FUNCTIONS OF THE SAME CLASS.

323. By substituting in the equation

$$T = \Omega + H$$

for η, ω, etc., as well as for t and H, their equivalent expressions, as partial derivatives of V, S, W, Q, U, and P, partial differential equations are obtained, the integrals of which give the values of these functions. To facilitate the expression of this substitution, T and Ω may be assumed to have such functional significations that

$$T(\eta,\omega) = T_{\eta,\,\omega},$$
$$\Omega\,(t,\eta) = \Omega.$$

The partial differential equations are, then,

$$T(\eta, D_\eta V) = \Omega\,(D_H V, \eta) + H,$$
$$T(\eta, D_\eta S) = \Omega - D_t S,$$
$$T(D_\omega W, \omega) = \Omega\,(- D_H W, D_\omega W) + H,$$
$$T(D_\omega Q, \omega) = \Omega\,(t, D_\omega Q) + D_t Q.$$

324. When the independent elements of position are the rectangular coördinates of the bodies, these equations become, by the notation of (54_{31}),

$$\Sigma_m\left(\frac{1}{m}\square V\right) = 2\,\Omega\,(D_H V, x) + 2\,H,$$

$$\Sigma_m\left(\frac{1}{m}\square S\right) = 2\,\Omega - 2\,D_t S,$$

$$\Sigma_m m\,(x'^2 + y'^2 + z'^2) = 2\,\Omega\left(- D_H W,\ \frac{1}{m} D_{x'} W\right) + 2\,H.$$

$$\Sigma_m m(x'^2 + y'^2 + z'^2) = 2\Omega\left(t, \frac{1}{m} D_{x'} Q\right) + 2 D_t Q.$$

325. Through the preceding investigations, the forms are developed by which every dynamical problem can be expressed in differential equations. It only remains, therefore, before applying these forms to especial problems, to consider those methods of integration which are best adapted to their discussion.

CHAPTER X.

INTEGRATION OF THE DIFFERENTIAL EQUATIONS OF MOTION.

326. In discussing the differential equations of motion, it might be permitted to suppose a previous knowledge of all that has been written upon the integral calculus. But since the profound philosophical views, with which this subject has been illuminated by Jacobi, have not yet passed from the original memoirs into the text-books, a development of them is required by the plan of the present work to facilitate its further progress.

I.

DETERMINANTS AND FUNCTIONAL DETERMINANTS.

327. If $(n+1)^2$ different quantities are given, which are represented by

$$a_k^{(i)},$$

in which every number from 0 to n can be substituted for k or for the number of accents denoted by i; and if all possible products of $(n+1)$ factors are formed similar to

$$\pm a a_1' a_2'' \ldots\ldots a_n^{(n)},$$

in each of which the same number is never repeated, either for k or for i; and if these products are successively formed by mutually interchanging two of the inferior numbers, and at the same time reversing the sign of the product; the sum of the products has been called by GAUSS *the determinant of the given quantities, and may be represented by*

$$\mathfrak{R}_n = \Sigma \pm a a_1' a_2'' \ldots\ldots a_n^{(n)}.$$

Thus, for example,

$$\mathfrak{R}_0 = \Sigma \pm a = a,$$
$$\mathfrak{R}_1 = \Sigma \pm a a_1' = a a_1' - a_1 a',$$
$$\mathfrak{R}_2 = \Sigma \pm a a_1' a_2'' = a a_1' a_2'' - a a_2' a_1'' + a_1 a_2' a'' - a_1 a' a_2'' + a_2 a' a_1'' - a_2 a_1' a''.$$

The same result might also have been produced by mutually interchanging the accents without disturbing the inferior numbers.

328. The sign of the determinant would be reversed, by reversing the sign of the product originally assumed as the basis of the subsequent changes.

329. If, for the different values of k, all the given quantities are equal, so that

$$a_k^{(i)} = a_{k'}^{(i)},$$

the determinate vanishes. For, by interchanging k and k' in all the terms, the sign of the determinant is reversed by the regular process of formation, whereas if k is substituted for k' and the reverse, no change is produced on account of the equality of the given terms. Hence

$$\mathfrak{R}_n = - \mathfrak{R}_n,$$

or

$$\mathfrak{R}_n = 0.$$

330. Whenever all those values of the given elements vanish, for which i is as great as m, while k is less than m, which condition may be denoted by the equation

$$a_{k<m}^{(i \geqq m-1)} = 0,$$

the form of the determinant may be simplified. For it is evident from inspection of the fundamental product,

$$(a a'_1 a''_2 \ldots\ldots a_{m-1}^{(m-1)})(a_m^{(m)} a_{m+1}^{(m+1)} \ldots\ldots a_n^{(n)}),$$

regarded as separated into two factors, that every elementary product, produced by an interchange between the inferior numbers, such as to transfer one of these numbers into the second factor, vanishes, and may be neglected. Hence

$$\begin{aligned}\mathfrak{R}_n &= \Sigma \pm a a'_1 a''_2 \ldots\ldots a_{m-1}^{(m-1)} . \Sigma \pm a_m^{(m)} a_{m+1}^{(m+1)} \ldots\ldots a_n^{(n)} \\ &= \mathfrak{R}_{m-1} \mathfrak{R}_{m,n},\end{aligned}$$

if

$$\mathfrak{R}_{m,n} = \Sigma \pm a_m^{(m)} a_{m+1}^{(m+1)} \ldots\ldots a_n^{(n)}.$$

331. When, in the preceding proposition, m is equal to n, so that

$$a_{k<n}^{(n)} = 0,$$

it becomes

$$\mathfrak{R}_n = \mathfrak{R}_{n-1} a_n^{(n)};$$

for, in this case,

$$\mathfrak{R}_{n,n} = \Sigma \pm a_n^{(n)} = a_n^{(n)}.$$

332. When, in addition to the preceding equation, the values of the elements vanish, for which m is equal to $n-1$, so that

$$a_{k<n-1}^{(i>n-1)} = 0,$$

the value of the determinant becomes

$$\mathfrak{R}_n = \mathfrak{R}_{n-1} a_n^{(n)} = \mathfrak{R}_{n-2} a_{n-1}^{(n-1)} a_n^{(n)}.$$

333. Whenever the equation (174_{10}) is true for all values of m, it may be written in the form

$$a_i^{(i>k)} = 0;$$

and the determinant is reduced to the single term

$$\mathfrak{R}_n = a a_1' a_2'' \ldots\ldots a_n^{(n)}.$$

334. If a determinant is formed from the given elements, with the omission of all those of which the number of accents is i, and those of which the inferior number is k, so that n is the number of factors of each elementary product, this determinant is the factor of $a_k^{(i)}$ in the expression of the determinant $\mathfrak{R}_n$. If, therefore, *this partial determinant* is denoted by $\mathfrak{A}_k^{(i)}$, the expression of the complete determinant is

$$\mathfrak{R}_n = \Sigma_k (\mathfrak{A}_k^{(i)} a_k^{(i)}) = \Sigma_i (\mathfrak{A}_k^{(i)} a_k^{(i)}).$$

The derivative of this expression is

$$D_{a_k^{(i)}} \mathfrak{R}_n = \mathfrak{A}_k^{(i)};$$

whence

$$\mathfrak{R}_n = \Sigma_k (a_k^{(i)} D_{a_k^{(i)}} \mathfrak{R}_n) = \Sigma_i (a_k^{(i)} D_{a_k^{(i)}} \mathfrak{R}_n).$$

335. The preceding notation gives

$$\mathfrak{A} = \mathfrak{R}_{1,n} = \Sigma \pm a_1' a_2'' \ldots\ldots a_n^{(n)}.$$

Hence the expression of $\mathfrak{A}_k^{(i)}$ can be deduced from that of $\mathfrak{A}$

by putting

$$i = k = 0\,;$$

and those of $-\mathcal{A}^{(i)}$, or of $-\mathcal{A}_k$, are deduced from that of $\mathcal{A}$ by putting

$$i = 0, \text{ or } k = 0.$$

336. If, in the third member of (175_{21}), $a_{k'}^{(i)}$ is substituted for $a_k^{(i)}$, the expression of the determinant is that which corresponds to the case of § 329. Hence,

$$0 = \Sigma_i(\mathcal{A}_k^{(i)} a_{k'}^{(i)}) = \Sigma_i(a_{k'}^{(i)} D_{a_k^{[i]}} \mathcal{R}_n)\,;$$

and, in the same way,

$$0 = \Sigma_k(\mathcal{A}_k^{(i)} a_k^{(i')}) = \Sigma_k(a_k^{(i')} D_{a_k^{[i]}} \mathcal{R}_n).$$

337. If a partial determinant is formed from the elements of $\mathcal{A}_k^{(i)}$, with the omission of those in which the number of accents is i', and those of which the inferior number is k', this determinant, taken with its proper sign, is the factor of $a_{k'}^{(i')}$ in the value of $\mathcal{A}_k^{(i)}$. If, then, it is denoted by $\mathcal{A}_{k,k'}^{(i,i')}$, the value of $\mathcal{A}_k^{(i)}$ is

$$\mathcal{A}_k^{(i)} = \Sigma_{i'}(\mathcal{A}_{k,k'}^{(i,i')} a_{k'}^{(i')}) = \Sigma_{k'}(\mathcal{A}_{k,k'}^{(i,i')} a_{k'}^{(i')}),$$

in which it must be observed that, from the definition

$$\mathcal{A}_{k,k'}^{(i,i)} = \mathcal{A}_{k,k}^{(i,i')} = 0.$$

These equations give

$$D_{a_{k'}^{[i']}} \mathcal{A}_k^{(i)} = D_{a_{k'}^{[i']}} D_{a_k^{[i]}} \mathcal{R}_n = \mathcal{A}_{k,k'}^{(i,i')},$$

$$\mathcal{R}_n = \Sigma_{i,i'}(\mathcal{A}_{k,k'}^{(i,i')} a_{k'}^{(i')} a_k^{(i)}) = \Sigma_{k,k'}(\mathcal{A}_{k,k'}^{(i,i')} a_{k'}^{(i')} a_k^{(i)})$$

$$= \Sigma_{i,i'}(a_k^{(i)} a_{k'}^{(i')} D_{a_{k'}^{[i']}} D_{a_k^{[i]}} \mathcal{R}_n) = \Sigma_{k,k'}(a_k^{(i)} a_{k'}^{(i')} D_{a_{k'}^{[i']}} D_{a_k^{[i]}} \mathcal{R}_n).$$

338. All the given elements which have k or k' for their inferior number, are excluded from the value of $\mathcal{A}_{k,k'}^{(i,i')}$, and, therefore,

this partial determinant is not affected by the interchange of k and k', by which the terms of the complete determinant, comprehended in

$$\mathcal{A}_{k,k'}^{(i,i')} a_k^{(i)} a_{k'}^{(i')},$$

are transformed into those comprehended in

$$-\mathcal{A}_{k,k'}^{(i,i')} a_{k'}^{(i)} a_k^{(i')}.$$

But this last aggregate of terms is also represented by

$$\mathcal{A}_{k',k}^{(i,i')} a_{k'}^{(i)} a_k^{(i')}.$$

Hence these partial determinants satisfy the equations

$$\mathcal{A}_{k,k'}^{(i,i')} = -\mathcal{A}_{k',k}^{(i,i')} = -\mathcal{A}_{k,k'}^{(i',i)} = \mathcal{A}_{k',k}^{(i',i)}.$$

The determinant may, therefore, be written in the form

$$\begin{aligned}\mathcal{R}_n &= \Sigma_{i,i'>i}\left(\mathcal{A}_{k,k'}^{(i,i')}\left(a_{k'}^{(i')} a_k^{(i)} - a_{k'}^{(i)} a_k^{(i')}\right)\right)\\ &= \Sigma_{k,k'>k}\left(\mathcal{A}_{k,k'}^{(i,i')}\left(a_{k'}^{(i')} a_k^{(i)} - a_k^{(i')} a_{k'}^{(i)}\right)\right).\end{aligned}$$

339. The solution of linear algebraic equations is easily accomplished by the aid of determinants. For if the given $(n+1)$ equations are

$$\begin{aligned}u &= at + a_1 t_1 + \ldots\ldots + a_n t_n = \Sigma_k (a_k t_k),\\ u' &= a' t + a_1' t_1 + \ldots\ldots + a_n' t_n = \Sigma_k (a_k' t_k),\\ u^{(i)} &= a^{(i)} t + a_1^{(i)} t_1 + \ldots\ldots + a_n^{(i)} t_n = \Sigma_k (a_k^{(i)} t_k),\\ u^{(n)} &= a^{(n)} t + a_1^{(n)} t_1 + \ldots\ldots + a_n^{(n)} t_n = \Sigma_k (a_k^{(n)} t_k);\end{aligned}$$

the sum, obtained by adding the products of the given equations, multiplied respectively by $\mathcal{A}_k$, $\mathcal{A}_k'$, ... $\mathcal{A}_k^{(n)}$, is

$$\begin{aligned}\mathcal{R}_n t_k &= \mathcal{A}_k u + \mathcal{A}_k' u' + \ldots\ldots \mathcal{A}_k^{(n)} u^{(n)} = \Sigma_i (\mathcal{A}_k^{(i)} u^{(i)})\\ &= \Sigma_i (u^{(i)} D_{a_k^{(i)}} \mathcal{R}_n).\end{aligned}$$

340. If all the quantities u, u', u'', etc., vanish, t, t_1, t_2, etc., must likewise vanish, unless the determinant vanishes. If, therefore, either of the quantities t, t_1, t_2, etc., does not vanish, when u, u', u'', etc., vanish, the determinant must also vanish, whence the equation (176_{13}) applies even when

$$i' = i,$$

or for all values of i'

$$0 = \Sigma_k(\mathcal{A}_k^{(i)} a_k^{(i')}) = \Sigma_k(t_k a_k^{(i')}).$$

Hence, it is evident that

$$t : t_1 : t_2 \ldots\ldots : t_n = \mathcal{A}^{(i)} : \mathcal{A}_1^{(i)} : \mathcal{A}_2^{(i)} \ldots\ldots : \mathcal{A}_n^{(i)}.$$

341. The process, by which the value of t_n was obtained, may be regarded as designed to eliminate the n quantities $t, t_1, t_2 \ldots\ldots t_{n-1}$ from the given equations. By precisely a similar process, the m quantities $t, t_1, t_2 \ldots\ldots t_{m-1}$ may be eliminated from the first $m+1$ of the given equations, and the form of the resulting equation must be

$$Bu + B'u' + \ldots\ldots + B^{(m)}u^{(m)} = C_m t_m + C_{m+1} t_{m+1} + \ldots\ldots + C_n t_n,$$

in which

$$B_{(m)} = \mathcal{R}_{m-1}, \quad C_m = \mathcal{R}_m.$$

In the same way, if

$$r_n = \Sigma \pm \mathcal{A} \mathcal{A}_1' \mathcal{A}_2'' \ldots\ldots \mathcal{A}_n^{(n)},$$
$$r_{m,n} = \Sigma \pm \mathcal{A}_m^{(m)} \mathcal{A}_{m+1}^{(m+1)} \ldots\ldots \mathcal{A}_n^{(n)},$$

the quantities $u^{(m+1)}, u^{(m+2)}, \ldots\ldots u^{(n)}$ may be eliminated from those of the equations (177_{30}) which give the values of $t_m, t_{m+1}, \ldots\ldots t_n$, and the form of the resulting equation is

$$Eu + E'u' + \dots\dots + E^{(m)}u^{(m)} = F_m t_m + F_{m+1} t_{m+1} + \dots\dots + F_n t_n,$$

in which

$$E^{(m)} = r_{m,n}, \quad F_m = \mathfrak{K}_n r_{m+1,n}.$$

But the two equations, obtained by these processes, must be identical in the ratios of their coefficients. Hence

$$\frac{B^{[m]}}{E^{[m]}} = \frac{C_m}{F_m},$$

or

$$\frac{\mathfrak{K}_{m-1}}{\mathfrak{K}_m} = \frac{r_{m,n}}{\mathfrak{K}_n r_{m+1,n}} = \frac{\mathfrak{K}_n^m r_{m,n}}{\mathfrak{K}_n^{m+1} r_{m+1,n}},$$

or by extending the series of ratios to all values of m,

$$\mathfrak{K}_0 : \mathfrak{K}_1 : \dots\dots : \mathfrak{K}_{n-1} = \mathfrak{K}_n r_{1,n} : \mathfrak{K}_n^2 r_{2,n} : \dots\dots : \mathfrak{K}_n^n r_{n,n}.$$

But it is easily seen that

$$r_{n,n} = \mathfrak{A}_n^{(n)} = \mathfrak{K}_{n-1},$$

and

$$r_{1,n} = D_{\mathfrak{A}} r_n;$$

whence

$$r_{1,n} = D_{\mathfrak{A}} r_n = a \mathfrak{K}_n^{n-1}.$$

A repetition of the same process, in a different order, upon the given equations gives

$$D_{\mathfrak{A}^{(i)}} r_n = a^{(i)} \mathfrak{K}_n^{n-1}.$$

Hence

$$r_n = \Sigma_i (\mathfrak{A}^{(i)} D_{\mathfrak{A}^{(i)}} r_n) = \mathfrak{K}_n^{n-1} \Sigma_i (a^i \mathfrak{A}^i) = \mathfrak{K}_n^n.$$

342. The ratio of the values of r_n and $r_{1,n}$ may be prefixed to the series of ratios of (179_{16}) in the form

$$\mathfrak{R}_n : \mathfrak{R}_0 = \mathfrak{R}_n r_n : \mathfrak{B}_n r_{1,n}.$$

The series of ratios gives, then,

$$\mathfrak{R}_n : \mathfrak{R}_m = r_n : \mathfrak{R}_m^m r_{m+1,n},$$

or

$$r_{m+1,n} = \mathfrak{R}_n^{n-m-1} \mathfrak{R}_m.$$

This investigation is derived from Jacobi.

343. The variation of a function of the quantities represented by $a_k^{(i)}$ is expressed by the formula

$$\delta U = \Sigma_{i,k} (D_{a_k^{(i)}} U \delta a_k^{(i)}).$$

If, then, the values of the quantities, denoted by $u^{(i)}$, are such that

$$u^{(i)} = \delta a_k^{(i)} + (i, k),$$

and if the corresponding values of $t, t_1, \ldots\ldots t_n$ are denoted by $t^{(k)}, t_1^k, \ldots\ldots t_n^{(k)}$, the expression of $t_k^{(k)}$ assumes the form

$$\mathfrak{R}_n t_k^{(k)} = \Sigma_i [D_{a_k^{(i)}} \mathfrak{R}_n (\delta a_k^{(i)} + (i, k))],$$

and therefore

$$\begin{aligned} \mathfrak{R}_n \Sigma_k t_k^{(k)} &= \Sigma_{i,k} [D_{a_k^{(i)}} \mathfrak{R}_n (\delta a_k^{(i)} + (i, k))] \\ &= \delta \mathfrak{R}_n + \Sigma_{i,k} (\mathfrak{A}_k^{(i)} (i, k)). \end{aligned}$$

344. If the given quantities are such that

$$a_k^{(i)} = a_i^{(k)},$$
$$(i, k) = -(k, i),$$

it is readily perceived that

$$\mathcal{A}_k^{(i)} = \mathcal{A}_i^{(k)},$$

$$\mathcal{A}_k^{(i)}(i,k) = -\mathcal{A}_i^{(k)}(k,i),$$

and

$$\Sigma_k t_k^{(k)} = \frac{\delta \mathfrak{R}_n}{\mathfrak{R}_n} = \delta \log. \mathfrak{R}_n,$$

which is given by JACOBI.

345. A system of equations, similar to those of § 339, represented by the form

$$v_k = \Sigma_i (a_k^{(i)} t_i),$$

gives, in the same way,

$$\mathfrak{R}_n t_i = \Sigma_k (v_k D_{a_k^{(i)}} \mathfrak{R}_n).$$

If

$$v_k = \delta a_k^{(i)} + (i,k),$$

an equation similar to (180_{26}) is derived,

$$\mathfrak{R}_n \Sigma_k t_k^{(k)} = \delta \mathfrak{R}_n + \Sigma_{i,k} (D_{a_k^{(i)}} \mathfrak{R}_n (i,k)).$$

346. Let the $(n+1)^2$ quantities, represented by $c_k^{(i)}$, be derived from the given elements $a_k^{(i)}$ and $b_k^{(i)}$ by the formula

$$c_k^{(i)} = a_i b_k + a_i' b_k' + \ldots\ldots a_i^{(p)} b_k^{(p)} = \Sigma_m (a_i^{(m)} b_k^{(m)}),$$

and let the determinant of these quantities be

$$\mathfrak{Q}_n = \Sigma \pm c c_1' c_2'' \ldots\ldots c_n^{(n)}.$$

If only one term is taken in each of the quantities $c_k^{(i)}$, the general term of $\mathfrak{Q}_n$ is represented by

$$\pm a_i^{(m)} a_{i'}^{(m')} a_{i''}^{(m'')} \ldots\ldots b_k^{(m)} b_{k'}^{(m')} b_{k''}^{(m'')} \ldots\ldots$$

A mutual interchange of the letters k, followed by a mutual interchange of the letters i in the resulting terms, produces all the terms of $\mathfrak{Q}_n$, which correspond to the same combination (M) of accents m, m^1, etc. A different combination of accents gives a different set of terms; and if

$$\mathfrak{R}_n^{(M)} = \Sigma \pm a^{(m)} a_1^{(m')} a_2^{(m'')} \ldots\ldots a_{(n)}^{(m^{[n]})},$$
$$\mathfrak{P}_n^{(M)} = \Sigma \pm b^{(m)} b_1^{(m')} b_1^{(m'')} \ldots\ldots b_{(n)}^{(m^{[n]})},$$

denote the determinants of the given elements corresponding to one of these combinations, the complete determinant is expressed by

$$\mathfrak{Q}_n = \Sigma_M (\mathfrak{R}_n^{(M)} \mathfrak{P}_n^{(M)}),$$

which is given by Jacobi.

347. In the case of

$$p = n,$$

there is only one combination (M) of the accents, so that in this case

$$\mathfrak{Q}_n = \mathfrak{R}_n \mathfrak{P}_n,$$

which was given by Cauchy.

When

$$p < n,$$

there is no combination (M), in which all the accents are different from each other, and, therefore, it follows from § 329 that, in this case

$$\mathfrak{Q} = 0,$$

and that, in all cases, the combination (M) must consist of accents which differ from each other.

348. In the special case of

$$a_k^{(i)} = b_k^{(i)},$$

which gives

$$c_k^{(i)} = c_i^{(k)},$$

the value of the determinant is reduced to

$$\mathfrak{Q}_n = \Sigma_M (\mathfrak{R}_n^{(M)})^2,$$

which, when

$$p = n$$

is reduced to

$$\mathfrak{Q}_n = \mathfrak{R}_n^2.$$

FUNCTIONAL DETERMINANTS.

349. If the given elements $a_k^{(i)}$ are the derivatives of $(n+1)$ functions $f, f_1, \ldots\ldots f_n$ of $(n+1)$ variables $x, x_1, \ldots\ldots x_n$, so that

$$a_k^{(i)} = D_{x_k} f_i,$$

the determinant of the elements is called *the functional determinant of the given functions.* Thus, in the present case, all the terms of the determinant

$$\mathfrak{R}_n = \Sigma \pm D_x f \, D_{x_1} f_1 \, D_{x_2} f_2 \ldots\ldots D_{x_n} f_n,$$

are obtained either by a mutual interchange of the variables, or by a mutual interchange of the functions, the interchange being accompanied in either case with a reversal of the sign, precisely as in deducing the terms of the ordinary determinant. The propositions, which have already been given in reference to determinants, are easily applied to functional determinants.

350. In the case in which all the functions, above the $(m+1)$st, are free from the first m variables, the condition of (174_9) is satisfied, so that the notation of (174_{22}) gives the equation (174_{20})

$$\mathfrak{R}_n = \mathfrak{R}_{m-1} \mathfrak{R}_{m,n}.$$

351. In the case in which every function is free from the variables of which the inferior number is less than that of the function itself, the equation (175_{10}) is satisfied, and the functional determinant, reduced to a single term, is

$$\mathfrak{R}_n = D_x f \, D_{x_1} f_1 \, D_{x_2} f_2 \, \ldots\ldots \, D_{x_n} f_n .$$

352. *If the given functions are not independent of each other, the determinant vanishes.* For if the equation, which denotes their mutual dependence, is expressed by

$$\Pi = 0,$$

its derivatives, with regard to the given variables, are represented by the equation

$$\Sigma_k \left(D_{f_k} \Pi \, D_{x_i} f_k \right) = 0 .$$

The equations, included in this form, are identical with the linear equations of § 339 when the values of u vanish and

$$t_k = D_{f_k} \Pi .$$

All these values of t cannot vanish, because the equation, which expresses the mutual dependence of the functions, must involve one or more of them; and, therefore, the determinant must vanish by § 340.

353. *If either of the given functions (f_i) contains any of the other functions, these functions may be regarded as constant in finding the functional determinant.* For each derivative of f_i is the sum of two parts, one of which is derived by direct differentiation with reference to the variable explicitly contained in the function, and the other part is obtained by indirect differentiation through the functions involved in f_i. The whole determinant may then be regarded as composed of two such portions. But the portion of the determinant obtained by the indirect differentiation of f_i is the

same as if f_i, not containing explicitly any variables, were simply a function of the other functions. This portion must, therefore, vanish, and the remaining portion of the determinant is that which is obtained by direct differentiation, conducted as if the functions, involved in f_i, were constant.

This proposition is applicable even where several of the given functions contain the remaining functions; but not when they mutually involve each other.

354. If the second of the given functions contains the first, if the third contains the first and second functions, and if, in general, each function contains all the previous functions, the preceding proposition is applicable. Hence if, by means of the first function, the first variable is eliminated from all the other functions; if, by means of the second function thus reduced, the second variable is eliminated from all the subsequent functions; and if this process is continued until each function is liberated from all the variables designated by an inferior number, although it may involve all the preceding functions; the determinant is reduced to a single term as in § 351. This will often afford a convenient method of obtaining the functional determinant.

355. In performing the successive eliminations, the operation must not be restricted to any prescribed order of the variables, but one of the variables, remaining in f_i, must occupy the place of x_i. Hence there is not one of the factors of the determinant in the form of § 351 which vanishes, unless a function be obtained from which all the variables are explicitly eliminated, or, in other words, unless one of the given functions is included in the others and can be derived from them, so that they are not independent of each other. *If, therefore, the given functions are mutually independent, their functional determinant does not vanish.*

356. If $F, F_1, \ldots\ldots F_n$ are given functions of $f, f_1, \ldots\ldots f_p$,

which are themselves functions of the variables $x, x_1, \ldots\ldots x_n$, the derivatives of the functions (F_i) with respect to the variables (x_i) are represented by the equation

$$D_{x_i} F_k = \Sigma_m (D_{f_m} F_k D_{x_i} f_m).$$

This equation coincides with (181_{24}), if the notation for $a_k^{(i)}$ is combined with the notation

$$c_k^{(i)} = D_{x_i} F_k,$$
$$b_m^{(k)} = D_{f_m} F_k.$$

The remaining notation and conclusions of §§ 346 and 347 may, therefore, be applied to this case. Hence, by (182_{18}) *the functional determinant of the independent functions (F_i), taken with respect to the same number of variables (x_i), which enter into (F_i) only as they are involved in the same number of independent functions (f_i) explicitly involved in (F_i), is obtained by multiplying the functional determinant of (F_i) taken with respect to (f_i) by the functional determinant of (f_i) taken with respect to (x_i).*

If the number $(p+1)$ of functions (f_i) exceeds the number $(n+1)$ of functions (F_i), the complete functional determinant of (F_i) is by (182_{11}) the sum of all the partial determinants of (F_i) obtained by every possible combination of $(n+1)$ of the functions (f_i).

If the number of functions (f_i) is less than that of the functions (F_i), the functional determinant vanishes, as in (182_{25}), which corresponds to the proposition that the number of independent functions cannot exceed the number of variables, by which they may be expressed.

357. In the case, in which

$$F_i = x_i,$$

all the derivatives of (F_i) with reference to the variables (x_i) vanish, except those included in the form

$$D_{x_i} F_i = D_{x_i} x_i = 1.$$

In this case, therefore,

$$\mathfrak{P}_n = \Sigma \pm D_f x D_{f_1} x \ldots\ldots D_{f_n} x_n,$$

is the functional determinant of (x_i) regarded as functions of (f_i), and the equation (182_{18}) becomes

$$\mathfrak{Q}_n = 1 = \mathfrak{P}_n \mathfrak{R}_n,$$

or *the functional determinant of* (x_i) *taken with respect to* (f_i) *is the reciprocal of the functional determinant of* (f_i) *taken with respect to* (x_i).

358. If in the linear equations of § 339, the values of (t) are expressed by the formula

$$t_k = D_{f_i} x_k,$$

either of the equations is represented by

$$u_m = \Sigma_k (D_{x_k} f_m D_{f_i} x_k) = D_{f_i} f_m = 0;$$

unless

$$m = i,$$

in which case

$$u_i = D_{f_i} f_i = 1.$$

This value substituted in (177_{30}) gives

$$\mathfrak{R}_n D_{f_i} x_k = D_{a_k^{(i)}} \mathfrak{R}_n = \mathfrak{A}_k^{(i)}.$$

359. If it is again assumed that

$$t_k^{(i)} = D_{f_k} \delta f_i,$$

the equations of § 345 give

$$v_k = \Sigma_m (a_k^{(m)} t_m^{(i)}) = \Sigma_m (D_{x_k} f_m D_{f_m} \delta f_i) = D_{x_k} \delta f_i$$
$$= \delta D_{x_k} f_i = \delta a_k^{(i)},$$

$$\Sigma_k t_k^{(k)} = \Sigma_k D_{f_k} \delta f_k = \frac{\delta \mathfrak{R}_n}{\mathfrak{R}_n} = \delta \log \mathfrak{R}_n.$$

360. By the same process, it may be proved that, if (f_i) are the variables and (x_i) the independent functions,

$$\Sigma_k D_{x_k} \delta x_k = \delta \log \mathfrak{P}_n = - \delta \log \mathfrak{R}_n.$$

But it must be observed, that in finding the derivatives of δx_k they are supposed to be expressed as functions of the original variables, precisely as in the preceding section the values of δf_k are supposed to be expressed in terms of f_k.

361. The equation (188_9) reduced to the form

$$- \delta \mathfrak{R}_n = \Sigma_k (\mathfrak{R}_n D_{x_k} \delta x_k),$$

may be added to the identical equation

$$\delta \mathfrak{R}_n = \Sigma_k (D_{x_k} \mathfrak{R}_n \delta x_k).$$

The sum is, by (187_{27}),

$$0 = \Sigma_k D_{x_k} (\mathfrak{R}_n \delta x_x)$$
$$= \Sigma_{k,i} D_{x_k} (\mathfrak{R}_n D_{f_i} x_k \delta f_i)$$
$$= \Sigma_{k,i} D_{x_k} (\mathfrak{A}_k^{(i)} \delta f_i).$$

362. In the case, in which the arbitrary variation δ is assumed such that

$$\delta f_i = 0,$$

except for the value

$$i = 0,$$

the preceding equation becomes

$$0 = \Sigma_k D_{x_k} \mathcal{A}_k.$$

If this equation is multiplied by f and added to the equation

$$\mathfrak{R}_n = \Sigma_k (\mathcal{A}_k D_{x_k} f),$$

the sum is

$$\mathfrak{R}_n = \Sigma_k D_{x_k} (f \mathcal{A}_k).$$

363. If the equations, by which the functions (f_i) depend upon the variables (x_i), are represented by

$$F_k = 0,$$

their derivatives are represented by

$$D_{x_i} F_x = - \Sigma_m (D_{f_m} F_k D_{x_i} f_m).$$

The comparison of this equation with (186_4) indicates that the concluding propositions of § 356 may be applied to this case, provided the negative sign is introduced as a factor of all the derivatives taken with respect to (f_i). Hence, if the number of the functions (f_i) is the same with that of the variables (x_i),

$$\Sigma \pm D_x F D_{x_1} F_1 \ldots\ldots D_{x_n} F_n = (-)^{n+1} \mathfrak{R}_n \Sigma \pm D_f F D_{f_1} F_1 \ldots\ldots D_{f_n} F_n,$$

and

$$\mathfrak{R}_n = (-)^{n+1} \frac{\Sigma \pm D_x F D_{x_1} F_1 \ldots\ldots D_{x_n} F_n}{\Sigma \pm D_f F D_{f_1} F_1 \ldots\ldots D_{f_n} F_n}.$$

364. If the number of the functions (f_i) exceeds that of the variables (x_i) and is $p + 1$ instead of $n + 1$, let (F_i^1) be the form of (F_i) when the last $p - n$ of the functions (f_i) are eliminated from it by means of the last $p - n$ of the given equations. In this case

it follows from the reasoning of § 354 that

$$\Sigma \pm D_x F D_{x_1} F_1 \ldots\ldots D_{x_n} F_n D_{f_{n+1}} F_{n+1} \ldots\ldots D_{f_p} F_p$$
$$= \Sigma \pm D_x F^1 D_{x_1} F_1^1 \ldots\ldots D_{x_n} F_n^1 \, \Sigma \pm D_{f_{n+1}} F_{n+1} D_{f_{n+2}} F_{n+2} \ldots\ldots D_{f_p} F_p,$$
$$\Sigma \pm D_f F D_{f_1} F_1 \ldots\ldots D_{f_p} F_p$$
$$= \Sigma \pm D_f F^1 D_{f_1} F_1^1 \ldots\ldots D_{f_n} F_n^1 \, \Sigma \pm D_{f_{n+1}} F_{n+1} D_{f_{n+2}} F_{n+2} \ldots\ldots D_{f_p} F_p.$$

But the equation (189_{26}) is applicable to this case if (F_i) is changed to (F_i^1), and, therefore, the introduction of a common factor into the terms of (189_{26}) gives, by means of the preceding equations,

$$\mathfrak{R}_n = (-)^{n+1} \frac{\Sigma \pm D_x F D_{x_1} F_1 \ldots\ldots D_{x_n} F_n D_{f_{n+1}} F_{n+1}}{\Sigma \pm D_f F D_{f_1} F_1 \ldots\ldots D_{f_p} F_p}.$$

365. There are various interesting and instructive relations between the partial determinants of functions which have been developed by Jacobi, and which will be found useful in discussing the theory of differential equations. If the number of the functions (f_i) as well as of the variables (x_i) is increased to $m + n + 1$, let

$$\mathfrak{B}_k^{(i)} = \Sigma \pm D_x f D_{x_1} f_1 \ldots\ldots D_{x_{n-1}} f_{n-1} D_{x_{n+k}} f_{n+i}.$$

If, then, from the function (f_{n+i}), all the variables $x, x_1, \ldots\ldots x_{n-1}$ are eliminated, and the functions $f, f_1 \ldots\ldots f_{n-1}$ introduced in their places, and the function (f_{n+i}) thus transformed is denoted by (f_{n+i}^1), the values of $\mathfrak{B}$ become

$$\mathfrak{B}_k^{(i)} = \mathfrak{R}_{n-1} D_{x_{n+k}} f_{n+i}^1.$$

The determinant of the $(m+1)^2$ functions $(B_k^{(i)})$ is, consequently,

$$\Sigma \pm \mathfrak{B} \mathfrak{B}_1' \mathfrak{B}_2'' \ldots\ldots \mathfrak{B}_m^{(m)} = \mathfrak{R}_{n-1}^{m+1} \Sigma \pm D_{x_n} f_n^1 D_{x_{n+1}} f_{n+1}^1 \ldots\ldots D_{x_{n+m}} f_{n+m}^1.$$

But it is obvious that

$$\mathfrak{R}_{n+m} = \mathfrak{R}_{n-1} \Sigma \pm D_{x_n} f^1 \ldots\ldots D_{x_{n+m}} f^1_{n+m};$$

whence

$$\Sigma \pm \mathfrak{B}\,\mathfrak{B}'_1 \ldots\ldots \mathfrak{B}^{(m)}_m = \mathfrak{R}^m_{n-1}\,\mathfrak{R}_{n+m}.$$

366. If $\mathfrak{C}^{(i)}_k$ denotes the value which $\mathfrak{R}_{n-1}$ assumes when all the derivatives relatively to x_i are changed into the derivatives relatively to x_{n+k}, it is evidently the factor of $D_{x_i} f_{n+i}$ in the value of $-\mathfrak{B}^{(i)}_k$. In the value, therefore, of the determinant

$$\Sigma \pm \mathfrak{B}\,\mathfrak{B}'_1 \ldots\ldots \mathfrak{B}^{(m)}_m,$$

the factor of $D_x f_n D_{x_1} f_{n+1} \ldots\ldots D_{x_n} f_{n+m}$ is

$$(-)^{m+1} \Sigma \pm \mathfrak{C}\,\mathfrak{C}'_1 \ldots\ldots \mathfrak{C}^{(m)}_m.$$

But the factor of the same quantity in $\mathfrak{R}_{n+m}$ is, by inspection,

$$(-)^{m+1} \Sigma \pm D_{x_n} f D_{x_{n+1}} f_1 \ldots\ldots D_{x_{n+m}} f_m D_{x_{m+1}} f_{m+1} \ldots\ldots D_{x_{n\ 1}} f_{n-1}$$
$$= (-)^{n(m+1} \Sigma \pm D_{x_{m+1}} f D_{x_{m+2}} f_1 \ldots\ldots D_{x_{m+n}} f_{n-1}.$$

It, therefore, follows from (191_5) that

$$\Sigma \pm \mathfrak{B}\,\mathfrak{B}^1_1 \ldots\ldots \mathfrak{B}^{(m)}_m$$
$$= (-)^n \mathfrak{R}^m_{n+1} \Sigma \pm D_{x_{m+1}} f D_{x_{m+2}} f_1 \ldots\ldots D_{x_{m+n}} f_{n-1}.$$

367. The factor of $D_{x_{i-1}} f_{n+i}$ in the value of $\mathfrak{B}^{(i)}_k$ is $-\mathfrak{C}^{(i-1)}_k$, and therefore the determinant

$$(-)^m \Sigma \pm \mathfrak{B}\,\mathfrak{C}_1 \mathfrak{C}'_2 \ldots\ldots \mathfrak{C}^{(m-1)}_m$$

is in (191_{12}) the factor of $D_x f_{n+1} D_{x_1} f_{n+2} \ldots\ldots D_{x_{m-1}} f_{n+m}$. But the factor of this same quantity in $\mathfrak{R}_{n+m}$ is, by inspection,

$$(-)^m \Sigma \pm D_{x_{n+1}} f D_{x_{n+2}} f_1 \ldots\ldots D_{x_{n+m}} f_{m-1} D_{x_m} f_m \ldots\ldots D_{x_n} f_n$$
$$= (-)^{m(n+1)} \Sigma \pm D_{x_m} f D_{x_{m+1}} f_1 \ldots\ldots D_{x_{m+n}} f_n.$$

Hence it follows from (191_5) that

$$\Sigma \pm \mathfrak{B}\, \mathfrak{C}_1 \mathfrak{C}'_2 \ldots\ldots \mathfrak{C}^{(m-1)}_m$$
$$= (-)^{n+1} \mathfrak{R}^m_{n-1} \Sigma \pm D_{x_m} f D_{x_{m+1}} f_1 \ldots\ldots D_{x_{m+n}} f_n.$$

368. By the same process it will be found that, in general,

$$\Sigma \pm \mathfrak{B}\, \mathfrak{B}'_1 \mathfrak{B}''_2 \ldots\ldots \mathfrak{B}^{(i-1)}_{i-1} \mathfrak{C}_i \mathfrak{C}'_{i+1} \ldots\ldots \mathfrak{C}^{(m-1)}_m$$
$$= (-)^{n+i} \mathfrak{R}^m_{n-1} \Sigma \pm D_{x_{m-i+1}} f D_{x_{m-i+2}} f_1 \ldots\ldots D_{x_{m+n}} f_{n+i-1}.$$

369. If the factor of $\mathfrak{B}_k$ in the value of (191_{29}) is denoted by $(-)^n \lambda_k$, this expression gives

$$\Sigma \pm \mathfrak{B}\, \mathfrak{C}_1 \mathfrak{C}'_2 \ldots\ldots \mathfrak{C}^{(m-1)}_m = \Sigma_k (\lambda_k \mathfrak{B}_k),$$

in which neither the quantities $(\mathfrak{C}^{(i)}_k)$, nor any function of them, such as λ_k, contain the derivatives of f_n. Hence the derivative of f_n, with respect to x_{n+k}, only occurs in this expression because it is in $\mathfrak{B}_k$, in which its coefficient is $\mathfrak{R}_{n-1}$, so that the term of the preceding expression which contains this derivative is $\lambda_k \mathfrak{R}_{n-1} D_{x_{n+k}} f_n$. If μ_k is the coefficient of the same derivative in

$$\Sigma \pm D_{x_m} f D_{x_{m+1}} f_1 \ldots\ldots D_{x_{m+n}} f_n,$$

the equation (192_8) gives

$$\lambda_k = (-)^{n+1} \mathfrak{R}^{m-1}_{n-1} \mu_k.$$

The comparison of (192_9) with this equation gives

$$\mathfrak{R}_{n-1} \Sigma \pm D_{x_m} f D_{x_{m+1}} f_1 \ldots\ldots D_{x_{m+n}} f_n = \Sigma_0^m{}_k (\mu_k \mathfrak{B}_k).$$

It is to be observed that, from their definitions, the functions μ_k and $\mathfrak{B}_k$ are both of them partial determinants of the same functions $f, f_1, \ldots . f_{n-1}$ the former being taken with respect to the variables $x_m, x_{m+1} \ldots . x_{n+m}$ excluding x_{n+k}, and the latter being taken with respect to the variables $x, x_1 \ldots . x_{n-1}$ and x_{n+k}.

In the case, therefore, in which m and n are equal, these two determinants are formed with respect to an entirely different set of variables, and each of the variables x_{n+k} is taken in succession from the set $x_n, x_{n+1} \ldots\ldots x_{2n}$ in forming μ_k and combined with the set $x, x_1 \ldots\ldots x_{n-1}$, in forming $\mathfrak{B}_k$.

370. The first member of (193_3) does not contain any derivative of f_n with respect to a variable of which the inferior number is less than m. The factor, therefore, of such a derivative as $D_x f_n$ in the second member vanishes identically; which is represented by the equation

$$\Sigma_k (\mu_k \Sigma \pm D_{x_{n+k}} f D_{x_1} f_1 D_{x_2} f_2 \ldots\ldots D_{x_{n-1}} f_{n-1}) = 0.$$

371. If in the equation (191_3)

$$n = 1,$$

this equation becomes, by writing $n-1$ for m,

$$\mathfrak{R}_n = \mathfrak{R}_0 \Sigma \pm D_{x_1} f_1^1 D_{x_2} f_2^1 \ldots\ldots D_{x_n} f_n^1.$$

But

$$\mathfrak{R}_0 = D_x f,$$

so that if x is supposed to be a function of the other variables and f to be equal to x, these equations are reduced to

$$\Sigma_k(\mathcal{A}_k D_{x_k} f) = \Sigma_k(\mathcal{A}_k D_{x_k} x) = \Sigma \pm D_{x_1} f_1^1 D_{x_2} f_2^1 \ldots\ldots D_{x_n} f_n^1$$
$$= \mathcal{A} + \Sigma_1^n{}_k(\mathcal{A}_k D_{x_k} x);$$

in which

$$\mathcal{A} = \mathcal{R}_{1,n},$$

and, by (176_3), $-\mathcal{A}_k$ is deduced from $\mathcal{A}$ by changing the derivatives relating to x_k into the derivatives relatively to x. This equation is derived from LAGRANGE.

372. In the greater portion of these formulæ upon functional determinants, the derivative taken with regard to either of the variables may be supposed to be frequently repeated, so that $D_{x_k}^h$ may be substituted for D_{x_k}, and h may even be zero. Thus if, in § 365, D_x^0 is substituted for D_x, and if

$$n = 1,$$

the equations of that section are reduced to

$$\mathcal{R}_{n-1} = D_x^0 f = f,$$
$$\mathcal{B}_{k-1}^{(i-1)} = \Sigma \pm f D_{x_k} f_i$$
$$= f D_{x_k} f_i - f_i D_{x_k} f$$
$$= f^2 D_{x_k} \frac{f_i}{f}.$$

Hence if

$$v_i = \frac{f_i}{f}$$

and if n is written for $m + 1$, the equation (190_{31}) becomes

$$\Sigma \pm D_{x_1} v_1 D_{x_2} v_2 \ldots\ldots D_{x_n} v_n = \frac{1}{f^{n+1}} \Sigma \pm f D_{x_1} f_1 D_{x_2} f_2 \ldots\ldots D_{x_n} f_n.$$

If each of the functions (f_i) is multiplied by t, the values of the functions (v_i) remain unchanged, and therefore the value of the determinant

$$\Sigma \pm f D_{x_1} f_1 D_{x_2} f_2 \ldots\ldots D_{x_n} f_n$$

is multiplied by t^{n+1}.

373. A system of functions (f_i) can always be found such that their determinant, with respect to the variables (x_i), may be equal to a given function Π of those variables. For, if all these functions except f_n are assumed at pleasure, and if f_n^1 represents the form of f_n when all the variables except x_n are eliminated and the remaining functions (f_i) are introduced in their place, the required determinant becomes

$$\mathfrak{R}_n = \mathfrak{R}_{n-1} D_{x_n} f_n^1 = \Pi.$$

Hence, f_n^1 is by (187_{10}) determined by the integration

$$f_n^1 = \int_{x_n} \frac{\Pi}{\mathfrak{R}_{n-1}} = \int_{x_n} (\mathfrak{P}_{n-1} \Pi),$$

in which it must be observed that the quantity under the sign of integration is expressed in terms of $f, f_1 \ldots\ldots f_{n-1}$ and x_n.

In the case of

$$\Pi = 1$$

this formula becomes

$$f_n^1 = \int_{x_n} \mathfrak{R}_{n-1}^{-1} = \int_{x_n} \mathfrak{P}_{n-1}.$$

The substance of all these investigations upon determinants is taken without important modifications from Jacobi.

MULTIPLE DERIVATIVES AND INTEGRALS.

374. The functional determinant is shown by JACOBI to be of singular use in the transformation of multiple derivatives and integrals. The expression of these functions is facilitated by the notation

$$D^{n-m+1}_{f_m\ldots} = D^{n-m+1}_{f_m, f_{m+1}} \ldots\ldots f_n,$$

and

$$\int^{n-m+1}_{f_m\ldots} = \int^{n-m+1}_{f_m, f_{m+1}} \ldots\ldots f_n.$$

If then

$$\Omega = D^{n+1}_{f\ldots} W,$$

a new variable x_n, which is a given function of all the variables, f_i may be substituted for either of them as f_n in W, and the new derivative is given by the formula

$$\Omega D_{x_n} f_n = D^n_{f\ldots} D_{x_n} W.$$

Another new variable x_{n-1} may next be introduced instead of f_{n-1} in the same way, and this process may be repeated of substituting successively for each variable f_i a new variable x_i, which shall be a function of all the other variables remaining in the derivative at the instant of the substitution of x_i, until, finally, an entirely new set of variables shall be introduced into the derivative. The final form is

$$\Omega D_x f D_{x_1} f_1 \ldots\ldots D_{x_n} f_n = D^{n+1}_{x\ldots} W.$$

From the comparison of this form with § 351, it appears that the factor of Ω is identical with the determinant of that section. From the reasoning of §§ 353 and 354, it follows that the determinant is not changed by substituting in either of the quantities (f_i) regarded as functions of the variables (x_i) the values of any or all the preceding functions in terms of these variables. But each of the functions (f_i) contains, in its present form, none of the succeeding functions; so that, after this substitution, it is expressed in terms of (x_i). Hence

$$\mathfrak{R}_n D_{f\ldots.}^{n+1} W = D_{x\ldots.}^{n+1} W.$$

375. The preceding equation gives, for the multiple integral

$$\int_{f\ldots.}^{n+1} \Omega = \int_{x\ldots.}^{n+1} (\mathfrak{R}_n \Omega),$$

in which the limiting values of (x_i) may be supposed to be constant, while those of (f_i) may not be constant. If then Π is determined by the integration

$$\Pi = \int_f \Omega,$$

so as to contain neither of the variables (x_i) except as they are involved in (f_i), it is by § 353 unnecessary to have regard to the derivatives of Π otherwise than as they are dependent upon f in finding the value of the determinant, which is the first member of the following equation, and which therefore becomes

$$\Sigma \pm D_x \Pi\, {}^{D_{x_1}} f_1\, {}^{D_{x_2}} f_2 \ldots\ldots {}^{D_{x_n}} f_n = \mathfrak{R}_n D_f \Pi = \mathfrak{R}_n \Omega.$$

But by (189_8)

$$\Sigma \pm D_x \Pi\, {}^{D_{x_1}} f_1\, {}^{D_{x_2}} f_2 \ldots\ldots {}^{D_{x_n}} f_n = \Sigma_k {}^{D_{x_k}} (\Pi \mathfrak{A}_k);$$

and, therefore,

$$\int_{f\ldots}^{n+1}\Omega = \Sigma_k \int_{x\ldots}^{n+1} D_{x_k}(\Pi \mathcal{A}_k) = \Sigma_k \int_{x..x_{k-1}, x_{k+1}..x_n}^{n} \lim_{x_k}(\Pi \mathcal{A}_k),$$

in which lim_{x_k} denotes that the function to which it is prefixed is referred to the limiting values of x_k, so that the difference of the values of the function at these two limits is represented by this notation.

But since

$$\mathcal{A} = \mathcal{R}_{1,n},$$

it is evident from (197_{13}) that

$$\int_{x_1\ldots}^{n} \lim_x(\Pi \mathcal{A}) = \lim_x \int_{f_1\ldots}^{n} \Pi;$$

and a similar equation may be given for each of the terms of the last member of (198_3), whereby this equation is reduced to

$$\int_{f\ldots}^{n+1}\Omega = \Sigma_k \lim_{x_k} \int_{f_1\ldots}^{n} \Pi.$$

The multiple integral of the $(n+1)$ th order is thus reduced to $2n+2$ multiple integrals of the n th order, and this reduction may be continued until the whole process is made to depend upon single integrals, of which one is performed with reference to f, and the number, performed with reference to any other of the variables (f_i), is

$$2^i (n+1) n \ldots\ldots (n+2-i).$$

II.

SIMULTANEOUS DIFFERENTIAL EQUATIONS AND LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

376. An equation

$$f = 0,$$

of which the derivative vanishes identically, by means of the simultaneous differential equations represented by

$$D x_i = X_i,$$

in which (X_i) are given functions of the variables (x_i), is called an *integral* of these equations. It is a *general integral* if it involves arbitrary constants, and a *particular integral* if it does not involve arbitrary constants. When it involves an arbitrary constant, it is more conveniently expressed in the form

$$f = \alpha,$$

in which α is an arbitrary constant.

377. A function f, which satisfies the linear partial differential equation of the first order

$$\Sigma_i (X_i D_{x_i} f) = 0,$$

is called a *solution* of this equation. By means of the notation

$$V_n = \Sigma_{i\,0}^{\;n} (X_i D_{x_i})$$

$$V_{m,n} = \Sigma_{i\,m}^{\;n} (X_i D_{x_i}),$$

this equation may be written

$$V_n f = 0.$$

378. *The first member of every integral, expressed in the form* (199_8) *or* (199_{19}), *of the simultaneous differential equations* (199_{11}) *is a solution of the partial differential equation* (200_2); *and, conversely, every solution of the partial differential equation* (200_2) *is the first member of an integral of the simultaneous differential equations* (199_8), *and its second member is any constant.* For the derivative of (199_8) or (199_{19}) vanishes by the substitution of (199_{11}), which gives

$$\Sigma_i ({}^{D_{x_i}} f D x_i) = V_n f = 0,$$

that is, f satisfies the equation (200_2). Reciprocally, the satisfying of this condition is all that is required in order that (199_{19}) may be an integral of (199_8).

379. If the equation (199_8) is solved relatively to x, so as to express x as a function of the other variables (x_i), the equation (199_{25}) becomes

$$X - V_{1,n} x = 0,$$

which is distinguished from (200_2), because the function x, of which the derivatives are taken, is involved in the functions (X_i), whereas f is not involved in these functions.

380. A solution of (200_2) which shall, for a given equation between the variables, become equal to a given function, may be determined by means of series. For this purpose, let the given equation be

$$t = \tau,$$

in which τ is constant, and t a function of the variables, and let the solution become a function φ of the variables when this equation is satisfied. If then t were assumed to be also one of the variables of the given equation, and such that in forming the simultaneous equations

$$Dt = 1,$$

by which the simultaneous equations become

$$D_t x_i = X_i;$$

and the given partial differential equation is

$$D_t f + V_n f = 0;$$

assume the functional notation

$$\square = -\int_\tau^t V_n,$$

and the integral of the partial differential equation with reference to t is

$$f - \varphi - \square f = 0,$$

which gives

$$(1 - \square) f = \varphi,$$

$$f = \frac{\varphi}{1 - \square} = (1 + \square + \square^2 + \text{etc.})\varphi$$

$$= \Sigma_0^\infty{}_i \, \square^i \varphi.$$

This value of f taken from Cauchy, expresses a true solution of the given equation if $\square^i \varphi$ is finite for all values of i and vanishes when i is infinite, which is always the case for sufficiently small values of $t - \tau$.

381. *There are* n *independent solutions of the partial differential equation* (200_2) *and no more than* n *independent solutions.*

First. The equation (200_2) has n independent solutions. It has been proved in the preceding section that it has one such solution. Let it then be assumed that m such independent solutions have been obtained, denoted by $f_n, f_{n-1} \ldots\ldots f_{n-m+1}$. These independent solutions may be substituted for the m variables, $x_n, x_{n-1} \ldots\ldots x_{n-m+1}$, with regard to which they are independent; and if $f_{x,f}$ denotes the value of f when expressed in terms of the new variables, the equations of substitution are represented by

$$D_{x_i} f = D_{x_i} f_{x,f} + \sum_{n-m+1}^{m}{}_i (D_{f_k} f_{x,f} D_{x_i} f_k).$$

But since

$$V_n f_k = 0,$$

the substitution of these equations in (200_2) reduces it to

$$V_{n-m} f_{x,f} = 0;$$

in which the functions f_k may be regarded as constant. This reduced equation has, then, a solution by the preceding section; its solution does not involve the variables $x_n, x_{n-1} \ldots\ldots x_{n-m+1}$, and is independent of the given m solutions. The given equation is then proved to have another solution independent of the given solutions; and this number may again be increased by the same process, until the n independent solutions are obtained, $f_1, f_2 \ldots\ldots f_n$.

Secondly. The equation (200_2) cannot have more than n independent solutions. For if there are $(n+1)$ solutions (f_i), each gives an equation represented by

$$V_n f_i = 0,$$

which may be regarded as a linear equation between the quantities

(X_i). By the usual process of elimination, if $\mathfrak{R}_n$ denotes the functional determinant of (f_i) with respect to the variables (x_i), these equations give, by § 340,

$$\mathfrak{R}_n X_i = 0.$$

But all the quantities X_i do not vanish, and, therefore,

$$\mathfrak{R}_n = 0,$$

or the $(n + 1)$ functions (f_i) are, by § 355, not independent of each other.

382. It is evident, from the preceding demonstration, *that any function of the solutions of the linear partial differential equation* (200_2) *is itself a solution of that equation.*

383. A system of finite equations, of which the derivatives are satisfied by the simultaneous equations (199_{12}), is called *a system of integral equations of the simultaneous differential equations.* This system is said to be *general,* when, by the successive elimination of the constants, it can be reduced to a form, in which each equation involves an arbitrary constant not included in the other equations, and it is *complete* when the number of finite equations is equal to that of the given differential equations. When reduced in the method just proposed, the general system is represented by

$$\varphi_i = \beta_i,$$

in which the functions (φ_i) are independent of the arbitrary constants (β_i). The *particular* system is represented by a set of similar equations, combined with other equations, which involve no arbitrary constants, and which are represented by

$$\psi_i = 0.$$

384. *Each equation of a general system of integral equations, reduced to the form* (203_{24}), *is an integral of the given simultaneous differential equations.* For the derivative of (203_{24}), when reduced to a finite equation by the substitution of the given differential equations, is independent of the arbitrary constants (β_i), and vanishes, therefore, independently of the equations themselves in which these constants are involved. When the system is general, therefore, the functions (φ_i) are functions of the solutions (f_i) of the partial differential equation (200_2).

385. If the system is particular, and if the number of the equations (203_{31}), which are free from arbitrary constants is $m - n$, the same number of variables can be eliminated, by their aid, from the functions (X_i) and (φ_i). The equations, to which (203_{24}) are thus reduced, are integrals of the simultaneous differential equations, represented by

$$Dx_i = X_i,$$

in which the variables (x_i), of which the number is m, are those which are not eliminated from (X_i) and (φ_i).

386. The system of equations (203_{31}) is, by itself, a particular system of integral equations of the given differential equations, which does not contain any arbitrary constant. For the derivative of either of them, involving no arbitrary constant, must be satisfied by means of the equations (199_{12}) and (203_{31}), without any aid from the equations (203_{24}). The derivative of each of the equations (203_{24}) is, for the same reason, satisfied by the same equations (199_{12}) and (203_{31}), without the assistance of the equations (203_{24}).

387. The functions (f_i) may be supposed to be introduced as the variables instead of the given variables (x_i). By this substitution, the proposed system of differential equations assumes the form

$$Dx = X,$$
$$Df_i = 0.$$

By this same substitution in the equations (203_{24}) and (203_{31}), the equations (203_{31}) may be readily reduced by processes of elimination to an equal number of equations of the form

$$f_i = F_i,$$

in which the functions (F_i) do not involve those of the functions (f_i) of which the values constitute the first members of these equations. Hence the derivatives of these equations, reduced to a finite form by the substitution of (205_2) become of the form

$$D(F_i - f_i) = XD_x F_i = 0,$$

or

$$D_x F_i = 0.$$

But this equation does not involve either of the functions (f_i) which are not contained in (F_i), and, therefore, cannot depend upon the equations (205_8). It is, therefore, identical, the functions (F_i) are independent of x, and the equations (203_{31}) from which they are derived, contain only the functions (f_i). The substitution in (203_{31}) of the arbitrary constants (α_i) for the functions (f_i) to which they are equivalent, reduces these equations to conditional equations between the arbitrary constants. *These equations* (203_{31}), *therefore, represent the conditional equations, to which the arbitrary constants of the integrals of* (199_{12}) *must be subject, in order that they may coincide with the particular system of integral equations, to which the equations* (203_{31}) *belong.* After the introduction of the functions (f_i), instead of the variables (x_i), into the functions (φ_i), these functions (φ_i) can, by the substitution of (205_8), be freed from all the functions (f_i) which are not contained in (F_i). The derivatives of (φ_i) when thus reduced become, by means of the equations (205_2), of the form

$$D\varphi_i = XD_x\varphi_i = 0,$$

which must vanish independently of the equations (205_8), and, therefore, the functions (φ_i) do not involve x. *Hence, by the substitution of* (x_i) *for* (f_i) *the equations* (203_{24}) *give the values of* (β_i) *in terms of* (α_i).

388. From any one given integral equation, denoted by

$$u = 0,$$

the whole system of integral equations, to which it belongs, can be readily obtained. For the finite equation, to which the derivative of this equation is reduced by the substitution of the given differential equations, is, from the very nature of the problem, another of the required system of integral equations. The derivative of this new equation gives a third integral equation, and the continuation of this process leads to *the final determination of the whole of the required system of integral equations.*

389. This process of deriving a system of integral equations from one of its component equations, affords the means of testing a proposed equation, and ascertaining whether it be an integral equation. For as great a number of independent integral equations is not admissible as that of the variables themselves; if, therefore, *the application of the process to a proposed equation conducts to a number of independent equations equal to that of the variables, it is a sufficient proof that the proposed equation is not an integral equation.*

390. When a system of integral equations contains *superfluous* arbitrary constants, that is, constants, which remain in the functions (φ_i), after the system is reduced to the form given in § 383; such

constants supply the means of obtaining other integral equations which are not contained in the given system. Thus if (206_{11}) denotes an integral equation, from which the proposed system may be supposed to be derived, so that, reciprocally, this equation may be derived from the proposed system, and, therefore,

$$u = F(\varphi_i - \beta_i, \psi_i) = 0,$$

in which F is any arbitrary function; and if the notation is adopted

$$V_\beta = \Sigma_i (\gamma_i D_{\beta_i} u),$$

in which arbitrary constants are denoted by (γ_i); the equation

$$V_\beta u = 0$$

is also an integral equation. For the equation

$$Du = 0$$

gives, by direct differentiation,

$$DV_\beta u = 0.$$

But it is obvious, from the form of (207_6), that the derivatives of u with reference to those of the constants (β_i), which are eliminated from the functions (φ_i) and to which these functions are equal, are, themselves, functions of $(\varphi_i - \beta_i)$ and ψ_i; whereas the derivatives of u with reference to the superfluous independent constants (β_i), which are contained in the functions (φ_i), are not merely functions of $(\varphi_i - \beta_i)$ and ψ_i. Hence the integral equation (207_{13}) is a new equation, if it contains the derivative of u with reference to either of the superfluous constants (β_i), and there are as many of these new equations as there are superfluous constants. But the number of independent integral equations thus obtained, is, of course, subject to the condition, that it cannot exceed the number (n) of the independent solutions of the equation (200_2).

391. Of all systems of integral equations, that, in which the arbitrary constants are the values which the variables themselves assume for a given value of one of them, deserves especial consideration. To simplify the discussion of this case, and place it in the position, in which it will best illustrate the problems of mechanics, the variable (x), of which the value is given, may denote the *time*, and the given time is the *epoch* or origin, at which the elements of the system of variables are given, and from which the variations are estimated. The values of the variables at this beginning of time may be termed their *initial* values, while those at any subsequent time are their *final* values. The differential equations express the laws of change, under which the variables pass from their initial to their final values, and are equally compatible with any proposed combination of initial values. *The initial values are, therefore, wholly arbitrary and independent. Their number is equal to that of the variables* (x_i), *and, consequently, equal to the whole number of independent arbitrary constants, which is required for the complete integral equations.*

The epoch is also arbitrary, and seems to introduce an additional arbitrary constant. But this constant is obviously superfluous; it corresponds to the arbitrary position of the problem in time, without involving any modification of the essential conditions; and is the complement of the arbitrary element, which is not expressed, and in reference to which the derivatives in the equations (199_{12}) are supposed to be taken.

392. The passage, down the stream of time, from the initial to the final values, conformably to the conditions of change expressed in the differential equations, may be imagined to be reversed and, in a retrograde transit, the same laws of change would, by their reverted action, restore the variables to their initial values. In the

direct action, the initial values constitute the cause, and the final values are the effect; whereas, in the reverted action, the final values become the cause of which the initial values are the effect. Hence it follows that, *in any integral equation between the final and the initial values of the variables, the final and initial values of each variable may be mutually interchanged, and the resulting equation, if not identical with the given equation, is a new integral equation. In making this change, the sign of the variable, which expresses the interval of time, must be reversed,* because the interval, which is positive with reference to the initial epoch, is negative with reference to the final epoch. If, indeed, the interval were expressed, by means of the initial value (x_0) and the final value (x) of the time, in the form ($x - x_0$), its sign is directly reversed by the mutual interchange of the initial and final values, which transforms its expression to ($x_0 - x$).

393. Let F^0 denote the form, which any function F of the final and initial values of the variables assumes after the mutual interchange of these values; and let

$$x_i^0 = \varphi_i,$$

represent the system of integral equations reduced so that the functions (φ_i) do not involve the initial values (x_i^0). The interchange of the initial and final values in this system, produces a system of integral equations in which each variable is expressed in terms of that one variable, which represents the time, and of the arbitrary constants which are the initial values of the variables. This new system is represented by

$$x_i = \varphi_i^0.$$

394. The discussion has, hitherto, been limited to differential equations of the first order, but it can, readily, be extended so as to

embrace those of higher orders. If, for instance, the equations are given in the form

$$D_t^{p_i} x_i = X_i,$$

in which the functions (X_i) may involve all the derivatives of the variables (x_i), which are of an order inferior to (p_i), each of these inferior derivatives may be regarded as an independent variable, expressed by the form

$$x_i^{(q)} = D_t^q x_i.$$

With this new system of variables the given equations are replaced by the differential equations of the first order, represented by

$$D x_i^{(q-1)} = x_i^{(q)},$$
$$D x_i^{(p_i-1)} = X_i,$$
$$Dt = 1.$$

The number of these differential equations of the first order is easily seen to be $(\Sigma_i p_i + 1)$.

395. When the differential equations are not given in the normal form (210_3), they can always be reduced to this form. For this purpose, each of the equations, which contains none of the highest derivatives of the variables, must be differentiated as many times, denoted by α_i, as are necessary to raise it to an order, which contains such derivatives. If the given equations are represented by

$$F_i = 0;$$

the equations, which are thus derived from them, may be expressed by

$$\varphi_i = D_t^{\alpha_i} F_i = 0;$$

in which α_i is zero, when it is applied to an equation which is not differentiated. Each of the derived equations contains at least one of the highest derivatives of the variables, which may be expressed by $D_t^{p_i+\alpha_i}x_i$. The functions (φ_i) should be independent functions of these derivatives; whenever this is not the case, such derivatives can be eliminated from the derived equations, and one or more resulting equations will be obtained in which they are not involved. The independence of the functions (φ_i) can, however, be directly tested by means of their determinant (185_{29}), which vanishes when it is taken with respect to quantities, for which these functions are not independent.

