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VOLUMEN PRIMUM

IN QUO METHODUS INTEGRANDI A PRIMIS PRINCIPIIS USQUE AD INTEGRATIONEM AEQUATIONUM DIFFERENTIALIUM PRIMI GRADUS PERTRACTATUR.

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PRAENOTANDA.

DE
CALCULO INTEGRALI
IN GENERE.

Definitio 1.

1.

Calculus integralis est methodus, ex data differentialium relatione inveniendi relationem ipsarum quantitatum: et operatio, qua hoc praestatur, integratio vocari solet.

Corollarium 1.

2. Cum igitur calculus differentialis ex data relatione quantitatum variabilium, relationem differentialium investigare doceat: calculus integralis methodum inversam suppeditat.

Corollarium 2.

3. Quemadmodum scilicet in Analysi perpetuo binae operationes sibi opponuntur, veluti subtractio additioni, divisio multiplicationi, extractio radicum evectioni ad potestates, ita etiam simili ratione calculus integralis calculo differentiali opponitur.

Corollarium 3.

4. Proposita relatione quacunque inter binas quantitates variabiles x et y, in calculo differentiali methodus traditur rationem differentialium $\partial y : \partial x$ investigandi: sin autem vicissim ex hac differentialium ratione ipsa quantitatum x et y relatio sit definienda, hoc opus calculo integrali tribuitur.

Scholion 1.

5. In calculo differentiali iam notavi, quaestionem de differentialibus non absolute sed relative esse intelligendam, ita ut, si y fuerit functio quaecunque ipsius x, non tam ipsum eius differentiale ∂y, quam eius ratio ad differentiale ∂x sit definienda. Cum enim omnia differentialia per se sint nihilo aequalia, quaecunque functio y fuerit ipsius x, semper est $\partial y = 0$, neque sic quicquam amplius absolute quaeri posset. Verum quaestio ita rite proponi debet, ut dum x incrementum capit infinite parvum adeoque evanescens ∂x, definiatur ratio incrementi functionis y, quod inde capiet, ad istud ∂x: etsi enim utrumque est $= 0$, tamen ratio certa inter ea intercedit, quae in calculo differentiali proprie investigatur. Ita si fuerit $y = x x$, in calculo differentiali ostenditur esse $\frac{\partial y}{\partial x} = 2 x$, neque hanc incrementorum rationem esse veram, nisi incrementum ∂x, ex quo ∂y nascitur, nihilo aequale statuatur. Verum tamen, hac vera differentialium notione observata, locutiones communes, quibus differentialia quasi absolute enunciantur, tolerari possunt, dummodo semper in mente saltem ad veritatem referantur. Recte ergo dicimus, si $y = xx$, fore $\partial y = 2x\partial x$, tam etsi falsum non esset, si quis diceret $\partial y = 3x\partial x$, vel $\partial y = 4x\partial x$, quoniam ob $\partial x = 0$ et $\partial y = 0$, hae aequalitates aeque subsisterent; sed prima sola rationi verae $\frac{\partial y}{\partial x} = 2 x$ est consentanea.

Scholion 2.

6. Quemadmodum calculus differentialis apud Anglos methodus fluxionum appellatur, ita calculus integralis ab iis methodus fluxionum inuersa vocari solet, quandoquidem a fluxionibus ad quantitates fluentes revertitur. Quas enim nos quantitates variabiles vocamus, eas Angli nomine magis idoneo quantitates fluentes vocant, et earum incrementa infinite parva seu evanescentia fluxiones nominant, ita vt fluxiones ipsis idem sint, quod nobis differentialia. Haec diversitas loquendi ita iam usu invaluit, ut conciliatio vix unquam sit expectanda; equidem Anglos in formulis loquendi luben-

ter imitarer, sed signa quibus nos utimur, illorum signis longe anteferenda videntur. Verum cum tot iam libri utraque ratione conscripti prodierint, huiusmodi conciliatio nullum usum esset habitura.

Definitio 2.

7. Cum functionis cuiuscunque ipsius x differentiale huiusmodi habeat formam $X\,\partial x$, proposita tali forma differentiali $X\,\partial x$, in qua X sit functio quaecunque ipsius x, illa functio, cuius differentiale est $= X\,\partial x$, huius vocatur integrale, et praefixo signo $\int$ indicari solet: ita vt $\int X\,\partial x$ eam denotet quantitatem variabilem, cuius differentiale est $= X\,\partial x$.

Corollarium 1.

8. Quemadmodum ergo propositae formulae differentialis $X\partial x$ integrale, seu ea functio ipsius x, cuius differentiale est $= X\,\partial x$, quae hac scriptura $\int X\,\partial x$ indicatur, investigari debeat, in calculo integrali est explicandum.

Corollarium 2.

9. Uti ergo littera ∂ signum est differentiationis, ita littera $\int$ pro signo integrationis utimur, sicque haec duo signa sibi mutuo opponuntur, et quasi se destruunt: scilicet $\int\partial X$ erit $= X$, quia ea quantitas denotatur cuius differentiale est ∂X, quae utique est X.

Corollarium 3.

10. Cum igitur harum ipsius x functionum

$$x^2,\; x^n,\; \surd(aa - xx)$$

differentialia sint

$$2x\,\partial x,\; nx^{n-1}\partial x,\; \frac{-x\,\partial x}{\surd(aa-xx)}$$

signo integrationis $\int$ adhibendo, patet fore:

$$\int 2x\partial x = xx;\; \int nx^{n-1}\partial x = x^n;\; \int\frac{-x\,\partial x}{\surd(aa-xx)} = \surd(aa-xx)$$

unde usus huius signi clarius perspicitur.

Scholion 1.

11. Hic unica tantum quantitas variabilis in computum ingredi videtur, cum tamen statuamus tam in calculo differentiali quam integrali, semper rationem duorum pluriumve differentialium spectari. Verum etsi hic una tantum quantitas variabilis x apparet, tamen revera duae considerantur; altera enim est ipsa illa functio, cuius differentiale sumimus esse $X\,\partial x$, quae si designetur littera y, erit $\partial y = X\,\partial x$, seu $\frac{\partial y}{\partial x} = X$, ita ut hic omnino ratio differentialium $\partial y : \partial x$ proponatur, quae est $= X$, indeque erit $y = \int X\,\partial x$: hoc autem integrale non tam ex ipso differentiali $X\,\partial x$, quod vtique est $= 0$, quam ex eius ratione ad ∂x inveniri est censendum. Caeterum hoc signum $\int$ vocabulo *summae* efferri solet, quod ex conceptu parum idoneo, quo integrale tanquam summa omnium differentialium spectatur, est natum; neque maiore iure admitti potest, quam vulgo lineae ex punctis constare concipi solent.

Scholion 2.

12. At calculus integralis multo latius quam ad huiusmodi formulas integrandas patet, quae unicam variabilem complectuntur. Quemadmodum enim hic functio unius variabilis x ex data differentialis forma investigatur; ita calculus integralis quoque extendi debet ad functiones duarum pluriumve variabilium investigandas, cum relatio quaedam differentialium fuerit proposita. Deinde calculus integralis non solum ad differentialia primi ordinis adstringitur, sed etiam praecepta tradere debet, quorum ope functiones tam unius quam duarum pluriumve variabilium investigari queant, cum relatio quaedam differentialium secundi altiorisve cuiusdam ordinis fuerit data. Atque hanc ob rem definitionem calculi integralis ita instruximus, vt omnes huiusmodi investigationes in se complecteretur; differentialia enim cuiusque ordinis intelligi debent, et voce relationis, quae inter ea proponatur, sum usus, ut latius pateret voce rationis, quae tantum duorum differentialium comparationem indicare videatur. Ex his ergo divisionem calculi integralis constituere poterimus.

Definitio 3.

13. Calculus integralis dividitur in duas partes, quarum prior tradit methodum, functionem unius variabilis inveniendi ex data quadam relatione inter eius differentialia tam primi quam altiorum ordinum.

Pars autem altera methodum continet, functionem duarum pluriumve variabilium inveniendi, cum relatio inter eius differentialia sive primi sive altioris cuiusdam gradus fuerit proposita.

Corollarium 1.

14. Prout ergo functio ex data differentialium relatione invenienda, vel vnicam variabilem complectitur, vel duas pluresve, inde calculus integralis commode in duas partes principales dispescitur, quibus exponendis duos libros destinamus.

Corollarium 2.

15. Semper igitur calculus integralis in inventione functionum vel unius vel plurium variabilium versatur, cum scilicet relatio quaepiam inter eius differentialia sive altioris cuiuspiam ordinis fuerit proposita.

Scholion.

16. Cum hic primam partem calculi integralis in investigatione functionum unicae variabilis ex data differentialium relatione constituamus, plures partes pro numero variabilium functionem ingredientium constitui debere videatur, ita ut pars secunda functiones duarum variabilium, tertia trium, quarta quatuor etc. complectatur. Verum pro his posterioribus partibus methodus fere eadem requiritur, ita ut si inventio functionum duas variabiles involventium fuerit in potestate, via ad eas, quae plures variabiles implicant, satis sit patefacta; unde inventionem eiusmodi functionum, quae duas pluresve variabiles continent, commode coniungimus, indeque

unicam partem calculi integralis constituimus, posteriori libro tractandam.

Caeterum haec altera pars in elementis adhuc nusquam est tractata, etiamsi eius usus in Mechanica ac praecipue in doctrina fluidorum maximi sit usus. Quocirca cum in hoc genere praeter prima rudimenta vix quicquam sit exploratum, noster secundus liber de calculo integrali admodum erit sterilis, ac praeter commemorationem eorum, quae adhuc desiderantur, parum erit expectandum; verum hoc ipsum ad scientiae incrementum multum conferre videtur.

Definitio 4.

17. Uterque de calculo integrali liber commode subdividitur in partes pro gradu differentialium, ex quorum relatione functionem quaesitam investigari oportet. Ita prima pars versatur in relatione differentialium primi gradus, secunda in relatione differentialium secundi gradus, quorsum etiam differentialia altiorum graduum ob tenuitatem eorum, quae adhuc sunt investigata, referri possunt.

Corollarium 1.

18. Uterque ergo liber constabit duabus partibus, in quarum priore relatio inter differentialia primi gradus proposita considerabitur, in posteriore vero eiusmodi integrationes occurrent, vbi relatio inter differentialia secundi altiorumve graduum proponitur.

Corollarium 2.

19. In primi ergo libri parte prima eiusmodi functio variabilis x invenienda proponitur, ut posita ea functione $= y$, et $\frac{\partial y}{\partial x} = p$, relatio quaecunque data inter has tres quantitates x, y et p adimpleatur: seu proposita quacunque aequatione inter has ternas quantitates, ut indoles functionis y seu aequatio inter x et y tantum, exclusa p, eruatur.

Corollarium 3.

20. Posterioris autem partis primi libri quaestiones ita erunt comparatae, ut posito $\frac{\partial y}{\partial x} = p$, $\frac{\partial p}{\partial x} = q$, $\frac{\partial q}{\partial x} = r$ etc. si proponatur aequatio quaecunque inter quantitates x, y, p, q, r etc. indoles functionis y per x, seu aequatio inter x et y eliciatur.

Scholion 1.

21. Quae adhuc in calculo integrali sunt elaborata maximam partem ad libri primi partem primam sunt referenda, in qua excolenda Geometrae imprimis operam suam collocarunt: pauca sunt quae in parte posteriore sunt praestita, et alter liber, quem secundum fecimus, etiamnunc fere vacuus est relictus. Prima autem pars libri primi, in qua potissimum nostra tractatio consumetur, denuo in plures sectiones distinguitur, pro modo relationis, quae inter quantitates x, y et $p = \frac{\partial y}{\partial x}$ proponitur. Relatio enim prae caeteris simplicissima est, quando $p = \frac{\partial y}{\partial x}$ aequatur functioni cuipiam ipsius x, qua posita $= X$, vt sit $\frac{\partial y}{\partial x} = X$ seu $\partial y = X \partial x$; totum negotium in integratione formulae differentialis $X \partial x$ absolvitur: huius operationis iam supra mentionem fecimus, quae vulgo sub titulo integrationis formularum differentialium simplicium, seu unicam variabilem involventium tractari solet. Eodem res rediret, si $p = \frac{\partial y}{\partial x}$ aequaretur functioni ipsius y tantum, quandoquidem quantitates x et y ita inter se reciprocantur, ut altera tanquam functio alterius spectari possit; haec ergo ad sectionem primam referentur. Sin autem $p = \frac{\partial y}{\partial x}$ aequetur expressioni ambas quantitates x et y involventi, aequatio habetur differentialis huius formae $P\partial x + Q\partial y = 0$, ubi P et Q sunt expressiones quaecunque ex x, y et constantibus conflatae. Quanquam autem Geometrae multum in huiusmodi aequationum integratione desudarunt, tamen vix ultra quosdam casus satis

particulares sunt progressi. Sin autem p magis complicate per x et y determinatur, ut eius valor explicite exhiberi nequeat, veluti si fuerit:

$$p^5 = xxp^3 - xyp + x^5 - y^5$$

ne via quidem constat tentanda, quomodo inde relatio inter x et y investigari queat: pauca ergo, quae hic tradere licebit, cum praecedentibus secundam sectionem primae partis libri primi occupabunt. Ita ex universa nostra tractatione magis patebit, quod adhuc in calculo integrali desideretur, quam quid iam sit expeditum, cum hoc prae illo ut minima quaedam particula sit spectandum.

Scholion 2.

22. In singulis partibus, quas enarravimus, fieri etiam solet, ut non solum vna quaedam functio, sed etiam simul plures investigentur, ita vt neutra sine reliquis definiri possit, quemadmodum in Algebra communi usu venit, ut ad solutionem problematis plures incognitae in calculum sint introducendae, quae deinceps per totidem aequationes determinentur. Veluti si eiusmodi binae functiones y et z ipsius x sint inveniendae, ut sit:

$$x\partial y + azz\partial x = 0, \text{ et } xx\partial z + bxy\partial y = c\partial y:$$

hinc novae subdivisiones nostrae tractationis constitui possent. Verum quia hic ut in Algebra communi totum negotium ad eliminationem unius litterae revocatur, ut deinceps duae tantum variabiles in una aequatione supersint, hinc tractatio non multiplicanda videtur.

Scholion 3.

23. In secundo libro calculi integralis, quo functio duarum pluriumve variabilium ex data differentialium relatione investigatur, multo maior quaestionum varietas locum habet. Sit enim z functio binarum variabilium x et t investiganda, et cum $\left(\frac{\partial z}{\partial x}\right)$ denotet ratio-

nem ejus differentialis ad ∂x, si sola x pro variabili habeatur, at $\left(\frac{\partial z}{\partial t}\right)$ rationem ejus differentialis ad ∂t, si sola t variabilis sumatur; prima pars ejusmodi continebit quaestiones, in quibus certa quaedam relatio inter quantitates x, t, z et $\left(\frac{\partial z}{\partial x}\right)$, $\left(\frac{\partial z}{\partial t}\right)$ proponitur, et quaestio huc redit, ut hinc aequatio inter solas quantitates x, t et z eruatur; inde enim qualis z sit functio ipsarum x et t, patebit. In secunda parte praeter has formulas $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial t}\right)$ etiam istae $\left(\frac{\partial \partial z}{\partial x \partial x}\right)$, $\left(\frac{\partial \partial z}{\partial x \partial t}\right)$ et $\left(\frac{\partial \partial z}{\partial t \partial t}\right)$, in computum ingredientur: quarum significatio ita est intelligenda, ut positis prioribus $\left(\frac{\partial z}{\partial x}\right) = p$ et $\left(\frac{\partial z}{\partial t}\right) = q$, ubi p et q iterum certae erunt functiones ipsorum x et t, futurum sit simili expressionis modo,

$$\left(\frac{\partial \partial z}{\partial x \partial x}\right) = \left(\frac{\partial p}{\partial x}\right); \quad \left(\frac{\partial \partial z}{\partial x \partial t}\right) = \left(\frac{\partial p}{\partial t}\right) = \left(\frac{\partial q}{\partial x}\right); \quad \left(\frac{\partial \partial z}{\partial t \partial t}\right) = \left(\frac{\partial q}{\partial t}\right).$$

Proposita ergo relatione inter has formulas et praecedentes, simulque ipsas quantitates x, t et z, aequatio inter ternas istas quantitates solas x, t et z erui debet. Hujusmodi quaestiones frequenter occurrunt in Mechanica et Hydraulica, quando motus corporum flexibilium et fluidorum indagatur; ex quo maxime est optandum, ut haec altera sectio secundi libri calculi integralis omni cura excolatur. Neque vero opus erit, ut hanc investigationem ad differentialia altiora extendamus, cum nullae adhuc quaestiones sint tractatae, quae tanta calculi incrementa desiderent.

Definitio 5.

24. Si functiones, quae in calculo integrali ex relatione differentialium quaeruntur, algebraice exhiberi nequeant, tum eae vocantur *transcendentes*, quandoquidem earum ratio vires Analyseos communis transcendit.

Corollarium 1.

25. Quoties ergo integratio non succedit, toties functio quae per integrationem quaeritur, pro transcendente est habenda. Ita

si formula differentialis $X \partial x$ integrationem non admittit, ejus integrale, quod ita indicari solet $\int X \partial x$, est functio transcendens ipsius x.

Corollarium 2.

26. Hinc intelligitur, si y fuerit functio transcendens ipsius x, vicissim fore x functionem transcendentem ipsius y, atque ex hac conversione novae functiones transcendentes oriuntur.

Corollarium 3.

27. Pro variis partibus et sectionibus calculi integralis nascuntur etiam plura genera functionum transcendentium, quorum adeo numerus in infinitum exsurgit: unde patet, quanta copia omnium quantitatum possibilium nobis adhuc sit ignota.

Scholion 1.

28. Jam ante quam in Analysin infinitorum penetravimus species quasdam functionum transcendentium cognoscere licuit. Primam suppeditavit doctrina logarithmorum: si enim y denotet logarithmum ipsius x, ut sit $y = lx$, erit y utique functio transcendens ipsius x, sicque logarithmi quasi primam speciem functionum transcendentium constituunt. Deinde cum ex aequatione $y = lx$ vicissim sit $x = e^y$, erit x utique etiam functio transcendens ipsius y: ac tales functiones vocantur *exponentiales*. Porro autem consideratio angulorum aliud genus aperuit: veluti si angulus, cujus sinus est $= s$, ponatur, $= \Phi$, ut sit $\Phi = \text{Arc. sin. } s$, nullum est dubium, quin Φ sit functio transcendens ipsius s, et quidem infinitiformis: hincque cum convertendo prodeat $s = \sin. \Phi$, erit etiam sinus s functio transcendens anguli Φ. Quanquam autem hae functiones transcendentes sine subsidio calculi integralis sunt agnitae, tamen in ipso quasi limine calculi integralis ad eas deducimur: earumque indoles ita nobis jam est perspecta, ut propemodum functionibus

algebraicis accenseri queant. Quare etiam perpetuo in calculo integrali, quoties functiones transcendentes ibi repertas ad logarithmos vel angulos revocare licet, eas tanquam algebraicas spectare solemus.

Scholion 2.

29. Cum calculus integralis ex inversione calculi differentialis oriatur, perinde ac reliquae methodi inversae ad notitiam novi generis quantitatum nos perducit. Ita si a tyrone primorum elementorum nihil praeter notitiam numerorum integrorum positivorum postulemus, apprehensa additione, statim atque ad operationem inversam, subtractionem scilicet, ducitur, notionem numerorum negativorum assequetur. Deinde multiplicatione tradita, cum ad divisionem progreditur, ibi notionem fractionum accipiet. Porro postquam evectionem ad potestates didicerit, si per operationem inversam extractionem radicum suscipiat, quoties negotium non succedit, ideam numerorum irrationalium adipiscetur, haecque cognitio per totam Analysin communem sufficiens censetur. Simili ergo modo calculus integralis, quatenus integratio non succedit, novum nobis genus quantitatum transcendentium aperit. Non enim, uti omnium differentialia exhiberi possunt, ita vicissim omnium differentialium integralia exhibere licet.

Scholion 3.

30. Neque vero statim ac primis conatus in integratione expedienda fuerint irriti, functiones quaesitae pro transcendentibus sunt habendae; fieri enim saepe solet, ut integrale etiam algebraicum nonnisi per operationes artificiosas obtineri queat. Deinde quando functio quaesita fuerit transcendens, sollicite videndum est, num forte ad species illas simplicissimas logarithmorum vel angulorum revocari possit, quo casu solutio algebraicae esset aequiparanda. Quod si minus successerit, formam tamen simplicissimam functionum transcendentium, ad quam quaesitam reducere liceat, indagari conve-

niet. Ad usum autem longe commodissimum est, ut valores functionum transcendentium vero proxime exhibentur, quem in finem insignis pars calculi integralis in investigationem serierum infinitarum impenditur, quae valores earum functionum contineant.

Theorema.

31. Omnes functiones per calculum integralem inventae sunt indeterminatae, ac requirunt determinationem ex natura quaestionis, cujus solutionem suppeditant, petendam.

Demonstratio.

31. Cum semper infinitae dentur functiones, quarum idem est differentiale, siquidem functionis $P + C$, quicunque valor constanti C tribuatur, differentiale idem est $= \partial P$: vicissim etiam proposito differentiali ∂P, integrale est $P + C$, ubi pro C quantitatem constantem quamcunque ponere licet: unde patet eam functionem, cujus differentiale datur $= \partial P$, esse indeterminatam, cum quantitatem constantem arbitrariam in se involvat. Idem etiam eveniat necesse est, si functio ex quacunque differentialium relatione sit determinanda, semperque complectetur quantitatem constantem arbitrariam, cujus nullum vestigium in relatione differentialium apparuit. Determinabitur ergo hujusmodi functio per calculum integralem inventa, dum constanti illi arbitrariae certus valor tribuitur, quem semper natura quaestionis, cujus solutio ad illam functionem perduxerat, suppeditabit.

Corollarium 1.

32. Si ergo functio y ipsius x ex relatione quapiam differentialium definitur, per constantem arbitrariam ingressam ita determinari potest, ut posito $x = a$ fiat $y = b$: quo facto functio erit determinata, et pro quovis valore ipsi x tributo functio y determinatum obtinebit valorem.

Corollarium 2.

33. Si ex relatione differentialium secundi gradus functio y definiatur, binas involvet constantes arbitrarias, ideoque duplicem determinationem admittit, qua effici potest, ut posito $x = a$, non solum y obtineat datum valorem b, sed etiam ratio $\frac{\partial y}{\partial x}$ dato valori e fiat aequalis.

Corollarium 3.

34. Si y sit functio binarum variabilium x et t ex relatione differentialium eruta, etiam constantem arbitrariam involvet, cujus determinatione effici poterit, ut posito $t = a$, aequatio inter y et x prodeat data, seu naturam datae cujuspiam curvae exprimat.

Scholion.

35. Ista functionum integralium, seu quae per calculum integralem sunt inventae, determinatio quovis casu ex natura quaestionis tractatae facile deducitur; neque ulla difficultate laborat, nisi forte praeter necessitatem solutio ad differentialia fuerit perducta, cum per Analysin communem erui potuisset: quo casu perinde atque in Algebra quasi radices inutiles ingeruntur. Cum autem haec determinatio tantum in applicatione ad certos casus instituatur, hic ubi integrandi methodum in genere tradimus, integralia in omni amplitudine conabimur; ita ut constantes per integrationem ingressae maneant arbitrariae, neque nisi conditio quaedam urgeat, eas determinabimus. Caeterum determinatio functionum ipsius x simplicissima est, quae casu $x = 0$, ipsae evanescentes redduntur.

Definitio 6.

36. Integrale *completum* exhiberi dicitur, quando functio quaesita omni extensione cum constante arbitraria repraesentatur. Quando autem ista constans jam certo modo est determinata, integrale vocari solet *particulare*.

CONSPECTUS
UNIVERSI OPERIS
DE
CALCULO INTEGRALI.

LIBER PRIOR: Tradit methodum investigandi functiones unius variabilis ex data quadam relatione differentialium, continetque duas partes:

Pars prior: Quando relatio illa data tantum differentialia primi gradus complectitur.

Pars posterior: Quando relatio illa data differentialia secundi altiorumve graduum complectitur.

LIBER POSTERIOR: Tradit methodum investigandi functiones duarum pluriumve variabilium ex data quadam relatione differentialium, continetque duas partes:

Pars prior: Seu Investigatio functionum duarum tantum variabilium ex data differentialium cujusvis gradus relatione.

Pars posterior: Seu Investigatio functionum trium variabilium ex data differentialium relatione.

CALCULI INTEGRALIS
LIBER PRIOR.

PARS PRIMA,

SEU

METHODUS INVESTIGANDI FUNCTIONES UNIUS VARIABILIS EX DATA RELATIONE QUACUNQUE DIFFERENTIALIUM PRIMI GRADUS.

SECTIO PRIMA,

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM.

CAPUT I.

DE
INTEGRATIONE FORMULARUM DIFFERENTIALIUM RATIONALIUM.

Definitio.

40.

Formula differentialis *rationalis* est, quando variabilis x, cujus functio quaeritur, differentiale ∂x multiplicatur in functionem rationalem ipsius x: seu si X designet functionem rationalem ipsius x, haec formula differentialis $X \partial x$ dicitur rationalis.

Corollarium 1.

41. In hoc ergo capite ejusmodi functio ipsius x quaeritur, quae si ponatur y, ut $\frac{\partial y}{\partial x}$ aequetur functioni rationali ipsius x seu posita tali functione $= X$, ut sit $\frac{\partial y}{\partial x} = X$.

Corollarium 2.

42. Hinc quaeritur ejusmodi functio ipsius x, cujus differentiale sit $= X \partial x$; hujus ergo integrale, quod ita indicari solet $\int X \partial x$, praebebit functionem quaesitam.

Corollarium 3.

43. Quodsi P fuerit ejusmodi functio ipsius x, ut ejus differentiale ∂P sit $= X \partial x$, quoniam quantitatis $P + C$ idem est differentiale, formulae propositae $X \partial x$ integrale completum est $P + C$.

Scholion 1.

44. Ad libri primi partem priorem hujusmodi referuntur quaestiones, quibus functiones solius variabilis x, ex data differentialium

primi gradus relatione quaeruntur. Scilicet si functio quaesita $= y$ et $\frac{\partial y}{\partial x} = p$, id praestari oportet, ut proposita aequatione quacunque inter ternas quantitates x, y et p, inde indoles functionis y, seu aequatio inter x et y, elisa littera p, inveniatur. Quaestio autem sic in genere proposita vires analyseos adeo superare videtur, ut ejus solutio nunquam expectari queat. In casibus igitur simplicioribus vires nostrae sunt exercendae, inter quos primum occurrit casus, quo p functioni cuipiam ipsius x puta X aequatur, ut sit $\frac{\partial y}{\partial x} = X$, seu $\partial y = X \partial x$, ideoque integrale $y = \int X \partial x$ requiratur, in quo primam sectionem collocamus. Verum et hic casus pro varia indole functionis X latissime patet, ac plurimis difficultatibus implicatur: unde in hoc capite ejusmodi tantum quaestiones evolvere instituimus, in quibus ista functio X est rationalis: deinceps ad functiones irrationales atque adeo transcendentes progressuri. Hinc ista pars commode in duas sectiones subdividitur, in quarum altera integratio formularum simplicium, quibus $p = \frac{\partial y}{\partial x}$ functioni tantum ipsius x aequatur, est tradenda, in altera autem rationem integrandi doceri conveniet, cum proposita fuerit aequatio quaecunque ipsarum x, y et p. Et cum in his duabus sectionibus, ac potissimum priore, a Geometris plurimum sit elaboratum, eae maximam partem totius operis complebunt.

Scholion 2.

45. Prima autem integrationis principia ex ipso calculo differentiali sunt petenda, perinde ac principia divisionis ex multiplicatione, et principia extractionis radicum ex ratione evectionis ad potestates sumi solent. Cum igitur si quantitas differentianda ex pluribus partibus constet, ut $P + Q - R$, ejus differentiale sit $\partial P + \partial Q - \partial R$, ita vicissim si formula differentialis ex pluribus partibus constet, ut $P\partial x + Q\partial x - R\partial x$, integrale erit $\int P\partial x + \int Q\partial x - \int R\partial x$, singulis scilicet partibus seorsim integrandis. Deinde cum quantitatis aP differentiale sit $a\partial P$, formulae differentialis $aP\partial x$ integrale erit $a\int P\partial x$: scilicet

per quam quantitatem constantem formula differentialis multiplicatur, per eandem integrale multiplicari debet. Ita si formula differentialis sit $aP\partial x + bQ\partial x + cR\partial x$, quaecunque functiones ipsius x litteris P, Q, R designentur, integrale erit $a\int P\partial x + b\int Q\partial x + c\int R\partial x$: ita ut integratio tantum in singulis formulis $P\partial x$, $Q\partial x$ et $R\partial x$, sit instituenda. Hocque facto insuper adjici debet constans arbitraria C, ut integrale completum obtineatur.

Problema 1.

46. Invenire functionem ipsius x, ut ejus differentiale sit $= a x^n \partial x$, seu integrare formulam differentialem $a x^n \partial x$.

Solutio.

Cum potestatis x^m differentiale sit $m x^{m-1} \partial x$, erit vicissim:

$$\int m x^{m-1}\partial x = m\int x^{m-1}\partial x = x^m, \text{ ideoque } \int x^{m-1}\partial x = \tfrac{1}{m} x^m.$$

Fiat $m - 1 = n$, seu $m = n + 1$, erit:

$$\int x^n \partial x = \frac{1}{n+1} x^{n+1}, \text{ et } a\int x^n \partial x = \frac{a}{n+1} x^{n+1}.$$

Unde formulae differentialis propositae $a x^n \partial x$ integrale completum erit $\frac{a}{n+1} x^{n+1} + C$, cujus ratio vel inde patet, quod ejus differentiale revera fit $= a x^n \partial x$. Atque haec integratio semper locum habet, quicunque numerus exponenti n tribuatur, sive positivus sive negativus, sive integer sive fractus, sive etiam irrationalis.

Unicus casus hinc excipitur, quo est exponens $n = -1$, seu haec formula $\frac{a\partial x}{x}$ integranda proponitur. Verum in calculo differentiali jam ostendimus, si $l x$ denotet logarithmum hyperbolicum ipsius x, fore ejus differentiale $= \frac{\partial x}{x}$; unde vicissim concludimus esse $\int \frac{\partial x}{x} = l x$, et $\int \frac{a\partial x}{x} = a\, l\, x$. Quare adjecta constante arbitraria, erit formulae $\frac{a\partial x}{x}$ integrale completum $= a l x + C = l x^a + C$: quod etiam pro C ponendo $l c$, ita exprimitur $l c x^a$.

Corollarium 1.

47. Formulae ergo differentialis $a x^n \partial x$ integrale semper est algebraicum, solo excepto casu quo $n = -1$, et integrale per logarithmos exprimitur, qui ad functionis transcendentes sunt referendi. Est scilicet $\int \frac{a \partial x}{x} = a l x + C = l c x^a$.

Corollarium 2.

48. Si exponens n numeros positivos denotet, sequentes integrationes utpote maxime obviae probe sunt tenendae:

$$\int a \partial x = a x + C;\ \int a x \partial x = \tfrac{a}{2} x x + C;\ \int a x^2 \partial x = \tfrac{a}{3} x^3 + C;$$

$$\int a x^3 \partial x = \tfrac{a}{4} x^4 + C;\ \int a x^4 \partial x = \tfrac{a}{5} x^5 + C;\ \int a x^5 \partial x = \tfrac{a}{6} x^6 + C;\ \text{etc.}$$

Corollarium 3.

49. Si n sit numerus negativus, posito $n = -m$, fit

$$\int \frac{a \partial x}{x^m} = \frac{a}{1-m} x^{1-m} + C = \frac{-a}{(m-1) x^{m-1}} + C;$$

unde hi casus simpliciores notentur:

$$\int \frac{a \partial x}{x^2} = \frac{-a}{x} + C;\ \int \frac{a \partial x}{x^3} = \frac{-a}{2 x x} + C;\ \int \frac{a \partial x}{x^4} = \frac{-a}{3 x^3} + C;$$

$$\int \frac{a \partial x}{x^5} = \frac{-a}{4 x^4} + C;\ \int \frac{a \partial x}{x^6} = \frac{-a}{5 x^5} + C;\ \text{etc.}$$

Corollarium 4.

50. Quin etiam si n denotet numeros fractos, integralia hinc obtinentur. Sit primo $n = \frac{m}{2}$, erit

$$\int a \partial x \sqrt{x^m} = \frac{2a}{m+2} x \sqrt{x^m} + C.$$

Unde casus notentur:

$$\int a \partial x \sqrt{x} = \tfrac{2a}{3} x \sqrt{x} + C;\ \int a x \partial x \sqrt{x} = \tfrac{2a}{5} x^2 \sqrt{x} + C;$$

$$\int a x x \partial x \sqrt{x} = \tfrac{2a}{7} x^3 \sqrt{x} + C;\ \int a x^3 \partial x \sqrt{x} = \tfrac{2a}{9} x^4 \sqrt{x} + C;\ \text{etc.}$$

Corollarium 5.

51. Ponatur etiam $n = \frac{-m}{2}$, et habebitur

$$\int \frac{a\,\partial x}{\sqrt{x^m}} = \frac{2a}{2-m} \cdot \frac{x}{\sqrt{x^m}} + C = \frac{-2a}{(m-2)\sqrt{x^{m-2}}} + C.$$

Unde hi casus notentur:

$$\int \frac{a\,\partial x}{\sqrt{x}} = 2a\sqrt{x} + C; \int \frac{a\,\partial x}{x\sqrt{x}} = \frac{-2a}{\sqrt{x}} + C;$$

$$\int \frac{a\,\partial x}{xx\sqrt{x}} = \frac{-2a}{3x\sqrt{x}} + C; \int \frac{a\,\partial x}{x^3\sqrt{x}} = \frac{-2a}{5x^2\sqrt{x}} + C; \text{ etc.}$$

Corollarium 6.

52. Si in genere ponamus $n = \frac{\mu}{\nu}$, fiet:

$$\int a x^{\frac{\mu}{\nu}} \partial x = \frac{\nu a}{\mu+\nu} x^{\frac{\mu+\nu}{\nu}} + C, \text{ seu per radicalia:}$$

$$\int a\,\partial x \sqrt[\nu]{x^\mu} = \frac{\nu a}{\mu+\nu} \sqrt[\nu]{x^{\mu+\nu}} + C.$$

Sin autem ponatur $n = \frac{-\mu}{\nu}$ habebitur:

$$\int \frac{a\,\partial x}{x^{\frac{\mu}{\nu}}} = \frac{\nu a}{\nu-\mu} x^{\frac{\nu-\mu}{\nu}} + C, \text{ seu per radicalia:}$$

$$\int \frac{a\,\partial x}{\sqrt[\nu]{x^\mu}} = \frac{\nu a}{\nu-\mu} \sqrt[\nu]{x^{\nu-\mu}} + C.$$

Scholion 1.

53. Quanquam in hoc capite functiones tantum rationales tractare institueram, tamen istae irrationalitates tam sponte se obtulerunt, ut perinde ac rationales tractari possint. Caeterum hinc quoque formulae magis complicatae integrari possunt, si pro x functiones alius cujuspiam variabilis z statuantur. Veluti si ponamus $x = f + gz$, erit $\partial x = g\,\partial z$: quare si pro a scribamus $\frac{a}{g}$, habebitur:

$$\int a\,\partial z\,(f+gz)^n = \frac{a}{(n+1)g}(f+gz)^{n+1} + C.$$

Casu autem singulari, quo $n = -1$:

$$\int \frac{a\,\partial z}{f+gz} = \frac{a}{g}\,l(f+gz) + C.$$

Tum si sit $n = -m$, fiet:

$$\int \frac{a\,\partial z}{(f+gz)^m} = \frac{-a}{(m-1)g(f+gz)^{m-1}} + C.$$

Ac posito $n = \frac{\mu}{\nu}$, prodit:

$$\int a\,\partial z\,(f+gz)^{\frac{\mu}{\nu}} = \frac{\nu a}{(\nu+\mu)g}(f+gz)^{\frac{\mu}{\nu}+1} + C.$$

Posito autem $n = -\frac{\mu}{\nu}$, obtinetur,

$$\int \frac{a\,\partial z}{(f+gz)^{\frac{\mu}{\nu}}} = \frac{\nu a(f+gz)}{(\nu-\mu)g(f+gz)^{\frac{\mu}{\nu}}} + C.$$

Scholion 2.

54. Caeterum hic insignis proprietas annotari meretur. Cum hic quaeratur functio y, ut sit $\partial y = ax^n\partial x$, si ponamus $\frac{\partial y}{\partial x} = p$, haec habebitur relatio $p = ax^n$, ex qua functio y investigari debet. Quoniam igitur est

$$y = \frac{a}{n+1}x^{n+1} + C,$$

ob $ax^n = p$, erit quoque $y = \frac{px}{n+1} + C$: sicque casum habemus, ubi relatio differentialium per aequationem quandam inter x, y et p proponitur, cuique jam novimus satisfieri per aequationem $y = \frac{a}{n+1}x^{n+1} + C$. Verum haec non amplius erit integrale completum pro relatione in aequatione $y = \frac{px}{n+1} + C$ contenta, sed tantum particulare, quoniam integrale illud non involvit novam constantem, quae in relatione differentiali non insit. Integrale autem completum est $y = \frac{a D}{n+1} x^{n+1} + C$: novam constantem D involvens: hinc enim fit $\frac{\partial y}{\partial x} = a D x^n = p$, ideoque $y = \frac{p x}{n+1} + C$. Etsi hoc non ad praesens institutum pertinet, tamen notasse juvabit.

Problema 2.

55. Invenire functionem ipsius x, cujus differentiale sit $= X \partial x$, denotante X functionem quamcunque rationalem integram ipsius x, seu definire integrale $\int X \partial x$.

Solutio.

Cum X sit functio rationalis integra ipsius x, in hac forma contineatur necesse est:

$$X = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \text{etc.}$$

unde per problema praecedens integrale quaesitum est

$$\int X \partial x = C + \alpha x + \tfrac{1}{2}\beta x^2 + \tfrac{1}{3}\gamma x^3 + \tfrac{1}{4}\delta x^4 + \tfrac{1}{5} x^5 + \tfrac{1}{6}\zeta x^6 + \text{etc.}$$

Atque in genere si sit $X = \alpha x^\lambda + \beta x^\mu + \gamma x^\nu + \text{etc.}$ erit

$$\int X \partial x = C + \frac{\alpha}{\lambda+1} x^{\lambda+1} + \frac{\beta}{\mu+1} x^{\mu+1} + \frac{\gamma}{\nu+1} x^{\nu+1} + \text{etc.}$$

ubi exponentes λ, μ, ν etc. etiam numeros tam negativos quam fractos significare possunt; dummodo notetur, si fuerit $\lambda = -1$, fore $\int \frac{\alpha \partial x}{x} = \alpha l x$, qui est unicus casus ad ordinem transcendentium referendus.

Problema 3.

56. Si X denotet functionem quamcunque rationalem fractam ipsius x, methodum describere, cujus ope formulae $X \partial x$ integrale investigari conveniat.

Solutio.

Sit igitur $X = \frac{M}{N}$, ita ut M et N futurae sint functiones integrae ipsius x, ac primo dispiciatur, num summa potestas ipsius x in numeratore M tanta sit, vel etiam major quam in denominatore N? quo casu ex fractione $\frac{M}{N}$ partes integrae per divisionem eliciantur, quarum integratio, cum nihil habeat difficultatis, totum negotium reducitur ad ejusmodi fractionem $\frac{M}{N}$, in cujus numeratore M summa potestas ipsius x minor sit quam denominatore N.

Tum quaerantur omnes factores ipsius denominatoris N, tam simplices si fuerint reales, quam duplices reales, vicem scilicet binorum simplicium imaginariorum gerentes; simulque videndum est, utrum hi factores omnes sint inaequales nec ne? pro factorum enim aequalitate alio modo resolutio fractionis $\frac{M}{N}$ in fractiones simplices est instituenda, quandoquidem ex singulis factoribus fractiones partiales nascuntur, quarum aggregatum fractioni propositae $\frac{M}{N}$ aequatur. Scilicet ex factore simplici $a+bx$ nascitur fractio $\frac{A}{a+bx}$; si bini sint aequales, seu denominator N factorem habeat $(a+bx)^2$, hinc nascuntur fractiones $\frac{A}{(a+bx)^2}+\frac{B}{a+bx}$; ex hujusmodi autem factore $(a+bx)^3$ hae tres fractiones

$$\frac{A}{(a+bx)^3}+\frac{B}{(a+bx)^2}+\frac{C}{a+bx}$$

et ita porro.

Factor autem duplex, cujus forma est $aa-2abx\cos.\zeta+bbxx$, nisi alius ipsi fuerit aequalis, dabit fractionem partialem $\frac{A+Bx}{aa-2abx\cos.\zeta+bbxx}$: si autem denominator N duos hujusmodi factores aequales involvat, inde nascuntur binae hujusmodi fractiones partiales:

$$\frac{A+Bx}{(aa-2abx\cos.\zeta+bbxx)^2}+\frac{C+Dx}{aa-2abx\cos.\zeta+bbxx}:$$

at si cubus adeo $(aa-2abx\cos.\zeta+bbxx)^3$ fuerit factor denominatoris N, ex eo oriuntur hujusmodi tres fractiones partiales:

$$\frac{A+Bx}{(aa-2abx\cos.\zeta+bbxx)^3}+\frac{C+Dx}{(aa-2abx\cos.\zeta+bbxx)^2}+\frac{E+Fx}{aa-2abx\cos.\zeta+bbxx}$$

et ita porro.

Cum igitur hoc modo fractio proposita $\frac{M}{N}$ in omnes suas fractiones simplices fuerit resoluta, omnes continebuntur in alterutra harum formarum,

$$\text{vel } \frac{A}{(a+bx)^n}, \text{ vel } \frac{A+Bx}{(aa-2abx\cos.\zeta+bbxx)^n},$$

ac singulos jam per ∂x multiplicatos integrari oportet, erit omnium horum integralium aggregatum valor functionis quaesitae $\int X\partial x = \int \frac{M}{N}\partial x$.

Corollarium 1.

57. Pro integratione ergo omnium hujusmodi formularum $\frac{M}{N}\partial x$, totum negotium reducitur ad integrationem hujusmodi binarum formularum:

$$\int \frac{A\partial x}{(a+bx)^n} \text{ et } \int \frac{(A+Bx)\partial x}{(aa-2abx\cos.\zeta+bbxx)^n},$$

dum pro n successive scribuntur numeri 1, 2, 3, 4 etc.

Corollarium 2.

58. Ac prioris quidem formae integrale jam supra (53) est expeditum, unde patet fore:

$$\int \frac{A\partial x}{a+bx} = \frac{A}{b} l(a+bx) + \text{Const.}$$

$$\int \frac{A\partial x}{(a+bx)^2} = \frac{-A}{b(a+bx)} + \text{Const.}$$

$$\int \frac{A\partial x}{(a+bx)^3} = \frac{-A}{2b(a+bx)^2} + \text{Const.}$$

et generatim:

$$\int \frac{A\partial x}{(a+bx)^n} = \frac{-A}{(n-1)b(a+bx)^{n-1}} + \text{Const.}$$

Corollarium 3.

59. Ad propositum ergo absolvendum nihil aliud superest, nisi ut integratio hujus formulae

$$\int \frac{(A+Bx)\partial x}{(aa-2abx\cos.\zeta+bbxx)^n}$$

••

doceatur, primo quidem casu $n = 1$, tum vero casibus $n = 2$, $n = 3$, $n = 4$, etc.

Scholion 1.

60. Nisi vellemus imaginaria evitare, totum negotium ex jam traditis confici posset: denominatore enim N in omnes suos factores simplices resoluto, sive sint reales sive imaginarii, fractio proposita semper resolvi poterit in fractiones partiales hujus formae $\frac{A}{a + bx}$, vel hujus $\frac{A}{(a + bx)^n}$, quarum integralia cum sint in promptu, totius formae $\frac{M}{N} \partial x$ integrale habetur. Tum autem non parum molestum foret binas partes imaginarias ita conjungere, ut expressio realis resultaret, quod tamen rei natura absolute exigit.

Scholion 2.

61. Hic utique postulamus, resolutionem cujusque functionis integrae in factores nobis concedi, etiamsi algebra neutiquam adhuc eo sit perducta, ut haec resolutio actu institui possit. Hoc autem in Analysi ubique postulari solet, ut quo longius progrediamur, ea quae retro sunt relicta, etiamsi non satis fuerint explorata, tanquam cognita assumamus: sufficere scilicet hic potest, omnes factores per methodum approximationum quantumvis prope assignari posse. Simili modo cum in calculo integrali longius processerimus, integralia omnium hujusmodi formularum $X \partial x$, quaecunque functio ipsius x littera X significetur, tanquam cognita spectabimus; plurimumque nobis praestitisse videbimur, si integralia magis abscondita ad eas formas reducere valuerimus: atque hoc etiam in usu practico nihil turbat, cum valores talium formularum $\int X \partial x$, quantumvis prope assignare liceat, uti in sequentibus ostendemus. Caeterum ad has integrationes, resolutio denominatoris N in suos factores absolute est necessaria, propterea quod singuli hi factores in expressionem integralis ingrediuntur: paucissimi sunt casus, iique maxime obvii, quibus ista resolutione carere possumus: veluti si proponatur haec

formula $\frac{x^{n-1}\partial x}{1+x^n}$, statim patet, posito $x^n = v$, eam abire in $\frac{\partial v}{n(1+v)}$, cujus integrale est $\frac{1}{n}l(1+v) = \frac{1}{n}l(1+x^n)$; ubi resolutione in factores non fuerat opus. Verum hujusmodi casus per se tam sunt perspicui, ut eorum tractatio nulla peculiari explicatione indigeat.

Problema 4.

62. Invenire integrale hujus formulae:

$$y = \int \frac{(A + Bx)\partial x}{aa - 2abx\cos.\zeta + bbxx}.$$

Solutio.

Cum numerator duabus constet partibus $A\partial x + Bx\partial x$, haec posterior $Bx\partial x$ sequenti modo tolli poterit. Cum sit

$$l(aa - 2abx\cos.\zeta + bbxx) = \int \frac{-2ab\partial x\cos.\zeta + 2bbx\partial x}{aa - 2abx\cos.\zeta + bbxx},$$

multiplicetur haec aequatio per $\frac{B}{2bb}$, et a proposita auferatur: sic enim prodibit

$$y - \frac{B}{2bb}l(aa - 2abx\cos.\zeta + bbxx) = \int \frac{\left(A + \frac{Ba\cos.\zeta}{b}\right)\partial x}{aa - 2abx\cos.\zeta + bbxx}:$$

ita ut haec tantum formula integranda supersit. Ponatur brevitatis gratia $A + \frac{Ba\cos.\zeta}{b} = C$, ut habeatur haec formula:

$$\int \frac{C\partial x}{aa - 2abx\cos.\zeta + bbxx},$$

quae ita exhiberi potest

$$\int \frac{Cdx}{aa\sin.\zeta^2 + (bx - a\cos.\zeta)^2}.$$

Statuatur $bx - a\cos.\zeta = av\sin.\zeta$, hincque $\partial x = \frac{a\partial v\sin.\zeta}{b}$: unde formula nostra erit:

$$\int \frac{Ca\partial v\sin\zeta : b}{aa\sin.\zeta^2(1 + vv)} = \frac{C}{ab\sin.\zeta}\int \frac{\partial v}{1 + vv}.$$

Ex calculo autem differentiali novimus esse:

$$\int \frac{\partial v}{1+vv} = \text{Arc. tang. } v = \text{Arc. tang. } \frac{bx - a\cos.\zeta}{a\sin.\zeta};$$

unde ob $C = \frac{Ab + Ba\cos.\zeta}{b}$, erit nostrum integrale

$$\frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta} \text{Arc. tang. } \frac{bx - a\cos.\zeta}{a\sin.\zeta}.$$

Quocirca formulae propositae $\frac{(A+Bx)\partial x}{aa - 2abx\cos.\zeta + bbxx}$ integrale est:

$$\frac{B}{2bb} l(aa - 2abx\cos.\zeta + bbxx) + \frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta} \text{Arc. tang. } \frac{bx - a\cos.\zeta}{a\sin.\zeta},$$

quod ut fiat completum, constans arbitraria C insuper addatur.

Corollarium 1.

63. Si ad Arc. tang. $\frac{bx - a\cos.\zeta}{a\sin.\zeta}$ addamus Arc tang. $\frac{\cos.\zeta}{\sin.\zeta}$, quippe qui in constante addenda contentus concipiatur, prodibit Arc. tang. $\frac{bx\sin.\zeta}{a - bx\cos.\zeta}$, sicque habebimus:

$$\int \frac{(A+Bx)\partial x}{aa - 2abx\cos.\zeta + bbxx} = \frac{B}{2bb} l(aa - 2abx\cos.\zeta + bbxx) + \frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta} \text{Arc. tang. } \frac{bx\sin.\zeta}{a - bx\cos.\zeta}$$

adjecta constante C.

Corollarium 2.

64. Si velimus ut integrale hoc evanescat, posito $x = 0$, constans C sumi debet $= -\frac{B}{2bb} l aa$, sicque fiet:

$$\int \frac{(A+Bx)\partial x}{aa - 2abx\cos.\zeta + bbxx} = \frac{B}{bb} l \frac{\sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a} + \frac{Ab + Ba\cos.\zeta}{abb\sin.\zeta} \text{Arc. tang. } \frac{bx\sin.\zeta}{a - bx\cos.\zeta}.$$

Pendet ergo hoc integrale partim a logarithmis, partim ab arcubus circularibus seu angulis.

Corollarium 3.

65. Si littera B evanescat, pars a logarithmis pendens evanescit, fitque

$$\int \frac{A\partial x}{aa - 2abx\cos.\zeta + bbxx} = \frac{A}{ab\sin.\zeta} \text{Arc. tang. } \frac{bx\sin.\zeta}{a - bx\cos.\zeta} + C$$

sicque per solum angulum definitur.

Corollarium 4.

66. Si angulus ζ sit rectus, ideoque $\cos.\zeta = 0$, et $\sin.\zeta = 1$, habebitur:

$$\int \frac{(A + Bx)\partial x}{aa + bbxx} = \frac{B}{bb}\, l \frac{\sqrt{(aa + bbxx)}}{a} + \frac{A}{ab} \text{ Arc. tang. } \frac{bx}{a} + C.$$

Si angulus ζ sit 60°, ideoque $\cos.\zeta = \frac{1}{2}$ et $\sin.\zeta = \frac{\sqrt{3}}{2}$, erit:

$$\int \frac{(A + Bx)\partial x}{aa - abx + bbxx} = \frac{B}{bb}\, l \frac{\sqrt{(aa - abx + bbxx)}}{a} + \frac{2Ab + Ba}{abb\sqrt{3}} \text{ Arc. tang. } \frac{bx\sqrt{3}}{2a - bx}.$$

At si $\zeta = 120^\circ$, ideoque $\cos.\zeta = -\frac{1}{2}$ et $\sin.\zeta = \frac{\sqrt{3}}{2}$ erit:

$$\int \frac{(A + Bx)\partial x}{aa + abx + bbxx} = \frac{B}{bb}\, l \frac{\sqrt{(aa + abx + bbxx)}}{a} + \frac{2Ab - Ba}{abb\sqrt{3}} \text{ Arc. tang. } \frac{bx\sqrt{3}}{2a + bx}.$$

Scholion 1.

67. Omnino hic notatu dignum evenit, quod casu $\zeta = 0$, quo denominator $aa - 2abx + bbxx$ fit quadratum, ratio anguli ex integrali discedat. Posito enim angulo ζ infinite parvo, erit $\cos.\zeta = 1$, et $\sin.\zeta = \zeta$; unde pars logarithmica fit $\frac{B}{bb}\, l \frac{a - bx}{a}$, et altera pars:

$$\frac{Ab + Ba}{abb\zeta} \text{ Arc. tang. } \frac{bx\zeta}{a - bx} = \frac{(Ab + Ba)x}{ab(a - bx)}$$

quia arcus infinite parvi $\frac{bx\zeta}{a - bx}$ tangens ipsi est aequalis; sicque haec pars fit algebraica. Quocirca erit

$$\int \frac{(A + Bx)\partial x}{(a - bx)^2} = \frac{B}{bb}\, l \frac{a - bx}{a} + \frac{(Ab + B\,a)x}{ab(a - bx)} + \text{Const.}$$

cujus veritas ex praecedentibus est manifesta: est enim

$$\frac{A + Bx}{(a - bx)^2} = -\frac{B}{b(a - bx)} + \frac{Ab + Ba}{b(a - bx)^2}.$$

Jam vero est

$$\int \frac{-B\partial x}{b(a - bx)} = \frac{B}{bb}\, l(a - bx) - \frac{B}{bb}\, l\, a = \frac{B}{bb}\, l \frac{a - bx}{a},$$

$$\int \frac{(Ab + Ba)\partial x}{b(a - bx)^2} = \frac{Ab + Ba}{bb(a - bx)} - \frac{(Ab + Ba)}{abb} = \frac{(Ab + B\,a)x}{ab(a - bx)},$$

siquidem utraque integratio ita determinetur ut, casu $x = 0$, integralia evanescant.

Scholion 2.

68. Simili modo, quo hic usi sumus, si in formula differentiali fracta $\frac{M\partial x}{N}$, summa potestas ipsius x, in numeratore M, uno gradu minor sit quam in denominatore N, etiam is terminus tolli poterit. Sit enim

$$M = Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{etc. et}$$
$$N = \alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \text{etc.}$$

ac ponatur $\frac{M\partial x}{N} = \partial y$: Cum jam sit

$$\partial N = n\alpha x^{n-1}\partial x + (n-1)\beta x^{n-2}\partial x + (n-2)\gamma x^{n-3}\partial x + \text{etc.}$$

erit:

$$\frac{A\partial N}{n\alpha N} = \frac{\partial x}{N}\left(Ax^{n-1} + \frac{(n-1)A\beta}{n\alpha}x^{n-2} + \frac{(n-2)A\gamma}{n\alpha}x^{n-3} + \text{etc.}\right)$$

quo valoro inde subtracto remanebit:

$$\partial y - \frac{A\partial N}{n\alpha N} = \frac{\partial x}{N}\left[\left(B - \frac{(n-1)A\beta}{n\alpha}\right)x^{n-2} + \left(C - \frac{(n-2)A\gamma}{n\alpha}\right)x^{n-3} + \text{etc.}\right]$$

Quare si brevitatis gratia ponatur:

$$B - \frac{(n-1)A\beta}{n\alpha} = \mathfrak{B};\ C - \frac{(n-2)A\gamma}{n\alpha} = \mathfrak{C};\ D - \frac{(n-3)A\delta}{n\alpha} = \mathfrak{D};\ \text{etc.}$$

obtinebitur:

$$y = \frac{A}{n\alpha}lN + \int\frac{\partial x(\mathfrak{B}x^{n-2} + \mathfrak{C}x^{n-3} + \mathfrak{D}x^{n-4} + \text{etc.})}{\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \text{etc.}} = \int\frac{M\partial x}{N}.$$

Hoc igitur modo omnes formulae differentiales fractae eo reduci possunt, ut summa potestas ipsius x in numeratore duobus pluribusve gradibus minor sit quam in denominatore.

Problema 5.

69. Formulam integralem $\int\frac{(A + \beta x)\partial x}{(aa - 2abx\cos.\zeta + bbxx)^{n+1}}$ ad aliam similem reducere, ubi potestas denominatoris sit uno gradu inferior.

Solutio.

Sit brevitatis gratia $aa - 2abx\cos.\zeta + bbxx = X$, ac ponatur $\int\frac{(A + Bx)\partial x}{X^{n+1}} = y$. Cum ob $\partial X = -2ab\partial x\cos.\zeta + 2bbx\partial x$, sit:

$$\partial\frac{C + Dx}{X^n} = -\frac{n(C + Dx)\partial X}{X^{n+1}} + \frac{D\partial x}{X^n}$$

ideoque:

$$\frac{C + Dx}{X^n} = \int\frac{2nb(C + Dx)(a\cos.\zeta - bx)\partial x}{X^{n+1}} + \int\frac{D\partial x}{X^n},$$

habebimus:

$$y + \frac{C + Dx}{X^n} = \int\frac{\partial x[A + 2nCab\cos.\zeta + x(B + 2nDab\cos.\zeta - 2nCbb) - 2nDbbxx]}{X^{n+1}}$$

$$+ \int\frac{D\partial x}{X^n}.$$

Jam in formula priori litterae C et D ita definiantur, ut numerator per X fiat divisibilis. Oportet ergo sit $= -2nDX\partial x$, unde nanciscimur:

$$A + 2nCab\cos.\zeta = -2nDaa, \text{ et}$$
$$B + 2nDab\cos.\zeta - 2nCbb = 4nDab\cos.\zeta,$$

seu $B - 2nCbb = 2nDab\cos.\zeta$; hincque

$$2nDa = \frac{B - 2nCbb}{b\cos.\zeta}.$$

At ex priori conditione est

$$2nDa = \frac{-A - 2nCab\cos.\zeta}{a}, \text{ quibus aequatis fit:}$$

$$Ba + Ab\cos.\zeta - 2nCabb\sin.\zeta^2 = 0, \text{ seu}$$

$$C = \frac{Ba + Ab\cos.\zeta}{2nabb\sin.\zeta^2}: \text{ unde}$$

$$B - 2nCbb = \frac{Ba\sin.\zeta^2 - Ba - Ab\cos.\zeta}{a\sin\zeta^2} = \frac{-Ab\cos.\zeta - Ba\cos.\zeta^2}{a\sin.\zeta^2},$$

ita ut reperiatur $D = \frac{-Ab - Ba\cos.\zeta}{2naab\sin.\zeta^2}$. Sumtis ergo litteris

$$C = \frac{Ba + Ab\cos.\zeta}{2naabb\sin.\zeta^2} \text{ et } D = \frac{-Ab - Ba\cos.\zeta}{2naab\sin.\zeta^2}, \text{ erit}$$

$$y + \frac{C + Dx}{X^n} = \int \frac{-2nD\partial x}{X^n} + \int \frac{B\partial x}{X^n} = -(2n-1)D\int \frac{\partial x}{X^n}:$$

ideoque

$$\int \frac{(A + Bx)\partial x}{X^{n+1}} = \frac{-C - Dx}{X^n} - (2n-1)D\int \frac{\partial X}{X^n}, \text{ sive}.$$

$$\int \frac{(A + Bx)\partial x}{X^{n+1}} = \frac{-Baa - Aab\cos.\zeta + (Abb + Bab\cos.\zeta)x}{2naabb\sin.\zeta^2 X^n}$$
$$+ \frac{(2n-1)(Ab + Ba\cos.\zeta)}{2naab\sin.\zeta^2}\int \frac{\partial x}{X^n}.$$

Quare si formula $\int \frac{\partial x}{X^n}$ constet, etiam integrale hoc $\int \frac{(A + Bx)\partial x}{X^{n+1}}$ assignari poterit.

Corollarium 1.

70. Cum igitur manente

$$X = aa - 2abx\cos.\zeta + bbxx, \text{ fit}$$

$$\int \frac{\partial x}{X} = \frac{1}{ab\sin.\zeta}\text{Arc. tang.}\frac{bx\sin.\zeta}{a - bx\cos.\zeta} + \text{Const. erit:}$$

$$\int \frac{(A + Bx)\partial x}{X^2} = \frac{-Baa - Aab\cos.\zeta + (Abb + Bab\cos.\zeta)x}{2aabb\sin.\zeta^2 X}$$
$$+ \frac{Ab + Ba\cos\zeta}{2a^3bb\sin.\zeta^3}\text{Arc. tang.}\frac{bx\sin.\zeta}{a - bx\cos.\zeta} + \text{Const.}$$

Ideoque posito $B = 0$ et $A = 1$, fiet

$$\int \frac{\partial x}{X^2} = \frac{-a\cos.\zeta + bx}{2aab\sin.\zeta^2 X} + \frac{1}{2a^3b\sin.\zeta^3}\text{Arc. tang.}\frac{bx\sin.\zeta}{a - bx\cos.\zeta} + \text{Const.}$$

Integrale ergo $\int \frac{(A + Bx)\partial x}{X^2}$ logarithmos non involvit.

Corollarium 2.

71. Hinc ergo cum sit:

$$\int \frac{\partial x}{X^3} = \frac{-a\cos.\zeta + bx}{4aab\sin.\zeta^2.X^2} + \frac{3}{4aa\sin.\zeta^2}\int \frac{\partial x}{X^2} + \text{Const.}$$

erit illum valorem substituendo:

$$\int \frac{\partial x}{X^3} = \frac{-a\cos.\zeta + bx}{4aab\sin.\zeta^2.X^2} + \frac{3(-a\cos.\zeta + bx)}{2.4a^4b\sin\zeta^4.X}$$
$$+ \frac{1.3}{2.4a^5b\sin.\zeta^5}\text{Arc. tang.}\frac{bx\sin.\zeta}{a - bx\cos.\zeta}.$$

Hincque porro concluditur:

$$\int\frac{\partial x}{X^4}=\frac{-a\cos.\zeta+bx}{6aab\sin.\zeta^2.X^3}+\frac{5(-a\cos.\zeta+bx)}{4.6a^4b\sin\zeta^4.X^2}+\frac{3.5(a-\cos\zeta+bx)}{2.4.6a^6b\sin.\zeta^6.X}$$
$$+\frac{1.3.5}{2.4.6a^7b\sin\zeta^7}\text{Arc. tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}.$$

Corollarium 3.

72. Sic ulterius progrediendo, omnium hujusmodi formularum integralia obtinebuntur:

$$\int\frac{\partial x}{X},\ \int\frac{\partial x}{X^2},\ \int\frac{\partial x}{X^3},\ \int\frac{\partial x}{X^4},\ \text{etc.}$$

quorum primum arcu circulari solo exprimitur, reliqua vero praeterea partes algebraicas continent.

Scholion.

73. Sufficit autem integralia $\int\frac{\partial x}{X^{n+1}}$ nosse, quia formula $\int\frac{(A+Bx)\partial x}{X^{n+1}}$ facile eo reducitur: ita enim repraesentari potest

$$\frac{1}{2bb}\int\frac{2Abb\partial x+2Bbbx\partial x-2Bab\partial x\cos.\zeta+2Bab\partial x\cos.\zeta}{X^{n+1}}$$

quae ob $2bbx\partial x-2ab\partial x\cos.\zeta=\partial X$, abit in hanc

$$\frac{1}{2bb}\int\frac{B\partial X}{X^{n+1}}+\frac{1}{b}\int\frac{(Ab+Ba\cos.\zeta)\partial x}{X^{n+1}}.$$

At $\int\frac{\partial X}{X^{n+1}}=-\frac{1}{nX^n}$, unde habebitur:

$$\int\frac{(A+Bx)\partial x}{X^{n+1}}=\frac{-B}{2nbbX^n}+\frac{Ab+Ba\cos.\zeta}{b}\int\frac{\partial x}{X^{n+1}};$$

unde tantum opus est nosse integralia $\int\frac{\partial X}{X^{n+1}}$, quae modo exhibuimus. Atque haec sunt omnia subsidia quibus indigemus ad omnes formulas fractas $\frac{M}{N}\partial x$ integrandas, dummodo M et N sunt functio-

Exemplum 2.

77. *Proposita formula differentiali* $\frac{x^{m-1}\partial x}{1+x^n}$, *siquidem exponens* m — 1 *minor sit quam* n, *integrale definire.*

In capite ultimo Institut. Calculi Differential. invenimus fractiones simplices, in quas haec fractio $\frac{x^m}{1+x^n}$ resolvitur, sumto π pro mensura duorum angulorum rectorum, in hac forma generali contineri:

$$\frac{2\sin\frac{(2k-1)\pi}{n}\sin.\frac{m(2k-1)\pi}{n} - 2\cos.\frac{m(2k-1)\pi}{n}\left(x-\cos.\frac{(2k-1)\pi}{n}\right)}{n\left(1-2x\cos.\frac{(2k-1)\pi}{n}+xx\right)}:$$

ubi pro k successive omnes numeros 1, 2, 3, etc. substitui convenit, quoad $2k-1$ numerum n superare incipiat. Hac ergo forma in ∂x ducta, et cum generali nostra

$\frac{(A+Bx)\partial x}{aa-2abx\cos.\zeta+bbxx}$ comparata, fit

$a=1$, $b=1$, $\zeta=\frac{(2k-1)\pi}{n}$; et

$$A=\frac{2}{n}\sin.\frac{(2k-1)\pi}{n}\sin.\frac{m(2k-1)\pi}{n}+\frac{2}{n}\cos.\frac{(2k-1)\pi}{n}\cos.\frac{m(2k-1)\pi}{n}:$$

seu $A=\frac{2}{n}\cos.\frac{(m-1)(2k-1)\pi}{n}$, et

$B=-\frac{2}{n}\cos.\frac{m(2k-1)\pi}{n}$, unde fit

$$Ab+Ba\cos.\zeta=\frac{2}{n}\sin.\frac{(2k-1)\pi}{n}\sin.\frac{m(2k-1)\pi}{n}:$$

ac propterea hujus partis integrale erit =

$$-\frac{2}{n}\cos.\frac{m(2k-1)\pi}{n}\,l\sqrt{\left(1-2x\cos.\frac{(2k-1)\pi}{n}+xx\right)}$$

$$+\frac{2}{n}\sin.\frac{m(2k-1)\pi}{n}\text{ Arc. tang. }\frac{x\sin.\frac{(2k-1)\pi}{n}}{1-x\cos.\frac{(2k-1)\pi}{n}}.$$

Ac si n numerus impar, praeterea accedit fractio $\frac{\pm\partial x}{n(1+x)}$ cujus integrale est $\pm\frac{1}{n}l(1+x)$: ubi signum superius valet, si m impar, inferius vero, si m par. Quocirca integrale quaesitum $\int \frac{x^{m-1}\partial x}{1+x^n}$ sequenti modo exprimetur:

$$-\frac{2}{n}\cos.\frac{m\pi}{n}\, l\sqrt{(1-2x\cos.\frac{\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{m\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{\pi}{n}}{1-x\cos.\frac{\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{3m\pi}{n}\, l\sqrt{(1-2x\cos.\frac{3\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{3m\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{3\pi}{n}}{1-x\cos.\frac{3\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{5m\pi}{n}\, l\sqrt{(1-2x\cos.\frac{5\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{5m\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{5\pi}{n}}{1-x\cos.\frac{5\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{7m\pi}{n}\, l\sqrt{(1-2x\cos.\frac{7\pi}{n}+xx)}+\frac{2}{n}\sin.\frac{7m\pi}{n}\,\text{Arc. tang.}\frac{x\sin.\frac{7\pi}{n}}{1-x\cos.\frac{7\pi}{n}}$$

etc.

secundum numeros impares ipso n minores, sicque totum obtinetur integrale si n fuerit numerus par; sin autem n sit numerus impar, insuper accedit haec pars $\pm\frac{1}{n}l(1+x)$, prout m sit numerus vel impar vel par: unde si $m=1$, accedit insuper $+\frac{1}{n}l(1+x)$.

Corollarium 1.

78. Sumamus $m=1$, ut habeatur forma $\int\frac{\partial x}{1+x^n}$, et pro variis casibus ipsius n adipiscimur:

I. $\int\frac{\partial x}{1+x}=l(1+x)$

II. $\int\frac{\partial x}{1+x^2}=\text{Arc. tang}\, x$

III. $\int\frac{\partial x}{1+x^3}=-\frac{2}{3}\cos.\frac{\pi}{3}\, l\sqrt{(1-2x\cos.\frac{\pi}{3}+xx)}+\frac{2}{3}\sin.\frac{\pi}{3}\,\text{Arc. tang.}\frac{x\sin.\frac{\pi}{3}}{1-x\cos.\frac{\pi}{3}}$

$+\frac{1}{3}l(1+x)$

$$\text{IV. } \int \frac{\partial x}{1+x^4} = \begin{cases} -\frac{2}{4}\cos.\frac{\pi}{4}\, l\sqrt{(1-2x\cos.\frac{\pi}{4}+xx)} + \frac{2}{4}\sin.\frac{\pi}{4}\,\text{Arc.tang.}\,\frac{x\sin.\frac{\pi}{4}}{1-x\cos.\frac{\pi}{4}} \\ -\frac{2}{4}\cos.\frac{3\pi}{4}\, l\sqrt{(1-2x\cos.\frac{3\pi}{4}+xx)} + \frac{2}{4}\sin.\frac{3\pi}{4}\,\text{Arc.tang.}\,\frac{x\sin.\frac{3\pi}{4}}{1-x\cos.\frac{3\pi}{4}} \end{cases}$$

$$\text{V. } \int \frac{\partial x}{1+x^5} = \begin{cases} -\frac{2}{5}\cos.\frac{\pi}{5}\, l\sqrt{(1-2x\cos.\frac{\pi}{5}+xx)} + \frac{2}{5}\sin.\frac{\pi}{5}\,\text{Arc.tang.}\,\frac{x\sin.\frac{\pi}{5}}{1-x\cos.\frac{\pi}{5}} \\ -\frac{2}{5}\cos.\frac{3\pi}{5}\, l\sqrt{(1-2x\cos.\frac{3\pi}{5}+xx)} + \frac{2}{5}\sin.\frac{3\pi}{5}\,\text{Arc.tang}\,\frac{x\sin.\frac{3\pi}{5}}{1-x\cos.\frac{3\pi}{5}} \\ +\frac{1}{5}\, l(1+x) \end{cases}$$

$$\text{VI. } \int \frac{\partial x}{1+x^6} = \begin{cases} -\frac{2}{6}\cos.\frac{\pi}{6}\, l\sqrt{(1-2x\cos.\frac{\pi}{6}+xx)} + \frac{2}{6}\sin.\frac{\pi}{6}\,\text{Arc.tang.}\,\frac{x\sin.\frac{\pi}{6}}{1-x\cos.\frac{\pi}{6}} \\ -\frac{2}{6}\cos.\frac{3\pi}{6}\, l\sqrt{(1-2x\cos.\frac{3\pi}{6}+xx)} + \frac{2}{6}\sin.\frac{3\pi}{6}\,\text{Arc.tang}\,\frac{x\sin\frac{3\pi}{6}}{1-x\cos.\frac{3\pi}{6}} \\ -\frac{2}{6}\cos.\frac{5\pi}{6}\, l\sqrt{(1-2x\cos.\frac{5\pi}{6}+xx)} + \frac{2}{6}\sin.\frac{5\pi}{6}\,\text{Arc.tang}\,\frac{x\sin.\frac{5\pi}{6}}{1-x\cos.\frac{5\pi}{6}} \end{cases}$$

Corollarium 2.

79. Loco sinuum et cosinuum valores, ubi commode fieri potest, substituendo, obtinemus:

$$\int \frac{\partial x}{1+x^3} = -\frac{1}{3} l\sqrt{(1-x+xx)} + \frac{1}{\sqrt{3}}\,\text{Arc. tang.}\,\frac{x\sqrt{3}}{2-x} + \frac{1}{3} l(1+x)$$

seu

$$\int \frac{\partial x}{1+x^3} = \frac{1}{3}\, l\, \frac{1+x}{\sqrt{(1-x+xx)}} + \frac{1}{\sqrt{3}}\,\text{Arc. tang.}\,\frac{x\sqrt{3}}{2-x}.$$

Deinde ob $\sin.\frac{\pi}{4} = \cos.\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \sin.\frac{3\pi}{4} = -\cos.\frac{3\pi}{4}$, fit

$$\int \frac{\partial x}{1+x^4} = +\frac{1}{2\sqrt{2}}\, l\, \frac{\sqrt{(1+x\sqrt{2}+xx)}}{\sqrt{(1-x\sqrt{2}+xx)}} + \frac{1}{2\sqrt{2}}\,\text{Arc. tang.}\,\frac{x\sqrt{2}}{1-xx},$$

tum vero

$$\int \frac{\partial x}{1+x^6} = \frac{1}{2\sqrt{3}}\, l\, \frac{\sqrt{(1+x\sqrt{3}+xx)}}{\sqrt{(1-x\sqrt{3}+xx)}} + \frac{1}{6}\,\text{Arc. tang.}\,\frac{3x(1-xx)}{1-4xx+x^4}.$$

Exemplum 2.

80. *Proposita formula differentiali* $\frac{x^{m-1}\,dx}{1-x^n}$, *siquidem exponens* m — 1 *sit minor quam* n, *ejus integrale definire.*

Functionis fractae $\frac{x^{m-1}}{1-x^n}$ pars, ex factore quocunque oriunda, hac forma continetur:

$$\frac{2\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}-2\cos.\frac{2mk\pi}{n}\left(x-\cos.\frac{2k\pi}{n}\right)}{n\left(1-2x\cos.\frac{2k\pi}{n}+xx\right)}$$

quae cum forma nostra $\frac{A+Bx}{aa-2abx\cos.\zeta+bbxx}$ comparata, dat $a=1$, $b=1$, $\zeta=\frac{2k\pi}{n}$;

$$A=\frac{2}{n}\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}+\frac{2}{n}\cos.\frac{2k\pi}{n}\cos.\frac{2mk\pi}{n},$$

$$B=-\frac{2}{n}\cos.\frac{2mk\pi}{n}: \text{ hincque}$$

$$Ab+Ba\cos.\zeta=\frac{2}{n}\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}.$$

Ex quo integrale hinc oriundum erit =

$$-\frac{2}{n}\cos.\frac{2km\pi}{n}\,l\sqrt{\left(1-2x\cos.\frac{2k\pi}{n}+xx\right)}$$

$$+\frac{2}{n}\sin.\frac{2km\pi}{n}\text{ Arc. tang. }\frac{x\sin.\frac{2k\pi}{n}}{1-x\cos.\frac{2k\pi}{n}}:$$

ubi pro k successive omnes numeri 0, 1, 2, 3, etc. substitui debent, quamdiu $2k$ non superat n. At casu $k=0$ fit integralis pars $=-\frac{1}{n}l(1-x)$: et quando n est numerus par, ultima pars oritur ex $2k=n$, quae ergo erit

$$-\frac{2}{n}\cos.m\pi\,l\sqrt{(1+2x+xx)}=-\frac{\cos.m\pi}{n}l(1+x):$$

ergo si m est par, erit $\cos.m\pi=+1$, at si m impar, fit $\cos.m\pi=-1$. Quocirca integrale $\int\frac{x^{m-1}\partial x}{1-x^n}$, hoc modo exprimitur:

$$-\frac{1}{n}l(1-x)$$

$$-\frac{2}{n}\cos.\frac{2m\pi}{n}l\surd(1-2x\cos.\frac{2\pi}{n}+xx)$$

$$+\frac{2}{n}\sin.\frac{2m\pi}{n}\text{ Arc. tang. }\frac{x\sin.\frac{2\pi}{n}}{1-x\cos.\frac{2\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{4m\pi}{n}l\surd(1-2x\cos.\frac{4\pi}{n}+xx)$$

$$+\frac{2}{n}\sin.\frac{4m\pi}{n}\text{ Arc. tang. }\frac{x\sin.\frac{4\pi}{n}}{1-x\cos.\frac{4\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{6m\pi}{n}l\surd(1-2x\cos.\frac{6\pi}{n}+xx)$$

$$+\frac{2}{n}\sin.\frac{6m\pi}{n}\text{ Arc. tang. }\frac{x\sin.\frac{6\pi}{n}}{1-x\cos.\frac{6\pi}{n}}$$

etc.

Corollarium.

81. Sit $m=1$, et pro n successive numeri 1, 2, 3, etc. substituantur, ut nanciscamur sequentes integrationes:

I. $\int\frac{\partial x}{1-x}=-l(1-x)$

II. $\int\frac{\partial x}{1-xx}=-\frac{1}{2}l(1-x)+\frac{1}{2}l(1+x)=\frac{1}{2}l\frac{1+x}{1-x}$

III. $\int\frac{\partial x}{1-x^3}=\begin{cases}-\frac{1}{3}l(1-x)-\frac{2}{3}\cos.\frac{2}{3}\pi\, l\surd(1-2x\cos.\frac{2}{3}\pi+xx)\\ \qquad +\frac{2}{3}\sin.\frac{2}{3}\pi\text{ Arc. tang. }\frac{x\sin.\frac{2}{3}\pi}{1-x\cos.\frac{2}{3}\pi}\end{cases}$

IV. $\int\frac{\partial x}{1-x^4}=\begin{cases}-\frac{1}{4}l(1-x)-\frac{2}{4}\cos.\frac{2}{4}\pi\, l\surd(1-2x\cos.\frac{2}{4}\pi+xx)\\ \qquad +\frac{2}{4}\sin.\frac{2}{4}\pi\text{ Arc. tang. }\frac{x\sin.\frac{2}{4}\pi}{1-x\cos.\frac{2}{4}\pi}\\ +\frac{1}{4}l(1+x)\end{cases}$

$$\text{V. } \int \frac{\partial x}{1-x^5} = \begin{cases} -\frac{1}{5} l(1-x) - \frac{2}{5}\cos.\frac{2}{5}\pi\, l\sqrt{(1-2x\cos.\frac{2}{5}\pi + xx)} \\ \quad + \frac{2}{5}\sin.\frac{2}{5}\pi \text{ Arc. tang. } \frac{x\sin.\frac{2}{5}\pi}{1-x\cos.\frac{2}{5}\pi} \\ \quad - \frac{2}{5}\cos.\frac{4}{5}\pi\, l\sqrt{(1-2x\cos.\frac{4}{5}\pi + xx)} \\ \quad + \frac{2}{5}\sin.\frac{4}{5}\pi \text{ Arc. tang. } \frac{x\sin.\frac{4}{5}\pi}{1-x\cos.\frac{4}{5}\pi} \end{cases}$$

$$\text{VI. } \int \frac{\partial x}{1-x^6} = \begin{cases} -\frac{1}{6} l(1-x) - \frac{2}{6}\cos.\frac{2}{6}\pi\, l\sqrt{(1-2x\cos.\frac{2}{6}\pi + xx)} \\ \quad + \frac{2}{6}\sin.\frac{2}{6}\pi \text{ Arc. tang. } \frac{x\sin.\frac{2}{6}\pi}{1-x\cos.\frac{2}{6}\pi} \\ + \frac{1}{6} l(1+x) - \frac{2}{6}\cos.\frac{4}{6}\pi\, l\sqrt{(1-2x\cos.\frac{4}{6}\pi + xx)} \\ \quad + \frac{2}{6}\sin.\frac{4}{6}\pi \text{ Arc. tang. } \frac{x\sin.\frac{4}{6}\pi}{1-x\cos.\frac{4}{6}\pi} \end{cases}$$

Exemplum 4.

82. *Proposita formula differentiali* $\frac{(x^{m-1} + x^{n-m-1})\,\partial x}{1+x^n}$, *existente* $n > m - 1$, *ejus integrale definire.*

Ex exemplo 2[do] patet, integralis partem quamcunque in genere esse, sumto i pro numero quocunque impare non majore quam n,

$$-\frac{2}{n}\cos.\frac{im\pi}{n}\, l\sqrt{\left(1 - 2x\cos.\frac{i\pi}{n} + xx\right)}$$

$$+\frac{2}{n}\sin.\frac{im\pi}{n} \text{ Arc. tang. } \frac{x\sin.\frac{i\pi}{n}}{1 - x\cos.\frac{i\pi}{n}}$$

$$-\frac{2}{n}\cos.\frac{i(n-m)\pi}{n}\, l\sqrt{\left(1 - 2x\cos.\frac{i\pi}{n} + xx\right)}$$

$$+\frac{2}{n}\sin.\frac{i(n-m)\pi}{n} \text{ Arc. tang. } \frac{x\sin.\frac{i\pi}{n}}{1 - x\cos.\frac{i\pi}{n}}:$$

Verum est

$$\cos.\frac{i(n-m)\pi}{n} = \cos.\left(i\pi - \frac{im\pi}{n}\right) = -\cos.\frac{im\pi}{n}, \text{ et}$$

$$\sin.\frac{i(n-m)\pi}{n} = \sin.\left(i\pi - \frac{im\pi}{n}\right) = +\sin.\frac{im\pi}{n}:$$

unde partes logarithmicae se destruent, eritque pars integralis in genere,

$$+\frac{4}{n}\sin.\frac{im\pi}{n}\text{ Arc. tang. }\frac{x\sin.\frac{i\pi}{n}}{1-x\cos.\frac{i\pi}{n}}.$$

Ponatur commoditatis ergo angulus $\frac{\pi}{n}=\omega$, eritque

$$\int\frac{(x^{m-1}+x^{n-m-1})\partial x}{1+x^n}=+\frac{4}{n}\sin.\ m\omega\text{ Arc. tang. }\frac{x\sin.\omega}{1-x\cos.\omega}$$
$$+\frac{4}{n}\sin.\ 3m\omega\text{ Arc. tang. }\frac{x\sin.3\omega}{1-x\cos.3\omega}$$
$$+\frac{4}{n}\sin.\ 5m\omega\text{ Arc. tang. }\frac{x\sin.5\omega}{1-x\cos.5\omega}$$
$$\vdots$$
$$+\frac{4}{n}\sin.\ im\omega\text{ Arc. tang. }\frac{x\sin.i\omega}{1-x\cos.i\omega}:$$

sumto pro i maximo numero impare, exponentem n non excedente. Si ipse numerus n sit impar, pars ex positione $i=n$ oriunda, ob $\sin. m\pi=0$, evanescet. Notetur ergo, hic totum integrale per meros angulos exprimi.

Corollarium.

83. Simili modo sequens integrale elicitur, ubi soli logarithmi relinquentur, manente $\frac{\pi}{n}=\omega$:

$$\int\frac{(x^{m-1}-x^{n-m-1})\partial x}{1+x^n}=-\frac{4}{n}\cos.\ m\omega l\sqrt{(1-2x\cos.\ \omega+xx)}$$
$$-\frac{4}{n}\cos.3m\omega l\sqrt{(1-2x\cos.3\omega+xx)}$$
$$-\frac{4}{n}\cos.5m\omega l\sqrt{(1-2x\cos.5\omega+xx)}$$
$$\vdots$$
$$-\frac{4}{n}\cos.\ im\omega l\sqrt{(1-2x\cos.i\omega+xx)}:$$

donec scilicet numerus impar i non superet exponentem n.

Exemplum 5.

84. *Proposita formula differentiali* $\frac{(x^{m-1} - x^{n-m-1})\,\partial x}{1 - x^n}$; *existente* $n > m - 1$; *ejus integrale definire.*

Ex exemplo 3^{tio} integralis pars quaecunque concluditur, siquidem brevitatis gratia $\frac{\pi}{n} = \omega$ statuamus:

$$-\frac{2}{n}\cos.\,2km\omega\, l\sqrt{(1 - 2x\cos.\,2k\omega + xx)}$$
$$+\frac{2}{n}\sin.\,2km\omega\,\text{Arc. tang.}\,\frac{x\sin.\,2k\omega}{1 - x\cos.\,2k\omega}$$
$$+\frac{2}{n}\cos.\,2k\,(n - m)\,\omega\, l\sqrt{(1 - 2x\cos.\,2k\omega + xx)}$$
$$-\frac{2}{n}\sin.\,2k\,(n - m)\,\omega\,\text{Arc. tang.}\,\frac{x\sin.\,2k\omega}{1 - x\cos.\,2k\omega}.$$

At est:

$$\cos.\,2k(n - m)\,\omega = \cos.\,(2k\pi - 2km\omega) = \cos.\,2km\omega, \text{ et}$$
$$\sin.\,2k(n - m)\,\omega = \sin.\,(2k\pi - 2km\omega) = -\sin.\,2km\omega:$$

unde ista pars generalis abit in: $\frac{4}{n}\sin.\,2km\omega\,\text{Arc. tang.}\,\frac{x\sin.\,2k\omega}{1 - x\cos.\,2k\omega}$. Quare hinc ista integratio colligitur:

$$\int\frac{(x^{m-1} - x^{n-m-1})\,\partial x}{1 - x^n} = +\frac{4}{n}\sin.\,2m\omega\,\text{Arc. tang.}\,\frac{x\sin.\,2\omega}{1 - x\cos.\,2\omega}$$
$$+\frac{4}{n}\sin.\,4m\omega\,\text{Arc. tang.}\,\frac{x\sin.\,4\omega}{1 - x\cos.\,4\omega}$$
$$+\frac{4}{n}\sin.\,6m\omega\,\text{Arc. tang.}\,\frac{x\sin.\,6\omega}{1 - x\cos.\,6\omega}$$
$$\text{etc.}$$

numeris paribus tamdiu ascendendo, quoad exponentem n non superent.

Corollarium.

85. Indidem etiam haec integratio absolvitur, manente $\frac{\pi}{n} = \omega$:

$$\int \frac{(x^{m-1} + x^{n-m-1})\,\partial x}{1 - x^n} = -\tfrac{2}{n}\, l(1 - x)$$
$$-\tfrac{4}{n}\cos.\, 2m\omega\, l\sqrt{(1 - 2x\cos.2\omega + xx)}$$
$$-\tfrac{4}{n}\cos.\, 4m\omega\, l\sqrt{(1 - 2x\cos 4\omega + xx)}$$
$$-\tfrac{4}{n}\cos.\, 6m\omega\, l\sqrt{(1 - 2x\cos.6\omega + xx)}$$
etc.

ubi etiam numeri pares non ultra terminum n sunt continuandi.

Exemplum 6.

86. *Proposita formula differentiali* $\partial y = \frac{\partial x}{x^3(1+x)(1-x^4)}$, *ejus integrale invenire.*

Functio fracta per ∂x affecta secundum denominatoris factores est $\frac{1}{x^3(1+x)^2(1-x)(1+xx)}$, quae in has fractiones simplices resolvitur:

$$\frac{1}{x^3} - \frac{1}{x^2} + \frac{1}{x} - \frac{1}{4(1+x)^2} - \frac{9}{8(1+x)} + \frac{1}{8(1-x)} + \frac{1+x}{4(1+xx)} = \frac{\partial y}{\partial x},$$

unde per integrationem elicitur:

$$y = -\frac{1}{2x^2} + \frac{1}{x} + lx + \frac{1}{4(1+x)} - \tfrac{9}{8} l(1+x) - \tfrac{1}{8} l(1-x)$$
$$+ \tfrac{1}{8} l(1 + xx) + \tfrac{1}{4}\,\text{Arc. tang. } x,$$

quae expressio in hanc formam transmutatur

$$y = C - \frac{2+2x+5xx}{4xx(1+x)} - l\frac{1+x}{x} + \tfrac{1}{8} l \frac{1+xx}{1-xx} + \tfrac{1}{4}\,\text{Arc. tang. } x.$$

Scholion.

87. Hoc igitur caput ita pertractare licuit, ut nihil amplius in hoc genere desiderari possit. Quoties ergo ejusmodi functio y ipsius x quaeritur, ut $\frac{\partial y}{\partial x}$ aequetur functioni rationali ipsius x, toties integratio nihil habet difficultatis, nisi forte ad denominatoris singu-

los factores eliciendos Algebrae praecepta non sufficiant: verum tum defectus ipsi Algebrae, non vero methodo integrandi, quam hic tractamus, est tribuendus. Deinde etiam potissimum notari convenit, semper, cum $\frac{\partial y}{\partial x}$ functioni rationali ipsius x aequale ponitur, functionem y, nisi sit algebraica, alias quantitates transcendentes non involvere praeter logarithmos et angulos: ubi quidem observandum est, hic perpetuo logarithmos hyperbolicos intelligi oportere, cum ipsius lx differentiale non sit $= \frac{\partial x}{x}$, nisi logarithmus hyperbolicus sumatur: at horum reductio ad vulgares est facillima, ita ut hinc applicatio calculi ad praxin nulli impedimento sit obnoxia. Quare progrediamur ad eos casus, quibus formula $\frac{\partial y}{\partial x}$ functioni irrationali ipsius x aequatur, ubi quidem primo notandum est, quoties ista functio per idoneam substitutionem ad rationalitatem perduci poterit, casum ad hoc caput revolvi. Veluti si fuerit $\partial y = \frac{(1+\sqrt{x}-\sqrt[3]{xx})\partial x}{1+\sqrt[3]{x}}$, evidens est, ponendo $x = z^6$, unde fit $\partial x = 6z^5 \partial z$, fore

$$\partial y = \frac{(1+z^3-z^4)}{1+zz} \cdot 6z^5 \partial z, \text{ ideoque}$$

$$\frac{\partial y}{\partial z} = -6z^7 + 6z^6 + 6z^5 - 6z^4 + 6zz - 6 + \frac{6}{1+zz},$$

unde integrale

$$y = -\tfrac{3}{4}z^8 + \tfrac{6}{7}z^7 + z^6 - \tfrac{6}{5}z^5 + 2z^3 - 6z + 6 \text{ Arc. tang. } z,$$

et restituto valore

$$y = -\tfrac{3}{4}x\sqrt[3]{x} + \tfrac{6}{7}x\sqrt[6]{x} + x - \tfrac{6}{5}\sqrt[6]{x^5} + 2\sqrt{x} - 6\sqrt[6]{x}$$
$$+ 6 \text{ Arc. tang. } \sqrt[6]{x} + C.$$

CAPUT II.

DE INTEGRATIONE FORMULARUM DIFFERENTIALIUM IRRATIONALIUM.

Problema 6.

88.

Proposita formula differentiali $\partial y = \frac{\partial x}{\sqrt{(\alpha + \beta x + \gamma x x)}}$, ejus integrale invenire.

Solutio.

Quantitas $\alpha + \beta x + \gamma x x$, vel habet duos factores reales vel secus.

I. Priori casu formula proposita erit hujusmodi $\partial y = \frac{\partial x}{\sqrt{(a + b x)(f + g x)}}$. Statuatur ad irrationalitatem tollendam

$$(a + b x)(f + g x) = (a + b x)^2 z z,$$

erit $x = \frac{f - a z z}{b z z - g}$, ideoque

$$\partial x = \frac{2(a g - b f) z \partial z}{(b z z - g)^2} \text{ et } \sqrt{(a + b x)(f + g x)} = -\frac{(a g - b f) z}{b z z - g}:$$

unde fit $\partial y = \frac{-2 \partial z}{b z z - g} = \frac{2 \partial z}{g - b z z}$, atque $z = \sqrt{\frac{f + g x}{a + b x}}$. Quare si litterae b et g paribus signis sunt affectae, integrale per logarithmos, sin autem signis disparibus, per angulos exprimetur.

II. Posteriori casu habebimus $\partial y = \frac{\partial x}{\sqrt{(a a - 2 a b x \cos. \zeta + b b x x)}}$. Statuatur

$$b b x x - 2 a b x \cos. \zeta + a a = (b x - a z)^2, \text{ erit}$$

$$-2 b x \cos. \zeta + a = -2 b x z + a z z \text{ et } x = \frac{a(1 - z z)}{2 b(\cos. \zeta - z)};$$

hinc $\partial x = \frac{a\partial z(1 - 2z\cos.\zeta + zz)}{ab(\cos.\zeta - z)^2}$, et

$$\sqrt{(aa - 2abx\cos.\zeta + bbxx)} = \frac{a(1 - 2z\cos.\zeta + zz)}{2(\cos.\zeta - z)}: \text{ ergo}$$

$$\partial y = \frac{\partial z}{b(\cos.\zeta - z)}, \text{ et } y = -\frac{1}{b} l(\cos.\zeta - z).$$

At est

$$z = \frac{bx - \sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a}, \text{ ideoque}$$

$$y = -\frac{1}{b} l \frac{a\cos.\zeta - bx + \sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a}, \text{ vel}$$

$$y = \frac{1}{b} l[-a\cos.\zeta + bx + \sqrt{(aa - 2abx\cos.\zeta + bbxx)}] + C.$$

Corollarium 1.

89. Casus ultimus latius patet, et ad formulam $\partial y = \frac{\partial x}{\sqrt{(\alpha + \beta x + \gamma xx)}}$, accomodari potest, dummodo fuerit γ quantitas positiva: namque ob $b = \sqrt{\gamma}$ et $a\cos.\zeta = \frac{-\beta}{2\sqrt{\gamma}}$, oritur,

$$y = \frac{1}{\sqrt{\gamma}} l [\frac{\beta}{2\sqrt{\gamma}} + x\sqrt{\gamma} + \sqrt{(\alpha + \beta x + \gamma xx)}] + C \text{ seu}$$

$$y = \frac{1}{\sqrt{\gamma}} l [\tfrac{1}{2}\beta + \gamma x + \sqrt{\gamma(\alpha + \beta x + \gamma xx)}] + C.$$

Corollarium 2.

90. Pro casu priori cum sit

$$\int \frac{2\partial z}{g - bzz} = \frac{1}{\sqrt{bg}} l \frac{\sqrt{g} + z\sqrt{b}}{\sqrt{g} - z\sqrt{b}} \text{ et}$$

$$\int \frac{2\partial z}{g + bzz} = \frac{2}{\sqrt{bg}} \text{ Arc. tang. } \frac{z\sqrt{b}}{\sqrt{g}},$$

habebimus hos casus:

$$\int \frac{\partial x}{\sqrt{(a+bx)(f+gx)}} = \frac{1}{\sqrt{bg}} l \frac{\sqrt{g(a+bx)} + \sqrt{b(f+gx)}}{\sqrt{g(a+bx)} - \sqrt{b(f+gx)}} + C$$

$$\int \frac{\partial x}{\sqrt{(bx-a)(f+gx)}} = \frac{1}{\sqrt{bg}} l \frac{\sqrt{g(bx-a)} + \sqrt{b(f+gx)}}{\sqrt{g(bx-a)} - \sqrt{b(f+gx)}} + C$$

$$\int \frac{\partial x}{\sqrt{(bx-a)(gx-f)}} = \frac{1}{\sqrt{bg}} l \frac{\sqrt{g(bx-a)} + \sqrt{b(gx-f)}}{\sqrt{g(bx-a)} - \sqrt{b(gx-f)}} + C$$

$$\int \frac{\partial x}{\sqrt{(a-bx)(f-gx)}} = \frac{-1}{\sqrt{bg}} l \frac{\sqrt{g(a-bx)} + \sqrt{b(f-gx)}}{\sqrt{g(a-bx)} - \sqrt{b(f-gx)}} + C$$

$$\int \frac{\partial x}{\sqrt{(a-bx)(f+gx)}} = \frac{2}{\sqrt{bg}} \text{ Arc. tang. } \frac{\sqrt{b(f+gx)}}{\sqrt{g(a-bx)}} + C$$

$$\int \frac{\partial x}{\sqrt{(a-bx)(gx-f)}} = \frac{2}{\sqrt{bg}} \text{ Arc. tang. } \frac{\sqrt{b(gx-f)}}{\sqrt{g(a-bx)}} + C.$$

Corollarium 3.

91. Harum sex integrationum quatuor priores omnes in casu Coroll. 1. continentur, binae autem postremae in hac formula $\partial y = \frac{\partial x}{\sqrt{(\alpha+\beta x-\gamma xx)}}$ continentur: sit enim pro penultima

$$af = \alpha, \; ag - bf = \beta, \; bg = \gamma,$$

unde colligitur

$$y = \frac{1}{\sqrt{\gamma}} \text{ Arc. tang. } \frac{2\sqrt{\gamma(\alpha+\beta x-\gamma xx)}}{\beta - 2\gamma x};$$

si scilicet ille arcus duplicetur. Per cosinum autem erit

$$y = \frac{1}{\sqrt{\gamma}} \text{ Arc. cos. } \frac{\beta - 2\gamma x}{\sqrt{(\beta\beta + 4\alpha\gamma)}} + C;$$

cujus veritas ex differentiatione patet.

Scholion 1.

92. Ex solutione hujus problematis patet etiam, hanc formulam latius patentem $\frac{X\partial x}{\sqrt{(\alpha+\beta x+\gamma xx)}}$, si X fuerit functio rationalis quaecunque ipsius x, per praecepta capitis praecedentis integrari posse. Introducta enim loco x variabili z, qua formula radicalis rationalis redditur, etiam X abibit in functionem rationalem ipsius z. Idem adhuc generalius locum habet, si posito $\sqrt{(\alpha+\beta x+\gamma xx)} = u$, fuerit X functio quaecunque rationalis binarum quantitatum x et u, tum enim per substitutionem adhibitam, quia tam pro x quam pro u formulae rationales ipsius z scribuntur, prodibit formula differentialis rationalis. Hoc idem etiam ita enunciari potest, ut dicamus, formulae $X\partial x$, si functio X nullam aliam irrationalem praeter $\sqrt{(\alpha+\beta x+\gamma xx)}$ involvat, integrale assignari posse, propterea quod ea, ope substitutionis, in formulam differentialem rationalem transformari potest.

Scholion 2.

93. Proposita autem formula differentiali quacunque irrationali, ante omnia videndum est, num ea ope cujuspiam substitutionis in rationalem transformari possit? quod si succedat, integratio per praecepta capitis praecedentis absolvi poterit: unde simul intelligitur, integrale nisi sit algebraicum, alias quantitates transcendentes non involvere praeter logarithmos et angulos. Quodsi autem nulla substitutio ad hoc idonea inveniri possit, ab integrationis labore est desistendum, quandoquidem integrale neque algebraice neque per logarithmos vel angulos exprimere valemus. Veluti si $X\partial x$ fuerit ejusmodi formula differentialis, quae nullo pacto ad rationalitatem reduci queat, ejus integrale $\int X\partial x$ ad novum genus functionum transcendentium erit referendum, in quo nihil aliud nobis relinquitur, nisi ut ejus valorem vero proxime assignare conemur. Admisso autem novo genere quantitatum transcendentium, innumerabiles aliae formulae eo reduci atque integrari poterunt. Imprimis igitur in hoc erit elaborandum, ut pro quolibet genere formula simplicissima notetur, qua concessa reliquarum formularum integralia definire liceat. Hinc deducimur ad quaestionem maximi momenti, quomodo integrationem formularum magis complicatarum ad simpliciores reduci oporteat. Quod antequam aggrediamur, alias ejusmodi formulas perpendamus, quae ope idoneae substitutionis ab irrationalitate liberari queant; quemadmodum jam ostendimus, quoties X fuerit functio rationalis quantitatum

$$x \text{ et } u = \sqrt{(\alpha + \beta x + \gamma xx)},$$

ita ut alia irrationalitas non ingrediatur praeter radicem quadratam hujusmodi formulae $\alpha + \beta x + \gamma xx$, toties formulam differentialem $X\partial x$ in rationalem transformari posse.

Problema 7.

94. Proposita formula differentiali $X\partial x(a + bx)^{\frac{\mu}{\nu}}$, in qua X denotet functionem quamcunque rationalem ipsius x, eam ab irrationalitate liberare.

Solutio.

Statuatur $a + bx = z^\nu$, ut fiat $(a + bx)^{\frac{\mu}{\nu}} = z^\mu$: tum quia $x = \frac{z^\nu - a}{b}$, facta hac substitutione, functio X abibit in functionem rationalem ipsius z, quae sit Z, et ob $\partial x = \frac{\nu}{b} z^{\nu-1} \partial z$, formula nostra differentialis induet hanc formam $\frac{\nu}{b} Z z^{\mu+\nu-1} \partial z$, quae cum sit rationalis, per caput superius integrari potest, et integrale, nisi sit algebraicum, per logarithmos et angulos exprimetur.

Corollarium 1.

95. Hac substitutione generalius negotium confici poterit, si posito $(a + bx)^{\frac{1}{\nu}} = u$, littera V denotet functionem quamcunque rationalem binarum quantitatum x et u; cum enim posito $x = \frac{u^\nu - a}{b}$, fiat V functio rationalis ipsius u, formula $V \partial x = \frac{\nu}{b} V u^{\nu-1} \partial u$, erit rationalis.

Corollarium 2.

96. Quin etiam si binae irrationalitates ejusdem quantitatis $a + bx$, scilicet $(a + bx)^{\frac{1}{\nu}} = u$ et $(a + bx)^{\frac{1}{n}} = v$, ingrediantur in formulam $X\partial x$, posito $a + bx = z^{n\nu}$ fit $x = \frac{z^{n\nu} - a}{b}$, $u = z^n$, et $v = z^\nu$; unde cum X fiat functio rationalis ipsius z, et $\partial x = \frac{n\nu}{b} z^{n\nu-1} \partial z$, hac substitutione formula $X \partial x$ evadet rationalis.

Corollarium 3.

97. Eodem modo intelligitur, si posito

$$(a + bx)^{\frac{1}{\lambda}} = u,\ (a + bx)^{\frac{1}{\mu}} = v,\ (a + bx)^{\frac{1}{\nu}} = t \text{ etc.}$$

littera X denotet functionem quamcunque rationalem quantitatum x, u, v, t etc. formulam differentialem $X\partial x$ rationalem reddi facto $a + bx = z^{\lambda\mu\nu}$; erit enim

$$x = \frac{z^{\lambda\mu\nu} - a}{b}; \quad u = z^{\mu\nu}; \quad v = z^{\lambda\nu}; \quad t = z^{\lambda\mu} \text{ etc. et}$$

$$\partial x = \frac{\lambda\mu\nu}{b} z^{\lambda\mu\nu - 1} \partial z.$$

Exemplum.

98. Proposita hac formula $\partial y = \frac{x\partial x}{\sqrt[3]{(1+x)} - \sqrt{(1+x)}}$, facto $1 + x = z^6$, reperitur $\partial y = -\frac{6z^3\partial z(1 - z^6)}{1 - z}$, seu

$$\partial y = -6\partial z(z^3 + z^4 + z^5 + z^6 + z^7 + z^8):$$

hincque integrando

$$y = C - \tfrac{3}{2}z^4 - \tfrac{6}{5}z^5 - z^6 - \tfrac{6}{7}z^7 - \tfrac{3}{4}z^8 - \tfrac{2}{3}z^9,$$

et restituendo

$$y = C - \tfrac{3}{2}\sqrt[3]{(1+x)^2} - \tfrac{6}{5}\sqrt[6]{(1+x)^5} - 1 - x - \tfrac{6}{7}(1+x)\sqrt[6]{(1+x)}$$
$$- \tfrac{3}{4}(1+x)\sqrt[3]{(1+x)} - \tfrac{2}{3}(1+x)\sqrt{(1+x)}$$

ita ut integrale adeo algebraice exhibeatur.

Problema 8.

99. Proposita formula differentiali $X\partial x\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}}$, denotante X functionem rationalem quamcunque ipsius x, eam ab irrationalitate liberare.

Solutio.

Posito $\frac{a+bx}{f+gx} = z^\nu$, fit $\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}} = z^\mu$, et

$$x = \frac{a - fz^\nu}{gz^\nu - b}, \text{ atque } \partial x = \frac{\nu(bf - ag)z^{\nu-1}\partial z}{(gz^\nu - b)^2};$$

sicque loco X prodibit functio rationalis ipsius z, qua posita $= Z$, erit formula nostra differentialis

$$= \frac{\nu (bf - ag) Z z^{\mu+\nu-1} \partial z}{(gz^{\nu} - b)^2},$$

quae cum sit rationalis, per praecepta Cap. I. integrari poterit.

Corollarium 1.

100. Posito $\left(\frac{a + bx}{f + gx}\right)^{\frac{1}{\nu}} = u$, si X fuerit functio quaecunque rationalis binarum quantitatum x et u, formula differentialis $X \partial x$ per substitutionem usurpatam in rationalem transformabitur, cujus propterea integratio constat.

Corollarium 2.

101. Si X fuerit functio rationalis tam ipsius x, quam quantitatum quotcunque hujusmodi

$$\left(\frac{a + bx}{f + gx}\right)^{\frac{1}{\lambda}} = u, \quad \left(\frac{a + bx}{f + gx}\right)^{\frac{1}{\mu}} = v, \quad \left(\frac{a + bx}{f + gx}\right)^{\frac{1}{\nu}} = t$$

tum formula differentialis $X \partial x$ rationalis reddetur, adhibita substitutione $\frac{a + bx}{f + gx} = z^{\lambda\mu\nu}$, unde fit

$$x = \frac{a - fz^{\lambda\mu\nu}}{gz^{\lambda\mu\nu} - b}; \text{ et } u = z^{\mu\nu}; \; v = z^{\lambda\nu}; \; t = z^{\lambda\mu}.$$

Scholion 1.

102. His casibus reductio ad rationalitatem ideo succedit, etiamsi plures formulae irrationales insint, quod eae omnes simul per eandem substitutionem rationales efficiantur, indeque etiam ipsa quantitas x per novam variabilem z rationaliter exprimetur. Sin autem differentiale propositum duas ejusmodi formulas irrationales contineat, quae non ambae simul ope ejusdem substitutionis rationa-

les reddi queant, etiamsi hoc in utraque seorsim fieri possit, reductio locum non habet, nisi forte ipsum differentiale in duas partes dispesci liceat, quarum utraque unam tantum formulam irrationalem complectatur. Veluti si proposita sit haec formula differentialis

$$\partial y = \frac{\partial x}{\sqrt{(1+xx)} - \sqrt{(1-xx)}}$$

ejus numeratorem ac denominatorem per $\sqrt{(1+xx)} + \sqrt{(1-xx)}$ multiplicando, fit

$$\partial y = \frac{\partial x \sqrt{(1+xx)}}{2xx} + \frac{\partial x \sqrt{(1-xx)}}{2xx},$$

cujus utraque pars seorsim rationalis reddi et integrari potest. Reperitur autem:

$$y = C - \frac{\sqrt{(1-xx)} + \sqrt{(1+xx)}}{2x} + \tfrac{1}{2} l[x + \sqrt{(1+xx)}]$$
$$- \tfrac{1}{2} \text{Arc. tang.} \frac{x}{\sqrt{(1-xx)}}.$$

Commodissime autem ibi irrationalitas tollitur, si in parte priori ponatur $\sqrt{(1+xx)} = px$, in posteriori $\sqrt{(1-xx)} = qx$. Etsi enim hinc sit

$$x = \frac{1}{\sqrt{(pp-1)}} \text{ et } x = \frac{1}{\sqrt{(1+qq)}},$$

tamen oritur rationaliter

$$\partial y = \frac{-pp\partial p}{2(pp-1)} - \frac{qq\partial q}{2(1+qq)}.$$

Scholion 2.

103. Circa formulas generales, quae ab irrationalitate liberari queant, vix quicquam amplius praecipere licet; dummodo hunc casum addamus, quo functio X binas hujusmodi formulas radicales $\sqrt{(a+bx)}$ et $\sqrt{(f+gx)}$ complectitur. Posito enim $(a+bx) = (f+gx)tt$, fit $x = \frac{a-ftt}{gtt-b}$, atque

$$\sqrt{(a+bx)} = \frac{t\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}}; \quad \sqrt{(f+gx)} = \frac{\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}};$$

et in formula differentiali unica tantum formula irrationalis erit $\sqrt{(gtt-b)}$, quae nova substitutione facile tolletur, per ea quae

Problemate 6. tradidimus. Ut igitur ad alia pergamus, imprimis considerari meretur haec formula differentialis

$$x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}},$$

cujus ob simplicitatem usus per universam analysin est amplissimus; ubi quidem sumimus litteras m, n, μ, ν numeros integros denotare, nisi enim tales essent, facile ad hanc formam reducerentur. Veluti si haberemus $x^{-\frac{1}{3}}\partial x(a+b\sqrt{x})^{\frac{\mu}{\nu}}$, statui oportet $x=u^6$, hinc $\partial x=6u^5\partial u$: unde prodit

$$6u^3\partial u(a+bu^3)^{\frac{\mu}{\nu}}.$$

Tum vero pro n valorem positivum assumere licet: si enim esset negativus, puta

$$x^{m-1}\partial x(a+bx^{-n})^{\frac{\mu}{\nu}},$$

ponatur $x=\frac{1}{u}$, fietque formula

$$-u^{-m-1}\partial u(a+bu^n)^{\frac{\mu}{\nu}},$$

similis principali; quae ergo quibus casibus ab irrationalitate liberari queat, investigemus.

Problema 9.

104. Definire casus, quibus formulam differentialem

$$x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}},$$

ad rationalitatem perducere liceat.

Solutio.

Primo patet, si fuerit $\nu=1$, seu $\frac{\mu}{\nu}$ numerus integer, formulam per se fore rationalem, neque substitutione opus esse. At si $\frac{\mu}{\nu}$ sit fractio, substitutione est utendum, eaque duplici.

I. Ponatur $a + bx^n = u^\nu$, ut fiat $(a + bx^n)^{\frac{\mu}{\nu}} = u^\mu$, erit

$$x^n = \frac{u^\nu - a}{b}, \text{ hinc } x^m = \left(\frac{u^\nu - a}{b}\right)^{\frac{m}{n}}, \text{ ideoque}$$

$$x^{m-1}\partial x = \frac{\nu}{nb} u^{\nu-1}\partial u \left(\frac{u^\nu - a}{b}\right)^{\frac{m-n}{n}};$$

unde formula nostra fiet

$$\frac{\nu}{nb} u^{\mu+\nu-1}\partial u \left(\frac{u^\nu - a}{b}\right)^{\frac{m-n}{n}}.$$

Hinc ergo patet, quoties exponens $\frac{m-n}{n}$ seu $\frac{m}{n}$ fuerit numerus integer sive positivus, sive negativus, hanc formulam esse rationalem.

II. Ponatur $a + bx^n = x^n z^\nu$, ut fiat

$$x^n = \frac{a}{z^\nu - b}, \text{ et } (a + bx^n)^{\frac{\mu}{\nu}} = \frac{a^{\frac{\mu}{\nu}} z^\mu}{(z^\nu - b)^{\frac{\mu}{\nu}}}; \text{ tum}$$

$$x^m = \frac{a^{\frac{m}{n}}}{(z^\nu - b)^{\frac{m}{n}}}, \text{ hinc } x^{m-1}\partial x = \frac{-\nu a^{\frac{m}{n}} z^{\nu-1}\partial z}{n(z^\nu - b)^{\frac{m}{n}+1}}.$$

Ideoque formula nostra erit

$$\frac{-\nu a^{\frac{m}{n}+\frac{\mu}{\nu}} z^{\mu+\nu-1}\partial z}{n(z^\nu - b)^{\frac{m}{n}+\frac{\mu}{\nu}+1}}.$$

Ex quo patet hanc formam fore rationalem, quoties $\frac{m}{n} + \frac{\mu}{\nu}$ fuerit numerus integer. Facile autem intelligitur, alias substitutiones huic scopo idoneas excogitari non posse.

Quare concludimus formulam irrationalem hanc

$$x^{m-1}\partial x (a + bx^n)^{\frac{\mu}{\nu}}$$

ab irrationalitate liberari posse, si fuerit vel $\frac{m}{n}$, vel $\frac{m}{n} + \frac{\mu}{\nu}$ numerus integer.

Corollarium 1.

105. Si sit $\frac{m}{n}$ numerus integer, casus per se est facilis; ponatur enim $m = in$, et sit $x^n = v$, erit $x^m = v^i$; ideoque formula nostra $\frac{i}{m} v^{i-1} \partial v (a + bv)^{\frac{\mu}{\nu}}$, quae per Problema 7. expeditur.

Corollarium 2.

106. At si $\frac{m}{n}$ non est numerus integer, ut reductio ad rationalitatem locum habeat, necesse est ut $\frac{m}{n} + \frac{\mu}{\nu}$ sit numerus integer: quod fieri nequit, nisi sit $\nu = n$, ideoque $m + \mu$ multiplum debet esse ipsius $n = \nu$.

Corollarium 3.

107. Quod si ergo haec formula

$$x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}},$$

ad rationalitatem reduci queat, etiam haec formula

$$x^{m \pm \alpha n - 1} \partial x (a + bx^n)^{\frac{\mu}{\nu} \pm \beta},$$

eandem reductionem admittet; quicunque numeri integri pro α et β assumantur. Unde ad casus reducibiles cognoscendos sufficit ponere $m < n$ et $\mu < \nu$.

Corollarium 4.

108. Si $m = 0$, haec formula $\frac{\partial x}{x} (a + bx^n)^{\frac{\mu}{\nu}}$, semper per casum primum ad rationalitatem reducitur, ponendo

$$x^n = \frac{u^\nu - a}{b};$$

transformatur enim in hanc

$$\frac{\nu b u^{\mu + \nu - 1} \partial u}{n (u^\nu - a)}.$$

Scholion 1.

109. Quoniam formula $x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, quoties est $m=in$, denotante i numerum integrum sive positivum sive negativum quemcunque, semper ad rationalitatem reduci potest, hicque casus per se sunt perspicui, reliquos casus hanc reductionem admittentes accuratius contemplari operae pretium videtur. Quem in finem statuamus $\nu = n$ et $m < n$, item $\mu < n$, ac necesse est ut sit $m+\mu = n$: unde sequentes formae in genere suo simplicissimae, quae quidem ad rationalitatem reduci queant, obtinentur.

I. $\partial x\,(a+bx^2)^{\frac{1}{2}}$;

II. $\partial x\,(a+bx^3)^{\frac{2}{3}}$; $x\partial x\,(a+bx^3)^{\frac{1}{3}}$;

III. $\partial x\,(a+bx^4)^{\frac{3}{4}}$; $xx\partial x\,(a+bx^4)^{\frac{1}{4}}$;

IV. $\partial x\,(a+bx^5)^{\frac{4}{5}}$; $x\partial x\,(a+bx^5)^{\frac{3}{5}}$; $x^2\partial x\,(a+bx^5)^{\frac{2}{5}}$;
$x^3\partial x\,(a+bx^5)^{\frac{1}{5}}$;

V. $\partial x\,(a+bx^6)^{\frac{5}{6}}$; $x^4\partial x\,(a+bx^6)^{\frac{1}{6}}$;

unde etiam hae reductionem admittent:

$x^{\pm 2\alpha}\partial x\,(a+bx^2)^{\frac{1}{2}\pm\beta}$;

$x^{\pm 3\alpha}\partial x\,(a+bx^3)^{\frac{2}{3}\pm\beta}$; $x^{1\pm 3\alpha}\partial x\,(a+bx^3)^{\frac{1}{3}\pm\beta}$;

$x^{\pm 4\alpha}\partial x\,(a+bx^4)^{\frac{3}{4}\pm\beta}$; $x^{2\pm 4\alpha}\partial x\,(a+bx^4)^{\frac{1}{4}\pm\beta}$;

$x^{\pm 5\alpha}\partial x\,(a+bx^5)^{\frac{4}{5}\pm\beta}$; $x^{1\pm 5\alpha}\partial x\,(a+bx^5)^{\frac{3}{5}\pm\beta}$;

$x^{2\pm 5\alpha}\partial x\,(a+bx^5)^{\frac{2}{5}\pm\beta}$; $x^{3\pm 5\alpha}\partial x\,(a+bx^5)^{\frac{1}{5}\pm\beta}$;

$x^{\pm 6\alpha}\partial x\,(a+bx^6)^{\frac{5}{6}\pm\beta}$; $x^{4\pm 6\alpha}\partial x\,(a+bx^6)^{\frac{1}{6}\pm\beta}$.

Scholion 2.

110. Verum etiamsi formula $x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}}$, ab irrationalitate liberari nequeat, tamen semper omnium harum formularum $x^{m\pm n\alpha-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$, integrationem ad eam reducere licet, ita ut illius integrali tanquam cognito spectato, etiam harum integralia assignari queant. Quae reductio cum in Analysi summam afferat utilitatem, eam hic exponere necesse erit. Caeterum hic affirmare haud dubitamus, praeter eos casus, quos reductionem ad rationalitatem admittere hic ostendimus, nullos alios existere, qui ulla substitutione adhibita ab irrationalitate liberari queant. Proposita enim hac formula $\frac{\partial x}{\sqrt{(a+bx^3)}}$, nulla functio rationalis ipsius z loco x poni potest, ut $a+bx^3$ extractionem radicis quadratae admittat: objici quidem potest, scopo satisfieri posse, etiamsi loco x functio irrationalis ipsius z substituatur, dummodo similis irrationalitas in denominatore $\sqrt{(a+bx^3)}$ contineatur, qua illa numeratorem ∂x afficiens destituatur: quemadmodum fit in hac formula $\frac{\partial x}{\sqrt[3]{(a+bx^3)}}$, adhibendo substitutionem

$$x=\frac{\sqrt[3]{a}}{\sqrt[3]{(z^3-b)}},$$

verum quod hic commode usu venit, nullo modo perspicitur, quomodo idem illo casu evenire possit. Hoc tamen minime pro demonstratione haberi volo.

Problema 10.

111. Integrationem formulae

$$\int x^{m+n-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}},$$

perducere ad integrationem huius formulae: $\int x^{m-1}\partial x(a+bx^n)^{\frac{\mu}{\nu}}$.

Solutio.

Consideretur functio $x^m (a + bx^n)^{\frac{\mu}{\nu} + 1}$, cujus differentiale cum sit

$$\left(max^{m-1}\partial x + mbx^{m+n-1}\partial x + \frac{n(\mu+\nu)}{\nu}bx^{m+n-1}\partial x\right)(a+bx^n)^{\frac{\mu}{\nu}},$$

erit

$$x^m (a + bx^n)^{\frac{\mu}{\nu}+1} = ma\int x^{m-1}\partial x\,(a + bx^n)^{\frac{\mu}{\nu}}$$
$$+ \frac{(m\nu + n\mu + n\nu)b}{\nu}\int x^{m+n-1}\partial x\,(a + bx^n)^{\frac{\mu}{\nu}}:$$

unde elicitur

$$\int x^{m+n-1}\partial x\,(a + bx^n)^{\frac{\mu}{\nu}} = \frac{\nu x^m (a + bx^n)^{\frac{\mu}{\nu}+1}}{(m\nu + n\mu + n\nu)b}$$
$$- \frac{m\nu a}{(m\nu + n\mu + n\nu)b}\int x^{m-1}\partial x\,(a + bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 1.

112. Cum inde quoque sit

$$\int x^{m-1}\partial x\,(a + bx^n)^{\frac{\mu}{\nu}} = \frac{x^m (a + bx^n)^{\frac{\mu}{\nu}+1}}{ma}$$
$$- \frac{(m\nu + n\mu + n\nu)b}{m\nu a}\int x^{m+n-1}\partial x\,(a + bx^n)^{\frac{\mu}{\nu}}$$

loco m scribamus $m - n$, et habebimus hanc reductionem:

$$\int x^{m-n-1}\partial x\,(a + bx^n)^{\frac{\mu}{\nu}} = \frac{x^{m-n}(a + bx^n)^{\frac{\mu}{\nu}+1}}{(m-n)a}$$
$$- \frac{(m\nu + n\mu)b}{(m-n)\nu a}\int x^{m-1}\partial x\,(a + bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 2.

113. Concesso ergo integrali $\int x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}}$, etiam harum formularum $\int x^{m \pm n - 1} \partial x (a + bx^n)^{\frac{\mu}{\nu}}$, similique modo ulterius progrediendo omnium harum formularum

$$\int x^{m \pm \alpha n - 1} \partial x (a + bx^n)^{\frac{\mu}{\nu}}$$

integralia exhiberi possunt.

Problema 11.

114. Integrationem formulae $\int x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}+1}$ ad integrationem hujus $\int x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}}$ perducere.

Solutio.

Functionis $x^m (a + bx^n)^{\frac{\mu}{\nu}+1}$ differentiale hoc modo exhiberi potest

$$\left(ma - \frac{(m\nu + n\mu + n\nu)a}{\nu}\right) x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}}$$
$$+ \frac{m\nu + n\mu + n\nu}{\nu} x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}+1},$$

unde concluditur

$$x^m (a + bx^n)^{\frac{\mu}{\nu}+1} = -\frac{(n\mu + n\nu)a}{\nu} \int x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}}$$
$$+ \frac{m\nu + n\mu + n\nu}{\nu} \int x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}+1},$$

quocirca habebimus:

$$\int x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}+1} = \frac{\nu x^m (a + bx^n)^{\frac{\mu}{\nu}+1}}{m\nu + n(\mu + \nu)}$$
$$+ \frac{n(\mu + \nu)a}{m\nu + n(\mu + \nu)} \int x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 1.

115. Deinde ex eadem aequatione elicimus:

$$\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}} = \frac{-\nu x^m (a+bx^n)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)a}$$
$$+\frac{m\nu+n(\mu+\nu)}{n(\mu+\nu)a}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1}.$$

Scribamus jam $\mu-\nu$ loco μ, ut nasciscamur hanc reductionem

$$\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{-\nu x^m (a+bx^n)^{\frac{\mu}{\nu}}}{n\mu a}$$
$$+\frac{m\nu+n\mu}{n\mu a}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}.$$

Corollarium 2.

116. Concesso ergo integrali $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, etiam harum formularum $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}\pm 1}$, et ulterius progrediendo, harum $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$ integralia exhiberi possunt, denotante β numerum integrum quemcunque.

Corollarium 3.

117. His cum praecedentibus conjunctis, ad integrationem $\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$, omnia haec integralia

$$\int x^{m\pm\alpha n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}\pm\beta}$$

revocari possunt, quae ergo omnia ab eadem functione transcendente pendent.

Scholion 1.

118. Ex formae $x^m (a + bx^n)^{\frac{\mu}{\nu}}$ differentiali ita disposito

$$m x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}} + \frac{n\mu}{\nu} b x^{m+n-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}-1}$$

deducimus hanc reductionem:

$$\int x^{m+n-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu x^m (a + bx^n)^{\frac{\mu}{\nu}}}{n\mu b}$$
$$- \frac{m\nu}{n\mu b} \int x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}}:$$

ac praeterea hanc inversam, pro m et μ scribendo $m - n$ et $\mu + \nu$:

$$\int x^{m-n-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}+1} = \frac{x^{m-n} (a + bx^n)^{\frac{\mu}{\nu}+1}}{m - n}$$
$$- \frac{n(\mu + \nu) b}{\nu (m - n)} \int x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}}.$$

Hinc scilicet una operatione absolvitur reductio, cum superiores formulae duplicem reductionem exigant; ex quo sex reductiones sumus nacti, omnino memorabiles, quas idcirco conjunctim conspectui exponamus.

$$\text{I. } \int x^{m+n-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}} = \frac{\nu x^m (a + bx^n)^{\frac{\mu}{\nu}+1}}{[m\nu + n(\mu + \nu)] b}$$
$$- \frac{m\nu a}{[m\nu + n(\mu + \nu)] b} \int x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}}$$

$$\text{II. } \int x^{m-n-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}} = \frac{x^{m-n} (a + bx^n)^{\frac{\mu}{\nu}+1}}{(m - n) a}$$
$$- \frac{(m\nu + n\mu) b}{(m - n)\nu a} \int x^{m-1} \partial x (a + bx^n)^{\frac{\mu}{\nu}}$$

$$\text{III. } \int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{-\nu x^m (a+bx^n)^{\frac{\mu}{\nu}+1}}{m\nu + n(\mu+\nu)} + \frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{IV. } \int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{-\nu x^m (a+bx^n)^{\frac{\mu}{\nu}}}{n\mu a} + \frac{m\nu+n\mu}{n\mu a}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{V. } \int x^{m+n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu x^m (a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b} - \frac{m\nu}{n\mu b}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}$$

$$\text{VI. } \int x^{m-n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}+1} = \frac{x^{m-n}(a+bx^n)^{\frac{\mu}{\nu}+1}}{m-n} - \frac{n(\mu+\nu)b}{\nu(m-n)}\int x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}}.$$

Scholion 2.

119. Circa has reductiones primo observandum est, formulam priorem algebraice esse integrabilem, si coëfficiens posterioris evanescat. Ita sit

$$\text{pro I. si } m=0 \ldots \int x^{n-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}} = \frac{\nu(a+bx^n)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)b}$$

$$\text{pro II. si } \frac{\mu}{\nu} = \frac{-m}{n} \ldots \int x^{m-n-1}\partial x\,(a+bx^n)^{\frac{-m}{n}} = \frac{x^{m-n}(a+bx^n)^{\frac{-m}{n}+1}}{(m-n)a}$$

pro IV. si $\frac{\mu}{\nu} = \frac{-m}{n} \ldots \int x^{m-1}\partial x (a+bx^n)^{\frac{-m}{n}-1} = \frac{x^m (a+bx^n)^{\frac{-m}{n}}}{ma}$

pro V. si $m = 0 \ldots \int x^{n-1}\partial x (a+bx^n)^{\frac{\mu}{\nu}-1} = \frac{\nu (a+bx^n)^{\frac{\mu}{\nu}}}{n\mu b}$

Deinde etiam casus notari merentur, quibus coëfficiens postremae formulae fit infinitus; tum enim reductio cessat, et prior formula peculiare habet integrale seorsim evolvendum.

In prima hoc evenit si $\frac{\mu+\nu}{\nu} = \frac{-m}{n}$, et formula

$$\int x^{m+n-1}\partial x (a+bx^n)^{\frac{-m}{n}-1},$$

posito $a+bx^n = x^n z^n$, seu $x^n = \frac{a}{z^n - b}$, abit in $-\frac{z^{-m-1}\partial z}{z^n - b}$, cujus integrale per caput primum definiri debet.

In secunda evenit si $m = n$, et formula $\int \frac{\partial x}{x}(a+bx^n)^{\frac{\mu}{\nu}}$, posito $a+bx^n = z^\nu$, seu $x^n = \frac{z^\nu - a}{b}$, abit in $\frac{\nu z^{\mu+\nu-1}\partial z}{n(z^\nu - a)}$.

In tertia evenit, si $\frac{\mu}{\nu} = \frac{-m}{n} - 1$, et formula

$$\int x^{m-1}\partial x (a+bx^n)^{\frac{-m}{n}},$$

posito $a+bx^n = x^n z^n$, seu $x^n = \frac{a}{z^n - b}$, abit in $\int \frac{-z^{-m-n-1}\partial z}{z^n - b}$, seu posito $z = \frac{1}{u}$, in

$$\int \frac{u^{m+2n-1}\partial u}{1 - bu^n} = \frac{-u^{m+n}}{(m+n)b} - \frac{u^m}{mbb} + \frac{1}{bb}\int \frac{u^{m-1}\partial u}{a - bu^n}.$$

In quarta evenit, si $\mu = 0$, et formula $\int \frac{x^{m-1}\partial x}{a+bx^n}$ per se est rationalis.

In quinta idem evenit, si $\mu = 0$.

In sexta autem, si $m = n$, et formula $\int \frac{\partial x}{x} (a + bx^n)^{\frac{\mu}{\nu}+1}$, posito $a + bx^n = z^\nu$, abit in $\frac{\nu}{n} \int \frac{z^{\mu+2\nu-1} \partial z}{z^\nu - a}$.

Exemplum 1.

120. *Invenire integrale hujus formulae* $\int \frac{x^{m-1} \partial x}{\sqrt{(1-xx)}}$, *pro numeris positivis exponenti* m *datis.*

Hic ob $a = 1$, $b = -1$, $n = 2$, $\mu = -1$, $\nu = 2$, prima reductio dat:

$$\int \frac{x^{m+1} \partial x}{\sqrt{(1-xx)}} = \frac{-x^m \sqrt{(1-xx)}}{m+1} + \frac{m}{m+1} \int \frac{x^{m-1} \partial x}{\sqrt{(1-xx)}}:$$

hinc prout pro m sumantur numeri vel impares vel pares, obtinebimus.

Pro numeris imparibus:

$$\int \frac{xx \partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{2} x \sqrt{(1-xx)} + \tfrac{1}{2} \int \frac{\partial x}{\sqrt{(1-xx)}}$$

$$\int \frac{x^4 \partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{4} x^3 \sqrt{(1-xx)} + \tfrac{3}{4} \int \frac{x^2 \partial x}{\sqrt{(1-xx)}}$$

$$\int \frac{x^6 \partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{6} x^5 \sqrt{(1-xx)} + \tfrac{5}{6} \int \frac{x^4 \partial x}{\sqrt{(1-xx)}}$$

Pro numeris paribus:

$$\int \frac{x^3 \partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{3} x^2 \sqrt{(1-xx)} + \tfrac{2}{3} \int \frac{x \partial x}{\sqrt{(1-xx)}}$$

$$\int \frac{x^5 \partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{5} x^4 \sqrt{(1-xx)} + \tfrac{4}{5} \int \frac{x^3 \partial x}{\sqrt{(1-xx)}}$$

$$\int \frac{x^7 \partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{7} x^6 \sqrt{(1-xx)} + \tfrac{6}{7} \int \frac{x^5 \partial x}{\sqrt{(1-xx)}}$$

etc.

Cum nunc sit $\int \frac{\partial x}{\sqrt{(1-xx)}} = \text{Arc. sin. } x$, et

$$\int \frac{x\partial x}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)},$$

habebimus sequentia integralia.

Pro ordine priore:

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \text{Arc. sin. } x$$

$$\int \frac{xx\partial x}{\sqrt{(1-xx)}} = -\tfrac{1}{2} x\sqrt{(1-xx)} + \tfrac{1}{2}\text{Arc. sin. } x$$

$$\int \frac{x^4\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{4}x^3 + \tfrac{1.3}{2.4}x\right)\sqrt{(1-xx)} + \tfrac{1.3}{2.4}\text{Arc. sin. } x$$

$$\int \frac{x^6\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{6}x^5 + \tfrac{1.5}{4.6}x^3 + \tfrac{1.3.5}{2.4.6}x\right)\sqrt{(1-xx)}$$
$$+ \tfrac{1.3.5}{2.4.6}\text{Arc. sin. } x$$

$$\int \frac{x^8\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{8}x^7 + \tfrac{1.7}{6.8}x^5 + \tfrac{1.5.7}{4.6.8}x^3 + \tfrac{1.3.5.7}{2.4.6.8}x\right)\sqrt{(1-xx)}$$
$$+ \tfrac{1.3.5.7}{2.4.6.8}\text{Arc. sin. } x.$$

Pro ordine posteriore:

$$\int \frac{x\partial x}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)}$$

$$\int \frac{x^3\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{3}x^2 + \tfrac{2}{3}\right)\sqrt{(1-xx)}$$

$$\int \frac{x^5\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{5}x^4 + \tfrac{1.4}{3.5}x^2 + \tfrac{2.4}{3.5}\right)\sqrt{(1-xx)}$$

$$\int \frac{x^7\partial x}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{7}x^6 + \tfrac{1.6}{5.7}x^4 + \tfrac{1.4.6}{3.5.7}x^2 + \tfrac{2.4.6}{3.5.7}\right)\sqrt{(1-xx)}.$$

Corollarium 1.

121. In genere ergo formula $\int \frac{x^{2i}\partial x}{\sqrt{(1-xx)}}$, si brevitatis gratia $\frac{1.3.5\ldots(2i-1)}{2.4.6\ldots\ldots 2i} = J$, habebimus hoc integrale

$$\int \frac{x^{2i}\partial x}{\sqrt{(1-xx)}} = J \text{ Arc. sin. } x$$

$$-J\left(x + \tfrac{2}{3}x^3 + \tfrac{2.4}{3.5}x^5 + \tfrac{2.4.6}{3.5.7}x^7 \ldots + \tfrac{2.4.6\ldots(2i-2)}{3.5.7\ldots(2i-1)}x^{2i-1}\right)\sqrt{(1-xx)}.$$

Corollarium 2.

122. Simili modo pro formula $\int \frac{x^{2i+1}\partial x}{\sqrt{(1-xx)}}$, si ponamus brevitatis ergo $\frac{2.4.6\ldots 2i}{3.5.7\ldots\ldots(2i+1)} = K$, habebimus hoc integrale:

$$\int \frac{x^{2i+1}\partial x}{\sqrt{(1-xx)}} = K$$

$$-K\left(1 + \tfrac{1}{2}x^2 + \tfrac{1.3}{2.4}x^4 + \tfrac{1.3.5}{2.4.6}x^6 + \ldots \tfrac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i}x^{2i}\right)\sqrt{(1-xx)}$$

ut integrale evanescat posito $x = 0$.

Exemplum 2.

123. *Invenire integrale formulae $\int \frac{x^{m-1}\partial x}{\sqrt{(1-xx)}}$, casibus quibus pro m numeri negativi assumuntur.*

Hic utendum est secunda reductione quae dat:

$$\int \frac{x^{m-3}\partial x}{\sqrt{(1-xx)}} = \frac{x^{m-2}\sqrt{(1-xx)}}{m-2} + \frac{m-1}{m-2}\int \frac{x^{m-1}\partial x}{\sqrt{(1-xx)}}:$$

unde patet si $m = 1$, fore $\int \frac{\partial x}{xx\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{x}$. Deinde si $m = 2$, formula $\int \frac{\partial x}{x\sqrt{(1-xx)}}$, facta substitutione $1 - xx = zz$, abit in $\frac{-\partial z}{1-zz}$ cujus integrale est

$$-\tfrac{1}{2}l\frac{1+z}{1-z} = -\tfrac{1}{2}l\frac{1+\sqrt{(1-xx)}}{1-\sqrt{(1-xx)}} = -l\frac{1+\sqrt{(1-xx)}}{x}:$$

unde duplicem seriem integrationum elicimus:

$$\int \frac{\partial x}{x\sqrt{(1-xx)}} = -l\frac{1+\sqrt{(1-xx)}}{x} = l\frac{1-\sqrt{(1-xx)}}{x};$$

$$\int \frac{\partial x}{x^3\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{2xx} + \tfrac{1}{2}\int \frac{\partial x}{x\sqrt{(1-xx)}}$$

$$\int \frac{\partial x}{x^5\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{4x^4} + \tfrac{3}{4}\int \frac{\partial x}{x^3\sqrt{(1-xx)}};$$

$$\int \frac{\partial x}{x^7\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{6x^6} + \tfrac{5}{6}\int \frac{\partial x}{x^5\sqrt{(1-xx)}}.$$

etc.

$$\int \frac{\partial x}{xx\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{x};$$

$$\int \frac{\partial x}{x^5\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{3x^3} + \tfrac{2}{3}\int \frac{\partial x}{xx\sqrt{(1-xx)}};$$

$$\int \frac{\partial x}{x^6\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{5x^5} + \tfrac{4}{5}\int \frac{\partial x}{x^4\sqrt{(1-xx)}}.$$

etc.

Hinc erit, ut in binis praecedentibus corrollariis

$$\int \frac{\partial x}{x^{2i+1}\sqrt{(1-xx)}} = lJ.\frac{1-\sqrt{(1-xx)}}{x} - J\left[\frac{1}{xx} + \frac{2}{3x^4} + \frac{2.4}{3.5x^6} + \ldots \right.$$

$$\left. \ldots\ldots + \frac{2.4\ldots.(2i-2)}{3.5\ldots\ldots(2i-1)x^{2i}}\right]\sqrt{(1-xx)};$$

$$\int \frac{\partial x}{x^{2i}\sqrt{(1-xx)}} = C - K\left[\frac{1}{x} + \frac{1}{2x^3} + \frac{1.3}{2.4x^5} + \ldots\ldots, \right.$$

$$\left. \ldots\ldots + \frac{1.3\ldots(2i-1)}{2.4\ldots.2i.x^{2i+1}}\right]\sqrt{(1-xx)}.$$

Scholion 1.

124. Hinc jam facile integralia formularum

$$\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}}$$

tam pro omnibus numeris m, quam pro imparibus μ assignari poterunt. Reductiones autem nostrae generales ad hunc casum accommodatae sunt:

$$\text{I. } \int x^{m+1}\,\partial x\,(1-xx)^{\frac{\mu}{2}} = \frac{-x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{m}{m+\mu+2}\int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}};$$

$$\text{II. } \int x^{m-3}\,\partial x\,(1-xx)^{\frac{\mu}{2}} = \frac{x^{m-2}(1-xx)^{\frac{\mu}{2}+1}}{m-2} + \frac{m+\mu}{m-2}\int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}};$$

$$\text{III. } \int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}+1} = \frac{x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{\mu+2}{m+\mu+2}\int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}};$$

$$\text{IV. } \int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m(1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m+\mu}{\mu}\int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}};$$

$$\text{V. } \int x^{m+1}\,\partial x\,(1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m(1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m}{\mu}\int x^{m-1}\,\partial x\,(1-xx)^{\frac{\mu}{2}};$$

$$\text{VI. } \int x^{m-3}\,\partial x\,(1-xx)^{\frac{\mu}{2}+1} = \frac{x^{m-2}(1-xx)^{\frac{\mu}{2}+1}}{m-2} + \frac{\mu+2}{m-2}\int x^{m-3}\,\partial x\,(1-xx)^{\frac{\mu}{2}}.$$

Posito enim $\mu = -1$, quatuor posteriores dant:

$$\int x^{m-1}\partial x\sqrt{(1-xx)}=\frac{x^m\sqrt{(1-xx)}}{m+1}+\frac{1}{m+1}\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

$$\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)^3}}=\frac{x^m}{\sqrt{(1-xx)}}-(m-1)\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

$$\int\frac{x^{m+1}\partial x}{\sqrt{(1-xx)^3}}=\frac{x^m}{\sqrt{(1-xx)}}-m\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

$$\int x^{m-3}\partial x\sqrt{(1-xx)}=\frac{x^{m-2}\sqrt{(1-xx)}}{m-2}+\frac{1}{m-2}\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}};$$

unde integrationes pro casibus $\mu=1$ et $\mu=-3$ eliciuntur, indeque porro reliqui.

Scholion 2.

125. Pro aliis formulis irrationalibus magis complicatis vix regulas dare licet, quibus ad formam simpliciorem reduci queant: et quoties ejusmodi formulae occurrant, reductio, si quam admittunt, plerumque sponte se offert. Veluti si formula fuerit hujusmodi $\int\frac{P\partial x}{Q^{n+1}}$, sive n sit numerus integer sive fractus, semper ad aliam hujus formae $\int\frac{S\partial x}{Q^n}$, quae utique simplicior aestimatur, reduci potest. Cum enim sit

$$\partial\frac{R}{Q^n}=\frac{Q\partial R-nR\partial Q}{Q^{n+1}}, \text{ posito } \int\frac{P\partial x}{Q^{n+1}}=y, \text{ erit}$$

$$y+\frac{R}{Q^n}=\int\frac{P\partial x+Q\partial R-nR\partial Q}{Q^{n+1}}.$$

Jam definiatur R ita, ut $P\partial x+Q\partial R-nR\partial Q$ per Q fiat divisibile, vel quia $Q\partial R$ jam factorem habet Q, ut fiat $P\partial x-nR\partial Q=QT\partial x$, prodibitque

$$y + \frac{R}{Q^n} = \int \frac{\partial R + T \partial x}{Q^n}, \text{ seu}$$

$$\int \frac{P \partial x}{Q^{n+1}} = -\frac{R}{Q^n} + \int \frac{\partial R + T \partial x}{Q^n}.$$

At semper functionem R ita definire licet, ut $P \partial x - n R \partial Q$ factorem Q obtineat, quod etsi in genere praestari nequit, tamen rem in exemplis tentando, mox perspicietur negotium semper succedere. Assumo autem hic P et Q esse functiones integras, ac talis quoque semper pro R erui poterit. Si forte eveniat, ut $\partial R + T \partial x = 0$, formula proposita algebraicum habebit integrale, quod hoc modo reperietur; contra autem haec forma ulterius reduci poterit in alias; ubi denominatoris exponens continuo unitate diminuatur; ac si n sit numerus integer, negotium tandem reducitur ad hujusmodi formam $\frac{V \partial x}{Q}$, quae sine dubio est simplicissima. Quamobrem cum in hoc capite vix quicquam amplius proferri possit, ad integrationem formularum irrationalium juvandam, methodum easdem integrationes per series infinitas perficiendi exponamus.

ADDITAMENTUM.

Problema.

Proposita formula $\partial y = [x + \sqrt{(1 + xx)}]^n \partial x$, *invenire ejus integrale.*

Solutio.

Posito $x + \sqrt{(1 + xx)} = u$, fit $x = \frac{uu - 1}{2u}$, et $\partial x = \frac{\partial u (uu + 1)}{2uu}$: unde formula nostra

$$\partial y = \tfrac{1}{2} u^{n-2} \partial u (uu + 1),$$

deoque ejus integrale

$$y = \frac{u^{n+1}}{2(n+1)} + \frac{u^{n-1}}{2(n-1)} + \text{Const.}$$

quod ergo semper est algebraicum nisi sit vel $n = 1$, vel $n = -1$.

Corollarium 1.

Patet etiam hanc formam latius patentem

$$\partial y = [x + \sqrt{(1 + xx)}]^n X \partial x$$

hoc modo integrari posse, dummodo X fuerit functio rationalis ipsius x. Posito enim $x = \frac{uu-1}{2u}$, pro X prodit functio rationalis ipsius u, quae sit $= U$, hincque fit

$$\partial y = \tfrac{1}{2} U u^{n-2} \partial u (uu + 1),$$

quae formula vel est rationalis, si n sit numerus integer, vel ad rationalitatem facile reducitur, si n sit numerus fractus.

Corollarium 2.

Cum sit $\sqrt{(1 + xx)} = \frac{uu+1}{2u}$; posito $\sqrt{(1 + xx)} = v$, etiam haec formula

$$\partial y = [x + \sqrt{(1 + xx)}]^n X \partial x$$

integrabitur, si X fuerit functio rationalis quaecunque quantitatum x et v. Facto enim $x = \frac{uu-1}{2u}$, functio X abit in functionem rationalem ipsius u, qua posita $= U$, habebitur ut ante $\partial y = \frac{1}{2} U u^{n-2} \partial u (uu + 1)$.

Exemplum.

Proposita sit formula

$$\partial y = [ax + b\sqrt{(1 + xx)}] [x + \sqrt{(1 + xx)}]^n \partial x.$$

Posito $x = \frac{uu-1}{2u}$, fit

$$\partial y = \left(\frac{a(uu-1) + b(uu+1)}{2u}\right) \times \tfrac{1}{2} u^{n-2} \partial u (uu + 1):$$

seu

$$\partial y = \tfrac{1}{4} u^{n-3} \partial u \left[a(u^4 - 1) + b(u^4 + 2uu + 1)\right],$$

cujus integrale est:

$$y = \frac{a+b}{4(n+2)} u^{n+2} + \frac{b}{2n} u^n + \frac{b-a}{4(n-2)} u^{n-2} + \text{Const.}$$

quae est algebraica, nisi sit vel $n = 2$, vel $n = -2$, vel etiam $n = 0$.

CAPUT III.

DE INTEGRATIONE FORMULARUM DIFFERENTIALIUM PER SERIES INFINITAS.

Problema 12.

126.

Si X fuerit functio rationalis fracta ipsius x, formulae differentialis $\partial y = X \partial x$ integrale per seriem infinitam exhibere.

Solutio.

Cum X sit functio rationalis fracta, ejus valor semper ita evolvi potest, ut fiat

$$X = A x^m + B x^{m+n} + C x^{m+2n} + D x^{m+3n} + E x^{m+4n} + \text{etc.}$$

ubi coëfficientes A, B, C, etc. seriem recurrentem constituent, ex denominatore fractionis determinandam. Multiplicentur ergo singuli termini per ∂x, et integrentur, quo facto integrale y per sequentem seriem exprimetur

$$y = \frac{A x^{m+1}}{m+1} + \frac{B x^{m+n+1}}{m+n+1} + \frac{C x^{m+2n+1}}{m+2n+1} + \text{etc.} + \text{Const.}$$

ubi si in serie pro X occurrat hujusmodi terminus $\frac{M}{x}$, inde in integrale ingredietur terminus $M l x$.

Scholion.

127. Cum integrale $\int X \partial x$, nisi sit algebraicum, per logarithmos et angulos exprimatur, hinc valores logarithmorum et angulorum per series infinitas exhiberi possunt. Cujusmodi series cum jam in Introductione plures sint traditae, non solum eaedem, sed etiam infinitae aliae hic per integrationem erui possunt. Hoc exem-

plis declarasse juvabit, ubi potissimum ejusmodi formulas evolvemus, in quibus denominator est binomium; tum vero etiam casus aliquot denominatore trinomio vel multinomio praeditos contemplabimur. Imprimis autem ejusmodi eligemus, quibus fractio in aliam, cujus denominator est binomius, transmutari potest.

Exemplum 1.

128. *Formulam differentialem* $\frac{\partial x}{a+x}$ *per seriem integrare.*

Sit $y=\int\frac{\partial x}{a+x}$, erit $y=l(a+x)+$ Const., unde integrali ita determinato, ut evanescat posito $x=0$, erit $y=l(a+x)-la$. Jam cum sit

$$\frac{1}{a+x}=\frac{1}{a}-\frac{x}{a^2}+\frac{xx}{a^3}-\frac{x^3}{a^4}+\frac{x^4}{a^5}-\text{etc.}$$

erit eadem lege integrale definiendo:

$$y=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\frac{x^5}{5a^5}-\text{etc.}$$

unde colligimus, uti quidem jam constat:

$$l(a+x)=la+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\text{etc.}$$

Corollarium 1.

129. Si capiamus x negativum, ut sit $\partial y=\frac{-\partial x}{a-x}$, eodem modo patebit esse:

$$l(a-x)=la-\frac{x}{a}-\frac{x^2}{2a^2}-\frac{x^3}{3a^3}-\frac{x^4}{4a^4}-\text{etc.}$$

hisque combinandis:

$$l(aa-xx)=2la-\frac{xx}{aa}-\frac{x^4}{2a^4}-\frac{x^6}{3a^6}-\frac{x^8}{4a^8}-\text{etc. et}$$

$$l\frac{a+x}{a-x}=\frac{2x}{a}+\frac{2x^3}{3a^3}+\frac{2x^5}{5a^5}+\frac{2x^7}{7a^7}+\text{etc.}$$

Corollarium 2.

130. Hae posteriores series eruuntur per integrationem formularum:

$$\frac{-2x\partial x}{aa-xx} = -2x\partial x\left(\frac{1}{aa}+\frac{xx}{a^4}+\frac{x^4}{a^6}+\text{etc.}\right) \text{ et}$$

$$\frac{2a\partial x}{aa-xx} = 2a\partial x\left(\frac{1}{aa}+\frac{xx}{a^4}+\frac{x^4}{a^6}+\text{etc.}\right).$$

Est autem $\int\frac{-2x\partial x}{aa-xx} = l(aa-xx) - laa$, et $\int\frac{2a\partial x}{aa-xx} = l\frac{a+x}{a-x}$, ita ut jam his formulis per series integrandis supersedere possimus.

Exemplum 2.

131. *Formulam differentialem* $\frac{a\partial x}{aa+xx}$ *per seriem integrare.*

Sit $\partial y = \frac{a\partial x}{aa+xx}$, et cum sit $y =$ Arc. tang. $\frac{x}{a}$, idem angulus serie infinita exprimetur. Quia enim habemus:

$$\frac{a}{aa+xx} = \frac{1}{a} - \frac{xx}{a^3} + \frac{x^4}{a^5} - \frac{x^6}{a^7} + \frac{x^8}{a^9} - \text{etc.}$$

erit integrando:

$$y = \text{Arc. tang.}\frac{x}{a} = \frac{x}{a} - \frac{x^3}{3a^3} + \frac{x^5}{5a^5} - \frac{x^7}{7a^7} + \text{etc.}$$

Exemplum 3.

132. *Integralia harum formularum* $\frac{\partial x}{1+x^3}$ *et* $\frac{x\partial x}{1+x^3}$ *per series exprimere.*

Cum sit $\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + x^{12} - \text{etc.}$ erit

$$\int\frac{\partial x}{1+x^3} = x - \tfrac{1}{4}x^4 + \tfrac{1}{7}x^7 - \tfrac{1}{10}x^{10} + \tfrac{1}{13}x^{13} - \text{etc. et}$$

$$\int\frac{x\partial x}{1+x^3} = \tfrac{1}{2}x^2 - \tfrac{1}{5}x^5 + \tfrac{1}{8}x^8 - \tfrac{1}{11}x^{11} + \tfrac{1}{14}x^{14} - \text{etc.}$$

Verum per §. 77. habemus per logarithmos et angulos:

$$\int \frac{\partial x}{1+x^3} = \tfrac{1}{3} l(1+x) - \tfrac{2}{3}\cos.\tfrac{\pi}{3}\, l\sqrt{(1-2x\cos.\tfrac{\pi}{3}+xx)}$$

$$+ \tfrac{2}{3}\sin.\tfrac{\pi}{3}\ \text{Arc. tang.}\ \frac{x\sin\frac{\pi}{3}}{1-x\cos.\frac{\pi}{3}}$$

$$\int \frac{x\,\partial x}{1+x^3} = -\tfrac{1}{3} l(1+x) - \tfrac{2}{3}\cos.\tfrac{2\pi}{3}\, l\sqrt{(1-2x\cos.\tfrac{\pi}{3}+xx)}$$

$$+ \tfrac{2}{3}\sin.\tfrac{2\pi}{3}\ \text{Arc. tang.}\ \frac{x\sin.\frac{\pi}{3}}{1-x\cos.\frac{\pi}{3}}$$

At est $\cos.\frac{\pi}{3} = \frac{1}{2}$; $\cos.\frac{2\pi}{3} = -\frac{1}{2}$; $\sin.\frac{\pi}{3} = \frac{\sqrt{3}}{2}$; $\sin.\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$; unde fit

$$\int \frac{\partial x}{1+x^3} = \tfrac{1}{3} l(1+x) - \tfrac{1}{3} l\sqrt{(1-x+xx)} + \tfrac{1}{\sqrt{3}}\text{Arc. tang}\ \frac{x\sqrt{3}}{2-x}$$

$$\int \frac{x\,\partial x}{1+x^3} = -\tfrac{1}{3} l(1+x) + \tfrac{1}{3} l\sqrt{(1-x+xx)} + \tfrac{1}{\sqrt{3}}\text{Arc. tang.}\ \frac{x\sqrt{3}}{2-x}$$

integralibus ut seriebus ita sumtis, ut evanescant posito $x=0$.

Corollarium 1.

133. His igitur seriebus additis, prodit

$$\tfrac{2}{\sqrt{3}}\text{Arc. tang.}\ \frac{x\sqrt{3}}{2-x} = x + \tfrac{1}{2}xx - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \tfrac{1}{8}x^8$$
$$- \tfrac{1}{10}x^{10} - \tfrac{1}{11}x^{11} + \text{etc.}$$

subtracta autem posteriori a priori, fit

$$\tfrac{2}{3} l \frac{1+x}{\sqrt{(1-x+xx)}} = x - \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{8}x^8$$
$$- \tfrac{1}{10}x^{10} + \tfrac{1}{11}x^{11} + \text{etc.}$$

cujus valor etiam est

$$\tfrac{1}{3} l \frac{(1+x)^2}{1-x+xx} = \tfrac{1}{3} l \frac{(1+x)^3}{1+x^3}.$$

Corollarium 2.

134. Cum sit $\int \frac{xx\partial x}{1+x^3} = \frac{1}{3} l(1+x^3)$, erit eodem modo

$$\frac{1}{3} l(1+x^3) = \frac{1}{3}x^3 - \frac{1}{6}x^6 + \frac{1}{9}x^9 - \frac{1}{12}x^{12} + \text{etc.}$$

qua serie illis adjecta omnes potestates ipsius x occurrent.

Exemplum 4.

135. *Integrale hoc $y = \int \frac{(1+xx)\partial x}{1+x^4}$ per seriem exprimere.*

Cum sit $\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + x^{16} - \text{etc.}$ erit

$$y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \frac{1}{11}x^{11} - \frac{1}{13}x^{13} - \frac{1}{15}x^{15} + \text{etc.}$$

Verum per §. 82 ubi $m=1$ et $n=4$, posito $\frac{\pi}{4}=\omega$, fit integrale idem:

$$y = \sin.\omega \text{ Arc. tang } \frac{x\sin.\omega}{1-x\cos.\omega}$$
$$+ \sin.3\omega \text{ Arc. tang. } \frac{x\sin.3\omega}{1-x\cos.3\omega}:$$

Atque ob $\frac{\pi}{4}=\omega=45^\circ$, est $\sin.\omega=\frac{1}{\sqrt{2}}$; $\cos.\omega=\frac{1}{\sqrt{2}}$; $\sin.3\omega=\frac{1}{\sqrt{2}}$; $\cos.3\omega=-\frac{1}{\sqrt{2}}$; hinc habebimus:

$$y = \frac{1}{\sqrt{2}} \text{ Arc. tang. } \frac{x}{\sqrt{2}-x} + \frac{1}{\sqrt{2}} \text{ Arc. tang. } \frac{x}{\sqrt{2}+x}$$
$$= \frac{1}{\sqrt{2}} \text{ Arc. tang. } \frac{x\sqrt{2}}{1-xx}.$$

Exemplum 5.

136. *Integrale hoc $y = \int \frac{(1+x^4)\partial x}{1+x^6}$ per seriem exprimere.*

Cum sit $\frac{1}{1+x^6} = 1 - x^6 + x^{12} - x^{18} + x^{24} - \text{etc.}$ erit

$$y = x + \tfrac{1}{5} x^5 - \tfrac{1}{7} x^7 - \tfrac{1}{11} x^{11} + \tfrac{1}{13} x^{13} + \tfrac{1}{17} x^{17} - \text{etc.}$$

At per §. 82. ubi $m = 1$, $n = 6$, et $\omega = \frac{\pi}{6} = 30^\circ$, est

$$y = \tfrac{2}{3} \sin.\omega \text{ Arc. tang. } \frac{x \sin.\omega}{1 - x\cos.\omega} + \tfrac{2}{3} \sin. 3\omega \text{ Arc. tang. } \frac{x \sin. 3\omega}{1 - x \cos. 3\omega}$$
$$+ \tfrac{2}{3} \sin. 5\omega \text{ Arc. tang. } \frac{x \sin. 5\omega}{1 - x\cos. 5\omega}:$$

est vero $\sin.\omega = \frac{1}{2}$; $\cos.\omega = \frac{\sqrt{3}}{2}$; $\sin. 3\omega = 1$; $\cos. 3\omega = 0$; $\sin. 5\omega = \frac{1}{2}$; $\cos. 5\omega = -\frac{\sqrt{3}}{2}$, ergo

$$y = \tfrac{1}{3} \text{ Arc. tang. } \frac{x}{2 - x\sqrt{3}} + \tfrac{2}{3} \text{ Arc. tang. } x + \tfrac{1}{3} \text{ Arc. tang. } \frac{x}{2 + x\sqrt{3}}:$$

seu

$$y = \tfrac{1}{3} \text{ Arc. tang. } \frac{x}{1 - xx} + \tfrac{2}{3} \text{ Arc.tang. } x = \tfrac{1}{3} \text{ Arc.tang. } \frac{3x(1 - xx)}{3 - 4xx + x^4}$$

Corollarium 1.

137. Sit $z = \int \frac{xx\partial x}{1 + x^6} = \frac{1}{3}x^3 - \frac{1}{9}x^9 + \frac{1}{15}x^{15} - \frac{1}{21}x^{21} + \text{etc.}$ at facto $x^3 = u$, est

$$z = \tfrac{1}{3}\int \frac{\partial u}{1 + uu} = \tfrac{1}{3} \text{ Arc. tang. } u = \tfrac{1}{3} \text{ Arc. tang. } x^3.$$

Hinc series hujusmodi mixta formatur:

$$x + \tfrac{n}{3} x^3 + \tfrac{1}{5} x^5 - \tfrac{1}{7} x^7 - \tfrac{n}{9} x^9 - \tfrac{1}{11} x^{11} + \tfrac{1}{13} x^{13} + \tfrac{n}{15} x^{15} + \tfrac{1}{17} x^{17} - \text{etc.}$$

cujus summa est $= \frac{1}{3} \text{ Arc. tang. } \frac{3x(1 - xx)}{1 - 4xx + x^4} + \frac{n}{3} \text{ Arc. tang. } x^3$.

Corollarium 2.

138. Si hic capiatur $n = -1$, binos angulos in unum colligendo, fit

$$\tfrac{1}{3} \text{ Arc. tang. } \frac{3x(1 - xx)}{1 - 4xx + x^4} - \tfrac{1}{3} \text{ Arc. tang. } x^3$$
$$= \tfrac{1}{3} \text{ Arc. tang. } \frac{3x - 4x^3 + 4x^5 - x^7}{1 - 4xx + 4x^4 - 3x^6}:$$

quae fractio per $1 - xx + x^4$ dividendo, reducitur ad $\frac{3x - x^3}{1 - 3xx}$, quae est tangens tripli anguli x pro tangente habentis, ita ut sit $\frac{1}{3}$ Arc. tang. $\frac{3x - x^3}{1 - 3xx} =$ Arc. tang. x, quod idem series inventa manifesto indicat.

Exemplum 6.

139. *Hanc formulam* $\partial y = \frac{(x^{m-1} + x^{n-m-1})\partial x}{1 + x^n}$, *per seriem integrare.*

Ob $\frac{1}{1 + x^n} = 1 - x^n + x^{2n} - x^{3n} + x^{4n} -$ etc. habebitur

$$y = \frac{x^m}{m} + \frac{x^{n-m}}{n-m} - \frac{x^{n+m}}{n+m} - \frac{x^{2n-m}}{2n-m} + \frac{x^{2n+m}}{2n+m} + \frac{x^{3n-m}}{3n-m} - \text{etc.}$$

Haec ergo series per §. 82. aggregatum aliquot arcuum circularium exprimit, quos ibi videre licet.

Corollarium.

140. Eodem modo proposita formula $\partial z = \frac{(x^{m-1} - x^{n-m-1})\partial x}{1 - x^n}$, ob $\frac{1}{1 - x^n} = 1 + x^n + x^{2n} + x^{3n} +$ etc. invenitur:

$$z = \frac{x^m}{m} - \frac{x^{n-m}}{n-m} + \frac{x^{n+m}}{n+m} - \frac{x^{2n-m}}{2n-m} + \frac{x^{2n+m}}{2n+m} - \frac{x^{3n-m}}{3n-m} + \text{etc.}$$

cuius valor §. 84. est exhibitus.

Exemplum 7.

141. *Hanc formulam* $\partial y = \frac{(1 + 2x)\partial x}{1 + x + xx}$, *per seriem integrare.*

Primo integrale est manifesto $y = l(1 + x + xx)$; ut autem in seriem convertatur, multiplicetur numerator et denominator per $1 - x$, ut fiat $\partial y = \frac{(1 + x - 2xx)\partial x}{1 - x^3}$. Cum nunc sit $\frac{1}{1 - x^3} = 1 + x^3 + x^6 + x^9 + x^{12} + \text{etc.}$ erit integrando:

$$y = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{2x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} - \frac{2x^9}{9} + \text{etc.}$$

Corollarium 1.

142. Eodem modo inveniri potest

$$y = l(1 + x + xx + x^3)$$

per seriem. Cum enim fiat $y + l(1 - x) = l(1 - x^4)$, erit

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \text{etc.}$$
$$- x^4 \qquad\qquad - \frac{x^8}{2}$$

sive

$$y = x + \frac{xx}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3x^8}{8} + \frac{x^9}{9} + \text{etc.}$$

Corollarium 2.

143. At fractio $\frac{1 + 2x}{1 + x + xx}$ per seriem recurrentem evoluta dat

$$1 + x - 2xx + x^3 + x^4 - 2x^5 + x^6 + x^7 - 2x^8 + \text{etc.}$$

unde per integrationem eadem series obtinetur, quae ante.

Exemplum 8.

144. *Hanc formulam* $\partial y = \frac{\partial x}{1 - 2x\cos.\zeta + xx}$ *per seriem integrare.*

Per §. 64. ubi $A = 1$, $B = 0$, $a = 1$, et $b = 1$, est hujus formulae integrale $y = \frac{1}{\sin.\zeta}$ Arc. tang. $\frac{x \sin.\zeta}{1 - x\cos.\zeta}$. At per seriem recurrentem reperimus

$$\frac{1}{1 - 2x\cos.\zeta + xx} = 1 + 2x\cos.\zeta + (4\cos.\zeta^2 - 1)xx$$
$$+ (8\cos.\zeta^3 - 4\cos.\zeta)x^3 + (16\cos.\zeta^4 - 12\cos.\zeta^2 + 1)x^4$$
$$+ (32\cos.\zeta^5 - 32\cos.\zeta^3 + 6\cos.\zeta)x^5 + \text{etc.}$$

qua serie per ∂x multiplicata et integrata, obtinetur quaesitum. Potestatibus autem ipsius $\cos.\zeta$ in cosinus angulorum multiplorum conversis, reperitur:

$$y = x + \tfrac{1}{2}xx(2\cos.\zeta) + \tfrac{1}{3}x^3(2\cos.2\zeta + 1)$$
$$+ \tfrac{1}{4}x^4(2\cos.3\zeta + 2\cos.\zeta) + \tfrac{1}{5}x^5(2\cos.4\zeta + 2\cos.2\zeta + 1)$$
$$+ \tfrac{1}{6}x^6(2\cos.5\zeta + 2\cos.3\zeta + 2\cos.\zeta) + \text{etc.}$$

Corollarium 1.

145. Si ponatur $\partial z = \frac{(1 - x\cos.\zeta)\partial x}{1 - 2x\cos.\zeta + xx}$, erit per §. 63. $A = 1$, $B = -\cos.\zeta$, $a = 1$ et $b = 1$, ideoque

$$z = -\cos.\zeta\, l\sqrt{(1 - 2x\cos.\zeta + xx)} + \sin.\zeta \text{ Arc. tang. } \frac{x\sin.\zeta}{1 - x\cos.\zeta}.$$

At per seriem

ob $\frac{1 - x\cos.\zeta}{1 - 2x\cos.\zeta + xx} = 1 + x\cos.\zeta + x^2\cos.2\zeta$
$+ x^3\cos.3\zeta + x^4\cos.4\zeta + \text{etc.}$ fit

$$z = x + \tfrac{1}{2}xx\cos.\zeta + \tfrac{1}{3}x^3\cos.2\zeta + \tfrac{1}{4}x^4\cos.3\zeta$$
$$+ \tfrac{1}{5}x^5\cos.4\zeta + \text{etc.}$$

Corollarium 2.

146. At quia $\partial z = \frac{\partial x(-x\cos.\zeta + \cos.\zeta^2 + \sin.\zeta^2)}{1 - 2x\cos.\zeta + xx}$, erit

$$z = -\cos.\zeta\, l\sqrt{(1 - 2x\cos.\zeta + xx)} + \sin.\zeta^2 \int \frac{\partial x}{1 - 2x\cos.\zeta + xx}.$$

Hinc ergo pro $y = \int \frac{\partial x}{1 - 2x\cos.\zeta + xx}$ alia reperitur series infinita cum logarithmo connexa, scilicet

$$y = \frac{\cos.\zeta}{\sin.\zeta^2}\, l\sqrt{(1 - 2x\cos.\zeta + xx)}$$
$$+ \frac{1}{\sin.\zeta^2}\left(x + \tfrac{1}{2}xx\cos.\zeta + \tfrac{1}{3}x^3\cos.2\zeta + \tfrac{1}{4}x^4\cos.3\zeta + \text{etc.}\right)$$

Problema 12.

147. Formulam differentialem irrationalem

$\partial y = x^{m-1}\partial x\,(a + bx^n)^{\frac{\mu}{\nu}}$ per seriem infinitam integrare.

Solutio.

Sit $a^{\frac{\mu}{\nu}} = c$, erit $\partial y = c x^{m-1}\partial x\left(1 + \frac{b}{a}x^n\right)^{\frac{\mu}{\nu}}$, ubi quidem assumimus c non esse quantitatem imaginariam. Cum igitur sit

$$\left(1+\frac{b}{a}x^n\right)^{\frac{\mu}{\nu}} = 1 + \frac{\mu b}{1\nu.a}x^n + \frac{\mu(\mu-\nu)bb}{1\nu.2\nu.aa}x^{2n} + \frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu.2\nu.3\nu.a^3}x^{3n} + \text{etc.}$$

erit integrando:

$$y = c\left(\frac{x^m}{m} + \frac{\mu b}{1\nu.a}\cdot\frac{x^{m+n}}{m+n} + \frac{\mu(\mu-\nu)bb}{1\nu.2\nu.aa}\cdot\frac{x^{m+2n}}{m+2n} + \frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu.2\nu.3\nu.a^3}\cdot\frac{x^{m+3n}}{m+3n} + \text{etc.}\right),$$

quae series in infinitum excurrit, nisi $\frac{\mu}{\nu}$ sit numerus integer positivus.

Sin autem casu, quo ν numerus par, a fuerit quantitas negativa, expressio nostra ita est repraesentanda,

$$\partial y = x^{m-1}\partial x\,(bx^n - a)^{\frac{\mu}{\nu}} = b^{\frac{\mu}{\nu}}\, x^{m+\frac{\mu n}{\nu}-1}\,\partial x\left(1 - \frac{a}{b}x^{-n}\right)^{\frac{\mu}{\nu}}.$$

Cum igitur sit

$$\left(1 - \frac{a}{b}x^{-n}\right)^{\frac{\mu}{\nu}} = 1 - \frac{\mu a}{1\nu.b}x^{-n} + \frac{\mu(\mu-\nu)a^2}{1\nu.2\nu.b^2}x^{-2n}$$
$$- \frac{\mu(\mu-\nu)(\mu-2\nu)a^3}{1\nu.2\nu.3\nu.b^3}x^{-3n} + \text{etc.}$$

erit integrando

$$y = b^{\frac{\mu}{\nu}} \left(\frac{\nu x^{m + \frac{\mu n}{\nu}}}{m\nu + \mu n} - \frac{\mu a}{1\nu . b} \cdot \frac{\nu x^{m + \frac{(\mu - \nu) n}{\nu}}}{m\nu + (\mu - \nu) n} + \frac{\mu (\mu - \nu) a^2}{1\nu . 2\nu . b^2} \cdot \frac{\nu x^{m + \frac{(\mu - 2\nu) n}{\nu}}}{m\nu + (\mu - 2\nu) n} - \text{etc.} \right)$$

Si a et b sint numeri positivi, utraque evolutione uti licet.

Exemplum 1.

148. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(1 - xx)}}$, *per seriem integrare.*

Primo ex superioribus patet esse $y = \text{Arc. sin. } x$ qui ergo angulus etiam per seriem infinitam exprimetur. Cum enim sit

$$\frac{1}{\sqrt{(1 - xx)}} = 1 + \frac{1}{2} x^2 + \frac{1.3}{2.4} x^4 + \frac{1.3.5}{2.4.6} x^6 + \frac{1.3.5.7}{2.4.6.8} x^8 + \text{etc.}$$

erit

$$y = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \frac{1.3.5.7}{2.4.6.8} \cdot \frac{x^9}{9} + \text{etc.}$$

utroque valore ita definito, ut evanescat posito $x = 0$.

Corollarium 1.

149. Si ergo sit $x = 1$, ob Arc. sin. $1 = \frac{\pi}{2}$, erit

$$\frac{\pi}{2} = 1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \frac{1.3.5}{2.4.6.7} + \frac{1.3.5.7}{2.4.6.8.9} + \text{etc.}$$

At si ponatur $x = \frac{1}{2}$, ob Arc. sin. $\frac{1}{2} = 30^\circ = \frac{\pi}{6}$, erit

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1.}{2.2^3.3} + \frac{1.3}{2.4.2^5.5} + \frac{1.3.5}{2.4.6.2^7.7} + \frac{1.3.5.7}{2.4.6.8.2^9.9} + \text{etc.}$$

cujus seriei decem termini additi dant 0,52359877, cujus sextuplum 3,14159262 tantum in octava figura a veritate discrepat.

Corollarium 2.

150. Proposita hac formula $\partial y = \frac{\partial x}{\sqrt{(x - xx)}}$ posito $x = uu$, fit

$$\partial y = \frac{2u\partial u}{\sqrt{(uu-u^4)}} = \frac{2\partial u}{\sqrt{(1-uu)}}:$$

ergo $y = 2$ Arc. sin. $u = 2$ Arc. sin. $\sqrt{x}$. Tum vero per seriem erit:

$$y = 2\left(u + \tfrac{1}{2}\cdot\frac{u^3}{3} + \frac{1.3}{2.4}\cdot\frac{u^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{u^7}{7} + \text{etc.}\right) \text{ seu}$$

$$y = 2\left(1 + \tfrac{1}{2}\cdot\frac{x}{3} + \frac{1.3}{2.4}\cdot\frac{xx}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{7} + \text{etc.}\right)\sqrt{x}.$$

Exemplum 2.

151. *Formulam $\partial y = \partial x\sqrt{(2ax - xx)}$ per seriem integrare.*

Posito $x = uu$, fit $\partial y = 2uu\partial u\sqrt{(2a - uu)}$: at per reductionem I. (§. 118.) est $n = 2$; $m = 1$; $a = 2a$; $b = -1$; $\mu = 1$; $\nu = 2$; unde

$$\int uu\partial u\sqrt{(2a - uu)} = -\tfrac{1}{4}u(2a - uu)^{\frac{3}{2}} + \tfrac{1}{2}a\int\partial u\sqrt{(2a - uu)}:$$

et per tertiam, sumendo $m = 1$, $a = 2a$, $b = -1$, $n = 2$, $\mu = -1$, $\nu = 2$, fit

$$\int\partial u\sqrt{(2a - uu)} = \tfrac{1}{2}u\sqrt{(2a - uu)} + a\int\frac{\partial u}{\sqrt{(2a - uu)}}:$$

at est

$$\int\frac{\partial u}{\sqrt{(2a - uu)}} = \text{Arc. sin. } \frac{u}{\sqrt{2a}} = \text{Arc. sin. } \frac{\sqrt{x}}{\sqrt{2a}}, \text{ ideoque}$$

$$\int uu\partial u\sqrt{(2a - uu)} = -\tfrac{1}{4}u(2a - uu)^{\frac{3}{2}} + \tfrac{1}{4}au\sqrt{(2a - uu)} + \tfrac{1}{2}aa \text{ Arc. sin. } \frac{\sqrt{x}}{\sqrt{2a}}$$
$$= \tfrac{1}{4}u(uu - a)\sqrt{(2a - uu)} + \tfrac{1}{2}aa \text{ Arc. sin. } \frac{\sqrt{x}}{\sqrt{2a}}.$$

Ergo $\quad y = \tfrac{1}{2}(x - a)\sqrt{(2ax - xx)} + aa \text{ Arc. sin. } \frac{\sqrt{x}}{\sqrt{2a}}.$

Pro serie autem invenienda est $\partial y = \partial x\sqrt{2ax}\left(1 - \frac{x}{2a}\right)^{\frac{1}{2}}$

$$= x^{\frac{1}{2}}\partial x\left(1 - \tfrac{1}{2}\cdot\frac{x}{2a} - \frac{1.1}{2.4}\cdot\frac{xx}{4aa} - \frac{1.1.3}{2.4.6}\cdot\frac{x^3}{8a^3} - \text{etc.}\right)\sqrt{2a}:$$

hincque integrando

$$y = \left(\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{2}\cdot\frac{2x^{\frac{5}{2}}}{5.2a} - \frac{1.1}{2.4}\cdot\frac{2x^{\frac{7}{2}}}{7.4aa} - \frac{1.1.3}{2.4.6}\cdot\frac{2x^{\frac{9}{2}}}{9.8a^3} - \text{etc.}\right)\sqrt{2a},$$

seu

$$y = \left(\tfrac{x}{3} - \tfrac{1}{2}\cdot\frac{x^2}{5.2a} - \frac{1.1}{2.4}\cdot\frac{x^3}{7.4aa} - \frac{1.1.3}{2.4.6}\cdot\frac{x^4}{9.8a^3} - \text{etc.}\right)2\sqrt{2ax}.$$

Corollarium 1.

152. Integrale facilius inveniri potest, ponendo $x = a - v$, unde fit $\partial y = -\partial v\sqrt{(aa - vv)}$, et per reductionem tertiam

$$\int \partial v\sqrt{(aa - vv)} = \tfrac{1}{2}v\sqrt{(aa - vv)} + \tfrac{1}{2}aa\int\frac{\partial v}{\sqrt{(aa - vv)}}, \text{ hinc}$$

$$y = C - \tfrac{1}{2}v\sqrt{(aa - vv)} - \tfrac{1}{2}aa\,\text{Arc. sin.}\,\tfrac{v}{a}, \text{ seu}$$

$$y = C - \tfrac{1}{2}(a - x)\sqrt{(2ax - xx)} - \tfrac{1}{2}aa\,\text{Arc. sin.}\,\tfrac{a-x}{a}.$$

Ut igitur fiat $y = 0$, posito $x = 0$, capi debet $C = \tfrac{1}{2}aa\,\text{Arc. sin.}\,1$, ita uta sit

$$y = -\tfrac{1}{2}(a - x)\sqrt{(2ax - xx)} + \tfrac{1}{2}aa\,\text{Arc. cos.}\,\tfrac{a-x}{a}.$$

Est vero

$$\text{Arc. sin.}\,\tfrac{\sqrt{x}}{\sqrt{2a}} = \tfrac{1}{2}\,\text{Arc. cos.}\,\tfrac{a-x}{a}.$$

Corollarium 2.

153. Si ponamus $x = \frac{a}{2}$, fit $y = \frac{-aa\sqrt{3}}{8} + \frac{\pi aa}{6}$, series autem dat

$$y = 2aa\left(\frac{1}{2.3} - \frac{1}{2.5.2^3} - \frac{1.1}{2.4.7.2^5} - \frac{1.1.3}{2.4.6.9.2^7} - \text{etc.}\right),$$

unde colligitur

$$\pi = \frac{3\sqrt{3}}{4} + 6\left(\tfrac{1}{3} - \frac{1}{2.5.2^2} - \frac{1.1}{2.4.7.2^4} - \frac{1.1.3}{2.4.6.9.2^6} - \text{etc.}\right)$$

at per superiorem est

$$\pi = 3\left(1 + \frac{1}{2.3.2^2} + \frac{1.3}{2.4.5.2^4} + \frac{1.3.5}{2.4.6.7.2^6} + \text{etc.}\right)\ (\S.\ 149.)$$

ex quarum combinatione plures aliae formari possunt.

Exemplum 3.

154. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(1+xx)}}$, *per seriem integrare.*

Integrale est $y = l[x + \sqrt{(1+xx)}]$, ita sumtum ut evanescat posito $x = 0$. At ob

$$\frac{1}{\sqrt{(1+xx)}} = 1 - \tfrac{1}{2}x^2 + \frac{1.3}{2.4}x^4 - \frac{1.3.5}{2.4.6}x^6 + \text{etc.}$$

erit idem integrale per seriem expressum:

$$y = x - \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \text{etc.}$$

Exemplum 4.

155. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(xx-1)}}$ *per seriem integrare.*

Integratio dat $y = l[x + \sqrt{(xx-1)}]$ quod evanescit posito $x = 1$. Jam ob

$$\frac{1}{\sqrt{(xx-1)}} = \frac{1}{x} + \frac{1}{2x^3} + \frac{1.3}{2.4.x^5} + \frac{1.3.5}{2.4.6x^7} + \text{etc.}$$

erit idem integrale:

$$y = C + lx - \frac{1}{2.2x^2} - \frac{1.3}{2.4.4x^4} - \frac{1.3.5}{2.4.6.6x^6} - \text{etc.}$$

quod ut evanescat posito $x = 1$, constans ita definitur, ut fiat:

$$y = lx + \frac{1}{2.2}\left(1 - \frac{1}{xx}\right) + \frac{1.3}{2.4.4}\left(1 - \frac{1}{x^4}\right) + \frac{1.3.5}{2.4.6.6}\left(1 - \frac{1}{x^6}\right) + \text{etc.}$$

Corollarium.

156. Posito $x = 1 + u$ fit

$$\partial y = \frac{\partial u}{\sqrt{(2u+uu)}} = \frac{\partial u}{\sqrt{2u}}\left(1 + \frac{u}{2}\right)^{-\frac{1}{2}} =$$

$$\frac{\partial u}{\sqrt{2u}}\left(1 - \frac{1}{2}\cdot\frac{u}{2} + \frac{1.3}{2.4}\cdot\frac{uu}{4} - \frac{1.3.5}{2.4.6}\cdot\frac{u^3}{8} + \text{etc}\right)$$

unde integrando habebitur:

$$y = \frac{1}{\sqrt{2}}\left(2\sqrt{u} - \frac{1}{2}\cdot\frac{2u^{\frac{3}{2}}}{2.3} + \frac{1.3}{2.4}\cdot\frac{2u^{\frac{5}{2}}}{5.4} - \frac{1.3.5}{2.4.6}\cdot\frac{2u^{\frac{7}{2}}}{7.8} + \text{etc.}\right) \text{ seu}$$

$$y = \left(1 - \frac{1u}{2.3.2} + \frac{1.3.uu}{2.4.5.4} - \frac{1.3.5.u^3}{2.4.6.7.8} + \text{etc.}\right)\sqrt{2u}.$$

Exemplum 5.

157. *Formulam* $\partial y = \frac{\partial x}{(1-x)^n}$ *per seriem integrare*

Per integrationem fit

$$y = \frac{1}{(n-1)(1-x)^{n-1}} - \frac{1}{n-1},$$

facto $y = 0$ si $x = 0$, seu

$$y = \frac{(1-x)^{-n+1} - 1}{n-1}.$$

Jam vero per seriem est

$$\partial y = \partial x\left(1 + nx + \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n+2)}{1.2.3}x^3 + \text{etc.}\right)$$

unde idem integrale ita exprimetur:

$$y = x + \frac{nx^2}{2} + \frac{n(n+1)x^3}{1.2.3} + \frac{n(n+1)(n+2)x^4}{1.2.3.4} + \text{etc.}$$

Hinc autem quoque manifesto fit

$$(n-1)y + 1 = \frac{1}{(1-x)^{n-1}}.$$

Scholion.

158. Haec autem cum sint nimis obvia, quam ut iis fusius inhaerere sit opus, aliam methodum series eliciendi exponam magis absconditam, quae saepe in Analysi eximium usum afferre potest.

Problema 13.

159. Proposita formula differentiali

$$\partial y = x^{m-1}\partial x\,(a+bx^n)^{\frac{\mu}{\nu}-1},$$

ejus integrale altera methodo in seriem convertere.

Solutio.

Ponatur $y = (a+bx^n)^{\frac{\mu}{\nu}}z$, erit

$$\partial y = (a+bx^n)^{\frac{\mu}{\nu}-1}\left[\partial z(a+bx^n)+\frac{n\mu}{\nu}bx^{n-1}z\partial x\right]:$$

unde fit

$$x^{m-1}\partial x = \partial z\,(a+bx^n)+\frac{n\mu}{\nu}bx^{n-1}z\partial x, \text{ seu}$$

$$\nu x^{m-1}\partial x = \nu\partial z\,(a+bx^n)+n\mu bx^{n-1}z\partial x.$$

Jam antequam seriem, qua valor ipsius z definiatur, investigemus, notandum est casu, quo x evanescit, fieri

$$\partial y = a^{\frac{\mu}{\nu}-1}x^{m-1}\partial x = a^{\frac{\mu}{\nu}}\partial z,$$

ut sit $\partial z = \frac{1}{a}x^{m-1}\partial x$. Statuamus ergo:

$$z = Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + \text{etc.}$$

eritque

$$\frac{\partial z}{\partial x} = mAx^{m-1}+(m+n)Bx^{m+n-1}+(m+2n)Cx^{m+2n-1}+\text{etc.}$$

Substituantur hae series loco z et $\frac{\partial z}{\partial x}$ in aequatione

$$\frac{\nu\partial z}{\partial x}(a+bx^n)+n\mu bx^{n-1}z-\nu x^{m-1}=0,$$

singulisque terminis secundum potestates ipsius x dispositis, orietur ista aequatio:

$$\left.\begin{array}{lll} m\nu aAx^{m-1} & +(m+n)\nu aBx^{m+n-1} & +(m+2n)\nu aCx^{m+2n-1}+\text{etc.} \\ -\nu & +m\nu bA & +(m+n)\nu bB \\ & +n\mu bA & +n\mu bB \end{array}\right\}=0:$$

unde singulis terminis nihilo aequalibus positis, coëfficientes ficti per sequentes formulas definientur:

$$m\nu a A - \nu = 0; \qquad \text{hinc } A = \frac{1}{ma};$$

$$(m+n)\nu a B + (m\nu + n\mu) b A = 0; \quad B = -\frac{(m\nu+n\mu)b}{(m+n)\nu a} A;$$

$$(m+2n)\nu a C + [(m+n)\nu + n\mu] b B = 0; \quad C = -\frac{[(m+n)\nu+n\mu]b}{(m+2n)\nu a} B;$$

$$(m+3n)\nu a D + [(m+2n)\nu + n\mu] b C = 0; \quad D = -\frac{[(m+2n)\nu+n\mu]b}{(m+3n)\nu a} C;$$

sicque quilibet coëfficiens facile ex praecedente reperitur. Tum vero erit:

$$y = (a + bx^n)^{\frac{\mu}{\nu}} (A x^m + B x^{m+n} + C x^{m+2n} + D x^{m+3n} + \text{etc.})$$

Solutio 2.

Quemadmodum hic seriem secundum potestates ipsius x ascendentem assumsimus, ita etiam descendentem constituere licet:

$$z = A x^{m-n} + B x^{m-2n} + C x^{m-3n} + D x^{m-4n} + \text{etc.}$$

ut sit

$$\frac{\partial z}{\partial x} = (m-n) A x^{m-n-1} + (m-2n) B x^{m-2n-1} + (m-3n) C x^{m-3n-1} + \text{etc.}$$

quibus seriebus substitutis prodit:

$$\left.\begin{array}{llll} +(m-n)\nu b A x^{m-1} & +(m-n)\nu a A x^{m-n-1} & +(m-2n)\nu a B x^{m-2n-1} & +(m-3n)\nu a C x^{m-3n-1} \\ +n\mu b A & +(m-2n)\nu b B & +(m-3n)\nu b C & +(m-4n)\nu b D \\ -\nu & +n\mu b B & +n\mu b C & +n\mu b D \end{array}\right\} = 0$$

Hinc ergo sequenti modo litterae A, B, C, etc. determinantur:

$$(m-n)\nu b A + n\mu b A - \nu = 0 \quad \text{ergo} \quad A = \frac{\nu}{(m-n)\nu + n\mu} \cdot \frac{1}{b};$$

$$(m-n)\nu a A + (m-2n)\nu b B + n\mu b B = 0, \quad B = \frac{-(m-n)\nu}{(m-2n)\nu + n\mu} \cdot \frac{a}{b} A;$$

$$(m-2n)\nu a B + (m-3n)\nu b C + n\mu b C = 0, \quad C = \frac{-(m-2n)\nu}{(m-3n)\nu + n\mu} \cdot \frac{a}{b} B;$$

$$(m-3n)\nu a C + (m-4n)\nu b D + n\mu b D = 0, \quad D = \frac{-(m-3n)\nu}{(m-4n)\nu + n\mu} \cdot \frac{a}{b} C;$$

ubi iterum lex progressionis harum litterarum est manifesta.

Corollarium 1.

160. Prior series ideo est memorabilis, quod casibus, quibus $(m+in)\nu + n\mu = 0$, seu $-\frac{m}{n} - \frac{\mu}{\nu} = i$, abrumpitur, atque

ipsum integrale algebraicum exhibet. Posterior vero abrumpitur, quoties $m - in = 0$ seu $\frac{m}{n} = i$, denotante i numerum integrum positivum.

Corollarium 2.

161. Utraque vero series etiam incommodo quodam laborat, quod non semper in usum vocari potest. Quando enim vel $m = 0$, vel $m + in = 0$, priori uti non licet: quando vero $(m + in)\nu + n\mu = 0$, seu $\frac{m}{n} + \frac{\mu}{\nu} = i$, usus posterioris tollitur, quia termini fierent infiniti.

Corollarium 3.

162. Hoc vero commode usu venit, ut quoties altera applicari nequit, altera certo in usum vocari possit, iis tantum casibus exceptis, quibus et $-\frac{m}{n}$ et $\frac{\mu}{\nu} + \frac{m}{n}$ sunt numeri integri positivi. Quia autem tum est $\nu = 1$, hi casus sunt rationales integri, nihilque difficultatis habent.

Corollarium 4.

163. Possunt etiam ambae series simul pro z conjungi hoc modo: Sit prior series $= P$, posterior vero $= Q$, ut capi possit tam $z = P$, quam $z = Q$. Binis autem conjungendis, erit $z = \alpha P + \beta Q$, dummodo sit $\alpha + \beta = 1$.

Scholion.

164. Inde autem, quod duas series pro z exhibemus, minime sequitur, has duas series inter se esse aequales, neque enim necesse est, ut valores ipsius y inde orti fiant aequales, dummodo quantitate constante a se invicem differant. Ita si prior series inventa per P, posterior per Q indicetur, quia ex illa fit $y = (a + bx^n)^{\frac{\mu}{\nu}} P$, ex hac vero $y = (a + bx^n)^{\frac{\mu}{\nu}} Q$, certo erit $(a + bx^n)^{\frac{\mu}{\nu}} (P - Q)$ quantitas constans, ideoque $P - Q = C(a + bx^n)^{-\frac{\mu}{\nu}}$. Utraque

scilicet series tantum integrale particulare praebet, quoniam nullam constantem involvit, quae non jam in formula differentiali contineatur. Interim tamen eadem methodo etiam valor completus pro z erui potest: praeter seriem enim assumtam P vel Q statui potest

$$z = P + \alpha + \beta x^n + \gamma x^{2n} + \delta x^{3n} + \varepsilon x^{4n} + \text{etc.}$$

ac substitutione facta, series P ut ante definitur, pro altera vero nova serie efficiendum est, ut sit

$$\left.\begin{matrix} n\nu a\beta x^{n-1} + 2n\nu a\gamma x^{2n-1} + 3n\nu a\delta x^{3n-1} + 4n\nu a\varepsilon x^{4n-1} \\ + n\mu b\alpha \quad + n\nu b\beta \quad + 2n\nu b\gamma \quad + 3n\nu b\delta \\ \quad + n\mu b\beta \quad + n\mu b\gamma \quad + n\mu b\delta \end{matrix}\right\} = 0,$$

unde ducuntur hae determinationes:

$$\beta = \frac{-\mu b}{\nu a}.\alpha;\ \gamma = \frac{-(\mu+\nu)b}{2\nu a}.\beta;\ \delta = \frac{-(\mu+2\nu)b}{3\nu a}.\gamma;$$
$$\varepsilon = \frac{-(\mu+3\nu)b}{4\nu a}.\delta \text{ etc.}$$

ita ut prodeat

$$z = P + \alpha\left(1 - \frac{\mu}{\nu}.\frac{b}{a}x^n + \frac{\mu(\mu+\nu)}{\nu.2\nu}.\frac{b^2}{a^2}x^{2n} - \frac{\mu(\mu+\nu)(\mu+2\nu)}{\nu.2\nu.3\nu}.\frac{b^3}{a^3}x^{3n} + \text{etc.}\right)$$

seu $z = P + \alpha\left(1 + \frac{b}{a}x^n\right)^{-\frac{\mu}{\nu}}$, hincque

$$y = P(a + bx^n)^{\frac{\mu}{\nu}} + \alpha a^{\frac{\mu}{\nu}};$$

quod est integrale completum quia constans α mansit arbitraria

Exemplum 1.

165. *Formulam* $\partial y = \frac{\partial x}{\sqrt{(1-xx)}}$ *hoc modo per seriem integrare.*

Comparatione cum forma generali instituta, sit $a = 1$, $b = -1$, $m = 1$, $n = 2$, $\mu = 1$, $\nu = 2$: unde posito $y = z\sqrt{(1-xx)}$ prima solutio

$$z = Ax + Bx^3 + Cx^5 + Dx^7 + \text{etc. praebet}$$
$$A = 1,\ B = \tfrac{2}{3}A;\ C = \tfrac{4}{5}B;\ D = \tfrac{6}{7}C;\ E = \tfrac{8}{9}D;\ \text{etc.}$$

unde colligimus:

$$y = \left(x + \tfrac{2}{3}x^3 + \tfrac{2.4}{3.5}x^5 + \tfrac{2.4.6}{3.5.7}x^7 + \text{etc.}\right)\sqrt{(1 - xx)},$$

quod integrale evanescit posito $x = 0$, est ergo $y = \text{Arc. sin. } x$. Altera methodus hic frustra tentatur, ob $\frac{m}{n} + \frac{\mu}{\nu} = 1$.

Corollarium 1.

166. Posito $x = 1$, videtur hinc fieri $y = 0$, ob $\sqrt{(1 - xx)} = 0$; at perpendendum est, fieri hoc casu seriei infinitae summam infinitam, ita ut nihil obstet, quo minus fit $y = \frac{\pi}{2}$. Si ponamus $x = \frac{1}{2}$, fit $y = 30° = \frac{\pi}{6}$, ideoque

$$\frac{\pi}{6} = \left(1 + \frac{2}{3.4} + \frac{2.4}{3.5.4^2} + \frac{2.4.6}{3.5.7.4^3} + \text{etc.}\right)\frac{\sqrt{3}}{4}.$$

Corollarium 2.

167. Simili modo proposita formula $\partial y = \frac{\partial x}{\sqrt{(1 + xx)}}$, reperitur:

$$y = \left(x - \tfrac{2}{3}x^3 + \tfrac{2.4}{3.5}x^5 - \tfrac{2.4.6}{3.5.7}x^7 + \text{etc.}\right)\sqrt{(1 + xx)}$$

estque $y = l[x + \sqrt{(1 + xx)}]$.

Exemplum 2.

168. *Formulam* $\partial y = \frac{\partial x}{x\sqrt{(1 - xx)}}$ *hoc modo per seriem integrare.*

Est ergo $m = 0$, $n = 2$, $\mu = 1$, $\nu = 2$, $a = 1$, et $b = -1$, utendum igitur est altera serie sumendo

$$z = \frac{y}{\sqrt{(1 - xx)}} = Ax^{-2} + Bx^{-4} + Cx^{+6} + Dx^{-8} + \text{etc.}$$

sitque

$$A = 1;\ B = \tfrac{2}{3}A;\ C = \tfrac{4}{5}B;\ D = \tfrac{6}{7}C;\ \text{etc.}$$

Hinc ergo colligimus:

$$y = \left(\frac{1}{xx} + \frac{2}{3x^4} + \frac{2.4}{3.5x^6} + \frac{2.4.6}{3.5.7x^8} + \text{etc.}\right)\sqrt{(1 - xx)}.$$

At integratio praebet $y = l\,\frac{1-\surd(1-xx)}{x}$, qui valores conveniunt, quia uterque evanescit posito $x = 1$.

Corollarium 1.

169. Cum autem haec series non convergat nisi capiatur $x > 1$; hoc autem casu formula $\surd(1-xx)$ fiat imaginaria, haec series nullius est usus.

Corollarium 2.

170. Si proponatur $\partial y = \frac{\partial x}{x\surd(xx-1)}$, eadem pro y series emergit per $\surd -1$ multiplicata, eritque

$$y = -\left(\frac{1}{xx} + \frac{2}{3x^4} + \frac{2.4}{3.5x^6} + \frac{2.4.6}{3.5.7x^8} + \text{etc.}\right)\surd(xx-1).$$

Posito autem $x = \frac{1}{u}$, erit $\partial y = \frac{-\partial u}{\surd(1-uu)}$, et $y = C - \text{Arc. sin. } u$, seu $y = C - \text{Arc. sin. } \frac{1}{x}$: ubi sumi oportet $C = 0$, quia series illa evanescit posito $x = \infty$: ita ut sit $y = -\text{Arc. sin. } \frac{1}{x}$, quae cum superiori convenit statuendo $\frac{1}{x} = v$.

Exemplum 3.

171. *Formulam* $\partial y = \frac{\partial x}{\surd(a + bx^4)}$ *hoc modo per seriem integrare.*

Est hic $m=1$, $n=4$, $\mu=1$, $\nu=2$, ideoque posito $y = z\surd(a+bx^4)$, prior resolutio dat

$$z = Ax + Bx^5 + Cx^9 + Dx^{13} + \text{etc.}$$

existente

$$A = \frac{1}{a};\ B = \frac{-3b}{5a}A;\ C = \frac{-7b}{9a}B;\ D = \frac{-11b}{13a}C;\ \text{etc.}$$

ita ut sit

$$y = \left(\frac{x}{a} - \frac{3bx^5}{5aa} + \frac{3.7b^2x^9}{5.9a^3} - \frac{3.7.11b^3x^{13}}{5.9.13a^4} + \text{etc.}\right)\surd(a + bx^4).$$

Hic autem quoque altera resolutio locum habet, ponendo

$$z = Ax^{-3} + Bx^{-7} + Cx^{-11} + Dx^{-15} + \text{etc.}$$

existente

$$A = \frac{-1}{b};\ B = \frac{-3a}{5b}A;\ C = \frac{-7a}{9b}B;\ D = \frac{-11a}{13b}C;\ \text{etc.}$$

unde colligitur:

$$y = -\left(\frac{1}{bx^3} - \frac{3a}{5b^2x^7} + \frac{3.7aa}{5.9b^3x^{11}} - \frac{3.7.11a^3}{5.9.13b^4x^{15}} + \text{etc.}\right)\sqrt{(a+bx^4)}$$

quarum serierum illa evanescit posito $x = 0$, haec vero posito $x = \infty$.

Corollarium 1.

172. Differentia ergo harum duarum serierum est constans, scilicet:

$$\left\{\begin{matrix} +\dfrac{x}{a} - \dfrac{3bx^5}{5aa} + \dfrac{3.7b^2x^9}{5.9a^3} - \dfrac{3.7.11b^3x^{13}}{5.9.13a^4} + \text{etc.} \\ +\dfrac{1}{bx^3} - \dfrac{3a}{5bbx^7} + \dfrac{3.7a^2}{5.9b^3x^{11}} - \dfrac{3.7.11a^3}{5.9.13b^4x^{15}} + \text{etc.}\end{matrix}\right\}\sqrt{(a+bx^4)} = \text{Const.}$$

Corollarium 2.

173. Has ergo binas series colligendo habebimus

$$\frac{a+bx^4}{abx^3} - \frac{3}{5}\cdot\frac{a^3+b^3x^{12}}{a^2b^2x^7} + \frac{3.7}{5.9}\cdot\frac{a^5+b^5x^{20}}{a^3b^3x^{11}} - \text{etc.} = \frac{C}{\sqrt{(a+bx^4)}}$$

ubi quicunque valor ipsi x tribuatur, pro C semper eadem quantitas obtinetur.

Corollarium 3.

174. Ita si $a = 1$ et $b = 1$, erit haec series in $\sqrt{(1+x^4)}$ ducta semper constans, scilicet

$$\left(\frac{1+x^4}{x^3} - \frac{3}{5}\cdot\frac{1+x^{12}}{x^7} + \frac{3.7}{5.9}\cdot\frac{1+x^{20}}{x^{11}} - \text{etc.}\right)\sqrt{(1+x^4)} = \text{C}.$$

Cum igitur posito $x = 1$, fiat

$$\text{C} = \left(1 - \tfrac{3}{5} + \tfrac{3.7}{5.9} - \tfrac{3.7.11}{5.9.13} + \text{etc.}\right) 2\sqrt{2},$$

huicque valori etiam illa series, quicunque valor ipsi x tribuatur, est aequalis.

Corollarium 4.

175. Haec postrema series signis alternantibus procedens, per differentias facile in aliam iisdem signis praeditam transformatur, unde eadem constans concluditur

$$\text{C} = \left(1 + \tfrac{1}{5} + \tfrac{1.3}{5.9} + \tfrac{1.3.5}{5.9.13} + \tfrac{1.3.5.7}{5.9.13.17} + \text{etc.}\right)\sqrt{2},$$

quae series satis cito convergit, eritque proxime $\text{C} = \frac{13}{7}$.

Scholion.

176. Ista methodus in hoc consistit, ut series quaedam indefinita fingatur, ejusque determinatio ex natura rei derivetur. Ejus usus autem potissimum cernitur in aequationibus differentialibus resolvendis; verum etiam in praesenti instituto saepe utiliter adhibetur. Ejusdem quoque methodi ope quantitates transcendentes reciprocae, veluti exponentiales et sinus cosinusve angulorum, per series exprimuntur, quae etsi jam aliunde sint cognitae, tamen earum investigationem per integrationem exposuisse juvabit, cum simili modo alia praeclara erui queant.

Problema 14.

177. Quantitatem exponentialem $y = a^x$ in seriem convertere.

Solutio.

Sumtis logarithmis, habemus $ly = xla$, et differentiando $\frac{\partial y}{y} = \partial x la$, seu $\frac{\partial y}{\partial x} = yla$: unde valorem ipsius y per seriem quaeri oportet. Cum autem integrale completum latius pateat, notetur nostro casu posito $x = 0$, fieri debere $y = 1$: quare fingatur haec pro y series:

$$y = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{etc.}$$

unde fit

$$\frac{\partial y}{\partial x} = A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.}$$

quibus substitutis in aequatione $\frac{\partial y}{\partial x} - yla = 0$, erit

$$\left.\begin{array}{l} A + 2Bx + 3Cx^2 + 4Dx^3 + 5Ex^4 + \text{etc.} \\ -la - Ala - Bla - Cla - Dla - \text{etc.} \end{array}\right\} = 0,$$

hincque coëfficientes ita determinantur:

$$A = la;\ B = \tfrac{1}{2}Ala;\ C = \tfrac{1}{3}Bla;\ D = \tfrac{1}{4}Cla \text{ etc.}$$

sicque consequimur:

$$y = a^x = 1 + \frac{xla}{1} + \frac{x^2(la)^2}{1.2} + \frac{x^3(la)^3}{1.2\,3} + \frac{x^4(la)^4}{1.2.3.4} + \text{etc.}$$

quae est ipsa series notissima in Introductione data.

Scholion.

178. Pro sinibus et cosinibus angulorum ad differentialia secundi gradus est descendendum, ex quibus deinceps series integrale referens elici debet. Cum autem gemina integratio duplicem determinationem requirat, series ita est fingenda, ut duabus conditionibus ex natura rei petitis satisfaciat. Verum haec methodus etiam ad alias investigationes extenditur, quae adeo in quantitatibus algebraicis versantur, a cujusmodi exemplo hic inchoëmus.

Problema 15.

179. Hanc expressionem $y = [x + \sqrt{(1 + xx)}]^n$ in seriem, secundum potestates ipsius x progredientem, convertere.

Solutio.

Quia est $ly = nl[x + \sqrt{(1 + xx)}]$ erit $\frac{\partial y}{y} = \frac{n\partial x}{\sqrt{(1+xx)}}$; jam ad signum radicale tollendum sumantur quadrata, erit

$(1+xx)\partial y^2 = nnyy\partial x^2$. Aequatio, sumto ∂x constante, denuo differentietur, ut per $2\partial y$ diviso prodeat

$$\partial\partial y(1+xx) + x\partial x\partial y - nny\partial x^2 = 0:$$

unde y per seriem elici debet. Primo autem patet, si sit $x = 0$ fore $y = 1$, ac si x infinite parvum, $y = (1+x)^n = 1 + nx$. Fingatur ergo talis series:

$$y = 1 + nx + Ax^2 + Bx^3 + Cx^4 + Dx^5 + Ex^6 + \text{etc.}$$

ex qua colligitur:

$$\frac{\partial y}{\partial x} = n + 2Ax + 3Bxx + 4Cx^3 + 5Dx^4 + 6Ex^5 + \text{etc. et}$$
$$\frac{\partial\partial y}{\partial x^2} = 2A + 6Bx + 12Cxx + 20Dx^3 + 30Ex^4 + \text{etc.}$$

Facta ergo substitutione adipiscimur:

$$\left.\begin{array}{llllllll} 2A + 6Bx + 12Cxx + 20Dx^3 + 30Ex^4 + 42Fx^5 + \text{etc.} \\ \qquad\qquad + 2A \quad + 6B \quad + 12C \quad + 20D \quad + \text{etc.} \\ + nx + 2A \quad + 3B \quad + 4C \quad + 5D \quad + \text{etc.} \\ - nn - n^3 - An^2 \quad - Bn^2 \quad - Cn^2 \quad - Dn^2 \quad + \text{etc.} \end{array}\right\} = 0$$

hincque derivantur sequentes determinationes

$$A = \frac{nn}{2};\ B = \frac{n(nn-1)}{2.3};\ C = \frac{A(nn-4)}{3.4};\ D = \frac{B(nn-9)}{4.5};\ \text{etc.}$$

ita ut sit

$$y = 1 + nx + \frac{nn}{1.2}x^2 + \frac{n(nn-1)}{1.2.3}x^3 + \frac{nn(nn-4)}{1.2.3.4}x^4 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5$$
$$+ \frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}x^6 + \frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7 + \text{etc.}$$

Corollarium 1.

180. Uti est $y = [x + \sqrt{(1+xx)}]^n$, si statuamus $z = [-x + \sqrt{(1+xx)}]^n$, pro z similis series prodit, in qua x tantum negative capitur, hinc ergo concluditur:

$$\frac{y+z}{2} = 1 + \frac{nn}{1.2}x^2 + \frac{nn(nn-4)}{1.2.3.4}x^4 + \frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}x^6 + \text{etc. et}$$
$$\frac{y-z}{2} = nx + \frac{n(nn-1)}{1.2.3}x^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5$$
$$+ \frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7 + \text{etc.}$$

Corollarium 2.

181. Si ponatur $x = \sqrt{-1} \,.\, \sin.\,\phi$, erit $\sqrt{(1+xx)} = \cos.\,\phi$; hincque

$$y = (\cos.\,\phi + \sqrt{-1} \,.\, \sin.\,\phi)^n = \cos.\,n\phi + \sqrt{-1} \,.\, \sin.\,n\phi, \text{ et}$$

$$z = (\cos.\,\phi - \sqrt{-1} \,.\, \sin.\,\phi)^n = \cos.\,n\phi - \sqrt{-1} \,.\, \sin.\,n\phi:$$

unde deducimus:

$$\cos.\,n\phi = 1 - \frac{nn}{1.2}\sin.\,\phi^2 + \frac{nn(nn-4)}{1.2.3.4}\sin.\,\phi^4 - \frac{nn(nn-4)(nn-16)}{1.2.3.4.5.6}\sin.\,\phi^6 + \text{etc.}$$

$$\sin.\,n\phi = n\sin.\,\phi - \frac{n(nn-1)}{1.2.3}\sin.\,\phi^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}\sin.\,\phi^5 - \frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}\sin.\,\phi^7 + \text{etc.}$$

Corollarium 3.

182. Hae series ad multiplicationem angulorum pertinent, atque hoc habent singulare, quod prior tantum casibus, quibus n est numerus par, posterior vero, quibus est numerus impar, abrumpatur.

Problema 16.

183. Proposito angulo ϕ, tam ejus sinum quam cosinum per seriem infinitam exprimere.

Solutio.

Sit $y = \sin.\,\phi$ et $z = \cos.\,\phi$, erit $\partial y = \partial\phi\sqrt{(1-yy)}$ et $\partial z = -\partial\phi\sqrt{(1-zz)}$. Sumantur quadrata

$$\partial y^2 = \partial\phi^2(1-yy) \text{ et } \partial z^2 = \partial\phi^2(1-zz):$$

differentietur sumto $\partial\phi$ constante, fietque

$$\partial\partial y = -y\partial\phi^2 \text{ et } \partial\partial z = -z\partial\phi^2,$$

sicque y et z ex eadem aequatione definiri oportet. Sed pro $y = \sin.\,\phi$ observandum est, si ϕ evanescat, fieri $y = \phi$; pro $z = \cos.\,\phi$ verum, si ϕ evanescat, fieri $z = 1 - \frac{1}{2}\phi\phi$, seu $z = 1 + 0\phi$. Fingatur ergo

$$y = \Phi + A\Phi^3 + B\Phi^5 + C\Phi^7 + \text{etc.}$$
$$z = 1 + \alpha\Phi^2 + \beta\Phi^4 + \gamma\Phi^6 + \delta\Phi^8 + \text{etc.}$$

fietque substitutione facta:

$$\left.\begin{array}{llll} 2.3\,A\,\Phi & + 4.5\,B\,\Phi^3 & + 6.7\,C\,\Phi^5 & + \text{etc.} \\ + 1 & + A & + B & \end{array}\right\} = 0 \text{ et}$$

$$\left.\begin{array}{llll} 1.2\,\alpha & + 3.4\,\beta\,\Phi^2 & + 5.6\,\gamma\,\Phi^4 & + \text{etc.} \\ + 1 & + \alpha & + \beta & \end{array}\right\} = 0:$$

unde colligimus:

$$A = \frac{-1}{2.3};\; B = \frac{-A}{4.5};\; C = \frac{-B}{6.7};\; D = \frac{-C}{8.9};\; \text{etc.}$$
$$\alpha = \frac{-1}{1.2};\; \beta = \frac{-\alpha}{3.4};\; \gamma = \frac{-\beta}{5.6};\; \delta = \frac{-\gamma}{7.8};\; \text{etc.}$$

unde series jam notissimae obtinentur:

$$\sin.\Phi = \frac{\Phi}{1} - \frac{\Phi^3}{1.2.3} + \frac{\Phi^5}{1.2.3.4.5} - \frac{\Phi^7}{1.2\ldots\ldots7} + \text{etc.}$$
$$\cos.\Phi = 1 - \frac{\Phi^2}{1.2} + \frac{\Phi^4}{1.2.3.4} - \frac{\Phi^6}{1.2\ldots\ldots7} + \text{etc}$$

Scholion.

184. Non opus erat ad differentialia secundi gradus descendere: sed ex formularum $y = \sin.\Phi$ et $z = \cos.\Phi$ differentialibus, quae sunt $\partial y = z\partial\Phi$ et $\partial z = -y\partial\Phi$, eaedem series facile reperiuntur. Fictis enim seriebus ut ante $y = \Phi + A\Phi^3 + B\Phi^5 + C\Phi^7 + \text{etc.}$ et $z = 1 + \alpha\Phi^2 + \beta\Phi^4 + \gamma\Phi^6 + \text{etc.}$ substitutione facta, obtinebitur:

ex priore

$$\left.\begin{array}{lllll} 1 & + 3\,A\,\Phi^2 & + 5\,B\,\Phi^4 & + 7\,C\,\Phi^6 & + \text{etc.} \\ -1 & - \alpha & - \beta & - \gamma & \end{array}\right\} = 0$$

ex posteriore

$$\left.\begin{array}{llll} 2\,\alpha\,\Phi & + 4\,\beta\,\Phi^3 & + 6\,\gamma\,\Phi^5 & + \text{etc.} \\ + 1 & + A & + B & \end{array}\right\} = 0:$$

unde colliguntur hae determinationes:

$$\alpha = -\tfrac{1}{2};\ A = \tfrac{\alpha}{3};\ \beta = \tfrac{-A}{4};\ B = \tfrac{\beta}{5};\ \gamma = \tfrac{-B}{6};\ C = \tfrac{\gamma}{7};$$

ideoque

$$\alpha = -\tfrac{1}{2};\ \beta = +\tfrac{1}{2.3.4};\ \gamma = -\tfrac{1}{2.3.4.5.6};\ \text{etc.}$$

$$A = -\tfrac{1}{2.3};\ B = +\tfrac{1}{2.3.4.5};\ C = -\tfrac{1}{2.3.4.6.6.7};\ \text{etc.}$$

qui valores cum praecedentibus conveniunt. Hinc intelligitur, quomodo saepe duae aequationes simul facilius per series evolvuntur, quam si alteram seorsim tractare velimus.

Problema 17.

185. Per seriem exprimere valorem quantitatis y, qui satisfaciat huic aequationi $\frac{m\,\partial y}{\sqrt{(a+byy)}} = \frac{n\,\partial x}{\sqrt{(f+gxx)}}$.

Solutio.

Integratio hujus aequationis suppeditat:

$$\tfrac{m}{\sqrt{b}}\, l\,[\sqrt{(a+byy)} + y\sqrt{b}] = \tfrac{n}{\sqrt{g}}\, l\,[\sqrt{(f+gxx)} + x\sqrt{g}] + C,$$

unde deducimus:

$$y = \frac{1}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)} + x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}$$

$$- \frac{a}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)} - x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}.$$

constantes h et k ita capiendo, ut sit $hk = f$. Hinc discimus, si x sumatur evanescens, fore

$$y = \frac{1}{2\sqrt{b}}\left(\frac{\sqrt{f} + x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - \frac{a}{2\sqrt{b}}\left(\frac{\sqrt{f} - x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}},\ \text{seu}$$

$$y = \frac{1}{2\sqrt{b}}\left[\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right]$$

$$+ \frac{nx}{2m\sqrt{f}}\left[\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} + a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right]$$

vel posito $y = A + Bx$, erit $B = \frac{n\sqrt{(AAb+a)}}{m\sqrt{f}}$, ita ut constans B definiatur ex constante

$$A = \frac{1}{2\sqrt{b}}\left[\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} - a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right]:$$

et vicissim

$$\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} = A\sqrt{b} + \sqrt{(a + bAA)}, \text{ atque}$$

$$a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}} = -A\sqrt{b} + \sqrt{(a + bAA)}.$$

Nunc ad seriem inveniendam, aequatio proposita, sumtis quadratis

$$mm(f + gxx)\,\partial y^2 = nn(a + byy)\,\partial x^2,$$

denuo differentietur, capto ∂x constante, ut facta divisione per $2\,\partial y$ prodeat:

$$mm\,\partial\partial y\,(f + gxx) + mmgx\,\partial x\,\partial y - nnby\,\partial x^2 = 0.$$

Jam pro y fingatur series:

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

qua substituta habebitur

$$\left.\begin{array}{rrrrr} 2mmfC & +\,6mmfDx & +\,12mmfEx^2 & +\,20mmfFx^3 & +\,\text{etc.} \\ & & +\,2mmgC & +\,6mmgD & +\,\text{etc.} \\ & +\,mmgbB & +\,2mmgC & +\,3mmgD & +\,\text{etc.} \\ -nnbA & -\,nnbB & -\,nnbC & -\,nnbD & -\,\text{etc.} \end{array}\right\} = 0.$$

Cum ergo A et B dentur, reliquae litterae ita determinantur:

$$C = \frac{nnb}{2mmf}A;$$

$$D = \frac{nnb - mmg}{2.3mmf}B; \quad E = \frac{nnb - 4mmg}{3.4mmf}C;$$

$$E = \frac{nnb - 9mmg}{4.5mmf}D; \quad G = \frac{nnb - 16mmg}{5.6mmf}E;$$

$$H = \frac{nnb - 25mmg}{6.7mmf}F; \quad J = \frac{nnb - 36mmg}{7.8mmf}G;$$

sicque series pro y erit cognita.

Exemplum 1.

186. *Functionem transcendentem* $c^{\text{Arc. sin.}\,x}$ *per seriem secundum potestates ipsius* x *progredientem exprimere.*

Ponatur $y = c^{\text{Arc. sin.}\,x}$, erit $ly = lc \cdot \text{Arc. sin.}\,x$, et $\frac{\partial y}{y} = \frac{\partial x\, lc}{\sqrt{(1-xx)}}$: hinc $\partial y^2 (1-xx) = yy\partial x^2 (lc)^2$ et differentiando $\partial\partial y\,(1-xx) - x\partial x\partial y - y\partial x^2 (lc)^2 = 0$. Observetur jam, posito x evanescente, fore $y = c^x = 1 + xlc$: hinc fingatur series $y = 1 + xlc + Ax^2 + Bx^3 + Cx^4 + Dx^5 +$ etc. qua substituta habebitur:

$$\left.\begin{array}{llllll} 1.2\,A & +\,2.3\,B\,x & +\,3.4\,C\,x^2 & +\,4.5\,D\,x^3 & +\,5.6\,E\,x^4 & \\ & & -\,1.2\,A & -\,2.3\,B & -\,3.4\,C & \\ & -\,lc & -\,2\,A & -\,3\,B & -\,4\,C & \text{etc.} \\ -\,(lc)^2 & -\,(lc)^3 & -\,A\,(lc)^2 & -\,B\,(lc)^2 & -\,C\,(lc)^2 & + \end{array}\right\} = 0.$$

Unde reliqui coëfficientes ita definiuntur:

$$A = \frac{(lc)^2}{1.2}; \qquad B = \frac{[1+(lc)^2]\,lc}{2.3};$$

$$C = \frac{4+(lc)^2}{3.4}A; \qquad D = \frac{9+(lc)^2}{4.5}B;$$

$$E = \frac{16+(lc)^2}{5.6}C; \qquad F = \frac{25+(lc)^2}{6.7}D; \text{ etc.}$$

Sit brevitatis gratia $lc = \gamma$, eritque

$$c^{\text{Arc. sin.}\,x} = 1 + \gamma x + \frac{\gamma\gamma}{1.2}x^2 + \frac{\gamma(1+\gamma\gamma)}{1.2.3}x^3 + \frac{\gamma\gamma(4+\gamma\gamma)}{1.2.3.4}x^4$$
$$+ \frac{\gamma(1+\gamma\gamma)(9+\gamma\gamma)}{1.2.3.4.5}x^5 + \frac{\gamma\gamma(4+\gamma\gamma)(16+\gamma\gamma)}{1.2.3.4.5.6}x^6 + \text{etc.}$$

Exemplum 2.

187. *Posito* $x = \sin.\,\Phi$, *invenire seriem secundum potestates ipsius* x *progredientem, quae sinum anguli* $n\Phi$ *exprimat.*

Ponatur $y = \sin.\,n\Phi$, ac notetur evanescente Φ, fieri $x = \Phi$ et $y = n\Phi = nx$, hoc est $y = 0 + nx$, quod est seriei quaesitae initium. Nunc autem est

$\partial\Phi = \frac{\partial x}{\sqrt{(1-xx)}}$, et $n\partial\Phi = \frac{\partial y}{\sqrt{(1-yy)}}$. Ergo

$\frac{\partial y}{\sqrt{(1-yy)}} = \frac{n\partial x}{\sqrt{(1-xx)}}$,

et sumtis quadratis

$$(1-xx)\,\partial y^2 = nn\,\partial x^2\,(1-yy):\ \text{hinc}$$

$$\partial\partial y\,(1-xx) - x\partial x\partial y + nny\partial x^2 = 0.$$

Quare fingatur haec series

$$y = nx + Ax^3 + Bx^5 + Cx^7 + Dx^9 + \text{etc.}$$

qua substituta habebitur:

$$\left.\begin{array}{lllll} 2.3\,A x & + 4.5\,B x^3 & + 6.7\,C x^5 & + 8.9\,D x^7 & \\ & - 2.3\,A & - 4.5\,B & - 6.7\,C & \\ -n & - 3\,A & - 5\,B & - 7\,C & \text{etc.} \\ +n^3 & + nnA & + nnB & + nnC & \end{array}\right\} = 0.$$

Unde hae determinationes colliguntur:

$$A = \frac{-n(nn-1)}{2.3};\ B = \frac{-(nn-9)A}{4.5};\ C = \frac{-(nn-25)B}{6.7};\ \text{etc.}$$

ita ut sit:

$$y = nx - \frac{n(nn-1)}{1.2.3}x^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5 - \frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7 + \text{etc.}$$

sive

$$\sin.n\Phi = n\sin.\Phi - \frac{n(nn-1)}{1.2.3}\sin.\Phi^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}\sin.\Phi^5 - \text{etc.}$$

Scholion.

188. Quia haec series tantum casibus, quibus n est numerus impar, abrumpitur, pro paribus notandum est, seriem commode exprimi posse per productum ex sin. Φ in aliam seriem, secundum cosinus ipsius Φ potestates progredientem. Ad quam inveniendam ponamus cos. $\Phi = u$, fitque sin. $n\Phi = z$ sin. $\Phi = z\sqrt{(1-uu)}$; unde ob $\partial\Phi = -\frac{\partial u}{\sqrt{(1-uu)}}$, erit differentiando

$$-\frac{n\partial u \cos.n\Phi}{\sqrt{(1-uu)}} = \partial z\sqrt{(1-uu)} - \frac{zu\partial u}{\sqrt{(1-uu)}},\ \text{seu}$$

$$-n\partial u\cos.n\Phi = \partial z(1-uu) - zu\partial u,$$

quae, sumto ∂u constante, denuo differentiata dat: $-\frac{nn\partial u^2 \sin. n\Phi}{\sqrt{(1-uu)}}$ $= \partial\partial z(1-uu) - 3u\partial u\partial z - z\partial u^2 = -nnz\partial u^2$, ob $\frac{\sin. n\Phi}{\sqrt{(1-uu)}} = z$. Quocirca series quaesita pro $z = \frac{\sin. n\Phi}{\sin. \Phi}$ ex hac aequatione erui debet

$$\partial\partial z(1-uu) - 3u\partial u\partial z - z\partial u^2 + nnz\partial u^2 = 0,$$

ubi notandum est, quia $u = \cos. \Phi$ evanescente u, quo casu sit $\Phi = 90^\circ$, fore vel $z = 0$, si n numerus par, vel $z = 1$, si $n = 4\alpha + 1$; vel $z = -1$, si $n = 4\alpha - 1$. Qui singuli casus seorsim sunt evolvendi: et quo principium cujusque seriei pateat, sit $\Phi = 90^\circ - \omega$, et evanescente ω, fit $u = \cos. \Phi = \omega$; $\sin. \Phi = 1$; $\sin. n\Phi = \sin. (90.n - n\omega) = z$.

Nunc pro casibus singulis:

I. si $n = 4\alpha$; fit $z = -\sin. n\omega = -nu$.
II. si $n = 4\alpha + 1$; fit $z = \cos. n\omega = 1$
III. si $n = 4\alpha + 2$; fit $z = \sin. n\omega = +nu$
IV. si $n = 4\alpha + 3$; fit $z = -\cos. n\omega = -1$

unde series jam satis notae deducuntur.

CAPUT IV.

DE INTEGRATIONE FORMULARUM LOGARITHMICARUM ET EXPONENTIALIUM.

Problema 18.

189. Si X designet functionem algebraicam ipsius x, invenire integrale formulae $X \partial x \, l x$.

Solutio.

Quaeratur integrale $\int X \partial x$, quod sit $= Z$, et cum quantitatis $Z l x$ differentiale sit $= \partial Z l x + \frac{Z \partial x}{x}$, erit $Z l x = \int \partial Z l x + \int \frac{Z \partial x}{x}$: ideoque

$$\int \partial Z \, l x = \int X \partial x \, l x = Z \, l x - \int \frac{Z \partial x}{x}.$$

Sicque integratio formulae propositae reducta est ad integrationem hujus $\frac{Z \partial x}{x}$, quae, si Z fuerit functio algebraica ipsius x non amplius logarithmum involvit, ideoque per praecedentes regulas tractari poterit. Sin autem $\int X \partial x$ algebraice exhiberi nequeat, hinc nihil subsidii nascitur, expedietque indicatione integralis $\int X \partial x \, l x$ acquiescere, ejusque valorem per approximationem investigare.

Nisi forte sit $X = \frac{1}{x}$, quo casu manifesto dat $\int \frac{\partial x}{x} l x = \frac{1}{2} (l x)^2 + C$.

Corollarium 1.

190. Eodem modo, si denotante V functionem quamcunque ipsius x, proposita sit formula $X \partial x \, l V$, erit existente $\int X \partial x = Z$, ejus integrale $= Z \, l V - \int \frac{Z \partial V}{V}$, sicque ad formulam algebraicam reducitur, si modo Z algebraice detur.

Corollarium 2.

191. Pro casu singulari $\frac{\partial x}{x} lx$ notare licet, si posito $lx = u$, fuerit U functio quaecunque algebraica ipsius u, integrationem hujus formulae $\frac{U\partial x}{x}$ non fore difficilem, quia ob $\frac{\partial x}{x} = \partial u$ abit in $U\partial u$, cujus integratio ad praecedentia capita refertur.

Scholion.

192. Haec reductio innititur isti fundamento, quod cum sit $\partial . xy = y\partial x + x\partial y$, hinc vicissim fiat $xy = \int y\partial x + \int x\partial y$, ideoque $\int y\partial x = xy - \int x\partial y$, ita ut hoc modo in genere integratio formulae $y\partial x$ ad integrationem formulae $x\partial y$ reducatur. Quod si ergo, proposita quacunque formula $V\partial x$, functio V in duos factores, puta $V = PQ$, resolvi queat, ita ut integrale $\int P\partial x = S$ assignari queat, ob $P\partial x = \partial S$, erit $V\partial x = PQ\partial x = Q\partial S$, hincque $\int V\partial x = QS - \int S\partial Q$. Hujusmodi reductio insignem usum affert, cum formula $\int S\partial Q$ simplicior fuerit quam proposita $\int V\partial x$, eaque insuper simili modo ad simpliciorem reduci queat. Interdum etiam commode evenit, ut hac methodo tandem ad formulam propositae similem perveniatur, quo casu integratio pariter obtinetur. Veluti si ulteriori reductione inveniremus $\int S\partial Q = T + n\int V\partial x$, foret utique $\int V\partial x = QS - T - n\int V\partial x$, hincque $\int V\partial x = \frac{QS - T}{n+1}$. Tum igitur talis reductio insignem praestat usum, cum vel ad formulam simpliciorem, vel ad eandem perducit. Atque ex hoc principio praecipuos casus, quibus formula $X\partial x lx$ vel integrationem admittit, vel per seriem commode exhiberi potest, evolvamus.

Exemplum 1.

193. *Formulae differentialis $x^n\partial x\, lx$ integrale invenire denotante n numerum quemcunque.*

Cum sit $\int x^n \partial x = \frac{1}{n+1} x^{n+1}$, erit

$$\int x^n \partial x\, lx = \frac{1}{n+1} x^{n+1} lx - \int \frac{1}{n+1} x^{n+1} \partial . lx$$

$$=\frac{1}{n+1}x^{n+1}\,lx-\frac{1}{n+1}\int x^n\partial x=\frac{1}{n+1}x^{n+1}\,lx-\frac{1}{(n+1)^2}x^{n+1};$$

ideoque

$$\int x^n\partial x\,lx=\frac{1}{n+1}x^{n+1}\left(lx-\frac{1}{n+1}\right).$$

Sicque haec formula absolute est integrabilis.

Corollarium 1.

194. Casus simpliciores, quibus n est numerus integer sive positivus sive negativus, tenuisse juvabit:

$$\int\partial x\,lx=x\,lx-x;\qquad \int\frac{\partial x}{xx}\,lx=-\frac{1}{x}\,lx-\frac{1}{x};$$

$$\int x\,\partial x\,lx=\tfrac{1}{2}xx\,lx-\tfrac{1}{4}xx;\quad \int\frac{\partial x}{x^3}\,lx=-\frac{1}{2xx}\,lx-\frac{1}{4xx};$$

$$\int x^2\,\partial x\,lx=\tfrac{1}{3}x^3\,lx-\tfrac{1}{9}x^3;\quad \int\frac{\partial x}{x^4}\,lx=-\frac{1}{3x^3}\,lx-\frac{1}{9x^3};$$

$$\int x^3\,\partial x\,lx=\tfrac{1}{4}x^4\,lx-\tfrac{1}{16}x^4;\quad \int\frac{\partial x}{x^5}\,lx=-\frac{1}{4x^4}\,lx-\frac{1}{16x^4}.$$

Corollarium 2.

195. Casum $\int\frac{\partial x}{x}\,lx=\frac{1}{2}(lx)^2$, qui est omnino singularis, jam supra annotavimus, sequitur vero etiam ex reductione ad eandem formulam. Namque per superiorem reductionem habemus

$$\int\frac{\partial x}{x}\,lx=lx\,.\,lx-\int lx\,.\,\partial\,lx=(lx)^2-\int\frac{\partial x}{x}\,lx:$$

hincque

$$2\int\frac{\partial x}{x}\,lx=(lx)^2,\ \text{consequenter}\ \int\frac{\partial x}{x}\,lx=\tfrac{1}{2}(lx)^2.$$

Exemplum 2.

196. *Formulae* $\frac{\partial x}{1-x}\,lx$ *integrale per seriem exprimere.*

Reductione ante adhibita parum lucramur, prodit enim:

$$\int\frac{\partial x}{1-x}\,lx=l\frac{1}{1-x}\,.\,lx-\int\frac{\partial x}{x}\,l\frac{1}{1-x}.$$

Cum autem sit

$$l\frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \text{etc. erit}$$

$$\int\frac{\partial x}{x}\, l\frac{1}{1-x} = x + \tfrac{1}{4}x^2 + \tfrac{1}{9}x^3 + \tfrac{1}{16}x^4 + \tfrac{1}{25}x^5 + \text{etc.}$$

ideoque

$$\int\frac{\partial x}{1-x}\, lx = l\frac{1}{1-x}\,.\,lx - x - \tfrac{1}{4}x^2 - \tfrac{1}{9}x^3 - \tfrac{1}{16}x^4 - \tfrac{1}{25}x^5 - \text{etc.}$$

quod integrale evanescit casu $x = 0$, etsi enim lx tum in infinitum abit, tamen $l\frac{1}{1-x} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \text{etc.}$ ita evanescit, ut etiam si per lx multiplicetur, in nihilum abeat, est enim in genere $x^n\, lx = 0$ posito $x = 0$, dum n numerus positivus.

Corollarium 1.

197. Si ponamus $1 - x = u$, fit

$$\frac{\partial x}{1-x}\, lx = -\frac{\partial u}{u}\, l(1-u) = \frac{\partial u}{u}\, l\frac{1}{1-u},$$

ideoque

$$\int\frac{\partial x}{1-x}\, lx = C + u + \tfrac{1}{4}u^2 + \tfrac{1}{9}u^3 + \tfrac{1}{16}u^4 + \tfrac{1}{25}u^5 + \text{etc.}$$

quae, ut etiam casu $x = 0$ seu $u = 1$, evanescat, capi debet

$$C = -1 - \tfrac{1}{4} - \tfrac{1}{9} - \tfrac{1}{16} - \tfrac{1}{25} - \text{etc.} = -\tfrac{1}{6}\pi\pi.$$

Corollarium 2.

198. Sumto ergo $1 - x = u$ seu $x + u = 1$, aequales erunt inter se hae expressiones:

$$-lx\,.\,lu - x - \tfrac{1}{4}x^2 - \tfrac{1}{9}x^3 - \tfrac{1}{16}x^4 - \text{etc.}$$

$$= -\tfrac{1}{6}\pi^2 + u + \tfrac{1}{4}u^2 + \tfrac{1}{9}u^3 + \text{etc.}$$

seu erit

$$\tfrac{1}{6}\pi^2 - lx\,.\,lu = x + u + \tfrac{1}{4}(x^2+u^2) + \tfrac{1}{9}(x^3+u^3)$$
$$+ \tfrac{1}{16}(x^4+u^4) + \text{etc.}$$

Corollarium 3.

199. Haec series maxime convergit, ponendo $x = u = \frac{1}{2}$: hoc ergo casu habebimus

$$\frac{1}{6}\pi\pi - (l2)^2 = 1 + \frac{1}{2.4} + \frac{1}{4.9} + \frac{1}{8.16} + \frac{1}{16.25} + \frac{1}{32.36} + \text{etc.}$$

Hujus ergo seriei

$$x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 + \frac{1}{25}x^5 + \text{etc.}$$

summa habetur non solum casu $x = 1$, quo est $= \frac{\pi\pi}{6}$, sed etiam casu $x = \frac{1}{2}$, quo est $= \frac{1}{12}\pi^2 - \frac{1}{2}(l\,2)^2$.

Corollarium 4.

200. Si ponamus $x = \frac{1}{3}$, et $u = \frac{2}{3}$, erit hujus seriei

$$1 + \frac{5}{3^2.4} + \frac{9}{3^3.9} + \frac{17}{3^4.16} + \frac{33}{3^5.25} + \frac{65}{3^6.36} + \text{etc.}$$

cujus terminus generalis $= \frac{1 + 2^n}{3^n\,nn}$, summa $= \frac{1}{6}\pi^2 - l\,3.\,l\frac{3}{2}$: neque vero hinc seriei $x + \frac{1}{4}x^2 + \frac{1}{9}x^3 + \frac{1}{16}x^4 + \text{etc.}$ binos casus $x = \frac{1}{3}$ et $x = \frac{2}{3}$ seorsim summare licet.

Exemplum 3.

201. *Formulae $\frac{\partial x}{(1-x)^2}\,l\,x$ integrale invenire, idemque in seriem convertere.*

Cum sit $\int \frac{\partial x}{(1-x)^2} = \frac{1}{1-x}$, erit

$$\int \frac{\partial x}{(1-x)^2}\,l\,x = \frac{1}{1-x}\,l\,x - \int \frac{\partial x}{x(1-x)}: \text{ at ob}$$

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}, \text{ fit } \int \frac{\partial x}{x(1-x)} = l\,x + l\frac{1}{1-x},$$

unde colligimus integrale

$$\int \frac{\partial x}{(1-x)^2}\,lx = \frac{lx}{1-x} - lx - l\frac{1}{1-x} = \frac{x\,lx}{1-x} - l\frac{1}{1-x},$$

ita sumtum, ut evanescat posito $x = 0$.

Jam pro serie commodissime invenienda, statuatur $1 - x = u$, et nostra formula fit

$$= \frac{-\partial u}{uu}\,l(1-u) = \frac{\partial u}{uu}\,l\frac{1}{1-u} = \frac{\partial u}{uu}(u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + \frac{1}{4}u^4 + \frac{1}{5}u^5 + \text{etc.})$$

Quocirca integrando nanciscimur:

$$\int \frac{\partial x}{(1-x)^2}\, lx = C + lu + \frac{u}{1.2} + \frac{uu}{2.3} + \frac{u^3}{3.4} + \frac{u^4}{4.5} + \text{etc.}$$

quae expressio ut etiam evanescat, facto $x = 0$ seu $u = 1$, oportet sit:

$$C = -\frac{1}{1.2} - \frac{1}{2.3} - \frac{1}{3.4} - \frac{1}{4.5} - \text{etc.} = -1.$$

Quare ob $x = 1 - u$, obtinebimus:

$$\frac{u}{1.2} + \frac{u^2}{2.3} + \frac{u^3}{3.4} + \frac{u^4}{4.5} + \text{etc.} = 1 - lu + \frac{(1-u)\,l(1-u)}{u} + lu$$

$$= 1 + \frac{(1-u)\,l(1-u)}{u}.$$

Corollarium 1.

202. Simili modo si $\partial y = \frac{\partial u}{u\sqrt{u}}\, l\frac{1}{1-u}$, erit

$$y = -\frac{2}{\sqrt{u}}\, l\frac{1}{1-u} + \int \frac{2\,\partial u}{(1-u)\sqrt{u}};$$

at posito $u = xx$, fit

$$\int \frac{2\,\partial u}{(1-u)\sqrt{u}} = 4\int \frac{\partial x}{1-xx} = 2\, l\frac{1+x}{1-x}. \quad \text{Ergo}$$

$$y = 2\, l\frac{1+\sqrt{u}}{1-\sqrt{u}} - \frac{2}{\sqrt{u}}\, l\frac{1}{1-u}.$$

At quia per seriem

$$\partial y = \frac{\partial u}{u\sqrt{u}}\left(u + \tfrac{1}{2}uu + \tfrac{1}{3}u^3 + \tfrac{1}{4}u^4 + \text{etc.}\right)$$

erit etiam

$$y = 2\sqrt{u} + \frac{2}{2.3}\, u\sqrt{u} + \frac{2}{3.5}\, u^2\sqrt{u} + \frac{2}{4.7}\, u^3\sqrt{u} + \text{etc.}$$

Corollarium 2.

203. Si ergo multiplicemus per $\frac{\sqrt{u}}{2}$, adipiscimur:

$$u + \frac{uu}{2.3} + \frac{u^3}{3.5} + \frac{u^4}{4.7} + \frac{u^5}{5.9} + \text{etc.} = \sqrt{u}\cdot l\frac{1+\sqrt{u}}{1-\sqrt{u}} + l(1-u);$$

quae summa est etiam

$$= (1+\sqrt{u})\, l(1+\sqrt{u}) + (1-\sqrt{u})\, l(1-\sqrt{u}).$$

Quare sumto $u = 1$, ob $(1-\sqrt{u})\, l(1-\sqrt{u}) = 0$, erit

$$1 + \frac{1}{2.3} + \frac{1}{3.5} + \frac{1}{4.7} + \frac{1}{5.9} + \frac{1}{6.11} + \text{etc.} = 2\,l2.$$

Problema 19.

204. Si P denotet functionem ipsius x, invenire integrale huius formulae $\partial y = \partial P (lx)^n$.

Solutio.

Per reductionem supra monstratam fit

$$y = P (lx)^n - \int P \partial \cdot (lx)^n = P (lx)^n - n \int \frac{P \partial x}{x} (lx)^{n-1}.$$

Hinc si sit $\int \frac{P \partial x}{x} = Q$, erit simili modo

$$\int \frac{P \partial x}{x} (lx)^{n-1} = Q (lx)^{n-1} - (n-1) \int \frac{Q \partial x}{x} (lx)^{n-2}.$$

Quo modo si ulterius progredimur, haecque integralia capere liceat

$$\int \frac{P \partial x}{x} = Q; \int \frac{Q \partial x}{x} = R; \int \frac{R \partial x}{x} = S; \int \frac{S \partial x}{x} = T; \text{ etc.}$$

obtinebimus integrale quaesitum:

$$\int \partial P (lx)^n = P (lx)^n - nQ (lx)^{n-1} + n (n-1) R (lx)^{n-2} - n (n-1)(n-2) S (lx)^{n-3} + \text{etc.}$$

ac si exponens n fuerit numerus integer positivus, integrale forma finita exprimetur.

Exemplum 1.

205. *Formulae $x^m \partial x (lx)^2$ integrale assignare.*

Hic est $n = 2$, et $P = \frac{x^{m+1}}{m+1}$; hinc $Q = \frac{x^{m+1}}{(m+1)^2}$, et $R = \frac{x^{m+1}}{(m+1)^3}$: unde colligimus

$$\int x^m \partial x (lx)^2 = x^{m+1} \left(\frac{(lx)^2}{m+1} - \frac{2 lx}{(m+1)^2} + \frac{2 \cdot 1}{(m+1)^3} \right),$$

quod integrale evanescit posito $x = 0$, dum sit $m + 1 > 0$.

Corollarium 1.

206. Hinc posito $x = 1$, fit $\int x^m \partial x (lx)^2 = \frac{2 \cdot 1}{(m+1)^3}$. Ex praecedentibus autem patet, si formula $\int x^m \partial x lx$ ita integretur, ut evanescat posito $x = 0$, tum facto $x = 1$, fieri $\int x^m \partial x lx = \frac{-1}{(m+1)^2}$.

Corollarium 2.

207. At si sit $m = -1$, ut habeatur $\frac{\partial x}{x}(lx)^2$, erit ejus integrale $\int \frac{\partial x}{x}(lx)^2 = \frac{1}{3}(lx)^3$, qui solus casus ex formula generali est excipiendus.

Exemplum 2.

208. *Formulae $x^{m-1}\partial x\,(lx)^3$ integrale assignare.*

Hic est $n = 3$ et $P = \frac{x^m}{m}$, hinc $Q = \frac{x^m}{m^2}$; $R = \frac{x^m}{m^3}$ et $S = \frac{x^m}{m^4}$: unde integrale quaesitum fit

$$\int x^{m-1}\partial x\,(lx)^3 = x^m\left(\frac{(lx)^3}{m} - \frac{3(lx)^2}{m^2} + \frac{3.2lx}{m^3} - \frac{3.2.1}{m^4}\right);$$

quod integrale evanescit, posito $x = 0$, dum sit $m > 0$.

Corollarium 1.

209. Quod si integrali ita sumto, ut evanescat posito $x = 0$, tum ponatur $x = 1$, erit:

$$\int x^{m-1}\partial x = \frac{1}{m};\ \int x^{m-1}\partial x\,lx = -\frac{1}{m^2};\ \int x^{m-1}\partial x\,(lx)^2 = +\frac{1.2}{m^3};\ \text{et}$$

$$\int x^{m-1}\partial x\,(lx)^3 = -\frac{1.2.3}{m^4}.$$

Corollarium 2.

210. Casu autem $m = 0$, erit integrale

$$\int \frac{\partial x}{x}(lx)^3 = \frac{1}{4}(lx)^4,$$

quod ita determinari nequit, ut evanescat posito $x = 0$; oporteret enim constantem infinitam adjici. Hoc autem integrale evanescit posito $x = 1$.

Exemplum 3.

211. *Formulae $x^{m-1}\partial x(lx)^n$ integrale assignare.*

Cum hic sit $P = \frac{x^m}{m}$; erit $Q = \frac{x^m}{m^2}$; $R = \frac{x^m}{m^3}$; $S = \frac{x^m}{m^4}$; etc. Hinc integrale quaesitum prodit

$$\int x^{m-1}\partial x\,(lx)^n = x^m\left(\frac{(lx)^n}{m} - \frac{n\,(lx)^{n-1}}{m^2} + \frac{n\,(n-1)\,(lx)^{n-2}}{m^3} - \frac{n\,(n-1)\,(n-2)\,(lx)^{n-3}}{m^4} + \text{etc.}\right).$$

Casu autem $m = 0$, est $\int\frac{\partial x}{x}(lx)^n = \frac{1}{n+1}(lx)^{n+1}$.

Corollarium 1.

212. Si $m > 0$ integrale assignatum evanescit, posito $x = 0$: deinceps ergo si sumatur $x = 1$, erit integrale

$$\int x^{m-1}\partial x\,(lx)^n = \pm\frac{1.\,2.\,3\,.\,.\,.\,.\,.\,.\,n}{m^{n+1}},$$

ubi signum $+$ valet, si n sit numerus par, inferius vero $-$ si n impar.

Corollarium 2.

213. Haec ergo ambiguitas tollitur, si loco lx scribatur $-l\frac{1}{x}$; tum enim integratione eodem modo instituta, positoque $x = 1$, fiet

$$\int x^{m-1}\partial x\,(l\tfrac{1}{x})^n = +\frac{1.\,2.\,3\,.\,.\,.\,.\,.\,.\,n}{m^{n+1}}.$$

Scholion.

214. Si exponens n sit numerus fractus, integrale inventum per seriem infinitam exprimitur, veluti si sit $n = -\frac{1}{2}$, reperitur

$$\int \frac{x^{m-1}\partial x}{\sqrt{lx}} = x^m \left(\frac{1}{m\sqrt{lx}} + \frac{1}{2m^2(lx)^{\frac{3}{2}}} + \frac{1.\,3}{4m^3(lx)^{\frac{5}{2}}} + \frac{1.\,3.\,5}{8m^4(lx)^{\frac{7}{2}}} + \text{etc.} \right),$$

quae etiam, quatenus initio x ab 0 ad 1 crescere sumitur, hoc modo repraesentari potest:

$$\int \frac{x^{m-1}\partial x}{\sqrt{l\frac{1}{x}}} = \frac{x^m}{\sqrt{l\frac{1}{x}}} \left(\frac{1}{m} + \frac{1}{2m^2 lx} + \frac{1.\,3}{4m^3(lx)^2} + \frac{1.\,3.\,5}{8m^4(lx)^3} + \text{etc.} \right).$$

Si exponens n sit negativus, etsi integer, tamen integrale inventum in infinitum progreditur: verum hoc casu alia ratione integrationem instituere licet, qua tandem reducitur ad hujusmodi formulam $\int \frac{T\partial x}{lx}$, cujus integratio nullo modo simplicior reddi potest. Hanc ergo reductionem sequenti problemate doceamus.

Problema 20.

215. Integrationem hujus formulae $\partial y = \frac{X\partial x}{(lx)^n}$ continuo ad formulas simpliciores reducere.

Solutio.

Formula proposita ita repraesentetur $\partial y = Xx \cdot \frac{\partial x}{x(lx)^n}$, et cum sit $\int \frac{\partial x}{x(lx)^n} = \frac{-1}{(n-1)(lx)^{n-1}}$, erit

$$y = \frac{-Xx}{(n-1)(lx)^{n-1}} + \frac{1}{n-1}\int \frac{1}{(lx)^{n-1}} \cdot \partial(Xx).$$

Quare si ponamus continuo

$$\partial . (\mathrm{X}x) = \mathrm{P}\partial x;\ \partial . (\mathrm{P}x) = \mathrm{Q}\partial x;\ \partial . (\mathrm{Q}x) = \mathrm{R}\partial x \text{ etc.}$$

erit hanc reductionem continuando:

$$y = \frac{-\mathrm{X}x}{(n-1)(lx)^{n-1}} - \frac{\mathrm{P}x}{(n-1)(n-2)(lx)^{n-2}}$$

$$- \frac{\mathrm{Q}x}{(n-1)(n-2)(n-3)(lx)^{n-3}} - \text{etc.}$$

donec tandem perveniatur ad hanc integralem

$$+ \frac{1}{(n-1)(n-2)\ldots 1}\int\frac{\mathrm{V}\partial x}{lx},$$

ita ut quoties n fuerit numerus integer positivus, integratio tandem ad hujusmodi formulam perducatur.

Exemplum 1.

216. *Formulae differentialis* $\partial y = \frac{x^{m-1}\partial x}{(lx)^2}$ *integrale investigare.*

Hic est $n = 2$ et $\mathrm{X} = x^{m-1}$, unde fit $\mathrm{P} = mx^{m-1}$, hincque integrale

$$y = \int\frac{x^{m-1}\partial x}{(lx)^2} = -\frac{x^m}{lx} + \frac{m}{1}\int\frac{x^{m-1}\partial x}{lx}.$$

At formulae $\frac{x^{m-1}\partial x}{lx}$ integrale exhiberi nequit, nisi casu $m = 0$, quo fit $\int\frac{\partial x}{x\,lx} = llx$. Verum si $m = 0$, formulae propositae integratio ne hinc quidem pendet: fit enim absolute $y = \int\frac{\partial x}{x(lx)^2} = -\frac{1}{lx} + \mathrm{C}$.

Exemplum 2.

217. *Formulae differentialis* $\partial y = \frac{x^{m-1}\partial x}{(lx)^n}$ *integrale investigare, casibus, quibus* n *est numerus integer positivus.*

Cum sit $X^{\circ} = x^{m-1}$, erit $P = \frac{\partial.(Xx)}{\partial x} = m x^{m-1}$, tum vero $Q = \frac{\partial.(Px)}{\partial x} = m^2 x^{m-1}$; $R = m^3 x^{m-1}$; $S = m^4 x^{m-1}$; etc. Quare integrale hinc ita formabitur, ut sit

$$y = \int \frac{x^{m-1} \partial x}{(lx)^n} = \frac{-x^m}{(n-1)(lx)^{n-1}} - \frac{m x^m}{(n-1)(n-2)(lx)^{n-2}}$$
$$- \frac{m^2 x^m}{(n-1)(n-2)(n-3)(lx)^{n-3}} - \text{etc.}$$
$$\ldots + \frac{m^{n-1}}{(n-1)(n-2)\ldots\ldots 1} \int \frac{x^{m-1} \partial x}{lx}.$$

Corollarium.

218. Pro n ergo successive numeros 1, 2, 3, 4, etc. substituendo, habebimus istas reductiones:

$$\int \frac{x^{m-1} \partial x}{(lx)^2} = \frac{-x^m}{lx} + \frac{m}{1} \int \frac{x^{m-1} \partial x}{lx}$$
$$\int \frac{x^{m-1} \partial x}{(lx)^3} = \frac{-x^m}{2(lx)^2} - \frac{m x^m}{2.1\, lx} + \frac{m^2}{2.1} \int \frac{x^{m-1} \partial x}{lx}$$
$$\int \frac{x^{m-1} \partial x}{(lx)^4} = \frac{-x^m}{3(lx)^3} - \frac{m x^m}{3.2(lx)^2} - \frac{m^2 x^m}{3.2.1\, lx} + \frac{m^3}{3.2.1} \int \frac{x^{m-1} \partial x}{lx}.$$

Scholion.

219. Hae ergo integrationes pendent a formula $\int \frac{x^{m-1} \partial x}{lx}$, quae posito $x^m = z$, ob $x^{m-1} \partial x = \frac{1}{m} \partial z$ et $lx = \frac{1}{m} lz$, reducitur ad hanc simplicissimam formam $\int \frac{\partial z}{lz}$, cujus integrale si assignari posset, amplissimum usum in Analysi esset allaturum, verum nullis adhuc artificiis, neque per logarithmos, neque angulos, exhiberi potuit: quomodo autem per seriem exprimi possit, infra ostendemus (§. 227). Videtur ergo haec formula $\int \frac{\partial z}{lz}$ singularem speciem func-

tionum transcendentium suppeditare, quae utique accuratiorem evolutionem meretur. Eadem autem quantitas transcendens in integrationibus formularum exponentialium frequenter occurrit, quas in hoc capite tractare instituimus, propterea quod cum logarithmicis tam arcte cohaerent, ut alterum genus facile in alterum converti possit: veluti ipsa formula modo considerata $\frac{\partial z}{lz}$, posito $lz = x$, ut sit $z = e^x$, et $\partial z = e^x \partial x$, transformatur in hanc exponentialem $e^x . \frac{\partial x}{x}$, cujus ergo integratio aeque est abscondita. Formulas igitur tractabiles evolvamus et ejusmodi quidem, quae non obvia substitutione ad formam algebraicam reduci possunt. Veluti si V fuerit functio quaecunque ipsius v, sitque $v = a^x$, formula $V \partial x$, ob $x = \frac{lv}{la}$ et $\partial x = \frac{\partial v}{v\,la}$, abit in $\frac{V \partial v}{v\,la}$, qua ratione variabilis v est algebraica. Hujusmodi ergo formulas $\frac{a^x \partial x}{\sqrt{(1 + a^{nx})}}$, quippe quae posito $a^x = v$, nihil habent difficultatis, hinc excludimus.

Problema 21.

220. Formulae differentialis $a^x X \partial x$, denotante X functionem quamcunque ipsius x, integrale investigare.

Solutio 1.

Cum sit $\partial . a^x = a^x \partial x la$, erit vicissim $\int a^x \partial x = \frac{1}{la} a^x$: quare si formula proposita in hos factores resolvatur, $X . a^x \partial x$, habebitur per reductionem:

$$\int a^x X \partial x = \frac{1}{la} a^x X - \frac{1}{la} \int a^x \partial X.$$

Quodsi ulterius ponamus $\partial X = P \partial x$, ut sit

$$\int a^x P \partial x = \frac{1}{la} a^x P - \frac{1}{la} \int a^x \partial P,$$

prodibit haec reductio

$$\int a^x X \partial x = \frac{1}{la} a^x X - \frac{1}{(la)^2} a^x P + \frac{1}{(la)^2} \int a^x \partial P.$$

Si porro ponamus $\partial P = Q \partial x$, habebitur haec reductio

$$\int a^x X \partial x = \frac{1}{l\,a} a^x X - \frac{1}{(la)^2} a^x P + \frac{1}{(la)^3} a^x Q - \frac{1}{(la)^3} \int a^x \partial Q;$$

sicque ulterius ponendo $\partial Q = R \partial x$, $\partial R = S \partial x$, etc. progredi licet, donec ad formulam vel integrabilem, vel in suo genere simplicissimam perveniatur.

Solutio. 2.

Alio modo resolutio formulae in factores institui potest; ponatur $\int X \partial x = P$ seu $X \partial x = \partial P$, et formula ita relata $a^x . \partial P$, habebitur

$$\int a^x X \partial x = a^x P - la \int a^x P \partial x;$$

simili modo si ponamus $\int P \partial x = Q$, obtinebimus

$$\int a^x X \partial x = a^x P - la . a^x Q + (la)^2 \int a^x Q \partial x.$$

Ponamus porro $\int Q \partial x = R$, et consequimur

$$\int a^x X \partial x = a^x P - la . a^x Q + (la)^2 . a^x R - (la)^3 \int a^x R \partial x,$$

hocque modo quousque lubuerit progredi licet, donec ad formulam vel integrabilem vel in suo genere simplicissimam perveniamus.

Corollarium 1.