When the functions (φ_i) *are independent with respect to the highest derivatives contained in them, the required normal equations* (210_3) *are obtained from the given equations and their successive derivatives of an order not higher than those of the derived equations* (210_{31}) *by the usual process of elimination.* For,

First, there is a sufficient number of equations, because the number of equations, added to the given equations by differentiation, is $\Sigma_i\alpha_i$ which is the same with the number of derivatives, superior to the order (p_i), the highest of which are to be retained in the normal equations.

Secondly, these equations are independent of each other in respect to the derivatives of the order (p_i), and of the superior orders, and, therefore, sufficient for the required elimination; because if any of the equations of the inferior orders were not independent, their derivatives, which are included in the group, (210_{31}) would not be independent of each other.

396. When the functions (φ_i) are not independent with respect to the highest derivatives contained in them, each of the equations of an inferior order, obtained from the derived equations by elimination, can be substituted for one of the derived equations,

which is necessarily involved in the elimination by which the reduced equation is obtained. If, therefore, one of the given equations is involved in the elimination, the order of the given equations is reduced by the substitution of the given equation. But if all the equations, necessarily involved in the elimination, were derived by differentiation from the given equations; and if a denotes the smallest number of successive differentiations, by which either of these derived equations was obtained; the reduced equation is obviously a derivative of the order (a) of an equation, which can be obtained by direct elimination from those of the given equations, which are of an order inferior by (a) to the derived equations, combined with the derivatives of the other given equations of an inferior order. This reduced equation of an inferior order may, then, be substituted for either of the given equations of a higher order, upon which its elimination necessarily depends. *In all cases, therefore, in which the functions* (φ_i) *are not independent with respect to the highest derivatives contained in them, the order of the given equations can be reduced by the substitution of an equation of an inferior order obtained by elimination between some of the given equations and the derivatives of others, which are of an inferior order.*

397. That the normal forms, obtained by the process of § 395, are, as it was remarked by Jacobi, those which are obtained with the least complexity of operation, is easily perceived without any attempt at demonstration. It is, also, obvious, by what modes of substitution other normal forms can be derived from these, which are equivalent to them in the aggregate order of differentiation, but differ in the distribution of the derivatives. Thus if either of the functions (X_i) is of an order inferior by (q_i) to that of the given equations, it is by (q_i) successive differentiations elevated to an order which contains one or more of the highest derivatives involved

in the normal forms. The (q_i)th derivative of the equation (210_3), after the values of the highest derivatives, given by the normal equations, are substituted in its second member, so that it is expressed in the form

$$D_t^{p_i+q_i} x_i = X_i^{(q_i)},$$

may take the place of this equation in the system of normal equations. If then $D_t^{p_{i'}-q_i} x_{i'}$ is one of the derivatives contained in (X_i), and if the normal equation (210_3) is reduced to the form

$$D_t^{p_{i'}-q_i} x_{i'} = X'_{i'},$$

it may take the place of the equation

$$D_t^{p_{i'}} x_{i'} = X_{i'}$$

in the group of normal equations. By means of (213_{14}) and its derivatives of an order inferior to the (q_i)th, all the other equations may be reduced so as only to contain derivatives of $(x_{i'})$ of an order inferior to the $(p_{i'} - q_i)$th. *The normal system is by this means transformed to another normal system, in which the highest derivative of one of the variables is increased, just as much as that of another of the variables is decreased.*

398. The repetition of the process of the preceding section may be so conducted that one or more of the variables shall finally disappear from the system of normal equations, and the number of equations will be simultaneously diminished to the same amount as that of the variables. The process may be continued, indeed, until only two variables remain, one of which is the variable (t), with respect to which the derivatives are taken; but the reduction to this form involves the greatest prolixity and complexity of computation. There are special cases, however, and particularly that of

linear differential equations, in which this mode of reduction is peculiarly advantageous.

The principal portion of this discussion of differential equations is the combined result of the investigations of Euler, Lagrange, Cauchy, and Jacobi; but an important addition to these researches is now to be developed, for which geometry is eminently indebted to Jacobi.

THE JACOBIAN MULTIPLIER OF DIFFERENTIAL EQUATIONS.

399. The function, which was called by Jacobi *the new multiplier*, in order to distinguish it from the *Eulerian multiplier*, but which, on account of its superior importance, is here distinguished simply as *the multiplier* of a linear partial differential equation of the first order represented by (200_2), is *that function which, multiplied by this equation, renders its first member an exact functional determinant* ($\mathfrak{R}_n$) *of the indefinite function* (f) *and of* n *undefined functions* (f_i) *with respect to the* ($n+1$) *variables* (x_i), *which are the independent variables of the given equation.* On account of the mutual relations of the partial differential equation (200_2) and the simultaneous differential equations (199_{12}), this same function may also be regarded as *a multiplier of the differential equations* (199_{12}); and, for the same reason, it may be considered as *a multiplier of the linear partial differential equation of the first order* (200_{20}) *of* n *independent variables.*

400. If either of the functions (f_i), or any function of these functions, is substituted for f, the determinant vanishes, by § 352, and the equation (200_2) is satisfied. *The functions* (f_i) *are, therefore,* n *independent solutions of the equation* (200_2).

401. If the multiplier of the equation (200_2) is denoted by $\mathcal{M}$, the condition, by which the multiplier is defined, is expressed by

the identical equation

$$\mathcal{M}\,\mathcal{V}_n f = \mathcal{R}_n.$$

The equality of the coefficients of $D_{x_i} f$ in the two members of this identity is, by the notation adopted in the theory of determinants, expressed by the formula

$$\mathcal{M}\,X_i = \mathcal{A}_i.$$

The substitution of this value of $\mathcal{A}_i$ in the equation (189_2) gives the equation

$$\Sigma_i D_{x_i}(\mathcal{M}\,X_i) = 0,$$

which is *a linear partial differential equation of the first order, by which the multiplier is analytically defined.*

402. The defining equation of the multiplier may by (199_{12}) be developed into the form

$$\Sigma_i (X_i D_{x_i}\mathcal{M} + \mathcal{M}\,D_{x_i}X_i) = \Sigma_i (D_{x_i}\mathcal{M}\,D_{x_i} + \mathcal{M}\,D_{x_i}X_i) = 0,$$

or

$$\mathcal{V}_n \mathcal{M} + \mathcal{M}\,\Sigma_i D_{x_i}X_i = D\,\mathcal{M} + \mathcal{M}\,\Sigma_i D_{x_i}X_i = 0.$$

This equation divided by $\mathcal{M}$ becomes

$$\mathcal{V}_n \log \mathcal{M} + \Sigma_i D_{x_i}X_i = D \log \mathcal{M} + \Sigma_i D_{x_i}X_i = 0.$$

If all the variables are regarded as functions of x, and if x is introduced in place of the element of variation, by means of the formula

$$Dx = X,$$

the preceding equation finally assumes the form

$$XD_x \log \mathcal{M} + \Sigma_i D_{x_i} X_i = 0;$$

which is *an equation involving common differentials, by which the multiplier is analytically defined.*

403. The equation (215_8) gives, by (194_8), when

$$i = 0,$$

the value of the multiplier in the form

$$\mathcal{M} = \frac{\mathfrak{R}_{1,n}}{X}.$$

404. If the values of (f_i) are expressed in terms of (x_i), by means of the equations (189_{12}), and if, by reason of the integrals (199_{10}), the constants (α_i) are substituted for (f_i), the value of the multiplier becomes

$$\mathcal{M} = (-)^n \frac{1}{X} \cdot \frac{\Sigma \pm D_{x_1} F_1 D_{x_2} F_2 \ldots\ldots D_{x_n} F_n}{\Sigma \pm D_{\alpha_1} F_1 D_{\alpha_2} F_2 \ldots\ldots D_{\alpha_n} F_n},$$

in which the sign may be rejected at pleasure.

405. In the particular case, in which the equations (189_{12}) assume the form

$$x_i = \varphi_i,$$

in which the functions (φ_i) involve the arbitrary constants (α_i), together with no other variable than x, the value of the multiplier is by (189_{12}) reduced to

$$\mathcal{M} = \frac{1}{X \Sigma \pm D_{\alpha_1} \varphi_1 D_{\alpha_2} \varphi_2 \ldots\ldots D_{\alpha_n} \varphi_n} = \frac{1}{X \Sigma \pm D_{\alpha_1} x_1 D_{\alpha_2} x_2 \ldots\ldots D_{\alpha_n} x_n}$$

$$= \frac{1}{X \Sigma \pm D_{f_1} x_1 D_{f_2} x_2 \ldots\ldots D_{f_n} x_n} = \frac{1}{X \mathfrak{P}_{1,n}};$$

which equation might have been directly deduced from (216_{11}) and (187_{10}).

406. If the functions (F_i) are given independent functions of (f_i), they are independent solutions of the equation (200_2) and give a multiplier $(\mathcal{M}_i)$ different from $\mathcal{M}$, and which is determined by the equation derived from (216_{11}),

$$X\mathcal{M}_i = \Sigma \pm D_{x_1}F_1 D_{x_2}F_2 \ldots\ldots D_{x_n}F_n.$$

This equation, by means of (186_{14}) and (216_{11}), assumes the form

$$\begin{aligned} X\mathcal{M}_i &= \mathcal{R}_{1,n}\Sigma \pm D_{f_1}F_1 D_{f_2}F_2 \ldots\ldots D_{f_n}F_n \\ &= X\mathcal{M}\Sigma \pm D_{f_1}F_1 D_{f_2}F_2 \ldots\ldots D_{f_n}F_n, \end{aligned}$$

which gives

$$\frac{\mathcal{M}_i}{\mathcal{M}} = \Sigma \pm D_{f_1}F_1 D_{f_2}F_2 \ldots\ldots D_{f_n}F_n.$$

The second member of this equation is a function of the functions (f_i), and may be an arbitrary function of these functions, so that it can have n independent values. The equation, therefore, serves to determine $n+1$ independent values of the multiplier $(\mathcal{M}_i)$, which is, by (215_{12}), the whole number of independent values of which it is susceptible. Hence, *the ratio of any two multipliers is a solution of the equation* (200_2). It also follows from this argument that *every solution of the equation* (215_{12}) *is a value of the multiplier.*

407. In the particular case, in which

$$\Sigma_i D_{x_i} X_i = 0,$$

one of the $n+1$ solutions of (215_{12}) is reduced to a constant, so that in this case, the constant must, contrary to the ordinary usage, be included among the solutions of the equation. The constant may be supposed to be unity, *and, therefore, one of the multipliers of the equation* (200_2) *is unity, when the condition* (217_{27}) *is fulfilled, and all the other multipliers are solutions of the equation* (200_2).

408. When the solutions (f_i) of the equation (200_2) are known, the corresponding value of the multiplier may be determined from (216_{11}). But it can be derived by a shorter process, when either of the solutions $(\mathcal{M}_i)$ of (215_{12}) is known, and also the initial value of $\mathcal{A}$. Thus if Π denotes the ratio of $\mathcal{M}_i$ to $\mathcal{M}$, the equation (216_{11}) gives by (194_7),

$$\Pi = \frac{\mathcal{M}_i X}{\mathcal{R}_{1,n}} = \frac{\mathcal{M}_i X}{\mathcal{A}}.$$

When the initial values are substituted in this equation with the notation of § 393, it becomes

$$\Pi^\circ = \frac{\mathcal{M}_i^\circ X^\circ}{\mathcal{A}^\circ}.$$

The value of Π° may, by the elimination of the variables (x_i°) be reduced to a function of the functions (f_i°); and, if in this expression the functions (f_i) are substituted for their initial values (f_i°), the value of Π is reproduced. For the function, which is obtained by this substitution, is a function of (f_i) and therefore a solution of the equation (200_2); and it is, moreover, that particular solution, of which the initial value is the given function Π°.

409. In the especial case, in which the initial values of (f_i) are the variables (x_i), the value of $\mathcal{A}^\circ$ is obviously reduced to unity and the equation (218_{11}) becomes

$$\Pi^\circ = \mathcal{M}_i^\circ X^\circ.$$

410. When, in the differential equations (199_{12}), the arbitrary

element of variation is assumed to be the variable x, the value of X is unity; and, in this case, the equation (218_{11}) becomes

$$\Pi^\circ = \frac{\mathcal{M}_i^\circ}{\mathcal{A}^\circ};$$

which in the case of the preceding section is reduced to

$$\Pi^\circ = \mathcal{M}_i^\circ;$$

and when, moreover, the equation (217_{27}) is satisfied, so that one of the multipliers is unity, this value is still further reduced to

$$\Pi^\circ = 1.$$

411. The arbitrary constants (α_i) may be substituted for the functions (f_i) in the equation (218_{11}), when it is regarded as resulting from the integrals of (199_{12}). By this substitution Π becomes a function of the arbitrary constants, which may be represented by C, and the equation gives, by means of (187_{10}),

$$\mathcal{P}_{1,n} = \Sigma \pm D_{a_1} X_1 D_{a_2} X_2 \ldots\ldots D_{a_n} X_n = \frac{C}{\mathcal{M}_i X}.$$

The logarithm of this equation becomes by the substitution of (216_2), and including C in the constants of integration,

$$\log \Sigma \pm D_{a_1} X_1 D_{a_2} X_2 \ldots\ldots D_{a_n} X_n = \log \frac{1}{X} + \log C - \log \mathcal{M}_i$$
$$= \log \frac{1}{X} + \int_x \frac{\Sigma_i D_{x_i} X_i}{X},$$

in which all the functions (X_i) can evidently be multiplied by any common factor, without disturbing the equality.

412. In the especial case of

$$X = 1$$

the preceding formula becomes

$$\log \Sigma \pm D_{a_1} X_1 D_{a_2} \ldots\ldots D_{a_n} X_n = \int_x \Sigma_i D_{x_i} X_i.$$

413. When simultaneous differential equations are transformed from one system of variables to another, the multiplier usually undergoes a change at the same time, but there are conditions, to which the arbitrary element of differentiation may be subjected, and under which the multiplier remains unchanged. Thus if the new system of variables is represented by (w_i), if the equations (199_{12}), in their new form, are represented by

$$D' w_i = W_i,$$

in which the accented sign of differentiation refers to the new arbitrary element of differentiation, and if

$$\frac{D'}{D} = G,$$

the values of (W_i) become, by (199_{28}) and the preceding formulæ of this section,

$$\begin{aligned} W_i &= G D w_i = G \Sigma_k (D_{x_k} w_i D x_k) \\ &= G \Sigma_k (X_k D_{x_k} w_i) = G V_n w_i. \end{aligned}$$

This value of (W_i), in combination with the formulæ (199_{28}) and (215_2), gives

$$\begin{aligned} \Sigma_i (W_i D_{w_i} f) &= G \Sigma_k [X_k \Sigma_i (D_{w_i} f D_{x_k} w_i)] \\ &= G \Sigma_k (X_k D_{x_k} f) \\ &= G V_n f = \frac{G \mathfrak{R}_n}{\mathfrak{M}}. \end{aligned}$$

If $\mathcal{N}$ is a multiplier of (220_{12}), the defining equation of (215_2) is, in respect to this multiplier,

$$\mathcal{N}\Sigma_i(W_i D_{w_i} f) = \Sigma \pm D_w f D_{w_1} f_1 \ldots\ldots D_{w_n} f_n.$$

The ratio of the equations (220_{27}) and (221_3), reduced by means of (186_{13}) and (187_{10}), gives

$$\begin{aligned} G\frac{\mathcal{N}}{\mathcal{M}} &= \frac{\Sigma \pm D_w f D_{w_1} f_1 \ldots\ldots D_{w_n} f_n}{\mathcal{R}_n} \\ &= \Sigma \pm D_w x D_{w_1} x_1 \ldots\ldots D_{w_n} x_n \\ &= (\Sigma \pm D_x w D_{x_1} w_1 \ldots\ldots D_{x_n} w_n)^{-1}. \end{aligned}$$

If, therefore, the multipliers $\mathcal{N}$ and $\mathcal{M}$ are equal, the value of G becomes G', if

$$\begin{aligned} G' &= \Sigma \pm D_w x D_{w_1} x_1 \ldots\ldots D_{w_n} x_n \\ &= (\Sigma \pm D_x w D_{x_1} w_1 \ldots\ldots D_{x_n} w_n)^{-1}. \end{aligned}$$

414. The equation (215_{25}), applied to the new system of variables (w_i), gives, by means of this equation and (220_{17}), if the multipliers are, for the instant, assumed to be equal,

$$\begin{aligned} \Sigma_i D_{w_i} W_i &= - D' \log \mathcal{M} = - G' D \log \mathcal{M} \\ &= G' \Sigma_i D_{x_i} X_i. \end{aligned}$$

415. If the arbitrary element of differentiation is supposed to be the same in both systems of variables, the values of G, W_i, and $\mathcal{N}$ become

$$\begin{aligned} G &= 1, \\ W_i &= V_n w_i, \\ \mathcal{N} &= G' \mathcal{M}. \end{aligned}$$

416. If the first $m+1$, only, of the variables (x_i) are exchanged for the new variables (w_i), which limitation is expressed by the formula

$$x_{i>m} = w_{i>m},$$

the value of G' is abbreviated to

$$\begin{aligned} G' &= \Sigma \pm D_w x D_{w_1} x_1 \ldots\ldots D_{w_m} x_m \\ &= (\Sigma \pm D_x w D_{x_1} w_1 \ldots\ldots\ldots D_{x_m} w_m)^{-1}. \end{aligned}$$

417. Hence if the arbitrary element of differentiation, common to the two systems, is one of the variables and is expressed by t, so that the remaining variables are still denoted by (x_i) and (w_i), the formula (221_{15}) continues to express the value of G'.

418. If the last $(n-m)$ of the variables (w_i) are solutions of the equation (200_2), the corresponding values of the functions (W_i) vanish by (220_{22}). If the multiplier is also supposed to remain unchanged, the partial differential equation (200_2), by which it is determined, is reduced to

$$\Sigma_0^m{}_i D_{w_i}(\mathcal{M} W_i) = 0.$$

The arbitrary constants (β_i) may, therefore, be substituted for the solutions (w_i), and the value of G' becomes

$$G' = \Sigma \pm D_w x D_{w_1} x_1 \ldots\ldots D_{w_m} x_m D_{\beta_{m+1}} x_{m+1} D_{\beta_{m+2}} x_{m+2} \ldots\ldots D_{\beta_n} x_n.$$

419. But if, instead of the equality of multipliers, the elements of differentiation are identical in the systems, the defining equation is expressed in the slightly different form of

$$\Sigma_0^m{}_i D_{w_i}(G' \mathcal{M} W_i) = 0,$$

in which the functions (W_i) and the multiplier $(\mathcal{N})$ are given by (221_{30}).

420. If the variables (w_i) which are retained, coincide with the original variables (x_i), the equation for the multiplier becomes

$$\Sigma_0^m{}_i D_{x_i} (G \mathcal{M} X_i) = 0,$$

in which

$$G = \Sigma \pm D_{\beta_{m+1}} x_{m+1} D_{\beta_{m+2}} x_{m+2} \ldots\ldots D_{\beta_n} x_n$$
$$= (\Sigma \pm D_{x_{m+1}} w_{m+1} D_{x_{m+2}} w_{m+2} \ldots\ldots D_{x_n} w_n)^{-1}.$$

By the formulæ of this and the two preceding sections the multiplier of the system of differential equations, to which a given system is reduced by means of any of its integrals, can be obtained from the multiplier of the given system. This will, soon, appear to be one of the most important properties of multipliers.

421. If the given differential equations are of an order, which is higher than the first order, and have the normal form (210_3), the equation (215_{25}), by which the multiplier is defined, is simplified by the consideration that

$$D_{x_i^{(q-1)}} x_i^{(q)} = 0.$$

The multiplier of the given equations, or of the equations (210_{15}), *by which they should be replaced, is, therefore, determined by the equation*

$$D \log \mathcal{M} + \Sigma_i D_{x_i^{(p_i-1)}} X_i = 0.$$

422. If the functions X_i do not involve $x_i^{(p_i-1)}$ or if, in general,

$$\Sigma_i D_{x_i^{(p_i-1)}} X_i = 0,$$

unity is one of the values of the multiplier of the given equations.

423. If the given equations have not the formal form, but have the form

$$\varphi_i = 0,$$

such that they involve no derivatives of a higher order than the normal forms, to which they are reducible by immediate elimination without differentiation, the equation for determining the multiplier assumes a simple symbolic form, by means of the notation

$$D_{x_k^{(p_k)}} \varphi_i = a_k^{(i)}$$
$$D_{x_k^{(p_k - 1)}} \varphi_i = \delta a_k^{(i)} = - u_k^{(i)}.$$

For it is to be observed that each of the subsidiary terms, of which the second term of the equation (223_{25}) is the aggregate, is to be obtained from the equations (224_3), by taking their derivatives relatively to $x_i^{(p_i - 1)}$ on the hypothesis that $x_k^{(p_k)}$ are functions of this variable, and thence determining, by elimination, the values of these subsidiary terms. Hence if

$$t_k^{(i)} = D_{x_i^{(p_i - 1)}} x_k^{(p_k)},$$

the derivatives of (224_3), relatively to $x_i^{(p_i - 1)}$ are represented by (177_{24}), provided the letters t of that equation are accented i times, and the number k is written below the u. From the comparison of (180_{18}) with (224_{11}), it appears that (i,k) vanishes in the present case, and that the sign of δ is to be reversed, whence the equation (180_{26}) becomes

$$\Sigma_k t_k^{(k)} = - \delta \log \mathcal{R}_n.$$

The equation (223_{25}) *by which the multiplier is determined, assumes the symbolical form*

$$D \log \mathcal{M} = - \Sigma_k t_k^{(k)} = \delta \log \mathcal{R}_n.$$

424. It may, sometimes, happen that the values of $a_k^{(i)}$ and $\delta a_k^{(i)}$ are such that the sum of $\delta a_k^{(i)}$, and of $\lambda D a_k^{(i)}$, in which λ is constant, is simpler than $\delta a_k^{(i)}$. In this case, if

$$\delta' = \delta + \lambda D,$$

the addition of

$$D \log \mathfrak{R}_n^\lambda = \lambda D \log \mathfrak{R}_n,$$

to the equation (224_{31}) gives *the symbolical form*

$$D \log (\mathfrak{M}\,\mathfrak{R}_n^\lambda) = \delta' \log \mathfrak{R}_n.$$

425. If the given differential equations have the form (210_{27}), so that they cannot be reduced to the normal form without differentiation, the equations (210_{31}), which are derived from them by differentiation, give, by direct elimination, a system of normal forms, which include, as a reduced system, the normal forms finally obtained by the process of § 395. The multiplier of the equations (210_{27}) is determined by the symbolic equation (224_{31}), or (225_{10}), provided that in the values (224_{10}) of $a_k^{(i)}$ and $\delta a_k^{(i)}$ from which $\delta \mathfrak{R}_n$ is constituted, the value of p_k is increased by α_k.

426. The values of $a_k^{(i)}$ and $\delta a_k^{(i)}$ may be determined directly from the equations (210_{27}). For this purpose, if λ is written instead of α in order to avoid the confusion which might arise from the use of α as an arbitrary constant, and if the ingenious notation, which is familiar to the German mathematicians, for the continued product of all the integers from 1 to λ inclusive,

$$\lambda ! = \lambda(\lambda - 1)(\lambda - 2) \ldots\ldots 3.2.1,$$

is adopted, the equations (210_{31}) are represented by

$$\varphi = D_t^\lambda F = 0;$$

and we find, by well-known formulæ,

$$D_{x^{(\kappa)}}\varphi = \sum_{\nu'}^{\kappa}{}_{\nu} D_t^{\lambda}\left[D_{x^{(\nu)}} F D_{x^{(\kappa)}} x^{(\nu)}\right]$$

$$= \sum_{\nu'}^{\kappa}{}_{\nu}\left[\frac{\lambda!}{(\varkappa-\nu)!\,(\lambda-\varkappa+\nu)!} D_t^{\lambda-\kappa+\nu} D_{x^{(\nu)}} F\right].$$

The inferior limit ν' is determined by the condition that neither ν' nor $\lambda - \varkappa + \nu'$ can be negative. Hence

if $\lambda + 1 > \varkappa,\ \nu' = 0,$

if $\lambda - 1 < \varkappa,\ \nu' = \varkappa - \lambda.$

In the former of these two cases the last term is

$$\frac{\lambda!}{\varkappa!\,(\lambda-\varkappa)!} D_t^{\lambda-\kappa} D_x F;$$

but in the latter case it is simply

$$D_{x^{(\kappa-\lambda)}} F.$$

It follows, then, from (224_{10}) that, since F_i does not contain any higher derivative of x_k than p_k,

$$a_k^{(i)} = D_{x_k^{(p_k)}} F_i,$$

$$\delta a_k^{(i)} = D_{x_k^{(p_k-1)}} F_i + \lambda_i D_t a_k^{(i)}.$$

427. The system of normal equations, derived by the process of § 395, is related to the system of normal forms, which has been discussed in the preceding sections, precisely as any reduced system of differential equations is related to that from which it is reduced by means of a portion of its integral equations. The integral equations are, in this case, the equations (210_{27}) and all their derivatives,

which are inferior to the final derivatives expressed by equations (210_{31}), the multiplier of the reduced equations is, consequently, obtained by dividing the multiplier $\mathfrak{M}$ of the equations (210_{31}) by the function G given by the expression (223_9). The functions (w), involved in the value of G, represent the first members of the integral equations (210_{27}) and their derivatives. But it follows from (226_4) and (226_{23}) that

$$D_{x_k^{(p_k+h)}} D_t^h F_i = D_{x_k^{(p_k)}} F = a_k^{(i)}.$$

The equations (210_{27}) may, now, be supposed to be arranged in an order conformable to the orders of the derivatives, by which they are brought to the form (210_{31}), so that those, of which the higher orders of derivative are taken, may precede the equations of which lower orders are taken. Instead of reducing the equations, by a single step, to the final system, the reduction may be accomplished by successive steps; and, at each step, the derivatives of the equations (210_{27}), which are admitted into the group of integrals, may be diminished by unity, while the number of accents of the eliminated variables is also diminished by unity. At the step denoted by h, therefore, the derivatives of those equations (210_{27}) are added to the group of integrals for which the orders of derivative (λ_i) are greater than h. At this step a factor (G_h) of G is also obtained, and all the derivatives of which it is composed are represented by the functions ($a_k^{(i)}$), in which the superior limit of k is the same with that of i. Hence if

$$\lambda_i - \lambda_{i+1} > 0,$$

the value of the factor of G is

$$G_{\lambda-i} = \mathfrak{N}_i^{-1}$$

but if

$$\lambda_i - \lambda_{i+1} = 0$$

this factor is

$$G_{\lambda-i} = G_{\lambda-i-1}.$$

The logarithm of the complete value of G is, therefore,

$$\log G = \Sigma_i [(\lambda_{i+1} - \lambda_i) \log \mathfrak{R}_i].$$

PRINCIPLE OF THE LAST MULTIPLIER.

428. The consideration of the case in which there are two variables, leads to a valuable principle of integration, discovered by Jacobi, and which he called *the principle of the last multiplier.* In the case of two variables, the equation (215_2) becomes

$$\mathcal{M} (X D_x f + X_1 D_{x_1} f) = D_x f D_{x_1} f_1 - D_{x_1} f D_x f_1,$$

which gives

$$D_x f_1 = \mathcal{M} X$$
$$D_{x_1} f_1 = - \mathcal{M} X_1.$$

Hence it is obvious that

$$D f_1 = \mathcal{M} (X D x - X_1 D x_1)$$

or, by integration,

$$f_1 = \int \mathcal{M} (X D x - X_1 D x_1),$$

so that *when the multiplier is known, this equation determines the integral of the two differential equations* (199_{12}) *of two variables, or that of the sin-*

gle equation to which they are equivalent,

$$XD_x x_1 - X_1 = 0,$$

and the multiplier is, in this case, identical with the well-known Eulerian multiplier.

429. *When all the integrals but one of a given system of differential equations* (199_{12}) *are known, of which the multiplier is also given, the last integral is determined by quadratures by the process of the preceding section;* because the multiplier of the two differential equations with two variables, to which the given system may, in this case, be reduced, is determined from the given multiplier by § 418. This is Jacobi's *principle of the last multiplier.*

430. *In the case of* § 380, *in which the element of variation* (t) *is one of the variables, if the functions* (X_i) *do not involve* (t), the equation (201_8) gives

$$t = \int_{x_i} X_i^{-1},$$

from which t can be determined by quadratures, when all the other integrals of the given equations are known, even if the multiplier is not known, provided that X_i is reduced to a function of x_i, by means of the known integrals.

If the multiplier is also known, and if it does not involve t, *the last of the integrals which do not involve* t *can be determined by the process of the preceding section, and, therefore, the two last integrals of the given equations can, in this case, be determined by quadratures.*

But if the given multiplier $(\mathcal{M})$ *involves* t, *a multiplier* $(\mathcal{M}_i)$, *which does not involve* t, *can be derived from all the integrals which do not involve* t, *and the quotient of these two multipliers gives by* § 406, *an integral involving* t, *and which takes the place of* (229_{17}); *so that, in this case, the last integral is determined in a finite form without integration.*

431. This proposition was shown by Jacobi to admit of the following generalization. *If all the functions* (X_i), *in which* i *is greater than* m, *are free from those of the variables* (x_i) *in which* i *is not greater than* m, *and if the remaining functions satisfy the equation*

$$\Sigma_0^m{}_i D_{x_i} X_i = 0,$$

two integrations can always be performed by quadratures, whenever a multiplier is known which does not involve the variables $(x_{i<m+1})$, *but when the given multiplier does involve either of these variables one integration can be performed by quadratures, and another integral is given, immediately, without any process of integration.* For if the given multiplier $\mathcal{M}$ involves only the variables $(x_{i>m})$, it not only satisfies the condition (215_8), but also on account of the equation (230_6)

$$\Sigma_{m+1}^n{}_i D_{x_i}(\mathcal{M} X_i) = 0,$$

and is, therefore, a multiplier of the portion of the equations (199_{12}) in which i is greater than m. This portion of the given equations can, therefore, be first integrated, independently of the remainder of the system, and the last integral of this portion will be obtained by quadratures, because its multiplier is given. But the last integral of the whole system may, also, be obtained by quadratures, because its multiplier is known; so that two of the integrals can be obtained by quadratures.

But if the given multiplier involves any of the variables $(x_{i<m+1})$, the separate integration of that portion of the equations (199_{12}) in which i is greater than m, gives a multiplier of this portion involving only the variables $(x_{i>m})$, which satisfies the equation (230_{15}); and by (230_6) it also satisfies the equation (215_8), so that it is a new multiplier of the given equation. The quotients of these two multipliers gives, by § 406, an integral involving $(x_{i<m+1})$, and which takes the place of the first of the two integrals, which are obtained by quadratures when the given multiplier involves only the variables $(x_{i>m})$.

PARTIAL MULTIPLIERS.

432. Additional to the systems of Eulerian and Jacobian multipliers, and inclusive of them, are those, of which I have given the investigation in GOULD's *Astronomical Journal*, and which I have called *partial multipliers.* The partial multipliers of the differential equations (199_{12}) are represented by $(\mathcal{M}_{i, k_1, k_2, \ldots\ldots k_m})$, in which $i, k_1, k_2, \ldots$ etc. are any different numbers, or by $(\mathcal{M}_{I, K})$, in which I and K denote groups of numbers; and they are defined by the equation

$$P\mathcal{M}_I = \Sigma \pm D_{x_{k_1}} f_1 D_{x_{k_2}} f_2 \ldots\ldots D_{x_{k_m}} f_m$$

in which P is any arbitrary function, $k_1, k_2 \ldots k_m$ are numbers not included in the groups I, and f_1, f_2, etc. are solutions of the equation (200_2). The notation $(\mathcal{M}^{(K)})$ may also be used to denote the multiplier, with the definition that if

$$\mathcal{M}^{(K)} = \mathcal{M}_I,$$

K denotes the group of numbers represented by (k_m).

433. The system of multipliers of (199_{12}), evidently, satisfies the system of differential equations, which are derived from (187_{10}), and represented by

$$0 = \Sigma_i D_{x_i} (P\mathcal{M}_{i, I})$$

in which i includes all the numbers not belonging to the group I.

434. The group of all the numbers not included in the group

(I) with the exception of any two, which may be selected at pleasure, may be denoted by H. The elimination of the corresponding values of X_h from the equations, obtained from (200_2) by the substitution of the various values of (f_i) gives the equations, which are represented by

$$\Sigma_k(\mathcal{M}^{(H,k)} X_k) = 0.$$

This system of equations combined with that of (231_{28}) defines, analytically, the system of partial multipliers.

435. In the formation of the multipliers, a careful regard must be had to their signs, conformably to the rule of formation of determinants, so that in general

$$\mathcal{M}_{i,I,k,K} = - \mathcal{M}_{k,I,i,K}.$$

436. In the special case, in which the group (i, I) of § 433 is reduced to a single number, and in which P is X, the preceding equations become

$$X \mathcal{M}_i = - \mathcal{A}_i,$$
$$- X \mathcal{M}_i + X_i \mathcal{M} = 0,$$
$$0 = \Sigma_i D_{x_i}(X \mathcal{M}_i) = \Sigma_i D_{x_i}(\mathcal{M} X_i);$$

so that, *the multiplier is, in this case, the Jacobian multiplier.*

437. In the case, in which the groups (i, I) of § 433 include the numbers of all the variables but one, and in which P is unity, the equations become

$$\mathcal{M}^{(i)} = D_{x_i} f_1,$$
$$D_{x_i} \mathcal{M}^{(k)} - D_{x_k} \mathcal{M}^{(i)} = 0,$$
$$\Sigma_i(\mathcal{M}^{(i)} X_i) = 0;$$

so that, *the system of multipliers is, in this case, that of the Eulerian multipliers amplified by* Lagrange.

438. The partial multipliers may be denoted as *the first, second, etc., to the last* corresponding to the degree of the determinant which is the second member of the equation (216_{11}). With this designation, the last multiplier coincides with the Jacobian multiplier and gives a last integral of the differential equations, while the first multipliers coincide with the Eulerian, of which each system gives a first integral of those equations. This proposition may be generalized, and it may be shown that *each system of multipliers determines an integral of the given equations by means of quadratures, and holds a place in the rank of multipliers similar to that held by the integral, in the rank of integrals.*

The investigation of the relations of the multipliers of different systems will be found to lead immediately to this proposition, after its truth has been established in the case of the Eulerian multipliers.

439. The deduction of an integral of a system of differential equations (199_{12}), by means of quadratures, from a given system of Eulerian multipliers, is quite a simple process. For the definition of these multipliers in § 437 gives

$$Df = \Sigma_i (\mathcal{M}^{(i)} D x_i).$$

If the quantities represented by (Q_i) are defined by the equation

$$Q_i = \int_{x_i} (\mathcal{M}^{(i)} - D_{x_i} \Sigma_k{}_0^{i-1} Q_k),$$

the required integral is

$$f = \Sigma_i Q_i = \alpha.$$

For the defining equation of Q_i gives

$$D_{x_i} \Sigma_k{}_0^{i} Q_k = \mathcal{M}^{(i)}.$$

Hence it is found by differentiation that (Q_i) is free from all the variables $(x_{k<i})$, for if this is supposed to be proved for $(Q_{k<i})$ it it seen, by (232_{27}), that

$$D_{x_h} D_{x_i} Q_i = D_{x_h}(\mathfrak{M}^{(i)} - D_{x_i} \overset{i-1}{\underset{0}{\Sigma_k}} Q_k) = D_{x_k} \mathfrak{M}^{(i)} - D_{x_i} D_{x_h} \overset{h}{\underset{0}{\Sigma_k}} Q_k$$
$$= D_{x_k} \mathfrak{M}^{(i)} - D_{x_i} \mathfrak{M}^{(h)} = 0.$$

The differential of (233_{27}) is, therefore,

$$Df = \Sigma_i (D_{x_i} \overset{i}{\underset{0}{\Sigma_k}} Q_k Dx_i) = \Sigma_i (\mathfrak{M}^{(i)} Dx_i),$$

which corresponds to the required differential (233_{19}).

440. When the differential equations (199_{12}) are transformed to other variables in the manner which is indicated in § 413, any multiplier of the new system is obtained by the following formula which corresponds to (231_{15}),

$$P' \mathfrak{N}_H = \Sigma \pm D_{w_{h_1}} f_1 D_{w_{h_2}} f_2 \ldots\ldots D_{w_{h_m}} f_m.$$

If, then, the functions (G) are defined by the equation

$$G_I^{(H)} = \Sigma \pm D_{w_{h_1}} x_{i_1} D_{w_{h_2}} x_{i_2} \ldots\ldots D_{w_{h_m}} x_{i_m}$$
$$= (\Sigma \pm D_{x_{i_1}} w_{h_1} D_{x_{i_2}} w_{h_2} \ldots\ldots D_{x_{i_m}} w_{h_m})^{-1},$$

the proposition (186_{20}) gives by (231_{15})

$$P' \mathfrak{N}_H = P \Sigma_I (\mathfrak{M}_I G_I^{(H)}).$$

441. If any of the solutions (f_i) of (200_2) are known, they can be assumed as new variables to take the place of either of the given variables, and the new multipliers must be determined by

the preceding equation. But it is evident that, in this case, the number of elements which compose each of the terms of $(\mathcal{N}_H)$ will be diminished by a number equal to that of the solutions, which are introduced as variables. Hence since m is the number of elements which compose each term of $(\mathcal{M}_I)$, if $(m - 1)$ is that of the known solutions, the number of elements of $(\mathcal{N}_H)$ may be reduced to one, in which case the multipliers $(\mathcal{N}_H)$ become Eulerian and give the mth solution of (200_2) or the mth integral of (199_{12}), by means of quadratures, which corresponds to the proposition of § 438.

III.

INTEGRALS OF THE DIFFERENTIAL EQUATIONS OF MOTION.

442. When the differential equations of motion are expressed in their utmost generality, there is no known integral which is sufficiently comprehensive to embrace them. But the equation (163_{14}) of living forces is an integral, which is applicable to all the great problems of physics, and holds the most important position in reference to investigations into the phenomena of the material world. There are other integrals of great generality, which might be investigated in this place, if they were not clothed with such a character of speciality, that they properly belong to some of the following chapters. The application of Jacobi's principle of the last multiplier to dynamic equations gives results of so general a character, that their investigation cannot appropriately be reserved for any chapter devoted to the consideration of special problems.

THE APPLICATION OF JACOBI'S PRINCIPLE OF THE LAST MULTIPLIER TO LAGRANGE'S CANONICAL FORMS.

443. It follows from the homogeneous nature of T (165_{10}), that each of LAGRANGE's equations (164_{12}), involves one or more of the quantities represented by (η''), and the system of these equations has, therefore, the form represented by (210_{30}). If, then, $(a_k^{(i)})$ denotes the coefficient of (η'_k) in the value of (ω_i), given by (165_5), this value becomes

$$\omega_i = \Sigma_k (a_k^{(i)} \eta'_k),$$

and that of T is by (165_{11})

$$T = \tfrac{1}{2} \Sigma_{k,i} (a_k^{(i)} \eta'_i \eta'_k),$$

so that the functions $(a_k^{(i)})$ only involve the quantities represented by (η) and the time (t), and satisfy the equations

$$a_k^{(i)} = a_i^{(k)}.$$

444. Each of LAGRANGE's equations may be expressed in the form

$$\varphi_i = D_t \Sigma_k (a_k^{(i)} \eta'_k) - \tfrac{1}{2} \Sigma_{h,k} (D_{\eta_i} a_k^{(h)} \eta'_h \eta'_k) - D_{\eta_i} \Omega = 0.$$

Hence, when Ω is only a function of (η_i) and t, the equations (224_{10}) become

$$D_{\eta''_k} \varphi_i = a_k^{(i)},$$

$$D_{\eta'_k} \varphi_i = \delta a_k^{(i)} = D_t a_k^{(i)} + \Sigma_h (D_{\eta_k} a_h^{(i)} \eta'_h - D_{\eta_i} a_h^{(k)} \eta'_h);$$

from which are easily derived the equations

$$D_{\eta''_k} \varphi_i = D_{\eta''_i} \varphi_k,$$
$$\tfrac{1}{2}(D_{\eta'_k} \varphi_i + D_{\eta'_i} \varphi_k) = D_t a_k^{(i)}.$$

The notation

$$(i,k) = \Sigma_h (D_{\eta_k} a_h^{(i)} \eta'_h - D_{\eta_i} a_h^{(k)} \eta'_h),$$

gives

$$(i,k) = -(k,i),$$
$$\delta a_k^{(i)} = D_t a_k^{(i)} + (i,k).$$

In the substitution of these values in (224_{31}), it is evident from (180_{18}), (180_{31}), and (181_6) that the functions (i,k) disappear, and since D takes the place of D_t, (224_{31}) becomes

$$D \log \mathcal{M} = D \log \mathcal{R}_n,$$

and, therefore, since the arbitrary constant may be neglected,

$$\mathcal{M} = \mathcal{R}_n,$$

which holds, even if the equations of condition involve the time.

In all dynamical problems, therefore, in which the forces are independent of the velocities of the moving bodies, a Jacobian multiplier is given directly by the equation (237_{19}), *so that the last integral can always be obtained by quadratures.*

445. Hence, by § 430, *in any dynamical problem, in which the forces and equations of condition are independent of the time as well as of the velocities of the bodies, the two last integrals can be obtained by quadratures.*

446. The substitution of

$$\xi_{3i} = x_i \sqrt{m_i}, \xi_{3i+1} = y_i \sqrt{m_i}, \xi_{3i+2} = z_i \sqrt{m_i},$$

in (164_{20}) and (162_{28}) gives

$$mv_i^2 = \xi'^2_{3i} + \xi'^2_{3i+1} + \xi'^2_{3i+2},$$

$$T = \tfrac{1}{2}\Sigma_i \xi'^2_i .$$

Hence if

$$b_k^{(h)} = D_{\eta_k} \xi_h,$$

the value of $a_k^{(i)}$ is by (236_{15}),

$$a_k^{(i)} = \Sigma_h (b_k^{(h)} b_i^{(h)}),$$

which, combined with §§ 346 and 348, gives

$$\mathcal{M} = \mathcal{R}_n = \Sigma_M (\mathcal{P}_n^{(M)})^2,$$

in which $\mathcal{P}_n^{(M)}$ denotes the functional determinant of a group (M) of ($n+1$) of the functions (ξ_i) relatively to the variables (η_i). It may be observed that if n_1 is the number of bodies of the system, and n_2 the number of conditional equations, the value of n is

$$n = 3n_1 - n_2 - 1.$$

447. If the conditional equations are represented by

$$\Pi = 0,$$

and if

$$D_{\xi_h} \Pi_i = c_h^{(i)};$$

their derivatives with reference to (η_i) are represented by

$$\Sigma_k (c_k^{(i)} b_k^{(h)}) = 0.$$

If then (H) denotes any group of n of the quantities (ξ_i), and

(H,h) denotes a group of $n+1$ of the same quantities in which the group (H) is included, the preceding equations give, by elimination, between all those in which i remains unchanged,

$$\Sigma_h (\mathfrak{P}_n^{(H,h)} c_h^{(i)}) = 0.$$

Since then the group (H,h) is also denoted by $(\mathcal{M})$, if the group of all the remaining quantities (ξ_i) is denoted by (N), if M' and N' are other groups of the same species, and if $(Q^{(N)})$ denotes the determinant of the corresponding values of $(c_h^{(i)})$, the preceding equations give, by elimination,

$$\mathfrak{P}_n^{(M)} \mathfrak{Q}^{(N')} = \mathfrak{P}_n^{(M')} \mathfrak{Q}^{(N)};$$

which, it is easily seen, may be extended to the case of any groups whatever (M and M_i), in which each includes $(n+1)$ of the quantities (ξ_i). If, therefore, some one group is arbitrarily selected and denoted by (M_0), the equation (238_{13}) becomes

$$\mathcal{M} = \left(\frac{\mathfrak{P}_n^{[M_0]}}{\mathfrak{Q}^{[N_0]}}\right)^2 \Sigma_N (\mathfrak{Q}^{(N)})^2.$$

448. If the derivatives of (η_i) relatively to (ξ_i) are denoted by

$$e_h^{(i)} = D_{\xi_h} \eta_i,$$

and if $\mathfrak{E}_n^{(M)}$ denotes the determinant of the values of $(e_h^{(i)})$, which correspond to those of $(b_h^{(i)})$ in $\mathfrak{P}_n^{(M)}$, the derivatives of (η_i) may first be taken with respect to (ξ_i), and if those of (ξ_i) are afterwards taken with respect to (η_i), they give by (186_{20})

$$1 = \Sigma \pm D_\eta \eta D_{\eta_1} \eta_1 D_{\eta_2} \eta_2 \ldots\ldots D_{\eta_n} \eta_n = \Sigma_M (\mathfrak{P}_n^{(M)} \mathfrak{E}_n^{(M)}).$$

Hence, if $\mathcal{N}$ denotes the determinant of all the quantities (Π_i) and (η_i) with reference to (ξ_i), the equation (239_{13}) gives

$$\mathcal{N} = \Sigma_M (\mathcal{Q}^{(N)} \mathcal{C}_n^{(M)}) = \frac{\mathcal{Q}^{(N_0)}}{\mathcal{P}_n^{(M_0)}} \Sigma_M (\mathcal{P}_n^{(M)} \mathcal{C}_n^{(M)}) = \frac{\mathcal{Q}^{(N_0)}}{\mathcal{P}_n^{(M_0)}},$$

which, substituted in (239_{20}) reduces it to

$$\mathcal{M} = \frac{\Sigma_N (\mathcal{Q}^{(N)})^2}{\mathcal{N}_2}.$$

449. If there are no equations of condition, the value of $\mathcal{M}$ is reduced to

$$\begin{aligned} \mathcal{M} &= \frac{1}{\mathcal{N}^2} = \mathcal{P}_n^2 \\ &= (\Sigma \pm D_\eta \xi \, D_{\eta_1} \xi_1 \ldots\ldots D_{\eta_n} \xi_n)^2 \\ &= (\Sigma \pm D_\xi \eta \, D_{\xi_1} \eta_1 \ldots\ldots D_{\xi_n} \eta_n)^{-2}. \end{aligned}$$

If in this case, therefore, the values of (η_i) *coincide with those of* (ξ_i), *the multiplier is reduced to unity.*

450. If the equations of motion were given in the system of § 310, in which the forces, represented by the equations of condition, are included in those of Ω, this system might, by means of the equations of condition, be reduced to that of Lagrange's canonical forms. In performing this reduction, the equations of condition hold the same relation to the differential equations, which the equations (210_{27}) hold to the equations (210_{31}), in performing the reduction of §§ 395 and 425. It is also obvious that

$$D_{\xi_i} \eta_h = D'_{\xi_i} \eta'_h.$$

Hence the divisor by which the multiplier of the first of these systems is reduced to that of the last, is by (228_8), (222_7), and the preceding sections

$$(\Sigma \pm D_{\xi}\eta\, D_{\xi_1}\eta_1 \ldots\ldots D_{\xi_n}\eta_n\, D_{\xi_{n+1}} \Pi\, D_{\xi_{n+2}} \Pi_1 \ldots\ldots D_{\xi_{3n_1}} \Pi_{n_2})^2 = \mathfrak{N}^2;$$

and, therefore, the multiplier of the system, previous to reduction, is by (240_8)

$$\mathfrak{M}\,\mathfrak{N}^2 = \Sigma_N(\mathfrak{Q}^{(N)})^2.$$

451. If the system of differential equations is given in Hamilton's form, (166_3), the equation (215_{25}) for the determination of the multiplier becomes

$$D \log \mathfrak{M} + \Sigma_i (D_{\eta_i} D_{\omega_i} - D_{\omega_i} D_{\eta_i})\, \Pi_{\eta,\omega} = D \log \mathfrak{M} = 0,$$

whence the multiplier of this system is unity.

CHAPTER XI.

MOTION OF TRANSLATION.

452. If the coördinates of the centre of gravity of a system are x_g, y_g, z_g, and if those of any other point are $x_g + x_i$, $y_g + y_i$, $z_g + z_i$, the value of T becomes, by (162_{28}) and (164_{20}) and the conditions of the centre of gravity (155_{19}),

$$\begin{aligned} T &= \tfrac{1}{2}\,(x_g'^2 + y_g'^2 + z_g'^2)\,\Sigma_i m_i + \tfrac{1}{2}\,\Sigma_i\,[m_i(x_i'^2 + y_i'^2 + z_i'^2)] \\ &= \tfrac{1}{2} v_g^2\, \Sigma_i m_i + \tfrac{1}{2}\,\Sigma_i (m_i v_i^2). \end{aligned}$$

Hence the motion of the centre of gravity is determined by the equation, derived from (164_{12}),

$$\Sigma_i m_i D_t x'_g = \Sigma_i m_i D_t^2 x_g = D_{x_g} \Omega,$$

and the corresponding equations for the other axes. The value of Ω may be restricted in this equation to the external forces and those which correspond to the external equations of condition, for the internal forces and equations of condition being dependent solely upon the relative positions of the bodies of the system, are functions of the differences of the corresponding coördinates of the bodies, from which x_g, y_g, z_g disappear.

The motion of the centre of gravity is, therefore, independent of the mutual connections of the parts of the system, and is the same as if all the forces were applied directly at this centre, provided they are unchanged in amount and direction.

453. Since the second member of (242_3) expresses the whole amount of force, acting upon the system and resolved in the direction of the axis of x, this equation expresses that *the motion of the centre of gravity in any direction depends upon the whole amount of external force acting in that direction.*

If, therefore, *the whole amount of external force acting in any direction vanishes, the velocity of the centre of gravity in that direction is uniform.*

MOTION OF A POINT.

454. When the system is reduced to a single point, it becomes a mass united at its centre of gravity, and the only possible motion is that of translation. The position of the point is determined by three coördinates, which, combined with their derivatives and with the time, constitute a system of seven variables, and require, in general, six integrals for the complete determination of the motion of

the point. The differential equations become, in this case, if the mass of the body is assumed to be the unit of mass,

$$D_t^2 x = D_x \Omega,$$

with the corresponding equations for the other axes.

A POINT MOVING UPON A FIXED LINE.

455. The two equations by which the line is defined are two equations of condition, which may be denoted by

$$\Pi = 0, \Pi_1 = 0.$$

Together with their derivatives, they take the place of four of the integrals of § 454. *Of the two remaining integrals, when Ω does not involve the time, both can be determined by quadratures by* § 445.

One of these integrals is, indeed, the equation of living forces (163_{13}), which becomes in this case

$$v^2 = 2(\Omega + \Pi) = (D_t s)^2.$$

The final integral is obtained from this integral by the equation

$$t = \int_s \frac{1}{\sqrt{(2\Omega + 2\Pi)}} = \int_\eta \frac{D_\eta s}{\sqrt{(2\Omega + 2\Pi)}}.$$

456. It follows from (243_{19}) that the velocity of a body only depends upon its initial velocity and the value of the potential at each point of its path; and this conclusion coincides with the proposition of § 58. *In whatever path, therefore, a body moves from one point to another, the increase or decrease of the square of its velocity may be meas-*

ured by that of the potential, when the equations of condition and the forces which act upon the system are, like the fixed forces of nature, independent of the time and the velocity of the body.

457. If there is any point upon the line, beyond which the decrease of the potential exceeds one half of the square of the initial velocity, the body cannot proceed beyond that point. *If there is, in each direction from the initial position of the body upon the line, a limiting point of this description, the motion of the body is restricted to the intervening space.* Since the body can only have the direction of its motion reversed at the limiting points where its velocity vanishes, it must oscillate back and forth upon the whole of the intervening portion of the line, according to the law expressed by the equation (243_{23}).

It is evident from the inspection of the equation (243_{23}), that the time which the body occupies in passing from any point (A) of the line to another point (B), must be the same with that which it occupied in the preceding oscillation in the reverse transit from the point (B) to the point (A); and, therefore, *the entire duration of oscillation must be invariable.*

458. If the line returns into itself, and if there is no point upon it for which the decrease of the potential is as great as the initial power of the body, *the body will continue to move through the whole circuit of the line, and will always return to the same point with the same velocity, so that the period of the circuit will be constant.*

459. When the forces and the equations of condition involve the time, the multiplier becomes by (238_{13})

$$\mathcal{M} = \Sigma_x (D_\eta x)^2$$

and *the last of the integrals, which are required to solve the problem, can be obtained by quadratures.*

THE MOTION OF A BODY UPON A LINE, WHEN THERE IS NO EXTERNAL FORCE. CENTRIFUGAL FORCE.

460. *When the line is fixed, and there is no external force,* Ω *vanishes in* (243_{19}), *and the velocity is, therefore, constant.*

461. In this case, the line may be regarded as the locus of a resisting force, which acts perpendicularly to the line. The plane of x and y may be supposed to be, for each instant, that of the curvature of the line at the position of the body, R may be the resisting force of the line, and ϱ its radius of curvature; and elementary considerations, combined with the equation (164_{25}), give

$$D_t x = D_t s \sin{}^{\rho}_{x} = v \sin{}^{\rho}_{x},$$

$$D_t^2 x = v \cos{}^{\rho}_{x} D_t{}^{\rho}_{x} = v^2 \cos{}^{\rho}_{x} D_s{}^{\rho}_{x} = \frac{v^2}{\varrho} \cos{}^{\rho}_{x} = R \cos{}^{\rho}_{x},$$

whence

$$R = \frac{v^2}{\varrho},$$

so that *the pressure against the line is measured by the quotient of the square of the velocity divided by the radius of curvature,* which is called *the centrifugal force of the body.*

462. If there are external forces, *the whole pressure upon the line is obtained by combining the action of all the external forces resolved perpendicularly to the line, with the centrifugal force.*

463. The centrifugal force cannot be used as a motive power in machinery, for the body moves perpendicularly to the direction of this force; and, therefore, the power communicated by it vanishes, because it is measured by the product of the intensity of the force multiplied by the space through which it acts.

464. If the line is not fixed in position, but has a motion of

translation, the same motion of translation may be attributed to the axes of coördinates, so that the coördinates of the moving origin at any time may be a_x, a_y, a_z, with reference to the fixed axes. If the coördinates of the body with reference to the moving axes are ξ_x, ξ_y, ξ_z, the value of $2\,T$ (164_{21}) becomes

$$\begin{aligned} 2\,T &= \Sigma_x(\xi_x' + a_x')^2 \\ &= D_t s^2 + 2\,w\,D_t s \cos{}^s_w + w^2, \\ &= s'^2 + 2\,w s' \cos{}^s_w + w^2 \end{aligned}$$

if w denotes the velocity of the motion of the origin, and s the length of the line passed over by the body. Hence LAGRANGE'S equation (164_{12}) gives

$$D_t\,(s' + w \cos{}^s_w) = D_s\,(s' w \cos{}^s_w) = s' w\, D_s \cos{}^s_w.$$

But, since the angles which s makes with the axes are independent of the time, the derivative is

$$\begin{aligned} D_t\,(w \cos{}^s_w) = D_t \Sigma_x\,(w_x \cos{}^s_x) &= \Sigma_x\,(w_x' \cos{}^s_x + s' w_x\, D_s \cos{}^s_x) \\ &= \Sigma_x(w_x' \cos{}^s_x) + s' D_s\,(w \cos{}^s_w), \end{aligned}$$

which reduces the preceding equation to

$$D_t s' = -\,\Sigma_x(w_x' \cos{}^s_x) = -\,W \cos{}^s_W,$$

if

$$W = \sqrt{(w_x'^2 + w_y'^2 + w_z'^2)}$$

denotes the acceleration of the line at each instant. Hence it is easy to see that *if the acceleration is perpendicular to the line, the relative velocity of the body to the line is not changed; but if the acceleration is in the direction of the line, the change of relative velocity is exactly equal to the* *acceleration, so that there is, in this case, no change in the actual velocity of the body in space.*

465. It follows, from the preceding investigation, that *if the motion of the line is uniform, the relative velocity of the body and the line remains constant.*

466. It is also apparent from this investigation that *even under the action of external forces, the relative motion of the body to the line may be computed, by regarding the acceleration of the line as a force acting upon the body in a direction opposite to its actual direction.*

467. If the line rotates about a fixed axis, which is assumed to be the axis of z, let

u be the projection of the radius vector upon the plane of xy,
φ the angle which u makes with the rotating axis of x, and
α the velocity of rotation,

and the value of $2T$ becomes

$$\begin{aligned} 2T &= u'^2 + z'^2 + u^2(\varphi' + \alpha)^2 \\ &= (D_t s)^2 + 2u^2\varphi'\alpha + u^2\alpha^2 \\ &= s'^2 + 2u\alpha s' \cos\theta + u^2\alpha^2, \end{aligned}$$

in which θ is the angle, which s makes with the elementary arc $u\,d\varphi$. Hence the derivatives of T are

$$\begin{aligned} D_{s}T &= s' + u\alpha\cos\theta, \\ D_{s}T &= s'\,D_s(u\alpha\cos\theta) + \alpha^2 u \cos{}^{u}_{s}; \end{aligned}$$

and the equation (164_{12}) becomes

$$D_t^2 s = -u\cos\theta\,\alpha' + \alpha^2 u \cos{}^{u}_{s}.$$

The former of the two terms which compose the second member of this equation, is the negative of the acceleration of the rotative velocity resolved in the direction of the arc of the rotating line. The latter term represents the centrifugal force, which corresponds at the body to the rotation (α), and which is also resolved in the direction of the moving arc. But the centrifugal force is purely relative in its character, and arises from the resistance of the body to accompany the curve in its change of motion occasioned by rotation. These terms combined show, then, that in this case, as well as in that of translation, and, consequently, *in every case the relative motion of the body to the line may be obtained by attributing to the body the negative of the acceleration of the line, which occurs at the position of the body; in the case of external forces, their action must be united to that which arises from the acceleration of the line.*

468. In the case of an uniform rotation about a fixed axis, the equation (247_{29}) becomes

$$D_t^2 s = \alpha^2 u \cos {}^u_s = \alpha^2 u D_s u.$$

The integral of the product of this equation, multiplied by $2 D_t s$, is

$$(D_t s)^2 = \alpha^2 (u^2 + A),$$

in which A is an arbitrary constant. Hence it is obvious that

$$\alpha t = \int_s \frac{1}{\sqrt{(u^2+A)}} = \int_u \frac{D_u s}{\sqrt{(u^2+A)}}.$$

469. When the constant (A) is *negative,* the value of u cannot be less than $\sqrt{-A}$; so that when the body approaches the axis, its velocity upon the line is constantly retarded, and vanishes, when its distance from the axis is reduced to $\sqrt{-A}$, after which the direction of the motion is reversed. If the portion of the line, upon which the body moves, extends at each extremity, so as to be at as small a dis-

tance as $\sqrt{-A}$ from the axis, *the body will oscillate upon it with a constant period of oscillation.*

470. When the constant (A) is *positive*, or when it is *negative*, and no portion of the line in the direction, towards which the body is moving, is at so small a distance as $\sqrt{-A}$ from the axis, the motion of the body upon the line will constantly retain the same direction. If, moreover, the curve returns into itself, *the body will always continue to move around it, with a constant period of revolution.*

471. When the constant (A) *vanishes*, the equation (248_{20}) gives

$$D_t s = \alpha u$$

$$\alpha t = \int_s \frac{1}{u} = \int_u \frac{D_u s}{u}.$$

If the curve, also, passes through the axis of rotation, the value of $D_u s$ may be supposed to be constant, while the body is very near the axis, and may be represented by β; so that the motion of the body in the vicinity of the axis is given by the equation

$$\alpha t = \beta \log u.$$

The second member of this equation becomes infinite when u vanishes, and, therefore, *the motion of the body, in this case, is infinitely slow in the immediate vicinity of the axis.*

472. When the rotating line is *straight*, let

p be the distance of its nearest approach to the axis of rotation, and
θ the angle which it makes with the plane of $x\,y$.

If then s is counted from the foot of the perpendicular, which joins the nearest points of the line and the axis of revolution, the value of u^2 is given by the equation

$$u^2 = p^2 + s^2 \cos^2 \theta ;$$

whence (248_{24}) becomes, in this case,

$$a t = \int \frac{1}{\sqrt{(p^2 + A + s^2 \cos^2 \theta)}}$$
$$= \frac{1}{\cos \theta} \log \left[s \cos \theta + \sqrt{(p^2 + A + s^2 \cos^2 \theta)} \right] - \frac{\log (p^2 + A)}{2 \cos \theta} ;$$

in which the arbitrary constant is determined so that t may vanish with s, and this equation is applicable when $(p^2 + A)$ is positive. In this case, the substitution of the notation

$$h^2 = p^2 + A,$$
$$\tan \varphi = \frac{h}{s \cos \theta},$$

reduces the preceding equation to

$$a t \cos \theta = \log \cot \tfrac{1}{2} \varphi .$$

But when $(p^2 + A)$ is negative, the substitution of the notation

$$k^2 = -(p^2 + A),$$
$$\sin \psi = \frac{h}{s \cos \theta},$$

and the determination of the arbitrary constant, so that t may vanish when s has its least possible value of $k \sec \theta$, reduce the equation (250_6) to

$$a t \cos \theta = \log \tan \tfrac{1}{2} \psi .$$

When $(p^2 + A)$ vanishes, the equation (250_6) is reduced to

$$a t \cos\theta = \log \frac{s}{s_0};$$

in which s_0 is the initial value of s. When p also vanishes, the surface described by the line is a right cone, and when it is developed into a plane, *the path, described by the body, becomes a logarithmic spiral.*

473. When the rotating line is *the circumference of a circle which is situated in the plane of rotation*, let

R denote the radius of the circle,

a the distance of the centre of the circle from the origin,

2φ the angle, which the radius of the circle, drawn to the body, makes with that which is drawn in a direction opposite to the origin,

and the equation (248_{24}) becomes

$$a t = \int_{\varphi} \frac{2R}{\sqrt{[A + (R+a)^2 - 4aR\sin^2\varphi]}}.$$

When $A + (R - a)^2$ is *positive*, which corresponds to the case of § 470, let

$$h^2 = A + (R + a)^2,$$
$$\sin^2 i = \frac{4aR}{h^2},$$

and by the notation of elliptic integrals of § 169, the equation (251_{18}) becomes

$$a t = \frac{2R}{h} \mathfrak{F}_i \varphi.$$

When i is so small that its fourth power may be rejected, this equation gives, by an easy reduction,

$$at = (1 + \tfrac{1}{4}\sin^2 i)\frac{2R\varphi}{h} - \frac{R}{4h}\sin^2 i \sin 2\varphi.$$

In this case, therefore, the time of describing the semicircumference, for which 2φ is greater than a quadrant, exceeds the time of describing that for which 2φ is less than a quadrant by

$$\frac{R}{h}\sin^2 i = \frac{4aR^2}{h^3} = \frac{4aR^2}{[A + (R+a)^2]^{\frac{3}{2}}}.$$

When $A + (R - a)^2$ is *negative*, which corresponds to the case of § 469, let

$$\sin^2 i = \frac{h^2}{4aR},$$

$$\sin\theta = \frac{\sin\varphi}{\sin i};$$

and the equation (251_{18}) becomes

$$\begin{aligned} at &= \frac{2R}{h}\int_{\phi}\sec\theta = \frac{2R\sin i}{h}\int_{\theta}\sec\varphi \\ &= \sqrt{\frac{R}{a}}\,\mathfrak{F}_i\theta. \end{aligned}$$

When i is so small that its square may be rejected, the duration of an oscillation becomes

$$T = \frac{\pi}{a}\sqrt{\frac{R}{a}}.$$

When the circumference passes through the axis of rotation, a is equal to R, and the time of the small oscillation becomes identical with that of the semi-revolution of the circle; but the time of a larger oscillation exceeds that of the semi-revolution.

When $A + (R - a)^2$ *vanishes,* the equation (251_{18}) becomes

$$a t = \sqrt{\frac{R}{a}} \int_0 \sec \varphi = \sqrt{\frac{R}{a}} \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\varphi).$$

When $A + (R - a)^2$ is *very small,* and its ratio to $4aR$ is denoted by δA, the equation (251_{18}) gives throughout the greater portion of the path, in which φ differs sensibly from $\frac{1}{2}\pi$, that is, in which the body is not near its point of closest approach to the axis of rotation, so that the square of δA may be neglected,

$$a t = \sqrt{\frac{R}{a}} \int_\phi \frac{1}{\sqrt{(\cos^2 \varphi + \delta A)}} = \sqrt{\frac{R}{a}} \int_\phi \sec \varphi - \tfrac{1}{2}\delta A \sqrt{\frac{R}{a}} \int_\phi \sec^3 \varphi$$

$$= (1 - \tfrac{1}{4}\delta A) \sqrt{\frac{R}{a}} \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2}\varphi) - \tfrac{1}{4}\delta A \sqrt{\frac{R}{a}} \tan \varphi \sec \varphi.$$

But in the vicinity of the point of nearest approach, let

$$\psi = \tfrac{1}{2}\pi - \varphi$$

be so small that its square is of the same order with δA, and the equation (251_{18}) gives

$$a t = -\sqrt{\frac{R}{a}} \int_\psi \frac{1}{\sqrt{(\psi^2 + \delta A)}} = -\sqrt{\frac{R}{a}} \operatorname{Sin}^{[-1]} \frac{\psi}{\sqrt{\delta A}}, \text{ when } \delta A \text{ is positive.}$$

$$= -\sqrt{\frac{R}{a}} \operatorname{Cos}^{[-1]} \frac{\psi}{\sqrt{(-\delta A)}}, \text{ when } \delta A \text{ is negative.}$$

474. *When the rotating line is wholly contained upon the surface of a cylinder of revolution of which the axis is the axis of revolution,* u is constant and the equation (247_{29}) becomes

$$D_\phi s \, D_t^2 s = -u^2 \alpha',$$

from which φ or s may be eliminated by the given equation of the curve.