221. Priori solutione semper uti licet, quia functiones P, Q, R, etc. per differentiationem functionis X eliciuntur, dum est

$$P = \frac{\partial X}{\partial x}; \quad Q = \frac{\partial P}{\partial x}; \quad R = \frac{\partial Q}{\partial x}; \text{ etc.}$$

Quare si X fuerit functio rationalis integra, tandem ad formulam pervenietur $\int a^x \partial x = \frac{1}{la} . a^x$; ideoque his casibus integrale absolute exhiberi potest.

Corollarium 2.

222. Altera solutio locum non invenit, nisi formulae $X \partial x$ integrale P assignari queat; neque etiam eam continuare licet, nisi

quatenus sequentes integrationes $\int P\partial x = Q$, $\int Q\partial x = R$, etc. succedunt.

Exemplum 1.

223. *Formulae $a^x x^n \partial x$ integrale definire, denotante* n *numerum integrum positivum.*

Cum sit $X = x^n$, solutione prima utentes habebimus

$$\int a^x x^n \partial x = \frac{1}{la} \cdot a^x x^n - \frac{n}{la} \int a^x x^{n-1} \partial x;$$

hinc ponendo pro n successive numeros 0, 1, 2, 3, etc., quia primo casu integratio constat, sequentia integralia eruemus:

$$\int a^x \partial x = \frac{1}{la} \cdot a^x$$

$$\int a^x x \partial x = \frac{1}{la} \cdot a^x x - \frac{1}{(la)^2} a^x$$

$$\int a^x x^2 \partial x = \frac{1}{la} \cdot a^x x^2 - \frac{2}{(la)^2} a^x x + \frac{2.1}{(la)^3} a^x$$

$$\int a^x x^3 \partial x = \frac{1}{la} \cdot a^x x^3 - \frac{3}{(la)^2} a^x x^2 + \frac{3.2}{(la)^3} a^x x - \frac{3.2.1}{(la)^4} a^x$$

etc.

unde in genere pro quovis exponente n concludimus

$$\int a^x x^n \partial x = a^x \left(\frac{x^n}{la} - \frac{n x^{n-1}}{(la)^2} + \frac{n(n-1) x^{n-2}}{(la)^3} - \frac{n(n-1)(n-2) x^{n-3}}{(la)^4} + \text{etc.} \right).$$

ad quam expressionem insuper constantem arbitrariam adjici oportet, ut integrale completum obtineatur.

Corollarium.

224. Si integrale ita determinari debeat, ut evanescat posito $x = 0$, erit

$$\int a^x \partial x = \frac{1}{la}\cdot a^x - \frac{1}{la}$$

$$\int a^x x \partial x = a^x\left(\frac{x}{la} - \frac{1}{(la)^2}\right) + \frac{1}{(la)^2}$$

$$\int a^x x^2 \partial x = a^x\left(\frac{x^2}{la} - \frac{2x}{(la)^2} + \frac{2.1}{(la)^3}\right) - \frac{2.1}{(la)^3}$$

$$\int a^x x^3 \partial x = a^x\left(\frac{x^3}{la} - \frac{3x^2}{(la)^2} + \frac{3.2x}{(la)^3} - \frac{3.2.1}{(la)^4}\right) + \frac{3.2.1}{(la)^4}$$

etc.

Exemplum 2.

225. *Formulae* $\frac{a^x \partial x}{x^n}$ *integrale investigare, si quidem* n *denotet numerum integrum positivum.*

Hic commode altera solutione utemur, ubi cum sit $X = \frac{1}{x^n}$, erit $P = \frac{-1}{(n-1)x^{n-1}}$; hincque resultat ista reductio

$$\int \frac{a^x \partial x}{x^n} = \frac{-a^x}{(n-1)x^{n-1}} + \frac{la}{n-1}\int \frac{a^x \partial x}{x^{n-1}}.$$

Perspicuum igitur est, posito $n = 1$ hinc nihil concludi posse; qui est ipse casus supra memoratus $\int \frac{a^x \partial x}{x}$, singularem speciem transcendentium functionum complectens, qua admissa integralia sequentium casuum exhibere poterimus:

$$\int \frac{a^x \partial x}{x^2} = C - \frac{a^x}{1x} + \frac{la}{1}\int \frac{a^x \partial x}{x}$$

$$\int \frac{a^x \partial x}{x^3} = C - \frac{a^x}{2x^2} - \frac{a^x la}{2.1x} + \frac{(la)^2}{2.1}\int \frac{a^x \partial x}{x}$$

$$\int \frac{a^x \partial x}{x^4} = C - \frac{a^x}{3x^3} - \frac{a^x la}{3.2x^2} - \frac{a^x (la)^2}{3.2.1x} + \frac{(la)^3}{3.2.1}\int \frac{a^x \partial x}{x}$$

unde in genere colligimus

$$\int \frac{a^x \partial x}{x^n} = C - \frac{a^x}{(n-1)x^{n-1}} - \frac{a^x\, la}{(n-1)(n-2)x^{n-2}}$$
$$- \frac{a^x (la)^2}{(n-1)(n-2)(n-3)x^{n-3}} - \ldots - \frac{a^x (la)^{n-2}}{(n-1)(n-2)\ldots 1.x}$$
$$+ \frac{(la)^{n-1}}{(n-1)(n-2)\ldots\ldots 1}\int \frac{a^x \partial x}{x}.$$

Corollarium 1.

226. Admissa ergo quantitate transcendente $\int \frac{a^x \partial x}{x}$, hanc formulam $a^x x^m \partial x$ integrare poterimus, sive exponens m fuerit numerus integer positivus, sive negativus. Illis quidem casibus integratio ab ista nova quantitate transcendente non pendet.

Corollarium 2.

227. At si m fuerit fractus numerus, neutra solutio negotium conficit, sed utraque seriem infinitam pro integrali exhibet. Veluti si sit $m = -\frac{1}{2}$, habebimus ex priore

$$\int \frac{a^x \partial x}{\sqrt{x}} = a^x \left(\frac{1}{la} + \frac{1}{2x(la)^2} + \frac{1.3}{4x^2(la)^3} + \frac{1.3.5}{8x^3(la)^4} + \text{etc.} \right) : \sqrt{x} + C,$$

ex posteriore autem;

$$\int \frac{a^x \partial x}{\sqrt{x}} = C + \frac{a^x}{\sqrt{x}} \left(\frac{2x}{1} - \frac{4x^2\, la}{1.3} + \frac{8x^3 (la)^2}{1.3.5} - \frac{16x^4 (la)^3}{1.3.5.7} + \text{etc.} \right).$$

Scholion 1.

228. Hinc quantitas transcendens $\int \frac{a^x \partial x}{x}$ per seriem exprimi potest secundum potestates ipsius x progredientem. Cum enim sit

$$a^x = 1 + xla + \frac{x^2(la)^2}{1.2} + \frac{x^3(la)^3}{1.2.3} + \text{etc. erit}$$

$$\int \frac{a^x \partial x}{x} = C + lx + \frac{xla}{1} + \frac{x^2(la)^2}{1.2.2} + \frac{x^3(la)^3}{1.2.3.3}$$
$$+ \frac{x^4(la)^4}{1.2.3.4.4} + \text{etc.}$$

Ac si pro a sumamus numerum, cujus logarithmus hyperbolicus est unitas, quem numerum littera e indicemus, habebimus

$$\int \frac{e^x \partial x}{x} = C + lx + \frac{x}{1} + \frac{1}{2} \cdot \frac{x^2}{1.2} + \frac{1}{3} \cdot \frac{x^3}{1.2.3} + \frac{1}{4} \cdot \frac{x^4}{1.2.3.4} + \text{etc.}$$

Atque hinc etiam ponendo $e^x = z$, ut sit $x = lz$, formulam supra memoratam $\frac{\partial z}{lz}$ per seriem integrare poterimus, eritque

$$\int \frac{\partial z}{lz} = C + llz + \frac{lz}{1} + \frac{1}{2} \cdot \frac{(lz)^2}{1.2} + \frac{1}{3} \cdot \frac{(lz)^3}{1.2.3} + \frac{1}{4} \cdot \frac{(lz)^4}{1.2.3.4} + \text{etc.}$$

quod integrale si debeat evanescere, sumto $z = 0$, constans C fit infinita, unde pro reliquis casibus nihil concludi potest. Idem incommodum locum habet, si evanescens reddamus casu $z = 1$, quia $llz = l0$ fit infinitum. Caeterum patet, si integrale sit reale, pro valoribus ipsius z unitate minoribus, ubi lz est negativus, tum pro valoribus unitate majoribus fieri imaginarium, et vicissim. Hinc ergo natura hujus functionis transcendentis parum cognoscitur.

Scholion 2.

229. Quando vel integratio non succedit, vel series ante inventae minus idoneae videntur, hinc quantitatem a^x in seriem resolvendo, statim sine aliis subsidiis formulae $a^x X \partial x$ integrale per seriem exhiberi potest, erit enim

$$\int a^x X \partial x = \int X \partial x + \frac{la}{1} \int X x \partial x + \frac{(la)^2}{1.2} \int X x^2 \partial x$$
$$+ \frac{(la)^3}{1.2.3} \int X x^3 \partial x + \text{etc.}$$

Ita si sit $X = x^n$, habebitur

$$\int a^x x^n \partial x = C + \frac{x^{n+1}}{n+1} + \frac{x^{n+2}\, la}{1\,(n+2)} + \frac{x^{n+3}\,(la)^2}{1.2\,(n+3)} + \frac{x^{n+4}\,(la)^3}{1.2.3\,(n+4)} + \text{etc.}$$

ubi notandum, si n fuerit numerus integer negativus, puta $n = -i$, loco $\frac{x^{n+i}}{n+i}$ scribi debere lx.

Exemplum 3.

230. *Formulae $\frac{a^x \partial x}{1-x}$ integrale per seriem infinitam exprimere.*

Per priorem solutionem obtinemus, ob

$$X = \frac{1}{1-x};\quad P = \frac{\partial X}{\partial x} = \frac{1}{(1-x)^2};\quad Q = \frac{\partial P}{\partial x} = \frac{1.2}{(1-x)^3};\quad R = \frac{\partial Q}{\partial x} = \frac{1.2.3}{(1-x)^4}\ \text{etc.}$$

hincque sequentem seriem:

$$\int \frac{a^x \partial x}{1-x} = a^x\left(\frac{1}{(1-x)\,la} - \frac{1}{(1-x)^2(la)^2} + \frac{1.2}{(1-x)^3(la)^3} - \frac{1.2.3}{(1-x)^4(la)^4} + \text{etc.}\right)$$

Aliae series reperiuntur, si vel a^x, vel fractio $\frac{1}{1-x}$ in seriem evolvatur. Commodissima autem videtur, quae seriem fingendo eruitur: brevitatis gratia pro a sumamus numerum e, ut $le = 1$, ac statuatur $\partial y = \frac{e^x \partial x}{1-x}$ seu

$$\frac{\partial y}{\partial x}(1-x) - 1 - x - \frac{x^2}{1.2} - \frac{x^3}{1.2.3} - \frac{x^4}{1.2.3.4} - \text{etc.} = 0.$$

Jam pro y fingatur haec series

$$y = \int \frac{e^x \partial x}{1-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

eritque facta substitutione

$$\left.\begin{array}{llllll} B & +2Cx & +3Dx^2 & +4Ex^3 & +5Fx^4 & +\text{etc.} \\ & -B & -2C & -3D & -4E & \\ -1 & -1 & -\frac{1}{2} & -\frac{1}{6} & -\frac{1}{24} & \end{array}\right\} = 0:$$

unde eliciuntur istae determinationes:

$$\begin{array}{l|l} B = 1 & E = \frac{1}{4}(1 + 1 + \frac{1}{2} + \frac{1}{6}) \\ C = \frac{1}{2}(1 + 1) & F = \frac{1}{5}(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}) \\ D = \frac{1}{3}(1 + 1 + \frac{1}{2}) & \quad \text{etc.} \end{array}$$

Problema 22.

231. Formulae differentialis $\partial y = x^{nx}\partial x$ integrale investigare, ac per seriem infinitam exprimere.

Solutio.

Commodius hoc praestari nequit, quam ut formula exponentialis x^{nx} in seriem infinitam convertatur, quae est

$$x^{nx} = 1 + nxlx + \frac{n^2x^2(lx)^2}{1.2} + \frac{n^3x^3(lx)^3}{1.2.3} + \frac{n^4x^4(lx)^4}{1.2.3.4} + \text{etc.}$$

qua per ∂x multiplicata, et singulis terminis integratis, erit:

$$\int \partial x = x;$$

$$\int x\partial x lx = x^2\left(\frac{lx}{2} - \frac{1}{2^2}\right);$$

$$\int x^2\partial x (lx)^2 = x^3\left(\frac{(lx)^2}{3} - \frac{2lx}{3^2} + \frac{2.1}{3^3}\right);$$

$$\int x^3\partial x (lx)^3 = x^4\left(\frac{(lx)^3}{4} - \frac{3(lx)^2}{4^2} + \frac{3.2lx}{4^3} - \frac{3.2.1}{4^4}\right);$$

$$\int x^4\partial x (lx)^4 = x^5\left(\frac{(lx)^4}{5} - \frac{4(lx)^3}{5^2} + \frac{4.3(lx)^2}{5^3} - \frac{4.3.2lx}{5^4} + \frac{4.3.2.1}{5^5}\right).$$

etc.

Quare si hae series substituantur, et secundum potestates ipsius lx disponantur, integrale quaesitum exprimetur per has innumerabiles series infinitas:

$$y = \int x^{nx} \partial x = + x\left(1 - \frac{nx}{2^2} + \frac{n^2x^2}{3^3} - \frac{n^3x^3}{4^4} + \frac{n^4x^4}{5^5} - \text{etc.}\right)$$
$$+ \frac{nx^2 lx}{1}\left(\frac{1}{2^1} - \frac{nx}{3^2} + \frac{n^2x^2}{4^3} - \frac{n^3x^3}{5^4} + \frac{n^4x^4}{6^5} - \text{etc.}\right)$$
$$+ \frac{n^2x^3(lx)^2}{1.2}\left(\frac{1}{3^1} - \frac{nx}{4^2} + \frac{n^2x^2}{5^3} - \frac{n^3x^3}{6^4} + \frac{n^4x^4}{7^5} - \text{etc.}\right)$$
$$+ \frac{n^3x^4(lx)^3}{1.2.3}\left(\frac{1}{4^1} - \frac{nx}{5^2} + \frac{n^2x^2}{6^3} - \frac{n^3x^3}{7^4} + \frac{n^4x^4}{8^5} - \text{etc.}\right)$$
$$\text{etc.}$$

quod integrale ita est sumtum, ut evanescat, posito $x = 0$.

Corollarium.

232. Hac ergo lege instituta integratione, si ponatur $x = 1$, valor integralis $\int x^{nx} \partial x$ huic seriei aequatur

$$1 - \frac{n}{2^2} + \frac{n^2}{3^3} - \frac{n^3}{4^4} + \frac{n^4}{5^5} - \frac{n^5}{6^6} + \text{etc.}$$

quae ob concinnitatem terminorum omnino est notatu digna.

Scholion.

233. Eodem modo reperitur integrale hujus formulae:

$$y = \int x^{nx} x^m \partial x = \int x^m \partial x\left(1 + nxlx + \frac{n^2x^2(lx)^2}{1.2} + \frac{n^3x^3(lx)^3}{1.2.3} + \text{etc.}\right)$$

erit enim singulis terminis integrandis:

$$\int x^m \partial x = \frac{x^{m+1}}{m+1};$$

$$\int x^{m+1} \partial x\,(lx) = x^{m+2}\left(\frac{lx}{m+2} - \frac{1}{(m+2)^2}\right);$$

$$\int x^{m+2} \partial x\,(lx)^2 = x^{m+3}\left(\frac{(lx)^2}{m+3} - \frac{2\,lx}{(m+3)^2} + \frac{2.1}{(m+3)^3}\right);$$

$$\int x^{m+3} \partial x\,(lx)^3 = x^{m+4}\left(\frac{(lx)^3}{m+4} - \frac{3\,(lx)^2}{(m+4)^2} + \frac{3.2lx}{(m+4)^3} - \frac{3.2.1}{(m+4)^4}\right);$$

$$\text{etc.}$$

Quod si ergo integrale ita determinetur, ut evanescat posito $x = 0$, tum vero statuatur $x = 1$, pro hoc casu valor formulae integralis $\int x^{nx} x^m \partial x$ exprimetur hac serie satis memorabili:

$$\frac{1}{m+1} - \frac{n}{(m+2)^2} + \frac{nn}{(m+3)^3} - \frac{n^3}{(m+4)^4} + \frac{n^4}{(m+5)^5} - \text{etc.}$$

quae uti manifestum est, locum habere nequit, quoties m est numerus integer negativus.

Alia exempla formularum exponentialium non adjungo, quia plerumque integralia nimis inconcinne exprimuntur, methodus autem eas tractandi hic sufficienter est exposita. Interim tamen singularem attentionem merentur formulae integrationem absolute admittentes, quae in hac forma continentur $e^x(\partial P + P \partial x)$ cujus integrale manifesto est $e^x P$. Hujusmodi autem casibus difficile est regulas tradere integrale inveniendi, et conjecturae plerumque plurimum est tribuendum. Veluti si proponeretur haec formula $\frac{e^x x \partial x}{(1+x)^2}$, facile est suspicari integrale, si datur, talem formam esse habiturum $\frac{e^x z}{1+x}$. Hujus ergo differentiale $\frac{e^x [\partial z(1+x) + xz\partial x]}{(1+x)^2}$ cum illo comparatum dat $\partial z(1+x) + xz\partial x = x\partial x$, ubi statim patet esse $z = 1$, quod nisi per se pateret, ex regulis difficulter cognosceretur. Quare transeo ad alterum genus formularum transcendentium jam in Analysin receptarum, quae vel angulos vel sinus, tangentesve angulorum complectuntur.

CAPUT V.

DE INTEGRATIONE FORMULARUM ANGULOS SINUSVE ANGULORUM IMPLICANTIUM.

Problema 23.

234.

Proposita formula differentiali $X\partial x$ Ang. sin. x, ejus integrale investigare.

Solutio.

Cum sit ∂ . Ang. sin. $x = \frac{\partial x}{\sqrt{(1-xx)}}$, formula proposita ita in factores discerpatur: Ang. sin. $x \times X\partial x$. Si jam $X\partial x$ integrationem patiatur, sitque $\int X\partial x = P$, erit nostrum integrale $\int X\partial x$ Ang. sin. $x =$ P Ang. sin. $x - \int \frac{P\partial x}{\sqrt{(1-xx)}}$; itaque opus reductum est ad integrationem formulae algebraicae, pro qua supra praecepta sunt tradita.

Caeterum si fuerit $X = \frac{1}{\sqrt{(1-xx)}}$, manifestum est integrale fore $\int \frac{\partial x}{\sqrt{(-xx)}}$ Ang. sin. $x = \frac{1}{2}$ (Ang. sin. $x)^2$; quo solo casu quadratum anguli in integrale ingreditur.

Exemplum 1.

235. *Hanc formulam* $\partial y = x^n \partial x$ Ang. sin. x *integrare.*

Cum sit $P = \int x^n \partial x = \frac{x^{n+1}}{n+1}$ habebimus

$$y = \frac{x^{n+1}}{n+1} \text{ Ang. sin. } x - \frac{1}{n+1} \int \frac{x^{n+1}\partial x}{\sqrt{(1-xx)}}.$$

Hinc pro variis valoribus ipsius n erunt integralia ope §. 120. eruta, ut sequentur:

$$\int \partial x \text{ Ang. sin. } x = x \text{ Ang. sin. } x + \sqrt{(1-xx)} - 1;$$

$$\int x \partial x \text{ Ang. sin. } x = \tfrac{1}{2} xx \text{ Ang. sin. } x + \tfrac{1}{4} x \sqrt{(1-xx)} - \tfrac{1}{4} \text{ Ang. sin. } x;$$

$$\int x^2 \partial x \text{ Ang. sin. } x = \tfrac{1}{3} x^3 \text{ Ang. sin. } x + \tfrac{1}{3}(\tfrac{1}{3} x^2 + \tfrac{2}{3}) \sqrt{(1-xx)} - \tfrac{1}{3} \cdot \tfrac{2}{3};$$

$$\int x^3 \partial x \text{ Ang. sin. } x = \tfrac{1}{4} x^4 \text{ Ang. sin. } x + \tfrac{1}{4}(\tfrac{1}{4} x^3 + \tfrac{1.3}{2.4} x) \sqrt{(1-xx)} - \tfrac{1}{4} \cdot \tfrac{1.3}{2.4} \text{ Ang. sin. } x;$$

quae ita sunt sumta, ut evanescant posito $x = 0$.

Exemplum 2.

236. *Hanc formulam* $\partial y = \frac{x \partial x}{\sqrt{(1-xx)}}$ Ang. sin. x *integrare.*

Cum sit $\int \frac{x \partial x}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)} = \text{P}$, erit integrale quaesitum $y = \text{C} - \sqrt{(1-xx)} \text{ Ang. sin. } x + \int \frac{\partial x \sqrt{(1-xx)}}{\sqrt{(1-xx)}}$, sicque habebitur:

$$y = \int \frac{x \partial x}{\sqrt{(1-xx)}} \text{ Ang. sin. } x = \text{C} - \sqrt{(1-xx)} \text{ Ang. sin } x + x.$$

Exemplum 3.

237. *Hanc formulam* $\partial y = \frac{\partial x}{(1-xx)^{\frac{3}{2}}}$ Ang. sin. x *integrare.*

Hic est $\text{P} = \int \frac{\partial x}{(1-xx)^{\frac{3}{2}}} = \frac{x}{\sqrt{(1-xx)}}$; unde fit

$$y = \frac{x}{\sqrt{(1-xx)}} \text{ Ang. sin. } x - \int \frac{x \partial x}{1-xx}, \text{ seu}$$

$$y = \int \frac{\partial x}{(1-xx)^{\frac{3}{2}}} \text{ Ang. sin. } x = \frac{x}{\sqrt{(1-xx)}} \text{ Ang. sin. } x + l\sqrt{(1-xx)},$$

quod integrale evanescit posito $x = 0$.

Scholion.

238. Simili modo integratur formula $\partial y = X\partial x$ Ang. cos. x. Cum enim sit ∂ . Ang. cos. $x = \frac{-\partial x}{\sqrt{(1-xx)}}$, si ponamus $\int X\partial x = P$, erit $y = P$ Ang. cos $x + \int \frac{P\partial x}{\sqrt{(1-xx)}}$. Quin etiam si proponatur formula $\partial y = X\partial x$ Ang. tang. x, quia est ∂. Ang. tang. $x = \frac{\partial x}{1+xx}$, posito $\int X\partial x = P$, erit hoc integrale:

$$y = \int X\partial x \text{ Ang. tang. } x = P \text{ Ang. tang. } x - \int \frac{P\partial x}{1+xx}.$$

Quoties ergo $\int X\partial x$ algebraice dari potest, toties integratio reducitur ad formulam algebraicam, sicque negotium confectum est habendum. Cum igitur in his formulis angulus, cujus sinus, cosinus, vel tangens erat $= x$, inesset, consideremus etiam ejusmodi formulas, in quas quadratum hujus anguli, altiorve potestas ingreditur.

Problema 24.

239. Denotet Φ angulum, cujus sinus tangensve est functio quaedam ipsius x, unde fiat $\partial\Phi = u\partial x$, propositaque sit haec formula $\partial y = X\partial x \,.\, \Phi^n$ quam integrare oporteat.

Solutio.

Sit $\int X\partial x = P$, ut habeamus $\partial y = \Phi^n \partial P$, eritque integrando $y = \Phi^n P - n\int \Phi^{n-1} Pu\partial x$. Jam simili modo sit $\int Pu\partial x = Q$, erit

$$\int \Phi^{n-1} Pu\partial x = \Phi^{n-1} Q - (n-1)\int \Phi^{n-2} Qu\partial x,$$

tum posito $\int Qu\partial x = R$, erit

$$\int \Phi^{n-2} Qu\partial x = \Phi^{n-2} R - (n-2)\int \Phi^{n-3} Ru\partial x.$$

Hocque modo potestas anguli Φ continuo deprimitur, donec tandem ad formulam ab angulo Φ liberam perveniatur: id quod semper eveniet, dummodo n sit numerus integer positivus, et haec integralia continuo sumere liceat $\int X\partial x = P$, $\int Pu\partial x = Q$, $\int Qu\partial x = R$, etc. quae integrationes, si non succedant, frustra integratio suscipitur.

Exemplum.

240. *Sit Φ angulus cujus sinus $= x$, ut sit $\partial\Phi = \frac{\partial x}{\sqrt{(1-xx)}}$, integrare formulam $\partial y = \Phi^n \partial x$.*

Erit ergo $X = 1$; $P = x$;

$$Q = \int \frac{P\partial x}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)};\quad R = \int \frac{Q\partial x}{\sqrt{(1-xx)}} = -x$$

$$S = \int \frac{R\partial x}{\sqrt{(1-xx)}} = \sqrt{(1-xx)};\quad T = x \text{ etc.}$$

quibus valoribus inventis reperietur:

$$y = \int \Phi^n \partial x = \Phi^n x + n\Phi^{n-1}\sqrt{(1-xx)} - n(n-1)\Phi^{n-2}x$$
$$- n(n-1)(n-2)\Phi^{n-3}\sqrt{(1-xx)} + \text{etc.}$$

Pro variis ergo valoribus exponentis n habebimus:

$$\int \Phi\partial x = \Phi x + \sqrt{(1-xx)} - 1;$$
$$\int \Phi^2\partial x = \Phi^2 x + 2\Phi\sqrt{(1-xx)} - 2.1x;$$
$$\int \Phi^3\partial x = \Phi^3 x + 3\Phi^2\sqrt{(1-xx)} - 3.2\Phi x - 3.2.1\sqrt{(1-xx)} + 6;$$
etc.

integralibus ita determinatis, ut evanescant posito $x = 0$.

Scholion.

241. Si sit $X\partial x = u\partial x = \partial\Phi$, formulae $\Phi^n\partial\Phi$ integrale est $\frac{1}{n+1}\Phi^{n+1}$; similique modo, si fuerit Φ functio quaecunque anguli Φ, formulae $\Phi u\partial x = \Phi\partial\Phi$ integratio nihil habet difficultatis. Multo latius patent formulae sinus, cosinusve angulorum et tangentes implicantes, quarum integratio per inversam Analysin amplissimum habet usum; cum praecipue Theoria Astronomiae ad hujusmodi formulas sit reducta. Prima autem fundamenta peti debent ex calculo differentiali, unde cum sit:

$$\partial.\sin.n\Phi = n\partial\Phi\cos.n\Phi;\quad \partial.\cos.n\Phi = -n\partial\Phi\sin.n\Phi;$$
$$\partial.\text{tang}.n\Phi = \frac{n\partial\Phi}{\cos.n\Phi^2};\quad \partial.\cot.n\Phi = \frac{-n\partial\Phi}{\sin.n\Phi^2};$$
$$\partial.\frac{1}{\sin.n\Phi} = \frac{-n\partial\Phi\cos.n\Phi}{\sin.n\Phi^2};\quad \partial.\frac{1}{\cos.n\Phi} = \frac{n\partial\Phi\sin.n\Phi}{\cos.n\Phi^2};$$

nanciscimur has integrationes elementares:

$$\int \partial \Phi \cos. n\Phi = \tfrac{1}{n} \sin. n\Phi; \quad \int \partial \Phi \sin. n\Phi = -\tfrac{1}{n} \cos. n\Phi;$$

$$\int \frac{\partial \Phi}{\cos. n\Phi^2} = \tfrac{1}{n} \text{tang.}\, n\Phi; \quad \int \frac{\partial \Phi}{\sin. n\Phi^2} = -\tfrac{1}{n} \cot. n\Phi;$$

$$\int \frac{\partial \Phi \cos. n\Phi}{\sin. n\Phi^2} = -\frac{1}{n \sin. n\Phi}; \quad \int \frac{\partial \Phi \sin. n\Phi}{\cos. n\Phi^2} = \frac{1}{n \cos. n\Phi};$$

unde statim hujusmodi formularum differentialium integratio

$$\partial \Phi (A + B\cos.\Phi + C\cos. 2\Phi + D\cos. 3\Phi + E\cos. 4\Phi + \text{etc.})$$

consequitur, cum integrale manifesto sit

$$A\Phi + B\sin.\Phi + \tfrac{1}{2} C \sin. 2\Phi + \tfrac{1}{3} D \sin. 3\Phi + \tfrac{1}{4} E \sin. 4\Phi + \text{etc.}$$

Deinde etiam in subsidium vocari convenit, quae in elementis de angulorum compositione traduntur: scilicet

$$\sin.\alpha.\sin.\beta = \tfrac{1}{2}\cos.(\alpha - \beta) - \tfrac{1}{2}\cos.(\alpha + \beta);$$

$$\cos.\alpha.\cos.\beta = \tfrac{1}{2}\cos.(\alpha - \beta) + \tfrac{1}{2}\cos.(\alpha + \beta);$$

$$\sin.\alpha.\cos.\beta = \tfrac{1}{2}\sin.(\alpha + \beta) + \tfrac{1}{2}\sin.(\alpha - \beta) = \tfrac{1}{2}\sin.(\alpha + \beta) - \tfrac{1}{2}\sin.(\beta - \alpha);$$

unde producta plurium sinuum et cosinuum in simplices sinus cosinusve resolvuntur.

Problema 25.

242. Formulae $\partial \Phi \sin. \Phi^n$ integrale investigare.

Solutio.

Repraesentetur in hos factores resoluta $\sin. \Phi^{n-1}. \partial \Phi \sin. \Phi$; et quia $\int \partial \Phi \sin. \Phi = -\cos. \Phi$, erit

$$\int \partial \Phi \sin. \Phi^n = -\sin. \Phi^{n-1} \cos. \Phi + (n-1) \int \partial \Phi \sin. \Phi^{n-2} \cos. \Phi^2.$$

Hinc ob $\cos. \Phi^2 = 1 - \sin. \Phi^2$, habebitur

$$\int \partial \Phi \sin. \Phi^n = -\sin. \Phi^{n-1} \cos. \Phi + (n-1) \int \partial \Phi \sin. \Phi^{n-2} - (n-1) \int \partial \Phi \sin. \Phi^n:$$

ubi cum postrema formula ipsi propositae sit similis, hinc colligitur ista reductio

$$\int \partial \phi \sin. \phi^n = -\tfrac{1}{n} \sin. \phi^{n-1} \cos. \phi + \tfrac{n-1}{n} \int \partial \phi \sin. \phi^{n-2},$$

qua integratio ad hanc formulam simpliciorem $\partial \phi \sin. \phi^{n-2}$ revocatur. Cum igitur casus simplissimi constent,

$$\int \partial \phi \sin. \phi^0 = \phi \text{ et } \int \partial \phi \sin. \phi = -\cos. \phi,$$

hinc via ad continuo majores exponentes n paratur:

$$\int \partial \phi \sin. \phi^0 = \phi$$
$$\int \partial \phi \sin. \phi = -\cos. \phi$$
$$\int \partial \phi \sin. \phi^2 = -\tfrac{1}{2} \sin. \phi \cos. \phi + \tfrac{1}{2} \phi$$
$$\int \partial \phi \sin. \phi^3 = -\tfrac{1}{3} \sin. \phi^2 \cos. \phi - \tfrac{2}{3} \cos. \phi$$
$$\int \partial \phi \sin. \phi^4 = -\tfrac{1}{4} \sin. \phi^3 \cos. \phi - \tfrac{1.3}{2.4} \sin. \phi \cos. \phi + \tfrac{1.3}{2.4} \phi$$
$$\int \partial \phi \sin. \phi^5 = -\tfrac{1}{5} \sin. \phi^4 \cos. \phi - \tfrac{1.4}{3.5} \sin. \phi^2 \cos. \phi - \tfrac{2.4}{3.5} \cos. \phi$$
$$\int \partial \phi \sin. \phi^6 = -\tfrac{1}{6} \sin. \phi^5 \cos. \phi - \tfrac{1.5}{4.6} \sin \phi^3 \cos. \phi - \tfrac{1.3.5}{2.4.6} \sin. \phi \cos. \phi + \tfrac{1.3.5}{2.4.6} \phi$$

etc.

Corollarium 1.

243. Quoties n est numerus impar, integrale per solum sinum et cosinum exhibetur, at si n est numerus par, integrale insuper ipsum augulum involvit, ideoque est functio transcendens.

Corollarium 2.

244. Casibus ergo quibus n est numerus impar, id imprimis notari convenit; etiamsi angulus seu arcus ϕ in infinitum crescat, integrale tamen nunquam ultra certum limitem excrescere posse, cum tamen si n sit numerus par, etiam in infinitum excrescat.

Scholion.

245. Simili modo formula $\partial \phi \cos. \phi^n$ tractatur, quae in hos factores resoluta $\cos. \phi^{n-1} . \partial \phi \cos. \phi$, praebet,

$$\int\partial\Phi\cos.\Phi^n = \cos.\Phi^{n-1}\sin.\Phi + (n-1)\int\partial\Phi\cos.\Phi^{n-2}\sin.\Phi^2$$
$$= \cos.\Phi^{n-1}\sin.\Phi + (n-1)\int\partial\Phi\cos.\Phi^{n-2} - (n-1)\int\partial\Phi\cos.\Phi^n;$$

unde cum postrema formula propositae sit similis, colligitur

$$\int\partial\Phi\cos.\Phi^n = \tfrac{1}{n}\sin.\Phi\cos.\Phi^{n-1} + \tfrac{n-1}{n}\int\partial\Phi\cos.\Phi^{n-2}.$$

Quare cum casibus $n=0$, et $n=1$ integratio sit in promptu, ad altiores potestates patet progressio;

$$\int\partial\Phi\cos.\Phi^0 = \Phi$$
$$\int\partial\Phi\cos.\Phi = \sin.\Phi$$
$$\int\partial\Phi\cos.\Phi^2 = \tfrac{1}{2}\sin.\Phi\cos.\Phi + \tfrac{1}{2}\Phi$$
$$\int\partial\Phi\cos.\Phi^3 = \tfrac{1}{3}\sin.\Phi\cos.\Phi^2 + \tfrac{2}{3}\sin.\Phi$$
$$\int\partial\Phi\cos.\Phi^4 = \tfrac{1}{4}\sin.\Phi\cos.\Phi^3 + \tfrac{1.3}{2.4}\sin.\Phi\cos.\Phi + \tfrac{1.3}{2.4}\Phi$$
$$\int\partial\Phi\cos.\Phi^5 = \tfrac{1}{5}\sin.\Phi\cos.\Phi^4 + \tfrac{1.4}{3.5}\sin.\Phi\cos.\Phi^2 + \tfrac{2.4}{3.5}\sin.\Phi$$
$$\int\partial\Phi\cos.\Phi^6 = \tfrac{1}{6}\sin.\Phi\cos.\Phi^5 + \tfrac{1.5}{4.6}\sin.\Phi\cos.\Phi^3$$
$$+ \tfrac{1.3.5}{2.4.6}\sin.\Phi\cos.\Phi + \tfrac{1.3.5}{2.4.6}\Phi$$

etc.

Problema 26.

246. Formulae $\partial\Phi\sin.\Phi^m\cos.\Phi^n$ integrale invenire.

Solutio.

Quo hoc facilius praestetur, consideremus factum $\sin\Phi^\mu\cos.\Phi^\nu$, quod differentiatum fit $\mu\partial\Phi\sin.\Phi^{\mu-1}\cos.\Phi^{\nu+1} - \nu\partial\Phi\sin.\Phi^{\mu+1}\cos.\Phi^{\nu-1}$. Jam prout vel in parte priori $\cos.\Phi^2 = 1 - \sin.\Phi^2$, vel in posteriori $\sin\Phi^2 = 1 - \cos.\Phi^2$ statuitur, oritur

$$\text{vel } \partial.\sin.\Phi^\mu\cos.\Phi^\nu = \mu\partial\Phi\sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1}$$
$$-(\mu+\nu)\,\partial\Phi\sin.\Phi^{\mu+1}\cos.\Phi^{\nu-1},$$
$$\text{vel } \partial.\sin.\Phi^\mu\cos.\Phi^\nu = -\nu\partial\Phi\sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1}$$
$$+(\mu+\nu)\,\partial\Phi\sin.\Phi^{\mu-1}\cos.\Phi^{\nu+1}.$$

Hinc igitur duplicem reductionem adipiscimur:

$$\text{I. } \int \partial\Phi \sin.\Phi^{\mu+1} \cos.\Phi^{\nu-1} = -\frac{1}{\mu+\nu} \sin.\Phi^{\mu} \cos.\Phi^{\nu} + \frac{\mu}{\mu+\nu} \int \partial\Phi \sin.\Phi^{\mu-1} \cos.\Phi^{\nu-1}$$

$$\text{II. } \int \partial\Phi \sin.\Phi^{\mu-1} \cos.\Phi^{\nu+1} = \frac{1}{\mu+\nu} \sin.\Phi^{\mu} \cos.\Phi^{\nu} + \frac{\nu}{\mu+\nu} \int \partial\Phi \sin.\Phi^{\mu-1} \cos.\Phi^{\nu-1}$$

Quare formula proposita $\int \partial\Phi \sin.\Phi^{m} \cos.\Phi^{n}$ successive continuo ad simpliciores potestates tam ipsius $\sin.\Phi$ quam ipsius $\cos.\Phi$ reducitur, donec alter vel penitus abeat, vel simpliciter adsit, quo casu integratio per se patet, cum sit

$$\int \partial\Phi \sin.\Phi^{m} \cos.\Phi = +\frac{1}{m+1} \sin.\Phi^{m+1} \text{ et}$$

$$\int \partial\Phi \sin.\Phi \cos.\Phi^{n} = -\frac{1}{n+1} \cos.\Phi^{n+1}.$$

Exemplum.

247. *Formulae* $\partial\Phi \sin.\Phi^{8} \cos.\Phi^{7}$ *integrale invenire.*

Per priorem reductionem ob $\mu = 7$ et $\nu = 8$, impetramus

$$\int \partial\Phi \sin.\Phi^{8} \cos.\Phi^{7} = -\frac{1}{15} \sin.\Phi^{7} \cos.\Phi^{8} + \frac{7}{15} \int \partial\Phi \sin.\Phi^{6} \cos.\Phi^{7};$$

istam per posteriorem reductionem tractemus:

$$\int \partial\Phi \sin.\Phi^{6} \cos.\Phi^{7} = \frac{1}{13} \sin.\Phi^{7} \cos.\Phi^{6} + \frac{6}{13} \int \partial\Phi \sin.\Phi^{6} \cos.\Phi^{5},$$

hoc modo ulterius progrediamur:

$$\int \partial\Phi \sin.\Phi^{6} \cos.\Phi^{5} = -\frac{1}{11} \sin.\Phi^{5} \cos.\Phi^{6} + \frac{6}{11} \int \partial\Phi \sin.\Phi^{4} \cos.\Phi^{5}$$

$$\int \partial\Phi \sin.\Phi^{4} \cos.\Phi^{5} = \frac{1}{9} \sin.\Phi^{5} \cos.\Phi^{4} + \frac{4}{9} \int \partial\Phi \sin.\Phi^{4} \cos.\Phi^{3}$$

$$\int \partial\Phi \sin.\Phi^{4} \cos.\Phi^{3} = -\frac{1}{7} \sin.\Phi^{3} \cos.\Phi^{4} + \frac{3}{7} \int \partial\Phi \sin.\Phi^{2} \cos.\Phi^{3}$$

$$\int \partial\Phi \sin.\Phi^{2} \cos.\Phi^{3} = \frac{1}{5} \sin.\Phi^{3} \cos.\Phi^{2} + \frac{2}{5} \int \partial\Phi \sin.\Phi^{2} \cos.\Phi$$

$$\int \partial\Phi \sin.\Phi^{2} \cos.\Phi = -\frac{1}{3} \sin.\Phi \cos.\Phi^{2} + \frac{1}{3} \int \partial\Phi \cos.\Phi (+\tfrac{1}{3} \sin.\Phi).$$

Ex his colligitur formulae propositae integrale

$$\int \partial \Phi \sin. \Phi^8 \cos. \Phi^7$$
$$= -\tfrac{1}{15} \sin. \Phi^7 \cos. \Phi^8 + \tfrac{1.7}{15.13} \sin. \Phi^7 \cos. \Phi^6 - \tfrac{1.7.6}{15.13.11} \sin. \Phi^5 \cos. \Phi^6$$
$$+ \tfrac{1.7.6.5}{15.13.11.9} \sin. \Phi^5 \cos. \Phi^4 - \tfrac{1.7.6.5.4}{15.13.11.9.7} \sin. \Phi^3 \cos. \Phi^4$$
$$+ \tfrac{1.7.6.5.4.3}{15.13.11.9.7.5} \sin. \Phi^3 \cos. \Phi^2 - \tfrac{1.7.6.5.4.3.2}{15.13.11.9.7.5.3} \sin. \Phi \cos. \Phi^2$$
$$+ \tfrac{1.7.6.5.4.3.2}{15.13.11.9.7.5.3} \sin. \Phi.$$

Scholion.

248. Quando autem hujusmodi casus occurunt, semper praestat productum $\sin. \Phi^m \cos. \Phi^n$ in sinus vel cosinus angulorum multiplorum resolvere, quo facto singulae partes facillime integrantur. Caeterum hic brevitatis gratia angulum simpliciter littera Φ indicavi, nihiloque res foret generalior, si per $\alpha\Phi + \beta$ exprimeretur, quemadmodum etiam ante haec expressio Ang. sin. x aeque late patet, ac si loco x functio quaecunque scriberetur. Contemplemur ergo ejusmodi formulas, in quibus sinus cosinusve denominatorem occupant, ubi quidem simplicissimae sunt

$$\text{I. } \frac{\partial \Phi}{\sin. \Phi}; \quad \text{II. } \frac{\partial \Phi}{\cos. \Phi}; \quad \text{III. } \frac{\partial \Phi \cos. \Phi}{\sin. \Phi}; \quad \text{IV. } \frac{\partial \Phi \sin. \Phi}{\cos. \Phi};$$

quarum integralia imprimis nosse oportet. Pro prima adhibeantur hae transformationes

$$\frac{\partial \Phi}{\sin. \Phi} = \frac{\partial \Phi \sin. \Phi}{\sin. \Phi^2} = \frac{\partial \Phi \sin. \Phi}{1 - \cos. \Phi^2} = \frac{-\partial x}{1 - xx} \text{ (posito } \cos. \Phi = x\text{)},$$

unde fit

$$\int \frac{\partial \Phi}{\sin. \Phi} = -\tfrac{1}{2} l \frac{1+x}{1-x} = -\tfrac{1}{2} l \frac{1 + \cos. \Phi}{1 - \cos. \Phi}.$$

Pro secunda

$$\frac{\partial \Phi}{\cos. \Phi} = \frac{\partial \Phi \cos. \Phi}{\cos. \Phi^2} = \frac{\partial \Phi \cos. \Phi}{1 - \sin. \Phi^2} = \frac{\partial x}{1 - xx} \text{ (posito } \sin. \Phi = x\text{)}$$

ergo

$$\int \frac{\partial \Phi}{\cos. \Phi} = \tfrac{1}{2} l \frac{1+x}{1-x} = \tfrac{1}{2} l \frac{1 + \sin. \Phi}{1 - \sin. \Phi}.$$

Tertiae et quartae integratio manifesto logarithmis conficitur: quare haec integralia probe notasse juvabit

I. $\int\frac{\partial\Phi}{\sin.\Phi} = -\tfrac{1}{2}\,l\,\frac{1+\cos.\Phi}{1-\cos.\Phi} = l\,\frac{\sqrt{(1-\cos.\Phi)}}{\sqrt{(1+\cos.\Phi)}} = l\,\text{tang.}\,\tfrac{1}{2}\Phi,$

II. $\int\frac{\partial\Phi}{\cos.\Phi} = \tfrac{1}{2}\,l\,\frac{1+\sin.\Phi}{1-\sin.\Phi} = l\,\frac{\sqrt{(1+\sin.\Phi)}}{\sqrt{(1-\sin.\Phi)}} = l\,\text{tang.}\,(45^\circ+\tfrac{1}{2}\Phi),$

III. $\int\frac{\partial\Phi\,\cos.\Phi}{\sin.\Phi} = l\,\sin.\Phi = \int\frac{\partial\Phi}{\text{tang.}\,\Phi} = \int\partial\Phi\,\cot.\Phi$

IV. $\int\frac{\partial\Phi\,\sin.\Phi}{\cos.\Phi} = -\,l\,\cos.\Phi = \int\partial\Phi\,\text{tang.}\,\Phi$

hincque sequitur III. + IV.

$$\int\frac{\partial\Phi}{\sin.\Phi\,\cos.\Phi} = l\,\frac{\sin.\Phi}{\cos.\Phi} = l.\,\text{tang.}\,\Phi.$$

Problema 27.

249. Formularum $\frac{\partial\Phi\,\sin.\Phi^m}{\cos.\Phi^n}$ et $\frac{\partial\Phi\,\cos.\Phi^m}{\sin.\Phi^n}$ integralia investigare.

Solutio.

Primo statim perspicitur, alteram formulam in alteram transmutari, posito $\Phi = 90^\circ - \psi$, quia tum fit $\sin.\Phi = \cos.\psi$ et $\cos.\Phi = \sin.\psi$, dummodo notetur fore $\partial\Phi = -\partial\psi$. Quare sufficit priorem tantum tractasse. Reductio autem prior §. 246. data, sumto $\mu+1=m$ et $\nu-1=-n$, praebet

$$\int\frac{\partial\Phi\,\sin.\Phi^m}{\cos.\Phi^n} = -\frac{1}{m-n}\cdot\frac{\sin.\Phi^{m-1}}{\cos.\Phi^{n-1}} + \frac{m-1}{m-n}\int\frac{\partial\Phi\,\sin.\Phi^{m-2}}{\cos.\Phi^n}$$

quo pacto in numeratore exponens ipsius $\sin.\Phi$ continuo binario deprimitur, ita ut tandem perveniatur vel ad $\int\frac{\partial\Phi}{\cos.\Phi^n}$ vel ad $\int\frac{\partial\Phi\,\sin.\Phi}{\cos\,\Phi^n} = \frac{1}{(n-1)\cos.\Phi^{n-1}}$: ideoque sola formula $\int\frac{\partial\Phi}{\cos.\Phi^n}$ tractanda supersit. Altera autem reductio ibidem tradita (246.) sumto $\mu-1=m$ et $\nu-1=-n$, dat

$$\int\frac{\partial\Phi\,\sin.\Phi^m}{\cos.\Phi^{n-2}} = \frac{1}{m-n+2}\cdot\frac{\sin.\Phi^{m+1}}{\cos.\Phi^{n-1}} - \frac{n-1}{m-n+2}\int\frac{\partial\Phi\,\sin.\Phi^m}{\cos.\Phi^n}:$$

unde colligitur

Scholion.

253. Pro reliquis casibus denominatoris totum negotium conficietur his reductionibus:

$$\int \frac{\partial \phi \sin. \phi^m}{\cos. \phi^2} = \frac{\sin. \phi^{m+1}}{\cos. \phi} - m \int \partial \phi \sin. \phi^m$$

$$\int \frac{\partial \phi \sin. \phi^m}{\cos. \phi^3} = \frac{1}{2} \cdot \frac{\sin. \phi^{m+1}}{\cos. \phi^2} - \frac{m-1}{2} \int \frac{\partial \phi \sin. \phi^m}{\cos. \phi}$$

$$\int \frac{\partial \phi \sin. \phi^m}{\cos. \phi^4} = \frac{1}{3} \cdot \frac{\sin. \phi^{m+1}}{\cos. \phi^3} - \frac{m-2}{3} \int \frac{\partial \phi \sin. \phi^m}{\cos. \phi^2}$$

$$\int \frac{\partial \phi \sin. \phi^m}{\cos. \phi^5} = \frac{1}{4} \cdot \frac{\sin. \phi^{m+1}}{\cos. \phi^4} - \frac{m-3}{4} \int \frac{\partial \phi \sin. \phi^m}{\cos. \phi^3}$$

etc.

Exemplum 2.

254. *Formulae* $\frac{\partial \phi}{\cos. \phi^n}$ *integrale assignare.*

Altera reductio ob $m = 0$ dat

$$\int \frac{\partial \phi}{\cos. \phi^n} = \frac{1}{n-1} \cdot \frac{\sin. \phi}{\cos. \phi^{n-1}} + \frac{n-2}{n-1} \int \frac{\partial \phi}{\cos. \phi^{n-2}};$$

quia jam casus simplicissimi

$$\int \partial \phi = \phi \text{ et } \int \frac{\partial \phi}{\cos. \phi} = l \operatorname{tang} (45^\circ + \tfrac{1}{2} \phi)$$

sunt cogniti, ad eos sequentes omnes revocabuntur:

$$\int \frac{\partial \phi}{\cos. \phi^2} = \frac{\sin. \phi}{\cos. \phi}$$

$$\int \frac{\partial \phi}{\cos. \phi^3} = \frac{1}{2} \cdot \frac{\sin. \phi}{\cos. \phi^2} + \frac{1}{2} \int \frac{\partial \phi}{\cos. \phi}$$

$$\int \frac{\partial \phi}{\cos. \phi^4} = \frac{1}{3} \cdot \frac{\sin. \phi}{\cos. \phi^3} + \frac{2}{3} \cdot \frac{\sin. \phi}{\cos. \phi}$$

$$\int\frac{\partial\Phi}{\cos.\Phi^5}=\tfrac{1}{4}\cdot\frac{\sin.\Phi}{\cos.\Phi^4}+\frac{1.3}{2.4}\cdot\frac{\sin.\Phi}{\cos.\Phi^2}+\frac{1.3}{2.4}\int\frac{\partial\Phi}{\cos.\Phi}$$

$$\int\frac{\partial\Phi}{\cos.\Phi^6}=\tfrac{1}{5}\cdot\frac{\sin.\Phi}{\cos.\Phi^5}+\frac{1.4}{3.5}\cdot\frac{\sin.\Phi}{\cos.\Phi^3}+\frac{2.4}{3.5}\cdot\frac{\sin.\Phi}{\cos.\Phi}$$

etc.

Corollarium 1.

255. Simili modo habebimus has integrationes:

$$\int\frac{\partial\Phi}{\sin.\Phi}=l\,\mathrm{tang.}\,\tfrac{1}{2}\Phi;\quad\int\frac{\partial\Phi}{\sin.\Phi^2}=-\frac{\cos.\Phi}{\sin.\Phi};$$

$$\int\frac{\partial\Phi}{\sin.\Phi^3}=-\tfrac{1}{2}\cdot\frac{\cos.\Phi}{\sin.\Phi^2}+\tfrac{1}{2}\int\frac{\partial\Phi}{\sin.\Phi}$$

$$\int\frac{\partial\Phi}{\sin.\Phi^4}=-\tfrac{1}{3}\cdot\frac{\cos.\Phi}{\sin.\Phi^3}-\tfrac{2}{3}\cdot\frac{\cos.\Phi}{\sin.\Phi}$$

$$\int\frac{\partial\Phi}{\sin.\Phi^5}=-\tfrac{1}{4}\cdot\frac{\cos.\Phi}{\sin.\Phi^4}-\frac{1.3}{2.4}\cdot\frac{\cos.\Phi}{\sin.\Phi^2}+\frac{1.3}{2.4}\int\frac{\partial\Phi}{\sin.\Phi}$$

etc.

Corollarium 2.

256. Deinde est

$$\int\frac{\partial\Phi\,\sin.\Phi}{\cos.\Phi^n}=\frac{1}{n-1}\cdot\frac{1}{\cos.\Phi^{n-1}};\ \text{et}$$

$$\int\frac{\partial\Phi\cos.\Phi}{\sin.\Phi^n}=\frac{-1}{n-1}\cdot\frac{1}{\sin.\Phi^{n-1}}.$$

Porro

$$\int\frac{\partial\Phi\,\sin.\Phi^2}{\cos.\Phi^n}=\int\frac{\partial\Phi}{\cos.\Phi^n}-\int\frac{\partial\Phi}{\cos.\Phi^{n-2}};$$

$$\int\frac{\partial\Phi\,\cos.\Phi^2}{\sin.\Phi^n}=\int\frac{\partial\Phi}{\sin.\Phi^n}-\int\frac{\partial\Phi}{\sin.\Phi^{n-2}}$$

et $$\int\frac{\partial\Phi\,\sin.\Phi^3}{\cos.\Phi^n}=\int\frac{\partial\Phi\,\sin.\Phi}{\cos.\Phi^n}-\int\frac{\partial\Phi\,\sin.\Phi}{\cos.\Phi^{n-2}};$$

$$\int\frac{\partial\Phi\cos.\Phi^3}{\sin.\Phi^n}=\int\frac{\partial\Phi\cos.\Phi}{\sin.\Phi^n}-\int\frac{\partial\Phi\cos.\Phi}{\sin.\Phi^{n-2}};$$

quibus reductionibus continuo ulterius progredi licet.

Problema 28.

257. Formulae $\frac{\partial\Phi}{\sin.\Phi^m\cos.\Phi^n}$ integrale investigare.

Solutio.

Reductiones supra adhibitas huc accommodare licet, sumendo in praecedente problemate m negative: ita erit

$$\int\frac{\partial\Phi}{\sin.\Phi^m\cos.\Phi^n}=+\frac{1}{m+n}\cdot\frac{1}{\sin.\Phi^{m+1}\cos.\Phi^{n-1}}$$
$$+\frac{m+1}{m+n}\int\frac{\partial\Phi}{\sin.\Phi^{m+2}\cos.\Phi^n},$$

unde loco m scribendo $m-2$, per conversionem fit

$$\int\frac{\partial\Phi}{\sin.\Phi^m\cos.\Phi^n}=-\frac{1}{m-1}\cdot\frac{1}{\sin.\Phi^{m-1}\cos.\Phi^{n-1}}$$
$$+\frac{m+n-2}{m-1}\int\frac{\partial\Phi}{\sin.\Phi^{m-2}\cos.\Phi^n}$$

Altera huic similis est

$$\int\frac{\partial\Phi}{\sin.\Phi^m\cos.\Phi^n}=\frac{1}{n-1}\cdot\frac{1}{\sin.\Phi^{m-1}\cos.\Phi^{n-1}}$$
$$+\frac{m+n-2}{n-1}\int\frac{\partial\Phi}{\sin.\Phi^m\cos.\Phi^{n-2}}$$

Cum jam in hoc genere formae simplicissimae sint:

$$\int\frac{\partial\Phi}{\sin.\Phi}=l.\ \text{tang.}\ \tfrac{1}{2}\Phi;\quad \int\frac{\partial\Phi}{\cos.\Phi}=l.\ \text{tang.}\ (45^\circ+\tfrac{1}{2}\Phi);$$

$$\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi}=l.\text{tang.}\Phi;\quad \int\frac{\partial\Phi}{\sin.\Phi^2}=-\cot.\Phi;\quad \int\frac{\partial\Phi}{\cos.\Phi^2}=\text{tang.}\ \Phi;$$

hinc magis compositas eliciemus:

$$\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi^2}=\frac{1}{\cos.\Phi}+\int\frac{\partial\Phi}{\sin.\Phi};$$

$$\int\frac{\partial\Phi}{\sin.\Phi^2\cos.\Phi}=-\frac{1}{\sin.\Phi}+\int\frac{\partial\Phi}{\cos.\Phi};$$

$$\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi^4}=\tfrac{1}{3}\cdot\frac{1}{\cos.\Phi^3}+\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi^2};$$

$$\int\frac{\partial\Phi}{\sin.\Phi^4\cos.\Phi}=-\tfrac{1}{3}\cdot\frac{1}{\sin.\Phi^3}+\int\frac{\partial\Phi}{\sin.\Phi^2\cos.\Phi};$$

$$\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi^6}=\tfrac{1}{5}\cdot\frac{1}{\cos.\Phi^5}+\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi^4};$$

$$\int\frac{\partial\Phi}{\sin.\Phi^6\cos.\Phi}=-\tfrac{1}{5}\cdot\frac{1}{\sin.\Phi^5}+\int\frac{\partial\Phi}{\sin.\Phi^4\cos.\Phi};$$

$$\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi^3}=\tfrac{1}{2}\cdot\frac{1}{\cos.\Phi^2}+\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi};$$

$$\int\frac{\partial\Phi}{\sin.\Phi^3\cos.\Phi}=-\tfrac{1}{2}\cdot\frac{1}{\sin.\Phi^2}+\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi};$$

$$\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi^5}=\tfrac{1}{4}\cdot\frac{1}{\cos.\Phi^4}+\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi^3};$$

$$\int\frac{\partial\Phi}{\sin.\Phi^5\cos.\Phi}=-\tfrac{1}{4}\cdot\frac{1}{\sin.\Phi^4}+\int\frac{\partial\Phi}{\sin.\Phi^3\cos.\Phi};$$

$$\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi^7}=\tfrac{1}{6}\cdot\frac{1}{\cos.\Phi^6}+\int\frac{\partial\Phi}{\sin.\Phi\cos.\Phi^5};$$

$$\int\frac{\partial\Phi}{\sin.\Phi^7\cos.\Phi}=-\tfrac{1}{6}\cdot\frac{1}{\sin.\Phi^6}+\int\frac{\partial\Phi}{\sin.\Phi^5\cos.\Phi};$$

etc.

$$\int\frac{\partial\phi}{\sin.\phi^2\cos.\phi^2}=\frac{1}{\sin.\phi\cos.\phi}+2\int\frac{\partial\phi}{\sin.\phi^2}=-\frac{1}{\sin\,\phi\cos.\phi}+2\int\frac{\partial\phi}{\cos.\phi^2}$$

$$\int\frac{\partial\phi}{\sin.\phi^2\cos.\phi^4}=\tfrac{1}{3}\cdot\frac{1}{\sin.\phi\cos.\phi^3}+\tfrac{4}{3}\int\frac{\partial\phi}{\sin.\phi^2\cos.\phi^2}$$

$$\int\frac{\partial\phi}{\sin.\phi^4\cos.\phi^2}=-\tfrac{1}{3}\cdot\frac{1}{\sin.\phi^3\cos.\phi}+\tfrac{4}{3}\int\frac{\partial\phi}{\sin.\phi^2\cos.\phi^2}.$$

Sicque formulae quantumvis compositae ad simpliciores, quarum integratio est in promtu, reducuntur.

Corollarium 1.

258. Ambo exponentes ipsius sin. ϕ et cos. ϕ simul binario minui possunt: erit enim per priorem reductionem

$$\int\frac{\partial\phi}{\sin.\phi^{\mu}\cos.\phi^{\nu}}=-\frac{1}{\mu-1}\cdot\frac{1}{\sin.\phi^{\mu-1}\cos.\phi^{\nu-1}}+\frac{\mu+\nu-2}{\mu-1}\int\frac{\partial\phi}{\sin.\phi^{\mu-2}\cos.\phi^{\nu}}$$

nunc haec formula per posteriorem ob $m=\mu-2$ et $n=\nu$ dat

$$\int\frac{\partial\phi}{\sin.\phi^{\mu-2}\cos.\phi^{\nu}}=\frac{1}{\nu-1}\cdot\frac{1}{\sin.\phi^{\mu-3}\cos.\phi^{\nu-1}}+\frac{\mu+\nu-4}{\nu-1}\int\frac{\partial\phi}{\sin.\phi^{\mu-2}\cos.\phi^{\nu-2}},$$

unde concluditur

$$\int\frac{\partial\phi}{\sin.\phi^{\mu}\cos.\phi^{\nu}}=-\frac{1}{\mu-1}\cdot\frac{1}{\sin.\phi^{\mu-1}\cos.\phi^{\nu-1}}+\frac{\mu+\nu-2}{(\mu-1)(\nu-1)}\cdot\frac{1}{\sin.\phi^{\mu-3}\cos.\phi^{\nu-1}}+\frac{(\mu+\nu-2)(\mu+\nu-4)}{(\mu-1)(\nu-1)}\int\frac{\partial\phi}{\sin.\phi^{\mu-2}\cos.\phi^{\nu-2}}$$

Corollarium 2.

259. Prioribus membris ad communem denominatorem reductis obtinebitur

$$\int \frac{\partial \Phi}{\sin.\Phi^\mu \cos.\Phi^\nu} = \frac{(\mu - 1)\sin.\Phi^2 - (\nu - 1)\cos.\Phi^2}{(\mu - 1)(\nu - 1)\sin.\Phi^{\mu-1}\cos.\Phi^{\nu-1}}$$
$$+ \frac{(\mu+\nu-2)(\mu+\nu-4)}{(\mu-1)(\nu-1)} \int \frac{\partial \Phi}{\sin.\Phi^{\mu-2}\cos.\Phi^{\nu-2}}$$

qua reductione semper ad calculum contrahendum uti licet, nisi vel $\mu = 1$ vel $\nu = 1$.

Scholion.

260. Hujusmodi formulae $\frac{\partial \Phi}{\sin.\Phi^m \cos.\Phi^n}$ etiam hoc modo maxime obvio ad simpliciores reduci possunt; dum numerator per $\sin.\Phi^2 + \cos.\Phi^2 = 1$ multiplicatur, unde fit

$$\int \frac{\partial \Phi}{\sin.\Phi^m \cos.\Phi^n} = \int \frac{\partial \Phi}{\sin.\Phi^{m-2}\cos.\Phi^n} + \int \frac{\partial \Phi}{\sin.\Phi^m \cos.\Phi^{n-2}}$$

quae eousque continuari potest, donec in denominatore unica tantum potestas relinquatur. Ita erit

$$\int \frac{\partial \Phi}{\sin.\Phi \cos.\Phi} = \int \frac{\partial \Phi \sin.\Phi}{\cos.\Phi} + \int \frac{\partial \Phi \cos.\Phi}{\sin.\Phi} = l\frac{\sin.\Phi}{\cos.\Phi}$$
$$\int \frac{\partial \Phi}{\sin.\Phi^2 \cos.\Phi^2} = \int \frac{\partial \Phi}{\sin.\Phi^2} + \int \frac{\partial \Phi}{\cos.\Phi^2} = \frac{\sin.\Phi}{\cos.\Phi} - \frac{\cos.\Phi}{\sin.\Phi}.$$

Quodsi proposita sit haec formula $\int \frac{\partial \Phi}{\sin.\Phi^n \cos.\Phi^n}$, in subsidium vocari potest, esse $\sin.\Phi \cos.\Phi = \frac{1}{2}\sin.2\Phi$, unde habetur $\int \frac{2^n \partial \Phi}{\sin.2\Phi^n} = 2^{n-1} \int \frac{\partial \omega}{\sin.\omega^n}$, posito $\omega = 2\Phi$, quae formula per superiora praecepta resolvitur. His igitur adminiculis observatis circa formulam $\partial \Phi \sin.\Phi^m \cos.\Phi^n$, si quidem m et n fuerint numeri integri sive positivi sive negativi, nihil amplius desideratur: sin autem fuerint numeri fracti, nihil admodum praecipiendum occurrit, quandoquidem casus, quibus integratio succedit, quasi sponte se produnt. Quemadmodum autem integralia, quae exhiberi nequeunt, per series exprimi conveniat, in capite sequente accuratius exponatur.

mus. Nunc vero formulas fractas consideremus, quarum denominator est $a + b\cos.\Phi$ ejusque potestas, tales enim formulae in Theoria Astronomiae frequentissime occurrunt.

Problema 29.

261. Formulae differentialis $\frac{\partial\Phi}{a+b\cos.\Phi}$ integrale investigare.

Solutio.

Haec investigatio commodius institui nequit, quam ut formula proposita ad formam ordinariam reducatur, ponendo $\cos\Phi = \frac{1-xx}{1+xx}$, ut rationaliter fiat $\sin.\Phi = \frac{2x}{1+xx}$, hincque $\partial\Phi\cos.\Phi = \frac{2\partial x(1-xx)}{(1+xx)^2}$, sicque $\partial\Phi = \frac{2\partial x}{1+xx}$. Quia igitur $a+b\cos.\Phi = \frac{a+b+(a-b)xx}{1+xx}$, erit formula nostra $\frac{\partial\Phi}{a+b\cos.\Phi} = \frac{2\partial x}{a+b+(a-b)xx}$, quae prout fuerit $a>b$ vel $a<b$, vel angulum vel logarithmum praebet.

Casu $a>b$ reperitur

$$\int\frac{\partial\Phi}{a+b\cos.\Phi} = \frac{2}{\sqrt{(aa-bb)}}\text{ Arc. tang. }\frac{(a-b)x}{\sqrt{(aa-bb)}}:$$

casu $a<b$ vero est

$$\int\frac{\partial\Phi}{a+b\cos.\Phi} = \frac{1}{\sqrt{(bb-aa)}}\,l\frac{\sqrt{(bb-aa)}+x(b-a)}{\sqrt{(bb-aa)}-x(b-a)}.$$

Nunc vero est

$$x = \sqrt{\frac{1-\cos.\Phi}{1+\cos.\Phi}} = \text{tang.}\tfrac{1}{2}\Phi = \frac{\sin.\Phi}{1+\cos.\Phi};$$

qua restitutione facta, cum sit

$$2\text{ Ang. tang. }\frac{(a-b)x}{\sqrt{(aa-bb)}} = \text{Ang. tang. }\frac{2x\sqrt{(aa-bb)}}{a+b-(a-b)xx}$$
$$= \text{Ang. tang. }\frac{2\sin.\Phi\sqrt{(aa-bb)}}{(a+b)(1+\cos.\Phi)-(a-b)(1-\cos.\Phi)}$$
$$= \text{Ang. tang. }\frac{\sin.\Phi\sqrt{(aa-bb)}}{a\cos.\Phi+b}.$$

Quocirca pro casu $a>b$ adipiscimur:

$$\int\frac{\partial\Phi}{a+b\cos.\Phi} = \frac{1}{\sqrt{(aa-bb)}}\text{ Ang. tang. }\frac{\sin.\Phi\sqrt{(aa-bb)}}{a\cos.\Phi+b},\text{ seu}$$
$$\int\frac{\partial\Phi}{a+b\cos.\Phi} = \frac{1}{\sqrt{(aa-bb)}}\text{ Ang. sin. }\frac{\sin.\Phi\sqrt{(aa-bb)}}{a+b\cos.\Phi},\text{ sive}$$
$$\int\frac{\partial\Phi}{a+b\cos.\Phi} = \frac{1}{\sqrt{(aa-bb)}}\text{ Ang. cos. }\frac{a\cos.\Phi+b}{a+b\cos.\Phi}.$$

Pro casu autem $a < b$:

$$\int \frac{\partial \Phi}{a + b \cos. \Phi} = \frac{1}{\sqrt{(bb - aa)}} l \frac{\sqrt{(b+a)}(1 + \cos. \Phi) + \sqrt{(b-a)}(1 - \cos. \Phi)}{\sqrt{(b+a)}(1 + \cos. \Phi) - \sqrt{(b-a)}(1 - \cos. \Phi)},$$

seu

$$\int \frac{\partial \Phi}{a + b \cos. \Phi} = \frac{1}{\sqrt{(bb - aa)}} l \frac{a \cos. \Phi + b + \sin. \Phi . \sqrt{(bb - aa)}}{a + b \cos. \Phi}.$$

At casu $b = a$, integrale est $= \frac{x}{a} = \frac{1}{a}$ tang. $\frac{1}{2} \Phi$, unde fit

$$\int \frac{\partial \Phi}{1 + \cos. \Phi} = \text{tang.} \tfrac{1}{2} \Phi = \frac{\sin. \Phi}{1 + \cos. \Phi},$$

quae integralia evanescunt facto $\Phi = 0$.

Corollarium 1.

262. Formulae autem $\frac{\partial \Phi \sin. \Phi}{a + b \cos. \Phi} = \frac{-\partial. \cos. \Phi}{a + b \cos. \Phi}$ integrale est $\frac{1}{b} l \frac{a + b}{a + b \cos. \Phi}$, ita sumtum, ut evanescat posito $\Phi = 0$; sicque habebimus:

$$\int \frac{\partial \Phi \sin. \Phi}{a + b \cos. \Phi} = \frac{1}{b} l \frac{a + b}{a + b \cos. \Phi}.$$

Corollarium 2.

263. Formula autem $\frac{\partial \Phi \cos. \Phi}{a + b \cos. \Phi}$ transformatur in $\frac{\partial \Phi}{b} - \frac{a \partial \Phi}{b(a + b \cos. \Phi)}$, unde integrale per solutionem problematis exhiberi potest:

$$\int \frac{\partial \Phi \cos. \Phi}{a + b \cos. \Phi} = \frac{\Phi}{b} - \frac{a}{b} \int \frac{\partial \Phi}{a + b \cos. \Phi}.$$

Scholion 1.

264. Integratione hac inventa, etiam hujus formulae $\frac{\partial \Phi}{(a + b \cos. \Phi)^n}$ integrale inveniri potest, existente n numero integro; quod fingendo integralis forma commodissime praestari videtur: ponatur

$$\int \frac{\partial \Phi}{(a + b \cos. \Phi)^2} = \frac{A \sin. \Phi}{a + b \cos. \Phi} + m \int \frac{\partial \Phi}{a + b \cos. \Phi};$$

ac reperitur

$\mathrm{A} = \frac{-b}{aa-bb}$, et $m = \frac{a}{aa-bb}$. Porro fingatur

$$\int \frac{\partial \Phi}{(a+b\cos.\Phi)^3} = \frac{(\mathrm{A}+\mathrm{B}\cos.\Phi)\sin.\Phi}{(a+b\cos.\Phi)^2} + m \int \frac{\partial \Phi}{(a+b\cos.\Phi)^2},$$

reperiturque

$$\mathrm{A} = \frac{-b}{aa-bb};\quad \mathrm{B} = \frac{-bb}{2a(aa-bb)};\quad m = \frac{2aa+bb}{2a(aa-bb)};$$

similique modo investigatio ad majores potestates continuari potest, labore quidem non parum taedioso. Sequenti autem modo negotium facillime expediri videtur.

Consideretur scilicet formula generalior $\frac{\partial \Phi (f+g\cos.\Phi)}{(a+b\cos.\Phi)^{n+1}}$, ac ponatur

$$\int \frac{\partial \Phi (f+g\cos.\Phi)}{(a+b\cos.\Phi)^{n+1}} = \frac{\mathrm{A}\sin.\Phi}{(a+b\cos.\Phi)^n} + \int \frac{\partial \Phi (\mathrm{B}+\mathrm{C}\cos.\Phi)}{(a+b\cos.\Phi)^n},$$

sumtisque differentialibus, ista prodibit aequatio:

$$f+g\cos.\Phi = \mathrm{A}\cos.\Phi(a+b\cos.\Phi) + n\mathrm{A}b\sin.\Phi^2$$
$$+ (\mathrm{B}+\mathrm{C}\cos.\Phi)(a+b\cos.\Phi);$$

quae ob $\sin.\Phi^2 = 1 - \cos.\Phi^2$ hanc formam induit

$$\left.\begin{array}{lll} -f & -g\cos.\Phi & +\mathrm{A}b\cos.\Phi^2 \\ +n\mathrm{A}b & +\mathrm{A}a\cos.\Phi & -n\mathrm{A}b\cos.\Phi^2 \\ +\mathrm{B}a & +\mathrm{B}b\cos.\Phi & +\mathrm{C}b\cos.\Phi^2 \\ & +\mathrm{C}a\cos.\Phi & \end{array}\right\} = 0;$$

unde singulis membris nihilo aequatis, elicitur:

$$\mathrm{A} = \frac{ag-bf}{n(aa-bb)};\quad \mathrm{B} = \frac{af-bg}{aa-bb}\ \text{et}\ \mathrm{C} = \frac{(n-1)(ag-bf)}{n(aa-bb)}.$$

Ita ut haec obtineatur reductio

$$\int \frac{\partial \Phi (f+g\cos.\Phi)}{(a+b\cos.\Phi)^{n+1}} = \frac{(ag-bf)\sin.\Phi}{n(aa-bb)(a+b\cos.\Phi)^n}$$
$$+ \frac{1}{n(aa-bb)} \int \frac{\partial \Phi [n(af-bg)+(n-1)(ag-bf)\cos.\Phi]}{(a+b\cos.\Phi)^n}$$

cujus ope tandem ad formulam $\int \frac{\partial \Phi (b+k\cos.\Phi)}{a+b\cos.\Phi}$ pervenitur, cujus integrale $= \frac{k}{b}\Phi + \frac{bb-ak}{b}\int \frac{\partial \Phi}{a+b\cos.\Phi}$ ex superioribus constat. Perspicuum autem est semper fore $k = 0$.

Scholion 2.

265. Occurrunt etiam ejusmodi formulae, in quas insuper quantitas exponentialis $e^{\alpha\Phi}$, angulum ipsum Φ in exponente gerens, ingreditur, quas quomodo tractari oporteat, ostendendum videtur, cum hinc methodus reductionum supra exposita maxime illustretur. Hic enim per illam reductionem ad formulam propositae similem pervenitur, unde ipsum integrale colligi poterit. In hunc finem notetur esse $\int e^{\alpha\Phi}\partial\Phi = \frac{1}{\alpha}e^{\alpha\Phi}$.

Problema 30.

266. Formulae differentialis $\partial y = e^{\alpha\Phi}\partial\Phi \sin.\Phi^n$ integrale investigare.

Solutio.

Sumto $e^{\alpha\Phi}\partial\Phi$ pro factore differentiali, erit

$$y = \frac{1}{\alpha}e^{\alpha\Phi}\sin.\Phi^n - \frac{n}{\alpha}\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^{n-1}\cos.\Phi:$$

simili modo reperitur

$$\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^{n-1}\cos.\Phi = \frac{1}{\alpha}e^{\alpha\Phi}\sin.\Phi^{n-1}\cos.\Phi$$
$$-\frac{1}{\alpha}\int e^{\alpha\Phi}\partial\Phi\,[(n-1)\sin.\Phi^{n-2}\cos.\Phi^2 - \sin.\Phi^n],$$

quae postrema formula, ob $\cos.\Phi^2 = 1 - \sin.\Phi^2$, reducitur ad has

$$(n-1)\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^{n-2} - n\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^n:$$

unde habebitur

$$\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^n = \frac{1}{\alpha}e^{\alpha\Phi}\sin.\Phi^n - \frac{n}{\alpha\alpha}e^{\alpha\Phi}\sin.\Phi^{n-1}\cos.\Phi$$
$$+\frac{n(n-1)}{\alpha\alpha}\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^{n-2} - \frac{nn}{\alpha\alpha}\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^n.$$

Quare hanc postremam formulam cum prima conjungendo, elicitur:

$$\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^n = \frac{e^{\alpha\Phi}\sin.\Phi^{n-1}(\alpha\sin.\Phi - n\cos.\Phi)}{\alpha\alpha+nn}$$
$$+\frac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^{n-2}.$$

Duobus ergo casibus integrale absolute datur, scilicet $n = 0$ et $n = 1$, eritque

$$\int e^{\alpha\Phi}\partial\Phi = \frac{1}{\alpha}e^{\alpha\Phi} - \frac{1}{\alpha}, \text{ et}$$

$$\int e^{\alpha\Phi}\partial\Phi \sin.\Phi = \frac{e^{\alpha\Phi}(\alpha\sin.\Phi - \cos.\Phi)}{\alpha\alpha + 1} + \frac{1}{\alpha\alpha + 1}$$

atque ad hos sequentes omnes, ubi n est numerus integer unitate major, reducuntur.

Corollarium 1.

267. Ita si $n = 2$, acquirimus hanc integrationem

$$\int e^{\alpha\Phi}\partial\Phi \sin.\Phi^2 = \frac{e^{\alpha\Phi}\sin.\Phi(\alpha\sin.\Phi - 2\cos.\Phi)}{\alpha\alpha + 4} + \frac{1.2}{\alpha(\alpha\alpha + 4)}e^{\alpha\Phi} - \frac{1.2}{\alpha(\alpha\alpha + 4)}:$$

at si sit $n = 3$, istam

$$\int e^{\alpha\Phi}\partial\Phi \sin\Phi^3 = \frac{e^{\alpha\Phi}\sin.\Phi^2(\alpha\sin.\Phi - 3\cos.\Phi)}{\alpha\alpha + 9} + \frac{2.3e^{\alpha\Phi}(\alpha\sin.\Phi - \cos.\Phi)}{(\alpha\alpha+1)(\alpha\alpha+9)} + \frac{2.3}{(\alpha\alpha+1)(\alpha\alpha+9)}$$

integralibus ita sumtis, ut evanescant, posito $\Phi = 0$.

Corollarium 2.

268. Si igitur determinatis hoc modo integralibus, statuatur $\alpha\Phi = -\infty$, ut $e^{\alpha\Phi}$ evanescat, erit in genere

$$\int e^{\alpha\Phi}\partial\Phi \sin.\Phi^n = \frac{n(n-1)}{\alpha\alpha + nn}\int e^{\alpha\Phi}\partial\Phi \sin.\Phi^{n-2};$$

hincque integralia pro isto casu $\alpha\Phi = -\infty$ erunt

$\int e^{\alpha\Phi}\partial\Phi = -\frac{1}{\alpha}$; $\int e^{\alpha\Phi}\partial\Phi\sin.\Phi = \frac{1}{\alpha\alpha+1}$;

$\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^2 = \frac{-1.2}{\alpha(\alpha\alpha+4)}$; $\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^3 = \frac{1.2.3}{(\alpha\alpha+1)(\alpha\alpha+9)}$;

$\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^4 = \frac{-1.2.3.4}{\alpha(\alpha\alpha+4)(\alpha\alpha+16)}$; $\int e^{\alpha\Phi}\partial\Phi\sin.\Phi^5 = \frac{1.2.3.4.5}{(\alpha\alpha+1)(\alpha\alpha+9)(\alpha\alpha+25)}$.

Corollarium 3.

269. Quare si proponatur haec series infinita

$$s = 1 + \frac{1.2}{\alpha\alpha+4} + \frac{1.2.3.4}{(\alpha\alpha+4)(\alpha\alpha+16)} + \frac{1.2.3.4.5.6}{(\alpha\alpha+4)(\alpha\alpha+16)(\alpha\alpha+36)} + \text{etc. erit}$$

$$s = -\alpha \int e^{\alpha\Phi}\partial\Phi (1 + \sin.\Phi^2 + \sin.\Phi^4 + \sin.\Phi^6 + \text{etc.})$$

seu $s = -\alpha \int \frac{e^{\alpha\Phi}\partial\Phi}{\cos.\Phi^2}$, posito post integrationem $\alpha\Phi = -\infty$.

Problema 31.

270. Formulae differentialis $e^{\alpha\Phi}\partial\Phi\cos.\Phi^n$ integrale investigare.

Solutio.

Simili modo procedendo ut ante, erit

$$e^{\alpha\Phi}\partial\Phi\cos.\Phi^n = \frac{1}{\alpha}e^{\alpha\Phi}\cos.\Phi^n + \frac{n}{\alpha}\int e^{\alpha\Phi}\partial\Phi\sin.\Phi\cos.\Phi^{n-1};$$

tum vero

$$\int e^{\alpha\Phi}\partial\Phi\sin.\Phi\cos.\Phi^{n-1} = \frac{1}{\alpha}e^{\alpha\Phi}\sin.\Phi\cos.\Phi^{n-1}$$
$$-\frac{1}{\alpha}\int e^{\alpha\Phi}\partial\Phi[\cos.\Phi^n - (n-1)\cos.\Phi^{n-2}\sin.\Phi^2],$$

quae postrema formula abit in $-(n-1)\int e^{\alpha\Phi}\partial\Phi\cos.\Phi^{n-2} + n\int e^{\alpha\Phi}\partial\Phi\cos.\Phi^n$, ita ut sit

$$\int e^{\alpha\Phi}\partial\Phi\cos.\Phi^n = \frac{1}{\alpha}e^{\alpha\Phi}\cos.\Phi^n + \frac{n}{\alpha\alpha}e^{\alpha\Phi}\sin.\Phi\cos.\Phi^{n-1}$$
$$+\frac{n(n-1)}{\alpha\alpha}\int e^{\alpha\Phi}\partial\Phi\cos.\Phi^{n-2} - \frac{nn}{\alpha\alpha}\int e^{\alpha\Phi}\partial\Phi\cos.\Phi^n,$$

unde colligimus

$$\int e^{\alpha\Phi}\partial\Phi\cos.\Phi^n = \frac{e^{\alpha\Phi}\cos.\Phi^{n-1}(\alpha\cos.\Phi + n\sin.\Phi)}{\alpha\alpha+nn}$$
$$+\frac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\Phi}\partial\Phi\cos.\Phi^{n-2}.$$

Hinc ergo casus simplicissimi sunt

$$\int e^{\alpha\Phi}\partial\Phi = \frac{1}{\alpha}e^{\alpha\Phi} + C;\ \int e^{\alpha\Phi}\partial\Phi\cos.\Phi = \frac{e^{\alpha\Phi}(\alpha\cos.\Phi + \sin.\Phi)}{\alpha\alpha+1} + C,$$

fore

$$A = 2.\frac{1.3.5\ldots\ldots(2\lambda-1)}{2.4.6\ldots\ldots 2\lambda} = \frac{2}{2^{2\lambda-1}}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\ldots\frac{4\lambda-2}{\lambda};$$

$B = \frac{\lambda-1}{\lambda+1}A$; $C = \frac{\lambda-2}{\lambda+2}B$; $D = \frac{\lambda-3}{\lambda+3}C$; $E = \frac{\lambda-4}{\lambda+4}D$; etc.

Pro paribus vero potestatibus est

$\cos.\phi^0 = 1$

$\cos.\phi^2 = \frac{1}{2} + \frac{1}{2}\cos.2\phi$

$\cos.\phi^4 = \frac{3}{8} + \frac{4}{8}\cos.2\phi + \frac{1}{8}\cos.4\phi$

$\cos.\phi^6 = \frac{10}{32} + \frac{15}{32}\cos.2\phi + \frac{6}{32}\cos.4\phi + \frac{1}{32}\cos.6\phi$

$\cos.\phi^8 = \frac{35}{128} + \frac{56}{128}\cos.2\phi + \frac{28}{128}\cos.4\phi + \frac{8}{128}\cos.4\phi + \frac{1}{128}\cos.8\phi$

In genere autem si ponatur:

$$\cos.\phi^{2\lambda} = \mathfrak{A} + \mathfrak{B}\cos.2\phi + \mathfrak{C}\cos.4\phi + \mathfrak{D}\cos.6\phi$$
$$+ \mathfrak{E}\cos.8\phi + \text{etc. erit}$$

$$\mathfrak{A} = \frac{1.3.5\ldots\ldots(2\lambda-1)}{2.4.6\ldots\ldots 2\lambda} = \frac{1}{2^{2\lambda-1}}\cdot\frac{6}{2}\cdot\frac{10}{3}\cdot\frac{14}{4}\ldots\ldots\frac{4\lambda-2}{\lambda}$$

$\mathfrak{B} = \frac{2\lambda}{\lambda+1}\mathfrak{A}$; $\mathfrak{C} = \frac{\lambda-1}{\lambda+2}\mathfrak{B}$; $\mathfrak{D} = \frac{\lambda-2}{\lambda+3}\mathfrak{C}$; $\mathfrak{E} = \frac{\lambda-3}{\lambda+4}\mathfrak{D}$; etc.

Quodsi nunc isti valores substituantur, erit $\frac{1}{1+n\cos.\phi} =$

$$\begin{array}{lllllll}
1 & -n\cos.\phi & +\frac{1}{2}nn\cos.2\phi & -\frac{1}{4}n^3\cos.3\phi & +\frac{1}{8}n^4\cos.4\phi & -\frac{1}{16}n^5\cos.5\phi & +\frac{1}{32}n^6\cos.6\phi \\
+\frac{1}{2}nn & -\frac{3}{4}n^3 & +\frac{4}{8}n^4 & -\frac{5}{16}n^5 & +\frac{6}{32}n^6 & -\frac{7}{64}n^7 & +\frac{8}{128}n^8 \\
+\frac{3}{8}n^4 & -\frac{10}{16}n^5 & +\frac{15}{32}n^6 & -\frac{21}{64}n^7 & +\frac{28}{128}n^8 & -\frac{36}{256}n^9 & \\
+\frac{10}{32}n^6 & -\frac{35}{64}n^7 & +\frac{56}{128}n^8 & -\frac{84}{256}n^9 & & & \\
+\frac{35}{128}n^8 & & & & & &
\end{array}$$

unde patet, si ponatur

$$\frac{1}{1+n\cos.\phi} = A - B\cos.\phi + C\cos.2\phi - D\cos.3\phi$$
$$+ E\cos.4\phi - \text{etc.}$$

est $A = 1 + \frac{1}{2} nn + \frac{3}{8} n^4 + \frac{10}{32} n^6 + \text{etc.}$ seu

$$A = 1 + \frac{1}{2} n^2 + \frac{1.3}{2.4} n^4 + \frac{1.3.5}{2.4.6} n^6 + \frac{1.3.5.7}{2.4.6.8} n^8 + \text{etc.}$$

sicque evidens est esse $A = \frac{1}{\sqrt{(1 - nn)}}$.

Simili modo est

$$B = n + \frac{3}{4} n^3 + \frac{10}{16} n^5 + \text{etc.} = \frac{2}{n} \left(\frac{1}{2} n^2 + \frac{1.3}{2.4} n^4 + \frac{1.3.5}{2.4.6} n^6 + \text{etc.}\right)$$

ideoque $B = \frac{2}{n} \left(\frac{1}{\sqrt{(1 - nn)}} - 1\right)$. Verum et hunc valorem et sequentes facilius hoc modo definire licet. Cum sit

$$\frac{1}{1 + n \cos. \Phi} = A - B \cos. \Phi + C \cos. 2\Phi - D \cos. 3\Phi + E \cos. 4\Phi - \text{etc.}$$

multiplicetur per $1 + n \cos. \Phi$, et quia

$\cos. \Phi \cos. \lambda \Phi = \frac{1}{2} \cos. (\lambda - 1) \Phi + \frac{1}{2} \cos. (\lambda + 1) \Phi$, fiet

$$\begin{array}{rllll} 1 = A & - B \cos. \Phi & + C \cos. 2\Phi & - D \cos. 3\Phi & + E \cos. 4\Phi - \text{etc.} \\ + An & - \frac{1}{2} Bn & + \frac{1}{2} Cn & - \frac{1}{2} Dn & \\ - \frac{1}{2} Bn & + \frac{1}{2} Cn & - \frac{1}{2} Dn & + \frac{1}{2} En & - \frac{1}{2} Fn \end{array}$$

unde quia A jam definivimus, reliqui coëfficientes ita determinantur:

$$B = \frac{2}{n} (A - 1); \qquad E = \frac{2D - Cn}{n}$$
$$C = \frac{2B - An}{n}; \qquad F = \frac{2E - Dn}{n}$$
$$D = \frac{2C - Bn}{n}; \qquad G = \frac{2F - En}{n}$$
etc.

His igitur coëfficientibus inventis, integrale facile assignatur; nam, cum sit $\int \partial \Phi \cos. \lambda \Phi = \frac{1}{\lambda} \sin. \lambda \Phi$, habebimus

$$\int \frac{\partial \Phi}{1 + n \cos. \Phi} = A \Phi - B \sin. \Phi + \frac{1}{2} C \sin. 2\Phi - \frac{1}{3} D \sin. 3\Phi + \frac{1}{4} E \sin. 4\Phi - \text{etc.}$$

quae series secundum sinus angulorum Φ, 2Φ, 3Φ, etc. progreditur, uti desiderabatur.

Corollarium 1.

273. Primo patet hanc resolutionem locum habere non posse, nisi n sit numerus unitate minor; si enim $n > 1$, singuli coëfficientes prodeunt imaginarii. Sin autem sit $n = 1$, ob $1 + \cos.\Phi = 2\cos.\tfrac{1}{2}\Phi^2$, erit integrale

$$\int \frac{\partial\Phi}{1+\cos.\Phi} = \int \frac{\tfrac{1}{2}\partial\Phi}{\cos.\tfrac{1}{2}\Phi^2} = \text{tang.}\ \tfrac{1}{2}\Phi.$$

Corollarium 2.

274. Cum sit $A = \frac{1}{\sqrt{(1-nn)}}$ et $B = \frac{2}{n}\left(\frac{1}{\sqrt{(1-nn)}} - 1\right)$, reliqui coëfficientes C, D, E, etc. seriem recurrentem constituunt, ita ut si bini contigui sint P et Q sequens futurus sit $\frac{2}{n}Q - P$. Hinc cum aequationis $zz = \frac{2}{n}z - 1$ radices sint $\frac{1 \pm \sqrt{(1-nn)}}{n}$, quilibet terminus in hac forma continetur

$$\alpha\left(\frac{1+\sqrt{(1-nn)}}{n}\right)^\lambda + \beta\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^\lambda.$$

Corollarium 3.

275. Quia autem in nostra lege non A sed 2A sumitur: posito $\lambda = 0$, prodire debet 2A ideoque $\alpha + \beta = \frac{2}{\sqrt{(1-nn)}}$, deinde facto $\lambda = 1$, fieri debet

$$\frac{\alpha+\beta}{n} + \frac{(\alpha-\beta)\sqrt{(1-nn)}}{n} = \frac{2-2(1-nn)}{n\sqrt{(1-nn)}},$$

unde $\alpha - \beta = -\frac{2}{\sqrt{(1-nn)}}$. Ergo $\alpha = 0$ et $\beta = \frac{2}{\sqrt{(1-nn)}}$, sicque quilibet terminus praeter A erit =

$$\frac{2}{\sqrt{(1-nn)}}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^\lambda.$$

Corollarium 4.

276. Coëfficientes ergo evoluti ita se habebunt:

$$A = \frac{1}{\sqrt{(1-nn)}}$$

$$B = \frac{2 - 2\sqrt{(1-nn)}}{n\sqrt{(1-nn)}}$$

$$C = \frac{4 - 2nn - 4\sqrt{(1-nn)}}{nn\sqrt{(1-nn)}}$$

$$D = \frac{8 - 6nn - 2(4 - nn)\sqrt{(1-nn)}}{n^3\sqrt{(1-nn)}}$$

$$E = \frac{16 - 16nn + 2n^4 - 2(8 - 4nn)\sqrt{(1-nn)}}{n^4\sqrt{(1-nn)}}$$

$$F = \frac{32 - 40nn + 10n^4 - 2(16 - 12nn + n^4)\sqrt{(1-nn)}}{n^5\sqrt{(1-nn)}}$$

$$G = \frac{64 - 96nn + 36n^4 - 2n^6 - 2(32 - 23nn + 6n^4)\sqrt{(1-nn)}}{n^6\sqrt{(1-nn)}}$$

Corollarium 5.

277. Quia $n < 1$, hi coëfficientes plerumque facilius determinantur per series primum inventas, scilicet:

$$A = 1 + \tfrac{1}{2}n^2 + \tfrac{1.3}{2.4}n^4 + \tfrac{1.3.5}{2.4.6}n^6 + \tfrac{1.3.5.7}{2.4.6.8}n^8 + \text{etc.}$$

$$B = n\left(1 + \tfrac{3}{4}n^2 + \tfrac{3.5}{4.6}n^4 + \tfrac{3.5.7}{4.6.8}n^6 + \tfrac{3.5.7.9}{4.6.8.10}n^8 + \text{etc.}\right)$$

$$C = \tfrac{1}{2}n^2\left(1 + \tfrac{3.4}{2.6}n^2 + \tfrac{3.4.5.6}{2.6.4.8}n^4 + \tfrac{3.4.5.6.7.8}{2.6.4.8.6.10}n^6 + \text{etc.}\right)$$

$$D = \tfrac{1}{4}n^3\left(1 + \tfrac{4.5}{2.8}n^2 + \tfrac{4.5.6.7}{2.8.4.10}n^4 + \tfrac{4.5.6.7.8.9}{2.8.4.10.6.12}n^6 + \text{etc.}\right)$$

$$E = \tfrac{1}{8}n^4\left(1 + \tfrac{5.6}{2.10}n^2 + \tfrac{5.6.7.8}{2.10.4.12}n^4 + \tfrac{5.6.7.8.9.10}{2.10.4.12.6.14}n^6 + \text{etc.}\right)$$

$$F = \tfrac{1}{16}n^6\left(1 + \tfrac{6.7}{2.12}n^2 + \tfrac{6.7.8.9}{2.12.4.14}n^4 + \tfrac{6.7.8.9.10.11}{2.12.4.14.6.16}n^6 + \text{etc.}\right)$$

etc.

Scholion.

278. Cum ex his valoribus sit

$$\int \frac{\partial \Phi}{1 + n\cos.\Phi} = A\Phi - B\sin.\Phi + \tfrac{1}{2}C\sin.2\Phi - \tfrac{1}{3}D\sin.3\Phi + \tfrac{1}{4}E\sin.4\Phi - \text{etc.}$$

in hac serie terminus primus $A\Phi$ imprimis est notandus, quod crescente angulo Φ continuo crescat, idque in infinitum usque, dum reliqui termini modo crescent modo descrescent: neque tamen certum limitem excedunt; nam sin. $\lambda\Phi$ neque supra $+1$ crescere, neque infra -1 decrescere potest. Cum deinde hoc integrale supra inventum sit

$$\frac{1}{\sqrt{(1-nn)}} \text{ Ang. cos. } \frac{n+\cos.\Phi}{1+n\cos.\Phi}$$

series illa huic angulo aequatur. Quare si hic angulus vocetur ω, ut sit $\partial\omega = \frac{\partial\Phi\sqrt{(1-nn)}}{1+n\cos.\Phi}$, erit $\cos.\omega = \frac{n+\cos.\Phi}{1+n\cos.\Phi}$, hincque $n+\cos.\Phi - \cos.\omega - n\cos.\Phi\cos.\omega = 0$, ex quo est vicissim $\cos.\Phi = \frac{\cos.\omega - n}{1-n\cos.\omega}$, quae formula cum ex illa nascatur sumto n negativo, erit

$$\partial\Phi = \frac{\partial\omega\sqrt{(1-nn)}}{1-n\cos.\omega}, \text{ et}$$

$$\frac{\Phi}{\sqrt{(1-nn)}} = A\omega + B\sin.\omega + \tfrac{1}{2}C\sin.2\omega + \tfrac{1}{3}D\sin.3\omega + \tfrac{1}{4}E\sin.4\omega + \text{etc.}$$

Quia vero est

$$\frac{\omega}{\sqrt{(1-nn)}} = A\Phi - B\sin.\Phi + \tfrac{1}{2}C\sin.2\Phi - \tfrac{1}{3}D\sin.3\Phi + \tfrac{1}{4}E\sin.4\Phi - \text{etc.}$$

ob $\frac{1}{\sqrt{(1-nn)}} = A$, habebimus:

$$0 = B(\sin.\omega - \sin.\Phi) + \tfrac{1}{2}C(\sin.2\omega + \sin.2\Phi) + \tfrac{1}{3}D(\sin.3\omega - \sin.3\Phi) + \text{etc.}$$

cujusmodi relationes notasse juvabit.

Problema 36.

279. Integrale formulae $\partial\Phi(1+n\cos.\Phi)^\nu$ per seriem, secundum sinus angulorum multiplorum ipsius Φ progredientem, exprimere.

Solutio.

Cum sit

$$(1+n\cos.\Phi)^\nu = 1 + \frac{\nu}{1}n\cos.\Phi + \frac{\nu(\nu-1)}{1.2}n^2\cos.\Phi^2 + \frac{\nu(\nu-1)(\nu-2)}{1.2.3}n^3\cos.\Phi^3 + \text{etc.}$$

si ponamus

$$(1 + n \cos. \Phi)^\nu = A + B \cos. \Phi + C \cos. 2\Phi + D \cos 3\Phi + E \cos. 4\Phi + \text{etc.}$$

erit per formulas supra indicatas:

$$A = 1 + \frac{\nu(\nu-1)}{1.2} \cdot \tfrac{1}{2} n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1.2.3.4} \cdot \frac{1.3}{2.4} n^4 + \frac{\nu(\nu-1)\ldots\ldots(\nu-6)}{1.2\ldots\ldots 6} \cdot \frac{1.3.5}{2.4.6} n^6 + \text{etc.}$$

$$B = 2n \left[\frac{\nu}{2} + \frac{\nu(\nu-1)(\nu-2)}{1.\;2.\;3} \cdot \frac{1.3}{2.4} n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{1.\;2.\;3.\;4.\;5} \cdot \frac{1.3.5}{2.4.6} n^4 + \text{etc.}\right]$$

quae series ita clarius exhibentur:

$$A = 1 + \frac{\nu(\nu-1)}{2.\;2.} n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2.\;2.\;4.\;4} n^4 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2.\;2.\;4.\;4.\;6.\;6} n^6 + \text{etc.}$$

$$\tfrac{1}{2} B = \frac{\nu}{2} + \frac{\nu(\nu-1)(\nu-2)}{2.\;2.\;4} n^3 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{2.\;2.\;4.\;4.\;6.} n^5 + \text{etc.}$$

Inventis autem his binis coëfficientibus A et B, reliqui ex his sequenti modo commodius determinari poterunt. Cum sit

$$\nu l(1 + n \cos. \Phi) = l[A + B \cos. \Phi + C \cos. 2\Phi + D \cos. 3\Phi + E \cos. 4\Phi + \text{etc.}]$$

sumantur differentialia, ac per $-\partial\Phi$ dividendo prodit

$$\frac{\nu n \sin. \Phi}{1 + n \cos. \Phi} = \frac{B \sin. \Phi + 2C \sin. 2\Phi + 3D \sin. 3\Phi + 4E \sin. 4\Phi + \text{etc.}}{A + B \cos. \Phi + C \cos. 2\Phi + D \cos. 3\Phi + E \cos. 4\Phi + \text{etc.}}$$

Jam per crucem multiplicando,

ob $\sin. \lambda\Phi \cos. \Phi = \tfrac{1}{2} \sin. (\lambda + 1)\Phi + \tfrac{1}{2} \sin. (\lambda - 1)\Phi$ et

$\sin. \Phi \cos. \lambda\Phi = \tfrac{1}{2} \sin. (\lambda + 1)\Phi - \tfrac{1}{2} \sin. (\lambda - 1)\Phi$,

pervenietur ad hanc aequationem:

$$\begin{array}{llllll}
0 = & B \sin. \Phi & + 2C \sin. 2\Phi & + 3D \sin. 3\Phi & + 4E \sin. 4\Phi & + 5F \sin. 5\Phi + \text{etc.} \\
 & & + \frac{1}{2} Bn & + \frac{2}{2} Cn & + \frac{3}{2} Dn & + \frac{4}{2} En \\
 & + \frac{2}{2} Cn & + \frac{3}{2} Dn & + \frac{4}{2} En & + \frac{5}{2} Fn & + \frac{6}{2} Gn \\
 & - \nu An & - \frac{\nu}{2} Bn & - \frac{\nu}{2} Cn & - \frac{\nu}{2} Dn & - \frac{\nu}{2} En \\
 & + \frac{\nu}{2} Cn & + \frac{\nu}{2} Dn & + \frac{\nu}{2} En & + \frac{\nu}{2} Fn & + \frac{\nu}{2} Gn
\end{array}$$

$$C = \frac{\nu(\nu-1)}{1.\ 2} \cdot \frac{n^2}{2}\left(1 + \frac{(\nu-2)(\nu-3)}{2.\ 6} n^2 + \frac{(\nu-4)(\nu-5)}{4.\ 8} P n^2 \right.$$
$$\left. + \frac{(\nu-6)(\nu-7)}{6.\ 10} P n^2 + \text{etc.}\right)$$
$$D = \frac{\nu(\nu-1)(\nu-2)}{1.\ 2.\ 3} \cdot \frac{n^3}{4}\left(1 + \frac{(\nu-3)(\nu-4)}{2.\ 8} n^2 + \frac{(\nu-5)(\nu-6)}{4.\ 10} P n^2 \right.$$
$$\left. + \frac{(\nu-7)(\nu-8)}{6.\ 12} P n^2 + \text{etc.}\right)$$
$$E = \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{1.\ 2.\ 3.\ 4} \cdot \frac{n^4}{8}\left(1 + \frac{(\nu-4)(\nu-5)}{2.\ 10} n^2 + \frac{(\nu-6)(\nu-7)}{4.\ 12} P n^2 \right.$$
$$\left. + \frac{(\nu-8)(\nu-9)}{6.\ 14} P n^2 + \text{etc.}\right)$$
$$F = \frac{\nu \ldots\ldots (\nu-4)}{1 \ldots\ldots 5} \cdot \frac{n^5}{16}\left(1 + \frac{(\nu-5)(\nu-6)}{2.\ 12} n^2 + \frac{(\nu-7)(\nu-8)}{4.\ 14} P n^2 \right.$$
$$\left. + \frac{(\nu-9)(\nu-10)}{6.\ 16} P n^2 + \text{etc.}\right)$$

etc.

ubi in qualibet serie littera P terminum praecedentem integrum denotat. Atque ope serierum istarum coëfficientes plerumque facilius inveniuntur, quam ex lege ante tradita, qua quisque ex binis praecedentibus determinatur. Quin haec lex defectu laborat, quod si ν fuerit numerus integer negativus praeter -1; quidam coëfficientes plane non definiantur, quos ergo ex his seriebus desumi oportet. Ita si fuerit

$$\nu = -2, \text{ erit } B = \nu A n = -2 A n, \text{ et}$$
$$C = \frac{3}{1} \cdot \frac{n^2}{2}\left(1 + \frac{4.5}{2.6} n^2 + \frac{4.5.6.7}{2.6.4.8} n^4 + \frac{4.5.6.7.8.9}{2.6.4.8.6.10} n^6 + \text{etc.}\right)$$

si sit $\nu = -3$, erit $C = -Bn$, et

$$D = -\frac{4.5}{1.2} \cdot \frac{n^3}{4}\left(1 + \frac{6.7}{2.8} n^2 + \frac{6.7.8.9}{2.8.4.10} n^4 + \frac{6.7.8.9.10.11}{2.8.4.10.6.12} n^6 + \text{etc.}\right)$$

si sit $\nu = -4$, erit $D = -Cn$, et

$$E = \frac{5.6.7}{1.2.3} \cdot \frac{n^4}{8}\left(1 + \frac{8.9}{2.10} n^2 + \frac{8.9.10.11}{2.10.4.12} n^4 + \frac{8.9.10.11.12.13}{2.10.4.12.6.14} n^6 + \text{etc.}\right)$$

si sit $\nu = -5$, erit $E = -Dn$, et

$$F = -\frac{6.7.8.9}{1.2.3.4} \cdot \frac{n^5}{16}\left(1 + \frac{10.11}{2.12} n^2 + \frac{10.11.12.13}{2.12.4.14} n^4 \right.$$
$$\left. + \frac{10.11.12.13.14.15}{2.12.4.14.6.16} n^6 + \text{etc.}\right)$$

et ita de reliquis.

Exemplum 1.

285. *Formulae* $\partial\Phi(1+n\cos.\Phi)^\nu$ *integrale evolvere, si* ν *sit numerus integer positivus.*

Posito $(1+n\cos.\Phi)^\nu = A + B\cos.\Phi + C\cos.2\Phi$
$+ D\cos.3\Phi + E\cos.4\Phi + \text{etc.}$

pro singulis valoribus exponentis ν habebimus:

1.) si $\nu = 1$; $A = 1$; $B = n$; $C = 0$; etc.
2.) si $\nu = 2$; $A = 1 + \frac{1}{2}n^2$; $B = 2n$; $C = \frac{1}{2}nn$; $D = 0$; etc.
3.) si $\nu = 3$; $A = 1 + \frac{3}{2}n^2$; $B = 3n(1 + \frac{1}{4}n^2)$; $C = \frac{3}{2}n^2$;
$D = \frac{1}{4}n^3$; $E = 0$; etc.
4.) si $\nu = 4$; $A = 1 + \frac{6}{2}n^2 + \frac{3}{8}n^4$; $B = 4n(1 + \frac{3}{4}n^2)$;
$C = 3n^2(1 + \frac{1}{6}n^2)$; $D = n^3$; $E = \frac{1}{8}n^4$; $F = 0$; etc.

Hi autem casus nihil habent difficultatis. Ad usum sequentem tantum juvabit primum terminum absolutum A notasse:

si $\nu = 1$; $A = 1$;

si $\nu = 2$; $A = 1 + \frac{2.1}{2.2}n^2$;

si $\nu = 3$; $A = 1 + \frac{3.2}{2.2}n^2$;

si $\nu = 4$; $A = 1 + \frac{4.3}{2.2}n^2 + \frac{4.3.2.1}{2.2.4.4}n^4$;

si $\nu = 5$; $A = 1 + \frac{5.4}{2.2}n^2 + \frac{5.4.3.2}{2.2.4.4}n^4$;

si $\nu = 6$; $A = 1 + \frac{6.5}{2.2}n^2 + \frac{6.5.4.3}{2.2.4.4}n^4 + \frac{6.5.4.3.2.1}{2.2.4.4.6.6}n^6$;

si $\nu = 7$; $A = 1 + \frac{7.6}{2.2}n^2 + \frac{7.6.5.4}{2.2.4.4}n^4 + \frac{7.6.5.4.3.2}{2.2.4.4.6.6}n^6$;

etc.

Exemplum 2.

286. *Formulae* $\frac{\partial\Phi}{(1+n\cos.\Phi)^\mu}$ *integrale per seriem evolvere.*

Posito $\frac{1}{(1+n\cos.\phi)^{\mu}} = A + B\cos.\phi + C\cos.2\phi$
$+ D\cos.3\phi + E\cos.4\phi + \text{etc.}$

ex praecedentibus formulis ponendo $\nu = -\mu$ erit

$$A = 1 + \frac{\mu(\mu+1)}{2.\ 2}n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2.\ 2.\ 4.\ 4}n^4$$
$$+ \frac{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)}{2.\ 2.\ 4.\ 4.\ 6.\ 6}n^6 + \text{etc.}$$

$$B = -\mu n\Big(1 + \frac{(\mu+1)(\mu+2)}{2.\ 4}n^2 + \frac{(\mu+3)(\mu+4)}{4.\ 6}Pn^2$$
$$+ \frac{(\mu+5)(\mu+6)}{6.\ 8}Pn^2 + \text{etc.}\Big);$$

$$C = \frac{\mu(\mu+1)}{1.\ 2}\cdot\frac{n^2}{2}\Big(1 + \frac{(\mu+2)(\mu+3)}{2.\ 6}n^2 + \frac{(\mu+4)(\mu+5)}{4.\ 8}Pn^2$$
$$+ \frac{(\mu+6)(\mu+7)}{6.\ 10}Pn^2 + \text{etc.}\Big)$$

$$D = -\frac{\mu(\mu+1)(\mu+2)}{1.\ 2.\ 3}\cdot\frac{n^3}{4}\Big(1 + \frac{(\mu+3)(\mu+4)}{2.\ 8}n^2 + \frac{(\mu+5)(\mu+6)}{4.\ 10}Pn^2$$
$$+ \frac{(\mu+7)(\mu+8)}{6.\ 12}Pn^2 + \text{etc.}\Big);$$

etc.

ubi ut ante in quaque serie P terminum praecedentem denotat. Hi autem coëfficientes ita a se invicem pendent, ut sit

$$B = \frac{-2(\mu-2)}{n}\int An\partial n - 2An \text{ et}$$

$$C = \frac{2B + 2\mu An}{(\mu-2)n}; \qquad D = \frac{4C + (\mu+1)Bn}{(\mu-3)n};$$
$$E = \frac{6D + (\mu+2)Cn}{(\mu-4)n}; \qquad F = \frac{8E + (\mu+3)Dn}{(\mu-5)n};$$
$$G = \frac{10F + (\mu+4)En}{(\mu-6)n}; \qquad H = \frac{12G + (\mu+5)Fn}{(\mu-7)n};$$

etc.

Ubi incommodo, quando μ est numerus integer, supra jam remedium est allatum. Hic igitur praecipue investigamus quomodo coëfficientes cujusque casus ex casu praecedente determinari queant, quod ita fieri poterit. Cum sit

$$\frac{1}{(1+n\cos.\phi)^{\mu}} = A + B\cos.\phi + C\cos.2\phi$$
$$+ D\cos.3\phi + \text{etc.}$$

ponatur

$$\frac{1}{(1+n\cos.\phi)^{\mu+1}} = A' + B'\cos.\phi + C'\cos.2\phi + D'\cos.3\phi + \text{etc.}$$

haec igitur series per $1+n\cos.\phi$ multiplicata in illam abire debet, est autem productum

$$\begin{array}{llll} A' & +B'\cos.\phi & +C'\cos.2\phi & +D'\cos.3\phi + \text{etc.} \\ & +A'n & +\tfrac{1}{2}B'n & +\tfrac{1}{2}C'n \\ +\tfrac{1}{2}B'n & +\tfrac{1}{2}C'n & +\tfrac{1}{2}D'n & +\tfrac{1}{2}E'n \end{array}$$

unde colligimus

$$B' = \frac{2(A-A')}{n}; \qquad C' = \frac{2(B-B')-2A'n}{n};$$
$$D' = \frac{2(C-C')-B'n}{n}; \qquad E' = \frac{2(D-D')-C'n}{n};$$

dummodo ergo coëfficiens A' constaret, sequentes B', C', D' etc. haberemus. Videamus igitur quomodo A' ex A determinari possit: quia est

$$A = 1 + \frac{\mu(\mu+1)}{2\cdot 2}n^2 + \frac{\mu(\mu+1)(\mu+2)(\mu+3)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.}$$
$$A' = 1 + \frac{(\mu+1)(\mu+2)}{2\cdot 2}n^2 + \frac{(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2\cdot 2\cdot 4\cdot 4}n^4 + \text{etc.}$$

tractetur n ut variabilis, ac prior series per n^μ multiplicata differentietur, ut prodeat

$$\frac{\partial A n^\mu}{\partial n} = \mu n^{\mu-1} + \frac{\mu(\mu+1)(\mu+2)n^{\mu+1}}{2\cdot 2} + \frac{\mu(\mu+1)(\mu+2)(\mu+3)(\mu+4)}{2\cdot 2\cdot 4\cdot 4}n^{\mu+3} + \text{etc.}$$

quae series manifesto est $=\mu n^{\mu-1}A'$; quocirca A' ita per A determinatur, ut sit

$$A' = \frac{\partial . A n^\mu}{\partial . n^\mu} = A + \frac{n\partial A}{\mu\partial n}.$$

Cum igitur pro casu $\mu = 1$ invenerimus

$$A = \frac{1}{\sqrt{(1-nn)}}; \text{ ob } \frac{\partial A}{\partial n} = \frac{n}{(1-nn)^{\frac{3}{2}}}, \text{ erit}$$

$$A' = \frac{1}{\sqrt{(1-nn)}} + \frac{nn}{(1-nn)^{\frac{3}{2}}} = \frac{1}{(1-nn)^{\frac{3}{2}}}.$$

Hic jam est valor ipsius A pro $\mu = 2$, unde ob

$$\frac{\partial A}{\partial n} = \frac{3n}{(1-nn)^{\frac{5}{2}}}, \text{ fiet pro } \mu = 3,$$

$$A = \frac{1}{(1-nn)^{\frac{3}{2}}} + \frac{3nn}{2(1-nn)^{\frac{5}{2}}} = \frac{1+\frac{1}{2}nn}{(1-nn)^{\frac{5}{2}}}.$$

Hoc modo si ulterius progrediamur, reperiemus:

$$\text{si } \mu = 1; \; A = \frac{1}{\sqrt{(1-nn)}};$$

$$\text{si } \mu = 2; \; A = \frac{1}{(1-nn)\sqrt{(1-nn)}};$$

$$\text{si } \mu = 3; \; A = \frac{1+\frac{1}{2}nn}{(1-nn)^2\sqrt{(1-nn)}};$$

$$\text{si } \mu = 4; \; A = \frac{1+\frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}};$$

$$\text{si } \mu = 5; \; A = \frac{1+3nn+\frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}.$$

Corollarium 1.

287. Eodem modo etiam reliqui coëfficientes B', C' etc. ex analogis B, C etc. definientur, eruntque omnes istae relationes inter se similes, scilicet uti est

$$A' = \frac{\partial . A n^\mu}{\partial . n^\mu} = A + \frac{n\partial A}{\mu\partial n}, \text{ ita erit}$$

$$B' = \frac{\partial . B n^\mu}{\partial . n^\mu} = B + \frac{n\partial B}{\mu\partial n};$$

$$C' = \frac{\partial . C n^\mu}{\partial . n^\mu} = C + \frac{n\partial C}{\mu\partial n};$$

etc.

Corollarium 2.

288. At ante invenimus $B' = \frac{2(A - A')}{n}$, unde fiet

$$B' = -\frac{2\partial A}{\mu\partial n} = B + \frac{n\partial B}{\mu\partial n},$$ hincque

$$\mu B\partial n + n\partial B + 2\partial A = 0:$$

multiplicetur per $n^{\mu-1}$ ut sit

$$\partial . B n^\mu + 2 n^{\mu-1}\partial A = 0,$$

unde integrando

$$B n^\mu = -2\int n^{\mu-1}\partial A = -2 n^{\mu-1} A + 2(\mu-1)\int A n^{\mu-2}\partial n:$$

ideoque

$$B = -\frac{2A}{n} + \frac{2(\mu-1)}{n^\mu}\int A n^{\mu-2}\partial n.$$

At ante habueramus

$$B = -2An - \frac{2(\mu-2)}{n}\int A n\partial n.$$

Corollarium 3.

289. His valoribus aequatis, obtinetur aequatio inter A et n, qua quantitas A per n determinatur, erit enim

$$n^{-\mu}\int n^{\mu-1}\partial A = An + \frac{(\mu-2)}{n}\int A n\partial n:$$

unde per duplicem differentiationem prodit,

$$(1-nn)\partial\partial A + \frac{\partial n\partial A}{n} - 2(\mu+1)n\partial n\partial A - \mu(\mu+1)A\partial n^2 = 0.$$

Scholion 1.

290. Si hos valores ipsius A cum superioribus, ubi μ erat numerus integer negativus inter se comparemus, eximiam convenientiam deprehendemus.

Pro superioribus.	Pro his formulis.
si $\nu=0$; $A=1$	si $\mu=1$; $A=\frac{1}{\sqrt{(1-nn)}}$
$\nu=1$; $A=1$	$\mu=2$; $A=\frac{1}{(1-nn)\sqrt{(1-nn)}}$
$\nu=2$; $A=1+\frac{1}{2}n^2$	$\mu=3$; $A=\frac{1+\frac{1}{2}nn}{(1-nn)^2\sqrt{(1-nn)}}$
$\nu=3$; $A=1+\frac{3}{2}n^2$	$\mu=4$; $A=\frac{1+\frac{3}{2}nn}{(1-nn)^3\sqrt{(1-nn)}}$
$\nu=4$; $A=1+3n^2+\frac{3}{8}n^4$	$\mu=5$; $A=\frac{1+3nn+\frac{3}{8}n^4}{(1-nn)^4\sqrt{(1-nn)}}$
	etc.

unde concludimus, si fuerit

$$(1+n\cos.\Phi)^{\nu}=A+B\cos.\Phi+C\cos.2\Phi+\text{etc.}$$
$$(1+n\cos.\Phi)^{-\nu-1}=\mathfrak{A}+\mathfrak{B}\cos.\Phi+\mathfrak{C}\cos.2\Phi+\text{etc.}$$

fore $\mathfrak{A}=\frac{A}{(1-nn)^{\nu}\sqrt{(1-nn)}}$.

Quare cum pro casibus, quibus ν est numerus integer positivus, valor ipsius A facile definiatur, etiam pro casibus, quibus est negativus, inde expedite assignabitur.

Scholion 2.

291. Cum pro casu $\mu=1$, supra valores singularum litterarum A, B, C, D etc. sint inventi, scilicet posito brevitatis gratia $\frac{1-\sqrt{(1-nn)}}{n}=m$,

$$A = \frac{1}{\sqrt{(1-nn)}};\quad B = \frac{2m}{\sqrt{(1-nn)}};\quad C = \frac{2mm}{\sqrt{(1-nn)}};\quad D = \frac{2m^3}{\sqrt{(1-nn)}};$$

et in genere pro termino quocunque $N = \frac{2m^\lambda}{\sqrt{(1-nn)}}$: si pro simili termino casu $\mu = 2$, scribamus N', erit $N' = \frac{\partial . N n}{n \partial n}$. Nunc autem est $\frac{\partial . N n}{\partial n} = \frac{2m^\lambda}{(1-nn)^{\frac{3}{2}}} + \frac{2\lambda n m^{\lambda-1} \partial m}{\partial n \sqrt{(1-nn)}}$: tum vero $\frac{\partial m}{\partial n} = \frac{m}{n\sqrt{(1-nn)}}$, unde colligimus

$$N' = \frac{2m^\lambda}{(1-nn)^{\frac{3}{2}}} + \frac{2\lambda m^\lambda}{1-nn} = \frac{2m^\lambda[1+\lambda\sqrt{(1-nn)}]}{(1-nn)\sqrt{(1-nn)}}.$$

Quare si statuamus:

$$\frac{1}{(1+n\cos.\Phi)^2} = A + B\cos.\Phi + C\cos.2\Phi + D\cos.3\Phi + E\cos.4\Phi + \text{etc.}$$

erit

$$A = \frac{1}{(1-nn)^{\frac{3}{2}}};\quad B = \frac{2m[1+\sqrt{(1-nn)}]}{(1-nn)^{\frac{3}{2}}};\quad C = \frac{2m^2[1+2\sqrt{(1-nn)}]}{(1-nn)^{\frac{3}{2}}};$$

$$D = \frac{2m^3[1+3\sqrt{(1-nn)}]}{(1-nn)^{\frac{3}{2}}};\ \text{etc.}$$

Verum si exponens μ fuerit numerus fractus, coefficientes A, B, C, D, E, etc. haud aliter, ac per series supra datas definiri posse videntur. Primus autem A modo peculiari vero proxime assignari potest, quemadmodum in problemate sequente docemus.

Problema 34.

292. Pro evolutione formulae $(1+n\cos.\Phi)^\nu$ in hujusmodi seriem $A + B\cos.\Phi + C\cos.2\Phi + D\cos.3\Phi + E\cos.4\Phi + \text{etc.}$ terminum absolutum A vero proxime definire.

Solutio.

Cum necessario sit $n < 1$, series quidem supra inventa pro A convergit, verum si n parum ab unitate deficiat, permultos terminos actu evolvi oportet, antequam valor ipsius A satis exacte prodeat, praecipue si ν fuerit numerus mediocriter magnus tam positivus quam negativus. Quoniam tamen posita evolutione hujus formulae $(1+n\cos.\phi)^{-\nu-1} = \mathfrak{A} + \mathfrak{B}\cos.\phi + \mathfrak{C}\cos.2\phi + \text{etc.}$

a termino $\mathfrak{A}$ ille A ita pendet, ut sit $A = (1-nn)^{\nu+\frac{1}{2}}\mathfrak{A}$; pro hoc termino A inveniendo duplicem habemus seriem

$$A = 1 + \frac{\nu(\nu-1)}{2.2}n^2 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2.2.4.4}n^4 + \frac{\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)(\nu-5)}{2.2.4.4.6.6}n^6 + \text{etc.}$$

$$A = (1-nn)^{\nu+\frac{1}{2}}\left(1 + \frac{(\nu+1)(\nu+2)}{2.2}n^2 + \frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)}{2.2.4.4}n^4 + \frac{(\nu+1)(\nu+2)(\nu+3)(\nu+4)(\nu+5)(\nu+6)}{2.2.4.4.6.6}n^6 + \text{etc.}\right)$$

quovis casu ea usurpari potest, quae magis convergit. Verum tamen quia reliqui coëfficientes B, C, D, E, etc. tandem convergere debent, hinc alia via ad valorem ipsius A appropinquandi patet. Quoniam enim hi coëfficientes alternatim per pares et impares potestates ipsius n definiuntur, sumto angulo quocunque α erit

$$(1+n\cos.\alpha)^\nu = A + B\cos.\alpha + C\cos.2\alpha + D\cos.3\alpha + E\cos.4\alpha + \text{etc. et}$$

$$(1-n\cos.\alpha)^\nu = A - B\cos.\alpha + C\cos.2\alpha - D\cos.3\alpha + E\cos.4\alpha - \text{etc.}$$

His igitur additis prodit

$$\tfrac{1}{2}(1+n\cos.\alpha)^\nu + \tfrac{1}{2}(1-n\cos.\alpha)^\nu = A + C\cos.2\alpha + E\cos.4\alpha + G\cos.6\alpha + \text{etc.}$$

ubi si pro α scribamus $90^\circ - \alpha$, erit

$$\tfrac{1}{2}(1+n\sin.\alpha)^\nu + \tfrac{1}{2}(1-n\sin.\alpha)^\nu = A - C\cos.2\alpha + E\cos.4\alpha - G\cos.6\alpha + \text{etc.}$$

unde his additis, semissis terminorum denuo tollitur. Formemus plures hujusmodi expressiones, ac ponamus brevitatis gratia:

$$\begin{array}{l|l} \cos. 4\alpha + \cos. 4\beta + \cos. 4\gamma = 0 & \cos. 16\alpha + \cos. 16\beta + \cos. 16\gamma = 0 \\ \cos. 8\alpha + \cos. 8\beta + \cos. 8\gamma = 0 & \cos. 20\alpha + \cos. 20\beta + \cos. 20\gamma = 0 \\ \cos. 12\alpha + \cos. 12\beta + \cos. 12\gamma = 0 & \cos. 24\alpha + \cos. 24\beta + \cos. 24\gamma = 3 \end{array}$$

unde colligitur

$$A = \tfrac{1}{3}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C}) + (24) - (48) + \text{etc.}$$

IV. Si haec determinatio non satis exacta videatur, addantur quatuor ejusmodi expressiones $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, sitque

$$4\alpha = \tfrac{\pi}{8};\ 4\beta = \tfrac{3\pi}{8};\ 4\gamma = \tfrac{5\pi}{8};\ 4\delta = \tfrac{7\pi}{8};$$

ac reperietur

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D} = 4A - 4(32) + 4(64) - \text{etc.}$$

ergo multo propius

$$A = \tfrac{1}{4}(\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D}).$$

Corollarium 1.

293. Ex invento autem valore A sequens B satis expedite reperitur, cum sit

$$B = \frac{2(\nu+2)}{n}\int A n \partial n - 2An.$$

Quatenus ergo in A ingreditur membrum $(1 \pm n\cos.\alpha)^\nu$, vel $(1 + nf)^\nu$, dum f omnes illos sinus et cosinus complectitur, inde pro B oritur

$$\frac{2(\nu+2)}{n}\int n\partial n(1+nf)^\nu - 2n(1+nf)^\nu = \frac{2 - 2(1-\nu nf)(1+nf)^\nu}{(\nu+1)nff}.$$

Corollarium 2.

294. Cognitis autem coëfficientibus A et B, quemadmodum sequentes omnes ex illis derivari possint, supra ostendimus. Iis vero inventis integratio formulae $\partial\phi(1 + n\cos.\phi)^\nu$ per se est manifesta.

Problema 35.

295. Integrale formulae $\partial\Phi\, l(1+n\cos.\Phi)$ per seriem secundum sinus angulorum Φ, 2Φ, 3Φ, etc. progredientem evolvere.

Solutio.

Cum sit

$$l(1+n\cos.\Phi) = n\cos.\Phi - \tfrac{1}{2}n^2\cos.\Phi^2 + \tfrac{1}{3}n^3\cos.\Phi^3 - \tfrac{1}{4}n^4\cos.\Phi^4 + \text{etc.}$$

erit his potestatibus ad simplices cosinus reductis

$$\begin{aligned} l(1+n\cos.\Phi) = {} & +n\cos.\Phi - \tfrac{1}{2}\cdot\tfrac{1}{2}n^2\cos.2\Phi + \tfrac{1}{3}\cdot\tfrac{1}{4}n^3\cos.3\Phi - \tfrac{1}{4}\cdot\tfrac{1}{8}n^4\cos.4\Phi \\ & -\tfrac{1}{2}\cdot\tfrac{1}{2}n^2 + \tfrac{1}{3}\cdot\tfrac{3}{4}n^3 - \tfrac{1}{4}\cdot\tfrac{4}{8}n^4 + \tfrac{1}{5}\cdot\tfrac{5}{16}n^5 \\ & -\tfrac{1}{4}\cdot\tfrac{3}{8}n^4 + \tfrac{1}{5}\cdot\tfrac{10}{16}n^5 - \tfrac{1}{6}\cdot\tfrac{15}{32}n^6 \\ & -\tfrac{1}{6}\cdot\tfrac{10}{32}n^6 + \tfrac{1}{7}\cdot\tfrac{35}{64}n^7 \\ & -\tfrac{1}{8}\cdot\tfrac{35}{128}n^8 \end{aligned}$$

Quare, si ponamus

$$l(1+n\cos.\Phi) = -A + B\cos.\Phi - C\cos.2\Phi + D\cos.3\Phi - \text{etc.}$$

erit

$$A = +\frac{1}{2}\cdot\frac{n^2}{2} + \frac{1.3}{2.4}\cdot\frac{n^4}{4} + \frac{1.3.5}{2.4.6}\cdot\frac{n^6}{6} + \frac{1.3.5.7}{2.4.6.8}\cdot\frac{n^8}{8} + \text{etc.}$$

considerato ergo numero n ut variabili, erit

$$\frac{n\partial A}{\partial n} = \tfrac{1}{2}nn + \frac{1.3}{2.4}n^4 + \frac{1.3.5}{2.4.6}n^6 + \text{etc.} = \frac{1}{\sqrt{(1-nn)}} - 1.$$

Hinc $\partial A = \frac{\partial n}{n\sqrt{(1-nn)}} - \frac{\partial n}{n}$, unde integratio praebet

$$A = l\frac{1-\sqrt{(1-nn)}}{n} - l\,n + C = l\frac{2-2\sqrt{(1-nn)}}{nn};$$

hoc enim modo evanescente n, fit $A = l\,1 = 0$. Tum vero erit

$$\tfrac{1}{2}B = \tfrac{1}{2}\cdot n + \frac{1.3}{2.4}\cdot\frac{n^3}{3} + \frac{1.3.5}{2.4.6}\cdot\frac{n^5}{5} + \text{etc.}$$

unde differentiatio praebet

$$\frac{nn\partial B}{2\partial n} = \tfrac{1}{2}nn + \frac{1.3}{2.4}n^4 + \frac{1.3.5}{2.4.6}n^6 + \text{etc.} = \frac{1}{\sqrt{(1-nn)}} - 1:$$

ergo $\frac{1}{2}\partial B = \frac{\partial n}{nn\sqrt{(1-nn)}} - \frac{\partial n}{nn}$, et integrando

$$\tfrac{1}{2}B = \frac{-\sqrt{(1-nn)}}{n} + \frac{1}{n} + C = \frac{1-\sqrt{(1-nn)}}{n}$$

integrali ita determinato, ut evanescat posito $n = 0$.

Quocirca pro binis primis terminis habemus:

$$A = l\frac{2-2\sqrt{(1-nn)}}{nn} \quad \text{et} \quad B = \frac{2-2\sqrt{(1-nn)}}{n},$$

ut sit $A = l\frac{B}{n}$. At pro reliquis differentiemus aequationem assumtam

$$\frac{-n\partial\phi \sin.\phi}{1+n\cos.\phi} = -B\partial\phi \sin.\phi + 2C\partial\phi \sin.2\phi$$
$$- 3D\partial\phi \sin.3\phi + 4E\partial\phi \sin.4\phi - \text{etc.}$$

seu

$$0 = \frac{n\sin.\phi}{1+n\cos.\phi} - B\sin.\phi + 2C\sin.2\phi$$
$$- 3D\sin.3\phi + 4E\sin.4\phi - \text{etc.}$$

Quare per $2 + 2n\cos.\phi$ multiplicando prodit:

$$\begin{array}{l} 0 = 2n\sin.\phi - 2B\sin.\phi + 4C\sin.2\phi - 6D\sin.3\phi + 8E\sin.4\phi - \text{etc.} \\ \qquad\qquad\qquad\qquad\qquad\quad - Bn \qquad\quad + 2Cn \qquad\quad - 3Dn \\ \qquad\qquad\quad + 2Cn \qquad\quad - 3Dn \qquad\quad + 4En \qquad\quad - 5Fn \end{array}$$

unde colligimus:

$$C = \frac{B-n}{n};\quad D = \frac{4C-Bn}{3n};\quad E = \frac{6D-2Cn}{4n};\quad F = \frac{8E-3Dn}{5n};$$

Cum igitur sit $B = \frac{2-2\sqrt{(1-nn)}}{n}$, erit $C = \frac{2-nn-2\sqrt{(1-nn)}}{nn}$, seu $C = \left(\frac{1-\sqrt{(1-nn)}}{n}\right)^2$: tum vero

$$D = \tfrac{2}{3}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^3;\quad E = \tfrac{2}{4}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^4;\quad F = \tfrac{2}{5}\left(\frac{1-\sqrt{(1-nn)}}{n}\right)^5;\ \text{etc.}$$

Hinc si brevitatis gratia ponamus $\frac{1-\sqrt{(1-nn)}}{n} = m$, erit

$$l(1+n\cos.\phi) = -l\frac{2m}{n} + \tfrac{2}{1}m\cos.\phi - \tfrac{2}{2}m^2\cos.2\phi$$
$$+ \tfrac{2}{3}m^3\cos.3\phi - \tfrac{2}{4}m^4\cos.4\phi + \text{etc.}$$

ideoque integrale quaesitum:

$$\int \partial \Phi\, l(1+n\cos.\Phi) = \text{Const.} - \Phi\, l\tfrac{2m}{n} + \tfrac{2}{1} m \sin.\Phi - \tfrac{2}{4} m^2 \sin.2\Phi + \tfrac{2}{9} m^3 \sin.3\Phi - \tfrac{2}{16} m^4 \sin.4\Phi + \tfrac{2}{25} m^5 \sin.5\Phi - \text{etc.}$$

Corollarium 1.

296. Quodsi ergo ponamus $n = 1$, erit $m = 1$ et

$$l(1+\cos.\Phi) = -l2 + \tfrac{2}{1}\cos.\Phi - \tfrac{2}{2}\cos.2\Phi + \tfrac{2}{3}\cos.3\Phi - \tfrac{2}{4}\cos.4\Phi + \text{etc.}$$

et

$$l(1-\cos.\Phi) = -l2 - \tfrac{2}{1}\cos.\Phi - \tfrac{2}{2}\cos.2\Phi - \tfrac{2}{3}\cos.3\Phi - \tfrac{2}{4}\cos.4\Phi - \text{etc.}$$

Cum jam sit

$$1+\cos.\Phi = 2\cos.\tfrac{1}{2}\Phi^2 \quad \text{et} \quad 1-\cos.\Phi = 2\sin.\tfrac{1}{2}\Phi^2, \text{ erit}$$

$$l\cos.\tfrac{1}{2}\Phi = -l2 + \cos.\Phi - \tfrac{1}{2}\cos.2\Phi + \tfrac{1}{3}\cos.3\Phi - \tfrac{1}{4}\cos.4\Phi + \text{etc. et}$$

$$l\sin.\tfrac{1}{2}\Phi = -l2 - \cos.\Phi - \tfrac{1}{2}\cos.2\Phi - \tfrac{1}{3}\cos.3\Phi - \tfrac{1}{4}\cos.4\Phi - \text{etc.}$$

hinc

$$l\,\text{tang.}\tfrac{1}{2}\Phi = -2\cos.\Phi - \tfrac{2}{3}\cos.3\Phi - \tfrac{2}{5}\cos.5\Phi - \tfrac{2}{7}\cos.7\Phi - \text{etc.}$$

CAPUT VII.

METHODUS GENERALIS
INTEGRALIA QUAECUNQUE PROXIME INVENIENDI.

Problema 36.

297.

Formulae integralis cujuscunque $y = \int X \partial x$ valorem vero proxime indagare.

Solutio.

Cum omnis formula integralis per se sit indeterminata, ea semper ita determinari solet, ut si variabili x certus quidam valor, puta a, tribuatur, ipsum integrale $y = \int X \partial x$ datum valorem, puta b, obtineat. Integratione igitur hoc modo determinata, quaestio huc redit, si variabili x alius quicunque valor ab a diversus tribuatur, valor, quem tum integrale y sit habiturum, definiatur. Tribuamus ergo ipsi x primo valorem parum ab a discrepantem, puta $x = a + \alpha$, ut α sit quantitas valde parva: et quia functio X parum variatur, sive pro x scribatur a sive $a + \alpha$, eam tanquam constantem spectare licebit. Hinc ergo formulae differentialis $X \partial x$ integrale erit $Xx +$ Const. $= y$; sed quia posito $x = a$, fieri debet $y = b$, et valor ipsius X quasi manet immutatus, erit $Xa +$ Const. $= b$, ideoque Const. $= b - Xa$, unde consequimur $y = b + X(x - a)$. Quare si ipsi x valorem $a + \alpha$ tribuamus, habebimus valorem convenientem ipsius y, qui sit $= b + \beta$; ac jam simili modo ex hoc casu definire poterimus y, si ipsi x tribuatur alius valor parum superans $a+\alpha$: posito igitur $a+\alpha$ loco x, valor ipsius X inde ortus denuo pro constante haberi poterit, indeque fiet $y=b+\beta+X(x-a-\alpha)$. Hanc igitur operationem continuare licet quousque lubuerit, cujus ratio quo melius perspiciatur, rem ita repraesentemus:

si $x=a$ fiat $X=A$ et $y=b$
si $x=a'$ . . $X=A'$ . . $y=b'=b+A(a'-a)$
si $x=a''$ . . $X=A''$ . . $y=b''=b'+A'(a''-a')$
si $x=a'''$ . . $X=A'''$ . . $y=b'''=b''+A''(a'''-a'')$
etc.

ubi valores a, a', a'', a''', etc. secundum differentias valde parvas procedere ponuntur. Erit ergo $b'=b+A(a'-a)$, quippe in quam abit formula inventa $y=b+X(x-a)$: fit enim $X=A$, quia ponitur $x=a$, tum vero tribuitur ipsi x valor $=a'$, cui respondet $y=b'$: simili modo erit $b''=b'+A'(a''-a')$, sum $b'''=b''+A''(a'''-a'')$ etc. uti supra posuimus. Restituendo ergo valores praecedentes habebimus:

$$b'=b+A(a'-a)$$
$$b''=b+A(a'-a)+A'(a''-a')$$
$$b'''=b+A(a'-a)+A'(a''-a')+A''(a'''-a'')$$
$$b''''=b+A(a'-a)+A'(a''-a')+A''(a'''-a'')+A'''(a''''-a''')$$
etc.

unde, si x quantumvis excedet a, series a', a'', a''', etc. crescendo continuetur ad x, et ultimum aggregatum dabit valorem ipsius y.

Corollarium 1.

298. Si incrementa, quibus x augetur, aequalia statuantur scilicet $=\alpha$, ut sit $a'=a+\alpha$, $a''=a+2\alpha$, $a'''=a+3\alpha$, etc. quibus valoribus pro x substitutis functio X abeat in A', A'',

**

A'''', etc. atque ultimus illorum, puta $a + n\alpha$, sit $= x$, horum vero X, erit

$$y = b + \alpha\,(A + A' + A'' + A''' \ldots\ldots + X).$$

Corollarium 2.

299. Valor ergo integralis y per summationem seriei A, A', A'' .... X, cujus termini ex formula X formantur, ponendo loco x successive a, $a + \alpha$, $a + 2\alpha \ldots a + n\alpha$; eruitur. Summa enim illius seriei per differentiam α multiplicata et ad b adjecta, dabit valorem ipsius y, qui ipsi $x = a + n\alpha$ respondet.

Corollarium 3.

300. Quo minores statuuntur differentiae, secundum quas valor ipsius x increscat, eo accuratius hoc modo valor ipsius y definitur. Siquidem termini seriei A, A', A'', etc. inde etiam secundum parvas differentias progrediantur; nisi enim hoc eveniat, illa determinatio nimis erit incerta.

Corollarium 4.

301. Haec ergo approximatio ex doctrina serierum ita explicatur:

Ex indicibus a, a', a'', a''' . . . . . x formetur
series A, A', A'', A''' X

cujus ergo terminus generalis X ex formula differentiali $\partial y = X\partial x$ datur. Tum in hac serie sit terminus ultimum praecedens 'X, respondens indici 'x; hincque nova formetur series

$$A(a' - a);\ A'(a'' - a');\ A''(a''' - a'') \ldots\ldots {}'X(x - {}'x),$$

cujus summa si ponatur $= S$, erit integrale $y = \int X\partial x = b + S$, proxime.

Scholion 1.

302. Hoc modo integratio vulgo explicari solet, ut dicatur, esse summatio omnium valorum formulae differentialis $X\partial x$, si variabili x successive omnes valores a dato quodam a usque ad x tribuantur, qui secundum differentiam ∂x procedunt, hanc differentiam autem infinite parvam accipi oportere. Similis igitur haec ratio integrationem repraesentandi est illi, qua in Geometria lineae ut aggregata infinitorum punctorum concipi solent, quae idea, quemadmodum si rite explicetur, admitti potest, ita etiam illi integrationis explicatio tolerari potest, dummodo ad vera principia, uti hic fecimus, revocetur, ut omni cavillationi occurratur. Ex methodo igitur exposita utique patet, integrationem per summationem vero proxime obtineri posse, neque vero exacte expediri, nisi differentiae infinite parvae, hoc est nullae, statuantur. Atque ex hoc fonte tam nomen integrationis, quae etiam summatio vocari solet, quam signum integralis $\int$ est natum, quae, re bene explicata, omnino retineri possunt.

Scholion 2.

303. Si pro singulis intervallis, in quae saltum ab a ad x distinximus, quantitates A, A', A'', A''', etc. revera essent constantes, integrale $\int X\partial x$ accurate impetraremus. Eatenus ergo error inest, quatenus pro singulis illis intervallis istae quantitates non sunt constantes. Ac pro primo quidem intervallo, quo variabilis x a termino a ad a' procedit, A est valor ipsius X termino a conveniens, alteri autem termino a' respondet A'; unde quatenus non est $A' = A$, eatenus error irrepit: cum igitur in istius intervalli initio sit $X = A$, in fine autem $X = A'$, conveniret potius medium quoddam inter A et A' assumi, id quod in correctione hujus methodi mox tradenda observabitur. Interim hic notasse juvabit, pari jure pro quovis intervallo valorem tam finalem quam initialem capi posse, ubi simul hoc perspicitur, si altero modo in excessu pecce-

tur, altero plerumque in defectu errari. Ex quo hinc binas expressiones eruere licet, quarum altera valorem ipsius y nimis magnum, altera nimis parvum sit praebitura, ita ut illae quasi limites veri valoris ipsius y constituant. Quemadmodum ergo rem repraesentavimus §. 301. valor ipsius $y = \int X \partial x$ intra hos duos limites continebitur

$$b + A(a'-a) + A'(a''-a') + A''(a'''-a'') \ldots\ldots + {}'X(x-{}'x) =$$
$$b + A'(a'-a) + A''(a''-a') + A'''(a'''-a'') \ldots\ldots + X(x-{}'x)$$

quibus cognitis, ad veritatem propius accedere licet.

Scholion 3.

304. Jam notavimus intervalla illa, per quae x successive increscere assumimus, ideo valde parva statui debere, ut valores respondentes A, A′, A″, etc. parum a se invicem discrepent: atque hinc potissimum judicari oportet, utrum illa intervalla $a'-a$, $a''-a'$, $a'''-a''$, etc. inter se aequalia an inaequalia capi conveniat. Ubi enim valor ipsius X, mutando x, parum mutatur, ibi intervalla, per quae x procedit, tuto majora constitui possunt; ubi autem evenit, ut ipsi x levi mutatione inducta, functio X vehementer varietur, ibi intervalla minima accipi debent. Veluti si sit $X = \frac{1}{\sqrt{(1-xx)}}$, perspicuum est, ubi x proxime ad unitatem accedit, quantumvis parvum intervallum, per quod x augeatur, accipiatur, functionem X maximam mutationem pati posse, quia tandem sumto $x = 1$, ea adeo in infinitum excrescit. His igitur casibus ista approximatione pro eo saltem intervallo, in cujus altero termino X fit infinita, uti non licet; sed huic incommodo facile remedium affertur, dum formula ope idoneae substitutionis in aliam transformatur, vel dum pro hoc saltem intervallo peculiaris integratio instituitur. Veluti si proposita sit formula $\frac{x \partial x}{\sqrt{(1-x^3)}}$, pro intervallo ab $x = 1 - \omega$ ad $x = 1$, illa methodo integrale non reperitur: at posito $x = 1 - z$, quia termini ipsius z sunt 0 et ω, erit z

quantitas minima, unde formula erit $\frac{\partial z(1-z)}{\sqrt{(3z-3z^3+z^3)}} = \frac{\partial z}{\sqrt{3z}}$, cujus integrale $\frac{2\sqrt{z}}{\sqrt{3}}$ pro intervallo illo praebet partem integralis $\frac{2\sqrt{\omega}}{\sqrt{3}}$. Quod artificium in omnibus hujusmodi casibus adhiberi potest; ipsam autem methodum descriptam aliquot exemplis illustrari opus est.

Exemplum 1.

305. *Integrale $y = \int x^n \partial x$ ita sumtum, ut evanescat posito $x = 0$, proxime exhibere.*

Hic est $a = 0$ et $b = 0$, tum $X = x^n$, jam valores ipsius x a 0 crescant per communem differentiam α, ut sint

indices $0,\ \alpha,\ 2\alpha,\ 3\alpha,\ 4\alpha \ldots\ x$
series $0,\ \alpha^n,\ 2^n\alpha^n,\ 3^n\alpha^n,\ 4^n\alpha^n \ldots\ x^n$

et terminus ultimum praecedens est $(x-\alpha)^n$, quare integralis $y = \int x^n \partial x = \frac{1}{n+1} x^{n+1}$ limites sunt

$$\alpha[0 + \alpha^n + 2^n\alpha^n + 3^n\alpha^n + \ldots + (x-\alpha)^n] \text{ et}$$
$$\alpha(\alpha^n + 2^n\alpha^n + 3^n\alpha^n + \ldots + x^n)$$

qui eo erunt arctiores, quo minus intervallum α accipiatur. Ita si $\alpha = 1$ erunt limites:

$$0 + 1 + 2^n + 3^n + 4^n \ldots + (x-1)^n \text{ et}$$
$$1 + 2^n + 3^n + 4^n + \ldots + x^n,$$

si sumatur $\alpha = \frac{1}{2}$, erunt limites

$$\frac{1}{2^{n+1}}[0 + 1 + 2^n + 3^n + 4^n + \ldots + (2x-1)^n] \text{ et}$$
$$\frac{1}{2^{n+1}}[1 + 2^n + 3^n + 4^n + \ldots + (2x)^n];$$

ac si in genere sit $\alpha = \frac{1}{m}$, erunt limites:

$$\frac{1}{m^{n+1}}[0 + 1 + 2^n + 3^n + 4^n + \ldots + (mx-1)^n] \text{ et}$$
$$\frac{1}{m^{n+1}}[1 + 2^n + 3^n + 4^n + \ldots + (mx)^n],$$

quorum hic illum superat excessu $\frac{x^n}{m}$; unde patet si numerus m sumatur infinitus, utrumque limitem verum integralis $y = \frac{1}{n+1} x^{n+1}$ esse praebiturum valorem.

Corollarium 1.

306. Seriei ergo $1 + 2^n + 3^n + 4^n + \ldots + (mx)^n$ summa eo propius ad $\frac{1}{n+1}(mx)^{n+1}$ accedit, quo major capiatur numerus m; quare posito $mx = z$, hujus progressionis

$$1 + 2^n + 3^n + 4^n + \ldots\ldots + z^n,$$

summa eo propius ad $\frac{1}{n+1} z^{n+1}$ accedit, quo major fuerit numerus z.

Corollarium 2.

307. Ex priore autem limite posito $mx = z$, eadem quantitas $\frac{1}{n+1} z^{n+1}$ proxime exhibet summam hujus seriei

$$0 + 1 + 2^n + 3^n + 4^n + \ldots\ldots + (z-1)^n,$$

unde medium sumendo erit accuratius:

$$1 + 2^n + 3^n + 4^n \ldots\ldots + (z-1)^n + \tfrac{1}{2} z^n = \frac{1}{n+1} z^{n+1}$$

seu addendo utrinque $\frac{1}{2} z^n$, habebimus

$$1 + 2^n + 3^n + 4^n \ldots\ldots + z^n = \frac{1}{n+1} z^{n+1} + \tfrac{1}{2} z^n,$$

proxime quod congruit cum iis, quae de vera hujus progressionis summa sunt cognita.

Exemplum 2.

308. *Integrale* $y = \int \frac{\partial x}{x^n}$ *ita sumtum, ut evanescat posito* $x = 1$, *proxime exhibere.*

Erit ergo $a = 1$ et $b = 0$, unde si ab a ad x intervallum progressionis statuatur $= \alpha$, erunt indices

$$a,\ a + \alpha,\ a + 2\alpha,\ a + 3\alpha,\ \ldots\ldots\ x,$$

et termini seriei

$$\frac{1}{a^n},\ \frac{1}{(a+\alpha)^n},\ \frac{1}{(a+2\alpha)^n},\ \frac{1}{(a+3\alpha)^n},\ \ldots\ldots\ \frac{1}{x^n},\ = \mathrm{X},$$

ubi terminus ultimum praecedens est $\frac{1}{(x-\alpha)^n} = \mathrm{X}'$. Cum nunc nostrum integrale sit $y = \frac{1}{n-1} - \frac{1}{(n-1)\,x^{n-1}}$, ejus valor intra hos limites continebitur:

$$\alpha\left[1 + \frac{1}{(1+\alpha)^n} + \frac{1}{(1+2\alpha)^n} + \frac{1}{(1+3\alpha)^n} + \ldots + \frac{1}{(x-\alpha)^n}\right] \text{ et}$$

$$\alpha\left[\frac{1}{(1+\alpha)^n} + \frac{1}{(1+2\alpha)^n} + \frac{1}{(1+3\alpha)^n} + \ldots\ldots + \frac{1}{x^n}\right].$$

Quare posito $\alpha = \frac{1}{m}$, erunt hi limites:

$$m^{n-1}\left[\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \ldots + \frac{1}{(mx-1)^n}\right] \text{ et}$$

$$m^{n-1}\left[\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \frac{1}{(m+4)^n} + \ldots + \frac{1}{(mx)^n}\right]$$

qui, quo major accipiatur numerus m, eo propius ad valorem integralis $\frac{1}{n-1} - \frac{1}{(n-1)\,x^{n-1}}$ accedunt. Notandum autem est, casu $n = 1$ integrale fore $= lx$.

Corollarium 1.

309. Quodsi ponamus $mx = m + z$; ut sit $x = \frac{m+z}{m}$, prodibunt hae progressiones:

$$m^{n-1}\left(\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\ldots+\frac{1}{(m+z-1)^n}\right) \text{ et}$$

$$m^{n-1}\left(\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\ldots+\frac{1}{(m+z)^n}\right)$$

quarum summa alterius major est, alterius minor quam

$$\frac{1}{n-1}-\frac{m^{n-1}}{(n-1)(m+z)^{n-1}}=\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)(m+z)^{n-1}}:$$

casu autem $n=1$, haec expressio abit in $l\left(1+\frac{z}{m}\right)$.

Corollarium 2.

310. Cum prior progressio major sit quam posterior, erit

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\ldots\ldots$$

$$\ldots\ldots+\frac{1}{(m+z-1)^n}>\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}}$$

$$\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\ldots\ldots$$

$$\ldots\ldots+\frac{1}{(m+z)^n}<\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}}:$$

addatur hic utrinque $\frac{1}{m^n}$, ibi vero $\frac{1}{(m+z)^n}$, et sumatur medium arithmeticum, erit exactius

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\ldots\ldots+\frac{1}{(m+z)^n}$$

$$=\frac{(2m+n-1)(m+z)^n-(2z+2m-n+1)m^n}{2(n-1)m^n(m+z)^n},$$

quae expressio casu $n=1$, abit in $l\left(1+\frac{z}{m}\right)+\frac{1}{2m}+\frac{1}{2(m+z)}$.

Corollarium 3.

311. Ponatur $z = mv$, et habebimus sequentis seriei summam proxime expressam:

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \ldots\ldots + \frac{1}{m^n(1+v)^n}$$
$$= \frac{(2m+n-1)(1+v)^n - 2m(1+v)+n-1}{2(n-1)m^n(1+v)^n},$$

et casu $n = 1$

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{m+mv} = l(1+v) + \frac{2+v}{2m(1+v)}:$$

unde si $v = 1$, erit proxime

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \ldots + \frac{1}{2^n m^n}$$
$$= \frac{2^n(2m+n-1) - 4m+n-1}{2^{n+1}(n-1)m^n}, \text{ et}$$

$$\frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{2m} = l2 + \frac{3}{4m}.$$

Corollarium 4.

312. Hinc nascitur regula, logarithmos quantumvis magnorum numerorum proxime assignandi, dum series vulgares tantum pro numeris parum ab unitate differentibus valent. Scribamus enim u pro $1 + v$, et habebimus

$$lu = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{mu} - \frac{1-u}{2mu}.$$

unde lu eo accuratius definitur, quo major sumatur numerus m.

Exemplum 3.

313. *Integrale $y = \int \frac{c\,\partial x}{cc+xx}$ ita sumtum, ut evanescat posito $x = 0$; proxime exprimere.*

Hoc integrale ut novimus, est $y = \text{Ang. tang.} \frac{x}{c}$, ad quem valorem proxime exhibendum, est $a = 0$, et $b = 0$; si ergo valor ipsius x ab 0 per differentiam constantem α crescere statuatur, ob $X = \frac{c}{cc + xx}$, erunt ejus valores

pro indicibus 0 α 2α . . . . . . x;

$$\frac{1}{c};\ \frac{c}{cc+\alpha\alpha};\ \frac{c}{cc+4\alpha\alpha};\ \ldots\ldots\ \frac{c}{cc+xx};$$

cujus terminus ultimum praecedens est $X = \frac{c}{cc+(x-\alpha)^2}$.

Quare integralis nostri $y =$ Ang. tang. $\frac{x}{c}$ valor proxime est

$$\alpha\left(\frac{1}{c} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \ldots\ldots + \frac{c}{cc+(x-\alpha)^2}\right);$$

alter vero proxime minor, quia hic est nimis magnus, est

$$\alpha\left(\frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \ldots\ldots + \frac{c}{cc+xx}\right).$$

Inter quos si medium capiatur, ibi $\alpha \cdot \frac{1}{c}$, hic vero $\alpha \cdot \frac{c}{cc+xx}$ adjiciendo, propius erit

$$\alpha\left(\frac{c}{cc} + \frac{c}{cc+\alpha\alpha} + \frac{c}{cc+4\alpha\alpha} + \frac{c}{cc+9\alpha\alpha} + \ldots\ldots + \frac{c}{cc+xx}\right)$$

$$= \text{Ang. tang.}\frac{x}{c} + \frac{\alpha}{2}\left(\frac{1}{c} + \frac{c}{c+xx}\right)$$

$$= \text{Ang. tang.}\frac{x}{c} + \frac{\alpha(2c \ + xx)}{2c(cc+xx)}.$$

Pro hoc ergo angulo valorem proxime verum habemus

$$\text{Ang tang.}\frac{x}{c} = \alpha c\left(\frac{1}{cc} + \frac{1}{cc+\alpha\alpha} + \frac{1}{cc+4\alpha\alpha} + \ldots + \frac{1}{cc+xx}\right)$$
$$- \frac{\alpha(2cc+xx)}{2c(cc+xx)},$$

qui eo minus a veritate discrepabit, quo minor fuerit α numerus ratione ipsius c. Quodsi ergo pro c numerum valde magnum sumamus, pro α unitatem accipere licet; unde posito $x = cv$, erit

$$\text{Ang. tang.}\, v = c\left(\frac{1}{cc} + \frac{1}{cc+1} + \frac{1}{cc+4} + \frac{1}{cc+9} + \ldots + \frac{1}{cc+ccvv}\right)$$
$$- \frac{(2+vv)}{2c(1+vv)},$$

idque eo exactius, quo major capiatur numerus c.

Corollarium 1

314. Si ponamus $c = 1$, quo casu error insignis esse debet, fiet

$$\text{Ang.tang.}v = 1 + \frac{1}{1+1} + \frac{1}{1+4} + \frac{1}{1+9} + \ldots + \frac{1}{1+vv} - \frac{(2+vv)}{2(1+vv)}.$$

Sit $v = 1$, erit Ang. tang. $1 = \frac{\pi}{4} = 1 + \frac{1}{2} - \frac{3}{4} = \frac{3}{4}$, hincque $\pi = 3$, quod non multum abhorret a vero; si ponamus $c = 2$, prodit

$$\text{Ang.tang.}v = 2\left(\frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+4} + \frac{1}{4+9} + \ldots + \frac{1}{4+4vv}\right) - \frac{(2+vv)}{4(1+vv)},$$

unde si $v = 1$, colligitur

$$\text{Ang. tang. } 1 = \frac{\pi}{4} = 2\left(\frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+4}\right) - \frac{3}{8} = \frac{23}{20} - \frac{3}{8} = \frac{31}{40},$$

sicque $\pi = \frac{31}{10} = 3,1$, propius accedens.

Corollarium 2.

315. Sit $c = 6$, eritque

$$\text{Ang.tang.}v = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4} + \ldots + \frac{1}{36+36vv}\right) - \frac{(2+vv)}{12(1+vv)},$$

unde si $v = \frac{1}{2}$ et $v = \frac{1}{3}$, oritur:

$$\text{Ang. tang. } \tfrac{1}{2} = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4} + \frac{1}{36+9}\right) - \frac{8}{22},$$

$$\text{Ang. tang. } \tfrac{1}{3} = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4}\right) - \frac{19}{120}.$$

At est Ang. tang $\frac{1}{2}$ + Ang. tang. $\frac{1}{3}$ = Ang. tang. $1 = \frac{\pi}{4}$. Ergo

$$\frac{\pi}{4} = 12\left(\frac{1}{36} + \frac{1}{37} + \frac{1}{40}\right) + \frac{2}{15} - \frac{37}{120} = \frac{1063}{1110} - \frac{7}{40} = \frac{695}{888},$$

seu $\pi = \frac{695}{222} = 3,1306$.

Corollarium 3.

316. Sin autem ibi statim ponamus $v = 1$, erit

$$\frac{\pi}{4} = 6\left(\frac{1}{36} + \frac{1}{37} + \frac{1}{40} + \frac{1}{45} + \frac{1}{52} + \frac{1}{61} + \frac{1}{72}\right) - \frac{3}{8},$$

unde fit $\pi = 3,13696$ multo propius veritati; plurium scilicet terminorum additio propius ad veritatem perducit.

Problema 37.

317. Methodum ad integralium valores appropinquandi ante expositam, perfectiorem reddere, ut minus a veritate aberretur.

Solutio.

Sit $y = \int X \partial x$ formula integralis proposita, cujus valorem jam constet esse $y = b$, si ponatur $x = a$, sive is fit datus per ipsam integrationis conditionem, sive jam per aliquot operationes inde derivatus; ac tribuamus jam ipsi x valorem parum superantem illum a, cui respondet $y = b$, tum vero fiat $X = A$, si ponatur $x = a$. In superiori autem methodo assumsimus, dum x parum supra a excrescit, manere X constantem $= A$, ideoque fore $\int X \partial x = A(x - a)$. At quatenus X non est constans, eatenus non est $\int X \partial x = X(x-a)$, sed revera habetur $\int X \partial x = X(x-a) - \int (x-a) \partial X$. Ponamus igitur $\partial X = P \partial x$, eritque $\int (x-a) \partial X = \int P(x-a) \partial x$, et si jam $P = \frac{\partial X}{\partial x}$, quamdiu x non multum a excedit, ut constantem spectemus, habebimus $\int P(x-a) \partial x = \frac{1}{2} P(x-a)^2$ sicque fiet $y = \int X \partial x = b + X(x-a) - \frac{1}{2} P(x-a)^2$, qui valor jam propius ad veritatem accedit, etsi pro X et P ii valores capiantur, quos induunt vel posito $x = a$, vel posito $x = a + \alpha$, majore scilicet valore, ad quem hac operatione x crescere statuimus: ex quo hinc prout vel $x = a$ vel $x = a + \alpha$ ponimus, geminos limites obtinebimus, inter quos veritas subsistit. Simili autem modo ulterius progredi poterimus: cum enim P non sit constans, erit $\int P(x-a) \partial x = \frac{1}{2} P(x-a)^2 - \frac{1}{2} \int (x-a)^2 \partial P$, unde si statuamus $\partial P = Q \partial x$, erit $\int (x-a)^2 \partial P = \int Q (x-a)^2 \partial x = \frac{1}{3} Q (x-a)^3$, si quidem Q ut quantitatem constantem spectemus, ita ut sit

$$y = \int X \partial x = b + X(x-a) - \tfrac{1}{2} P(x-a)^2 + \tfrac{1}{2} \cdot \tfrac{1}{3} Q(x-a)^3.$$

Eadem ergo methodo si ulterius procedamus, ponendo

$$X = \frac{\partial y}{\partial x};\ P = \frac{\partial X}{\partial x};\ Q = \frac{\partial P}{\partial x};\ R = \frac{\partial Q}{\partial x};\ S = \frac{\partial R}{\partial x};\ \text{etc.}$$

inveniemus

$$y = b + X(x-a) - \tfrac{1}{2} P(x-a)^2 + \tfrac{1}{2.3} Q(x-a)^3 - \tfrac{1}{2.3.4} R(x-a)^4 + \tfrac{1}{2.3.4.5} S(x-a)^5 - \text{etc.}$$

quae series vehementer convergit, si modo x non multum superet a, atque adeo si in infinitum continuetur, verum valorem ipsius y exhibebit, siquidem in functionibus X, P, Q, R, etc. valor extremus $x = a + \alpha$ substituatur. Nisi autem eam seriem in infinitum extendere velimus, praestabit per intervalla procedere tribuendo ipsi x successive valores a, a', a'', a''', a'''', etc. ac tum pro singulis valores litteris X, P, Q, R, S, etc. convenientes quaeri oportet, qui sint, ut sequuntur:

$$\begin{array}{llllllll}
\text{si fuerit } x = a, & a', & a'', & a''', & a^{IV}, & a^{V}, & \text{etc.} \\
\text{fiat } \quad X = A, & A', & A'', & A''', & A^{IV}, & A^{V}, & \text{etc.} \\
\frac{\partial X}{\partial x} = P = B, & B'', & B', & B''', & B^{IV}, & B^{V}, & \text{etc.} \\
\frac{\partial P}{\partial x} = Q = C, & C', & C'', & C''', & C^{IV}, & C^{V}, & \text{etc.} \\
\frac{\partial Q}{\partial x} = R = D, & D', & D'', & D''', & D^{IV}, & D^{V}, & \text{etc.}
\end{array}$$

etc.

tum vero sit

$$y = b,\ b',\ b'',\ b''',\ b^{IV},\ b^{V},\ \text{etc.}$$

quibus constitutis erit, ut ex antecedentibus colligere licet:

$$b' = b + A'(a' - a) - \tfrac{1}{2}B'(a' - a)^2 + \tfrac{1}{6}C'(a' - a)^3 - \tfrac{1}{24}D'(a' - a)^4 + \text{etc.}$$

$$b' = b' + A''(a'' - a') - \tfrac{1}{2}B''(a'' - a')^2 + \tfrac{1}{6}C''(a'' - a')^3 - \tfrac{1}{24}D''(a'' - a')^4 + \text{etc.}$$

$$b''' = b'' + A'''(a''' - a'') - \tfrac{1}{2}B'''(a''' - a'')^2 + \tfrac{1}{6}C'''(a''' - a'')^3 - \tfrac{1}{24}D'''(a''' - a'')^4 + \text{etc.}$$

$$b^{IV} = b''' + A^{IV}(a^{IV} - a''') - \tfrac{1}{2}B^{IV}(a^{IV} - a''')^2 + \tfrac{1}{6}C^{IV}(a^{IV} - a''')^3 - \tfrac{1}{24}D^{IV}(a^{IV} - a''')^4 + \text{etc.}$$

etc.

quae expressiones eousque continuentur, donec pro valore ipsius x quantumvis ab initiali a discrepante, valor ipsius y obtineatur.

Corollarium 1.

318. Haec igitur approximandi methodus eo utitur Theoremate, cujus veritas jam in calculo differentiali est demonstrata, quod si y ejusmodi fuerit functio ipsius x, quae posito $x = a$, fiat $= b$, ac statuatur

$$\frac{\partial y}{\partial x} = \mathrm{X},\quad \frac{\partial \mathrm{X}}{\partial x} = \mathrm{P},\quad \frac{\partial \mathrm{P}}{\partial x} = \mathrm{Q},\quad \frac{\partial \mathrm{Q}}{\partial x} = \mathrm{R},\ \text{etc.}$$

fore generaliter:

$$y = b + (x - a) - \tfrac{1}{2}\mathrm{P}(x - a)^2 + \tfrac{1}{6}\mathrm{Q}(x - a)^3 - \tfrac{1}{24}\mathrm{R}(x - a)^4 + \tfrac{1}{120}\mathrm{S}(x - a)^5 - \text{etc.}$$

Corollarium 2.

319. Si hanc seriem in infinitum continuare vellemus, non opus esset, valorem ipsius x parum tantum ab a diversum assumere. Verum quo ista series magis convergens reddatur, expedit saltum ab a ad x in intervalla dispesci, et pro singulis operationem hic descriptam institui.

Corollarium 3.

320. Si valores ipsius x ab a per differentias constantes $= \alpha$ crescere faciamus, sitque ultimus $a + n\alpha = x$, ita ut

si fuerit $x =$	$a,$	$a + \alpha,$	$a + 2\alpha,$	$a + 3\alpha,$	$\ldots\ldots x$
fiat $\mathrm{X} =$	$\mathrm{A},$	$\mathrm{A}',$	$\mathrm{A}'',$	$\mathrm{A}''',$	$\ldots\ldots \mathrm{X}$
$\frac{\partial \mathrm{X}}{\partial x} = \mathrm{P} =$	$\mathrm{B},$	$\mathrm{B}',$	$\mathrm{B}'',$	$\mathrm{B}''',$	$\ldots\ldots \mathrm{P}$
$\frac{\partial \mathrm{P}}{\partial x} = \mathrm{Q} =$	$\mathrm{C},$	$\mathrm{C}',$	$\mathrm{C}'',$	$\mathrm{C}''',$	$\ldots\ldots \mathrm{Q}$
$\frac{\partial \mathrm{Q}}{\partial x} = \mathrm{R} =$	$\mathrm{D},$	$\mathrm{D}',$	$\mathrm{D}'',$	$\mathrm{D}''',$	$\ldots\ldots \mathrm{R}$
			etc.		

indeque $y = b,\quad b',\quad b'',\quad b''',\quad \ldots\ldots y,$

erit pro valore $x = x$ omnes series colligendo:

$$
\begin{aligned}
y = b &+ \alpha\,(A + A'' + A''' + \dots\dots + X)\\
&- \tfrac{1}{2}\alpha^2\,(B' + B'' + B'' + \dots\dots + P)\\
&+ \tfrac{1}{6}\alpha^3\,(C' + C' + C'' + \dots\dots + Q)\\
&- \tfrac{1}{24}\alpha^4\,(D' + D' + D'' + \dots\dots + R)\\
&\quad \text{etc.}
\end{aligned}
$$

Scholion 1.

321. Demonstratio theorematis Corollario 1. memorati, cui haec methodus approximandi innititur, ex natura differentialium ita instruitur: Sit y functio ipsius x, quae posito $x = a$, fiat $y = b$; et quaeramus valorem ipsius y, si x utcunque excedat a: incipiamus a valore ipsius maximo, qui est x, et per differentialia descendamus; atque ex differentialibus patet:

si fuerit x	fore y
$x - \partial x$	$y - \partial y + \partial\partial y - \partial^3 y + \partial^4 y - \text{etc.}$
$x - 2\partial x$	$y - 2\partial y + 3\partial\partial y - 4\partial^3 y + 5\partial^4 y - \text{etc.}$
$x - 3\partial x$	$y - 3\partial y + 6\partial\partial y - 10\partial^3 y + 15\partial^4 y - \text{etc.}$
.	.
.	.
.	.
.	.
$x - n\partial x$	$y - n\partial y + \frac{n(n+1)}{1.2}\partial\partial y - \frac{n(n+1)(n+2)}{1.2.3}\partial^3 y + \frac{n(n+1)(n+2)(n+3)}{1.2.3.4}\partial^4 y - \text{etc.}$

Ponamus nunc $x - n\partial x = a$, erit $n = \frac{x-a}{\partial x}$, ideoque numerus infinitus; tum vero valor pro y resultans per hypothesin esse debet $= b$, quamobrem habebimus

$$b = y - \frac{(x-a)\partial y}{\partial x} + \frac{(x-a)^2\partial\partial y}{1.2\partial x^2} - \frac{(x-a)^3\partial^3 y}{1.2.3\partial x^3} + \frac{(x-a)^4\partial^4 y}{1.2.3.4\partial x^4} - \text{etc.}$$

Quod si jam statuamus

$$\frac{\partial y}{\partial x} = X, \quad \frac{\partial X}{\partial x} = P, \quad \frac{\partial P}{\partial x} = Q, \quad \frac{\partial Q}{\partial x} = R, \text{ etc.}$$

reperimus ut ante:

$$y = b + X(x-a) - \tfrac{1}{2}P(x-a)^2 + \tfrac{1}{6}Q(x-a)^3 - \tfrac{1}{24}R(x-a)^4 + \text{etc.}$$

Unde patet, si x quam minime superet a, sufficere statui $y = b + X(x-a)$, quod est fundamentum approximationis primum propositae, illius scilicet limitis, quo X ex valore majore ipsius x definitur.

Scholion 2.

322. Quemadmodum hoc ratiocinium nobis alterum tantum limitem supra assignatum patefecit, ita ad alterum limitem hoc ratiocinium nos manuducet. Scilicet, uti ante ab x ad a descendimus, ita nunc ab a ad x ascendamus,

si abeat a	tum b abibit in
in $a + \partial a$	$b + \partial b$
$a + 2\partial a$	$b + 2\partial b + \partial\partial b$
$a + 3\partial a$	$b + 3\partial b + 3\partial\partial b + \partial^3 b$
.	.
.	.
.	.
$a + n\partial a$	$b + n\partial b + \frac{n(n-1)}{1.2}\partial\partial b + \frac{n(n-1)(n-2)}{1.2.3}\partial^3 b + \text{etc.}$

Sit jam $a + n\partial a = x$, seu $n = \frac{x-a}{\partial a}$, et valor ipsius b fiet $= y$; Sint autem A, B, C, D, etc. valores superiorum functionum X, P, Q, R, etc. si loco x scribatur a, eritque pro praesenti casu $A = \frac{\partial b}{\partial a}$; $B = \frac{\partial\partial b}{\partial a^2}$; $C = \frac{\partial^3 b}{\partial a^3}$; etc. Quocirca habebimus

$$y = b + A(x-a) + \tfrac{1}{2}B(x-a)^2 + \tfrac{1}{6}C(x-a)^3 + \tfrac{1}{24}D(x-a)^4 + \text{etc.}$$

quae series superiori praeter signa omnino est similis; ac si x parum excedat a, ut $b + A(x - a)$ satis exacte valorem ipsius y indicet, hinc alter limes supra assignatus nascitur. Quodsi autem progressum ab a ad x, ut supra §. 320. in intervalla aequalia secundum differentiam α dispescamus, et termini in singulis seriebus ultimos praecedentes notentur per ′X, ′P, ′Q, ′R, etc. habebimus pro y quasi alterum limitem

$$\begin{aligned} y = b &+ \alpha\,(A + A' + A'' + \ldots\ldots + {}'X) \\ &+ \tfrac{1}{2}\alpha^2\,(B + B' + B'' + \ldots\ldots + {}'P) \\ &+ \tfrac{1}{6}\alpha^3\,(C + C' + C'' + \ldots\ldots + {}'Q) \\ &+ \tfrac{1}{24}\alpha^4\,(D + D' + D'' + \ldots\ldots + {}'R) \\ &\qquad \text{etc.} \end{aligned}$$

ita ut etiam in hac methodo emendata binos habebimus limites, inter quos verus valor ipsius y contineatur. Propius ergo valorem assequemur, si inter hos limites medium arithmeticum capiamus; unde prodibit

$$\begin{aligned} y = b &+ \alpha\,(A + A' + A'' + \ldots\ldots + X) - \tfrac{1}{2}\alpha\,(A + X) + \tfrac{1}{4}\alpha^2\,(B - P) \\ &+ \tfrac{1}{6}\alpha^3(C + C' + C'' + \ldots\ldots + Q) - \tfrac{1}{12}\alpha^3(C + Q) + \tfrac{1}{48}\alpha^4\,(D - R) \\ &+ \tfrac{1}{120}\alpha^5(E + E' + E'' + \ldots\ldots + S) - \tfrac{1}{240}\alpha^5(E + S) + \tfrac{1}{1440}\alpha^6\,(F - T) \\ &\qquad \text{etc.} \end{aligned}$$

Atque hinc superiores approximationes tantum addendo membrum $\frac{1}{4}\alpha^2\,(B - P)$, non mediocriter corrigentur.

Exemplum 1.

323. *Logarithmum cujusvis numeri* x *proxime exprimere.*

Hic igitur est $y = \int\frac{\partial x}{x}$, quod integrale ita capitur, ut evanescat posito $x = 1$: erit ergo $a = 1$, $b = 0$ et $X = \frac{1}{x}$. Sumamus jam, ab unitate ad x per intervalla $= \alpha$ ascendi, et cum sit $P = \frac{\partial X}{\partial x} = -\frac{1}{xx}$; $Q = \frac{\partial P}{\partial x} = \frac{2}{x^3}$; $R = \frac{\partial Q}{\partial x} = -\frac{6}{x^4}$; pro indicibus

$$
\begin{array}{llllll}
x = & 1; & 1+\alpha; & 1+2\alpha; & 1+3\alpha; \ldots\ldots & x, \text{ erit} \\
X = & 1; & \frac{1}{1+\alpha}; & \frac{1}{1+2\alpha}; & \frac{1}{1+3\alpha}; \ldots\ldots & \frac{1}{x} \\
P = & -1; & \frac{1}{(1+\alpha)^2}; & \frac{1}{(1+2\alpha)^2}; & \frac{1}{(1+3\alpha)^2}; \ldots\ldots & -\frac{1}{xx} \\
Q = & 2; & \frac{2}{(1+\alpha)^3}; & \frac{2}{(1+2\alpha)^3}; & \frac{2}{(1+3\alpha)^3}; \ldots\ldots & +\frac{2}{x^3} \\
R = & -6; & \frac{6}{(1+\alpha)^4}; & \frac{6}{(1+2\alpha)^4}; & \frac{6}{(1+3\alpha)^4}; \ldots\ldots & -\frac{6}{x^4}
\end{array}
$$

etc.

unde adipiscimur

$$
\begin{aligned}
lx = \alpha\,[1 &+ \frac{1}{1+\alpha} + \frac{1}{1+2\alpha} + \frac{1}{1+3\alpha} + \ldots\ldots + \frac{1}{x}] \\
&- \tfrac{1}{2}\alpha\left(1+\frac{1}{x}\right) - \tfrac{1}{4}\alpha\alpha\left(1-\frac{1}{xx}\right) \\
+\tfrac{1}{2}\alpha^3[1 &+ \frac{1}{(1+\alpha)^3} + \frac{1}{(1+2\alpha)^3} + \frac{1}{(1+3\alpha)^3} + \ldots\ldots + \frac{1}{x^3}] \\
&- \tfrac{1}{6}\alpha^3\left(1+\frac{1}{x^3}\right) - \tfrac{1}{8}\alpha^4\left(1-\frac{1}{x^4}\right) \\
+\tfrac{1}{6}\alpha^5[1 &+ \frac{1}{(1+\alpha)^5} + \frac{1}{(1+2\alpha)^5} + \frac{1}{(1+3\alpha)^5} + \ldots\ldots + \frac{1}{x^5}] \\
&- \tfrac{1}{10}\alpha^5\left(1+\frac{1}{x^5}\right) - \tfrac{1}{12}\alpha^6\left(1-\frac{1}{x^6}\right)
\end{aligned}
$$

etc.

Quare si sumamus $\alpha = \frac{1}{m}$, erit:

$$
\begin{aligned}
lx = \frac{1}{m} &+ \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots + \frac{1}{mx} \\
&- \frac{(x+1)}{2mx} - \frac{(xx-1)}{4mmxx}
\end{aligned}
$$

$$+\tfrac{1}{3}\left[\frac{1}{m^3}+\frac{1}{(m+1)^3}+\frac{1}{(m+2)^3}+\dots\dots+\frac{1}{(mx)^3}\right]$$

$$-\frac{(x^3+1)}{6m^3x^3}-\frac{(x^4-1)}{8m^4x^4}$$

$$+\tfrac{1}{5}\left[\frac{1}{m^5}+\frac{1}{(m+1)^5}+\frac{1}{(m+2)^5}+\dots\dots+\frac{1}{(mx)^5}\right]$$

$$-\frac{(x^5+1)}{10m^5x^5}-\frac{(x^6-1)}{12m^6x^6}$$

etc.

Corollarium.

324. Si hae progressiones in infinitum continuentur, erit postremarum partium summa

$$= -\tfrac{1}{2}l\frac{m}{m-1}-\tfrac{1}{2}l\frac{mx+1}{mx} = -\tfrac{1}{2}l\frac{mx+1}{(m-1)x},$$

primarum vero $=\frac{1}{2}l\frac{m+1}{m-1}$: unde cum sit

$$lx+\tfrac{1}{2}l\frac{mx+1}{(m-1)x}+\tfrac{1}{2}l\frac{m-1}{m+1}=\tfrac{1}{2}l\frac{x(mx+1)}{m+1},$$

erit

$$l\frac{x(mx+1)}{m+1}=2\left(\frac{1}{m+1}+\frac{1}{m+2}+\frac{1}{m+3}+\dots\dots+\frac{1}{mx}\right)$$

$$+\tfrac{2}{3}\left(\frac{1}{(m+1)^3}+\frac{1}{(m+2)^3}+\frac{1}{(m+3)^3}+\dots\dots+\frac{1}{m^3x^3}\right)$$

$$+\tfrac{2}{5}\left(\frac{1}{(m+1)^5}+\frac{1}{(m+2)^5}+\frac{1}{(m+3)^5}+\dots\dots+\frac{1}{m^5x^5}\right)$$

etc.

quae expressio adeo, si in infinitum continuetur, verum valorem log. $\frac{x(mx+1)}{m+1}$ praebet.

Exemplum 2.

325. *Arcum circuli cujus tangens est* $=\frac{x}{c}$ *hac methodo proxime exprimere.*

Quaestio igitur est de integrali $y = \int \frac{c\,\partial x}{cc+xx}$, quod posito $x = 0$ evanescit; eritque $a = 0$, et $b = 0$, tum vero

$$X = \frac{c}{cc+xx};\quad P = \frac{\partial X}{\partial x} = \frac{-2cx}{(cc+xx)^2};\quad Q = \frac{\partial P}{\partial x} = \frac{-2c(cc-3xx)}{(cc+xx)^3};$$

$$R = \frac{\partial Q}{\partial x} = \frac{6cx(3cc-4xx)}{(cc+xx)^4};\quad S = \frac{\partial R}{\partial x} = \frac{6c(3c^4-33ccxx+20x^4)}{(cc+xx)^5};\ \text{etc.}$$

quae formae in infinitum continuatae dant

$$y = \frac{cx}{cc+xx} + \frac{cx^3}{(cc+xx)^2} - \frac{cx^3(cc-3xx)}{3(cc+xx)^3} - \frac{cx^5(3cc-4xx)}{4(cc+xx)^4}$$
$$+ \frac{cx^5(3c^4-33ccxx+20x^4)}{20(cc+xx)^5} + \text{etc.}$$

Verum si x per intervalla $= 1$, ut sit $\alpha = 1$, crescere ponamus, erit

$$A = \frac{c}{cc};\quad B = 0;\quad C = \frac{-2c^3}{c^6};\quad D = 0;$$
$$A' = \frac{c}{cc+1};\quad B' = \frac{-2c}{(cc+1)^2};\quad C' = \frac{-2c(cc-3)}{(cc+1)^3};\quad D' = \frac{6c(3cc-4)}{(cc+1)^4};$$
$$A'' = \frac{c}{cc+4};\quad B'' = \frac{-4c}{(cc+4)^2};\quad C'' = \frac{-2c(cc-12)}{(cc+4)^3};\quad D'' = \frac{12c(3cc-16)}{(cc+4)^4};$$
$$A''' = \frac{c}{cc+9};\quad B''' = \frac{-6c}{(cc+9)^2};\quad C''' = \frac{-2c(cc-27)}{(cc+9)^3};\quad D''' = \frac{18c(3cc-36)}{(cc+9)^4};$$

⋮

$$X = \frac{c}{cc+xx};\quad P = \frac{-2cx}{(cc+xx)^2};\quad Q = \frac{-2c(cc-3xx)}{(cc+xx)^3};\quad R = \frac{6cx(3cc-4xx)}{(cc+xx)^4};$$

hincque

$$y = c\left(\frac{1}{cc} + \frac{1}{cc+1} + \frac{1}{cc+4} + \frac{1}{cc+9} + \ldots + \frac{1}{cc+xx}\right)$$
$$- \frac{1}{2c} - \frac{c}{2(cc+xx)} + \frac{cx}{2(cc+xx)^2}$$
$$- \frac{c}{3}\left(\frac{1}{c^4} + \frac{cc-3}{(cc+1)^3} + \frac{cc-12}{(cc+4)^3} + \frac{cc-27}{(cc+9)^3} + \ldots + \frac{cc-3xx}{(cc+xx)^3}\right)$$
$$+ \frac{1}{6c^3} + \frac{c(cc-3xx)}{6(cc+xx)^3} - \frac{cx(3cc-4xx)}{8(cc+xx)^4}$$
etc.

Corollarium.

326. Posito ergo $c = x = 4$, ut fiat
$y = \text{Ang. tang. } 1 = \frac{\pi}{4}$, erit

$$\frac{\pi}{4} = \frac{1}{4} + \frac{4}{17} + \frac{4}{20} + \frac{4}{25} + \frac{1}{8} - \frac{1}{8} - \frac{1}{16} + \frac{1}{128}$$
$$- \frac{4}{3}\left(\frac{1}{256} + \frac{13}{17^3} + \frac{4}{20^3} - \frac{11}{25^3} - \frac{32}{32^3}\right) + \frac{1}{384} - \frac{1}{1536} + \frac{1}{128 \cdot 256},$$

cujus valor non multum a veritate discedit; sed haec exempla tantum illustrationis causa afferro, non ut approximatio facilior, quam aliae methodi suppeditant, inde expectetur.

Exemplum 3.

327. *Integrale* $y = \int \frac{e^{-\frac{1}{x}} \partial x}{x}$ *ita sumtum, ut evanescet posito* $x = 0$, *vero proxime assignare.*

Per reductiones supra expositas est

$$\int \frac{e^{-\frac{1}{x}} \partial x}{x} = e^{-\frac{1}{x}} x - \int e^{-\frac{1}{x}} \partial x;$$

et pars $e^{-\frac{1}{x}} x$ evanescit, posito $x = 0$. Quaeramus ergo integrale $z = \int e^{-\frac{1}{x}} \partial x$, quia eo invento habetur $y = e^{-\frac{1}{x}} x - z$; ac supra jam observavimus, alias methodos approximandi in hoc exemplo frustra tentari. Cum igitur, posito $x = 0$, evanescat z, erit $a = 0$ et $b = 0$, tum vero $X = e^{-\frac{1}{x}}$; hincque $P = \frac{\partial X}{\partial x} = e^{-\frac{1}{x}} \frac{1}{xx}$; $Q = \frac{\partial P}{\partial x} = e^{-\frac{1}{x}} \left(\frac{1}{x^4} - \frac{2}{x^3}\right)$; $R = \frac{\partial Q}{\partial x} = e^{-\frac{1}{x}} \left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right)$; $S = \frac{\partial R}{\partial x} = e^{-\frac{1}{x}} \left(\frac{1}{x^8} - \frac{12}{x^7} + \frac{36}{x^6} - \frac{24}{x^5}\right)$ etc., quibus valoribus in infinitum continuatis, erit

$$z = e^{-\frac{1}{x}} \left[\begin{array}{l} x - \frac{1}{2} + \frac{1}{6} x^3 \left(\frac{1}{x^4} - \frac{2}{x^3}\right) - \frac{1}{24} x^4 \left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right) \\ + \frac{1}{120} x^5 \left(\frac{1}{x^8} - \frac{12}{x^7} + \frac{36}{x^6} - \frac{24}{x^5}\right) - \text{etc.} \end{array} \right] \text{ seu}$$

$$z = e^{-\frac{1}{x}} \left[\begin{array}{l} x - \frac{1}{2} + \frac{1}{6}\left(\frac{1}{x} - 2\right) - \frac{1}{24}\left(\frac{1}{xx} - \frac{6}{x} + 6\right) + \frac{1}{120}\left(\frac{1}{x^3} - \frac{12}{x^2} + \frac{36}{x} - 24\right) \\ - \frac{1}{720}\left(\frac{1}{x^4} - \frac{20}{x^3} + \frac{120}{x^2} - \frac{240}{x} + 120\right) + \text{etc.} \end{array} \right]$$

quae series parum convergit, quicunque valor ipsi x tribuitur. Per intervalla igitur a 0 usque ad x ascendamus, ponendo pro x successive 0, α, 2α, 3α, etc. ubi notandum fore $A = 0$, $B = 0$, $C = 0$, $D = 0$, etc. ac regula nostra praebet:

$$z = \alpha\left(e^{-\frac{1}{\alpha}} + e^{-\frac{1}{2\alpha}} + e^{-\frac{1}{3\alpha}} + \ldots + e^{-\frac{1}{x}}\right) - \tfrac{1}{2}\alpha e^{-\frac{1}{x}} - \tfrac{1}{4}\alpha^2 e^{-\frac{1}{x}} \frac{1}{xx}$$
$$+ \tfrac{1}{6}\alpha^3\left[e^{-\frac{1}{\alpha}}\left(\frac{1}{\alpha^4} - \frac{2}{\alpha^3}\right) + e^{-\frac{1}{2\alpha}}\left(\frac{1}{16\alpha^4} - \frac{2}{8\alpha^3}\right) + e^{-\frac{1}{3\alpha}}\left(\frac{1}{81\alpha^4} - \frac{2}{27\alpha^3}\right) \ldots\right.$$
$$\left. + e^{-\frac{1}{x}}\left(\frac{1}{x^4} - \frac{2}{x^3}\right)\right] - \tfrac{1}{12}\alpha^3 e^{-\frac{1}{x}}\left(\frac{1}{x^4} - \frac{2}{x^3}\right) - \tfrac{1}{48}\alpha^4 e^{-\frac{1}{x}}\left(\frac{1}{x^6} - \frac{8}{x^5} + \frac{6}{x^4}\right).$$

Si hinc valorem ipsius z pro casu $x = 1$ determinare velimus, et pro α fractionem parvam $\frac{1}{n}$ assumamus, habebimus:

$$z = \tfrac{1}{n}\left(e^{-\frac{n}{1}} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + e^{-\frac{n}{4}} + \ldots + e^{-\frac{n}{n}}\right) + \frac{1}{2ne} - \frac{1}{4nne}$$
$$+ \tfrac{1}{6}\left[e^{-\frac{n}{1}}\left(\frac{n-2}{1}\right) + e^{-\frac{n}{2}}\left(\frac{n-4}{16}\right) + e^{-\frac{n}{3}}\left(\frac{n-6}{81}\right) + \ldots + e^{-\frac{n}{n}}\left(\frac{n-2n}{n^4}\right)\right]$$
$$+ \frac{1}{12n^3 e} - \frac{1}{48n^4 e}.$$

Si hic pro n sumatur numerus mediocriter magnus vel uti 10, valor ipsius z ad partem millionesimam unitatis exactus reperitur, ac vicies exactior prodiret, si pro n sumeremus 20.

Scholion 1.

328. Hoc exemplum sufficiat eximium usum hujus methodi approximandi ostendisse. Interim tamen occurrunt casus, quibus ne hac quidem methodo uti licet, etiamsi totum spatium, per quod variabilis x crescit, in minima intervalla dividamus. Evenit hoc, quando functio X pro quopiam intervallo, dum variabili x certus quidam valor tribuitur, in infinitum excrescit, cum tamen ipsa quantitas integralis $y = \int X\partial x$ hoc casu non fiat infinita: veluti si fuerit $y = \int \frac{\partial x}{\sqrt{(a-x)}}$, ubi $X = \frac{1}{\sqrt{(a-x)}}$, quae posito $x = a$ fit infinita, integrale vero $y = C - 2\sqrt{(a-x)}$ hoc casu est finitum.

Hoc autem semper usu venit, quoties hujusmodi factor $a - x$ in denominatore habet exponentem unitate minorem, tum enim idem factor in integrali in numeratorem transit; sin autem ejusdem factoris exponens in denominatore est unitas, vel adeo unitate major, tum etiam ipsum integrale casu $x = a$ fit infinitum, quo casu quia approximatio cessat, hic tantum de iis sermo est, ubi exponens unitate est minor; quoniam tum approximatio revera turbatur. Verum huic incommodo facile medela afferri potest, cum enim differentiale ejusmodi formam sit habiturum $\frac{X \partial x}{(a - x)^{\lambda : \mu}}$, existente $\lambda < \mu$, ponatur $a - x = z^{\mu}$, ut sit $x = a - z^{\mu}$ et $\partial x = - \mu z^{\mu - 1} \partial z$, et differentiale nostrum erit $= - \mu X z^{\mu - \lambda - 1} \partial z$, quod casu $x = a$ seu $z = 0$, non amplius fit infinitum. Vel quod eodem redit, pro iis intervallis, quibus functio X fit infinita, integratio seorsim revera instituatur, ponendo $x = a \pm \omega$, tum enim formula $X \partial x$ satis fiet simplex ob ω valde parvum, ut integratio nihil habeat difficultatis. Veluti si valorem ipsius $y = \int \frac{x x \partial x}{\sqrt{(a^4 - x^4)}}$ per intervalla ab $x = 0$ usque ad $x = a - \alpha$, jam simus consecuti, pro hoc ultimo intervallo ponamus $x = a - \omega$, et integrari oportebit $\frac{(a - \omega)^2 \partial \omega}{\sqrt{(4 a^3 \omega - 6 a a \omega \omega + 4 a \omega^3 - \omega^4)}}$, quod ob ω valde parvum abit in

$$\frac{\partial \omega \sqrt{a}}{2 \sqrt{\omega}} \left(1 - \frac{\omega}{2 a} + \frac{7 \omega \omega}{8 a a}\right),$$

cujus integrale, sumto $\omega = \alpha$, est

$$\sqrt{a \alpha} - \frac{\alpha \sqrt{\alpha}}{6 \sqrt{a}} + \frac{7 \alpha \alpha \sqrt{\alpha}}{4 a a \sqrt{a}},$$

quod si ad plures terminos continuetur, non solum pro ultimo intervallo sed pro duobus pluribusve postremis, ponendo $\omega = 2 \alpha$ vel $\omega = 3 \alpha$ adhiberi potest. Pro quibus enim intervallis denominator jam fit satis parvus, praestat hac methodo uti, quam ea quae ante est exposita.

Scholion 2.

329. Interdum etiam illud incommodum occurrit, ut denominator duobus casibus evanescat, veluti si fuerit $y = \int \frac{X \partial x}{\sqrt{(a - x)(x - b)}}$,

ubi variabilis x semper inter limites b et a contineri debet, ita ut cum a b ad a creverit, deinceps iterum ab a ad b decrescat; interea autem integrale y continuo crescere pergat, cujus igitur valor per intervalla commode determinari non potest. Hoc ergo casu in subsidium vocetur haec substitutio $x = \frac{1}{2}(a + b) - \frac{1}{2}(a - b)\cos.\Phi$, qua fit $\partial x = + \frac{1}{2}(a - b)\partial\Phi \sin.\Phi$, et $(a - x)(x - b) = [\frac{1}{2}(a - b) + \frac{1}{2}(a - b)\cos.\Phi][\frac{1}{2}(a - b) - \frac{1}{2}(a - b)\cos.\Phi]$, seu $(a - x)(x - b) = \frac{1}{4}(a - b)^2 \sin.\Phi^2$: unde oritur $y = \int X\partial\Phi$, quae nullo amplius incommodo laborat, cum angulum Φ continuo ulterius aequabiliter augere licet. Hoc etiam ad casus patet, ubi bini factores in denominatore non eundem habent exponentem, veluti si fuerit $y = \int \frac{X\partial x}{\sqrt[2\lambda]{(a - x)^\mu (x - b)^\nu}}$, ita ut μ et ν sint minores quam 2λ, quem exponentem parem suppono. Si jam μ et ν non sint aequales sed $\nu < \mu$, ad aequalitatem reducantur hoc modo, $y = \int \frac{X\partial x \sqrt[2\lambda]{(x - b)^{\mu-\nu}}}{\sqrt[2\lambda]{(a - x)^\mu (x - b)^\mu}}$. Quodsi jam ut ante ponatur $x = \frac{1}{2}(a + b) - \frac{1}{2}(a - b)\cos.\Phi$, obtinebitur

$$y = \left(\frac{a-b}{2}\right)^{\frac{2\lambda-\mu-\nu}{2\lambda}} \int X\partial\Phi \sin.\Phi^{\frac{\lambda-\mu}{\lambda}} (1 - \cos.\Phi)^{\frac{\mu-\nu}{2\lambda}},$$

ubi angulum Φ quousque libuerit continuare et methodo per intervalla procedente uti licet. Quibus observatis vix quicquam amplius hanc methodum approximandi remorabitur.

CAPUT VIII.

DE VALORIBUS INTEGRALIUM QUOS CERTIS TANTUM CASIBUS RECIPIUNT.

Problema 38.

330.

Integralis $\int \frac{x^m \partial x}{\surd (1-xx)}$ valorem, quem posito $x=1$ recipit, assignare, integrali scilicet ita determinato, ut evanescat posito $x=0$.

Solutio.

Pro casibus simplicissimis, quibus $m=0$ vel $m=1$, habemus posito $x=1$, post integrationem

$$\int \frac{\partial x}{\surd (1-xx)} = \frac{\pi}{2} \text{ et } \int \frac{x \partial x}{\surd (1-xx)} = 1.$$

Deinde supra §. 119. vidimus esse in genere

$$\int \frac{x^{m+1} \partial x}{\surd (1-xx)} = \frac{m}{m+1} \int \frac{x^{m-1} \partial x}{\surd (1-xx)} - \frac{1}{m+1} x^m \surd (1-xx):$$

casu ergo $x=1$ erit

$$\int \frac{x^{m+1} \partial x}{\surd (1-xx)} = \frac{m}{m+1} \int \frac{x^{m-1} \partial x}{\surd (1-xx)},$$

unde a simplicissimis ad majores exponentis m valores progrediendo obtinebimus;

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2}$$

$$\int \frac{x^2 \partial x}{\sqrt{(1-xx)}} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int \frac{x^4 \partial x}{\sqrt{(1-xx)}} = \frac{1.3}{2.4} \cdot \frac{\pi}{2}$$

$$\int \frac{x^6 \partial x}{\sqrt{(1-xx)}} = \frac{1.3.5}{2.4.6} \cdot \frac{\pi}{2}$$

$$\int \frac{x^8 \partial x}{\sqrt{(1-xx)}} = \frac{1.3.5.7}{2.4.6.8} \cdot \frac{\pi}{2}$$

$$\vdots$$

$$\int \frac{x^{2n} \partial x}{\sqrt{(1-xx)}} = \frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n} \cdot \frac{\pi}{2}$$

$$\int \frac{x \partial x}{\sqrt{(1-xx)}} = 1$$

$$\int \frac{x^3 \partial x}{\sqrt{(1-xx)}} = \frac{2}{3}$$

$$\int \frac{x^5 \partial x}{\sqrt{(1-xx)}} = \frac{2.4}{3.5}$$

$$\int \frac{x^7 \partial x}{\sqrt{(1-xx)}} = \frac{2.4.6}{3.5.7}$$

$$\int \frac{x^9 \partial x}{\sqrt{(1-xx)}} = \frac{2.4.6.8}{3.5.7.9}$$

$$\vdots$$

$$\int \frac{x^{2n+1} \partial x}{\sqrt{(1-xx)}} = \frac{2.4.6 \ldots 2n}{3.5.7 \ldots (2n+1)}$$

Corollarium 1.

331. Integrale ergo $\int \frac{x^m \partial x}{\sqrt{(1-xx)}}$, posito $x = 1$, algebraice exprimitur casibus, quibus exponens m est numerus integer impar; casibus autem, quibus est par, quadraturam circuli involvit; semper enim π designat peripheriam circuli, cujus diameter $= 1$.

Corollarium 2.

332. Si binas postremas formulas in se multiplicemus prodit:

$$\int \frac{x^{2n} \partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^{2n+1} \partial x}{\sqrt{(1-xx)}} = \frac{1}{2n+1} \cdot \frac{\pi}{2}.$$

posito scilicet $x = 1$, quam veram esse patet, etiamsi n non sit numerus integer.

Corollarium 3.

333. Haec ergo aequalitas subsistet, si ponamus $x = z^{\nu}$, iisdem conditionibus, quia sumto $x = 0$ vel $x = 1$ fit $z = 0$ vel $z = 1$. Erit ergo

$$\nu\nu \int \frac{z^{2n\nu+\nu-1}\,\partial z}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{2n\nu+2\nu-1}\,\partial z}{\sqrt{(1-z^{2\nu})}} = \frac{1}{2n+1} \cdot \frac{\pi}{2},$$

et posito $2n\nu + \nu - 1 = \mu$, fiet posito $z = 1$

$$\int \frac{z^{\mu}\,\partial z}{\sqrt{(1-z^{2\nu})}} \cdot \int \frac{z^{\pi+\nu}\,\partial z}{\sqrt{(1-z^{2\nu})}} = \frac{1}{\nu(\mu+1)} \cdot \frac{\pi}{2}.$$

Scholion 1.

334. Quod tale productum binorum integralium exhiberi queat, eo magis est notatu dignum, quod aequalitas haec subsistit, etiamsi neutra formula neque algebraice neque per π exhiberi queat. Veluti si $\nu = 2$ et $\mu = 0$, fit

$$\int \frac{\partial z}{\sqrt{(1-z^4)}} \cdot \int \frac{zz\partial z}{\sqrt{(1-z^4)}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4},$$

similique modo:

$\nu = 3,\ \mu = 0$ fit $\int \frac{\partial z}{\sqrt{(1-z^6)}} \cdot \int \frac{z^3\partial z}{\sqrt{(1-z^6)}} = \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6}$;

$\nu = 3,\ \mu = 1$ fit $\int \frac{z\partial z}{\sqrt{(1-z^6)}} \cdot \int \frac{z^4\partial z}{\sqrt{(1-z^6)}} = \frac{1}{6} \cdot \frac{\pi}{2} = \frac{\pi}{12}$;

$\nu = 4,\ \mu = 0$ fit $\int \frac{\partial z}{\sqrt{(1-z^8)}} \cdot \int \frac{z^4\partial z}{\sqrt{(1-z^8)}} = \frac{1}{4} \cdot \frac{\pi}{2} = \frac{\pi}{8}$;

$\nu = 4,\ \mu = 2$ fit $\int \frac{zz\partial z}{\sqrt{(1-z^8)}} \cdot \int \frac{z^6\partial z}{\sqrt{(1-z^8)}} = \frac{1}{12} \cdot \frac{\pi}{2} = \frac{\pi}{24}$;

$\nu = 5,\ \mu = 0$ fit $\int \frac{\partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^5\partial z}{\sqrt{(1-z^{10})}} = \frac{1}{5} \cdot \frac{\pi}{2} = \frac{\pi}{10}$;

$\nu = 5,\ \mu = 1$ fit $\int \frac{z\partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^6\partial z}{\sqrt{(1-z^{10})}} = \frac{1}{20} \cdot \frac{\pi}{2} = \frac{\pi}{40}$;

$\nu = 5, \mu = 2$ fit $\int \frac{zz\partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^7 \partial z}{\sqrt{(1-z^{10})}} = \frac{1}{15} \cdot \frac{\pi}{2} = \frac{\pi}{30}$;

$\nu = 5, \mu = 3$ fit $\int \frac{z^3 \partial z}{\sqrt{(1-z^{10})}} \cdot \int \frac{z^8 \partial z}{\sqrt{(1-z^{10})}} = \frac{1}{20} \cdot \frac{\pi}{2} = \frac{\pi}{40}$;

quae Theoremata sine dubio omni attentione sunt digna.

Scholion 2.

335. Facile hinc etiam colligitur valor integralis $\int \frac{x^m \partial x}{\sqrt{(x-xx)}}$ posito $x = 1$, si enim scribamus $x = zz$, fiet hoc integrale $2\int \frac{z^{2m} \partial z}{\sqrt{(1-zz)}}$; quocirca pro casu $x = 1$ nanciscimur sequentes valores:

$$\int \frac{\partial x}{\sqrt{(x-xx)}} = \pi$$
$$\int \frac{x \partial x}{\sqrt{(x-xx)}} = \tfrac{1}{2} \cdot \pi$$
$$\int \frac{x^2 \partial x}{\sqrt{(x-xx)}} = \frac{1.3}{2.4} \cdot \pi$$
$$\int \frac{x^3 \partial x}{\sqrt{(x-xx)}} = \frac{1.3.5}{2.4.6} \cdot \pi$$
$$\int \frac{x^4 \partial x}{\sqrt{(x-xx)}} = \frac{1.3.5.7}{2.4.6.8} \pi;$$
$$\int \frac{x^5 \partial x}{\sqrt{(x-xx)}} = \frac{1.3.5.7.9}{2.4.6.8.10} \pi;$$
$$\vdots$$
$$\int \frac{x^m \partial x}{\sqrt{(x-xx)}} = \frac{1.3.5 \ldots (2m-1)}{2.4.6 \ldots \quad 2m} \pi.$$

Hinc ergo integralium hujusmodi formulas involventium, quae magis sunt complicata, valores, quos posito $x = 1$ recipiunt, per series succincte exprimi possunt, quem usum aliquot exemplis declaremus.

Exemplum 1.

336. *Valorem integralis* $\int \frac{\partial x}{\sqrt{(1-x^4)}}$, *posito* $x = 1$, *per seriem exhibere.*

Integrali detur haec forma $\int \frac{\partial x}{\sqrt{(1-xx)}} \cdot (1+xx)^{-\frac{1}{2}}$, ut habeamus

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} = \int \frac{\partial x}{\sqrt{(1-xx)}} \left(1 - \tfrac{1}{2} xx + \tfrac{1.3}{2.4} x^4 - \tfrac{1.3.5}{2.4.6} x^6 + \tfrac{1.3.5.7}{2.4.6.8} x^8 - \text{etc.}\right)$$

singulis ergo terminis pro casu $x = 1$ integratis, orietur

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} = \frac{\pi}{2} \left(1 - \tfrac{1}{4} + \tfrac{1.9}{4.16} - \tfrac{1.9.25}{4.16.36} + \tfrac{1.9.25.49}{4.16.36.64} - \text{etc.}\right)$$

Corollarium.

337. Simili modo pro eodem casu $x = 1$ reperitur:

$$\int \frac{x \partial x}{\sqrt{(1-x^4)}} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.} = \frac{\pi}{4}$$

$$\int \frac{xx \partial x}{\sqrt{(1-x^4)}} = \frac{\pi}{2} \left(\tfrac{1}{2} - \tfrac{1^2.3}{2^2.4} + \tfrac{1^2.3^2.5}{2^2.4^2.6} - \tfrac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8} + \text{etc.}\right)$$

$$\int \frac{x^3 \partial x}{\sqrt{(1-x^4)}} = \tfrac{2}{3} - \tfrac{4}{3.5} + \tfrac{6}{5.7} - \tfrac{8}{7.9} + \tfrac{10}{9.11} - \text{etc.}$$

est autem $\int \frac{x^3 \partial x}{\sqrt{(1-x^4)}} = \frac{1}{2} - \frac{1}{2}\sqrt{(1-x^4)}$, ideoque $= \frac{1}{2}$, posito $x = 1$, unde haec postrema series $= \frac{1}{2}$, quod manifestum est.

Exemplum 2.

338. *Valorem integralis* $\int \partial x \sqrt{\frac{1+axx}{1-xx}}$, *casu* $x = 1$, *per seriem exhibere.*

Cum sit

$$\sqrt{(1+axx)} = 1 + \tfrac{1}{2} axx - \tfrac{1.1}{2.4} a^2 x^4 + \tfrac{1.1.3}{2.4.6} a^3 x^6 - \text{etc.}$$

erit per $\int \frac{\partial x}{\sqrt{(1-xx)}}$ multiplicando et integrando

$$\int \partial x \sqrt{\frac{1+axx}{1-xx}} = \frac{\pi}{2} \left(1 + \tfrac{1.1}{2.2} a - \tfrac{1.1.1.3}{2.2.4.4} a^2 + \tfrac{1.1.1.3.3.5}{2.2.4.4.6.6} a^3 - \text{etc.}\right)$$

unde peripheriam ellipsis cognoscere licet.

Exemplum 3.

339. *Valorem integralis* $\int \frac{\partial x}{\sqrt{x(1-xx)}}$, *casu* $x = 1$, *per seriem exhibere.*

Repraesentetur haec formula ita $\int \frac{\partial x (1+x)^{-\frac{1}{2}}}{\sqrt{(x-xx)}}$, ut sit $=$

$$\int \frac{\partial x}{\sqrt{(x-xx)}} \left(1 - \tfrac{1}{2} x + \tfrac{1.3}{2.4} x^2 - \tfrac{1.3.5}{2.4.6} x^3 + \text{etc.}\right);$$

unde series haec obtinetur:

$$\frac{\partial x}{\sqrt{x(1-xx)}} = \pi\left(1 - \tfrac{1}{4} + \tfrac{1.9}{4.16} - \tfrac{1.9.25}{4.16.36} + \text{etc.}\right)$$

quae ab exemplo primo haud differt: quod non mirum, cum posito $x = zz$, haec formula ad illam reducatur.

Problema 39.

340. Valorem integralis $\int x^{m-1}\partial x(1-xx)^{n-\frac{1}{2}}$, quod posito $x = 0$ evanescat, definire casu $x = 1$.

Solutio.

Reductiones supra §. 118. datae praebent pro hoc casu

$$\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}+1} = \frac{x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2}$$

$$+ \frac{\mu+2}{m+\mu+2}\int x^{m-1}\partial x(1-xx)^{\frac{\mu}{2}}:$$

sumto ergo $\mu = 2n - 1$, erit

$$\int x^{m-1}\partial x(1-xx)^{n+\frac{1}{2}} = \frac{2n+1}{m+2n+1}\int x^{m-1}\partial x(1-xx)^{n-\frac{1}{2}}$$

posito $x = 1$. Cum igitur in praecedente problemate valor $\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}}$ sit assignatus, quem brevitatis gratia ponamus $= \mathrm{M}$, hinc ad sequentes progrediamur:

$$\int\frac{x^{m-1}\partial x}{\sqrt{(1-xx)}} = \mathrm{M};$$

$$\int x^{m-1}\partial x(1-xx)^{\frac{1}{2}} = \frac{1}{m+1}\mathrm{M};$$

$$\int x^{m-1}\partial x(1-xx)^{\frac{3}{2}} = \frac{1\;.\;3}{(m+1)(m+3)}\mathrm{M};$$

$$\int x^{m-1}\partial x(1-xx)^{\frac{5}{2}} = \frac{1\;.\;3\;.\;5}{(m+1)(m+3)(m+5)};$$

et in genere

$$\int x^{m-1}\partial x\,(1-xx)^{n-\frac{1}{2}} = \frac{1.\ \ 3.\ \ 5.\ \ldots\ (2n-1)}{(m+1)(m+3)(m+5)\ldots(m+2n-1)}\,\mathrm{M}.$$

Jam duo casus sunt perpendendi, prout $m-1$ est vel numerus par vel impar: si enim

$m-1$ sit par, erit $\mathrm{M} = \frac{1.3.5\ldots\ldots(m-2)}{2.4.6\ldots\ldots(m-1)}\cdot\frac{\pi}{2};$

$m-1$ sit impar, erit $\mathrm{M} = \frac{2.4.6\ldots\ldots(m-2)}{3.5.7\ldots\ldots(m-1)}.$

Hinc sequentes deducuntur valores:

$$\int\partial x\sqrt{(1-xx)} = \frac{\pi}{4}$$
$$\int x^2\partial x\sqrt{(1-xx)} = \frac{1}{4}\cdot\frac{\pi}{4}$$
$$\int x^4\partial x\sqrt{(1-xx)} = \frac{1.3}{4.6}\cdot\frac{\pi}{4}$$
$$\int x^6\partial x\sqrt{(1-xx)} = \frac{1.3.5}{4.6.8}\cdot\frac{\pi}{4}$$

$$\int x\,\partial x\sqrt{(1-xx)} = \frac{1}{3}$$
$$\int x^3\partial x\sqrt{(1-xx)} = \frac{1}{3}\cdot\frac{2}{5}$$
$$\int x^5\partial x\sqrt{(1-xx)} = \frac{1}{3}\cdot\frac{2.4}{5.7}$$
$$\int x^7\partial x\sqrt{(1-xx)} = \frac{1}{3}\cdot\frac{2.4.6}{5.7.9}$$

$$\int\partial x\,(1-xx)^{\frac{3}{2}} = \frac{3\pi}{16}$$
$$\int x^2\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1}{6}\cdot\frac{3\pi}{16}$$
$$\int x^4\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1.3}{6.8}\cdot\frac{3\pi}{16}$$
$$\int x^6\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1.3.5}{6.8.10}\cdot\frac{3\pi}{16}$$

$$\int x\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1}{5}$$
$$\int x^3\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1}{5}\cdot\frac{2}{7}$$
$$\int x^5\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1}{5}\cdot\frac{2.4}{7.9}$$
$$\int x^7\partial x\,(1-xx)^{\frac{3}{2}} = \frac{1}{5}\cdot\frac{2.4.6}{7.9.11}$$

$$\int\partial x\,(1-xx)^{\frac{5}{2}} = \frac{5\pi}{32}$$
$$\int x^2\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1}{8}\cdot\frac{5\pi}{32}$$
$$\int x^4\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1.3}{8.0}\cdot\frac{5\pi}{32}$$
$$\int x^6\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1.3.5}{8.10.12}\cdot\frac{5\pi}{32}$$

$$\int x\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1}{7}$$
$$\int x^3\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2}{9}$$
$$\int x^5\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2.4}{9.11}$$
$$\int x^7\partial x\,(1-xx)^{\frac{5}{2}} = \frac{1}{7}\cdot\frac{2.4.6}{9.11.13}$$

etc.

Problema 40.

341. Valores integralium $\int \frac{x^m \partial x}{\sqrt[3]{(1-x^3)}}$ et $\int \frac{x^m \partial x}{\sqrt[3]{(1-x^3)^2}}$, posito $x = 1$, assignare.

Solutio.

Ponamus pro casibus simplicissimis:

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = A; \int \frac{x \partial x}{\sqrt[3]{(1-x^3)}} = B; \int \frac{xx \partial x}{\sqrt[3]{(1-x^3)}} = C;$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = B'; \int \frac{x \partial x}{\sqrt[3]{(1-x^3)^2}} = B'; \int \frac{xx \partial x}{\sqrt[3]{(1-x^3)^2}} = C'$$

et ex reductione prima §. 118. posito $a = 1$ et $b = -1$, pro casu $x = 1$ habemus

$$\int x^{m+n-1} \partial x (1 - x^n)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu + n\mu + n\nu} \int x^{m-1} \partial x (1 - x^n)^{\frac{\mu}{\nu}},$$

ergo pro priori ubi $n = 3$, $\nu = 3$ et $\mu = -1$,

$$\int x^{m+2} \partial x (1 - x^3)^{-\frac{1}{3}} = \frac{m}{m+2} \int x^{m-1} \partial x (1 - x^3)^{-\frac{1}{3}}$$

et pro posteriori, ubi $n = 3$, $\nu = 3$ et $\mu = -2$

$$\int x^{m+2} \partial x (1 - x^3)^{-\frac{2}{3}} = \frac{m}{m+1} \int x^{m-1} \partial x (1 - x^3)^{-\frac{2}{3}}$$

hinc obtinemus pro forma priori:

$\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = A$	$\int \frac{x \partial x}{\sqrt[3]{(1-x^3)}} = B$	$\int \frac{xx \partial x}{\sqrt[3]{(1-x^3)}} = C$
$\int \frac{x^3 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1}{3} A$	$\int \frac{x^4 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2}{4} B$	$\int \frac{x^5 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3}{5} C$
$\int \frac{x^6 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4}{3.6} A$	$\int \frac{x^7 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5}{4.7} B$	$\int \frac{x^8 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3.6}{5.8} C$
$\int \frac{x^9 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4.7}{3.6.9} A$	$\int \frac{x^{10} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5.8}{4.7.10} B$	$\int \frac{x^{11} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3.6.9}{5.8.11} C$
$\int \frac{x^{12} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4.7.10}{3.6.9.12} A$	$\int \frac{x^{13} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5.8.11}{4.7.10.13} B$	$\int \frac{x^{14} \partial x}{\sqrt[3]{(1-x^3)}} = \frac{3.6.9.12}{5.8.11.14} C$

etc.

at pro forma posteriori

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = A' \qquad \int \frac{x\partial x}{\sqrt[3]{(1-x^3)^2}} = B' \qquad \int \frac{xx\partial x}{\sqrt[3]{(1-x^3)^2}} = C'$$

$$\int \frac{x^3\partial x}{\sqrt[3]{(1-x^3)^2}} = \text{[illegible]} \qquad \int \frac{x^4\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{2}{3}B' \qquad \int \frac{x^5\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{3}{4}C'$$

$$\int \frac{x^6\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4}{2.5}A' \qquad \int \frac{x^7\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5}{3.6}B' \qquad \int \frac{x^8\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6}{4.7}C'$$

$$\int \frac{x^9\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4.7}{2.5.8}A' \qquad \int \frac{x^{10}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5.8}{3.6.9}B' \qquad \int \frac{x^{11}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6.9}{4.7.10}C'$$

$$\int \frac{x^{12}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4.7.10}{2.5.8.11}A' \qquad \int \frac{x^{13}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5.8.11}{3.6.9.12}B' \qquad \int \frac{x^{14}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6.9.12}{4.7.10.13}C'$$

unde concludimus fore generaliter:

$$\int \frac{x^{3n}\partial x}{\sqrt[3]{(1-x^3)}} = \frac{1.4.7\ldots(3n-2)}{3.6.9\ldots3n}A \qquad \int \frac{x^{3n}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1.4.7\ldots(3n-2)}{2.5.8\ldots(3n-1)}A'$$

$$\int \frac{x^{3n+1}\partial x}{\sqrt[3]{(1-x^3)}} = \frac{2.5.8\ldots(3n-1)}{4.7.10\ldots(3n+1)}B \qquad \int \frac{x^{3n+1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2.5.8\ldots(3n-1)}{3.6.9\ldots3n}B'$$

$$\int \frac{x^{3n+2}\partial x}{\sqrt[3]{(1-x^3)}} = \frac{3.6.9\ldots3n}{5.8.11\ldots(3n+2)}C \qquad \int \frac{x^{3n+2}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{3.6.9\ldots3n}{4.7.10\ldots(3n+1)}C'$$

notandum autem est esse $C = \frac{1}{2}$ et $C' = 1$.

Corollarium 1.

342. Hae formulae variis modis combinari possunt, ut egregia Theoremata inde oriantur, erit scilicet;

$$\int \frac{x^{3n}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n+2}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{AC'}{3n+1} = \frac{1}{3n+1}\int \frac{\partial x}{\sqrt[3]{(1-x^3)}}.$$

$$\int \frac{x^{3n+1}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{AB}{3n+1} = \frac{1}{3n+1}\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x\partial x}{\sqrt[3]{(1-x^3)}}$$

$$\int \frac{x^{3n+2}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n+1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2B'C}{3n+2} = \frac{1}{3n+2}\int \frac{x\partial x}{\sqrt[3]{(1-x^3)^2}}.$$

Corollarium 2.

343. Quia nunc ratio exponentium ad ternarium non amplius in computum ingreditur, erit generaliter:

$$\int \frac{x^{\lambda-1}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda+1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{\partial x}{\sqrt[3]{(1-x^3)}}.$$

$$\int \frac{x^{\lambda}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda-1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{x\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$$

$$\int \frac{x^{\lambda}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda-1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{x\partial x}{\sqrt[3]{(1-x^3)^2}}.$$

quare ex binis postremis consequimur

$$\int \frac{x\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = \int \frac{x\partial x}{\sqrt[3]{(1-x^3)^2}}$$

Corollarium 3.

344. Ponatur $x = z^n$ et $\lambda n = m$, et nostra Theoremata sequentes induent formas:

$$\int \frac{z^{m-1}\partial z}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{m+n-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}} = \frac{1}{m}\int \frac{z^{n-1}\partial z}{\sqrt[3]{(1-z^{3n})}}$$

$$\int \frac{z^{m+n-1}\partial z}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{m-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}} = \frac{n}{m}\int \frac{z^{2n-1}\partial z}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{n-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}}$$

$$= \frac{1}{m}\int \frac{z^{2n-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}}.$$

Problema 41.

345. Dato integrali $\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$, assignare integrale hujus formulae $\int \frac{x^{m+\lambda n-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$, posito $x = 1$.

Solutio.

Ut integrale sit finitum necesse est, ut m et k sint numeri positivi. Cum igitur per reductionem generalem sit

$$\int x^{m+n-1}\partial x(1-x^n)^{\frac{\mu}{\nu}} = \frac{m\nu}{m\nu+n(\mu+\nu)}\int x^{m-1}\partial x(1-x^n)^{\frac{\mu}{\nu}};$$

ponatur $\nu = n$ et $\mu = k-n$; ut sit $\mu+\nu = k$, erit

$$\int \frac{x^{m+n-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \frac{m}{m+k}\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}.$$

Ponatur ergo hujus formulae valor, quia datur, $= A$, haecque reductio repetita continuo dabit, posito brevitatis gratia P pro $(1-x^n)^{\frac{n-k}{n}}$,

$$\int \frac{x^{m-1}\partial x}{P} = A$$

$$\int \frac{x^{m+n-1}\partial x}{P} = \frac{m}{m+k}A$$

$$\int \frac{x^{m+2n-1}\partial x}{P} = \frac{m(m+n)}{(m+k)(m+n+k)}A$$

$$\int \frac{x^{m+3n-1}\partial x}{P} = \frac{m(m+n)(m+2n)}{(m+k)(m+n+k)(m+2n+k)}A$$

$$\int \frac{x^{m+\alpha n-1}\partial x}{P} = \frac{m(m+n)(m+2n)\ldots\ldots[m+(\alpha-1)n]}{(m+k)(m+n+k)(m+2n+k)\ldots\ldots[m+(\alpha-1)n+k]}A.$$

$$\int \frac{x^{3n+2}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{3n+1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2B'C}{3n+2} = \frac{1}{3n+2}\int \frac{x\partial x}{\sqrt[3]{(1-x^3)^2}}.$$

Corollarium 2.

343. Quia nunc ratio exponentium ad ternarium non amplius in computum ingreditur, erit generaliter:

$$\int \frac{x^{\lambda-1}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda+1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{\partial x}{\sqrt[3]{(1-x^3)}}$$

$$\int \frac{x^{\lambda}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda-1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{x\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$$

$$\int \frac{x^{\lambda}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\lambda-1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\lambda}\int \frac{x\partial x}{\sqrt[3]{(1-x^3)^2}}.$$

quare ex binis postremis consequimur

$$\int \frac{x\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = \int \frac{x\partial x}{\sqrt[3]{(1-x^3)^2}}$$

Corollarium 3.

344. Ponatur $x = z^n$ et $\lambda n = m$, et nostra Theoremata sequentes induent formas:

$$\int \frac{z^{m-1}\partial z}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{m+n-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}} = \frac{1}{m}\int \frac{z^{n-1}\partial z}{\sqrt[3]{(1-z^{3n})}}$$

$$\int \frac{z^{m+n-1}\partial z}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{m-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}} = \frac{n}{m}\int \frac{z^{2n-1}\partial z}{\sqrt[3]{(1-z^{3n})}} \cdot \int \frac{z^{n-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}}$$

$$= \frac{1}{m}\int \frac{z^{2n-1}\partial z}{\sqrt[3]{(1-z^{3n})^2}}.$$

Hinc si sumamus $m + k = n$, seu $m = n - k$, ob $\int \frac{x^{n-1}\partial x}{(1-x^n)^{\frac{k}{n}}} = \frac{1-(1-x^n)^{\frac{n-k}{n}}}{n-k} = \frac{1}{n-k}$, posito $x = 1$, erit

$$\int \frac{x^{\mu-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} \cdot \int \frac{x^{\mu+k-1}\partial x}{(1-x^n)^{\frac{k}{n}}} = \frac{1}{\mu}\int \frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \frac{\pi}{\mu n \sin. \frac{k\pi}{n}}.$$

Ac posito $x = z^{\nu}$, tum vero $\mu n = p$, $\nu n = q$, et $k = \lambda n$, habebitur:

$$\int \frac{z^{p-1}\partial z}{(1-z^q)^{1-\lambda}} \cdot \int \frac{z^{p+\lambda q-1}\partial z}{(1-z)^{\lambda}} = \frac{n}{p}\int \frac{z^{(1-\lambda)q-1}\partial z}{(1-z^q)^{1-\lambda}}.$$

Scholion 1.

349. Theoremata particuliaria, quae hinc consequuntur, ita se habebunt:

I. $n=2$; $k=1$; $\int \frac{x^{\mu-1}\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^{\mu}\partial x}{\sqrt{(1-xx)}} = \frac{1}{\mu}\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2\mu}$

II. $n=3$; $k=1$; $\int \frac{x^{\mu-1}\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x^{\mu}\partial x}{\sqrt[3]{(1-x^3)}} = \frac{1}{\mu}\int \frac{x\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{2\pi}{3\mu\sqrt{3}}$

$n=3$; $k=2$; $\int \frac{x^{\mu-1}\partial x}{\sqrt[3]{(1-x^3)}} \cdot \int \frac{x^{\mu+1}\partial x}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{\mu}\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\mu\sqrt{3}}$

III. $n=4$; $k=1$; $\int \frac{x^{\mu-1}\partial x}{\sqrt[4]{(1-x^4)^3}} \cdot \int \frac{x^{\mu}\partial x}{\sqrt[4]{(1-x^4)}} = \frac{1}{\mu}\int \frac{xx\partial x}{\sqrt[4]{(1-x^4)^3}} = \frac{\pi}{2\mu\sqrt{2}}$

$n=4$; $k=2$; $\int \frac{x^{\mu-1}\partial x}{\sqrt{(1-x^4)}} \cdot \int \frac{x^{\mu+1}\partial x}{\sqrt{(1-x^4)}} = \frac{1}{\mu}\int \frac{x\partial x}{\sqrt{(1-x^4)}} = \frac{\pi}{4\mu}$

$n=4$; $k=3$; $\int \frac{x^{\mu-1}\partial x}{\sqrt[4]{(1-x^4)}} \cdot \int \frac{x^{\mu+2}\partial x}{\sqrt[4]{(1-x^4)^3}} = \frac{1}{\mu}\int \frac{\partial x}{\sqrt[4]{(1-x^4)}} = \frac{\pi}{2\mu\sqrt{2}}$

etc.

Ubi notandum est, formulam $\int \frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$ ad rationalitatem reduci posse. Ponatur enim $\frac{x^n}{1-x^n} = z^n$, seu $x^n = \frac{z^n}{1+z^n}$, unde $\frac{\partial x}{x} = \frac{\partial z}{z(1+z^n)}$. Quare cum formula nostra sit $= \int \left(\frac{x^n}{1-x^n}\right)^{\frac{n-k}{n}} \cdot \frac{\partial x}{x}$, evadet ea $= \int \frac{z^{n-k-1}\partial z}{1+z^n}$, cujus integrale ita determinari debet, ut evanescat posito $x=0$ ideoque $z=0$; tum verò posito $x=1$, hoc est $z=\infty$ dabit valorem, quo hic utimur. Mox autem ostendemus valorem hujus integralis $\int \frac{z^{n-k-1}\partial z}{1+z^n}$, posito $z=\infty$, ideoque et hujus $\int \frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$ per angulos exprimi posse, quorum valores hic statim apposui. Deinde etiam notari meretur formulae $\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}}$ haec transformatio oriunda, posito $1-x^n = z^n$, quae praebet $-\int \frac{z^{k-1}\partial z}{(1-z^n)^{\frac{n-m}{n}}}$ ita integranda, ut evanescat posito $x=0$ seu $z=1$, tum vero statui debet $x=1$ seu $z=0$. Quod eodem redit, ac si mutato signo haec formula $\int \frac{z^{k-1}\partial z}{(1-z^n)^{\frac{n-m}{n}}}$ ita integretur, ut evanescat, posito $z=0$, tum vero ponatur $z=1$. Cum jam nihil impediat quo minus loco z scribamus x, habebimus hoc insigne Theorema:

$$\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}\partial x}{(1-x^n)^{\frac{n-m}{n}}},$$

ita ut in hujusmodi formula exponentes m et k inter se commutare liceat, pro casu scilicet $x = 1$. Ita pro praecedente formula ad rationalitatem reducibili, ubi $m = n - k$, erit

$$\int \frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}\partial x}{(1-x^n)^{\frac{k}{n}}},$$

unde sequitur etiam fore, posito $z = \infty$,

$$\int \frac{z^{n-k-1}\partial z}{1+z^n} = \int \frac{z^{k-1}\partial z}{1+z^n}.$$

Scholion 2.

350. Hinc etiam formularum magis compositarum integralia pro casu $x = 1$, per series concinnas exprimi possunt. Cum enim in reductione superiori, posito $m + k = \mu$ seu $k = \mu - m$, sit

$$\int \frac{x^{m+n-1}\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}} = \frac{m}{\mu}\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}},$$

si habeatur hujusmodi formula differentialis

$$\partial y = \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}}(A + Bx^n + Cx^{2n} + Dx^{3n} + \text{etc.})$$

quam ita integrari oporteat, ut y evanescat posito $x = 0$, ac requiratur valor ipsius y casu $x = 1$, erit si hoc casu fieri ponamus $\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}} = O$, iste valor $=$

$$O\left(A + \frac{m}{\mu}B + \frac{m(m+n)}{\mu(\mu+n)}C + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}D + \text{etc.}\right)$$

Vicissim ergo proposita hac serie

$$A + \frac{m}{\mu}B + \frac{m(m+n)}{\mu(\mu+n)}C + \frac{m(m+n)(m+2n)}{\mu(\mu+n)(\mu+2n)}D + \text{etc.}$$

cujus summa aequabitur huic formulae integrali

$$\frac{1}{0}\int\frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m+n-\mu}{n}}}(A+Bx^n+Cx^{2n}+Dx^{3n}+\text{etc.})$$

si post integrationem ponatur $x=1$. Quod si ergo eveniat, ut hujus seriei $A+Bx^n+Cx^{2n}+$ etc. summa assignari, indeque integratio absolvi queat, obtinebitur summa illius seriei.

Problema 42.

351. Integralis hujus formulae $\frac{x^{m-1}\partial x}{1+x^n}$ ita determinatum, ut posito $x=0$ evanescat, valorem casu $x=\infty$ assignare.

Solutio.

Hujus formulae integrale jam supra §. 77. exhibuimus, et quidem ita determinatum, ut posito $x=0$ evanescat, quod posito brevitatis gratia $\frac{\pi}{n}=\omega$, ita se habet:

$$-\frac{2}{n}\cos.m\omega\, l\sqrt{(1-2x\cos.\omega+xx)}+\frac{2}{n}\sin.m\omega.\text{Arc.tang.}\frac{x\sin.\omega}{1-x\cos.\omega}$$
$$-\frac{2}{n}\cos.3m\omega\, l\sqrt{(1-2x\cos.3\omega+xx)}+\frac{2}{n}\sin.3m\omega.\text{Arc.tang.}\frac{x\sin.3\omega}{1-x\cos.3\omega}$$
$$-\frac{2}{n}\cos.5m\omega\, l\sqrt{(1-2x\cos.5\omega+xx)}+\frac{2}{n}\sin.5m\omega.\text{Arc.tang.}\frac{x\sin.5\omega}{1-x\cos.5\omega}$$
$$\vdots$$
$$-\frac{2}{n}\cos.\lambda m\omega\, l\sqrt{(1-2x\cos.\lambda\omega+xx)}+\frac{2}{n}\sin.\lambda m\omega.\text{Arc.tang.}\frac{x\sin.\lambda\omega}{1-x\cos.\lambda\omega}$$

ubi λ denotat maximum numerum imparem exponente n minorem, ac si n fuerit ipse numerus impar, insuper accedit pars $\pm\frac{1}{n}l(1+x)$, prout m fuerit vel numerus impar, vel par; illo scilicet casu signum $+$, hoc vero signum $-$ valet. Hic igitur quaeritur istius integralis valor, qui prodit posito $x = \infty$. Primo ergo partes logarithmos implicantes expendamus, et quia ob $x = \infty$ est

$l\sqrt{(1 - 2x\cos.\lambda\omega + xx)} = l(x - \cos.\lambda\omega) = lx + l(1 - \frac{\cos.\lambda\omega}{x}) = lx,$

ob $\frac{\cos.\lambda\omega}{x} = 0$; unde partes logarithmicae praebent:

$$-\frac{2lx}{n}(\cos.m\omega + \cos.3m\omega + \cos.5m\omega + \ldots + \cos.\lambda m\omega)$$
$$(\pm\frac{lx}{n}, \text{ si } n \text{ impar}).$$

Ponamus hanc seriem cosinuum

$$\cos.m\omega + \cos.3m\omega + \cos.5m\omega + \ldots + \cos.\lambda m\omega = s,$$

eritque per $2\sin.m\omega$ multiplicando

$$2s\sin.m\omega = \sin.2m\omega + \sin.4m\omega + \sin.6m\omega \ldots + \sin.(\lambda + 1)m\omega$$
$$- \sin.2m\omega - \sin.4m\omega - \sin.6m\omega,$$

unde fit $s = \frac{\sin.(\lambda + 1)m\omega}{2\sin.m\omega}$. Quare si n sit numerus par, erit $\lambda = n - 1$, sicque partes logarithmicae fiunt

$$-\frac{lx}{n}\cdot\frac{\sin.nm\omega}{\sin.m\omega} = -\frac{lx}{n}\cdot\frac{\sin.m\pi}{\sin.m\omega}, \text{ ob } n\omega = \pi.$$

At propter m numerum integrum, est $\sin.m\pi = 0$, unde hae partes evanescunt. Sin autem sit n numerus impar, est $\lambda = n - 2$, et summa partium logarithmicarum fit

$$-\frac{lx}{n}\cdot\frac{\sin.(n-1)m\omega}{\sin.m\omega} \pm \frac{lx}{n};$$

at $\sin.(n-1)m\omega = \sin.(m\pi - m\omega) = \pm\sin.m\omega$, ubi signum superius valet, si m sit numerus impar, contra vero inferius, quod idem de altera ambiguitate est tenendum, ita ut habeamus $\mp\frac{lx}{n}\cdot\frac{\sin.m\omega}{\sin.m\omega} \pm \frac{lx}{n} = 0$. Perpetuo ergo partes logarithmicae se mutuo tollunt; quod etiam inde est perspicuum, quod alioquin integrale foret infinitum, cum tamen manifesto debeat esse finitum.

Relinquuntur ergo soli anguli, quos in unam summam colligamus; consideretur ergo Arc. tang. $\frac{x\sin.\lambda\omega}{1 - x\cos.\lambda\omega}$, qui arcus casu $x = 0$ evanescit, tum vero casu $x = \frac{1}{\cos.\lambda\Phi}$ fit quadrans, ulterius ergo aucta x quadrantem superabit, donec facto $x = \infty$, ejus tangens fiat $= -\frac{\sin.\lambda\omega}{\cos.\lambda\omega} = -\text{tang.}\,\lambda\omega = \text{tang.}\,(\pi - \lambda\omega)$, ideoque ipse arcus $= \pi - \lambda\omega$, ex quo hi arcus junctim sumti dabunt:

$$\frac{2}{n}[(\pi - \omega)\sin.m\omega + (\pi - 3\omega)\sin.3m\omega + (\pi - 5\omega)\sin.5m\omega + \ldots$$
$$\ldots\ldots + (\pi - \lambda\omega)\sin.\lambda m\omega]:$$

unde duas series adipiscimur

$$\frac{2\pi}{n}(\sin.m\omega + \sin.3m\omega + \sin.5m\omega + \ldots\ldots$$
$$\ldots\ldots + \sin.\lambda m\omega) = \frac{2\pi}{n}p;$$
$$\frac{-2\omega}{n}(\sin.m\omega + 3\sin.3m\omega + 5\sin.5m\omega + \ldots\ldots$$
$$\ldots\ldots + \lambda\sin.\lambda m\omega) = \frac{-2\omega}{n}q;$$

quas seorsim investigemus, ac pro posteriori quidem cum ante habuissemus

$$\cos.m\omega + \cos.3m\omega + \cos.5m\omega + \ldots\ldots$$
$$\ldots\ldots + \cos.\lambda m\omega = s = \frac{\sin.(\lambda+1)m\omega}{2\sin.m\omega},$$

si angulum ω ut variabilem spectemus, differentiatio praebet

$$-m\partial\omega(\sin.m\omega + 3\sin.3m\omega + 5\sin.5m\omega + \ldots + \lambda\sin.\lambda m\omega)$$
$$= \frac{(\lambda+1)m\partial\omega\cos.(\lambda+1)m\omega}{2\sin.m\omega} - \frac{m\partial\omega\sin.(\lambda+1)m\omega\cos.m\omega}{2\sin.m\omega^2}$$

ergo

$$-q = \frac{(\lambda+1)\cos.(\lambda+1)m\omega}{2\sin.m\omega} - \frac{\sin.(\lambda+1)m\omega\cos.m\omega}{2\sin.m\omega^2}, \text{ seu}$$
$$-q = \frac{\lambda\cos.(\lambda+1)m\omega}{2\sin.m\omega} - \frac{\sin.\lambda m\omega}{2\sin.m\omega^2}.$$

Pro altera serie

$$p = \sin.m\omega + \sin.3m\omega + \sin.5m\omega + \ldots + \sin.\lambda m\omega,$$

multiplicemus utrinque per $2\sin.m\omega$, fietque

$$2p\sin.m\omega = 1 - \cos.2m\omega - \cos.4m\omega - \cos.6m\omega - \ldots - \cos.(\lambda+1)m\omega$$
$$+ \cos.2m\omega + \cos.4m\omega + \cos.6m\omega$$

sicque erit $p = \frac{1 - \cos.(\lambda+1)m\omega}{2\sin.m\omega}$.

Quodsi jam fuerit n numerus par, erit $\lambda = n - 1$, indeque

$\cos. (\lambda + 1) m\omega = \cos. nm\omega = \cos. m\pi$, et

$\sin. (\lambda + 1) m\omega = \sin. m\pi = 0$, ergo

$$p = \frac{1 - \cos. m\pi}{2 \sin. m\omega} \text{ et } -q = \frac{n \cos. m\pi}{2 \sin. m\omega};$$

hincque omnes arcus junctim sumti

$$\frac{2\pi}{n} \cdot \frac{(1 - \cos. m\pi)}{2 \sin. m\omega} + \frac{2\omega}{n} \cdot \frac{n \cos. m\pi}{2 \sin. m\omega} = \frac{\pi}{n \sin. m\omega}, \text{ ob } n\omega = \pi.$$

Sit nunc n numerus impar, erit $\lambda = n - 2$, indeque

$\cos. (\lambda + 1) m\omega = \cos. (m\pi - m\omega)$, et

$\sin. (\lambda + 1) m\omega = \sin. (m\pi - m\omega)$, seu

$\cos. (\lambda + 1) m\omega = \cos. m\pi \cos. m\omega$, et

$\sin. (\lambda + 1) m\omega = - \cos. m\pi \sin. m\omega$, ergo

$$p = \frac{1 - \cos. m\pi \cos. m\omega}{2 \sin. m\omega} \text{ et } -q = \frac{(n-1) \cos. m\pi \cos. m\omega}{2 \sin. m\omega} + \frac{\cos. m\pi \cos. m\omega}{2 \sin. m\omega}:$$

unde summa omnium angulorum

$$\frac{\pi (1 - \cos. m\pi \cos. m\omega)}{n \sin. m\omega} + \frac{\omega (n-1) \cos. m\pi \cos. m\omega}{n \sin. m\omega} + \frac{\omega \cos. m\pi \cos. m\omega}{n \sin. m\omega},$$

quae ob $n\omega = \pi$ reducitur ad $\frac{\pi}{n \sin. m\omega}$.

Sive ergo exponens n sit positivus sive negativus, posito $x = \infty$ habemus

$$\int \frac{x^{m-1} \partial x}{1 + x^n} = \frac{\pi}{n \sin. m\omega} = \frac{\pi}{n \sin. \frac{m\pi}{n}}.$$

Corollarium 1.

252. Hinc ergo erit formula supra memorata (349)

$$\int \frac{z^{n-k-1} \partial z}{1 + z^n} = \int \frac{z^{k-1} \partial z}{1 + z^n} = \frac{\pi}{n \sin. \frac{(n-k)\pi}{n}} = \frac{\pi}{n \sin. \frac{k\pi}{n}}, \text{ posito } z = \infty.$$

Unde sequitur fore etiam formulam, cui hanc aequari ostendimus:

$$\int \frac{x^{n-k-1}\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \int \frac{x^{k-1}\partial x}{(1-x^n)^{\frac{k}{n}}} = \frac{\pi}{n \sin. \frac{k\pi}{n}}, \text{ posito } x = 1.$$

Corollarium 2.

353. Percurramus casus simpliciores, pro utroque formularum genere, posito $z = \infty$ et $x = 1$;

$$\int \frac{\partial z}{1+zz} = \int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2 \sin. \frac{1}{2}\pi} = \frac{\pi}{2};$$

$$\int \frac{\partial z}{1+z^3} = \int \frac{z\partial z}{1+z^3} = \int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = \int \frac{x\partial x}{\sqrt[3]{(1+x^3)^2}}$$

$$= \frac{\pi}{3 \sin. \frac{1}{3}\pi} = \frac{2\pi}{3\sqrt{3}};$$

$$\int \frac{\partial z}{1+z^4} = \int \frac{zz\partial z}{1+z^4} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)}} = \int \frac{xx\partial x}{\sqrt[4]{(1-x^4)^3}}$$

$$= \frac{\pi}{4 \sin. \frac{1}{4}\pi} = \frac{\pi}{2\sqrt{2}};$$

$$\int \frac{\partial z}{1+z^6} = \int \frac{z^4\partial z}{1+z^6} = \int \frac{\partial x}{\sqrt[6]{(1-x^6)}} = \int \frac{x^4\partial x}{\sqrt[6]{(1-x^6)^5}}$$

$$= \frac{\pi}{6 \sin. \frac{1}{6}\pi} = \frac{\pi}{3}.$$

Corollarium 3.

354. Cum sit

$$\frac{1}{(1-x^n)^{\frac{k}{n}}} = 1 + \frac{k}{n}x^n + \frac{k(k+n)}{n.2n}x^{2n} + \frac{k(k+n)(k+2n)}{n.2n.3n}x^{3n} + \text{etc.}$$

erit per $x^{k-1}\partial x$ multiplicando, tum integrando, ac $x = 1$ ponendo

$$\frac{\pi}{n \sin. \frac{k\pi}{n}} = \frac{1}{k} + \frac{k}{n(k+n)} + \frac{k(k+n)}{n.2n(k+2n)} + \frac{k(k+n)(k+2n)}{n.2n.3n.(k+3n)} + \text{etc.}$$

et loco k scribendo $n - k$ erit quoque

$$\frac{\pi}{n \sin. \frac{k\pi}{n}} = \frac{1}{n-k} + \frac{n-k}{n(2n-k)} + \frac{(n-k)(2n-k)}{n.2n.(3n-k)} + \frac{(n-k)(2n-k)(3n-k)}{n.2n.3n(4n-k)} \text{ etc.}$$

Scholion.

355. Pro formulis quantitates transcendentes continentibus supra jam praecipuos valores, quos integralia dum variabili certus quidam valor tribuitur, recipiunt, evolvimus; ita ut non opus sit hujusmodi formulas hic denuo examinare. Hinc autem intelligitur, eos valores integralis $\int X \partial x$ prae reliquis esse notatu dignos, ac plerumque multo succinctius exprimi posse, qui ejusmodi valoribus variabilis x respondent, quibus functio X vel fit infinita vel in nihilum abit. Ita integralia formularum $\int \frac{x^{m-1}\partial x}{(1-x^n)^{\frac{\mu}{\nu}}}$ et $\int \frac{z^{m-1}\partial z}{1+z^n}$, valores prae reliquis memorabiles recipiunt, si fiat $x = 1$ et $z = \infty$, ubi illius denominator evanescit, hujus vero fit infinitus. Caeterum omni attentione dignum est, quod hic ostendimus, formulae integralis $\int \frac{z^{m-1}\partial z}{1+z^n}$ valorem casu $z = \infty$ tam concinne exprimi, ut sit $\frac{\pi}{n \sin. \frac{m}{n}\pi}$, cujus demonstratio cum per tot ambages sit adstructa, merito suspicionem excitat, eam via multo faciliori confici posse, etiamsi modus nondum perspiciatur. Id quidem manifestum est, hanc demonstrationem ex ratione sinuum angulorum multiplorum peti oportere; et quoniam in Introductione sin. $\frac{m}{n}\pi$ per productum infinitorum factorum expressi, mox videbimus, inde eandem veritatem

multo facilius deduci posse, etiamsi ne hanc quidem viam pro maxime naturali haberi velim. Sequens autem caput hujusmodi investigationi destinavi, quo valores integralium, quos uti in hoc capite certo quodam casu recipiunt, per producta infinita seu ex innumeris factoribus constantia exprimere docebo; quandoquidem hinc insignia subsidia in Analysin redundant, pluraque alia incrementa inde expectari possunt.

CAPUT IX.

DE EVOLUTIONE INTEGRALIUM PER PRODUCTA INFINITA.

Problema 43.

356.

Valorem hujus integralis $\int \frac{\partial x}{\sqrt{(1-xx)}}$, quem casu $x = 1$ recipit, in productum infinitum evolvere.

Solutio.

Quemadmodum supra formulas altiores ad simplicem reduximus, ita hic formulam $\int \frac{\partial x}{\sqrt{(1-xx)}}$ continuo ad altiores perducamus. Ita cum posito $x = 1$ sit

$$\int \frac{x^{m-1}\partial x}{\sqrt{(1-xx)}} = \frac{m+1}{m}\int \frac{x^{m+1}\partial x}{\sqrt{(1-xx)}}, \text{ erit}$$

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{2}{1}\int \frac{xx\partial x}{\sqrt{(1-xx)}} = \frac{2.4}{1.3}\int \frac{x^4\partial x}{\sqrt{(1-xx)}}$$

$$= \frac{2.4.6}{1.3.5}\int \frac{x^6\partial x}{\sqrt{(1-xx)}} \text{ etc.}$$

unde concludimus fore indefinite:

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{2.4.6.8 \ldots\ldots 2i}{1.3.5.7\ldots\ldots(2i-1)}\int \frac{x^{2i}\partial x}{\sqrt{(1-xx)}}$$

atque adeo etiam si pro i sumatur numerus infinitus. Nunc simili modo a formula $\int \frac{x\partial x}{\sqrt{(1-xx)}}$ ascendamus, reperiemusque

$$\int \frac{x\partial x}{\sqrt{(1-xx)}} = \frac{3.\,5.\,7.\,9\ldots\ldots(2i+1)}{2.\,4.\,6.\,8\ldots\ldots\quad 2i}\int \frac{x^{2i+1}\partial x}{\sqrt{(1-xx)}},$$

atque observo, si i sit numerus infinitus, formulas istas

$$\int \frac{x^{2i}\partial x}{\sqrt{(1-xx)}} \text{ et } \int \frac{x^{2i+1}\partial x}{\sqrt{(1-xx)}}$$

rationem aequalitatis esse habituras. Ex reductione enim principali perspicuum est, si m sit numerus infinitus, fore

$$\int \frac{x^{m-1}\partial x}{\sqrt{(1-xx)}} = \int \frac{x^{m+1}\partial x}{\sqrt{(1-xx)}} = \int \frac{x^{m+3}\partial x}{\sqrt{(1-xx)}}$$

atque adeo in genere $\int \frac{x^{m+\mu}\partial x}{\sqrt{(1-xx)}} = \int \frac{x^{m+\nu}\partial x}{\sqrt{(1-xx)}}$ quantumvis magna fuerit differentia inter μ et ν, modo finita. Cum igitur sit $\int \frac{x^{2i}\partial x}{\sqrt{(1-xx)}} = \frac{x^{2i+1}\partial x}{\sqrt{(1-xx)}}$, si ponamus:

$$\frac{2.\,4.\,6.\ldots\ldots 2i}{1.\,3.\,5.\ldots\ldots(2i-1)} = M \text{ et } \frac{3.\,5.\,7.\,9.\ldots\ldots(2i+1)}{2.\,4.\,6.\,8.\ldots\ldots 2i} = N, \text{ erit}$$

$$\int \frac{\partial x}{\sqrt{(1-xx)}} : \int \frac{x\partial x}{\sqrt{(1-xx)}} = M : N = \frac{M}{N} : 1, \text{ posito } x = 1.$$

At est $\int \frac{x\partial x}{\sqrt{(1-xx)}} = 1$ et $\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2}$,
unde colligitur $\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{M}{N}$, quia producta M et N ex aequali factorum numero constant, si primum factorem $\frac{2}{1}$ producti M per primum factorem $\frac{3}{2}$ producti N, secundum $\frac{4}{3}$ illius, per secundum $\frac{5}{4}$ hujus et ita porro dividamus, fiet

$$\frac{M}{N} = \frac{2.\,2}{1.\,3} \cdot \frac{4.\,4}{3.\,5} \cdot \frac{6.\,6}{5.\,7} \cdot \frac{8.\,8}{7.\,9} \cdot \text{etc.}$$

unde obtinemus pro casu $x = 1$, per productum infinitum,

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{2.\,2}{1.\,3} \cdot \frac{4.\,4}{3.\,5} \cdot \frac{6.\,6}{5.\,7} \cdot \frac{8.\,8}{7.\,9} \cdot \text{etc.} = \frac{\pi}{2}.$$

Corollarium 1.

357. Pro valore ergo ipsius π idem productum infinitum elicuimus, quod olim jam Wallisius invenerat, et cujus veritatem

in Introductione confirmavimus, diversissimis viis incedentes, erit itaque

$$\pi = 2 \cdot \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdot \frac{8.8}{7.9} \cdot \text{etc.}$$

Corollarium 2.

358. Nihil interest, quonam ordine singuli factores in hoc producto disponantur, dummodo nulli relinquantur. Ita aliquot ab initio seorsim sumendo, reliqui ordine debito disponi possunt; veluti

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2.4}{3.3} \cdot \frac{4.6}{5.5} \cdot \frac{6.8}{7.7} \cdot \frac{8.10}{9.9} \cdot \text{etc. vel}$$

$$\frac{\pi}{2} = \frac{2.4}{1.3} \times \frac{2.6}{3.5} \cdot \frac{4.8}{5.7} \cdot \frac{6.10}{7.9} \cdot \frac{8.12}{9.11} \cdot \text{etc. vel}$$

$$\frac{\pi}{2} = \frac{2}{3} \times \frac{2.4}{1.5} \cdot \frac{4.6}{3.7} \cdot \frac{6.8}{5.9} \cdot \frac{8.10}{7.11} \cdot \text{etc. vel}$$

$$\frac{\pi}{2} = \frac{2.4}{3.5} \times \frac{2.6}{1.7} \cdot \frac{4.8}{3.9} \cdot \frac{6.10}{5.11} \cdot \frac{8.12}{7.13} \cdot \text{etc.}$$

Scholion.

359. Fundamentum ergo hujus evolutionis in hoc consistit, quod valor integralis $\int \frac{x^{i+\alpha}\partial x}{\sqrt{(1-xx)}}$, denotante i numerum infinitum, idem sit, utcunque numerus finitus α varietur. Atque hoc quidem ex reductione

$$\int \frac{x^{i-1}\partial x}{\sqrt{(1-xx)}} = \frac{i+1}{i} \int \frac{x^{i+1}\partial x}{\sqrt{(1-xx)}}$$

manifestum est, si pro α valores binario differentes assumantur. Deinde autem nullum est dubium, quin hoc integrale $\int \frac{x^{i+1}\partial x}{\sqrt{(1-xx)}}$ inter haec $\int \frac{x^{i}\partial x}{\sqrt{(1-xx)}}$ et $\int \frac{x^{i+2}\partial x}{\sqrt{(1-xx)}}$, quasi limites contineatur, qui cum sint inter se aequales necesse est omnes formulas intermedias iisdem quoque esse aequales. Atque hoc latius patet ad formulas magis complicatas, ita ut denotante i numerum infinitum sit

$$\int \frac{x^{i+\alpha}\,\partial x}{(1-x^n)^k} = \int \frac{x^i\,\partial x}{(1-x^n)^k}.$$

Cum enim sit

$$\int \frac{x^{m+n-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}} = \frac{m}{m+k}\int \frac{x^{m-1}\,\partial x}{(1-x^n)^{\frac{n-k}{n}}}$$

hae formulae posito $m = \infty$ sunt aequales; unde illarum quoque aequalitas casibus, quibus $\alpha = n$, vel $\alpha = 2n$, vel $\alpha = 3n$ etc. perspicitur; sin autem α medium quempiam valorem teneat formulae, ipsius quoque valor medium quoddam tenere debet inter valores aequales, ideoque ipsis erit aequalis. Hoc igitur principio stabilito sequens problema resolvere poterimus.

Problema 44.

360. Rationem horum duorum integralium

$$\int x^{m-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}} \text{ et } \int x^{\mu-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}},$$

casu $x = 1$, per productum infinitorum factorum exprimere.

Solutio.

Cum sit

$$\int x^{m-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}} = \frac{m+k}{n}\int x^{m+n-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}},$$

casu $x = 1$, valor istius integralis ad integrale infinite remotum reducetur hoc modo:

$$\int x^{m-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}}$$

$$= \frac{(m+k)(m+k+n)(m+k+2n)\ldots\ldots(m+k+in)}{m\quad(m+n)\quad(m+2n)\ldots\ldots(m+in)}\int x^{m+in+n-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}},$$

ubi i numerum infinitum denotare assumimus. Simili autem modo pro altera formula proposita erit

$$\int x^{\mu-1}\partial x(1-x^n)^{\frac{k-n}{n}}$$
$$= \frac{(\mu+k)(\mu+k+n)(\mu+k+2n)\ldots\ldots(\mu+k+in)}{\mu\quad(\mu+n)\quad(\mu+2n)\ldots\ldots(\mu+in)}\int x^{\mu+in+n-1}\partial x(1-x^n)^{\frac{k-n}{n}},$$

atque hae postremae formulae integrales ob exponentes infinitos, aequales erunt, non obstante inaequalitate numerorum m et μ: tum vero bina haec producta infinita pari factorum numero constant. Quare si singuli per singulos, hoc est primus per primum, secundus per secundum dividantur, ratio binorum integralium propositorum ita exprimetur:

$$\frac{\int x^{m-1}\partial x(1-x^n)^{\frac{k-n}{n}}}{\int x^{\mu-1}\partial x(1-x^n)^{\frac{k-n}{n}}} = \frac{\mu(m+k)}{m(\mu+k)}\cdot\frac{(\mu+n)(m+k+n)}{(m+n)(\mu+k+n)}\cdot\frac{(\mu+2n)(m+k+2n)}{(m+2n)(\mu+k+2n)}\text{ etc.}$$

si quidem ambo integralia ita determinentur, ut posito $x=0$ evanescant, tum vero statuatur $x=1$; litteris autem m, μ, n, k numeros positivos denotari necesse est.