475. When the velocity of rotation is constant, the second member of (253_{28}) vanishes, and the velocity of the body is consequently uniform.

476. When the curve is a *helix*, the value of $D_\phi s$ is constant, and the equation (253_{29}) gives

$$D_t s = \frac{u^2}{D_\phi s}(A - \alpha),$$

$$s = \frac{u^2}{D_\phi s}(At - \textstyle\int_t \alpha);$$

in which A is an arbitrary constant.

477. When the acceleration is uniform, α' is constant, and the integral of (253_{28}) gives

$$(D_t s)^2 = u^2 \alpha' (A - \varphi),$$

$$u^2 \alpha' t = \int_s \frac{1}{\sqrt{(A - \varphi)}};$$

in which A is an arbitrary constant.

MOTION OF A HEAVY BODY UPON A FIXED LINE. THE SIMPLE PENDULUM.

478. When the line is fixed, and the force which acts upon the body is that of gravity at the surface of the earth, represented by g, and the axis of z is assumed to be the vertical, directed downwards, the equations (243_{19-32}) give

$$v^2 = 2gz + 2H,$$

$$t = \int_s \frac{1}{\sqrt{(2gz + 2H)}} = \int_z \frac{D_z s}{\sqrt{(2gz + 2H)}}.$$

479. If the curve is contained upon the surface of a cylinder of which the axis is vertical, the motion of the body is the same as

it would be upon the plane curve, obtained by the development of the cylinder into a vertical plane; because the value of $D_z s$ is not changed by this development.

480. If the fixed line is straight, the equation (254_{28}) becomes

$$\cos{}^{s}_{z}\, gt = \sqrt{(2gz + 2H)} - \sqrt{(2H)} = v - v_0,$$

if v_0 is the initial velocity of the body.

481. If there is no initial velocity, the preceding equations become

$$\cos{}^{s}_{z}\, gt = \sqrt{(2gs \cos{}^{s}_{z})} = v,$$

or

$$\cos{}^{s}_{z} = \frac{v}{gt} = \frac{2s}{gt^2} = \frac{v^2}{2gs} = \frac{z}{s}.$$

482. If the curve is the circumference of a circle, the centre of the circle may be assumed as the origin of coördinates. If then the axis of z_1 is the intersection of the plane of the circle with the vertical plane, which is drawn perpendicular to it through the origin, and if R is the radius of the circle, and

$$2\varphi = ({}^{r}_{z_1}),$$

the equation (254_{28}) becomes

$$t = \int_\phi \frac{2R}{\sqrt{(2gR\cos{}^{z}_{z_1}\cos 2\varphi + 2H)}} = \int_\phi \frac{2R}{\sqrt{(2H + 2gR\cos{}^{z}_{z_1} - 4gR\cos{}^{z}_{z_1}\sin^2\varphi)}}.$$

If then H is greater than $gR\cos{}^{z}_{z_1}$, which is similar to the case of § 470, let

$$h^2 = 2H + 2gR\cos{}^{z}_{z_1},$$

$$4gR\cos{}^{z}_{z_1} = h^2 \sin^2 i;$$

and the preceding equation becomes

$$t = \frac{2R}{h} \mathfrak{F}_i \varphi .$$

When i is quite small, this equation admits the same reduction with that given (251_{31}—252_3).

If H is smaller than $gR\cos^z_{z_1}$, which is similar to the case of § 469, let

$$h^2 = 4gR\cos^z_{z_1}\sin^2 i$$

$$\sin\theta = \frac{\sin\varphi}{\sin i},$$

and the equation (255_{24}), becomes, by the same reduction with that given in (252_{17}),

$$t = \sqrt{\left(\frac{R}{g\cos^z_{z_1}}\right)} \mathfrak{F}_i \theta,$$

which when i is small gives *for the time of oscillation of the simple pendulum in an oblique plane*

$$T = \pi \sqrt{\frac{R}{g\cos^z_{z_1}}} .$$

If H is just equal to $gR\cos^z_{z_1}$, the equation (255_{24}) becomes

$$t = \sqrt{\left(\frac{R}{g\cos^z_{z_1}}\right)} \log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\varphi\right).$$

The case in which H differs but little from $gR\cos^z_{z_1}$, may be subjected to the same treatment with that adopted in (255_{5-23}).

MOTION OF A HEAVY BODY UPON A MOVING LINE.

483. If the heavy body moves upon a line, which has a motion of translation in space, the equation of motion becomes, by the form of argument and notation adopted in § 464,

$$D_t^2 s = - W \cos_w^s + g \cos_z^s.$$

484. If the motion of the line is uniformly accelerated and invariable in direction, the motion of the body upon the line is the same which it would be if the line were fixed, and the force a constant force which coincided in amount and direction with the resultant of g and $-W$. Thus if the line moves vertically downwards with an accelerated velocity, equal to that of a heavy falling body, the body moves upon the line with an uniform velocity.

485. If the line is *straight*, and if the motion of translation follows a law, dependent exclusively upon the time, so that if

A_t denotes the law, by which the line moves in the direction of its length, the acceleration in the direction of the line is

$$- W \cos_W^s = D_t^2 A_t;$$

and the value of s becomes

$$s = a + bt + \tfrac{1}{2} g t^2 \cos_z^s + A_t,$$

in which a and b are arbitrary constants. The absolute motion of the point in any direction in space, as that of the axis of x_1, is represented by the equation

$$x_1 = (s - A_t) \cos_{x_1}^s + p \cos_{x_1}^p,$$

in which p denotes the perpendicular upon the line from the origin. If the line is vertical, and limited in its motion to the vertical plane of $x_1\, z_1$, and if the axis of z_1 is vertical, the equations which determine the position of the point in space are

$$x_1 = p,$$
$$z_1 = a + bt + \tfrac{1}{2} g t_2.$$

When p increases uniformly so that p' is constant, these equations give

$$x_1 = p' t,$$
$$z_1 = a + \frac{p}{p'} x_1 + \frac{g}{2p'^2} x_1^2,$$

so that the path of the body in space is a parabola, of which the axis is vertical.

486. *If the line moves with an uniform motion in a straight line,* the equation (257_8) gives

$$D_t^2 s = g \cos {}^s_z.$$

The integral of the product if this equation multiplied by $2\, D_t s$ is

$$(D_t\, s)^2 = \int_t 2\, g \cos {}^s_z D_t s = 2 g \int_t D_t\, z = 2 g z + a,$$

in which a is an arbitrary constant. Hence if

V denotes the velocity of the translation of the line,

the square of the velocity of the point in space is

$$(D_t s_1)^2 = [\sqrt{(2 g z + a)} - V \cos {}^V_s]^2 + (V \sin {}^V_s)^2$$
$$= 2 g z + a + V^2 - 2\, V \cos {}^V_s \sqrt{(2 g z + a)}.$$

The augmentation of the power of the moving body above its initial power is, then,

$$P = \tfrac{1}{2}(D_t s_1)^2 - \tfrac{1}{2}(D_t s^\circ)^2 = g(z - z^\circ) - V(v \cos {}^V_s - v^\circ \cos {}^V_{s^\circ}).$$

If the body had moved through the same path upon a fixed curve, the increase of power would have been

$$Q = g(z - z_\circ) + gVt \cos {}^V_z.$$

If P is greater than Q, the excess of P above Q is the power acquired by the body from the accelerating motion of the line. But if Q exceeds P, the excess of Q above P is the power communicated by the body to the line, which involves the theory of many machines, of which heavy bodies are the moving forces. If, for example, the line moves horizontally, the power communicated by the weight is

$$Q - P = V(v \cos {}^V_s - v^\circ \cos {}^V_{s^\circ}).$$

If, moreover, the initial velocity of the body, relatively to the line, vanishes, the expression of the communicated power is reduced to

$$Q - P = V \cos {}^V_s \surd [2g(z - z^\circ)];$$

and when the direction of the line at its extremity coincides with that of its translation, this expression is still further reduced to

$$Q - P = V \surd [2g(z - z^\circ)].$$

487. *If the line is the circumference of a vertical circle*, of which the radius is R, and if φ is the angular distance of the body from the lowest point of the circumference, the equation of motion (257_8) becomes

$$RD_t^2\varphi = -W \cos {}^s_W - g \sin \varphi.$$

When the motion of the line is in a vertical direction this equation becomes

$$R D_t^2 \varphi = -(W + g) \sin \varphi ;$$

which, when φ is very small, is reduced to

$$R D_t^2 \varphi = -(W + g) \varphi ,$$

The integral of this equation is

$$\varphi = A \sin \left(t \sqrt{\tfrac{g}{R}} + b \right),$$

in which A and b may be determined by the equations

$$2 \sqrt{(g R)}\, D_t \log A = - W \sin 2 \left(t \sqrt{\tfrac{g}{R}} + b \right),$$

$$2 \sqrt{(g R)}\, D_t b = W A \left[1 - \cos 2 \left(t \sqrt{\tfrac{g}{R}} + b \right) \right];$$

which give

$$\begin{aligned} \sqrt{(g R)}\, D_t (A \sin b) &= - A\, W \sin \left(t \sqrt{\tfrac{g}{R}} + b \right) \sin \left(t \sqrt{\tfrac{g}{R}} \right) \\ &= \tfrac{1}{2} A\, W \left[\cos \left(2t \sqrt{\tfrac{g}{R}} + b \right) - \cos b \right] \\ &= A\, W \left[\sin^2 \tfrac{1}{2} b - \sin^2 \left(t \sqrt{\tfrac{g}{R}} + \tfrac{1}{2} b \right] ; \\ \sqrt{(g R)}\, D_t (A \cos b) &= - A\, W \sin \left(t \sqrt{\tfrac{g}{R}} + b \right) \cos \left(t \sqrt{\tfrac{g}{R}} \right) \\ &= - \tfrac{1}{2} A\, W \left[\sin \left(2t \sqrt{\tfrac{g}{R}} + b \right) + \sin b \right] ; \end{aligned}$$

when W is very small in comparison with g, A and B may be assumed to be constant in the first integration of the second members of these equations.

When W is dependent upon the position of the body in such a way, that, if $\mathfrak{P}$ is a function of time,

$$W\varphi = A\,W \sin\left(t\sqrt{\tfrac{g}{R}} + b\right) = \mathfrak{T};$$

the preceding equations give

$$\sqrt{(gR)}\,A \sin b = -\int_t \left(\mathfrak{T} \sin\left(t\sqrt{\tfrac{g}{R}}\right)\right),$$

$$\sqrt{(gR)}\,A \cos b = -\int_t \left(\mathfrak{T} \cos\left(t\sqrt{\tfrac{g}{R}}\right)\right).$$

If, for example,

$$\mathfrak{T} = 2h \sin mt;$$

these integrals become

$$\sqrt{(gR)}\,A \sin b = \frac{h}{m+\sqrt{\frac{g}{R}}} \cos\left(mt + t\sqrt{\tfrac{g}{R}}\right) - \frac{h}{m-\sqrt{\frac{g}{R}}} \cos\left(mt - t\sqrt{\tfrac{g}{R}}\right)$$

$$+ \frac{2\sqrt{(gR)}}{m^2 R - g}$$

$$\sqrt{(gR)}\,A \cos b = -\frac{h}{m+\sqrt{\frac{g}{R}}} \sin\left(mt + t\sqrt{\tfrac{g}{R}}\right) - \frac{h}{m-\sqrt{\frac{g}{R}}} \sin\left(mt - t\sqrt{\tfrac{g}{R}}\right)$$

in which the arbitrary constants are determined so that A and b vanish with the time.

488. *If the line rotates about the vertical axis* of z, the equation of motion becomes, by the analysis and notation of § 467,

$$D_t^2 s = -u \cos\theta\,\alpha' + \alpha^2 u \cos{}_s^u + g \cos{}_z^s$$

$$= -u \cos\theta\,\alpha' + \alpha^2 u D_s u + g\,D_s z.$$

489. *When the rotation about the vertical axis is uniform,* this equation becomes

$$D_t^2 s = \alpha^2 u D_s u + g D_s z.$$

The integral of the product of this equation multiplied by

$2\,D_t s$ is

$$(D_t s)^2 = \alpha^2 u^2 + 2gz + a,$$

in which a is an arbitrary constant.

490. *When the rotating line is straight* and passes at a distance p from the axis, if s is counted from the foot of the perpendicular (p) upon the line, the equation becomes

$$\begin{aligned}(D_t s)^2 &= \alpha^2 s^2 \sin^2 \tfrac{s}{z} + 2gs \cos \tfrac{s}{z} + \alpha^2 p^2 + a \\ &= \left(\alpha s \sin \tfrac{s}{z} + \tfrac{g}{\alpha} \cot \tfrac{s}{z}\right)^2 + a + \alpha^2 p^2 - \left(\tfrac{g}{\alpha} \cot \tfrac{s}{z}\right)^2,\end{aligned}$$

of which the integral is easily found to be

$$\alpha t \sin \tfrac{s}{z} = \log\left(\alpha^2 s \sin^2 \tfrac{s}{z} + 2g \cos \tfrac{s}{z} + 2\alpha \sin \tfrac{s}{z} D_t s\right) + b,$$

in which b is an arbitrary constant.

491. The integral, in this case, can be just as readily obtained from the equation (261_{29}) which becomes a linear differential equation. Its direct integral is

$$s - \frac{g \cos \frac{s}{z}}{\alpha^2 \sin^2 \frac{s}{z}} = A c^{\alpha t \sin \frac{s}{z}} + B c^{-\alpha t \sin \frac{s}{z}}$$

in which A and B are arbitrary constants. This form is identical with that given by VIEILLE in his solution of the particular case of this problem, in which p vanishes.

492. If $$a < \left(\tfrac{g}{\alpha} \cot \tfrac{s}{z}\right)^2 - \alpha^2 p^2$$

the value of s must be such as to render the second member of (262_9) positive; that is, the limiting values, between which the body cannot be contained, are defined by the equation

$$\alpha s_0 \sin \tfrac{s}{z} = -\tfrac{g}{\alpha} \cot \tfrac{s}{z} \pm \sqrt{\left[\left(\tfrac{g}{\alpha} \cot \tfrac{s}{z}\right)^2 - \alpha^2 p^2 - a\right]}.$$

The velocity of the body upon the line vanishes at these limits. If the initial direction of the motion of the body is towards these limits, it will approach them with a diminishing velocity; and when it arrives at the nearest limit, the direction of motion will be reversed, and it will thenceforth continue to move away from the limits.

If $$a = -\alpha^2 p^2$$

one of the limits is at the foot of the perpendicular (p), and the other limit is above this foot, at the point for which

$$s = -\frac{2g}{\alpha} \cot \tfrac{s}{z}.$$

If $$a < -\alpha^2 p^2,$$

one of the limits is above the foot of the perpendicular, while the other is below it. But if

$$a > -\alpha^2 p^2$$

while it satisfies the condition (262_{25}), both the limits are above the foot of the perpendicular.

493. If $$a > \left(\frac{g}{\alpha} \cos \tfrac{s}{z}\right)^2 - \alpha^2 p^2,$$

the motion will always continue in the same direction along the line, $(a + \alpha^2 p^2)$ will express the square of the velocity of the body upon the line when it is at the foot of the perpendicular. The point of least velocity upon the line will be determined by the equation

$$s = -\frac{g \cos \tfrac{s}{z}}{\alpha^2 \sin^2 \tfrac{s}{z}};$$

and the least velocity will be

$$D_t s = \sqrt{\left[a + \alpha^2 p^2 - \left(\frac{g \cot \tfrac{s}{z}}{\alpha}\right)^2\right]}.$$

494. If $$a = \left(\frac{g}{\alpha}\cot\frac{s}{z}\right)^2 - \alpha^2 p^2$$

the direction of the motion along the line is not subject to reversal; for, in this case, the equation (262_9) becomes

$$D_t s = \alpha s \sin\frac{s}{z} + \frac{g}{\alpha}\cot\frac{s}{z};$$

of which the integral is

$$\alpha t \sin\frac{s}{z} = \log\left(\frac{\alpha^2 s \sin^2\frac{s}{z}}{g\cos\frac{s}{z}} + 1\right).$$

The time of reaching the point, at which

$$s = -\frac{g\cos\frac{s}{z}}{\alpha^2\sin^2\frac{s}{z}},$$

that is, the point, at which the velocity vanishes, becomes infinite; or in other words, the body never reaches this point, at which its direction of motion is to be reversed; or if the body is placed at this point without any initial velocity along the line, it will remain stationary upon the line.

495. *If the rotating line is the circumference of a circle,* of which the radius is R, let the origin be assumed so that the centre of the circle may be upon a level with the foot of the perpendicular (p), let fall from the origin upon the plane of the circle. Let then

k denote the distance of the centre of the circle from the foot of the perpendicular,

φ the angular distance upon the circumference of the body from the lowest point of the circumference,

and the values of z and u, in equation (262_2), are given by the equation

$$z = R \cos \varphi \sin {}^{p}_{z} + p \cos {}^{p}_{z},$$

$$u^2 = (k + R \sin \varphi)^2 + (p \sin {}^{p}_{z} - R \cos \varphi \cos {}^{p}_{z})^2$$

$$= k^2 + R^2 + p^2 \sin^2 {}^{p}_{z} + 2 k R \sin \varphi - p R \sin 2 {}^{p}_{z} \cos \varphi - R^2 \sin^2 {}^{p}_{z} \cos^2 \varphi,$$

whence equation (262_2) becomes

$$R^2 (D_t \varphi)^2 = a + \alpha^2 (k^2 + R^2 + p^2 \sin^2 {}^{p}_{z}) + 2 g p \cos {}^{p}_{z} + 2 \alpha^2 k R \sin \varphi$$
$$+ 2 (g - \alpha^2 p \cos {}^{p}_{z}) R \sin {}^{p}_{z} \sin \varphi - \alpha^2 R^2 \sin^2 {}^{p}_{z} \cos^2 \varphi.$$

The points of maximum and minimum velocity along the arc are, therefore, determined by the equation

$$\alpha^2 k R \cos \varphi_1 - (g - \alpha^2 p \cos {}^{p}_{z}) R \sin {}^{p}_{z} \sin \varphi_1 + \alpha^2 R^2 \sin^2 {}^{p}_{z} \sin \varphi_1 \cos \varphi_1 = 0,$$

and are, consequently, at the intersections of the circumference with the equilateral hyperbola, which is described in the plane and passes through the centre of the circle, of which one of the asymptotes is horizontal, and the polar coördinates (r_2, φ_2) of the centre, with reference to the centre of the circle, are given by the equations,

$$r_2 \sin \varphi_2 = - k \operatorname{cosec}^2 {}^{p}_{z},$$

$$r_2 \cos \varphi_2 = \frac{g}{\alpha^2} \operatorname{cosec} {}^{p}_{z} - p \cot {}^{p}_{z}.$$

This hyperbola cannot cut the circumference in less than two points; and there are four points of intersection when the distance from the centre of the circle to the nearest point of the branch of the hyperbola, which does not pass through it, is less than the radius of the circle. The polar coördinates (r_3, φ_3) of this nearest point of the second branch of the hyperbola are given by the equations

$$\tan \varphi_3 = \sqrt[3]{\tan \varphi_2},$$

$$r_3 = r_2 \cos \varphi_2 \sec^3 \varphi_3.$$

496. When the body is originally placed at one of the points of maximum or minimum velocity, without any initial velocity along the circle, it remains stationary upon the curve; but its position upon the curve is one of stable equilibrium, when it is placed at a point of maximum velocity, and a position of unstable equilibrium, when it is placed at a point of minimum velocity. When the body is originally placed upon the curve, without any initial velocity along the line, at a point different from these points of maximum or minimum velocity, it oscillates about that point of greatest velocity from which it is not separated by a point of least velocity; its oscillations embrace both the points of greatest velocity, when the velocity is sufficient to carry it through either of the points of least velocity, that is, when the velocity, which corresponds to the initial point in the general equation (265_7), is less than that which corresponds to one of the points of least velocity. When the initial velocity of the body is greater than the excess, which is given by equation (265_7) of the velocity at the initial point above the least of the minimum velocities, the body constantly moves, in the same direction, through the entire circumference.

497. The case in which the initial velocity of the body is just equal to the excess, which is given by equation (265_7) of the velocity at the initial point above either of the minimum velocities, admits of integration. In this case, it is easy to express the equation (265_7) in the form

$$R\,(D_t\,\varphi)^2 = 2\,\alpha^2\,k\,(\sin\varphi - \sin\varphi_1) - \alpha^2\,R\,\sin^2\tfrac{p}{z}\,(\cos^2\varphi - \cos^2\varphi_1) \\ + 2\,(g - \alpha^2\,p\,\cos\tfrac{p}{z})\,\sin\tfrac{p}{z}\,(\cos\varphi - \cos\varphi_1),$$

which by means of (265_{20}) assumes the form

$$R\,(D_t\varphi)^2 = \alpha^2\sin^2\tfrac{p}{z}\,[2\,r_2\cos(\varphi + \varphi_2) - 2\,r_2\cos(\varphi_1 + \varphi_2) \\ - R\cos^2\varphi + R\cos^2\varphi_1]\,.$$

The condition for the determination of the point of minimum velocity gives also the equation

$$2 r_2 \sin(\varphi_1 + \varphi_2) = R \sin 2\varphi_1,$$

which substituted in the previous equation with the notation

$$\Phi = \tfrac{1}{2}(\varphi - \varphi_1)$$
$$H = \frac{\sin(\varphi_1 - \varphi_2)}{\sin(\varphi_1 + \varphi_2)}$$

gives

$$(D_t \Phi)^2 = \tfrac{1}{2} \alpha^2 \sin^2 \tfrac{r}{z} \sin^2 \Phi \left[\cos 2(\Phi + \varphi_1) - H\right].$$

If, therefore, H is negative and absolutely greater than unity, that is, if φ_1 is not in the same quadrant with φ_2, the value of Φ is unlimited; but if H is less than unity, the limits of Φ are given by the equation

$$\cos 2(\Phi + \varphi_1) = H.$$

The integral of the equation (267_{11}) is

$$\alpha t \sin \tfrac{r}{z} \sqrt{(\tfrac{1}{2}\cos 2\varphi_1 - \tfrac{1}{2} H)}$$
$$= \log \frac{\sin(\Phi + \varphi_1)\sqrt{(\cos 2\varphi_1 - H)} - \sin\varphi_1 \sqrt{[\cos 2(\Phi + \varphi_1) - H]}}{\cos(\Phi + \varphi_1)\sqrt{(\cos 2\varphi_1 - H)} + \cos\varphi_1 \sqrt{[\cos 2(\Phi + \varphi_1) - H]}}.$$

498. *If the rotating line is a parabola, of which the transverse axis is vertical,* let

q be the distance from the vertex to the focus of the parabola,

and let the origin of coördinates be assumed to be upon a level with the vertex, and let

k denote the distance of the vertex from the foot of the perpendicular (p) let fall from the origin upon the plane of the parabola.

If the axis of x_1 is the horizontal line, which is drawn in the plane of the parabola through its vertex, and if the vertex is the origin of x_1, the values of z and u are given by the equations

$$4\,qz = x_1^2,$$
$$u^2 = p^2 + (k + x_1)^2;$$

and the equation (262_2) is reduced to

$$(D_t s)^2 = \alpha^2 p^2 + \alpha^2 (k + x_1)^2 + \frac{g x_1^2}{2q} + a$$
$$= \left(\frac{x_1^2}{4q^2} + 1\right)(D_t x_1)^2.$$

The integral of this equation, in its general form, can be obtained by elliptic functions. The point of least velocity along the curve is determined by the equation

$$2\,\alpha^2 (k + x_1) + \frac{g}{q} x_1 = 0;$$

but there is no such point, when

$$q = -\frac{g}{2\,\alpha^2}.$$

When this latter condition is satisfied, and also

$$k = 0,$$

the velocity of the body along the curve is constant.

When k vanishes and

$$a = \alpha^2 (4\,q^2 - p^2) + 2gq,$$

the equation (268_{10}) becomes

$$D_t x_1 = \pm 2q \sqrt{\left(\alpha^2 + \frac{g}{2q}\right)},$$

so that, in this case, the horizontal velocity of the body upon the plane of the parabola is constant.

499. In the especial case, in which the initial velocity is that which corresponds to the vanishing of the minimum velocity, let

x_2 be the value of x_1 for this point of minimum velocity,

and the integral of the equation of motion is

$$2\,q\,t\,\sqrt{\left(\alpha^2+\frac{g}{2\,q}\right)}=\sqrt{(x_1^2+4\,q^2)}+x_2\log\left[x_1+\sqrt{(x_1^2+4\,q^2)}\right]$$
$$+\sqrt{(x_2^2+4\,q^2)}\log\frac{(x_2^2+4\,q^2)+x_2\,(x_1-x_2)-\sqrt{[(x_2^2+4\,q^2)\,(x_1^2+4\,q^2)]}}{x_1-x_2}.$$

500. *When the axis* (h) *of rotation is not vertical,* the equation of motion is still reduced to the form (261_{24}), and *when the rotation is uniform,* it becomes

$$D_t^2\,s=\alpha^2\,u\cos{}_s^u+g\cos{}_s^z=\tfrac{1}{2}\,\alpha^2\,D_s u^2+g\cos{}_s^z.$$

501. *When the rotating line about the inclined axis is straight,* if the point of the axis of rotation which is nearest to the rotating line is assumed as the origin, let

p be the perpendicular upon the line from the origin,

let s be counted from the foot of the perpendicular (p), and the time from the instant, when the plane of the directions of the axis and the rotating line is vertical. The values of u and $\cos{}_s^z$ are given by the equations

$$u^2=p^2+s^2\sin^2{}_s^h,$$
$$\cos{}_s^z=\cos{}_s^h\cos{}_z^h+\sin{}_s^h\sin{}_z^h\cos(\alpha\,t);$$

which reduce the equation (269_{18}) for this case to

$$D_t^2 s = \alpha^2 s \sin^2 {}^h_s + g \cos {}^h_s \cos {}^h_z + g \sin {}^h_s \sin {}^h_z \cos (\alpha t).$$

The integral of this equation is

$$s = A c^{\alpha t \sin {}^h_s} + B c^{-\alpha t \sin {}^h_s} - \frac{g \cos {}^h_s \cos {}^h_z}{\alpha^2 \sin^2 {}^h_s} - \frac{g \sin {}^h_s \sin {}^h_z}{\alpha^2 (1 + \sin^2 {}^h_s)} \cos (\alpha t),$$

in which A and B are arbitrary constants.

502. *If in the general case of the rotation of a plane curve about the inclined axis* the time is computed from the instant, when the plane of the curve is vertical the expression of $({}^z_s)$ is given by the formula

$$\cos {}^z_s = \cos {}^h_s \cos {}^h_z + \sin {}^h_s \sin {}^h_z \cos \alpha t.$$

MOTION OF A BODY UPON A LINE IN OPPOSITION TO FRICTION, OR THROUGH A RESISTING MEDIUM.

503. The forces of nature, which resist the motions of bodies, are of various kinds and subject to different laws. While their philosophical discussion must be reserved to its appropriate place, it is sufficient for the present purpose, to recognize them as forces, which are opposed to the motion of bodies, and which depend in general upon the relative motions of the body and of the origin of the resistance, whether this origin be solid or fluid.

504. If either of the resisting forces is denoted by S_1, and if $({}^s_{s_1})$ denotes the angle which the direction of its action makes with the path of the body, the resistance to the motion of the body in its path will be expressed by $S_1 \cos {}^s_{s_1}$, which may be immediately introduced into the equation of motion.

505. *If the body moves upon a fixed line,* the equation of motion (243_{19}) becomes

$$D_t s' = D_s \Omega + \Sigma_1 (S_1 \cos {}^{s}_{s_1}).$$

If there is, likewise, no motion in the resisting medium, all the forces of resistance can be combined in one, which is directly opposed to the motion of the body, and the preceding equation assumes the form

$$D_t s' = D_s \Omega - S.$$

506. *If there is no external force,* these equations become

$$D_t s' = \Sigma_1 (S_1 \cos {}^{s}_{s_1}),$$
$$D_t s' = - S.$$

507. The integral of the latter of these equations is

$$t = - \int_{s'} \frac{1}{S}.$$

Let S have the form

$$S = a + b s' + e s'^2,$$

in which a and e are positive, in the case of nature, and

$$b + \sqrt{(4ae)} > 0,$$

because S is always positive when s' is positive. The corresponding integral of (271_{17}) is

$$t = A - \frac{2}{\sqrt{(4ae - b^2)}} \tan^{[-1]} \frac{2es' + b}{\sqrt{(4ae - b^2)}}$$
$$= A + \frac{1}{\sqrt{(b^2 - 4ae)}} \log \frac{2es' + b + \sqrt{(b^2 - 4ae)}}{2es' + b - \sqrt{(b^2 - 4ae)}}$$

in which A is an arbitrary constant, and the former integral corresponds to the case of $b^2 < 4ae$, while the latter corresponds to $b^2 > 4ae$. The velocity vanishes after the time t_0 given by the equation

$$t_0 = A - \frac{2}{\sqrt{(4ae - b^2)}} \tan^{[-1]} \frac{b}{\sqrt{(4ae - b^2)}}$$
$$= A + \frac{1}{\sqrt{(4b^2 - 4ae)}} \log \frac{b + \sqrt{(b^2 - 4ae)}}{b - \sqrt{(b^2 - 4ae)}}.$$

These values are infinite in form, when

$$b^2 = 4ae;$$

but, in this case, the integral is

$$t = A + \frac{4a}{b(bs' + 2a)} = A + \frac{2}{2es' + b},$$

so that the velocity vanishes, when

$$t_0 = A + \frac{2}{b} = A + \frac{1}{\sqrt{(ae)}}.$$

These values become infinite in form when both b and e vanish, but, in this case, which includes that of friction upon a straight path, the integral is

$$t = A - \frac{s'}{a} = \frac{s'_0 - s'}{a};$$

and the instant, at which the velocity vanishes is determined by the equation

$$t_0 = \frac{s'_0}{a}.$$

When a vanishes, the value of t_0 is actually infinite, so that the velocity of the body can never be wholly destroyed by any such form of resistance. It would seem, from the preceding equations, that the direction of motion would be reversed after the time (t_0). But this conclusion, which is absurd, because it would give a resistance the power of creating motion, arises from the defective forms of notation which do not express the solution of continuity corresponding to the abrupt ceasing of the friction at the instant of the suspension of motion.

508. *When the resistance is simply that of friction arising from the pressure of the moving body upon the line, to which its motion is restricted,* let

p denote the direction of the perpendicular to the fixed line, which is drawn in the common plane of the direction of the external force and of that of the line,

dv the elementary angle made by two successive radii of curvature to the fixed line, and

a the coefficient of friction,

and the equation of motion becomes by (245_{18})

$$D_t s' = D_s \Omega - a D_p \Omega - \frac{a s'^2}{D_v s} = D_s \Omega - a D_p \Omega - a s' v'.$$

509. When there is no external force, this equation becomes

$$D_t s' = -a s' v';$$

the integral of which is

$$\log s' = A - a v,$$

in which A is an arbitrary constant. Another integration gives

$$t = \int_s c^{av - A} = \int_v \left(D_v s c^{av - A}\right) = \int_v \left(\varrho c^{av - A}\right)$$

in which c is the Naperian base, and ϱ the radius of curvature of the fixed line.

510. *If the fixed line is the involute of the circle,* and if its equation is

$$\rho = R\nu,$$

the equation (273_{28}) becomes

$$t = \frac{R}{a^2}\left(a\nu - 1\right) e^{a\nu - A} + B,$$

in which B is an arbitrary constant.

511. *If the fixed line is the logarithmic spiral,* and if its equation is

$$\rho = R e^{b\nu},$$

the equation (273_{28}) becomes

$$t = \frac{R}{a+b} e^{(a+b)\nu - A} + B,$$

in which B is an arbitrary constant.

512. *If the fixed line is the cycloid,* and if its equation is

$$\rho = 4R\sin\nu,$$

the equation (273_{28}) becomes

$$t = \frac{4R}{a^2+1}\left(a\sin\nu - \cos\nu\right) e^{a\nu - A} + B$$

in which B is an arbitrary constant.

513. *When the resistance of the line is constant, and the resisting medium is moving with an uniform velocity in an invariable direction, and the resistance arising from it is proportional to the velocity in the medium,* let

a be the constant resistance of the line,

h the resistance of the medium for the unit of velocity, and

b the velocity of the medium,

and if the direction of the motion of the medium is assumed for that of the axis of x, the equation of motion becomes

$$D_t s' = D_s \Omega - a - h s'_1 \cos\,^s_{s_1}$$
$$= D_s \Omega - a + h (b \cos\,^s_x - s'),$$

in which it is carefully to be observed that the sign of a must be reversed simultaneously with the direction of motion.

514. *When the fixed line is straight and there is no external force* the integral of the equation (275_5) becomes

$$\log\left(s' - b \cos\,^s_x + \frac{a}{h}\right) = A - ht$$

in which A is an arbitrary constant. When

$$a < b h \cos\,^s_x,$$

the velocity of the body will never be destroyed, but will constantly approximate to

$$b \cos\,^s_x - \frac{a}{h}.$$

But when

$$a > bh \cos\,^s_x,$$

the velocity will vanish after the time t_0, determined by the equation

$$\log\left(\frac{a}{h} - b \cos\,^s_x\right) = A - ht_0.$$

If the initial velocity of the body had been negative, the equation of motion would have assumed the form

$$\log\left(-s' + b \cos\,^s_x + \frac{a}{h}\right) = -A + ht;$$

so that the velocity would have vanished after the time t_0, determined by the equation

$$\log\left(b\cos\tfrac{s}{x}+\tfrac{a}{h}\right)=-A+h\,t_0.$$

The body would then have remained at rest unless the condition (275_{14}) had been satisfied, in which case its subsequent motion would be defined by the equation (275_{11}).

515. *When a heavy body moves upon a fixed straight line, and the resistances consist of a constant resistance, arising from the friction along the line, and also of a resistance arising from a resisting medium, which has a uniform motion in the direction of the fixed line; and when the resistance of the medium is proportional to the square of the velocity of the body in the medium,* let

a be the constant of friction,

b the velocity of the medium, and

h the resistance of the medium for the unit of velocity.

The line may be assumed to be vertical without diminishing the generality of the investigation and the equation of motion will be

$$D_t s'=g-a-h(s'-b)^2,$$

in which the signs of a and h must be reversed simultaneously with those of s' and $(s'-b)$ respectively. The equation of motion has precisely the same form with that of § 507, so that the forms of the integral are the same which are there given, but the constants are not subject to the restrictions of that section.

If, then, the initial velocity is upward and exceeds that of the medium, when the medium is also moving upwards, the ascending velocity decreases by the law expressed in the equation

$$s'-b=\sqrt{\frac{g+a}{h}}\tan\left[(t-\tau)\sqrt{(h(g+a))}\right],$$

in which τ is an arbitrary constant. This law of ascent continues until the body is brought to rest when the medium is not moving upwards. But when the medium is moving upwards, it continues until the instant (τ), when the velocity of the body is the same with that of the medium. After this instant, the velocity decreases by the law

$$s' - b = \sqrt{\frac{g+a}{h}} \operatorname{Tan} [(t-\tau) \sqrt{(h(g+a))}];$$

which continues forever if

$$g + a < hb^2$$

and the velocity constantly approximates to that, which is determined by the equation

$$g + a = h(s' - b)^2.$$

But when

$$g + a > hb^2,$$

the body is brought to a state of rest, in which it continues permanently if

$$g - a < hb^2.$$

But if the motion of the medium is upward, and

$$g - a > hb^2,$$

the body moves from the state of rest with an increasing descending velocity of which the law is expressed by the equation

$$s' - b = \sqrt{\frac{g-a}{h}} \operatorname{Tan} [(t-\tau_1) \sqrt{(g-a)}],$$

in which τ_1 must be determined so that the instant of rest coincides with that given by the equation (277_8). The increasing velocity continually approximates to that which is determined by the equation

$$g - a = h(s' - b)^2.$$

The state of rest to which the body is brought, when the medium is not moving upwards, is permanent if

$$a - g > h b^2.$$

But if, on the contrary,

$$a - g < h b^2$$

the body moves from the state of rest with an increasing descending velocity, of which the law is expressed by the equation

$$s' - b = \sqrt{\frac{g-a}{h}} \tan \left[(t - \tau_1) \sqrt{(h(g-a))}\right],$$

when

$$g > a,$$

in which τ_1 must be determined so that the instant of rest coincides with that given by the equation (276_{30}). This law of motion continues until the instant τ_1, when the downward velocity of the body becomes the same with that of the medium; and after this instant, the law of increasing velocity of descent is expressed by the equation (277_{29}); so that the velocity continually approximates to that which is determined by the equation (278_4).

But when the body begins to descend from the state of rest, and

$$g < a,$$

the law of descent is expressed by the equation

$$s' - b = \sqrt{\frac{a-g}{h}} \operatorname{Cot}[(\tau_1 - t)\sqrt{(h(a-g))}],$$

so that the increasing velocity constantly approximates to that which is determined by the equation

$$a - g = h(s' - b)^2.$$

If the initial velocity is downward, and exceeds that determined by the equation (278_4), the decreasing velocity when

$$g > a$$

is expressed by the equation

$$s' - b = \sqrt{\frac{g-a}{h}} \operatorname{Cot}[(t - \tau)\sqrt{(h(g-a))}],$$

in which τ is an arbitrary constant. If, therefore, the motion of the medium is downward, or if it is upward and the condition (277_{24}) is satisfied, the decreasing velocity continually approximates to that which is determined by the equation (277_{29}). But if the motion of the medium is upward and the condition (277_{21}) is satisfied, the body is brought to a state of rest which is permanent if the condition (277_{17}) is also satisfied. If, however, the condition (277_{11}) is satisfied by the upward motion of the medium, the body leaves the state of rest and ascends with an increasing velocity, which is defined by the equation

$$s' - b = \sqrt{\frac{g+a}{h}} \operatorname{Cot}[(t - \tau_1)\sqrt{(h(g+a))}],$$

in which τ_1 must be determined so that the instant of rest coincides with that which is given by the equation (279_{15}). The ascending velocity continually approximates to that which is determined by the equation (277_{15}).

If the initial velocity is downward, and exceeds that of the medium, when the medium is also moving downwards, the descending velocity, when

$$g < a,$$

decreases by the law, expressed in the equation

$$s' - b = \sqrt{\frac{a-g}{h}} \tan\left[(\tau - t)\sqrt{(h(a-g))}\right],$$

in which τ is an arbitrary constant. This law of descent continues until the body is brought to rest, when the medium is not moving downwards; but when the medium is moving downwards, the law continues until the instant τ, when the velocity of the body is the same with that of the medium. After this instant, the law of decreasing velocity becomes

$$s' - b = \sqrt{\frac{a-g}{h}} \operatorname{Tan}\left[(\tau - t)\sqrt{(h(a-g))}\right],$$

which continues until the body is brought to rest, when the condition (278_9) is satisfied. But when, on the contrary, the condition (278_{12}) is satisfied, the body continues to move forever with the law of decreasing velocity expressed in (280_{19}), and the velocity continually approximates to that, which is determined by the equation (279_7). When the body has been brought to the state of rest, the condition and laws of leaving it are the same with those defined in (279_{23-31}), when

$$g > a.$$

THE SIMPLE PENDULUM IN A RESISTING MEDIUM.

516. *When the curve is the circumference of a vertical circle, the problem is that of the simple pendulum in a resisting medium. If the arc of vibration is supposed to be so small that its powers, which are higher than the square may be neglected, and if the resistance arising from the medium is supposed to be proportional to the velocity, and to be combined with a constant friction,* let

a be the friction, and

h the resistance of the medium for the unit of velocity,

and the equation of motion becomes, by adopting the notation of § 487,

$$D_t^2 \varphi = -\frac{g}{R}\varphi \mp a - h D_t \varphi,$$

in which the sign which precedes a, must be the reverse of that of $D_t \varphi$. The integral of this equation is

$$\varphi = \pm \frac{Ra}{g} + \frac{\varphi'_0}{k} e^{-\frac{1}{2}ht} \sin kt,$$

in which

$$k = \sqrt{\frac{g}{R}} \cos \alpha,$$

$$\tfrac{1}{2} h = \sqrt{\frac{g}{R}} \sin \alpha,$$

and the arbitrary constants have been determined so that the initial angular velocity (φ'_0) shall be the maximum velocity, and, therefore, the initial value of φ is

$$\pm \frac{Ra}{g}.$$

517. The equation (281_{21}) only applies to the first vibration and for the $(m+1)^{st}$ vibration, the correct equation is

$$\varphi = \pm \frac{Ra}{g} + \frac{\varphi'_m}{k} c^{-\frac{1}{2}h(t-\tau_m)} \sin k(t-\tau_m),$$

in which τ_m is the instant of the maximum angular velocity (φ'_m) of that vibration and the doubtful sign is alternately positive and negative for the successive oscillations, so that the position of maximum velocity is always upon the descending portion of the oscillation.

518. The angular velocity of vibration is expressed by the equation

$$\varphi' = \varphi'_m c^{-\frac{1}{2}h(t-\tau_m)} \frac{\cos[k(t-\tau_m)+\alpha]}{\cos\alpha},$$

and it vanishes for the instants

$$t = \tau_m \mp \frac{\pi}{2k} - \frac{\alpha}{k}$$

which correspond to the beginning and end of the oscillation. The whole time of oscillation is, therefore,

$$T = \frac{\pi}{k} = \pi \sqrt{\frac{R}{g}} \sec\alpha,$$

which is invariable, although it exceeds the time of vibration in a vacuum, in consequence of the factor, sec α.

519. The angular deviations of the pendulum from the vertical at the beginning and end of the oscillation are given by the equation

$$\varphi = \mp \frac{Ra}{g} \mp \varphi'_m \sqrt{\frac{R}{g}} c^{(\alpha \pm \frac{1}{2}\pi)\tan\alpha};$$

whence the whole arc of the $(m+1)^{st}$ vibration is

$$\Phi_m = 2\varphi'_m \sqrt{\frac{R}{g}}\, c^{\alpha \tan \alpha} \operatorname{Cos}(\tfrac{1}{2}\pi \tan \alpha).$$

520. The angular deviations of the pendulum from the vertical at the end of one vibration and the beginning of the next are identical, but the deviation from the point of maximum velocity is, on account of the change in the position of this point, diminished by the quantity

$$\frac{2Ra}{g}.$$

The successive values of the maximum velocity are therefore connected by the equation

$$\varphi'_m c^{(\alpha - \frac{1}{2}\pi)\tan \alpha} - 2a\sqrt{\frac{R}{g}} = \varphi'_{m+1} c^{(\alpha + \frac{1}{2}\pi)\tan \alpha},$$

or

$$\varphi'_{m+1} = \varphi'_m c^{-\pi \tan \alpha} - 2a\sqrt{\frac{R}{g}}\, c^{-(\alpha + \frac{1}{2}\pi)\tan \alpha}.$$

The general expression for the maximum velocity is then found to be

$$\varphi'_m = \varphi'_0 c^{-m\pi \tan \alpha} - 2a\sqrt{\frac{R}{g}}\, c^{-(\alpha + \frac{1}{2}\pi)\tan \alpha} \left(\frac{c^{-m\pi \tan \alpha} - 1}{c^{-\pi \tan \alpha} - 1}\right);$$

which, on account of the smallness of a and α, may be reduced to

$$\varphi'_m = \varphi'_0 c^{-m\pi \tan \alpha} - 2ma\sqrt{\frac{R}{g}}.$$

The corresponding value of the arc of vibration is

$$\Phi_m = \Phi_0 c^{-m\pi \tan \alpha} - \frac{4aR}{g} c^{-\frac{1}{2}\pi \tan \alpha} \operatorname{Cos}(\tfrac{1}{2}\pi \tan \alpha) \frac{c^{-m\pi \tan \alpha} - 1}{c^{-\pi \tan \alpha} - 1}.$$

or

$$\Phi_m = \Phi_0 c^{-m\pi \tan \alpha} - \frac{4maR}{g}.$$

The law of the diminution of the arc of vibration and of the maximum of velocity is, therefore, such that either of these quantities consists of two terms, one of which is dependent upon the portion of the resistance, which is proportional to the velocity, and decreases in geometrical ratio, while the other is principally dependent upon the constant friction and decreases, sensibly, in arithmetical ratio. The vibration ceases when the second term of either of these quantities surpasses the first.

521. *If the resistance is proportional to the square of the velocity,* and if h is its value for the unit of velocity, the equation of the motion of the pendulum is

$$D_t^2\varphi = -\frac{g}{R}\sin\varphi - h\,(D_t\varphi)^2.$$

If one of the first integrals of this equation is supposed to be (254_{26}), in which, however, H is not constant but variable, the differential of (254_{26}) gives, by means of this equation and (254_{26}),

$$\begin{aligned} D_t H &= R^2\,D_t\varphi\,D_t^2\varphi + g\,R\sin\varphi\,D_t\varphi = -h\,R^2\,(D_t\varphi)^3 \\ &= -2\,h\,D_t\varphi\,(g\,R\cos\varphi + H), \\ D_\phi H &= -2\,g\,h\,R\cos\varphi - 2\,h\,H; \end{aligned}$$

and the integral of this last equation if

$$\tan\mu = 2\,h,$$

is

$$H = A\,e^{-\varphi\tan\mu} - g\,R\sin\mu\sin(\varphi+\mu),$$

in which A is an arbitrary constant. The equation (254_{26}) is then reduced to

$$R^2\,(D_t\varphi)^2 = 2\,A\,e^{-\varphi\tan\mu} + 2\,g\,R\cos\mu\cos(\varphi+\mu);$$

of which the integral is

$$t-\tau = \int_\phi \frac{R}{\sqrt{[2\,A\,e^{-\phi\tan\mu} + 2\,g\,R\cos\mu\cos(\varphi+\mu)]}}.$$

The signs which precede the quantities h and μ must be reversed in the alternate oscillations.

522. The angle of greatest deviation from the vertical for the $(m+1)$st oscillation is determined by the equation

$$-\frac{A_m}{gR\cos\mu} = c^{-\varphi_m\tan\mu}\cos(\varphi_m-\mu)$$
$$= c^{\varphi_{m+1}\tan\mu}\cos(\varphi_{m+1}+\mu).$$

If Δ is adopted as the sign of finite differences, this equation gives, when μ is so small that its square may be neglected,

$$\Delta\,[\cos\varphi_m-(\sin\varphi_m-\varphi_m\cos\varphi_m)\,\mu] = 2\,(\sin\varphi_m-\varphi_m\cos\varphi_m)\,\mu.$$

When the oscillations of the pendulum are so small that the fourth power of φ_m may be neglected, and also the product of μ by $\varphi_m^2\,\Delta\varphi_m$, this equation is reduced to

$$\Delta\varphi_m = -\tfrac{2}{3}\,\mu\,\varphi_m^2;$$

of which the approximate integral is

$$\varphi_m = \frac{\varphi_0}{1+\frac{2}{3}\mu_m\varphi_0}.$$

523. The substitution of (285_5) reduces (284_{27}) to the form

$$\frac{R}{2g\cos\mu}(D_t\varphi)^2 = \cos(\varphi+\mu) - c^{-(\varphi+\varphi_m)\tan\mu}\cos(\varphi_m-\mu),$$

which, when μ is so small that its square may be neglected, becomes

$$\frac{R}{2g}(D_t\varphi)^2 = \cos(\varphi+\mu) - \cos(\varphi_m-\mu) + \cos\varphi_m\,(\varphi+\varphi_m)\,\mu$$
$$= \cos\varphi - \cos\varphi_m - \mu\,[\sin\varphi+\sin\varphi_m-(\varphi+\varphi_m)\cos\varphi_m].$$

When the oscillations are very small, this equation may be still further reduced to

$$\frac{R}{g}(D_t\varphi)^2 = \varphi_m^2 - \varphi^2 + \tfrac{1}{3}\mu(\varphi^3 - 3\varphi\varphi_m^2 - 2\varphi_m^3)$$

$$= (\varphi_m^2 - \varphi^2)\left[1 - \tfrac{1}{3}\mu\left(\varphi + \frac{2\varphi_m^2}{\varphi_m - \varphi}\right)\right];$$

which gives

$$\sqrt{\frac{g}{R}} = \frac{D_t\varphi}{\sqrt{(\varphi_m^2 - \varphi^2)}}\left[1 + \tfrac{1}{6}\mu\left(\varphi + \frac{2\varphi_m^2}{\varphi_m - \varphi}\right)\right].$$

The integral of this equation is

$$\varphi = \varphi_m \sin\left[\sqrt{\frac{g}{R}}(t - \tau) - \tfrac{1}{6}\mu\frac{\varphi_m + \varphi}{\varphi_m - \varphi}\sqrt{(\varphi_m^2 - \varphi^2)}\right].$$

The time of the descending semioscillation, deduced from this equation, is

$$t_m - t_{m'} = \tfrac{1}{2}\pi\sqrt{\frac{g}{R}}\left(1 + \frac{\mu\varphi_m}{3\pi}\right).$$

The time of the preceding semioscillation is obtained by reversing the sign of μ, which gives

$$t_{m'} - t_{m-1} = \tfrac{1}{2}\pi\sqrt{\frac{R}{g}}\left(1 - \frac{\mu\varphi_m}{3\pi}\right),$$

and the time of the whole oscillation is, therefore, the same as if the pendulum vibrated in a vacuum. The preceding formulæ and conclusions coincide, substantially, with those which are given by Poisson.

524. *If the law of the resistance to the motion of the pendulum may be expressed as a function of the time,* let

$\mathfrak{T}$ denote the resistance,

and the motion of the pendulum in a small arc is expressed by the formulæ (260_9) and (261_7). If $\mathfrak{T}$ is a periodic function, which has the same period with that of the vibration of the pendulum, it may be expressed in the form

$$\mathfrak{T} = h_0 + 2 \sum_1^\infty {}_i \left[h_i \cos\left(i t \sqrt{\tfrac{g}{R}} + \beta_i \right) \right];$$

and, if the variable portions of $A \sin b$ and $A \cos b$ are denoted by δ, these equations give

$$g \delta (A \sin b) = h_0 \left(1 - \cos\left(t \sqrt{\tfrac{g}{R}}\right)\right) - h_1 t \sqrt{\tfrac{g}{R}} \sin \beta_1 - \tfrac{1}{2} h_1 \cos\left(2 t \sqrt{\tfrac{g}{R}} + \beta_1\right)$$
$$+ \tfrac{1}{2} h_1 \cos \beta_1 + \sum_1^\infty {}_i \left[\frac{h_i}{i-1} \cos\left((i-1) t \sqrt{\tfrac{g}{R}} + \beta_i\right) \right.$$
$$\left. - \frac{h_2}{i+1} \cos\left((i+1) t \sqrt{\tfrac{g}{R}} + \beta_i\right) - \frac{2 h_i}{i^2 - 1} \cos \beta_i \right],$$

$$g \delta (A \cos b) = - h_0 \sin\left(t \sqrt{\tfrac{g}{R}}\right) - h_1 t \sqrt{\tfrac{g}{R}} \cos \beta_1 - \tfrac{1}{2} h_1 \sin\left(2 t \sqrt{\tfrac{g}{R}} + \beta_1\right)$$
$$+ \tfrac{1}{2} h_1 \sin \beta_1 - \sum_1^\infty {}_i \left[\frac{h_i}{i-1} \sin\left((i-1) t \sqrt{\tfrac{g}{R}} + \beta_i\right) \right.$$
$$\left. + \frac{h_i}{i-1} \sin\left((i+1) t \sqrt{\tfrac{g}{R}} + \beta_i\right) - \frac{2 i h_i}{i^2 - 1} \sin \beta_i \right];$$

which vanish with t.

525. *If the vibrations of the pendulum cause the medium to oscillate, the period of the oscillations of the medium is probably the same with that of the pendulum, but the successive phases of the motion of the medium are likely to lag somewhat behind those of the pendulum.* Hence the relative velocity of the pendulum to the medium may be expressed by the equation

$$V = v A \cos\left(t \sqrt{\tfrac{g}{R}} + b + \beta\right),$$

in which A and b may be regarded as constant for a single vibration.

If, then, the resistance of the medium is *proportional to the relative velocity*, the value of $\mathfrak{T}$ assumes the form

$$\mathfrak{T} = 2hA\cos\left(t\sqrt{\frac{g}{R}} + b + \beta\right);$$

and the equations (287_{8-18}) give

$$\frac{g}{hA}\delta(A\sin b) = -t\sqrt{\frac{g}{R}}\sin(b+\beta) - \tfrac{1}{2}\cos\left(2t\sqrt{\frac{g}{R}} + b + \beta\right) + \tfrac{1}{2}\cos(b+\beta),$$

$$\frac{g}{hA}\delta(A\cos b) = -t\sqrt{\frac{g}{R}}\cos(b+\beta) - \tfrac{1}{2}\sin\left(2t\sqrt{\frac{g}{R}} + b + \beta\right) + \tfrac{1}{2}\sin(b+\beta),$$

whence

$$\frac{g}{h}\delta\log A = -t\sqrt{\frac{g}{R}}\cos\beta - \tfrac{1}{2}\sin\left(2t\sqrt{\frac{g}{R}} + 2b + \beta\right) + \tfrac{1}{2}\sin(2b+\beta),$$

$$\frac{g}{h}\delta b = -t\sqrt{\frac{g}{R}}\sin\beta - \tfrac{1}{2}\cos\left(2t\sqrt{\frac{g}{R}} + 2b + \beta\right) + \tfrac{1}{2}\cos(2b+\beta).$$

If T is the time of vibration of the pendulum, the changes of A and b in a single vibration are given by the formula

$$\varDelta\log A = -\frac{h}{g}T\sqrt{\frac{g}{R}}\cos\beta = -\pi\frac{h}{g}\cos\beta,$$

$$\varDelta b = -\frac{h}{g}T\sqrt{\frac{g}{R}}\sin\beta = -\pi\frac{h}{g}\sin\beta.$$

If the resistance is proportional to the square of the velocity, the value of $\mathfrak{T}$ assumes the form

$$\mathfrak{T} = 2kA^2 + 2kA^2\cos\left(2t\sqrt{\frac{g}{R}} + 2b + 2\beta\right),$$

in which the sign of k must be reversed, when the direction of the relative motion of the body to the medium is reversed. This value of $\mathfrak{T}$ gives

$$\frac{g}{kA^2}\delta(A\sin b) = 2 - 2\cos\left(t\sqrt{\frac{g}{R}}\right) + \cos\left(t\sqrt{\frac{g}{R}} + 2b + 2\beta\right)$$
$$- \tfrac{1}{3}\cos\left(3t\sqrt{\frac{g}{R}} + 2b + 2\beta\right) - \tfrac{2}{3}\cos(2b + 2\beta),$$

$$\frac{g}{kA^2}\delta(A\cos b) = -2\sin\left(t\sqrt{\frac{g}{R}}\right) - \sin\left(t\sqrt{\frac{g}{R}} + 2b + 2\beta\right)$$
$$- \tfrac{1}{3}\sin\left(3t\sqrt{\frac{g}{R}} + 2b + 2\beta\right) + \tfrac{4}{3}\sin(2b + 2\beta);$$

whence

$$\frac{g}{kA^2}\delta A = 2\sin b - 2\sin\left(t\sqrt{\frac{g}{R}} + b\right) + \sin\left(t\sqrt{\frac{g}{R}} + b + 2\beta\right)$$
$$- \tfrac{1}{3}\sin\left(3t\sqrt{\frac{g}{R}} + 3b + 2\beta\right) + \tfrac{1}{3}\sin(3b + 2\beta) - \sin(b + 2\beta),$$

$$\frac{g}{kA}\delta b = 2\cos b - 2\cos\left(t\sqrt{\frac{g}{R}} + b\right) + \cos\left(t\sqrt{\frac{g}{R}} + b + 2\beta\right)$$
$$- \tfrac{1}{3}\cos\left(3t\sqrt{\frac{g}{R}} + 3b + 2\beta\right) + \tfrac{1}{3}\cos(3b + 2\beta) - \cos(b + 2b).$$

The changes of A and b in a vibration are found, by having regard to the reversal of the sign of k which corresponds to that of V, to be

$$g\Delta A = -\tfrac{16}{3}kA^2\cos\beta,$$
$$g\Delta b = -\tfrac{16}{3}kA\sin\beta.$$

If the law of the resistance is similar to that of friction so as to be constant if the medium is at rest, it must, when the medium is in motion, be proportional to the quotient of the relative motion of the body through the medium divided by the velocity of the body. The form of $\mathfrak{T}$ is, then,

$$\mathfrak{T} = \frac{a\cos\left(t\sqrt{\frac{g}{R}}+b+\beta\right)}{\cos\left(t\sqrt{\frac{g}{R}}+b\right)}$$

in which the sign of a must be reversed, when the direction of the relative motion of the body to the medium is reversed. This value of $\mathfrak{T}$ gives

$$\frac{g}{a}\delta(A\sin b) = -\cos\left(t\sqrt{\frac{g}{R}}+\beta\right) - \sin\beta\cos b\log\tan\left(\tfrac{1}{4}\pi + \frac{t\sqrt{\frac{g}{R}}+b}{2}\right),$$

$$\frac{g}{a}\delta(A\cos b) = -\sin\left(t\sqrt{\frac{g}{R}}+\beta\right) + \sin\beta\sin b\log\tan\left(\tfrac{1}{4}\pi + \frac{t\sqrt{\frac{g}{R}}+b}{2}\right);$$

whence

$$\delta A = -\frac{a}{g}\sin\left(t\sqrt{\frac{g}{R}}+b+\beta\right),$$

$$\delta b = -\frac{a}{Ag}\cos\left(t\sqrt{\frac{g}{R}}+b+\beta\right) - \frac{a}{Ag}\sin\beta\log\tan\left(\tfrac{1}{4}\pi + \frac{t\sqrt{\frac{g}{R}}+b}{2}\right).$$

The changes of A and b in a vibration are

$$\Delta A = -\frac{2a}{g},$$

$$\Delta b = \frac{2a}{Ag}\sin\beta\log\tan\tfrac{1}{2}\beta.$$

The combination of these values give

$$\Delta A = -\frac{2a}{g} - \pi A\frac{h}{g}\cos\beta - \tfrac{16}{3}\frac{k}{g}A^2\cos\beta,$$

$$\Delta b = \frac{2a}{Ag}\sin\beta\log\tan\tfrac{1}{2}\beta - \pi\frac{h}{g}\sin\beta - \tfrac{16}{3}\frac{k}{g}A\sin\beta.$$

The change of b is exhibited in the motion of the pendulum by a change in the time of vibration, which differs from that which it would be in a vacuum. The difference is

$$\varDelta T = -\varDelta b \sqrt{\frac{R}{g}} = -\frac{T}{\pi} \varDelta b.$$

526. The vibration of the pendulum may be regarded as affected by the medium not only in consequence of its direct action as resistance, but also indirectly, because a portion of the medium may be regarded as composing a part of the moving body, and its motion is sustained by the action of gravitation upon the body. If, then,

> q denotes the ratio of the mass of that portion of the medium which moves with the body to the mass of the body,

the motion of q may be assumed to have a period identical with that of the body, and an amplitude of excursion proportional to that of the body, so that its velocity may be of the form

$$V' = h' A \sqrt{\frac{g}{R}} \cos\left(t \sqrt{\frac{g}{R}} + b - \beta'\right).$$

The resistance, then, arising from the preservation of the motion of q may be expressed in $\mathfrak{T}$ by the form

$$\mathfrak{T} = q D_t V' = -\frac{q h' A g}{R} \sin\left(t \sqrt{\frac{g}{R}} + b - \beta'\right).$$

The similarity of this form to that of (288_4) shows that the corresponding influence upon A and b may be expressed by the equations

$$\varDelta \log A = -\frac{\pi q h'}{2R} \sin \beta',$$

$$\varDelta b = -\frac{\pi q h'}{2R} \cos \beta'.$$

The importance of this form of resistance was first noticed by DUBUAT and has been investigated experimentally by DUBUAT, BESSEL, and BAILY. The formulæ (290_{27}) and (291_{29}) may be adopted as a guide in the conduct of these and similar investigations.

527. In the application of the preceding formulæ to the reduction of experiments, the quantities a, h, k, and q are inversely proportional to the density of the body, and directly proportional to the density of the medium, and for bodies of similar forms they are nearly in an inverse ratio to their linear dimensions. For pendulums of different lengths, k is proportional to the length of the pendulum, and h to the time of vibration. If H_1 denotes the resistance of the medium which is proportional to the velocity for the unit of weight and the unit of surface, and if H_2 denotes the resistance which is proportional to the square of the velocity for the same unit of weight and surface, the values of h and k, for the units of weight and surface, are

$$h = \frac{g}{2\pi} T H_1,$$
$$k = \tfrac{1}{4} g R H_2.$$

528. The best experiments which have been made with the pendulum are almost wholly free from any constant term of resistance, so that, in their discussion, this term may be neglected which reduces the formula (290_{26}) to the form

$$\Delta A = -\tfrac{1}{2} T H_1 A \cos\beta - \tfrac{4}{3} R H_2 A^2 \cos\beta,$$

of which the approximate integral is

$$\log\left(1 + \frac{3 T H_1}{8 R H_2 A_m}\right) - \log\left(1 + \frac{3 T H_1}{8 R H_2 A_0}\right) = \tfrac{1}{2} m T H_1 \cos\beta.$$

529. In order to illustrate these formulæ, they may be applied to some of the experiments which have been actually made,

and in which the diminution of the arc of vibration has been observed. For this purpose the observations of NEWTON, DUBUAT, BORDA, BESSEL, and BAILY are selected, and the formula (292_{28}) is found to be applicable to all these experiments, although the values of H_1 and H_2 are different for the different experiments. The unit of length which is here adopted is the meter, the unit of weight is the chiliogramme, and that of time is the mean solar second. The measures and weights are, however, given in the form in which they were actually observed.

530. In NEWTON'S first series of experiments upon the diminution of the oscillations of a pendulum, a wooden sphere of $6\frac{7}{8}$ English inches in diameter, weighing $57\frac{7}{22}$ ounces, of about 0.56 specific gravity, and suspended by a fine wire so as to give $10\frac{1}{2}$ feet for the length of the pendulum, was vibrated until the arc of descent was diminished one fourth or one eighth of its initial extent, and the number of vibrations was recorded. From the reduction of these observations, I have obtained for the values of H_1 and H_2

$$H_1 = 0.0223 \sec \beta,$$
$$H_2 = 0.4473 \sec \beta.$$

In NEWTON'S second series of experiments, a leaden sphere of 2 inches in diameter, weighing $26\frac{1}{4}$ pounds, and suspended so as to give $10\frac{1}{2}$ feet for the length of the pendulum, was vibrated in the same way as in the former series. From the reduction of these observations, I have obtained

$$H_1 = 0.2044 \sec \beta,$$
$$H_2 = 0.701 \sec \beta.$$

To test the accuracy of these reductions, and their conformity with the given observations, I have computed the lengths of the observed arcs of vibration, and have placed them in the following table for comparison.

COMPARISON OF NEWTON'S EXPERIMENTS UPON VIBRATIONS OF THE PENDULUM WITH COMPUTATION.

WOODEN SPHERE.				LEADEN SPHERE.			
m	Computed A_m	Observed A_m	$C—O$	m	Computed A_m	Observed A_m	$C—O$
	in.	in.	in.		in.	in.	in.
0	64.08	64	.08	0	64.03	64	.03
$9\frac{2}{3}$	56.02	56	.02	30	56.04	56	.04
$22\frac{2}{3}$	47.91	48	—.09	70	47.92	48	—.08
0	31.86	32	—.14	0	31.92	32	—.08
$18\frac{1}{2}$	27.92	28	—.08	53	28.00	28	0.
$41\frac{2}{3}$	24.19	24	.19	121	24.07	24	.07
0	15.99	16	—.01	0	16.01	16	.01
$35\frac{1}{2}$	14.01	14	.01	$90\frac{1}{2}$	13.99	14	—.01
$83\frac{1}{2}$	11.99	12	—.01	204	11.99	12	—.01
0	8.04	8	.04	0	8.05	8	.05
69	7.01	7	.01	140	7.01	7	.01
$162\frac{1}{2}$	5.95	6	—.05	318	5.95	6	—.05
0	4.01	4	.01	0	4.03	4	.03
121	3.50	$3\frac{1}{2}$	0.	193	3.49	$3\frac{1}{2}$	—.01
272	2.99	3	—.01	420	2.97	3	—.03
0	1.98	2	—.02	0	2.04	2	.04
164	1.74	$1\frac{3}{4}$	—.01	228	1.74	$1\frac{3}{4}$	—.01
374	1.52	$1\frac{1}{2}$	.02	518	1.46	$1\frac{1}{2}$	—.04
				0	1.00	1	0.
				226	.88	$\frac{7}{8}$	0.
				510	.75	$\frac{3}{4}$	0.

With these values of H_1 and H_2, a minute arc of vibration of the wooden sphere would be reduced one eighth part in 446 vibrations, and one fourth part in 961 vibrations, and a minute arc of vibration of the leaden sphere would be reduced one eighth part in 290 vibrations, and one fourth part in 625 vibrations.