Corollarium 1.

361. Si differentia numerorum m et μ aequetur multiplo ipsius n, in producto invento infiniti factores se destruunt, relinqueturque factorum numerus finitus, uti si $\mu=m+n$ habebitur:

$$\frac{(m+n)(m+k)}{m(m+k+n)}\cdot\frac{(m+2n)(m+k+n)}{(m+n)(m+k+2n)}\cdot\frac{(m+3n)(m+k+2n)}{(m+2n)(m+k+3n)}\text{ etc.}$$

quod reducitur ad $\frac{m+k}{m}$.

Corollarium 2.

362. Valor autem illius producti necessario est finitus, id quod tam ex formulis integralibus, quarum rationem exprimit, patet, quam inde, quod in singulis factoribus numeratores et denominatores sunt alternatim majores et minores.

Corollarium 3.

363. Si ponamus $m = 1$, $\mu = 3$, $n = 4$ et $k = 2$, erit

$$\frac{\int \frac{\partial x}{\sqrt{(1-x^4)}}}{\int \frac{x x \partial x}{\sqrt{(1-x^4)}}} = \frac{3 \cdot 3}{1 \cdot 5} \cdot \frac{7 \cdot 7}{5 \cdot 9} \cdot \frac{11 \cdot 11}{9 \cdot 13} \cdot \frac{15 \cdot 15}{13 \cdot 17} \text{ etc.}$$

supra autem invenimus productum harum binarum formularum esse $= \frac{\pi}{4}$.

Problema 45.

364. Valorem hujus integralis $\int x^{m-1} \partial x (1 - x^n)^{\frac{k-n}{n}}$, quem posito $x = 1$ recipit, per productum infinitum exprimere.

Solutio.

Cum in problemate praecedente ratio hujus integralis ad hoc alterum $\int x^{\mu-1} \partial x (1 - x^n)^{\frac{k-n}{n}}$ per productum infinitum sit assignata, in hoc exponens μ ita accipiatur, ut integrale exhiberi possit. Capiatur ergo $\mu = n$, et integrale fit =

$$C - \frac{1}{k}(1 - x^n)^{\frac{k}{n}} = \frac{1 - (1 - x^n)^{\frac{k}{n}}}{k}$$

ita determinatum, ut posito $x = 0$ evanescat: ponatur nunc, ut conditio postulat, $x = 1$, et quia hoc integrale erit $= \frac{1}{k}$, habebimus fosmulae propositae integrale casu $x = 1$, ita expressum

$$\int x^{m-1} \partial x (1 - x^n)^{\frac{k-n}{n}} = \frac{1}{k} \cdot \frac{n(m+k)}{m(k+n)} \cdot \frac{2n(m+k+n)}{(m+n)(k+2n)} \cdot \frac{3n(m+k+2n)}{(m+2n)(k+3n)} \text{ etc.}$$

quod singulos factores partiendo ita repraesentari potest

$$\int x^{m-1} \partial x (1 - x^n)^{\frac{k-n}{n}} = \frac{n}{mk} \cdot \frac{2n(m+k)}{(m+n)(k+n)} \cdot \frac{3n(m+k+n)}{(m+2n)(k+2n)} \cdot \frac{4n(m+k+2n)}{(m+3n)(k+3n)} \text{ etc.}$$

Corollarium 1.

365. Cum in hac expressione litterae m et k sint permutabiles, sequitur etiam, haec integralia posito $x=1$ inter se esse aequalia:

$$\int x^{m-1}\partial x\,(1-x^n)^{\frac{k-n}{n}}=\int x^{k-1}\partial x\,(1-x^n)^{\frac{m-n}{n}}$$

quam aequalitatem jam supra §. 349. elicuimus.

Corollarium 2.

366. Cum formulae nostrae valor, si $m=n-k$, aequalis sit valori hujus $\int\frac{z^{k-1}\partial z}{1+z^n}$ posito $z=\infty$, si ob $m+k=n$ statuamus $m=\frac{n+\alpha}{2}$ et $k=\frac{n-\alpha}{2}$, habebimus:

$$\int\frac{x^{m-1}\partial x}{(1-x^n)^{\frac{n+\alpha}{2n}}}\int=\frac{x^{k-1}\partial x}{(1-x^n)^{\frac{n-\alpha}{2n}}}=\int\frac{z^{k-1}\partial z}{1+z^n}=\int\frac{z^{m-1}\partial z}{1+z^n}$$

$$=\frac{4n}{nn-\alpha\alpha}\cdot\frac{2.4nn}{9nn-\alpha\alpha}\cdot\frac{4.6nn}{25nn-\alpha\alpha}\cdot\frac{6.8nn}{49nn-\alpha\alpha}\text{ etc.}$$

Quod productum etiam hoc modo exponi potest

$$\frac{2}{n-\alpha}\cdot\frac{2n.2n}{(n+\alpha)(3n-\alpha)}\cdot\frac{4n.4n}{(3n+\alpha)(5n-\alpha)}\cdot\frac{6n.6n}{(5n+\alpha)(7n-\alpha)}\text{ etc.}$$

quod ergo etiam exprimit valorem ipsius $\frac{\pi}{n\sin.\frac{m\pi}{n}}=\frac{\pi}{n\cos.\frac{\alpha\pi}{2n}}$ per §. 351.

Corollarium 3.

367. Vel si simpliciter ponamus $k=n-m$, fiet

$$\int\frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m}{n}}}=\int\frac{x^{n-m-1}\partial x}{(1-x^n)^{\frac{n-m}{n}}}=\int\frac{z^{m-1}\partial z}{1+z^n}=\int\frac{z^{n-m-1}\partial z}{1+z^n}$$

$$=\frac{1}{n-m}\cdot\frac{nn}{m(2n-m)}\cdot\frac{4nn}{(n+m)(3n-m)}\cdot\frac{9nn}{(2n+m)(4n-m)}\text{ etc.}$$

quae ex forma primum inventa oritur. Haec ergo aequalitas subsistit, si ponatur $x=1$ et $z=\infty$.

Scholion 1.

368. In Introductione autem pro multiplicatione angulorum inveneram

$$\sin.\frac{m\pi}{n}=\frac{m\pi}{n}\left(1-\frac{mm}{nn}\right)\left(1-\frac{mm}{4nn}\right)\left(1-\frac{mm}{9nn}\right)\left(1-\frac{mm}{16nn}\right)\text{ etc.}$$

et cum $\sin.\frac{(n-m)\pi}{n}=\sin.\frac{m\pi}{n}$, ob $n-m=k$, erit etiam

$$\sin.\frac{m\pi}{n}=\frac{k\pi}{n}\left(1-\frac{kk}{nn}\right)\left(1-\frac{kk}{4nn}\right)\left(1-\frac{kk}{9nn}\right)\left(1-\frac{kk}{16nn}\right)\text{ etc.}$$

quae reducitur ad hanc formam

$$\sin.\frac{m\pi}{n}=\frac{k\pi}{n}\cdot\frac{(n-k)(n+k)}{nn}\cdot\frac{(2n-k)(2n+k)}{4nn}\cdot\frac{(3n-k)(3n+k)}{9nn}\text{ etc.}$$

et pro k suo valore restituto

$$\sin.\frac{m\pi}{n}=\frac{\pi}{n}(n-m)\cdot\frac{m(2n-m)}{nn}\cdot\frac{(n+m)(3n-m)}{4nn}\cdot\frac{(2n+m)(4n-m)}{9nn}\text{ etc.}$$

unde manifeste pro $\frac{\pi}{n\sin.\frac{m\pi}{n}}$ idem reperitur productum, quod valorem nostrorum integralium erprimit, sicque novam habemus demonstrationem pro Theoremate illo eximio supra per multas ambages evicto, esse

$$\int\frac{x^{m-1}\partial x}{(1-x^n)^{\frac{m}{n}}}=\int\frac{x^{n-m-1}\partial x}{(1-x^n)^{\frac{n-m}{n}}}=\int\frac{z^{m-1}\partial z}{1+z^n}=\int\frac{z^{n-m-1}\partial z}{1+z^n}$$

$$=\frac{\pi}{n\sin.\frac{m\pi}{n}}.$$

Scholion 2.

369. Quo nostra formula latius pateat, ponamus $\frac{k}{n}=\frac{\mu}{\nu}$ seu $k=\frac{\mu n}{\nu}$, et nanciscemur $\int x^{m-1}\partial x\,(1-x^n)^{\frac{\mu}{\nu}-1}$

$$= \frac{\nu}{m\mu} \cdot \frac{2(m\nu + n\mu)}{(m+n)(\mu+\nu)} \cdot \frac{3[m\nu + n(\mu+\nu)]}{(m+2n)(\mu+2\nu)} \cdot \frac{4[m\nu + n(\mu+2\nu)]}{(m+3n)(\mu+3\nu)} \cdot \text{etc.}$$

$$= \frac{\nu}{m\mu} \cdot \frac{2(m\nu+n\mu)}{(m+n)(\mu+\nu)} \cdot \frac{3(m\nu+n\mu+n\nu)}{(m+2n)(\mu+2\nu)} \cdot \frac{4(m\nu+n\mu+2n\nu)}{(m+3n)(\mu+3\nu)} \cdot \frac{5(m\nu+n\mu+3n\nu)}{(m+4n)(\mu+4\nu)} \text{ etc.}$$

in qua expressione litterae m, n et μ, ν sunt permutabiles, praeterquam in primo factore, qui cum reliquis lege continuitatis non connectitur; ac si per n multiplicemus, permutabilitas erit perfecta, unde concludimus fore

$$n\int x^{m-1}\partial x\,(1-x^n)^{\frac{\mu}{\nu}-1} = \nu\int x^{\mu-1}\partial x\,(1-x^\nu)^{\frac{m}{n}-1}$$

quae aequalitas casu $\nu = n$ ad supra observatam reducitur. Caeterum juvabit casus praecipuos perpendisse, quos ex valoribus μ et ν desumamus.

Exemplum 1.

370. *Sit* $\mu = 1$ *et* $\nu = 2$, *fietque*

$$\int \frac{x^{m-1}\partial x}{\sqrt{(1-x^n)}} = \frac{2}{m} \cdot \frac{2(2m+n)}{3(m+n)} \cdot \frac{3(2m+3n)}{5(m+2n)} \cdot \frac{4(2m+5n)}{7(m+3n)} \text{ etc.}$$

$$= \frac{2}{n}\int \frac{\partial x}{\sqrt[n]{(1-x^2)^{n-m}}}$$

quae expressio ita commodius repraesentatur:

$$\int \frac{x^{m-1}\partial x}{\sqrt{(1-x^n)}} = \frac{2}{m} \cdot \frac{4(2m+n)}{3(2m+2n)} \cdot \frac{6(2m+3n)}{5(2m+4n)} \cdot \frac{8(2m+5n)}{7(2m+6n)} \text{ etc.}$$

unde sequentes casus specialissimi deducuntur:

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = 2 \cdot \frac{2.4}{3.3} \cdot \frac{4.6}{5.5} \cdot \frac{6.8}{7.7} \text{ etc.} = \int \frac{\partial x}{\sqrt{(1-xx)}}$$

$$\int \frac{\partial x}{\sqrt{(1-x^3)}} = 2 \cdot \frac{4.5}{3.8} \cdot \frac{6.11}{5.14} \cdot \frac{8.17}{7.20} \cdot \frac{10.23}{9.26} \text{ etc.} = \frac{2}{3}\int \frac{\partial x}{\sqrt[3]{(1-x^2)^2}}$$

$$\int \frac{x\partial x}{\sqrt{(1-x^3)}} = 1 \cdot \frac{4.7}{3.10} \cdot \frac{6.13}{5.16} \cdot \frac{8.19}{7.22} \cdot \frac{10.25}{9.28} \text{ etc.} = \frac{2}{3}\int \frac{\partial x}{\sqrt[3]{(1-x^2)}}$$

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} = 2 \cdot \frac{4\cdot 3}{3\cdot 5} \cdot \frac{6\cdot 7}{5\cdot 9} \cdot \frac{8\cdot 11}{7\cdot 13} \cdot \frac{10\cdot 15}{9\cdot 17} \text{ etc.} = \tfrac{1}{2}\int \frac{\partial x}{\sqrt[4]{(1-x^8)^3}}$$

$$\int \frac{x\,\partial x}{\sqrt{(1-x^4)}} = 1 \cdot \frac{4\cdot 4}{3\cdot 6} \cdot \frac{6\cdot 8}{5\cdot 10} \cdot \frac{8\cdot 12}{7\cdot 14} \cdot \frac{10\cdot 16}{9\cdot 18} \text{ etc.} = \tfrac{1}{2}\int \frac{\partial x}{\sqrt{(1-x^2)}}$$

sive $$= 1 \cdot \frac{2\cdot 4}{3\cdot 3} \cdot \frac{4\cdot 6}{5\cdot 5} \cdot \frac{6\cdot 8}{7\cdot 7} \cdot \frac{8\cdot 10}{9\cdot 9} \text{ etc.}$$

$$\int \frac{xx\,\partial x}{\sqrt{(1-x^4)}} = \tfrac{2}{3} \cdot \frac{4\cdot 5}{3\cdot 7} \cdot \frac{6\cdot 9}{5\cdot 11} \cdot \frac{8\cdot 13}{7\cdot 15} \cdot \frac{10\cdot 17}{9\cdot 19} \text{ etc.} = \tfrac{1}{2}\int \frac{\partial x}{\sqrt[4]{(1-x^8)}}$$

$$\int \frac{x^3\,\partial x}{\sqrt{(1-x^4)}} = \tfrac{2}{4} \cdot \frac{4\cdot 6}{3\cdot 8} \cdot \frac{6\cdot 10}{5\cdot 12} \cdot \frac{8\cdot 14}{7\cdot 16} \cdot \frac{10\cdot 18}{9\cdot 20} \text{ etc.} = \tfrac{1}{2}$$

Exemplum 2.

371. *Sit* $\mu = 1$ *et* $\nu = 3$, *fietque*

$$\int \frac{x^{m-1}\,\partial x}{\sqrt[3]{(1-x^n)^2}} = \frac{3}{m} \cdot \frac{2(3m+n)}{4(m+n)} \cdot \frac{3(3m+4n)}{7(m+2n)} \cdot \frac{4(3m+7n)}{10(m+3n)} \text{ etc.}$$

$$= \frac{3}{n}\int \frac{\partial x}{\sqrt[n]{(1-x^3)^{n-m}}}:$$

unde sequentes casus specialissimi deducuntur:

$$\int \frac{\partial x}{\sqrt[3]{(1-x^2)^2}} = \tfrac{3}{1} \cdot \frac{2\cdot 5}{4\cdot 3} \cdot \frac{3\cdot 11}{7\cdot 5} \cdot \frac{4\cdot 17}{10\cdot 7} \cdot \frac{5\cdot 23}{13\cdot 9} \text{ etc.} = \tfrac{3}{2}\int \frac{\partial x}{\sqrt{(1-x^3)}}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{3}{1} \cdot \frac{2\cdot 6}{4\cdot 4} \cdot \frac{3\cdot 15}{7\cdot 7} \cdot \frac{4\cdot 24}{10\cdot 10} \cdot \frac{5\cdot 33}{13\cdot 13} \text{ etc.} = \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$$

sive $$= \tfrac{3}{1} \cdot \frac{2\cdot 6}{4\cdot 4} \cdot \frac{5\cdot 9}{7\cdot 7} \cdot \frac{8\cdot 12}{10\cdot 10} \cdot \frac{11\cdot 15}{13\cdot 13} \text{ etc.}$$

$$\int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}} = \tfrac{3}{2} \cdot \frac{2\cdot 9}{4\cdot 5} \cdot \frac{3\cdot 18}{7\cdot 8} \cdot \frac{4\cdot 27}{10\cdot 11} \cdot \frac{5\cdot 36}{13\cdot 14} \text{ etc.} = \int \frac{\partial x}{\sqrt[3]{(1-x^3)}}$$

sive $$= \tfrac{3}{2} \cdot \frac{3\cdot 6}{4\cdot 5} \cdot \frac{6\cdot 9}{7\cdot 8} \cdot \frac{9\cdot 12}{10\cdot 11} \cdot \frac{12\cdot 15}{13\cdot 14} \text{ etc.}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^4)^2}} = \tfrac{3}{1} \cdot \frac{2\cdot 7}{4\cdot 5} \cdot \frac{3\cdot 19}{7\cdot 9} \cdot \frac{4\cdot 31}{10\cdot 13} \cdot \frac{5\cdot 43}{13\cdot 17} \text{ etc.} = \tfrac{3}{4}\int \frac{\partial x}{\sqrt[4]{(1-x^3)^3}}$$

$$\int \frac{xx\,\partial x}{\sqrt[3]{(1-x^4)^2}} = 1 \cdot \frac{2\cdot 13}{4\cdot 7} \cdot \frac{3\cdot 25}{7\cdot 11} \cdot \frac{4\cdot 37}{10\cdot 15} \cdot \frac{5\cdot 49}{13\cdot 19} \text{ etc.} = \tfrac{3}{4}\int \frac{\partial x}{\sqrt[4]{(1-x^3)}}$$

Exemplum 3.

372. *Sit* $\mu = 2$ *et* $\nu = 3$, *fietque*

$$\int \frac{x^{m-1}\,\partial x}{\sqrt[3]{(1-x^n)}} = \frac{3}{2m}\cdot\frac{2(3m+2n)}{5(m+n)}\cdot\frac{3(3m+5n)}{8(m+2n)}\cdot\frac{4(3m+8n)}{11(m+3n)}\text{ etc.}$$

$$= \frac{3}{n}\int \frac{x\,\partial x}{\sqrt[n]{(1-x^3)^{n-m}}}:$$

unde sequentes casus speciales deducuntur:

$$\int \frac{\partial x}{\sqrt[3]{(1-x^2)}} = \tfrac{3}{2}\cdot\frac{2.\ 7}{5.\ 3}\cdot\frac{3.\ 13}{8.\ 5}\cdot\frac{4.\ 19}{11.\ 7}\cdot\frac{5.\ 25}{14.\ 9}\text{. etc.} = \tfrac{3}{2}\int\frac{x\,\partial x}{\sqrt{(1-x^3)}}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = \tfrac{3}{2}\cdot\frac{2.\ 9}{5.\ 4}\cdot\frac{3.\ 18}{8.\ 7}\cdot\frac{4.\ 27}{11.\ 10}\cdot\frac{5.\ 36}{14.\ 13}\text{. etc.} = \int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}.$$

$$\text{sive} = \frac{3.\ 3}{2.\ 4}\cdot\frac{6.\ 6}{5.\ 7}\cdot\frac{9.\ 9}{8.\ 10}\cdot\frac{12.\ 12}{11.\ 13}\text{. etc.}$$

$$\int \frac{x\,\partial x}{\sqrt[3]{(1-x^3)}} = \tfrac{3}{4}\cdot\frac{2.\ 12}{5.\ 5}\cdot\frac{3.\ 21}{8.\ 8}\cdot\frac{4.\ 30}{11.\ 11}\cdot\frac{5.\ 39}{14.\ 14}\text{. etc.} = \int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)}}$$

$$\text{sive} = \tfrac{3}{4}\cdot\frac{4.\ 6}{5.\ 5}\cdot\frac{7.\ 9}{8.\ 8}\cdot\frac{10.\ 12}{11.\ 11}\cdot\frac{13.\ 15}{14.\ 14}\text{. etc.}$$

$$\int \frac{\partial x}{\sqrt[3]{(1-x^4)}} = \tfrac{3}{2}\cdot\frac{2.\ 11}{5.\ 5}\cdot\frac{3.\ 23}{8.\ 9}\cdot\frac{4.\ 35}{11.\ 13}\cdot\frac{5.\ 47}{14.\ 17}\text{. etc.} = \tfrac{3}{4}\int\frac{x\,\partial x}{\sqrt[4]{(1-x^3)^3}}$$

$$\int \frac{xx\,\partial x}{\sqrt[3]{(1-x^4)}} = \tfrac{1}{2}\cdot\frac{2.\ 17}{5.\ 7}\cdot\frac{3.\ 29}{8.\ 11}\cdot\frac{4.\ 41}{11.\ 15}\cdot\frac{5.\ 53}{14.\ 19}\text{. etc.} = \tfrac{3}{4}\int\frac{x\,\partial x}{\sqrt[4]{(1-x^3)}}.$$

Exemplum 4.

373. *Sit* $\mu = 1$ *et* $\nu = 4$, *fietque*

$$\int \frac{x^{m-1}\,\partial x}{\sqrt[4]{(1-x^n)^3}} = \frac{4}{m}\cdot\frac{2(4m+n)}{5(m+n)}\cdot\frac{3(4m+5n)}{9(m+2n)}\cdot\frac{4(4m+9n)}{13(m+3n)}\text{ etc.}$$

$$= \frac{4}{n}\int\frac{\partial x}{\sqrt[n]{(1-x^4)^{n-m}}}:$$

unde sequentes casus speciales prodeunt:

$$\int \frac{\partial x}{\sqrt[4]{(1-x^2)^3}} = \frac{4}{1} \cdot \frac{2.6}{5.3} \cdot \frac{3.14}{9.5} \cdot \frac{4.22}{13.7} \cdot \frac{5.30}{17.9} \cdot \text{etc.} = 2\int \frac{\partial x}{\sqrt{(1-x^4)}}$$

$$\text{seu} \quad = \frac{4}{1} \cdot \frac{4.3}{3.5} \cdot \frac{6.7}{5.9} \cdot \frac{8.11}{7.13} \cdot \frac{10.15}{9.17} \cdot \text{etc.}$$

$$\int \frac{\partial x}{\sqrt[4]{(1-x^3)^3}} = \frac{4}{1} \cdot \frac{2.7}{5.4} \cdot \frac{3.19}{9.7} \cdot \frac{4.31}{13.10} \cdot \frac{5.43}{17.13} \cdot \text{etc.} = \frac{4}{3}\int \frac{\partial x}{\sqrt[3]{(1-x^4)^2}}$$

$$\int \frac{x\,\partial x}{\sqrt[4]{(1-x^3)^3}} = \frac{2}{1} \cdot \frac{2.11}{5.5} \cdot \frac{3.23}{9.8} \cdot \frac{4.35}{13.11} \cdot \frac{5.47}{17.14} \cdot \text{etc.} = \frac{4}{3}\int \frac{\partial x}{\sqrt[3]{(1-x^4)}}$$

$$\int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} = \frac{4}{1} \cdot \frac{2.8}{5.5} \cdot \frac{3.24}{9.9} \cdot \frac{4.40}{13.13} \cdot \frac{5.56}{17.17} \cdot \text{etc.} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}}$$

$$\text{seu} \quad = \frac{4}{1} \cdot \frac{4.4}{5.5} \cdot \frac{6.12}{9.9} \cdot \frac{8.20}{13.13} \cdot \frac{10.28}{17.17} \cdot \text{etc.}$$

$$\text{seu} \quad = \frac{4}{1} \cdot \frac{2.8}{5.5} \cdot \frac{6.12}{9.9} \cdot \frac{10.16}{13.13} \cdot \frac{14.20}{17.17} \cdot \text{etc.}$$

$$\int \frac{x x\,\partial x}{\sqrt[4]{(1-x^4)^3}} = \frac{4}{3} \cdot \frac{2.16}{5.7} \cdot \frac{3.32}{9.11} \cdot \frac{4.48}{13.15} \cdot \frac{5.64}{17.19} \cdot \text{etc.} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)}}$$

$$\text{seu} \quad = \frac{4}{3} \cdot \frac{4.8}{5.7} \cdot \frac{6.16}{9.11} \cdot \frac{8.24}{13.15} \cdot \frac{10.32}{17.19} \cdot \text{etc.}$$

$$\text{seu} \quad = \frac{4}{3} \cdot \frac{4.8}{5.7} \cdot \frac{8.12}{9.11} \cdot \frac{12.16}{13.15} \cdot \frac{16.20}{17.17} \cdot \text{etc.}$$

Atque in his et praecedentibus jam casus $\mu = 3$ et $\nu = 4$ est contentus.

Scholion.

374. Caeterum hae formulae, in quas litteras μ et ν introduxi, latius non patent quam primum consideratae, series enim pendent a binis fractionibus $\frac{m}{n}$ et $\frac{\mu}{\nu}$, quae cum semper ad communem denominatorem revocari queant, formulas

$$\int \frac{x^{m-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-k}}} = \int \frac{x^{k-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}$$

perpendisse sufficiet. Cum igitur earum valor casu $x = 1$ aequetur huic producto

$$\frac{1}{k} \cdot \frac{n(m+k)}{m(k+n)} \cdot \frac{2n(m+k+n)}{(m+n)(k+2n)} \cdot \frac{3n(m+k+2n)}{(m+2n)(k+3n)} \cdot \text{etc.}$$

si in singulis membris factores numeratorum permutemus, et membra aliter partiamur, idem productum hanc induet formam

$$\frac{m+k}{mk}\cdot\frac{n(m+k+n)}{(m+n)(k+n)}\cdot\frac{2n(m+k+2n)}{(m+2n)(k+2n)}\cdot\frac{3n(m+k+3n)}{(m+3n)(k+3n)}\cdot\text{etc.}$$

quae ad memoriam magis accommodata videtur. Simili modo cum sit:

$$\int\frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}=\int\frac{x^{q-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

$$=\frac{p+q}{p\,q}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\frac{3n(p+q+3n)}{(p+3n)(q+3n)}\cdot\text{etc.}$$

illam formam per hanc dividendo, erit

$$\frac{\int x^{m-1}\,\partial x\,(1-x^n)^{\frac{k-n}{n}}}{\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q-n}{n}}}$$

$$=\frac{pq(m+k)}{mk(p+q)}\cdot\frac{(p+n)(q+n)(m+k+n)}{(m+n)(k+n)(p+q+n)}\cdot\frac{(p+2n)(q+2n)(m+k+2n)}{(m+2n)(k+2n)(p+q+2n)}\cdot\text{etc.}$$

cujus omnia membra eadem lege continentur. Hinc autem eximiae comparationes hujusmodi formularum deduci possunt, quae quo facilius commemorari queant, brevitatis causa sequenti scriptionis compendio utar.

Definitio.

375. Formulae integralis $\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q-n}{n}}$ valorem, quem posito $x=1$ recipit, brevitatis gratia hoc signo $\left(\frac{p}{q}\right)$ indicemus, ubi quidem exponentem n, quem in comparatione plurium hujusmodi formularum eundem esse assumo, subintelligi oportet.

Corollarium 1.

376. Primum igitur patet esse $\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right)$, et utramque formulam esse

$$=\frac{p+q}{p.q}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\text{etc.}$$

$$\left(\frac{a}{b}\right)\left(\frac{a+b}{d}\right)=\left(\frac{b+d}{a}\right)\left(\frac{b}{d}\right).$$

II. Quia $r=b$ non differt a praecedenti ob a et b permutabiles, statuatur $r=p+q$, fietque

$$abc\,(d+p+q)=pq\,(a+b)\,(c+d).$$

Quoniam r ipsi c aequari nequit, factor $d+p+q$ neque ipsi p, neque q, neque $c+d$ aequalis poni potest, relinquitur ergo $d+p+q=a+b$, et $abc=pq\,(c+d)$, ubi quia c ipsi $c+d$ aequari nequit, ac p et q pari conditione gaudent, fiat $p=c$; erit $q=a+b-c-d$, et $ab=(c+d)(a+b-c-d)$; unde $a=c+d$; $q=b$; $p=c$; $r=b+c$; $s=d$; sicque conficitur:

$$\left(\frac{c+d}{b}\right)\left(\frac{c}{d}\right)=\left(\frac{c}{b}\right)\left(\frac{b+c}{d}\right).$$

Corollarium 1.

380. Hae solutiones eodem fere redeunt, indeque tria producta binarum formularum, aequalia eruuntur:

$$\left(\frac{c}{d}\right)\left(\frac{c+d}{b}\right)=\left(\frac{c}{b}\right)\left(\frac{b+c}{d}\right)=\left(\frac{b}{d}\right)\left(\frac{b+d}{c}\right)$$

vel in litteris p, q, r,

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right).$$

Corollarium 2.

381. Si hae formulae in producta infinita evolvantur, reperietur

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\frac{p+q+r}{pqr}\cdot\frac{nn\,(p+q+r+n)}{(p+n)\,(q+n)\,(r+n)}\cdot\frac{4nn\,(p+q+r+2n)}{(p+2n)\,(q+2n)\,(r+2n)}\ \text{etc.}$$

unde patet, tres litteras p, q, r, utcunque inter se permutari posse, atque hinc ternas illas formulas concludere licet.

Corollarium 3.

382. Restituamus ipsas formulas integrales, et sequentia tria producta erunt inter se aequalia

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \cdot \int \frac{x^{p+q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} =$$
$$\int \frac{x^{q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{q+r-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}} =$$
$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{p+r-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}.$$

Corollarium 4.

383. Hic casus notatu dignus, quo $p+q=n$, tum enim ob

$$\left(\frac{p+q}{r}\right)=\left(\frac{n}{r}\right)=\frac{1}{r} \text{ et } \left(\frac{p}{q}\right)=\frac{\pi}{n \sin.\frac{p\pi}{n}},$$

haec tria producta fient $=\frac{\pi}{nr \sin.\frac{p\pi}{n}}$. Erit scilicet

$$\int \frac{x^{n-p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{n-p+r-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}} = \int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}} \cdot \int \frac{x^{p+r-1}\partial x}{\sqrt[n]{(1-x^n)^{p}}}$$
$$= \frac{\pi}{nr \sin.\frac{p\pi}{n}}.$$

Scholion.

384. Triplex ista proprietas productorum ex binis formulis maxime est notatu digna, ac pro variis numeris loco p, q, r substituendis obtinebuntur sequentes aequalitates speciales:

p	q	r	
1		2	$\left(\frac{1}{1}\right)\left(\frac{2}{2}\right)=\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)$
1	2	2	$\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{2}{2}\right)\left(\frac{4}{1}\right)$
1	2	3	$\left(\frac{2}{1}\right)\left(\frac{3}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{1}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{2}\right)$
1		3	$\left(\frac{1}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)$
2	2	3	$\left(\frac{2}{2}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)$
1	3	3	$\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{1}\right)$
2	3	3	$\left(\frac{3}{2}\right)\left(\frac{5}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{2}\right)$
1		4	$\left(\frac{1}{1}\right)\left(\frac{4}{2}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)$
1	2	4	$\left(\frac{2}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{1}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{2}\right)$
1	3	4	$\left(\frac{3}{1}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{3}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{1}\right)$
1	4	4	$\left(\frac{4}{1}\right)\left(\frac{5}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{1}\right)$
2	2	4	$\left(\frac{2}{2}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)$
2	3	4	$\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{2}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{3}\right)$
2	4	4	$\left(\frac{4}{2}\right)\left(\frac{6}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{2}\right)$
3	3	4	$\left(\frac{3}{3}\right)\left(\frac{6}{4}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{3}\right)$
3	4	4	$\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{3}\right).$

Quae formulae pro omnibus numeris n valent, ac si numeri majores quam n occurrant, eos ad minores reduci posse supra vidimus.

Problema 47.

385. Invenire producta diversa ex ternis hujusmodi formulis, quae inter se sint aequalia.

Solutio.

Consideretur productum $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)$, quod evolutum praebet:

$$\frac{p+q+r+s}{pqrs}\cdot\frac{n^3(p+q+r+s+n)}{(p+n)(q+n)(r+n)(s+n)}\ \text{etc.}$$

quod eundem valorem retinere evidens est, quomodocunque quatuor litterae inter se commutentur. Tum vero eadem evolutio prodit ex hoc producto: $\left(\frac{p}{q}\right)\left(\frac{r}{s}\right)\left(\frac{p+q}{r+s}\right)$, ubi eadem permutatio locum habet. Aequalia ergo sunt inter se omnia haec producta:

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{p}{s}\right)\left(\frac{p+s}{q}\right)\left(\frac{p+q+s}{r}\right);$$
$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+s}{r}\right);\ \left(\frac{p}{q}\right)\left(\frac{p+r}{s}\right)\left(\frac{p+r+s}{q}\right);\ \left(\frac{p}{s}\right)\left(\frac{p+s}{r}\right)\left(\frac{p+r+s}{q}\right);$$
$$\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)\left(\frac{p+q+r}{s}\right);\ \left(\frac{q}{s}\right)\left(\frac{q+s}{p}\right)\left(\frac{p+q+s}{r}\right);\ \left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right);$$
$$\left(\frac{q}{r}\right)\left(\frac{q+r}{s}\right)\left(\frac{q+r+s}{p}\right);\ \left(\frac{q}{s}\right)\left(\frac{q+s}{r}\right)\left(\frac{q+r+s}{p}\right);\ \left(\frac{r}{s}\right)\left(\frac{r+s}{q}\right)\left(\frac{q+r+s}{p}\right).$$

Producta alterius formae ope praecedentis proprietatis hinc sponte fluunt: est enim

$$\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

Deinde vero etiam hoc productum $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)$ evolutum pro primo membro dat: $\frac{(p+q+r)(p+r+s)}{pqrs(p+r)}$, in quo tam p et r, quam q et s inter se permutare licet, ita ut sit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right).$$

Scholion.

386. Quantumvis late haec patere videantur, tamen nullas novas comparationes suppeditant, quae non jam in praecedenti contineantur. Postrema enim aequalitas

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right)$$

oritur ex multiplicatione harum

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)$$
$$\left(\frac{p}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right).$$

Priorum vero formatio ex hoc exemplo patebit,

aequalitas $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right)$

oritur ex multiplicatione harum

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r+s}\right)=\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right)$$
$$\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

Istae autem comparationes praecipue utiles sunt ad valores diversarum formularum ejusdem ordinis seu pro dato numero n invicem reducendos, ut integratio ad paucissimas revocetur, quibus datis reliquae per eas definiri queant.

Problema 48.

387. Formulas simplicissimas exhibere, ad quas integratio omnium casuum in forma $\left(\frac{p}{q}\right) = \int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$ contentorum reduci queat.

Solutio.

Primo est $\left(\frac{n}{p}\right) = \frac{1}{p}$, unde habentur hi casus

$$\left(\frac{n}{1}\right) = 1;\ \left(\frac{n}{2}\right) = \frac{1}{2};\ \left(\frac{n}{3}\right) = \frac{1}{3};\ \left(\frac{n}{4}\right) = \frac{1}{4};\ \left(\frac{n}{5}\right) = \frac{1}{5} \text{ etc.}$$

Deinde est $\left(\frac{p}{n-p}\right) = \frac{\pi}{n \sin. \frac{p\pi}{n}}$, unde omnium harum formularum valores sunt cogniti, quas indicemus:

$$\left(\frac{n-1}{1}\right) = \alpha;\ \left(\frac{n-2}{2}\right) = \beta;\ \left(\frac{n-3}{3}\right) = \gamma;\ \left(\frac{n-4}{4}\right) = \delta \text{ etc.}$$

Verum hi non sufficiunt ad reliquos omnes expediendos, praeterea tanquam cognitos spectari oportet hos:

$$\left(\frac{n-2}{1}\right) = A;\ \left(\frac{n-3}{2}\right) = B;\ \left(\frac{n-4}{3}\right) = C;\ \left(\frac{n-5}{4}\right) = D \text{ etc.}$$

atque ex his reliqui omnes determinari poterunt ope aequationum supra demonstratarum; unde potissimum has notasse juvabit:

$$\left(\frac{n-a}{a}\right)\left(\frac{n}{b}\right) = \left(\frac{n-a}{b}\right)\left(\frac{n-a+b}{a}\right);$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-a-b}{b}\right) = \left(\frac{n-b}{b}\right)\left(\frac{n-a-b}{a}\right);$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-b-1}{b}\right)\left(\frac{n-a-b}{a-1}\right) = \left(\frac{n-b}{b}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-b}{a}\right).$$

Ex harum prima posito $a = b + 1$ invenitur

$$\left(\frac{n-1}{a}\right) = \left(\frac{n-a}{a}\right)\left(\frac{n}{a-1}\right) : \left(\frac{n-a}{a-1}\right),$$

ubi $\left(\frac{n}{a-1}\right) = \frac{1}{a-1}$, ideoque per formulas assumtas definitur $\left(\frac{n-1}{a}\right)$.

Ex secunda posito $b = 1$ deducitur

$$\left(\frac{n-a-1}{1}\right) = \left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right) : \left(\frac{n-a}{a}\right).$$

Ex tertia posito $b = 1$ invenitur

$$\left(\frac{n-a-1}{a-1}\right) = \left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right) : \left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$$

sicque reperiuntur omnes formulae $\left(\frac{n-a-2}{a}\right)$, et ex his porro ponendo $b = 2$ in tertia

$$\left(\frac{n-a-2}{a-1}\right) = \left(\frac{n-2}{2}\right)\left(\frac{n-a}{n-1}\right)\left(\frac{n-a-2}{a}\right) : \left(\frac{n-a}{a}\right)\left(\frac{n-3}{2}\right)$$

unde reperiuntur formae $\left(\frac{n-a-3}{a}\right)$, et ita porro omnes $\left(\frac{n-a-b}{a}\right)$, quippe quae forma omnes complectitur. Labor autem per priores aequationes non mediocriter contrahitur. Inventa enim $\left(\frac{n-a-2}{a}\right)$ ex prima colligitur

$$\left(\frac{n-2}{a+2}\right) = \left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right) : \left(\frac{n-a-2}{a}\right)$$

ex secunda vero

$$\left(\frac{n-a-2}{3}\right) = \left(\frac{n-2}{2}\right)\left(\frac{n-a-2}{a}\right) : \left(\frac{n-a}{a}\right)$$

similique modo ex inventis formulis $\left(\frac{n-a-3}{a}\right)$ derivantur hae

$$\left(\frac{n-3}{a+3}\right) = \left(\frac{n-a-3}{a+3}\right)\left(\frac{n}{a}\right) : \left(\frac{n-a-3}{a}\right)$$
$$\left(\frac{n-a-3}{3}\right) = \left(\frac{n-3}{3}\right)\left(\frac{n-a-3}{a}\right) : \left(\frac{n-a}{a}\right).$$

Corollarium 1.

388. Ex aequatione $\left(\frac{n-1}{a}\right) = \frac{1}{a-1}\left(\frac{n-a}{a}\right) : \left(\frac{n-a}{a-1}\right)$ definiuntur

$$\left(\frac{n-1}{2}\right) = \frac{\beta}{1\,A};\ \left(\frac{n-1}{3}\right) = \frac{\gamma}{2\,B};\ \left(\frac{n-1}{4}\right) = \frac{\delta}{3\,C};\ \left(\frac{n-1}{5}\right) = \frac{\varepsilon}{4\,D};\ \text{etc.}$$

Ex aequatione vero $\left(\frac{n-a-1}{1}\right) = \left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right) : \left(\frac{n-a}{a}\right)$ hae formulae

$$\left(\frac{n-2}{1}\right)=\frac{\alpha A}{\alpha};\ \left(\frac{n-3}{1}\right)=\frac{\alpha B}{\beta};\ \left(\frac{n-4}{1}\right)=\frac{\alpha C}{\gamma};\ \left(\frac{n-5}{1}\right)=\frac{\alpha D}{\delta};\ \text{etc.}$$

Corollarium 2.

389. Aequatio

$$\left(\frac{n-a-1}{a-1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$$

praebet

$$\left(\frac{n-3}{1}\right)=\frac{\alpha AB}{\beta A};\ \left(\frac{n-4}{2}\right)=\frac{\alpha BC}{\gamma A};\ \left(\frac{n-5}{3}\right)=\frac{\alpha CD}{\delta A};\ \left(\frac{n-6}{4}\right)=\frac{\alpha DE}{\varepsilon A}\ \text{etc.}$$

unde reperiuntur pro $\left(\frac{n-2}{a+2}\right)=\left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-2}{a}\right)$ istae formulae

$$\left(\frac{n-2}{3}\right)=\frac{\gamma\beta A}{1\alpha AB};\ \left(\frac{n-2}{4}\right)=\frac{\delta\gamma A}{2\alpha BC};\ \left(\frac{n-2}{5}\right)=\frac{\varepsilon\delta A}{3\alpha CD};\ \left(\frac{n-2}{6}\right)=\frac{\zeta\varepsilon A}{4\alpha DE}\ \text{etc.}$$

atque etiam istae

$$\left(\frac{n-a-2}{2}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right),\ \text{quae sunt}$$

$$\left(\frac{n-3}{2}\right)=\frac{\beta\alpha AB}{\alpha\beta A};\ \left(\frac{n-4}{2}\right)=\frac{\beta\alpha BC}{\beta\gamma A};\ \left(\frac{n-5}{2}\right)=\frac{\beta\alpha CD}{\gamma\delta A};\ \left(\frac{n-6}{2}\right)=\frac{\beta\alpha DE}{\delta\varepsilon A}\ \text{etc.}$$

Corollarium 3.

390. Tum aequatio

$$\left(\frac{n-a-2}{a-1}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-3}{2}\right)\ \text{dat}$$

$$\left(\frac{n-4}{1}\right)=\frac{\alpha\beta ABC}{\beta\gamma AB};\ \left(\frac{n-5}{2}\right)=\frac{\alpha\beta BCD}{\gamma\delta AB};\ \left(\frac{n-6}{3}\right)=\frac{\alpha\beta CDE}{\delta\varepsilon AB};\ \left(\frac{n-7}{4}\right)=\frac{\alpha\beta DEF}{\varepsilon\zeta AB}$$

hinc $\left(\frac{n-3}{a+3}\right)=\left(\frac{n-a-3}{a+3}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-3}{a}\right)$ praebet

$$\left(\frac{n-3}{4}\right)=\frac{\beta\gamma\delta AB}{1\alpha\beta ABC};\ \left(\frac{n-3}{5}\right)=\frac{\gamma\delta\varepsilon AB}{2\alpha\beta BCD};\ \left(\frac{n-3}{6}\right)=\frac{\delta\varepsilon\zeta AB}{3\alpha\beta CDE}\ \text{etc.}$$

atque ex $\left(\frac{n-a-3}{3}\right)=\left(\frac{n-3}{3}\right)\left(\frac{n-a-3}{a}\right):\left(\frac{n-a}{a}\right)$ deducuntur

$$\left(\frac{n-5}{3}\right)=\frac{\alpha\beta\gamma BCD}{\beta\gamma\delta AB};\ \left(\frac{n-6}{3}\right)=\frac{\alpha\beta\gamma CDE}{\gamma\delta\varepsilon AB};\ \left(\frac{n-7}{3}\right)=\frac{\alpha\beta\gamma DEF}{\delta\varepsilon\zeta AB}\ \text{etc.}$$

Exemplum 1.

391. *Casus in hac forma* $\int\frac{x^{p-1}\partial x}{\sqrt[2]{(1-x^2)^{2-q}}}=\left(\frac{p}{q}\right)$ *contentos, ubi* n $=2$, *evolvere, ubi est* $\left(\frac{p+2}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right)$.

Manifestum est has formulas omnes vel algebraice vel per angulos expediri, his tamen regulis utentes, quia numeri p et q binarium superare non debent, unam formulam a circulo pendentem habemus $\left(\frac{1}{1}\right) = \frac{\pi}{2 \sin. \frac{\pi}{2}} = \frac{\pi}{2} = \alpha$, unde nostri casus erunt:

$\left(\frac{2}{1}\right) = 1$; $\left(\frac{2}{2}\right) = \frac{1}{2}$

$\left(\frac{1}{1}\right) = \alpha$.

Exemplum 2.

392. *Casus in hac forma* $\int \frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)^{3-q}}} = \left(\frac{p}{q}\right)$ *contentos, ubi* n $= 3$, *evolvere, ubi est* $\left(\frac{p+3}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right)$.

Hic casus principales, ad quos caeteri reducuntur, sunt

$$\left(\frac{2}{1}\right) = \frac{\pi}{3 \sin. \frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}} = \alpha \text{ et } \left(\frac{1}{1}\right) = A = \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}},$$

qua concessa erunt reliqui:

$\left(\frac{3}{1}\right) = 1$; $\left(\frac{3}{2}\right) = \frac{1}{2}$; $\left(\frac{3}{3}\right) = \frac{1}{3}$

$\left(\frac{2}{1}\right) = \alpha$; $\left(\frac{2}{2}\right) = \frac{\alpha}{A}$

$\left(\frac{1}{1}\right) = A$.

Exemplum 3.

393. *Casus in hac forma* $\int \frac{x^{p-1}\partial x}{\sqrt[4]{(1-x^4)^{4-q}}} = \left(\frac{p}{q}\right)$ *contentos, ubi* n $= 4$, *evolvere, ubi est* $\left(\frac{p+4}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right)$.

A circulo pendent hae duae

$$\left(\frac{3}{1}\right) = \frac{\pi}{4 \sin. \frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}} = \alpha \text{ et } \left(\frac{2}{2}\right) = \frac{\pi}{4 \sin. \frac{2\pi}{4}} = \frac{\pi}{4} = \beta,$$

praeterea vero una transcendente singulari opus est $\left(\frac{2}{1}\right)=\mathrm{A}$, unde reliquae ita determinantur:

$$\left(\tfrac{4}{1}\right)=1;\ \left(\tfrac{4}{2}\right)=\tfrac{1}{2};\ \left(\tfrac{4}{3}\right)=\tfrac{1}{3};\ \left(\tfrac{4}{4}\right)=\tfrac{1}{4}$$
$$\left(\tfrac{3}{1}\right)=\alpha;\ \left(\tfrac{3}{2}\right)=\tfrac{\beta}{\mathrm{A}};\ \left(\tfrac{3}{3}\right)=\tfrac{\alpha}{2\mathrm{A}}$$
$$\left(\tfrac{2}{1}\right)=\mathrm{A};\ \left(\tfrac{2}{2}\right)=\beta$$
$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha\mathrm{A}}{\beta}.$$

Exemplum 4.

394. *Casus in hac forma* $\int\frac{x^{p-1}\,\partial x}{\sqrt[5]{(1-x^5)^{5-q}}}=\left(\frac{p}{q}\right)$ *contentos, ubi* $n=5$, *evolvere, ubi est* $\left(\frac{p+5}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right)$.

A circulo pendent hae duae formulae:

$$\left(\tfrac{4}{1}\right)=\frac{\pi}{5\sin.\frac{\pi}{5}}=\alpha \text{ et } \left(\tfrac{3}{2}\right)=\frac{\pi}{5\sin.\frac{2\pi}{5}}=\beta,$$

praeter quas duas novas transcendentes assumi oportet

$$\left(\tfrac{3}{1}\right)=\mathrm{A} \text{ et } \left(\tfrac{2}{2}\right)=\mathrm{B},$$

per quas omnes sequenti modo determinantur

$$\left(\tfrac{5}{1}\right)=1;\ \left(\tfrac{5}{2}\right)=\tfrac{1}{2};\ \left(\tfrac{5}{3}\right)=\tfrac{1}{3};\ \left(\tfrac{5}{4}\right)=\tfrac{1}{4};\ \left(\tfrac{5}{5}\right)=\tfrac{1}{5}$$
$$\left(\tfrac{4}{1}\right)=\alpha;\ \left(\tfrac{4}{2}\right)=\tfrac{\beta}{\mathrm{A}};\ \left(\tfrac{4}{3}\right)=\tfrac{\beta}{2\mathrm{B}};\ \left(\tfrac{4}{4}\right)=\tfrac{\alpha}{3\mathrm{A}};$$
$$\left(\tfrac{3}{1}\right)=\mathrm{A};\ \left(\tfrac{3}{2}\right)=\beta;\ \left(\tfrac{3}{3}\right)=\tfrac{\beta\mathrm{B}}{\alpha\mathrm{B}}$$
$$\left(\tfrac{2}{1}\right)=\tfrac{\alpha\mathrm{B}}{\beta};\ \left(\tfrac{2}{2}\right)=\mathrm{B}$$
$$\left(\tfrac{1}{1}\right)=\tfrac{\alpha\mathrm{A}}{\beta}$$

Exemplum 5.

395. *Casus in hac forma* $\int\frac{x^{p-1}\,\partial x}{\sqrt[6]{(1-x^6)^{6-q}}}=\left(\frac{p}{q}\right)$ *contentos, ubi* $n=6$, *evolvere.*

A circulo pendent hae tres formulae:

$$\left(\tfrac{5}{1}\right) = \frac{\pi}{6\sin.\frac{\pi}{6}} = \frac{\pi}{3} = \alpha;\quad \left(\tfrac{4}{2}\right) = \frac{\pi}{6\sin.\frac{2\pi}{6}} = \frac{\pi}{3\sqrt{3}} = \beta;$$

$$\left(\tfrac{3}{3}\right) = \frac{\pi}{6\sin.\frac{3\pi}{6}} = \frac{\pi}{6} = \gamma$$

tum vero assumantur hae duae transcendentes:

$$\left(\tfrac{4}{1}\right) = A \text{ et } \left(\tfrac{3}{2}\right) = B$$

atque per has omnes sequenti modo determinantur

$$\left(\tfrac{6}{1}\right) = 1;\ \left(\tfrac{6}{2}\right) = \tfrac{1}{2};\ \left(\tfrac{6}{3}\right) = \tfrac{1}{3};\ \left(\tfrac{6}{4}\right) = \tfrac{1}{4};\ \left(\tfrac{6}{5}\right) = \tfrac{1}{5};\ \left(\tfrac{6}{6}\right) = \tfrac{1}{6}$$

$$\left(\tfrac{5}{1}\right) = \alpha;\ \left(\tfrac{5}{2}\right) = \frac{\beta}{A};\ \left(\tfrac{5}{3}\right) = \frac{\gamma}{2B};\ \left(\tfrac{5}{4}\right) = \frac{\beta}{3B};\ \left(\tfrac{5}{5}\right) = \frac{\alpha}{4A}$$

$$\left(\tfrac{4}{1}\right) = A;\ \left(\tfrac{4}{2}\right) = \beta;\ \left(\tfrac{4}{3}\right) = \frac{\beta\gamma}{\alpha B};\ \left(\tfrac{4}{4}\right) = \frac{\beta\gamma A}{2\alpha BB}$$

$$\left(\tfrac{3}{1}\right) = \frac{\alpha B}{\beta};\ \left(\tfrac{3}{2}\right) = B;\ \left(\tfrac{3}{3}\right) = \gamma$$

$$\left(\tfrac{2}{1}\right) = \frac{\alpha B}{\gamma};\ \left(\tfrac{2}{2}\right) = \frac{\alpha BB}{\gamma A}$$

$$\left(\tfrac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

Scholion.

396. Has determinationes quousque libuerit, continuare licet, in quibus praecipue notari debent casus novas transcendentium species introducentes; quorum primus occurrit si $n = 3$, estque $\left(\tfrac{1}{1}\right) = \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$, cujus valorem per productum infinitum supra vidimus esse

$$= \frac{3}{1} \cdot \frac{2 \cdot 6}{4 \cdot 4} \cdot \frac{5 \cdot 9}{7 \cdot 7} \cdot \frac{8 \cdot 12}{10 \cdot 10} \text{ etc.}$$

quod ex formula $\left(\tfrac{1}{1}\right)$, ob $n = 3$, etiam est

$$\frac{2}{1 \cdot 1} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{9 \cdot 11}{10 \cdot 10} \cdot \frac{12 \cdot 14}{13 \cdot 13} \text{ etc.}$$

Deinde ex classe $n = 4$ nascitur haec nova forma transcendens:

$$\left(\tfrac{2}{1}\right) = \int \frac{x\partial x}{\sqrt[4]{(1-x^4)^3}} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} = \int \frac{\partial x}{\sqrt{(1-x^4)}},$$

quae aequatur huic producto infinito

$$\frac{3}{1.2}\cdot\frac{4.7}{5.6}\cdot\frac{8.11}{9.10}\cdot\frac{12.15}{13.14}\cdot\frac{16.19}{17.18}\text{ etc.} = \frac{3}{2}\cdot\frac{2.7}{5.3}\cdot\frac{4.11}{9.5}\cdot\frac{6.15}{13.7}\cdot\frac{8.19}{17.9}\text{ etc.}$$

Ex classe $n = 5$ impetramus duas novas formulas transcendentes

$$\left(\frac{3}{1}\right) = \int\frac{x^2\partial x}{\sqrt[5]{(1-x^5)^4}} = \int\frac{\partial x}{\sqrt[5]{(1-x^5)^2}} = \frac{4}{1.3}\cdot\frac{5.9}{6.8}\cdot\frac{10.14}{11.13}\cdot\frac{15.19}{16.18}\text{ etc. et}$$

$$\left(\frac{2}{2}\right) = \int\frac{x\partial x}{\sqrt[5]{(1-x^5)^3}} = \frac{4}{2.2}\cdot\frac{5.9}{7.7}\cdot\frac{10.14}{12.12}\cdot\frac{15.19}{17.17}\text{ etc.}$$

ita ut sit

$$\left(\frac{3}{1}\right):\left(\frac{2}{2}\right) = \frac{2.2}{1.3}\cdot\frac{7.7}{6.8}\cdot\frac{12.12}{11.13}\cdot\frac{17.17}{16.18}\text{ etc.}$$

Classis $n = 6$ has duas formulas transcendentes suppeditat:

$$1.\ \left(\frac{4}{1}\right) = \int\frac{x^3\partial x}{\sqrt[6]{(1-x^6)^5}} = \int\frac{\partial x}{\sqrt[3]{(1-x^6)}} = \frac{1}{2}\int\frac{y\partial y}{\sqrt[6]{(1-y^3)^5}}$$

$$2.\ \left(\frac{3}{2}\right) = \int\frac{x^2\partial x}{\sqrt[3]{(1-x^6)^2}} = \int\frac{x\partial x}{\sqrt{(1-x^6)}} = \frac{1}{2}\int\frac{\partial y}{\sqrt{(1-y^3)}} = \frac{1}{3}\int\frac{\partial z}{\sqrt[3]{(1-zz)^2}}$$

sumto $y = xx$ et $z = x^3$. Notandum autem est inter has et primam $\int\frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = 2\int\frac{y\partial y}{\sqrt[6]{(1-y^6)^4}} = 2\left(\frac{2}{2}\right)$ relationem dari, quae est $2\gamma\left(\frac{4}{1}\right)\left(\frac{2}{2}\right) = \alpha\left(\frac{3}{3}\right)\left(\frac{3}{2}\right)$, ita ut prima admissa, hic altera sufficiat.

CALCULI INTEGRALIS
LIBER PRIOR.

PARS PRIMA,

SEU

METHODUS INVESTIGANDI FUNCTIONES UNIUS VARIABILIS EX DATA RELATIONE QUACUNQUE DIFFERENTIALIUM PRIMI GRADUS.

SECTIO SECUNDA,

DE

INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM.

CAPUT I.

DE SEPARATIONE VARIABILIUM.

Definitio.

§. 397.

In aequatione differentiali *separatio variabilium* locum habere dicitur, cum aequationem ita in duo membra dispescere licet, ut in utroque unica tantum variabilis cum suo differentiali insit.

Corollarium 1.

398. Quando igitur aequatio differentialis ita est comparata, ut ad hanc formam $X\partial x = Y\partial y$ reduci possit, in qua X functio sit solius x et Y solius y, tum ea aequatio separationem variabilium admittere dicitur.

Corollarium 2.

399. Quodsi P et X functiones ipsius x tantum, at Q et Y functiones ipsius y tantum denotent, haec aequatio $PY\partial x = QX\partial y$ separationem variabilium admittit, nam per XY divisa abit in $\frac{P\partial x}{X} = \frac{Q\partial y}{Y}$, in qua variabiles sunt separatae.

Corollarium 3.

400. In forma ergo generali $\frac{\partial y}{\partial x} = V$, separatio variabilium locum habet, si V ejusmodi fuerit functio ipsarum x et y, ut in duos factores resolvi possit, quorum alter solam variabilem x, alter

solam y contineat. Si enim sit $V = XY$, inde prodit aequatio separata $\frac{\partial y}{Y} = X\partial x$.

Scholion.

401. Posita differentialium ratione $\frac{\partial y}{\partial x} = p$, in hac sectione ejusmodi relationem inter x, y et p cosiderare instituimus, qua p aequetur functioni cuicunque ipsarum x et y. Hic igitur primum eum casum contemplamur, quo ista functio in duos factores resolvitur, quorum alter est functio tantum ipsius x et alter ipsius y, ita ut aequatio ad hanc formam reduci possit $X\partial x = Y\partial y$, in qua binae variabiles a se invicem separatae esse dicuntur. Atque in hoc casu formulae simplices ante tractatae continentur, quando $Y = 1$, ut sit $\partial y = X\partial x$, et $y = \int X\partial x$, ubi totum negotium ad integrationem $X\partial x$ revocatur. Haud majorem autem habet difficultatem aequatio separata $X\partial x = Y\partial y$, quam perinde ac formulas simplices tractare licet, id quod in sequente problemate osteudemus.

Problema 49.

402. Aequationem differentialem, in qua variabiles sunt separatae, integrare, seu aequationem inter ipsas variabiles invenire.

Solutio.

Aequatio separationem variabilium admittens semper ad hanc formam $Y\partial y = X\partial x$ reducitur; ubi $X\partial x$ tanquam differentiale functionis cujusdam ipsius x et $Y\partial y$ tanquam differentiale functionis cujusdam ipsius y spectari potest, cum igitur differentialia sint aequalia eorum integralia quoque aequalia esse, vel quantitate constante differre necesse est. Integrentur ergo per praecepta superioris sectionis seorsim ambae formulae, seu quaerantur integralia $\int Y\partial y$ et $\int X\partial x$, quibus inventis erit utique $\int Y\partial y = \int X\partial x + \text{Const.}$ qua aequatione relatio finita inter quantitates x et y exprimetur.

Corollarium 1.

403. Quoties ergo aequatio differentialis separationem variabilium admittit, toties integratio per eadem praecepta, quae supra de formulis simplicibus sunt tradita, absolvi potest.

Corollarium 2.

404. In aequatione integrali $\int Y \partial y = \int X \partial x +$ Const. vel ambae functiones $\int Y \partial y$ et $\int X \partial x$ sunt algebraicae, vel altera algebraica, altera vero transcendens, vel ambae transcendentes, sicque relatio inter x et y vel erit algebraica, vel transcendens.

Scholion.

405. In separatione variabilium a nonnullis totum fundamentum resolutionis aequationum differentialium constitui solet, ita ut cum aequatio proposita separationem variabilium non admittit, idonea substitutio sit investiganda, cujus beneficio novae variabiles introductae separationem patiantur. Totum ergo negotium huc reducitur, ut proposita aequatione differentiali quacunque, ejusmodi substitutio seu novarum variabilium introductio doceatur, ut deinceps separatio variabilium locum sit habitura. Optandum utique esset, ut hujusmodi methodus, pro quovis casu idoneam substitutionem inveniendi, aperiretur; sed nihil omnino certi in hoc negotio est compertum, dum pleraeque substitutiones, quae adhuc in usu fuerunt, nullis certis principiis innituntur. Deinde autem variabilium separatio non tanquam verum fundamentum omnis integrationis spectari potest, propterea quod in aequationibus differentialibus secundi altiorisve gradus nullum usum praestat; infra autem aliud principium latissime patens sum expositurus. In hoc capite interim praecipuas integrationes ope separationis variabilium administratas exponere operae pretium videtur; quandoquidem in hoc arduo negotio, quam plurimas methodos cognoscere, plurimum interest.

Problema 50.

406. Aequationem differentialem $P\partial x = Q\partial y$, in qua P et Q sint functiones homogeneae ejusdem dimensionum numeri ipsarum x et y, ad separationem variabilium reducere; ejusque integrale invenire.

Solutio.

Cum P et Q sint functiones homogeneae ipsarum x et y ejusdem dimensionum numeri, erit $\frac{P}{Q}$ functio homogenea nullius dimensionis, quae ergo posito $y = ux$ abit in functionem ipsius u. Ponatur igitur $y = ux$, abeatque $\frac{P}{Q}$ in U functionem ipsius u, ita ut sit $\partial y = U\partial x$. Sed ob $y = ux$, fit $\partial y = u\partial x + x\partial u$, qua substitutione nostra aequatio induet hanc formam $u\partial x + x\partial u = U\partial x$, inter binas variabiles x et u, quae manifesto sunt separabiles. Nam dispositis terminis ∂x continentibus ad unam partem, habetur

$$x\partial u = (U - u)\partial x, \text{ ideoque } \frac{\partial x}{x} = \frac{\partial u}{U - u},$$

quae integrata dat $lx = \int \frac{\partial u}{U - u}$, ita ut jam ex variabili u determinetur x, unde porro cognoscitur $y = ux$.

Corollarium 1.

407. Quodsi ergo integrale $\int \frac{\partial u}{U - u}$ etiam per logarithmos exprimi possit, ita ut lx aequetur logarithmo functionis cujuspiam ipsius u; habebitur aequatio algebraica inter x et u, ideoque pro u posito valore $\frac{x}{y}$, aequatio algebraica inter x et y.

Corollarium 2.

408. Cum sit $y = ux$, erit $ly = lu + lx$, ideoque cum sit $lx = \int \frac{\partial u}{U - u}$, erit

$$ly = lu + \int \frac{\partial u}{U - u} = \int \frac{\partial u}{u} + \int \frac{\partial u}{U - u};$$

quibus integralibus in unum reductis, fit $ly = \int \frac{U \partial u}{u(U - u)}$. Verum hic notandum est, non in utraque integratione pro lx et ly constantem arbitrariam adjicere licere; statim enim atque alteri integrali est adjecta, simul constans alteri adjicienda definitur, cum esse debeat $ly = lx + lu$.

Corollarium 3.

409. Cum sit

$$\int \frac{\partial u}{U - u} = \int \frac{\partial u - \partial U + \partial U}{U - u} = \int \frac{\partial U}{U - u} - \int \frac{\partial U - \partial u}{U - u},$$

ob hoc posterius membrum per logarithmos integrabile, erit $lx = \int \frac{\partial U}{U - u} - l(U - u)$, seu $lx(U - u) = \int \frac{\partial U}{U - u}$. Perinde ergo est, sive haec formula $\int \frac{\partial u}{U - u}$ sive $\int \frac{\partial U}{U - u}$ integretur.

Scholion.

410. Quoniam haec methodus ad omnes aequationes homogeneas patet, neque etiam ob irrationalitatem, quae forte in functionibus P et Q inest, impeditur, imprimis est aestimanda, plurimumque aliis methodis anteferenda, quae tantum ad aequationes nimis speciales sunt accomodatae. Atque hinc etiam discimus omnes aequationes, quae ope cujusdam substitutionis ad homogeneitatem revocari possunt, per eandem methodum tractari posse. Veluti si proponatur haec aequatio $\partial z + zz\partial x = \frac{a\partial x}{xx}$, statim patet posito $z = \frac{1}{y}$, eam ad hanc homogeneam $-\frac{\partial y}{yy} + \frac{\partial x}{yy} = \frac{a\partial x}{xx}$, seu $xx\partial y = \partial x(xx - ayy)$ reduci. Caeterum non difficulter perspicitur, utrum aequatio proposita hujusmodi substitutione ad homogeneitatem perduci queat? Plerumque, quoties quidem fieri potest, sufficit has positiones $x = u^m$ et $y = v^n$ tentasse, ubi facile judicabitur, num exponentes m et n ita assumere liceat, ut ubique idem dimensionum numerus prodeat, magis enim complicatis sub-

stitutionibus in hoc genere vix locus conceditur, nisi forte quasi sponte se prodant. Methodum autem integrandi hic expositam aliquot exemplis illustrasse juvabit.

Exemplum 1.

411. *Proposita aequatione differentiali homogenea* $x\partial x + y\partial y = my\partial x$, *ejus integrale invenire.*

Cum ergo hinc sit $\frac{\partial y}{\partial x} = \frac{my - x}{y}$, posito $y = ux$ fit $\frac{my - x}{y} = \frac{mu - 1}{u}$, ideoque ob $\partial y = u\partial x + x\partial u$, erit

$$u\partial x + x\partial u = \frac{(mu - 1)}{u}\partial x, \text{ hincque}$$

$$\frac{\partial x}{x} = \frac{u\partial u}{mu - 1 - uu} = \frac{-u\partial u}{1 - mu + uu}, \text{ seu}$$

$$\frac{\partial x}{x} = \frac{-u\partial u + \frac{1}{2}m\partial u}{1 - mu + uu} - \frac{\frac{1}{2}m\partial u}{1 - mu + uu};$$

unde integrando

$$lx = -\tfrac{1}{2}l(1 - mu + uu) - \tfrac{1}{2}m\int\frac{\partial u}{1 - mu + uu} + \text{Const.}$$

ubi tres casus sunt considerandi, prout $m > 2$, vel $m < 2$, vel $m = 2$.

1.) Sit $m > 2$, et $1 - mu + uu$ hujusmodi formam habebit $(u - a)(u - \frac{1}{a})$, ut fit $m = a + \frac{1}{a} = \frac{aa + 1}{a}$, et ob

$$\frac{\partial u}{(u - a)(u - \frac{1}{a})} = \frac{a}{aa - 1}\cdot\frac{\partial u}{u - a} - \frac{a}{aa - 1}\cdot\frac{\partial u}{u - \frac{1}{a}}, \text{ fiet}$$

$$lx = -\tfrac{1}{2}l(1 - mu + uu) - \frac{(aa + 1)}{2(aa - 1)}\,l\cdot\frac{u - a}{u - \frac{1}{a}} + \text{C}, \text{ seu}$$

$$lx\sqrt{(1 - mu + uu)} + \frac{aa + 1}{2(aa - 1)}\,l\cdot\frac{au - aa}{au - 1} = lc,$$

et restituto valore $u = \frac{y}{x}$, aequatio integralis erit

$$l\sqrt{(xx - mxy + yy)} + \frac{aa + 1}{2(aa - 1)}\,l\cdot\frac{ay - aax}{ay - x} = lc, \text{ seu}$$

$$\left(\frac{ay - aax}{ay - x}\right)^{\frac{aa+1}{2(aa-1)}} \sqrt{(xx - mxy + yy)} = C.$$

2.) Sit $m < 2$ seu $m = 2\cos.\alpha$, erit

$$\int \frac{\partial u}{1 - 2u\cos.\alpha + uu} = \frac{1}{\sin.\alpha}\text{ Ang. tang. }\frac{u\sin.\alpha}{1 - u\cos.\alpha}:$$

unde

$$lx\sqrt{(1 - mu + uu)} = C - \frac{\cos.\alpha}{\sin.\alpha}\text{ Ang. tang. }\frac{u\sin.\alpha}{1 - u\cos.\alpha},\text{ seu}$$

$$l\sqrt{(xx - mxy + yy)} = C - \frac{\cos.\alpha}{\sin.\alpha}\text{ Ang. tang. }\frac{y\sin.\alpha}{x - y\cos.\alpha}.$$

3.) Sit $m = 2$, erit $\int \frac{\partial u}{(1-u)^2} = \frac{1}{1-u}$, hincque

$lx(1 - u) = C - \frac{1}{1-u}$, seu $l(x - y) = C - \frac{x}{x-y}$.

Exemplum 2.

412. *Proposita aequatione differentiali homogenea*

$$\partial x(\alpha x + \beta y) = \partial y(\gamma x + \delta y)$$

ejus integrale invenire.

Posito $y = ux$, erit $u\partial x + x\partial u = \partial x \cdot \frac{\alpha + \beta u}{\gamma + \delta u}$, ideoque

$$\frac{\partial x}{x} = \frac{\partial u(\gamma + \delta u)}{\alpha + \beta u - \gamma u - \delta uu} = \frac{\partial u(\delta u + \frac{1}{2}\gamma - \frac{1}{2}\beta) + \partial u(\frac{1}{2}\gamma + \frac{1}{2}\beta)}{\alpha + (\beta - \gamma)u - \delta uu},$$

unde integrando

$$lx = C - l\sqrt{[\alpha + (\beta - \gamma)u - \delta uu]} + \frac{1}{2}(\beta + \gamma)\int \frac{\partial u}{\alpha + (\beta - \gamma)u - \delta uu}:$$

ubi iidem casus, qui ante, sunt considerandi, prout scilicet denominator $\alpha + (\beta - \gamma)u - \delta uu$ vel duos factores habet reales et inaequales, vel aequales, vel imaginarios.

Exemplum 3.

413. *Proposita aequatione differentiali homogenea*

$$x\partial x + y\partial y = x\partial y - y\partial x$$

ejus integrale invenire.

Cum hinc sit $\frac{\partial y}{\partial x} = \frac{x+y}{x-y}$, posito $y = ux$, fit $u\partial x + x\partial u = \frac{1+u}{1-u}\partial x$, seu $x\partial u = \frac{1+uu}{1-u}\partial x$, unde colligitur $\frac{\partial x}{x} = \frac{\partial u - u\partial u}{1+uu}$, et integrando

$$lx = \text{Ang. tang. } u - l\surd(1+uu) + C, \text{ seu}$$
$$l\surd(xx+yy) = C + \text{Ang. tang. } \frac{y}{x}.$$

Exemplum 4.

414. *Proposita aequatione differentiali homogenea*

$$xx\partial y = (xx - ayy)\partial x$$

ejus integrale invenire.

Hic ergo est $\frac{\partial y}{\partial x} = \frac{xx - ayy}{xx}$, et posito $y = ux$, prodit $u\partial x + x\partial u = (1 - auu)\partial x$, ideoque $\frac{\partial x}{x} = \frac{\partial u}{1 - u - auu}$ et $lx = \int \frac{\partial u}{1 - u - auu}$, cujus evolutioni non opus est immorari.

Exemplum 5.

415. *Proposita aequatione differentiali homogenea*

$$x\partial y - y\partial x = \partial x\surd(xx+yy)$$

ejus integrale invenire.

Erit ergo $\frac{\partial y}{\partial x} = \frac{y + \surd(xx+yy)}{x}$, unde posito $y = ux$, fit $u\partial x + x\partial u = [u + \surd(1+uu)]\partial x$, seu $x\partial u = \partial x\surd(1+uu)$; ita ut sit $\frac{\partial x}{x} = \frac{\partial u}{\surd(1+uu)}$, cujus integrale est

$$lx = la + l[u + \surd(1+uu)] = la + l\left(\frac{y + \surd(xx+yy)}{x}\right),$$

seu $lx = la + l\frac{x}{\surd(xx+yy) - y}$, unde colligitur $x = \frac{ax}{\surd(xx+yy) - y}$, seu $\surd(xx+yy) = a + y$, hincque $xx = aa + 2ay$.

Scholion.

416. Huc etiam functiones transcendentes numerari possunt, modo afficiant functiones nullius dimensionis ipsarum x et y, quia

posito $y = ux$ simul in functiones ipsius u abeunt. Ita si in aequatione $P\partial x = Q\partial y$, praeterquam quod P et Q sunt functiones homogeneae ejusdem dimensionum numeri, insint hujusmodi formulae

$$l\frac{\sqrt{(xx+yy)}}{x};\ e^{y:x};\ \text{Ang. sin.}\frac{x}{\sqrt{(xx+yy)}};\ \cos.\frac{nx}{y};\ \text{etc.}$$

methodus exposita pari successu adhiberi potest, quia posito $y = ux$, ratio $\frac{\partial y}{\partial x}$, aequatur functioni solius novae variabilis u.

Problema 51.

417. Aequationem differentialem primi ordinis

$$\partial x(\alpha + \beta x + \gamma y) = \partial y(\delta + \varepsilon x + \zeta y)$$

ad separationem variabilium revocare et integrare.

Solutio.

Ponatur $\alpha + \beta x + \gamma y = t$ et $\delta + \varepsilon x + \zeta y = u$, ut fiat $t\partial x = u\partial y$. At inde colligimus

$$x = \frac{\zeta t - \gamma u + \alpha\zeta + \gamma\delta}{\beta\zeta - \gamma\varepsilon} \text{ et } y = \frac{\beta u - \varepsilon t + \alpha\varepsilon - \beta\delta}{\beta\zeta - \gamma\varepsilon},$$

hincque $\partial x : \partial y = \zeta\partial t - \gamma\partial u : \beta\partial u - \varepsilon\partial t$, unde nanciscimur hanc aequationem

$$\zeta t\partial t - \gamma t\partial u = \beta u\partial u - \varepsilon u\partial t, \text{ seu}$$
$$\partial t(\zeta t + \varepsilon u) = \partial u(\beta u + \gamma t),$$

quae cum sit homogenea et cum exemplo §. 412. conveniat, integratio jam est expedita.

Verum tamen casus existit, quo haec reductio ad homogeneitatem locum non habet, cum fuerit $\beta\zeta - \gamma\varepsilon = 0$, quoniam tum introductio novarum variabilium t et u tollitur. Hic ergo casus peculiarem requirit solutionem, quae ita instituatur; quoniam tum aequatio proposita ejusmodi formam est habitura

$$\alpha\partial x + (\beta x + \gamma y)\partial x = \delta\partial y + n(\beta x + \gamma y)\partial y$$

ponamus $\beta x + \gamma y = z$, erit $\frac{\partial y}{\partial x} = \frac{\alpha + z}{\delta + nz}$. At $\partial y = \frac{\partial z - \beta \partial x}{\gamma}$, ergo $\frac{\partial z - \beta \partial x}{\gamma} = \frac{\alpha + z}{\delta + nz} \partial x$, ubi variabiles manifesto sunt separabiles, fit enim $\partial x = \frac{\partial z (\delta + nz)}{\alpha\gamma + \beta\delta + (\gamma + n\beta) z}$, cujus integratio logarithmos involvit, nisi sit $\gamma + n\beta = 0$, quo casu algebraice dat $x = \frac{2\delta z + nzz}{2(\alpha\gamma + \beta\delta)} + C$.

Corollarium 1.

418. Aequatio ergo differentialis primi ordinis, uti vocatur, in genere ad homogeneitatem reduci nequit, sed casus, quibus $\beta\zeta = \gamma\varepsilon$, inde excipi debent, qui etiam ad aequationem separatam omnino diversam deducunt.

Corollarium 2.

419. Si in his casibus exceptis sit $n = 0$, seu haec proposita sit aequatio $\partial y = \partial x (\alpha + \beta x + \gamma y)$, posito $\beta x + \gamma y = z$, ob $\delta = 1$, haec oritur aequatio $\partial x = \frac{\partial z}{\alpha\gamma + \beta + \gamma z}$, cujus integrale est

$$\gamma x = l\frac{\beta + \alpha\gamma + \gamma z}{C} = l\frac{\beta + \alpha\gamma + \beta\gamma x + \gamma\gamma y}{C}, \text{ seu}$$
$$\beta + \gamma(\alpha + \beta x + \gamma y) = C e^{\gamma x}.$$

Problema 52.

420. Proposita aequatione differentiali hujusmodi:

$$\partial y + P y \partial x = Q dx$$

in qua P et Q sint functiones quaecunque ipsius x, altera autem variabilis y cum suo differentiali nusquam plus una habeat dimensionem, eam ad separationem variabilium perducere et integrare.

Solutio.

Quaeratur ejusmodi functio ipsius x, quae sit X, ut facta substitutione $y = Xu$ aequatio prodeat separabilis: Tum autem oritur

$$\begin{aligned} X\partial u + u\partial X & = Q\partial x \\ + PXu\partial x & \end{aligned}$$

quam aequationem separationem admittere evidens est, si fuerit $\partial X + PX\partial x = 0$, seu $\frac{\partial X}{X} = -P\partial x$, unde integratio dat $lX = -\int P\partial x$ et $X = e^{-\int P\partial x}$; hac ergo pro X sumta functione, aequatio nostra transformata erit $X\partial u = Q\partial x$, seu $\partial u = \frac{Q\partial x}{X} = e^{\int P\partial x} Q\partial x$, unde cum P et Q sunt functiones datae ipsius x, erit $u = \int e^{\int P\partial x} Q\partial x = \frac{y}{X}$. Quocirca aequationis propositae integrale est $y = e^{-\int P\partial x} \int e^{\int P\partial x} Q\partial x$.

Corollarium 1.

421. Resolutio ergo hujus aequationis $\partial y + Py\partial x = Q\partial x$ duplicem requirit integrationem, alteram formulae $\int P\partial x$, alteram formulae $\int e^{\int P\partial x} Q\partial x$. Sufficit autem in posteriori constantem arbitrariam adjecisse, cum valor ipsius y plus una non recipiat. Etiamsi enim in priori loco $\int P\partial x$ scribatur $\int P\partial x + C$, formula pro y manet eadem.

Corollarium 2.

422. Dum ergo formula $P\partial x$ integratur, sufficit ejus integrale particulare sumi, ideoque constanti ingredienti ejusmodi valorem tribui convenit, ut integralis forma fiat simplicissima.

Scholion.

423. En ergo aliud aequationum genus non minus late patens quam praecedens homogenearum, quod ad separationem variabilium perduci, hocque modo integrari potest. Inde autem in Analysin maxima utilitas redundat, cum hic litterae P et Q functiones quascunque ipsius x denotent. Hoc ergo modo manifestum est, tractari posse hanc aequationem $R\partial y + Py\partial x = Q\partial x$, si etiam R func-

tionem quamcunque ipsius x denotet, facta enim divisione per R forma proposita prodit, modo loco P et Q scribatur $\frac{P}{R}$ et $\frac{Q}{R}$, ita ut integrale futurum sit

$$y = e^{-\int \frac{P\partial x}{R}} \int \frac{e^{\int \frac{P\partial x}{R}} Q\partial x}{R}$$

Ad hujus problematis illustrationem quaedam exempla adjiciamus.

Exemplum 1.

424. *Proposita aequatione differentiali*

$$\partial y + y\partial x = a x^n \partial x$$

ejus integrale invenire.

Cum hic sit $P = 1$ et $Q = ax^n$, erit $\int P\partial x = x$, et aequatio integralis fiet

$$y = e^{-x} \int e^x x^n \partial x,$$

quae si n sit numerus integer positivus, evadet

$$y = e^{-x} [e^x (x^n - n x^{n-1} + n(n-1) x^{n-2} - \text{etc.}) + C] \quad (\S\ 223.)$$

qua evoluta prodit

$$y = C e^{-x} + x^n - n x^{n-1} + n(n-1) x^{n-2} - n(n-2)(n-3) x^{n-3} + \text{etc.}$$

unde pro simplicioribus valoribus ipsius n,

si $n = 0$, erit $y = C e^{-x} + 1$;
si $n = 1$, erit $y = C e^{-x} + x - 1$;
si $n = 2$, erit $y = C e^{-x} + x^2 - 2x + 2.1$;
si $n = 3$, erit $y = C e^{-x} + x^3 - 3x^2 + 3.2x - 3.2.1$;
etc.

Corollarium 1.

425. Si ergo constans C sumatur $= 0$, habebitur integrale particulare

$$y = x^n - n x^{n-1} + n(n-1) x^{n-2} - n(n-1)(n-2) x^{n-3} + \text{etc.}$$

quod ergo est algebraicum, dummodo n sit numerus integer positivus.

Corollarium 2.

426. Si integrale ita determinari debeat, ut posito $x = 0$, valor ipsius y evanescat, constans C aequalis sumi debet ultimo termino constanti signo mutato, unde id semper erit transcendens.

Exemplum 2.

427. *Proposita aequatione differentiali* $(1 - xx)\partial y + xy\partial x = a\partial x$ *ejus integrale invenire.*

Aequatio ista per $1 - xx$ divisa ad hanc formam reducitur $\partial y + \frac{xy\partial x}{1-xx} = \frac{a\partial x}{1-xx}$, ita ut sit $P = \frac{x}{1-xx}$; $Q = \frac{a}{1-xx}$; hinc $\int P\partial x = -l\sqrt{(1-xx)}$, et $e^{\int P\partial x} = \frac{1}{\sqrt{(1-xx)}}$, ex quo integrale reperitur:

$$y = \sqrt{(1-xx)} \int \frac{a\partial x}{(1-xx)^{\frac{3}{2}}} = \left(\frac{ax}{\sqrt{(1-xx)}} + C\right)\sqrt{(1-xx)};$$

quocirca integrale quaesitum erit

$$y = ax + C\sqrt{(1-xx)}$$

quod si ita determinari debeat, ut posito $x = 0$ evanescat, sumi oportet $C = 0$, eritque $y = ax$.

Exemplum 3.

428. *Proposita aequatione differentiali* $\partial y + \frac{ny\partial x}{\sqrt{(1+xx)}} = a\partial x$, *ejus integrale invenire.*

Cum hic sit $P = \frac{n}{\sqrt{(1+xx)}}$ et $Q = a$, erit

$\int P\partial x = n\,l[x + \sqrt{(1+xx)}]$ et

$$e^{\int P\partial x} = [x + \sqrt{(1 + xx)}]^n, \text{ et}$$
$$e^{-\int P\partial x} = [\sqrt{(1 + xx)} - x]^n:$$

unde integrale quaesitum erit

$$y = [\sqrt{(1 + xx)} - x]^n \int a\partial x\, [x + \sqrt{(1 + xx)}]^n,$$

ad quod evolvendum ponatur $x + \sqrt{(1 + xx)} = u$, et fiet $x = \frac{uu - 1}{2u}$, hinc $\partial x = \frac{\partial u (1 + uu)}{2uu}$, ergo

$$\int u^n \partial x = \frac{u^{n-1}}{2(n-1)} + \frac{u^{n+1}}{2(n+1)} + C.$$

Nunc quia $[\sqrt{(1 + xx)} - x]^n = u^{-n}$, erit

$$y = Cu^{-n} + \frac{au^{-1}}{2(n-1)} + \frac{au}{2(n+1)} \text{ sive}$$

$$y = C[\sqrt{(1 + xx)} - x]^n + \frac{a}{2(n-1)}[\sqrt{(1 + xx)} - x]$$
$$+ \frac{a}{2(n+1)}[\sqrt{(1 + xx)} + x]$$

quae expressio ad hanc formam reducitur

$$y = C[\sqrt{(1 + xx)} - x]^n + \frac{na}{nn-1}\sqrt{(1 + xx)} - \frac{ax}{nn-1},$$

si integrale ita determinari debeat, ut posito $x = 0$ fiat $y = 0$, sumi oportet $C = -\frac{na}{nn-1}$.

Problema 53.

429. Proposita aequatione differentiali

$$\partial y + Py\partial x = Qy^{n+1}\partial x,$$

ubi P et Q denotent functiones quascunque ipsius x, eam ad separationem variabilium reducere et integrare.

Solutio.

Haec aequatio posito $\frac{1}{y^n} = z$ statim ad formam modo tractatam reducitur, nam ob $\frac{\partial y}{y} = -\frac{\partial z}{nz}$, aequatio nostra per y divisa, scilicet $\frac{\partial y}{y} + P\partial x = Q y^n \partial x$, statim abit in $-\frac{\partial z}{nz} + P\partial x = \frac{Q\partial x}{z}$, seu $\partial z - nPz\partial x = -nQ\partial x$, cujus integrale est

$$z = -e^{n\int P\partial x}\int e^{-n\int P\partial x}\, nQ\partial x,\ \text{ideoque}$$

$$\frac{1}{y^n} = -ne^{n\int P\partial x}\int e^{-n\int P\partial x}\, Q\partial x.$$

Tractari autem potest ut praecedens, quaerendo hujusmodi functionem X, ut facta substitutione $y = Xu$ prodeat aequatio separabilis: prodit autem

$$X\partial u + u\partial X + PXu\partial x = X^{n+1} u^{n+1} Q\partial x.$$

Fiat ergo $\partial X + PX\partial x = 0$, seu $X = e^{-\int P\partial x}$, eritque

$$\frac{\partial u}{u^{n+1}} = X^n Q\partial x = e^{-n\int P\partial x} Q\partial x,$$

et integrando

$$-\frac{1}{n u^n} = \int e^{-n\int P\partial x} Q\partial x.$$

Jam quia $u = \frac{y}{X} = e^{\int P\partial x} y$, habebitur ut ante

$$\frac{1}{y^n} = -ne^{n\int P\partial x}\int e^{-n\int P\partial x} Q\partial x.$$

Scholion.

430. Hic ergo casus a praecedente non differre est censendus, ita ut hic nihil novi sit praestitum. Atque haec duo genera sunt fere sola, quae quidem aliquanto latius pateant, in quibus separatio variabilium obtineri queat. Caeteri casus, qui ope cujusdam substitutionis ad variabilium separationem praeparari possunt, plerumque sunt nimis speciales, quam ut insignis usus inde expectari possit. Interim tamen aliquot casus prae caeteris hic exponamus.

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Problema 54.

431. Proposita hac aequatione differentiali

$$\alpha y\partial x + \beta x\partial y + x^m y^n(\gamma y\partial x + \delta x\partial y) = 0,$$

eam ad separationem variabilium reducere, et integrare.

Solutio.

Tota aequatione per xy divisa, nanciscimur hanc formam:

$$\frac{\alpha\partial x}{x} + \frac{\beta\partial y}{y} + x^m y^n\left(\frac{\gamma\partial x}{x} + \frac{\delta\partial y}{y}\right) = 0,$$

unde statim has substitutiones $x^\alpha y^\beta = t$ et $x^\gamma y^\delta = u$ insigni usu non esse carituras colligimus: inde enim fit

$$\frac{\alpha\partial x}{x} + \frac{\beta\partial y}{y} = \frac{\partial t}{t} \text{ et } \frac{\gamma\partial x}{x} + \frac{\delta\partial y}{y} = \frac{\partial u}{u},$$

hincque aequatio nostra $\frac{\partial t}{t} + x^m y^n . \frac{\partial u}{u} = 0$. At ex substitutione sequitur

$$x^{\alpha\delta-\beta\gamma} = t^\delta u^{-\beta}, \text{ et } y^{\alpha\delta-\beta\gamma} = u^\alpha t^{-\gamma}, \text{ ideoque}$$

$$x = t^{\frac{\delta}{\alpha\delta-\beta\gamma}} u^{\frac{-\beta}{\alpha\delta-\beta\gamma}}, \text{ et } y = t^{\frac{-\gamma}{\alpha\delta-\beta\gamma}} u^{\frac{\alpha}{\alpha\delta-\beta\gamma}};$$

quibus substitutis fit

$$\frac{\partial t}{t} + t^{\frac{\delta m-\gamma n}{\alpha\delta-\beta\gamma}} u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}} \frac{\partial u}{u} = 0, \text{ ideoque}$$

$$t^{\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}-1}\partial t + u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}-1}\partial u = 0,$$

cujus aequationis integrale est

$$\frac{t^{\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}}}{\gamma n-\delta m} + \frac{u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}}}{\alpha n-\beta m} = C.$$

Ubi tantum superest ut restituantur valores $t = x^\alpha y^\beta$ et $u = x^\gamma y^\delta$. Caeterum notetur, si fuerit vel $\gamma n - \delta m = 0$ vel $\alpha n - \beta m = 0$, loco illorum membrorum vel lt vel lu scribi debere.

Scholion.

432. Ad aequationem propositam ducit quaestio, qua ejusmodi relatio inter variabiles x et y quaeritur, ut fiat

$$\int y\partial x = axy + bx^{m+1}y^{n+1};$$

ad hanc enim resolvendam differentialia sumi debent, quo prodit

$$y\partial x = ax\partial y + ay\partial x + bx^m y^n[(m+1)y\partial x + (n+1)x\partial y],$$

qua aequatione cum nostra forma comparata, est

$$\alpha = a - 1,\ \beta = a,\ \gamma = (m+1)b, \text{ et } \delta = (n+1)b; \text{ ergo}$$
$$\alpha\delta - \beta\gamma = (n-m)ab - (n+1)b$$
$$\alpha n - \beta m = (n-m)a - n, \text{ et } \gamma n - \delta m = (n-m)b,$$

unde aequatio integralis fit manifesta.

Problema 55.

433. Proposita hac aequatione differentiali

$$y\partial y + \partial y(a + bx + nxx) = y\partial x(c + nx),$$

eam ad separationem variabilium reducere, et integrare.

Solutio.

Cum hinc sit $\frac{\partial y}{\partial x} = \frac{y(c+nx)}{y+a+bx+nxx}$, tentetur haec substitutio $\frac{y(c+nx)}{y+a+bx+nxx} = u$, seu $y = \frac{u(a+bx+nxx)}{c+nx-u}$, fierique debet $\partial y = u\partial x$, seu

$$\frac{\partial y}{y} = \frac{u\partial x}{y} = \frac{\partial x(c+nx-u)}{a+bx+nxx}:$$

at ex logarithmis colligitur

$$\frac{\partial y}{y} = \frac{\partial u}{u} + \frac{\partial x(b+2nx)}{a+bx+nxx} - \frac{n\partial x + \partial u}{c+nx-u} = \frac{\partial x(c+nx-u)}{a+bx+nxx},$$

quae contrahitur in

$$\frac{\partial u(c+nx) - nu\partial x}{u(c+nx-u)} = \frac{\partial x(c-b-nx-u)}{a+bx+nxx}, \text{ seu}$$
$$\frac{\partial u(c+nx)}{u(c+nx-u)} = \frac{\partial x(na+cc-bc+(b-2c)u+uu)}{(c+nx-u)(a+bx+nxx)},$$

quae per $c + nx - u$ multiplicata manifesto est separabilis, proditque

$$\frac{\partial x}{(a + bx + nxx)(c + nx)} = \frac{\partial u}{u(na + cc - bc + (b - 2c)u + uu)},$$

cujus ergo integratio per logarithmos et angulos absolvi potest. Casu autem hic vix praevidendo evenit, ut haec substitutio ad votum successerit, neque hoc problema magnopere juvabit.

Problema 56.

434. Propositam hanc aequationem differentialem

$$(y - x)\partial y = \frac{n\partial x(1 + yy)\sqrt{(1 + yy)}}{\sqrt{(1 + xx)}},$$

ad separationem variabilium reducere, et integrare.

Solutio.

Ob irrationalitatem duplicem vix ullo modo patet, cujusmodi substitutione uti conveniat. Ejusmodi certe quaeri convenit, qua eidem signo radicali non ambae variabiles simul implicentur. Ad hunc scopum commoda videtur haec substitutio $y = \frac{x - u}{1 + xu}$, qua fit $y - x = \frac{-u(1 + xx)}{1 + xu}$, $1 + yy = \frac{(1 + xx)(1 + uu)}{(1 + xu)^2}$, et $\partial y = \frac{\partial x(1 + uu) - \partial u(1 + xx)}{(1 + xu)^2}$: atque his valoribus in nostra aequatione substitutis, prodit

$$-u\partial x(1 + uu) + u\partial u(1 + xx) = n\partial x(1 + uu)\sqrt{(1 + uu)},$$

quae manifesto separationem variabilium admittit: colligitur scilicet

$$\frac{\partial x}{1 + xx} = \frac{u\partial u}{(1 + uu)[n\sqrt{(1 + uu)} + u]},$$

quae aequatio posito $1 + uu = tt$, concinnior redditur

$$\frac{\partial x}{1 + xx} = \frac{\partial t}{t[nt + \sqrt{(tt - 1)}]},$$

et ope positionis $t = \frac{1 + ss}{2s}$, sublata irrationalitate,

$$\frac{\partial x}{1 + xx} = -\frac{2\partial s(1 - ss)}{(1 + ss)[n + 1 + (n - 1)ss]} = \frac{2\partial s}{1 + ss} - \frac{2n\partial s}{n + 1 + (n - 1)ss},$$

cujus integratio nulla amplius laborat difficultate.

Scholion.

435. In hoc casu praecipue substitutio $y = \frac{x-u}{1+xu}$ notari meretur, qua duplex irrationalitas tollitur: unde operae pretium erit videre, quid hac substitutione generaliori praestari possit $y = \frac{\alpha x+u}{1+\beta xu}$; inde autem fit

$$\alpha - \beta yy = \frac{(\alpha-\beta uu)(1-\alpha\beta xx)}{(1+\beta xu)^2},\quad y - \alpha x = \frac{u(1-\alpha\beta xx)}{1+\beta xu},\ \text{et}$$
$$\partial y = \frac{\partial x(\alpha-\beta uu)+\partial u(1-\alpha\beta xx)}{(1+\beta xu)^2};$$

ac jam facile perspicitur, in cujusmodi aequationibus haec substitutio usum afferre possit; ejus scilicet beneficio haec duplex irrationalitas $\frac{\sqrt{(\alpha-\beta yy)}}{\sqrt{(1-\alpha\beta xx)}}$ reducitur ad hanc simplicem $\frac{\sqrt{(\alpha-\beta uu)}}{1+\beta xu}$, quam porro facile rationalem reddere licet. Atque hic fere sunt casus, in quibus reductio ad separabilitatem locum invenit, quibus probe perpensis, aditus facile patebit ad reliquos casus, qui quidem etiamnum sunt tractati; unicam vero adhuc investigationem apponam circa casus, quibus haec aequatio $\partial x + yy\partial x = ax^m\partial x$ separationem variabilium admittit, quandoquidem ad hujusmodi aequationes frequenter pervenitur, atque haec ipsa aequatio olim inter Geometras omni studio est agitata.

Problema 57.

436. Pro aequatione $\partial y + yy\partial x = ax^m\partial x$ valores exponentis m definire, quibus eam ad separationem variabilium reducere licet.

Solutio.

Primo haec aequatio sponte est separabilis casu $m = 0$, tum enim ob $\partial y = \partial x(a - yy)$, fit $\partial x = \frac{\partial y}{a-yy}$. Omnis ergo investigatio in hoc versatur, ut ope substitutionum alii casus ad hunc reducantur.

Ponamus $y = \frac{b}{z}$, et fit $-b\partial z + bb\partial x = ax^m zz\partial x$, quae forma ut propositae similis evadat, statuatur $x^{m+1} = t$, ut sit

$$x^m \partial x = \frac{\partial t}{m+1}, \text{ et } \partial x = \frac{t^{\frac{-m}{m+1}} \partial t}{m+1}, \text{ eritque}$$

$$b\partial z + \frac{azz\partial t}{m+1} = \frac{bb}{m+1} t^{\frac{-m}{m+1}} \partial t,$$

quae sumto $b = \frac{a}{m+1}$, ad similitudinem propositae propius accedit, ut sit $\partial z + zz\partial t = \frac{a}{(m+1)^2} t^{\frac{-m}{m+1}} \partial t$. Si ergo haec esset separabilis, ipsa proposita ista substitutione separabilis fieret et vicissim; unde concludimis, si aequatio proposita separationem admittat casu $m = n$, eam quoque esse admissuiram casu $m = -\frac{n}{n+1}$. Hinc autem ex casu $m = 0$ alius non repertur.

Ponamus $y = \frac{1}{x} - \frac{z}{xx}$, ut sit

$$\partial y = -\frac{\partial x}{xx} - \frac{\partial z}{xx} + \frac{2z\partial x}{x^3}, \text{ et}$$

$$yy\partial x = \frac{\partial x}{xx} - \frac{2z\partial x}{x^3} + \frac{zz\partial x}{x^4},$$

unde prodit

$$-\frac{\partial z}{xx} + \frac{zz\partial x}{x^4} = ax^m \partial x, \text{ seu}$$

$$\partial z - \frac{zz\partial x}{xx} = -ax^{m+2}\partial x:$$

sit nunc $x = \frac{1}{t}$ et fit $\partial z + zz\partial t = at^{-m-4}\partial t$, quae cum propositae sit similis, discimus, si separatio succedat casu $m = n$, etiam succedere casu $m = -n - 4$.

Ex uno ergo casu $m = n$ consequimur duos, scilicet $m = -\frac{n}{n+1}$ et $m = -n - 4$. Cum igitur constet casus $m = 0$, hinc formulae alternatim adhibitae praebent sequentes

$m = -4$; $m = -\frac{4}{3}$; $m = -\frac{8}{3}$; $m = -\frac{8}{5}$;
$m = -\frac{12}{5}$; $m = -\frac{12}{7}$; $m = -\frac{16}{7}$; etc.

qui casus omnes in hac formula $m = \frac{-4i}{2i \pm 1}$ continentur.

Corollarium 1.

437. Quodsi ergo fuerit vel $m = \frac{-4i}{2i+1}$, vel $m = \frac{-4i}{2i-1}$, aequatio $\partial y + yy\partial x = ax^m \partial x$ per aliquot substitutiones repetitas tandem ad formam $\partial u + uu\partial v = c\partial v$, cujus separatio et integratio constat, reduci potest.

Corollarium 2.

438. Scilicet si fuerit $m = \frac{-4i}{2i+1}$, aequatio

$$\partial y + yy\partial x = ax^m \partial x$$

per substitutiones $x = t^{\frac{1}{m+1}}$ et $y = \frac{a}{(m+1)z}$ reducitur ad hanc $\partial z + zz\partial t = \frac{a}{(m+1)^2} t^n \partial t$, ubi $n = \frac{-4i}{2i-1}$, qui casus uno gradu inferior est censendus.

Corollarium 3.

439. Sin autem fuerit $m = \frac{-4i}{2i-1}$, aequatio

$$\partial y + yy\partial x = ax^m \partial x$$

per has substitutiones $x = \frac{1}{t}$ et $y = \frac{1}{x} - \frac{z}{xx}$ seu $y = t - ttz$, reducitur ad hanc $\partial z + zz\partial t = at^n \partial t$, in qua est

$$n = \frac{-4(i-1)}{2i-1} = \frac{-4(i-1)}{2(i-1)+1},$$

qui casus denuo uno gradu inferior est.

Corollarium 4.

440. Omnes ergo casus separabiles hoc modo inventi, pro exponente m dant numeros negativos intra limites 0 et -4

contentos, ac si i sit numerus infinitus, prodit casus $m = -2$, qui autem per se constat, cum aequatio $\partial y + yy\partial x = \frac{a\partial x}{xx}$, posito $y = \frac{1}{z}$, fiat homogenea.

Scholion 1.

441. Aequatio haec $\partial y + yy\partial x = ax^m \partial x$ vocari solet Riccatiana ab Auctore Comite *Riccati*, qui primus casus separabiles proposuit. Hic quidem eam in forma simplicissima exhibui, cum eo haec $\partial y + Ayyt^\mu \partial t = Bt^\lambda \partial t$, ponendo $At^\mu \partial t = \partial x$ et $At^{\mu+1} = (\mu + 1)x$, statim reducatur. Caeterum etsi binae substitutiones, quibus hic sum usus, sunt simplicissimae, tamen magis compositis adhibendis nulli alii casus separabiles deteguntur: ex quo hoc omnino memorabile est visum, hanc aequationem rarissime separationem admittere, tametsi numerus casuum, quibus hoc praestari queat, revera sit infinitus. Caeterum haec investigatio ab exponente ad simplicem coëfficientem traduci potest; posito enim $y = x^{\frac{m}{2}} z$, prodit $\partial z + \frac{mz\partial x}{2x} + x^{\frac{m}{2}} zz\partial x = ax^{\frac{m}{2}} \partial x$, ubi si fiat $x^{\frac{m}{2}} \partial x = \partial t$, et $x^{\frac{m+2}{2}} = \frac{m+2}{2} t$, erit $\frac{\partial x}{x} = \frac{2\partial t}{(m+2)t}$, hineque

$$\partial z + \frac{mz\partial t}{(m+2)t} + zz\partial t = a\partial t,$$

quae ergo aequatio, quoties fuerit $\frac{m}{m+2} = \pm 2i$, seu numerus par, tam positivus, quam negativus, separabilis reddi potest, ita ut haec aequatio

$$\partial z \pm \frac{2iz\partial t}{t} + zz\partial t = a\partial t$$

semper sit integrabilis. Si praeterea ponatur $z = u - \frac{m}{2(m+2)t}$, oritur

$$\partial u + uu\partial t = a\partial t - \frac{m(m+4)\partial t}{4(m+2)^2 tt},$$

et pro casibus separabilitatis $m = \frac{-4i}{2i \pm 1}$, habebitur

$$\partial u + uu\partial t = a\partial t + \frac{i(i \pm 1)\partial t}{tt}.$$

Uberiorem autem hujus aequationis evolutionem, quandoquidem est maximi momenti, in sequentibus docebo; ubi integratione aequationum differentialium per series infinitas sum acturus, hinc enim facilius casus separabiles eruemus, simulque integralia assignare poterimus.

Scholion 2.

442. Ampliora praecepta circa separationem variabilium, quae quidem usum sint habitura, vix tradi posse videntur, unde intelligitur in paucissimis aequationibus differentialibus hanc methodum adhiberi posse. Progrediar igitur ad aliud principium explicandum, unde integrationes haurire liceat, quod multo latius patet, dum etiam ad aequationes differentiales altiorum graduum accommodari potest, ita ut in eo verus ac naturalis fons omnium integrationum contineri videatur. Istud autem principium in hoc consistit, quod proposita quacunque aequatione differentiali inter duas variabiles, semper detur functio quaedam, per quam aequatio multiplicata fiat integrabilis. Aequationis scilicet omnia membra ad eandem partem disponi oportet, ut talem formam obtineat $P\partial x + Q\partial y = 0$; ac tum dico semper dari functionem quandam variabilium x et y, puta V, ut facta multiplicatione, formula $VP\partial x + VQ\partial y$ integrabilis existat, seu ut verum sit differentiale ex differentiatione cujuspiam functionis binarum variabilium x et y natum. Quodsi enim haec functio ponatur $= S$, ut sit $\partial S = VP\partial x + VQ\partial y$, quia est $P\partial x + Q\partial y = 0$, erit etiam $\partial S = 0$, ideoque $S = \text{Const.}$ quae ergo aequatio erit integrale idque completum aequationis differentialis $P\partial x + Q\partial y = 0$. Totum ergo negotium ad inventionem illius multiplicatoris V redit.

CAPUT II.

DE INTEGRATIONE AEQUATIONUM OPE MULTIPLICATORUM.

Problema 58.

443.

Propositam aequationem differentialem examinare, utrum per se sit integrabilis nec ne?

Solutio.

Dispositis omnibus aequationis terminis ad eandem partem signi aequalitatis, ut hujusmodi habeatur forma $P\partial x + Q\partial y = 0$, aequatio per se erit integrabilis, si formula $P\partial x + Q\partial y$ fuerit verum differentiale functionis cujuspiam binarum variabilium x et y. Hoc autem evenit, uti in calculo differentiali ostendimus, si differentiale ipsius P, sumta sola y variabili, ad ∂y eandem habeat rationem, ac differentiale ipsius Q, sumta sola x variabili, ad ∂x: seu adhibito signandi modo, quo in Calculo differentiali sumus usi, si fuerit $\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right)$. Nam si Z sit ea functio, cujus differentiale est $P\partial x + Q\partial y$, erit hoc signandi modo $P = \left(\frac{\partial Z}{\partial x}\right)$ et $Q = \left(\frac{\partial Z}{\partial y}\right)$: hinc ergo sequitur $\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial\partial Z}{\partial x\partial y}\right)$ et $\left(\frac{\partial Q}{\partial x}\right) = \left(\frac{\partial\partial Z}{\partial y\partial x}\right)$. At est $\left(\frac{\partial\partial Z}{\partial x\partial y}\right) = \left(\frac{\partial\partial Z}{\partial y\partial x}\right)$, unde colligitur $\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right)$. Quare proposita aequatione differentiali $P\partial x + Q\partial y = 0$, utrum ea per se sit integrabilis nec ne? hoc modo dignoscetur: Quaerantur per differentiationem valores $\left(\frac{\partial P}{\partial y}\right)$ et $\left(\frac{\partial Q}{\partial x}\right)$, qui si fuerint inter se aequales, aequatio per se erit integrabilis; sin autem hi valores sint inaequales, aequatio non erit per se integrabilis.

Corollarium 1.

444. Omnes ergo aequationes differentiales, in quibus variabiles sunt a se invicem separatae, per se sunt integrabiles: habebunt enim hujusmodi formam $X\partial x + Y\partial y = 0$, ut X sit functio solius x et Y solius y, eritque propterea

$$\left(\frac{\partial X}{\partial y}\right) = 0 \text{ et } \left(\frac{\partial Y}{\partial x}\right) = 0.$$

Corollarium 2.

445. Vicissim igitur, si proposita aequatione differentiali $P\partial x + Q\partial y = 0$, fuerit $\left(\frac{\partial P}{\partial y}\right) = 0$ et $\left(\frac{\partial Q}{\partial x}\right) = 0$, variabiles in ea erunt separatae; littera enim P erit functio tantum ipsius x et Q tantum ipsius y. Unde aequationes separatae quasi primum genus aequationum per se integrabilium constituunt.

Corollarium 3.

446. Evidens autem est, fieri posse, ut sit $\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right)$, etiamsi neuter horum valorum sit nihilo aequalis. Dantur ergo aequationes per se integrabiles, licet variabiles in iis non sint separatae.

Scholion.

447. Criterium hoc, quo aequationes per se integrabiles agnoscimus, maximi est momenti in hac, quam tradere suscipimus, methodo integrandi. Quodsi enim aequatio deprehendatur per se integrabilis, ejus integrale per praecepta jam exposita inveniri potest; sin autem aequatio non fuerit per se integrabilis, semper dabi-

tur quantitas, per quam si ea multiplicetur, fiat per se integrabilis; unde totum negotium eo revocabitur, ut proposita aequatione quacunque per se non integrabili, inveniatur multiplicator idoneus, qui eam reddat per se integrabilem; qui si semper inveniri posset, nihil amplius in hac methodo integrandi esset desiderandum. Verum haec investigatio rarissime succedit, ac vix adhuc latius patet, quam ad eas aequationes, quas ope separationis variabilium jam tractare docuimus; interim tamen non dubito hanc methodum praecedenti longe praeferre, cum ad naturam aequationum magis videatur accommodata, atque etiam ad aequationes differentiales altiorum graduum pateat, in quibus separatio variabilium nullius est usus.

Problema 59.

448. Aequationis differentialis, quam per se integrabilem esse constat, integrale invenire.

Solutio.

Sit aequatio differentialis $P\partial x + Q\partial y = 0$, in qua cum sit $\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right)$, erit $P\partial x + Q\partial y$ differentiale cujuspiam functionis binarum variabilium x et y, quae sit Z, ut sit $\partial Z = P\partial x + Q\partial y$. Cum ergo habeamus hanc aequationem $\partial Z = 0$, erit integrale quaesitum $Z = C$. Totum negotium ergo huc redit, ut ista functio Z eruatur, quod cum sciamus esse $\partial Z = P\partial x + Q\partial y$ haud difficulter praestabitur. Nam quia sumta tantum x variabili, et altera y ut constante spectata, est $\partial Z = P\partial x$, habemus hic formulam differentialem simplicem unicam variabilem x involventem, quae per praecepta superioris sectionis integrata dabit $Z = \int P\partial x + \text{Const.}$ ubi autem notandum est, in hac constante quantitatem hic pro constanti habitam y utcunque inesse posse; unde ejus loco scribatur Y, ut sit $Z = \int P\partial x + Y$. Deinde simili modo x pro constante habeatur, spectata sola y ut variabili, et cum sit $\partial Z = Q\partial y$, erit quoque $Z = \int Q\partial y + \text{Const.}$ quae constans autem quantitatem x

olvet, ita ut sit functio ipsius x, qua posita X, erit $Z = \int Q\partial y + X$.
nquam autem neque hic functio X neque ibi functio Y determi-
ur, tamen quia esse debet $\int P\partial x + Y = \int Q\partial y + X$, hinc
ique determinabitur. Cum enim sit $\int P\partial x - \int Q\partial y = X - Y$,
e quantitas $\int P\partial x - \int Q\partial y$ semper in ejusmodi binas partes
inguetur, quarum altera est functio ipsius x tantum, et altera
us y tantum, unde valores X et Y sponte cognoscuntur.

Corollarium 1.

449. Cum sit $Q = \left(\frac{\partial Z}{\partial y}\right)$, duplici integratione ne opus qui-
est. Invento enim integrali $\int P\partial x$, id iterum differentietur,
ta sola y variabili, prodeatque $V\partial y$, unde necesse est fiat
$y + \partial Y = Q\partial y$, ideoque

$$\partial Y = Q\partial y - V\partial y = (Q - V)\,\partial y.$$

Corollarium 2.

450. Aequationum ergo per se integrabilium $P\partial x + Q\partial y = 0$
gratio ita perficietur. Quaeratur integrale $\int P\partial x$ spectata y
stante, idque rursus differentietur spectata sola y variabili, unde
leat $V\partial y$: tum $Q - V$ erit functio ipsius y tantum; unde quae-
r $Y = \int (Q - V)\,\partial y$, eritque aequatio integralis $\int P\partial x + Y =$
s.

Corollarium 3.

451. Vel quaeratur $\int Q\partial y$ spectata x constante, quod inte-
e rursus differentietur sumta x variabili, y autem constante,
e prodeat $U\partial x$: tum certe erit $P - U$ functio ipsius x tantum;
e quaeratur $X = \int (P - U)\,\partial x$, eritque aequatio integralis quae-
$\int Q\partial y + X = \text{Const.}$

Corollarium 4.

452. Ex rei natura patet, perinde esse utra via procedatur, necesse enim est ad eandem aequationem integralem perveniri, si quidem aequatio differentialis proposita per se fuerit integrabilis. Tum autem certe eveniet, ut priori casu $Q - V$ sit functio solius y, posteriori autem $P - U$ functio solius x.

Scholion.

453. Haec methodus integrandi etiam tentari posset, antequam exploratum esset, num aequatio integrabilis existat; si enim vel in modo Corollarii 2. eveniret, ut $Q - V$ esset functio ipsius y tantum, vel in modo Corollarii 3. ut $P - U$ esset functio ipsius x tantum, hoc ipsum indicio foret, aequationem esse per se integrabilem. Verum tamen praestat ante omnia scrutari, an aequatio integrabilis sit per se nec ne; seu an sit $\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right)$? quoniam hoc examen sola differentiatione absolvitur. Exempla igitur aliquot aequationum per se integrabilium afferamus, quo non solum methodus integrandi, sed etiam insignes illae proprietates, quas commemoravimus, clarius intelligantur.

Exemplum 1.

454. *Aequationem per se integrabilem*

$$\partial x\,(\alpha x + \beta y + \gamma) + \partial y\,(\beta x + \delta y + \varepsilon) = 0,$$

integrare.

Cum hic sit

$$P = \alpha x + \beta y + \gamma \text{ et } Q = \beta x + \delta y + \varepsilon, \text{ erit}$$
$$\left(\frac{\partial P}{\partial y}\right) = \beta \text{ et } \left(\frac{\partial Q}{\partial x}\right) = \beta,$$

qua aequalitate integrabilitas per se confirmatur. Quaeratur ergo per Corollarium 2, spectata y ut constante,

$$\int P\partial x = \tfrac{1}{2}\alpha xx + \beta yx + \gamma x,$$ erit

$V\partial y = \beta x\partial y$, et $(Q - V)\,\partial y = \partial y\,(\delta y + \varepsilon) = \partial Y$,

ideoque $Y = \tfrac{1}{2}\delta yy + \varepsilon y$, unde integrale erit

$$\tfrac{1}{2}\alpha xx + \beta yx + \gamma x + \tfrac{1}{2}\delta yy + \varepsilon y = C.$$

Modo autem Corrollarii 3. spectata x constante, erit

$$\int Q\partial y = \beta xy + \tfrac{1}{2}\delta yy + \varepsilon y,$$

quae, spectata y constante, praebet $U\partial x = \beta y\partial x$, hincque

$$(P - U)\,\partial x = (\alpha x + \gamma)\,\partial x, \text{ et } X = \tfrac{1}{2}\alpha xx + \gamma x,$$

unde $\int Q\partial y + X = C$ integrale dat ut ante. Hinc simul etiam intelligitur esse

$$\int P\partial x - \int Q\partial y = \tfrac{1}{2}\alpha xx + \gamma x - \tfrac{1}{2}\delta yy - \varepsilon y,$$

quae in duas functiones $X - Y$ sponte dispescitur.

Exemplum 2.

455. *Aequationem per se integrabilem*

$$\frac{\partial y}{y} = \frac{x\partial y - y\partial x}{y\surd(xx+yy)}, \text{ seu } \frac{\partial x}{\surd(xx+yy)} + \frac{\partial y}{y}\left(1 - \frac{x}{\surd(xx+yy)}\right) = 0$$

integrare.

Cum hic sit

$$P = \frac{1}{\surd(xx+yy)} \text{ et } Q = \frac{1}{y} - \frac{x}{y\surd(xx+yy)},$$

pro charactere integrabilitatis per se cognoscendo est

$$\left(\frac{\partial P}{\partial y}\right) = \frac{-y}{(xx+yy)^{\frac{3}{2}}} \text{ et } \left(\frac{\partial Q}{\partial x}\right) = \frac{-y}{(xx+yy)^{\frac{3}{2}}},$$

qui bini valores utique sunt aequales. Jam pro integrali inveniendo, utamur regula Corrollarii 2. et habebimus

$$\int P\partial x = l[x + \surd(xx+yy)] \text{ et } V\partial y = \frac{y\partial y}{[x+\surd(xx+yy)]\surd(xx+yy)},$$

seu supra et infra per $\surd(xx+yy) - x$ multiplicando,

$$V = \frac{\sqrt{(xx+yy)}-x}{y\sqrt{(xx+yy)}} = \frac{1}{y} - \frac{x}{y\sqrt{(xx+yy)}},$$

unde $Q - V = 0$, et $Y = \int (Q - V)\,\partial y = 0$, sicque integrale quaesitum $l[x + \sqrt{(xx+yy)}] = \text{Const.}$

Per regulam Corollarii 3. habemus

$$\int Q\partial y = ly - x\int \frac{\partial y}{y\sqrt{(xx+yy)}},$$

at posito $y = \frac{1}{z}$, est

$$\int \frac{\partial y}{y\sqrt{(xx+yy)}} = -\int \frac{\partial z}{\sqrt{(xxzz+1)}} = -\frac{1}{x} l[xz - \sqrt{(xxzz+1)}],$$

ergo

$$\int Q\partial y = ly + l\frac{x+\sqrt{(xx+yy)}}{y} = l[x+\sqrt{(xx+yy)}],$$

vnde $U\partial x = \frac{\partial x}{\sqrt{(xx+yy)}}$; hinc $(P - U)\,\partial x = 0$.

Exemplum 3.

456. *Aequationem per se integrabilem*

$$(xx + yy - aa)\,\partial y + (aa + 2xy + xx)\,\partial x = 0,$$

integrare.

Hic ergo est

$$P = aa + 2xy + xx, \text{ et } Q = xx + yy - aa,$$

unde $\left(\frac{\partial P}{\partial y}\right) = 2x$ et $\left(\frac{\partial Q}{\partial x}\right) = 2x$, quae aequalitas integrabilitatem per se innuit. Tum vero est

$$\int P\partial x = aax + xxy + \tfrac{1}{3}x^3 \text{ et } V\partial y = xx\partial y,$$

unde $(Q - V)\,\partial y = (yy - aa)\,\partial y$ et $Y = \frac{1}{3}y^3 - aay$.
Ergo integrale

$$aax + xxy + \tfrac{1}{3}x^3 + \tfrac{1}{3}y^3 - aay = \text{Const.}$$

Altero modo est

$\int Q\partial y = xxy + \frac{1}{3}y^3 - aay$, hineque
$U\partial x = 2xy\partial x$, ergo
$(P - U)\,\partial x = (aa + xx)\,\partial x$ et $X = aax + \frac{1}{3}x^3$,

unde integrale oritur ut ante.

Scholion.

457. In his exemplis licuit, integrale $\int P \partial x$ actu exhibere, indeque ejus differentiale $V \partial y$, sumta sola y variabili, assignare. Quodsi autem hoc integrale $\int P \partial x$ evolvi nequeat, haud liquet quomodo inde differentiale $V \partial y$ elici possit, quandoquidem formula $\int P \partial x$ in se spectata constantem quamcunque, quae etiam y in se implicet, complectitur. Tum igitur quomodo procedendum sit, videamus. Ponamus $Z = \int P \partial x + Y$, et cum quaeratur $\left(\frac{\partial . \int P \partial x}{\partial y}\right) = V$, ob $\int P \partial x = Z - Y$, erit $V = \left(\frac{\partial Z}{\partial y}\right) - \frac{\partial Y}{\partial y}$. At est $\left(\frac{\partial Z}{\partial x}\right) = P$, ergo $\left(\frac{\partial \partial Z}{\partial x \partial y}\right) = \left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial V}{\partial x}\right)$, ob $\left(\frac{\partial Z}{\partial y}\right) = V + \frac{\partial Y}{\partial y}$. Hinc erit $V = \int \partial x \left(\frac{\partial P}{\partial y}\right)$, quare quantitas V invenitur per integrationem hujus formulae $\int \partial x \left(\frac{\partial P}{\partial y}\right)$, in qua y ut constans spectatur, postquam in valore $\left(\frac{\partial P}{\partial y}\right)$ inveniendo sola y variabilis esset assumta. Verum cum hic denuo constans cum y implicetur, hinc illa functio Y quam quaerimus non determinatur. Ratio hujus incommodi manifesto in ambiguitate integralium $\int P \partial x$ et $\int \partial x \left(\frac{\partial P}{\partial y}\right)$ est sita, dum utraque functiones arbitrarias ipsius y recipit. Remedium ergo afferetur, si utrumque integrale certa quadam conditione determinetur. Ita quando integrale $\int P \partial x$ ita accipi ponimus, ut evanescat posito $x = f$, ubi quidem constantem f pro lubitu accipere licet, tum eadem lege alterum integrale $\int \partial x \left(\frac{\partial P}{\partial y}\right)$ capiatur. Quo facto erit $Q - \int \partial x \left(\frac{\partial P}{\partial y}\right)$ functio ipsius y tantum, et aequationis $P \partial x + Q \partial y = 0$ integrale erit

$$\int P \partial x + \int \partial y \left[Q - \int \partial x \left(\frac{\partial P}{\partial y}\right)\right] = \text{Const.}$$

dummodo ambo integralia $\int P \partial x$ et $\int \partial x \left(\frac{\partial P}{\partial y}\right)$, in quibus y ut constans tractatur, ita determinentur, ut evanescant, dum in utraque ipsi x idem valor f tribuitur. Quare hinc istam colligimus regulam:

Regula pro integratione aequationis per se integrabilis

$$P\partial x + Q\partial y = 0, \text{ in qua } \left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right).$$

458. Quaerantur integralia $\int P\partial x$ et $\int \partial x \left(\frac{\partial P}{\partial y}\right)$, spectando y ut constantem, ita ut ambo evanescant, dum ipsi x certus quidam valor, puta $x = f$, tribuitur. Tum erit $Q - \int \partial x \left(\frac{\partial Q}{\partial y}\right)$ functio ipsius y tantum, quae sit $= Y$, et integrale quaesitum erit $\int P\partial x + \int Y \partial y = \text{Const.}$

Vel quod eodem redit, quaerantur integralia $\int Q\partial y$ et $\int \partial y \left(\frac{\partial Q}{\partial x}\right)$, spectando x ut constantem, ita ut ambo evanescant, dum ipsi y certus quidem valor, puta $y = g$, tribuitur: tum $P - \int \partial y \left(\frac{\partial Q}{\partial x}\right)$ erit functio ipsius x tantum, qua posita $= X$, erit integrale quaesitum $\int Q\partial y + \int X \partial x = \text{Const.}$

Demonstratio.

Veritatem hujus regulae ex praecedentibus perspicere licet, si cui forte precario assumsisse videamur, ambas formulas $\int P\partial x$ et $\int \partial x \left(\frac{\partial P}{\partial y}\right)$ eadem lege determinari debere, ut dum ipsi x certus quidam valor puta $x = f$ tribuitur, ambae evanescant. Sed ne forte quis putet, alteram integrationem pari jure secundum aliam legem determinari posse, hanc demonstrationem addo. Prima quidem integratio ab arbitrio nostro pendet, quam ergo ita determinari assumamus, ut integrale $\int P\partial x$ evanescat posito $x = f$, quo facto dico, alterum integrale $\int \partial x \left(\frac{\partial P}{\partial y}\right)$ necessario per eandem conditionem determinari oportere. Sit enim $\int P\partial x = Z$, eritque Z ejusmodi functio ipsarum x et y, quae evanescit posito $x = f$; habebit ergo factorem $f - x$, vel ejus quampiam potestatem positivam $(f - x)^\lambda$, ita ut sit $Z = (f - x)^\lambda T$. Nunc quia $\int \partial x \left(\frac{\partial P}{\partial y}\right)$ exprimit valorem ipsius $\left(\frac{\partial Z}{\partial y}\right)$, erit $\int \partial x \left(\frac{\partial P}{\partial y}\right) = (f - x)^\lambda \left(\frac{\partial T}{\partial y}\right)$, ex quo manifestum est hoc integrale etiam evanescere posito $x = f$, ita ut hujus integralis determinatio non amplius arbitrio nostro relinquatur. Hoc posito erit utique aequationis per se integrabilis $P\partial x + Q\partial y = 0$ integrale $\int P \partial x + \int Y \partial y =$ Const., existente $Y = Q - \int \partial x \left(\frac{\partial P}{\partial y}\right)$; nam posito $\int P \partial x = Z$, quatenus scilicet in hac integratione y pro constante habetur, ut habeatur haec aequatio $Z + \int Y \partial y =$ Const. quam esse integrale quaesitum vel ex ipsa differentiatione patebit. Cum enim sit

$$\partial Z = P \partial x + \partial y \left(\frac{\partial Z}{\partial y}\right) = P \partial x + \partial y \int \partial x \left(\frac{\partial P}{\partial y}\right),$$

erit aequationis inventae differentiale

$$P \partial x + \partial y \int \partial x \left(\frac{\partial P}{\partial y}\right) + Y \partial y = 0,$$

sed $Y = Q - \int \partial x \left(\frac{\partial P}{\partial y}\right)$, unde prodit $P \partial x + Q \partial y = 0$, quae est ipsa aequatio differentialis proposita. Quod autem sit $Q - \int \partial x \left(\frac{\partial P}{\partial y}\right)$ functio ipsius y tantum, inde sequitur, quoniam aequatio differentialis per se est integrabilis.

Theorema.

459. Pro omni aequatione, quae per se non est integrabilis semper datur quantitas, per quam ea multiplicata, redditur integrabilis.

Demonstratio.

Sit $P \partial x + Q \partial y = 0$ aequatio differentialis, et concipiamus ejus integrale completum, quod erit aequatio quaedam inter x et y, in quam constans quantitas arbitraria ingrediatur. Ex hac aequatione eruatur haec ipsa constans arbitraria, ut prodeat hujusmodi aequatio: Const. = *functioni cuidam ipsarum* x *et* y, quae differentiata praebeat $0 = M \partial x + N \partial y$, quae aequatio jam a constante illa, arbitraria per integrationem ingressa est libera, ideoque necesse est ut haec aequatio differentialis conveniat cum proposita, alioquin integrale suppositum non esset verum. Oportet ergo, ut relatio inter ∂x et ∂y utrinque prodeat eadem, unde erit $\frac{P}{Q} = \frac{M}{N}$, ideoque $M = LP$ et $N = LQ$. Sed quia $M\partial x + N\partial y$ est verum differentiale ex differentiatione cujuspiam functionis ipsarum x et y ortum, est $\left(\frac{\partial M}{\partial y}\right) = \left(\frac{\partial N}{\partial x}\right)$. Quare pro aequatione $P\partial x + Q\partial y = 0$ dabitur certo quidam multiplicator L, ut sit $\left(\frac{\partial . LP}{\partial y}\right) = \left(\frac{\partial . LQ}{\partial x}\right)$, seu ut aequatio per L multiplicata fiat per se integrabilis.