531. Dubuat vibrated in water a sphere of 2.645 French inches in diameter, weighing in air 40068 grains, and in water 36448 grains, and suspended so that the length of the pendulum

was 36.714 inches; he observed the arc of descent at each successive oscillation. From these observations, I have obtained a result which corresponds with his own in respect to the law of diminution of oscillation, and which gives for the values of H_1 and H_2 in water

$$H_1 = 0,$$
$$H_2 = 378.7 \sec \beta.$$

DUBUAT also vibrated in air a paper sphere of 4.0416 inches in diameter, weighing in air 155 grains, with a density 11.33 times as great as that of air, and suspended by a fine thread so that the length of the pendulum was 36.714 inches. From these observations, I have deduced

$$H_1 = 0,$$
$$H_2 = 0.37 \sec \beta.$$

The following table contains the comparison of DUBUAT'S experiments with the computations derived from the values of H_1 and H_2.

COMPARISON OF DUBUAT'S EXPERIMENTS UPON THE DIMINUTION OF THE ARC OF VIBRATION OF A PENDULUM WITH COMPUTATION.

SPHERE IN WATER.				SPHERE IN AIR.			
m	Computed A_m	Observed A_m	$C—O$	m	Computed A_m	Observed A_m	$C—O$
	in.	in.	in.		in.	in.	in.
0	12.00	12.00	0.	0	11.90	12.00	—.10
1	9.21	9.25	—.04	1	10.10	10.00	.10
2	7.47	7.42	.05	2	8.77	8.70	.07
3	6.28	6.25	.03	3	7.75	7.79	—.04
4	5.42	5.33	.09	4	6.94	6.96	—.02
5	4.77	4.75	.02				
6	4.25	4.25	0.				
7	3.84	3.83	.01				
8	3.50	3.48	.02				
9	3.22	3.23	—.01				
10	2.97	2.98	—.01				

532. Borda vibrated a platinum sphere of $16\frac{1}{6}$ lines in diameter, weighing with the wire and screw 9963 grains, and suspended by a wire so that the length of the pendulum was 3.95497 metres. These observations give for the values of H_1 and H_2 in air

$$H_1 = 0.10722 \sec \beta,$$
$$H_2 = 0.6267 \sec \beta.$$

In his observations for determining the length of the seconds pendulum, this same pendulum was vibrated by Borda, and the lengths of its arcs of vibration were observed. From the mean of these observations, I have obtained the values of H_1 and H_2,

$$H_1 = 0.11214 \sec \beta,$$
$$H_2 = 0.6564 \sec \beta.$$

Borda vibrated the same sphere with a smaller wire, so that the weight was reduced to 9958 grains, and the length increased to 3.95597 metres. From these observations I have derived

$$H_1 = 0.1134 \sec \beta,$$
$$H_2 = 0.590 \sec \beta.$$

The comparison of Borda's experiments with the computations based upon these values of H_1 and H_2 is contained in the following tables.

COMPARISON OF BORDA'S OBSERVATIONS UPON THE DIMINISHED VIBRATIONS OF THE PENDULUM WITH COMPUTATION.

First Experiment with direct reference to the Diminution of the Arc of Vibration.

m	Computed A_m	Observed A_m	$C-O$	m	Computed A_m	Observed A_m	$C-O$
0	120′.0	120′.0	0′.	12600	4′.2	4′.1	0′.1
1800	61.2	61.2	0.	14400	2.8	2.7	.1
3600	35.6	35.4	.2	16200	1.9	1.8	.1
5400	22.1	21.9	.2	18000	1.3	1.2	.1
7200	14.2	14.1	.1	19800	0.9	0.8	.1
9000	9.4	9.4	0.	21600	0.6	0.5	.1
10800	6.2	6.3	—.1	36000	0.002	Very minute.	

Experiments for determining the Length of the Second's Pendulum with the Pendulum used in the First Experiment.

m Mean Value.	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	64′	64′	0′	67′	67′	0′	63′	63′	0′
2169	32½	32	½	34	34	0	32	32	0
4338	18	19	—1	18½	19	—½	18	18	0
6507	10½	11	—½	11	11	0	10½	11	—½
8676	6½	7	—½	6½	7	—½	6	6	0
0	60	60	0	61	61	0	64½	64½	0
2169	31	31	0	31½	31½	0	32½	32½	0
4338	17½	17	½	17½	18	—½	18	17½	½
6507	10	10	0	10	10	0	10½	10	½
8676				6	6	0	6½	6	½
0	63	63	0	68	68	0	61	61	0
2169	32	32	0	34	34½	—½	31½	31½	0
4338	18	18	0	19	19½	—½	17½	17	½
6507	10½	10	½	11	11½	—½	10	10	0
8676	6	6¼	—¼	6½	7	—½	6	6	0
0	59½	59½	0	57½	57½	0	62	62	0
2169	30½	30½	0	30	30	0	31½	31½	0
4338	17	17	0	17	17	0	18	17	1
6507	10	10	0	9½	10	—½	10½	10	½
8676				6	6	0	6	6	0
0	67	67	0	65	65	0	63	63	0
2169	34	34	0	33	33½	—½	32	32	0
4338	18½	19	—½	18½	18½	0	18	17½	½
6507	11	11	0	10½	11	—½	10½	10	½
8676	6½	6	½	6½	6½	0	6	6	0
0	71	71	0	59½	59½	0			
2169	35	34½	½	31	31	0			
4338	19½	19	½	17½	17	½			
6507	11	11	0	10	10	0			
8676	7	7	0	6	6	0			

Experiments for determining the Length of the Second's Pendulum with the Second Pendulum.

m	Comp'd A_m	Observ'd A_m	$C—O$	m	Comp'd A_m	Observ'd A_m	$C—O$	m	Computed A_m	Observed A_m	$C—O$
0	55½′	55½′	0′	0	79′	79′	0′	0	111′	110′	1′
1575	34½	35	—½	1538	47	47	0	1445	64½	64½	0
3150	22½	23	—½	3114	30	30	0	2970	40½	40	½
4725	15	16	—1	4690	19½	20	—½	4495	26	26	0
6300	10	10½	—½	6266	13	14	—1	6020	17½	18	—½
7875	7	7½	—½	7842	9	9½	—½	7545	12	12	0
9450	5	5	0	9418	5	6½	—½	9070	8	8½	—½

533. In Bessel's experiments made for the determination of the length of the second's pendulum of Königsberg, a brass sphere of 24.164 lines in diameter, weighing $0^k.695364$ was suspended so that the length of the pendulum was 1305.3 lines. From his observations with this pendulum, I have found these values of H_1 and H_2.

$$H_1 = 0.05698 \sec \beta,$$
$$H_2 = 0.529 \sec \beta.$$

The same sphere was also vibrated with a length of pendulum of 441.8 lines, from the observations of which I have deduced

$$H_1 = 0.0452 \sec \beta,$$
$$H_2 = 0.587 \sec \beta.$$

Bessel also vibrated an ivory sphere, weighing $0^k.15112$, and having a diameter of 24.094 lines, with each of the preceding lengths of pendulum. From his observations with this sphere and the long pendulum, I have obtained

$$H_1 = 0.05517 \sec \beta,$$
$$H_2 = 0.512 \sec \beta;$$

and from his observations with the short pendulum,

$$H_1 = 0.0509 \sec \beta,$$
$$H_2 = 0.282 \sec \beta.$$

In Bessel's experiments for the determination of the length of the second's pendulum at Berlin, a hollow cylinder was vibrated, of which the diameter of the base was 15.305 lines, and the altitude

15.296 lines, weighing, with its appendages, when it was filled with lead, $0^{k}.67920$, and when it was empty, $0^{k}.22595$. It was suspended in two different modes, in one of which the length of the pendulum was 1304.8 lines, when the cylinder was filled, and 1303.8 lines, when it was empty; and, in the other mode of suspension the length was 440.9 lines when the cylinder was filled, and 440.7 lines when it was empty. From his observations with this pendulum, I have obtained the following values of H_1 and H_2.

When the cylinder was full, and the suspension was long, the values were

$$H_1 = 0.08544 \sec \beta,$$
$$H_2 = 0.733 \sec \beta;$$

when it was full, and the suspension short, they were

$$H_1 = 0.07026 \sec \beta,$$
$$H_2 = 0.724 \sec \beta.$$

When the cylinder was empty, and the suspension long, the values were

$$H = 0.09578 \sec \beta,$$
$$H = 0.559 \sec \beta;$$

when it was empty, and the suspension short, they were

$$H_1 = 0.07003 \sec \beta,$$
$$H_2 = 0.270 \sec \beta.$$

In order to compare the theory of these values with experiment, all the values of observation have been recomputed, and the comparisons are contained in the following tables.

COMPARISON OF BESSEL'S OBSERVED ARCS OF VIBRATION OF THE PENDULUM WITH THE COMPUTED ARCS.

1. *Experiments with the Brass Sphere and Long Suspension.*

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	38.3	38.3	0	39.0	39.0	0	39.5	39.5	0
500	33.7	33.8	—.1	34.2	34.2	0	34.6	34.6	0
1000	29.7	29.8	—.1	30.2	30.1	.1	30.5	30.5	0
1500	26.4	26.4	0	26.8	26.8	0	27.1	26.8	.3
2000	23.5	23.6	—.1	23.9	23.8	.1	24.1	23.9	.2
2500	21.0	20.9	.1	21.3	21.3	0	21.6	21.6	0
3000	18.8	18.8	0	19.1	19.2	—.1	19.3	19.3	0
3500	16.9	16.9	0	17.2	17.2	0	17.4	17.3	—.1
4000	15.3	15.4	—.1	15.5	15.5	0	15.7	15.7	0
0	39.7	39.9	—.2	39.0	39.3	—.3	39.6	39.7	—.1
500	34.8	34.6	.2	34.2	34.1	.1	34.7	34.8	—.1
1000	30.7	30.4	.3	30.2	30.0	.2	30.6	30.5	.1
1500	27.2	27.1	.1	26.8	26.4	.4	27.1	26.9	.2
2000	24.2	24.1	.1	23.9	23.5	.4	24.2	24.0	.2
2500	21.6	21.5	.1	21.3	20.9	.4	21.6	21.4	.2
3000	19.4	19.3	.1	19.1	18.5	.6	19.4	19.3	.1
3500	17.4	17.3	.1	17.2	16.4	.8	17.4	17.3	.1
4000	15.7	15.5	.2	15.5	14.6	.9	15.7	15.5	.2
0	38.6	38.6	0	40.0	40.3	—.3	40.1	39.9	.2
500	33.9	33.9	0	35.1	34.9	.2	35.1	35.2	—.1
1000	29.9	29.9	0	30.9	30.8	.1	31.0	31.0	0
1500	26.5	26.6	—.1	27.4	27.2	.2	27.4	27.5	—.1
2000	23.7	23.6	.1	24.4	24.2	.2	24.4	24.4	0
2500	21.1	21.2	—.1	21.8	21.8	0	21.8	21.9	—.1
3000	19.0	19.0	0	19.5	19.5	0	19.6	19.6	0
3500	17.1	17.1	0	17.5	17.4	.1	17.6	17.6	0
4000	15.4	15.4	0	15.8	15.6	.2	15.8	15.9	—.1
0	39.1	39.1	0	39.3	39.2	.1	38.8	38.5	.3
500	34.3	34.3	0	34.5	34.5	0	34.0	34.0	0
1000	30.3	30.3	0	30.4	30.5	—.1	30.1	30.2	—.1
1500	26.9	26.9	0	27.0	27.1	—.1	26.7	27.0	—.3
2000	23.9	23.9	0	24.0	24.2	—.2	23.8	24.0	—.2
2500	21.4	21.4	0	21.5	21.8	—.3	21.3	21.5	—.2
3000	19.2	19.3	—.1	19.3	19.4	—.1	19.1	19.3	—.2
3500	17.2	17.3	—.1	17.3	17.5	—.2	17.2	17.4	—.1
4000	15.5	15.6	—.1	15.6	15.7	—.1	15.5	15.5	0

1. *Experiments with the Brass Sphere and Long Suspension.* — Continued.

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	39.1	39.1	0	37.8	37.7	.1	39.6	39.7	—.1
500	34.3	34.2	.1	33.2	33.3	—.1	34.7	34.7	0
1000	30.3	30.2	.1	29.4	29.3	.1	30.6	30.6	0
1500	26.9	27.0	—.1	26.1	26.2	—.1	27.1	27.1	0
2000	23.9	24.0	—.1	23.2	23.4	—.2	24.2	24.1	.1
2500	21.4	21.5	—.1	20.8	20.9	—.1	21.6	21.6	0
3000	19.2	19.2	0	18.7	18.7	0	19.4	19.4	0
3500	17.2	17.3	—.1	16.8	16.7	.1	17.4	17.3	.1
4000	15.5	15.5	0	15.1	15.2	—.1	15.6	15.6	.1
0	39.0	39.0	0	41.7	41.6	.1	39.4	39.4	0
500	34.2	34.1	.1	36.5	36.6	—.1	34.6	34.5	.1
1000	30.2	30.1	.1	32.1	32.3	—.2	30.5	30.4	.1
1500	26.8	26.5	.3	28.4	28.6	—.2	27.0	27.1	—.1
2000	23.9	24.0	—.1	25.3	25.5	—.2	24.1	24.1	0
2500	21.3	21.4	—.1	22.5	22.7	—.2	21.5	21.6	—.1
3000	19.1	19.2	—.1	20.2	20.3	—.1	19.3	19.4	—.1
3500	17.2	17.2	0	18.1	18.1	0	17.3	17.3	0
4000	15.5	15.5	0	16.3	16.3	0	15.6	15.6	0
0	39.2	39.4	—.2	38.6	38.6	0	38.5	39.3	—.8
500	34.4	34.4	0	33.9	33.4	.5	33.8	34.2	—.4
1000	30.3	30.3	0	29.9	29.9	0	29.8	30.0	—.2
1500	26.9	26.9	0	26.5	26.7	—.2	26.5	26.3	.2
2000	24.0	24.0	0	23.7	23.6	.1	23.6	23.2	.4
2500	21.4	21.5	—.1	21.1	21.4	—.3	21.1	20.5	.6
3000	19.2	19.2	0	19.0	19.2	—.2	18.9	18.3	.6
3500	17.2	17.1	.1	17.1	17.2	—.1	17.0	16.3	.7
4000	15.6	15.4	.2	15.4	15.4	0	15.3	14.5	.8
0	40.0	39.9	.1	39.9	39.6	.3	39.3	39.0	.3
500	35.1	34.9	.2	35.0	35.0	0	34.5	34.5	0
1000	30.9	30.9	0	30.8	30.9	—.1	30.4	30.5	—.1
1500	27.4	27.5	—.1	27.3	27.5	—.2	27.0	27.1	—.1
2000	24.4	24.4	0	24.3	24.3	0	24.0	24.1	—.1
2500	21.8	21.9	—.1	21.7	21.7	0	21.5	21.5	0
3000	19.5	19.7	—.2	19.5	19.5	0	19.3	19.3	0
3500	17.5	17.6	—.1	17.5	17.3	.2	17.3	17.3	0
4000	15.8	15.8	0	15.8	15.5	.3	15.6	15.5	.1
0	39.7	39.8	—.1	38.9	38.8	.1	38.7	38.7	0
500	34.8	34.8	0	34.1	34.0	.1	34.0	34.3	—.3
1000	30.7	30.7	0	30.1	30.2	—.1	30.0	29.9	.1
1500	27.2	27.2	0	26.7	26.9	—.2	26.6	26.4	.2
2000	24.2	24.2	0	23.8	24.0	—.2	23.7	23.5	.2
2500	21.7	21.8	—.1	21.3	21.4	—.1	21.2	21.1	.1
3000	19.4	19.4	0	19.1	19.2	—.1	19.0	18.9	.1
3500	17.4	17.4	0	17.2	17.2	0	17.1	16.8	.3
4000	15.7	15.6	.1	15.5	15.4	.1	15.4	15.2	.2

1. *Experiments with the Brass Sphere and Long Suspension.* — Continued.

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	38.7	38.7	0	39.3	39.3	0	39.1	39.2	—.1
500	34.0	34.1	—.1	34.5	34.7	—.2	34.3	34.2	.1
1000	30.0	30.0	0	30.4	30.2	.2	30.3	30.3	0
1500	26.6	26.6	0	27.0	27.0	0	26.9	27.0	—.1
2000	23.7	23.6	.1	24.0	24.1	—.1	23.9	23.9	0
2500	21.2	21.2	0	21.5	21.5	0	21.4	21.4	0
3000	19.0	19.0	0	19.3	19.3	0	19.2	19.3	—.1
3500	17.1	16.9	.2	17.3	17.3	0	17.2	17.2	0
4000	15.4	15.3	.1	15.6	15.5	.1	15.5	15.5	0
0	39.0	39.0	0	39.8	39.7	.1			
500	34.2	34.1	.1	34.9	34.9	0			
1000	30.2	30.1	.1	30.8	30.8	0			
1500	26.8	26.8	0	27.3	27.2	.1			
2000	23.9	23.7	.2	24.3	24.3	0			
2500	21.3	21.2	0	21.7	21.7	0			
3000	19.1	19.2	—.1	19.4	19.4	0			
3500	17.2	17.2	0	17.5	17.4	.1			
4000	15.5	15.4	.1	15.7	15.6	.1			

2. *Experiments with the Brass Sphere and the Short Suspension.*

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	14.4	14.65	—.2	13.2	13.5	—.3	12.4	12.4	0
560	13.5	13.7	—.2	12.4	12.7	—.3	11.7	11.6	.1
1120	12.7	12.8	—.1	11.7	11.9	—.2	11.0	10.9	.1
1680	12.0	11.9	.1	11.0	11.0	.0	10.3	10.2	.1
2240	11.3	11.0	.3	10.4	10.3	.1	9.7	9.6	.1
2800	10.6	10.3	.3	9.7	9.7	0	9.2	9.0	.2
3360	10.0	9.6	.4	9.2	9.0	.2	8.6	8.5	.1
3920	9.4	8.9	.5	8.6	8.4	.2	8.1	8.0	.1
4480	8.8	8.3	.5	8.1	7.9	.2	7.6	7.5	.1
5040	8.3	7.8	.5	7.6	7.4	.2	7.2	7.1	.1
5600	7.8	7.3	.5	7.2	7.0	.2	6.8	6.7	.1
0	12.2	12.3	—.1	11.5	11.6	—.1	12.2	12.2	0
560	11.5	11.5	0	10.9	10.9	0	11.5	11.5	0
1120	10.8	10.8	0	10.3	10.3	0	10.8	10.9	—.1
1680	10.2	10.1	.1	9.7	9.7	0	10.2	10.3	—.1
2240	9.6	9.5	.1	9.1	9.1	0	9.6	9.7	—.1
2800	9.0	8.9	.1	8.6	8.6	0	9.0	9.1	—.1
3360	8.5	8.4	.1	8.1	8.15	—.1	8.5	8.5	0
3920	8.0	8.0	0	7.6	7.7	—.1	8.0	8.1	—.1
4480	7.5	7.5	0	7.1	7.3	—.2	7.5	7.7	—.2
5040	7.1	7.0	.1	6.7	6.9	—.2	7.1	7.2	—.1
5600	6.7	6.5	.2	6.3	6.4	—.2	6.7	6.8	—.1

2. *Experiments with the Brass Sphere and the Short Suspension.* — Continued.

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	12.5	12.3	.2	12.8	12.7	.1	12.9	12.8	.1
560	11.8	11.7	.1	12.0	11.95	.1	12.1	12.0	.1
1120	11.1	11.0	.1	11.3	11.3	0	11.4	11.3	.1
1680	10.4	10.4	0	10.7	10.7	0	10.8	10.7	.1
2240	9.8	9.8	0	10.0	10.15	—.1	10.1	10.2	—.1
2800	9.2	9.2	0	9.5	9.5	0	9.5	9.7	—.2
3369	8.7	8.75	—.1	8.9	8.9	0	9.0	9.1	—.1
3920	8.2	8.3	—.1	8.4	8.4	0	8.4	8.6	—.2
4480	7.7	7.9	—.2	7.9	8.0	—.1	7.9	8.1	—.2
5040	7.2	7.45	—.2	7.4	7.6	—.2	7.5	7.7	—.2
5600	6.8	7.0	—.2	7.0	7.2	—.2	7.0	7.2	—.2
0	13.0	12.9	.1	10.9	10.9	0	13.4	13.2	.2
560	12.2	12.1	.1	10.4	10.3	.1	12.6	12.5	.1
1120	11.5	11.4	.1	9.7	9.7	0	11.9	11.8	.1
1680	10.8	10.8	0	9.2	9.2	0	11.2	11.2	0
2240	10.2	10.3	—.1	8.6	8.7	—.1	10.5	10.6	—.1
2800	9.6	9.8	—.2	8.1	8.2	—.1	9.9	9.9	0
3360	9.0	9.2	—.2	7.6	7.7	—.1	9.3	9.3	0
3920	8.5	8.8	—.3	7.2	7.2	0	8.8	8.8	0
4480	8.0	8.2	—.2	6.8	6.8	0	8.3	8.3	0
5040	7.5	7.8	—.3	6.4	6.4	0	7.8	7.85	—.1
5600	7.1	7.4	—.3	6.0	6.0	0	7.3	7.45	—.1
0	13.3	13.3	0	11.1	11.3	—.2	12.4	12.5	—.1
560	12.5	12.5	0	10.5	10.5	0	11.7	11.7	0
1120	11.8	11.8	0	9.9	9.8	.1	11.0	10.9	.1
1680	11.1	11.1	0	9.3	9.3	0	10.3	10.2	.1
2240	10.4	10.5	—.1	8.8	8.8	0	9.7	9.6	.1
2800	9.8	9.8	0	8.3	8.3	0	9.2	9.0	.2
3360	9.2	9.2	0	7.8	7.8	0	8.6	8.5	.1
3920	8.7	8.7	0	7.3	7.2	.1	8.1	8.0	.1
4480	8.2	8.2	0	6.9	6.8	.1	7.6	7.6	0
5040	7.7	7.7	0	6.5	6.4	.1	7.2	7.2	0
5600	7.3	7.3	0	6.1	6.1	0	6.8	6.8	0
0	11.7	11.8	—.1						
560	11.1	11.1	0						
1120	10.4	10.4	0						
1680	9.8	9.8	0						
2240	9.3	9.2	.1						
2800	8.7	8.7	0						
3360	8.2	8.2	0						
3920	7.7	7.7	0						
4480	7.3	7.3	0						
5040	6.8	6.8	0						
5600	6.4	6.4	0						

3. *Experiments with the Ivory Sphere and Long Suspension.*

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	36.5	36.4	.1	38.9	38.9	0	38.7	38.6	.1
500	21.5	22.0	—.5	22.7	22.7	0	22.6	22.7	—.1
1000	13.5	13.2	.3	14.2	14.3	—.1	14.1	14.3	—.2
0	38.9	38.9	0	37.9	37.8	.1	37.9	37.9	0
500	22.7	22.9	—.2	22.2	22.6	—.4	22.2	22.4	—.2
1000	14.2	14.5	—.3	13.9	14.3	—.4	13.9	14.0	—.1
0	39.1	39.2	—.1	37.4	37.5	—.1	38.5	38.5	0
500	22.7	22.4	.3	21.9	21.7	.2	22.5	22.3	.2
1000	14.2	13.7	.5	13.7	12.9	.8	14.0	14.2	—.2
0	38.4	38.4	0	37.0	37.1	—.1	37.3	37.3	0
500	22.4	22.0	.4	21.7	21.1	.6	21.9	21.8	.1
1000	14.0	14.0	0	13.6	13.4	.2	13.7	13.9	—.2
0	37.2	37.3	—.1	36.8	36.8	0	37.1	36.9	.2
500	21.8	21.7	.1	21.6	21.7	—.1	21.8	22.1	—.3
1000	13.7	13.8	—.1	13.6	13.4	.2	13.7	13.9	—.2
0	34.7	34.7	0						
500	20.5	20.6	—.1						
1000	13.3	13.0	.3						

4. *Experiments with the Ivory Sphere and Short Suspension.*

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	12.3	12.3	0	13.6	13.6	0	13.9	14.0	—.1
650	9.3	9.3	0	10.1	10.0	.1	10.3	10.1	.2
1300	7.1	7.2	—.1	7.6	7.8	—.2	7.8	7.8	0
1950	5.4	5.7	—.3	5.8	5.9	—.1	5.9	5.8	.1
2600	4.2	4.3	—.1	4.4	4.3	.1	4.5	4.3	.2
0	13.0	13.1	—.1	14.8	14.9	—.1	14.3	14.3	0
650	9.9	9.9	0	10.9	10.9	0	10.6	10.7	—.1
1300	7.5	7.5	0	8.2	8.0	.2	8.0	8.0	0
1950	5.7	5.7	0	6.2	6.0	.2	6.0	6.0	0
2600	4.5	4.5	0	4.8	4.5	.3	4.6	4.6	0
0	12.9	13.1	—.2	14.0	14.0	0	13.2	13.0	.2
650	9.6	9.6	0	10.4	10.4	0	19.8	19.9	—.1
1300	7.2	7.0	.2	7.8	8.0	—.2	7.3	7.4	—.1
1950	5.5	5.4	.1	5.9	6.1	—.2	5.6	5.9	—.3
2600	4.2	4.1	.1	4.5	4.5	0	4.3	4.4	—.1
0	13.3	13.1	.2	16.0	16.0	0	16.8	16.8	0
650	9.8	10.0	—.2	11.8	11.8	0	12.4	12.5	—.1
1300	7.4	7.3	.1	8.9	8.8	.1	9.3	9.4	—.1
1950	5.6	5.8	—.2	6.7	6.8	—.1	7.1	7.1	0
2600	4.3	4.5	—.2	5.2	5.2	0	5.4	5.5	—.1

4. *Experiments with the Ivory Sphere and Short Suspension.* — Continued.

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	16.6	16.6	0	17.8	18.0	—.2	16.3	16.3	0
650	12.2	12.1	.1	13.0	13.0	0	12.0	12.2	—.2
1300	9.2	9.2	0	9.7	9.5	.2	9.0	9.1	—.1
1950	7.0	7.0	0	7.3	7.1	.2	6.8	7.0	—.2
2600	5.3	5.5	—.2	5.6	5.5	.1	5.2	5.2	0
0	16.1	16.0	.1						
650	11.8	12.0	—.2						
1300	8.8	9.0	—.2						
1950	6.7	6.8	—.1						
2600	5.1	5.1	0						

5. *Experiments with the Full Cylinder and Long Suspension.*

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	39.8	39.8	0	41.5	41.2	.3	38.3	38.9	—.6
500	35.8	35.9	—.1	37.3	37.3	0	34.5	34.3	.2
1000	32.4	32.2	.2	33.6	33.8	—.2	31.2	30.8	.4
1500	29.4	29.4	0	30.5	30.7	—.2	28.4	27.8	.6
2000	26.7	26.6	.1	27.7	27.8	—.1	25.8	25.2	.6
2500	24.3	24.3	0	25.2	25.3	—.1	23.6	22.9	.7
3000	26.3	22.0	.3	23.0	23.2	—.2	21.6	20.6	1.0
3500	20.4	20.3	.1	21.1	21.3	—.2	19.8	18.7	1.1
4000	18.7	19.0	—.3	19.3	19.5	—.2	18.2	17.1	1.1
0	39.4	39.6	—.2	41.0	41.5	—.5	41.7	41.8	—.1
500	35.5	35.3	.2	36.9	36.9	0	37.5	37.6	—.1
1000	32.1	31.9	.2	33.3	32.9	.4	33.8	33.5	.3
1500	29.1	29.0	.1	30.1	29.9	.2	30.6	30.5	.1
2000	26.5	26.2	.3	27.4	27.1	.3	27.8	27.7	.1
2500	24.1	24.0	.1	25.0	24.6	.4	25.3	25.4	—.1
3000	22.1	22.0	.1	22.8	22.5	.3	23.1	23.2	—.1
3500	20.2	20.0	..2	20.9	20.6	.3	21.2	21.2	0
4000	18.6	18.3	.3	19.1	19.0	.1	19.4	19.3	.1
0	39.5	39.2	.3	40.2	40.3	—.1	42.7	42.7	0
500	35.6	35.6	0	36.2	36.1	.1	38.3	38.1	.2
1000	32.1	32.4	—.3	32.7	32.7	0	34.5	34.6	—.1
1500	29.2	29.4	—.2	29.6	30.0	—.4	31.2	31.4	—.2
2000	26.5	26.6	—.1	26.9	27.1	—.2	28.4	28.3	.1
2500	24.2	24.3	—.1	24.5	24.7	—.2	25.8	25.9	—.1
3000	22.1	22.3	—.2	22.4	22.6	—.2	23.6	23.6	0
3500	20.3	20.5	—.2	20.6	20.7	—.1	21.6	21.6	0
4000	18.6	18.7	—.1	18.9	18.9	0	19.8	19.9	—.1

5. *Experiments with the Full Cylinder and Long Suspension.* — Continued.

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	42.3	42.5	—.2	43.2	43.1	.1	42.0	41.8	.2
500	38.0	37.9	.1	38.8	38.8	0	37.7	38.4	—.7
1000	34.2	34.0	.2	34.9	35.0	—.1	34.0	34.0	0
1500	31.0	31.1	—.1	31.6	31.6	0	30.8	30.9	—.1
2000	28.1	28.1	0	28.7	28.6	.1	28.0	28.1	—.1
2500	25.6	25.5	.1	26.1	26.1	0	25.5	25.7	—.2
3000	23.4	23.5	—.1	23.8	23.9	—.1	23.3	23.5	—.2
3500	21.4	21.5	—.1	21.8	21.7	.1	21.3	21.5	—.2
4000	19.6	19.4	.2	20.0	20.0	0	19.5	19.6	—.1
0	41.5	41.4	.1	41.4	41.2	.2	41.6	41.4	.2
500	37.3	37.2	.1	37.2	37.3	—.1	37.4	37.4	0
1000	33.6	33.5	.1	33.6	33.6	0	33.7	33.8	—.1
1500	30.5	30.5	0	30.4	30.4	0	30.5	30.6	—.1
2000	27.7	27.9	—.2	27.6	27.6	0	27.7	27.9	—.2
2500	25.2	25.3	—.1	25.2	25.2	0	25.3	25.5	—.2
3000	23.0	23.1	—.1	23.0	23.1	—.1	23.1	23.2	—.1
3500	21.1	21.3	—.2	21.0	21.2	—.2	21.1	21.3	—.2
4000	19.3	19.4	—.1	19.3	19.1	.2	19.5	19.6	—.1
0	41.4	41.1	.3	40.5	40.3	.2	39.3	39.4	—.1
500	37.2	37.2	0	36.4	36.5	—.1	35.4	35.3	.1
1000	33.6	33.6	0	32.9	33.0	—.1	32.0	32.0	0
1500	30.4	30.5	—.1	29.8	29.8	0	29.0	29.0	0
2000	27.6	27.7	—.1	27.1	27.0	.1	26.4	26.4	0
2500	25.2	25.2	0	24.7	24.6	.1	24.1	24.1	0
3000	23.0	23.1	—.1	22.6	22.5	.1	22.0	22.1	—.1
3500	21.0	21.2	—.2	20.7	20.7	0	20.2	20.3	—.1
4000	19.3	19.5	—.2	19.0	19.0	0	18.5	18.5	0
0	38.0	38.3	—.3	39.6	39.5	.1	42.0	41.8	.2
500	34.1	33.5	.6	35.6	35.6	0	37.7	37.9	—.2
1000	30.9	30.4	.5	32.2	32.1	.1	34.0	34.2	—.2
1500	28.0	27.8	.2	29.2	29.3	—.1	30.8	31.0	—.2
2000	25.5	25.3	.2	26.6	26.7	—.1	28.0	28.2	—.2
2500	23.2	23.0	.2	24.2	24.3	—.1	25.5	25.6	—.1
3000	21.3	21.3	0	22.2	22.3	—.1	23.3	23.4	—.1
3500	19.6	19.6	0	20.3	20.4	—.1	21.3	21.5	—.2
4000	18.0	17.8	.2	18.6	18.7	—.1	19.5	19.6	—.1
0	42.1	42.0	.1	41.7	41.7	0	40.6	40.6	0
500	37.8	37.9	—.1	37.4	37.4	0	36.5	36.6	—.1
1000	34.1	34.1	0	33.8	33.9	—.1	33.0	33.0	0
1500	30.9	30.9	0	30.6	30.6	0	29.9	30.0	—.1
2000	28.0	28.1	—.1	27.8	27.9	—.1	27.2	27.2	0
2500	25.5	25.7	—.2	25.3	25.5	—.2	24.8	24.8	0
3000	23.3	23.3	0	23.1	23.2	—.1	22.6	22.7	—.1
3500	21.3	21.3	0	21.2	21.4	—.2	20.7	20.8	—.1
4000	19.5	19.5	0	19.4	19.6	—.2	19.0	18.9	.1

6. *Experiments with the Full Cylinder and Short Suspension.*

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	12.4	12.45	0	12.1	12.15	0	13.2	13.15	0
730	11.7	11.6	.1	11.4	11.35	0	12.5	12.5	0
1460	11.0	11.0	0	10.8	10.65	.1	11.8	11.95	—.1
2190	10.4	10.4	0	10.2	10.25	—.1	11.2	11.1	.1
2920	9.9	9.8	.1	9.6	9.55	.1	10.5	10.55	—.1
3650	9.3	9.3	0	9.1	9.1	0	9.9	10.0	—.1
4380	8.8	8.85	0	8.6	8.6	0	9.4	9.45	—.1
5110	8.3	8.35	0	8.1	8.15	0	8.9	8.9	0
5840	7.9	7.85	0	7.7	7.65	0	8.4	8.5	—.1
0	13.0	13.0	0	12.1	12.05	0	12.6	12.55	0
730	12.3	12.25	0	11.5	11.7	—.2	12.0	12.0	0
1460	11.6	11.55	0	10.9	11.05	—.1	11.3	11.3	0
2190	10.9	11.05	—.1	10.4	10.5	—.1	10.8	10.75	0
2920	10.3	10.3	0	9.8	9.95	—.1	10.2	10.1	.1
3650	9.8	9.75	0	9.5	9.45	.1	9.7	9.65	.1
4380	9.2	9.2	0	8.9	9.05	—.1	9.2	9.25	0
5110	8.7	8.8	—.1	8.5	8.6	—.1	8.8	8.8	0
5840	8.2	8.2	0	8.0	8.05	0	8.4	8.35	0
0	12.4	12.4	0	12.4	12.4	0	13.5	13.5	0
730	11.8	11.65	.1	11.8	11.75	0	12.8	12.75	.1
1460	11.2	11.15	.1	11.2	11.2	0	12.2	12.35	—.1
2190	10.6	10.55	.1	10.6	10.6	0	11.6	11.65	0
2920	10.1	10.2	—.1	10.1	10.05	.1	11.0	11.0	0
3650	9.6	9.6	0	9.6	9.55	.1	10.5	10.55	0
4380	9.2	9.15	0	9.2	9.15	.1	10.0	10.0	0
5110	8.7	8.7	0	8.7	8.65	.1	9.5	9.55	0
5840	8.3	8.35	0	8.4	8.3	.1	9.1	9.1	0
0	13.0	12.95	0	13.6	13.6	0	13.9	14.0	—.1
730	12.4	12.4	0	12.9	13.0	—.1	13.2	13.2	0
1460	11.7	11.85	—.1	12.3	12.05	.2	12.5	12.65	—.1
2190	11.2	11.3	—.1	11.6	11.5	.1	11.9	11.95	0
2920	10.6	10.8	—.2	11.1	11.0	.1	11.3	11.4	—.1
3650	10.1	10.1	0	10.5	10.55	0	10.7	10.55	.2
4380	9.7	9.7	0	10.0	10.0	0	10.2	10.25	0
5110	9.2	9.2	0	9.5	9.55	0	9.7	9.8	—.1
5840	8.8	8.9	—.1	9.0	9.0	0	9.3	9.2	.1

7. *Experiments with the Empty Cylinder and Long Suspension.*

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	37.5	37.7	—.2	38.2	37.8	.4	37.6	37.6	0
500	28.1	27.7	.4	28.6	28.8	—.2	28.2	28.3	—.1
1000	21.4	21.0	.4	21.8	22.0	—.2	21.5	21.4	.1
1500	16.5	16.6	—.1	16.8	16.7	.1	16.6	16.7	—.1
2000	12.9	13.1	—.2	13.1	13.0	.1	12.9	13.0	—.1
0	38.0	37.9	.1	39.2	38.8	.4	38.9	38.8	.1
500	28.4	28.6	—.2	29.3	29.4	—.1	29.1	29.2	—.1
1000	21.7	21.6	.1	22.3	22.4	—.1	22.1	22.3	—.2
1500	16.7	16.8	—.1	17.2	17.2	0	17.1	17.2	—.1
2000	13.0	13.0	0	13.4	13.4	0	13.3	13.2	.1
0	40.3	40.4	—.1	40.8	40.8	0	38.0	38.0	0
500	30.0	29.9	.1	30.4	30.4	0	28.4	28.4	0
1000	22.8	22.5	.3	23.1	23.1	0	21.7	21.7	0
1500	17.6	17.8	—.2	17.7	17.9	—.2	16.7	16.8	—.1
2000	13.7	13.9	—.2	13.8	13.9	—.1	13.0	13.0	0
0	37.9	37.9	0	40.2	40.2	0	39.9	40.0	—.1
500	28.4	28.4	0	30.0	30.0	0	29.7	29.6	.1
1000	21.6	21.5	.1	22.8	22.9	—.1	22.6	22.5	.1
1500	16.7	16.8	—.1	17.5	17.6	—.1	17.4	17.3	.1
2000	13.0	13.0	0	13.6	13.9	—.3	13.6	13.6	0
0	39.8	40.0	—.2	39.2	39.1	.1	40.7	40.5	.2
500	29.7	29.7	0	29.3	29.6	—.3	30.3	30.4	—.1
1000	22.6	22.4	.2	22.3	22.3	0	23.0	23.1	—.1
2500	17.4	16.7	.7	17.2	17.2	0	17.7	17.6	.1
2000	13.5	13.2	.3	13.4	13.4	0	13.8	13.7	.1
0	40.4	40.4	0	40.1	40.2	—.1	40.4	40.4	0
500	30.1	30.3	—.2	29.9	29.9	0	30.1	30.2	—.1
1000	22.9	22.7	.2	22.7	22.7	0	22.9	22.7	.2
1500	17.6	17.5	.1	17.5	17.5	0	17.6	17.6	0
2000	13.7	13.7	0	13.6	13.6	0	13.7	13.6	.1
0	38.9	38.9	0	38.6	38.8	—.2	38.7	38.7	0
500	29.1	29.1	0	28.8	28.6	.2	28.9	28.8	.1
1000	22.1	21.9	.2	22.0	21.9	.1	22.0	22.2	—.2
1500	17.0	17.0	0	16.9	16.9	0	17.0	17.1	—.1
2000	13.3	13.4	—.1	13.2	13.2	0	13.2	13.4	—.2
0	39.5	39.5	0	38.1	38.0	.1	38.5	38.5	0
500	29.5	29.8	—.3	28.5	28.6	—.1	28.8	29.0	—.2
1000	22.4	22.4	0	21.7	21.8	—.1	21.9	21.9	0
1500	17.3	17.2	.1	16.8	16.6	.2	16.9	16.8	.1
2000	13.4	13.4	0	13.0	12.9	.1	13.2	13.0	.2

7. *Experiments with the Empty Cylinder and Long Suspension.* — Continued.

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	39.2	38.8	.4	38.5	38.4	.1	39.0	39.0	0
500	29.3	29.5	—.2	28.8	29.0	—.2	29.1	29.2	—.1
1000	22.3	22.4	—.1	21.9	21.9	0	22.2	22.2	0
1500	17.2	17.2	0	16.9	16.9	0	17.1	17.2	—.1
2000	13.4	13.6	—.2	13.2	13.2	0	13.3	13.4	—.1
0	39.9	39.9	0	38.4	38.4	0	38.9	38.9	0
500	29.8	29.8	0	28.7	28.8	—.1	29.1	29.2	—.1
1000	22.6	22.8	—.2	21.9	22.0	—.1	22.1	22.1	0
1500	17.4	17.6	—.2	16.9	17.0	—.1	17.1	17.1	0
2000	13.6	13.6	0	13.1	13.1	0	13.3	13.2	.1
0	38.6	38.6	0	39.1	39.1	0	36.8	36.8	0
500	28.9	28.9	0	29.2	29.3	—.1	27.6	27.6	0
1000	22.0	21.9	.1	22.2	22.1	.1	21.1	21.1	0
1500	16.9	16.8	.1	17.1	17.0	.1	16.3	16.3	0
2000	13.2	13.1	.1	13.3	13.1	.2	12.6	12.6	0
0	36.3	36.3	0	38.3	38.3	0	39.4	39.3	.1
500	27.3	27.4	—.1	28.6	28.7	—.1	29.4	29.5	—.1
1000	20.8	21.0	—.2	21.8	21.8	0	22.4	22.4	0
1500	16.1	16.1	0	16.8	16.8	0	17.2	17.3	—.1
2000	12.5	12.5	0	13.1	13.1	0	13.4	13.5	—.1
0	40.1	40.1	0	39.4	39.3	.1	38.8	38.8	0
500	29.9	29.9	0	29.4	29.5	—.1	29.0	29.4	—.4
1000	22.7	22.6	.1	22.4	22.4	0	22.1	22.0	.1
1500	17.5	17.4	.1	17.2	17.3	—.1	17.0	17.0	0
2000	13.6	13.5	.1	13.4	13.5	—.1	13.2	13.3	—.1
0	40.0	40.0	0	38.7	38.7	0	38.8	38.9	—.1
500	29.8	30.0	—.2	28.9	28.9	0	29.0	28.9	.1
1000	22.7	22.5	.2	22.0	21.9	.1	22.1	22.0	.1
1500	17.4	17.3	.1	17.0	16.9	.1	17.0	17.1	—.1
2000	13.6	13.6	0	13.2	13.2	0	13.2	13.3	—.1
0	39.1	39.1	0	38.5	38.5	0	38.7	38.7	0
500	29.2	29.1	.1	28.8	28.8	0	28.9	29.0	—.1
1000	22.2	22.1	.1	21.9	21.8	.1	22.0	22.0	0
1500	17.1	17.1	0	16.9	16.9	0	17.0	16.9	.1
2000	13.3	13.3	0	13.2	13.1	.1	13.2	13.2	0
0	37.7	37.7	0	38.8	38.8	0	38.1	38.2	—.1
500	28.2	28.0	.2	29.0	29.0	0	28.5	28.5	0
1000	21.5	21.5	0	22.1	22.0	.1	21.7	21.5	.2
1500	16.6	16.7	—.1	17.0	16.9	.1	16.8	16.6	.2
2000	12.9	12.9	0	13.2	13.1	.1	13.0	12.9	.1

8. *Experiments with the Empty Cylinder and the Short Suspension.*

m	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$	Computed A_m	Observed A_m	$C—O$
0	11.4	11.4	0	12.2	12.3	—.1	13.3	13.3	0
800	9.5	9.4	.1	10.1	10.15	0	11.0	10.95	0
1600	7.9	7.7	2	8.5	8.45	0	9.1	9.05	0
2400	6.6	6.5	1	7.1	7.05	0	7.6	7.55	0
3200	5.5	5.5	.0	6.0	5.9	.1	6.3	6.3	0
4000	4.7	4.6	.1	5.0	4.9	.1	5.3	5.3	0
4800	4.0	3.9	.1	4.2	4.1	.1	4.4	4.5	—.1
0	13.3	13.3	0	13.4	13.4	0	12.1	12.1	0
800	11.0	11.1	—.1	11.3	11.3	0	10.2	10.25	0
1600	9.1	9.15	0	9.5	9.4	.1	8.7	8.65	0
2400	7.6	7.8	—.2	8.0	8.05	0	7.3	7.4	—.1
3200	6.3	6.2	.1	6.8	6.7	.1	6.2	6.25	0
4000	5.3	5.4	—.1	5.8	5.7	.1	5.3	5.3	0
4800	4.4	4.45	0	4.9	4.9	0	4.5	4.5	0
0	12.3	12.2	.1	13.0	13.0	0	13.2	13.15	0
800	10.5	10.5	0	11.0	11.0	0	11.2	11.15	0
1600	8.9	9.0	—.1	9.4	9.3	.1	9.5	9.6	—.1
2400	7.6	7.85	—.2	8.0	8.0	0	8.1	8.0	.1
3200	6.5	6.7	—.2	6.9	6.9	0	6.9	6.85	0
4000	5.6	5.75	—.2	5.9	5.9	0	5.9	5.95	0
4800	4.8	4.95	—.1	5.0	5.1	—.1	5.1	5.1	0
0	13.0	12.95	0	14.1	14.25	—.1	12.9	12.9	0
800	11.0	11.05	0	11.9	11.9	0	10.9	10.95	0
1600	9.4	9.4	0	10.1	10.1	0	9.3	9.15	.1
2400	8.0	8.05	—.1	8.6	8.65	—.1	7.9	7.85	0
3200	6.8	6.9	—.1	7.3	7.35	—.1	6.7	6.85	—.1
4000	5.8	5.85	0	6.2	5.95	.3	5.7	5.85	—.1
4800	5.0	5.0	0	5.3	5.25	.1	4.9	4.9	0

In the computation of these values, there has been no regard to the resistance arising from the wires of suspension. The difference between the values of H_2 may be attributed to the uncertainty of the observations, and those of H_1 may, perhaps, be accounted for, in the same way. The value of H_2 is nearly ten times as great as that which is given by the observations of Borda upon the resistance of the atmosphere. It must, therefore, be doubtful, whether the observed diminution of the arcs of vibration of the pendulum is, wholly or principally, due to the medium in which it vibrates, or to some more latent cause. This doubt is much increased by the discussion of the observations of BAILY.

534. In BAILY's experiments, various pendulums, which were mostly spheres and cylinders, were vibrated in the receiver of an air-pump, with the air either at its ordinary pressure, or at the small density of about one thirtieth of an atmosphere. For the full and exact description of the pendulums the original memoir must be consulted, but the following brief description is sufficient for the present purpose. Numbers 1, 2, 3, and 4 are spheres of platina, lead, brass, and ivory, all of the same diameter, which is somewhat less than 1½ inches, and of which the weights with their vibrating appendages are, respectively 9050, 4648, 3217, and 776½ grains. Nos. 5, 6, and 7 are spheres of lead, brass, and ivory, all of the same diameter, which is 2.06 inches, and of which the weights are respectively, 13019, 9302, and 2066½ grains. Nos. 8 and 9 are the same spheres of lead and ivory with those of Nos. 5 and 7, but suspended from a wire passing over a small cylinder instead of from a knife edge. In Nos. 10, 11, 12, and 13 the vibrating mass was a brass cylinder, of which the diameter of the base is 2.06 inches, the altitude 2.06 inches, and the weight 14190 grains; in Nos. 10 and 13 the axis of the cylinder coincides with that of the pendulum rod, but the rod of No. 13, which was also adopted in Nos. 11 and 12, was a thick brass wire 0.185 inch in diameter, 37½ inches long, and weighing 2050 grains; in Nos. 11 and 12 the axis of the cylinder was horizontal, in No. 11 it was perpendicular to the plane of vibration, and in No. 12 it was in the plane of vibration. No. 14 is a cylinder of lead, of which the diameter of the base is 2.06 inches, the altitude 4 inches, the weight 34500 grains, and the axis coincident with the rod of the pendulum. In Nos. 15, 16, 17, 18, and 19 the vibrating mass was a hollow cyl-

inder of the same position and external dimensions with No. 14; in No. 15 both ends were open; in No. 16 the top was open and the bottom closed; in No. 17 the top was closed and the bottom open; in No. 18 both ends were closed; in No. 19 an inner sliding tube was removed so as to reduce the weight; and the weights, with the inclosed air, were, respectively, 8497, 8922, 8622, 9048, and 7250 grains. No. 20 is a lens of lead 2.06 inches in diameter, an inch thick in the middle, with a flat circumference of about a quarter of an inch wide, and a weight of 6505 grains. No. 21 is a solid copper cylindrical rod of 0.41 inch in diameter, 58.8 inches long, and weighing 16810 grains. In Nos. 25, 26, 27, 28, 29, 30, 31, 32, 33, and 34, the vibrating masses were convertible pendulums, formed of plane bars, and they are vibrated successively with each of their points of suspension, which were knife edges; in Nos. 25 and 26 the bar was brass, two inches wide, three eighths of an inch thick, 62.2 inches long, and weighing 121406 grains; in Nos. 27 and 28 it was copper of the same width with the brass bar, half an inch thick, 62.5 inches long, and weighed 155750 grains; in Nos. 29 and 30, it was iron of the same width and thickness with the copper bar, 62.1 inches long, and weighed 140547 grains; in Nos. 31, 32, 33, and 34 it was a doubly convertible brass bar, three quarters of an inch thick, 62 inches long, and weighed 231437 grains. In Nos. 35, 36, 37, and 38, a doubly convertible pendulum, made of a brass cylindrical tube of 1½ inches in diameter, 56 inches long, and weighing 81047 grains was vibrated upon a knife edge with all four of its planes of suspension. No. 39 is a mercurial pendulum. Nos. 40 and 41 are clock pendulums in which the vibrating mass was a leaden cylinder 1.8 inches in diameter, 13.5 inches long, and weighing 93844 grains; in No. 40 it was suspended from a spring, by a cylindrical rod of deal of three eighths of an inch in diameter, and in No. 41 by a flat rod of deal one inch wide, 0.14 inch thick in the middle of its width and bevelled on each side to a thin edge, which was opposed to the direction of its motion.

In the discussion of Baily's experiments, the value of H_2 is neglected, because it is of small influence, and the arcs of vibration, being usually given only for the beginning and end of the experiment, are just sufficient to determine one of the quantities H_1 and H_2; and the values of H_1 are not reduced to the same density of air. The ratio of the value of H_1 for the ordinary state of the air to its value in the exhausted receiver, varies from 1.9 to 4.2, instead of being about 30, which it should be if it were proportional to the density of the air; the value of this ratio in the following table is expressed by J. The total resistance to the motion of the pendulum, supposed to be proportional to the velocity is, for the unit of velocity, expressed by H_1'' in the table; and this same resistance, reduced to the unit of weight, is expressed by H_1'.

The observation of the arcs of vibration in Baily's experiments is limited to the initial and final arcs, and the direct comparison of the computed and observed arcs is, consequently, quite unnecessary, and cannot contribute to verify the accuracy of the hypothesis upon which the computation is based. The only two cases in which an intermediate arc was observed with Nos. 6 and 14 seem to sustain the hypothesis; for they differ from it slightly, but in opposite directions.

The diversity of the values of H_1 indicates that the resisting force of the motion to the pendulum demands a new experimental investigation, conducted with a direct object to its determination; and that, until such an investigation has been made, the length of the seconds pendulum must be regarded as liable to an unknown error.

Values of H_1 in Baily's Experiments upon the Vibrations of Pendulums.

No. of Pendulums.	Barometer.	H_1	H_1'	H_1''	J
1	0.7689	.0673	.000077	.000132	2.68
3	0.7646	.0702	.000080	.000384	2.62
2	0.7523	.0662	.000075	.000250	2.55
4	0.7660	.0561	.000063	.001272	2.71
6	0.7638	.0570	.000123	.000204	2.74
7	0.7630	.0538	.000116	.000864	2.62
5	0.7644	.0627	.000128	.000161	3.18
9	0.7682	.0589	.000127	.000945	2.86
8	0.7677	.1021	.000219	.000261	2.92
10	0.7652	.0651	.000179	.000194	3.42
11	0.7637	.0558	.000270	.000256	2.62
12	0.7623	.0603	.000290	.000277	3.33
13	0.7552	.0571	.000235	.000262	2.98
18	0.7491	.0535	.000285	.000484	3.27
15	0.7554	.0658	.000350	.000635	4.10
16	0.7495	.0595	.000292	.000505	2.95
17	0.7584	.0558	.000297	.000531	3.39
14	0.7747	.0592	.000315	.000140	4.22
19	0.7620	.0510	.000272	.000578	3.33
20	0.7620	.0656	.000065	.000156	2.09
21	0.7575	.0661	.000742	.000682	2.72
25	0.7522	.0789	.005606	.000333	3.32
26	0.7465	.0756	.004782	.000319	3.74
31	0.7522	.1555	.003666	.000245	3.32
32	0.7520	.1581	.003673	.000245	3.55
34	0.7529	.1661	.003772	.000251	3.72
33	0.7535	.1417	.003480	.000232	3.13
35	0.7595	.0739	.003091	.000589	3.48
36	0.7627	.0660	.002763	.000526	3.31
37	0.7577	.0701	.002931	.000558	3.39
38	0.7564	.0659	.002760	.000526	2.97
39	0.7622		.001396	.000209	1.87
41	0.7573	.0664	.001260	.000207	2.52
40	0.7589	.0769	.001299	.000213	2.39

Values of H_1 *in Baily's Experiments upon the Vibrations of Pendulums.* — Continued.

No. of Pendulums.	Barometer.	H_1	H_1'	H_1''	J
1	0.0288	.0251	.000028	.000049	2.68
3	0.0294	.0267	.000031	.000146	2.62
2	0.0265	.0259	.000030	.000098	2.55
4	0.0347	.0284	.000024	.000470	2.71
6	0.0268	.0285	.000044	.000074	2.74
7	0.0270	.0282	.000044	.000330	2.62
5	0.0290	.0275	.000042	.000050	3.18
9	0.0360	.0282	.000044	.000331	2.86
8	0.0299	.0348	.000075	.000089	2.92
10	0.0239	.0190	.000052	.000057	3.42
11	0.0478	.0213	.000103	.000098	2.62
12	0.0348	.0182	.000088	.000083	3.33
13	0.0370	.0192	.000092	.000089	2.98
18	0.0300	.0164	.000087	.000148	3.27
15	0.0271	.0164	.000097	.000148	4.10
16	0.0266	.0186	.000099	.000171	2.95
17	0.0362	.0165	.000088	.000157	3.39
14	0.0298	.0139	.000074	.000033	4.22
19	0.0305	.0154	.000083	.000174	3.33
20	0.0305	.0313	.000031	.000074	2.09
21	0.0288	.0244	.000274	.000251	2.72
25	0.0313	.0238	.001505	.000101	3.32
26	0.0325	.0202	.001277	.000086	3.74
31	0.0414	.0469	.001105	.000074	3.32
32	0.0391	.0439	.001034	.000069	3.55
34	0.0410	.0431	.001014	.000067	3.72
33	0.0463	.0472	.001111	.000074	3.13
35	0.0384	.0213	.000888	.000170	3.48
36	0.0367	.0200	.000834	.000160	3.31
37	0.0422	.0206	.000859	.000166	3.39
38	0.0412	.0222	.000930	.000178	2.97
39	0.0477		.000747	.000112	1.87
41	0.0457	.0263	.000498	.000083	2.52
40	0.0434	.0320	.000543	.000089	2.39

THE TAUTOCHRONE.

535. The consideration of the pendulum leads, directly, to the investigation of that curve, upon which the duration of the vibration is independent of the length of the arc of oscillation. Such a curve is called a *tautochrone*, and is readily determined when the body is only subject to the action of fixed forces.

536. If the force which acts in the direction of the motion of the body is denoted by S, the equation of its motion is

$$D_t^2 s = S.$$

In the case in which S is a function of s, let s_0 denote the point, at which the velocity vanishes, or the extremity of the arc of vibration. Hence

$$v^2 = 2\int_{s_0}^{s} {}_s S = 2\,(\Omega - \Omega_0)\,;$$

and if the origin of coördinates is at the point of maximum velocity, the time of vibration is determined by the equation

$$T = \int_0^{s_0} {}_s \frac{\sqrt{2}}{\sqrt{(\Omega - \Omega_0)}}.$$

If $$h - \frac{s}{s_0},$$

if Ω is a function of s expressed by Ω_s, and if s is written instead of s_0, the value of T becomes

$$T = \int_0^1 {}_h \frac{s\sqrt{2}}{\sqrt{(\Omega_{sh} - \Omega_s)}}.$$

In order that the special value of the arc may disappear from this integral, it is obvious that Ω_s has the form

$$\Omega_s = A - B s^2,$$

which reduces the value of T to

$$T = \int_0^1 {}_h \frac{\sqrt{2}}{\sqrt{B}\sqrt{(1-h^2)}} = \sqrt{\frac{\pi}{2B}}.$$

The tangential force along the curve is, therefore,

$$S = D_s \Omega = -2Bs.$$

537. If F denotes the actual force, which acts upon the body in the direction of f, the preceding equation gives for *an equation of the tautochrone*

$$F \cos \tfrac{f}{s} = -2Bs = F D_s f,$$

or

$$A - Bs^2 = \int_f F.$$

538. *In the case in which the body is restricted to move upon a curve which rotates uniformly about a fixed axis,* the equations and notation of § 468 combined with the previous section, give for *the equation of the tautochrone*

$$A - Bs^2 = \tfrac{1}{2} \alpha^2 u^2,$$

which may assume the form

$$\frac{s^2}{a^2} + \frac{u^2}{b^2} = 1,$$

in which a and b are constants.

539. *When the revolving curve is a plane curve, and situated in the same plane with the axis of revolution,* the notation

$$b = a \cot i,$$
$$s = a \sin \theta = a \sin \varphi \sin i,$$

and that of elliptic functions give

$$u = b \cos \theta,$$

$$z = \frac{a}{\sin i} \mathcal{E}_i \varphi - b \cos i \, \mathcal{F}_i \varphi;$$

and if τ is the inclination of the curve to the axis of rotation, its value is

$$\sin \tau = - \cot i \tan \theta.$$

The maximum of u is b, but its least value, corresponding to

$$\tau = \pm \tfrac{1}{2} \pi,$$

or

$$\theta = \mp i,$$

is

$$u = b \cos i;$$

and the corresponding value of s is

$$s = \mp a \sin i.$$

The curve consists of several branches, which form cusps by their mutual contact at their extremities, and it resembles the cycloid in its general character.

540. *In the case of a heavy body moving upon a plane vertical curve,* let ν denote the angle which the radius of curvature ϱ makes with its horizontal projection, and the equation (317_{14}) gives

$$s = - \frac{F}{2B} \cos \nu,$$

$$\varrho = \frac{F}{2B} \sin \nu;$$

which is the equation of the cycloid referred to its radius of curvature and angle of direction, so that *the cycloid is the tautochrone of a free heavy body in a vacuum. The same curve, drawn upon the de-*

veloped surface, is the tautochrone of a heavy body, moving upon a vertical cylinder.

541. Every curve may be regarded as being upon the surface of its vertical cylinder of projection; and, therefore, *the tautochrone of a heavy body moving in a vacuum upon any surface whatever, is the intersection of the surface with such a vertical cylinder, that the intersection is a cycloid upon the developed vertical cylinder.* The determination of the tautochrone upon any surface is thus reduced to a problem of pure geometry. If the axis of z is the upward vertical, and if z_0 is the height of the lowest point of the curve above the origin, the equation (317_{16}) becomes, in the present case,

$$z = \frac{B}{g} s^2 + z_0.$$

542. *If a heavy body is restricted to move upon a cylinder of which the axis is horizontal,* and of which the equation of the base is

$$\varrho_1 = na \cos \nu_1 \sin^{n-1} \nu_1,$$

in which ν_1 is the angle, which the radius of curvature, denoted by ϱ_1, makes with the upward vertical; and when the cylinder is developed into a vertical plane, if y is the height of the moving body above the horizontal line, which corresponds to the lowest side of the undeveloped cylinder, the value of y is

$$y = a \sin^n \nu_1.$$

The force of gravity, resolved in a direction tangential to the cylinder, is

$$g \sin \nu_1 = g \sqrt[n]{\frac{y}{a}};$$

so that *the present problem corresponds to that of a body moving in a ver-*

tical plane, and subject to a force which is fixed in direction, and proportional to some power of the height above a given level. The equation (319_{13}) gives *for the equation of the tautochrone*

$$\frac{B}{g}s^2+z_0=\int_{\nu_1}\left(\sin\nu_1\,D_{\nu_1}y\right)=\frac{n}{n+1}a\sin^{n+1}\nu_1=\frac{n\,a}{n+1}\left(\frac{y}{a}\right)^{(1+\frac{1}{n})}.$$

543. If ν denotes the angle which the radius of curvature (ϱ) of the tautochrone makes with the upward vertical in the developed cylinder, the equation (317_{14}) gives

$$\sin\nu\sin\nu_1=\frac{2B}{g}s,$$

which, substituted in (320_5), reduces *the equation of the tautochrone* to

$$\frac{B}{g}s^2+z_0=\frac{n}{n+1}a\left(\frac{2Bs}{g\sin\nu}\right)^{n+1}.$$

544. When z_0 vanishes in the problem of the preceding section, the *equation of the tautochrone becomes*

$$s=b\sin^{\frac{n+1}{n-1}}\nu=\sqrt{\frac{nag}{(n+1)B}}\left(\frac{y}{a}\right)^{\frac{n+1}{2n}},$$

or

$$\varrho=\frac{n+1}{n-1}b\sin^{\frac{2}{n-1}}\nu\cos\nu;$$

in which

$$(2b)^{n-1}=\frac{n+1}{4na}\left(\frac{g}{B}\right)^n;$$

so that *the tautochrone on the developed cylinder of* § 542 *is of the same trigonometric class of curves with the base of the cylinder, when it passes through the lowest side of the undeveloped cylinder.* This case is impossible, when n is included between positive and negative unity; for when n is negative and, independently of its sign, less than unity, s becomes infinite when y vanishes, but when n is positive and less than unity, the derivative of (320_{19}), which is

$$D_y s = \operatorname{cosec} \nu = \tfrac{1}{2}\sqrt{\frac{a g}{B}\left(\frac{y}{a}\right)^{\frac{1-n}{2n}}},$$

gives the impossible result that $\operatorname{cosec} \nu$ vanishes with y.

545. *The differential equation of the tautochrone, in the case of* § 542, *referred to rectangular coördinates upon the developed cylinder,* is readily obtained from the equations of § 542, which give

$$k^2\left(\frac{y}{a}\right)^{\frac{2}{n}} = \left(\left(\frac{y}{a}\right)^{1+\frac{1}{n}} - \frac{n+1}{n\,a} z_0\right)(D_y x^2 + 1),$$

in which

$$k^2 = \frac{g}{4B}\frac{n+1}{n a},$$

and the axis of x is horizontal.

In the case of § 544, in which z_0 vanishes, this equation becomes

$$D_y x^2 + 1 = k^2\left(\frac{y}{a}\right)^{\frac{1}{n}-1}.$$

546. In the case in which n is unity, that is, *in which the base of the cylinder is a cycloid, the equation of the tautochrone on the developed cylinder, becomes*

$$\frac{B}{g} s^2 + z_0 = \frac{y^2}{2a}.$$

When z_0 vanishes, this curve is reduced to a straight line, but in all other cases, its form, if it is infinitely extended in the plane of the developed cylinder, resembles the hyperbola. By the adoption of the notation

$$\sin^2 i = \frac{2aB}{g},$$

$$y = \sqrt{(2 a z_0)} \sec \varphi,$$

and that of elliptic functions, its equation may be expressed in the forms

$$s = \sqrt{\frac{g z_0}{B}} \tan \varphi,$$

$$x = \sqrt{\frac{g z_0}{B}} (\cos \theta \tan \varphi + \mathfrak{F}_i \varphi - \mathfrak{E}_i \varphi).$$

547. *If a heavy body is restricted to move upon a surface of revolution about a vertical axis,* of which the equation of the meridian curve is that of (319_{17}). If y is the distance of the body on the meridian curve from the lowest point of the surface, the value of y is given by the equation (319_{25}), and the force of gravity, resolved in a direction tangential to the meridian curve is expressed by (319_{29}), so that the present problem resembles that of a body moving in a plane, and subject to a force, which is directed towards a fixed point in the plane, and is proportional to some power of the distance from that point. *The equation* (317_{14}) *of the tautochrone,* gives

$$B s^2 = \frac{g}{m+1} \frac{y^{m+1} - y_0^{m+1}}{a^m},$$

in which m is the reciprocal of n, and y_0 the value of y at the lowest point of the tautochrone.

548. When m vanishes, the surface of revolution is a *right cone,* and the equation (322_{19}) becomes

$$B s^2 = g (y - y_0).$$

By means of the notation

$$\sin^2 \theta = \frac{4B}{g} (y - y_0),$$

$$\sec \beta = 1 + \frac{8 B y^0}{g};$$

the angle (φ) which y makes with y_0 in the developed cone is given by the formula

$$\tan\left[(\vartheta + \tfrac{1}{2}\varphi)\tan\tfrac{1}{2}\beta\right] = \frac{\tan\theta}{\tan\frac{1}{2}\beta};$$

so that *the polar equation of this tautochrone upon the developed cone is expressed by the combination of* (322_{28}) *and* (323_3).

549. When y_0 vanishes, β also vanishes, and the equation (323_3) becomes

$$\vartheta + \tfrac{1}{2}\varphi + \cot\vartheta = 0.$$

550. When m is unity, the surface of revolution is cycloidal and the equation (322_{19}), becomes

$$aBs^2 = \tfrac{1}{2}g\,(y^2 - y_0^2),$$

which becomes the meridian curve itself, when y_0 vanishes.

551. In the case given in (322_{14}), *of a body moving in a plane and subject to a force, which is directed towards a fixed point in the plane, and is proportional to some power* (m) *of the distance from that point, the equation of the tautochrone may be given in the form*

$$s^2 = A\,(r^{m+1} - r_0{}^{m+1}),$$

in which the attracting point is the origin of polar coördinates. *The polar differential equation is*

$$r^2 D_r\varphi^2 + 1 = \frac{\frac{1}{4}(1+m)^2 A\,r^{2m}}{r^{m+1} - r_0{}^{m+1}}.$$

552. *If the attraction or repulsion of the point had been any function whatever of the distance from the origin, the equation of the tautochrone would have assumed the form*

$$s^2 = Fr - Fr_0,$$

in which F denotes the function of which the derivative expresses the given law of attraction. This equation may therefore assume the form

$$x^2 + y^2 = r^2 = S_1,$$

in which S_1 is a function of S. If then ν is the angle which the radius of curvature makes with the axis of x, the derivatives of this equation are

$$2x \sin\nu - 2y \cos\nu = S_1',$$

$$(2x \cos\nu + 2y \sin\nu)\, D_s\nu = S_1'' - 2\,;$$

whence

$$2x\, D_s\nu = S_1' \sin\nu\, D_s\nu + (S_1'' - 2) \cos\nu,$$

$$2y\, D_s\nu = - S_1' \cos\nu\, D_s\nu + (S_1'' - 2) \sin\nu,$$

$$4\, S_1\, D_s\nu^2 = S_1'^2\, D_s\nu^2 + (S'' - 2)^2,$$

$$\varrho^2 = D_\nu s^2 = \frac{4\, S_1 - S_1'^2}{(S_1'' - 2)^2} = S_2,$$

which is *the equation of the tautochrone expressed in terms of the radius of curvature and the arc.*

553. *The polar differential equation of the tautochrone in the case of the preceding section is*

$$r^2\, D_r\, \varphi^2 + 1 = \frac{\frac{1}{4}(F'\, r)^2}{Fr - F r_0},$$

which is the same equation with that which is given by Puiseux.

554. The derivative of (324_{16}) relatively to ν is

$$2\, D_\nu\, \varrho = S_2',$$

so that the elimination of s between (324_{16}) and (324_{27}) gives *the differential equation of this tautochrone in terms of the radius of curvature and the angle of its direction.*

555. In the case of § 552, when

$$S_1 = a s + b,$$

the value of S_2 is

$$S_1 = a s + b - \tfrac{1}{4} a^2.$$

The equation (324_{27}) becomes therefore

$$D_v \varrho = \tfrac{1}{2} a,$$

and *the equation of the tautochrone is*

$$\varrho = \tfrac{1}{2} a v,$$

which is that of the involute of the circle. This case corresponds to that in which the law of the central force is of the form

$$B r (r^2 - r_0^2).$$

556. In the case of § 552, when

$$S_1 = a (s + b)^2,$$

the value of S_2 is

$$S_2 = \varrho^2 = m^2 (s + b)^2,$$

in which $$m^2 = \frac{a}{1 - a},$$

so that a must be positive and less than unity. *The equation of the tautochrone is, then,*

$$\varrho = R e^{m v},$$

which is that of the logarithmic spiral. This case corresponds to that in which the law of the central attraction is of the form

$$\frac{r - r_0}{a},$$

that is, *in which the force is proportional to the distance of the body from the circumference of the circle described from the origin as the centre with a radius equal to that of the initial position of the body.* This case is discussed by Puiseux.

557. In the case of § 552, *when the force is proportional to the distance from the origin.* The equation (323_{31}) assumes the form

$$s^2 = \frac{r^2 - r_0^2}{a},$$

which, with the value of m in (325_{22}), reduces S_1 and S_2 to

$$S_1 = a s^2 + r_0^2,$$

$$\varrho^2 = S_2 = m^2 s^2 + \left(\frac{r_0}{1-a}\right)^2.$$

The equation of the tautochrone is, therefore,

$$D_\nu \varrho = m^2 s = m \sqrt{\left[\varrho^2 - \left(\frac{r_0}{1-a}\right)^2\right]}$$

of which the integral is

$$\varrho = \frac{r_0}{1-a} \operatorname{Cos}(m \nu)$$

in which the arbitrary constant is determined so that ν may vanish with s.

The second derivative of this equation gives, for the radius of curvature of the second evolute of the tautochrone

$$\varrho'' = m^2 \varrho$$

so that *the second evolute is similar to the tautochrone itself.*

In the case in which m *is real,* which corresponds to that in which a is positive and less than unity, this curve runs off to infinity in each direction, with a constantly increasing radius of curvature.

In the case in which m *is imaginary,* the substitution of

$$-n^2 = m^2,$$

reduces the equation of the tautochrone to the form

$$\varrho = \frac{r_0}{1-a} \cos(nv),$$

which is the equation of an epicycloid. The epicycloid is formed by the external rotation of one circle upon another, when n *is less than unity, in which case* a *is negative and the force is repulsive; but the epicycloid is formed by internal rotation, when* n *is greater than unity, which corresponds to the case when* a *is positive and greater than unity.* In either of these cases, the initial velocity must not be more than sufficient to carry the body to either of the cusps.

In the case in which a *is infinite, the tautochrone is reduced to a straight line.*

The example of this section is discussed by Puiseux.

558. The example of the preceding section embraces the case of any force, which is a function of a distance from the origin, in the immediate vicinity of the point of greatest velocity. *The form of the tautochrone, near the point of greatest velocity, in the example of* § 552, *is typified, therefore, by the epicycloid, or by the curve of equation* (326_{21}).

559. The investigation of the tautochrone in a resisting medium is postponed to the general case of the chronic curves.

THE BRACHYSTOCHRONE.

560. The curve upon which a body moves in the least possible time from one given point to another, is called the *brachystochrone.*

561. The investigation of the general case of a brachystochrone which is confined to any surface or limited by any condition, may be conducted by means of rectangular coördinates. The time of transit from the first to the last of the given points may be expressed by the equation

$$T = \int_x \frac{D_x s}{v},$$

which is to be a minimum. This condition gives, for each of the other axes, the equation

$$D_y \left(\frac{D_x s}{v}\right) - D_x D_{y'} \left(\frac{D_x s}{v}\right) = 0.$$

562. When the body is only subject to the action of fixed forces, v does not involve either y' or z', and the preceding equation becomes

$$\frac{D_y v}{v^2} + D_s \left(\frac{\cos \frac{y}{s}}{v}\right) = 0,$$

or by (316_{17}),

$$D_y \Omega + v^3 D_s \left(\frac{\cos \frac{y}{s}}{v}\right) = 0.$$

563. If the plane of xy is assumed, at each instant, to be that in which the body moves, and if the axis of y is taken normal

to the path of the body, the preceding equation becomes, if ϱ expresses the radius of curvature of the path

$$\frac{v^2}{\varrho} = D_y \Omega,$$

so that *the centrifugal force of the body is equal to the normal pressure, and the whole pressure upon the brachystochrone is double the centrifugal force.* This proposition was discovered by Euler.

564. *When the normal pressure vanishes, the radius of curvature is infinite,* which corresponds in general to *a point of contrary flexure. When there is no force acting upon the body throughout its path, the brachystochrone is reduced to a straight line.*

565. Any conditions to which the path must be subject, whether elementary such as that it is confined to a given surface, or integral such as that its whole length is given, must be combined with the general condition of brachystochronity by the usual methods of the calculus of variations.

566. *If the only force which acts upon the body is directed to a given point, and if the path is subject to no conditions,* let the plane of xz be assumed to be that which passes through the centre of action and the initial element of the path. In this case the equation (328_{27}) gives

$$\cos\tfrac{y}{s} = 0, \quad \tfrac{y}{s} = \tfrac{1}{2}\pi,$$

or *the brachystochrone is contained in a plane which passes through the centre of action.*

567. The preceding case includes that in which the centre of action is removed to an infinite distance, so that, *in the case of parallel forces, the free brachystochrone is contained in a plane, which is parallel to the direction of the forces.*

568. *When the body is acted upon by no forces, or only by those which are normal to its path and do not tend to change its velocity, the*

equation (328_{13}) *shows that the brachystochrone is the shortest line which can be drawn under the given conditions.*

569. When the force is directed towards a fixed centre, the equation (329_9), combined with (316_{18}) gives, if the centre is adopted as the origin

$$\frac{D_r \Omega}{\Omega - \Omega_0} = \frac{2}{\varrho \sin \frac{r}{s}},$$

If p is the perpendicular let fall from the origin upon the tangent to the curve, this equation becomes

$$\frac{D_r \Omega}{\Omega - \Omega_0} = \frac{2\, D_r p}{r \sin \frac{r}{s}} = \frac{2\, D_r p}{p},$$

of which the integral is

$$\Omega - \Omega_0 = \left(\frac{p}{p_1}\right)^2,$$

which is *the equation of the brachystochrone referred to the radius vector and the perpendicular from the origin upon the tangent as the coördinates.* This form is given by Euler.

570. *When the force in the preceding case, is proportional to the distance from the origin* so that Ω has the form

$$\Omega = a r^2,$$

the equation (330_{14}) becomes

$$a\,(r^2 - r_0^2) = \left(\frac{p}{p_1}\right)^2 = \left(\frac{r \sin \frac{r}{s}}{p_1}\right)^2,$$

of which the derivative gives

$$a\,\varrho = \frac{p}{p_1^2}.$$

If ν is the angle which ϱ makes with the fixed axis, the derivative of this last equation gives, by means of the preceding equation

$$a p_1^2 D_\nu \varrho = r \cos \tfrac{r}{s} = \sqrt{[a p_1^2 (1 - a p_1^2) \varrho^2 + r_0^2]},$$

which becomes

$$D_\nu \varrho = m \sqrt{\left(\varrho^2 + \frac{r_0^2}{m^2 a^2 p_1^4}\right)},$$

if

$$m^2 = \frac{1 - a p_1^2}{a p_1^2}.$$

The integral of this equation is

$$\varrho = \frac{r_0}{m a p_1^2} \operatorname{Sin}(m \nu)$$

so that its *second evolute is similar to the brachystochrone itself.*

When m is real, which corresponds to the case of a repulsive force, and $a p_1^2$ less than unity, *this brachystochrone is a spiral which has a cusp at the point at which ν vanishes.*

When m is imaginary, the substitution of (327_5) reduces (331_{11}) to the real form

$$\varrho = \frac{r}{n a p_1^2} \sin(n \nu)$$

so that *in this case, the brachystochrone is an epicycloid which is formed by internal rotation when the force is attractive, and by external rotation when the force is repulsive.* This case is given by Euler.

571. *When the forces are parallel,* the equation (329_3) gives, if the axis of z is supposed to be in the direction of the forces

$$\frac{D_z \Omega}{\Omega - \Omega_0} = \frac{2}{\varrho \sin \tfrac{z}{s}} = 2 \cot \tfrac{z}{s} D_z \tfrac{z}{s},$$

of which the integral is

$$\Omega - \Omega_0 = a \sin^2 \tfrac{z}{s},$$

in which a is an arbitrary constant, *and this is the equation of the brachystochrone referred to the coördinates, which are* z *and the inclination of the curve to the axis of* z; *and the equation, referred to* ϱ *and* $\frac{z}{s}$ *as coördinates, is obtained by eliminating* z *between* (331_{27}) *and* (331_{31}).

572. *In the case of a constant force,* the preceding equation assumes the forms

$$g\,(z - z_0) = a \sin^2 \tfrac{z}{s},$$

$$\varrho = \frac{2a}{g} \sin \tfrac{z}{s};$$

so that, *in this case, the brachystochrone is a cycloid.*

573. *When the parallel forces are proportional to the distance from a given line,* which may be adopted for the axis of x, the value of Ω has the form

$$\Omega = b\, z^2;$$

whence *the equation of the brachystochrone is*

$$\varrho = \frac{a \sin \frac{z}{s}}{\sqrt{(b^2 z_0^2 + a\, b \sin^2 \frac{z}{s})}}.$$

When the force is repulsive, or when it is attractive, but

$$z_0 > \sqrt{-\tfrac{a}{b}},$$

this curve consists of branches, which are united by cusps, and resemble the cycloid in general form; but when the force is attractive, and

$$z_0 < \sqrt{-\tfrac{a}{b}},$$

this curve consists of branches which are still united by external cusps; but the middle point of each branch is upon the axis of x, *and is a point* *of inflexion, and the interval between two successive points of inflexion, expressed by elliptic integrals, is*

$$\sqrt{\left(-\frac{a}{b}\right)}\left[\mathcal{E}_i(\tfrac{1}{2}\pi)-\mathcal{F}_i(\tfrac{1}{2}\pi)\right],$$

in which

$$\sin i = z_0 \sqrt{-\frac{b}{a}},$$

In the case of the attractive force, and

$$z_0 = \sqrt{-\frac{a}{b}},$$

the equation of the brachystochrone becomes

$$\varrho = z_0 \tan \tfrac{z}{s},$$

which consists of two infinite branches joined by an external cusp, and the axis of x *is an asymptote to each of the branches.*

574. When the body is subjected to move upon a given surface, the force by which it is retained upon the surface is perpendicular to its path, and must be united with the second member of equation (329_3). Hence it follows *that the centrifugal force of the body in the direction of the tangent plane to the surface, upon which it is confined, is equal to the normal force which acts in this plane normal to the brachystochrone.*

At the beginning of the motion when the velocity is zero, there is no centrifugal force, so that *the initial direction of the brachystochrone upon the surface coincides with that of the tangential force.*

575. If the first and last points of the brachystochrone are so situated upon the given surface, that a line can be drawn through them, which coincides throughout with the direction of the tangential force to the surface, *this line is the brachystochrone.*

Hence, *the brachystochrone upon the surface of revolution is the meridian line, when both its extremities are upon the same meridian line, and the force is directed to a point upon the axis of revolution, or is parallel to this axis.*

576. In the general case of a surface of revolution and a force which is directed to a point upon the axis of revolution, let

σ denote the arc of the meridian curve measured from the pole,
u the perpendicular from the surface upon the axis,
ϱ_T the radius of curvature of the projection of the brachystochrone upon the tangent plane to the surface,

and the proposition (333_{20}) is expressed by the equation

$$\frac{v^2}{\varrho_T} = D_s \Omega \tan {}^{\sigma}_{s}$$

which gives

$$\frac{D_s \Omega}{\Omega - \Omega_0} = \frac{D_s (v^2)}{v^2} = \frac{2 \cot {}^{\sigma}_{s}}{\varrho_T}.$$

But the equations

$$D_s u = \cos {}^{u}_{s} = \cos {}^{\sigma}_{s} \cos {}^{\sigma}_{u},$$

$$D_s {}^{\sigma}_{s} = \frac{1}{\varrho_T} - \frac{\sin {}^{\sigma}_{s} \cos {}^{\sigma}_{u}}{u},$$

give

$$D_s (u \sin {}^{\sigma}_{s}) = \frac{u \cos {}^{\sigma}_{s}}{\varrho_T},$$

and if A is an arbitrary constant,

$$D_s \log v = D_s \log (u \sin {}^{\sigma}_{s})$$
$$A\, v = u \sin {}^{\sigma}_{s} = u^2 D_s {}^{u}_{x}$$
$$A\, v^2 = u\, v \sin {}^{\sigma}_{s} = u^2 D_t {}^{u}_{x}$$

so that *the area described by the projection of the radius vector upon* *the plane of* x y *is proportional to the square of the velocity of the body.*

577. The equation (334_{28}) gives

$$D_\sigma s = \sec \tfrac{\sigma}{s} = \frac{u}{\sqrt{(u^2 - A^2 v^2)}} = \frac{u}{\sqrt{[u^2 - 2A^2(\Omega - \Omega_0)]}},$$

$$D_\sigma \tfrac{u}{x} = \frac{\tan \tfrac{\sigma}{s}}{u} = \frac{Av}{u\sqrt{(u^2 - A^2 v^2)}} = \frac{A}{u}\sqrt{\frac{2(\Omega - \Omega_0)}{u^2 - 2A^2(\Omega - \Omega_0)}}.$$

578. If ϕ is the angle which the radius rector makes with the axis, the preceding values give

$$D_\phi s = \sqrt{\frac{u^2[r^2 + (D_\phi r)^2]}{u^2 - 2A^2(\Omega - \Omega_0)}},$$

$$D_\phi \tfrac{u}{x} = \frac{A}{u}\sqrt{\frac{2(\Omega - \Omega_0)[r^2 + (D_\phi r)^2]}{u^2 - 2A^2(\Omega - \Omega_0)}}.$$

When the forces are parallel these equations give

$$D_z s = \frac{u D_z \sigma}{\sqrt{[u^2 - 2A^2(\Omega - \Omega_0)]}}$$

$$D_z \tfrac{u}{x} = \frac{A D_z \sigma}{u}\sqrt{\frac{2(\Omega - \Omega_0)}{u^2 - 2A^2(\Omega - \Omega_0)}}.$$

579. Upon the surface of revolution which is determined by the equation

$$Bv = u$$

in which B is an arbitrary constant, the value of $\tfrac{\sigma}{s}$ is by (334_{28}) constant, so that *upon this surface the brachystochrone makes a constant angle with the meridian curve.* In the case in which

$$A = B$$

the brachystochrone becomes perpendicular to the meridian, and is a small circle, of which the plane is horizontal.

Whatever is the value of B, the point at which v vanishes, coincides with that at which u vanishes, so that at the pole of this surface the *veloeity vanishes.*

Upon any other surfaee of revolution about the same axis, the ineli-nation of the brachystoehrone to the meridian arc is the same with the corresponding inelinatiôn upon the surface of equation (335_{22}), *at the common circle of interscetion of these two surfaees. Hence the limit of the brachystoehrone upon a given surfaee of revolution is its cirele of interseetion with the surface of equation*

$$A v = u,$$

and the braehystoehrone extends over that portion of the given surfaee, whieh is exterior to the given surfaee, by which the limits are thus defined.

580. In the case of a heavy body, the surface of equation (335_{22}) is a paraboloid of revolution. *When the veloeity of a heavy body upon any paraboloid of revolution, of whieh the axis is vertieal and direeted downwards, is just suffieient to carry it to the vertex, the brachystoehrone makes a constant angle with the meridian eurve; but when the veloeity is too small to earry the body to the vertex, the braehystoehrone is a eurve whieh makes an increasing angle with the meridian as it deseends, and may sometimes become perpendieular to the meridian; and when the veloeity is more than suffieient to carry the body to the vertex of the paraboloid, the braehystochrone is an infinite eurve, which is horizontal at its highest point, and diminishes its angle with the meridian as it deseends.*

If the equation of the paraboloid is

$$u^2 = 4 p z$$

in which the axis of z is the downward vertical, the equation (334_{28}) becomes

$$\sin \tfrac{\sigma}{s} = A \sqrt{\left[\tfrac{g}{2p}\left(1 - \tfrac{z_0}{z}\right)\right]}.$$

If z_0 is positive and

$$p > \tfrac{1}{2} A^2 g,$$

the substitution of

$$\sin^2 \alpha = \frac{A^2 q}{2p},$$

$$q = z_0 \tan^2 \alpha,$$

$$\operatorname{Cos} \varphi = \pm \frac{2z + p + q}{p - q},$$

gives

$$s = \tfrac{1}{2} \sec \alpha \, (p - q) \, (\varphi \pm \operatorname{Sin} \varphi),$$

in which the upper signs correspond to the case in which p is greater than q, and the lower to that in which p is less than q. In the case in which p is greater than q, the substitution of

$$\cos^2 \psi = \frac{z - z_0}{z + p},$$

$$\sin^2 i = \frac{p - q}{p + z_0},$$

gives

$$\frac{x}{u} = -\tan \alpha \sqrt{\left(1 + \frac{z_0}{p}\right)} \Big[\mathfrak{F}_i \psi - \mathfrak{E}_i \psi - \cot \psi \sqrt{(1 - \sin^2 i \sin^2 \psi)}$$
$$- \frac{z_0}{p + z_0} \mathfrak{P}_i \left(- \frac{p}{p + z_0}, \psi \right) \Big].$$

When p is smaller than q, the substitution of

$$\cos^2 \psi = \frac{z - z_0}{z + q},$$

$$\sin^2 i = \frac{q - p}{q + z_0},$$

gives

$$\frac{u}{x} = \tan \alpha \sqrt{\left(\frac{q + z_0}{p}\right)} \Big[\mathfrak{E}_i \psi - \frac{p}{q} \mathfrak{F}_i \psi + \cot \psi \sqrt{(1 - \sin^2 i \sin^2 \psi)}$$
$$+ \frac{p z_0}{q (q + z_0)} \mathfrak{P}_i \left(- \frac{q}{q + z_0}, \psi \right) \Big].$$

When

$$p = q,$$

the arc is

$$s = \sec\alpha\,(z - z_0),$$

so that *its inclination to the axis is constantly equal to* α, and the brachystochrone is defined by the equations

$$z = z_0 \sec^2\varphi,$$

$$\tfrac{u}{x} = \tan\alpha \sqrt{\tfrac{z_0}{p}}(\tan\varphi - \varphi).$$

When

$$p = \tfrac{1}{2} A^2 g,$$

the arc, measured from its cusp, is

$$s = \frac{2}{3\sqrt{z_0}}\left[(z+p)^{\frac{3}{2}} - (z_0+p)^{\frac{3}{2}}\right].$$

and if

$$\tan\gamma = \sqrt{\frac{z_0}{p}},$$

the brachystochrone is defined by the equations

$$p + z = p \sec^2\gamma \operatorname{Cos}^2\varphi,$$

$$\frac{\operatorname{Tan}\varphi}{\tan\gamma} = \tan\left(\frac{\operatorname{Sin} 2\varphi}{2\sin 2\gamma} + \frac{\varphi}{\tan 2\gamma} - \frac{u}{x}\right).$$

when

$$p < \tfrac{1}{2} A^2 g,$$

in which case the brachystochrone has a lower limit at which it is horizontal, the substitution of

$$\sec^2\alpha = \frac{A^2 g}{2p},$$

$$q = \frac{z_0}{\sin^2\alpha},$$

$$\cos\varphi = \frac{2z + p - q}{p + q},$$

gives, at the lowest point of the curve, where φ vanishes

$$z = q,$$

and for the value of s, measured from the lowest point,

$$s = \tfrac{1}{2}(p+q)\cot\alpha\,(\sin\varphi + \varphi).$$

The substitution of

$$\tan^2\psi = \frac{q-z}{z-z_0},$$

$$\sin^2 i = \frac{q-z_0}{p+q}.$$

gives

$$\tfrac{u}{x} = \frac{z_0}{\sin\alpha\sqrt{[p(p+q)]}}\left[\mathfrak{F}_i\,\psi - \frac{z_0(p+q)}{q^2}\,\mathfrak{E}\,\psi + \frac{p}{q}\,\mathfrak{P}_i\left(-\frac{q-z_0}{q}, \psi\right)\right].$$

When z_0 is negative, in which case the condition (336_{30}) is satisfied, the substitution of the equations (337_{2-5}) with the lower sign gives the corresponding value of (337_7) for the arc measured from its upper limit, which corresponds to the vanishing of φ.

When

$$-z_0 < p,$$

the substitution of

$$\cos^2\psi = \frac{z+q}{z+p},$$

$$\sin^2 i = \frac{p+z_0}{p-q}$$

gives

$$\tfrac{u}{x} = \tan\alpha\sqrt{\left(\frac{p-q}{p}\right)}\Big[\mathfrak{E}_i\,\psi - \mathfrak{F}_i\,\psi + \cos\psi\sqrt{(\cot^2\psi + \cos^2 i)}$$
$$+ \frac{z_0}{p-q}\,\mathfrak{P}_i\left(-\frac{p}{p-q}, \psi\right)\Big].$$

When

$$-z_0 > p$$

the substitution of

$$\cos^2\psi = \frac{z+q}{z-z_0},$$

$$\sin^2 i = \frac{p+z_0}{q+z_0},$$

gives

$$\tfrac{u}{x} = \tan\alpha\sqrt{\left(\frac{-z_0-q}{p}\right)}\Big[\mathcal{E}_i\,\psi - \mathcal{F}_i\,\psi + \cos\psi\,\sqrt{(\cos^2\psi+\cos^2 i)}$$
$$+\frac{p_0}{z_0+q}\,\mathcal{P}_i\left(-\frac{z_0}{z_0+q},\psi\right)\Big].$$

When

$$-z_0 = p,$$

the brachystochrone is defined by the equation

$$\tfrac{u}{x} = \tan\alpha\left[\sqrt{\left(\frac{z+q}{p}\right)} + \tfrac{1}{2}\sqrt{\frac{p}{q}}\log\frac{\sqrt{(z+q)}-\sqrt{q}}{\sqrt{(z+q)}+\sqrt{q}}\right].$$

581. *In the case of the heavy body upon the paraboloid of revolution in which the axis is vertical and directed upwards, the brachystochrone forms an increasing angle with the meridian as it descends and is perpendicular to the meridian at its lowest point.* In this case, the inclination to the meridian is determined by the equation

$$\sin\tfrac{\sigma}{s} = A\sqrt{\left[\frac{g}{2p}\left(\frac{z_0}{z}+1\right)\right]},$$

if (336_{25}) is the equation of the paraboloid. By the substitution of

$$\sin^2\alpha = \frac{A^2 g}{2p},$$

$$q = z_0\tan^2\alpha,$$

$$\operatorname{Cos}\varphi = \frac{2z+p-q}{p+q},$$

φ vanishes at the lowest point where

$$z = q,$$

and the value of the arc, measured from the lowest point, is

$$s = \tfrac{1}{2}(p+q)\sec\alpha\,(\varphi + \mathrm{Sin}\,\varphi).$$

The substitution of

$$\tan^2\psi = \frac{z_0 - z}{z - q},$$

$$\sin^2 i = \frac{z_0 - q}{p + z_0},$$

gives

$$\tfrac{y}{x} = \tan\alpha\sqrt{\left(\frac{p+z_0}{p}\right)}\left[\mathcal{E}_i\,\psi - \frac{z_0}{p+z_0}\mathcal{F}_i\,\psi - \frac{p}{p+z_0}\mathcal{P}_i\left(-\frac{z_0-q}{z_0},\psi\right)\right].$$

582. *In the case of a heavy body upon a vertical right cone,* if the vertex of the cone is assumed as the origin, and if

α is the angle which the side of the cone makes with the axis,

$$r_1 = \frac{A^2 g\cos\alpha}{\sin^2\alpha},$$

ϑ = the angle which r makes with the axis upon the developed cone,

the inclination to the meridian, the derivative of the arc and of ϑ are

$$\sin\tfrac{\sigma}{s} = \frac{\sqrt{[2r_1(r-r_0)]}}{r},$$

$$D_r s = \frac{r}{\sqrt{[r^2 - 2r_1(r-r_0)]}},$$

$$D_r\vartheta = \frac{\sqrt{[2r_1(r-r_0)]}}{r\sqrt{[r^2 - 2r_1(r-r_0)]}}.$$

When

$$2r_0 > r_1,$$

the substitution of

$$\sin^2 i = \frac{r_1}{2r_0},$$

$$\tan\psi = \frac{r_1\cot i}{r - r_1},$$

$$r = r_0\sec^2\tfrac{1}{2}\varphi,$$

gives

$$\cos\tfrac{\sigma}{s} = \cos i \sec(\psi - i),$$
$$s = r_0 \sin^2 i (\operatorname{cosec}\psi - \operatorname{cosec} 2i) - r_1 \log(\tan\tfrac{1}{2}\psi \cot i),$$
$$\theta = -\sin^{[-1]}(\sin i \sin\varphi) + \sin i\, \mathfrak{F}_i \varphi,$$

in which the arc is measured from the cusps, at which point

$$\theta = \varphi = \tfrac{\sigma}{s} = 0, \quad \psi = 2i.$$

This brachystochrone extends to infinity from the cusp without ever becoming perpendicular to the side of the cone. The greatest angle which it makes with the side is *i*, and at this point of least inclination to the side

$$\psi = i, \quad r = 2r_0, \quad \varphi = \tfrac{1}{2}\pi,$$
$$\theta = -i + \sin i\, \mathfrak{F}_i(\tfrac{1}{2}\pi).$$

When

$$2r_0 = r_1,$$

the brachystochrone is defined by the equation

$$\tfrac{1}{2}\theta = \tan^{[-1]}\sqrt{\left(\frac{r}{r_0} - 1\right)} - \operatorname{Cot}^{[-1]}\sqrt{\left(\frac{r}{r_0} - 1\right)},$$

and the length of the arc, measured from the point of least inclination to the side, is

$$s = r - 2r_0 + r_0 \log\left(\frac{r}{2r_0} - 1\right)^2.$$

When r_0 is positive and

$$2r_0 < r_1,$$

the substitution of

$$\operatorname{Sec}^2\beta = \frac{2r_0}{r_1},$$
$$\operatorname{Tan}\psi = \pm\frac{r_1 \operatorname{Tan}\beta}{r - r_1},$$

gives

$$s = r_1 \operatorname{Tan} \beta \operatorname{Cosec} \psi \mp r_1 \log \operatorname{Tan} \tfrac{1}{2} \psi,$$

in which the arc is measured upon each branch from the point at which it is horizontal and the upper sign belongs to the lower branch and the reverse. *The upper branch is finite, while the lower branch is infinite,* and the value of ψ extends on the upper branch from 2β to infinity, and on the lower branch from infinity to zero. For the upper branch the substitution of

$$\sin i = e^{-2\beta},$$
$$r - r_0 = r_0 \sin i \sin^2 \varphi,$$

gives

$$\theta = 2(1 + \sin i)\left[\mathfrak{F}_i \varphi - \mathfrak{P}_i (\sin i, \varphi)\right].$$

Upon the lower branch the substitution of

$$r - r_0 = \frac{r_0}{\sin i \sin^2 \psi},$$

gives

$$\theta = 2(1 + \sin i)\, \mathfrak{P}_i (\sin i, \psi).$$

When r_0 *vanishes, the equation of the brachystochrone upon the developed cone is*

$$r = 2 r_1 \sec^2 \tfrac{1}{2} \theta,$$

and the length of the arc is

$$s = 2 r_1 \tan \tfrac{1}{2} \theta \sec \tfrac{1}{2} \theta + 2 r_1 \log \tan \left(\tfrac{1}{4} \pi + \tfrac{1}{4} \theta\right).$$

When r_0 is negative, the substitution of

$$\operatorname{Cosec}^2 \beta = -\frac{2 r_0}{r_1} = \frac{-4 \sin i}{(1 + \sin i)^2},$$

$$\operatorname{Cot} \psi = \pm \frac{r - r_1}{- r_1 \operatorname{Cos} \beta},$$

gives

$$s = r_1 \operatorname{Cot} \beta \operatorname{Cosec} \psi - r_1 \log \operatorname{Tan} \tfrac{1}{2} \psi,$$

in which the order of the signs and of the value of ψ is the same as in (343_4) with reference to the branches. *The upper and finite branch of the brachystochrone lies in this case upon the upper and inverted portion of the cone.* The formulæ $(343_{11}, 343_{16-19})$, apply to this case, in which it must, however, be noticed that the $\sin i$ is negative.

583. *When the solid of revolution upon which the heavy body moves, is the ellipsoid* of which the equation is

$$\left(\frac{z}{A_z}\right)^2 + \left(\frac{u}{A_u}\right)^2 = 1,$$

the inclination to the meridian is determined by the equation

$$\sin \tfrac{\sigma}{s} = \frac{A A_z \sqrt{[2g(z - z_0)]}}{A_u \sqrt{(A_z^2 - z^2)}}.$$

The problem naturally divides itself into two cases. *In the first case the velocity is more than sufficient to carry the body to the highest point of the ellipsoid, the brachystochrone is a continuous curve which is horizontal at its highest and lowest limits, and which, always running round the ellipsoid, is most inclined to the meridian curve at the point*

$$z_3 = z_0 + \sqrt{(z_0^2 - A_z^2)}.$$

In the second case, the velocity is not sufficient to carry the body up to the highest point of the ellipsoid, and the brachystochrone is horizontal at its lowest point, but has cusps for its upper points. In each of these cases the length of the arc can be found by means of elliptic functions. If in the first case $-z_1$ and z_2 are the coördinates of the upper and lower limits, or of the common intersections of the ellipsoid with the paraboloid of revolution of which the equation is

$$u^2 = 2A^2 g(z - z_0),$$

and if in the second case z_2 refers to the intersection of the ellipsoid with the paraboloid, while $-z_1$ is the coördinate of the intersection of this paraboloid, inverted at the horizontal plane of $u\,x$, with the hyperboloid of revolution, of which the equation is

$$\left(\frac{z}{A_z}\right)^2 - \left(\frac{u}{A_u}\right)^2 = 1,$$

the derivative of the arc is

$$D_z\, s = \frac{1}{A_z}\sqrt{\frac{A_z^4 + (A_u^2 - A_z^2)\, z^2}{(z+z_1)\,(z_2-z)}}.$$

In the first case, when the ellipsoid is prolute, and

$$z_1^2 = \frac{A_z^4}{A_z^2 - A_u^2},$$

the substitution of

$$\text{Cos}\,\varphi = \frac{2\,z - z_2 - z_1}{z_2 - z_1},$$

gives

$$s = \frac{A_z}{2\,z_1}\,(z_2 - z_1)\,(\text{Sin}\,\varphi + \varphi).$$

When the ellipsoid is a sphere, of which the radius is R, the hyperbola (345_6) becomes equilateral, and the length of the arc, measured from the lowest point, is determined by the equation

$$\cos\frac{s}{R} = \frac{2\,z + z_1 - z_2}{z_1 + z_2}.$$

In the first case (344_{17}), the substitution of

$$\psi = \frac{z}{r},$$

$$\cos i = \frac{\sin\psi_1}{\sin\psi_2},$$

gives, for the sphere,

$$\frac{u}{x}=\frac{2\cos^2\frac{1}{2}\psi_1}{\sin\psi_2}\mathcal{P}_i\left(\frac{\cos\psi_2+\cos\psi_1}{1-\cos\psi_2},\frac{s}{2R}\right)+\frac{2\sin^2\frac{1}{2}\psi_1}{\sin\psi_2}\mathcal{P}_i\left(-\frac{\cos\psi_2+\cos\psi_1}{1+\cos\psi_2},\frac{s}{2R}\right)$$

$$=\frac{4\cos^2\frac{1}{2}\psi_1}{\sin\psi_2}\mathcal{P}_i\left(\frac{\cos\psi_2+\cos\psi_1}{1-\cos\psi_2},\frac{s}{2R}\right)-\frac{\cos\psi_1-\cos\psi_2}{\sin\psi_2}\mathcal{F}_i\left(\frac{s}{2R}\right)$$

$$-\tan^{[-1]}\frac{\cos\psi_2+\cos\psi_1}{\sqrt{\left(\sin^2\psi_2\operatorname{cosec}^2\frac{s}{2R}+\sin^2\psi_1\sec^2\frac{s}{2R}\right)}}.$$

In the second case (344_{23}), the substitution of

$$\cos 2\varphi=\frac{2z-z_2-z_0}{z_2-z_0},$$

$$\sin^2 i=\frac{z_2-z_0}{z_2+z_1},$$

gives, for the sphere,

$$\frac{u}{x}=\frac{z_1+R}{R\sin\psi_2}\mathcal{P}_i\left(\frac{\cos\psi_2-\cos\psi_0}{1-\cos\psi_2},\varphi\right)-\frac{z_1-R}{R\sin\psi_2}\mathcal{P}_i\left(\frac{\cos\psi_0-\cos\psi_2}{1+\cos\psi_2},\varphi\right)$$

$$=\frac{z_1+R}{R\cos\psi_2}\left[\mathcal{P}_i\left(\frac{\cos\psi_2-\cos\psi_0}{1-\cos\psi_2},\varphi\right)-\mathcal{P}_i\left(\frac{R(1-\cos\psi_2)}{z_1+R\cos\psi_2},\varphi\right)\right]$$

$$+\frac{\cos\psi_0-\cos\psi_2}{\sin\psi_2}\mathcal{F}_i\varphi-\operatorname{cosec} i\tan^{[-1]}\frac{\sin i\sin\varphi}{\sqrt{(1+\cos^2 i\tan^2\varphi)}}.$$

In the case in which

$$z_0=-R,$$

the brachystochrone is defined by the equation

$$\frac{u}{x}=\tan\tfrac{1}{2}\psi_2\operatorname{Tan}^{[-1]}\sin\frac{s}{2R}+\tan^{[-1]}\frac{\sin\frac{s}{2R}}{\tan\frac{1}{2}\psi_2}.$$

584. In the case of a heavy body upon any surface whatever, it follows from (329_3) that

$$\frac{v^2}{\varrho_T}=\frac{2g(z-z_0)}{\varrho_T}=g\cos\frac{z}{\rho_T}.$$

If, then, N_τ is the normal to the brachystochrone drawn in the tangent plane, and extended to meet the horizontal plane from which the body must fall to acquire its velocity, the preceding equation gives

$$N_\tau = (z - z_0) \sec \frac{z}{\rho_\tau} = \tfrac{1}{2} \varrho_\tau,$$

or *the tangential radius of curvature of the brachystochrone is twice the tangential normal which extends to the horizontal plane of evanescent velocity.* This proposition is given by Jellett.

585. *When the force is parallel to the axis and proportional to the distance from a plane which is perpendicular to the axis, the surface of revolution of equation* (335_{22}) *is an ellipsoid when the force is attractive towards the plane, and it is an hyperboloid of two sheets when the force is repulsive from the plane.*

586. *When the force is directed towards a fixed point and proportional to the distance from the point, the surface of equation* (335_{22}) *is an ellipsoid if the force is attractive, but if the force is repulsive, the surface may be an ellipsoid or it may be an hyperboloid of two sheets.*

587. When the force is directed towards a fixed point, and inversely proportional to the square of the distance from the point, the surface of revolution of equation (335_{22}) is defined by an equation of the form

$$u^2 = A\left(\frac{1}{r} - \frac{1}{r_0}\right).$$

588. Other conditions might be combined with that of the brachystochrone. *Thus if the total length of the arc is given, the normal pressure to the brachystochrone is*

$$D_y \Omega = \frac{v^2 + b v^3}{\varrho} = \frac{v^2}{\varrho}(1 + b v),$$

in which b is an arbitrary constant, and is dependent, for its value, upon the given length of the arc. This constant is generally infinite, when the brachystochrone is a straight line.

589. Under the condition of the preceding section, *the equation of the brachystochrone, in the case of* § 569, *referred to the coördinates of* (330_{17}) *is*

$$\Omega - \Omega_0 = \tfrac{1}{2}\left(\frac{p}{p_2 - b p}\right)^2.$$

In the case of § 570, this equation gives

$$2 a \varrho = \frac{p_2 p}{(p_2 - b p)^3}.$$

590. *In the case of the parallel forces of* § 571, (347_{28}) *gives*

$$\Omega - \Omega_0 = \tfrac{1}{2}\left(\frac{a \sin \frac{z}{s}}{1 - b a \sin \frac{z}{s}}\right)^2.$$

When the force is constant, this equation gives

$$\varrho = \frac{a^2 \sin \frac{z}{s}}{g\,(1 - b\, a \sin \frac{z}{s})^3};$$

so that when

$$b\, a > 1,$$

the curve has points of contrary flexure.

591. *In the case of* § 576, *and with the condition of* § 589, *the equation of the brachystochrone has the form*

$$\frac{A v}{1 + b v} = u \sin \tfrac{\sigma}{s} = u^2 D_s \tfrac{u}{x}.$$

The inclination of the curve to the meridian arc is therefore constant upon the surface of revolution, which is defined by the equation

$$B v = u\,(1 + b\, v),$$

and this surface has the same relation to other surfaces of revolution in *respect to the brachystochrone formed under the present conditions with those which are indicated for the surface of* § 579.

In the case of a heavy body, the equation of this defining surface of revolution is

$$2g(z-z_0)=\left(\frac{u}{B-bu}\right)^2.$$

592. If the condition is a mechanical one, such that *the total expenditure of action, defined as in* § 308, *shall be given, the normal pressure to the brachystochrone is*

$$D_y\Omega=\frac{v^2}{\varrho}\cdot\frac{1+bv^2}{1-bv^2},$$

in which b is an arbitrary constant, and is dependent, for its value, upon the given expenditure of action. When this constant is infinite, the normal pressure is equal and opposed to the centrifugal force.

It is apparent, from the preceding equation, that under the action of finite forces, this brachystochrone cannot be a continuous curve, in one portion of which the direction of the normal pressure coincides with that of the centrifugal force, and is opposed to it in another portion.

593. Under the condition of the preceding section, *the equation of the brachystochrone, in the case of* § 569, *referred to the coördinates of* (330_{17}) *is*

$$\frac{\Omega-\Omega_0}{[1+2b(\Omega-\Omega_0)]^2}=\left(\frac{p}{p_1}\right)^2.$$

In the case of § 570, this equation gives

$$\varrho=\frac{[1+2ba(r^2-r_0^2)]^2}{1-2ba(r^2-r_0^2)}\cdot\frac{\sqrt{(r^2-r_0^2)}}{p_1\sqrt{a}}.$$

594. *In the case of the parallel forces of* § 571, (349_{24}) *gives*

$$\frac{\Omega-\Omega_0}{[1+2b(\Omega-\Omega_0)]^2}=\sin^2\frac{z}{s}.$$

When the force is constant, this equation gives

$$\varrho = 2\sqrt{[a(z-z_0)]}\,\frac{[1+2bg(z-z_0)]^2}{1-2bg(z-z_0)}.$$

595. *In the case of* § 576 *and with the condition of* § 592, *the equation of the brachystochrone has the form*

$$\frac{Av}{1+bv^2} = u\sin\tfrac{\sigma}{s} = u^2 D_s\tfrac{u}{x}.$$

The inclination of the curve to the meridian arc is, therefore, constant upon the surface of revolution, which is defined by the equation

$$Bv = u(1+bv^2),$$

and this surface involves, for the present case, the properties of the defining surface of § 579.

In the case of a heavy body, the equation of this defining surface of revolution is

$$2B^2g(z-z_0) = u^2[1+2bg(z-z_0)]^2.$$

596. *The brachystochrone in a medium of constant resistance* is entitled to special consideration. In this case, it is convenient to introduce the length of the arc as the independent variable. The equation of motion along the curve is

$$v^2 = 2\Omega - 2ks,$$

in which k is the constant of resistance. This equation must be combined with the equation

$$(D_s x)^2 + (D_s z)^2 = 1.$$

If $\frac{1}{2}\mu_1$ and $\frac{1}{2}\mu$ are the respective multipliers of these equations in the method of variations, the *brachystochrone is defined by the differential equations*

$$\mu_1 = \frac{1}{v^3},$$

$$D_s \mu = \frac{D_s \Omega}{v^3} = \frac{v D_s v + k}{v^3};$$

and by the following expression of the normal pressure directed in the opposite way to the centrifugal force

$$D_z \Omega \sin \nu + D_x \Omega \cos \nu = \frac{\mu v^3}{\varrho}.$$

When k vanishes, the value of μ is

$$\mu = -\frac{1}{v},$$

and, therefore, *the value of* μ *is the negative of the reciprocal of the expression which is obtained for* v *when there is no resisting medium, and which is independent of the magnitude of the fixed force.*

597. *When the force is directed towards a fixed centre,* the notation of § 569 gives by (330_{15}) for the value of μ,

$$\mu = -\frac{p_1}{p\sqrt{2}}.$$

598. *When the forces are parallel,* the equation (331_{31}) gives μ in the form

$$\mu = -\frac{\mu_0}{\cos \nu}.$$

599. From the preceding equations, *the equation of the brachystochrone of a heavy body in a medium of constant resistance has the form*

$$\varrho = \frac{R \sin \nu}{[1 - h \cos(\nu - \nu_0)]^3},$$

in which R, h, and ν_0 are arbitrary constants.

600. *In a medium of which the law of resistance is expressed as a given function of the velocity,* the derivative equation of motion is

$$v\, D_s v = D_s \Omega - V,$$

in which V is a given function of v. The differential equations, by which the brachystochrone is defined, become, if $\frac{1}{2}\mu$ and μ_1 are the multipliers of (350_{29}) and (352_4),

$$D_s(\mu \sin \nu) = D_x \Omega\, D_s \mu_1,$$

$$-D_s(\mu \sin \nu) = D_z \Omega\, D_s \mu_1,$$

$$-\frac{1}{v^2} - v\, D_s \mu_1 + \mu_1 D_v V = 0.$$

The reduction of these equations gives

$$D_s \mu = D_s \Omega\, D_s \mu_1 = D_s\left(\frac{1}{v} + \mu_1 V\right),$$

$$\mu = \frac{1}{v} + \mu_1 V;$$

and the expression of the normal pressure to the brachystochrone becomes

$$D_z \Omega \sin \nu + D_x \Omega \cos \nu = \frac{\mu}{\varrho D_s \mu_1} = \frac{D_s \Omega}{\varrho D_s \log \mu} = \frac{v^3 \mu}{\varrho v^2 \mu_1 D_v V - \varrho}$$

$$= \frac{\mu_1 V v^3 + v^2}{\varrho \mu_1 v^2 D_v V - \varrho}.$$

601. *When the forces are parallel* to the axis of z, the equations (352_9) and (352_{17}) give

$$\mu = \frac{a}{\sin \nu},$$

$$V \mu_1 = \frac{a}{\sin \nu} - \frac{1}{v}.$$

602. These equations give for *the brachystochrone of a heavy body in a resisting medium,*

$$g\,\mu_1 = -a\cot\nu + b = \frac{g\,a}{V\sin\nu} - \frac{g}{Vv},$$

by which v is determined in terms of ν. The substitution of this value of v in the equation

$$\frac{v\,D_\nu\,v}{\varrho} = -g\cos\nu - V,$$

gives *the equation of the brachystochrone in terms of* ϱ *and* ν. The preceding formulæ include the results obtained by Jellett in his investigation of this particular case.

When V is inversely proportional to the velocity, the equation of the brachystochrone may assume the form

$$\varrho = \frac{2\,h\,[h\cos 2\,(\nu - \alpha) + k]^2 \sin 2\,(\nu - \alpha)}{m + g\cos\nu\,[h\cos 2\,(\nu - \alpha) + k]}.$$

When V is proportional to the square of the velocity and has the form

$$V = k\,v^2,$$

the equation of the brachystochrone is derived from the elimination of v between the equations

$$\cos(\nu - \alpha) = \frac{g\cos\alpha}{k\,v^2} - \frac{g\sin\nu}{h\,k\,v^3},$$

$$\frac{\cos\alpha}{k\,\varrho\sin\nu}\left(\frac{g\cos\nu}{v^2} - k\right) = \left[\frac{5\,g\cos\alpha}{k\,v^2} - 3\cos(\nu - \alpha)\right]\left(\frac{g\cos\nu}{v^2} + k\right).$$

603. In these cases of the brachystochrone in a resisting medium, it is apparent that the condition (329_6) is usually violated, and that Euler, consequently, erred in extending this proposition to the case of the resisting medium.

604. The determination of the form of the curve constitutes the principal feature of the general problem of the brachystochrone. But the nature of the curve may be given, and the problem is then reduced to one of maxima and minima, in which the various parameters of the curve are to be determined. EULER has shown that there is a peculiar analytic difficulty in some problems of this class. A single example will illustrate this species of inquiry.

Let the given curve be the circumference of a circle, of which the plane is vertical, and let the ball start from a state of rest at the upper point. If, then, 2α is the angle which the line, joining the two points, makes with the horizontal line, and if $2i$ is the angle which the radius drawn to the upper point makes with the vertical, the equation for determining i is

$$\sec i\left[\mathcal{E}_i(\tfrac{1}{2}\pi)-\mathcal{E}_i(2\alpha-i)\right]-\left[\cot 2(i-\alpha)+\cos i\right]$$
$$\left[\mathcal{F}_i(\tfrac{1}{2}\pi)-\mathcal{F}_i(2\alpha-i)\right]+\frac{\cos 2\alpha}{\cos i}\sqrt{\frac{\sin 2(i-\alpha)}{\sin 2\alpha}}=0.$$

THE HOLOCHRONE.

605. A curve, in which the time of descent along a given arc, is a given function of the arc, or of its defining elements may be called a *holochrone.*

606. The problem of the holochrone becomes simple, *when the forces are fixed, and the time of descent is proportional to a given power of the arc.* Thus, if the time of descent is expressed by

$$T_s = A s^n,$$

in which s is the length of the arc. Let

$$B=\frac{\pm(1-n)}{A^2}\left[\int_h^1 {}_0 (1-h^{2-2n})^{-\frac{1}{2}}\right]^2$$

in which the upper sign corresponds to the case, in which n is less than unity, and the lower to that in which n exceeds unity. The force along the curve is

$$S = -B s^{1-2n}.$$

When

$$n = 1,$$

the force along the curve is

$$S = -\frac{\pi}{2 A_2 s} = -\frac{B}{s}.$$

607. When the force is that of gravity, the equation of the holochrone of the preceding problem assumes the form

$$g \sin \tau = -B s^{1-2n}.$$

608. *If the time of descent admits of being developed according to integral ascending powers of* s, the developed expressions of S and Ω_s are obtained from the formulæ

$$\Omega_s = s^2 P_s,$$
$$S = D_s \Omega_s;$$

in which the successive terms of P_s are obtained from the equations represented by

$$\frac{1}{\sqrt{2}} D_s^m T_{s=0} = \int_0^{\frac{1}{2}\pi}{}_\phi \left[\cos \varphi \, V_s^m \left(P_{s=0}^{-\frac{1}{2}}\right)\right].$$

The second member of this equation is to be developed in form precisely as if V were the symbol of derivation, and in the result there must be substituted for $P_{s=0}$ and $V_s^m P_{s=0}$, the values

$$P_{s=0} = \cos \varphi \, P_{s=0},$$
$$V_s^m P_{s=0} = (1 - \sin^{m+2} \varphi) \, D_s^m P_{s=0}.$$

609. *When the forces are fixed, and the time of descent is a given function of the initial value of the potential,* the problem of the holochrone can be solved by the method applied by Abel to the case of a heavy body. If A is the final value of the potential, in which the arbitrary constant is determined so that the potential may vanish with the velocity, the time of transit expressed as a function of A, assumes the form

$$T_A = \frac{1}{\sqrt{2}} \int_{\Omega\,0}^{A} \frac{D_\Omega s}{\sqrt{(A-\Omega)}}.$$

The integral, relatively to A of the product of this expression, multiplied by

$$\frac{1}{\pi\sqrt{(\Omega - A)}},$$

is

$$\frac{1}{\pi}\int_{A\,0}^{\Omega} \frac{T_A}{\sqrt{(\Omega-A)}} = \frac{1}{\pi\sqrt{2}}\int_{A\,0}^{\Omega}\left[\frac{1}{\sqrt{[\Omega-A]}}\int_{\Omega\,0}^{A}\frac{D_\Omega s}{\sqrt{(A-\Omega)}}\right].$$

But the notation

$$\Gamma h = \int_{x\,0}^{1} (-\log x)^{h-1},$$

with the familiar equation

$$\int_{x\,0}^{1} \frac{x^{a-1}}{(1-x)^n} = \frac{\Gamma a\, \Gamma(1-n)}{\Gamma(a+1-n)},$$

gives, by a ready reduction

$$\int_{A\,0}^{x}\left[\frac{1}{(\Omega-A)^{1-n}}\int_{\Omega\,0}^{A}\frac{\Omega^{a-1}}{(A-\Omega)^n}\right] = \frac{x^a}{a}\,\Gamma n\,\Gamma(1-n) = \frac{\pi x^a}{a \sin n\pi}.$$

Note. — The notation (356_{20}) is substituted for that of (91_{24}), which was unwisely introduced instead of the usual form, which is here restored.

If the product of this equation multiplied by $\alpha\varphi(\alpha)$ is integrated relatively to α, and if the function f_x of x is defined by the equation

$$\int_a (\varphi(\alpha)\, x^a) = f_x,$$

so that

$$\int_a (\alpha\,\varphi(\alpha)\, x^{a-1}) = D_x f_x,$$

the integral gives

$$\frac{\sin n\pi}{\pi}\int_A{\overset{x}{\underset{0}{}}}\left[\frac{1}{(\Omega-A)^{1-n}}\cdot\int_\Omega{\overset{A}{\underset{0}{}}}\frac{D_\Omega f_\Omega}{(A-\Omega)^n}\right]=f_x,$$

which, when

$$n=\tfrac{1}{2},$$

gives by (356_{16})

$$s=\frac{\sqrt{2}}{\pi}\int_A{\overset{\Omega}{\underset{0}{}}}\frac{T_A}{\sqrt{(\Omega-A)}},$$

The general relations between s and Ω complete the solution, and indicate the form of coördinates in which the solution should be finally exhibited.

610. If the forces are parallel to the axis of z, Ω is a function of z, and the elimination of z between (357_{15}) and the equation

$$\cos{}^s_z = D_z\, s,$$

gives this holochrone expressed in terms of the length and direction of the arc.

611. If the forces are directed towards a fixed point, which is assumed to be the origin of coördinates, the elimination of r between (357_{15}) and

$$\cos{}^s_r = D_r\, s,$$

gives this holochrone expressed in terms of the length of the arc and its inclination to the radius vector.

612. If T_A, developed according to powers of A, is expressed by

$$T_A = \Sigma_\mu (C'_\mu A^\mu),$$

it is evident that

$$s = \sqrt{\frac{2\,\Omega}{\mu}}\ \Sigma_\mu \left[\frac{\Gamma(\mu+1)}{\Gamma(\mu+\frac{1}{2})}\, C_\mu \Omega^\mu\right].$$

613. *An interesting case of this potential holochrone is obtained, when the body is supposed to approach the point of maximum potential along a given curve, and the required curve is to be such that the whole time of oscillation shall be a given function of the maximum potential.* If s_1 denotes the given arc, the time of oscillation has the form

$$T_A = \frac{1}{\sqrt{2}} \int_0^A {}_{\Omega}\, \frac{D_\Omega (s+s_1)}{\sqrt{(A-\Omega)}};$$

so that, by the process of § 609,

$$s + s_1 = \frac{\sqrt{2}}{\pi} \int_0^\Omega {}_A\, \frac{T_A}{\sqrt{(\Omega - A)}}.$$

In order that the two curves may be continuous, the direction of the given curve must coincide with that of the level surface at the point of maximum potential. But this direction may be given by an infinitesimal bend at the extremity of the curve, so that this is not a practical limitation of the problem.

614. If the given time of oscillation is constant, the equation (358_{18}) assumes the form

$$B\,(s+s_1)^2 = \Omega\,;$$

and *the compound curve becomes a peculiar species of tautochrone*, which was investigated by Euler in the case of heavy bodies.

615. *When the forces are not wholly fixed but may depend upon*

the velocity, the problem of the holochrone becomes, to a certain extent, indeterminate. For, if

$$W = 0,$$

is an assumed equation between s, t and v, such that t and s vanish together, but when v vanishes, the resulting equation between s and t assumes a given form corresponding to the given condition of the holochrone, *the derivative of this equation gives, for the expression of the force along the curve,*

$$R = -\frac{D_t W + v D_s W}{D_v W},$$

from which the time is to be eliminated by means of the assumed equation.

616. In most problems, in which the forces are dependent upon the velocity, *the form of* R *is not unlimited, but is usually so restricted that*

$$R = R_s + R_v,$$

in which R_s *is a function of* s *and represents the action of the fixed forces, while* R_v *is a function of* v *and represents the resistances, to which the body is subject.* In this form of the problem, geometers have not made much progress towards its solution, although the case of the tautochrone, exhibited in this aspect, has been the occasion of much discussion and many difficult memoirs.

617. If the equation (359_3) solved with reference to t, acquires the form

$$t = T_{s,v}$$

the expression for R is

$$R = \frac{1 - v\, D_s T_{s,v}}{D_v T_{s,v}},$$

which is essentially identical with Lagrange's *most general formula in the case of the tautochrone.*

618. If the equation (359_3), solved with reference to v, acquires the form

$$v = V_{s,t},$$

the expression for R is

$$R = v\, D_s\, V_{s,t} + D_t\, V_{s,t},$$

which formula comprises LAPLACE'S *general form of solving the tautochrone.*

619. If the equation (359_3), solved with reference to s, acquires the form

$$s = S_{v,t},$$

the expression for R is

$$R = \frac{v - D_t\, S_{v,t}}{D_v\, S_{v,t}}.$$

620. When the equation (359_3) is presented in the form

$$T + S + V = 0,$$

in which T, S, and V are respectively functions of t, s, and v, the value of R is

$$R = -\frac{D_t\, T + v\, D_s\, S}{D_v\, V}.$$

But $D_t\, T$ is a function of t and, therefore, of $S + V$; it may, indeed, be any arbitrary function of $S + V$, so that if ψ denotes this arbitrary function, R becomes

$$R = -\frac{\psi(S + V) + v\, D_s\, S}{D_v\, V}.$$

621. When, in the preceding section, S is changed into $-\log S$ and

$$V = \log v,$$

the value of R may be presented in the form

$$R = v\,\psi\left(\frac{v}{S}\right) + v^2 D_s \log S\,;$$

which is the same with *a familiar formula of* Lagrange *for the case of the tautochrone.*

622. The cases, in which the formula (361_3) assumes the form (359_{15}) are easily investigated. For this purpose let

$$z = \frac{v}{S},$$

$$\chi = z\,\psi(z)\,;$$

and the derivatives of (361_3) give

$$D_v R = D_z \chi + 2v\, D_s \log S = D_v R_v,$$

$$D_S D_v R = -\frac{z}{S} D_z^2 \chi + 2z\, S\, D_S D_s \log S = 0\,;$$

whence

$$D_z^2 \chi = 2\, S^2 D_S D_s \log S = 2\,a,$$

in which a is any constant. Hence

$$\chi = a z^2 + b z + e,$$

in which b and e are constants introduced by integration. The value of R is, then,

$$R = e\,S + b\,v + (a + D_s S)\frac{v^2}{S}\,;$$

so that, if h and H are constants, the final values of S and R are

$$S = M e^{hs} + \frac{a}{h},$$

$$R = e\,S + b\,v + h\,v^2\,;$$

and this formula of LAGRANGE *is restricted to the resisting medium, in which the resistance has the form*

$$a + b\,v + h\,v^2$$

which was first remarked by FONTAINE.

The form of T, in this case, may be derived from the equations

$$c^{-T} = z,$$

$$-D_t\,T = \frac{\chi}{z} = a\,z + b + \frac{e}{z} = b + e\,c^T + a\,c^{-T};$$

which give

$$\sqrt{(2\,e\,a)}\cos\left[(\tau - t)\sqrt{(2\,e\,a - b^2)}\right] = \frac{2\,e\,a + b\,(e\,c^T + a\,c^{-T})}{b + e\,c^T + a\,c^{-T}}$$

$$= \frac{2\,e\,a\,v\,S + b\,e\,S^2 + b\,a\,v^2}{b\,v\,S + e\,S^2 + a\,v^2}.$$

When v vanishes this equation becomes

$$\sqrt{(2\,e\,a)}\cos\left[(\tau - t)\sqrt{(2\,e\,a - b^2)}\right] = b,$$

so that the interval $\tau - t$ is independent of the length of the arc, and the curve is a tautochrone if τ is also independent of s, which is the case when S vanishes with s, that is, when

$$M = -\frac{a}{h}.$$

This condition is always observed, if the direction of the curve coincides with that of the level surface at its termination, so that in every case, *this holochrone is essentially tautochronous.*

623. If, instead of (359_{25}) we suppose

$$T = T_{s,\,v},$$

and if ψ denotes an arbitrary function, the value of R has the form

$$R = \frac{\psi\,(T_{s,\,v}) - v\,D_s\,T_{s,\,v}}{D_v\,T_{s,\,v}}.$$

When

$$T_{s,v} = S\,V + S_1$$

in which S and S_1 are functions of s, and V is a function of v, the value of R becomes

$$R = \frac{1}{S\,V'}\left[\psi\,(S\,V + S_1) - v\,(S'\,V + S_1')\right],$$

which includes LAGRANGE'S formula. Forms of this kind may be indefinitely multiplied, without diminishing the difficulty of obtaining such as are new and not included in the investigations of § 622.

624. A curious case of the holochrone is introduced, when the form of R is

$$R = R_s + R_v + v^2\,S_1,$$

in which S_1 is a function of s. The only case of (361_3), which can assume this form is easily proved to be that of (361_{31}) when S is left undetermined. If, then, the factor of v^2, diminished by a constant, is inversely proportional to the radius of curvature, *the form of the resistance*, by including in it part of the term $e\,S$, *is that of* (362_3) *increased by a term proportional to the friction upon the curve.*

If the fixed force, in this case, is that of gravity, and the axis of z is vertical, and if v is the inclination of the radius of curvature to the axis of z, the first and last terms of R give, if k is the constant of friction,

$$S = -\frac{g \sin v + k\,g \cos v}{e},$$

$$a + D_s\,S = a - \frac{g \cos v - k\,g \sin v}{e\,\varrho} = -\left(h + \frac{k}{\varrho}\right) S,$$

$$\frac{1}{\varrho} = \frac{a\,e - h\,g \sin v - k\,h\,g \cos v}{(1 - k^2)\,g \cos v};$$

so that the curve determined by (361_{29}) *is included in this form.* This is a generalization of Bertrand's similar investigation with regard to the cycloid.

THE TACHYTROPE.

625. A curve in which the law of the velocity is given may be called a *tachytrope.*

626. *When the law of the velocity is given in an equation between the velocity, the space, and the time, the formulæ of* § 615 *are directly applicable to the complete solution of the problem; and all the subsequent transformations of these formulæ may be applied to the present case.*

627. When the time is not involved in the equation (359_3), but the portion R_v of the force R is given, the other portion R_s is determined by the equation

$$R_s = -\frac{v\, D_s W}{D_v W} - R_v,$$

from which v is to be eliminated by the given equation (359_{35}). Euler has solved various cases of this tachytrope.

628. One of the simple examples, solved by Euler, is when, in the case of a heavy body,

$$R_v = -k\, v^m,$$

and the velocity is to depend upon the arc in the same form as if the body descended in a vacuum upon an inclined straight line, so that the equation (359_3) acquires the form

$$v^2 = h\, s,$$

whence

$$g \sin \nu = R_s = \tfrac{1}{2}\, h + k\, (h\, s)^{\frac{1}{2} m}.$$

When

$$m = 2,$$

this equation becomes

$$g \sin \nu = \tfrac{1}{2} h + k h s,$$

or the required *tachytrope is a cycloid.*

629. Another simple and interesting example of this problem was proposed by KLINGSTIERNA and solved by CLAIRAUT. It is that of a heavy body in a medium, of which the resistance is proportional to the square of the velocity, approaching the origin with a velocity equal to that which it would have acquired by falling in the same medium through a height equal to the distance of the body from the origin measured upon the curve. In this case

$$R_v = k v^2,$$

$$v^2 = \frac{g}{k} (1 - c^{-2ks});$$

whence the equation of the tachytrope is

$$D_s z = 2 c^{-2ks} - 1,$$

of which the integral is

$$k (z + s) = 1 - c^{-2ks}.$$

630. A simple example of the problem of § 627 is that in which the velocity is uniform. In this case

$$R_s = - R_v = \text{a constant} = D_s \Omega,$$

so that *in the case of a heavy body this tachytrope is a straight line; in that of a constant force directed towards a fixed point, it is a logarithmic spiral; and in every case the sine of the angle, at which it intersects each level surface, is inversely proportional to the fixed force which acts at the point of intersection.*

631. When the given forces are parallel to the axis of z, and the given equation (359_3) is expressed in terms of v and z, the equation of the tachytrope is

$$(D_z \Omega \sin \nu + R_v) D_v W + D_z W v \sin \nu = 0,$$

from which v is eliminated by means of the given equation. Euler has solved several cases of this tachytrope.

632. If, in this case, the curve is to be such, that the velocity shall have a constant ratio to that which it would have acquired in a vacuum, the equation (366_5) assumes the form

$$D_z \Omega \sin \nu = - \frac{R_v}{1 - a}.$$

If the resistance is proportional to the square of the velocity, so that R_v has the form

$$R_v = - k v^2 = - 2 k a (\Omega + H),$$

the equation of the tachytrope is

$$\sin \nu D_z \log (\Omega + H) = \frac{2 k a}{1 - a}.$$

633. *When the given forces are directed towards the origin, and the given equation* (359_3) *is expressed in terms of* v *and* r, *the equation of the tachytrope, in a medium of given resistance is*

$$(D_r \Omega \cos_s^r + R_v) D_v W + D_r W v \cos_s^r = 0$$

from which v is eliminated by means of the given equation.

634. If, in this case, the curve is to be such that the velocity shall have a constant ratio to that which it would have acquired in a vacuum, the equation (366_{24}) assumes the form

$$D_r \Omega \cos_s^r = - \frac{R_v}{1 - a}.$$

If the resistance has the form (366_{16}), the equation of the tachytrope is

$$\cos{}^r_s D_r \log(\Omega + \Pi) = \frac{2ak}{1-a}.$$

635. *When the law of the velocity, in a medium of known resistance, is given in a given direction, such for instance as that of the axis of* x, *and so given that*

$$v \cos{}^s_x = W_{s,x},$$

in which $W_{s,x}$ is a given function of s and x, the equation of the tachytrope is derived from the equation

$$(D_s \Omega + R_v) \cos{}^s_x - \frac{v^2}{\varrho} \sin{}^s_x = v D_s W_{s,x} + v \cos{}^s_x D_x W_{s,x};$$

from which v is eliminated by the given equation.