Corollarium 1.

460. Pro omni ergo aequatione $P\partial x + Q\partial y = 0$ datur ejusmodi functio L ut sit $\left(\frac{\partial . LP}{\partial y}\right) = \left(\frac{\partial . LQ}{\partial x}\right)$, ideoque evolvendo:

$$L\left(\frac{\partial P}{\partial y}\right) + P\left(\frac{\partial L}{\partial y}\right) = L\left(\frac{\partial Q}{\partial x}\right) + Q\left(\frac{\partial L}{\partial x}\right) \text{ seu}$$

$$L\left[\left(\frac{\partial P}{\partial y}\right) - \left(\frac{\partial Q}{\partial x}\right)\right] = Q\left(\frac{\partial L}{\partial x}\right) - P\left(\frac{\partial L}{\partial y}\right)$$

quae functio L si fuerit inventa, aequatio differentialis $LP\partial x + LQ\partial y = 0$ per se erit integrabilis.

Corollarium 2.

461. In aequatione proposita loco Q tuto unitatem scribere licet, quia omnis aequatio hac forma $P\partial x + \partial y = 0$ repraesentari potest. Hinc inventio multiplicatoris L, qui eam reddat per se integrabilem, pendet a resolutione hujus aequationis:

$$L\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial L}{\partial x}\right) - P\left(\frac{\partial L}{\partial y}\right),$$

ubi notandum est esse

$$\partial L = \partial x\left(\frac{\partial L}{\partial x}\right) + \partial y\left(\frac{\partial L}{\partial y}\right).$$

Scholion.

462. Quoniam hic quaeritur functio binarum variabilium x et y, quarum relatio mutua minime spectatur, quam involvit aequa-

sic $P\partial x + Q\partial y = 0$, haec investigatio in nostrum librum secundum incurrit ubi hujusmodi functio ex data quadam differentialium relatione indagare debet. In hac enim investigatione non attendimus ad aequationem propositam, qua formula $P\partial x + Q\partial y$ nihilo aequalis reddi debet, sed absolute quaeritur multiplicator L, per quem formula $P\partial x + Q\partial y$ multiplicata abeat in verum differentiale cujuspiam functionis finitae, quae sit Z, ita ut habeatur $\partial Z = LP\partial x + LQ\partial y$. Quo multiplicatore L invento tum demum aequalitas $P\partial x + Q\partial y = 0$ spectatur, indeque concluditur functionem Z quantitati constanti aequari oportere. Cum igitur minime expectari queat, ut methodum tradamus hujusmodi multiplicatores pro quavis aequatione differentiali proposita inveniendi, eos casus percurramus, quibus talis multiplicator constat, undecunque sit repertus. Interim tamen ad pleniorem usum hujus methodi notasse juvabit, statim atque unum multiplicatorem pro quapiam aequatione differentiali cognoverimus, ex eo facile innumerabiles alios deduci posse, qui pariter aequationem propositam per se integrabilem reddant.

Problema 60.

463. Dato uno multiplicatore L qui aequationem $P\partial x + Q\partial y = 0$ per se integrabilem reddat, invenire innumerabiles alios multiplicatores, qui idem officium praestent.

Solutio.

Cum ergo $L(P\partial x + Q\partial y)$ sit differentiale verum cujuspiam functionis Z, quaeratur per superiora praecepta haec functio Z, ita ut sit $L(P\partial x + Q\partial y) = \partial Z$: et nunc manifestum est, hanc formulam ∂Z integrationem etiam esse admissuram, si per functionem quamcunque ipsius Z quam ita $\Phi : Z$ indicemus, multiplicetur. Cum igitur etiam integrabilis sit haec formula $(P\partial x + Q\partial y) L\Phi : Z$, erit quoque $L\Phi : Z$ multiplicator aequationis propositae $P\partial x + Q\partial y = 0$, qui eam reddat integrabilem.

Quare invento uno multiplicatore L, quaeratur per integrationem $Z = \int L (P\partial x + Q\partial y)$, ac tum expressio $L\Phi:Z$, ubi pro $\Phi:Z$ functio quaecunque ipsius Z assumi potest, dabit infinitos alios multiplicatores idem officium praestantes.

Scholion.

464. Tametsi sufficiat pro quavis aequatione differentiali unicum miltiplicatorem cognovisse, tamen occurrunt casus, quibus perquam utile est, plures imo infinitos multiplicatores in promtu habere. Veluti si aequatio proposita in duas partes commode discerpatur, hujusmodi $(P\partial x + Q\partial y) + (R\partial x + S\partial y) = 0$ atque omnes multiplicatores constent, quibus utraque pars seorsim $P\partial x + Q\partial y$ et $R\partial x + S\partial y$ reddatur integrabilis, inde interdum communis multiplicator utramque integrabilem reddens concludi potest. Sit enim $L\Phi:Z$ expressio generalis pro omnibus multiplicatoribus formulae $P\partial x + Q\partial y$ et $M\Phi:V$ expressio generalis pro omnibus multiplicatoribus formulae $R\partial x + S\partial y$, et quoniam $\Phi:Z$ et $\Phi:V$ functiones quascunque quantitatum Z et V denotant, si eas ita capere liceat, ut fiat $L\Phi:Z = M\Phi:V$ habebitur multiplicator idoneus pro aequatione $P\partial x + Q\partial y + R\partial x + S\partial y = 0$. Intelligitur autem hoc iis tantum casibus praestari posse, quibus multiplicator pro tota aequatione, etiam singulas ejus partes seorsim sumtas integrabiles reddat. Quare cavendum est, ne huic methodo nimium tribuatur, et quando ea non succedit, aequatio pro irresolubili habeatur, evenire enim utique potest, ut tota aequatio habeat multiplicatorem, qui singulis ejus partibus non conveniat. Ita proposita aequatione $P\partial x + Q\partial y = 0$, multiplicator partem $P\partial x$ seorsim integrabilem reddens manifesto est $\frac{X}{P}$, denotante X functionem quamcunque ipsius x, et multiplicator partem alteram $Q\partial y$ integrabilem reddens est $\frac{Y}{Q}$: etiamsi autem neutiquam fieri possit, ut sit $\frac{X}{P} = \frac{Y}{Q}$ seu $\frac{P}{Q} = \frac{X}{Y}$, nisi casibus per se obviis, tamen tota formula $P\partial x + Q\partial y$ certo semper habet multiplicatorem, quo ea integrabilis reddatur.

Exemplum 1.

465. *Invenire omnes multiplicatores, quibus formula $\alpha y \partial x + \beta x \partial y$ integrabilis redditur.*

Primus multiplicator sponte se offert $\frac{1}{xy}$, qui praebet $\frac{\alpha \partial x}{x} + \frac{\beta \partial y}{y}$, cujus integrale est $\alpha l x + \beta l y = l x^\alpha y^\beta$. Hujus ergo functio quaecunque $\Phi : x^\alpha y^\beta$ in $\frac{1}{xy}$ ducta, dabit multiplicatorem idoneum, cujus itaque forma generalis est $\frac{1}{xy} \Phi : x^\alpha y^\beta$. Functio enim quantitatis $x^\alpha y^\beta$ etiam est functio logarithmi ejusdem quantitatis. Nam si P fuerit functio ipsius p, et Π functio ipsius P, etiam Π est functio ipsius p et vicissim.

Corollarium.

466. Si pro functione sumatur potestas quaecunque $x^{n\alpha} y^{n\beta}$, formula $\alpha y \partial x + \beta x \partial y$ integrabilis redditur, si multiplicetur per $x^{n\alpha - 1} y^{n\beta - 1}$, quo quidem casu integrale sponte patet, est enim $\frac{1}{n} x^{n\alpha} y^{n\beta}$.

Exemplum 2.

467. *Invenire omnes multiplicatores, qui hanc formulam $X y \partial x + \partial y$ integrabilem reddant.*

Primus multiplicator $\frac{1}{y}$ sponte se offert, unde cum sit $\int (X \partial x + \frac{\partial y}{y}) = \int X \partial x + l y$ seu $l e^{\int X \partial x} y$, omnes functiones hujus quantitatis, seu hujus $e^{\int X \partial x} y$ per y divisae, dabunt multiplicatores idoneos. Unde expressio generalis pro omnibus multiplicatoribus erit $= \frac{1}{y} \Phi : e^{\int X \partial x} y$.

Corollarium.

468. Pro formula $X y \partial x + \partial y$ multiplicator quoque est $e^{\int X \partial x}$, qui est functio ipsius x tantum; quo ergo cum etiam for-

mula $\mathfrak{X}\partial x$, denotante $\mathfrak{X}$ functionem quamcunque ipsius x, integrabilis reddatur, ille multiplicator etiam huic formulae $\partial y + Xy\partial x + \mathfrak{X}\partial x$ conveniet.

Problema 61.

469. Proposita aequatione $\partial y + Xy\partial x = \mathfrak{X}\partial x$, in qua X et $\mathfrak{X}$ sint functiones quaecunque ipsius x, invenire multiplicatorem idoneum, eamque integrare.

Solutio.

Cum alterum membrum $\mathfrak{X}\partial x$ per functionem quamcunque ipsius x multiplicatum fiat integrabile, dispiciatur num etiam prius membrum $\partial y + Xy\partial x$ per hujusmodi multiplicatorem integrabile reddi possit. Quod cum praestet multiplicator $e^{\int X\partial x}$, hoc adhibito habebitur aequatio integralis quaesita

$$e^{\int X\partial x} y = \int e^{\int X\partial x} \mathfrak{X}\partial x, \text{ sive}$$
$$y = e^{-\int X\partial x} \int e^{\int X\partial x} \mathfrak{X}\partial x,$$

uti jam supra invenimus.

Corollarium 1.

470. Patet etiam si loco y adsit functio quaecunque ipsius y, ut habeatur haec aequatio $\partial Y + YX\partial x = \mathfrak{X}\partial x$, eam per multiplicatorem $e^{\int X\partial x}$ reddi integrabilem, et integrale fore:

$$e^{\int X\partial x} Y = \int e^{\int X\partial x} \mathfrak{X}\partial x.$$

Corollarium 2.

471. Quare etiam haec aequatio $\partial y + yX\partial x = y^n \mathfrak{X}\partial x$, quia per y^n divisa abit in $\frac{\partial x}{y^n} + \frac{X\partial x}{y^{n-1}} = \mathfrak{X}\partial x$, ubi posito

$\frac{1}{y^{n-1}} = Y$, ob $-\frac{(n-1)\partial y}{y^n} = \partial Y$, seu $\frac{\partial y}{y^n} = -\frac{\partial Y}{n-1}$, prodit $-\frac{\partial Y}{n-1} + YX\partial x = \mathfrak{X}\partial x$, seu

$$\partial Y - (n-1)YX\partial x = -(n-1)\mathfrak{X}\partial x,$$

qui per multiplicatorem $e^{-(n-1)\int X\partial x}$ fit integrabilis: ejusque integrale erit

$$e^{-(n-1)\int X\partial x}Y = -(n-1)\int e^{-(n-1)\int X\partial x}\mathfrak{X}\partial x, \text{ sive}$$

$$\frac{1}{y^{n-1}} = -(n-1)e^{(n-1)\int X\partial x}\int e^{-(n-1)\int X\partial x}\mathfrak{X}\partial x.$$

Scholion.

472. Cum pro membro $\partial y + yX\partial x$ multiplicator generalis sit $\frac{1}{y}\Phi : e^{\int X\partial x}y$, sumta loco functionis potestate, multiplicator idoneus erit $e^{m\int X\partial x}y^{m-1}$, integrale praebens $\frac{1}{m}e^{m\int X\partial x}y^m$. Efficiendum ergo est, ut etiam idem multiplicator alterum membrum $y^n\mathfrak{X}\partial x$ reddat integrabile; quod evenit sumendo $m-1=-n$, seu $m=1-n$, ex quo hujus membri integrale fit $\int e^{m\int X\partial x}\mathfrak{X}\partial x$, ita ut aequatio integralis quaesita obtineatur:

$$\frac{1}{1-n}e^{(1-n)\int X\partial x}y^{1-n} = \int e^{(1-n)\int X\partial x}\mathfrak{X}\partial x,$$

quae cum modo inventa prorsus congruit.

Problema 62.

473. Proposita aequatione differentiali

$$\alpha y\partial x + \beta x\partial y = x^m y^n(\gamma y\partial x + \delta x\partial y).$$

invenire multiplicatorem idoneum, qui eam integrabilem reddat, ipsumque integrale assignare.

Solutio.

Consideretur utrumque membrum seorsim; ac pro priori vidimus $\alpha y \partial x + \beta x \partial y$ omnes multiplicatores idoneos contineri in hac forma $\frac{1}{xy} \Phi : x^{\alpha} y^{\beta}$. Pro altera parte

$$x^m y^n (\gamma y \partial x + \delta x \partial y),$$

primus multiplicator est $\frac{1}{x^{m+1} y^{n+1}}$, quo prodit $\frac{\gamma \partial x}{x} + \frac{\delta \partial y}{y}$, cujus integrale est $l x^{\gamma} y^{\delta}$: ergo forma generalis pro ejus multiplicatoribus est $\frac{1}{x^{m+1} y^{n+1}} \Phi : x^{\gamma} y^{\delta}$. Quo nunc hi duo multiplicatores pares reddantur, loco functionum sumantur potestates, fiatque

$$x^{\mu\alpha - 1} y^{\mu\beta - 1} = x^{\nu\gamma - m - 1} y^{\nu\delta - n - 1},$$

unde statui oportet $\mu\alpha = \nu\gamma - m$ et $\mu\beta = \nu\delta - n$; hincque colligitur:

$$\mu = \frac{\gamma n - \delta m}{\alpha\delta - \beta\gamma} \text{ et } \nu = \frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma}.$$

Quocirca multiplicator erit

$$x^{\mu\alpha - 1} y^{\mu\beta - 1} = x^{\nu\gamma - m - 1} y^{\nu\delta - n - 1},$$

unde aequatio nostra induit hanc formam

$$x^{\mu\alpha - 1} y^{\mu\beta - 1} (\alpha y \partial x + \beta x \partial y) = x^{\nu\gamma - 1} y^{\nu\delta - 1} (\gamma y \partial x + \delta x \partial y);$$

ubi utrumque membrum per se est integrabile, ideoque integrale quaesitum

$$\frac{1}{\mu} x^{\mu\alpha} y^{\mu\beta} = \frac{1}{\nu} x^{\nu\gamma} y^{\nu\delta} + \text{Const.}$$

quod convenit cum eo, quod capite praecedente est inventum.

Corollarium 1.

471. Posito ergo brevitatis gratia

$$\mu = \frac{\gamma n - \delta m}{\alpha\delta - \beta\gamma} \text{ et } \nu = \frac{\alpha n - \beta m}{\alpha\delta - \beta\gamma},$$

aequationis differentialis

$$\alpha y \partial x + \beta x \partial y = x^m y^n (\gamma y \partial x + \delta x \partial y)$$

integrale completum est

$$\tfrac{1}{\mu} x^{\mu\alpha} y^{\mu\beta} = \tfrac{1}{\nu} x^{\nu\gamma} y^{\nu\delta} + \text{Const.}$$

Corollarium 2.

475. Si eveniat, ut sit $\mu = 0$, seu $\gamma n = \delta m$, integrale ad logarithmos reducetur, eritque

$$l x^\alpha y^\beta = \tfrac{1}{\nu} x^{\nu\gamma} y^{\nu\delta} + \text{Const.}$$

Sin autem sit $\nu = 0$, seu $\alpha n = \beta m$, erit integrale

$$\tfrac{1}{\mu} x^{\mu\alpha} y^{\mu\beta} = l x^\gamma y^\delta + \text{Const.}$$

Scholion.

476. Hinc autem casus excipi videntur, quo $\alpha\delta = \beta\gamma$, quia tum ambo numeri μ et ν fiunt infiniti. Verum si $\delta = \frac{\beta\gamma}{\alpha}$ aequatio nostra hanc induit formam:

$$\alpha y \partial x + \beta x \partial y = \tfrac{\gamma}{\alpha} x^m y^n (\alpha y \partial x + \beta x \partial y), \text{ seu}$$

$$(\alpha y \partial x + \beta x \partial y)(1 - \tfrac{\gamma}{\alpha} x^m y^n) = 0,$$

quae cum habeat duos factores, duplex solutio ex utroque seorsim ad nihilum reducto, derivatur. Prior scilicet nascitur ex $\alpha y \partial x + \beta x \partial y = 0$, cujus integrale est $x^\alpha y^\beta =$ Const. alter vero factor per se dat aequationem finitam $1 - \frac{\gamma}{\alpha} x^m y^n = 0$, quarum solutionum utraque satisfacit. Atque hoc in genere tenendum est de omnibus aequationibus differentialibus, quas in factores resolvere licet, ubi perinde atque in aequationibus finitis singuli factores praebent solutiones. Plerumque autem factores finiti statim, antequam integratio suscipitur, per divisionem tolli solent, quandoquidem non ex natura rei, sed per operationes institutas demum accessisse censentur, ita ut perinde ac in Algebra saepe fieri solet, ad solutiones inutiles essent perducturi.

Problema 63.

477. Proposita aequatione differentiali homogenea, multiplicatorem idoneum invenire, qui eam integrabilem reddat, indeque ejus integrale eruere.

Solutio.

Sit $P\partial x + Q\partial y = 0$ aequatio proposita, in qua P et Q sint functiones homogeneae n dimensionum ipsarum x et y, ac quaeramus multiplicatorem L, qui sit etiam functio homogenea, cujus dimensionum numerus sit λ. Cum jam formula $L(P\partial x + Q\partial y)$ sit integrabilis, erit integrale functio $\lambda + n + 1$ dimensionum ipsarum x et y, quae functio si ponatur Z, erit ex natura functionum homogenearum

$$LPx + LQy = (\lambda + n + 1)Z.$$

Quare si λ sumatur $= -n - 1$, quantitas $LPx + LQy$ erit vel $= 0$, vel constans, unde obtinemus $L = \frac{1}{Px + Qy}$, qui ergo est multiplicator idoneus pro nostra aequatione. Idem quoque ex separatione variabilium colligitur: posito enim $y = ux$; fiet $P = x^n U$ et $Q = x^n V$, existentibus U et V functionibus u ipsius tantum, et ob $\partial y = u\partial x + x\partial u$

$$\text{erit } P\partial x + Q\partial y = x^n U\partial x + x^n Vu\partial x + x^n Vx\partial u,$$
$$\text{seu } P\partial x + Q\partial y = x^n(U + Vu)\partial x + x^{n+1}V\partial u.$$

At haec formula per $x^{n+1}(U + Vu)$ divisa fit integrabilis, ideoque et formula nostra $P\partial x + Q\partial y$ divisa per

$$x^{n+1}(U + Vu) = Px + Qy,$$

restitutis valoribus $U = \frac{P}{x^n}$, $V = \frac{Q}{x^n}$ et $u = \frac{y}{x}$, fiet integrabilis; seu multiplicator idoneus est $\frac{1}{Px + Qy}$, unde haec aequatio $\frac{P\partial x + Q\partial y}{Px + Qy} = 0$, semper per se est integrabilis.

Jam ad integrale ipsius inveniendum, integretur formula $\int \frac{P\partial x}{Px + Qy}$ spectando y ut constantem, ac determinetur certa ratione ut evanescat posito $x = f$. Tum posito brevitatis causa $\frac{P}{Px + Qy} = R$, sumatur valor $\left(\frac{\partial R}{\partial y}\right)$, et eadem lege quaeratur integrale $\int \partial x \left(\frac{\partial R}{\partial y}\right)$, spectando iterum y ut constantem. Tum erit $\frac{Q}{Px + Qy} - \int \partial x \left(\frac{\partial R}{\partial y}\right)$ functio ipsius y tantum seu

$$\frac{Q}{Px + Qy} - \int \partial x \left(\frac{\partial R}{\partial y}\right) = Y:$$

atque hinc erit integrale quaesitum

$$\int \frac{P\partial x}{Px + Qy} + \int Y \partial y = \text{Const.}$$

Corollarium 1.

478. Cum ergo formula $\frac{P\partial x + Q\partial y}{Px + Qy}$ sit per se integrabilis, si brevitatis gratia ponamus

$$\frac{P}{Px + Qy} = R \text{ et } \frac{Q}{Px + Qy} = S,$$

necesse est sit $\left(\frac{\partial R}{\partial y}\right) = \left(\frac{\partial S}{\partial x}\right)$. At est

$$\left(\frac{\partial R}{\partial y}\right) = \left[Qy\left(\frac{\partial P}{\partial y}\right) - Py\left(\frac{\partial Q}{\partial y}\right) - PQ\right] : (Px + Qy)^2 \text{ et}$$

$$\left(\frac{\partial S}{\partial x}\right) = \left[Px\left(\frac{\partial Q}{\partial x}\right) - Qx\left(\frac{\partial P}{\partial x}\right) - PQ\right] : (Px + Qy)^2$$

Quamobrem habebitur

$$Qy\left(\frac{\partial P}{\partial y}\right) - Py\left(\frac{\partial Q}{\partial y}\right) = Px\left(\frac{\partial Q}{\partial x}\right) - Qx\left(\frac{\partial P}{\partial x}\right).$$

Corollarium 2.

479. Haec aequalitas etiam ex natura functionum homogenearum concluditur. Cum enim P et Q sint functiones n dimensionum ipsarum x et y, ob

$$\partial P = \partial x \left(\frac{\partial P}{\partial x}\right) + \partial y \left(\frac{\partial P}{\partial y}\right) \text{ et } \partial Q = \partial x \left(\frac{\partial Q}{\partial x}\right) + \partial y \left(\frac{\partial Q}{\partial y}\right) \text{ erit}$$

$$nP = x \left(\frac{\partial P}{\partial x}\right) + y \left(\frac{\partial P}{\partial y}\right) \text{ et } nQ = x \left(\frac{\partial Q}{\partial x}\right) + y \left(\frac{\partial Q}{\partial y}\right).$$

Aequalitas autem inventa est

$$Q \left[x \left(\frac{\partial P}{\partial x}\right) + y \left(\frac{\partial P}{\partial y}\right)\right] = P \left[x \left(\frac{\partial Q}{\partial x}\right) + y \left(\frac{\partial Q}{\partial y}\right)\right],$$

quae hinc abit in identicam $nPQ = nPQ$.

Corollarium 3.

480. Si aequatio homogenea $P\partial x + Q\partial y = 0$ fuerit per se integrabilis, et P et Q sint functiones -1 dimensionis, erit $Px + Qy$ numerus constans. Veluti cum

$$\frac{x\partial x + y\partial y}{xx + yy} = 0,$$

hujusmodi sit aequatio, si loco ∂x et ∂y scribantur x et y, prodit $\frac{xx + yy}{xx + yy} = 1$.

Scholion.

481. In calculo differentiali ostendimus, si V fuerit functio homogenea n dimensionum ipsarum x et y, ponaturque $\partial V = P\partial x + Q\partial y$, fore $Px + Qy = nV$. Quare si $P\partial x + Q\partial y$ fuerit formula integrabilis, et P et Q functiones homogeneae $n-1$ dimensionum, integrale statim habetur, erit enim $V = \frac{1}{n}(Px + Qy)$, neque ad hoc ulla integratione est opus. Interim tamen videmus hinc excipi oportere casum quo $n = 0$, uti fit in nostra aequatione per multiplicatorem integrabili reddita $\frac{P\partial x + Q\partial y}{Px + Qy} = 0$, ubi ∂x et ∂y multiplicantur per functiones -1 dimensionis, neque enim hic integrale sine integratione obtineri potest. Ratio autem hujus excep-

tionis in hoc est sita, quod formulae integrabilis $P\,\partial x + Q\,\partial y$, in qua P et Q sunt functiones homogeneae $n-1$ dimensionum, integrale tum tantum sit functio homogenea n dimensionum, quando n non est $=0$, hoc enim solo casu fieri potest, ut integrale non sit functio nullius dimensionis, quemadmodum fit in hac formula differentiali $\frac{x\,\partial x + y\,\partial y}{xx+yy}$, quippe cujus integrale est $\frac{1}{2}\,l\,(xx+yy)$. Quocirca, quod formula $\frac{P\,\partial x + Q\,\partial y}{Px+Qy}$ sit integrabilis, hoc peculiari modo demonstravimus, ex ratione separabilitatis deducto. Interim tamen sine ullo respectu, unde hoc cognoverimus, id in praesenti negotio maxime est notatu dignum, omnes aequationes homogeneas $P\partial x + Q\partial y = 0$, per multiplicatorem $\frac{1}{Px+Qy}$ per se reddi integrabiles. Methodus igitur desideratur, cujus beneficio hunc multiplicatorem a priori invenire liceret; qua methodo sane maxima incrementa in Analysin importarentur. Quamdiu autem eousque pertingere non licet, plurimum intererit hujusmodi multiplicatores pro pluribus casibus probe notasse; quod cum jam in duobus aequationum generibus praestiterimus, pro reliquis aequationibus, quas supra integrare docuimus, multiplicatores investigemus; ipsa autem reductio ad separationem nobis hos multiplicatores patefaciet, uti in sequente problemate docebimus.

Problema 64.

482. Proposita aequatione differentiali, quam ad separationem variabilium reducere liceat, invenire multiplicatorem, per quem ea per se integrabilis reddatur.

Solutio.

Sit $P\,\partial x + Q\,\partial y = 0$, quae certa quadam substitutione, dum loco x et y aliae binae variabiles t et u introducuntur, ad separationem accommodetur: ponamus ergo facta hac substitutione fieri $P\,\partial x + Q\,\partial y = R\,\partial t + S\,\partial u$, nunc autem hanc formulam

$R\partial t + S\partial u$, si per V dividatur, separari, ita ut in hac formula $\frac{R\partial t + S\partial u}{V}$ quantitas $\frac{R}{V}$ sit functio solius t, et $\frac{S}{V}$ functio solius u. Cum igitur formula $\frac{R\partial t + S\partial u}{V}$ per se sit integrabilis, etiam integrabilis erit haec $\frac{P\partial x + Q\partial y}{V}$ quippe illi aequalis, siquidem in V variabiles x et y restituantur. Hinc ergo ex reductione ad separabilitatem aequationis $P\partial x + Q\partial y = 0$ discimus, multiplicatorem quo ea integrabilis reddatur, esse $\frac{1}{V}$, sicque quas aequationes ad separationem variabilium perducere licet, pro iisdem multiplicatorem, qui illas integrabiles reddat, assignare possumus.

Corollarium 1.

483. Methodus ergo per multiplicatores integrandi aequationes differentiales aeque late patet ac prior methodus, ope separationis variabilium; propterea quod ipsa separatio pro quavis aequatione, ubi succedit, multiplicatorem suppeditat.

Corollarium 2.

484. Contra autem methodus per multiplicatores integrandi latius patet altera, si pro ejusmodi aequationibus multiplicatores assignare liceat, quae quomodo ad separationem perduci debeant, non constet.

Scholion.

485. Etsi autem ex reductione ad separationem idoneum multiplicatorem elicere licet, tamen nondum intelligitur, quomodo cognito multiplicatore, separatio variabilium institui debeat, quare etiam ob hanc rationem methodus per multiplicatores integrandi alteri longe praeferenda videtur. Quamvis enim hactenus ipsa separatio nos ad inventionem multiplicatorum perduxerit, nullum tamen est dubium quin detur via multiplicatores inveniendi, nullo respectu ad separationem habito, licet haec via etiamnum nobis sit incognita. Ea au-

tem paullatim planior reddetur, si pro quamplurimis aequationibus multiplicatores idoneos cognoverimus, ex quo quos adhuc ex separatione eruere licet, indagemus in subjunctis exemplis.

Exemplum 1.

486. *Proposita aequatione differentiali primi ordinis*

$$\partial x(\alpha x+\beta y+\gamma)+\partial y(\delta x+\varepsilon y+\zeta)=0,$$

pro ea multiplicatorem idoneum assignare.

Haec aequatio ad separationem praeparatur ponendo primo

$$\alpha x+\beta y+\gamma=r \text{ et } \delta x+\varepsilon y+\zeta=s,$$

ideóque

$$\alpha\partial x+\beta\partial y=\partial r \text{ et } \delta\partial x+\varepsilon\partial y=\partial s,$$

unde oritur

$$\partial x=\frac{\varepsilon\partial r-\beta\partial s}{\alpha\varepsilon-\beta\delta} \text{ et } \partial y=\frac{\alpha\partial s-\delta\partial r}{\alpha\varepsilon-\beta\delta},$$

hincque aequatio nostra omisso denominatore utpote constante, erit

$$\varepsilon r\partial r-\beta r\partial s+\alpha s\partial s-\delta s\partial r=0,$$

quae cum sit homogenea, per $\varepsilon rr-(\beta+\delta)sr+\alpha ss$ divisa, fit integrabilis. Quod idem ex separatione colligitur, posito enim $r=su$, prodit

$$\varepsilon ssu\partial u+\varepsilon suu\partial s-\beta su\partial s+\alpha s\partial s-\delta ss\partial u-\delta su\partial s=0 \text{ seu}$$

$$ss\partial u(\varepsilon u-\delta)+s\partial s(\varepsilon uu-\beta u-\delta u+\alpha)=0,$$

quae divisa per $ss(\varepsilon uu-\beta u-\delta u+\alpha)$ separatur. Quare multiplicator nostrae aequationis propositae est

$$\frac{1}{ss(\varepsilon uu-\beta u-\delta u+\alpha)}=\frac{1}{\varepsilon rr-\beta rs-\delta rs+\alpha ss}=\frac{1}{r(\varepsilon r-\beta s)+s(\alpha s-\delta r)},$$

qui restitutis valoribus fit

$$\frac{1}{(\alpha x+\beta y+\gamma)[(\alpha\varepsilon-\beta\delta)x+\gamma\varepsilon-\beta\zeta]+(\delta x+\varepsilon y+\zeta)[(\alpha\varepsilon-\beta\delta)y+\alpha\zeta-\gamma\delta]}:$$

atque evolutione facta

$$\frac{1}{(\alpha\varepsilon-\beta\delta)[\alpha xx+(\beta+\delta)x\gamma+\varepsilon\gamma\gamma+\gamma x+\zeta y]+\alpha\zeta\zeta-(\beta+\delta)\gamma\zeta+\gamma\gamma\varepsilon+[\alpha\gamma\varepsilon-(\beta-\delta)\alpha\zeta-\gamma\delta\delta]x+[\alpha\varepsilon\zeta+(\beta-\delta)\gamma\varepsilon-\beta\beta\zeta]y.}$$

Quare per se integrabilis erit haec aequatio

$$\frac{\partial x(\alpha x+\beta y+\gamma)+\partial y(\delta x+\varepsilon y+\zeta)}{(\alpha\varepsilon-\beta\delta)[\alpha xx+(\beta+\delta)xy+\varepsilon yy+\gamma x+\zeta y]+Ax+By+C}=0$$

existente

$$A=\alpha\gamma\varepsilon-(\beta-\delta)\alpha\zeta-\gamma\delta\delta$$
$$B=\alpha\varepsilon\zeta+(\beta-\delta)\gamma\varepsilon-\beta\beta\zeta$$
$$C=\alpha\zeta\zeta-(\beta+\delta)\gamma\zeta+\gamma\gamma\varepsilon.$$

Corollarium.

487. Etiamsi forte fiat $\alpha\varepsilon-\beta\delta=0$, hic multiplicator non turbatur, cum tamen separatio non succedat hac quidem operatione. Sit enim $\alpha=ma$, $\beta=mb$, $\delta=na$, $\varepsilon=nb$, ut habeatur haec aequatio

$$\partial x[m(ax+by)+\gamma]+\partial y[n(ax+by)+\zeta]=0$$

ob $A=a(na-mb)(m\zeta-n\gamma)$
$B=b(na-mb)(m\zeta-n\gamma)$ et
$C=(m\zeta-n\gamma)(a\zeta-b\gamma)$,

omisso factore communi, multiplicator est

$$\frac{1}{(na-mb)(ax+by)+a\zeta-b\gamma},$$

ita ut haec aequatio per se sit integrabilis

$$\frac{(ax+by)(m\partial x+n\partial y)+\gamma\partial x+\zeta\partial y}{(na-mb)(ax+by)+a\zeta-b\gamma}=0.$$

Exemplum 2.

488. *Proposita aequatione differentiali*

$$y\partial x(c+nx)-\partial y(y+a+bx+nxx)=0,$$

multiplicatorem idoneum invenire.

Fiat substitutio $\frac{y(c+nx)}{y+a+bx+nxx}=u$, seu $y=\frac{u(a+bx+nxx)}{c+nx-u}$, ut contrahatur aequatio nostra in hanc formam

$$y\partial x(c+nx)-\frac{y\partial y(c+nx)}{u}=0,$$

seu $\frac{y(c+nx)}{u}(u\partial x-\partial y)=0$, vel $\frac{yy(c+nx)}{u}\left(\frac{\partial y}{y}-\frac{u\partial x}{y}\right)=0$; probe enim cavendum est, ne hic ullus factor omittatur. At facta substitutione reperitur

$$\frac{\partial y}{y}-\frac{u\partial x}{y}=\frac{\partial u}{u}+\frac{\partial x(b+2nx)}{a+bx+nxx}+\frac{\partial u-n\partial x}{c+nx-u}-\frac{\partial x(c+nx-u)}{a+bx+nxx}$$
$$=\frac{\partial u(c+nx)}{u(c+nx-u)}-\frac{\partial x(na+cc-bc+(b-2c)u+uu)}{(c+nx-u)(a+bx+nxx)}.$$

Unde aequatio nostra induet hanc formam

$$\frac{yy(c+nx)^2}{u(c+nx-u)}\left(\frac{\partial u}{u}-\frac{\partial x(na+cc-bc+(b-2c)u+uu)}{(a+bx+nxx)(c+nx)}\right)=0,$$

quae ergo separabitur ducta in hunc multiplicatorem

$$\frac{u(c+nx-u)}{yy(c+nx)^2(na+cc-bc+(b-2c)u+uu)}$$

tum enim prodit

$$\frac{\partial u}{u(na+cc-bc+(b-2c)u+uu)}-\frac{\partial x}{(a+bx+nxx)(c+nx)}=0.$$

Quo igitur multiplicatorem quaesitum consequamur, ibi loco u tantum opus est suum valorem restituere tum autem reperitur multiplicator

$$\frac{a+bx+nxx}{n(a+bx+nxx)y^3+(a+bx+nxx)[2na-bc+n(b-2c)x]yy+(na+cc-bc)(a+bx+nxx)^2y},$$

qui reducitur ad hanc formam

$$\frac{1}{ny^3+(2na-bc)yy+n(b-2c)xyy+(na+cc-bc)(a+bx+nxx)y}.$$

Exemplum 3.

489. *Proposita aequatione differentiali*

$$\frac{n\partial x(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}+(x-y)\partial y=0,$$

invenire multiplicatorem qui eam integrabilem reddat.

Posuimus supra (435.) $y = \frac{x-u}{1+xu}$, seu $u = \frac{x-y}{1+xy}$, unde fit $x - y = \frac{u(1+xx)}{1+xu}$, et $1 + yy = \frac{(1+xx)(1+uu)}{(1+xu)^2}$, hincque nostra aequatio hanc induit formam

$$\frac{n\partial x(1+xx)(1+uu)^{\frac{3}{2}}}{(1+xu)^3} + \frac{u\partial x(1+xx)(1+uu) - u\partial u(1+xx)^2}{(1+xu)^3} = 0,$$

quae primo multiplicata per $(1+xu)^3$, tum divisa per

$$(1+xx)^2(1+uu)[u+n\sqrt{(1+uu)}]$$

separatur. Quare aequationis nostrae multiplicator erit

$$\frac{(1+xu)^3}{(1+xx)^2(1+uu)[u+n\sqrt{(1+uu)}]},$$

qui primo ob $1+uu = \frac{(1+yy)(1+xu)^2}{1+xx}$, abit in

$$\frac{1+xu}{(1+xx)(1+yy)[u+n\sqrt{(1+uu)}]}.$$

Nunc ob $u = \frac{x-y}{1+xy}$, est $\sqrt{(1+uu)} = \frac{\sqrt{(1+xx)(1+yy)}}{1+xy}$ et $1+xu = \frac{1+xx}{1+xy}$, ideoque noster multiplicator colligitur:

$$\frac{1}{(1+yy)[x-y+n\sqrt{(1+xx)(1+yy)}]},$$

ita ut per se sit integrabilis haec aequatio

$$\frac{n\partial x(1+yy)\sqrt{(1+yy)} + (x-y)\partial y\sqrt{(1+xx)}}{(1+yy)[x-y+n\sqrt{(1+xx)(1+yy)}]\sqrt{(1+xx)}} = 0,$$

cujus integrationi non immoror, cum jam supra integrale exhibuerim.

Exemplum 4.

490. *Aliud exemplum memoratu dignum suppeditat haec aequatio*

$$y\partial x - x\partial y + ax^n y\partial y(x^n+b)^{\frac{1}{n}} = 0,$$

quae si hac forma repraesentetur

$$x\partial y - y\partial x + \tfrac{1}{b}x^{n+1}\partial y = \tfrac{1}{b}x^{n+1}\partial y + ax^n y\partial y(x^n+b)^{\frac{1}{n}}$$

evenit, ut utrumque integrabile existat, si ducatur in hunc multiplicatorem

$$\frac{y^{n-1}}{x^{n+1}+abx^n y(x^n+b)^{\frac{1}{n}}}:$$

ad quem inveniendum ex separatione variabilium, adhibeatur haec substitutio non adeo obvia $\frac{x}{(x^n+b)^{\frac{1}{n}}}=vy$, unde fit

$x^n=\frac{bv^n y^n}{1-v^n y^n}$, et hinc aequatio

$$\frac{y\partial x-x\partial y}{(x^n+b)^{\frac{1}{n}}}+ax^n y\partial y=0 \text{ abit in hanc}$$

$$\frac{yy\partial v+v^{n+1}y^{n+1}\partial y+abv^n y^{n+1}\partial y}{1-v^n y^n}=0,$$

quae multiplicata per $\frac{1-v^n y^n}{yyv^n(ab+v)}$ separatur

$$\frac{\partial v}{v^n(ab+v)}+y^{n-1}\partial y=0,$$

unde idem ille multiplicator colligitur.

Exemplum 5.

491. *Proposita aequatione differentiali*

$$\partial y+yy\partial x-\frac{a\partial x}{x^4}=0$$

invenire multiplicatorem, quo ea integrabilis reddatur.

Secundum §. 436. ponatur $x=\frac{1}{t}$ et ob $\partial x=-\frac{\partial t}{tt}$, nostra formula erit $\partial y-\frac{yy\partial t}{tt}+att\partial t$, in qua porro statuatur $y=t-ttz$, et prodibit $-tt(\partial z+zz\partial t-a\partial t)$, quae per $tt(zz-a)$ divisa separatur, ergo et nostra aequatio divisa per $tt(zz-a)=\frac{(t-y)^2-at^4}{tt}$

$= (1 - xy)^2 - \frac{a}{xx}$ fiet integrabilis, ex quo multiplicator erit $= \frac{xx}{xx(1-xy)^2 - a}$, et aequatio per se integrabilis $\frac{x^4 \partial y + x^4 yy \partial x - a \partial x}{x^4(1-xy)^2 - axx} = 0$. Spectetur jam x ut constans, eritque ex ∂y natum integrale

$$\frac{1}{2\surd a} \cdot l \frac{\surd a + x(1-xy)}{\surd a - x(1-xy)} + X,$$

pro quo ut valor ipsius X obtineatur, differentietur denuo, ac prodibit

$$\frac{2xy\partial x - \partial x}{xx(1-xy)^2 - a} + \partial X = \frac{x^4 yy \partial x - a\partial x}{x^4(1-xy)^2 - axx};$$

unde

$$\partial X = \frac{x^4 yy \partial x - a\partial x - 2x^3 y \partial x + xx\partial x}{x^4(1-xy)^2 - axx} = \frac{\partial x}{xx},$$

et $X = -\frac{1}{x} + C$; quare aequatio integralis completa erit

$$l \frac{\surd a + x(1-xy)}{\surd a - x(1-zy)} = \frac{2\surd a}{x} + C.$$

Scholion.

492. En ergo plures casus aequationum differentialium pro quibus multiplicatores novimus, ex quorum contemplatione haec insignis investigatio non parum adjuvari videtur. Quanquam autem adhuc longe absumus a certa methodo, pro quovis casu multiplicatores idoneos inveniendi; hinc tamen formas aequationum colligere poterimus, ut per datos multiplicatores integrabiles reddantur; quod negotium cum in hac ardua doctrina maximam utilitatem allaturum videatur, in sequente capite aequationes investigabimus, quibus dati multiplicatores conveniant? exempla scilicet hic evoluta idoneas multiplicatorum formas nobis suppeditant, quibus nostram investigationem superstruere licebit.

CAPUT III.

DE INVESTIGATIONE AEQUATIONUM DIFFERENTIALIUM QUAE PER MULTIPLICATORES DATAE FORMAE INTEGRABILES REDDANTUR.

Problema 65.

493.

Definire functiones P et Q ipsius x, ut aequatio differentialis $Py\partial x + (y + Q)\partial y = 0$, per multiplicatorem $\frac{1}{y^3 + Myy + Ny}$, ubi M et N sunt functiones ipsius x, fiat integrabilis.

Solutio.

Necesse igitur est, ut factoris ipsius ∂x, qui est $\frac{Py}{y^3 + Myy + Ny}$, differentiale ex variabilitate ipsius y natum, aequale sit differentiali factoris ipsius ∂y, qui est $\frac{y + Q}{y^3 + Myy + Ny}$, dum sola x variabilis sumitur. Horum valorum aequalium, neglecto denominatore communi, aequalitas dat

$$-2Py^3 - PMy^2 = (y^3 + Myy + Ny)\frac{\partial Q}{\partial x} - (y + Q)\frac{(yy\partial M + y\partial N)}{\partial x},$$

quae secundum potestates ipsius y ordinata praebet

$$\begin{aligned} 0 = {} & 2Py^3\partial x + PMy^2\partial x \\ & + y^3\partial Q \quad + My^2\partial Q + Ny\partial Q \\ & - y^3\partial M \quad - y^2\partial N \\ & \qquad\qquad - Qy^2\partial M - Qy\partial N \end{aligned}$$

unde singulis potestatibus seorsim ad nihilum perductis, nanciscimur primo $N\partial Q - Q\partial N = 0$, seu $\frac{\partial N}{N} = \frac{\partial Q}{Q}$, ex cujus integratione sequitur $N = \alpha Q$. Tum binae reliquae conditiones sunt,

$$\text{I. } 2P\partial x + \partial Q - \partial M = 0 \text{ et}$$

$$\text{II. } PM\partial x + M\partial Q - \alpha\partial Q - Q\partial M = 0;$$

unde I. M — II. 2 suppeditat

$$-M\partial Q - M\partial M + 2\alpha\partial Q + 2Q\partial M = 0, \text{ seu}$$

$$\partial Q + \frac{2Q\partial M}{2\alpha - M} = \frac{M\partial M}{2\alpha - M},$$

quae per $(2\alpha - M)^2$ divisa et integrata dat

$$\frac{Q}{(2\alpha - M)^2} = \int \frac{M\partial M}{(2\alpha - M)^3} = -\int \frac{\partial M}{(2\alpha - M)^2} + 2\alpha \int \frac{\partial M}{(2\alpha - M)^3},$$

seu

$$\frac{Q}{(2\alpha - M)^2} = \frac{-1}{2\alpha - M} + \frac{\alpha}{(2\alpha - M)^2} + \beta = \frac{M - \alpha}{(2\alpha - M)^2} + \beta.$$

Erit ergo

$$Q = M - \alpha + \beta(2\alpha - M)^2,$$

hincque

$$2P\partial x = \partial M - \partial Q = + 2\beta\partial M(2\alpha - M):$$

sicque pro M functionem quamcunque ipsius x sumere licet. Capiatur ergo $M = 2\alpha - X$, erit $P\partial x = -\beta X\partial X$, et $Q = \alpha - X + \beta XX$ atque $N = \alpha\alpha - \alpha X + \alpha\beta XX$. Quocirca pro hac aequatione

$$-\beta y X\partial X + \partial y(\alpha - X + \beta XX + y) = 0$$

habemus hunc multiplicatorem

$$\frac{1}{y^3 + (2\alpha - X)yy + \alpha(\alpha - X + \beta XX)y},$$

quo ea integrabilis redditur.

Corollarium 1.

494. Tribuatur aequationi haec forma

$$\partial y(y + A + BV + CVV) - CyV\partial V = 0$$

eritque $\alpha = A$; $X = -BV$; $\beta XX = \beta BBVV = CVV$: ergo $\beta = \frac{C}{BB}$, unde multiplicator fiet

$$\frac{1}{y^3 + (2A + BV)yy + A(A + BV + CVV)y}.$$

Corollarium 2.

495. Si hic sumatur $V = a + x$, obtinebitur aequatio similis illi, quam supra §. 488. integravimus, et multiplicator quoque cum eo, quem ibi dedimus, convenit. Hic autem multiplicator commodius hac forma exhibetur

$$\frac{1}{y(y+A)^2 + BVy(y+A) + ACVVy}.$$

Corollarium 3.

496. Si ponamus $y + A = z$, nostra aequatio erit

$$\partial z(z + BV + CVV) - C(z - A)V\partial V = 0,$$

cui convenit multiplicator $\frac{1}{(z-A)(zz + BVz + ACVV)}$; ita ut per se integrabilis sit haec aequatio

$$\frac{\partial z(z + BV + CVV) - C(z - A)V\partial V}{(z-A)(zz + BVz + ACVV)} = 0.$$

Scholion.

497. Quemadmodum hic aequationis $Py\partial x + (y + Q)\partial y = 0$ multiplicatorem assumsimus $= \frac{y^{-1}}{yy + My + N}$, ita generalius ejus loco sumere poterimus $\frac{y^{n-1}}{yy + My + N}$, ut haec aequatio

$$\frac{Py^n\partial x + (y^n + Qy^{n-1})\partial y}{yy + My + N} = 0$$

per se debeat esse integrabilis, qua comparata cum forma $R\partial x + S\partial y = 0$, ut sit $\left(\frac{\partial R}{\partial y}\right) = \left(\frac{\partial S}{\partial x}\right)$, habebimus

$$(n-2)Py^{n+1}+(n-1)PMy^n+nPNy^{n-1}=(yy+My+N)y^{n-1}\frac{\partial Q}{\partial x}$$
$$-(y^n+Qy^{n-1})\left(\frac{y\partial M}{\partial x}+\frac{\partial N}{\partial x}\right),$$

sive ordinata aequatione

$$\left.\begin{matrix} (n-2)Py^{n+1}\partial x+ & (n-1)PMy^n\partial x+ & nPNy^{n-1}\partial x \\ -y^{n+1}\partial Q & -My^n\partial Q & -Ny^{n-1}\partial Q \\ +y^{n+1}\partial M & +y^n\partial N & +y^{n-1}Q\partial N \\ & +y^nQ\partial M & \end{matrix}\right\}=0;$$

unde singulis membris ad nihilum reductis, fit

I. $(n-2)P\partial x=\partial Q-\partial M$

II. $(n-1)MP\partial x=M\partial Q-Q\partial M-\partial N$

III. $nNP\partial x=N\partial Q-Q\partial N.$

Sit $P\partial x=\partial V$, eritque ex prima $Q=A+M+(n-2)V$, quo valore in secunda substituto prodit

$$M\partial V+(n-2)V\partial M+A\partial M+\partial N=0$$

et tertia fit

$$2N\partial V+(n-2)V\partial N+M\partial N-N\partial M+A\partial N=0:$$

unde eliminando ∂V reperitur

$$(n-2)V+A=\frac{MM\partial N-MN\partial M-2N\partial N}{2N\partial M-M\partial N}.$$

Verum si hinc vellemus V elidere, in aequationem differentio-differentialem illaberemur. Casus tamen quo $n=2$ expediri potest.

Exemplum.

498. *Sit in evolutione hujus casus $n=2$, ut per se integrabilis esse debeat haec aequatio*

$$\frac{y[Py\partial x+(y+Q)\partial y]}{yy+My+N}=0.$$

Ac primo esse oportet $Q=A+M$, tum vero

$$2AN\partial M-AM\partial N=M(M\partial N-N\partial M)-2N\partial N,$$

quam ergo aequationem integrare debemus, quae cum in nulla jam tractatarum contineatur, videndum est, quomodo tractabilior reddi queat. Ponatur ergo $M=Nu$, ut fiat

$M\partial N - N\partial M = - NN\partial u$, et
$2N\partial M - M\partial N = 2NN\partial u + Nu\partial N$, hinc
$2ANN\partial u + ANu\partial N + N^3 u\partial u + 2N\partial N = 0$, sive

$$\frac{2\partial N}{NN} + \frac{Au\partial N}{NN} + \frac{2A\partial u}{N} + u\partial u = 0:$$

statuatur porro $\frac{1}{N} = v$, seu $N = \frac{1}{v}$, habebitur

$$-2\partial v - Au\partial v + 2Av\partial u + u\partial u = 0, \text{ seu}$$

$$\partial v - \frac{2Av\partial u}{2+Au} = \frac{u\partial u}{2+Au}.$$

Ubi variabilis v unicam habet dimensionem, et hanc ob rem patet, hanc aequationem integrabilem reddi, si dividatur per $(2+Au)^2$; prodibitque

$$\frac{v}{(2+Au)^2} = \int\frac{u\partial u}{(2+Au)^3} = \frac{C}{AA} - \frac{1-Au}{AA(2+Au)^2},$$

ideoque $v = \frac{C(2+Au)^2 - 1 - Au}{AA}$. Sumto ergo pro u functione quacunque ipsius x, erit

$$N = \frac{AA}{C(2+Au)^2 - 1 - Au}, \text{ et } M = \frac{AAu}{C(2+Au)^2 - 1 - Au},$$

atque $Q = \frac{AC(2+Au)^2 - A}{C(2+Au)^2 - 1 - Au}$. Jam ex tertia aequatione adipiscimur $2NP\partial x = N\partial Q - Q\partial N$, seu $2P\partial x = N\partial . \frac{Q}{N}$, at $\frac{Q}{N} = \frac{C(2+Au)^2 - 1}{A}$, unde $\partial . \frac{Q}{N} = 2C\partial u(2+Au)$, ideoque

$$P\partial x = \frac{AAC\partial u(2+Au)}{C(2+Au)^2 - 1 - Au}.$$

Quocirca aequatio nostra per se integrabilis est

$$\frac{AACyy\partial u(2+Au) + y\partial y[C(2+Au)^2 y - (1+Au)y + AC(2+Au)^2 - A)}{C(2+Au)^2 yy - (1+Au)yy + AAuy + AA} = 0,$$

quae posito $Au + 2 = t$, induet hanc formam

$$y . \frac{ACyt\partial t + y\partial y(Ctt - t + 1) + A\partial y(Ctt - 1)}{Cttyy - (t-1)yy + A(t-2)y + AA} = 0.$$

Hinc autem posito $A = \alpha$; $C = \frac{\alpha\gamma}{\beta\beta}$ et $t = -\frac{\beta x}{\alpha}$, invenimus

$$y . \frac{\alpha\gamma xy\partial x + y\partial y(\alpha + \beta x + \gamma xx) - \alpha\partial y(\alpha - \gamma xx)}{(\alpha + \beta x + \gamma xx)yy - \alpha(2\alpha + \beta x)y + \alpha^3} = 0.$$

Corollarium 1.

499. Hoc igitur modo integrari potest haec aequatio

$$\alpha\gamma xy\partial x+y\partial y(\alpha+\beta x+\gamma xx)-\alpha\partial y(\alpha-\gamma xx)=0,$$

quae quomodo ad separationem reduci debeat, non statim patet. Est autem multiplicator idoneus

$$\frac{\gamma}{(\alpha+\beta x+\gamma xx)yy-\alpha(2\alpha+\beta x)y+\alpha^3}.$$

Corollarium 2.

500. Hic multiplicator etiam hoc modo exprimi potest, ut ejus denominator in factores resolvatur

$$\frac{(\alpha+\beta x+\gamma xx)y}{[(\alpha+\beta x+\gamma xx)y-(\alpha+\frac{1}{2}\beta x)+\alpha x\sqrt{(\frac{1}{4}\beta\beta-\alpha\gamma)}][(\alpha+\beta x+\gamma xx)y-\alpha(\alpha+\frac{1}{2}\beta x)-\alpha x\sqrt{(\frac{1}{4}\beta\beta}$$

Corollarium 3.

501. Si ergo ponamus

$$(\alpha+\beta x+\gamma xx)y-\alpha(\alpha+\frac{1}{2}\beta x)=\alpha z,$$

erit multiplicator

$$\frac{\alpha+\frac{1}{2}\beta x+z}{[z+x\sqrt{(\frac{1}{4}\beta\beta-\alpha\gamma)}][z-x\sqrt{(\frac{1}{4}\beta\beta-\alpha\gamma)}]}.$$

At ob $y=\dfrac{\alpha\alpha+\frac{1}{2}\alpha\beta x+\alpha z}{\alpha+\beta x+\gamma xx}$, aequatio nostra erit

$$\gamma xy\partial x+\partial y(z+\tfrac{1}{2}\beta x+\gamma xx)=0.$$

At est

$$\partial y=\frac{-\frac{1}{2}\alpha(\alpha\beta+4\alpha\gamma x+\beta\gamma xx)\partial x-\alpha z\partial x(\beta+2\gamma x)+\alpha\partial z(\alpha+\beta x+\gamma xx)}{(\alpha-\beta x+\gamma xx)^2}$$

hoc autem valore substituto prodit aequatio nimis complicata.

Problema 66.

502. Invenire aequationem differentialem hujus formae

$$yP\partial x + (Qy + R)\partial y = 0$$

in qua P, Q et R sint functiones ipsius x, ut ea integrabilis evadat per hunc multiplicatorem $\frac{y^m}{(1 + Sy)^n}$, ubi S est etiam functio ipsius x.

Solutio.

Quia ∂x per $\frac{y^{m+1}P}{(1 + Sy)^n}$ et ∂y per $\frac{Qy^{m+1} + Ry^m}{(1 + Sy)^n}$ multiplicatur, oportet sit

$$(m+1)Py^m(1 + Sy) - nPSy^{m+1}$$
$$= \frac{(1 + Sy)(y^{m+1}\partial Q + y^m\partial R) - ny\partial S(Qy^{m+1} + Ry^m)}{\partial x},$$

qua evoluta aequatione erit

$$\left.\begin{array}{lll} (m+1)Py^m\partial x + & (m+1-n)PSy^{m+1}\partial x - & y^{m+2}S\partial Q \\ -y^m\partial R & -y^{m+1}\partial Q & +ny^{m+2}Q\partial S \\ & -y^{m+1}S\partial R & \\ & +ny^{m+1}R\partial S & \end{array}\right\} = 0$$

hinc fit $P\partial x = \frac{\partial R}{m+1}$ et $S\partial Q = nQ\partial S$, ideoque $Q = AS^n$ et $\partial Q = nAS^{n-1}\partial S$, quibus in membro medio substitutis fit

$$\frac{m+1-n}{m+1}S\partial R - nAS^{n-1}\partial S - S\partial R + nR\partial S = 0, \text{ seu}$$

$$-\frac{S\partial R}{m+1} - AS^{n-1}\partial S + R\partial S = 0, \text{ ideoque}$$

$$\partial R - \frac{(m+1)R\partial S}{S} = -(m+1)AS^{n-2}\partial S,$$

quae per S^{m+1} divisa et integrata praebet

$$\frac{R}{S^{m+1}} = B - \frac{(m+1) A S^{n-m-2}}{n-m-2}.$$

Ponamus $A = (m+2-n) C$, ut sit $Q = (m+2-n) C S^n$, et $R = B S^{m+1} + (m+1) C S^{n-1}$, ideoque

$$P \partial x = B S^m \partial S + (n-1) C S^{n-2} \partial S.$$

Quocirca habebimus hanc aequationem

$$y \partial S [B S^m + (n-1) C S^{n-2}] + \partial y [(m+2-n) C S^n y + B S^{m+1} + (m+1) C S^{n-1}] = 0,$$

quae multiplicata per $\frac{y^m}{(1+Sy)^n}$ fit integrabilis, ubi pro S functionem quamcunque ipsius x capere licet.

Corollarium 1.

503. Integrari ergo poterit haec aequatio

$$B y S^m \partial S + B S^{m+1} \partial y + (n-1) C y S^{n-2} \partial S + (m+1) C S^{n-1} \partial y + (m+2-n) C S^n y \partial y = 0,$$

quae sponte resolvitur in has duas partes

$$B S^m (y \partial S + S \partial y) + C S^{n-2} [(n-1) y \partial S + (m+1) S \partial y + (m+2-n) S^2 y \partial y] = 0,$$

quarum utraque seorsim per $\frac{y^m}{(1+Sy)^n}$ multiplicata fit integrabilis.

Corollarium 2.

504. Prior pars $B S^m (y \partial S + S \partial y)$ integrabilis redditur per hunc multiplicatorem $\frac{1}{S^m} \Phi : Sy$; est enim haec formula

$B(y\partial S + S\partial y)\,\Phi : Sy$ per se integrabilis. Unde pro hac parte multiplicator erit $S^{\lambda - m} y^{\lambda} (1 + Sy)^{\mu}$, qui utique continet assumtum $\frac{y^m}{(1+Sy)^n}$, si quidem capiatur $\lambda = m$ et $\mu = -n$. Est vero

$$\int \frac{y^m}{(1+Sy)^n} . BS^m (y\partial S + S\partial y) = B \int \frac{v^m \partial v}{(1+v)^n},$$

posito $Sy = v$.

Corollarium 3.

506. Pro altera parte, quae posito $S = \frac{1}{v}$ abit in

$$\frac{C}{v^n}[-(n-1)y\partial v + (m+1)v\partial y + (m+2-n)y\partial y],$$

habebimus

$$-\frac{(n-1)Cy}{v^n}\left(\partial v - \frac{(m+1)v\partial y}{(n-1)y} - \frac{(m+2-n)\partial y}{(n-1)}\right) =$$

$$-\frac{(n-1)Cy^{\frac{m+n}{n-1}}}{v^n}\left(y^{\frac{-m-1}{n-1}}\partial v - \frac{m+1}{n-1} y^{\frac{-m-n}{n-1}} v\partial y - \frac{m+2-n}{n-1} y^{\frac{-m-1}{n-1}}\partial y\right)$$

$$= -\frac{(n-1)Cy^{\frac{m+n}{n-1}}}{v^n}\partial . \left(y^{\frac{-m-1}{n-1}} v + y^{\frac{n-m-2}{n-1}}\right).$$

Ideoque haec altera pars ita repraesentabitur

$$-(n-1)CS^n y^{\frac{m+n}{n-1}} \partial . \frac{1+Sy}{y^{\frac{m+1}{n-1}} S}.$$

Multiplicator ergo hanc partem integrabilem reddens erit in genere

$$\frac{1}{S^n y^{\frac{m+n}{n-1}}} \Phi : \frac{1+Sy}{Sy^{\frac{m+1}{n-1}}}.$$

Corollarium 4.

506. Pro altera ergo parte multiplicator erit

$\frac{(1+Sy)^{\mu}}{S^{n+\mu}\, y^{\frac{m+n+\mu(m+1)}{n-1}}}$, quo haec pars fit:

$$-(n-1)\,C.\frac{(1+Sy)^{\mu}}{S^{\mu}\, y^{\frac{\mu(m+1)}{n-1}}}\,\partial.\frac{1+Sy}{y^{\frac{m+1}{n-1}}\,S},$$

cujus integrale est

$$-\frac{(n-1)\,C z^{\mu+1}}{\mu+1}, \text{ posito } z=\frac{1+Sy}{y^{\frac{m+1}{n-1}}\,S}.$$

Corollarium 5.

507. Jam multiplicator pro prima parte

$$S^{\lambda-m}\, y^{\lambda}\,(1+Sy)^{\mu}$$

congruens reddetur cum multiplicatore alterius partis modo exhibito, si sumatur $\lambda=m$ et $\mu=-n$, unde resultat multiplicator communis $\frac{y^m}{(1+Sy)^n}$, hincque posito $Sy=v$ et $\frac{1+Sy}{y^{\frac{m+1}{n-1}}\,S}=z$, nostrae aequationis integrale erit:

$$B\int\frac{v^m\,\partial v}{(1+v)^n}+C z^{1-n}=D \text{ sive}$$

$$B\int\frac{v^m\,\partial v}{(1+v)^n}+\frac{C S^{n-1}\, y^{m+1}}{(1+Sy)^{n-1}}=D.$$

Scholion.

508. Aequatio ergo, quam hoc problemate integrare didicimus, per principia jam supra stabilita tractari potest, dum pro binis ejus partibus seorsim multiplicatores quaeruntur, iique inter

se congruentes redduntur, cujus methodi hic insignem usum declaravimus. Possemus etiam multiplicatori hanc formam dare $\frac{y^m}{(1+Sy+Tyy)^n}$, ita ut haec aequatio

$$\frac{y^m\,[yP\partial x+(Qy+R)\,\partial y]}{(1+Sy+Tyy)^n}=0$$

per se debeat esse integrabilis, et calculo ut ante instituto inveniemus

$$y^m\left\{\begin{matrix}+(m+1)P\partial x\\ -\partial R\end{matrix}\right\}+y^{m+1}\left(\begin{matrix}+(m+1-n)PS\partial x\\ -\partial Q\\ -S\partial R\\ +nR\partial S\end{matrix}\right)+y^{m+2}\left(\begin{matrix}+(m+1-2n)PT\partial x\\ -S\partial Q\\ -T\partial R\\ +nQ\partial S\\ +nR\partial T\end{matrix}\right)$$
$$+y^{m+3}\left\{\begin{matrix}-T\partial Q\\ +nQ\partial T\end{matrix}\right\}=0,$$

unde ex ultimo membro $-T\partial Q+nQ\partial T=0$ concludimus $Q=AT^n$, et ex primo $P\partial x=\frac{\partial R}{m+1}$, qui valores in binis mediis substituti praebent

$$R\partial S-\frac{S\partial R}{m+1}-AT^{n-1}\,\partial T=0 \text{ et}$$
$$R\partial T-\frac{2T\partial R}{m+1}+AT^n\,\partial S-AST^{n-1}\,\partial T=0,$$

quarum illa fit integrabilis per se si $m=-2$, haec vero integrari potest si $m=2n-1$, fit enim

$$R\partial T-\frac{T\partial R}{n}+AT^{n-1}\,(T\partial S-S\partial T)=0, \text{ seu}$$
$$\frac{nR\partial T-T\partial R}{nT^{n+1}}+\frac{A\,(T\partial S-S\partial T)}{TT}=0,$$

cujus integrale est $\frac{-R}{nT^n}+\frac{AS}{T}=\frac{-B}{n}$; hincque

$$R=BT^n+nAT^{n-1}\,S.$$

Praeterea vero notari meretur casus $m = -1$, quem cum illis in subjunctis exemplis evolvamus.

Exemplum 1.

509. *Definire hanc aequationem*

$$yP\partial x + (Qy + R)\partial y = 0,$$

ut multiplicata per $\frac{1}{y(1 + Sy + Tyy)^n}$ *fiat per se integrabilis.*

Ob $m = -1$, habemus statim $\partial R = 0$, ideoque $R = C$: tum est ut ante $Q = AT^n$ et $\partial Q = nAT^{n-1}\partial T$, unde binae reliquae determinationes erunt:

$$-PS\partial x + AT^{n-1}\partial T + C\partial S = 0$$
$$-2PT\partial x - AST^{n-1}\partial T + AT^n\partial S + C\partial T = 0,$$

hinc eliminando $P\partial x$ prodit

$$ASST^{n-1}\partial T - 2AT^n\partial T - AT^n S\partial S$$
$$+ 2CT\partial S - CS\partial T = 0.$$

Statuatur hic $SS = Tv$, ut fiat

$$2T\partial S - S\partial T = TS\left(\frac{2\partial S}{S} - \frac{\partial T}{T}\right) = \frac{TS\partial v}{v} = \frac{T\partial v\sqrt{T}}{\sqrt{v}},$$

eritque

$$\tfrac{1}{2}AT^n v\partial T - 2AT^n\partial T - \tfrac{1}{2}AT^{n+1}\partial v + \frac{CT\partial v\sqrt{T}}{\sqrt{v}} = 0,$$

seu hoc modo

$$-\tfrac{1}{2}AT^{n+2}\partial . \frac{v-4}{T} + \frac{CT\partial v\sqrt{T}}{\sqrt{v}} = 0,$$

cujus prior pars integrabilis redditur per multiplicatorem

$$\frac{1}{T^{n+2}}\Phi : \frac{v-4}{T},$$

posterior vero per $\frac{1}{T\sqrt{T}}\Phi:v$, unde communis multiplicator erit $\frac{1}{T(v-4)^{n+\frac{1}{2}}\sqrt{T}}$, hincque aequatio elicitur integralis haec

$$\frac{AT^{n-\frac{1}{2}}}{(2n-1)(v-4)^{n-\frac{1}{2}}}+C\int\frac{\partial v}{(v-4)^{n+\frac{1}{2}}\sqrt{v}}=D,$$

unde T definitur per v; tum vero est $S=\sqrt{Tv}$, $R=C$,

$$Q=AT^n,\text{ et }P\partial x=\frac{C\partial S-AT^{n-1}\partial T}{S}.$$

Corollarium 1.

510. Casu quo est $n=\frac{1}{2}$, ob $\frac{1}{0}z^0=lz$, habetur

$$\tfrac{1}{2}Al\frac{T}{v-4}+C\int\frac{\partial v}{(v-4)\sqrt{v}}=\tfrac{1}{2}D,\text{ seu}$$
$$\tfrac{1}{2}Al\frac{T}{v-4}-\tfrac{1}{2}Cl\frac{\sqrt{v}+2}{\sqrt{v}-2}=\tfrac{1}{2}D:$$

unde posito $v=4uu$ et $C=\lambda A$, erit

$$l\frac{T}{1-uu}-\lambda l\frac{1+u}{1-u}=\text{Const. seu}$$
$$T=E(1-uu)\left(\frac{1+u}{1-u}\right)^{\lambda}.\text{ Hinc porro}$$
$$S=2u\sqrt{T}=2u\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)},\text{ et}$$

$R=C=\lambda A$; tum $Q=A\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}$, atque

$$P\partial x=\frac{\lambda A\partial u}{u}+\frac{\lambda A\partial T}{2T}-\frac{A\partial T}{2Tu}.$$

At est $\frac{\partial T}{T}=\frac{-2u\partial u+2\lambda\partial u}{1-uu}$. Ergo $P\partial x=\frac{A\partial u(1+\lambda\lambda-2\lambda u)}{1-uu}$

Quocirca pro hac aequatione

$$\frac{Ay\partial u(1+\lambda\lambda-2\lambda u)}{1-uu}+A\partial y\left[\lambda+y\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}\right]=0$$

multiplicator erit

$$\frac{1}{y\sqrt{\left[1+2uy\left(\frac{1+u}{1-u}\right)^{\frac{\lambda}{2}}\sqrt{E(1-uu)}+Eyy(1-uu)\left(\frac{1+u}{1-u}\right)^{\lambda}\right]}}.$$

Corollarium 2.

511. Casu quo $n = -\frac{1}{2}$ habemus

$$-\frac{A(v-4)}{2T}+2C\sqrt{v}=-2D, \text{ seu } T=\frac{A(v-4)}{4D+4C\sqrt{v}}.$$

Ponamus $v = 4uu$, ut sit $T=\frac{A(uu-1)}{D+2Cu}$, tum fit

$$S = 2u\sqrt{T} = 2u\sqrt{\frac{A(uu-1)}{D+2Cu}},$$

$$R = C,\ Q=\sqrt{\frac{A(D+2Cu)}{uu-1}}, \text{ et}$$

$$P\partial x = \frac{C\partial u}{u}+\frac{C\partial T}{2T}-\frac{A\partial T}{2TTu}=\frac{\partial u(C+Du+Cuu)(Cu^3-3Cu-D)}{u(uu-1)^2(D+2Cu)},$$

unde tam aequatio quam multiplicator definitur.

Exemplum 2.

512. *Definire aequationem*

$$yP\partial x+(Qy+R)\partial y=0,$$

ut multiplicata per $\frac{1}{y^2(1+Sy+Tyy)^n}$, *fiat per se integrabilis.*

Ob $m = -2$, ex superioribus habemus:

$$RS=\frac{A}{n}T^n+B, \text{ seu } R=\frac{AT^n}{nS}+\frac{B}{S},$$

qui valor in altera aequatione substitutus praebet

$$\frac{(2n+1)AT^n\partial T}{nS}-\frac{2AT^{n+1}\partial S}{nSS}+AT^n\partial S-AST^{n-1}\partial T$$
$$+\frac{B\partial T}{S}-\frac{2BT\partial S}{SS}=0,$$

quae in has tres partes distinguatur

$$\frac{AS}{nT^n}\left(\frac{(2n+1)T^{2n}\partial T}{S^2}-\frac{2T^{2n+1}\partial S}{S^3}\right)+AT^{n+1}\left(\frac{\partial S}{T}-\frac{S\partial T}{TT}\right)$$
$$+BS\left(\frac{\partial T}{SS}-\frac{2T\partial S}{S^3}\right)=0,\text{ seu}$$

$$\frac{AS}{nT^n}\partial\cdot\frac{T^{2n+1}}{SS}+AT^{n+1}\partial\cdot\frac{S}{T}+BS\partial\cdot\frac{T}{SS}=0.$$

Statuamus ad abbreviandum

$$\frac{T^{2n+1}}{SS}=p,\ \frac{S}{T}=q\ \text{ et }\ \frac{T}{SS}=r,$$

fiet $S=\frac{1}{qr}$, $T=\frac{1}{qqr}$, hinc $p=\frac{1}{q^{4n}r^{2n-1}}$; nostraque aequatio ita se habebit

$$\frac{A}{nq\sqrt{pr}}\partial p+\frac{A\sqrt{p}}{qqr\sqrt{r}}\partial q+\frac{B}{qr}\partial r=0,\text{ seu}$$
$$\frac{A\sqrt{r}}{n\sqrt{p}}\partial p+\frac{A\sqrt{p}}{q\sqrt{r}}\partial q+B\partial r=0.$$

Quas tres partes seorsim consideremus, ac prima fit integrabilis multiplicata per $\frac{\sqrt{p}}{\sqrt{r}}\Phi:p$, secunda vero per $\frac{q\sqrt{r}}{\sqrt{p}}\Phi:q$, tertia tandem per $\Phi:r$. Ut bini primi conveniant, ponatur

$$\frac{\sqrt{p}}{\sqrt{r}}\cdot p^\lambda=\frac{q\sqrt{r}}{\sqrt{p}}\cdot q^\mu\ \text{ seu }\ p^{\lambda+1}=q^{\mu+1}r,\text{ hinc}$$
$$p=q^{\frac{\mu+1}{\lambda+1}}r^{\frac{1}{\lambda+1}}=q^{-4n}r^{-2n+1}.$$

Fit ergo

$$\lambda+1=-\frac{1}{2n-1}\ \text{ et }\ \mu+1=-4n(\lambda+1)=\frac{4n}{2n-1};\text{ sicque}$$
$$\mu=\frac{2n+1}{2n-1}\ \text{ et }\ \lambda=-\frac{2n}{2n-1}.$$

Multiplicetur ergo aequatio per $\frac{q^{\frac{4n}{2n-1}}\sqrt{r}}{\sqrt{p}}=q^{2n+\frac{4n}{2n-1}}r^n$, ac prodibit

$$\frac{A}{n} p^\lambda \partial p + A q^\mu \partial q + B q^{2n + \frac{4n}{2n-1}} r^n \partial r = 0,$$

seu

$$A \partial . \left(\frac{p^{\lambda+1}}{n(\lambda+1)} + \frac{q^{\mu+1}}{\mu+1} \right) + B q^{\frac{4nn+2n}{2n-1}} r^n \partial r = 0,$$

vel

$$\frac{(2n-1)A}{4n} \partial . q^{\frac{4n}{2n-1}} (1 - 4r) + B q^{\frac{4nn+2n}{2n-1}} r^n \partial r = 0.$$

Multiplicetur per $q^{\frac{4\nu n}{2n-1}} (1 - 4r)^\nu$, ut prodeat

$$\frac{(2n-1)A}{4n} . q^{\frac{4\nu n}{2n-1}} (1 - 4r)^\nu \partial . q^{\frac{4n}{2n-1}} (1 - 4r)$$
$$+ B q^{\frac{4nn+2n+4\nu n}{2n-1}} r^n \partial r (1 - 4r)^\nu = 0.$$

Fiat ergo $4\nu + 4n + 2 = 0$ seu $\nu = -n - \frac{1}{2}$, et ambo membra integrari poterunt, eritque

$$\frac{(2n-1)A}{4n(\nu+1)} q^{\frac{4n(\nu+1)}{2n-1}} (1 - 4r)^{\nu+1} + B \int r^n \partial r (1 - 4r)^\nu = \text{Const.}$$

at est $\nu + 1 = -n + \frac{1}{2} = \frac{-2n+1}{2}$, sicque habebitur

$$-\frac{A}{2n} q^{-2n} (1 - 4r)^{\frac{-2n+1}{2}} + B \int \frac{r^n \partial r}{(1 - 4r)^{\frac{2n+1}{2}}} = \text{Const.}$$

Dabitur ergo q per r, eritque $S = \frac{1}{qr}$, $T = \frac{s}{q}$, tum $R = \frac{AT^n}{nS} + \frac{B}{S}$, $Q = AT^n$ et $P \partial x = -\partial R$.

Corollarium 1.

513. Si sit $n = -\frac{1}{2}$, erit $Aq + \frac{2Br\sqrt{r}}{3} = \frac{C}{3}$, seu $q = \frac{C - 2Br\sqrt{r}}{3A}$; hincque

$$S = \frac{3A}{Cr - 2Br^2\sqrt{r}}, \quad T = \frac{9AA}{r(C - 2Br\sqrt{r})^2}, \quad Q = \frac{C\sqrt{r} - 2Brr}{3} \text{ et}$$

$$R = \frac{Q + nB}{nS} = \frac{B - 2Q}{S} = \frac{r(C - 2Br\sqrt{r})(3B - 2C\sqrt{r} + 4Brr)}{9A} \text{ seu}$$

$$R = \frac{3BCr - 2CCr\sqrt{r} - 6BBrr\sqrt{r} + 8BCr^3 - 8BBr^4\sqrt{r}}{9A},$$

Corollarium 2.

514. Ponamus eodem casu $r = uu$, erit

$$S = \frac{3A}{Cuu - 2Bu^5}, \quad T = \frac{9AA}{uu(C - 2Bu^3)^2}, \quad Q = \frac{u(C - 2Bu^3)}{3}, \text{ et}$$

$$R = \frac{3BCu^2 - 2CCu^3 - 6BBu^5 + 8BCu^6 - 8BBu^9}{9A}, \text{ hincque}$$

$$P\partial x = \frac{-6BCu + 6CCuu + 30BBu^4 - 48BCu^5 + 72BBu^8}{9A}\partial u,$$

eritque aequatio $yP\partial x + (Qy + R)\partial y = 0$ integrabilis, si multiplicetur per

$$\frac{\sqrt{(1 + Sy + Tyy)}}{yy} = \frac{1}{yy}\sqrt{\left(1 + \frac{3Ay}{uu(C - 2Bu^3)} + \frac{9AAyy}{uu(C - 2Bu^3)^2}\right)}.$$

Exemplum 3.

515. *Definire aequationem*

$$yP\partial x + (Qy + R)\partial y = 0,$$

quae multiplicata per $\frac{y^{2n-1}}{(1 + Sy + Tyy)^n}$ *fiat per se integrabilis.*

Hic est $m = 2n - 1$, $Q = AT^n$, et $P\partial x = \frac{\partial R}{2n}$; tum vero ex superioribus $R = nAT^{n-1}S + BT^n$, ac superest aequatio

$$R\partial S - \frac{S\partial R}{2n} - AT^{n-1}\partial T = 0,$$

quae loco R substituto valore invento, abit in

$$(2n-1)AT^{n-1}S\partial S - (n-1)AT^{n-2}SS\partial T - 2AT^{n-1}\partial T$$
$$+ 2BT^n\partial S - BT^{n-1}S\partial T = 0, \text{ seu}$$
$$(2n-1)ATS\partial S - (n-1)ASS\partial T - 2AT\partial T$$
$$+ 2BTT\partial S - BTS\partial T = 0.$$

Prius membrum posito $SS = u$ abit in

$$(n-\tfrac{1}{2})AT\partial u - (n-1)Au\partial T - 2AT\partial T, \text{ seu}$$

$$(n-\tfrac{1}{2})AT\left(\partial u - \frac{(n-1)u\partial T}{(n-\frac{1}{2})T} - \frac{2\partial T}{n-\frac{1}{2}}\right), \text{ sive}$$

$$\tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}\left(\frac{\partial u}{T^{\frac{2n-2}{2n-1}}} - \frac{2(n-1)u\partial T}{(2n-1)T^{\frac{4n-3}{2n-1}}} - \frac{4\partial T}{(2n-1)T^{\frac{2n-2}{2n-1}}}\right)$$

$$= \tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}\partial.\left(\frac{u}{T^{\frac{2n-2}{2n-1}}} - 4T^{\frac{1}{2n-1}}\right), \text{ vel}$$

$$\tfrac{1}{2}(2n-1)AT^{\frac{4n-3}{2n-1}}\partial.T^{\frac{1}{2n-1}}\left(\frac{SS}{T}-4\right) + \frac{BT^3}{S}\partial.\frac{SS}{T} = 0, \text{ seu}$$

$$(2n-1)AT^{\frac{-1}{2n-1}}\partial.T^{\frac{1}{2n-1}}\left(\frac{SS}{T}-4\right) + \frac{2BT}{S}\partial.\frac{SS}{T} = 0.$$

Ponatur $\frac{SS}{T} = p$ et

$$T^{\frac{1}{2n-1}}\left(\frac{SS}{T}-4\right) = q = T^{\frac{1}{2n-1}}(p-4),$$

ut sit $T^{\frac{1}{2n-1}} = \frac{q}{p-4}$, unde

$$T = \frac{q^{2n-1}}{(p-4)^{2n-1}} \text{ et } S = \sqrt{\frac{pq^{2n-1}}{(p-4)^{2n-1}}}.$$

Ergo

$$\frac{(2n-1)A(p-4)\partial q}{q} + \frac{2B\sqrt{q^{2n-1}}}{\sqrt{p(p-4)^{2n-1}}}\partial p = 0$$

sive

$$\frac{(2n-1)A\partial q}{q^{n+\frac{1}{2}}} + \frac{2B\partial p:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}} = 0,$$

quae integrata praebet

$$\frac{-2A}{q^{n-\frac{1}{2}}} + 2B\int\frac{\partial p:\sqrt{p}}{(p-4)^{n+\frac{1}{2}}} = 2C,$$

et facto $\frac{p}{p-4} = vv$, seu $p = \frac{4vv}{vv-1}$, fiet

$$\frac{+A}{q^{n-\frac{1}{2}}} - \frac{B}{4^{n-1}} \int \partial v\, (vv-1)^{n-1} = C.$$

Scholion.

516. Haec fusius non prosequor, quia ista exempla eum in finem potissimum attuli, ut methodus supra tradita aequationes differentiales tractandi exerceretur; in his enim exemplis casus non parum difficiles se obtulerunt, quos ita per partes resolvere licuit, ut pro singulis multiplicatores idonei quaererentur, ex iisque multiplicator communis definiretur; nunc igitur alia aequationum genera, quae per multiplicatores integrabiles reddi queant, investigemus.

Problema 67.

517. Ipsius x functiones P, Q, R, S definire, ut haec aequatio $(Py + Q)\,\partial x + y\,\partial y = 0$, per hunc multiplicatorem $(yy + Ry + S)^n$ integrabilis reddatur.

Solutio.

Necesse igitur est, sit

$$\left(\frac{\partial \cdot (Py + Q)(yy + Ry + S)^n}{\partial y}\right) = \left(\frac{\partial \cdot y\,(yy + Ry + S)^n}{\partial x}\right)$$

unde colligitur per $(yy + Ry + S)^{n-1}$ dividendo

$$P(yy + Ry + S) + n(Py + Q)(2y + R) = \frac{ny\,(y\,\partial R + \partial S)}{\partial x}$$

seu

$$\left.\begin{array}{lll} (2n+1)Pyy\partial x & +(n+1)PRy\partial x & +PS\partial x \\ -nyy\partial R & +2nQy\partial x & +nQR\partial x \\ & -ny\partial S & \end{array}\right\} = 0.$$

**

Hinc ergo concluditur $P\partial x = \frac{n\partial R}{2n+1}$, et

$$\frac{(n+1)R\partial R}{2n+1} + 2Q\partial x - \partial S = 0,$$

$$\frac{S\partial R}{2n+1} + QR\partial x = 0, \text{ porroque}$$

$$Q\partial x = \frac{-S\partial R}{(2n+1)R} = \frac{-(n+1)R\partial R}{2(2n+1)} + \frac{\partial S}{2}: \text{ ergo}$$

$$\partial S + \frac{2S\partial R}{(2n+1)R} = \frac{(n+1)R\partial R}{2n+1},$$

quae per $R^{\frac{2}{2n+1}}$ multiplicata et integrata, dat

$$R^{\frac{2}{2n+1}} S = C + \tfrac{1}{4} R^{\frac{4n+4}{2n+1}}, \text{ hincque}$$

$$S = \tfrac{1}{4}RR + CR^{\frac{-2}{2n+1}}, \text{ atque}$$

$$Q\partial x = \frac{-R\partial R}{4(2n+1)} - \frac{C}{2n+1} R^{\frac{-2n-3}{2n+1}} \partial R, \text{ et } P\partial x = \frac{n\partial R}{2n+1}:$$

unde aequationem obtinemus

$$(ny - \tfrac{1}{4}R - CR^{\frac{-2n-3}{2n+1}})\partial R + (2n+1)y\partial y = 0,$$

quae integrabilis redditur per hunc multiplicatorem

$$(yy + Ry + \tfrac{1}{4}RR + CR^{\frac{-2}{2n+1}})^n.$$

Corollarium 1.

518. Casu quo $n = -\frac{1}{2}$, fit $\partial R = 0$ et $R = A$, et reliquae aequationes sunt

$$(n+1)AP\partial x + 2nQ\partial x - n\partial S = 0 \text{ et}$$

$$PS\partial x + nAQ\partial x = 0.$$

Ergo $P\partial x = \frac{AQ\partial x}{2S} = \frac{2Q\partial x - \partial S}{A}$, ideoque

$$(AA - 4S)Q\partial x = -2S\partial S, \text{ seu}$$

$$Q\partial x = \frac{-2S\partial S}{AA-4S} \text{ et } P\partial x = \frac{-A\partial S}{AA-4S}$$

sicque haec aequatio $\frac{(Ay+2S)\partial S}{4S-AA}+y\partial y=0$ integrabilis redditur per hunc multiplicatorem $\frac{1}{\sqrt{(yy+Ay+S)}}$.

Corollarium 2.

519. Si hic ponamus $A=2a$ et $S=x$, haec aequatio $\frac{(ay+x)\partial x+2y\partial y(x-aa)}{(x-aa)\sqrt{(yy+2ay+x)}}=0$ per se est integrabilis, unde integrale inveniri potest hujus aequationis

$$x\partial x+ay\partial x+2xy\partial y-2aay\partial y=0,$$

quae divisa per $(x-aa)\sqrt{(yy+2ay+x)}$ fit integrabilis.

Corollarium 3.

520. Ad integrale inveniendum, sumatur primo x constans, et partis $\frac{2y\partial y}{\sqrt{(yy+2ay+x)}}$ integrale est

$$2\sqrt{(yy+2ay+x)}+2al[a+y-\sqrt{(yy+2ay+x)}]+X,$$

cujus differentiale sumto y constante

$$\frac{\partial x}{\sqrt{(yy+2ay+x)}}-\frac{a\partial x:\sqrt{(yy+2ay+x)}}{a+y-\sqrt{(yy+2ay+x)}}+\partial X,$$

si alteri aequationis parti $\frac{(ay+x)\partial x}{(x-aa)\sqrt{(yy+2ay+x)}}$ aequetur, reperitur $\partial X=\frac{a\partial x}{aa-x}$ et $X=-al(aa-x)$. Ex quo integrale completum erit

$$\sqrt{(yy+2ay+x)}+al\frac{a+y-\sqrt{(yy+2ay+x)}}{\sqrt{(aa-x)}}=C.$$

Corollarium 4.

521. Memoratu dignus est etiam casus $n=-1$, qui scripto a loco $C+\frac{1}{4}$ praebet hanc aequationem

$$(y+aR)\partial R+y\partial y=0,$$

quae divisa per $yy+Ry+aRR$ fit integrabilis, haec autem aequatio est homogenea.

Scholion.

522. Potest etiam aequationis

$$(Py + Q)\,\partial x + y\,\partial y = 0$$

multiplicator statui $(y+R)^m (y+S)^n$, fierique debet

$$\left(\frac{\partial.(Py+Q)(y+R)^m(y+S)^n}{\partial y}\right) = \left(\frac{\partial.y(y+R)^m(y+S)^n}{\partial x}\right);$$

unde reperitur

$$\begin{aligned}&P\partial x(y+R)(y+S) + m\partial x(Py+Q)(y+S)\\ &\quad + n\partial x(Py+Q)(y+R) = my(y+S)\partial R + ny(y+R)\partial S,\end{aligned}$$

quae evolvitur in

$$\left.\begin{array}{ll}(m+n+1)Pyy\partial x & +(n+1)PRy\partial x + PRS\partial x\\ -myy\partial R & +(m+1)PSy\partial x + mQS\partial x\\ -nyy\partial S & +(m+n)Qy\partial x + nQR\partial x\\ & -mSy\partial R\\ & -nRy\partial S\end{array}\right\} = 0$$

unde colligitur

$$P\partial x = \frac{m\partial R + n\partial S}{m+n+1} \text{ et } Q\partial x = \frac{-PRS\partial x}{mS+nR} = \frac{-RS(m\partial R + n\partial S)}{(m+n+1)(mS+nR)},$$

hincque

$$\frac{(m\partial R + n\partial S)[(n+\ )R + (m+1)S]}{m+n+1} - \frac{(m+n)RS(m\partial R + n\partial S)}{(m+n+1)(mS+nR)} - mS\partial R - nR\partial S = 0,$$

seu

$$\begin{array}{l}+m(n+1)R\partial R - mnR\partial S\\ +n(m+1)S\partial S - mnS\partial R\end{array} - \frac{m(m+n)RS\partial R - n(m+n)RS\partial S}{mS+nR} = 0,$$

quae reducitur ad hanc formam

$$\left.\begin{array}{l}+(n+1)RR\partial R + (m-n-1)RS\partial R - mSS\partial R\\ +(m+1)SS\partial S + (n-m-1)RS\partial S - nRR\partial S\end{array}\right\} = 0,$$

quae cum sit homogenea, dividatur per

$$(n+1)R^3 + (m-2n-1)R^2S + (n-2m-1)RS^2 + (m+1)S^3,$$

seu per

$$(R - S)^2 [(n+1) R + (m+1) S]$$

ut fiat integrabilis. At ipsa illa aequatio per R — S divisa, erit

$$(n+1) R \partial R + m S \partial R - (m+1) S \partial S = 0.$$

Dividatur per

$$(R - S) [(n+1) R + (m+1) S]$$

et resolvatur in fractiones partiales, erit

$$\frac{\partial R}{m+n+2}\left(\frac{m+n+1}{R-S} + \frac{n+1}{(n+1)R+(m+1)S}\right)$$
$$+ \frac{\partial S}{m+n+2}\left(\frac{m+n+1}{S-R} + \frac{m+1}{(n+1)R+(m+1)S}\right) = 0$$

seu

$$\frac{(m+n+1)(\partial R - \partial S)}{R-S} + \frac{(n+1)\partial R + (m+1)\partial S}{(n+1)R+(m+1)S} = 0;$$

unde integrando obtinemus,

$$(R - S)^{m+n+1} [(n+1) R + (m+1) S] = C.$$

Sit $R - S = u$, erit

$$(n+1) R + (m+1) S = \frac{C}{u^{m+n+1}},$$

hincque

$$R = \frac{(m+1) u}{m+n+2} + \frac{a}{u^{m+n+1}}, \text{ et}$$

$$S = \frac{-(n+1) u}{m+n+2} + \frac{a}{u^{m+n+1}},$$

tum vero

$$P \partial x = \frac{(m-n) \partial u}{m+n+2} - \frac{(m+n) a \partial u}{u^{m+n+2}}, \text{ et}$$

$$Q \partial x = \frac{\partial u}{u}\left(\frac{a}{u^{m+n+1}} + \frac{(m+1) u}{m+n+2}\right)\left(\frac{a}{u^{m+n+1}} - \frac{(n+1) u}{m+n+2}\right).$$

Corollarium 1.

623. Hinc ergo integrari potest ista aequatio

Scholion.

522. Potest etiam aequationis

$$(Py + Q)\,\partial x + y\,\partial y = 0$$

multiplicator statui $(y + R)^m (y + S)^n$, fierique debet

$$\left(\frac{\partial . (Py + Q)(y + R)^m (y + S)^n}{\partial y}\right) = \left(\frac{\partial . y (y + R)^m (y + S)^n}{\partial x}\right);$$

unde reperitur

$$\begin{aligned} &P\,\partial x (y + R)(y + S) + m\,\partial x (Py + Q)(y + S) \\ &\quad + n\,\partial x (Py + Q)(y + R) = my (y + S)\,\partial R + ny (y + R)\,\partial S, \end{aligned}$$

quae evolvitur in

$$\left.\begin{array}{ll} (m+n+1)Pyy\,\partial x & +(n+1)PRy\,\partial x + PRS\,\partial x \\ -myy\,\partial R & +(m+1)PSy\,\partial x + mQS\,\partial x \\ -nyy\,\partial S & +(m+n)Qy\,\partial x + nQR\,\partial x \\ & -mSy\,\partial R \\ & -nRy\,\partial S \end{array}\right\} = 0$$

unde colligitur

$$P\,\partial x = \frac{m\,\partial R + n\,\partial S}{m + n + 1} \text{ et } Q\,\partial x = \frac{-PRS\,\partial x}{mS + nR} = \frac{-RS(m\,\partial R + n\,\partial S)}{(m+n+1)(mS + nR)},$$

hincque

$$\frac{(m\,\partial R + n\,\partial S)[(n+\ )R + (m+1)S]}{m+n+1} - \frac{(m+n)RS(m\,\partial R + n\,\partial S)}{(m+n+1)(mS+nR)} - mS\,\partial R - nR\,\partial S = 0,$$

seu

$$\begin{aligned} &+ m(n+1)R\,\partial R - mnR\,\partial S - \frac{m(m+n)RS\,\partial R - n(m+n)RS\,\partial S}{mS + nR} = 0, \\ &+ n(m+1)S\,\partial S - mnS\,\partial R \end{aligned}$$

quae reducitur ad hanc formam

$$\left.\begin{array}{l} +(n+1)RR\,\partial R + (m-n-1)RS\,\partial R - mSS\,\partial R \\ +(m+1)SS\,\partial S + (n-m-1)RS\,\partial S - nRR\,\partial S \end{array}\right\} = 0,$$

quae cum sit homogenea, dividatur per

$$(n+1)R^3 + (m-2n-1)R^2 S + (n-2m-1)RS^2 + (m+1)S^3,$$

seu per

$$(R - S)^2 [(n + 1) R + (m + 1) S]$$

ut fiat integrabilis. At ipsa illa aequatio per R — S divisa, erit

$$(n + 1) R \partial R + m S \partial R - (m + 1) S \partial S = 0.$$

Dividatur per

$$(R - S) [(n + 1) R + (m + 1) S]$$

et resolvatur in fractiones partiales, erit

$$\frac{\partial R}{m+n+2}\left(\frac{m+n+1}{R-S} + \frac{n+1}{(n+1)R+(m+1)S}\right)$$
$$+ \frac{\partial S}{m+n+2}\left(\frac{m+n+1}{S-R} + \frac{m+1}{(n+1)R+(m+1)S}\right) = 0$$

seu

$$\frac{(m+n+1)(\partial R - \partial S)}{R-S} + \frac{(n+1)\partial R + (m+1)\partial S}{(n+1)R+(m+1)S} = 0;$$

unde integrando obtinemus,

$$(R - S)^{m+n+1} [(n + 1) R + (m + 1) S] = C.$$

Sit $R - S = u$, erit

$$(n + 1) R + (m + 1) S = \frac{C}{u^{m+n+1}},$$

hincque

$$R = \frac{(m + 1) u}{m + n + 2} + \frac{a}{u^{m+n+1}}, \text{ et}$$

$$S = \frac{-(n + 1) u}{m + n + 2} + \frac{a}{u^{m+n+1}},$$

tum vero

$$P \partial x = \frac{(m - n) \partial u}{m + n + 2} - \frac{(m + n) a \partial u}{u^{m+n+2}}, \text{ et}$$

$$Q \partial x = \frac{\partial u}{u}\left(\frac{a}{u^{m+n+1}} + \frac{(m+1) u}{m+n+2}\right)\left(\frac{a}{u^{m+n+1}} - \frac{(n+1) u}{m+n+2}\right).$$

Corollarium 1.

523. Hinc ergo integrari potest ista aequatio

$$y\partial y + y\partial u\left(\frac{m-n}{m+n+2} - \frac{(m+n)a}{u^{m+n+2}}\right)$$
$$+\frac{\partial u}{u}\left(\frac{aa}{u^{2m+2n+2}} + \frac{(m-n)a}{(m+n+2)\,u^{m+n}} - \frac{(m+1)(n+1)uu}{(m+n+2)^2}\right) = 0,$$

quippe quae per se fit integrabilis, si multiplicetur per

$$\left(y + \frac{a}{u^{m+n+1}} + \frac{(m+1)u}{m+n+2}\right)^m \left(y + \frac{a}{u^{m+n+1}} - \frac{(n+1)u}{m+n+2}\right)^n.$$

Corollarium 2.

524. Sit $m = n$, et aequatio nostra erit

$$y\partial y - \frac{2nay\partial u}{u^{2n+2}} + \frac{aa\partial u}{u^{4n+3}} - \tfrac{1}{4}u\partial u = 0,$$

cujus multiplicator est $[(y + \frac{a}{u^{2n+1}})^2 - \frac{1}{4}uu]^n$. Quare si ponamus $y = z - \frac{a}{u^{2n+1}}$, aequatio prodit

$$z\partial z - \frac{a\partial z}{u^{2n+1}} + \frac{az\partial u}{u^{2n+2}} - \tfrac{1}{4}u\partial u = 0,$$

quae integrabilis fit multiplicata per $(zz - \frac{1}{4}uu)^n$. Vel ponatur $z = \frac{1}{2}y$ et $a = \frac{1}{2}b$, erit aequatio

$$y\partial y - u\partial u - \frac{b\partial y}{u^{2n+1}} + \frac{by\partial u}{u^{2n+2}} = 0,$$

et multiplicator $(yy - uu)^n$.

Corollarium 3.

525. Si $m = -n$, prodit haec aequatio

$$y\partial y - ny\partial u + \frac{aa\partial u}{u^3} + \tfrac{1}{4}(nn-1)u\partial u - \frac{na\partial u}{u} = 0,$$

quae integrabilis redditur multiplicata per

$$[y+\tfrac{a}{u}-\tfrac{1}{2}(n+1)u)]^n[y+\tfrac{a}{u}-\tfrac{1}{2}(n-1)u]^{-n}.$$

Posito autem $y+\frac{a}{u}=z$, predit haec aequatio

$$z\partial z-nz\partial u+\tfrac{1}{4}(nn-1)u\partial u-\frac{a\partial z}{u}+\frac{az\partial u}{uu}=0,$$

quam integrabilem reddit hic multiplicator

$$[z-\tfrac{1}{2}(n+1)u]^n[z-\tfrac{1}{2}(n-1)u]^{-n}.$$

Corollarium 4.

526. Ponamus hic $z=uv$, et habebitur ista aequatio

$$uuv\partial v+u\partial u[vv-nv+\tfrac{1}{4}(nn-1)]=a\partial v,$$

quae si multiplicetur per $\left(\frac{v-\frac{1}{2}(n+1)}{u-\frac{1}{2}(n-1)}\right)$, utramque membrum fiet integrabile. Posito enim $\frac{v-\frac{1}{2}(n+1)}{v-\frac{1}{2}(n-1)}=s$, seu

$$v=\frac{n+1-(n-1)s}{2(1-s)},$$

oritur

$$\frac{s^{n+1}u\partial u}{(1-s)^2}+\frac{n+1-(n-1)s}{2(1-s)^3}uus^n\partial s=\frac{as^n\partial s}{(1-s)^2},$$

cujus integrale est

$$\frac{s^{n+1}uu}{2(1-s)^2}=a\int\frac{s^n\partial s}{(1-s)^2}.$$

Scholion.

527. Quo nostram aequationem in genere concinniorem reddamus, ponamus $m=-\lambda-1+\mu$ et $n=-\lambda-1-\mu$, ut sit $m+n+2=-2\lambda$, fietque aequatio

$$y\partial y-y\partial u[\tfrac{\mu u}{\lambda}-2(\lambda+1)au^{2\lambda}]$$

$$+ u\partial u\left(\frac{\mu\mu - \lambda\lambda}{4\lambda\lambda} - \frac{\mu}{\lambda} a u^{2\lambda} + aa u^{4\lambda}\right) = 0,$$

quae per hunc multiplicatorem integrabilis redditur

$$\left(y + a u^{2\lambda+1} - \frac{(\mu-\lambda)u}{2\lambda}\right)^{\mu-\lambda-1}\left(y + a u^{2\lambda+1} - \frac{(\mu+\lambda)u}{2\lambda}\right)^{-\mu-\lambda-1}.$$

Ponatur $y + a u^{2\lambda+1} = uz$, et orietur haec aequatio

$$uz\partial z - a u^{2\lambda+1}\partial z + \partial u\left(zz - \frac{\mu}{\lambda} z + \frac{\mu\mu - \lambda\lambda}{4\lambda\lambda}\right) = 0,$$

cui respondet multiplicator

$$u^{-2\lambda-1}\left(z + \frac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda-1}\left(z - \frac{\lambda-\mu}{2\mu}\right)^{-\mu-\lambda-1}.$$

Reperitur autem integrale

$$C = a\int\partial z\left(z + \frac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda-1}\left(z - \frac{\lambda-\mu}{2\lambda}\right)^{-\mu-\lambda-1}$$

$$+ \frac{1}{2\lambda u^{2\lambda}}\left(z + \frac{\lambda-\mu}{2\lambda}\right)^{\mu-\lambda}\left(z - \frac{\lambda-\mu}{2\lambda}\right)^{-\mu-\lambda},$$

quod ergo convenit huic aequationi differentiali

$$z\partial z + \frac{\partial u}{u}\left(z + \frac{\lambda-\mu}{2\lambda}\right)\left(z - \frac{\lambda-\mu}{2\lambda}\right) = a u^{2\lambda}\partial z.$$

Problema 68.

528. Ipsius x functiones P, Q, R et X definire, ut haec aequatio $\partial y + yy\partial x + X\partial x = 0$ integrabilis reddatur per hunc multiplicatorem $\frac{1}{Pyy + Qy + R}$.

Solutio.

Debet ergo esse

$$\frac{1}{\partial y}\partial\cdot\frac{yy + X}{Pyy + Qy + R} = \frac{1}{\partial x}\partial\cdot\frac{1}{Pyy + Qy + R};$$

hincque

$$2y(Pyy + Qy + R) - (yy + X)(2Py + Q)$$
$$= -\frac{yy\partial P - y\partial Q - \partial R}{\partial x}:$$

ergo fieri debet

$$\left.\begin{array}{l} + Qyy\partial x + 2Ry\partial x - QX\partial x \\ + yy\partial P - 2PXy\partial x + \partial R \\ + y\partial Q \end{array}\right\} = 0.$$

Quare habetur $Q = -\frac{\partial P}{\partial x} = \frac{\partial R}{X\partial x}$; et $X = -\frac{\partial R}{\partial P}$. Sumto ergo ∂x constante est $\partial Q = -\frac{\partial\partial P}{\partial x}$, unde fieri oportet

$$2R\partial x + \frac{2P\partial R\partial x}{\partial P} - \frac{\partial\partial P}{\partial x} = 0, \text{ seu}$$

$$R\partial P + P\partial R = \frac{\partial P\partial\partial P}{2\partial x^2},$$

cujus integratio praebet $PR = \frac{\partial P^2}{4\partial x^2} + C$, hinc $R = \frac{\partial P^2}{4P\partial x^2} + \frac{C}{P}$, tum

$$Q = -\frac{\partial P}{\partial x}, \text{ et } X = \frac{C}{PP} + \frac{\partial P^2}{4PP\partial x^2} - \frac{\partial\partial P}{2P\partial x^2}.$$

Ponamus $P = SS$, ut S sit functio quaecunque ipsius x, obtinebimusque

$$P = SS,\ Q = -\frac{2S\partial S}{\partial x},\ R = \frac{C}{SS} + \frac{\partial S^2}{\partial x^2}, \text{ et } X = \frac{C}{S^4} - \frac{\partial\partial S}{S\partial x^2},$$

quibus sumtis valoribus, per se integrabilis erit haec aequatio

$$\frac{\partial y + yy\partial x + X\partial x}{Pyy + Qy + R} = 0.$$

Scholion.

529. Haec solutio commodius institui poterit, si multiplicatori tribuatur haec forma $\frac{P}{yy + 2Qy + R}$, ut fieri debeat

$$\frac{1}{\partial y}\partial\cdot\frac{P(yy + X)}{yy + 2Qy + R} = \frac{1}{\partial x}\partial\cdot\frac{P}{yy + 2Qy + R}.$$

unde oritur

$$\left.\begin{array}{l} 2PQyy\partial x + 2PRy\partial x - 2PQX\partial x \\ - yy\partial P \quad - 2PXy\partial x - R\partial P \\ \qquad\quad - 2Qy\partial P \quad + P\partial R \\ \qquad\quad + 2Py\partial Q \end{array}\right\} = 0,$$

ubi ex singulis commode definitur $\frac{\partial P}{P}$: scilicet

$$\frac{\partial P}{P} = 2Q\partial x = \frac{R\partial x - X\partial x + \partial Q}{Q} = \frac{\partial R - 2QX\partial x}{R}.$$

Hinc colligitur $2Q(R+X)\partial x = \partial R$, unde nunc ipsum elementum ∂x definiamus, $\partial x = \frac{\partial R}{2Q(R+X)}$, quo valore substituto adipiscimur

$$\frac{Q\partial R}{R+X} = \frac{(R-X)\partial R}{2Q(R+X)} + \partial Q \text{ seu}$$

$$2QQ\partial R = R\partial R - X\partial R + 2QR\partial Q + 2QX\partial Q:$$

unde colligimus

$$X = \frac{2QQ\partial R - 2QR\partial Q - R\partial R}{2Q\partial Q - \partial R}, \text{ et } R + X = \frac{2(QQ-R)\partial R}{2Q\partial Q - \partial R},$$

hinc $\partial x = \frac{2Q\partial Q - \partial R}{4Q(QQ-R)}$, atque $\frac{\partial P}{P} = \frac{2Q\partial Q - \partial R}{2(QQ-R)}$; ideoque

$$P = A\sqrt{(QQ-R)}.$$

Fiat $QQ - R = S$, ac reperietur

$$\partial x = \frac{\partial S}{4QS}, \; X = \frac{4QS\partial Q}{\partial S} - QQ - S, \; R = QQ - S,$$

atque $P = A\sqrt{S}$. Quocirca habebimus hanc aequationem

$$\partial y + \frac{yy\partial S}{4QS} + \partial Q - \frac{(QQ+S)\partial S}{4QS} = 0,$$

quae integrabilis redditur per hunc multiplicatorem

$$\frac{\sqrt{S}}{yy + 2Qy + QQ - S} = \frac{\sqrt{S}}{(y+Q)^2 - S}.$$

Ad ejus integrale inveniendum, sumantur Q et S constantes, prodibitque

$$\int \frac{\partial y\sqrt{S}}{(y+Q)^2 - S} = \frac{1}{2}\, l\, \frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}} + V,$$

existente V certa functione ipsius S vel Q. Jam differentietur haec forma sumta y constante, proditque

$$\frac{\partial Q\sqrt{S} - \frac{(Q+y)\partial S}{2\sqrt{S}}}{(y+Q)^2 - S} + \partial V = \frac{yy\partial S + 4QS\partial Q - QQ\partial S - S\partial S}{4Q[(y+Q)^2 - S]\sqrt{S}},$$

ideoque

$$\partial V = \frac{yy\partial S + 2Qy\partial S + QQ\partial S - S\partial S}{4Q[(y+Q)^2 - S]\sqrt{S}} = \frac{\partial S}{4Q\sqrt{S}}.$$

Ex quo aequationis nostrae integrale est

$$\frac{1}{2}\, l\, \frac{y+Q-\sqrt{S}}{y+Q+\sqrt{S}} + \frac{1}{4}\int \frac{\partial S}{Q\sqrt{S}} = C.$$

Corollarium 1.

530. Singularis est casus, quo $R = QQ$, fit enim

$$\frac{\partial P}{P} = 2Q\partial x = \frac{QQ\partial x - X\partial x + \partial Q}{Q} = \frac{2\partial Q - 2X\partial x}{Q},$$

unde has duas aequationes elicimus

$$QQ\partial x + X\partial x - \partial Q = 0 \text{ et } QQ\partial x + X\partial x - \partial Q = 0,$$

quae cum inter se conveniant, erit

$$X\partial x = \partial Q - QQ\partial x, \text{ et } lP = 2\int Q\partial x.$$

Corollarium 2.

531. Sumto ergo Q negativo, ut habeamus hanc aequationem

$$\partial y + yy\partial x - \partial Q - QQ\partial x = 0,$$

haec integrabilis redditur per hunc multiplicatorem

$$\frac{e^{-2\int Q\partial x}}{(y-Q)^2}. \text{ Et integrale erit}$$

$$\frac{-1}{y-Q} e^{-2\int Q\partial x} + V = \text{Const.}$$

ubi V est functio ipsius x, ad quam definiendam, differentietur sumta y constante

$$\frac{-\partial Q}{(y-Q)^2} e^{-2\int Q\partial x} + \frac{2Q\partial x}{y-Q} e^{-2\int Q\partial x} + \partial V = \frac{yy\partial x - \partial Q - QQ\partial x}{(y-Q)^2} e^{-2\int Q\partial x},$$

unde fit $V = \int e^{-2\int Q\partial x}\partial x$, ita ut integrale fit

$$\int e^{-2\int Q\partial x}\partial x - \frac{e^{-2\int Q\partial x}}{y-Q} = C.$$

Corollarium 3.

532. Proposita ergo aequatione

$$\partial y + yy\partial x + X\partial x = 0,$$

si ejus integrale particulare quoddam constet $y = Q$, ut sit

$$\partial Q + QQ\partial x + X\partial x = 0,$$

ideoque

$$\partial y + yy\partial x - \partial Q - QQ\partial x = 0,$$

multiplicator pro ea erit $\frac{1}{(y-Q)^2} e^{-2\int Q\partial x}$, et integrale completum

$$C e^{2\int Q\partial x} + \frac{1}{(y-Q)} = e^{2\int Q\partial x} \int e^{-2\int Q\partial x} \partial x.$$

Scholion.

533. Aequatio autem in praecedente scholio inventa

$$\partial y + \frac{yy\partial S}{4QS} + \partial Q - \frac{(QQ+S)\partial S}{4QS} = 0,$$

non multum habet in recessu, posito enim $y + Q = z$ prodit

$$\partial z - \frac{z\partial S}{2S} + \frac{\partial S(zz - S)}{4QS} = 0,$$

in qua, ut bini priores termini in unum contrahantur, ponatur $z = v\sqrt{S}$, reperieturque

$$\partial v\sqrt{S} + \frac{vv\partial S}{4Q} - \frac{\partial S}{4Q} = 0, \text{ seu } \frac{\partial v}{vv-1} + \frac{\partial S}{4Q\sqrt{S}} = 0,$$

quae cum sit separata integrale erit $\frac{1}{2} l \frac{1+v}{1-v} = \frac{1}{4}\int \frac{\partial S}{Q\sqrt{S}}$, ubi est $v = \frac{y+Q}{\sqrt{S}}$.

Aequatio autem in ipsa solutione inventa

$$\partial y + yy\partial x + \frac{C\partial x}{S^4} - \frac{\partial\partial S}{S\partial x} = 0,$$

ubi S est functio quaecunque ipsius x, et $\frac{\partial\partial S}{\partial x} = \partial \cdot \frac{\partial S}{\partial x}$, magis ardua videtur, dum per se fit integrabilis, si dividatur per

$$SSyy - \frac{2Sy\partial S}{\partial x} + \frac{\partial S^2}{\partial x^2} + \frac{C}{SS} = \left(Sy - \frac{\partial S}{\partial x}\right)^2 + \frac{C}{SS}.$$

At sumto x constante integrale reperitur

$$\frac{1}{\sqrt{C}} \text{Arc. tang.} \frac{SSy\partial x - S\partial S}{\partial x\sqrt{C}} + V = \text{Const.}$$

nunc ergo ad functionem V inveniendam, sumatur differentiale posita y constante, quod est

$$\frac{2\,\mathrm{S}y\partial\mathrm{S} - \frac{\mathrm{S}\partial\partial\mathrm{S}}{\partial x} - \frac{\partial\mathrm{S}^2}{\partial x}}{\mathrm{SS}(\mathrm{S}y - \frac{\partial\mathrm{S}}{\partial x})^2 + \mathrm{C}} + \partial\mathrm{V},$$

et aequari debet alteri parti

$$\frac{\frac{\mathrm{C}\partial x}{\mathrm{S}^4} - \frac{\partial\partial\mathrm{S}}{\mathrm{S}\partial x} + yy\partial x}{(\mathrm{S}y - \frac{\partial\mathrm{S}}{\partial x})^2 + \frac{\mathrm{C}}{\mathrm{SS}}} = \frac{\frac{\mathrm{C}\partial x}{\mathrm{SS}} - \frac{\mathrm{S}\partial\partial\mathrm{S}}{\partial x} + \mathrm{SS}yy\partial x}{\mathrm{SS}(\mathrm{S}y - \frac{\partial\mathrm{S}}{\partial x})^2 + \mathrm{C}}.$$

Ergo

$$\partial\mathrm{V} = \frac{\mathrm{SS}yy\partial x - 2\mathrm{S}y\partial\mathrm{S} + \frac{\partial\mathrm{S}^2}{\partial x} + \frac{\mathrm{C}\partial x}{\mathrm{SS}}}{\mathrm{SS}(\mathrm{S}y - \frac{\partial\mathrm{S}}{\partial x})^2 + \mathrm{C}} = \frac{\partial x}{\mathrm{SS}}.$$

Quocirca integrale completum est

$$\frac{1}{\sqrt{\mathrm{C}}}\,\text{Arc. tang.}\,\frac{\mathrm{SS}y\partial x - \mathrm{S}\partial\mathrm{S}}{\partial x\sqrt{\mathrm{C}}} + \int\frac{\partial x}{\mathrm{SS}} = \mathrm{D}.$$

Quod si sumamus $\mathrm{S} = x$, hujus aequationis

$$\partial y + yy\partial x + \frac{\mathrm{C}\partial x}{x^4} = 0,$$

integrale completum est

$$\frac{1}{\sqrt{\mathrm{C}}}\,\text{Arc. tang.}\,\frac{xxy - x}{\sqrt{\mathrm{C}}} - \frac{1}{x} = \mathrm{D}.$$

Sin autem fit $\mathrm{S} = x^n$,

ob $\frac{\partial\mathrm{S}}{\partial x} = nx^{n-1}$ et $\partial\,.\,\frac{\partial\mathrm{S}}{\partial x} = n(n-1)\,x^{n-2}\,\partial x$,

integrari poterit haec aequatio

$$\partial y + yy\partial x + \frac{\mathrm{C}\partial x}{x^{4n}} - \frac{n(n-1)\,\partial x}{xx} = 0,$$

integrale enim erit

$$\frac{1}{\sqrt{\mathrm{C}}}\,\text{Arc. tang.}\,\frac{x^{2n}y - nx^{2n-1}}{\sqrt{\mathrm{C}}} - \frac{1}{(2n-1)\,x^{2n-1}} = \mathrm{D}.$$

Supra autem invenimus hanc aequationem

$$\partial y + yy\partial x + Cx^m \partial x = 0,$$

ad separationem reduci posse, quoties fuerit $m = \frac{-4i}{2i+1}$; iisdem ergo casibus functionem S assignare licebit, ut fiat $\frac{C}{S4} - \frac{\partial\partial S}{S\partial x^2} = Cx^m$; quod cum ad aequationes differentiales secundi gradus pertineat, hic non attingemus.

Problema 69.

534. Definire functiones P et Q ambarum variabilium x et y, ut aequatio differentialis $P\partial x + Q\partial y = 0$, divisa per $Px + Qy$ fiat per se integrabilis.

Solutio.

Cum formula $\frac{P\partial x + Q\partial y}{Px + Qy}$ debeat esse integrabilis, statuamus $Q = PR$, ut habeamus $\frac{\partial x + R\partial y}{x + Ry}$, sitque $\partial R = M\partial x + N\partial y$. Quare fieri oportet

$$\frac{1}{\partial y}\partial \cdot \frac{1}{x + Ry} = \frac{1}{\partial x}\partial \cdot \frac{R}{x + Ry},$$

unde nanciscimur $\frac{-R - Ny}{(x + Ry)^2} = \frac{Mx - R}{(x + Ry)^2}$ seu $N = -\frac{Mx}{y}$; hinc fit $\partial R = M\partial x - \frac{Mx\partial y}{y} = My \cdot \frac{y\partial x - x\partial y}{yy}$, quae formula cum debeat esse integrabilis, necesse est sit My functio ipsius $\frac{x}{y}$, quia $\frac{y\partial x - x\partial y}{yy} = \partial \cdot \frac{x}{y}$: atque ex hac integratione prodit $R = \Phi : \frac{x}{y}$, seu quod eodem redit, R erit functio nullius dimensionis ipsarum x et y. Quocirea cum $\frac{Q}{P} = R$, manifestum est huic conditioni satisfieri, si P et Q fuerint functiones homogeneae ejusdem dimensionum numeri ipsarum x et y; hoc ergo modo eandem integrationem aequationum homogenearum sumus assecuti, quam in capite superiori docuimus.

Corollarium 1.

535. Cum igitur $\frac{\partial t + R\partial u}{t + Ru}$ sit integrabile, si fuerit $R = \Phi : \frac{t}{u}$, seu $R = \frac{t}{u}\Phi : \frac{t}{u}$, erit etiam haec formula

$\frac{\frac{\partial t}{t}+\frac{\partial u}{u}\Phi:\frac{t}{u}}{1+\Phi:\frac{t}{u}}$ integrabilis, quae ita repraesentari potest

$$\frac{\frac{\partial t}{t}+\frac{\partial u}{u}\Phi:(\int\frac{\partial t}{t}-\int\frac{\partial u}{u})}{1+\Phi:(\int\frac{\partial t}{t}-\int\frac{\partial u}{u})},$$

ubi littera Φ denotat functionem quamcunque quantitatis suffixae.

Corollarium 2.

536. Ponatur $\frac{\partial t}{t}=\frac{\partial x}{X}$ et $\frac{\partial u}{u}=\frac{\partial y}{Y}$, atque haec formula

$$\frac{\frac{\partial x}{X}+\frac{\partial y}{Y}\Phi:(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y})}{1+\Phi:(\int\frac{\partial x}{X}-\int\frac{\partial y}{X})}=\frac{\partial x+\frac{X\partial y}{Y}\Phi:(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y})}{X+X\Phi:(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y})}$$

erit per se integrabilis. Quare posito $R=\frac{X}{Y}\Phi:(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y})$, haec formula $\frac{\partial x+R\partial y}{X+RY}$ erit per se integrabilis, quaecunque functio sit X ipsius x, et Y ipsius y.

Corollarium 3.

537. Quare si quaerantur functiones P et Q, ut haec aequatio $P\partial x+Q\partial y=0$ fiat integrabilis, si dividatur per $PX+QY$, existente X functione quacunque ipsius x, et Y ipsius y, decet esse $\frac{Q}{P}=\frac{X}{Y}\Phi:(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y})$.

Corollarium 4.

538. Quare si signa Φ et ψ functiones quascunque indicent, fueritque

$$P=\frac{V}{X}\Phi:(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y})\text{ et }Q=\frac{V}{Y}\psi:(\int\frac{\partial x}{X}-\int\frac{\partial y}{Y}),$$

haec aequatio $P\partial x+Q\partial y=0$ integrabilis reddetur, si dividatur per $PX+QY$.

Scholion.

539. Hinc ergo innumerabiles aequationes proferri possunt, quas integrare licebit, etiamsi alioquin difficillime pateat, quomodo

eae ad separationem variabilium reduci queant. Verum haec investigatio proprie ad librum secundum Calculi Integralis est referenda, cujus jam egregia specimina hic habentur; definivimus enim functionem R binarum variabilium x et y ex certa conditione inter M et N proposita scilicet $Mx + Ny = 0$ seu $x\left(\frac{\partial R}{\partial x}\right) + y\left(\frac{\partial R}{\partial y}\right) = 0$, hoc est ex certa differentialium conditione.

CAPUT IV.

DE
INTEGRATIONE PARTICULARI AEQUATIONUM DIFFERENTIALIUM.

Definitio.

540.

Integrale particulare aequationis differentialis est relatio variabilium aequationi satisfaciens, quae nullam novam quantitatem constantem in se complectitur. Opponitur ergo integrali completo, quod constantem in differentiali non contentam involvit, in quo tamen contineatur necesse est.

Corollarium 1.

541. Cognito ergo integrali completo, ex eo innumerabilia integralia particularia exhiberi possunt, prout constanti illi arbitrariae alii atque alii valores determinati tribuuntur.

Corollarium 2.

542. Proposita ergo aequatione differentiali inter variabiles x et y, omnes functiones ipsius x, quae loco y substitutae aequationi satisfaciunt, dabunt integralia particularia, nisi forte sint completa.

Corollarium 3.

543. Cum omnis aequatio differentialis ad hanc formam $\frac{\partial y}{\partial x} = V$ revocetur, existente V functione quacunque ipsarum x et y,

si ejusmodi constet relatio inter x et y, unde pro $\frac{\partial y}{\partial x}$ et V resultent valores aequales, ea pro integrali particulari erit habenda.

Scholion 1.

544. Interdum facile est integrale particulare quasi divinatione colligere; veluti si proposita sit haec aequatio.

$$aa\partial y + yy\partial x = aa\partial x + xy\partial x.$$

Statim liquet ei satisfieri ponendo $y = x$, quae relatio cum non solum nullam novam constantem, sed ne eam quidem a, quae in ipsa aequatione differentiali continetur, implicet, utique est integrale particulare: unde nihil pro integrali completo colligere licet. Saepe numero quidem cognitio integralis particularis ad inventionem completi viam patefacit, quemadmodum in hoc ipso exemplo usu venit, in quo si statuamus $y = x + z$ fit

$$a^2\partial x + a^2\partial z + x^2\partial x + 2xz\partial x + z^2\partial x = a^2\partial x + x^2\partial x + xz\partial x, \text{ seu}$$

$$aa\partial z + xz\partial x + zz\partial x = 0,$$

quae aequatio posito $z = \frac{aa}{v}$ abit in hanc

$$\partial v - \frac{xv\partial x}{aa} = \partial x,$$

quae per $e^{-\int\frac{x\partial x}{aa}} = e^{\frac{-xx}{2aa}}$ multiplicata fit integrabilis, et dat

$$e^{\frac{-xx}{2aa}} v = \int e^{\frac{-xx}{2aa}} \partial x, \text{ seu } v = e^{\frac{xx}{2aa}} \int e^{\frac{-xx}{2aa}} \partial x,$$

quod ergo est maxime transcendens, cum tamen simplicissimum illud particulare involvat: scilicet si constans integratione $\int e^{\frac{-xx}{2aa}} \partial x$ invecta sumatur infinita, fit $v = \infty$ et $z = 0$, unde $y = x$. Interdum autem integrale particulare parum juvat ad completum investigandum, veluti si habeatur haec aequatio.

$$a^3\partial y + y^3\partial x = a^3\partial x + x^3\partial x.$$

i manifesto satisfacit $y = x$, posito autem $y = x + z$ prodit

$$a^3 \partial z + 3 x x z \partial x + 3 x z z \partial x + z^3 \partial x = 0,$$

jus resolutio haud facilior videtur, quam illius.

Scholion 2.

645. In his exemplis integrale particulare statim in oculos urrit, dantur autem casus quibus difficilius perspicitur; et quanam raro inde via pateat ad integrale completum perveniendi, tan saepenumero plurimum interest integrale particulare nosse, n eo nonnunquam totum negotium confici possit. Jam enim anidvertimus in omnibus problematibus, quorum solutio ad aequaiem differentialem perducitur, constantem arbitrariam per integraiem invectam ex ipsis conditionibus, cuique problemati adjunctis, erminari, ita ut semper integrali tantum particulari sit opus; are si eveniat, ut hoc ipsum integrale particulare cognosci possit, e subsidio completi, solutio problematis exhiberi poterit, etiamsi egratio aequationis differentialis non sit in potestate. Quibus ercasibus sine integratione vera solutio inveniri est censenda; propea quod proprie loquendo nulla aequatio differentialis integrari stimatur, nisi ejus integrale completum assignetur. Quocirca utile eos casus perpendere, quibus integrale particulare exhibere licet.

Scholion 3.

646. Maximi autem est momenti hic animadvertisse, non nes valores aequationi cuipiam differentiali satisfacientes pro ejus egrali particulari haberi posse. Veluti si habeatur haec aequatio $= \frac{\partial x}{\sqrt{(a-x)}}$, seu $\frac{\partial x}{\partial y} = \sqrt{(a-x)}$, posito $x = a$ fit tam $\sqrt{(a-x)} = 0$, m $\frac{\partial x}{\partial y} = 0$, ita ut aequatio $x = a$ illi differentiali satisfaciat, n tamen nequaquam ejus sit integrale particulare. Integrale namcompletum est $y = C - 2\sqrt{(a-x)}$ seu $a - x = \frac{1}{4}(C-y)^2$, le quicunque valor constanti C tribuatur, nunquam sequitur $- x = 0$. Simili modo huic aequationi

$$\partial y = \frac{x\partial x + y\partial y}{\sqrt{(xx + yy - aa)}}$$

satisfacit haec aequatio finita $xx + yy = aa$, quae tamen inter integralia particularia admitti nequit, propterea quod in integrali completo $y = C + \sqrt{(xx + yy - aa)}$ neutiquam continetur. Quare ad integrale particulare non sufficit, ut eo aequationi differentiali satisfiat, sed insuper hanc conditionem adjungi oportet, ut in integrali completo contineatur; ex quo investigatio integralium particularium maxime est lubrica, nisi simul integrale completum innotescat; hoc autem cognito supervacuum esset methodo peculiari in integralia particularia inquirere. Tum enim potissimum juvat ad investigationem integralium particularium confugere, quando integrale completum elicere non licet. Quo igitur hinc fructum percipere queamus, criteria tradi conveniet, ex quibus valores, qui aequationi cuipiam differentiali satisfaciunt, dijudicare liceat, utrum sint integralia particularia, nec ne? Etiamsi scilicet omnia integralia sint ejusmodi valores, qui aequationi differentiali satisfaciant, tamen non vicissim omnes valores, qui satisfaciunt, sunt integralia. Quod cum parum adhuc sit animadversum, operam dabo, ut hoc argumentum dilucide evolvam.

Problema 70.