636. When the velocity in the given direction is uniform, these equations become

$$v \cos{}^s_x = a,$$

$$\varrho = \frac{a^2 \sin{}^s_x}{(D_s \Omega + R_v) \cos^3{}^s_x}.$$

637. *When the given force is that of gravity, and* β *is the inclination of the given line to the vertical, the equation of this tachytrope becomes*

$$\varrho = \frac{a^2 \sin{}^s_x}{(g \cos(\beta - {}^s_x) + R_v) \cos^3{}^s_x}.$$

This problem is solved by Euler in the case in which the given direction is horizontal and in that in which it is vertical. A special solution is obtained upon the hypothesis of a constant velocity; in this case, the tachytrope is a straight line determined by the condition

$$g \cos(\beta - {}^s_x) + R_v = 0.$$

638. *When there is no resisting medium, the equation* (367_{24}) *of the tachytrope becomes*

$$\varrho = \frac{a^2 \sin \frac{s}{x}}{g \cos^3 \frac{s}{x} \cos (\beta - \frac{s}{x})}.$$

When the line is horizontal

$$\beta = \tfrac{1}{2}\pi,$$

and the equation becomes

$$\varrho = \frac{a^2}{g \cos^3 \frac{s}{x}},$$

so that *the tachytrope of this case is a parabola.*

When the line is vertical

$$\beta = 0,$$

and the equation becomes

$$\varrho = \frac{a^2 \sin \frac{s}{x}}{g \cos^4 \frac{s}{x}},$$

so that *the tachytrope of this case is the evolute of the parabola.*

With the notation

$$b^2 = \frac{g \sin \beta}{2 a^2}$$

the equation (368_3), expressed in rectangular coördinates is

$$2 b \sqrt{(x + y \cot \beta)} - b x = 2 \cot \beta \log [\cot \beta + b \sqrt{(x + y \cot \beta)}].$$

639. If the resistance is proportional to the velocity, so that

$$R_v = -k v,$$

and if the direction of the line in which the velocity is given is such that

$$g \cos \beta = k a,$$

the *equation of the tachytrope of a heavy body is*

$$x \sin \beta - y \cos \beta = \frac{a^2}{g} e^{-\frac{k x}{a}}.$$

THE TACHYSTOTROPE.

640. The curve on which the final velocity in a given resisting medium is a maximum, may be called a *tachystotrope.*

641. *In a medium in which the law of resistance is expressed as it is in* § 600, the notation of that section gives for the differential equations of the tachystotrope

$$D_s(\mu \sin \nu) = D_x \Omega\, D_s \mu_1,$$
$$-D_s(\mu \cos \nu) = D_z \Omega\, D_s \mu_1,$$
$$v\, D_s \mu_1 = \mu_1\, D_v\, V.$$

The reduction of these equations gives

$$D_s \mu = D_s \Omega\, D_s \mu_1 = D_s(\mu_1 V),$$
$$\mu = \mu_1 V,$$

and the expression of the normal pressure to the tachystotrope becomes

$$D_z \Omega \sin \nu + D_x \Omega \cos \nu = \frac{\mu}{\varrho\, D_s \mu_1} = \frac{D_s \Omega}{\varrho\, D_s \log \mu} = \frac{v \mu}{\varrho\, \mu_1 D_v V},$$
$$= \frac{v}{\varrho\, D_v \log V}.$$

642. In the case in which the law of the resistance is expressed by the formula

$$V = k\, v^m,$$

the normal pressure becomes

$$D_\rho \Omega = \frac{v^2}{m \varrho},$$

so that *the normal pressure has a constant ratio to the centrifugal force,* which result was obtained by Euler in the case of a heavy body.

643. *When the resistance is constant, the tachystotrope is a straight line.*

644. *When the forces are parallel to the axis of* z, the equations (369_9) and (359_{16}) give

$$\mu = \mu_1 V = \frac{a}{\sin v}.$$

645. *The equation of the tachystotrope of a heavy body is obtained, therefore, by the elimination of* v *between the equations*

$$V = \frac{g\,a}{b \sin v - a \cos v},$$

$$\frac{v\,D_v v}{\varrho} = -g \cos v - \frac{g\,a}{b \sin v - a \cos v}.$$

646. *When* V *has the form* (369_{24}), *the equation of the tachystotrope of a heavy body is*

$$\frac{a^2}{(b \sin v - a \cos v)^2} = k^2 (m\,g\,\varrho \sin v)^m.$$

THE BARYTROPE AND THE TAUTOBARYD.

647. The curve, in which the law of pressure is given, may be called a *barytrope*, and that barytrope, in which the pressure is everywhere the same, may be called a *tautobaryd.*

648. When the pressure is a given function of the arc, which may be denoted by S, its equivalent expression, if F is the fixed force which acts in the direction f, is

$$\frac{v^2}{\varrho} - F \sin{}^f_\rho = S;$$

and the differential equation of the barytrope is

$$2\,R = D_s\,[\varrho\,(S + F \cos{}^f_\rho)] = 2\,F \sin{}^f_\rho + 2\,R_v.$$

649. *In the case of a heavy body, if the axis of* z *is vertical, the differential equation of this barytrope becomes*

$$(S + g \cos \nu)\, D_s \varrho = - \varrho\, D_s S + 3 g \sin \nu + 2 R_v,$$

from which v may be eliminated by means of the equation

$$v^2 = \varrho\,(S + g \sin \nu).$$

In this case, the differential equation of the tautobaryd is

$$(a + g \cos \nu)\, D_s \varrho = 3 g \sin \nu + 2 R_v.$$

650. *When the resistance is constant, the equation of the barytrope of* § 648 *is*

$$\varrho\,(S + F \cos{}^{f}_{\rho}) = 2\,(\Omega + H) + 2\, s\, R_v.$$

In the case of the heavy body, this equation becomes

$$S\, D_\nu s + g \cos \nu\, D_\nu s = 2 g z + 2 H + 2 s R_v;$$

and *that of the tautobaryd is*

$$(a + g \cos \nu)^{b+3} \varrho = A\,[g + a \cos \nu + \sin \nu \sqrt{(g^2 - a^2)}]^b,$$

if A is an arbitrary constant,

$$g > a,$$

and

$$b = \frac{2 R_v}{\sqrt{(g^2 - a^2)}}.$$

But if

$$g < a,$$

the *equation of the tautobaryd* is

$$\log \left[\frac{\varrho}{A}(a + g \cos \nu)^3\right] = \frac{2 R_v}{\sqrt{(a^2 - g^2)}} \cos^{[-1]} \frac{g + a \cos \nu}{a + g \cos \nu}.$$

When there is no resistance, the tautobaryd of the heavy body is defined by the equation

$$\varrho = \frac{A}{(A + g\cos\nu)^3}.$$

651. When S vanishes, there is no pressure against the barytrope, and this curve is that on which the body moves freely. Thus the equation of the barytrope of the heavy body becomes, under this condition,

$$\varrho = \frac{A}{(\cos\nu)^3},$$

which is that of a parabola.

652. *When the curve of the barytrope is given, the equations* (370_{27}) *and* (370_{31}), *determine the law of the fixed force when that of the resistance is known, or, reciprocally, that of the resistance, when the fixed force is known.*

653. *When the forces are parallel to the axis of* z, the equation (370_{31}) becomes

$$D_s(S\varrho) + \frac{D_s(F\varrho\cos^3\nu)}{\cos^2\nu} = 2R_\nu,$$

which is applicable when the curve is given.

When there is no resistance, this equation gives

$$F\varrho\cos^3\nu = -\int_s[\cos^2\nu\, D_s(S\varrho)] = -\int_\nu[\cos^2\nu\, D_\nu(S\varrho)].$$

654. *In the case of parallel forces,* when the tautobaryd is a circle, and there is no resistance, the fixed force has the form

$$F = \frac{b}{\varrho\cos^3\nu},$$

in which b and F must vanish, if ν can become a right angle.

When the fixed force is that of gravity, and the tautobaryd is a circle, the expression of the resistance is

$$R_v = -\tfrac{3}{2} g \left(\frac{v^2}{\varrho} - a\right).$$

655. *In the case of parallel forces,* when the tautobaryd is a cycloid of which the base makes an angle α with the direction of the parallel forces, and when there is no resistance, the equation of the cycloid being

$$\varrho = 2 R \sin(\nu - \alpha),$$

the expression of the force is

$$F = \frac{a \sin(\nu - \alpha) + \frac{1}{6} a \sin(3\nu - \alpha) + \frac{1}{2} a \sin(\nu + \alpha) + b}{2 \sin(\nu - \alpha) \cos^3 \nu}.$$

When b vanishes and α is a right angle, this expression is reduced to

$$F = \tfrac{1}{3} a \operatorname{cosec} \nu,$$

which coincides with EULER's solution of this example.

THE SYNCHRONE.

656. The surface or curve which is the locus, at any instant, of all the bodies which start simultaneously from a given point with a given velocity, and move upon paths which are related by a given law, is called *a synchrone,* and the given starting point may be called *its dynamic pole.* This class of loci was first discussed by JOHN BERNOULLI.

657. If an integral of the motion of the body along one of the paths to the synchrone is obtained in the form

$$W = 0,$$

in which W is a function of the time, of the arc of the path, and of the parameters by which the relationship of the paths is expressed; *this equation is the required equation of the synchrone, if the time is assumed to be constant; and it is referred to the system of coördinates, consisting of the described arc and the given parameters.*

658. If the only force is that of a resisting medium, and if the form of the path is given, and also the position of the dynamic pole upon it, but not its direction in space, *the synchrone is obviously the surface of a sphere, of which the dynamic pole is the centre.*

659. If the body moves, without external force and without resistance, upon a straight line, which rotates uniformly about a given axis passing through the dynamic pole, *the synchrone is a surface of revolution about the same axis, and it is defined by the polar equation* (250_{20}) *or* (251_3) *when* p *vanishes and* t *is constant.*

660. *When the fixed forces are directed towards a point, or when they are parallel, the synchrone of bodies moving upon straight lines, is a surface of revolution, of which the axis is the line of action which passes through the dynamic pole.*

661. *In the rectilinear motion of a heavy body, it is obvious from* (255_{13}), *that the polar equation of the synchrone has the form*

$$r = a \cos{}^{r}_{z} + b,$$

which becomes a sphere, when b *vanishes, that is, when the initial velocity vanishes.*

662. *In the rectilinear motion of a heavy body through a medium, of which the resistance is proportional to the square of the velocity, the polar equation of the synchrone has the form,*

$$A\, c^{r} = \mathrm{Cos}\,(B \cos^{\frac{1}{2}}{}^{r}_{z}).$$

THE SYNTACHYD.

663. The surface or curve which is the locus of all the points, at which bodies have the same velocity, when they move from a given point, with a given velocity, upon paths which are related by a given law, may be called a *syntachyd.*

664. If an integral of the motion of a body along one of the paths which proceed to the syntachyd is obtained in the form

$$W = 0,$$

in which W is a function of the velocity, of the described arc, and of the parameters, this equation is that of the syntachyd in the same form of coördinates with those in which the synchrone of § 657 is expressed.

665. *In the case of § 658, the syntachyd coincides with the synchrone.*

666. *In the cases of §§ 659 and 660, the syntachyd is a surface of revolution about the same axis with the synchrone.*

667. *When the action is exclusively that of fixed forces, the syntachyd is a level surface.*

668. *When a heavy body moves upon a straight line, on which there is a constant friction, and through a medium of which the resistance is proportional to the square of the velocity, the equation of the syntachyd is*

$$e^{-2hr} - A = B \cos\left(\tfrac{r}{z} + \alpha\right),$$

in which the notation of § 515 is adopted, A and B are constants and

$$a = g \tan \alpha\,.$$

669. *When a heavy body moves upon a straight line, on which the friction is constant and through a medium of which the resistance is*

proportional to the velocity, the equation of the syntachyd has the form

$$\log\left[A - \cos\left(\tfrac{z}{r} + \alpha\right)\right] = B - \frac{A + Cr}{\cos\left(\tfrac{z}{r} + \alpha\right)}.$$

670. *When the body moves upon a line on which the friction is constant and through a medium of which the resistance is proportional to the square of the velocity, the equation of the syntachyd, expressed in the form of coördinates of* § 657, *is*

$$\left(\tfrac{1}{2} v^2 + A\right) e^{2ks} = \int_s \left(D_s \Omega\, e^{2sk}\right),$$

which coincides with Jacobi's investigation of this case of motion.

A POINT MOVING UPON A FIXED SURFACE.

671. Among the various forms, in which the motion of a point upon a fixed surface, with fixed forces, can be discussed, that of the principle of least action is here selected. In this case, therefore, the whole amount of action, denoted by

$$V = \int_s v,$$

is to be a minimum. If, then, the equation of the surface is

$$L = 0,$$

if rectangular coördinates are adopted, if μ_1 is the multiplier of the preceding equation of the surface, and μ that of the conditional equation

$$x'^2 + y'^2 + z'^2 = 1,$$

the equation of the path of the body, with reference to either axis, is

$$D_x v + \mu_1 D_x L - D_s (\mu x') = 0.$$

The sum of these three equations, multiplied respectively by x', y', and z', is

$$D_s v = D_s \mu,$$

or

$$v = \mu.$$

Whence

$$D_x v + \mu_1 D_x L - D_s (v x') = 0.$$

672. If the tangent plane to the given surface is assumed, at each instant, to be that of xy, and if the axis of y is taken normal to the path of the body, the preceding equation becomes, if ϱ_1 denotes the radius of curvature of the projection of the path upon the tangent plane,

$$\frac{v^2}{\varrho_1} = - D_y \Omega;$$

so that *the centrifugal force of the body in the direction of the surface to which it is restricted is equal to the normal pressure upon the path in the direction of the tangent plane.*

673. *When the direction of the force is normal to the surface, which is the case with the level surface, or when there is no force, the path of the body is the shortest line which can be drawn upon the surface, and coincides with the brachystochrone.*

674. *When the velocity is constant, the equation* (377_{13}) *expresses the condition that the body may move upon the intersection of a level surface with the given surface.* In this case ϱ_1 is the radius of curvature of this intersection, and $D_y \Omega$ is the whole force in the direction of the tangent plane to the surface.

675. When the velocity is a given function of the parameter of the level surface, the equation (377_{13}), with the notation of the preceding section, expresses the equation of a surface over which the body moves upon the intersection of this surface with the level surface.

676. *When the force is directed towards the origin, and the given surface is a plane passing through the axis*, the equation (377_{13}), combined with (316_{18}), gives in the notation of § 569

$$\frac{D_r \Omega}{\Omega - \Omega_0} = -\frac{2}{\varrho \sin \frac{r}{s}} = -\frac{2 D_r p}{p},$$

of which the integral is

$$\Omega - \Omega_0 = \frac{2 p_1^4}{p^2} = \tfrac{1}{2} v^2 = \tfrac{1}{2} D_t s^2.$$

Whence, if φ is the angle which r makes with the axis,

$$\tfrac{1}{2} r^2 D_t \varphi = p_1^2.$$

But $\frac{1}{2} r^2 d\varphi$ is the elementary area described by the radius vector in the instant dt, and it, therefore, follows that *the area described by the radius vector is proportional to the time.*

The equation (378_{11}), combined with that of living forces, gives

$$D_t s^2 = D_t r^2 + r^2 D_t \varphi^2 = D_t r^2 + \frac{4 p_1^4}{r^2} = 2 (\Omega - \Omega_0),$$

$$D_r \varphi = \frac{D_t \varphi}{D_t r} = \frac{2 p_1^2}{r \sqrt{[(2 r^2 (\Omega - \Omega_0) - 4 p_1^4]}};$$

Whence

$$\varphi = \int_r \frac{2 p_1^2}{r \sqrt{[2 r^2 (\Omega - \Omega_0) - 4 p_1^4]}},$$

which is *the polar equation of the path of the body.* That this equation can be obtained by integration by quadratures, is a simple case of the principle of the last multiplier.

677. When the potential of the force has the form

$$\Omega = \frac{a}{r^{2n}},$$

and the initial velocity is such that

$$\Omega_0 = 0,$$

or

$$v^2 = 2\,\Omega = \frac{2\,a}{r^{2\,n}},$$

the *polar equation of the path of the body is*

$$p_1^2\, r^{n-1} = \sqrt{(\tfrac{1}{2}\,a)}\,\sin\,[(n-1)\,(\varphi - \alpha)],$$

which was given by RICCATI.

If

$$n = \tfrac{1}{2},$$

the law of the force is that of gravitation, and the path is a parabola of which the origin is the focus.

If

$$n = 1,$$

the attractive force is inversely proportional to the cube of the radius vector, and the path is a logarithmic spiral, which was proved by NEWTON.

If

$$n = \tfrac{3}{2},$$

the attractive force is inversely proportional to the fourth power of the radius vector, and the path is the epicycloid formed by the exterior rotation of a circle upon an equal circle, which was proved by STADER.

If

$$n = 2,$$

the attractive force is inversely proportional to the fifth power of the radius vector, and the path is the circumference of a circle, which was proved by NEWTON.

If

$$n = \tfrac{5}{2},$$

the attractive force is inversely proportional to the sixth power of the radius vector, and the path may be called the trifolia of STADER, by whom it was investigated.

If

$$n = 3,$$

the attractive force is inversely proportional to the seventh power of the radius vector, and the path is the lemniscate of James Bernoulli, which was proved by Stader.

If

$$n = -1,$$

the repulsive force is proportional to the radius vector, and the path is an equilateral hyperbola.

When

$$n < 1,$$

r becomes infinite when $(\varphi - \alpha)$ vanishes, which was remarked by Stader.

678. When the values of Ω, Ω_0 and p_1 are such that, if R is an integral function of an integral root of r,

$$R = \sqrt{[2 r^2 (\Omega - \Omega_0) - 4 p_1^4]},$$

the expression of φ in (378_{22}) admits of integration. For if the integral root of r is denoted by

$$r_1 = \sqrt[m]{r},$$

and if the notation of the residual calculus is adopted, the equation (378_{22}) becomes

$$\varphi = 2 m p_1^2 \int_{r_1} \frac{1}{r_1 R} = 2 m p_1^2 \mathcal{E}_{r_1} \frac{\log (\sqrt[m]{r} - r_1)}{(r_1 R)}.$$

679. An example of the preceding section occurs, when m is unity and

$$R = a r^2 + b r + c,$$

which corresponds to

$$\Omega = \tfrac{1}{2} a^2 r^2 + a b r + \frac{b e}{r} + \frac{e^2 + 4 p_1^4}{2 r_2},$$
$$\Omega_0 = - \tfrac{1}{2} b^2 - a e,$$

and an attractive force of the form

$$- a^2 r - a b + \frac{b e}{r^2} + \frac{e^2 + 4 p_1^4}{r^3}.$$

In this case the value of φ is

$$\varphi - \alpha = \frac{p_1^2}{e} \log \frac{r^2}{R} - \frac{2 b p_1^2}{e \sqrt{(4 a e - b^2)}} \tan^{[-1]} \frac{2 a r + b}{\sqrt{(4 a e - b^2)}},$$
$$= \frac{p_1^2}{e} \log \frac{r^2}{R} + \frac{2 b p_1^2}{e \sqrt{(b^2 - 4 a e)}} \operatorname{Tan}^{[-1]} \frac{2 a r + e}{\sqrt{(b^2 - 4 a e)}}.$$

When b vanishes, these expressions become

$$\Omega = \tfrac{1}{2} a^2 r^2 + \frac{e^2 + 4 p_1^4}{2 r^2},$$
$$\Omega_0 = - a e,$$

the attractive force is

$$- a^2 r + \frac{e^2 + 4 p_1^4}{r^3},$$

and the equation of the path is

$$a + \frac{e}{r^2} = c^{- \frac{e}{p_1^2} (\varphi - \alpha)}.$$

When e vanishes, the expressions become

$$\Omega = \tfrac{1}{2} a^2 r^2 + a b r + \frac{2 p_1^4}{r^2},$$
$$\Omega_0 = - \tfrac{1}{2} b^2,$$

the repulsive force is

$$a^2 r + a b - \frac{4 p_1^4}{r^3},$$

and the equation of the path is

$$\frac{b^2}{2\,a\,p_1^2}(\varphi - \alpha) = \log\left(a + \frac{b}{r}\right) - \frac{b}{r}.$$

When $b^2 - 4\,a\,e$ vanishes, the equation of the path is

$$\frac{b^2}{8\,a\,p_1^2}(\varphi - \alpha) = \frac{b}{2\,a\,r + b} - \log\left(2\,a + \frac{b}{r}\right).$$

680. Another example of § 678 occurs when

$$R = a\,r + b + \frac{e}{r},$$

which corresponds to

$$\Omega_0 = -\tfrac{1}{2}\,a^2,$$

and an attractive force of the form

$$\frac{a\,b}{r^2} + \frac{b^2 + 2\,a\,e + 4\,p_1^4}{r^3} + \frac{3\,b\,e}{r^4} + \frac{2\,e^2}{r^5}.$$

The equation of the path is

$$2\,a\,r + b = \sqrt{(4\,a\,e - b^2)}\tan\left[\frac{\sqrt{(4\,a\,e - b^2)}}{4\,p_1^2}(\varphi - \alpha)\right]$$

$$= \sqrt{(b^2 - 4\,a\,e)}\,\mathrm{Tan}\left[\frac{\sqrt{(b^2 - 4\,a\,e)}}{4\,p_1^2}(\varphi - \alpha)\right].$$

When a vanishes, the value of Ω_0 vanishes, the attractive force is

$$\frac{b^2 + 4\,p_1^4}{r^3} + \frac{3\,b\,e}{r^4} + \frac{2\,e^2}{r^5},$$

and the equation of the path is

$$\log(b\,r + e) = \frac{b}{2\,p_1^2}(\varphi - \alpha).$$

When $b^2 - 4\,a\,e$ vanishes, the equation of the path is

$$2\,a\,r + b = \frac{4\,p_1^2}{\alpha - \varphi}.$$

681. Another example of § 678 occurs when

$$R = A r_1^n + B,$$

in which case

$$\Omega - \Omega_0 = \frac{2 p_1^2 + \frac{1}{2} B^2}{r^2} + \frac{A B}{r^{2 - \frac{n}{m}}} + \frac{\frac{1}{2} A^2}{r^{2(1 - \frac{n}{m})}}.$$

and the equation of the path is

$$\frac{n B}{2 m p_1^2} (\alpha - \varphi) = \log \left(1 + \frac{B}{A} r^{-\frac{n}{m}}\right).$$

682. *The forms, in which* (378_{22}) *admits of explicit integration without any special determination of* Ω_0 *and* p, *are included in the general expression*

$$\Omega = a r^h + \frac{b}{r^2},$$

in which h *is two, or the negative of unity, so that* Ω *only consists of two terms, of which one is*

$$\frac{b}{r^2},$$

and the general form of the central force consists, therefore, of two terms of which one is inversely proportional to the cube of the radius vector, and the other may be either directly proportional to the radius vector, or inversely proportional to the square of the radius vector.

683. In general, it is apparent that the addition of a term to the central force, which is inversely proportional to the cube of the radius vector, does not augment the difficulty of determining the path of the body. *In any equation of a path of a body described under the action of central forces, which is expressed by the elements* $\varphi - \alpha$, r *and* t, *and which may also involve the constant* p_1, *the multiplication of the angle* $\varphi - \alpha$, *and of* p_1^2 *by the factor*

$$B = \sqrt{\left(1 - \frac{b}{4 p_1^2}\right)},$$

gives the equation of the path, when the central force is increased by the term

$$\frac{b}{r^3}.$$

684. When there is no force the path is a straight line, so that *when the central force is inversely proportional to the cube of the radius vector*, the polar equation of the path is

$$r \cos [B(\varphi - \alpha)] = B p_1^2 \sqrt{\frac{2}{-\Omega_0}}.$$

If the force is repulsive, B exceeds unity, the path is convex to the origin, and its convexity increases with the increase of the repulsive force until it terminates in a straight line. *If the force is attractive*, and B^2 positive, it is less than unity, the path is concave to the origin but of infinite extent, and the concavity increases with the increase of the attractive force until it terminates in *the reciprocal spiral of* Archimedes. If the force is attractive, B^2 negative and Ω_0 positive, the equation of the path is

$$r \operatorname{Cos} [B(\varphi - \alpha)\sqrt{-1}] = B p_1^2 \sqrt{\frac{2}{-\Omega_0}},$$

so that the greatest distance of the path from the origin is limited, and the path is a spiral about the origin in which it terminates, at each extremity, through infinitely compressed coils. If the force is attractive, and B^2 and Ω_0 negative, the equation of the path is

$$r \operatorname{Sin} [B(\varphi - \alpha)\sqrt{-1}] = B p_1^2 \sqrt{\frac{2}{\Omega_0}},$$

so that the curve extends to an infinite distance from the origin at one extremity, and terminates in an infinitely condensed coil about the origin at the other extremity. In these three cases, the formula for the time which corresponds to (384_9) is

$$t = -\frac{Bp_1^2}{\Omega_0} \tan\left[B(\varphi - \alpha)\right],$$

the formula for (384_{20}) is

$$t = \frac{Bp_1^2\sqrt{-1}}{\Omega_0} \operatorname{Tan}\left[B(\varphi - \alpha)\sqrt{-1}\right],$$

and that for (384_{27}) is

$$t = \frac{Bp_1^2\sqrt{-1}}{\Omega_0} \operatorname{Cot}\left[B(\varphi - \alpha)\sqrt{-1}\right].$$

This law of central force has been discussed by several geometers, and, with peculiar regard to the special cases of the problem, by Stader, whose results coincide substantially with those of this section.

685. *When the central force is proportional to the radius vector, the path is a conic section of which the centre is at the origin. It is an ellipse, if the force is attractive, and an hyperbola, if the force is repulsive.* In the case of the ellipse, if a point were to start from the extremity of the major axis at the same instant with the body, and move upon the circumference of which this axis is the diameter, with such an uniform velocity as to complete its circuit synchronously with the body, the body and the point are always upon a straight line which is perpendicular to the major axis. For different ellipses, the time of description is proportional to the square root of the area. In the case of the hyperbola, if a catenary is drawn through the extremity of the transverse axis, in such a position that this axis is the direction of gravity, while its extremity is the lowest point of the catenary, and of such a magnitude that the radius of curvature of the catenary at this point is equal to the semi-transverse axis, and if a body starts upon the

catenary simultaneously with the given body, and proceeds in such a way as to recede uniformly from the transverse axis with a velocity equal to that of the given body at its nearest approach to the origin, the line which joins the two bodies will always remain perpendicular to the transverse axis of the hyperbola.

686. *When in addition to the term, which is proportional to the radius vector, the central force has a term inversely proportional to the cube of the radius vector, the path can be derived from the preceding section by the principle of* § 683.

When the term which is proportional to the radius vector is attractive and expressed by

$$a\,r,$$

the polar equation of the curve is

$$\begin{aligned}\frac{4\,B^2 p_1^4}{r^2} + \Omega_0 &= \sqrt{[\Omega_0^2 - 4\,a\,B^2 p_1^4]}\cos[2\,B\,(\varphi - \alpha)] \\ &= \sqrt{[\Omega_0^2 - 4\,a\,B^2 p_1^4]}\,\mathrm{Cos}\,[2\,B\,(\varphi - \alpha)\sqrt{-1}] \\ &= \sqrt{[4\,a\,B^2 p_1^4 - \Omega_0^2]}\,\mathrm{Sin}\,[2\,B\,(\varphi - \alpha)\sqrt{-1}].\end{aligned}$$

When a is positive, therefore, the path does not extend to infinity, although when B^2 is negative it is compressed at each extremity into an infinite coil. But when a is negative, the term proportional to the radius vector is repulsive, and the curve extends to infinity if B^2 is positive; but if B^2 is negative the curve is limited if Ω_0 is negative, or it may necessarily extend to infinity if Ω_0 is positive.

In the special case of

$$\frac{\tan(2\,n\,B\pi)}{2\,B} = \frac{p_1^2}{\Omega_0}\sqrt{-a},$$

the curve is asymptotic to itself.

687. *When the central force is inversely proportional to the square*

of the radius vector which is the law of gravitation, the path is a conic section, of which the origin is the focus. When the force is attractive, the path is an ellipse if Ω_0 is positive, a parabola if Ω_0 vanishes, and it is that branch of the hyperbola which contains the focus, if Ω_0 is negative. But when the force is repulsive, the path is that branch of the hyperbola which does not contain the focus. The farther consideration of this law of force is reserved, in this connection, for the *Celestial mechanics.*

688. *When in addition to the term, which is inversely proportional to the square of the radius vector, the central force has a term inversely proportional to the cube of the radius vector, the path can be derived from the preceding section by the principle of* § 683.

If the term of central force, which is inversely proportional to the square of the radius vector is

$$-\frac{a}{r^2},$$

the polar equation of the path is

$$\begin{aligned}\frac{4\,B^2 p_1^4}{r} - a &= \sqrt{(a^2 - 8\,\Omega_0 B^2 p_1^4)}\cos\left[B(\varphi - \alpha)\right] \\ &= \sqrt{(a^2 - 8\,\Omega_0 B^2 p_1^4)}\,\mathrm{Cos}\left[B(\varphi - \alpha)\sqrt{-1}\right] \\ &= \sqrt{(8\,\Omega_0 B^2 p_1^4 - a^2)}\,\mathrm{Sin}\left[B(\varphi - \alpha)\sqrt{-1}\right],\end{aligned}$$

when Ω_0 is positive, therefore, the curve is finite; it returns into itself if B^2 is positive, but if B^2 is negative it terminates at each extremity in an infinitely compressed coil about the origin. When Ω_0 is negative, one portion at least of the path extends to an infinite distance from the origin; if, moreover, a is positive and B^2 negative, but such that

$$a^2 > 8\,\Omega_0\,B^2 p_1^4,$$

another portion of the path is finite and terminates in the origin, through an infinitely compressed coil, while the two infinite portions commence in such a coil; if the negative B^2 is such that

$$a^2 < 8\,\Omega_0\,B^2 p_1^4,$$

or if a is negative as well as Ω_0, the curve only consists of the portion which extends from the coil to infinity. The time may be computed by the three formulæ, which correspond to the three forms of (387_{18}),

$$\begin{matrix}\tan\\ \mathrm{Tan}\end{matrix}\left[\frac{\sqrt{(2\,\Omega_0)}}{a}\left(t\,\Omega_0+\sqrt{\left(\frac{a^2}{16\,B^2 p_1^4}-\tfrac{1}{2}\,\Omega_0\right)}\,r\sin\left[B\,(\varphi-\alpha)\right]\right)\right]$$
$$=\frac{a-\sqrt{(a^2-8\,\Omega_0\,B^2 p_1^4)}}{\sqrt{(8\,\Omega_0\,B^2 p_1^4)}}\tan\left[\tfrac{1}{2}\,B\,(\varphi-\alpha)\right],$$

$$\begin{matrix}\tan\\ \mathrm{Tan}\end{matrix}\left[\frac{\sqrt{(2\,\Omega_0)}}{a}\left(t\,\Omega_0+\sqrt{\left(\frac{a^2}{16\,B^2 p_1^4}+\tfrac{1}{2}\,\Omega_0\right)}\,r\,\mathrm{Sin}\left[B\,(\varphi-\alpha)\sqrt{-1}\right]\right)\right]$$
$$=\frac{a-\sqrt{(a^2-8\,\Omega_0\,B^2 p_1^4)}}{\sqrt{(-8\,\Omega_0\,B^2 p_1^4)}}\,\mathrm{Tan}\left[\tfrac{1}{2}\,B\,(\varphi-\alpha)\sqrt{-1}\right],$$

$$\begin{matrix}\tan\\ \mathrm{Tan}\end{matrix}\left[\frac{\sqrt{(2\,\Omega_0)}}{a}\left(t\,\Omega_0+\sqrt{\left(\frac{a^2}{16\,B^2 p_1^4}-\tfrac{1}{2}\,\Omega_0\right)}\,r\,\mathrm{Cos}\left[B\,(\varphi-\alpha)\sqrt{-1}\right]\right)\right]$$
$$=\frac{a-\sqrt{(8\,\Omega_0\,B^2 p_1^4-a^2)}}{\sqrt{(-8\,\Omega_0\,B^2 p_1^4)}}\,\mathrm{Cot}\left[\tfrac{1}{2}\,B\,(\varphi-\alpha)\sqrt{-1}\right];$$

the upper of the double forms of the first member applies to the case in which Ω_0 is positive, and the lower to that in which Ω_0 is negative. This case was partially developed by Clairaut.

689. The principle of § 683 may be extended to § 677, and among the resulting curves, that in which n is 2, deserves to be noticed from its simplicity, the equation of this case is

$$p_1^2\,r=\frac{\sqrt{\tfrac{1}{2}\,a}}{B}\sin\left[B\,(\varphi-\alpha)\right]=\frac{\sqrt{\tfrac{1}{2}\,a}}{B\sqrt{-1}}\,\mathrm{Sin}\left[B\,(\varphi-\alpha)\sqrt{-1}\right].$$

690. *The law of central force, for which the integrals, involved in the equations of motion, can be expressed by the elliptic forms without*

any special determination of Ω_0 *and* p_1, *may be reduced to two general forms of algebraic polynomial besides other fractional forms.*

The first of these forms is

$$F = b_4 r^{4m-3} + b_3 r^{3m-3} + b_2 r^{2m-3} + b_1 r^{m-3} + b\, r^{-3},$$

in which m is either 2, 1, $\frac{2}{3}$, or $\frac{1}{2}$.

The second form is

$$F = b_4 r^{2m-3} + b_3 r^{m-3} + b_2 r^{-3} + b_1 r^{-m-3} + b\, r^{-2m-3},$$

in which m is either 1 or 2. In each of these cases the term which is inversely proportional to r must be omitted.

691. *In the first case of the preceding section,* when m is unity, the equation (378_{22}) acquires the form

$$\varphi = \int_r \frac{2\, p_1^2}{r\sqrt{(a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a)}}.$$

It is obvious from inspection that whenever

$$a_4 = b_4,$$

is positive, a portion of the curve extends to infinity; but whenever a_4 is negative, the curve is of finite extent. It is also apparent that whenever

$$a = -4\, B^2 p_1^4,$$

is positive, a portion of the curve terminates in an infinitely compressed coil about the origin, that no portion of the curve can approach the origin except through such a coil, and that when a is negative, the curve does not pass through the origin.

If all the roots of the equation

$$a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a = 0,$$

are imaginary, a_4 and a must be positive, and the curve extends continuously from the origin to infinity. If the moduli of the roots are h and h_1, and the arguments α and α_1, and if the following notation is adopted

$$p+q=\frac{2\,a_4\,(h^2+h_1^2)^2-4\,a}{2\,a_1-a_3\,(h^2+h_1^2)},$$

$$p\,q=\frac{2\,a\,a_3-(h^2+h_1^2)\,a_1\,a_4}{a_4\,[2\,a_1-a_3\,(h^2+h_1^2)]},$$

$$A^2=(p-h)^2+4\,p\,h\sin^2\tfrac{1}{2}\,\alpha,$$

$$A_1^2=(p-h_1)^2+4\,p\,h_1\sin^2\tfrac{1}{2}\,\alpha_1,$$

$$B^2=(q-h)^2+4\,q\,h\sin^2\tfrac{1}{2}\,\alpha,$$

$$B_1^2=(q-h_1)^2+4\,q\,h_1\sin^2\tfrac{1}{2}\,\alpha_1,$$

$$\tan\theta=\frac{B_1\,(r-p)}{A_1\,(q-r)},$$

$$\cos i=\frac{A_1\,B}{A\,B_1},$$

and if θ_0 is the value of θ when r vanishes, the equation of the curve is

$$\frac{A\,A_1 p^2 q^2\sqrt{(1-\sin^2 i\sin^2\theta_0)}}{2\,p_1^2\,(p-q)^2\cos^2\theta_0}(\varphi-\alpha)=\frac{\sqrt{(1-\sin^2 i\sin^2\theta_0)}}{(p-q)\cos^2\theta_0}(p\cos^2\theta_0+q\sin^2\theta_0)\,\mathfrak{E}_i\theta$$
$$-\frac{q^2}{p\tan\theta_0}\sqrt{(1-\sin^2 i\sin^2\theta_0)}\,\mathfrak{P}_i\,(-\operatorname{cosec}^2\theta_0,\,\theta)$$
$$-\log\frac{\sqrt{(1-\sin^2 i\sin^2\theta)}-\sqrt{(1-\sin^2 i\sin^2\theta_0)}}{\sqrt{(\sin^2\theta-\sin^2\theta_0)}};$$

and the expression of the time is

$$\frac{A\,A_1\,q^2\tan\theta_0\,(t-\tau)}{p^2\,(q-p)\,(q-r)}\sqrt{\frac{p^2+q^2\cos^2 i\tan^2\theta_0}{p^2\cot^2\theta_0+q^2}}=\frac{q\,(p\cot^2\theta_0+q)\sqrt{(p^2+q^2\cos^2 i\tan^2\theta_0)}}{(q+r)\,(p^2\cot^2\theta_0+q^2)^{\frac{3}{2}}}$$
$$+\frac{q\,(p^2+p\,q)\sqrt{(p^2+q^2\cos^2 i\tan^2\theta_0)}}{(q-r)\tan^2\theta_0\,(p^2\cot^2\theta_0+q^2)^{\frac{3}{2}}}$$
$$+\log\frac{\sqrt{[(p^2\cot^2\theta_0+q^2)\,(1-\sin^2 i\sin^2\theta)]}-\sqrt{(p^2\cot^2\theta_0+q^2\cos^2 i)}}{\sqrt{(p^2\cot^2\theta_0\sin^2\theta-q^2\cos^2\theta)}}.$$

The elliptic integrals disappear when

$$a_3 = a_1 = 0,$$

which case has already been discussed in § 686. They also disappear when the imaginary roots are equal, in which case

$$\frac{a_3^2}{a_4} = \frac{a_1^2}{a} = 4\,a_2 - 8\sqrt{(a\,a_4)}\,;$$

so that if

$$R = 2\,a_3\,a_4\,r^2 + a_3^2\,r + 2\,a_1\,a_4,$$

the expressions for φ and t are

$$\varphi - \alpha = \frac{p_1^2}{\sqrt{a}} \log\frac{r^2}{R} - \frac{2\,a_3^2\,p_1^2}{\sqrt{[a_1\,a_3\,a_4\,(16\,a\,a_4 - a_1\,a_3)]}} \tan^{[-1]} \frac{(4\,a_4\,r + a_3)\sqrt{a_1}}{\sqrt{[a_3\,(16\,a\,a_4 - a_1\,a_3)]}},$$

$$t - \tau = \frac{1}{2\sqrt{a_4}} \log R - \sqrt{\left(\frac{a_1\,a_3}{a_4\,(16\,a\,a_4 - a_1\,a_3)}\right)} \tan^{[-1]} \frac{(4\,a_4\,r + a_3)\sqrt{a_1}}{\sqrt{[a_3\,(16\,a\,a_4 - a_1\,a_3)]}}.$$

When two of the roots of the equation (389_{29}) are real and two are imaginary, if both the real roots are negative, a_4 and a must be positive, and the curve extends continuously from the origin to infinity. If one of the real roots, denoted by r_1, is positive, and the other, denoted by r_2, is negative, and if a_4 is positive, the curve extends to infinity at each extremity, and r_1 is its least distance from the origin; but if a_4 is negative, the curve is finite, terminates at each extremity in the origin, and r_1 is its greatest distance from the origin. If both the real roots are positive and if a_4 is also positive, the curve consists of two portions, one of which extends to infinity at each extremity, and the greater real root r^1 is its least distance from the origin, while the other portion is finite, terminates at each extremity in the origin, and r^2 is its greatest distance from the origin; but if a_4 is negative the curve consists of a continuous portion of which r_1 is the greatest, and r_2 the least distance from the origin. If h is the modulus and a the argument of one of the imaginary roots, the following notation may be adopted.

$$r_1 - h\, c^{\alpha\sqrt{-1}} = A \tan \tfrac{1}{2}\varepsilon \; c^{\beta\sqrt{-1}},$$
$$r_1 - h\, c^{-\alpha\sqrt{-1}} = A \tan \tfrac{1}{2}\varepsilon \; c^{-\beta\sqrt{-1}},$$
$$r_2 - h\, c^{\alpha\sqrt{-1}} = A \cot \tfrac{1}{2}\varepsilon \; c^{\beta_1\sqrt{-1}},$$
$$r_2 - h\, c^{-\alpha\sqrt{-1}} = A \cot \tfrac{1}{2}\varepsilon \; c^{-\beta_1\sqrt{-1}},$$
$$r_1 \cot \tfrac{1}{2}\varepsilon = l \cot \tfrac{1}{2}\gamma,$$
$$r_2 \tan \tfrac{1}{2}\varepsilon = l \tan \tfrac{1}{2}\gamma.$$

When a_4 is positive, if θ and i are determined by the equations

$$i = \tfrac{1}{2}\pi - \tfrac{1}{2}(\beta - \beta_1)$$
$$\tan \tfrac{1}{2}\theta \cot \tfrac{1}{2}\varepsilon = \sqrt{\left(\frac{r - r_2}{r - r_1}\right)},$$

or

$$\frac{r \sin\gamma}{l \sin\varepsilon} = \frac{\cos\gamma - \cos\theta}{\cos\varepsilon - \cos\theta} = \frac{\sin\frac{1}{2}(\theta+\gamma)\sin\frac{1}{2}(\theta-\gamma)}{\sin\frac{1}{2}(\theta+\varepsilon)\sin\frac{1}{2}(\theta-\varepsilon)},$$

the equation of the curve is

$$\frac{lA\sin\varepsilon(\varphi-\alpha)\sqrt{a_4}}{2p_1^2} = \frac{1-\cos\varepsilon\cos\gamma}{\sin\gamma}\mathfrak{F}_i\theta - \cot\gamma\,(\cos\gamma-\cos\varepsilon)\,\mathfrak{P}_i(-\sin^2\gamma\sin^2 i, \theta)$$
$$-\frac{\cos\gamma-\cos\varepsilon}{\sqrt{(1-\sin^2 i\sin^2\gamma)}}\,\mathrm{Tan}^{[-1]}\frac{\sin\gamma\sin\theta\sqrt{[1-\sin^2 i\sin^2\gamma)\,(1-\sin^2 i\sin^2\theta)]}}{1+\cos\gamma\cos\theta+\sin^2 i\sin^2\gamma\sin^2\theta},$$

and the value of $(t - \tau)$ is derived from that of $(\varphi - \alpha)$ by multiplying by $\frac{l^2}{2p_1^2}$ and interchanging γ and ε. It is apparent that ε is obtuse and exceeds γ, and that upon the finite portion of the curve θ extends from zero to γ, while upon the infinite portion, it extends from ε to π.

When a_4 is negative, if θ and i are determined by the equations

$$i = \tfrac{1}{2}(\beta - \beta_1),$$
$$\tan \tfrac{1}{2}\theta \cot \tfrac{1}{2}\varepsilon = \sqrt{\left(\frac{r - r_2}{r_1 - r}\right)},$$

or

$$\frac{r \sin \gamma}{l \sin \varepsilon} = \frac{1 - \cos \gamma \cos \theta}{1 - \cos \varepsilon \cos \theta},$$

the equation of the curve is

$$\frac{l A \sin \varepsilon \sin 2\gamma\, (\varphi - \alpha) \sqrt{-a_4}}{4 p_1^2} = \sin^2 \gamma\, \mathfrak{F}_i \theta + (\cos \gamma - \cos \varepsilon)\, \mathfrak{P}_i (\cot \gamma, \theta)$$
$$+ \frac{(\cos \gamma - \cos \varepsilon) \cos \gamma}{\sqrt{(\cot^2 \gamma + \sin^2 i)}} \tan^{[-1]} \sqrt{\left(\frac{\cot^2 \gamma + \sin^2 i}{\cot^2 \theta + \cos^2 i}\right)},$$

and the value of $(t - \tau)$ is derived from that of $(\varphi - \alpha)$ by multiplying by $\frac{l^2}{2 p_1^2}$ and interchanging γ and ε.

The elliptic integrals disappear when the two real roots are equal. In this case, a_4 is positive, and the curve is continuous from the origin to infinity. With the notation

$$R^2 = r^2 + h^2 - 2 r h \cos \alpha = (r - h \cos \alpha)^2 + h^2 \sin^2 \alpha,$$
$$R_1^2 = r_1^2 + h^2 - 2 r_1 h \cos \alpha = (r_1 - h \cos \alpha)^2 + h^2 \sin^2 \alpha,$$

the equation of the curve is

$$\frac{r_1 (\varphi - \alpha) \sqrt{a_4}}{2 p_1^2} = \frac{1}{h} \operatorname{Tan}^{[-1]} \frac{h - r \cos \alpha}{R} - \frac{1}{R_1} \operatorname{Tan}^{[-1]} \frac{h^2 - h \cos \alpha\, (r + r_1) + r r_1}{R R_1},$$

and the expression of the time is given by the equation

$$(t - \tau) \sqrt{a_4} = \operatorname{Sin}^{[-1]} \frac{r - h \cos \alpha}{h \sin \alpha} - \frac{r_1}{R_1} \operatorname{Tan}^{[-1]} \frac{h^2 - h \cos \alpha\, (r + r_1) + r r_1}{R R_1}.$$

When a_4 vanishes, if r_1 is the real root of the equation (389_{29}), the curve consists of a single portion which extends from the origin to infinity when r_1 is negative, in which case a_3 is positive. But if r_1 and a_3 are both positive, the portion extends to infinity, and r_1 is its least distance from the origin; if r_1 is positive while a_3 is negative, each extremity of the curve terminates in the origin, and r_1 is its greatest distance from the origin.

When a_3 is positive, if θ and i are determined by the equations

$$i = \tfrac{1}{2}\beta,$$

$$\tan^2 \tfrac{1}{2}\theta = \frac{r - r_1}{A \tan \frac{1}{2}\varepsilon} = \frac{r - r_1}{B^2},$$

the equation of the curve is

$$(r_1 - B^2)\frac{B(\varphi - \alpha)\sqrt{a_3}}{2p_1^2} = -\mathfrak{F}_i\theta - \frac{r_1 + B^2}{2r_1}\mathfrak{P}_i\left[\frac{(r_1 - B^2)^2}{4B^2 r_1}, \theta\right]$$
$$+ \frac{B(r_1 - B^2)^2}{\sqrt{[r_1(r_1^2 + B^4 - 2B^2 r_1 \cos 2i)]}}\tan^{[-1]}\frac{\sin\theta\sqrt{(r_1^2 + B^4 - 2B^2 r_1 \cos 2i)}}{2B\sqrt{[r_1(1 - \sin^2 i \sin^2\theta)]}},$$

and the expression for the time is

$$(t - \tau)\sqrt{a_3} = \left(\frac{r_1}{B} + B\right)\mathfrak{F}_i\theta - 2B\,\mathfrak{E}_i\theta + 2B\tan\tfrac{1}{2}\theta\sqrt{(1 - \sin^2 i \sin^2\theta)}.$$

When a_3 is negative, if θ and i are determined by the equations

$$i = \tfrac{1}{2}(\pi - \beta),$$

$$\tan^2\tfrac{1}{2}\theta = \frac{r_1 - r}{B^2},$$

the equation of the curve is

$$(r_1 + B^2)\frac{B(\varphi - \alpha)\sqrt{-a_3}}{2p_1^2} = -\mathfrak{F}_i\theta - \frac{r_1 - B^2}{2r_1}\mathfrak{P}_i\left[-\frac{(r_1 + B^2)^2}{4B^2 r_1}, \theta\right]$$
$$+ \frac{B(r_1 + B^2)^2}{\sqrt{[r_1(r_1^2 + B^4 + 2B^2 r_1 \cos 2i)]}}\operatorname{Tan}^{[-1]}\frac{\sin\theta\sqrt{(r_1^2 + B^4 + 2B^2 r_1\cos 2i)}}{2B\sqrt{[r_1(1 - \sin^2 i\sin^2\theta)]}},$$

and the expression for the time is

$$(t - \tau)\sqrt{-a_3} = \left(\frac{r_1}{B} - B\right)\mathfrak{F}_i\theta + 2B\,\mathfrak{E}_i\theta - 2B\tan\tfrac{1}{2}\theta\sqrt{(1 - \sin^2 i\sin^2\theta)}.$$

When all the roots of the equation (389_{29}) are real, if, beginning with the greatest, they are arranged in the order of algebraic magnitude, they may be denoted by r_1, r_2, r_3, and r_4. If they are all negative, the curve consists of a single portion which extends from the origin to infinity. But if r_1 is the only positive root, the curve consists of a single branch, which extends by the same law as that expressed in (391_{20}). If r_1 and r_2 are positive, while the other two roots are negative, the curve consists of one or two portions, according to the same principles which distinguish the forms of (391_{25}). If r_4 is the only negative root, and if a_4 is positive, the curve consists of two portions, one of which extends to infinity, and r_1 is its least distance from the origin, while the other portion is finite and limited by the circumferences described about the origin as centre, with r_2 and r_3 as radii; but if a_4 is negative, one portion terminates, at each extremity, in the origin, and r_3 is its greatest radius vector, while the other portion is contained between the limiting circumferences of which r_1 and r_2 are the radii. If all the roots are positive and if a_4 is also positive, the curve consists of three portions, one of which extends to infinity and r_1 is its least distance from the origin, a second portion is limited by the circumferences of which r_2 and r_3 are the radii, and the third portion passes through the origin at each extremity, and r_4 is its greatest radius vector; if a_4 is negative, the curve consists of two portions, one of which is limited by the circumferences of which r_1 and r_2 are the radii, and the other by the circumferences of which r_3 and r_4 are the radii.

When a_4 is positive, the following notation may be adopted.

$$r_1 - r_3 = A \tan \tfrac{1}{2} \varepsilon \tan \tfrac{1}{2} \eta,$$
$$r_1 - r_2 = A \cot \tfrac{1}{2} \varepsilon \tan \tfrac{1}{2} \eta_1,$$
$$r_2 - r_4 = A \tan \tfrac{1}{2} \varepsilon \cot \tfrac{1}{2} \eta,$$
$$r_3 - r_4 = A \cot \tfrac{1}{2} \varepsilon \cot \tfrac{1}{2} \eta_1,$$
$$i = \tfrac{1}{2} \pi - \varepsilon;$$

which give

$$\tan^2 \tfrac{1}{2} \varepsilon = \frac{\tan \eta}{\tan \eta_1},$$

or
$$\cos \varepsilon = \frac{\sin(\eta_1 - \eta)}{\sin(\eta_1 + \eta)}.$$

For the portion of the curve, which is contained between the circumferences of which r_2 and r_3 are the radii, the notation

$$r_2 \sin \tfrac{1}{2} \eta = l \sin \tfrac{1}{2} \varkappa,$$
$$r_3 \cos \tfrac{1}{2} \eta = l \cos \tfrac{1}{2} \varkappa,$$
$$\tan^2 (\tfrac{1}{4} \pi - \tfrac{1}{2} \theta) = \frac{\tan \frac{1}{2} \eta_1 (r - r_3)}{\tan \frac{1}{2} \eta (r_2 - r)},$$

gives
$$\frac{r}{l} = \frac{\sin \frac{1}{2} (\eta_1 + \varkappa) + \sin \frac{1}{2} (\eta_1 - \varkappa) \sin \theta}{\sin \frac{1}{2} (\eta_1 + \eta) + \sin \frac{1}{2} (\eta_1 - \eta) \sin \theta}.$$

The equation of the curve is, then,

$$\frac{A(\varphi - \alpha)\sqrt{a_4}}{2 p_1^2 l \cos i} = \frac{\sin \frac{1}{2} (\eta_1 - \eta)}{\sin \frac{1}{2} (\eta_1 - \varkappa)} \mathfrak{F}_i \theta + \frac{\sin \eta_1 \sin \frac{1}{2} (\eta_1 - \varkappa)}{\sin \frac{1}{2} (\eta_1 - \varkappa) \sin \frac{1}{2} (\eta_1 + \varkappa)} \mathfrak{P}_i \left[-\frac{\sin^2 \frac{1}{2} (\eta_1 - \varkappa)}{\sin^2 \frac{1}{2} (\eta_1 + \varkappa)}, \theta \right]$$
$$- \frac{\sin \frac{1}{2} (\eta - \varkappa) \sqrt{(\sin \eta_1 \operatorname{cosec} \varkappa)}}{\sqrt{[\sin \eta_1 \sin \varkappa - \sin^2 i \sin^2 \frac{1}{2} (\eta_1 + \varkappa)]}} \tan^{[-1]} \frac{\sqrt{(1 - \cos^2 i \tan^2 \theta)}}{\sqrt{[1 - \sin^2 i \sin^2 \frac{1}{2} (\eta_1 + \varkappa) \operatorname{cosec} \eta_1 \operatorname{cosec} \varkappa]}},$$

and the expression for the time may be obtained from this value of $(\varphi - \alpha)$ by interchanging $\varkappa$ and η and multiplying by $\frac{2 p_1^2}{l^2}$.

The nature of the motion through the space exterior to the circumference of which r_1 is radius, and within the circumference of which r_4 is radius, may be derived from equations (396_{5-15}) by changing r_3 to r_1 and r_2 to r_4 and augmenting each of the angles η and $\varkappa$ by the magnitude π.

When a_4 is negative, the following notation may be adopted,

$$r_1 - r_3 = A \tan \tfrac{1}{2} \varepsilon \tan \tfrac{1}{2} \eta,$$
$$r_2 - r_4 = A \tan \tfrac{1}{2} \varepsilon \cot \tfrac{1}{2} \eta,$$
$$r_1 - r_4 = A \cot \tfrac{1}{2} \varepsilon \tan \tfrac{1}{2} \eta_1,$$
$$r_2 - r_3 = A \cot \tfrac{1}{2} \varepsilon \cot \tfrac{1}{2} \eta_1,$$
$$i = \tfrac{1}{2} \pi - \tfrac{1}{2} \varepsilon.$$

The nature of the motion between the circumferences of which

r_3 and r_4 are the radii may, then, be expressed by the equations (396_{5-15}), provided that r_3 is changed to r_4, and r_2 to r_3 and the sign of a_4 is reversed. The character of the motion between the circumferences of which r_1 and r_2 are the radii, may be expressed by the same equations with the change of r_1 to r_2, and of r_2 to r_4, the reversal of the sign of a_4, and the increase of each of the angles η and $\varkappa$ by π.

The elliptic integrals disappear when two of the roots are equal; in this case, if r_1 denotes one of the equal roots, and if R^2 is the quotient of the division of the first number of (389_{29}) by $(r-r_1)^2$ so that the form of R^2 is

$$R^2 = h_2 r^2 + h_1 r + h,$$

the notation may be adopted

$$R_1^2 = h_2 r_1^2 + h_1 r_1 + h,$$
$$2h + h_1 r = 2R\sqrt{-h}\tan(\vartheta\sqrt{-h}) = -2R\sqrt{h}\ \mathrm{Tan}(\vartheta\sqrt{h}),$$
$$h_1 + 2h_2 r = 2R\sqrt{-h_2}\tan(\vartheta_2\sqrt{-h_2}) = -2R\sqrt{h_2}\ \mathrm{Tan}(\vartheta_2\sqrt{h_2}),$$
$$\frac{h_1 r_1 + (2h_2 r_1 + h_1) r}{2RR_1} = -\sqrt{-1}\tan(R_1\vartheta_1\sqrt{-1}) = \mathrm{Tan}(R_1\vartheta_1);$$

the equation of the curve is

$$\frac{r_1}{2p_1^2}(\varphi - \alpha) = \vartheta + \vartheta_1,$$

and the expression of the time is

$$t - \tau = \vartheta_2 + r_1\vartheta_1.$$

When a_4 vanishes, if a_3 is positive, the notation may be adopted

$$r_1 - r_2 = B^2 \tan^2 \tfrac{1}{2}\varepsilon,$$
$$r_1 - r_3 = B^2 \cot^2 \tfrac{1}{2}\varepsilon,$$
$$r_2 \cos^2 \tfrac{1}{2}\varepsilon = l \cos^2 \tfrac{1}{2}\varkappa,$$
$$r_3 \sin^2 \tfrac{1}{2}\varepsilon = l \sin^2 \tfrac{1}{2}\varkappa,$$
$$i = \tfrac{1}{2}\pi - \varepsilon;$$

and for the portion of the curve contained between the circumferences of which r_2 and r_3 are the radii,

$$\tan^2(\tfrac{1}{4}\pi - \tfrac{1}{2}\vartheta) = \tan^2 \tfrac{1}{2}\varepsilon \frac{r - r_1}{r_2 - r}.$$

The equation of this portion of the curve is, then,

$$\frac{(\varphi - \alpha)\sqrt{a_3}}{2 p_1^2 l \cos i} = \frac{\cos\varepsilon}{\cos\varkappa} \mathfrak{F}_i \vartheta + \left(1 - \frac{\cos\varepsilon}{\cos\varkappa}\right) \mathfrak{P}_i(-\cos^2\varkappa, \vartheta)$$
$$+ \frac{\cos\varkappa - \cos\varepsilon}{\sin\varkappa\sqrt{(\cos^2 i - \cos^2\varkappa)}} \tan^{[-1]} \sqrt{\left(\frac{1 + \cos^2 i \tan^2\vartheta}{\cos^2 i - \sin^2 i \cot^2\varkappa}\right)},$$

and the expression of the time is obtained from this value of $(\varphi - \alpha)$ by interchanging ε and $\varkappa$ and multiplying by $\frac{2 p_1^2}{l^2}$.

Upon the portion of the curve exterior to the circumference, of which r_1 is radius, the notation

$$r - r_1 = B^2 \tan^2(\tfrac{1}{4}\pi - \tfrac{1}{2}\vartheta) = \frac{1 - \sin\theta}{1 + \sin\theta} B^2,$$

gives for the equation of the curve

$$\frac{(\varphi - \alpha)\sqrt{a_3}}{2 p_1^2 \cos i} = \frac{\mathfrak{F}_i\vartheta}{r_1 - B^2} - \frac{2B^2}{(r_1 - B^2)(r_1 + B^2)} \mathfrak{P}_i\left[-\left(\frac{r_1 - B^2}{r_1 + B^2}\right)^2, \vartheta\right]$$
$$+ \frac{B}{\sqrt{[r_1(4 r_1 B^2 \cos^2 i - (r_1 - B^2)^2 \sin^2 i)]}} \tan^{[-1]} \frac{2B\sqrt{(r_1 + r_1\cos^2 i \tan^2\varphi)}}{\sqrt{[4 r_1 B^2 \cos^2 i - (r_1 - B^2)^2 \sin^2 i]}},$$

and for the expression of the time

$$(t - \tau)\sec i = (r_1 + B^2)\mathfrak{F}_i\vartheta - \frac{2B^2}{\cos^2 i}\mathfrak{E}_i\vartheta + \frac{2B^2\sin\theta}{\cos^2 i}\sqrt{(1 + \cos^2 i \tan^2\vartheta)}$$
$$+ 2B^2 \operatorname{Tan}^{[-1]}\sqrt{(1 + \cos^2 i \tan^2\vartheta)}.$$

If a_3 is negative, the notation may be adopted

$$r_1 - r_3 = B^2 \tan^2 \tfrac{1}{2}\varepsilon,$$
$$r_2 - r_3 = B^2 \cot^2 \tfrac{1}{2}\varepsilon;$$

which, combined with that obtained from (397_{27-31}) by changing r_3 into r_2 and r_2 into r_1, gives (398_7) for the equation of the portion of the curve contained between the circumferences of which r_1 and r_2 are the radii, while the expression of the time is derived by the process of (398_{13}). But with the notation obtained from (398_{16}) by changing r_1 into r_3 and reversing the sign of B^2, the equations (398_{19}) and (398_{25}) become the equation of the curve and the expression of the time, upon the portion which is contained within the circumference of which r_3 is the radius.

The form of the central force which corresponds to the discussion of this section is

$$F = b_4 r + b_3 + \frac{b_1}{r^2} + \frac{b}{r^3}.$$

692. If m is 2 in the first class of § 690, the expression of the central force is

$$F = b_4 r^5 + b_3 r^3 + b_2 r + \frac{b}{r^3},$$

and the forms of the equation of the curve are obtained from those of § 691 by changing r into r^2, and $(\varphi - \alpha)$ into $2(\varphi - \alpha)$. But the expressions of the time require, moreover, the substitution for (390_{26}) of

$$\frac{2 A_1 \sqrt{a_4}}{p-q}(t-\tau) = \mathfrak{F}_i \theta,$$

for (391_{14}) of

$$t - \tau = \sqrt{\left(\frac{4 a a_3}{a_1(16 a a_4 - a_1 a_3)}\right)} \tan^{[-1]} \frac{4 a_4 r^2 + a_3}{\sqrt{[a_3(16 a a_4 - a_1 a_3)]}},$$

for (392_{23}) of

$$(t-\tau)\sqrt{a_4} = \tfrac{1}{2} \mathfrak{F}_i \theta,$$

for (393_9) of

$$(t-\tau)\sqrt{-a_4} = \tfrac{1}{2} \mathfrak{F}_i \theta,$$

for (393_{23}) of

$$(t-\tau)\sqrt{a_4}=\frac{1}{2R_1}\,\mathrm{Tan}^{[-1]}\frac{(h\cos\alpha-r_1^2)\,r^2+(h\cos\alpha-r)\,r_1^2}{R\,R_1},$$

for (394_{12}) of

$$(t-\tau)\sqrt{a_3}=\tfrac{1}{2}\mathfrak{F}_i\,\theta,$$

for (394_{25}) of

$$(t-\tau)\sqrt{-a_3}=\tfrac{1}{2}\mathfrak{F}_i\,\theta,$$

for (396_{18}) with the form of (396_{23}) of

$$(t-\tau)\sqrt{a_4}=\tfrac{1}{2}\cos i\,\mathfrak{F}_i\,\theta,$$

for (397_5) of

$$(t-\tau)\sqrt{(-a_4)}=\tfrac{1}{2}\cos i\,\mathfrak{F}_i\,\theta,$$

for (397_{24}) of

$$t-\tau=\tfrac{1}{2}\,\theta,$$

for (398_{11}) and (398_{25}) of

$$(t-\tau)\,a_3=\tfrac{1}{2}\cos i\,\mathfrak{F}_i\,\theta,$$

and for (399_{2-6}) of

$$(t-\tau)\sqrt{-a_3}=\tfrac{1}{2}\cos i\,\mathfrak{F}_i\,\theta.$$

693. In the special case of § 692, in which F is reduced to its first term, so that

$$F=b_4\,r^5,$$

two of the roots of (389_{29}) are real and two are imaginary, so that the only portion of § 691, which is applicable to this case, is from (391_{15}) to (393_{23}). In this case, moreover, one of the real roots is positive and the other is negative if b_4 is positive, so that the curve extends to infinity; but if b_4 is negative, both of the real roots must be positive, so that the circumferences which correspond to these roots are the limits of the curve, and Ω_0 is negative and satisfies the condition

$$-\Omega_0>\tfrac{4}{3}p_1^3\sqrt[4]{(-2\,b_4)}.$$

694. In the special case of § 692, in which F is reduced to its second term, so that

$$F = b_3 r^3,$$

the equation (389_{29}) has no imaginary roots of r^2 when

$$\frac{4\,\Omega_0^3}{b_3} > 27\,p_1^8.$$

When b_3 is positive, there is only one real root, so that the curve extends to infinity from the circumference, which is defined by this root. When b_3 is negative, all the roots must be real, and the two roots, which are positive, define the circumferences which limit the extent of the curve.

695. If m is $\frac{2}{3}$ in the first class of § 690, the expression of the central force is

$$F = b_4 r^{-\frac{1}{3}} + b_2 r^{-\frac{5}{3}} + b_1 r^{-\frac{7}{3}} + b\, r^{-3},$$

and the forms of the equation of the curve are obtained from those of § 691 by changing r into $r^{\frac{2}{3}}$, and $\varphi - \alpha$ into $\frac{2}{3}(\varphi - \alpha)$. But the formulæ for the time are more complicated, although they are still reducible to elliptic integrals. If, indeed,

$$z = r^{\frac{2}{3}},$$

the expression for the time assumes the form

$$t - \tau = \int_z \frac{\frac{3}{2} z^2}{\sqrt{[a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a]}}.$$

696. In the special case of § 695, in which F is reduced to its first term, so that

$$F = b_4 r^{-\frac{1}{3}},$$

the conditions of the form of the curve are the same with those expressed in (400_{23-30}), but instead of (400_{31}), the limitation of Ω_0 when b_4 is negative is

$$-\Omega_0 > 2 p_1 \sqrt[4]{(-4 b_4)^3}.$$

697. In the special case of § 695, in which F is reduced to its second term, so that

$$F = b_2 r^{-\frac{5}{3}},$$

the equation (389_{29}) has no imaginary roots of $\sqrt[3]{r^2}$, when

$$b_2^3 < -4 p_1^4 \Omega_0^2.$$

In the special case, in which F is reduced to its third term, so that

$$F = b_1 r^{-\frac{7}{3}},$$

the equation (389_{29}) has no imaginary roots, when

$$64 p_1^8 < -\frac{b_1^3}{\Omega_0}.$$

In each of these cases, when Ω_0 is negative, there is only one real positive root, so that the curve extends to infinity from the circumference which is defined by this root. When Ω_0 is positive all the roots must be real, and the two roots, which are positive, define the circumferences, which limit the extent of the curve.

698. If m is $\frac{1}{2}$ in the first class of § 690, the expression of the central force is

$$F = b_3 r^{-\frac{3}{2}} + b_2 r^{-2} + b_1 r^{-\frac{5}{2}} + b\, r^{-3},$$

and the forms of the equation of the curve are obtained from those of § 691 by changing r into $\sqrt{r}$ and $(\varphi - \alpha)$ into $\frac{1}{2}(\varphi - \alpha)$. But if

$$z = \sqrt{r},$$

the expression of the time assumes the form

$$t - \tau = \int_z \frac{2 z^3}{\sqrt{(a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a)}}.$$

699. In the special case of § 698, in which F is reduced to its first term, so that

$$F = b_3 r^{-\frac{3}{2}},$$

and in that, in which it is reduced to its third term, so that

$$F = b_1 r^{-\frac{5}{2}},$$

two of the roots of (389_{29}) are real for $\sqrt{r}$, and two are imaginary, so that the only portion of § 691, which is applicable to this case, is from (391_{15}) to (393_{24}). In this case, moreover, one of the real roots is positive and the other is negative if Ω_0 is negative, so that the curve extends to infinity; but if Ω_0 is positive, both of the real roots must be positive, so that the circumferences, which correspond to these roots, are the limits of the curve, and in the former of these cases b_3 is negative and

$$-b_3 > \tfrac{4}{3} p_1 \sqrt[4]{(6 \Omega_0^3)},$$

while in the latter case b_1 is negative and

$$-b_1 > 4 p_1^3 \sqrt[4]{(\tfrac{3}{2} \Omega_0)}.$$

700. In the second class of § 690, when m is unity, the equation (378_{22}) of the curve assumes the form

$$\varphi - \alpha = \int_r \frac{2 p_1^2}{\sqrt{(a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a)}},$$

so that it can always be obtained from the expressions of $(t - \tau)$ in § 692, by multiplying either of those expressions by $4 p_1^2$. When, in this class, the curve terminates in the origin, it does not usually pass through the condensed coil of § 691. The formula for the time is

$$t - \tau = \int_r \frac{r^2}{\sqrt{(a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a)}}.$$

The form of the force, which corresponds to this case, is

$$F = b_3 r^{-2} + b_2 r^{-3} + b_1 r^{-4} + b\, r^{-5}.$$

701. In the special case of § 700, in which F is reduced to its third term, so that

$$F = \frac{b_1}{r^4},$$

one of the roots of (389_{29}) is zero, and the condition that all the roots are real is

$$-\frac{4}{3\,\Omega_0} > \frac{2\,b_1^2}{p_1^{12}}.$$

When Ω_0 is negative, if b_1 is positive, the curve extends to infinity, in the space exterior to the circumference of which the positive root of (389_{29}) is the radius; but if b_1 is negative, the curve extends from the origin to infinity, if two of the roots are imaginary, but if all the roots are real, one portion is exterior to the circumference of which the greater positive root is radius and extends to infinity, while the other portion is contained within the circumference of which the smaller positive root is the radius, and this portion passes through the origin. When Ω_0 is positive, b_1 is negative, and the curve passes through the origin, and is contained within the circumference of which the positive root of (389_{29}) is the radius. This case of force has been analyzed by Stader.

702. In the special case of § 700, in which F is reduced to its last term, so that

$$F = \frac{b}{r^5},$$

all the roots of (389_{29}) are imaginary when Ω_0 and b are both positive. When Ω_0 is positive, therefore, b must be negative and the curve is contained within the circumference of which the positive root of (389_{29}) is the radius. When Ω_0 is negative, if b is positive the curve extends to infinity in the space exterior to the circumference of which the positive root is radius; but if b is negative, the curve consists of two portions, one of which extends to infinity in the space exterior to the circumference of which the greater real root is radius, while the other portion passes through the origin and is contained within the circumference of which the smaller root is radius; or it extends from the origin to infinity.

703. When m is 2 in the second class of § 690, the form of the force is

$$F = b_4 r + b_2 r^{-3} + b_1 r^{-5} + b r^{-7},$$

and the equation of the curve can be obtained in each case from that of § 692, by multiplying $(t - \tau)$ by $2 p_1^2$, and changing $t - \tau$ into $\varphi - \alpha$, and r into r^2.

If

$$z = r^2,$$

the formula for the time is

$$t - \tau = \int_z \frac{z^2}{\sqrt{[a_4 z^4 + a_3 z^3 + a_2 z_2 + a_1 z + a]}}.$$

704. In the special case of

$$F = \frac{b}{r^7},$$

there are two imaginary roots of r^2 when

$$\frac{1}{b} > - \frac{9\,\Omega_0^2}{64 p_1^{12}}.$$

When Ω_0 is negative, if b is positive the curve extends to infinity in the space exterior to the circumference of which the real root of (389_{29}) is the radius; but if b is negative, and if all the roots of (389_{29}) are also real and two of them positive, the curve consists of two portions, one of which extends to infinity in the space exterior to the circumference of which the greater positive root is radius, while the other portion passes through the origin and is contained within the circumference of which the smaller positive root is radius; but if neither of the roots is positive when b and Ω_0 are both negative, the curve consists of a single portion which extends from the origin to infinity. When Ω_0 is positive, b must be negative and the curve consists of a single portion which passes through the origin and is contained within the circumference of which the positive root is radius. This law of force has been analyzed by Stader.

705. Another class of central force, in which the integration can be performed by elliptic integrals, corresponds to the form of the potential

$$\Omega = \frac{b_4 r^{4m} + b_3 r^{3m} + b_2 r^{2m} + b_1 r^m + b}{r^2 (r^m + h)^2},$$

in which m may be either 1 or 2. If, in these forms

$$\begin{aligned} z &= r^m, \\ Z^2 &= a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a \\ &= (2 b_4 r^{4m} + 2 b_3 r^{3m} + 2 b_2 r^{2m} + 2 b_1 r + 2 b) \\ &\quad - (2 \Omega_0 r^2 + 4 p_1^4)(r^m + h)^2, \end{aligned}$$

the equation of the curve assumes the form

$$\frac{m}{2 p_1^2}(\varphi - \alpha) = \int_z \frac{z+h}{z Z};$$

and the expression of the time is

$$m(t - \tau) = \int_z \frac{(z+h) z^{\frac{2}{m}-1}}{Z}.$$

706. The following graphic construction gives an easy geometrical process for tracing the various cases of limitation of the extent of the path described under the action of a central force, and especially for finding by inspection the effect of the values of Ω_0 and p_1 upon the limits of the curve. If

$$x = \frac{1}{r^2},$$

construct the curve of which the equation is

$$y = \Omega,$$

which may be called *the potential curve*, draw the straight line of which the equation is

$$y = 2 p_1^4 x + \Omega_0,$$

and the points of intersection of the straight line with the potential curve give the values of x for the limits of the path of the body. The path corresponds to those portions of the potential curve which lie upon that side of the straight line, which is positive with respect to the direction of the axis of y.

707. A term of Ω may be omitted in the preceding construction which is inversely proportional to the square of the radius vector, and its negative may be combined with that term of the equation of the straight line which determines its direction. The omitted term corresponds to a term of the force which is inversely proportional to the cube of the radius vector, and which may be represented by (383_{17}); and the corresponding equation of the straight line is

$$y = (2 p_1^4 + b)\, x + \Omega_0.$$

708. It is evident from the preceding construction, that *if the potential curve has no point of contrary flexure, and if its convexity is turned*

in the direction of the positive axis of y, *the path of the body can only consist of a single portion which may have either an outer or an inner limit, or it may have neither or both.* This case includes all forces of the form

$$F = b_1 r^m + \frac{b}{r^3},$$

in which b_1 and $m + 3$ have the same sign.

But if the potential curve has no point of contrary flexure, and if its convexity is turned in the direction of the negative axis of y, *the path of the body may consist of a single portion which has either an outer or an inner limit, or it may have neither, or it may consist of two separate portions of which one has only an outer and the other only an inner limit.* This case includes all forces of the form (408_5), in which b_1 and $m + 3$ have different signs.

709. Those portions of the potential curve, in which y and x simultaneously increase, correspond to the distances from the centre of action, at which the force is attractive, so that the convexity of the path of the body is turned away from the origin. The portions of the potential curve, in which y decreases with the increase of x, correspond to the distances from the centre of action, at which the force is repulsive, so that the convexity of the path of the body is turned towards the origin. Any point, therefore, at which the potential curve is parallel to the axis of x, and the ordinate is either a maximum or a minimum, corresponds to a distance from the origin, at which the central force changes from attraction to repulsion, and the path of the body has a point of contrary flexure.

710. If for an infinitesimal value of r denoted by i, Ω assumes the form

$$\Omega = k\, i^n,$$

the path of the body cannot pass through the origin if $n + 2$ is positive or if k is negative, except in the former case, when p_1 vanishes and n is positive while Ω_0 is negative, or n is negative while k is positive; but if k is positive and $n+2$ negative, the external portion of the path passes through the origin, and after passing through the origin, the continuity of curvature is destroyed and the path becomes a straight line.

711. If for an infinite value of r, denoted by the reciprocal of i, Ω assumes the form (408_{28}), the path of the body cannot extend to infinity when n and k are both negative, or when n and Ω_0 are both positive, or when n vanishes and

$$\Omega_0 > k;$$

but the external portion of the path extends to infinity when n is negative and k positive, or when n is positive and Ω_0 negative, or when n vanishes and

$$\Omega_0 < k.$$

712. If a line is drawn parallel to the axis of x at the distance Ω_0 from this axis, and assumed as a new axis of x_1, and if y_1 and y_2 are the corresponding ordinates, respectively, of the straight line (407_{14}) and of the potential curve, the value of the angle, which the path of the body makes with the radius vector, is given by the equation

$$\sin {}^{r}_{s} = \sqrt{\frac{y_1}{y_2}} = \frac{\sqrt{(y_1 y_2)}}{y_2},$$

which admits of simple geometrical construction. If z_2 denotes the subtangent of the potential curve upon the axis of x_1, the projection of the radius of curvature of the path of the body upon its radius vector is

$$\varrho \sin {}^{r}_{s} = \frac{r z_2}{x},$$

which is constructed without difficulty. By the combination of these two constructions, the path of the body may be obtained with sufficient exactness for most purposes of general discussion.

713. When the origin is infinitely remote from the body, *the forces of* § 676 *are parallel, and the plane of motion is parallel to the direction of action,* and the equation (378_5) gives, if the axis of z is supposed to have the same direction with the force,

$$\frac{D_z \Omega}{\Omega - \Omega_0} = -\frac{2}{\varrho \sin \frac{z}{s}} = -2 \cot \tfrac{z}{s} D_z \tfrac{z}{s},$$

of which the integral is

$$\Omega - \Omega_0 = \frac{a}{\sin^2 \frac{z}{s}},$$

in which a is an arbitrary constant, which is always positive, and *this is the equation of the path of the body referred to the same coördinates with those of* § 571.

714. *In the case of a constant force,* the preceding equation assumes the forms

$$g(z - z_0) = \frac{a}{\sin^2 \frac{z}{s}},$$

$$\varrho = \frac{2a}{g \sin^3 \frac{s}{z}};$$

so that, in this case, *the path is a parabola.*

715. The velocity in the direction of the axis of x is

$$v \sin \tfrac{z}{s} = \sin \tfrac{z}{s} \sqrt{(2\Omega - 2\Omega_0)} = \sqrt{(2a)},$$

so that *this velocity is constant,* and

$$x - x_0 = \sqrt{(2a)}\,(t - \tau).$$

The equation of the curve, expressed in rectangular coördinates, is

$$x - x_0 = \int_z \left(\frac{\Omega - \Omega_0}{a} - 1\right)^{-\frac{1}{2}}.$$

716. If a potential curve is constructed by the equation (407_{10}), in which y may be changed into x, and Ω retained as a function of z, the limits of the path of the body are defined by the intersection of the potential curve with a line drawn parallel to the axis of z at the distance $(\Omega_0 + a)$ from this axis. The portions of the potential curve which correspond to the path, lie in a positive direction from the intersecting line.

717. If the force of § 713 has the form

$$F = b_1 z + b,$$

the equation of the path is

$$b_1 z + b = \sqrt{(b^2 + 2 b_1 \Omega_0 + 2 b_1 a)} \sin\left[(x_0 - x)\sqrt{\left(-\frac{b_1}{2a}\right)}\right]$$

$$= \sqrt{(b^2 + 2 b_1 \Omega_0 + 2 b_1 a)} \operatorname{Cos}\left[(x - x_0)\sqrt{\frac{b_1}{2a}}\right]$$

$$= \sqrt{-(b^2 + b_1 \Omega_0 + 2 b_1 a)} \operatorname{Sin}\left[(x_0 - x)\sqrt{\frac{b_1}{2a}}\right].$$

718. If the force of § 713 has such a form that

$$\Omega_0 = \frac{b_1 z + b}{(z+h)^2},$$

the notation

$$b_1 = 2(k_1 + h)(\Omega_0 + a),$$
$$b = (k^2 + h^2)(\Omega_0 + a),$$

gives, for the equation of the path,

$$\frac{k_1 - z}{\sqrt{(k^2 + k_1^2)}} = \sin \frac{(x_0 - x)\sqrt{(\Omega_0 + a)} - \sqrt{(k^2 + 2 k_1 z - z^2)}}{k_1 + h},$$

which is easily transformed into the forms, which are appropriate when the radicals become imaginary.

719. *In the case of a surface of revolution, and a force which is*

directed to a point upon the axis of revolution, the notation of § 576 gives

$$\frac{A}{v} = u \sin \tfrac{\sigma}{s} = u^2 D_s \tfrac{u}{x},$$

$$A = u\, v \sin \tfrac{\sigma}{s} = u^2 D_t \tfrac{u}{x},$$

so that *the elementary area described by the projection of the radius vector upon the plane of* x y *is constant.*

720. The notation of § 578 gives

$$D_\phi s = \sqrt{\frac{2\, u^2 (\Omega - \Omega_0)\, [r^2 + (D_\phi r)^2]}{2\, u^2 (\Omega - \Omega_0) - A^2}},$$

$$D_\phi \tfrac{u}{x} = \frac{A}{u} \sqrt{\frac{r^2 + (D_\phi r)^2}{2\, u\, (\Omega - \Omega_0) - A^2}},$$

and, in the case of parallel forces

$$D_z s = D_z \sigma \sqrt{\frac{2\, u^2 (\Omega - \Omega_0)}{2\, u^2 (\Omega - \Omega_0) - A^2}},$$

$$D_z \tfrac{u}{x} = \frac{A\, D_z \sigma}{u \sqrt{[2\, u^2 (\Omega - \Omega_0) - A^2]}}.$$

721. Upon the surface of revolution which is defined by the equation

$$u\, v = B,$$

the path of the body makes a constant angle with the meridian curve. In the case of

$$B = A,$$

the path is perpendicular to the meridian, and is *a circle of which the plane is horizontal.*

Whatever is the value of B, for the point at which v vanishes u is infinite, while v is infinite when u vanishes.

Upon any other surface of revolution about the same axis, the inclination of the path of the body to the meridian arc is the same with *the corresponding inclination upon the surface of equation* (412_{21}) *at the common circle of intersection of these two surfaces. Hence the limits of the path upon the given surface of revolution are its intersections with the surface of equation*

$$u\,v = A,$$

and the path extends over that portion of the given surface, which is exterior to this surface by which the limits are defined.

722. In the case of a heavy body the equation (412_{21}) becomes

$$u^2\,z = \frac{B^2}{2\,g}.$$

723. *In the case of a heavy body upon a vertical right cone, if the body moves upon the inverted part of the cone, the path has an upper and a lower limit; but if it moves upon the part, which is below the vertex, the path has an upper limit from which it extends downwards to infinity.* In this case, if the notation of (341_{13}) and (341_{16}) is adopted, if two of the roots of the equation

$$r^2\,(r - r_0) = \frac{A^2}{2\,g\sin^2\alpha\cos\alpha},$$

are imaginary, which corresponds to

$$(-\tfrac{2}{3}\,r_0)^3 < \frac{A^2}{g\sin^2\alpha\cos\alpha},$$

if h is the modulus and β the argument of one of the imaginary roots, and if r_1 is the real root, the notation

$$r_1 - h\,e^{\beta\sqrt{-1}} = B^2\,e^{2i\sqrt{-1}},$$

$$r_1 - h\,e^{-\beta\sqrt{-1}} = B^2\,e^{-2i\sqrt{-1}},$$

$$r - r_1 = B^2\tan^2\tfrac{1}{2}\,\varphi,$$

gives for the equation of the path upon the developed cone

$$\sin^2 \alpha\,(r_1 - B^2)\frac{B(\theta-\theta_0)}{A}\sqrt{(g\cos\alpha)} = \mathfrak{F}_i\varphi - \frac{r_1+B^2}{2\,r_1}\mathfrak{P}_i\left(\frac{(r_1-B^2)^2}{4\,r_1 B^2},\varphi\right)$$

$$-\frac{2\,B}{\sqrt{(2\,r_1 - 2\,B^2\cos 2\,i)}}\tan^{[-1]}\frac{2\,B\sqrt{(\cot^2\varphi+\cos^2 i)}}{\sqrt{(2\,r_1-2\,B^2\cos 2\,i)}};$$

and the position of the body at any instant is defined by the equation

$$B\,(t-\tau)\sqrt{(2\,g\cot\alpha)} = (r_1+B^2)\,\mathfrak{F}_i\varphi - 2\,B^2\,\mathfrak{E}_i\varphi$$
$$+\,2\,B^2\tan\tfrac{1}{2}\varphi\sqrt{(1-\sin^2 i\sin^2\varphi)}.$$

If all the roots of (413_{20}) are real, and denoted in the order of decreasing magnitude by r_1, r_2, and r_3, and if

$$r_1 - r_2 = B^2\tan^2\beta,$$
$$r_1 - r_3 = B^2\cot^2\beta,$$
$$i = 2\,\beta + \tfrac{1}{2}\pi,$$
$$r - r_1 = \beta^2\tan^2(\tfrac{1}{4}\pi - \tfrac{1}{2}\varphi),$$

the equation of the path upon the lower portion of the developed cone is

$$\frac{\sin^2\alpha\,B\sqrt{(g\cos a)}}{A\cos i}(\theta-\theta_0) = \frac{1}{r_1-B^2}\mathfrak{F}_i\varphi - \frac{2\,B^2}{r_1^2-B^4}\mathfrak{P}_i\left[-\left(\frac{r_1-B^2}{r_1+B^2}\right)^2,\varphi\right]$$

$$+\frac{B}{[r_1(r_1+B^2)^2\cos^2 i - 4\,r_1^2 B^2]}\tan^{[-1]}\sqrt{\frac{4\,r_1 B^2(1+\cos^2 i\tan^2\varphi)}{(r_1+B^2)^2\cos^2 i - 4\,r_1 B^2}},$$

and the position of the body at any instant is defined by the equation

$$(t-\tau)\cos i\sqrt{(2\,g\cos\alpha)} = \left(\frac{r_1}{B}+B\right)\cos^2 i\,\mathfrak{F}_i\varphi - 2\,B\,\mathfrak{E}_i\varphi$$
$$+\left(\frac{r_1}{B}+B\sin\varphi\right)\sqrt{(1+\cos^2 i\tan^2\varphi)}.$$

The equation of the path of the body upon the upper portion of the cone is determined by the combination of the equations (414_{13-16}) with

$$\sin\varphi \sin i = \frac{r_1 - r - B^2}{r_1 - r + B^2},$$

$$\frac{\sin^2\alpha\, B\sqrt{(g\cos\alpha)}}{A\cos i}(\vartheta - \vartheta_0) = -\frac{1}{r_1 + B^2}\mathfrak{F}_i\,\varphi - \frac{2B^2}{r_1^2 - B^4}\mathfrak{P}_i\left[-\left(\frac{r_1 + B^2}{r_1 - B^2}\right)^2, \varphi\right]$$

$$-\frac{B}{\sqrt{[r_1(r_1+B^2)^2\cos^2 i - 4r_1^2 B^2]}}\tan^{[-1]}\sqrt{\frac{(1+\cos^2 i\tan^2\varphi)[(r_1+B^2)^2\cos^2 i - 4r_1 B^2]}{4r_1 B^2}},$$

and the position of the body at any instant is defined by the equation

$$(t-\tau)\cos i\sqrt{(2g\cos\alpha)} = -\left(\frac{r_1}{B} + B\right)\cos^2 i\,\mathfrak{F}_i\,\varphi$$

$$+ 2B\,\mathfrak{E}_i\,\varphi - 2B\sin i\cos\varphi\sqrt{\frac{1-\sin i\sin\varphi}{1+\sin i\sin\varphi}}.$$

The path of the body upon the upper portion of the cone may be expressed in a somewhat more simple form by the equations

$$\sin^2 i = \frac{r_2 - r_3}{r_1 - r_3},$$

$$r = r_2\sin^2\varphi + r_3\cos^2\varphi,$$

$$\sin\alpha\sqrt{(2g\cos\alpha)}(\vartheta - \vartheta_0 = \frac{2A}{r_3\sqrt{r_1 - r_3}}\mathfrak{P}_i\left(\frac{r_2 - r_3}{r_3}, \varphi\right),$$

and the corresponding formula for the position of the body at any instant is

$$\sqrt{(2g\cos\alpha)}(t-\tau) = \frac{2r_1\mathfrak{F}_i\,\varphi}{\sqrt{(r_1 - r_3)}} - 2\sqrt{(r_1 - r_3)}\,\mathfrak{E}_i\,\varphi.$$

In the special case, in which the roots r_2 and r_3 are equal, the path upon the upper portion is a horizontal circle, and the equation of the path upon the lower portion is

$$\tfrac{1}{2}(\vartheta - \vartheta_0) = \tan^{[-1]}\sqrt{\left(-\tfrac{1}{3} - \frac{r}{r_0}\right)} - \sqrt{3}\tan^{[-1]}\sqrt{\left(-1 - \frac{3r}{r_0}\right)},$$

while the position of the point at any instant is defined by the equation

$$\sqrt{(2g\cos\alpha)}\,(t-\tau)=2\sqrt{(r+\tfrac{1}{3}r_0)}+\tfrac{4}{3}\sqrt{(-r_0)}\tan^{[-1]}\sqrt{\left(-\tfrac{1}{3}-\frac{r}{r_0}\right)}.$$

724. *In the case of a heavy body upon the surface of a vertical paraboloid of revolution, of which the axis is directed downwards, the path has an upper limit, from which it proceeds downwards to infinity.* If (336_{25}) is the equation of the paraboloid, and if z_1 and $-q$ are the roots of the equation

$$8pgz(z-z_0)=A^2,$$

the path of the body when

$$p<q,$$

is defined by the equations

$$z-z_1=(z_1+p)\tan^2\varphi,$$
$$q-p=(q+z_1)\sin^2 i,$$
$$\frac{u}{x}=\frac{A\cos i}{2pz_1}\sqrt{\frac{q+z_1}{2g}}\ \mathfrak{P}_i\left(\frac{p}{z_1},\varphi\right),$$

and the position of the body at any instant is given by the equation

$$\tfrac{1}{2}\cos i\,(t-\tau)\sqrt{\frac{2g}{q+z_1}}=\cos^2 i\,\mathfrak{F}_i\varphi-\mathfrak{E}_i\varphi+\sqrt{(\cos^2 i\tan^2\varphi+\sin^2 i\sin^2\varphi)}.$$

But when

$$p>q,$$

the path is defined by the equations

$$z-z_1=(z_1+p)\cot^2\varphi,$$
$$p-q=(z_1+p)\sin^2 i,$$
$$\frac{u}{x}=\frac{A}{2p\sqrt{[2g(z_1+p)]}}\ \mathfrak{P}_i\left(-\frac{p}{z_1+p},\varphi\right),$$

and the position of the body at any instant is given by the equation

$$(t-\tau)\sqrt{[\tfrac{1}{2}g(z_1+p)]} = \mathfrak{F}_i\,\varphi - \mathfrak{E}_i\,\varphi - \sqrt{(\cot^2\varphi - \sin^2 i \cos^2\varphi)}.$$

In the especial case of

$$p = q,$$

the path of the body is the parabola, which is formed by the intersection of the paraboloid with the vertical plane, of which the equation is

$$u \cos {}^{u}_{x} = \sqrt{(4\,p^2 + u_0^2)},$$

and the position of the body at any instant is defined by the equation

$$\frac{(t-\tau)\sqrt{(p\,g)}}{\sqrt{(8\,p^2+2\,u_0^2)}} = \tan {}^{u}_{x}.$$

725. *In the case of a heavy body upon the surface of a vertical paraboloid, of which the axis is directed upward, the path has an upper and a lower limit.* If p is negative, (336_{25}) is the equation of this paraboloid, and if $-z_1$ and $-z_2$ are the roots of the equation (416_{11}), they correspond to the limits of the path. The path of the body is defined by the formulæ

$$z = -z_1\cos^2\varphi - z_2\sin^2\varphi,$$

$$(z_2 - p)\sin^2 i = z_2 - z_1,$$

$${}^{u}_{x} = p\sqrt{\frac{p\,z}{z_2\,(p-z_2)}}\,\mathfrak{P}_i\left(-\frac{z_2-z_1}{z_2},\varphi\right) - \sqrt{\frac{p\,z_1 z_2}{p-z_2}}\,\mathfrak{F}_i\,\varphi\,;$$

and the time is given by the equation

$$t - \tau = \sqrt{\frac{z\,(z_1-p)}{g}}\,\mathfrak{F}_i\,\varphi.$$

THE SPHERICAL PENDULUM.

726. When the surface upon which the body moves is that of a sphere, the problem becomes that of *the spherical pendulum*. In this case, the path has an upper and a lower limit. If the centre of the sphere is the origin, if R is the radius of the sphere, the limits of the path correspond to the roots of the equation

$$2g(R^2 - z^2)(z - z_0) - A^2 = 0.$$

If the roots of this equation are z_1, z_2, and $-p$, and if the notation is adopted

$$z = z_1 \cos^2 \varphi + z_2 \sin^2 \varphi,$$
$$(p + z_1) \sin^2 i = z_1 - z_2,$$

the path is defined by the formula

$$\tfrac{u}{x}\sqrt{[2g(p+z_1)]} = \frac{A}{R - z_1} \mathfrak{P}_i\left(\frac{z_1 - z_2}{R - z_1}, \varphi\right) + \frac{A}{R + z_1} \mathfrak{P}_i\left(\frac{z_2 - z_1}{R + z_2}, \varphi\right),$$

and the time by the formula

$$t\sqrt{(2g)} = \frac{2R}{\sqrt{(p + z_1)}} \mathfrak{F}_i \varphi.$$

727. From the equation (418_{10}), it is easily inferred that

$$(z_1 + z_2)(p^2 - R^2) = \frac{A^2}{2g},$$
$$z_1 z_2 + R^2 = p(z_1 + z_2),$$
$$z_0 = z_1 + z_2 - p,$$

that the sum of z_1 and z_2 is always positive, and that p exceeds R.

728. It is apparent from the inspection of (418_{21}) that, if the mutual ratios of R_1 and the roots of (418_{10}) are unchanged, the

time of oscillation of the pendulum is proportional to the square root of its length.

729. If the length of the pendulum and the sum of z_1 and p are given, it is evident from (418_{21}) that the time of oscillation increases with the increase of i, and is a minimum when i vanishes, that is, when

$$z_1 = z_2,$$

in which case the path of the pendulum is a horizontal circle. The time of oscillation in this case is

$$T = \frac{2\pi R}{\sqrt{[2g(p+z_1)]}}.$$

The mutual relation of p and z_1, which is here given by the equation (418_{26}), is

$$2p = z_1 + \frac{R^2}{z_1},$$

whence

$$\frac{2\pi R}{T\sqrt{g}} = \sqrt{\left(3z_1 + \frac{R^2}{z_1}\right)}.$$

This value is a minimum, when

$$z_1\sqrt{3} = R,$$

in which case

$$T = \pi\sqrt{\frac{2R}{g\sqrt{3}}} = \pi\sqrt{\frac{2z_1}{g}},$$

which is, therefore, the greatest time of vibration *when the path of the pendulum is a horizontal circle.*

It is easy to see that i cannot vanish for all values of the sum of p and z_1, but that its least value is determined by the equation

$$\sin^2 2i = 4 - \frac{12R^2}{(p+z_1)^2},$$

whenever

$$4\,R^2 > (p + z_1)^2 > 3\,R^2.$$

It is also evident that the least value of the sum of p and z_1 which corresponds to any assumed value of i is given by (419_{30}), so that *for any value of* i, *the greatest time of vibration is*

$$T = \sqrt{\frac{R}{g}}\sqrt[4]{\left(\frac{7 + \cos 4\,i}{6}\right)}\,\mathfrak{F}_i\,(\tfrac{1}{2}\,\pi),$$

which increases with i, *and is infinite when* i *becomes a right angle.*

When i is an octant, the value of $p + z_1$ in (419_{30}) is a maximum, and the corresponding values of $p + z_1$ and T are

$$p + z_1 = 2\,R$$
$$T = \sqrt{\frac{R}{g}}\,\mathfrak{F}_{\frac{1}{4}\pi}\,(\tfrac{1}{2}\,\pi).$$

730. In the discussion of the form of the path of the pendulum, it is convenient to adopt the notation

$$U_\phi = \tfrac{u}{x}.$$

In the case of (419_7), the equations of § 726 and 727 give

$$\frac{A^2}{2\,g} = 2\,z_1\,(p^2 - R^2) = \frac{(R^2 - z_1^2)^2}{2\,z_1},$$
$$U_\pi = \frac{2\,\pi\,R}{\sqrt{(3\,z_1^2 + R^2)}}.$$

When z_1 vanishes

$$U_\pi = 2\,\pi,$$

and T is the time of a complete revolution. When

$$z_1 = R,$$
$$U_\pi = \pi,$$

and T is the time of a semi-revolution. *The time of a complete revolution, when the pendulum moves in a horizontal circle is*

$$T_1 = 2\pi \sqrt{\frac{z_1}{g}},$$

so that it is proportional to the square root of the distance of the plane of revolution from the centre of the sphere.

731. *When the path of the pendulum deviates slightly from a horizontal circle,* so that i is very small, the notation

$$z_1 + z_2 = 2\, z_3 = 2\, R \cos \theta_3,$$

gives

$$z_1 - z_2 = (p + z_3)\, i^2 = \frac{R^2 + 3\, z_3^2}{2\, z_3}\, i^2,$$

$$\frac{A^2}{2\, g} = \frac{R^2 - z_3^2}{2\, z^3},$$

$$z = z_3 + \frac{R^2 + 3\, z_3^2}{2\, z_3}\, i^2 \cos 2\, \varphi,$$

$$U_\pi = \frac{2\, \pi\, R}{\sqrt{(R^2 + 3\, z_3^2)}}\, (1 - \tfrac{1}{2}\, i^2) = \frac{2\, \pi}{\sqrt{(1 + 3 \cos^2 \theta_3)}}\, (1 - \tfrac{1}{2}\, i^2).$$

732. When the path of the pendulum deviates slightly from a great circle, so that the sum of z_1 and z_2 is small, p is large and i is small, the formulæ become, by neglecting the fourth and higher powers of i

$$p = \frac{z_1 - z_2}{i^2}\, (1 - \tfrac{1}{2}\, i^2),$$

$$p + z_1 = \frac{z_1 - z_2}{i^2},$$

$$\frac{A^2}{2\, g} = (z_1 - z_2)\, [R^2 - (z_1 - z_2)^2]\left(\frac{1}{i^2} - \tfrac{1}{2}\right),$$

$$z = \tfrac{1}{2}\, (z_1 - z_2) \cos 2\, \varphi + \frac{R^2 - (z_1 - z_2)^2}{4\, (z_1 - z_2)}\, i^2,$$

$$U_\pi = 2\, \pi\, ;$$

so that *the vibration corresponds to a complete revolution of the pendulum.*

733. When the pendulum passes very near the lower point of the sphere, so that z_1 differs but little from R, the neglect of this difference and its higher powers gives

$$z_2 = R \cos 2\,i,$$
$$p = R + \tan^2 i\,(R - z_1),$$
$$\frac{R^2}{2\,g\,h} = 4\,R^2\,(R - z_1)\sin^2 i,$$
$$z = R - 2\,R\sin^2 i \sin^2 \varphi - (R - z_1)\cos^2\varphi,$$
$$U_\pi = \pi + \left[\operatorname{cosec} i\, \mathfrak{F}_i\,(\tfrac{1}{2}\pi) - \frac{4\cos 2\,i}{\sin^2 2\,i}\,\mathfrak{E}_i\,(\tfrac{1}{2}\pi)\right]\sqrt{\left(2 - \frac{2\,z_1}{R}\right)};$$

so that *the vibration corresponds to a little more than a semi-revolution of the pendulum.*

734. *In the general case the vibration of the pendulum corresponds to an arc of revolution which exceeds a semi-revolution, but is less than an entire revolution.* When the velocity at the highest point is quite small, the case of § 733 occurs, but the arc of revolution, which corresponds to a vibration, increases with the increase of velocity at the highest point. When the highest point is below the level of the centre of the sphere, the case of § 731 gives the highest limit of the velocity at this point; but when the highest point is upon or above the level of the centre, the greatest velocity extends to infinity, which limit corresponds to the case of § 732.

735. The azimuth of the pendulum at any instant, is derived from the equation of § 726 in a form suitable for computation by means of the following formulæ;

$$z = R\cos\theta,$$
$$p = R\sec\alpha = \frac{R\,(\cos\theta_1\cos\theta_2 + 1)}{\cos\theta_1 + \cos\theta_2},$$

$$\frac{A^2}{2g} = R^3 \tan^2 \alpha \, (\cos \theta_1 + \cos \theta_2) = \frac{R^3 \sin^2 \theta_1 \sin^2 \theta_2}{\cos \theta_1 + \cos \theta_2},$$

$$\cos^2 i = \frac{1 + \cos \alpha \cos \theta_2}{1 + \cos \alpha \cos \theta_1},$$

$$\sin \eta = \frac{\sin \frac{1}{2} \theta_1}{\sin \frac{1}{2} \theta_2},$$

$$\cos i \sin \eta_1 = \frac{\cos \frac{1}{2} \theta_2}{\cos \frac{1}{2} \theta_1},$$

$$\tan \mu = \frac{1 + \cos \theta_1 \cos \theta_2}{\sin \theta_1 \cos \theta_2},$$

$$\tan \mu_1 = \cos \alpha \cos \theta_2 \tan \mu = \frac{\cos \theta_1 + \cos \theta_2}{\sin \theta_1},$$

$$\cos \mu = \cos \theta_2 \cos \mu_1,$$

$$\text{co}\, i = \tfrac{1}{2} \pi - i,$$

$$\tan \lambda_1 = \frac{\sqrt{(\cot^2 \varphi - \sin^2 i \cos^2 \varphi)}}{\cos i \cos \mu_1 \sin \theta_2},$$

$$\tan \lambda_2 = \frac{\sin \mu \tan \theta_2 \tan \frac{1}{2} \theta_2 \tan \lambda}{\tan \theta_1 \cos \mu_1},$$

$$\tan \lambda_3 = \frac{\tan \lambda}{\tan^2 \eta \, (1 + \cos^2 i \tan^2 \varphi)},$$

$$\tan \lambda_4 = \frac{\tan^2 i \cos^2 \varphi \cos \theta_2 \tan \mu \tan \frac{1}{2} \theta_1 \tan \lambda}{\cos \mu \cos \eta \cos \eta_1 + (1 - \cos \mu \cos \eta \cos \eta_1) \sin^2 i \sin^2 \varphi},$$

$$U_\phi = \frac{\cos i \sin \mu \tan \theta_1}{\tan \theta_2} \left[\mathfrak{F}_i \left(-\frac{\cos^2 i \cos^2 \mu \tan^2 \theta_2}{\tan^2 \theta_1}, \varphi \right) - \mathfrak{F}_i \, \varphi \right]$$
$$+ \cos i \cos \mu \tan \theta_2 \, \mathfrak{F}_i \, \varphi + \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4,$$

and the arc of revolution for the complete vibration is

$$\tfrac{1}{2} U_\pi = \pi + [\mathfrak{F}_i(\tfrac{1}{2}\pi) - \mathfrak{E}_i(\tfrac{1}{2}\pi)] \, \mathfrak{F}_{\text{co}\,i} \mu - \mathfrak{F}_i(\tfrac{1}{2}\pi) \, \mathfrak{E}_{\text{co}\,i} \mu$$
$$+ \cos i \cos \mu \tan \theta_2 \, \mathfrak{F}_i(\tfrac{1}{2}\pi).$$

These formulæ do not appear to differ from those of Gudermann, although the reduction is more extended. They give with equal facility the area of spherical surface which is described by the arc of a great circle, which joins the extremity of the pendulum to the lower point of the sphere.

MOTION OF A FREE POINT.

736. When a material point is unconstrained by any condition, and is free to obey the action of any force whatever, its motion in any direction is simply defined by the equation

$$D_t^2\, x = D_x\, \Omega.$$

737. If the coördinates are assumed to be of the partial polar form in which

$z =$ the distance from the plane of $x\, y$,
$\varrho =$ the distance from the axis of z,
$\varphi =$ the inclination of ϱ to the axis of x,

the value of T (162_{28}) is, for the unit of mass,

$$T = \tfrac{1}{2}\, z'^2 + \tfrac{1}{2}\, \varrho'^2 + \tfrac{1}{2}\, \varrho^2\, \varphi'^2.$$

The corresponding values of ω (165_4) are

$$\omega = z',$$
$$\omega_1 = \varrho',$$
$$\omega_2 = \varrho^2\, \varphi';$$

so that ω_2 *is the double of the projection of the instantaneous area, which is described by the radius vector of the point, upon the plane of* x y.

The equation (166_2) gives, then,

$$H_{\eta,\, w} = \tfrac{1}{2}\, z'^2 + \tfrac{1}{2}\, \varrho'^2 + \frac{\omega_2^2}{2\, \varrho^2} - \Omega,$$

$$D_t^2\, \varrho = \frac{\omega_2^2}{\varrho^3} + D_\rho\, \Omega = \varrho\, \varphi'^2 + D_\rho\, \Omega,$$

$$D_t\, \omega_2 = D_\phi\, \Omega.$$

It is apparent from (39_{10}) that *the second member of the last equation* *is the moment, with reference to the axis of* z, *of all the forces which act upon the point.*

738. *If the forces are proportional to the distances from the centres from which they emanate, the body moves as if it were under the influence of a single force, acting by the same law with an intensity equal to the sum of the intensities of the given forces, and emanating from a centre which is the centre of gravity of the given centres regarded as masses proportional to the intensities of their action.* For, if the notation employed in § 128 is adopted, and if m denotes the sum of the intensities of action, the value of the potential is

$$\Omega = \tfrac{1}{2} m r^2 + \int_m \varrho^2 = \tfrac{1}{2} m r^2 + K,$$

in which K is a constant and can be absorbed into the constant H with which Ω is connected in the equations of motion.

It follows from § 685, that *the path of the body is, in this case, a conic section, of which the centre of gravity is the centre.*

739. *If all the forces are directed towards a fixed line, the area described by the projection of the radius vector upon a plane perpendicular to the fixed line is proportional to the time of description.* For the instantaneous area is in this case constant by the equation (424_{29}), in which the fixed line may be assumed for the axis of z, so that the second member shall vanish.

740. In the example of the preceding section, a peculiar system of coördinates may be advantageously adopted. This system consists of the sum of the distances from two fixed points of the given line, the difference of these distances, and the angle which is made by a plane passing through a fixed line, with a fixed plane which includes this line. If, then,

$2p =$ the sum of the distances of the body from the two fixed points,

$2\,q =$ the difference of these distances,

$\varphi =$ the angle which the plane including the body and the fixed line makes with the fixed plane,

$2\,a =$ the distance of the fixed points from each other,

$2\,\psi =$ the angle which the two lines, which are drawn from the body to the fixed points, make with each other,

$k =$ the perpendicular drawn from the body to the fixed lines,

$p_1^2 = p^2 - a^2,$

$q_1^2 = a^2 - q^2,$

the values of k, ψ, and T are

$$k = \frac{p_1 q_1}{a},$$

$$\tan\psi = \frac{q_1}{p_1},$$

$$T = \frac{p'^2}{2\cos^2\psi} + \frac{q'^2}{2\sin^2\psi} + \tfrac{1}{2} k^2 \varphi'^2 = \tfrac{1}{2}(p_1^2 + q_1^2)\left(\frac{p'^2}{p_1^2} + \frac{q'^2}{q_1^2}\right) + \frac{p_1^2 q_1^2 \varphi'^2}{2\,a^2}.$$

The corresponding differential equations derived from LEGRANGE's canonical forms (164_{12}) are

$$D_t\left(\frac{p_1^2 q_1^2 \varphi'}{a^2}\right) = D_t(k^2\varphi') = 0,$$

$$D_p\,\Omega = \left(1 + \frac{q_1^2}{p_1^2}\right) p'' - p\,\frac{q'^2}{q_1^2} - \frac{p\,q_1^2\varphi'^2}{a^2} + \frac{2\,p_1^2 q\,p'\,q' + p\,q_1^2 p'^2}{p_1^4},$$

$$D_q\,\Omega = \left(1 + \frac{p_1^2}{q_1^2}\right) q'' + q\,\frac{p'^2}{p_1^2} + \frac{q\,p_1^2\varphi'^2}{a^2} + \frac{2\,q_1^2 p\,p'\,q' + q\,p_1^2 q'}{q_1^4}.$$

The integral of (426_{20}) is

$$k^2\varphi^1 = B,$$

in which B is arbitrary, and this equation expresses the proposition of § 739, and gives

$$\varphi' = \frac{B}{k^2} = \frac{B a^2}{p_1^2 q_1^2}.$$

A second integral of these equations, corresponding to the principle of living forces, is

$$T = \Omega + H.$$

The sum of the equations obtained by multiplying (426_{22}) by $p_1^2 p'$, (426_{24}) by ($-q_1^2 q'$) and (427_3) by $2(p p' + q q')$ is

$$\tfrac{1}{2} D_t [(p^2 - q^2)(p'^2 - q'^2)] = D_t [(p^2 + q^2 - a^2)(\Omega + H)]$$
$$-(q^2 p' D_p \Omega + p^2 q' D_q \Omega).$$

This equation is integrable, whenever Ω satisfies the condition

$$2 q D_p \Omega - 2 p D_q \Omega = (p^2 - q^2) D^2_{p,q} \Omega,$$

which, by the substitution of

$$x = \frac{1}{p+q}, \qquad y = \frac{1}{p+q},$$

may be transformed into

$$x^4 D_x^2 \Omega = y^4 D_y^2 \Omega.$$

If, then,

$$\Omega_1 = x^2 D_x \Omega - y^2 D_y \Omega - (x - y) \Omega$$
$$= -\left(D_q \Omega - \frac{2 q \Omega}{p^2 - q^2}\right) = -\frac{D_q [(p^2 - q^2) \Omega]}{p^2 - q^2},$$

the equation (427_{17}) or (427_{11}) becomes

$$0 = x^2 D_x \Omega_1 + Y^2 D_y \Omega_1 - (x + y) \Omega_1$$
$$= -\left(D_p \Omega + \frac{2 p \Omega}{p^2 - q^2}\right) = -\frac{D_p [(p^2 - q^2) \Omega]}{p^2 - q^2}.$$

If, then, P and Q are arbitrary functions of p and q respectively, the general value of Ω is

$$\Omega = \frac{P + Q}{p^2 - q^2};$$

and, if

$$P_1^2 = 2H(p^4 - a^4) + 2(P + C)(p^2 - a^2) - a^2 B^2,$$
$$Q_1^2 = 2H(q^4 - a^4) + 2(Q - C)(a^2 - q^2) - a^2 B^2,$$

in which C is an arbitrary constant, the integral of (427_8) is

$$\tfrac{1}{2}(p^2 - q^2)^2 (p'^2 - q'^2) = P_1^2 - Q_1^2,$$

while (427_3) may be written in the form

$$(p^2 - q^2)^2 (q_1^2 p'^2 + p_1^2 q'^2) = q_1^2 P_1^2 + p_1^2 Q_1^2.$$

It is easy to deduce from these equations

$$(p^2 - q^2)\, p' = P_1$$
$$(p^2 - q^2)\, q' = Q_1,$$
$$\int_p \frac{1}{P_1} = \int_q \frac{1}{Q_1},$$
$$t = \int_p \frac{p^2}{P_1} - \int_q \frac{q^2}{Q_1},$$
$$\varphi = a^2 B \int_p \frac{1}{p_1^2 P_1} + a^2 B \int_q \frac{1}{q_1^2 Q_1}.$$

This solution is published by me in Gould's *Astronomical Journal.*

741. It is evident from the linear form of the equations with reference to Ω, that all special values of Ω may be combined into a more general value by addition or subtraction.

742. The integrals in the values (428_{15-20}) assume the elliptic form, when P and Q have the forms

$$P = A_0 + A_1 p + A_2 p^2 + \frac{A_3}{a - p} + \frac{A_4}{p + a},$$
$$Q = B_0 + B_1 q + B_2 q^2 + \frac{B_3}{a - q} + \frac{B_4}{a + q},$$

and it is apparent that, in the expressions of the integrals, the constants, A_0, A_2, B_0 and B_2 can be combined with H, C, and B. The elliptic integrals become circular, when

$$H = -A_2 = B_2,$$
$$A_1 = B_1 = 0,$$

as well as in other cases which do not seem to be of especial interest.

743. When P and Q have the forms

$$P = A_1 p, \quad Q = B_1 q,$$

the value of the potential is

$$\Omega = \frac{A_1 + B_1}{2(p-q)} + \frac{A_1 - B_1}{2(p+q)},$$

so that, in this case, *the forces are equivalent to two emanating from the fixed points with the same law of force as that of gravitation*, which case has been integrated by Euler, Lagrange, and Jacobi, and the forms of Lagrange's integrals are identical with those of (428_{15-20}).

744. When P and Q have the forms

$$P = Ap^4, \quad Q = -Aq^4,$$

the value of the potential is

$$\Omega = A(p^2 + q^2),$$

so that, in this case, *the force is equivalent to a single force emanating from a point which is midway between the two fixed points, and the law of force is proportional to the distance from the centre of force*, and this case is integrated by Euler and Lagrange.

745. When P and Q have the form

$$\Omega = \frac{A}{p_1^2} = \frac{A}{p^2 - a^2}, \quad Q = \frac{A}{q_1^2} = \frac{A}{a^2 - q^2},$$

the value of the potential is

$$\Omega = \frac{A}{p_1^2 q_1^2} = \frac{A}{a^2 k^2},$$

so that, in this case, *the force is equivalent to a force which emanates from an infinite axis of uniform extent, and is inversely proportional to the cube of the distance from the axis.*

746. When the curve is given upon which a material point moves freely, the law of the fixed force is restricted within certain limits which it may be interesting to investigate. The geometrical conditions of the force are simply that it must be directed in the osculating plane of the curve, and the normal force must be equal to the centrifugal force of the body.

By the adoption of the notation

$$\Omega_1 = \Omega + H = T,$$

the equality of the force in the direction of the normal N, or of the radius of curvature ϱ to the centrifugal force is expressed by the equation

$$D_N \Omega_1 = \frac{\Omega_1}{\varrho}.$$

747. Since the preceding equation is linear, *all the special values of Ω_1 by which it is satisfied, may be combined into a new value by addition or subtraction. Previously to this addition, each value of Ω_1 may be multiplied by a factor,* which may represent the mass of the body, and if the factor is denoted by m, the value of $m\,\Omega_1$ will correspond to the whole force acting upon the mass, and it is, then, evident that, if M denotes the mass upon which the combined forces act, and V its velocity, the combined power is

$$M\,V^2 = \Sigma\,(m\,v^2),$$

which expresses *a condition identical with the theorem of* Bonnet.

748. If a special value of Ω_1 is represented by Ω_0, and if Ω_2 satisfies the equation

$$D_N \Omega_2 = 0,$$

so that it is the potential of a force, to the level surfaces of which the given curve is a perpendicular trajectory, *the complete value of* Ω_1 is

$$\Omega_1 = \Omega_0 f(\Omega_2),$$

in which f *is an arbitrary function.* It is apparent, then, that Ω_1 has an endless variety of possible forms in every special case. But each form corresponds to an arbitrary value of one of the constants of the given curve, or of some combination of those constants.

749. If the given curve is the parabola, of which the equation is

$$(y - y_0)^2 = 2p(x - x_0),$$

the values, which correspond to the arbitrary value of x_0, are

$$\Omega_2 = \log(y - y_0) + \frac{x}{p},$$
$$\Omega_2 = \frac{1}{p^2 + (y - y_0)^2},$$

while those, which correspond to the arbitrary value of y_0, are

$$\Omega_2 = y + \tfrac{4}{3}\sqrt{\left(\frac{(x - x_0)^3}{2p}\right)},$$
$$\Omega_0 = 2(x - x_0) + p;$$

and it is interesting to observe that when, in this case, the arbitrary function of Ω_2 is assumed to be constant, the value of the force is independent of x_0 and p as well as of y_0.

The values, which correspond to the arbitrary value of p, are

$$\Omega_2 = (y - y_0)^2 + 2(x - x_0)^2,$$
$$\Omega_0 = (y - y_0)^2 + 4(x - x_0)^2.$$

750. If the given curve is the conic section, of which the equation is

$$e \cos(\varphi - \varphi_0) = \frac{P}{r} - 1,$$

the values, which correspond to arbitrary values of φ_0 are

$$\Omega_2 = \varphi + \frac{R}{P} - P \tan^{[-1]} \frac{r+P}{R} - \frac{1}{2\sqrt{(1-e^2)}} \sin^{[-1]} \frac{2P - (1-e^2)r}{2Pe},$$

$$\Omega_0 = 1 - e^2 - \frac{2P}{r},$$

in which

$$R^2 = e^2 r^4 - (Pr - r^2)^2.$$

When the arbitrary function of Ω_2 is assumed to be constant, the force is independent of e and P as well as of φ_0, and its law is identical with that of gravitation.

751. If the given curve is the cycloid determined by the equations

$$y - y_0 = R(1 - \cos\theta),$$

$$x - x_0 = R(\theta - \sin\theta);$$

the values which correspond to arbitrary values of x_0 are

$$\Omega_2 = x + R(\theta + \sin\theta),$$

$$\Omega_0 = \frac{1}{y - y_0},$$

in which θ is to be regarded as the function of y, which is determined by (432_{18}).

752. *If the given curve is a circle of which the centre is the origin while the radius is arbitrary, the potential of the force is an arbitrary homogeneous function of the reciprocals of* x *and* y, *which is of the second degree.*

This peculiar result is the more worthy of attention because it can be extended to the sphere, so that *the potential of a force by which a body may move upon a sphere of a given centre but of an arbitrary radius, is likewise an arbitrary homogeneous function of the second degree of the reciprocals of the rectangular coördinates, of which the centre of the sphere is the origin.*

These problems are fruitful of new subjects of interesting geometric speculation.

CHAPTER XII.

MOTION OF ROTATION.

753. If the coördinates of the points of a system are the partial polar coördinates of § 737, and if φ_0 is supposed to refer to some point of the system, that is, to an axis connected with the system, from which the corresponding angles θ are measured, so that the value of φ is

$$\varphi = \varphi_0 + \theta,$$

that of T becomes

$$T = \tfrac{1}{2}\,\Sigma_i\,[m_i\,(x_i'^2 + \varrho_i'^2 + \varrho_i^2\,\varphi_i'^2)].$$

Hence the equation (164_{12}) gives

$$D_\theta\,\Omega = D_t\,\Sigma_i\,(m_i\,\varrho_i^2\,\varphi_i'),$$

the second member of which is the derivative of double the sum of the products obtained by multiplying each element of mass by the area described by the projection of the radius vector upon the plane perpendicular to the axis of rotation. If this area is designated as the *rotation-area for the axis*, it follows from (39_{23}) that *the derivative of the rotation-area for the axis is equal to the sum of the moments of the forces with reference to that axis.* It is obvious that the mutual actions of the system may be neglected in obtaining the sum of the moments.

If, then, all the external forces which act upon a system are directed towards an axis, the rotation-area for that axis will be described with a uniform motion, which is the principle of the *Conservation of Areas.*

754. The rotation-area for an axis may be exhibited geometrically by a portion of the axis which is taken proportional to the area, and it is evident from the theory of projections that rotation-areas for different axes may be combined by the same laws with which forces applied to a point, and rotations are combined, so that there is a corresponding *parallelopiped of rotation-areas. There is, then, for every system an axis of resultant rotation-area, with reference to which the rotation is a maximum, and the rotation-area for any other axis is the corresponding projection of the resultant rotation-area. The rotation-area vanishes, therefore, for an axis which is perpendicular to the axis of resultant rotation-area.*

ROTATION OF A SOLID BODY.

755. In the rotation of a solid body, the axis of rotation does not usually coincide with that of resultant or maximum rotation-area; and the relations of these two axes is of fundamental importance in the investigation of the rotation. The determination of these relations depends directly upon the moment of inertia. *The moment of inertia of a body or system of bodies upon an axis is the sum*

of the products obtained by multiplying each element of mass by the square of its distance from the axis.

The distorting moment with reference to two rectangular axes is the sum of the products obtained by multiplying each element of mass by the products of its distances from the two corresponding coördinate planes.

Let then

$m =$ the mass of the body,

$\varrho_p =$ the distance of the element dm from the axis of p, which passes through the origin,

$\frac{1}{m} I_p^2 =$ the reciprocal of the moment of inertia for the axis of p,

$m J_p =$ the distorting motion of inertia for the two axes which form a rectangular system with the axis of p,

which gives

$$\frac{m}{I_p^2} = \int_m \varrho_p^2.$$

If, then, φ_q is the angle which the axis of p makes with the direction of q, the moment divided by the mass, becomes

$$\frac{1}{I_p^2} = \frac{1}{m}\int_m \left(r^2 \sin^2 \varphi_r\right) = \frac{1}{m}\int_m \left(r^2 - r^2 \cos^2 \varphi_r\right),$$

$$= \frac{1}{m}\int_m \left[\Sigma_x x^2 - \left(\Sigma_x (x \cos \varphi_x)\right)^2\right],$$

$$= \Sigma_x \left(\frac{\cos^2 \varphi_x}{I_x^2} - 2 J_x \cos \varphi_y \cos \varphi_z\right).$$

If I_p is set off upon the axis from the origin, its extremity lies upon a finite surface of the second degree, which is, therefore, an ellipsoid, and may be called *the inverse ellipsoid of inertia.* If the axes of this ellipsoid are assumed for the axes of coördinates, the values of J must vanish for each of these axes, that is, *there is no* *distorting inertia for these axes which may be called the principal axes of inertia.*

756. When a body rotates about an axis, the rotation-area for an axis, which is perpendicular to that of rotation, is obviously proportional to the distorting inertia for these two axes. *There is, therefore, no rotation-area for a principal axis of inertia proceeding from rotation about either of the other two axes of inertia.*

757. If θ'_p is the velocity of rotation about the axis of p, the corresponding velocity of rotation about the principal axis of x is

$$\theta'_x = \theta'_p \cos \varphi_x,$$

and the corresponding rotation-area is

$$\frac{m\,\theta'_p \cos \varphi_x}{I_x^2};$$

the cosines of the angles, which the axis of resultant rotation-area makes with the principal axes, are then proportional to

$$\frac{\cos \varphi_x}{I_x^2}, \quad \frac{\cos \varphi_y}{I_y^2}, \text{ and } \frac{\cos \varphi_z}{I_z^2},$$

so that this axis coincides with the perpendicular to the tangent plane of the ellipsoid which is drawn at the extremity of the axis of rotation. *The plane of maximum rotation-area is, therefore, conjugate to the diameter of the ellipsoid which is the axis of rotation,* which theorem is given by Poinsot.

758. If the reciprocal of the perpendicular let fall from the origin upon the tangent plane of the ellipsoid is set off upon the perpendicular, its extremity lies upon a second ellipsoid, which may be called the *ellipsoid of inertia,* and of which *the principal axes are the reciprocals of the principal axes of the ellipsoid of* § 755, *and are proportional to the square root of the principal moments of inertia.*

759. It is apparent *that the tangent plane to the ellipsoid of inertia*

which is drawn at the extremity of the axis of maximum rotation-area is perpendicular to the axis of rotation.

It is also evident, *that the axis of rotation is one of the principal axes of the section of the inverse ellipsoid of inertia, which is made by a plane passing through the axis of inertia and perpendicular to the common plane of the axis of rotation and of maximum rotation-area, while the latter axis is one of the principal axes of the section of the ellipsoid of inertia, which is made by a passing plane through this axis perpendicular to this same common plane.*

760. Although the fixed axes of coördinates may be assumed at any instant to coincide with the principal axes of inertia, the axes of inertia are nevertheless in constant motion from the fixed axes, and at the end of the instant dt, after coincidence, the axes of rotation, which coincided at the beginning of the instant with the fixed axes of y and z, will not remain perpendicular to the fixed axis of x, but will deviate from perpendicularity by the respective angles

$$\theta'_z \, dt \text{ and } -\theta'_y \, dt.$$

The rate of increase of the rotation-area for the fixed axis of x, which arises from the external forces is, therefore,

$$\frac{1}{m} D_{\theta_x} \Omega = \frac{D_t \theta'_x}{I_x^2} - \theta'_z \, \theta'_y \left(\frac{1}{I_y^2} - \frac{1}{I_z^2}\right),$$

which represents *the well-known equations given by* Euler *for the rotation of a solid body.*

If the rotation-area for the axis of p is denoted by $m A'_p$, the preceding equation may assume the form

$$\frac{1}{m} D_{\theta_x} \Omega = D_t A'_x - A'_y A'_z (I_z^2 - I_y^2).$$

761. If the equation (437_{22}) is multiplied by $2\,\theta'_x$ and added

to the corresponding products for the other axes, the integral of the sum is

$$\frac{\Omega + H}{m} = \Sigma_x \frac{\theta_x'^2}{I_x^2},$$

which is simply *the equation of living forces.* If p is the semidiameter of the inverse ellipsoid of inertia, about which the solid is revolving at the instant, the preceding equation may be reduced to

$$\frac{\Omega + H}{m} = \theta_p'^2 \Sigma_x \frac{\cos^2 \varphi_x}{I_x^2} = \frac{\theta_p'^2}{p^2}.$$

ROTATION OF A SOLID BODY WHICH IS SUBJECT TO NO EXTERNAL ACTION.

762. If a solid body is subject to no external force, the centre of gravity may be assumed for the origin. In this case the first member of (437_{22}) or (437_{29}) vanishes, and the equation (438_9) becomes

$$\frac{\theta_p'^2}{p^2} = \frac{H}{m} = h^2,$$

or

$$\theta_p' = h p,$$

so that the velocity of rotation is proportional to the diameter of the inverse ellipsoid which is the axis of instantaneous rotation, which is given by Poinsot.

763. It follows from §§ 757 and 762, that, if q is the perpendicular let fall upon the tangent plane which is drawn to the inverse ellipsoid at the extremity of the axis of rotation, q is the axis of maximum rotation-area, which is invariable when there is no external force, and that

$$A_p' = A_q' \cos \varphi_q = \frac{\theta_p'}{p^2} = \frac{h^2}{\theta_p'},$$

$$\theta'_q = \theta'_p \cos \varphi_p = h\, p \cos \varphi_p = h\, q = \frac{h^2}{A'_q};$$

so that *the velocity of rotation about the axis of maximum rotation-area, as well as the distance of the tangent plane which is drawn to the inverse ellipsoid of inertia at the extremity of the axis of rotation are invariable during the motion of the solid,* which are propositions given by Poinsot. They might have been deduced with facility from the geometrical theorem of § 759, without the aid of the equation of living forces, which might on the contrary have been derived, in the present case, as an inference from these theorems, and this was the elegant process of Poinsot.

If the solid body has no translation, the inverse ellipsoid remains constantly tangent to the same plane which is that of maximum rotation-area, and which touches the ellipsoid at the extremity of the axis of rotation. It is apparent, then, that in the motion of the solid, *the ellipsoid rolls upon the fixed plane of maximum rotation-area, without any sliding;* which is Poinsot's mode of conceiving this motion.

764. *The instantaneous axis moves within the body in such a way as to describe the surface of the cone of the second degree,* of which the equation is

$$\Sigma_x \left[\frac{x^2}{I_x^2}\left(\frac{1}{I_x^2} - \frac{1}{q^2}\right)\right] = 0.$$

The base of this cone is an ellipse perpendicular to the greatest axis of the inverse ellipsoid when q *is larger than the middle axis, or perpendicular to the least axis, when* q *is less than the middle axis; and in either case the centre of the ellipse is upon the axis to which it is perpendicular.*

When q is equal to either the greatest or the least axis, this axis becomes the permanent axis of rotation; but when q is equal to the middle axis, the cone is reduced to a plane which corresponds to one of the plane circular sections of the ellipsoid of inertia.

The axis of maximum rotation-area describes within the body the cone of the second degree of which the equation is

$$\Sigma_x \left[(I_x^2 - q^2) x^2 \right] = 0.$$

The common plane of the instantaneous axis of rotation and of the axis of maximum rotation-area is obviously normal to the surface of the cone described in the body by the axis of maximum rotation-area, which defines the relative position of these two axes at each instant.

765. The position of the axis of maximum rotation-area is fixed in space, and, therefore, the path of the instantaneous axis of rotation in space is determined by the preceding property, and a distinct geometrical idea of the cone described by the instantaneous axis in space, is obtained by conceiving the cone described in the body by the axis of maximum rotation-area to be compressed into a line carrying with it the cone described by the instantaneous axis, in such a way as not to change the relative inclination of the two axes or the surface of the cone of the instantaneous axis.

The algebraic definition of the cone of the instantaneous axis in space is obtained by assuming the axes of the inverse ellipsoid to be arranged in the order of magnitude as x, y, z, in which the cone of the axis of rotation has the axis of x as its central axis, and adopting the notation

$$\cos E_z = \frac{I_y}{I_x},$$

$$M_x^2 = I_z^2 + I_y^2 - \frac{I_y^2 I_z^2}{q^2} = q^2 - q^2 \left(1 - \frac{I_z^2}{q^2}\right)\left(1 - \frac{I_y^2}{q^2}\right),$$

and similar equations for the other axes, in which it is unimportant that the angles E_x may be imaginary, but it should be observed that M_y is the largest of the quantities, M_x, M_y, and M_z; the following notation is also to be adopted.

$$\sin^2 E = -\sin^2 E_x \sin^2 E_y \sin^2 E_z,$$

$$N = \left(1 - \frac{I_x^2}{q^2}\right)\left(1 - \frac{I_y^2}{q^2}\right)\left(1 - \frac{I_z^2}{q^2}\right)$$

$$= \sqrt{\left[\left(1 - \frac{M_x^2}{q^2}\right)\left(1 - \frac{M_y^2}{q^2}\right)\left(1 - \frac{M_z^2}{q^2}\right)\right]},$$

$$\cos \eta = \frac{M_x}{M_y},$$

$$\cos \eta' = \frac{M_z}{M_y},$$

$$\sin i = \frac{\sin \eta}{\sin \eta'},$$

$$\sin \varphi = \frac{\sqrt{(M_y^2 - p^2)}}{M_y \sin \eta},$$

$$\sin \theta = \sin i \sin \varphi;$$

which give for each axis, if x, y, z denote the extremity of the instantaneous axis upon the surface of the inverse ellipsoid,

$$I_z^2 x^2 \sin^2 E = I_x^2 (p^2 - M_x^2) \sin^2 E_x,$$

$$\frac{Dx}{x} = \frac{p\,Dp}{p^2 - M_x^2},$$

$$I_z^2 D x^2 \sin^2 E_x = \frac{p^2 I_x^2 \sin^2 E_x D p^2}{p^2 - M_x^2}.$$

If, then, ψ is the angle which the plane of p and q makes with a fixed plane passing through q, the cone of the instantaneous axis in space is defined by the equation

$$p^2 \sin^2 \varphi_q D\psi^2 + \frac{Dp^2}{\sin^2 \varphi_q} = Dx^2 + Dy^2 + Dz^2,$$

or

$$D_\phi \psi = \frac{q\,p\,D_\phi p\,(p^2 - q^2 + q^2 N)}{(p^2 - q^2)\sqrt{[(p^2 - M_x^2)(M_y^2 - p^2)(p^2 - M_z^2)]}}$$

$$= \frac{q \sec \theta}{M_y \sin \eta'} + \frac{(q^2 + M_y^2 - M_y^2 \sin^2 \eta \sin^2 \varphi)\, M_y \sin \eta'}{q^3 N \sec \theta};$$

which gives by the use of elliptic integrals

$$\psi = \frac{q}{M_y \sin \eta'} \mathfrak{F}_i \varphi + \frac{q^3 N}{(q^2 + M_y^2) M_y \sin \eta'} \mathfrak{P}_i \left(-\frac{M_y^2 \sin^2 \eta}{q^2 + M_y^2}, \varphi\right).$$

766. The velocity with which the instantaneous axis moves in the body is readily obtained from the equations of the preceding section, which give in combination with (437_{22})

$$p D_t p = \Sigma_x (x D_t x) = -h x y z \Sigma_x \frac{\sin^2 E_x I_x^2}{I_z^2}$$

$$= -\frac{h x y z}{I_x^2 I_y^2 I_z^2} \Sigma_x (I_x^4 I_y^2 \sin^2 E_x) = h \sqrt{[(p^2 - M_x^2)(M_y^2 - p^2)(p^2 - M_z^2)]};$$

whence

$$h D_\phi t = \frac{\sec \theta}{M_y \sin \eta'},$$

and by elliptic integrals

$$M_y \sin \eta' h (t - \tau) = \mathfrak{F}_i \varphi.$$

767. In the especial case of

$$q = I_y,$$

the axis of maximum rotation-area describes one of the circular sections of the ellipsoid of inertia, and the equations of § 765 become

$$M_x = M_z = I_y,$$

$$N = 0,$$

$$\cos \eta = \cos \eta' = \frac{I_y}{M_y},$$

$$i = \tfrac{1}{2} \pi,$$

$$h I_y (t - \tau) = \psi,$$

$$\sqrt{(M_y^2 - p^2)} = M_y \sin \eta \operatorname{Tan} [h (t - \tau) M_y \sin \eta]$$

$$= \sqrt{(M_y^2 - I_y^2 \sec^2 \varphi_q)} = M_y \sin \eta \operatorname{Tan} \left(\frac{\psi M_y \sin \eta}{I_y}\right).$$

The greatest value of p is, then, M_y, which corresponds to

$$t - \tau = o\,;$$

and its least value is I_y, which corresponds to

$$t - \tau = \pm \infty\,.$$

If the axis of rotation, therefore, coincides with the mean axis of the ellipsoid of inertia at the commencement of the motion, its position will be permanent in the ellipsoid, although it is affected with an element of instability; but, in all other instances of the present case, the axis of rotation describes the spiral of which (442_{30}) *is the equation, and is constantly approaching the mean axis at such a diminishing rate of velocity that it never reaches this axis.*

768. When the ellipsoid of inertia is one of revolution, the cones, described by the instantaneous axis in the body and in space, are both of them cones of revolution, so that the simplicity of this case requires no further illustration; but it may be observed, that, when the ellipsoid is oblate, the moving cone rolls externally upon the stationary, but internally when the ellipsoid is prolate.