547. Si in aequatione differentiali $\partial y = \frac{\partial x}{Q}$, functio Q evanescat posito $x = a$, determinare quibus casibus haec aequatio $x = a$ sit integrale particulare aequationis differentialis propositae?

Solutio.

Cum sit $Q = \frac{\partial x}{\partial y}$, posito $x = a$ fit tam $Q = 0$ quam $\frac{\partial x}{\partial y} = 0$, unde hic valor $x = a$ aequationi differentiali propositae $\partial y = \frac{\partial x}{Q}$ utique satisfacit, neque tamen hinc sequitur eum esse integrale. Hoc solum scilicet non sufficit, sed insuper requiritur, ut aequatio

$x=a$ in integrali completo contineatur, si quidem constanti per integrationem invectae certus quidam valor tribuatur. Ponamus ergo P esse integrale formulae $\frac{\partial x}{Q}$, ut integrale completum sit $y=C+P$; cui aequationi ponendo $x=a$ satisfieri nequit, nisi posito $x=a$ fiat $P=\infty$, tum enim sumta constante C pariter infinita, positione $x=a$ quantitas y manet indeterminata, ideoque si posito $x=a$ fiat $P=\infty$, tum demum aequatio $x=a$ pro integrali particulari erit habenda. En ergo criterium, ex quo dignoscere licet, utrum valor $x=a$ aequationi differentiali $\partial y=\frac{\partial x}{Q}$ satisfaciens simul sit ejus integrale particulare nec ne? scilicet tum demum erit integrale, si posito $x=a$ non solum fiat $Q=0$, sed etiam integrale $P=\int\frac{\partial x}{Q}$ abeat in infinitum. Quod quo clarius exponamus, quoniam posito $x=a$ fit $Q=0$, ponamus $Q=(a-x)^n R$, denotante n numerum quemcunque positivum, et cum aequatio

$$\partial y=\frac{\partial x}{Q}=\frac{\partial x}{(a-x)^n R}$$

induere queat hanc formam

$$\partial y=\frac{\alpha\,\partial x}{(a-x)^n}+\frac{\beta\,\partial x}{(a-x)^{n-1}}+\frac{\gamma\,\partial x}{(a-x)^{n-2}}+\ldots+\frac{S\,\partial x}{R},$$

ratio illius infiniti P pendebit a termino $\int\frac{\alpha\,\partial x}{(a-x)^n}$ qui si posito $x=a$ evadat infinitus, etiam integrale $P=\int\frac{\partial x}{Q}$ erit infinitum, utcunque se habeant reliqua membra. At est

$$\int\frac{\alpha\,\partial x}{(a-x)^n}=\frac{\alpha}{(n-1)(a-x)^{n-1}},$$

quae expressio fit infinita posito $x=a$, dummodo $n-1$ sit numerus positivus, vel etiam $n=1$. Quare dummodo exponens n non sit unitate minor, posito $Q=(a-x)^n R$ aequatio $x=a$ pro integrali particulari erit habenda.

Corollarium 1.

548. Quoties ergo posito $Q = (a - x)^n R$ exponens n est unitate minor, aequationi $\partial y = \frac{\partial x}{Q}$ non convenit integrale particulare $x = a$, etiamsi hoc modo aequationi differentiali satisfiat.

Corollarium 2.

549. Si exponens n est unitate minor, formula $\frac{\partial Q}{\partial x}$ fit infinita posito $x = a$; unde novum criterium adipiscimur: Scilicet proposita aequatione $\partial y = \frac{\partial x}{Q}$, si posito $x = a$ fiat quidem $Q = 0$, at $\frac{\partial Q}{\partial x} = \infty$, tum valor $x = a$ non est integrale particulare illius aequationis.

Corollarium 3.

550. His igitur casibus exclusis, aequationis $\partial y = \frac{\partial x}{Q}$, ubi posito $x = a$ fit $Q = 0$, integrale particulare semper erit $x = a$, nisi eodem casu $x = a$ fiat $\frac{\partial Q}{\partial x} = \infty$; hoc est quoties valor formulae $\frac{\partial Q}{\partial x}$ fuerit vel finitus vel evanescat.

Scholion 1.

551. Haec conclusio inversioni propositionum hypotheticarum innixa licet videri queat suspecta ac regulis Logicae adversa, verum totum ratiocinium regulis apprime est consentaneum, cum a sublatione consequentis ad sublationem antecedentis concludat. Quoties enim posito $Q = (a - x)^n R$ exponens n est unitate minor, toties $\frac{\partial Q}{\partial x}$ fit $= \infty$ posito $x = a$. Quare si posito $x = a$ non fiat $\frac{\partial Q}{\partial x} = \infty$, ideoque ejus valor vel finitus vel evanescat, tum certe exponens n non est unitate minor, erit ergo vel major unitate vel ipsi aequalis, utroque autem casu integrale $P = \int \frac{\partial x}{Q}$ posito $x = a$ fit infinitum, ideoque aequatio $x = a$ est integrale particulare.

Quare si in aequatione differentiali $\partial y = \frac{\partial x}{Q}$, posito $x = a$, fiat $Q = 0$, examinatur valor $\frac{\partial Q}{\partial x}$ pro casu $x = a$, qui si fuerit vel finitus vel evanescat, aequatio $x = a$ est integrale particulare; sin autem is sit infinitus, ea inter integralia locum non habet, etiamsi aequationi differentiali satisfiat. Eadem regula quoque locum habet, si aequatio differentialis fuerit hujusmodi $\partial y = \frac{P\partial x}{Q}$ seu $\frac{\partial y}{\partial x} = \frac{P}{Q}$, ac posito $x = a$ fiat $Q = 0$, quaecunque fuerit P functio ipsarum x et y; quin etiam necesse non est, ut Q sit functio solius variabilis x, sed simul alteram y utcunque implicare potest.

Scholion 2.

552. Demonstratio quidem inde est petita, quod quantitas Q, quae posito $x = a$ evanescit, factorem implicet potestatem quampiam ipsius $a - x$, quod in functionibus algebraicis est manifestum. Verum in functionibus transcendentibus eadem regula locum habet, cum potestate talibus dignitatibus aequivaleant. Veluti si sit $\partial y = \frac{\partial x}{lx - la}$, ubi $Q = lx - la = l\frac{x}{a}$, fitque $Q = 0$ posito $x = a$, quaeratur $\frac{\partial Q}{\partial x} = \frac{1}{x}$, quae formula cum non fiat infinita posito $x = a$, integrale particulare erit $x = a$. Quod etiam valet pro aequatione $\partial y = \frac{P\partial x}{lx - la}$, dummodo P non fiat $= 0$ posito $x = a$. Sit enim $P = \frac{1}{x}$, erit integrando $y = C + l(lx - la)$ et $l\frac{x}{a} = e^{y - C}$. Sumta jam constante $C = \infty$, fit $l\frac{x}{a} = 0$, ideoque $x = a$, quod ergo est integrale particulare. Simili modo si sit $\partial y = P\partial x : (e^{\frac{x}{a}} - e)$, ubi $Q = e^{\frac{x}{a}} - e$, ideoque posito $x = a$ fit $Q = 0$; quia $\frac{\partial Q}{\partial x} = \frac{1}{a} e^{\frac{x}{a}}$, hincque posito $x = a$ fit $\frac{\partial Q}{\partial x} = \frac{e}{a}$, erit $x = a$ etiam integrale particulare. Sumatur $P = e^{\frac{x}{a}}$ ut integratio succedat, et quia $y = C + a\,l(e^{\frac{x}{a}} - e)$, hincque $e^{\frac{x}{a}} =$

$e + e^{\frac{y-C}{a}}$, statuatur $C = \infty$, erit $e^{\frac{x}{a}} = e$, ideoque $x = a$, quo
ergo manifesto est integrale particulare.

Exemplum 1.

553. *Proposita aequatione differentiali $\partial y = \frac{P\partial x}{\sqrt{S}}$ in qu S evanescat posito $x = a$, definire casus, quibus aequatio $x =$ est ejus integrale particulare.*

Cum hic sit $\sqrt{S} = Q$, erit $\partial Q = \frac{\partial S}{2\sqrt{S}}$: ergo ut integral particulare sit $x = a$, necesse est, ut posito $x = a$ fiat $\frac{\partial Q}{\partial x} = \frac{\partial S}{2\partial x\sqrt{}}$ quantitas finita. Hinc eodem casu quantitas $\frac{\partial S^2}{S\partial x^2}$ fieri debet finita, unde cum S evanescat, etiam $\frac{\partial S^2}{\partial x^2}$ ac proinde $\frac{\partial S}{\partial x}$ evanescere debet: Tum autem posito $x = a$ illius fractionis valor est $\frac{2\partial S\partial\partial S}{\partial S\partial x^2} = \frac{2\partial\partial S}{\partial x^2}$, quem ergo finitum esse oportet, vel $= 0$. Quare ut aequati $x = a$, sit integrale particulare aequationis propositae, hae conditiones requiruntur, primo ut posito $x = a$ fiat $S = 0$. Secund ut fiat $\frac{\partial S}{\partial x} = 0$, ac tertio ut hujus formulae $\frac{\partial\partial S}{\partial x^2}$ valor prodeat ve finitus, vel $= 0$, dummodo ne fiat infinite magnus. Si S sit functio rationalis, haec eo redeunt, ut S factorem habeat $(a - x)^2$ ve potestatem altiorem.

Scholion.

554. Haec resolutio usum habet in motu corporis ad centrum virium attracti dignoscendo, num in circulo fiat. Si enim distantia corporis a centro ponatur $= x$, et vis centripeta huic distantiae conveniens $= X$, pro tempore t talis reperitur aequatio $\partial t = \frac{x\partial x}{\sqrt{(Exx - c^4 - 2\alpha xx\int X\partial x)}}$, ubi E est constans per praecedentem integrationem ingressa, cujus valor quaeritur, ut hinc aequationi satisfaciat valor $x = a$, quo casu corpus in circulo revolvetur.

ic ergo est $S = E x x - c^4 - 2 \alpha x x \int X \partial x$, vel sumi potest $= E - \frac{c^4}{xx} - 2 \alpha \int X \partial x$. Non solum ergo haec quantitas, sed iam ejus differentiale $\frac{\partial S}{\partial x} = \frac{2 c^4}{x^3} - 2 \alpha X$ evanescere debet posito $= a$, neque tamen differentio-differentiale $\frac{\partial \partial S}{\partial x^2} = - \frac{6 c^4}{x^4} - \frac{2 \alpha \partial X}{\partial x}$ infinitum abire debet. Inde ergo constans a erit valor ipsius x, t hac aequatione $\alpha x^3 X = c^4$ resultans, qui est radius circuli, quo corpus revolvi poterit, dummodo constans E, a qua celeris pendet, ita fuerit comparata, ut posito $x = a$ fiat $E = \frac{c^4}{aa} + \alpha \int X \partial x$; nisi forte eodem casu expressio $\frac{6 c^4}{x^4} + \frac{2 \alpha \partial X}{\partial x}$ seu salm haec $\frac{\partial X}{\partial x}$ fiat infinita. Hoc enim si eveniret motus in circulo lleretur; ad quod ostendendum ponamus $X = b + \sqrt{(a - x)}$, $\frac{\partial X}{\partial x} = - \frac{1}{2 \sqrt{(a - x)}}$ fiat infinitum posito $x = a$, et aequatio $x^3 X = c^4$ dabit $\alpha a^3 b = c^4$. Tum vero ob

$$\int X \partial x = b x - \tfrac{2}{3} (a - x)^{\frac{3}{2}} \text{ erit}$$

$$E = \alpha a b + 2 \alpha a b = 3 \alpha a b,$$

ostraque aequatio fit

$$t = \frac{x \partial x}{\sqrt{[3 \alpha a b x x - \alpha a^3 b - 2 \alpha b x^3 + \frac{4}{3} \alpha x x (a - x)^{\frac{3}{2}}]}}$$

ui valor $x = a$ certe non convenit tanquam integrale. Fit enim

$$S = \alpha (a - x) [- a a b - a b x + 2 b x x + \tfrac{4}{3} x x \sqrt{(a - x)}]$$

ujus factor cum non sit $(a - x)^2$ sed tantum $(a - x)^{\frac{3}{2}}$, integrale articulare $x = a$ locum habere nequit.

Exemplum 2.

555. *Proposita aequatione differentiali* $\partial y = \frac{P \partial x}{\sqrt[n]{S^m}}$ *in qua*

S evanescat posito $x=a$, invenire casus quibus integrale particulare est $x=a$.

Cum fiat $S=0$ posito $x=a$, concipere licet $S=(a-x)^\lambda R$, eritque denominator $\sqrt[n]{S^m}=(a-x)^{\frac{\lambda m}{n}} R^{\frac{m}{n}}$, unde patet aequationem $x=a$ fore integrale particulare aequationis propositae, si fuerit $\frac{\lambda m}{n}$ numerus positivus unitate major, seu saltem unitati aequalis, hoc est, si sit vel $\lambda=\frac{n}{m}$ vel $\lambda>\frac{n}{m}$, quae dijudicatio si S sit functio algebraica, facillime instituitur. Sin autem sit transcendens, ut exponens λ in numeris exhiberi nequeat, uti licebit altera regula: scilicet, cum sit $\sqrt[n]{S^m}=Q$, erit $\frac{\partial Q}{\partial x}=\frac{m S^{\frac{m-n}{n}} \partial S}{n \partial x}$, cujus valor debet esse finitus vel nullus posito $x=a$, siquidem integrale sit $x=a$. Sit igitur quoque necesse est hoc casu quantitas $\frac{S^{m-n} \partial S^n}{\partial x^n}$ finita. Quaeratur ergo hujus formulae valor casu $x=a$, qui si prodeat infinite magnus, aequatio $x=a$ non erit integrale, sin autem sit vel finitus vel nullus, erit ea certe integrale particulare aequationis propositae. Hic duo constituendi sunt casus, prout fuerit vel $m>n$ vel $m<n$.

I. Si $m>n$, quia posito $x=a$ fit $S^{m-n}=0$, nisi eodem casu fiat $\frac{\partial S}{\partial x}=\infty$, certe erit $x=a$ integrale. Sin autem fiat $\frac{\partial S}{\partial x}=\infty$, utrumque evenire potest, ut sit integrale et ut non sit. Ad quod dignoscendum ponatur $\frac{\partial x}{\partial S}=T$, ut nostra formula evadat $\frac{S^{m-n}}{T^n}$, cujus tam numerator, quam denominator evanescit posito $x=a$, ex quo ejus valor reducitur ad

$$\frac{(m-n) S^{m-n-1} \partial S}{n T^{n-1} \partial T}=\frac{-(m-n) S^{m-n-1} \partial S^{n+2}}{n \partial x^n \partial \partial S},$$

ıi si sit vel finitus vel nullus, integrale erit $x = a$. Simili modo terius progredi licet distinguendo casus $m > n + 1$ et $m < n + 1$.

II. Si $m < n$, formula nostra erit $\frac{\partial S^n}{S^{n-m} \partial x^n}$, cujus valor t fiat finitus, necesse est ut sit $\frac{\partial S}{\partial x} = 0$, ac praeterea, quia numeıtor ac denominator posito $x = a$ evanescit, formulae nostrae vaır erit

$$\frac{n \partial S^{n-1} \partial \partial S}{(n-m) S^{n-m-1} \partial S \partial x^n} = \frac{n \partial S^{n-2} \partial \partial S}{(n-m) S^{n-m-1} \partial x^n},$$

uem finitum esse oportet.

Facillime autem judicium absolvetur, ponendo statim $x = a + \omega$, um enim posito $x = a$ fiat $S = 0$, hac substitutione quantitas S ımper resolvi poterit in hujusmodi formam

$$P \omega^\alpha + Q \omega^\beta + R \omega^\gamma + \text{etc.}$$

ujus tantum unus terminus $P \omega^\alpha$ infimam potestatem ipsius ω comlectens spectetur; ac si fuerit vel $\alpha = \frac{n}{m}$ vel $\alpha > \frac{n}{m}$, aequatio $: = a$ certe erit integrale particulare.

Scholion.

556. Haec ultima methodus est tutissima, ac semper etiam a formulis transcendentibus optimo successu adhiberi potest. Sciicet proposita aequatione $\partial y = \frac{P \partial x}{Q}$, in qua posito $x = a$ fiat $Q = 0$, neque vero etiam numerator P evanescat: statuatur $x = a + \omega$, et quantitas ω spectetur ut infinite parva; ut omnes jus potestates prae infima evanescant, atque quantitas Q hujusmoli formam $R \omega^\lambda$ accipiet, ex qua patebit nisi exponens λ unitate uerit minor, aequationem $x = a$ certe fore integrale particulare aequationis propositae. Veluti si habeamus $\partial y = \frac{\partial x}{\sqrt{(1 + \cos. \frac{\pi x}{a})}}$, cu-

tuor dantur integralia particularia $a+x=0$, $a-x=0$, $b+y=0$, $b-y=0$. Integrale completum vero est

$$\frac{m}{2}\,l\,\frac{a+x}{a-x}=\tfrac{1}{2}\,l\,C+\frac{n}{2}\,l\,\frac{b+y}{b-y},\ \text{seu}$$

$$\left(\frac{a+x}{a-x}\right)^m=C\left(\frac{b+y}{b-y}\right)^n,\ \text{vel}$$

$$(a+x)^m(b-y)^n=C(a-x)^m(b+y)^n,$$

unde illa sponte fluunt.

Corollarium 5.

562. Hinc patet si fuerit $\partial y=\dfrac{P\partial x}{(a+x)^\alpha(b+x)^\beta(c+x)^\gamma}$, integralia particularia fore $a+x=0$, $b+x=0$, $c+x=0$, si modo exponentes α, β, γ etc. non fuerint unitate minores. Quare si Q sit functio rationalis ipsius x, proposita aequatione $\partial y=\frac{P\partial x}{Q}$, omnes factores ipsius Q nihilo aequales positi, praebent integralia particularia.

Scholion 1.

563. Hoc etiam pro factoribus imaginariis valet, etiamsi inde parum lucri nanciscamur. Si enim proposita sit aequatio $\partial y=\frac{a\partial x}{aa+xx}$, ex denominatore $aa+xx$ oriuntur integralia particularia $x=a\sqrt{-1}$ et $x=-a\sqrt{-1}$, quae ex integrali completo, quod est $y=C+\text{Ang. tang.}\ \frac{x}{a}$ minus sequi videntur. Verum posito $x=a\sqrt{-1}$ notandum est, esse Ang. tang. $\sqrt{-1}=\infty\sqrt{-1}$, unde si constanti C similis forma signo contrario affecta tribuatur, altera quantitas y manet indeterminata, etiamsi ponatur $x=a\sqrt{-1}$, quae positio propterea pro integrali particulari est habenda. Est enim in genere

$$\text{Ang. tang.}\ u\sqrt{-1}=\int\frac{\partial u\sqrt{-1}}{1-uu}=\frac{\sqrt{-1}}{2}\,l\,\frac{1+u}{1-u},$$

nde posito $u = +1$ vel $u = -1$, prodit $\infty\sqrt{-1}$, quod ins-itum in causa est, ut integralia assignata locum habeant. Quo-irca in genere affirmare licet, si fuerit $\partial y = \frac{P\partial x}{Q}$, denominatorque Q factorem habeat $(a+x)^\lambda$, cujus exponens λ unitate non sit mi-or, semper aequationem $a+x=0$ fore integrale particulare. Si autem λ sit unitate minor etsi positivus, non erit $a+x=0$ inte-rale particulare, etiamsi posito $x=-a$ aequationi differentiali atisfaciat.

Scholion 2.

564. Insigne hoc est paradoxon a nemine adhuc, quantum nihi quidem constat, observatum, quod aequationi differentiali ejus-nodi valor satisfacere queat, qui tamen ejus non sit integrale; at-ue adeo vix patet, quomodo haec cum solita integralium idea con-iliari possint. Quoties enim proposita aequatione differentiali ejus-nodi relationem variabilium exhibere licet, quae ibi substituta satis-aciat, seu aequationem identicam producat, vix cuiquam in mentem enit dubitare, an illa relatio pro integrali saltem particulari sit ha-enda, cum tamen hinc proclive sit in errorem delabi. Veluti etiam-i huic aequationi $\partial y\sqrt{(aa-xx-yy)} = x\partial x + y\partial y$ satisfaciat aec aequatio finita $xx+yy=aa$, tamen enormem errorem com-nitteremus, si eam pro integrali particulari habere vellemus, prop-erea quod ea in integrali completo $y = C - \sqrt{(aa-xx-yy)}$ eutiquam continetur. Quamobrem etsi omne integrale aequationi ifferentiali satisfacere debet, tamen non vicissim concludere licet, mnem aequationem finitam, quae satisfaciat, ejus esse integrale; erum praeterea requiritur, ut ea certa quadam proprietate sit prae-ita, cujusmodi hic exposuimus, et qua demum efficitur, ut in in-egrali completo contineatur. Hoc autem minime adversatur verae itegralium notioni, quam hic stabilivimus, neque hujusmodi dubium nquam in integralia per certas regulas inventa cadere potest; sed antum in ejusmodi integralibus, quae divinando quasi sumus asse-

cuti, locum habet. Saepe numero autem, quando integratio non succedit, divinationi plurimum tribui solet, tum igitur maxime cavendum est, ne relationem quampiam satisfacientem temere pro integrali particulari proferamus. Quod cum jam in aequationibus separatis simus assecuti, quomodo in omnibus aequationibus differentialibus hujusmodi errores vitari oporteat, sedulo investigemus.

Problema 72.

565. Si quaepiam relatio inter binas variabiles satisfaciat aequationi differentiali, definire utrum ea sit integrale particulare, nec ne?

Solutio.

Sit $P\partial x = Q\partial y$ aequatio differentialis proposita, ubi P et Q sint functiones quaecunque ipsarum x et y, cui satisfaciat relatio quaepiam inter x et y, ex qua fiat $y = X$, functioni scilicet cuidam ipsius x, ita ut si loco y ubique scribatur X, revera prodeat $P\partial x = Q\partial y$ seu $\frac{\partial y}{\partial x} = \frac{P}{Q}$. Quaeritur ergo utrum hic valor $y = X$ pro integrali aequationis propositae haberi possit nec ne? Ad hoc dijudicandum ponatur $y = X + \omega$, fietque $\frac{\partial X}{\partial x} + \frac{\partial \omega}{\partial x} = \frac{P}{Q}$ ubi notetur si esset $\omega = 0$, fore $\frac{\partial X}{\partial x} = \frac{P}{Q}$. Quare ob ω expressio $\frac{P}{Q}$ hac substitutione reducetur ad $\frac{\partial X}{\partial x}$ una cum quantitate ita per ω affecta, ut evanescat posito $\omega = 0$. In hoc negotio sufficit ω ut particulam infinite parvam spectasse, cujus ergo potestates altiores prae infima negligere liceat. Ponamus igitur hinc fieri $\frac{P}{Q} = \frac{\partial X}{\partial x}$ $+ S\omega^{\lambda}$, habebiturque $\frac{\partial \omega}{\partial x} = S\omega^{\lambda}$ seu $\frac{\partial \omega}{\omega^{\lambda}} = S\partial x$. Ex superioribus jam perspicuum est, tum demum fore $y = X$ integrale particulare, seu $\omega = 0$, cum exponens λ fuerit unitate aequalis vel major: similis enim hic est ratio ac supra, qua requiritur, ut integra-

$\int S \partial x = \int \frac{\partial \omega}{\omega^\lambda}$ fiat infinitum casu proposito, quo $\omega = 0$, hoc autem non evenit, nisi λ sit unitati aequalis, vel > 1. Quodsi ergo aequationi $P \partial x = Q \partial y$ seu $\frac{\partial y}{\partial x} = \frac{P}{Q}$ satisfaciat valor $y = X$, statuatur $y = X + \omega$, spectata particula ω infinite parva, et investigetur hinc forma $\frac{Q}{P} = \frac{\partial X}{\partial x} + S \omega^\lambda$, ex qua nisi sit $\lambda < 1$ concludetur, illum valorem $y = X$ esse integrale particulare aequationis propositae.

Scholion.

566. Cum ω tractetur ut quantitas infinite parva, valor ipsius $\frac{P}{Q}$ posito $y = X + \omega$ per differentiationem commodissime inveniri posse videtur. Cum enim $\frac{P}{Q}$ sit functio ipsarum x et y, statuamus

$$\partial \cdot \frac{P}{Q} = M \partial x + N \partial y,$$

et quia posito $y = X$, fractio $\frac{P}{Q}$ abit in $\frac{\partial X}{\partial x}$ per hypothesin, si loco y scribatur $X + \omega$, ea in $\frac{\partial X}{\partial x} + N \omega$ transibit, unde ob exponentem ipsius ω unitatem sequeretur, aequationem $y = X$ semper esse integrale particulare, quod tamen secus evenire potest. Ex quo patet differentiationem loco substitutionis adhiberi non posse; quod quo clarius ostendatur, ponamus esse $\frac{P}{Q} = \sqrt{(y - X)} + \frac{\partial X}{\partial x}$, unde posito $y = X + \omega$ manifesto oritur $\frac{P}{Q} = \frac{\partial X}{\partial x} + \sqrt{\omega}$. At differentiatione utentes ponendo

$$\partial \cdot \frac{P}{Q} = M \partial x + N \partial y,$$

fiet $N = \frac{1}{2\sqrt{(y - X)}}$, hincque $\frac{P}{Q} = \frac{\partial X}{\partial x} + N \omega$, quae expressio ab illa discrepat. Illa scilicet aequationem $y = X$ ex integralium numero removet, haec vero admittere videtur. Verum et hic notandum est quantitatem N ipsam potestatem ipsius ω negative involvere, unde potestas ω deprimatur. Quare ne hanc rationem spectare opus sit, semper praestat vera substitutione uti, differentiatione seposita. Hoc observato haud difficile erit omnes valores, qui aequationi cuipiam differentiali satisfaciunt, dijudicare, utrum sint vera integralia nec ne?

Exemplum 1.

567. *Cum huic aequationi*

$$\partial x (1 - y^m)^n = \partial y (1 - x^m)^n,$$

manifesto satisfaciat $y = x$, utrum sit ejus integrale particulare nec ne? definire.

Ponatur $y = x + \omega$, et spectato ω ut quantitate minima, est $y^m = x^m + m x^{m-1} \omega$, et

$$(1 - y^m)^n = (1 - x^m - m x^{m-1} \omega)^n$$
$$= (1 - x^m)^n - m n x^{m-1} \omega (1 - x^m)^{n-1},$$

unde aequatio $\frac{\partial y}{\partial x} = \frac{(1 - y^m)^n}{(1 - x^m)^n}$ abit in

$$1 + \frac{\partial \omega}{\partial x} = 1 - \frac{m n x^{m-1} \omega}{1 - x^m},$$

seu $\frac{\partial \omega}{\omega} = - \frac{m n x^{m-1} \partial x}{1 - x^m}$; ubi cum ω habeat dimensionem integram, aequatio $y = x$ certe est integrale particulare aequationis differentialis propositae.

Exemplum 2.

568. *Cum huic aequationi*

$$a \partial y - a \partial x = \partial x \sqrt{(y y - x x)},$$

satisfaciat valor $y = x$, investigare, utrum is fit ejus integrale particulare nec ne?

Ponatur $y = x + \omega$ et sumta ω quantitate infinite parva cum sit $\sqrt{(yy - xx)} = \sqrt{2x\omega}$, erit $a\partial\omega = \partial x\sqrt{2x\omega}$ seu $\frac{a\partial\omega}{\sqrt{\omega}} = \partial x\sqrt{2x}$. Quoniam igitur hic $\partial\omega$ dividitur per potestatem ipsius ω, cujus exponens est unitate minor, sequitur valorem $y = x$ non esse integrale particulare aequationis propositae, etiamsi ei satisfaciat. Scilicet si ejus integrale completum exhibere liceret, pateret, quomodocunque constans arbitraria per integrationem ingressa definiretur, in ea aequationem $y = x$ non contentum iri.

Scholion.

569. Hinc nova ratio intelligitur, cur dijudicatio integralis ab exponente ipsius ω pendeat. Cum enim in exemplo proposito facto $y = x + \omega$ prodeat $\frac{a\partial\omega}{\sqrt{\omega}} = \partial x\sqrt{2x}$, erit integrando $2a\sqrt{\omega} = C + \frac{2}{3}x\sqrt{2x}$. Verum per hypothesin ω est quantitas infinite parva, hinc autem utcunque definiatur constans C, quantitas ω obtinet valorem finitum, qui adeo quantumvis magnus evadere potest, quod cum hypothesi adversetur, necessario sequitur aequationem $y = x$ integrale esse non posse; hocque semper evenire debere, quoties $\partial\omega$ prodit divisum per potestatem ipsius ω, cujus exponens unitate est minor. Contra vero patet, si facta substitutione exposita prodeat $\frac{\partial\omega}{\omega} = R\partial x$, ut posito $\int R\partial x = lS$ fiat $l\omega = lC + lS$, seu $\omega = CS$, sumta constante C evanescente utique ipsam quantitatem ω evanescere, quod idem evenit si prodeat $\frac{\partial\omega}{\omega^\lambda} = R\partial x$, existente $\lambda > 1$. Erit enim $\frac{1}{(\lambda - 1)\omega^{\lambda-1}} = C - S$ seu $(\lambda-1)\omega^{\lambda-1} = \frac{1}{C-S}$, unde sumto $C = \infty$, quantitas ω revera fit evanescens, ut hypothesis exigit.

Caeterum aequatio hujus exempli, posito $x = pp - qq$ et $y = pp + qq$, ab irrationalitate liberatur, fitque $4aq\partial q = 4pq(p\partial p - q\partial q)$, sive $a\partial q = pp\partial p - pq\partial q$, quae nullo modo tractari posse videtur; neque ergo ejus integrale completum exhiberi potest. Cui aequationi cum non amplius satisfacit $x = y$ seu $q = 0$, hinc quoque concludendum est, valorem $y = x$ non esse integrale particulare.

Exemplum 3.

570. *Cum huic aequationi*

$$aa\partial y - aa\partial x = \partial x(yy - xx),$$

satisfaciat valor $y = x$, investigare, utrum is sit ejus integrale particulare nec ne?

Ponatur $y = x + \omega$ spectata ω ut quantitate infinite parva, et ob $yy - xx = 2x\omega$ aequatio nostra hanc induet formam $aa\partial\omega = 2x\omega\partial x$, seu $\frac{aa\partial\omega}{\omega} = 2x\partial x$. Quia igitur hic $\partial\omega$ dividitur per potestatem primam ipsius ω, aequatio $y = x$ utique erit integrale particulare aequationis propositae, atque adeo etiam in integrali completo continetur. Hoc enim invenitur ponendo $y = x - \frac{aa}{u}$, quo fit

$$\frac{a^4\partial u}{uu} = \partial x\left(\frac{a^4}{uu} - \frac{2aax}{u}\right), \text{ seu } \partial u + \frac{2ux\partial u}{aa} = \partial x.$$

Multiplicetur per $e^{\frac{xx}{aa}}$, et integrale prodit

$$e^{\frac{xx}{aa}}u = C + \int e^{\frac{xx}{aa}}\partial x, \text{ hincque}$$

$$y = x - aae^{\frac{xx}{aa}} : (C + \int e^{\frac{xx}{aa}}\partial x).$$

Quodsi ergo constans C capiatur infinita, fit $y = x$.

Scholion.

571. Si in hac aequatione ut supra ponatur $x = pp - qq$ et $y = pp + qq$, oritur $aa\partial q = ppq(p\partial p - q\partial q)$, cui satisfacit $q = 0$, unde casus $y = x$ nascitur. At facta hac transformatione difficulter patet, quomodo ejus integrale inveniri oporteat. Si quidem superiorem reductionem perpendamus, intelligemus hanc aequationem integrabilem reddi si multiplicetur per $e^{(pp-qq):aa} : q^3$, quod cum per se haud facile pateat, consultum erit hac substitutione uti $pp - qq = rr$, qua fit $pp = qq + rr$ et $p\partial p - q\partial q = r\partial r$, unde aequatio abit in $aa\partial q = qr\partial r(qq + rr)$, seu $\frac{aa\partial q}{q^3} = r\partial r + \frac{r^3\partial r}{qq}$, quae posito $\frac{1}{qq} = s$ facile integratur. Quoties ergo licet ejusmodi relationem inter variabiles colligere, quae aequationi differentiali satisfaciat, hoc modo judicari poterit, utrum ea relatio pro integrali particulari sit habenda, nec ne? Pro inventione autem hujusmodi integralium particularium regulae vix tradi possunt; quae enim habentur regulae, aeque ad integralia completa invenienda patent. Ita quae supra circa aequationes separatas observavimus, ob id ipsum quod sunt separatae, via simul ad integrale completum est patefacta. Simili modo si altera methodus per factores succedat, plerumque ex ipsis factoribus, quibus aequatio integrabilis redditur, integralia particularia concludi possunt; quaemadmodum in sequentibus propositionibus declarabimus.

Theorema.

572. Si aequatio differentialis $P\partial x + Q\partial y = 0$ per functionem M multiplicata reddatur integrabilis, integrale particulare erit $M = 0$, nisi eodem casu P vel Q abeat in infinitum.

Demonstratio.

Ponamus u esse factorem ipsius M, et ostendendum est aequationem $u = 0$ esse integrale particulare aequationis propositae, Cum u aequetur certae functioni ipsarum x et y, definiatur inde altera variabilis y, ut aequatio prodeat inter binas variabiles x et u, quae sit $R\partial x + S\partial u = 0$, unde posito multiplicatore $M = Nu$, integrabilis erit haec forma

$$NRu\partial x + NSu\partial u = 0.$$

Quodsi jam neque R neque S per u dividatur, quo casu posito $u = 0$ neque P neque Q abit in infinitum, integrale utique per u erit divisibile. Nam sive id colligatur ex termino $NRu\partial x$ spectata u ut constante, sive ex termino $NSu\partial u$ spectata x constante, integrale prodit factorem u implicans, si quidem in integratione constans omittatur. Unde concludimus integrale completum hujusmodi formam esse habiturum $V = uC$. Quare si haec constans C nihilo aequalis capiatur, integrale particulare erit $u = 0$, iis scilicet casibus exceptis, quibus functiones R et S jam ipsae per u essent divisae, ideoque ratiocinium nostrum vim suam amitteret. His ergo casibus exclusis, quoties aequatio $P\partial x + Q\partial y = 0$ per functionem M multiplicata fit per se integrabilis, eaque functio M factorem habeat u, integrale particulare erit $u = 0$, quod similiter de singulis factoribus functionis M valet.

Scholion.

573. Limitatio adjecta absolute est necessaria, cum ea neglecta universum ratiocinium claudicet. Quod quo facilius intelligatur, consideremus hanc aequationem

$$\frac{a\partial x}{y-x} + \partial y - \partial x = 0,$$

quae per $y - x$ multiplicata manifesto fit integrabilis: ponamus ergo hunc multiplicatorem $y - x = u$, seu $y = x + u$, unde nostra aequatio erit $\frac{a\partial x}{u} + \partial u = 0$, quae per u multiplicata, abit in $a\partial x + u\partial u = 0$: ubi cum pars $a\partial x$ non per u sit multiplicata, neutiquam concludere licet integrale per u fore divisibile, quippe quod est $ax + \frac{1}{2}uu$. Hinc patet, si modo pars ∂x per

u esset multiplicata, etiamsi altera pars ∂u factore u careret, tamen integrale per u divisibile fore, veluti evenit in $u\partial x + x\partial u$, cujus integrale xu utique factorem habet u. Ex quo intelligitur, si formula $Pu\partial x + Q\partial u$ fuerit per se integrabilis, dummodo Q non dividatur per u vel per potestatem ejus prima altiorem, etiam integrale, omissa scilicet constante, fore per u divisibile.

Theorema.

574. Si aequatio differentialis $P\partial x + Q\partial y = 0$ per functionem M divisa evadat per se integrabilis, integrale particulare erit $M = 0$, nisi posito $M = 0$ vel P vel Q evanescat.

Demonstratio.

Habeat divisor M factorem u, ut sit $M = Nu$, et ostendi oportet, integrale particulare futurum $u = 0$, id quod de singulis factoribus divisoris M, si quidem plures habeat, est tenendum. Cum igitur u sit functio ipsarum x et y, definiatur inde altera y per x et u, ut prodeat hujusmodi aequatio $R\partial x + S\partial u = 0$, quae ergo per Nu divisa per se erit integrabilis. Quaeri igitur oportet integrale formulae $\frac{R\partial x}{Nu} + \frac{S\partial u}{Nu}$, ubi assumimus neque R neque S per u multiplicari, neque hoc modo factorem u ex denominatore tolli. Quod si jam hoc integrale ex solo membro $\frac{R\partial x}{Nu}$ colligatur, spectando u ut constantem, prodit id $\frac{1}{u}\int\frac{R\partial x}{N} + \Phi : u$; sin autem ex altero membro $\frac{S\partial u}{Nu}$ sumta x constante colligatur, quia S non factorem habet u, id semper ita erit comparatum, ut posito $u = 0$, fiat infinitum. Ex quo integrale, quod sit V, ita erit comparatum, ut fiat $= \infty$ posito $u = 0$, quare cum integrale completum futurum sit $V = C$, huic aequationi, sumta constante C infinita, satisfit ponendo $u = 0$. Concludimus itaque, si divisor $M = Nu$ reddat aequationem differentialem $P\partial x + Q\partial y = 0$ per se integrabilem, ex quolibet divisoris M factore u obtineri integrale particulare $u = 0$, nisi forte posito $u = 0$, quantitates P et Q, vel R et S evanescant.

Corollarium 1.

575. Si aequatio $P\partial x + Q\partial y = 0$ fuerit homogenea, ea ut supra (§. 477.) vidimus integrabilis redditur, si dividatur per $Px + Qy$, quare integrale ejus particulare erit $Px + Qy = 0$. Quae aequatio cum etiam sit homogenea, factores habebit formae $\alpha x + \beta y$, quorum quisque nihilo aequatus dabit integrale particulare.

Corollarium 2.

576. Pro hac aequatione

$$y\partial x(c + nx) - \partial y(y + a + bx + nxx) = 0$$

divisorem, quo integrabilis redditur, supra §. 488. exhibuimus, unde integrale particulare concluditur $y = 0$, tum vero

$$nyy + (2na - bc)y + n(b - 2c)xy + (na + cc - bc)(a + bx + nxx) = 0,$$

cujus radices sunt

$$ny = \tfrac{1}{2}bc - na + n(c - \tfrac{1}{2}b)x \pm (c + nx)\sqrt{(\tfrac{1}{4}bb - na)}.$$

Corollarium 3.

577. Pro hac aequatione differentiali

$$\frac{n\partial x(1 + yy)\sqrt{(1 + yy)}}{\sqrt{(1 + xx)}} + (x - y)\partial y = 0$$

divisorem, quo integrabilis redditur, supra §. 489. dedimus, unde integrale particulare concludimus

$$x - y + n\sqrt{(1 + xx)(1 + yy)} = 0, \text{ seu}$$

$$yy - 2xy + xx = nn + nnxx + nnyy + nnxxyy,$$

ex quo porro fit $y = \frac{x \pm n(1 + xx)\sqrt{(1 - nn)}}{1 - nn(1 + xx)}$.

Corollarium 1.

578. Pro hac aequatione differentiali

$$\partial y + yy\partial x - \frac{a\partial x}{x^4} = 0$$

multiplicatorem supra §. 491. invenimus $\frac{xx}{xx(1-xy)^2-a}$, unde integrale particulare concludimus $xx(1-xy)^2 - a = 0$, hincque $x(1-xy) = \pm\sqrt{a}$, seu $y = \frac{1}{x} \pm \frac{\sqrt{a}}{xx}$, ita ut bina habeamus integralia particularia, quae autem imaginaria evadunt, si a fuerit quantitas negativa.

Scholion.

579. Haec fere sunt omnia, quae circa tractationem aequationum differentialium adhuc sunt explorata, nonnulla tamen subsidia evolutio aequationum differentialium secundi gradus infra suppeditabit. Huc autem commode referri possunt, quae circa comparationem certarum formularum transcendentium haud ita pridem sunt investigata. Quemadmodum enim logarithmi et arcus circulares, etsi sunt quantitates transcendentes, inter se comparari atque adeo aeque ac quantitates algebraicae in calculo tractari possunt, ita similem comparationem inter certas quantitates transcendentes altioris generis instituere licet, quae scilicet continentur in formula hac

$$\int \frac{\partial x}{\sqrt{(A + Bx + Cx^2 + Dx^3 + Ex^4)}},$$

ubi etiam numerator rationalis veluti $\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 +$ etc. addi potest. Quod argumentum cum sit maxime arduum, atque adeo vires Analyseos superare videatur, nisi certa ratione expediatur, in Analysin inde haud spernenda incrementa redundant; imprimis autem resolutio aequationum differentialium non mediocriter perfici videtur. Cum enim proposita fuerit hujusmodi aequatio

$$\frac{\partial x}{\sqrt{(A + Bx + Cx^2 + Dx^3 + Ex^4)}} = \frac{\partial y}{\sqrt{(A + By + Cy^2 + Dy^3 + Ey^4)}},$$

statim quidem patet ejus integrale particulare $x = y$, verum integrale completum maxime transcendens fore videtur, cum utraque formula per se neque ad logarithmos, neque ad arcus circulares reduci queat. Quare eo magis erit mirandum, quod integrale completum per aequationem adeo algebraicam inter x et y exhiberi possit. Quo autem methodus ad haec sublimia ducens clarius perspiciatur, eam primo ad quantitates transcendentes notas, hac formula $\int \frac{\partial x}{\sqrt{(A + Bx + Cxx)}}$ contentas applicemus, deinceps ejus usum in formulis illis magis complexis ostensuri.

CAPUT V.

DE COMPARATIONE QUANTITATUM TRANSCENDENTIUM IN FORMA $\int \frac{P\partial x}{\sqrt{(A+2Bx+Cxx)}}$ CONTENTARUM.

Problema 73.

580.

Proposita inter x et y hac aequatione algebraica

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy = 0$$

invenire formulas integrales formae praescriptae, quae inter se comparari queant.

Solutio.

Differentietur aequatio proposita, et ex ejus differentiali

$$2\beta\partial x + 2\beta\partial y + 2\gamma x\partial x + 2\gamma y\partial y + 2\delta x\partial y + 2\delta y\partial x = 0$$

colligetur haec aequatio

$$\partial x(\beta + \gamma x + \delta y) + \partial y(\beta + \gamma y + \delta x) = 0.$$

Statuatur $\beta + \gamma x + \delta y = p$ et $\beta + \gamma y + \delta x = q$, atque ex priori erit

$$pp = \beta\beta + 2\beta\gamma x + 2\beta\delta y + \gamma\gamma xx + 2\gamma\delta xy + \delta\delta yy,$$

a qua subtrahatur aequatio proposita per γ multiplicata

$$0 = \alpha\gamma + 2\beta\gamma x + 2\gamma\beta y + \gamma\gamma xx + \gamma\gamma yy + 2\gamma\delta xy,$$

fietque

$$pp = \beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)y + (\delta\delta - \gamma\gamma)yy.$$

Similique modo reperietur

$$qq = \beta\beta - \alpha\gamma + 2\beta(\delta - \gamma)x + (\delta\delta - \gamma\gamma)xx,$$

unde erit $p\partial x + q\partial y = 0$. Cum jam sit p functio ipsius y, et q similis functio ipsius x, ponatur

$$\beta\beta - \alpha\gamma = \mathrm{A},\ \beta(\delta - \gamma) = \mathrm{B},\ \text{et}\ \delta\delta - \gamma\gamma = \mathrm{C};$$

unde colligitur

$$\delta - \gamma = \frac{\mathrm{B}}{\beta}\ \text{et}\ \delta + \gamma = \frac{\mathrm{C}}{\delta - \gamma} = \frac{\beta\mathrm{C}}{\mathrm{B}},$$

hincque

$$\delta = \frac{\mathrm{BB} + \beta\beta\mathrm{C}}{2\mathrm{B}\beta}\ \text{et}\ \gamma = \frac{\beta\beta\mathrm{C} - \mathrm{BB}}{2\mathrm{B}\beta};$$

prima vero dat

$$\alpha = \frac{\beta\beta - \mathrm{A}}{\gamma} = \frac{2\mathrm{B}\beta(\beta\beta - \mathrm{A})}{\beta\beta\mathrm{C} - \mathrm{BB}}.$$

Quibus valoribus pro α, γ, δ assumtis, aequatio $\frac{\partial x}{q} + \frac{\partial y}{p} = 0$ abit in hanc

$$\frac{\partial x}{\sqrt{(\mathrm{A} + 2\mathrm{B}x + \mathrm{C}xx)}} + \frac{\partial y}{\sqrt{(\mathrm{A} + 2\mathrm{B}y + \mathrm{C}yy)}} = 0;$$

cui ergo aequationi differentiali satisfacit aequatio

$$\frac{2\mathrm{B}\beta(\beta\beta - \mathrm{A})}{\beta\beta\mathrm{C} - \mathrm{BB}} + 2\beta(x + y) + \frac{\beta\beta\mathrm{C} - \mathrm{BB}}{2\mathrm{B}\beta}(xx + yy)$$
$$+ \frac{\mathrm{BB} + \beta\beta\mathrm{C}}{\mathrm{B}\beta}xy = 0,$$

quae cum contineat constantem novam β, erit adeo integrale completum aequationis differentialis inventae.

Neque vero opus est, ut formulae illae ipsis litteris A, B, C aequentur, sed sufficit ut ipsis sint proportionales, unde fit

$$\frac{\beta\beta - \alpha\gamma}{\beta(\delta - \gamma)} = \frac{\mathrm{A}}{\mathrm{B}}\ \text{et}\ \frac{\delta + \gamma}{\beta} = \frac{\mathrm{C}}{\mathrm{B}}.$$

Ergo

$$\delta = \frac{\beta\mathrm{C}}{\mathrm{B}} - \gamma\ \text{et}\ \alpha = \frac{\beta\beta}{\gamma} - \frac{\beta\mathrm{A}}{\gamma\mathrm{B}}(\delta - \gamma),\ \text{seu}$$
$$\alpha = \frac{\beta\beta}{\gamma} - \frac{\beta\beta\mathrm{AC}}{\gamma\mathrm{BB}} + \frac{2\beta\mathrm{A}}{\mathrm{B}}.$$

Quare aequationis differentialis

$$\frac{\partial x}{\sqrt{(A+2Bx+Cxx)}}+\frac{\partial y}{\sqrt{(A+2By+Cyy)}}=0$$

integrale completum est

$$\beta\beta(CB-AC)+2\beta\gamma AB+2\beta\gamma BB(x+y)+\gamma\gamma BB(xx+yy)$$
$$+2\gamma B(\beta C-\gamma B)xy=0,$$

ubi ratio $\frac{\beta}{\gamma}$ constantem arbitrariam exhibet.

Corollarium 1.

581. Ex aequatione proposita radicem extrahendo fit

$$y=\frac{-\beta-\delta x+\sqrt{(\beta\beta+2\beta\delta x+\delta\delta xx-\alpha\gamma-2\beta\gamma x-\gamma\gamma xx)}}{\gamma},$$

seu loco α et δ substitutis valoribus,

$$y=-\frac{\beta}{\gamma}-\frac{(\beta C-\gamma B)}{\gamma B}x+\sqrt{\left(\frac{\beta\beta C-2\beta\gamma B}{\gamma\gamma BB}\right)(A+2Bx+Cxx)}.$$

Corollarium 2.

582. Si ergo $x=0$, fit

$$y=-\frac{\beta}{\gamma}+\sqrt{\frac{\beta\beta AC-2\beta\gamma AB}{\gamma\gamma BB}},$$

ponatur hic valor $=a$, ut sit

$$\gamma Ba+\beta B=\sqrt{(\beta\beta AC-2\beta\gamma AB)};$$

unde sumtis quadratis oritur

$$\gamma\gamma BBaa+2\beta\gamma BBa+\beta\beta BB=\beta\beta AC-2\beta\gamma AB,$$

hincque

$$\frac{\gamma}{\beta}=\frac{-A-Ba+\sqrt{A(A+2Ba+Caa)}}{Baa},\ \text{seu}$$
$$\frac{\beta}{\gamma}=\frac{B(A+Ba+\sqrt{A(A+2Ba+Caa)}}{AC-BB}.$$

Scholion 1.

283. Ut aequatio assumta

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy=0$$

satisfaciat aequationi differentiali

$$\frac{\partial x}{\sqrt{(A+2Bx+Cxx)}}+\frac{\partial y}{\sqrt{(A+2By+Cyy)}}=0,$$

necesse est ut sit

$$\beta\beta-\alpha\gamma=mA,\ \beta(\delta-\gamma)=mB \text{ et } \delta\delta-\gamma\gamma=mC,$$

unde fit

$$\beta+\gamma y+\delta x=\sqrt{m(A+2Bx+Cxx)} \text{ et}$$
$$\beta+\gamma x+\delta y=\sqrt{m(A+2By+Cyy)}.$$

At ex datis A, B, C, litterarum α, β, γ, δ et m tres tantum definiuntur; quare cum binae maneant indeterminatae, aequatio assumta, etiamsi per quemvis coëfficientium dividatur, unam tamen constantem continet novam, ex quo ea pro integrali completo erit habenda. Quare etsi aequationis differentialis neutra pars integrationem algebraice admittit, tamen integrale completum algebraice exhiberi potest. Loco constantis arbitrariae is valor ipsius y introduci potest, quem recipit posito $x=0$: cum autem evenire possit, ut hic valor fiat imaginarius, conveniet istam constantem ita definiri, ut posito $x=a$ fiat $y=b$, quo pacto ad omnes casus applicatio fieri poterit. Hinc erit

$$\frac{\beta+\gamma b+\delta a}{\beta+\gamma a+\delta b}=\sqrt{\frac{A+2Ba+Caa}{A+2Bb+Cbb}},$$

unde colligitur

$$\beta=\frac{(\gamma a+\delta b)\sqrt{(A+2Ba+Caa)}-(\gamma b+\delta a)\sqrt{(A+2Bb+Cbb)}}{-\sqrt{(A+2Ba+Caa)}+\sqrt{(A+2Bb+Cbb)}} \text{ et}$$

$$\sqrt{m(A+2Ba+Caa)}=\frac{(\delta-\gamma)(b-a)\sqrt{(A+2Ba+Caa)}}{\sqrt{(A+2Bb+Cbb)}-\sqrt{(A+2Ba+Caa)}}$$

seu

$$\sqrt{m}=\frac{(\delta-\gamma)(b-a)}{\sqrt{(A+2Bb+Cbb)}-\sqrt{(A+2Ba+Caa)}}.$$

Ponatur brevitatis gratia

$$\sqrt{(A+2Ba+Caa)}=\mathfrak{A} \text{ et } \sqrt{(A+2Bb+Cbb)}=\mathfrak{B},$$

ut sit

$$\sqrt{m}=\frac{(\delta-\gamma)(b-a)}{\mathfrak{B}-\mathfrak{A}} \text{ et}$$

$$\beta=\frac{\mathfrak{A}(\gamma a+\delta b)-\mathfrak{B}(\gamma b+\delta a)}{\mathfrak{B}-\mathfrak{A}},$$

aequatio $\beta(\delta-\gamma)=m\mathrm{B}$ induet hanc formam

$$\mathfrak{A}(\gamma a+\delta b)-\mathfrak{B}(\gamma b+\delta a)=\frac{\mathrm{B}(\delta-\gamma)(b-a)^2}{\mathfrak{B}-\mathfrak{A}}$$

e fit

$$\left.\begin{array}{l}+\gamma\mathfrak{A}\mathfrak{B}-\gamma\mathrm{A}-\gamma\mathrm{B}(a+b)-\gamma\mathrm{C}(aa-ab+bb)\\+\delta\mathfrak{A}\mathfrak{B}-\delta\mathrm{A}-\delta\mathrm{B}(a+b)-\delta\mathrm{C}ab\end{array}\right\}=0.$$

tuatur ergo

$$\gamma=n\mathfrak{A}\mathfrak{B}-n\mathrm{A}-n\mathrm{B}(a+b)-n\mathrm{C}ab$$

$$\delta=n\mathrm{A}+n\mathrm{B}(a+b)+n\mathrm{C}(aa-ab+bb)-n\mathfrak{A}\mathfrak{B}$$

$$\sqrt{m}=\frac{n(b-a)(\mathfrak{A}^2+\mathfrak{B}^2-2\mathfrak{A}\mathfrak{B})}{\mathfrak{B}-\mathfrak{A}}=n(b-a)(\mathfrak{B}-\mathfrak{A})$$

$$\beta=n\mathrm{B}(b-a)^2,\text{ ergo }\delta-\gamma=\frac{m}{n(b-a)^2},$$

le cum sit $\delta+\gamma=n\mathrm{C}(b-a)^2$, erit utique $\delta\delta-\gamma\gamma=m\mathrm{C}$.
ɔerest ut fiat $\alpha\gamma=\beta\beta-m\mathrm{A}$, hoc est

$$\alpha\gamma=nn\mathrm{B}\mathrm{B}(b-a)^4-nn\mathrm{A}(b-a)^2(\mathfrak{B}-\mathfrak{A})^2\text{ seu}$$

$$\alpha\gamma=nn(b-a)^2[\mathrm{B}\mathrm{B}(b-a)^2-\mathrm{A}(\mathfrak{B}-\mathfrak{A})^2].$$

l cum posito $x=a$ fiat $y=b$, erit quoque

$$\alpha=-2\beta(a+b)-\gamma(aa+bb)-2\delta ab,$$

ıcque

$$\alpha=n(a-b)^2[\mathrm{A}-\mathrm{B}(a+b)-\mathrm{C}ab-\mathfrak{A}\mathfrak{B}];$$

de aequatio nostra assumta est

$$\begin{array}{l}(b-a)^2[\mathrm{A}-\mathrm{B}(a+b)-\mathrm{C}ab-\mathfrak{A}\mathfrak{B}]+2\mathrm{B}(b-a)^2(x+y)\\\quad-[\mathrm{A}+\mathrm{B}(a+b)+\mathrm{C}ab-\mathfrak{A}\mathfrak{B}](xx+yy)\\\quad\quad+2[\mathrm{A}+\mathrm{B}(a+b)+\mathrm{C}(aa-ab+bb)-\mathfrak{A}\mathfrak{B}]xy=0.\end{array}$$

Scholion 2.

184. Si ponatur $\beta=0$, ut aequatio sit

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0, \text{ erit}$$

$$y = \frac{-\delta x + \sqrt{(-\alpha\gamma + (\delta\delta - \gamma\gamma)xx)}}{\gamma}.$$

Posito ergo $-\alpha\gamma = m\mathrm{A}$ et $\delta\delta - \gamma\gamma = m\mathrm{C}$, ut sit

$$\gamma y + \delta x = \sqrt{m(\mathrm{A} + \mathrm{C}xx)}, \text{ erit}$$

$$\frac{\partial x}{\sqrt{(\mathrm{A} + \mathrm{C}xx)}} + \frac{\partial y}{\sqrt{(\mathrm{A} + \mathrm{C}yy)}} = 0,$$

cujus aequationis integrale completum erit ipsa aequatio assumta, pro qua habebitur $\frac{\mathrm{C}}{\mathrm{A}} = \frac{\gamma\gamma - \delta\delta}{\alpha\gamma}$, seu $\delta = \sqrt{(\gamma\gamma - \frac{\alpha\gamma\mathrm{C}}{\mathrm{A}})}$. Sin autem posito $x = 0$ fieri debeat $y = b$, ob $\gamma b = \sqrt{m\mathrm{A}}$, erit $\gamma = \frac{\sqrt{m\mathrm{A}}}{b}$; tum $\alpha = -b\sqrt{m\mathrm{A}}$ et $\delta = \sqrt{(\frac{m\mathrm{A}}{bb} + m\mathrm{C})}$. Habebitur ergo haec aequatio

$$\frac{y\sqrt{m\mathrm{A}}}{b} + \frac{x\sqrt{m(\mathrm{A} + \mathrm{C}bb)}}{\mathrm{A}} = \sqrt{m(\mathrm{A} + \mathrm{C}xx)},$$

quae praebet

$$y = -x\sqrt{\frac{\mathrm{A} + \mathrm{C}bb}{\mathrm{A}}} + b\sqrt{\frac{\mathrm{A} + \mathrm{C}xx}{\mathrm{A}}},$$

quae est integrale completum aequationis illius differentialis. Quare si x capiatur negative, hujus aequationis differentialis

$$\frac{\partial x}{\sqrt{(\mathrm{A} + \mathrm{C}xx)}} = \frac{\partial y}{\sqrt{(\mathrm{A} + \mathrm{C}yy)}},$$

integrale completum est

$$y = x\sqrt{\frac{\mathrm{A} + \mathrm{C}bb}{\mathrm{A}}} + b\sqrt{\frac{\mathrm{A} + \mathrm{C}xx}{\mathrm{A}}}.$$

Quodsi simili modo calculus in genere tractetur, aequationis differentialis

$$\frac{\partial x}{\sqrt{(\mathrm{A} + 2\mathrm{B}x + \mathrm{C}xx)}} + \frac{\partial y}{\sqrt{(\mathrm{A} + 2\mathrm{B}y + \mathrm{C}yy)}} = 0,$$

si brevitatis gratia ponatur $\sqrt{(\mathrm{A} + 2\mathrm{B}b + \mathrm{C}bb)} = \mathfrak{B}$, erit integrale completum

$$y\left(\sqrt{\mathrm{A}} + \frac{\mathrm{B}b}{\sqrt{\mathrm{A}} - \mathfrak{B}}\right) + x\left(\mathfrak{B} + \frac{\mathrm{B}b}{\sqrt{\mathrm{A}} - \mathfrak{B}}\right)$$

$$= \frac{\mathrm{B}bb}{\sqrt{\mathrm{A}} - \mathfrak{B}} + b\sqrt{(\mathrm{A} + 2\mathrm{B}x + \mathrm{C}xx)};$$

unde casus praecedens manifesto sequitur, si ponatur $\mathrm{B} = 0$.

Verum ope levis substitutionis hae formulae, ubi adest B, ad illum casum ubi $B = 0$ reduci possunt.

Problema 74.

585. Si $\Pi : z$ significet eam functionem ipsius z, quae oritur ex integratione formulae $\int \frac{\partial z}{\sqrt{(1 + Czz)}}$, integrale hoc ita sumto, ut evanescat posito $z = 0$, comparationem inter hujusmodi functiones instituere.

Solutio.

Consideretur haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(A + Cxx)}} = \frac{\partial y}{\sqrt{(A + Cyy)}}$$

unde cum sit per hypothesin

$$\int \frac{\partial x}{\sqrt{(A + Cxx)}} = \Pi : x \text{ et } \int \frac{\partial y}{\sqrt{(A + Cyy)}} = \Pi : y,$$

utroque integrali ita sumto, ut evanescat illud posito $x = 0$, hoc vero posito $y = 0$, integrale completum erit

$$\Pi : y = \Pi : x + C.$$

Ante autem vidimus, hoc integrale esse

$$y = x \sqrt{\frac{A + Cbb}{A}} + b \sqrt{\frac{A + Cxx}{A}},$$

ubi posito $x = 0$ fit $y = b$, quare cum $\Pi : 0 = 0$, erit

$$\Pi : y = \Pi : x + \Pi : b;$$

cui ergo aequationi transcendentali satisfacit haec algebraica

$$y = x \sqrt{\frac{A + Cbb}{A}} + b \sqrt{\frac{A + Cxx}{A}}.$$

Simili modo sumto b negative, haec aequatio

$$\Pi : y = \Pi : x - \Pi : b$$

convenit cum hac

$$y = x \sqrt{\frac{A + Cbb}{A}} - b \sqrt{\frac{A + Cxx}{A}},$$

sicque tam summa, quam differentia duarum hujusmodi functionum

es

per similem functionem exprimi potest. Hic jam nullo habito discrimine inter quantitates variabiles et constantes, dum $\Pi:z$ functionem determinatam ipsius z significat, scilicet

$$\Pi:z=\int\frac{\partial z}{\sqrt{(A+Czz)}},$$

quae ut assumsimus evanescat posito $z=0$, ut hoc signandi modo recepto sit

$$\Pi:r=\Pi:p+\Pi:q,$$

debet esse

$$r=p\sqrt{\frac{A+Cqq}{A}}+q\sqrt{\frac{A+Cpp}{A}};$$

ut vero sit

$$\Pi:r=\Pi:p-\Pi:q,$$

debet esse

$$r=p\sqrt{\frac{A+Cqq}{A}}-q\sqrt{\frac{A+Cpp}{A}},$$

utrinque autem sublata irrationalitate prodit inter p, q, r haec aequatio

$$p^4+q^4+r^4-2ppqq-2pprr-2qqrr=\frac{4Cppqqrr}{A},$$

cujus forma hanc suppeditat proprietatem, ut si p, q, r sint latera cujusdam trianguli, eique circumscribatur circulus, cujus diameter vocetur $=T$, semper sit $A+CTT=0$. Illa autem aequatio ob plures quas complectitur radices, satisfacit huic relationi

$$\Pi:p\pm\Pi:q\pm\Pi:r=0.$$

Corollarium 1.

586. Hinc statim deducitur nota arcuum circularium comparatio, ponendo $A=1$ et $C=-1$. Tum enim fit

$$\Pi:z=\int\frac{\partial z}{\sqrt{(1-zz)}}=\text{Ang. sin. } z,$$

hincque ut sit

$$\text{Ang. sin. } r = \text{Ang. sin. } p + \text{Ang. sin. } q,$$

portet esse

$$r = p\sqrt{(1-qq)} + q\sqrt{(1-pp)},$$

: ut sit

$$\text{Ang. sin. } r = \text{Ang. sin } p - \text{Ang. sin. } q,$$

:bet esse

$$r = p\sqrt{(1-qq)} - q\sqrt{(1-pp)},$$

i constat.

Corollarium 2.

587. Si sit $A = 1$ et $C = 1$, erit

$$\Pi : z = \int \frac{\partial z}{\sqrt{(1+zz)}} = l[z + \sqrt{(1+zz)}],$$

ıde ut sit

$$l[r+\sqrt{(1+rr)}] = l[p+\sqrt{(1+pp)}] + l[q+\sqrt{(1+qq)}],$$

it

$$r = p\sqrt{(1+qq)} + q\sqrt{(1+pp)};$$

: autem sit

$$l[r+\sqrt{(1+rr)}] = l[p+\sqrt{(1+pp)}] - l[q+\sqrt{(1+qq)}],$$

it

$$r = p\sqrt{(1+qq)} - q\sqrt{(1+pp)},$$

i ex indole logarithmorum sponte liquet.

Corollarium 3.

588. Si ponamus in priori formula generali $q = p$; ut sit

$\Pi : r = 2\,\Pi : p$, erit

$$r = 2p\sqrt{\frac{A+Cpp}{A}}.$$

Iinc porro si fiat

$q = 2p\sqrt{\frac{A+Cpp}{A}}$, erit

$$\Pi : r = \Pi : p + 2\,\Pi : p = 3\,\Pi : p,$$

sumto

$$r = p\sqrt{\frac{A+Cqq}{A}} + q\sqrt{\frac{A+Cpp}{A}}.$$

Est vero

$$\sqrt{\frac{A+Cqq}{A}} = \sqrt{[1+\frac{4Cpp}{A}(1+\frac{Cpp}{A})]} = 1+\frac{2Cpp}{A},$$

unde ut sit

$$\Pi : r = 3\,\Pi : p \text{ fit}$$

$$r = p(1+\frac{2Cpp}{A}) + 2p(1+\frac{Cpp}{A}) = 3p + \frac{4Cp^3}{A}.$$

Scholion.

589. Quo haec multiplicatio facilius continuari queat, praeter relationem aequationi

$$\Pi : r = \Pi : p + \Pi : q$$

respondentem, quae est

$$r = p\sqrt{\frac{A+Cqq}{A}} + q\sqrt{\frac{A+Cpp}{A}},$$

notetur aequatio

$$\Pi : p = \Pi : r - \Pi : q,$$

cui respondet relatio

$$p = r\sqrt{\frac{A+Cqq}{A}} - q\sqrt{\frac{A+Crr}{A}}; \text{ unde fit}$$

$$\sqrt{\frac{A+Crr}{A}} = \frac{r}{q}\sqrt{\frac{A+Cqq}{A}} - \frac{p}{q} = \frac{p}{q}\left(\frac{A+Cqq}{A}\right) + \sqrt{\left(\frac{A+Cpp}{A}\right)\left(\frac{A+Cqq}{A}\right)} - \frac{p}{q}, \text{ seu}$$

$$\sqrt{\frac{A+Crr}{A}} = \frac{Cpq}{A} + \sqrt{\left(\frac{A+Cpp}{A}\right)\left(\frac{A+Cqq}{A}\right)}.$$

Quare ut sit

$$\Pi : r = \Pi : p + \Pi : q,$$

habemus non solum

$$r = p\sqrt{(1+\frac{C}{A}qq)} + q\sqrt{(1+\frac{C}{A}pp)},$$

sed etiam

$$\sqrt{(1+\frac{C}{A}rr)} = \frac{C}{A}pq + \sqrt{(1+\frac{C}{A}pp)(1+\frac{C}{A}qq)}.$$

nus brevitatis gratia $\sqrt{(1+\frac{C}{A}pp)}=P$, et sumto $q=p$ ut sit

$$\Pi:r=2\,\Pi:p,\ \text{erit}$$

$$r=2\,Pp \text{ et } \sqrt{(1+\frac{C}{A}rr)}=\frac{C}{A}pp+PP,$$

alor ipsius r pro q sumtus dabit

$$\Pi:r=3\,\Pi:p,$$

nte

$$r=\frac{C}{A}p^3+3\,PPp, \text{ et}$$

$$\sqrt{(1+\frac{C}{A}rr)}=\frac{3C}{A}Ppp+P^3.$$

valor ipsius r denuo pro q sumtus, dabit

$$\Pi:r=4\,\Pi:p,$$

nte

$$r=\frac{4C}{A}Pp^3+4\,P^3p, \text{ et}$$

$$\sqrt{(1+\frac{C}{A}rr)}=\frac{CC}{AA}p^4+\frac{6C}{A}PPpp+P^4.$$

q substituatur hic valor ipsius r, ut prodeat

$$\Pi:r=5\,\Pi:p,$$

nte

$$r=\frac{CC}{AA}p^5+\frac{10C}{A}PPp^3+5\,P^4p, \text{ et}$$

$$\sqrt{(1+\frac{C}{A}rr)}=\frac{5CC}{AA}Pp^4+\frac{10C}{A}P^3pp+P^5.$$

: hinc generatim concludere licet, ut sit

$$\Pi:r=n\,\Pi:p,$$

debere

$$r\sqrt{\frac{C}{A}}=\tfrac{1}{2}(P+p\sqrt{\tfrac{C}{A}})^n-\tfrac{1}{2}(P-p\sqrt{\tfrac{C}{A}})^n, \text{ et}$$

$$\sqrt{(1+\frac{C}{A}rr)}=\tfrac{1}{2}(P+p\sqrt{\tfrac{C}{A}})^n+\tfrac{1}{2}(P-p\sqrt{\tfrac{C}{A}})^n, \text{ seu}$$

$$r=\frac{\sqrt{A}}{2\sqrt{C}}(P+p\sqrt{\tfrac{C}{A}})^n-\frac{\sqrt{A}}{2\sqrt{C}}(P-p\sqrt{\tfrac{C}{A}})^n.$$

igitur relatio inter p et r satisfaciet huic aequationi differentiali

$$\frac{\partial r}{\sqrt{(A+Crr)}}=\frac{n\,\partial p}{\sqrt{(A+Cpp)}},$$

dum meminerimus esse $P = \sqrt{(1 + \frac{C\phi\phi}{A})}$.

Problema 76.

590. Si ponatur $\int \frac{\partial z}{\sqrt{(A + Czz)}} = \Pi : z$, integrali ita sumto ut evanescat posito $z = f$, unde $\Pi : z$ fit functio determinata ipsius x, comparationem inter hujusmodi functiones instituere.

Solutio.

Consideretur haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(A + Cxx)}} + \frac{\partial y}{\sqrt{(A + Cyy)}} = 0,$$

unde integrando fit

$$\Pi : x + \Pi : y = \text{Const.}$$

Integrale autem sit quoque

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0,$$

quod ut locum habeat necesse est, sit

$$-\alpha\gamma = Am, \text{ et } \delta\delta - \gamma\gamma = Cm:$$

tum vero erit

$$\gamma x + \delta y = \sqrt{m(A + Cyy)}, \text{ et } \gamma y + \delta x = \sqrt{m(A + Cxx)}.$$

Ponamus constantem integratione ingressam ita definiri, ut posito $x = a$ fiat $y = b$, et integrale erit

$$\Pi : x + \Pi : y = \Pi : a + \Pi : b.$$

Pro forma autem algebraica invenienda, sit brevitatis gratia

$$\sqrt{(A + Caa)} = \mathfrak{A} \text{ et } \sqrt{(A + Cbb)} = \mathfrak{B},$$

eritque

$$\gamma a + \delta b = \mathfrak{B}\sqrt{m} \text{ et } \gamma b + \delta a = \mathfrak{A}\sqrt{m};$$

unde colligitur

$$\gamma = \frac{\mathfrak{A}b - \mathfrak{B}a}{bb - aa}\sqrt{m} \text{ et } \delta = \frac{\mathfrak{B}b - \mathfrak{A}a}{bb - aa}\sqrt{m}.$$

Quocirca aequatio integralis algebraica erit

$$(\mathfrak{A}b - \mathfrak{B}a)x + (\mathfrak{B}b - \mathfrak{A}a)y = (bb - aa)\sqrt{(A + Cyy)}$$

seu

$$(\mathfrak{A}b - \mathfrak{B}a)y + (\mathfrak{B}b - \mathfrak{A}a)x = (bb - aa)\sqrt{(A + Cxx)}.$$

Hinc y per x ita definitur, ut sit

$$y = \frac{(\mathfrak{A}a - \mathfrak{B}b)x + (bb - aa)\sqrt{(A + Cxx)}}{\mathfrak{A}b - \mathfrak{B}a},$$

quae fractio supra et infra per $\mathfrak{A}b + \mathfrak{B}a$ multiplicando, ob

$$\mathfrak{A}\mathfrak{A}bb - \mathfrak{B}\mathfrak{B}aa = A(bb - aa) \text{ et}$$

$$(\mathfrak{A}a - \mathfrak{B}b)(\mathfrak{A}b + \mathfrak{B}a) = (\mathfrak{A}\mathfrak{A} - \mathfrak{B}\mathfrak{B})ab - \mathfrak{A}\mathfrak{B}(bb - aa) = -(bb - aa)(Cab + \mathfrak{A}\mathfrak{B}),$$

abit in

$$y = -\frac{(Cab + \mathfrak{A}\mathfrak{B})x}{A} + \frac{(\mathfrak{A}b + \mathfrak{B}a)\sqrt{(A + Cxx)}}{A}.$$

Hinc porro colligitur

$$(bb - aa)\sqrt{(A + Cyy)} = (\mathfrak{A}b - \mathfrak{B}a)x - \frac{(\mathfrak{B}b - \mathfrak{A}a)^2 x}{\mathfrak{A}b - \mathfrak{B}a} + \frac{(\mathfrak{B}b - \mathfrak{A}a)(bb - aa)}{\mathfrak{A}b - \mathfrak{B}a}\sqrt{(A + Cxx)},$$

seu

$$\sqrt{(A + Cyy)} = -\frac{C(bb - aa)}{\mathfrak{A}b - \mathfrak{B}a}x + \frac{\mathfrak{B}b - \mathfrak{A}a}{\mathfrak{A}b - \mathfrak{B}a}\sqrt{(A + Cxx)};$$

ubi iterum supra et infra multiplicando per $\mathfrak{A}b + \mathfrak{B}a$, fit

$$\sqrt{(A + Cyy)} = -\frac{C(\mathfrak{A}b + \mathfrak{B}a)}{A}x + \frac{(Cab + \mathfrak{A}\mathfrak{B})}{A}\sqrt{(A + Cxx)}.$$

Necesse autem est valorem formulae $\sqrt{(A + Cyy)}$ hoc modo potius definiri quam extractione radicis, qua ambiguitas implicaretur. Quocirca haec aequatio transcendens

$$\Pi : r + \Pi : s = \Pi : p + \Pi : q$$

praebet sequentem determinationem algebraicam, si quidem brevitatis gratia ponamus $\sqrt{(A + Cpp)} = P$, $\sqrt{(A + Cqq)} = Q$ et $\sqrt{(A + Crr)} = R$, scilicet ut sit

$\Pi : s = \Pi : p + \Pi : q - \Pi : r$, erit

$$s = \frac{-PQr - Cpqr + PRq + QRp}{A} \text{ et}$$

$$\surd(A + Css) = \frac{-CPqr - CQpr + CRpq + PQR}{A}, \text{ seu}$$

$$\surd(A + Css) = \frac{PQR + C(Rpq - Pqr - Qpr)}{A}.$$

Corollarium 1.

591. Quoniam est per hypothesin $\Pi : f = 0$, si ponamus brevitatis gratia $\surd(A + Cff) = F$, et $r = f$, ut sit $R = F$, hae aequatio

$$\Pi : s = \Pi : p + \Pi : q$$

praebet

$$s = \frac{F(Pq + Qp) - PQf - Cfpq}{A}, \text{ et}$$

$$\surd(A + Css) = \frac{FPQ + CFpq - Cf(Pq + Qp)}{A}.$$

Corollarium 2.

592. Si ponamus $q = f$ et $Q = F$, ut sit $\Pi : q = 0$, haec aequatio

$$\Pi : s = \Pi : p - \Pi : r$$

praebet

$$s = \frac{F(Rp - Pr) + fPR - Cfpr}{A} \text{ et}$$

$$\surd(A + Css) = \frac{FPR - CFpr + Cf(Rp - Pr)}{A}.$$

Corollarium 3.

593. Si sit $C = 0$ et $A = 1$, erit

$$\Pi : z = \int \partial z = z - f,$$

quia integrale ita capi debet, ut evanescat posito $x = f$. Tum ergo erit $P = 1$, $Q = 1$ et $R = 1$; unde ut sit

$$\Pi:s = \Pi:p + \Pi:q - \Pi:r,$$

i $s = p + q - r$, oportet esse

$$s = -r + q + p \text{ et } \sqrt{(1 + 0ss)} = 1,$$

per se constat.

Corollarium 4.

594. Si sumatur $A = 1$ et $C = -1$, fiatque $\Pi:z = \text{Ang. cos.}$ ut sit $f = 1$, erit

$$\text{Arc. cos. } s = \text{Arc. cos. } p + \text{Arc. cos. } q - \text{Arc. cos. } r,$$

fuerit

$$s = pqr - PQr + PRq + QRp \text{ et}$$

$$\sqrt{(1 - ss)} = PQR + Pqr + Qpr - Rpq,$$

le sumto $r = 1$, ut sit $R = 0$, et $\text{Arc. cos. } r = 0$, erit $s = pq - PQ$ $\sqrt{(1 - ss)} = Pq + Qp$.

Scholion.

595. Hinc notae regulae pro cosinibus deducuntur, quas ius non prosequor. Verum casus facillimus, quo $A = 0$ et $C = 1$, cque fit $\Pi:z = \int\frac{\partial z}{z} = lz$, existente $f = 1$, insigni difficultate premi etur, ob expressiones pro s et $\sqrt{(A + Czz)} = z$ in infinitum abe- es. Cui incommodo ut occurratur, primo quidem numerus A ut nite parvus spectetur, eritque

$$P = \sqrt{(pp + A)} = p + \frac{A}{2p}, \; Q = q + \frac{A}{2q}, \; R = r + \frac{A}{2r}.$$

are ut fiat $ls = lp + lq - lr$, reperitur

$$As = -r\left(p + \frac{A}{2p}\right)\left(q + \frac{A}{2q}\right) - pqr$$
$$+ q\left(p + \frac{A}{2p}\right)\left(r + \frac{A}{2r}\right) + p\left(q + \frac{A}{2q}\right)\left(r + \frac{A}{2r}\right);$$

singulis membris evolutis

$$As = -\frac{Aqr}{2p} - \frac{Apr}{2q} + \frac{Aqr}{2p} + \frac{Apq}{2r} + \frac{Apr}{2q} + \frac{Apq}{2r}$$

seu $s = \frac{pq}{r}$, uti natura logarithmorum exigit. Caeterum ex formulis inventis haud difficulter multiplicatio hujusmodi functionum transcendentium colligitur, veluti ut sit $\Pi : y = n\Pi : x$, relatio inter x et y algebraice assignari poterit.

Problema 76.

596. Si ponatur $\Pi : z = \int \frac{\partial z (L + Mzz)}{\sqrt{(A + Czz)}}$, sumto hoc integrali ita ut evanescat posito $z = 0$, comparationem inter hujusmodi functiones transcendentes investigare.

Solutio.

Statuatur inter binas variabiles x et y ista relatio

$$\alpha + \gamma (xx + yy) + 2\delta xy = 0,$$

unde fit

$$y = \frac{-\delta x + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma) xx]}}{\gamma}.$$

Ponatur $-\alpha\gamma = Am$ et $\delta\delta - \gamma\gamma = Cm$, ut sit

$$\gamma y + \delta x = \sqrt{m(A + Cxx)} \text{ et}$$

$$\gamma x + \delta y = \sqrt{m(A + Cyy)}.$$

At illam aequationem differentiando fit

$$\partial x (\gamma x + \delta y) + \partial y (\gamma y + \delta x) = 0, \text{ seu}$$

$$\frac{\partial x}{\sqrt{(A + Cxx)}} + \frac{\partial y}{\sqrt{(A + Cyy)}} = 0.$$

Jam statuatur

$$\frac{\partial x (L + Mxx)}{\sqrt{(A + Cxx)}} + \frac{\partial y (L + Myy)}{\sqrt{(A + Cyy)}} = \partial V \sqrt{m},$$

ut sit integrando

$$\Pi : x + \Pi : y = \text{Const.} + V\sqrt{m}.$$

Cum igitur sit

$$\frac{\partial y}{\sqrt{(A + Cyy)}} = \frac{-\partial x}{\sqrt{(A + Cxx)}}, \text{ erit}$$

$$\partial V \sqrt{m} = \frac{M \partial x (xx - yy)}{\sqrt{(A + Cxx)}},$$

:que ob

$y = \frac{\sqrt{m(A+Cxx)} - \delta x}{\gamma}$, erit

$$-yy = \frac{1}{\gamma\gamma}(\gamma\gamma xx - mA - mCxx - \delta\delta xx + 2\delta x \sqrt{m(A + Cxx)}).$$

$\gamma\gamma - \delta\delta = -mC$, ergo

$$\partial V \sqrt{m} = \frac{M \partial x (2\delta x \sqrt{m(A+Cxx)} - mA - 2mCxx)}{\gamma\gamma \sqrt{(A+Cxx)}},$$

ıs integrale commode capi potest, dum fit

$$V \sqrt{m} = \frac{\delta M xx \sqrt{m}}{\gamma\gamma} - \frac{Mmx}{\gamma\gamma} \sqrt{(A + Cxx)},$$

.e formula ob

$\sqrt{m(A + Cxx)} = \gamma y + \delta x$, abit in

$$V \sqrt{m} = \frac{\delta M xx - \gamma M xy - \delta M xx}{\gamma\gamma} \sqrt{m} = -\frac{Mxy}{\gamma} \sqrt{m}.$$

ɔcirca habebimus

$$\Pi : x + \Pi : y = \text{Const.} - \frac{Mxy}{\gamma} \sqrt{m},$$

stente

$$\gamma y + \delta x = \sqrt{m(A + Cxx)} \text{ et } \gamma x + \delta y = \sqrt{m(A + Cyy)},$$

praeterea

$$-\alpha\gamma = Am \text{ et } \delta\delta - \gamma\gamma = Cm.$$

constantem definiendam sumamus, posito $x = 0$ fieri $y = b$, sit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mxy}{\gamma} \sqrt{m}.$$

n vero est

$$\gamma b = \sqrt{mA} \text{ et } \delta b = \sqrt{(mA + mCbb)},$$

o

$$\gamma = \frac{\sqrt{mA}}{b} \text{ et } \delta = \frac{\sqrt{(mA + mCbb)}}{b}$$

ıc ergo concludimus, si fuerit

$$y \sqrt{A} + x \sqrt{(A + Cbb)} = b \sqrt{(A + Cxx)},$$

quod eodem redit

$x \sqrt{A} + y \sqrt{(A + Cbb)} = b \sqrt{(A + Cyy)}$, fore

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}};$$

denotante Π ejusmodi functionem quantitatis suffixae, ut sit

$$\Pi : z = \int \frac{\partial z (L + Mzz)}{\sqrt{(A + Czz)}},$$

integrali hoc ita sumto, ut evanescat posito $z = 0$. Natura harum functionum stabilita, ac sublato discrimine inter quantitates constantes ac variabiles, erit

$$\Pi : r = \Pi : p + \Pi : q + \frac{Mpqr}{\sqrt{A}},$$

si fuerit

$$q\sqrt{A} + p\sqrt{(A + Crr)} = r\sqrt{(A + Cpp)} \text{ et}$$
$$p\sqrt{A} + q\sqrt{(A + Crr)} = r\sqrt{(A + Cqq)}$$

unde fit

$$r = \frac{p\sqrt{(A + Cqq)} + q\sqrt{(A + Cpp)}}{\sqrt{A}} \text{ et}$$
$$\sqrt{(A + Crr)} = \frac{Cpq + \sqrt{(A + Cpp)(A + Cqq)}}{\sqrt{A}}.$$

Corollarium 1.

597. Sumto z negativo est

$$\Pi : -z = -\Pi : z,$$

unde capiendo quantitates p et q negative, fiet

$$\Pi : p + \Pi : q + \Pi : r = \frac{Mpqr}{\sqrt{A}},$$

si fuerit

$$p\sqrt{A} + q\sqrt{(A + Crr)} + r\sqrt{(A + Cqq)} = 0 \text{ seu}$$
$$q\sqrt{A} + p\sqrt{(A + Crr)} + r\sqrt{(A + Cpp)} = 0 \text{ seu}$$
$$r\sqrt{A} + p\sqrt{(A + Cqq)} + q\sqrt{(A + Cpp)} = 0 \text{ vel}$$
$$Cpq - \sqrt{A(A + Crr)} + \sqrt{(A + Cpp)(A + Cqq)} = 0$$

ex qua formatur haec relatio

$$Cpqr + p\sqrt{(A + Cqq)(A + Crr)} + q\sqrt{(A + Cpp)(A + Crr)}$$
$$+ r\sqrt{(A + Cpp)(A + Cqq)} = 0.$$

Corollarium 2.

598. Hac ergo methodo tres hujusmodi functiones $\Pi : z$ exhiberi possunt, quarum summam algebraice exprimere licet; quod autem de summa ostendimus, valet quoque de summa binarum lemta tertia.

Corollarium 3.

599. Si ponamus $L = A$ et $M = C$, functio proposita $\Pi : z = \int \partial z \sqrt{(A + Czz)}$, exprimit aream curvae, cujus abscissae z convenit applicata $\sqrt{(A + Czz)}$; et summa trium hujusmodi arearum ita algebraice dabitur:

$$\Pi : p + \Pi : q + \Pi : r = \frac{Cpqr}{\sqrt{A}}$$

si inter p, q, r superior relatio statuatur.

Scholion.

600. Haec proprietas inde est nata, quod differentiale ∂V integrationem admisit. Cum nempe esset

$$\partial V \sqrt{m} = \frac{M \partial x (xx - yy)}{\sqrt{(A + Cxx)}}, \text{ ob}$$

$$\sqrt{m(A + Cxx)} = \gamma y + \delta x, \text{ erit}$$

$$\partial V = \frac{M \partial x (xx - yy)}{\gamma y + \delta x},$$

cujus integrale commode ex aequatione assumta

$$\alpha + \gamma (xx + yy) + 2 \delta xy = 0$$

definiri potest. Ponatur enim

$$xx + yy = tt \text{ et } xy = u, \text{ erit}$$

$$\alpha + \gamma tt + 2 \delta u = 0$$

et differentialibus sumendis

$$x\partial x + y\partial y = t\partial t; \; x\partial y + y\partial x = \partial u \text{ et } \gamma t\partial t + \delta\partial u = 0;$$

ex binis prioribus colligitur

$(xx-yy)\partial x = xt\partial t - y\partial u$, et ob $t\partial t = -\frac{\delta\partial u}{\gamma}$, erit

$$(xx-yy)\partial x = -\frac{\partial u}{\gamma}(\delta x + \gamma y),$$

ita ut sit

$$\frac{\partial x(xx-yy)}{\gamma y + \delta x} = -\frac{\partial u}{\gamma},\ \text{hincque}\ \partial V = -\frac{M\partial u}{\gamma},$$

unde manifesto sequitur

$$V = -\frac{Mu}{\gamma} = -\frac{Mxy}{\gamma},$$

uti in solutione operosius eruimus. Verum hac operatione commode uti licebit in sequente problemate, ubi formulas magis complexas sumus contemplaturi.

Problema 77.

604. Si ponatur

$$\Pi : z = \int \frac{\partial z(L + Mz^2 + Nz^4 + Oz^6 + etc.)}{\sqrt{(A + Czz)}},$$

integrali hoc ita sumto ut evanescat posito $z = 0$, comparationem inter hujusmodi functiones transcendentes investigare.

Solutio.

Posita ut ante inter variabiles x et y hac relatione

$$\alpha + \gamma(xx + yy) + 2\delta xy = 0,$$

sit

$$-\alpha\gamma = Am \text{ et } \delta\delta - \gamma\gamma = Cm,$$

fietque

$$\gamma y + \delta x = \sqrt{m(A + Cxx)} \text{ et } \gamma x + \delta y = \sqrt{m(A + Cyy)},$$

sumtisque differentialibus

$$\frac{\partial x}{\sqrt{(A + Cxx)}} + \frac{\partial y}{\sqrt{(A + Cyy)}} = 0.$$

Jam statuatur

$$\frac{\partial x(L + Mx^2 + Nx^4 + Ox^6)}{\sqrt{(A + Cxx)}} + \frac{\partial y(L + My^2 + Ny^4 + Oy^6)}{\sqrt{(A + Cyy)}} = \partial V\sqrt{m}$$

ut sit
$$\Pi : x + \Pi : y = \text{Const.} + V\sqrt{m}.$$
At ob $\frac{\partial y}{\sqrt{(A+Cyy)}} = -\frac{\partial x}{\sqrt{(A+Cxx)}}$, ista aequatio abit in
$$\frac{\partial x[M(xx-yy)+N(x^4-y^4)+O(x^6-y^6)]}{\sqrt{(A+Cxx)}} = \partial V\sqrt{m},$$
et ob $\sqrt{m(A+Cxx)} = \gamma y + \delta x$, in hanc
$$\frac{\partial x(xx-yy)[M+N(xx+yy)+O(x^4+xxyy+y^4)]}{\gamma y+\delta x} = \partial V.$$
Sit nunc $xx+yy=tt$ et $xy=u$, ut habeatur
$$\alpha+\gamma tt+2\delta u=0 \text{ et } \gamma t\partial t+\delta\partial u=0,$$
seu $t\partial t = -\frac{\delta\partial u}{\gamma}$,
atque ob
$$x\partial x+y\partial y=t\partial t \text{ et } x\partial y+y\partial x=\partial u$$
hinc colligimus
$$(xx-yy)\partial x = xt\partial t - y\partial u = -\frac{\partial u}{\gamma}(\gamma y+\delta x),$$
ideoque
$$\frac{\partial x(xx-yy)}{\gamma y+\delta x} = -\frac{\partial u}{\gamma},$$
unde habebimus
$$\partial V = -\frac{\partial u}{\gamma}[M+N(xx+yy)+O(x^4+xxyy+y^4)].$$
At est
$$xx+yy=tt=\frac{-\alpha-\delta u}{\gamma} \text{ et}$$
$$x^4+xxyy+y^4=t^4-uu.$$
Notetur autem esse $\frac{\partial u}{\gamma} = -\frac{t\partial t}{\delta}$, unde concludimus
$$\partial V = -\frac{M\partial u}{\gamma}+\frac{Nt^3\partial t}{\delta}+\frac{Ot^5\partial t}{\delta}+\frac{Ouu\partial u}{\gamma},$$
sicque prodit integrando
$$V = -\frac{Mu}{\gamma}+\frac{Nt^4}{4\delta}+\frac{Ot^6}{6\delta}+\frac{Ou^3}{3\gamma}.$$
Quodsi jam ponamus fieri $y=b$ si $x=0$, erit $\gamma=\frac{\sqrt{mA}}{b}$, $\delta=\frac{\sqrt{m(A+Cbb)}}{b}$
et $\alpha=-b\sqrt{mA}$, tum vero

$$y\sqrt{A} + x\sqrt{(A + Cbb)} = b\sqrt{(A + Cxx)}$$
$$x\sqrt{A} + y\sqrt{(A + Cbb)} = b\sqrt{(A + Cyy)} \text{ et}$$
$$b\sqrt{A} = x\sqrt{(A + Cyy)} + y\sqrt{(A + Cxx)}.$$

Hinc cum sit

$$V = -\frac{Mbxy}{\sqrt{mA}} + \frac{Nb(xx + yy)^2}{4\sqrt{m(A + Cbb)}} + \frac{Ob(xx + yy)^3}{6\sqrt{m(A + Cbb)}} + \frac{Obx^3y^3}{3\sqrt{mA}},$$

nostra relatio, cui satisfaciunt praecedentes determinationes, inter functiones transcendentes, erit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} + \frac{Nb(xx+yy)^2}{4\sqrt{(A+Cbb)}} + \frac{Ob(xx+yy)^3}{6\sqrt{(A+Cbb)}}$$
$$+ \frac{Obx^3y^3}{3\sqrt{A}} - \frac{Nb^5}{4\sqrt{(A + Cbb)}} - \frac{Ob^7}{6\sqrt{(A + Cbb)}}:$$

ubi notandum est esse in rationalibus

$$-b\sqrt{A} + \frac{(xx + yy)\sqrt{A}}{b} + \frac{2xy\sqrt{(A + Cbb)}}{b} = 0, \text{ seu}$$
$$xx + yy = bb - \frac{2xy\sqrt{(A + Cbb)}}{\sqrt{A}}.$$

Hinc colligitur

$$(xx + yy)^2 - b^4 = -\frac{4bbxy\sqrt{(A + Cbb)}}{\sqrt{A}} + \frac{4xxyy(A + Cbb)}{A} \text{ et}$$
$$(xx+yy)^3 - b^6 = -\frac{6b^4xy\sqrt{(A+Cbb)}}{\sqrt{A}} + \frac{12bbxxyy(A+Cbb)}{A}$$
$$- \frac{8x^3y^3(A+Cbb)^{\frac{3}{2}}}{A\sqrt{A}};$$

ita ut nostra aequatio sit

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} - \frac{Nb^3xy}{\sqrt{A}} + \frac{Nbxxyy}{A}\sqrt{(A + Cbb)} - \frac{Ob^5xy}{\sqrt{A}}$$
$$+ \frac{2Ob^3xxyy}{A}\sqrt{A + Cbb)} - \frac{Obx^3y^3}{3A\sqrt{A}}(3A + 4Cbb).$$

Corollarium 1.

602. Si ponamus $b = r$, $x = -p$, $y = -q$, erit nostra aequatio

$$\Pi : p + \Pi : q + \Pi : r = \frac{pqr}{\sqrt{A}}(M + Nrr + Or^4)$$
$$- \frac{ppqq\sqrt{(A + Crr)}}{A}(Nr + 2Or^3) + \frac{Op^3q^3r}{3A\sqrt{A}}(3A + 4Crr),$$

stente $pp + qq = rr - \frac{2pq}{\sqrt{A}}\sqrt{(A + Crr)}$, unde fit
$\frac{\sqrt{(A+Crr)}}{\sqrt{A}} = \frac{rr - pp - qq}{2pq}$.

Corollarium 2.

603. Substituto hoc valore pro $\frac{\sqrt{(A+Crr)}}{\sqrt{A}}$, sequens obtineur aequatio, in quam ternae quantitates p, q, r aequaliter ingrentur

$$\Pi : p + \Pi : q + \Pi : r = \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2\sqrt{A}}(pp + qq + rr) + \frac{Opqr}{3\sqrt{A}}(p^4 + q^4 + r^4 + ppqq + pprr + qqrr)$$

satisfaciunt formulae supra datae, vel haec rationalis

$$\frac{4Cppqqrr}{A} = p^4 + q^4 + r^4 - 2ppqq - 2pprr - 2qqrr.$$

Corollarium 3.

604. Si numeratori formulae integralis adhuc adjecissemus minum Pz^8, ut esset

$$\Pi : z = \int \frac{\partial z (L + Mz^2 + Nz^4 + Oz^6 + Pz^8)}{\sqrt{(A + Czz)}},$$

aequationem modo inventam adhuc accessisset terminus

$$\frac{qr}{A}(p^6 + q^6 + r^6 + ppq^4 + ppr^4 + p^4qq + p^4rr + q^4rr + qqr^4 + \tfrac{4}{3}ppqqrr).$$

Scholion.

605. Istae relationes quoque ex superioribus reductionibus rivari possunt, cum enim inde sit $\Pi : z = E\int \frac{\partial z}{\sqrt{(A+Czz)}}$ + *antitate algebraica*, si hic pro z successive quantitates p, q, r bstituamus, ita a se invicem pendentes, ut ante declaravimus, erit

$$\int \frac{\partial p}{\sqrt{(A+Cpp)}} + \int \frac{\partial q}{\sqrt{(A+Cqq)}} + \int \frac{\partial r}{\sqrt{(A+Crr)}} = 0:$$

ide concludimus

$$\Pi : p + \Pi : q + \Pi : r = f : p + f : q + f : r,$$

;notante f functionem quandam algebraicam quantitatis suffixae;

atque summa harum trium functionum rediret ad expressionem ante inventam, si modo relationis inter p, q, r datae ratio habeatur: scilicet inde littera C eliminari deberet. Haec autem reductio ingentem laborem requireret. Hic vero imprimis methodum, qua hic sum usus, spectari convenit, quae cum sit prorsus singularis, ad magis arduam deducere videtur. Certe comparatio functionum transcendentium, quam in capite sequente sum traditurus, vix alia methodo investigari posse videtur, unde hujus methodi utilitas in sequenti capite potissimum cernetur.

CAPUT VI.

DE

COMPARATIONE QUANTITATUM TRANSCENDENTIUM CONTENTARUM IN FORMA

$$\int \frac{P \partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$$

Problema 78.

606.

Proposita relatione inter x et y hac

$$\alpha + \gamma(xx + yy) + 2\delta xy + \zeta xxyy = 0,$$

inde elicere functiones transcendentes formae praescriptae, quas inter se comparare liceat.

Solutio.

Ex proposita aequatione definiatur utraque variabilis

$$y = \frac{-\delta x + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)xx - \gamma\zeta x^4]}}{\gamma + \zeta xx} \text{ et}$$

$$x = \frac{-\delta y + \sqrt{[-\alpha\gamma + (\delta\delta - \gamma\gamma - \alpha\zeta)yy - \gamma\zeta y^4]}}{\gamma + \zeta yy},$$

quae radicalia ad formam praescriptam revocentur ponendo

$$-\alpha\gamma = Am, \delta\delta - \gamma\gamma - \alpha\zeta = Cm \text{ et } -\gamma\zeta = Em;$$

unde fit

$$\alpha = -\frac{Am}{\gamma}, \zeta = -\frac{Em}{\gamma} \text{ et } \delta\delta = Cm + \gamma\gamma + \frac{AEmm}{\gamma\gamma}.$$

Erit ergo

$$\gamma y + \delta x + \zeta xxy = \surd m (A + Cxx + Ex^4)$$
$$\gamma x + \delta y + \zeta xyy = \surd m (A + Cyy + Ey^4).$$

Ipsa autem aequatio proposita, si differentietur, dat

$$\partial x (\gamma x + \delta y + \zeta xyy) + \partial y (\gamma y + \delta x + \zeta xxy) = 0$$

ubi illi valores substituti praebent

$$\frac{\partial x}{\surd(A + Cxx + Ex^4)} + \frac{\partial y}{\surd(A + Cyy + Ey^4)} = 0.$$

Vicissim ergo proposita hac aequatione differentiali, ei satisfaciet haec aequatio finita

$$-Am + \gamma\gamma(xx + yy) + 2xy\surd(\gamma^4 + Cm\gamma\gamma + AEmm)$$
$$- Emxxyy = 0,$$

seu ponendo $\frac{\gamma\gamma}{m} = k$, haec

$$-A + k(xx + yy) + 2xy \surd(kk + kC + AE) - Exxyy = 0,$$

quae cum involvat constantem k, in aequatione differentiali non contentam, simul erit integrale completum. Hinc autem fit

$$ky + x\surd(kk + kC + AE) - Exxy = \surd k(A + Cxx + Ex^4) \text{ et}$$
$$kx + y\surd(kk + kC + AE) - Exyy = \surd k(A + Cyy + Ey^4).$$

Corollarium 1.

607. Constans k ita assumi potest, ut posito $x = 0$, fiat $y = b$, oritur autem

$$kk = \surd Ak \text{ et } b\surd(kk + kC + AE) = \surd k(A + Cbb + Eb^4),$$

ergo

$$k = \frac{A}{bb} \text{ et } \surd(kk + kC + AE) = \frac{1}{bb}\surd A(A + Cbb + Eb^4),$$

ideoque habebimus

$$Ay + x\surd A(A + Cbb + Eb^4) - Ebbxxy = b\surd A(A + Cxx + Ex^4)$$

et

$$Ax + y\surd A(A + Cbb + Eb^4) - Ebbxyy = b\surd A(A + Cyy + Ey^4).$$

Corollarium 2.

608. Haec igitur relatio finita inter x et y erit integrale completum aequationis differentialis

$$\frac{\partial x}{\sqrt{(A+Cxx+Ex^4)}}+\frac{\partial y}{\sqrt{(A+Cyy+Ey^4)}}=0,$$

quod rationaliter inter x et y expressum erit

$$A(xx+yy-bb)+2xy\sqrt{A(A+Cbb+Eb^4)}-Ebbxxyy=0.$$

Corollarium 3.

609. Hinc ergo y ita per x exprimetur, ut sit

$$y=\frac{b\sqrt{A(A+Cxx+Ex^4)}-x\sqrt{A(A+Cbb+Eb^4)}}{A-Ebbxx},$$

atque ex hoc valore elicitur

$$\sqrt{\frac{A+Cyy+Ey^4}{A}}$$
$$=\frac{(A+Ebbxx)\sqrt{(A+Cbb+Eb^4)(A+Cxx+Ex^4)}-2AEbx(bb+xx)-Cbx(A+Ebbxx)}{(A-Ebbxx)^2}.$$

Corollarium 4.

610. Hinc constantem b pro lubitu determinando infinita integralia particularia exhiberi possunt, quorum praecipua sunt: 1) sumendo $b=0$, unde fit $y=-x$; 2) sumendo $b=\infty$, unde fit $y=\frac{\sqrt{A}}{x\sqrt{E}}$. 3) Si $A+Cbb+Eb^4=0$, hincque $bb=\frac{-C+\sqrt{(CC-4AE)}}{2E}$, unde fit $y=\frac{b\sqrt{A(A+Cxx+Ex^4)}}{A-Ebbxx}$.

Scholion.

611. Hic jam usus istius methodi, qua retrogrediendo ab aequatione finita ad aequationem differentialem pervenimus, luculenter perspicitur. Cum enim integratio formulae $\int\frac{\partial x}{\sqrt{(A+Cxx+Ex^4)}}$ nullo modo neque per logarithmos neque per arcus circulares perfici posset, mirum sane est talem aequationem differentialem adeo algebraice integrari posse; quae quidem in praecedente capite ope ejusdem methodi sunt tradita, etiam methodo ordinaria erui possunt, dum singulae formulae differentiales vel per logarithmos vel arcus circulares exprimuntur, quorum deinceps comparatio ad ae-

quationem algebraicam reducitur. Verum quia hic talis integratio plane non locum invenit, nulla certe alia methodus patet, qua idem integrale, quod hic exhibuimus, investigari posset. Quare hoc argumentum diligentius evolvamus.

Problema 79.

612. Si $\Pi:z$ denotet ejusmodi functionem ipsius z, ut sit $\Pi:z = \int \frac{\partial z}{\sqrt{(A + Czz + Ez^4)}}$, integrali ita sumto ut evanescat posito $z = 0$, comparationem inter hujusmodi functiones investigare.

Solutio.

Posita inter binas variabiles x et y relatione supra definita, vidimus fore

$$\frac{\partial x}{\sqrt{(A + Cxx + Ex^4)}} + \frac{\partial y}{\sqrt{(A + Cyy + Ey^4)}} = 0.$$

Hinc cum posito $x = 0$ fiat $y = b$, elicitur integrando

$$\Pi:x + \Pi:y = \Pi:b.$$

Cum jam nullum amplius discrimen inter variabiles x, y et constantem b intercedat, statuamus $x = p$, $y = q$, et $b = -r$, ut sit $\Pi:b = -\Pi:r$, atque haec relatio inter functiones transcendentes

$$\Pi:p + \Pi:q + \Pi:r = 0$$

per sequentes formulas algebraicas exprimetur,

$$(A - Epprr)q + p\sqrt{A(A + Crr + Er^4)} + r\sqrt{A(A + Cpp + Ep^4)} = 0$$

seu

$$(A - Eppqq)r + q\sqrt{A(A + Cpp + Ep^4)} + p\sqrt{A(A + Cqq + Eq^4)} = 0$$

seu

$$(A - Eqqrr)p + r\sqrt{A(A + Cqq + Eq^4)} + q\sqrt{A(A + Crr + Er^4)} = 0$$

quae oriuntur ex hac aequatione

$$A(pp + qq - rr) - Eppqqrr + 2pq\sqrt{A(A + Crr + Er^4)} = 0.$$

Haec vero ad rationalitatem perducta sit

$$AA(p^4+q^4+r^4-2ppqq-2pprr-2qqrr)$$
$$-2AEppqqrr(pp+qq+rr)-4ACppqqrr$$
$$+EEp^4q^4r^4=0,$$

quae autem ob pluralitatem radicum satisfacit omnibus signorum variationibus in superiori aequatione transcendente.

Corollarium 1.

613. Sumamus r negative, ut fiat

$$\Pi:r=\Pi:p+\Pi:q,$$

eritque

$$y=\frac{p\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Cpp+Ep^4)}}{A-Eppqq};$$

unde colligitur fore

$$\sqrt{\frac{A+Crr+Er^4}{A}}$$
$$=\frac{(A+Eppqq)\sqrt{(A+Cpp+Ep^4)(A+Cqq+Eq^4)}+2AEpq(pp+qq)+Cpq(A+Eppqq)}{(A-Eppqq)^2},$$

Corollarium 2.

614. Quodsi ergo ponamus $q=p$, ut sit

$$\Pi:r=2\,\Pi:p,$$

erit

$$r=\frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4},$$

atque

$$\sqrt{\frac{A+Crr+Er^4}{A}}=\frac{AA+2ACpp+6AEp^4+2CEp^6+EEp8}{(A-Ep^4)^2}.$$

Hoc igitur modo functio assignari potest aequalis duplo similis functionis.

Corollarium 3.

615. Si ponatur $q=\frac{2p\sqrt{A(A+Cpp+Ep^4)}}{A-Ep^4}$ et

$$\sqrt{A(A+Cqq+Eq^4)}=\frac{A(AA+2ACpp+6AEp^4+2CEp^6+EEp^8)}{(A-Ep^4)^2},$$

ut sit $\Pi:q=2\Pi:p$, fiet ex primo Coroll. $\Pi:r=3\Pi:p$, Tum igitur erit

$$r = \frac{p(3AA + 4ACpp + 6AEp^4 - EEp^8)}{AA - 6AEp^4 - 4CEp^6 - 3EEp^8}.$$

Scholion 1.

616. Nimis operosum est hanc functionum multiplicationem ulterius continuare, multoque minus legem in earum progressione deprehendere licet. Quodsi ponamus brevitatis gratia

$$\sqrt{A(A + Cpp + Ep^4)} = AP \text{ et } A - Ep^4 = A\mathfrak{P},$$

ut sit

$$Cpp = APP - A - Ep^4 \text{ et } Ep^4 = A(1 - \mathfrak{P}),$$

hae multiplicationes usque ad quadruplum ita se habebunt, scilicet si statuamus

$$\Pi : r = 2\,\Pi : p;\ \Pi : s = 3\,\Pi : p \text{ et } \Pi : t = 4\,\Pi : p$$

reperietur:

$$r = \frac{2Pp}{\mathfrak{P}},\ s = \frac{p(4PP - \mathfrak{P}\mathfrak{P})}{\mathfrak{P}\mathfrak{P} - 4PP(1 - \mathfrak{P})},\ t = \frac{4pP\mathfrak{P}[2PP(2 - \mathfrak{P}) - \mathfrak{P}\mathfrak{P}]}{\mathfrak{P}^4 - 16P^4(1 - \mathfrak{P})}.$$

Quodsi simili modo ponamus

$$\sqrt{A(A + Crr + Er^4)} = AR \text{ et } A - Er^4 = A\mathfrak{R},$$

erit

$$R = \frac{2PP(2 - \mathfrak{P}) - \mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}} \text{ et } \mathfrak{R} = \frac{\mathfrak{P}^4 - 16P^4(1 - \mathfrak{P})}{\mathfrak{P}^4};$$

unde pro quadruplicatione fit

$$t = \frac{2Rr}{\mathfrak{R}},\ T = \frac{2RR(2 - \mathfrak{R}) - \mathfrak{R}\mathfrak{R}}{\mathfrak{R}\mathfrak{R}},\ \mathfrak{T} = \frac{\mathfrak{R}^4 - 16R^4(1 - \mathfrak{R})}{\mathfrak{R}^4}.$$

Quare si pro octuplicatione statuamus $\Pi : z = 8\,\Pi : p$ erit

$$z = \frac{2Tt}{\mathfrak{T}} = \frac{4rR\mathfrak{R}[2RR(2 - \mathfrak{R}) - \mathfrak{R}\mathfrak{R}]}{\mathfrak{R}^4 - 16R^4(1 - \mathfrak{R})}.$$

Hinc intelligitur quomodo in continua duplicatione versari oporteat, neque tamen legem progressionis observare licet. Caeterum cognitio hujus legis ad incrementum Analyseos maxime esset optanda, ut inde generatim relatio inter z et p, pro aequalitate $\Pi : z = n\,\Pi : p$ definiri posset, quaemadmodum hoc in capite praecedente successit;

hinc enim eximias proprietates circa integralia formae $\int \frac{\partial z}{\sqrt{(A + Czz + Ez^4)}}$ cognoscere liceret, quibus scientia analytica haud mediocriter promoveretur.

Scholion 2.

617. Modus maxime idoneus in legem progressionis inquirendi, videtur, si ternos terminos se ordine excipientes contemplemur hoc modo

$$\Pi : x = (n - 1)\, \Pi : p, \; \Pi : y = n\, \Pi : p, \; \Pi : z = (n + 1)\, \Pi : p;$$

ubi cum sit

$$\Pi : x = \Pi : y - \Pi : p \text{ et } \Pi : z = \Pi : y + \Pi : p, \text{ erit}$$

$$x = \frac{y \sqrt{A(A + Cpp + Ep^4)} - p\sqrt{A(A + Cyy + Ey^4)}}{A - Eppyy}$$

$$z = \frac{y \sqrt{A(A + Cpp + Ep^4)} + p\sqrt{A(A + Cyy + Ey^4)}}{A - Eppyy};$$

unde concludimus

$$(A - Eppyy)(x + z) = 2y\sqrt{A(A + Cpp + Ep^4)}.$$

Ponamus ut ante

$$\sqrt{A(A + Cpp + Ep^4)} = AP \text{ et } A - Ep^4 = A\mathfrak{P},$$

et quia singulae quantitates x, y, z factorem p simpliciter involvunt, sit

$$x = pX, \; y = pY \text{ et } z = pZ;$$

erit

$$[1 - (1 - \mathfrak{P})YY](X + Z) = 2PY$$

seu

$$Z = \frac{2PY}{1 - (1 - \mathfrak{P})YY} - X,$$

cujus formulae ope ex binis terminis contiguis X et Y sequens Z haud difficulter invenitur. Quod quo facilius appareat, ponatur $2P = Q$ et $1 - \mathfrak{P} = \Omega$, ut sit $Z = \frac{QY}{1 - \Omega YY} - X$. Jam progressio quaesita ita se habebit

1) 1;

2) $\frac{Q}{\mathfrak{P}}$;

3) $\frac{QQ - \mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P} - QQ\mathfrak{Q}}$;

4) $\frac{Q^3\mathfrak{P}(1+\mathfrak{Q}) - 2Q\mathfrak{P}^3}{\mathfrak{P}^4 - Q^4\mathfrak{Q}}$;

5) $\frac{\mathfrak{P}^6 - 3QQ\mathfrak{P}^4 + Q^4\mathfrak{P}^2(1+2\mathfrak{Q}) - Q^6\mathfrak{Q}\mathfrak{Q}}{\mathfrak{P}^6 - 3QQ\mathfrak{P}^4\mathfrak{Q} + Q^4\mathfrak{P}\mathfrak{P}\mathfrak{Q}(2+\mathfrak{Q}) - Q^6\mathfrak{Q}}$ etc.

Quaestio ergo huc redit, ut investigetur progressio, ex data relatione inter ternos terminos successivos X, Y, Z, quae sit $Z = \frac{QY}{1-\mathfrak{Q}YY} - X$; existente termino primo $= 1$ et secundo $= \frac{Q}{1-\mathfrak{Q}}$.

Problema 80.

618. Si $\Pi : z$ ejusmodi denotet functionem ipsius z, ut sit $\Pi : z = \int \frac{\partial z (L + Mzz + Nz^4)}{\sqrt{(A + Czz + Ez^4)}}$, integrali ita sumto ut evanescat posito $z = 0$, comparationem inter hujusmodi functiones transcendentes investigare.

Solutio.

Stabilita inter binas variabiles x et y hac relatione, ut sit

$$Ay + \mathfrak{B}x - Ebbxxy = b\sqrt{A(A + Cxx + Ex^4)} \text{ seu}$$

$$Ax + \mathfrak{B}y - Ebbxyy = b\sqrt{A(A + Cyy + Ey^4)},$$

sive sublata irrationalitate

$$A(xx + yy - bb) + 2\mathfrak{B}xy - Ebbxxyy = 0,$$

existente brevitatis gratia $\mathfrak{B} = \sqrt{A(A + Cbb + Eb^4)}$, erit uti ante vidimus

$$\frac{\partial x}{\sqrt{(A + Cxx + Ex^4)}} + \frac{\partial y}{\sqrt{(A + Cyy + Ey^4)}} = 0.$$

Ponamus igitur

$$\frac{\partial x (L + Mxx + Nx^4)}{\sqrt{(A + Cxx + Ex^4)}} + \frac{\partial y (L + Myy + Ny^4)}{\sqrt{(A + Cyy + Ey^4)}} = b\,\partial V \sqrt{A},$$

ut sit nostro signandi more

$$\Pi : x + \Pi : y = \text{Const.} + b\,V\sqrt{A},$$

ubi constans ita definiri debet, ut posito $x = 0$ fiat $y = b$. Quaestio ergo ad inventionem functionis V revocatur; quem in finem loco ∂y valore ex priori aequatione substituto, erit

$$b\,\partial V\sqrt{A} = \frac{\partial x\,[M(xx-yy)+N(x^4-y^4)]}{\sqrt{(A+Cxx+Ex^4)}};$$

verum quia

$$b\sqrt{A(A+Cxx+Ex^4)} = Ay + \mathfrak{B}x - Ebbxxy,$$

habebimus

$$\partial V = \frac{\partial x\,(xx-yy)\,[M+N(xx+yy)]}{Ay+\mathfrak{B}x-Ebbxxy}.$$

Sumamus jam aequationem rationalem

$$A\,(xx+yy-bb)+2\mathfrak{B}xy - Ebbxxyy = 0,$$

et ponamus

$$xx+yy = tt \text{ et } xy = u,$$

ut sit

$$A\,(tt-bb)+2\mathfrak{B}u - Ebbuu = 0,$$

ideoque

$$At\partial t = -\mathfrak{B}\partial u + Ebbu\partial u.$$

Cum porro sit

$$x\partial x + y\partial y = t\partial t \text{ et } x\partial y + y\partial x = \partial u,$$

erit

$$(xx-yy)\,\partial x = xt\partial t - y\partial u$$

seu

$$A\,(xx-yy)\partial x = -\partial u\,(Ay+\mathfrak{B}x-Ebbxxy),$$

ita ut sit

$$\frac{\partial x\,(xx-yy)}{Ay+\mathfrak{B}x-Ebbxxy} = -\frac{\partial u}{A},$$

ex quo deducitur

$$\partial V = -\frac{\partial u}{A}(M+Ntt),$$

et ob

$$tt = bb - \frac{2\mathfrak{B}u}{A} + \frac{Ebbuu}{A}, \text{ erit}$$

$$\partial V = -\frac{\partial u}{AA}(AM + ANbb - 2\mathfrak{B}Nu + ENbbuu):$$

unde integrando elicitur

$$V = -\frac{Mu}{A} - \frac{Nbbu}{A} + \frac{\mathfrak{B}Nuu}{AA} - \frac{ENbbu^3}{3AA}.$$

Hoc ergo valore substituto, ob $u = xy$, habebimus

$$\Pi : x + \Pi : y = \Pi : b - \frac{Mbxy}{\sqrt{A}} - \frac{Nb^3xy}{\sqrt{A}} + \frac{\mathfrak{B}Nbx^2y^2}{A\sqrt{A}} - \frac{ENb^3x^3y^3}{3A\sqrt{A}}.$$

Cum autem sit

$$\mathfrak{B}xy = \tfrac{1}{2}Abb - \tfrac{1}{2}A(xx + yy) + Ebbxxyy$$

erit

$$\Pi x + \Pi y = \Pi b - \frac{Mbxy}{\sqrt{A}} - \frac{Nbxy}{2A\sqrt{A}}[A(bb + xx + yy) - \tfrac{1}{3}Ebbxxyy]$$

cui ergo aequationi satisfit per formulas algebraicas supra exhibitas, quibus relatio inter x, y et b exprimitur. Quodsi ergo statuatur haec aequatio

$$\Pi : p + \Pi : q + \Pi : r$$
$$= \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}}[A(pp + qq + rr) - \tfrac{1}{3}Eppqqrr]$$

ea efficitur sequenti relatione inter p, q, r constituta

$$(A - Eppqq)r + p\sqrt{A(A + Cqq + Eq^4)} + q\sqrt{A(A + Cpp + Ep^4)} = 0$$

seu

$$(A - Epprr)q + p\sqrt{A(A + Crr + Er^4)} + r\sqrt{A(A + Cpp + Ep^4)} = 0$$

seu

$$(A - Eqqrr)p + q\sqrt{A(A + Crr + Er^4)} + r\sqrt{A(A + Cqq + Eq^4)} = 0$$

sive per simplicem irrationalitatem

$$A(pp + qq + rr) + 2pq\sqrt{A(A + Crr + Er^4)} - Eppqqrr = 0$$

seu

$$A(pp + rr - qq) + 2pr\sqrt{A(A + Cqq + Eq^4)} - Eppqqrr = 0$$

seu

$$A(qq + rr - pp) + 2qr\sqrt{A(A + Cpp + Ep^4)} - Eppqqrr = 0$$

penitusque irrationalitate sublata

$$EEp^4q^4r^4 - 2AEppqqrr(pp + qq + rr) - 4ACppqqrr$$
$$+ AA(p^4 + q^4 + r^4 - 2ppqq - 2pprr - 2qqrr) = 0.$$

Corollarium 1.

619. Sit $q = r = s$, ut habeamus hanc aequationem

$$\Pi : p + 2\,\Pi : s = \frac{Mpss}{\sqrt{A}} + \frac{Npss}{2A\sqrt{A}}[A(pp + 2ss) - \tfrac{1}{3}Epps^4]$$

cui satisfacit haec relatio

$$(A - Es^4)p + 2s\sqrt{A(A + Css + Es^4)} = 0.$$

Corollarium 2.

620. Sumamus s negative, et loco p sustituamus ibi hunc valorem, ut habeamus

$$2\,\Pi : s + \Pi : q + \Pi : r + \frac{Mpss}{\sqrt{A}} + \frac{Npss}{2A\sqrt{A}}[A(pp + 2ss) - \tfrac{1}{3}Epps^4]$$

$$= \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}}[A(pp + qq + rr) - \tfrac{1}{3}Eppqqrr)$$

existente

$$p = \frac{2s\sqrt{A(A + Css + Es^4)}}{A - Es^4},$$

unde fit

$$\sqrt{A(A + Cpp + Ep^4)} = \frac{A(A + Css + Es^4)^2 + A(4AE - CC)s^4}{(AE - 5^4)^2}$$

qui valores in superioribus formulis substitui debent.

Corollarium 3.

621. Hoc modo effici poterit, ut partes algebraicae evanescant, atque functiones transcendentes solae inter se comparentur. Veluti si esset $N = 0$, statui oporteret $ss = qr$, ut fieret

$$2\,\Pi : s + \Pi : q + \Pi : r = 0.$$

At posito $ss = qr$, fit

$$p = \frac{2\sqrt{Aqr(A + Cqr + Eqqrr)}}{A - Eqqrr}.$$

Est vero etiam

$$p = \frac{-q\sqrt{A(A + Crr + Er^4)} - r\sqrt{A(A + Cqq + Eq^4)}}{A - Eqqrr},$$

quibus valoribus aequatis, oritur haec aequatio

$$(AA + EEq^4r^4)(qq - 6qr + rr) - 8Cqqrr(A + Eqqrr)$$
$$- 2AEqqrr(qq + 10qr + rr) = 0.$$

Scholion.

622. Si $\Pi : z$ exprimat arcum cujuspiam lineae curvae respondentem abscissae vel cordae z, hinc plures arcus ejusdem curvae inter se comparare licet, ut vel differentia binorum arcuum fiat algebraica, vel arcus exhibeantur datam rationem inter se tenentes. Hoc modo ejusmodi insignes curvarum proprietates eruuntur, quarum ratio aliunde vix perspici queat. Comparatio quidem arcuum circularium ex elementis nota per caput praecedens, ut vidimus, facile expeditur, unde etiam comparatio arcuum parabolicorum derivatur. Ex hoc autem capite comparatio arcuum ellipticorum et hyperbolicorum simili modo institui potest; cum enim in genere arcus sectionis conicae tali formula exprimatur $\int \partial x \sqrt{\frac{a+bxx}{c+exx}}$, haec transformata in istam $\int \frac{\partial x(a+bxx)}{\sqrt{[ac+(ae+bc)xx+bex^4]}}$, per praecepta tradita tractari potest, ponendo $A = ac$, $C = ae + bc$, et $E = be$, $L = a$, $M = b$ atque $N = 0$. Haec autem investigatio ad formulas, quarum denominator est

$$\sqrt{(A + 2Bz + Czz + Dz^3 + Ez^4)}$$

extendi potest, similisque est praecedenti, quam idcirco hic sum expositurus, unde simul patebit, hunc esse ultimum terminum, quousque progredi liceat. Formulae enim integrales magis complicatae, ubi post signum radicale altiores potestates ipsius z occurrunt, vel ipsum signum radicale altiorem dignitatem involvit, hoc modo non videntur inter se comparari posse, paucissimis casibus exceptis, qui per quampiam substitutionem ad hujusmodi formam reduci queant.

Problema 81.

623. Si $\Pi : z$ ejusmodi functionem ipsius z denotet, ut sit

$$\Pi : z = \frac{\partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}},$$

hujusmodi functiones inter se comparare.

Solutio.

Inter binas variabiles x et y statuatur relatio hac aequatione expressa

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy = 0,$$

unde cum fiat

$$yy = \frac{-2y(\beta + \delta x + \varepsilon xx) - \alpha - 2\beta x - \gamma xx}{\gamma + 2\varepsilon x + \zeta xx},$$

erit radice extracta

$$y = \frac{-\beta - \delta x - \varepsilon xx + \sqrt{[(\beta + \delta x + \varepsilon xx)^2 - (\alpha + 2\beta x + \gamma xx)(\gamma + 2\varepsilon x + \zeta xx)]}}{\gamma + 2\varepsilon x + \zeta xx}.$$

Reducatur signum radicale ad formam propositam, ponendo

$$\beta\beta - \alpha\gamma = Am, \quad \beta\delta - \alpha\varepsilon - \beta\gamma = Bm,$$
$$\delta\delta - 2\beta\varepsilon - \alpha\zeta - \gamma\gamma = Cm, \quad \delta\varepsilon - \beta\zeta - \gamma\varepsilon = Dm,$$
$$\varepsilon\varepsilon - \gamma\zeta = Em;$$

unde ex sex coëfficientibus α, β, γ, δ, ε, ζ, quinque definiuntur, atque ad sextum insuper accedit littera m, ita ut aequatio assumta adhuc constantem arbitrariam involvat. Inde ergo si brevitatis gratia ponamus

$$\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)} = X \text{ et}$$
$$\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)} = Y,$$

habebimus

$$\beta + \gamma y + \delta x + \varepsilon xx + 2\varepsilon xy + \zeta xxy = X\sqrt{m} \text{ et}$$
$$\beta + \gamma x + \delta y + \varepsilon yy + 2\varepsilon xy + \zeta xyy = Y\sqrt{m}.$$

At aequatio assumta per differentiationem dat

$$+ \partial x(\beta + \gamma x + \delta y + 2\varepsilon xy + \varepsilon yy + \zeta xyy)$$
$$+ \partial y(\beta + \gamma y + \delta x + \varepsilon xx + 2\varepsilon xy + \zeta xxy) = 0,$$

quae expressiones quia cum superioribus conveniunt, dant

$$Y\,\partial x \sqrt{m} + X\,\partial y \sqrt{m} = 0, \text{ seu } \frac{\partial x}{X} + \frac{\partial y}{Y} = 0:$$

unde integrando colligimus

$$\Pi : x + \Pi : y = \text{Const.}$$

quae constans, si posito $x = 0$ fiat $y = b$, erit $= \Pi : 0 + \Pi : b$; vel in genere, si posito $x = a$ fiat $y = b$, ea erit $= \Pi : a + \Pi : b$. Quodsi ergo litterae α, β, γ, δ, ε, ζ per conditiones superiores definiantur, aequatio assumta algebraica inter x et y erit integrale completum hujus aequationis differentialis

$$\frac{\partial x}{\sqrt{(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}} + \frac{\partial y}{\sqrt{(A + 2By + Cyy + 2Dy^3 + Ey^4)}} = 0.$$

Corollarium 1.

624. Ad has litteras α, β, γ, δ, ε, ζ definiendas, sumantur primo aequationes binae ad dextram positae, quae sunt

$$(\delta - \gamma)\beta - \alpha\varepsilon = Bm \text{ et } (\delta - \gamma)\varepsilon - \zeta\beta = Dm,$$

unde quaerantur binae β et ε, reperieturque

$$\beta = \frac{(\delta - \gamma)B + \alpha D}{(\delta - \gamma)^2 - \alpha\zeta} m \text{ et } \frac{(\delta - \gamma)D + \zeta B}{(\delta - \gamma)^2 - \alpha\zeta} m.$$

Corollarium 2.

625. Sit brevitatis gratia $\delta - \gamma = \lambda$ seu $\delta = \gamma + \lambda$, erit

$$\beta = \frac{D\alpha + B\lambda}{\lambda\lambda - \alpha\zeta} m \text{ et } \varepsilon = \frac{B\zeta + D\lambda}{\lambda\lambda - \alpha\zeta} m.$$

Jam ex conditione prima et ultima oritur

$$\beta\beta\zeta - \alpha\varepsilon\varepsilon = (A\zeta - E\alpha) m,$$

ubi illi valores substituti praebent

$$\frac{BB\zeta - DD\alpha}{\lambda\lambda - \alpha\zeta} m = A\zeta - E\alpha,$$

unde fit

$$m = \frac{(\lambda\lambda - \alpha\zeta)(A\zeta - E\alpha)}{BB\zeta - DD\alpha}.$$

At ex prima et ultima sequitur

$$DD\beta\beta - BB\varepsilon\varepsilon + \gamma(BB\zeta - DD\alpha) = (ADD - BBE)\,m$$

unde colligitur

$$\gamma = \frac{[(A\zeta - E\alpha)(ADD - BBE)\lambda\lambda + 2BD(A\zeta - E\alpha)\lambda + ABB\zeta\zeta - DDE\alpha\alpha]}{(BB\zeta - DD\alpha)^2}.$$

Corollarium 3.

626. Superest tertia aequatio

$$2\gamma\lambda + \lambda\lambda - 2\beta\varepsilon - \alpha\zeta = Cm$$

quae, cum pro m substituto valore sit

$$\beta = \frac{(A\zeta - E\alpha)(D\alpha + B\lambda)}{BB\zeta - DD\alpha} \text{ et } \varepsilon = \frac{(A\zeta - E\alpha)(B\zeta + D\lambda)}{BB\zeta - DD\alpha},$$

si isti valores substituantur, commode inde colligitur

$$\lambda = \frac{C(A\zeta - E\alpha)(BB\zeta - DD\alpha) - 2BD(A\zeta - E\alpha)^2 - (BB\zeta - DD\alpha)^2}{2(A\zeta - A\alpha)(ADD - BBE)}.$$

Scholion.

627. Quia his valoribus uti non licet, quoties fuerit $ADD - BBE = 0$, aliam resolutionem huic incommodo non obnoxiam tradam. Posito $\delta = \gamma + \lambda$, sit insuper $\lambda\lambda = \alpha\zeta + \mu$, ut primae formulae fiant

$$\beta = \frac{m}{\mu}(D\alpha + B\lambda) \text{ et } \varepsilon = \frac{m}{\mu}(B\zeta + D\lambda).$$

Jam prima et ultima junctis prodit

$$A\zeta - E\alpha = \frac{m}{\mu}(BB\zeta - DD\alpha)$$

qua aequatione ratio inter α et ζ definitur, quae cum sufficiat, erit

$$\alpha = \mu A - BBm \text{ et } \zeta = \mu E - DDm,$$

hincque

$$\lambda\lambda = \mu + (\mu A - BBm)(\mu E - DDm):$$

unde colligimus

$$\gamma = \frac{mm}{\mu\mu}[2BD\lambda + (ADD - BBE)\mu] - \frac{2BBDDm^3}{\mu\mu} - \frac{m}{\mu}.$$

Valores α et ζ in formula Corollarii 3. substituti dant

$$\lambda = \frac{\mu\mu}{2m} + BDm - \tfrac{1}{2}C\mu,$$

cujus quadratum illi valori $\alpha\zeta + \mu$ aequatum, perducit ad hanc aequationem

$$\mu(\mu - Cm)^2 + 4(BD - AE)mm\mu$$
$$+ 4(ADD - BCD + BCE)m^3 = 4mm,$$

ad quam resolvendam ponatur $\mu = Mm$, fietque

$$m = \frac{4}{M(M-C)^2 + 4M(BD - AE) + 4(ADD - BCD + BBE)},$$

atque hic est M constans illa arbitraria pro integrali completo requisita. Hoc modo omnes litterae α, β, γ, δ, ε, ζ eodem denominatore affecti prodibunt, quo omisso habebimus

$$\alpha = 4(AM - BB),\ \beta = 2B(M - C) + 4AD,\ \gamma = 4AE - (M - C)^2,$$
$$\zeta = 4(EM - DD),\ \varepsilon = 2D(M - C) + 4BE,$$
$$\delta = MM - CC = 4(AE + BD),$$

quibus inventis aequatio nostra canonica

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy$$
$$+ 2\varepsilon xy(x + y) + \zeta xxyy$$

si brevitatis gratia ponamus

$$M(M - C)^2 + 4M(BD - AE) + 4(ADD - BCD + BBE) = \Delta,$$

resoluta dabit

$$\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx) =$$
$$\pm 2\sqrt{\Delta(A + 2Bx + Cxx + 2Dx^3 + Ex^4)}$$
$$\beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy) =$$
$$\pm 2\sqrt{\Delta(A + 2By + Cyy + 2Dy^3 + Ey^4)},$$

quae ergo est integrale completum hujus aequationis differentialis

$$0 = \frac{\partial x}{\pm\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} + \frac{\partial y}{\pm\sqrt{A+2By+Cyy+2Dy^3+Ey^4)}}.$$

Scholion.

628. Cum hic ab idonea coëfficientium determinatione totum negotium pendeat, operae praetium erit, eam luculentius exponere. Posito igitur statim

$$\delta = \gamma + \lambda \text{ et } \lambda\lambda - \alpha\zeta = Mm,$$

quinque conditiones adimplendae sunt:

$$\text{I. } \beta\beta - \alpha\gamma = Am;$$
$$\text{II. } \varepsilon\varepsilon - \gamma\zeta = Em;$$
$$\text{III. } \beta\lambda - \alpha\varepsilon = Bm;$$
$$\text{IV. } \varepsilon\lambda - \beta\zeta = Dm;$$
$$\text{V. } Mm + 2\gamma\lambda - 2\beta\varepsilon = Cm.$$

Hinc ex tertia et quarta combinando deducitur

$$m(B\lambda + D\alpha) = \beta(\lambda\lambda - \alpha\zeta) = \beta Mm, \text{ ergo } \beta = \frac{B\lambda + D\alpha}{M},$$
$$m(D\lambda + B\zeta) = \varepsilon(\lambda\lambda - \alpha\zeta) = \varepsilon Mm \text{ ergo } \varepsilon = \frac{D\lambda + B\zeta}{M}.$$

Jam ex prima et secunda elidendo γ, oritur

$$m(A\zeta - E\alpha) = \beta\beta\zeta - \varepsilon\varepsilon\alpha = \frac{BB\zeta - DD\alpha}{M}.m$$

hincque

$$\zeta(AM - BB) = \alpha(EM - DD);$$

quare statuatur

$$\alpha = n(AM - BB) \text{ et } \zeta = n(EM - DD).$$

Tum vero indidem est

$$E\beta\beta - E\alpha\gamma = A\varepsilon\varepsilon - A\gamma\zeta, \text{ seu}$$
$$\gamma(A\zeta - E\alpha) = A\varepsilon\varepsilon - E\beta\beta:$$

pro qua tractanda cum sit, pro α et ζ substitutis valoribus,

$$\beta = nAD + \frac{B}{M}(\lambda - nBD) \text{ et } \varepsilon = nBE + \frac{D}{M}(\lambda - nBD),$$

sit brevitatis ergo $\lambda - nBD = nMN$, ut habeamus

$$\beta = n(AD + BN) \text{ et } \varepsilon = n(BE + DN),$$

et quia

$$A\zeta - E\alpha = n\,(BBE - ADD)$$

atque

$$A\varepsilon\varepsilon - E\beta\beta = nn(ABBEE + ADDNN - AADDE - BBENN), \text{ seu}$$

$$A\varepsilon\varepsilon - E\beta\beta = nn(BBE - ADD)(AE - NN) \text{ fiet,}$$

$$\gamma = n\,(AE - NN).$$

Cum autem sit

$$\lambda = n\,(BD + MN) \text{ et}$$

$$\lambda\lambda = nn(AM - BB)(EM - DD) + Mm, \text{ erit}$$

$$Mm = nn[\,2BDMN + MMNN - AEMM + M(ADD + BBE)\,]$$

seu

$$m = nn(2BDN + MNN - AEM + ADD + BBE).$$

Denique aequatio quinta $\beta\varepsilon - \gamma\lambda = \frac{1}{2}m(M - C)$ evoluta praebet

$$\beta\varepsilon - \gamma\lambda = nn[(AD + BN)(BE + DN) - (AE - NN)(BD + MN)]$$
$$- nnN(2BDN + MNN - AEM + ADD + BBE) = Nm,$$

unde fit $N = \frac{1}{2}(M - C)$, ac propterea

$$m = nn[BD(M - C) + \tfrac{1}{4}M(M - C)^2 - AEM + ADD + BBE].$$

Hincque sumendo $n = 4$ superiores valores obtinentur.

Exemplum 1.

629. *Invenire integrale completum hujus aequationis differentialis*

$$\frac{\partial p}{\pm\sqrt{(a + bp)}} + \frac{\partial q}{\pm\sqrt{(a + bq)}} = 0.$$

Hic est $x = p$, $y = q$, $A = a$, $B = \frac{1}{2}b$, $C = 0$, $D = 0$, $E = 0$; unde fiunt coëfficientes

$$\alpha = 4\,aM - bb,\quad \beta = bM,\quad \gamma = -MM,$$
$$\zeta = 0,\quad \varepsilon = 0,\quad \delta = MM,$$

et $\Delta = M^3$, unde integrale completum erit

$$bM + MMp - MMq = \pm 2M\sqrt{M(a + bp)}, \text{ seu}$$

$$b+M(p-q)=\pm 2\sqrt{M(a+bp)}, \text{ vel}$$
$$b+M(q-p)=\pm 2\sqrt{M(a+bq)};$$

quae signa ambigua radicalium cum signis in aequatione differentiali convenire debent.

Exemplum 2.

630. *Invenire integrale completum hujus aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a+bp^2)}}+\frac{\partial q}{\pm\sqrt{(a+bq^2)}}=0.$

Sumto $x=p$ et $y=q$, erit $A=a$, $B=0$, $C=b$, $D=0$, ergo

$$\alpha=4aM,\ \beta=0,\ \gamma=-(M-b)^2,$$
$$\zeta=0,\quad \varepsilon=0,\ \delta=MM-bb,$$

atque $\Delta=M(M-b)^2$;

unde integrale completum in his aequationibus continebitur:

$$(MM-bb)p-(M-b)^2q=\pm 2(M-b)\sqrt{M(a+bpp)}, \text{ seu}$$
$$(M+b)p-(M-b)q=\pm 2\sqrt{M(a+bpp)} \text{ et}$$
$$(M+b)q-(M-b)p=\pm 2\sqrt{M(a+bqq)}.$$

Exemplum 3.

631. *Invenire integrale completum hujus aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a+bp^3)}}+\frac{\partial q}{\pm\sqrt{(a+bq^3)}}=0.$

Sumto $x=p$, $y=q$, erit $A=a$, $B=0$, $C=0$, $D=\frac{1}{4}b$, $E=0$, ergo

$$\alpha=4aM,\ \beta=2ab,\ \gamma=-MM;$$
$$\zeta=-bb,\ \varepsilon=bM,\ \delta=MM, \text{ et}$$
$$\Delta=M^3+abb;$$

unde integrale completum

$$2ab+MMp+bMpp+q(-MM+2bMp-bbpp)=$$
$$\pm 2\sqrt{(M^3+abb)(a+bp^3)}$$

sive

$$2ab + Mp(M+bp) - q(M-bp)^2 = \pm 2\sqrt{(M^3+abb)(a+bp^3)}$$

et

$$2ab + Mq(M+bq) - p(M-bq)^2 = \pm 2\sqrt{(M^3+abb)(a+bq^3)}.$$

Exemplum 4.

632. *Invenire integrale completum hujus aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a+bp^4)}} + \frac{\partial q}{\pm\sqrt{(a+bq^4)}} = 0$.

Posito $x=p$, $y=q$, erit $A=a$, $B=0$, $C=0$, $D=0$, $E=b$, ergo

$$\alpha = 4aM,\ \beta = 0,\ \gamma = 4ab - MM,$$
$$\zeta = 4bM,\ \varepsilon = 0,\ \delta = MM + 4ab, \text{ et}$$
$$\Delta = M^3 - 4abM;$$

unde integrale completum

$$(MM + 4ab)p + q(4ab - MM + 4bMpp) = \pm 2\sqrt{M(MM-4ab)(a+bp^4)}$$

$$(MM + 4ab)q + p(4ab - MM + 4bMqq) = \pm 2\sqrt{M(MM-4ab)(a+bq^4)}.$$

Exemplum 5.

633. *Invenire integrale completum hujus aequationis differentialis* $\frac{\partial p}{\pm\sqrt{(a+bp^6)}} + \frac{\partial q}{\pm\sqrt{(a+bq^6)}} = 0$.

Ponatur $x = pp$ et $y = qq$, atque aequatio nostra generalis induet, posito $A = 0$, hanc formam

$$\frac{\partial p}{\pm\sqrt{(2B + Cpp + 2Dp^4 + Ep^6)}} + \frac{\partial q}{\pm\sqrt{(2B + Cqq + 2Dq^4 + Eq^6)}} = 0.$$

Fieri ergo oportet $B = \frac{1}{2}a$, $C = 0$, $D = 0$ et $E = b$; unde coëfficientes ita determinantur

$$\alpha = -aa,\ \beta = aM,\ \gamma = -MM,$$
$$\zeta = 4bM,\ \varepsilon = 2ab,\ \delta = MM, \text{ et}$$
$$\Delta = M^3 + aab;$$

ergo integrale completum

$$aM + MMpp + 2abp^4 + qq(-MM + 4abpp + 4bMp^4) = \pm 2p\sqrt{(M^3+aab)(a+bp^6)}$$

sive

$$aM + MMqq + 2abq^4 + pp(-MM + 4abqq + 4bMq^4) = \pm 2q\sqrt{(M^3+aab)(a+bq^6)}.$$

Corollarium.

634. Si sumatur constans $M = -\sqrt[3]{aab}$, ut sit $M^3 + aab = 0$, prodibit integrale particulare, quod ita se habebit

$$pp = \frac{qq\sqrt[3]{b} + \sqrt[3]{a}}{2qq\sqrt[3]{b} - \sqrt[3]{a}} \cdot \sqrt[3]{\frac{a}{b}} \text{ seu } qq = \frac{pp\sqrt[3]{b} + \sqrt[3]{a}}{2pp\sqrt[3]{b} - \sqrt[3]{a}} \cdot \sqrt[3]{\frac{a}{b}}$$

quod aequationi differentiali utique satisfacit.

Problema 82.

635. Proposita hac aequatione differentiali

$$\frac{\partial p}{\pm\sqrt{(a+bpp+cp^4+ep^6)}} + \frac{\partial q}{\pm\sqrt{(a+bqq+cq^4+eq^6)}} = 0$$

ejus integrale completum algebraice assignare.

Solutio.

Aequatio praecedens differentialis algebraice integrata ad hanc formam reducitur, ponendo $x = pp$ et $y = qq$, atque $A = 0$; prodibit enim

$$\frac{\partial p}{\pm\sqrt{(2B+Cpp+2Dp^4+Ep^6)}} + \frac{\partial q}{\pm\sqrt{(2B+Cqq+2Dq^4+Eq^6)}} = 0.$$

Quare tantum opus est ut fiat

$$A = 0,\ B = \tfrac{1}{2}a,\ C = b,\ D = \tfrac{1}{2}c,\ E = e,$$

unde coëfficientes α, β, γ, δ, ε, ζ ita definientur

$$\alpha = -aa, \qquad \beta = a(M - b), \qquad \gamma = -(M - b)^2,$$

$$\zeta = 4eM - cc, \; \varepsilon = c(M - b) + 2ae, \; \delta = -MM - bb + ae,$$

$$\Delta = M(M - b)^2 + acM - abc + aae =$$
$$(M - b)^3 + b(M - b)^2 + ac(M - b) + aae;$$

hincque integrale completum ob constantem M ab arbitrio nostro pendentem, erit

$$\beta + \delta pp + \varepsilon p^4 + qq(\gamma + 2\varepsilon pp + \zeta p^4) =$$
$$\pm 2p\sqrt{\Delta(a + bp^2 + cp^4 + ep^6)}$$

$$\beta + \delta qq + \varepsilon q^4 + pp(\gamma + 2\varepsilon qq + \zeta q^4) =$$
$$\pm 2q\sqrt{\Delta(a + bq^2 + cq^4 + eq^6)},$$

quae binae quidem aequationes inter se conveniunt, sed ob ambiguitatem signorum in ipsa aequatione differentiale ambae notari debent, ambiguitate inde sublata. Utrinque autem haec aequatio rationalis resultat

$$0 = \alpha + 2\beta(pp + qq) + \gamma(p^4 + q^4) + 2\delta ppqq$$
$$+ 2\varepsilon ppqq(pp + qq) + \zeta p^4 q^4.$$

Corollarium 1.

636. Si constans M ita sumatur, ut fiat $\Delta = 0$, obtinetur integrale particulare hujus formae $qq = \frac{E + Fpp}{G + Hpp}$, quod etiam a posteriori cognoscere licet. Ut enim satisfaciat sumi debet

$$aG^3 + bEGG + cEEG + eE^3 = 0;$$

unde ratio E : G definitur, tum vero invenitur $F = -G$ et denique

$$H = \frac{-cEG - 2eEE}{aG} = \frac{2aGG + 2bEG + cEE}{aE}.$$

Corollarium 2.

637. Constans M ita mutetur, ut sit $M - b = \frac{e}{ff}$, fietque

$$\alpha = -aa, \qquad \beta = \frac{aa}{ff}, \qquad \gamma = -\frac{aa}{f^4},$$
$$\zeta = 4be - cc + \frac{4ae}{ff}, \; \varepsilon = \frac{ac}{ff} + 2ae, \; \delta = \frac{aa}{f^4} + \frac{2ab}{ff} + ac, \text{ et}$$
$$\Delta = \frac{aa}{f^6}(a + bff + cf^4 + ef^6),$$

et aequatio integralis erit

$$aaff + a(a + 2bff + cf^4)pp + aff(c + 2eff)p^4$$
$$- qq[aa - 2aff(c + 2eff)pp + ff(ccff - 4beff - 4ae)p^4]$$
$$= \pm 2afp\sqrt{(a + bff + cf^4 + ef^6)(a + bpp + cp^4 + ep^6)};$$

unde patet posito $p = 0$ fore $qq = ff$.

Corollarium 3.

638. Haec aequatio facile in hanc formam transmutatur

$$aff(a + bpp + cp^4 + ep^6) + app(a + bff + cf^4 + ef^6)$$
$$- qq(a - cffpp)^2 - aeffpp(ff - pp)^2 + 4effppqq(aff + app + bffpp)$$
$$= \pm 2fp\sqrt{a(a + bff + cf^4 + ef^6)a(a + bpp + cp^4 + ep^6)};$$

unde statim patet si sit $e = 0$, fore hanc aequationem, radicem extrahendo

$$f\sqrt{a(a + bpp + cp^4)} \pm p\sqrt{a(a + bff + cf^4)} = q(a - cffpp)$$

quae est integralis completa hujus differentialis

$$\frac{\partial p}{\pm\sqrt{(a + bpp + cp^4)}} + \frac{\partial q}{\pm\sqrt{(a + bqq + cq^4)}} = 0$$

prorsus ut supra jam invenimus.

Corollarium 4.

639. Simili modo patet in genere, quando e non evanescit, integrale completum ita commodius exprimi posse

$$[f\sqrt{a(a + bpp + cp^4 + ep^6)} \pm p\sqrt{a(a + bff + cf^4 + ef^6)}]^2 =$$
$$qq(a - cffpp)^2 + aeffpp(ff - pp)^2 - 4effppqq(aff + app + bffpp),$$

quae ergo cum posito $p = 0$ fiat $q = f$, respondet huic functionum transcendentium relationi

$$\pm \Pi : p \pm \Pi : q = \pm \Pi : 0 \pm \Pi : f.$$

Scholion 1.

640. Genera igitur functionum transcendentium, quas hoc modo perinde atque arcus circulares inter se comparare licet, in his binis formulis integralibus continentur

$$\int \frac{\partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}} \text{ et } \int \frac{\partial z}{\sqrt{(a + bzz + cz^4 + ez^6)}}$$

neque haec methodus ad alias formas magis complexas extendi posse videtur. Neque etiam posterior in denominatore potestates impares ipsius z admittit: nisi forte simplex substitutio reductioni ad illam formam sufficiat. Facile autem patet hujusmodi formam

$$\int \frac{\partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4 + Fz^5 + Gz^6)}},$$

hac methodo tractari certe non posse, si enim coëfficientes ita essent comparati, ut radicis extractio succederet, talis formula $\int \frac{\partial z}{a + bz + czz + ez^3}$ prodiret, cujus integratio, cum tam logarithmos quam arcus circulares involvat, fieri omnino nequit, ut plures hujusmodi functiones algebraice inter se comparentur. Caeterum prior formula latius patet quam posterior, cum haec ex illa nascatur posito $A = 0$, si zz loco z scribatur. De priori autem notari meretur, quod eandem formam servet, etiamsi transformetur hac substitutione $z = \frac{\alpha + \beta y}{\gamma + \delta y}$; prodit enim

$$\int \frac{(\beta\gamma - \alpha\delta)\partial y}{\left\{\sqrt{\begin{array}{l}[A(\gamma + \delta y)^4 + 2B(\alpha + \beta y)(\gamma + \delta y)^3 + C(\alpha + \beta y)^2(\gamma + \delta y)^2 \\ + 2D(\alpha + \beta y)^3(\gamma + \delta y) + E(\alpha + \beta y)^4]\end{array}}\right\}},$$

ex quo intelligitur quantitates α, β, γ, δ, ita accipi posse, ut potestates impares evanescant. Vel etiam ita definiri poterunt, ut terminus primus et ultimus evanescat, tum enim posito $y = uu$, iterum forma a potestatibus imparibus immunis nascitur.

Scholion 2.

641. Sublatio autem potestatum imparium ita commodissime instituitur. Cum formula

$$A + 2Bz + Czz + 2Dz^3 + Ez^4$$

certe semper habeat duos factores reales, ita exhibeatur formula integralis

$$\int \frac{\partial z}{\surd(a + 2bz + czz)(f + 2gz + hzz)},$$

quae posito $z = \frac{\alpha + \beta y}{\gamma + \delta y}$, abit in

$$\int \frac{(\beta\gamma - \alpha\delta)\partial y}{\left\{\begin{array}{l}\surd[a(\gamma+\delta y)^2 + 2b(\alpha+\beta y)(\gamma+\delta y) + c(\alpha+\beta y)^2][f(\gamma+\delta y)^2 \\ + 2g(\alpha+\beta y)(\gamma+\delta y) + h(\alpha+\beta y)^2]\end{array}\right\}},$$

ubi denominatoris factores evoluti sunt

$$(a\gamma\gamma + 2b\alpha\gamma + c\alpha\alpha) + 2(a\gamma\delta + b\alpha\delta + b\beta\gamma + c\alpha\beta)y$$
$$+ (a\delta\delta + 2b\beta\delta + c\beta\beta)yy$$
$$(f\gamma\gamma + 2g\alpha\gamma + h\alpha\alpha) + 2(f\gamma\delta + g\alpha\delta + g\beta\gamma + h\alpha\beta)y$$
$$+ (f\delta\delta + 2g\beta\delta + h\beta\beta)yy$$

quodsi jam utroque terminus medius evanescens reddatur, fit

$$\frac{\delta}{\beta} = \frac{-b\gamma - c\alpha}{a\gamma + b\alpha} = \frac{-g\gamma - h\alpha}{f\gamma + g\alpha},$$

hincque

$$bf\gamma\gamma + (bg + cf)\alpha\gamma + cg\alpha\alpha = ag\gamma\gamma + (ah + bg)\alpha\gamma + bh\alpha\alpha$$

seu

$$\gamma\gamma = \frac{(ah - cf)\alpha\gamma + (bh - cg)\alpha\alpha}{bf - ag},$$

unde fit

$$\frac{\gamma}{\alpha} = \frac{ah - cf + \surd[(ah - cf)^2 + 4(bf - ag)(bh - cg)]}{2(bf - ag)}.$$

Hinc sufficere posset eas tantum formulas, in quibus potestates impares desunt, tractasse, id quod initio hujus capitis fecimus, sed si insuper numerator accedat, haec reductio non amplius locum habet.

Problema 83.

642. Denotante n numerum integrum quemcunque, invenire integrale completum algebraice expressum hujus aequationis differentialis

$$\frac{\partial y}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} = \frac{n\,\partial x}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}}.$$

Solutio.

Per functiones transcendentes integrale completum est

$$\Pi : y = n\,\Pi : x + \text{Const.}$$

At ut idem algebraice expressum eruamus, posito $M - C = L$, sit per formulas supra (627.) inventas

$\alpha = 4(AC - BB + AL)$, $\beta = 4AD + 2BL$, $\gamma = 4AE - LL$,

$\zeta = 4(CE - DD + EL)$, $\varepsilon = 4BE + 2DL$, $\delta = 4AE + 4BD + 2CL + LL$,

et

$$\Delta = L^3 + CL^2 + 4(BD - AE) + 4(ADD + BBE - ACE).$$

Quibus positis si fuerit

$$\beta + \delta p + \varepsilon pp + q(\gamma + 2\varepsilon p + \zeta pp) = 2\sqrt{\Delta}(A + 2Bp + Cp^2 + 2Dp^3 + Ep^4)$$

$$\beta + \delta q + \varepsilon qq + p(\gamma + 2\varepsilon q + \zeta qq) = -2\sqrt{\Delta}(A + 2Bq + Cq^2 + 2Dq^3 + Eq^4)$$

erit $\Pi : q = \Pi : p + \text{Const.}$

Cum autem hac duae aequationes inter se conveniant, et in hac rationali contineantur

$$\alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq = 0$$

si sumamus, posito $p = a$ fieri $q = b$, constans illa L ita definiri debet, ut sit

$$\alpha + 2\beta(a+b) + \gamma(aa+bb)$$
$$+ 2\delta ab + 2\varepsilon ab(a+b) + \zeta aabb = 0,$$

eritque

$$\Pi : q = \Pi : p + \Pi : b - \Pi : a;$$

ubi jam nullum inest discrimen inter constantes et variabiles. Ponamus ergo $p = b$, ut sit

$$\Pi : q = 2\,\Pi : p - \Pi : a$$

atque huic aequationi superiores aequatioues algebraicae conveniunt, si modo quantitas L ita definiatur, ut sit

$$\alpha + 2\beta(a+p) + \gamma(aa+pp)$$
$$+ 2\delta ap + 2\varepsilon ap(a+p) + \zeta aapp = 0,$$

unde deducitur

$$\tfrac{1}{2}L(a-p)^2 = A + B(a+p) + Cap + Dap(a+p) + Eaapp$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}.$$

Hoc ergo valore pro L constituto, indeque litteris α, β, γ, δ, ε, ζ per superiores formulas rite definitis, si jam p et q ut variabiles, a vero ut constantem spectemus, erit haec aequatio

$$\alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq = 0,$$

integrale completum hujus aequationis differentialis

$$\frac{\partial q}{\sqrt{(A+2Bq+Cqq+2Dq^3+Eq^4)}} = \frac{2\partial p}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

Postquam hoc modo q per p definivimus, determinetur r per hanc aequationem

$$\alpha + 2\beta(q+r) + \gamma(qq+rr) + 2\delta qr + 2\varepsilon qr(q+r) + \zeta qqrr = 0,$$

erit

$$\Pi : r - \Pi : q = \Pi : p - \Pi : a,$$

quoniam, posito $q = a$ et $r = p$, littera L, quae in valores α, β, γ, δ, ε, ζ ingreditur, perinde definitur ut ante. Quare cum sit

$$\Pi : q = 2\,\Pi : p - \Pi : a, \text{ erit } \Pi : r = 3\,\Pi : p - 2\,\Pi : a;$$

unde sumto a constante, illa aequatio algebraica inter q et r, dum q per praecedentem aequationem ex p definitur, erit integrale completum hujus aequationis differentialis

$$\frac{\partial r}{\sqrt{(A+2Br+Crr+2Dr^3+Er^4)}} = \frac{3\partial p}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

Hoc valore ipsius r per p invento, quaeratur s per hanc aequationem

$$\alpha + 2\beta(r+s) + \gamma(rr+ss) + 2\delta rs + 2\varepsilon rs(r+s) + \zeta rrss = 0,$$

retinente L semper valorem primo assignatum, eritque

$$\Pi:s - \Pi:r = \Pi:p - \Pi:a, \text{ seu } \Pi:s = 4\,\Pi:p - 3\,\Pi:a$$

unde ista aequatio algebraica erit integrale completum hujus aequationis differentialis

$$\frac{\partial s}{\sqrt{(A+2Bs+Css+2Ds^3+Es^4)}} = \frac{4\partial p}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

Cum hoc modo quousque libuerit progredi liceat, perspicuum est, ad integrale completum hujus aequationis differentialis inveniendum

$$\frac{\partial z}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}} = \frac{n\partial p}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

sequentes operationes institui oportere.

1.) Quaeratur quantitas L, ut sit

$$\tfrac{1}{2}L(p-a)^2 = A + B(a+p) + Cap + Dap(a+p) + Eaapp$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}$$

2.) Hinc determinentur litterae α, β, γ, δ, ε, ζ, per has formulas

$$\alpha = 4(AC-BB+AL),\ \beta = 4AD+2BL,\ \gamma = 4AE-LL,$$
$$\zeta = 4(CE-DD+EL),\ \varepsilon = 4BE+2DL,\ \delta = 4AE+4BD+2CL+LL.$$

3.) Formetur series quantitatum $p, q, r, s, t, \ldots\ldots z$, quarum prima sit p, secunda q, tertia r etc. ultima vero ordine n sit z, quae successive per has aequationes determinentur

$$\alpha+2\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq=0$$
$$\alpha+2\beta(q+r)+\gamma(qq+rr)+2\delta qr+2\varepsilon qr(q+r)+\zeta qqrr=0$$
$$\alpha+2\beta(r+s)+\gamma(rr+ss)+2\delta rs+2\varepsilon rs(r+s)+\zeta rrss=0$$
etc.

donec ad ultimam z perveniatur.

4.) Relatio quae hinc concluditur inter p et z erit integrale completum aequationis differentialis propositae, et littera a vicem erit constantis arbitrariae per integrationem ingressae.

Corollarium.

643. Hinc etiam integrale completum inveniri potest hujus aequationis differentialis

$$\frac{m\partial y}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}=\frac{n\partial x}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}},$$

designantibus m et n numeros integros. Statuatur enim utrumque membrum $=\frac{\partial u}{\sqrt{(A+2Bu+Cuu+2Du^3+Eu^4)}}$, et quaeratur relatio tam inter x et u, quam inter y et u; unde elisa u orietur aequatio algebraica inter x et y.

Scholion.

644. Ne hic extractio radicis in singulis aequationibus repetenda ambiguitatem creet, loco uniuscujusque uti conveniet binis per extractionem jam erutis. Scilicet ut ex prima valor q rite per p definiatur, primo quidem habemus

$$q=\frac{-\beta-\delta p-\varepsilon pp+2\sqrt{\Delta(A+2Bp+Cpp+2Dp^3+Ep^4)}}{\gamma+2\varepsilon p+\zeta pp},$$

tum vero capi debet

$$2\sqrt{\Delta(A+2Bq+Cqq+2Dq^3+Eq^4)}=$$
$$-\beta-\delta q-\varepsilon qq-p(\gamma+2\varepsilon q+\zeta qq):$$

similique modo in relatione inter binas sequentes quantitates investiganda erit procedendum. Caeterum adhuc notari convenit numeros integros m et n positivos esse debere, neque hanc investigationem ad negativos extendi, propterea quod formula differentialis $\frac{\partial z}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$, posito z negativo, naturam suam mutat. Interim tamen cum hanc aequalitatem

$$\Pi : x + \Pi : y = \text{Const.}$$

supra algebraice expresserimus, ejus ope quoque iii casus resolvi possunt, ubi est m, vel n numerus negativus: si enim fuerit

$$\Pi : z = n\,\Pi : p + \text{Const.}$$

quaeratur y, ut sit

$$\Pi : y + \Pi : z = \text{Const.}$$

eritque

$$\Pi : y = -n\,\Pi : p + \text{Const.}$$

Problema. 84.

645. Si $\Pi : z$ ejusmodi functionem transcendentem ipsius z denotet, ut sit

$$\Pi : z = \int \frac{\partial z(\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}zz + \mathfrak{D}z^3 + \mathfrak{E}z^4)}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}},$$

comparationem inter hujusmodi functiones investigare.

Solutio.

Ex coëfficientibus A, B, C, D, E, una cum constante arbitraria L determinentur sequentes valores

$\alpha = 4(AC - BB + AL)$, $\beta = 4AD + 2BL$, $\gamma = 4AE - LL$,
$\zeta = 4(CE - DD + EL)$, $\varepsilon = 4BE + 2DL$, $\delta = 4AE + 4BD + 2CL + LL$,

et inter binas variabiles x et y haec constituatur relatio

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy = 0,$$

eritque

$$\frac{\partial x}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} + \frac{\partial y}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} = 0,$$

pro qua sine ambiguitate habetur

$$\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx) = 2\sqrt{\Delta(A+2Bx+Cxx+2Dx^3+Ex^4)}$$
$$\beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy) = 2\sqrt{\Delta(A+2By+Cyy+2Dy^3+Ey^4)}$$

existente

$$\Delta = L^3 + CL^2 + 4(BD - AE)L + 4(ADD + BBE - ACE).$$

re si ponamus

$$\frac{\partial x(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4)}{\surd(A+2Bx+Cxx+2Dx^3+Ex^4)}+\frac{\partial y(\mathfrak{A}+\mathfrak{B}y+\mathfrak{C}y^2+\mathfrak{D}y^3+\mathfrak{E}y^4)}{\surd(A+2By+Cyy+2Dy^3+Ey^4)}=2\partial V\surd\Delta,$$

it

$$\Pi : x + \Pi : y = \text{Const.} + 2V\surd\Delta, \text{ erit}$$

$$\frac{\partial x[\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y^4)]}{\surd(A+2Bx+Cxx+2Dx^3+Ex^4)}=2\partial V\surd\Delta, \text{ seu}$$

$$\partial V=\frac{\partial x[\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y^4)]}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}.$$

atur nunc $x+y=t$ et $xy=u$, et quia $\partial x+\partial y=\partial t$ $x\partial y+y\partial x=\partial u$, erit $\partial x=\frac{x\partial t-\partial u}{x-y}$, seu $(x-y)\partial x$ $x\partial t-\partial u$, tum vero est $x=\frac{1}{2}t+\surd(\frac{1}{4}tt-u)$. At his itionibus aequatio assumta induit hanc formam

$$\alpha+2\beta t+\gamma tt+2(\delta-\gamma)u+2\varepsilon tu+\zeta uu=0,$$

e fit differentiando

$$\partial t(\beta+\gamma t+\varepsilon u)+\partial u(\delta-\gamma+\varepsilon t+\zeta u)=0, \text{ ergo}$$

$$\partial t=\frac{-\partial u(\delta-\gamma+\varepsilon t+\zeta u)}{\beta+\gamma t+\varepsilon u}, \text{ et}$$

$$x\partial t-\partial u=\frac{-\partial u[\beta+\gamma t+\varepsilon u+(\delta-\gamma)x+\varepsilon tx+\zeta ux]}{\beta+\gamma t+\varepsilon u} \text{ sive}$$

$$x\partial t-\partial u=\frac{-\partial u[\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)]}{\beta+\gamma t+\varepsilon u}$$

que habebimus

$$\frac{\partial x(x-y)}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}=\frac{-\partial u}{\beta+\gamma t+\varepsilon u}; \text{ ergo}$$

$$\partial V=\frac{-\partial u[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{\beta+\gamma t+\varepsilon u} \text{ seu}$$

$$\partial V=\frac{+\partial t[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{\delta-\gamma+\varepsilon t+\zeta u}.$$

vero aequatione illa resoluta

$$t=\frac{-\beta-\varepsilon u+\surd[\beta\beta-\alpha\gamma+2(\gamma\gamma+\beta\varepsilon-\gamma\delta)u+(\varepsilon\varepsilon-\gamma\zeta)uu]}{\gamma} \text{ seu}$$

$$t=\frac{-\beta-\varepsilon u+2\surd\Delta(A+Lu+Euu)}{\gamma},$$

e conficitur

$$\partial V=\frac{-\partial u[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{2\surd\Delta(A+Lu+Euu)},$$

oque

$$\Pi : x+\Pi : y=\text{Const.}-\int\frac{\partial u[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)]}{\surd(A+Lu+Euu)}.$$

Vel cum reperiatur

$$u = \frac{-(\delta-\gamma)-\varepsilon t+\sqrt{[(\delta-\gamma)^2-\alpha\zeta+2[(\delta-\gamma)\varepsilon-\beta\zeta]t+(\varepsilon\varepsilon-\gamma\zeta)tt]}}{\gamma}$$

quae expressio abit in hanc

$$u = \frac{-(\delta-\gamma)-\varepsilon t+2\sqrt{\Delta(L+2Dt+Ett)}}{\zeta}:$$

unde fit

$$\partial V = \frac{\partial t[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-\varkappa)+\mathfrak{E}t(tt-2\varkappa)]}{2\sqrt{\Delta(L+C+2Dt+Ett)}}$$

sicque habebimus per t

$$\Pi : x + \Pi : y = \text{Const.} + \int\frac{\partial t[\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-\varkappa)+\mathfrak{E}t(tt-2\varkappa)]}{\sqrt{(L+C+2Dt+Ett)}},$$

quae expressio, nisi sit algebraica, certe vel per logarithmos, vel arcus circulares exhiberi potest. Tum vero post integrationem tantum opus est, ut loco t restituatur ejus valor $x+y$.

Corollarium 1.

646. Si velimus, ut posito $x=a$ fiat $y=b$; constans L ita debet definiri, ut sit

$$\tfrac{1}{2}L(b-a)^2 = A+B(a+b)+Cab+Dab(a+b)+Eaabb$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bb+Cbb+2Db^3+Eb^4)},$$

tum igitur constans nostra erit $=\Pi : a + \Pi : b$, integrali postremo ita sumto, ut evanescat posito $t=a+b$.

Corollarium 2.

647. Eodem modo etiam differentia functionum $\Pi : x - \Pi : y$ exprimi potest, mutando alterutrius formulae radicalis signum, quo pacto formularum differentialium signum alterius convertetur.

Corollarium 3.

648. Qvantitas V comparationi harum functionum inserviens, erit algebraica, si haec formula differentialis

$$\frac{\partial t[\mathfrak{B}+\mathfrak{C}\zeta t+\mathfrak{D}(\delta-\gamma+\varepsilon t+\zeta tt)+\mathfrak{E}(2(\delta-\gamma)+2\varepsilon t+\zeta tt)t]}{\zeta\sqrt{(L+C+2Bt+Ett)}}$$

integrationem admittat; quia altera pars $\frac{-2\partial t\sqrt{\Delta}}{\zeta}(\mathfrak{D}+2\mathfrak{E}t)$ per se est integrabilis.

Scholion.

649. Hoc ergo argumentum plane novum de comparatione jusmodi functionum transcendentium tam copiose pertractavimus, iam praesens institutum postulare videbatur. Quando autem ejus plicatio ad comparationem arcuum curvarum, quorum longitudo jusmodi functionibus exprimitur, erit facienda, uberiori evolutione it opus, ubi contemplatio singularium proprietatum, quae hoc mo- erveniunt, eximium usum afferre poterit. Commode autem hoc gumentum ad doctrinam de resolutione aequationum differentialium ferri videtur, siquidem inde ejusmodi aequationum integralia com- eta et quidem algebraice exhiberi possunt, quae aliis methodis istra indagantur. Hunc igitur huic sectionis finem faciet metho- s generalis omnium aequationum differentialium integralia proxime terminandi.

CAPUT VII.

DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM PER APPROXIMATIONEM.

Problema 85.

650.

Proposita aequatione differentiali quacunque, ejus integrale completum vero proxime assignare.

Solutio.

Sint x et y binae variabiles, inter quas aequatio differentialis proponitur, atque haec aequatio hujusmodi habebit formam ut sit $\frac{\partial y}{\partial x} = V$, existente V functione quaecunque ipsarum x et y. Jam cum integrale completum desideretur, hoc ita est interpretandum, ut dum ipsi x certus quidem valor puta $x = a$ tribuitur, altera variabilis y datum quemdam valorem puta $y = b$ adipiscatur. Quaestionem ergo primo ita tractemus, ut investigemus valorem ipsius y, quando ipsi x valor paulisper ab a discrepans tribuitur, seu posito $x = a + \omega$, ut quaeramus y. Cum autem ω sit particula minima, etiam valor ipsius y minime a b discrepabit; unde dum x ab a usque ad $a + \omega$ tantum mutatur, quantitatem V interea tanquam constantem spectare licet. Quare posito $x=a$ et $y=b$ fiat $V=A$, et pro hac exigua mutatione habebimus $\frac{\partial y}{\partial x}=A$, ideoque integrando $y=b+A(x-a)$, ejusmodi scilicet constante adjecta, ut posito $x=a$ fiat $y=b$. Statuamus ergo $x=a+\omega$, fietque $y=b+A\omega$. Quemadmodum ergo hic ex valoribus initio datis $x=a$ et $y=b$, proxime sequentes $x=a+\omega$ et $y=b+A\omega$ invenimus, ita ab his simili modo per intervalla minima ulterius progredi licet, quoad tandem ad valores a primitivis quantumvis remotos perveniatur. Quae operationes quo clarius ob oculos ponantur, sequenti modo successive instituantur.

Ipsius	Valores successivi
x	$a,\ a',\ a'',\ a''',\ a^{IV},\ \ldots\ {}'x,\ x$
y	$b,\ b',\ b'',\ b''',\ b^{IV},\ \ldots\ {}'y,\ y$
V	$A,\ A',\ A'',\ A''',\ A^{IV},\ \ldots\ {}'V,\ V.$

Scilicet ex primis $x=a$ et $y=b$ datis, habetur $V=A$; tum vero pro secundis erit $b'=b+A(a'-a)$, differentia $a'-a$ minima pro lubitu assumta. Hinc ponendo $x=a'$ et $y=b'$ colligitur $V=A'$, indeque pro tertiis obtinebitur $b''=b'+A'(a''-a')$, ubi posito $x=a''$ et $y=b''$ invenitur $V=A''$. Jam pro quartis habebimus $b'''=b''+A''(a'''-a'')$, hincque ponendo $x=a'''$ et $y=b'''$, colligemus $V=A'''$, sicque ad valores a primitivis quantumvis remotos progredi licebit. Series autem prima valores ipsius x successivos exhibens pro lubitu accipi potest, dummodo per intervalla minima ascendat vel etiam descendat.

Corollarium 1.

651. Pro singulis ergo intervallis minimis calculus eodem modo instituitur, sicque valores, a quibus sequentia pendent, obtinentur. Hoc ergo modo singulis pro x assumtis valoribus, valores respondentes ipsius y, assignari possunt.

Corollarium 2.

652. Quo minora accipiuntur intervalla, per quae valores ipsius x progredi assumuntur, eo accuratius valores pro singulis eliciuntur. Interim tamen errores in singulis commissi, etiamsi sint multo minores, ob multitudinem coacervantur.

Corollarium 3.

653. Errores autem in hoc calculo inde oriuntur, quod in singulis intervallis ambas quantitates x et y ut constantes spectemus, sicque functio V pro constante habeatur. Quo magis ergo valor ipsius V a quovis intervallo ad sequens immutatur, eo majores errores sunt pertimescendi.

Scholion 1.

654. Hoc incommodum imprimis occurrit, ubi valor ipsius V vel evanescit vel in infinitum excrescit, etiamsi mutationes ipsis x et y accidentes sint satis parvae. His autem casibus errores saltim enormes sequenti modo evitabuntur: sit pro initio hujusmodi intervalli $x = a$ et $y = b$, tum vero in ipsa aequatione proposita ponatur $x = a + \omega$ et $y = b + \psi$, ut sit $\frac{\partial \psi}{\partial \omega} = V$, in V autem ita fiat substitutio $x = a + \omega$ et $y = b + \psi$, ut quantitates ω et ψ tanquam minimae spectentur, rejiciendo scilicet altiores potestates prae inferioribus, hoc enim modo plerumque integratio pro his intervallis actu institui poterit. Hac autem emendatione vix unquam erit opus, nisi termini ex ipsis valoribus a et b nati se destruant. Veluti si habeatur haec aequatio $\frac{\partial y}{\partial x} = \frac{aa}{xx - yy}$, ac pro initio debeat esse $x = a$ et $y = a$; jam pro intervallo hinc incipiente ponatur $x = a + \omega$ et $y = a + \psi$ habebiturque $\frac{\partial \psi}{\partial \omega} = \frac{aa}{2a\omega - 2a\psi}$, seu $2\omega\partial\psi - 2\psi\partial\psi = a\partial\omega$, seu $\partial\omega - \frac{2\omega\partial\psi}{a} = \frac{-2\psi\partial\psi}{a}$, quae per $e^{\frac{-2\psi}{a}} = 1 - \frac{2\psi}{a}$ multiplicata et integrata praebet

$$\left(1 - \frac{2\psi}{a}\right)\omega = \frac{-2}{a}\int\left(1 - \frac{2\psi}{a}\right)\psi\partial\psi = -\frac{\psi\psi}{a},$$

quia posito $\omega = 0$ fieri debet $\psi = 0$. Hinc ergo habetur $\omega = \frac{-\psi\psi}{a-2\psi} = \frac{-\psi\psi}{a}$, seu $a(a'-a) = -(b'-b)^2$, existente $b = a$; unde colligitur pro sequente intervallo $b' = b + \sqrt{-a(a'-a)}$, quo casu patet valorem x non ultra a augeri posse, quia y fieret imaginarium.

Scholion 2.

655. Passim traduntur regulae aequationum differentialium integralia per series infinitas exprimendi, quae autem plerumque hoc vitio laborant, ut integralia tantum particularia exhibeant, praeterquam quod series illae certo tantum casu convergant, neque ergo aliis casibus ullum usum praestent. Veluti si proposita sit aequatio $\partial y + y\,\partial x = a x^n \partial x$, jubemur hujusmodi seriem in genere fingere

$$y = A x^{\alpha} + B x^{\alpha+1} + C x^{\alpha+2} + D x^{\alpha+3} + E x^{\alpha+4} + \text{etc.}$$

qua substituta fit

$$\left.\begin{array}{l} \alpha A x^{\alpha-1} + (\alpha+1) B x^{\alpha} + (\alpha+2) C x^{\alpha+1} + (\alpha+3) D x^{\alpha+2} + \text{etc.} \\ \qquad\qquad + \quad A \quad + \quad B \quad + \quad C \quad + \text{etc.} \\ - a x^n \end{array}\right\} = 0$$

Statuatur ergo $\alpha - 1 = n$, seu $\alpha = n + 1$, eritque $A = \frac{a}{n+1}$, tum vero reliquis terminis ad nihilum reductis

$$B = \frac{-A}{n+2},\ C = \frac{-B}{n+3},\ D = \frac{-C}{n+4},\ \text{etc.}$$

sicque habebitur haec series

$$y = \frac{a x^{n+1}}{n+1} - \frac{a x^{n+2}}{(n+1)(n+2)} + \frac{a x^{n+3}}{(n+1)(n+2)(n+3)} - \frac{a x^{n+4}}{(n+1)(n+2)(n+3)(n+4)} \text{ etc.}$$

Verum hoc integrale tantum est particulare, quoniam evanescente x, simul y evanescit, nisi $n+1$ sit numerus negativus; tum vero haec series non convergit, nisi x capiatur valde parvum. Quamobrem hinc minime cognoscere licet valores ipsius y, qui respondeant valoribus quibuscunque ipsius x. Hoc autem vitio non laborat methodus, quam hic adumbravimus, cum primo integrale completum praebeat, dum scilicet pro dato ipsius x valore datum ipsi y valorem tribuit, tum vero per intervalla minima procedens, semper proxime ad veritatem accedat, et quousque libuerit progredi liceat. Sequenti autem modo haec methodus magis perfici poterit.

Problema 86.

656. Methodum praecedentem, aequationes differentiales proxime integrandi, magis perficere, ut minus a veritate aberret.

Solutio.

Proposita aequatione integranda $\frac{\partial y}{\partial x} = V$, error methodi supra expositae inde oritur, quod per singula intervalla functio V ut constans spectetur, cum tamen revera mutationem subeat, praecipue nisi intervalla statuantur minima. Variabilitas autem ipsius V per quodvis intervallum simili modo in computum duci potest, quo in sectione praecedente §. 321. usi sumus. Scilicet si jam ipsi x conveniat y, ex natura differentialium ipsi $x - n\partial x$ vidimus convenire

$$y - n\partial y + \frac{n(n+1)}{1.2.}\partial\partial y - \frac{n(n+1)(n+2)}{1.2.3.}\partial^3 y + \text{etc.}$$

qui valor sumto n infinito erit

$$y - n\partial y + \frac{nn\partial\partial y}{1.2} - \frac{n^3\partial^3 y}{1.2.3} + \frac{n^4\partial^4 y}{1.2.3.4} - \text{etc.}$$

Statuatur jam $x - n\partial x = a$ et

$$y - n\partial y + \frac{nn\partial\partial y}{1.2} - \frac{n^3\partial^3 y}{1.2.3} + \frac{n^4\partial^4 y}{1.2.3.4} - \text{etc.} = b,$$

hicque valores in quovis intervallo ut primi spectentur, dum extremi per x et y indicantur. Cum igitur sit $n = \frac{x-a}{\partial x}$. fiet

$$y = b + \frac{(x-a)\partial y}{\partial x} - \frac{(x-a)^2\partial\partial y}{1.2\partial x^2} + \frac{(x-a)^3\partial^3 y}{1.2.3\partial x^3} - \frac{(x-a)^4\partial^4 y}{1.2.3.4\partial x^4} + \text{etc.}$$

quae expressio, si x non multum superat a, valde convergit, ideoque admodum est idonea ad valorem y proxime inveniendum. Verum ad singulos terminos hujus seriei evolvendos, notari oportet esse $\frac{\partial y}{\partial x} = V$, hincque $\frac{\partial \partial y}{\partial x^2} = \frac{\partial V}{\partial x}$. Cum autem V sit functio ipsarum x et y, si ponamus $\partial V = M \partial x + N \partial y$, ob $\frac{\partial y}{\partial x} = V$, erit $\frac{\partial \partial y}{\partial x^2} = M + NV$, seu exprimendi modo jam supra exposito $\frac{\partial \partial y}{\partial x^2} = (\frac{\partial V}{\partial x}) + V(\frac{\partial V}{\partial y})$, quae expressio uti nata est ex praecedente $\frac{\partial y}{\partial x} = V$, ita ex ea nascetur sequens

$$\frac{\partial^3 y}{\partial x^3} = (\frac{\partial \partial V}{\partial x^2}) + (\frac{\partial V}{\partial x})(\frac{\partial V}{\partial y}) + 2V(\frac{\partial \partial V}{\partial x \partial y}) + V(\frac{\partial V}{\partial y})^2 + VV(\frac{\partial \partial V}{\partial y^2}).$$

Quoniam vero ipse valor ipsius y nondum est cognitus, hoc modo saltem obtinetur aequatio algebraica, qua relatio inter x et y exprimitur, nisi forte sufficiat in terminis posuisse $y = b$.

Altera autem operatio §. 322. exposita valorem ipsius y, qui ipsi x in fine cujusque intervalli respondet, explicite determinabit, cum in initio ejusdem intervalli fuerit $x = a$ et $y = b$. Cum enim hinc posito $x = a + n \partial a$, si quidem a et b ut variabiles spectemus, fiat

$$y = b + n \partial b + \frac{n(n-1)}{1.2} \partial \partial b + \frac{n(n-1)(n-2)}{1.2.3} \partial^3 b + \text{etc.}$$

quia est $n = \frac{x-a}{\partial a}$, ideoque numerus infinitus, erit

$$y = b + \frac{(x-a)\partial b}{\partial a} + \frac{(x-a)^2 \partial \partial b}{1.2 \partial a^2} + \frac{(x-a)^3 \partial^3 b}{1.2.3 \partial a^3} + \text{etc.}$$

Est vero $\frac{\partial b}{\partial a} = V$, siquidem in functione V scribatur $x = a$ et $y = b$; tum vero iisdem pro x et y valoribus substitutis, erit

$$\frac{\partial \partial b}{\partial a^2} = (\frac{\partial V}{\partial x}) + V(\frac{\partial V}{\partial y}) \text{ et}$$

$$\frac{\partial^3 b}{\partial a^3} = (\frac{\partial \partial V}{\partial x^2}) + 2V(\frac{\partial \partial V}{\partial x \partial y}) + VV(\frac{\partial \partial V}{\partial y^2}) + (\frac{\partial V}{\partial y})[(\frac{\partial V}{\partial x}) + V(\frac{\partial V}{\partial y})],$$

unde sequentes simili modo formari oportet. Sit igitur postquam, scripserimus $x = a$ et $y = b$,

$$\frac{\partial y}{\partial x} = A, \frac{\partial \partial y}{\partial x^2} = B, \frac{\partial^3 y}{\partial x^3} = C, \frac{\partial^4 y}{\partial x^4} = D, \text{ etc.}$$

ac valori $x = a + \omega$ conveniet iste valor

$$y = b + A\omega + \tfrac{1}{2} B\omega^2 + \tfrac{1}{6} C\omega^3 + \tfrac{1}{24} D\omega^4 + \text{etc.}$$

qui duo valores jam pro sequente intervallo erunt initiales, ex quibus simili modo finales erui oportet.

Corollarium 1.

667. Quoniam hic variabilitatis functionis V rationem habuimus, intervalla jam majora statuere licet, ac si illas formulas A, B, C, D, etc. in infinitum continuare vellemus, intervalla quantumvis magna assumi possent, tum autem pro y oriretur series infinita.

Corollarium 2.

668. Si seriei inventae tantum binos terminos primos sumamus, ut sit $y = b + A\omega$, habebitur determinatio praecedens, unde simul patet errorem ibi commissum sequentibus terminis junctim sumtis aequari.

Corollarium 3.

669. Etiamsi autem seriei inventae plures terminos capiamus, consultum tamen non erit intervalla nimis magna constitui, ut ω valorem modicum obtineat, praecipue si quantitates B, C, D, etc. evadant valde magnae.

Scholion.

660. Maximo incommodo hae operationes turbantur, si quando horum coëfficientium A, B, C, D, etc. quidam in infinitum excrescant. Evenit autem hoc tantum in certis intervallis, ubi ipsa quantitas V vel in nihilum abit vel in infinitum, cui incommodo, quemadmodum sit occurrendum, jam innuimus et mox accuratius ostendemus. Caeterum calculus pro singulis intervallis pari modo instituitur, ita ut cum ejus ratio pro intervallo primo fuerit inventa, quod incipit a valoribus pro lubitu assumtis $x = a$ et $y = b$, ex-

dem pro sequentibus intervallis sit valitura. Cum enim pro fine intervalli primi fiat

$$x = a + \omega = a' \text{ et}$$

$$y = b + A\omega + \tfrac{1}{2}B\omega^2 + \tfrac{1}{6}C\omega^3 + \tfrac{1}{24}D\omega^4 + \text{etc.} = b',$$

hi erunt valores initiales pro intervallo secundo, ex quibus simili modo finales elici oportet; hic scilicet calculus innitetur perinde litteris a' et b', ac prior litteris a et b, id quod clarius ex exemplis subjunctis patebit.

Exemplum 1.

661. *Aequationis differentialis* $\partial y = \partial x(x^n + cy)$ *integrale completum proxime investigare.*

Cum hic sit $V = \frac{\partial y}{\partial x} = x^n + cy$, erit differentiando

$$\frac{\partial\partial y}{\partial x^2} = nx^{n-1} + cx^n + ccy,$$

sicque porro

$$\frac{\partial^3 y}{\partial x^3} = n(n-1)x^{n-2} + ncx^{n-1} + ccx^n + c^3 y$$

$$\frac{\partial^4 y}{\partial x^4} = n(n-1)(n-2)x^{n-3} + n(n-1)cx^{n-2} + nccx^{n-1} + c^3x^n + c^4y$$

etc.

Quodsi ergo ponamus valori $x = a$, convenire $y = b$, alii cuicunque valori $x = a + \omega$ conveniet

$$y = b + \omega(cb + a^n) + \tfrac{1}{2}\omega^2(ccb + ca^n + na^{n-1})$$
$$+ \tfrac{1}{6}\omega^3[c^3b + cca^n + nca^{n-1} + n(n-1)a^{n-2}]$$
$$+ \tfrac{1}{24}\omega^4[c^4b + c^3a^n + ncca^{n-1} + n(n-1)ca^{n-2} + n(n-1)(n-2)a^{n-3}]$$

etc.

quae series sumta quantitate ω satis parva, quantumvis promte convergit, sicque posito $a + \omega = a'$ et respondente valore ipsius $y = b'$, hinc simili modo ad sequentes perveniemus, quam operationem, quousque lubuerit, continuare licet.

Exemplum 2.

662. *Aequationis differentialis $\partial y = \partial x\,(xx + yy)$ integrale completum proxime investigare.*

Cum hic sit $\frac{\partial y}{\partial x} = V = xx + yy$, erit continuo differentiando

$$\frac{\partial\partial y}{\partial x^2} = 2x + 2xxy + 2y^3 \text{ et}$$

$$\frac{\partial^3 y}{\partial x^3} = 2 + 4xy + 2x^4 + 8xxyy + 6y^4$$

$$\frac{\partial^4 y}{\partial x^4} = 4y + 12x^3 + 20xyy + 16x^4y + 40xxy^3 + 24y^5$$

$$\frac{\partial^5 y}{\partial x^5} = 40x^2 + 24y^2 + 104x^3y + 120xy^3 + 16x^6 + 136x^4y^2 + 240x^2y^4 + 120y^6.$$

Quare si initio sit $x = a$ et $y = b$, erit

$$A = aa + bb$$

$$B = 2a + 2aab + 2b^3$$

$$C = 2 + 4ab + 2a^4 + 8aabb + 6b^4$$

$$D = 4b + 12a^3 + 20abb + 16a^4b + 40aab^3 + 24b^5$$

$$E = 40a^2 + 24b^2 + 104a^3b + 120ab^3 + 16a^6 + 136a^4b^2 + 240a^2b^4 + 120b^6,$$

unde valori cuicunque alii $x = a + \omega$ conveniet

$$y = b + A\omega + \tfrac{1}{2}B\omega^2 + \tfrac{1}{6}C\omega^3 + \tfrac{1}{24}D\omega^4 + \tfrac{1}{120}E\omega^5 + \text{etc.}$$

atque ex talibus binis valoribus, qui sint $x = a'$ et $y = b'$, denuo sequentes elici possunt.

Scholion.

663. Quoniam totum negotium ad inventionem horum coëfficientium A, B, C, D, etc. redit, observo eosdem sine differentiatione inveniri posse, id quod in hoc postremo exemplo $\frac{\partial y}{\partial x} =$

$xx + yy$ ita praestabitur. Cum statuamus posito $x = a$ fieri $y = b$, ponamus in genere $x = a + \omega$ et $y = b + \psi$, et nostra aequatio induet hanc formam

$$\frac{\partial \psi}{\partial \omega} = aa + bb + 2a\omega + \omega\omega + 2b\psi + \psi\psi$$

et quia evanescente ω simul evanescit ψ, sumamus

$$\psi = \alpha\omega + \beta\omega^2 + \gamma\omega^3 + \delta\omega^4 + \varepsilon\omega^5 + \text{etc.}$$

hocque valore substituto prodibit

$$\alpha + 2\beta\omega + 3\gamma\omega^2 + 4\delta\omega^3 + 5\varepsilon\omega^4 + \text{etc.} =$$

$$aa + bb + 2a\omega + \omega\omega$$
$$+ 2\alpha b\omega + 2\beta b\omega^2 + 2\gamma b\omega^3 + 2\delta b\omega^4 + \text{etc.}$$
$$+ \alpha^2\omega^2 \quad + 2\alpha\beta\omega^3 + 2\alpha\gamma\omega^4 + \text{etc.}$$
$$+ \beta\beta\omega^4$$

singulis ergo terminis ad nihilum reductis fiet

$$\alpha = aa + bb,\ 2\beta = 2\alpha b + 2a,\ 3\gamma = 2\beta b + \alpha\alpha + 1,$$
$$4\delta = 2\gamma b + 2\alpha\beta,\ 5\varepsilon = 2\delta b + 2\alpha\gamma + \beta\beta$$
$$6\zeta = 2\varepsilon b + 2\alpha\delta + 2\beta\gamma,\ \text{etc.}$$

unde iidem valores qui supra per differentiationem eliciuntur. Vti haec methodus simplicior est praecedente, ita etiam hoc illi praestat, quod semper in usum vocari possit, cum illa interdum frustra applicetur, veluti in exemplis allatis evenit, si valores initiales a et b evanescant, ubi plerique coëfficientes in nihilum abirent. Quod idem incommodum jam supra animadvertimus, cum adeo evenire possit, ut omnes coëfficientes vel evanescant, vel in infinitum abeant. Verum hoc nonnisi in certis intervallis usu venit, pro quibus ergo calculum peculiari modo institui conveniet; reliquis autem intervallis methodus hic exposita per differentiationem procedens commodius adhiberi videtur, quippe quae saepe facilius instituitur quam substitutio, certisque regulis continetur, semper locum habentibus etiam in aequationibus transcendentibus. Quare pro singularibus illis intervallis praecepta tradere oportebit.

Problema 87.

664. Si in integratione aequationis $\frac{\partial y}{\partial x} = V$ pro quopiam intervallo eveniat, ut quantitas V vel evanescat, vel fiat infinita, integrationem pro isto intervallo instituere.

Solutio.

Sit pro initio intervalli, quod contemplamur $x = a$ et $y = b$, quo casu cum V vel evanescat vel in infinitum abeat, ponamus $\frac{\partial y}{\partial x} = \frac{P}{Q}$, ita ut posito $x = a$ et $y = b$, vel P vel Q vel utrumque evanescat. Statuamus ergo ut ab his terminis ulterius progrediamur, $x = a + \omega$ et $y = b + \psi$, fictque $\frac{\partial y}{\partial x} = \frac{\partial \psi}{\partial \omega}$: atque tam P quam Q erit functio ipsarum ω et ψ, quarum altera saltem evanescat, facto $\omega = 0$ et $\psi = 0$. Jam ad rationem inter ω et ψ proxime saltem investigandam, ponatur $\psi = m\,\omega^n$, erit $\frac{\partial \psi}{\partial \omega} = m\,n\,\omega^{n-1}$, hincque $m\,n\,Q\,\omega^{n-1} = P$; ubi P et Q ob $\psi = m\,\omega^n$ meras potestates ipsius ω continebunt, quarum tantum minimas in calculo retinuisse sufficit, cum altiores prae his ut evanescentes spectari queant. Infimae ergo potestates ipsius ω inter se aequales reddantur, simulque ad nihilum redigantur; unde tam exponens n quam coëfficiens m determinabitur. Si deinde relationem inter ω et ψ exactius cognoscere velimus, inventis m et n, ad altiores potestates ascendamus ponendo

$$\psi = m\,\omega^n + M\,\omega^{n+\mu} + N\,\omega^{n+\nu} \text{ etc.}$$

hincque simili modo sequentes partes definientur, quousque ob magnitudinem intervalli seu particulae ω necessarium visum fuerit.

Corollarium 2.

665. Si posito $x = a$ et $y = b$, neque P neque Q evane-

scat, substitutione adhibita reperietur $\frac{\partial \psi}{\partial \omega} = \frac{A + etc.}{\alpha + etc.}$, hincque proxime $\alpha \partial \psi = A \partial \omega$ et $\psi = \frac{A}{\alpha} \omega$, qui est primus terminus praecedentis approximationis, quo invento reliqui ut ante se habebunt.

Corollarium 2.

666. Si α tantum evanescat, habebitur

$$\frac{\partial \psi}{\partial \omega} (M \omega^{\mu} + N \psi^{\nu} \text{ etc.}) = A$$

proxime: unde posito $\psi = m \omega^n$ fit

$$A = m n \omega^{n-1} (M \omega^{\mu} + N m^{\nu} \omega^{n \nu});$$

quod autem non valet, nisi sit $\nu (1 - \mu) > \mu$ seu $\nu > \frac{\mu}{1-\mu}$. Sin autem sit $\nu < \frac{\mu}{1-\mu}$, statui debet $n - 1 + n \nu = 0$ seu $n = \frac{1}{1+\nu}$, altero termino ut infima potestate spectata. At si fuerit $\nu = \frac{\mu}{1-\mu}$, ambo termini pro paribus potestatibus erunt habendi, fietque $n = 1 - \mu$ ut $A = m n (M + N m^{\nu})$, unde m definiri debet.

Scholion.

667. In genere hic vix quicquam praecipere licet, sed quovis casu oblato haud difficile est omnia, quae ad solutionem perducunt, perspicere. Si quidem omnes exponentes essent integri, regula illa *Newtoniana*, qua ope parallelogrammi resolutio aequationum instruitur, hic in usum vocari posset; tum vero exponentium fractorum ad integros reductio satis est nota. Verum hujusmodi casus tam raro occurrunt, ut inutile foret in praeceptis prolixum esse, quae quovis casu ab exercitatio facile conduntur. Veluti si perveniatur ad hanc aequationem $\frac{\partial \psi}{\partial \omega} (\alpha \sqrt{\omega} + \beta \psi) = \gamma$, ex superioribus patet primam operationem dare $\psi = m \sqrt{\omega}$, unde fit $\frac{1}{2} m (\alpha + \beta m) = \gamma$, unde m innotescit idque duplici modo. Quin etiam haec aequatio, posito $\sqrt{\omega} = p$, ad homogeneitatem reducitur,

ideoque revera integrari potest: verum haec vix unquam usum habitura fusius non prosequor, sed, quod adhuc in hac parte pertractandum restat exponam, quomodo ejusmodi aequationes differentiales resolvi oporteat, in quibus differentialium ratio puta $\frac{\partial y}{\partial x} = p$ vel plures obtinet dimensiones, vel adeo transcendenter ingreditur, quo absoluto partem secundam, in qua differentialia altiorium graduum occurrunt, aggrediar.

CALCULI INTEGRALIS

LIBER PRIOR.

PARS PRIMA,

SEU

METHODUS INVESTIGANDI FUNCTIONES UNIUS VARIABILIS EX DATA RELATIONE QUACUNQVE DIFFERENTIALIUM PRIMI GRADUS.

SECTIO TERTIA.

DE

RESOLUTIONE AEQUATIONAM DIFFERENTIALIUM MAGIS COMPLICATARUM.

DE
RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM IN QUIBUS DIFFERENTIALIA AD PLURES DIMENSIONES ASSURGUNT, VEL ADEO TRANSCENDENTER IMPLICANTUR.

Problema 88.

668.

Posita differentialium relatione $\frac{\partial y}{\partial x} = p$, si proponatur aequatio quaecunque inter binas quantitates x et p, relationem inter ipsas variabiles x et y investigare.

Solutio.

Cum detur aequatio inter p et x, concessa aequationum resolutione, ex ea quaeratur p per x, ac reperietur functio ipsius x, quae ipsi p erit aequalis. Pervenietur ergo ad hujusmodi aequationem $p = X$, existente X functione quapiam ipsius x tantum. Quare cum sit $p = \frac{\partial y}{\partial x}$, habebimus $\partial y = X \partial x$, sicque quaestio ad sectionem primam est reducta, unde formulae $X \partial x$ integrale investigari oportet, quo facto integrale quaesitum erit $y = \int X \partial x$.

Si aequatio inter x et p data ita fuerit comparata, ut inde facilius x per p definiri possit, quaeratur x, prodeatque $x = P$, existente P functione quadam ipsius p. Hac igitur aequatione differentiata erit $\partial x = \partial P$, hincque $\partial y = p \partial x = p \partial P$, unde integrando elicitur $y = \int p \partial P$, seu $y = p P - \int P \partial p$. Hinc ergo ambae variabiles x et y per tertiam p ita determinantur, ut sit

$$x = P \text{ et } y = pP - \int P \partial p,$$

unde relatio inter x et y est manifesta.

Si neque p commode per x, neque x per p definiri queat, saepe effici potest, ut utraque commode per novam quantitatem u definiatur; ponamus ergo inveniri $x = U$ et $p = V$, ut U et V sint functiones ejusdem variabilis u. Hinc ergo erit $\partial y = p \partial x = V \partial U$, et $y = \int V \partial U$, sicque x et y per eandem novam variabilem u exprimuntur.

Corollarium 1.

669. Simili modo resolvetur casus, quo aequatio quaecunque inter p et alteram variabilem y proponitur, quoniam binas variabiles x et y inter se permutare licet. Tum autem sive p per y, sive y per p, sive utraque per novam variabilem u definiatur, notari oportet esse $\partial x = \frac{\partial y}{p}$.

Corollarium 2.

670. Cum $\sqrt{(\partial x^2 + \partial y^2)}$ exprimat elementum arcus curvae, cujus coordinatae rectangulae sunt x et y, si ratio

$$\frac{\sqrt{(\partial x^2 + \partial y^2)}}{\partial x} = \sqrt{(1 + pp)}, \text{ seu } \frac{\sqrt{(\partial x^2 + \partial y^2)}}{\partial y} = \frac{\sqrt{(1 + pp)}}{p},$$

aequetur functioni vel ipsius x vel ipsius y, hinc relatio inter x et y inveniri poterit.

Corollarium 3.

671. Quoniam hoc modo relatio inter x et y per integrationem invenitur, simul nova quantitas constans introducitur, quocirca illa relatio pro integrali completo erit habenda.

Scholion 1.

672. Hactenus ejusmodi tantum aequationes differentiales exa-

mini subjicimus, quibus posito $\frac{\partial y}{\partial x} = p$, ejusmodi relatio inter ternas quantitates x, y et p proponitur, unde valor ipsius p commode per x et y exprimi potest, ita ut $p = \frac{\partial y}{\partial x}$ aequetur functioni cuipiam ipsarum x et y. Nunc igitur ejusmodi relationes inter x, y et p considerandae veniunt, ex quibus valorem ipsius p vel minus commode, vel plane non, per x et y definire liceat; atque hic simplicissimus casus sine dubio est, quando in relatione proposita altera variabilis x seu y plane deest, ita ut tantum relatio inter p et x vel p et y proponatur; quem casum in hoc problemate expedivimus. Solutionis autem vis in eo versatur, ut proposita aequatione inter x et p, non littera p per x, nisi forte hoc facile praestari queat, sed potius x per p, vel etiam utraque per novam variabilem u definiatur. Veluti si proponatur haec aequatio

$$x \partial x + a \partial y = b \surd (\partial x^2 + \partial y^2),$$

quae posito $\frac{\partial y}{\partial x} = p$, abit in hanc

$$x + ap = b \surd (1 + pp),$$

hinc minus commode definiretur p per x. Cum autem sit

$$x = b \surd (1 + pp) - ap, \text{ ob } y = \int p \partial x = px - \int x \partial p,$$

erit

$$y = bp \surd (1 + pp) - app - b \int \partial p \surd (1 + pp) + \tfrac{1}{2} app;$$

sicque relatio inter x et y constat. Sin autem perventum fuerit ad talem aequationem

$$x^3 \partial x^3 + \partial y^3 = a x \partial x^2 \partial y \text{ seu } x^3 + p^3 = apx,$$

hic neque x per p neque p per x commode definire licet; ex quo pono $p = ux$, unde fit $x + u^3 x = au$, hincque $x = \frac{au}{1 + u^3}$ et $p = \frac{auu}{1 + u^3}$. Jam ob $\partial x = \frac{a \partial u (1 - 2u^3)}{(1 + u^3)^2}$, colligitur $y = aa \int \frac{uu \partial u (1 - 2u^3)}{(1 + u^3)^3}$, ac reducendo hanc formam ad simpliciorem

$$y = \tfrac{1}{6} aa . \frac{2u^3 - 1}{(1 + u^3)^2} - aa \int \frac{uu \partial u}{(1 + u^3)^2} \text{ seu}$$

$$y = \tfrac{1}{6}aa . \frac{2u^3 - 1}{(1 + u^3)^2} + \tfrac{1}{3}aa . \frac{1}{1 + u^3} + \text{Const.}$$

Scholion 2.

673. Cum igitur hunc casum, quo aequatio vel inter x et p vel inter y et p proponitur, generatim expedire licuerit, videndum est quibus casibus evolutio succedat, quando omnes tres quantitates x, y et p in aequatione proposita insunt. Ac primo quidem observo, dummodo binae variabiles x et y ubique eundem dimensionum numerum adimpleant, quomodocunque praeterea quantitas p ingrediatur, resolutionem semper ad casus ante tractatos revocari posse; tales scilicet aequationes perinde tractare licet, atque aequationes homogeneas, ad quod genus etiam merito referuntur, cum dimensiones a differentialibus natae ubique debeant esse pares, et indicium ex solis quantitatibus finitis x et y peti oporteat. Quae ergo dummodo ubique eundem dimensionum numerum constituant, aequatio pro homogenea erit habenda, veluti est

$$xx\partial y - yy\sqrt{(\partial x^2 + \partial y^2)} = 0 \text{ seu}$$

$$pxx - yy\sqrt{(1 + pp)} = 0.$$

Deinde etiam ejusmodi aequationes evolutionem admittunt, in quibus altera variabilis x vel y plus una dimensione nusquam habet, utcunque praeterea differentialium ratio $p = \frac{\partial y}{\partial x}$ ingrediatur. Hos ergo casus hic accuratius explicemus.

Problema 89.

674. Posito $p = \frac{\partial y}{\partial x}$, si in aequatione inter x, y et p proposita binae variabiles x et y ubique eundem dimensionum numerum compleant, invenire relationem inter x et y, quae illius aequationis sit integrale completum.

Solutio.

Cum in aequatione inter x, y et p proposita binae variabiles

x et y ubique eundem dimensionum numerum constituant, si ponamus $y = u x$, quantitas x inde per divisionem tolletur, habebiturque aequatio inter duas tantum quantitates u et p, qua earum relatio ita definietur, ut vel u per p, vel p per u determinari possit. Jam ex positiene $y = u x$ sequitur $\partial y = u \partial x + x \partial u$, cum igitur sit $\partial y = p \partial x$, erit $p \partial x - u \partial x = x \partial u$, ideoque $\frac{\partial x}{x} = \frac{\partial u}{p - u}$. Quia itaque p per u datur, formula differentialis $\frac{\partial u}{p - u}$ unicam variabilem complectens per regulas primae sectionis integretur, eritque $l x = \int \frac{\partial u}{p - u}$, sicque x per u determinatur; et cum sit $y = u x$, ambae variabiles x et y per eandem tertiam variabilem u determinantur, et quia illa integratio constantem arbitrariam inducit, haec relatio inter x et y erit integrale completum.

Corollarium 1.

675. Cum sit $\frac{\partial x}{x} = \frac{\partial u}{p - u}$, erit etiam $l x = - l (p - u) + \int \frac{\partial p}{p - u}$, quae formula commodior est, si forte ex aequatione inter p et u proposita, quantitas u facilius per p definitur.

Corollarium 2.

676. Quodsi integrale $\int \frac{\partial u}{p - u}$ vel $\int \frac{\partial p}{p - u}$ per logarithmos exprimi possit, ut sit $\int \frac{\partial u}{p - u} = l \mathrm{U}$, erit $l x = l \mathrm{C} + l \mathrm{U}$; hincque $x = \mathrm{C\, U}$, et $y = \mathrm{C\, U}\, u$; unde relatio inter x et y algebraice dabitur: et cum sit $u = \frac{y}{x}$, haec tertia variabilis u facile eliditur.

Scholion.

677. Eandem hanc resolutionem supra in aequationibus homogeneis ordinariis docuimus, quae ergo ob dimensiones differentialium non turbatur; quin etiam succedit, etiamsi ratio differentialium

$\frac{\partial y}{\partial x} = p$ transcendenter ingrediatur. Hoc modo scilicet resolutio ad integrationem aequationis differentialis separatae $\frac{\partial x}{x} = \frac{\partial u}{p - u}$ perducitur, quemadmodum etiam supra per priorem methodum negotium fuit expeditum. Altera vero methodus, qua supra usi sumus, quaerendo factorem qui aequationem differentialem reddat per se integrabilem, hic plane locum non habet, cum per differentiationem aequationis finitae nunquam differentialia ad plures dimensiones exsurgere queant. Non ergo hoc modo invenitur aequatio finita inter x et y, quae differentiata ipsam aequationem propositam reproducat, sed quae saltem cum ea conveniat, et quidem non obstante arbitraria illa constante, quae per integrationem ingressa, integrale completum reddit.

Exemplum 1.

678. *Si in aequationem propositam neutra variabilium x et y ipsa ingrediatur, sed tantum differentialium ratio $\frac{\partial y}{\partial x} = p$, integrale completum assignare.*

Posito ergo $\frac{\partial y}{\partial x} = p$, aequatio proposita solam variabilem p cum constantibus complectetur, unde ex ejus resolutione, prout plures involvat radices, orietur $p = \alpha$, $p = \beta$, $p = \gamma$ etc. Jam ob $p = \frac{\partial y}{\partial x}$, ex singulis radicibus integralia completa elicientur, quae erunt

$$y = \alpha x + a, \quad y = \beta x + b, \quad y = \gamma x + c, \text{ etc.}$$

quae singula aequationi propositae aeque satisfaciunt. Quae si velimus omnia una aequatione finita complecti, erit integrale completum

$$(y - \alpha x - a)(y - \beta x - b)(y - \gamma x - c) \text{ etc.} = 0,$$

quae uti apparet non unam novam constantem, sed plures a, b, c, etc. comprehendit, tot scilicet, quot aequatio differentialis plurium dimensionum habuerit radices.

Corollarium 1.

679. Ita aequationis differentialis $\partial y^2 - \partial x^2 = 0$ seu $p - 1 = 0$, ob $p = +1$ et $p = -1$, duo habemus integra- $y = x + a$ et $y = -x + b$, quae in unum collecta dant $(y - x - a)(y + x - b) = 0$, seu

$$yy - xx - (a + b)y - (a - b)x + ab = 0.$$

Corollarium 2.

680. Proposita aequatione $\partial y^3 + \partial x^3 = 0$ seu $p^3 + 1 = 0$, radices $p = -1$, $p = \frac{1+\sqrt{-3}}{2}$, et $p = \frac{1-\sqrt{-3}}{2}$, erit vel $y = -x + a$, vel $y = \frac{1+\sqrt{-3}}{2} x + b$, vel $y = \frac{1-\sqrt{-3}}{2} x + c$, quae collecta praebent

$$y^3 + x^3 - (a + b + c)yy + (a - \tfrac{1-\sqrt{-3}}{2} b - \tfrac{1+\sqrt{-3}}{2} c)xy$$
$$+ (-a + \tfrac{1-\sqrt{-3}}{2} b + \tfrac{1+\sqrt{-3}}{2} c)xx + (ab + ac + bc)y$$
$$+ (bc - \tfrac{1-\sqrt{-3}}{2} ac - \tfrac{1+\sqrt{-3}}{2} ab)x - abc = 0,$$

quae aequatio etiam ita exhiberi potest

$$y^3 + x^3 - fyy - gxy - hxx + Ay + Bx + C = 0,$$

ubi constantes A, B, C, ita debent esse comparatae, ut aequatio haec resolutionem in tres simplices admittat.

Exemplum 2.

681. *Proposita aequatione differentiali*

$$y\partial x - x\sqrt{(\partial x^2 + \partial y^2)} = 0,$$

eius integrale completum invenire.

Posito $\frac{\partial y}{\partial x} = p$, fit $y - x\sqrt{(1 + pp)} = 0$; sit ergo $y = ux$, erit $u = \sqrt{(1 + pp)}$, et $\frac{\partial x}{x} = \frac{\partial u}{p - u}$, unde per alteram formulam cujus integrale est $\frac{yy}{x} + x = 2a$, ut ante, nisi quod altera solutio $x = 0$ hinc non eliciatur. Verum cum aequatio illa quadrata posito $n = 1$, subito abeat in simplicem, altera radix perit, quae reperitur ponendo $n = 1 - \alpha$, quo fit

$$yy - 2pxy = xx - 2\alpha xx - 2\alpha ppxx,$$

ideoque px infinitum, rejectis ergo terminis prae reliquis evanescentibus est $-pxy = xx - 2\alpha ppxx$, quae divisibilis per x, alteram praebet solutionem $x = 0$. Talis quidem resolutio succedit, quando valorem p per radicis extractionem elicere licet; sed si aequatio ad plures dimensiones ascendat, vel adeo transcendens fiat, methodo hic exposita carere non possumus.

Exemplum 4.

684. *Proposita aequatione*

$$x\partial y^3 + y\partial x^3 = \partial y\partial x\sqrt{xy(\partial x^2 + \partial y^2)},$$

ejus integrale completum investigare.

Posito $\frac{\partial y}{\partial x} = p$, et $y = ux$, nostra aequatio induet hanc formam $p^3 + u = p\sqrt{u(1 + pp)}$, unde conficitur

$$\frac{\partial x}{x} = \frac{\partial u}{p - u}, \text{ seu } lx = \int\frac{\partial u}{p - u} = -l(p - u) + \int\frac{\partial p}{p - u}.$$

Inde autem est

$$\sqrt{u} = \tfrac{1}{2}p\sqrt{(1 + pp)} + \tfrac{1}{2}p\sqrt{(1 - 4p + pp)},$$

et quadrando

$$u = \tfrac{1}{2}pp - p^3 + \tfrac{1}{2}p^4 + \tfrac{1}{2}pp\sqrt{(1 + pp)(1 - 4p + pp)},$$

hincque

$$p - u = \tfrac{1}{2}p(1 + pp)(2 - p) - \tfrac{1}{2}pp\sqrt{(1 + pp)(1 - 4p + pp)},$$

unde colligimus

$$\frac{\partial p}{p - u} = \frac{\partial p(2 - p)}{2p(1 - p + pp)} + \frac{\partial p\sqrt{(1 - 4p + pp)}}{2(1 - p + pp)\sqrt{(1 + pp)}}.$$

In quorum membrorum posteriore, si ponatur $\sqrt{\frac{1 - 4p + pp}{1 + pp}} = q$, ob

$p = \frac{2 + \sqrt{[4 - (1 - qq)^2]}}{1 - qq}$, $\partial p = \frac{4q\partial q\,[2 + \sqrt{(4 - (1 - qq)^2)}]}{(1 - qq)^2 \sqrt{[4 - (1 - qq)^2]}}$, et

$$1 - p + pp = \frac{(3 + qq)\,[2 + \sqrt{(4 - (1 - qq)^2)}]}{(1 - qq)^2}$$

obtinebitur

$$\int \frac{\partial p}{p - u} = \tfrac{1}{2} \int \frac{\partial p\,(2 - p)}{p\,(1 - p + pp)} + 2 \int \frac{qq\partial q}{(3 + qq)\sqrt{[4 - (1 - qq)^2]}},$$

ubi membrum posterius neque per logarithmos, neque arcus circulares integrari potest.

Exemplum 5.

685. *Invenire relationem inter x et y, ut posito $s = \int \sqrt{(\partial x^2 + \partial y^2)}$, fiat $ss = 2xy$.*

Cum sit $s = \sqrt{2xy}$, erit

$$\partial s = \sqrt{(\partial x^2 + \partial y^2)} = \frac{x\partial y + y\partial x}{\sqrt{2xy}},$$

hincque posito $\frac{\partial y}{\partial x} = p$ et $y = ux$, fiet $\sqrt{(1 + pp)} = \frac{p + u}{\sqrt{2u}}$, seu $u = \sqrt{2u(1 + pp)} - p$, et radice extracta

$$\sqrt{u} = \sqrt{\frac{1 + pp}{2}} + \frac{1 - p}{\sqrt{2}} = \frac{1 - p + \sqrt{(1 + pp)}}{\sqrt{2}},$$

quare

$$u = 1 - p + pp + (1 - p)\sqrt{(1 + pp)}, \text{ et}$$

$$p - u = -(1 - p)[1 - p + \sqrt{(1 + pp)}].$$

Ergo

$$\int \frac{\partial p}{p - u} = \int \frac{\partial p}{2p(1 - p)} [1 - p - \sqrt{(1 + pp)}] = \tfrac{1}{2} lp - \tfrac{1}{2} \int \frac{\partial p \sqrt{(1 + pp)}}{p(1 - p)}.$$

At posito $p = \frac{1 - qq}{2q}$, fit

$$\int \frac{\partial p \sqrt{(1 + pp)}}{p(1 - p)} = \int \frac{-\partial q(1 + qq)^2}{q(1 - qq)(qq + 2q - 1)} = + \int \frac{\partial q}{q} - 2\int \frac{\partial q}{1 - qq} - 4\int \frac{\partial q}{(q + 1)^2 - 2}$$

$$= + lq - l\frac{1 + q}{1 - q} + \sqrt{2}\, l \frac{\sqrt{2} + 1 + q}{\sqrt{2} - 1 - q},$$

hincque

$$\int \frac{\partial p}{p-u} = \tfrac{1}{2} l p - \tfrac{1}{2} l q + \tfrac{1}{2} l \frac{1+q}{1-q} - \frac{1}{\sqrt{2}} l \frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$
$$= l\left(\frac{1+q}{2q}\right) - \frac{1}{\sqrt{2}} l \frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}.$$

Jam

$$p - u = \frac{(1+q)(1-2q-qq)}{2q} = + \frac{(1+q)[2-(1+q)^2]}{2q},$$

sicque habetur

$$l x = C - l(1+q) + l q - l[2-(1+q)^2] + l\left(\frac{1+q}{q}\right)$$
$$- \frac{1}{\sqrt{2}} l \frac{\sqrt{2}+1+q}{\sqrt{2}-1-q} = l a - l[2-(1+q)^2] + \frac{1}{\sqrt{2}} l \frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$

ubi est $u = \frac{y}{x} = \frac{1}{2}(1+q)^2$, et $1+q = \sqrt{\frac{2y}{x}}$, unde

$$x = \frac{ax}{x-y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}} \text{ seu } x - y = a\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}, \text{ vel}$$
$$(\sqrt{x}+\sqrt{y})^{1+\frac{1}{\sqrt{2}}} = a(\sqrt{x}-\sqrt{y})^{\frac{1}{\sqrt{2}}-1}.$$

Est ergo aequatio inter x et y interscendens, uti vocari solet.

Scholion.

680. Facilius haec resolutio absolvitur quaerendo statim ex aequatione

$$u + p = \sqrt{2u(1+pp)}, \text{ seu } uu + 2up + pp = 2u + 2upp$$

valorem ipsius p, qui fit

$$p = \frac{u + \sqrt{(uu - 4uu + 2u + 2u^3 - uu)}}{2u-1}, \text{ seu } p = \frac{u + (1-u)\sqrt{2u}}{2u-1}, \text{ et}$$
$$p - u = \frac{(1-u)(2u + \sqrt{2u})}{2u-1} = \frac{(1-u)\sqrt{2u}}{\sqrt{2u}-1}.$$

Quare

$$l x = \int \frac{\partial u}{p-u} = \int \frac{\partial u(\sqrt{2u}-1)}{(1-u)\sqrt{2u}} = C - l(1-u) - \int \frac{\partial u}{(1-u)\sqrt{2u}}.$$

Sit u [illegible] $v v$, eritque

$$\int \frac{\partial u}{(1-u)\sqrt{2u}} = \frac{1}{\sqrt{2}} \int \frac{2\partial v}{1-vv} = \frac{1}{\sqrt{2}} l \frac{1+v}{1-v},$$

hincque

$$lx = la - l(1 - u) - \frac{1}{\sqrt{2}}\, l\, \frac{1+\sqrt{u}}{1-\sqrt{u}}.$$

Unde ob $u = \frac{y}{x}$, reperitur $x = \frac{ax}{x-y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}$, ut ante. Quare si curva desideretur coordinatis rectangulis x et y determinanda, ut ejus arcus s sit $= \sqrt{2xy}$, erit aequatio ejus naturam definiens

$$(\sqrt{x} + \sqrt{y})^{\frac{1}{\sqrt{2}}+1} = a(\sqrt{x} - \sqrt{y})^{\frac{1}{\sqrt{2}}-1}.$$

Caeterum evidens est simili modo quaestionem resolvi posse, si arcus s functioni cuicunque homogeneae unius dimensionis ipsarum x et y aequetur, seu si proponatur aequatio quaecunque homogenea inter x, y et s, id quod sequenti problemate ostendisse operae erit pretium.

Problema 90.

687. Si fuerit $s = \int \sqrt{(\partial x^2 + \partial y^2)}$, atque aequatio proponatur homogenea quaecunque inter x, y et s, in qua scilicet hae tres variabiles x, y et s, ubique eundem dimensionum numerum constituant, invenire aequationem finitam inter x et y.

Solutio.

Ponatur $y = ux$ et $s = vx$, ut hac substitutione ex aequatione homogenea proposita variabilis x elidatur, et aequatio obtineatur inter binas u et v, unde v per u definiri possit. Tum vero sit $\partial y = p\,\partial x$, eritque

$\partial s = \partial x \sqrt{(1 + pp)}$, unde fit

$p\partial x = u\partial x + x\partial u$, et $\partial x\sqrt{(1+pp)} = v\partial x + x\partial v$,

ergo

$$\frac{\partial x}{x} = \frac{\partial u}{p-u} = \frac{\partial v}{\sqrt{(1+pp)} - v}.$$

Quia nunc v datur per u, sit $\partial v = q\,\partial u$, ut habeatur

$$\sqrt{(1+pp)} = v + pq - qu,$$

et sumtis quadratis

$$1 + pp = (v - qu)^2 + 2pq(v - qu) + ppqq,$$

unde elicitur

$$p = \frac{q(v-qu) + \sqrt{[(v-qu)^2 - 1 + qq]}}{1-qq} \text{ et}$$

$$p - u = \frac{qv - u + \sqrt{[(v-qu)^2 - 1 + qq]}}{1-qq}.$$

Quare hinc deducimus

$$\frac{\partial x}{x} = \frac{\partial u(1-qq)}{qv - u + \sqrt{[(v-qu)^2 - 1 + qq]}} = \frac{\partial u(qv - u - \sqrt{[(v-qu)^2 - 1 + qq]})}{1 + uu - vv},$$

unde cum v et q dentur per u, inveniri potest x per eandem u: at ob $q\partial u = \partial v$ fiet

$$lx = la - l\sqrt{(1 + uu - vv)} - \int \frac{\partial u \sqrt{[(v-qu)^2 - 1 + qq]}}{1 + uu - vv},$$

tum vero est $y = ux$, seu posito $\frac{y}{x}$ loco u habebitur aequatio quaesita inter x et y.

Corollarium 1.

688. Cum s exprimat arcum curvae coordinatis rectangulis x et y respondentem, sic definitur curva, cujus arcus aequatur functioni cuicunque unius dimensionis ipsarum x et y; quae ergo erit algebraica, si integrale

$$\int \frac{\partial u \sqrt{[(v-qu)^2 - 1 + qq]}}{1 + uu - vv}$$

per logarithmos exhiberi potest.

Corollarium 2.

689. Simili modo resolvi poterit problema, si s ejusmodi formulam integralem exprimat, ut sit $\partial s = Q\partial x$, existente Q functione quacunque quantitatum p, u et v. Tum autem ex aequalitate $\frac{\partial x}{x} = \frac{\partial u}{p-u} = \frac{\partial v}{Q-v}$ valorem ipsius p elici oportet, et quia v per u datur, erit $lx = \int \frac{\partial u}{p-u}$.

Exemplum 1.

690. Si debeat esse $s = \alpha x + \beta y$, erit $v = \alpha + \beta u$, et $q = \frac{\partial v}{\partial u} = \beta$, hinc $v - qu = \alpha$, ergo

$$lx = la - l\sqrt{[1 + uu - (\alpha + \beta u)^2]} - \int \frac{\partial u \sqrt{(\alpha\alpha + \beta\beta - 1)}}{1 + uu - (\alpha + \beta u)^2},$$

quae postrema pars est

$$-\int \frac{\partial u \sqrt{(\alpha\alpha - \beta\beta - 1)}}{1 - \alpha\alpha - 2\alpha\beta u + (1 - \beta\beta)uu} = (\alpha\alpha + \beta\beta - 1)^{\frac{1}{2}} \int \frac{\partial u}{\alpha\alpha + 1 - 2\alpha\beta u + (\beta\beta - 1)uu},$$

quae transformatur in

$$\int \frac{(\beta\beta - 1)\,\partial u \sqrt{(\alpha\alpha + \beta\beta - 1)}}{[u(\beta\beta - 1) + \alpha\beta - \sqrt{(\alpha\alpha + \beta\beta - 1)}][u(\beta\beta - 1) + \alpha\beta + \sqrt{(\alpha\alpha + \beta\beta - 1)}]}$$

$$= \tfrac{1}{2} l \frac{(\beta\beta - 1)u + \alpha\beta - \sqrt{(\alpha\alpha + \beta\beta - 1)}}{(\beta\beta - 1)u + \alpha\beta + \sqrt{(\alpha\alpha + \beta\beta - 1)}}.$$

Quare posito $u = \frac{y}{x}$, aequatio integralis quaesita est, sumtis quadratis,

$$\frac{xx + yy - (\alpha x + \beta y)^2}{aa} = \frac{(\beta\beta - 1)y + \alpha\beta x - x\sqrt{(\alpha\alpha + \beta\beta - 1)}}{(\beta\beta - 1)y + \alpha\beta x + x\sqrt{(\alpha\alpha + \beta\beta - 1)}}.$$

At posito

$$(\beta\beta - 1)y + \alpha\beta x - x\sqrt{(\alpha\alpha + \beta\beta - 1)} = P$$

$$(\beta\beta - 1)y + \alpha\beta x + x\sqrt{(\alpha\alpha + \beta\beta - 1)} = Q$$

est

$$PQ = (\beta\beta - 1)^2 yy + 2\alpha\beta(\beta\beta - 1)xy + (\alpha\alpha - 1)(\beta\beta - 1)xx$$
$$= (\beta\beta - 1)[(\alpha x + \beta y)^2 - xx - yy],$$

unde mutata constante fit $\frac{PQ}{bb} = \frac{P}{Q}$, ergo vel $P = 0$ vel $Q = b$; solutio ergo in genere est

$$(\beta\beta - 1)y + \alpha\beta x \pm x\sqrt{(\alpha\alpha + \beta\beta - 1)} = c,$$

quae est aequatio pro linea recta.

Exemplum 2.

691. Si debeat esse $s = \frac{nyy}{x}$, erit $v = nuu$ et $q = 2nu$; unde $1 + uu - vv = 1 + uu - nnu^4$ et $v - qu = -nuu$, ergo

$$lx = la - l\sqrt{(1 + uu - nnu^4)} - \int \frac{\partial u \sqrt{(nnu^4 - 1 + 4nnuu)}}{1 + uu - nnu^4}$$

quae formula autem per logarithmos integrari nequit.

Exemplum 3.

692. Si debeat esse $ss = xx + yy$, erit $v = \sqrt{(1 + uu)}$ et $q = \frac{u}{\sqrt{(1 + uu)}}$, unde fit $1 + uu - vv = 0$, solutionem ergo ex primis formulis repeti convenit, unde fit

$$v - qu = \frac{1}{\sqrt{(1 + uu)}},$$

$$qq - 1 = \frac{-1}{1 - uu}, \text{ et } qv - u = 0;$$

ergo $p - u = 0$, seu $\frac{\partial y}{\partial x} - \frac{y}{x} = 0$, ita ut prodeat $y = nx$.

Exemplum 4.

693. Si debeat esse $ss = yy + nxx$, seu $v = \sqrt{(uu + n)}$, et $q = \frac{u}{\sqrt{(uu + n)}}$, erit $1 + uu - vv = 1 - n$, $v - qu = \frac{n}{\sqrt{(uu + n)}}$ et $qq - 1 = \frac{-n}{uu + n}$. Quare habebitur

$$lx = la - l\sqrt{(1 - n)} - \frac{1}{1 - n}\int \frac{\partial u \sqrt{(nn + n)}}{\sqrt{(uu + n)}}$$

$$= lb + \frac{\sqrt{n}}{\sqrt{(n - 1)}}\, l[u + \sqrt{(uu + n)}],$$

hincque

$$\frac{x}{b} = \left(\frac{y + \sqrt{(yy + nxx)}}{x}\right)^{\sqrt{\frac{n - 1}{n}}}.$$

Quoties ergo $\frac{n}{n - 1}$ est numerus quadratus, aequatio inter x et y prodit algebraica. Sit $\sqrt{\frac{n}{n - 1}} = m$, erit $n = \frac{mm}{mm - 1}$ et $ss = yy + \frac{mmxx}{mm - 1}$, cui conditioni satisfit hac aequatione algebraica

$$x^{m + 1} = b\left[y + \sqrt{\left(yy + \frac{mmxx}{mm - 1}\right)}\right]^m$$

quae tranformatur in

$$x^{\frac{2}{m}} - 2b^{\frac{1}{m}} x^{\frac{1 - m}{m}} y = \frac{mm}{mm - 1} b^{\frac{2}{m}}, \text{ seu}$$

$$y = \frac{(mm-1)x^{\frac{2}{m}} - mmb^{\frac{2}{m}}}{2(mm-1)b^{\frac{2}{m}}x^{\frac{1-m}{m}}}$$

Corollarium.

694. Ponamus $m = \frac{1}{n}$, ac si fuerit

$$y = \frac{b^{2n} + (nn-1)x^{2n}}{2(nn-1)b^n x^{n-1}}, \text{ erit}$$

$$ss = yy - \frac{xx}{nn-1}, \text{ seu } s = \sqrt{\left(yy - \frac{xx}{nn-1}\right)}.$$

Quare si

$$y = \frac{b^4 + 3x^4}{6bbx}, \text{ est } s = \sqrt{\left(yy - \frac{xx}{3}\right)}.$$

Problema 91.

695. Si posito $\frac{\partial y}{\partial x} = p$, ejusmodi detur aequatio inter x, y et p, in qua altera variabilis y unicam tantum habeat dimensionem, invenire relationem inter binas variabiles x et y.

Solutio.

Hinc ergo y aequabitur functioni cuipiam ipsarum x et p, unde differentiando fiet $\partial y = P\partial x + Q\partial p$. Cum igitur sit $\partial y = p\partial x$, habebitur haec aequatio differentialis $(P-p)\partial x + Q\partial p = 0$, quam integrari oportet. Quoniam tantum duas continet variabiles x et p, et differentialia simpliciter involvit, ejus resolutio per methodos supra expositas est tentanda.

Primo ergo resolutio succedet, si fuerit $P = p$, ideoque $\partial y = p\partial x + Q\partial p$. Quod evenit, si y per x et p ita determinetur, ut sit $y = px + \Pi$, denotante Π functionem quamcunque ipsius p. Tum ergo erit $Q = x + \frac{\partial \Pi}{\partial p}$, et cum solutio ab ista aequatione $Q\,\partial p = 0$ pendeat, erit vel $\partial p = 0$, hineque $p = \alpha$, seu $y = \alpha x + \beta$, ubi altera constantium α et β per ipsam aequationem propositam determinatur, dum posito $p = \alpha$ fit $\beta = \Pi$; vel erit $Q = 0$, ideoque $x = -\frac{\partial \Pi}{\partial p}$, et $y = -\frac{p\,\partial \Pi}{\partial p} + \Pi$, ubi ergo utraque solutio est algebraica, si modo Π fuerit functio algebraica ipsius p.

Secundo, aequatio $(P - p)\partial x + Q\partial p = 0$, resolutionem admittet, si altera variabilis x cum suo differentiali ∂x unam dimensionem non superet. Evenit hoc si fuerit $y = Px + \Pi$, dum P et Π sunt functiones ipsius p tantum, tum enim erit $P = P$ et $Q = \frac{x\,\partial P}{\partial p} + \frac{\partial \Pi}{\partial p}$, hincque haec habeatur aequatio integranda

$$(P - p)\partial x + x\partial P + \partial \Pi = 0 \text{ seu } \partial x + \frac{x\,\partial P}{P - p} = -\frac{\partial \Pi}{P - p},$$

quae per $e^{\int \frac{\partial p}{P - p}}$ multiplicata dat

$$e^{\int \frac{\partial p}{P - p}} x = -\int e^{\int \frac{\partial P}{P - p}} \frac{\partial \Pi}{P - p}.$$

Sive ponatur $\frac{\partial P}{P - p} = \frac{\partial R}{R}$, erit aequatio integralis

$$R x = C - \int \frac{R\,\partial \Pi}{P - p} = C - \int \frac{\partial \Pi\,\partial R}{\partial P};$$

unde fit

$$x = \frac{C}{R} - \frac{1}{R}\int \frac{\partial \Pi\,\partial R}{\partial P}, \text{ et}$$

$$y = \frac{CP}{R} + \Pi - \frac{P}{R}\int \frac{\partial \Pi\,\partial R}{\partial P}.$$

Tertio resolutio nullam habebit difficultatem, si denotantibus X et V functiones quascunque ipsius x, fuerit $y = X + Vp$. Tum enim erit

$$\partial y = p\,\partial x = \partial X + V\,\partial p + p\,\partial V,$$

ideoque

$$\partial p + p\left(\frac{\partial V - \partial x}{V}\right) = -\frac{\partial X}{V},$$

sit $\frac{\partial x}{V} = \frac{\partial R}{R}$, ut R sit etiam functio ipsius x, erit

$$\frac{V}{R}p = C - \int \frac{\partial X}{R}, \text{ seu } p = \frac{CR}{V} - \frac{R}{V}\int \frac{\partial X}{R}, \text{ et}$$

$$y = X + CR - R\int\frac{\partial X}{R},$$

quae aequatio relationem inter x et y exprimit.

Quarto aequatio $(P - p)\partial x + Q\partial p = 0$ resolutionem admittit si fuerit homogenea. Cum ergo terminus $p\partial x$ duas contineat dimensiones, hoc evenit, si totidem dimensiones et in reliquis terminis insint. Unde perspicuum est, P et Q esse debere functiones homogeneas unius dimensionis ipsarum x et p. Quare si y ita per x et p definiatur, ut y aequetur functioni homogeneae duarum dimensionum ipsarum x et p, resolutio succedet. Quodsi enim fuerit $\partial y = P\partial x + Q\partial p$, aequatio solutionem continens $(P - p)\partial x + Q\partial p = 0$, erit homogenea, fietque per se integrabilis, si dividatur per $(P - p)x + Qp$.

Corollarium 1.

696. Pro casu quarto si ponatur $y = zz$, aequatio proposita debet esse homogenea inter tres variabiles x, z et p. Unde si proponatur aequatio homogenea quaecunque inter x, z et p, in qua hae ternae litterae x, z et p ubique eundem dimensionum numerum constituant, problema semper resolutionem admittit.

Corollarium 2.

697. Simili modo conversis variabilibus, si ponatur $x = vv$ et $\frac{\partial x}{\partial y} = q$, ut sit $p = \frac{1}{q}$; ac proponatur aequatio homogenea quaecunque inter y, v et q, problema itidem resolvi potest.

Scholion.

698. Pro casu quarto, ut aequatio $(P - p)\partial x + Q\partial p = 0$ fiat homogenea, conditiones magis amplificari possunt. Ponatur enim $x = v^{\mu}$ et $p = q^{\nu}$, sitque facta substitutione haec aequatio

$$\mu(P - q^{\nu})v^{\mu-1}\partial u + \nu Q q^{\nu-1}\partial q = 0.$$

homogenea inter v et q, eritque P functio homogenea ν dimensionum, et Q functio homogenea μ dimensionum. Cum jam sit

$$\partial y = P\,\partial x + Q\,\partial p = \mu\,P\,v^{\mu-1}\,\partial v + \nu\,Q\,q^{\nu-1}\,\partial q,$$

erit y functio homogenea $\mu + \nu$ dimensionum. Quare posito $y = z^{\mu+\nu}$ problema resolutionem admittit, si inter x, y et p ejusmodi relatio proponatur, ut positio $y = z^{\mu+\nu}$, $x = v^{\mu}$ et $p = q^{\nu}$ habeatur aequatio homogenea inter ternas quantitates z, v et q, ita ut dimensionum ab iis formatarum numerus ubique sit idem. Ac si proposita fuerit hujusmodi aequatio homogenea inter z, v et q, solutio problematis ita expedietur. Cum sit $\partial y = p\,\partial x$, erit

$$(\mu + \nu)\,z^{\mu+\nu-1}\,\partial z = \mu\,v^{\mu-1}\,q^{\nu}\,\partial v;$$

ponatur jam $z = r\,q$ et $v = s\,q$, et aequatio proposita tantum duas litteras r et s continebit, ex qua alteram per alteram definire licet, tum autem per has substitutiones prodibit haec aequatio

$$(\mu + \nu)\,r^{\mu+\nu-1}\,q^{\mu+\nu-1}\,(r\,\partial q + q\,\partial r) = \mu\,s^{\mu-1}\,q^{\mu+\nu-1}\,(s\,\partial q + q\,\partial s),$$

ex qua oritur

$$\frac{\partial q}{q} = \frac{\mu\,s^{\mu-1}\,\partial s - (\mu + \nu)\,r^{\mu+\nu-1}\,\partial r}{(\mu + \nu)\,r^{\mu+\nu} - \mu\,s^{\mu}},$$

quae est aequatio differentialis separata, quoniam s per r datur. Quin etiam bini casus allati manifesto continentur in formulis $y = z^{\mu+\nu}$, $x = v^{\mu}$ et $p = q^{\nu}$; prior scilicet si $\mu = 1$ et $\nu = 1$, posterior vero si $\mu = 2$ et $\nu = -1$. Hos igitur casus perinde ac praecedentes exemplis illustrari conveniet, quorum primus praecipue est memorabilis, cum per differentiationem aequationis propositae $y = px + \Pi$ statim praebeat aequationem integralem quaesitam, neque integratione omnino sit opus, siquidem alteram solutionem ex $\partial p = 0$ natam excludamus.

Exemplum 1.

699. *Proposita aequatione differentiali*

$$y\,\partial x - x\,\partial y = a\sqrt{(\partial x^2 + \partial y^2)}$$

ejus integrale invenire.

Posito $\frac{\partial y}{\partial x} = p$ fit $y - px = a\sqrt{(1 + pp)}$, quae aequatio differentiata, ob $\partial y = p\,\partial x$, dat $-x\,\partial p = \frac{ap\,\partial p}{\sqrt{(1+pp)}}$, quae cum sit divisibilis per ∂p praebet primo $p = \alpha$, hincque $y = \alpha x + a\sqrt{(1 + \alpha\alpha)}$. Alter vero factor suppeditat $x = \frac{-ap}{\sqrt{(1+pp)}}$, hincque

$$y = \frac{-app}{\sqrt{(1+pp)}} + a\sqrt{(1 + pp)} = \frac{a}{\sqrt{(1+pp)}},$$

unde fit $xx + yy = aa$, quae est etiam aequatio integralis, sed quia novam constantem non involvit, non pro completo integrali haberi potest. Integrale autem completum duas aequationes complectitur. Scilicet

$$y = \alpha x + a\sqrt{(1 + \alpha\alpha)} \text{ et } xx + yy = aa,$$

quae in hac una comprehendi possunt

$$[(y - \alpha x)^2 - aa(1 + \alpha\alpha)](xx + yy - aa) = 0.$$

Scholion.

700. Nisi hoc modo operatio instituatur, solutio hujus quaestionis fit satis difficilis. Si enim aequationem differentialem $y\partial x - x\partial y = a\sqrt{(\partial x^2 + \partial y^2)}$ quadrando ab irrationalitate liberemus, indeque rationem $\frac{\partial y}{\partial x}$ per radicis extractionem definiamus, fit

$$(xx - aa)\partial y - xy\partial x = \pm a\partial x\sqrt{(xx + yy - aa)}$$

quae aequatio per methodos cognitas difficulter tractatur. Multiplicator quidem inveniri potest utrumque membrum per se integrabile reddens; prius enim membrum $(xx - aa)\partial y - xy\partial x$ divisum per $y(xx - aa)$ fit integrabile, integrali existente $= l\frac{y}{\sqrt{(xx - aa)}}$: unde in genere multiplicator id integrabile reddens est

$$\frac{1}{y(xx-aa)}\Phi : \frac{y}{\sqrt{(xx-aa)}}$$

quae functio ita determinari debet, ut eodem multiplicatore quoque alterum membrum $a\partial x\sqrt{(xx + yy - aa)}$ fiat integrabile. Talis autem multiplicator est:

$$\frac{1}{y(xx-aa)}\cdot\frac{y}{\sqrt{(xx+yy-aa)}}=\frac{1}{(xx-aa)\sqrt{(xx+yy-aa)}}$$

quo fit

$$\frac{(xx-aa)\partial y-xy\partial x}{(xx-aa)\sqrt{(xx+yy-aa)}}=\frac{\pm a\partial x}{xx-aa}.$$

Jam ad integrale prioris membri investigandum, spectetur x ut constans, eritque integrale

$$=l[y+\sqrt{(xx+yy-aa)}]+X,$$

denotante X functionem quampiam ipsius x, ita comparatam, ut sumta jam y constante fiat

$$\frac{x\partial x}{[y+\sqrt{(xx+yy-aa)}]\sqrt{(xx+yy-aa)}}+\partial X=\frac{-xy\partial x}{(xx-aa)\sqrt{(xx+yy-aa)}}$$

seu

$$\frac{-x\partial x[y-\sqrt{(xx+yy-aa)}]}{(xx-aa)\sqrt{(xx+yy-aa)}}+\partial X=\frac{-xy\partial x}{(xx-aa)\sqrt{(xx+yy-aa)}},$$

unde fit

$$\partial X=\frac{-x\partial x}{xx-aa} \text{ et } X=l\frac{C}{\sqrt{(xx-aa)}}.$$

Quare integrale quaesitum est

$$l[y+\sqrt{(xx+yy-aa)}]+l\frac{C}{\sqrt{(xx-aa)}}=\pm\tfrac{1}{2}l\frac{a+x}{a-x},$$

unde fit

$$y+\sqrt{(xx+yy-aa)}=\alpha(x\pm a), \text{ hincque}$$
$$xx-aa=\alpha\alpha(x\pm a)^2-2\alpha(x\pm a)y, \text{ vel}$$
$$x\mp a=\alpha\alpha(x\pm a)-2\alpha y$$

quae autem tantum est altera binarum aequationum integralium, altera autem aequatio integralis $xx+yy=aa$ jam quasi per divisionem de calculo sublata est censenda. Caeterum eadem solutio aequationis

$$(aa-xx)\partial y+xy\partial x=\pm a\partial x\sqrt{(xx+yy-aa)}$$

facilius instituitur ponendo $y=u\sqrt{(aa-xx)}$, unde fit

$$(aa-xx)^{\frac{3}{2}}\partial u=\pm a\partial x\sqrt{(aa-xx)(uu-1)} \text{ seu}$$
$$\frac{\partial u}{\sqrt{(uu-1)}}=\frac{\pm a\partial x}{aa-xx},$$

cui quidem satisfit sumendo $u=1$, neque tamen hic casus in aequatione integrali continetur, uti supra jam ostendimus. Ex quo suspicari liceret alteram solutionem $xx + yy = aa$ adeo esse excludendam, quod tamen secus se habere deprehenditur; si ipsam aequationem primariam $\frac{y\partial x - x\partial y}{\sqrt{(\partial x^2 + \partial y^2)}} = a$ perpendamus. Si enim x et y sint coordinatae rectangulae lineae curvae, formula $\frac{y\partial x - x\partial y}{\sqrt{(\partial x^2 + \partial y^2)}}$ exprimit perpendiculum ex origine coordinatarum in tangentem dimissum, quod ergo constans esse debet. Hoc autem evenire in circulo, origine in centro constituta, dum aequatio fit $xx + yy = aa$, per se est manifestum. Atque hinc realitas harum solutionum, quae minus congruae videri poterant, confirmatur, etiamsi earum ratio haud satis clare perspicitur.

Exemplum 2.

701. *Proposita aequatione differentiali*
$$y\partial x - x\partial y = \frac{a(\partial x^2 + \partial y^2)}{\partial x}$$
ejus integrale invenire.

Posito $\partial y = p\partial x$, fit $y - px = a(1 + pp)$, et differentiando $-x\partial p = 2ap\partial p$; unde concluditur vel $\partial p = 0$, et $p = \alpha$, hincque $y = \alpha x + a(1 + \alpha\alpha)$, vel $x = -2ap$ et $y = a(1 - pp)$, sicque, ob $p = \frac{-x}{2a}$, habebitur $4ay = 4aa - xx$, quae aequatio ad geometriam translata illam conditionem omnino adimplet.

Ex aequatione autem proposita radicem extrahendo reperitur
$$2a\partial y + x\partial x = \partial x\sqrt{(xx + 4ay - 4aa)},$$
quae posito $y = u(4aa - xx)$, abit in
$$2a\partial u(4aa - xx) - x\partial x(4au - 1)$$
$$= \partial x\sqrt{(4aa - xx)(4au - 1)},$$
haecque posito $4au - 1 = tt$, in
$$t\partial t(4aa - xx) - ttx\partial x = t\partial x\sqrt{(4aa - xx)},$$
quae cum sit divisibilis per t, concludere licet $t = 0$, ideoque $u = \frac{1}{4a}$, atque hinc $4ay = 4aa - xx$.

**

Exemplum 3.

702. *Proposita aequatione differentiali*

$$y\,\partial x - x\,\partial y = a\sqrt[3]{(\partial x^3 + \partial y^3)},$$

ejus integrale assignare.

Haec aequatio more consueto, si rationem $\frac{\partial y}{\partial x}$ inde extrahere vellemus, vix tractari posset. Posito autem $\partial y = p\,\partial x$ fit $y - px = a\sqrt[3]{(1+p^3)}$, et differentiando $x\,\partial p = \frac{-a\,p\,p\,\partial p}{\sqrt[3]{(1+p^3)^2}}$, unde duplex conclusio deducitur, vel $\partial p = 0$ et $p = \alpha$, sicque $y = \alpha x + a\sqrt[3]{(1+\alpha^3)}$, vel

$$x = \frac{-a\,p\,p}{\sqrt[3]{(1+p^3)^2}} \text{ et } y = \frac{a}{\sqrt[3]{(1+p^3)^2}},$$

unde fit $pp = -\frac{x}{y}$, et ob $y^3(1+p^3)^2 = a^3$, erit $p^3 = \frac{a\sqrt{a}}{y\sqrt{y}} - 1$, hincque $\frac{(a\sqrt{a} - y\sqrt{y})^2}{y^3} = -\frac{x^3}{y^3}$, seu $x^3 + (a\sqrt{a} - y\sqrt{y})^2 = 0$.

Exemplum 4.

703. *Proposita aequatione differentiali*

$$y\,\partial x - n\,x\,\partial y = a\sqrt{(\partial x^2 + \partial y^2)},$$

ejus integrale invenire.

Posito $\partial y = p\,\partial x$, habetur $y - npx = a\sqrt{(1+pp)}$, unde differentiando elicitur

$$(1-n)p\,\partial x - n\,x\,\partial p = \frac{a\,p\,\partial p}{\sqrt{(1+pp)}}, \text{ sive}$$

$$\partial x - \frac{n\,x\,\partial p}{(1-n)\,p} = \frac{a\,\partial p}{(1-n)\sqrt{(1+pp)}},$$

quae per $p^{\frac{n}{n-1}}$ multiplicata et integrata praebet

$$p^{\frac{n}{n-1}}x = \frac{a}{1-n}\int\frac{p^{\frac{n}{n-1}}\partial p}{\sqrt{(1+pp)}}.$$

Hinc deducimus casus sequentes, integrationem admittentes

si $n = \frac{3}{2}$; $p^3 x = C - \frac{2}{3} a (pp - \frac{2}{1}) \sqrt{(1 + pp)}$,
si $n = \frac{5}{4}$; $p^5 x = C - \frac{4}{5} a (p^3 - \frac{4}{3} p^3 + \frac{4.2}{3.1}) \sqrt{(1 + pp)}$,
si $n = \frac{7}{6}$; $p^7 x = C - \frac{6}{7} a (p^6 - \frac{6}{5} p^4 + \frac{6.4}{5.3} p^2 - \frac{6.4.2}{5.3.1}) \sqrt{(1 + pp)}$,
ac si $n = \frac{2\lambda + 1}{2\lambda}$, erit $y = px + a\sqrt{(1 + pp)} + \frac{px}{2\lambda}$, et

$$x = \frac{C}{p^{2\lambda+1}} -$$
$$\frac{2\lambda a}{(2\lambda+1)p}\left(1 - \frac{2\lambda}{(2\lambda-1)pp} + \frac{2\lambda(2\lambda-2)}{(2\lambda-1)(2\lambda-3)p^4} - \text{etc.}\right)\sqrt{(1+pp)}.$$

Quodsi ergo sumatur $\lambda = \infty$, ut sit $n = 1$, erit

$$y = px + a\sqrt{(1 + pp)}, \text{ et } x = \frac{C}{p^{2\lambda+1}} - \frac{ap}{\sqrt{(1 + pp)}},$$

unde si constans C sit $= 0$, statim sequitur solutio superior $xx + yy = aa$. At si constans C non evanescat, minimum discrimen in quantitate p infinitam varietatem ipsi x inducit. Quantumvis ergo x varietur, quantitas p ut constans spectari potest, unde posito $p = \alpha$, altera solutio $y = \alpha x + a\sqrt{(1 + \alpha\alpha)}$ obtinetur. Hinc ergo dubium supra, circa exemplum 1. natum, non mediocriter illustratur.

Exemplum 5.

704. *Proposita aequatione differentiali*

$$A \partial y^n = (B x^\alpha + C y^\beta) \partial x^n$$

existente $n = \frac{\alpha\beta}{\alpha - \beta}$, *ejus integrale investigare.*

Posito $\frac{\partial y}{\partial x} = p$ erit $A p^n = B x^\alpha + C y^\beta$. Ponamus jam $p = q^{\alpha\beta}$, $x = v^{\beta n}$ et $y = z^{\alpha n}$, ut habeamus hanc aequationem homogeneam $A q^{\alpha\beta n} = B v^{\alpha\beta n} + C z^{\alpha\beta n}$, quae positis $z = rq$ et $v = sq$, abit in $A = B s^{\alpha\beta n} + C r^{\alpha\beta n}$. Cum vero sit

$$\partial y = \alpha n z^{\alpha n - 1} \partial z = \alpha n r^{\alpha n - 1} q^{\alpha n - 1}(r\partial q + q\partial r) \text{ et}$$
$$p\partial x = \beta n v^{\beta n - 1} q^{\alpha\beta} \partial v = \beta n s^{\beta n - 1} q^{\alpha\beta + \beta n - 1}(s\partial q + q\partial s),$$

erit

$$\alpha r^{\alpha n - 1}(r\partial q + q\partial r) = \beta s^{\beta n - 1} q^{\alpha\beta + \beta n - \alpha n}(s\partial q + q\partial s).$$

Est vero per hypothesin $\alpha\beta+\beta n-\alpha n=0$, unde oritur

$$\alpha r^{\alpha n}\partial q+\alpha r^{\alpha n-1}q\partial r=\beta s^{\beta n}\partial q+\beta s^{\beta n-1}q\partial s,$$

hincque

$$\frac{\partial q}{q}=\frac{\alpha r^{\alpha n-1}\partial r-\beta s^{\beta n-1}\partial s}{\beta s^{\beta n}-\alpha r^{\alpha n}}.$$

At est

$$s^{\beta n}=\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}},\text{ hincque}$$

$$\beta s^{\beta n-1}\partial s=-\frac{\beta C}{B}r^{\alpha\beta n-1}\partial r\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}},$$

unde fit

$$\frac{\partial q}{q}=\frac{\alpha r^{\alpha n-1}\partial r+\frac{\beta C}{B}r^{\alpha\beta n-1}\partial r\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}}}{\beta\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}}-\alpha r^{\alpha n}}.$$

Facilius autem calculus hoc modo instituetur; sumto $A=1$, erit

$$p=\frac{\partial y}{\partial x}=(B x^{\alpha}+C y^{\beta})^{\frac{1}{n}},$$

sit $y=x^{\frac{\alpha}{\beta}}u$, fiet

$$x^{\frac{\alpha}{\beta}}\partial u+\frac{\alpha}{\beta}x^{\frac{\alpha-\beta}{\beta}}u\partial x=x^{\frac{\alpha}{n}}\partial x(B+C u^{\beta})^{\frac{1}{n}},$$

quae aequatio, cum sit $\frac{\alpha}{n}=\frac{\alpha-\beta}{\beta}$, abit in hanc

$$\beta x\partial u+\alpha u\partial x=\beta\partial x(B+C u^{\beta})^{\frac{1}{n}},$$

unde fit

$$\frac{\partial x}{x}=\frac{\beta\partial u}{\beta(B+C u^{\beta})^{\frac{1}{n}}-\alpha u},$$

sicque x per u determinatur, et quia $u = x^{-\frac{\alpha}{\beta}} y$, habebitur aequatio inter x et y.

Scholion.

705. Hoc igitur modo operationem institui conveniet, quando inter binas variabiles x et y una cum differentialium ratione $\frac{\partial y}{\partial x} = p$, ejusmodi relatio proponitur, ex qua valor ipsius p commode elici non potest. Tum ergo calculum ita tractari oportet, ut per differentiationem ponendo $\partial y = p \partial x$ vel $\partial x = \frac{\partial y}{p}$, tandem perveniatur ad aequationem differentialem simplicem inter duas tantum variabiles, quem in finem etiam saepe idoneis substitutionibus uti necesse est. Atque hucusque fere Geometris in resolutione aequationum differentialium primi gradus etiamnum pertingere licuit, vix enim ulla via integralia investigandi adhuc quidem adhibita hic praetermissa videtur. Num autem multo majorem calculi integralis promotionem sperare liceat? vix equidem affirmaverim, cum plurima extent inventa, quae ante vires ingenii humani superare videbantur.

Cum igitur calculum integralem in duos libros sim partitus, quorum prior circa relationem binarum tantum variabilium, posterior vero ternarum pluriumve versatur, atque jam libri primi partem priorem in differentialibus primi ordinis constitutam hic pro viribus exposuerim, ad ejus alteram partem progredior, in qua binarum variabilium relatio ex data differentialium secundi altiorisve ordinis conditione requiritur.

Corrigenda.

pag.	*lin.*	*loco:*	*lege.*
48	7 asc.	$\sqrt{\frac{f+gx}{a-bx}}$	$\sqrt{\frac{f+gx}{a+bx}}$
81	9	$3-4xx+x^4$	$1-4xx+x^4$
104	3 asc.	$E=$	$F=$
119	*ultima*	(§. 227)	(§. 228)
179	*ultima*	A', A'	A', A''
180	8 asc.	a, a	a, a'
182	13	A', A'	A', A''
—	15	$a'-a'$	$a''-a'$
201	18	*in numeratore* $\alpha-\omega$	$\alpha-\omega$
205	9	*in numeratore* $z^{m+\nu}$	$z^{\mu+\nu}$
208	10	$=$	$=-$
208	*ultima*	$\frac{1.3.5.}{(m+1)(m+3)(m+5)}$;	$\frac{1.3.5.}{(m+1)(m+3)(m+5)}M$;
209	4 asc.	*in exponente* $\frac{3}{2}$	$\frac{5}{2}$
210	7	$\int\frac{\partial x}{\sqrt[3]{(1-x^3)^2}}=B'$	$\int\frac{\partial x}{\sqrt[3]{(1-x^3)^2}}=A'$
221	2 asc.	252.	352.
222	7	$\int\frac{x\,\partial x}{\sqrt[3]{(1+x^3)^2}}$	$\int\frac{x\,\partial x}{\sqrt[3]{(1-x^3)^2}}$
231	11	$\int=$	$=\int$
261	6	ratio $\frac{\partial y}{\partial x}$,	ratio $\frac{\partial y}{\partial x}$
272	8	concludimis	concludimus
—	9	admissuiram	admissuram
—	10	repertur	reperitur
304	11	*in denominatore* $1-zy$	$1-xy$