769. This analysis, which is substantially the same with one of the forms of Poinsot, comprehends the principal conclusions of Euler, Lagrange, and Laplace, and may be extended to the case in which the origin is any fixed point of the solid.

THE GYROSCOPE AND THE TOP.

770. When the solid is subject to any accelerated force, and its gyration is about a fixed point, which may be assumed as the origin, and when the ellipsoid of inertia with reference to this

point is an ellipsoid of revolution about the axis of z, the corresponding Eulerian equation is

$$m\, D_t\, \theta'_2 = I_z^2\, D_{\theta_2}\, \Omega.$$

771. If the body is also symmetrical about the axis of z, the preceding equation becomes

$$D_t\, \theta'_z = 0,$$

$$\theta'_z = n,$$

so that the rotation about the axis of z *is uniform.*

772. If the force is that of gravitation, the problem becomes that of the *gyroscope.* If g is the direction of gravitation, h that of a horizontal axis, which is perpendicular to the axis of z, and has that direction about which the rotation from g to z is positive, if

$$\xi = {}^g_z,$$

$$\chi = {}^h_x,$$

if l is the distance of the centre of gravity from the origin, l_g the projection of l upon g, and if the gyration of the body is resolved into the three rotations, χ' about the axis of z, ξ' about the axis of h, and ψ' about the axis of g, the rotation about the principal axes are

$$\theta'_z = \psi' \cos \xi + \chi';$$

$$\theta'_x = \psi' \cos {}^g_x + \xi' \cos \chi,$$

$$\theta'_y = \psi' \cos {}^g_y - \xi' \sin \chi.$$

These equations give

$$\theta'_x \cos {}^g_x + \theta'_y \cos {}^g_y = \psi' \sin^2 \xi = \psi' \left(1 - \frac{l_g^2}{l^2}\right).$$

The area about the axis of g is evidently described uniformly by the principles of § 753, so that if

$$a = \frac{I_x}{I_z} \sqrt{l},$$

and if l_3 is a constant, we have

$$l^2 (\theta'_x \cos{}^g_x + \theta'_y \cos{}^g_y) + a^2 l \theta'_z \cos{}^g_z = (l^2 - l_g^2) \psi' + n a^2 l_g = n a^2 l_3.$$

The equation (438_3) gives, in the present case,

$$(l^2 - l_g^2)^2 \psi'^2 + l^2 l_g'^2 = h^2 (l^2 - l_g^2)(l_g - l_4),$$

provided the constants h and l_4 are determined by the equations

$$h^2 = 2 g l^2 I_x^2,$$

$$-2 g l_4 = \left(\frac{H}{m} - \frac{n^2}{2 I_z^2}\right),$$

The elimination of ψ' from the equations (445_7) and (445_{10}) gives

$$l^2 l_g'^2 = h^2 (l^2 - l_g^2)(l_g - l_4) - n^2 a^4 (l_3 - l_g)^2.$$

773. The limiting values of l_g correspond to the vanishing values of l'_g, and, therefore, reduce the second member of (445_{19}) to zero. If these values are denoted by l_1, l_2, and $-p$, it is evident, from the form of the equation, that p is greater than l, while l_1 and l_2 are included between $-l$ and $+l$. The equations for the spherical pendulum of §§ 726 and 727 may be directly applied to the gyroscope by changing z into l_g and z_0, z_1, and z_2 into l_0, l_1, and l_2, which give, by (418_{24-27}),

$$l_0 = l_4 - \frac{n^2 a^4}{h^2},$$

$$R^2 = l^2 + \frac{2 n^2 a^4}{h^2} l_3,$$

$$A^2 = \frac{n^2 a^4}{l^2 I_x^2} (l^2 - 2 l_0 l_3 + l_3^2).$$

With this notation and the equations derived from (418_{13-14}), the expression of the time is

$$\tfrac{1}{2} I_x (t - \tau) \sqrt{[2 g (p + l_1)]} = \mathfrak{F}_i \varphi,$$

and the equation of the path described by the axis of this body in space is

$$\frac{l I_x \psi}{n a^2} \sqrt{[2g(p + l_1)]} = \frac{l_3 - l}{l - l_1} \mathfrak{P}_i \left(\frac{l_1 - l_2}{l - l_1}, \varphi\right) + \frac{l_3 + l}{l + l_1} \mathfrak{P}_i \left(\frac{l_2 - l_1}{l + l_1}, \varphi\right),$$

which admits of reduction by the process of § 735.

774. When the velocity n vanishes, the gyroscope is reduced to a case of the spherical pendulum of which the length is

$$R = \frac{1}{l I_x^2}.$$

775. When the two roots l_1 and l_2 are equal, *the path of the gyroscope is a horizontal circle.* The values of l_3, and of the velocity of rotation can be determined for this case by the equations

$$l_1 - l_3 = \frac{(l^2 - l_1^2)(l_1 - l_4)}{l^2 + 2 l_1 l_4 - 3 l_1^2},$$

$$\frac{a^4 n^2}{h^2} = \frac{(l^2 + 2 l_1 l_4 - 3 l_1^2)^2}{(l^2 - l_1^2)(l_1 - l_4)}.$$

The denominator of the value of $(l_1 - l_3)$ can be written in the form

$$l^2 + l_1 l_4 - 3 l_1^2 = 3 (l_5 - l_1)(l_1 + l_6),$$

in which l_5 and l_6 are positive quantities. If, then, l_1 is greater than l_5, ψ' is positive; but when l_1 is contained between l_5 and l_6, ψ' is negative; and l_1 can never be contained between $-l_6$ and $-l$.

776. When the values of l_1 and l_4 are equal, ψ' vanishes at the same time with ξ', and we shall also find

$$l_3 = l_1 = l_4,$$

and the equation for determining l_2 and p is

$$h^2 (l^2 - l_2^2) = n^2 a^4 (l_2 - l_1).$$

This is, approximately, the ordinary case of the gyroscope, and it is evident that in this case the values of l_1 and l_2 cannot be equal, unless

$$l_1 = l,$$

so that *the centre of the gyroscope cannot under these circumstances describe a horizontal circle,* which coincides with the conclusion of Major J. G. Barnard.

If, however, in this case, n *is very large, it is obvious that the difference between* l_2 *and* l_1 *is quite small,* for this difference is

$$l_2 - l_1 = \frac{h^2 (l^2 - l_2^2)}{n^2 a^4},$$

which is also one of the results obtained by Major Barnard.

777. When l_3 is algebraically greater than l_4, it is also algebraically included between l_1 and l_2, so that ψ' is positive at the upper limit, and negative at the lower limit. But when l_3 is algebraically smaller than l_4, it is also algebraically smaller than either l_1 or l_2, so that, in this case, ψ' is always negative.

778. When l_1 is equal to l, it is also equal to l_3, that is,

$$l_1 = l = l_3.$$

The velocity of rotation which corresponds to this case is determined by the equation

$$\frac{n^2 a^4}{h^2} = \frac{(l + l_2)(l_2 - l_4)}{l - l_2} = \frac{(l - p)(p + l_4)}{p + l},$$

which gives

$$p = l + \frac{2l(l_2 - l_4)}{l - l_2},$$

$$\sin i = \frac{(l - l_2)^2}{2l(l - l_4)},$$

$$\psi = \sqrt{\left[\frac{(l + l_2)(l_2 - l_4)}{2l(l - l_4)}\right]} \mathfrak{P}_i\left(\frac{l_2 - l}{2l}, \varphi\right).$$

779. When l_2 is equal to $-l$, it is also equal to l_3, that is,

$$l_1 = -l = l_3.$$

In this case l_4 is algebraically less than $-l$, and the velocity of rotation which corresponds to this case is given by the equation

$$\frac{n^2 a^4}{h^2} = \frac{(l - l_1)(l_1 - l_4)}{l + l_1} = \frac{(l + p)(p + l_4)}{p - l},$$

which gives

$$p = l - \frac{2l(l + l_4)}{l + l_1},$$

$$\psi = \frac{2na^2 l}{h(l - l_1)\sqrt{(p + l_1)}} \mathfrak{P}_i\left(-\frac{l + l_1}{l - l_1}, \varphi\right).$$

780. When p is equal to $-l$, it is also equal to l_3, that is,

$$p = -l = l_3.$$

In this case l_4 is algebraically greater than $-l$, and the velocity of rotation which corresponds to this case is given by the equation

$$\frac{n^2 a^4}{h^2} = \frac{(l - l_1)(l_1 - l_4)}{l + l_1} = \frac{(l - l_2)(l_2 - l_4)}{l + l_2},$$

which gives

$$l_2 = \frac{2l(l + l_4)}{l + l_1} - l.$$

781. If, in the preceding case, l_4 is equal to the negative of l, it will also be equal to l_2, that is,

$$l_4 = l_2 = -l,$$

and, in this case, the elliptic integrals disappear from the equations, so that they become

$$\frac{n^2 a^4}{h_2} = l - l_1,$$

$$l_g = l_1 - (l_1 + l) \operatorname{Tan}^2 \left[\frac{h}{2l}(t - \tau)\sqrt{(l + l_1)}\right],$$

$$\psi = -\frac{l\sqrt{(l - l_1)}}{2(l + l_1)}\left[\operatorname{Sin}\left[\frac{h}{l}(t - \tau)\sqrt{(l + l_1)}\right] + \frac{h}{l}(t - \tau)\sqrt{(l + l_1)}\right];$$

and although the axis is constantly approaching the upper vertical, after passing the lower limit, it never reaches the upper limit; and if it begins at the upper limit it never recedes from it.

782. In the simplest form of the problem of *the spinning of the top, the extremity of the body is a point in the axis of revolution, which is restricted to move, without friction, in a horizontal plane.* In this case, the equation (444_9) is still applicable, as well as (445_7), provided that the moments of inertia are referred to the centre of gravity of the top, and that l denotes the distance from the centre of gravity to the point in the horizontal plane.

The equation (438_3) gives, in this case, with the notation of § 772,

$$(l^2 - l_g^2)\,\psi'^2 + l^2(1 + l^2 I_x^2 - I_x^2 l_g^2)\,l'^2 = h^2(l^2 - l_g^2)(l_g - l_4);$$

and if ψ' is eliminated by means of (445_7),

$$l^2(1 + l^2 I_x^2 - I_x^2 l_g^2)\,l_g'^2 = h^2(l^2 - l_g^2)(l_g - l_4) - n^2 a^4 (l_3 - l_g)^2.$$

The comparison of this equation with (445_{19}), shows that the limits of motion are the same as in the case of the gyroscope, and under the condition of the equality of l_1 and l_2, the extremity of the axis of the body describes a horizontal circle. The expressions of the time and of the azimuth of the axis are not, however, capable of expression by means of elliptic integrals, except in special cases, of which that of § 781 is one, and another corresponds to the case of

$$p^2 = l^2 + \frac{1}{I_x^2}.$$

783. When the horizontal plane, to which the extremity of the top is restricted, is not smooth, the problem is usually more complicated, although when the friction brings the lower extremity to the case of rest, it reassumes the form of the gyroscope, and this is the modification of the problem which has been investigated by Poisson. *In this case of the gyroscope, however, the friction becomes an interesting feature of the problem, and has a peculiar effect upon the limits to which the motion is subjected.* Instead of the equation (444_9), *the rotation about the axis of the body decreases uniformly*, which is expressed by the equation

$$\theta'_z = n - n_2 t.$$

The area described about the vertical axis is also described in this case, at a uniformly decreasing rate, which gives instead of (445_7),

$$(l^2 - l_g^2)\,\psi' + (n - n' t)\, a^2 l_g = n a^2 (l_3 - l'_3 t).$$

The power of the system is reduced by the friction about the body-axis, which is proportional to the angle χ, and by the friction about the vertical axis, which is proportional to ψ. If, then, the mean values of ψ' and ξ for a small interval of time are denoted by ψ'_m and ξ_m, the equation of the preservation of power may be reduced to

$$(l^2 - l_g^2)\,\psi'^2 + l^2 l_g'^2 = h^2 (l^2 - l_g^2)\,(l_g + l_0 + l'_0 t),$$

in which

$$2\,g\,l'_0\,t = \frac{n'}{I_z^2}\int_t (\psi' \cos\xi) - \frac{n\,a^2\,l'_3}{l^2\,I_x^2}\,\psi$$

$$= \frac{n'}{I_z^2}\,\psi'_m\,t\cos\xi_m - \frac{n\,a^2\,l'_3}{l^2\,I_x^2}\,\psi'_m\,t.$$

The combination of this equation with (450_{23}) gives

$$l^2\,l'^2_g = h^2\,(l^2 - l_g^2)\,(l_g + l_0 + l'_0\,t) - n^2\,a^4\left[l_3 - l'_3\,t + \left(\frac{n'}{n}\,t - 1\right)l_g\right]^2.$$

It is obvious from this equation, that if the friction about the body-axis vanishes, the height, to which the gyroscope ascends, diminishes at each oscillation. If, however, the friction about the vertical axis is destroyed, the height, to which the gyroscope ascends at each oscillation, increases when the body-axis is directed upwards in its mean position; but this height diminishes when it corresponds to a position in which the centre of gravity is below the fixed extremity of the axis. In all intermediate positions, and when both the frictions remain, the increase or decrease of ascent depends upon the peculiar relations of the various constants.

In the spinning of the top, the rounded point rolls upon the supporting plane, which induces an acceleration about the vertical axis which is the reverse of friction, and this is the principal cause of the ready rising of a top into the vertical position of apparent repose, known as *the sleeping of the top.*

THE DEVIL ON TWO STICKS AND THE CHILD'S HOOP.

784. Contrasted with the motion of the gyroscope is that of a solid of revolution of which, instead of a fixed point of the axis, the circumference of a section drawn through the centre of

gravity, perpendicular to the axis is restricted to move upon a point. A convenient type of this class of motion may be found in the familiar toy called *the devil on two sticks.* If the friction is neglected in this case, and the notation adopted from the preceding problem of the gyroscope, the rotation about the body-axis is found to be constant, and the equation for the preservation of area about the vertical axis is, by a slight reduction,

$$\psi' \sin^2 \xi + n \cos \xi = B,$$

in which B is an arbitrary constant. The principle of power gives, by reduction, the equation

$$\psi'^2 \sin^2 \xi + \xi'^2 = H + a \sin \xi,$$

in which H is an arbitrary constant, and a is a constant which depends upon the form of the solid and the radius of the confined circumference.

785. The combination of (452_9) and (452_{14}) gives

$$\sin^2 \xi \, \xi'^2 = (H + a \sin \xi) \sin^2 \xi - (B - n \cos \xi)^2,$$

from which it is obvious that, in the general case, $\sin \xi$ cannot vanish, that is, *the body-axis cannot become vertical.*

786. When B vanishes, and H is greater than a, we have the ordinary case of the devil on two sticks, and, in this case, there are three real values of $\sin \xi$, for which the second member of (452_{20}) vanishes. Two of these values of $\sin \xi$ are contained between positive and negative unity, and one of them is positive, while the other is negative; they give the limit of the motion of the axis, and correspond respectively to the cases in which the centre of gravity is below or above the point of dispersion, which latter is of course the actual case of the toy. In either case, *the end*

of the body-axis describes a curve which is similar in form to the figure 8, and the apparent want of rotation about the vertical axis arises from the repeated change in the direction of rotation which occurs at each successive return of the body-axis to the horizontal position.

787. When B vanishes, and H is greater than a, but satisfies the inequality,

$$H > 3\left(\tfrac{1}{2} n a\right)^{\frac{2}{3}} - n^2,$$

the three values of $\sin \xi$, for which the second member of (452_{20}) vanishes, are all contained between positive and negative unity. The positive value is the upper limit of the inferior position of the centre of gravity, as in the preceding case, and as it would be if the inequality of this section were not satisfied, so that both the negative values were to become imaginary or equal. But the two negative values are the limits of motion, when the centre of gravity is higher than the point of suspension, and in this case *the body-axis describes a waving curve, and continues to rotate in one and the same direction about the vertical axis, without ever becoming horizontal, which phenomenon usually occurs in the devil on two sticks, at the beginning of the game, and before it has attained a sufficiently rapid rotation to assume a horizontal position.*

When H satisfies the equation

$$H = 3\left(\tfrac{1}{2} n a\right)^{\frac{2}{3}} - n^2,$$

the two negative limits of $\sin \xi$ are equal, and correspond to a gyration of the body-axis about the vertical axis in a right cone. The motion which corresponds to a positive limit of $\sin \xi$ in this case can be expressed by means of elliptic integrals.

788. Whenever H satisfies the inequality

$$H > B^2 + a,$$

the body-axis may become horizontal with the centre of gravity above the point of suspension, and in this position its gyration is positive or negative in conformity with the sign of B. If, moreover, n is greater than B, the vibration of the body-axis from the horizontal position extends so far as to reverse the direction of the gyration about the vertical axis; but if n is less than B, the direction of this gyration remains unchanged.

When H satisfies the inequality

$$H < B^2 + a,$$

the body-axis cannot become horizontal with the centre of gravity above the point of suspension.

789. The case of

$$B = \pm n,$$

constitutes an exception to the conclusion of § 785, and it is obvious that in this case the body-axis may, and generally will, become vertical.

790. The case of *a hoop rolling upon a horizontal plane,* is included in that of any rolling solid of revolution, but which is so formed that a circumference of the section of § 784 is restricted to roll upon a horizontal plane. The rolling condition is geometrically satisfied by the restriction that the point of contact with the plane is stationary during the instant of contact. If the notation of the preceding sections is retained, and if l is the radius of the rolling circumference, the velocities of the centre of gravity in the directions of the body-axis and of a horizontal perpendicular to the body-axis are, respectively,

$$l\,\xi' \text{ and } l\,\theta'_z.$$

The equation of the power of the system multiplied by $2\,I_x^2$, and divided by m becomes, then,

$$\psi'^2 \sin^2 \xi + A\,\xi'^2 + \theta_z'^2 = H + a \sin \xi + B\,n^2,$$

in which

$$A = 1 + l^2 I_x^2,$$

$$B = l^2 I_x^2 + \frac{I_x^2}{I_z^2},$$

$$a = 2\,g\,l\,I_x^2,$$

n is the initial velocity of rotation about the body axis, and H is arbitrary.

The application of Lagrange's canonical forms to the preceding expression of the power gives the equations

$$D_t\, v_z = 0,$$

$$D_t\,(\psi' \sin^2 \xi + B\, v_z \cos \xi) = 0.$$

and by integration and reduction

$$v_z = n,$$

$$\psi' \sin^2 \xi + B\,n \cos \xi = C,$$

$$A \sin^2 \xi\; \xi'^2 = (H + a \sin \xi) \sin^2 \xi - (C - B\,n \cos \xi)^2;$$

and it is obvious that these expressions coincide in form with these which were obtained in the investigation of *the devil on two sticks, so that the various inferences made in that problem are applicable to the motion of the hoop.* The analysis of the present problem is identical with that which was adopted by Nulty.

791. When the hoop is gyrating with its plane in a position which is nearly horizontal, the cube and higher powers of $\sin \xi$ may be neglected, in which case the equation (455_{21}) gives the integral

$$\xi^2 = \xi_0^2 + b^2 t^2,$$

in which it is sufficient to observe that ξ_0 is the initial value of ξ and b is constant, *so that the hoop constantly tends, by its inertia, to rise from this position, which, combined with the irregular action of friction, accounts for the peculiar forms of gyration, which frequently accompany the fall of the hoop.*

ROTARY PROGRESSION, NUTATION, AND VARIATION.

792. The positions of the axis of rotation and of maximum rotation-area may be referred to a fixed axis, and the change of inclination to this fixed axis may be called *nutation,* while the gyration about it is called *progression;* and the change in the magnitude of the rotation, or of the maximum rotation-area may be called *variation.*

793. It is obvious from the simple principles of the computation of rotation-areas, that an accelerative force which tends to give a rotation-area about an axis perpendicular to the axis of maximum rotation-area, does not cause a variation of the rotation-area, but only a motion of the axis so as to incline it in the direction of the accelerative axis. *Hence if the accelerative axis is perpendicular to the fixed axis as well as to the axis of maximum rotation-area, progression is produced; if it is in the common plane of the fixed axis and axis of maximum rotation-area, while it is perpendicular to the latter axis, nutation is produced; if it is in the direction of the axis of maximum rotation-area, variation is produced.*

The three directions of the accelerative axis, which correspond to the respective production of progression, nutation, and variation are mutually rectangular; so that it is easy to determine the relative tendency of a given force to these different modes of action. This neat analysis is derived from Poinsot.

794. If the accelerative axis is constantly perpendicular to the fixed axis, and also to the axis of maximum rotation-area, the motion will be wholly that of progression, of which mode of action a fixed type is presented in the precession of the equinoxes, the discussion of which problem must be reserved for the CELESTIAL MECHANICS. If the accelerative axis is constantly in the plane of the fixed axis and of the axis of maximum rotation-area, while it is perpendicular to the latter axis, the motion is exclusively that of nutation, and this form of action is well exhibited in the friction at the point of the top as it rolls upon the horizontal plane.

ROLLING AND SLIDING MOTION.

795. A special example of the case of rolling motion has been considered in the hoop, and *the mode of analysis which was there adopted can be applied to the general investigation,* as it has been done by NULTY. Thus, let the axes of x, y, z have the same directions with the principal axes of the rolling solid, let x_g, y_g, and z_g denote the coördinates of the centre of gravity of the solid, and x_τ, y_τ, and z_τ those of its point of contact with the surface upon which it rolls. The condition of rolling without sliding gives the equation

$$x'_g = (y_\tau - y_g)\,\theta'_z - (z_\tau - z_g)\,\theta'_y,$$

with the similar equations for the other axes. The expression of the power is

$$T = \tfrac{1}{2} m \Sigma_x \Big[\theta'^2 \Big[\frac{1}{I_x^2} + (y_\tau - y_g)^2 + (z_\tau - z_q)^2\Big] - 2(y_\tau - y_g)(z_\tau - z_g)\,\theta'_y\theta'_z)\Big],$$

from which the equations of nutation can be readily obtained by LAGRANGE's canonical forms.

796. If the solid slides upon the surface, it still remains in

contact with the surface, so that the point of contact does not move in the direction of the normal to the surface. If the direction of the normal is denoted by N, *this condition is expressed by the equation*

$$\Sigma_x \left[[x'_g - (y_\tau - y_g)\,\theta'_z - (z_\tau - z_g)\,\theta'_y] \cos {}^N_x \right] = 0,$$

which is given by ANDERSON. *This is the only condition, to which the motion is subject, in the case of perfect sliding motion.*

797. *When the sliding is accompanied with friction, the friction may be regarded as a force proportional to the pressure applied at the point of the solid, which is in contact with the surface, in a direction opposite to that of its motion.*

When the velocity of rasure is destroyed by friction, the motion ceases to be sliding and becomes a rolling motion, *in which form it continues as long as the force of friction exceeds the accelerative force in the direction of friction.*

CHAPTER XIII.

MOTION OF SYSTEMS.

798. The motion of every system is necessarily subject to *the Law of Power,* expressed in § 58, to *the law of the motion of the centre of gravity* of § 452, and *to the law of areas* of § 753. These three principles not only apply to the whole system, but to each portion of it considered as a system in itself.

799. The various forces which act upon a system are often quite different in the magnitude of their effects, so that they may

be considered from this point of view as different orders of force. In a first investigation, all but the forces of the first order may be neglected; and in subsequent approximations the forces of the inferior orders may be successively introduced, *as disturbing forces*, and their various effects may be determined as *perturbations* of corresponding orders.

800. The separation of the system into partial systems is closely connected with this subdivision of the forces, for it may easily be seen that the forces, which are of chief importance in the whole system, or some portion of it, are least active in other portions of this system. Whenever, for instance, the parts of any portion are so isolated from the rest of the system, that their relative changes of position are of small influence out of the portion, they should be treated by themselves as a partial system, and, relatively to all the other parts, may be considered as condensed upon their common centre of gravity.

LAGRANGE'S METHOD OF PERTURBATIONS.

801. The method of perturbations which originated with Lagrange, and which depends upon *the variation of arbitrary constants*, deserves the first consideration from its surpassing elegance; and it is the natural introduction to the other modes of investigation.

Suppose, then, that a complete system of integral equations is obtained, when all the forces but those of the first order are neglected, and let one of these equations involving a single arbitrary constant be denoted by (199_{20}). Let

Ω denote the potential of the forces of the first order, and

Ψ that of the forces of the inferior orders, and the equations of motion (166_{2-3}) assume the forms

$$\omega' = - D_\eta \Pi + D_\eta \Psi,$$
$$\eta' = D_\omega H.$$

If the constant member of (199_{20}) is now assumed to vary, its derivative is

$$D_t \alpha_i = \Sigma_\eta (D_\omega f_i D_\eta \Psi) = \Sigma_k [\Sigma_\eta (D_\omega f_i D_\eta f_k) D_{f_k} \Psi],$$

for by (199_{25})

$$0 = \Sigma_\eta (D_\eta f_i D_\omega \Pi - D_\omega f_i D_\eta H).$$

The condition that Ψ does not involve ω gives algebraically,

$$\Sigma_k (D_\omega f_k D_{f_k} \Psi) = 0 ;$$

and the notation

$$A_k^{[i]} = D_\eta f_k D_\omega f_i - D_\eta f_i D_\omega f_k,$$
$$B_k^{[i]} = \Sigma_\eta A_k^{[i]},$$

gives in combination with (460_8)

$$D_t \alpha_i = \Sigma_k (B_k^{[i]} D_{f_k} \Psi),$$

in which α may be substituted for its equal f in the second member.

802. The integrals of (460_{2-4}) obtained with the omission of the forces of the inferior orders, admit of arbitrary variation of the arbitrary constants, so that if such variations taken with reference to arbitrary elements which may be denoted by $\varkappa$ and λ, the corresponding variations of (460_{2-4}) with the omission of the terms dependent upon Ψ are

$$D_t D_\kappa \omega = - D_\eta D_\kappa H,$$
$$D_t D_\kappa \eta = D_\omega D_\kappa H,$$

and similar equations for λ which give

$$\Sigma_\eta (D_\lambda \eta\, D_t D_\kappa \omega + D_\kappa \omega\, D_t D_\lambda \eta - D_\kappa \eta\, D_t D_\lambda \omega - D_\lambda \omega\, D_t D_\kappa \eta)$$
$$= D_t \Sigma_\eta (D_\lambda \eta\, D_\kappa \omega - D_\kappa \eta\, D_\lambda \omega)$$
$$= \Sigma_\eta (D_\lambda D_\omega H D_\kappa \omega + D_\lambda D_\eta H D_\kappa \eta - D_\kappa D_\omega H D_\lambda \omega - D_\kappa D_\eta H D_\lambda \eta)$$
$$= D_\lambda D_\kappa H - D_\kappa D_\lambda H = 0,$$

so that if

$$C_\kappa^{[\lambda]} = \Sigma_\eta (D_\lambda \eta\, D_\kappa \omega - D_\kappa \eta\, D_\lambda \omega),$$

$C_\kappa^{[\lambda]}$ does not involve the time explicitly.

803. If $\varkappa$ is the element of actual variation of the arbitrary constants when the inferior forces are introduced, which element may be expressed as the time when it is so connected with the arbitrary constants, as not to cause ambiguity, the variations of the equations (460_{2-4}), give

$$D_\kappa \omega = D_\eta \Psi,$$
$$D_\kappa \eta = 0,$$
$$D_\lambda \Psi = \Sigma_\eta (D_\lambda \eta\, D_\kappa \omega - D_\kappa \eta\, D_\lambda \omega) = C_\kappa^{[\lambda]},$$

so that $D_\lambda \Psi$ does not involve the time explicitly. When $\varkappa$ and λ are changed to α_i and α_k, it is sufficient to retain i and k in the notation $C_k^{[i]}$, so that it is apparent from (461_{18}) that

$$D_{\alpha_k} \Psi = \Sigma_i (C_k^{[i]} D_t \alpha_i).$$

By elimination from the equations represented in the preceding form, the value of $D_t \alpha_i$ can be obtained identical with that of (460_{20}), so that it is evident that $B_k^{[i]}$ does not contain the time explicitly. It is also apparent that

$$\Sigma_k (B_k^{[i]} C_k^{[i']}) = 0,$$

except when

$$i = i',$$

in which case

$$\Sigma_k (B_k^{[i]} C_k^{[i]}) = 1.$$

804. The independence of $B_k^{[i]}$ of the time in an explicit form, renders it possible to compute its value for any instant, and the value thus obtained is universally true. Thus in the especial case in which the arbitrary constants are the initial values of η, ω, etc., the values of $B_k^{[i]}$, computed for the initial instant, are easily seen to vanish when the k and i refer to different points of the system; but when k and i refer to the η_0 and ω_0 of the same point, the value of $B_k^{[i]}$ is positive or negative unity, so that

$$D_t \eta_0 = -D_{\omega_0} \Psi,$$

$$D_t \omega_0 = D_{\eta_0} \Psi.$$

In the case of rectangular coördinates these equations become, for either axis,

$$D_t x_0 = -D_{x'_0} \Psi,$$

$$D_t x'_0 = D_{x_0} \Psi.$$

805. The especial variation of the constant H may be derived from the equation (171_7) which gives

$$D_t H = \Sigma_\eta (D_\eta \Psi D_t \eta) = D_{t_\eta} \Psi,$$

provided that t_η is intended to express the t which is involved in any of the quantities denoted by η. This development of the variation of the arbitrary constants is taken from Lagrange.

LAPLACE'S METHOD OF PERTURBATION.

806. The values of ω, η, etc., can be substituted from the first integrals directly in the first form of the second member of

(460_7), and the integral values of α_i which are then obtained can be introduced into ω, η, etc., as a second approximation to their values. This mode of analysis is especially useful when the equations of the first form are linear with reference to η, ω, and their derivatives. For in this case it is apparent that the functions denoted by f_i are linear with reference to η, ω, which may be demonstrated in the following manner. Let η_i, ω_i, etc., be special values of η, ω, of which there must be as many independent values as there are equations expressed by (460_{2-4}). The arbitrary constants α_i may then be such that the complete values of η and ω are

$$\eta = \Sigma_i (\alpha_i \eta_i),$$
$$\omega = \Sigma_i (\alpha_i \omega_i);$$

whence the values of α_i assume, by elimination, the linear form in reference to η, ω, etc. The values of $D_\omega f_i$, are then functions of t, and do not involve η, ω, etc. If $D_\eta \Psi$ represent forces, which are also functions of the time, the integrals of (460_7) can be completely obtained. By the substitution of these values of α_i thus obtained in the expressions of η, the complete values of η are obtained, which often admit of useful modification, and the success of the method depends upon the skill with which this modification is effected.

807. A special case of frequent occurrence in the problems of celestial mechanics is one in which

$$\omega = \eta',$$
$$H = \omega \eta' + \tfrac{1}{2} a^2 \eta^2.$$

The value of the integral in this case is, for a first approximation,

$$\eta = \alpha \cos a t + \alpha_1 \sin a t,$$

whence

$$\omega = -a\,\alpha \sin a\,t + a\,\alpha_1 \cos a\,t,$$

$$\alpha = \eta \cos a\,t - \frac{\omega}{a} \sin a\,t = f,$$

$$\alpha_1 = \eta \sin a\,t + \frac{\omega}{a} \cos a\,t = f_1.$$

The values of the constants obtained by integration of (460_{2-4}), are increased to

$$\alpha - \frac{1}{a}\int_t (D_\eta\, \Psi \sin a\,t),$$

and

$$\alpha_1 + \frac{1}{a}\int_t (D_\eta\, \Psi \cos a\,t\,;$$

so that the complete value of η is

$$\eta = \alpha \cos at + \alpha_1 \sin at - \frac{\cos a\,t}{a}\int_t (D_\eta\, \Psi \sin a\,t) + \frac{\sin a\,t}{a}\int_t (D_\eta\, \Psi \cos at).$$

808. The disturbed motion of the ordinary projectile exhibits an easy example of change of form. In this case, by the introduction of rectangular coördinates in which the axis of x is horizontal, and that of y vertical, the equations are

$$D_t\, x' = D_x\, \Psi,$$

$$D_t\, y' = -g + D_y\, \Psi,$$

whence

$$x = \alpha\, t + \alpha_1 + t \int_t D_x\, \Psi - \int_t (t\, D_x\, \Psi)$$

$$= \alpha\, t + \alpha_1 + \int_t^2 D_x\, \Psi,$$

$$y = -\tfrac{1}{2} g\, t^2 + \alpha_2\, t + \alpha_3 + t \int_t D_y\, \Psi - \int_t (t\, D_y\, \Psi)$$

$$= -\tfrac{1}{2} g\, t^2 + \alpha_2\, t + \alpha_3 + \int_t^2 D_y\, \Psi.$$

HANSEN'S METHOD OF PERTURBATIONS.

809. If V_t denotes any function of the time and of the arbitrary constants in the undisturbed orbit, its value in the disturbed orbit may be obtained, from the integration of the equation

$$D_t\, V_t = \Sigma_i\, (D_{a_i}\, V_\tau\, D_t\, a_i),$$

by the substitution of t for τ after the integration is performed. In the performance of the integration, the arbitrary constants are to be regarded as variable, and the value of V_t in the undisturbed orbit is to be taken for the initial value of V_τ. *This introduction of τ for* t *constitutes the first principle of* HANSEN'S *method of perturbations.*

810. The application of this method to the example of § 808, gives, for the values of x and y

$$x = a\, t + a_1 + \int_0^t{}_t \left[(\tau - t)\, D_x\, \Psi \right],$$

$$y = -\tfrac{1}{2} g\, t^2 + a_2 t + a_3 + \int_0^t{}_t \left[(\tau - t)\, D_y\, \Psi \right].$$

811. In the example of § 807, the value of η given by this method is

$$\eta = a \cos a\, t + a_1 \sin a\, t - \frac{1}{a} \int_0^t{}_\tau \left[\sin a\, (\tau - t)\, D_\eta\, \Psi_\tau \right],$$

in which the form of notation is slightly modified so that no subsequent change of τ to t is necessary. A case, which often occurs in connection with this example, is worthy of notice; it is when

$$D_\eta\, \Psi = h \cos\, (m\, t + \varepsilon),$$

in which case the value of η is

$$\eta = \alpha \cos at + \alpha_1 \sin at + \frac{h}{a^2 - m^2} \cos (mt + \varepsilon).$$

In the special case of

$$m = a,$$

this value of η becomes

$$\eta = \alpha \cos at + \alpha_1 \sin at + \frac{1}{2a} th \sin (at + \varepsilon).$$

812. If the function V increases with the time from negative to positive infinity, so that for all values of t

$$D_t V > 0,$$

there is an instant which may be denoted by z, for which the undisturbed value of V coincides with its disturbed value for the instant denoted by t. The corresponding value of z_τ is a function of both t and τ, which may be introduced into V_τ instead of τ, but after this substitution all the changes in the value of V_τ must arise from those of z_τ, so that

$$D_t V_\tau = D_{z_\tau} V_{z_\tau} D_t z_\tau,$$

$$D_\tau V_\tau = D_{z_\tau} V_{z_\tau} D_\tau z_\tau,$$

and the differential equation for the determination of z_τ is

$$D_t z_\tau = \frac{1}{D_{z_\tau} V_{z_\tau}} D_t V_\tau = \frac{D_\tau z_\tau}{D_\tau V_\tau} D_t V_t$$

$$= \frac{D_\tau z_\tau}{D_\tau V_\tau} \Sigma_i (D_{a_i} V_\tau D_t a_i).$$

In the integration of this equation, τ must be taken as the initial value of z_τ, whence, for the first approximation,

$$D_\tau z_\tau = 1.$$

After the integration is performed, the value of z is derived from that of z_τ by changing τ to t.

813. The disturbed value of any other function, U may be partially obtained by the substitution of z for t, and, since

$$D_\tau U_\tau = D_\tau U_{z_\tau} + D_{z_\tau} U_{z_\tau} D_\tau z_\tau,$$

the residual portion is obtained from the equation

$$D_t U_{z_\tau} = D_t U_\tau - D_{z_\tau} U_{z_\tau} D_t z_\tau,$$

$$= \Sigma_i \left[\left(D_{a_i} U_\tau - \frac{D_\tau U_\tau}{D_\tau V_\tau} D_{a_i} V_\tau \right) D_t \alpha_i \right] + \frac{D_\tau U_{z_\tau}}{D_\tau z_\tau} D_t z_\tau,$$

by changing τ for t after the integration is performed, and completing the integration, so that U_z may be the value of U_{z_τ} when τ vanishes.

This introduction of the disturbed time, which is denoted by z, *constitutes the second principle of* HANSEN'S *method of perturbations,* and upon the skilful use of the two principles thus developed, combined with an appropriate choice of coördinates, depends the success of this highly ingenious and original method.

814. It is obvious that, in the first approximation,

$$D_\tau U_{z_\tau} = 0,$$

so that the last term of (467_{11}) disappears for this approximation.

815. If V is such a function that it can be expressed in terms of η, etc., without involving ω, etc., or t, it follows from § 801, that the second member of (465_8) vanishes, when τ is changed to t, so that this must also be the case with the second member of the equation derived from (466_{27}),

$$\frac{D_t z_\tau}{D_\tau z_\tau} = \Sigma_i \left(\frac{D_{a_i} V_\tau}{D_\tau V_\tau} D_t \alpha_i \right).$$

The value of the first member of this equation can therefore be obtained by the integration of the equation

$$D_\tau\left(\frac{D_t z_\tau}{D_\tau z_\tau}\right) = \Sigma_i\left(\frac{D_\tau V_\tau D_{a_i} D_\tau V_\tau - D_{a_i} V_\tau D_\tau^2 V_\tau}{(D_\tau V_\tau)^2} D_t a_i\right),$$

provided that the integration is completed in conformity with the previous condition.

816. If one of the arbitrary constants, which may be denoted by β is so involved in V that

$$D_\beta V = 1 + K D_t V,$$

in which K does not involve the time, or if the form of V is

$$V = \beta + f_{\beta + Kt},$$

the corresponding term of the second member of (468_4) is

$$-\frac{D_\tau^2 V_\tau}{(D_\tau V_\tau)^2} D_t \beta.$$

The corresponding term of the second member of (467_{11}), if U has the same form with $V - \beta$ in (468_{14}), is

$$-\frac{D_\tau U_\tau}{D_\tau V_\tau} D_t \beta.$$

817. If one of the arbitrary constants, which may be denoted by γ is so involved in U that $U - \gamma$ may be expressed as a function of V without explicitly involving γ or t, the corresponding term of (467_{11}) is reduced to

$$D_t \gamma.$$

818. The further development of the methods of perturbations depends upon the peculiarities of the problem to which they

are applied. But the example, to which they are most appropriate, is that from which they have derived their origin, the motions of the bodies of the solar system, so that their ampler discussion is reserved for the Celestial Mechanics.

SMALL OSCILLATIONS.

819. When the motion of a system is restricted to small oscillations about a position of equilibrium, the quantities η, etc., may be supposed to be so small that the terms of T and Ω, which are of more than two dimensions in reference to these quantities and their derivatives, may be neglected.

The value of T may, then, by (165_8), be expressed in the form

$$T = \Sigma_{k,i}(T_k^{[i]}\eta'_k\eta'_i),$$

in which the quantities denoted by $T_k^{[i]}$, are constant.

If the values of η, etc., are supposed to vanish for the position of equilibrium, the derivative of Ω with reference to either of these variables vanishes for the same position, so that Ω must have the form

$$\Omega = \Omega_0 + \Sigma_{k,i}(\Omega_k^{[i]}\eta_k\eta_i),$$

in which the quantities, denoted by $\Omega_k^{[i]}$, are constant.

The equations of motion, derived from Lagrange's canonical forms, are, therefore, represented by

$$\Sigma_i[(T_k^{[i]}D_t^2 - \Omega_k^{[i]})\eta_i] = 0,$$

that is, *they constitute a system of linear differential equations with constant coefficients.*

820. It follows from the linear forms of these equations, that

the various systems of values by which they are satisfied, can be combined by addition into a new system. This is the mathematical expression of the important physical law of the possibility of *the superposition of small oscillations.*

821. With the notation

$$a_k^{[i]} = T_k^{[i]} D_t^2 - \Omega_k^{[i]},$$

the equation (469_{28}) assumes the form

$$\Sigma_i (a_k^{[i]} \eta_i) = 0.$$

If, then, there are m of the quantities η, etc., if $-n^2$ is one of the values of D_t^2 which satisfies the equation, expressed in the notation of determinants,

$$\mathfrak{R}_m = 0,$$

any system of values of η_i is expressed by the equation

$$\eta_i = E_i \sin (nt + \varepsilon_n),$$

in which ε_n is an arbitrary constant, and the constants E_i have a common arbitrary factor. The mutual ratios of the quantities E_i are determined from the equations derived from (470_{10}) by the substitution of $-n^2$ for D_t^2, and E for η. Hence, by § 340, E_i is determined in the form

$$E_i = E_n D_a^{[i]} \mathfrak{R}_m,$$

in which E_n is an arbitrary constant.

822. By the combination of all the values of n, the complete value of η_i is

$$\eta_i = \Sigma_n [E_n D_a^{[i]} \mathfrak{R}_m \sin (nt + \varepsilon_n)];$$

but it is evident that only those values of n should be retained for which the values n^2 given by (470_{14}) are real, positive, and unequal. For all other values of n^2, the time t would be introduced into the value of η_i in such a way that it would indefinitely increase. It is plain, therefore, that the only values of n, which can be retained in (470_{28}), are those which correspond to elements of stability, so that if the elements η are selected with a due regard to the conditions of equilibrium, those which correspond to the unstable equilibrium will disappear of themselves with the rejection of the corresponding values of n.

When the position of equilibrium is stable with reference to all of its elements, all the m *values of* n^2 *are real, positive, and unequal.*

823. The forms of T and Ω of § 819, lead, by inspection, to the equations

$$T_k^{[i]} = T_i^{[k]},$$
$$\Omega_k^{[i]} = \Omega_i^{[k]},$$

and the equation (469_{28}) gives, for each value of n,

$$\Sigma_i \left[(T_k^{[i]} n^2 + \Omega_k^{[i]}) E_i^{[n]} \right] = 0\,;$$

if n written as an accent indicates a special value of n, to which the functional form is applicable. If ξ_n is determined by the notation

$$\xi_n = \Sigma_i \Sigma_k (T_k^{[i]} E_k^{[n]} \eta_i),$$

and if the equations, represented by (469_{28}), are added together after being multiplied by $E_k^{[n]}$, the sum is

$$D_t^2 \xi_n + n^2 \xi_n = 0.$$

If, moreover, T_n denotes the value of T when η'_i is changed to $E_i^{[n]}$, the value of ξ_n, given by integration, is

$$\xi_n = T_n \sin(nt + \varepsilon_n).$$

The elements ξ thus obtained, correspond to the independent elements of stability which affect the position of equilibrium, and embody the true analysis of the various forms of oscillation of which the system is susceptible. When the different values of n have a common divisor, the oscillation is evidently periodic.

This investigation of the theory of small oscillation coincides, in substance, with that of LAGRANGE.

824. The importance and variety of the forms, in which oscillation and vibration are physically exhibited, give peculiar interest to the mechanical discussion of this subject. But the mode of analysis is so dependent upon the form of the phenomena, that the special researches are reserved for the chapters to which they are appropriate.

A SYSTEM MOVING IN A RESISTING MEDIUM.

825. When a system moves in a resisting medium, the law of resistance may be regarded as dependent upon the velocity, so as to be the same for all the bodies, but it may vary by a constant factor from one body to the other. If this constant factor for the mass m_i is denoted by k_i, and if V_i is the function of the velocity v_i, the resistance to the mass m_i moving with the velocity v_i is $k_i\, V_i$. If, then, rectangular coördinates are adopted, the equations of motion assume the form

$$D_t^2\, x_i = \frac{1}{m_i}\, D_{x_i}\, \Omega - k_i\, V_i \frac{x_i'}{v_i}.$$

The corresponding form of the equation for the determination of the Jacobian multiplier is, by §§ 402 and 451,

$$D_t \log \mathcal{M} = \Sigma_i \left[k_i\, \Sigma_x\, D_{x_i} \frac{V_i x_i'}{v_i} \right].$$

This equation becomes, when the motion is unrestricted in space,

$$D_t \log \mathcal{M} = \Sigma_i \left[k_i \left(2 \frac{V_i}{v^i} + D_{v_i} V_i\right)\right];$$

when the motion is in a plane,

$$D_t \log \mathcal{M} = \Sigma_i \left[k_i \left(\frac{V_i}{v_i} + D_{v_i} V_i\right)\right];$$

when the motion is in a straight line,

$$D_t \log \mathcal{M} = \Sigma_i (k_i D_{v_i} V_i).$$

826. It is evident from the linear form of these equations, that *the multiplier can be separated into factors, each of which shall independently correspond to a term of* V_i.

827. When the resistance is constant, and the motion in a straight line; or when the resistance is inversely proportional to the velocity, and the motion is in a plane; or when the resistance is inversely proportional to the square of the velocity, and the motion is unrestricted in space, *the multiplier becomes unity.* In either case of motion, a term of the corresponding form may be added to the resistance without affecting the multiplier.

828. *When the resistance is proportional to the velocity*, the value of the multiplier in the case of unrestricted motion is

$$\mathcal{M} = c^{3t\,\Sigma_i k_i};$$

in the case of motion in a plane it is

$$\mathcal{M} = c^{2t\,\Sigma_i k_i};$$

and in the case of the straight line it is

$$\mathcal{M} = c^{t\,\Sigma_i k_i}.$$

All these results, with regard to the multiplier, are derived from JACOBI.

829. *When the resistance is proportional to the square of the velocity*, the value of the multiplier for motion, which is unrestricted in space, is

$$\mathcal{M} = c^{4\,\Sigma_i (k_i s_i)};$$

for motion in a plane, it is

$$\mathcal{M} = c^{3\,\Sigma_i (k_i s_i)};$$

and for motion in a straight line, it is

$$\mathcal{M} = c^{2\,\Sigma_i (k_i s_i)}.$$

830. The sum of the equations (472_{26}), multiplied by $m_i\, x'_i$, is

$$D_t (T - \Omega) = -\Sigma_i (k_i\, m_i\, V_i\, v_i).$$

When V_i has the form

$$V_i = 1 + \frac{a_i}{v_i},$$

the integral of this equation is

$$T - \Omega = - \Sigma_i [k_i\, m_i\, (s_i + a_i\, t)].$$

831. When there are no external forces acting upon the system, the sum of the equations (472_{26}) for each axis multiplied by m_i, if x_g refers to the centre of gravity, is

$$\Sigma_i\, m_i\, D_t^2\, x_g = - \Sigma_i \left(m_i\, k_i\, V_i \frac{x'_i}{v_i}\right).$$

If the resistance is proportional to the velocity, the integral of this equation is

$$\Sigma_i\, m_i\, (D_t\, x_g - A) = - \Sigma_i\, (m_i\, k_i\, x_i),$$

in which A is an arbitrary constant. If k_i has the same value for all the bodies, the complete integral is

$$k\,x_g - A = B\,c^{-kt},$$

in which B is an arbitrary constant.

832. The introduction of polar coördinates, and the substitution of $A_z^{[i]}$ for the product of the area described by m_i about the axis of z, multiplied by the mass m_i, give for the corresponding equations of motion

$$D_t^2\,A_z^{[i]} = D_{\theta_z^{[i]}}\,\Omega - k_i\,\frac{V_i}{v_i}\,D_t\,A_z^{[i]}.$$

When there are no external forces, the sum of these equations is

$$D_t^2\,\Sigma_i\,A_z^{[i]} = -\,\Sigma_i\left(k_i\,\frac{V_i}{v_i}\,D_t\,A_z^{[i]}\right).$$

When the resistance is proportional to the velocity, the integral of this equation is

$$D_t\,\Sigma_i\,A_z^{[i]} = C - \Sigma_i\,(k_i\,A_z^{[i]}),$$

in which C is an arbitrary constant, which vanishes if the area vanishes with the time. If k_i has the same value for all the bodies the next integral is

$$\Sigma_i\,A_z^{[i]} = B\,(1 - c^{-kt}).$$

So that *the rotation-area instead of being proportional to the time is proportional to*

$$1 - c^{-kt},$$

but the position of the axis of maximum rotation-area is not affected by this uniform mode of resistance, which proposition is from JACOBI.

THE CONCLUSION.

833. In the beginning, the creating spirit embodied, in the material universe, those laws and forms of motion, which were best adapted to the instruction and development of the created intellect. The relations of the physical world to man as developed in space and time, as ordered in proximate simplicity and remote complication, in the immediate supply of bodily wants by the mechanic arts, and the infinite promise of spiritual enjoyment by the contemplation and study of unlimited change and variety of phenomena, are admirably adapted to stimulate and encourage the action and growth of the mind. True philosophy begins with the actual, but may not remain there; it yields sympathetically to the projectile force of nature, and earnestly forces its path into the possible, and even into speculations upon the impossible. But whenever the initial impetus is exhausted, the philosopher may not be content to remain stationary, or merely to turn upon his axis. He, then, descends to the world of sensible phenomena for new instruction and a stronger impulse. Let such be our method. In the present volume the attempt has been made to concentrate the more important and abtruser speculations of analytic mechanics clothed in the most recent forms of analysis, and to make a few additions, which may not be rejected as unworthy of their position. Much, undoubtedly, remains imperfect and unfinished, for it cannot be otherwise in a science which is susceptible of infinite improvement; and much must soon become antiquated and obsolete as the science advances, and especially when we shall have received the full benefit of the remarkable machinery of HAMILTON'S *Quaternions*. But it is time to return to nature, and learn from her actual solutions the recondite analysis of the more obscure problems of

celestial and physical mechanics. In these researches there is one lesson, which cannot escape the profound observer. Every portion of the material universe is pervaded by the same laws of mechanical action, which are incorporated into the very constitution of the human mind. The solution of the problem of this universal presence of such a spiritual element is obvious and necessary. There is one God, and Science is the knowledge of Him.

APPENDIX.

NOTE A.

ON THE FORCE OF MOVING BODIES.

It is remarkable, that, notwithstanding the convincing arguments of Leibnitz, the force of moving bodies is almost universally introduced into systems of *analytic mechanics* as being proportional to the velocity, instead of to the square of the velocity. Some philosophers, in quite an unphilosophic spirit, have stigmatized the early discussions of this subject as a war of words, as if the eminent geometers who entered into it could have been so deficient in their powers of logic and analysis. The great objection to the proportionality of the force to the velocity is derived from the necessity which it involves of regarding force in one direction as being the negative of that which is in the opposite direction. On this account, when a body or system rotates without any motion of translation, its aggregate force vanishes, so that such a motion would seem capable of being produced without any expenditure of force, and this statement has actually been made in some works upon astronomy. Leibnitz proposed as test propositions the transfer of motion from body to body in various forms, in all of which he supposed the whole force to be transferred from one body to another of a different weight without any external action. But it is evident from the law of preservation of momentum that such a transfer is impossible, and, therefore, this test cannot be practically applied. If, however, in the case of the impact of an elastic body upon a heavier one at rest, the striking body is held fast, as soon as it comes to instantaneous rest by the transfer of all its motion to the other body, the subsequent action of the elasticity must finally cause the body which is struck to move forward with a velocity inversely proportional to the square root of its mass. The external effort applied to the system in this case to hold the body at rest, arises from the force with which the elastic spring of the bodies is compressed, and is therefore an evidence of such a compression, and a proof that there has been an expenditure of force in its production, although the momentum of the system is not changed until the body is held. If, again, a spherical ball were to be impelled into a cylindrical tube of the same diameter, which terminates in another cylinder of a different diameter, but which containing a ball that exactly fits it, and if the included air acts as a compressed spring, it is easy to imagine such a mutual proportion of the parts and weights that the second ball shall leave the cylinder at the very instant when the first ball arrivcs at a state of rest, and when the air has rcturned to its initial density. In this case the whole living force of the first ball passes without increase or diminution into the second ball, and the momentum is not preserved. It is true that an external force is required to keep the cylinders in place, but this is a mere pressure, which is no more entitled to be regarded as an active force than is the centrifugal force, or any of the modifying forces which are represented analytically by equations of condition. Seeing, then, that by admitting the square of the velocity to be the true measure of the force of a moving body, the fiction of negative force is wholly avoided, and the fundamental principles of mechanical problems are reduced to their utmost simplicity, there secms to be sufficient reason to reverse the modern decisions, and return to the higher philosophy of LEIBNITZ.

NOTE B.

ON THE THEORY OF ORTHOGRAPHIC PROJECTIONS.

For the convenience of students, the theory of orthographic projections is here condensed into a few simple formulæ.

The projection of a line a upon another line b is

$$a_b = a \cos {}^{a}_{b}.$$

If many successive lines represented by a_i, are so united that each line begins where the previous line ended, and if the last line terminates where the first began, the sum of the projections is

$$\Sigma_i \, (a_i \cos {}^{b}_{a_i}) = 0.$$

If there are four of these lines, and if the three first are mutually rectangular and parallel to the axes of x, y, and z, this equation becomes

$$\Sigma_x \, (a_x \cos {}^{b}_{x}) + a_4 \cos {}^{b}_{a_4} = 0.$$

But it is evident that a_x is the projection of $-a_4$ upon the axis of x, whence

$$a_x = - a_4 \cos {}^{x}_{a_4},$$

and if the subjacent 4 is now omitted as unnecessary, this equation gives

$$\cos {}^{b}_{a} = \Sigma_x \, (\cos {}^{a}_{x} \cos {}^{b}_{x}),$$

of which the equation

$$1 = \Sigma_x \cos^2 {}^{a}_{x},$$

is a particular case.

These equations may be applied to the projections of plane areas, if each area is represented in a linear form by the length of a line which is drawn perpendicular to it.

ERRATA.

Page	For	Read
12_{9}	axes	axis.
$15_{4\text{-}10}$ and 15_{20}	The signs of the second members should be reversed.	
15_{24}	acute	right.
26_{8}	λ	λ_1.
30_{17}	these	those.
40_{18}	*resultant*	*resultant moment.*
40_{23}	different lines	opposite directions.
41_{22}	force	resultant of the forces.
42_{27}	O' with reference to O	O with reference to O'.
51_{1}	x,	x_1.
51_{4}	y	η.
55_{14}	POINT UPON A DISTANT MASS	MASS UPON A DISTANT POINT.
57_{13}	4	$\frac{4}{3}$.
57_{22} and 57_{28}	$\frac{1}{2}$	$\frac{3}{2}$.
*59_{2}	$\cos^{f}$	$\cos \frac{f}{N}$.
59_{21} and 60_{12}	four	eight.
59_{23} and 60_{21}	two	four.
73_{16}	surface	surfaces.
83_{2}	b_y	b_x.
85_{6}	D_{Φ}	D_{ϕ}.
85_{21}	$\frac{1}{4}$	4.
86_{11}	4π	$4\pi K$.
86_{22}	$\frac{1}{4}$ and $\frac{1}{2}$	1 and 2.
88_{10}	$-A_z^2)$	$+A_z^2)$.
90_{18} and 90_{20}	H^{m-1}	H^{n-1}.
90_{27}	$(\cos (m-1)$	$\cos (m-1)\eta$.
91_{25}	See note on page 356.	
98_{24}	89_{7}	89_{22}.
99_{10}	r_n	r^n.
100_{7}	independent of	dependent upon.
101_{12} twice	$+r$	$+2r$.

* This correction only applies to some copies.

Page	For	Read
$107_{1, 9, 16, 18}$	3	6.
107_{11}	A_x^2	$A_{x'}^2$.
107_{29}	(104_{19}) to the form (107_{16})	(104_9) to the form (107_1).
111_{10}	Σ_n	Σ_m.
111_{13}	$-\Sigma$	Σ.
111_{13}	$(-r)^{-(m-1)}$	$(-r)^{-(m-2)}$.
$111_{17, 20}$	$\frac{k}{\Gamma m r^{m-1}}$	$\frac{k}{\Gamma m r^{m-2}}$.
111_{19}	72	24.
117_{19}	42_{12}	42_{22}.
117_{22}	Δp_x	Δu_x.
117_{16}	for another point of the body which is near the former point	arising from this motion.
117_{21}	dele $\Delta x =$.	
120_7	$\frac{1}{2} p$	$\frac{1}{2} \pi$.
120_{20}	similar to	like.
121_1	119_6	118_3.
122_{27}	π	$\frac{1}{2} \pi$.
123_5	above	about.
126_{11}	put	but.
127_8	*in order*	*in order that.*
127_{24}	*place*	*plane.*
139_{16}	tan	cot.
140_{23}	c^y	$Bc^{\frac{y}{B}}$.
140_{25}	c^y	$c^{\frac{y}{B}}$.
141_{16-20}	$\sec^2 \frac{s}{x}$	$\sec \frac{s}{x}$.
148_{15}	$\cot \varepsilon$	$\tan \varepsilon$.
148_{22}	take $F' K$	take $F' L$.
149_3	$+ \sin^2 \eta$	$- \cos^2 \eta$.
149_6	$\cos^2 \omega = \cos^2 \alpha$	$\cot^2 \omega = \cot^2 \alpha$.
149_7	$\sec^2 \omega$	$\cos^2 \omega$.
149_{18}	$\cos^2\theta \sin^2\eta \sin^2\varphi \cos^2\theta + \sin^2\eta \sin^2\theta$	$\cos^2\theta - \sin^2\eta \sin^2\varphi \cos^2\theta + \sin^2\eta \sin^2\theta \cos^2\varphi$.
149_{19}	$\sin^2 \eta \sin \varphi$	$\sin^2 \eta \sin^2 \varphi$.
150_{28}	$-$	$+$.
150_{30} twice	$\theta' +$	$\theta' -$.
150_{31}	$+ \tan \beta$	$- \tan \beta$.
151_1	$\theta' -$	$\theta' +$.
151_3	$\theta' +$	$\theta' -$.
151_5	$- \cos^2 \beta_1 \varphi'$	$- \cos^2 \beta, \varphi'$.
151_5	$\varphi' +$	$\varphi' -$.
151_{29}	$\cot \varphi'$	$(\cot \varphi' + \frac{1}{2} \varphi')$.

Page	*For*	*Read*
152_{24}	$a' - y'_0$	$\frac{a' - y'_0}{\sqrt{\cos 2\beta}}$.
152_{29}	$-$	$+$.
152_{31}	$\varphi'' -$	$\varphi'' +$.
156_{25}	*prolate*	*oblate.*
157_{1}	*oblate*	*prolate.*
173_{25}	for the	for two.
173_{29}	determinate	determinant.
175_{3}	$i > n - 1$	$i > n - 2$.
175_{9}	$a_i^{i > k}$	$a_k^{i > k}$
178_{23}	$B_{(m)}$	$B^{(m)}$.
180_{4}	$\mathfrak{B}_n$	$\mathfrak{R}_n$.
180_{7}	$\mathfrak{R}^m_m$	$\mathfrak{R}^m_n$.
190_{28}	B	$\mathfrak{B}$.
191_{2}	f^1	f^1_n.
191	The sections 366 to 369 should be limited by the condition that	$m < n - 1$.
191_{24}	$\mathfrak{B}\,\mathfrak{B}'_1 \ldots \mathfrak{B}^{(m)}_m$	$\mathfrak{C}\,\mathfrak{C}'_1 \ldots \mathfrak{C}^{(m)}_m$.
191_{25}	$(-)^n\,\mathfrak{R}^m_{n+1}$	$(-)^{(n+1)(m+1)}\,\mathfrak{R}^m_{n-1}$.
192_{10}	$(-)^{n+1}$	$(-)^{mn}$.
192_{13}	$m - 1$	$m - i$.
192_{15}	$(-)^{n+i}$	$(-)^{mn+(i+1)(m+n+1)}$.
200_{16}	199_8	199_{11}.
202_{10}	$\overset{m}{\Sigma}$	$\overset{n}{\Sigma}$.
203_{3}	340	339.
215_{19}	D_{x_i}	$D x_i$.
$219_{19, 24}$, 220_3	$X_1, X_2 \ldots X_n$	$x_1, x_2 \ldots x_n$.
222_{18}	200_2	215_{12}.
224_{1}	formal	normal.
226_{11}	$\lambda - 1$	λ.
227_{2}	(210_{31}), the	(210_{31}). The.
227_{9}	F	F_i.
228_{20}	D_x	D_{x_1}.
228_{21}	$D^1_{x_1}$	D_x.
231_{25}	187_{10}	189_3.
233_{3}	216_{11}	231_{14}.
$234_{5,7}$	$D_{x_k}\,\mathcal{M}^{(i)}$	$D_{x_h}\,\mathcal{M}^{(i)}$.
238_{29}	Σ_k	Σ_h.
239	$\mathcal{M}$	M.

Page	*For*	*Read*
239_9	Q	$\mathfrak{Q}$.
247_4	*uniform*	*uniform and constant in direction.*
247_{25}	D_s	$D_{s'}$.
250_{25}	h	k.
258_{12}	$\frac{p}{p'}$	$\frac{b}{p'}$.
261_1	$\mathfrak{P}$	$\mathfrak{T}$.
262_{13}	a^2	$2\,a^2$.
262_{20}	$s -$	$s +$.
263_{21}	cos	cot.
265_9	$\sin\varphi$	$\cos\varphi$.
277_{29}	$\sqrt{(g-a)}$	$\sqrt{[h\,(g-a)]}$.
279_3^1	Cot	Tan.
281_{21}	$k\,t$	$(k\,t - 2\,\alpha)$.
281_{31}	$\pm\frac{R\,a}{g}$	$\pm\frac{R\,a}{g} - \frac{\varphi_0'}{k}\sin 2\,a$.
282_2	correct	nearly correct.
$282_{14,18}$	$+\alpha$	$-\alpha$.
285_{20}	μ_m	$\mu\,m$.
$287_{10,16}$	$\sum_1$	$\sum_2$.
287_{12}	h_2	h_i.
289_{19}	$b + 2\,b$	$b + 2\,\beta$.
317_6	$\sqrt{\pi}$	π.
328–354	Brachystochrone	Brachistochrone.
328_9	is confined	is not confined.
330_4	329_9	329_3.
348_{22}	589	588.
352_{11}	$\sin\nu$	$\cos\nu$.
364_{19}	359_{35}	359_3.
369–370	Tachystotrope	Tachistotrope.
377_{30}	level	given.
393_{23}	$\sin^{[-4]}$	$\sin^{[-1]}$.
426_{27}	φ^1	φ'.
431_{19}	Ω_2	Ω_0.
472_{31}	D_{x_i}	$D_{x_i'}$.

ALPHABETICAL INDEX.

Q.

R.

S.

T.

V.

W.

Fig. 1.

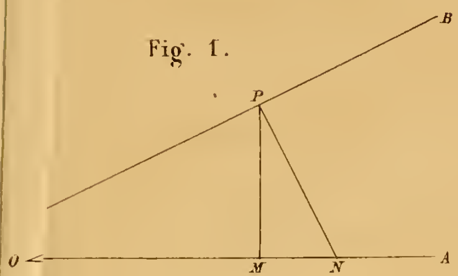

Fig. 2.

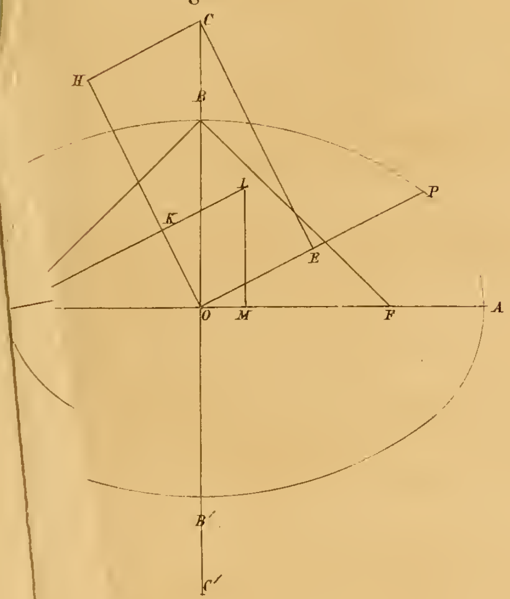

Fig. 1.

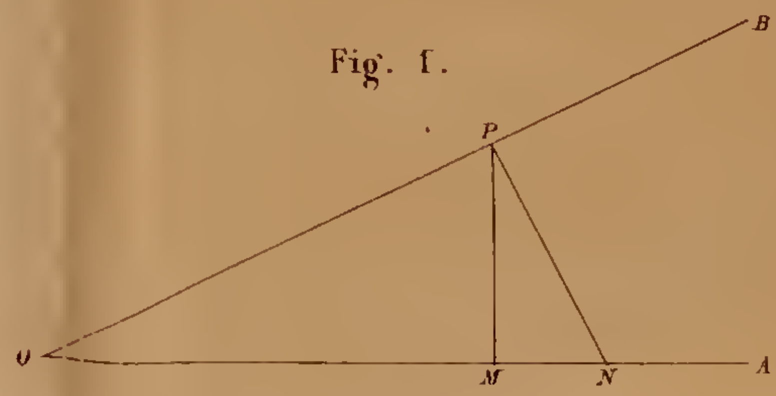

Fig. 2.